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Nonlinear Fourier Methods for Ocean Waves

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Abstract

Multiperiodic Fourier series solutions of *integrable nonlinear wave equations* are applied to the study of ocean waves for scientific and engineering purposes. These series can be used to compute analytical formulae for the *stochastic properties* of nonlinear equations, in analogy to the standard approach for linear equations. Here I emphasize analytically computable results for the *correlation functions*, *power spectra* and *coherence functions* of a *nonlinear random process* associated with an integrable nonlinear wave equation. The multiperiodic Fourier series have the advantage that the *coherent structures* of soliton physics are encoded in the formulation, so that *solitons*, *breathers*, *vortices*, etc. are contained in the *temporal evolution* of the nonlinear power spectrum and phases. I illustrate the method for the Korteweg-deVries and nonlinear Schrödinger equations. Applications of the method to the analysis of data are discussed.

Keywords: nonlinear integrable wave equations; finite gap theory; periodic inverse scattering transform; nonlinear Fourier analysis; nonlinear ocean waves; nonlinear numerical methods; nonlinear time series analysis.

1. Multiperiodic Fourier Series as Solutions of Nonlinear Integrable Wave Equations

I give an overview of *multiperiodic Fourier series solutions of integrable, nonlinear wave equations*. I discuss how these series can be used as practical tools for the study of nonlinear ocean waves in one and two dimensions. This perspective has arisen from the pure mathematical algebraic-geometric construction of *single valued, multiperiodic, meromorphic Fourier series* from *Riemann theta functions* [1-3, 23-24, 29, 32]. Modern interest in this area of research has occurred because recent developments have shown how theta functions can be used to solve *nonlinear integrable wave equations* [4, 24], a field known as *finite gap theory* (FGT) or the *periodic inverse scattering transform*. Many of these developments have been used for oceanographic

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applications [8, 28]. Here I address how multiperiodic, meromorphic Fourier series constructed from theta functions can themselves also be generically useful as mathematical and data analysis tools. The resultant formulation I refer to as *nonlinear Fourier analysis*, where the *Stokes wave* is a single degree of freedom component of the nonlinear Fourier theory, as opposed to the sine wave of linear Fourier series. *Coherent structures* such as solitons, breathers and vortices are all constructed naturally from the Stokes waves by increasing the nonlinearity and/or by phase locking two Stokes components. Nonlinear interactions amongst the Stokes waves are also formulated. The nonlinear Fourier methods are applicable to many aspects of the study of ocean waves from scientific, engineering, numerical, and data analysis perspectives. Sections 1 and 2 give some historical perspective, while Sections 3-6 discuss applications of physically relevant nonlinear wave equations.

Why do I take the path of periodicity/quasiperiodicity offered by FGT for solving the *spectral structure* of *nonlinear wave equations* describing ocean waves? Why would this approach lead to *multiperiodic Fourier series* for their description rather than ordinary *trigonometric Fourier series*? Here are a few of the reasons:

(1) To analyze *time series data* one most often assumes the data to be *periodic* and *discrete*. The fast Fourier transform (FFT), mathematically a discrete Fourier transform, is used for numerical computations.
(2) To develop a natural theory for the *nonlinear Fourier analysis of wave motion* founded on *Stokes wave basis functions* that *interact nonlinearly* with each other.
(3) To have *data analysis* and *analytical approaches* that include *coherent structures in the nonlinear Fourier analysis*. These include *Stokes waves, solitons, breathers and vortices*.
(4) To develop a full theory of *nonlinear random wave trains* for describing *stochastic ocean waves for nonlinear integrable systems*.
(5) To develop a method which naturally extends to *nonintegrability of perturbed (or higher order) nonlinear systems by finding ordinary differential equations that vary adiabatically in the Riemann spectrum of the solitons, breathers, etc*. This is because nearly integrable systems can often be treated with the slow time evolution of their FGT spectra.

An introduction to some of this material is already presented in Osborne (2010), but much of the work presented here is a new approach for *nonlinear*, *stochastic ocean waves*, in which the *correlation function* and *power spectrum*, together with other stochastic properties, can be computed analytically from the multiperiodic series.

Anticipating later results below, we consider the possibility that there exist *spectral solutions* of *integrable, nonlinear wave equations* that have the form of *multidimensional, quasiperiodic Fourier series*

$$u(x,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} \tag{1}$$

We will see that these are regarded, from the point of view of algebraic geometry, as the *most general, single valued, multiply periodic meromorphic functions of N variables with 2N periods (Baker-Mumford Theorem)*. Here the wavenumbers $\mathbf{k}$, the frequencies $\boldsymbol{\omega}$ and the phases $\boldsymbol{\phi}$ are vectors of dimension N (the genus). The summation index $\mathbf{n}$ is over the integer lattice $\mathbb{Z}^N$. Eq. (1) can be written as a nested summation:

$$u(x,t) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} u_{n_1,n_2\dots n_N} e^{i\sum_{i=1}^{N} n_i k_i x - i\sum_{i=1}^{N} n_i \omega_i t + i\sum_{i=1}^{N} n_i \phi_i} \tag{2}$$

I discuss below how series of this type can be constructed with the aid of the *Baker-Mumford Theorem* and from *finite gap theory using theta functions*. But first I consider a number of properties of these series.

1.1. Reduction of Multiperiodic Fourier Series to Ordinary Trigonometric Series

Assume the solution of an integrable, nonlinear partial differential equation is given by (1). For *spatially periodic boundary conditions* $u(x,t) = u(x+L,t)$ the multidimensional, quasiperiodic Fourier series (1) can be written as a trigonometric series with time varying coefficients [28]:

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(t) e^{ik_n x}, \qquad k_n = \frac{2\pi n}{L} \tag{3}$$

$$u_n(t) = \sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ k_n=\mathbf{n}\bullet\mathbf{k}\}} u_{\mathbf{n}} e^{-i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}}, \qquad \mathbf{n}\bullet\mathbf{k} = \sum_{i=1}^{N} n_i k_i \tag{4}$$

Thus the solution to a nonlinear integrable partial differential equation (1) is reduced to a *Fourier (trigonometric) series* (3) with *time varying coefficients* $u_n(t)$ represented by *quasiperiodic Fourier series* (4). The coefficients $u_n(t)$ are of course quasiperiodic in time due to the incommensurable frequencies. Traditionally $u_n(t)$ are computed by a set of ordinary differential equations obtained by inserting (3) into a particular *integrable* nonlinear wave equation. Eq. (4) solves these ordinary differential equations.

Herein I seek to show utility for the series of type (1) that might help in better understanding some aspects of the nonlinear behavior of ocean waves. Computer codes for the computation of (1) are often assumed to have spatially periodic boundary conditions so that (3, 4) hold [28]: In this case (3) can be computed by the fast Fourier transform (FFT) and (4) contains the coherent structures such as solitons in the specific form for the $u_{\mathbf{n}}$ in terms of the Riemann spectrum, see the developments for the Korteweg-deVries (KdV) equation below. Extension of (1) and (3, 4) to two dimensions (2D) is conceptually simple [28]:

$$u(x,y,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\cdot\boldsymbol{\kappa}x + i\mathbf{n}\cdot\boldsymbol{\lambda}y - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}} \tag{1a}$$

and

$$u(x,y,t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u_{mn}(t) e^{i\kappa_m x + i\lambda_n y} \tag{3a}$$

where

$$u_{mn}(t) = \sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ \kappa_m=\mathbf{n}\cdot\boldsymbol{\kappa},\ \lambda_n=\mathbf{n}\cdot\boldsymbol{\lambda}\}} u_{\mathbf{n}} e^{-i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}}, \qquad \mathbf{n}\cdot\boldsymbol{\kappa} = \sum_{i=1}^{N} n_i\kappa_i, \quad \mathbf{n}\cdot\boldsymbol{\lambda} = \sum_{i=1}^{N} n_i\lambda_i \tag{4a}$$

This is appropriate for integrable equations in 2D such as the Kadomtsev-Petviashvili (KP) and 2+1 Gardner equations.

In the field of oceanography we traditionally use (3, 3a) for Fourier analysis applications and a set of temporal ordinary differential equations are determined for the $u_{mn}(t)$ rather than the series (4, 4a) [9, 35]. For an integrable system (4, 4a) solves the requisite ordinary differential equations, but for a nonintegrable system one might, at the suggestion of Poincaré [29], provide time dependence for the $u_{\mathbf{n}} \to u_{\mathbf{n}}(t)$ in (4, 4a) (see further discussion below).

1.2. Perspective in Terms of Stokes Waves and Their Interactions

The above multidimensional, quasiperiodic Fourier series can be expressed as:

$$u(x,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{k}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}} = \sum_{n=1}^{N} S_n(x,t) + \text{pairwise nonlinear interactions} \tag{5}$$

By construction the above solution of some nonlinear, integrable wave equation is a *spectral theory of Stokes waves*, $S_n(x,t)$. To see this we note that a single Stokes wave has the form:

$$S_n(x,t) = \sum_{m=-\infty}^{\infty} u_m^n e^{ik_m^n x - i[\omega_m^n(k_m) + \Delta\omega_m^n]t + i\phi_m^n} \tag{6}$$

The *nth* Stokes wave $S_n(x,t)$ has a set of amplitudes u_m^n, wavenumbers k_m^n, frequencies ω_m^n and phases ϕ_m^n. The dispersion relation is represented by $\omega_m^n(k_m)$. Here $\Delta\omega_m^n(k_m^n)$ is the *frequency shift* first identified by Stokes [34]. From the above multidimensional Fourier series represented by *N* nested sums (2), we see that each of these sums, in the absence of the others, is a single Stokes wave, together with the usual Stokes "amplitude dependent frequency correction," but here computed for a particular nonlinear integrable wave equation. The interactions are accounted for by pairwise cross terms among the individual Stokes waves in (2) [28].

We see that the multiperiodic Fourier series have all of the Stokes wave harmonics, together with all the "cross harmonics" among the Stokes waves. The Stokes wave components u_m^n form the "bound modes" in terms of the index *m* (phase locked together) and the Stokes waves $S_n(x,t)$ themselves are the "free modes" that can move relative to one another and undergo pairwise scattering from the others in the nonlinear dynamics.

2. Construction of Meromorphic Fourier Series from Theta Functions

2.1. The Baker-Mumford Theorem

The formal construction of multiperiodic, meromorphic Fourier series is a problem from 19th century mathematics, see Baker [1-3]. A more modern approach is due to Mumford [23]. I summarize the

Baker-Mumford Theorem – The most general, single-valued, multiply periodic, meromorphic functions of N variables with $2N$ sets of periods (obeying the necessary relations, see Baker [3], p. 224), can be expressed by means of *theta functions*. The construction is obtained in *only three ways*:

$$\text{(i)} \qquad \partial_{xx} \ln \theta(\mathbf{z}) \tag{7}$$

$$\text{(ii)} \qquad \partial_x \ln\left(\frac{\theta(\mathbf{z}+\mathbf{a})}{\theta(\mathbf{z})}\right) \tag{8}$$

$$\text{(iii)} \qquad \frac{\prod_{n=1}^{N} \theta(\mathbf{z}-\mathbf{a}_n)}{\prod_{n=1}^{N} \theta(\mathbf{z}-\mathbf{b}_n)}, \qquad \sum_{n=1}^{N} \mathbf{a}_n = \sum_{n=1}^{N} \mathbf{b}_n \tag{9}$$

where the *theta functions* have the form

$$\theta(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^N} \theta_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{z}}, \qquad \theta_{\mathbf{n}} = e^{-\frac{1}{2}\mathbf{n}\bullet\tilde{\mathbf{B}}\mathbf{n}} \tag{10}$$

and $\tilde{\mathbf{B}}$ is the *period matrix*. Note that, according to this theorem, one describes the most *general* meromorphic functions from theta functions in only three ways, although a construction method is not provided in the theorem. As will be seen below $\mathbf{z} = \mathbf{k}x - \boldsymbol{\omega}t + \boldsymbol{\phi}$ for solutions of nonlinear, integrable wave equations. The utility of this theorem is therefore quite striking and has a number of *practical* uses as seen below.

Furthermore, the implication is that each of the three forms (7)-(9) for constructing *multiply periodic, meromorphic functions* using theta functions can be written explicitly in terms *of multidimensional, quasiperiodic Fourier series*. This means explicitly that for practical applications one must compute:

$$\partial_{xx} \ln \theta(x,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} \tag{11}$$

$$\frac{\theta(x,t|\mathbf{B},\boldsymbol{\phi}^-)}{\theta(x,t|\mathbf{B},\boldsymbol{\phi}^+)} = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} \tag{12}$$

$$\frac{\prod_{n=1}^{N} \theta(\mathbf{z}-\mathbf{a}_n|\mathbf{B})}{\prod_{n=1}^{N} \theta(\mathbf{z}-\mathbf{b}_n|\mathbf{B})} = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} \tag{13}$$

where the parameters in the series on the right anticipate the physics of the solutions of a particular wave equation: the wavenumber $\mathbf{k}$, frequency $\boldsymbol{\omega}$, and phase $\boldsymbol{\phi}$ vectors and the coefficients $u_{\mathbf{n}}$ that are determined from the *theta function parameters* (specifically the period matrix $\tilde{\mathbf{B}}$) of a *particular nonlinear wave equation*. The Baker-Mumford theorem does not tell us how to make this construction, but I adapt a method of [40] for this purpose. Specific examples are given below for the KdV and nonlinear Schrödinger (NLS) equations.

In a further *tour de force* Baker [2] essentially derived the KdV hierarchy and KP equation by using the bilinear differential operator D (today we attribute this to Hirota [17]), identities of Pfaffians, symmetric functions, hyperelliptic σ-functions and $\wp$-functions. The identification between Baker's differential equations and the soliton equations means that Baker essentially discovered the KdV hierarchy and the KP equation over one hundred years ago. Baker used bilinear forms in his work, although he did not address soliton solutions.

Reevaluation of Baker's work from the perspective of modern soliton theory can be found in a very interesting paper by Matsutani [21].

2.2. Relationship of Theta Functions to Nonlinear Partial Differential Equations

Mumford [20] discusses the relationship of theta functions to the solution of several important nonlinear wave equations. He uses Fay's trisecant identity [12], which takes the form:

$$\sum_{n=1}^{3} c_i \theta(\mathbf{z}+\mathbf{a}_n)\theta(\mathbf{z}+\mathbf{b}_n) = 0 \tag{14}$$

Details of the coefficient values c_i are given in the above papers. Fay's identity is a fundamental identity of theta functions that holds for *period matrices of algebraic curves*, but *not* for *period matrices on general Abelian varieties*. While the limitation to algebraic curves restricts the class of wave equations one can study, Mumford was nevertheless able to construct theta function solutions to a number of equations of mathematical physics using theta functions. These include the KdV, the KP, the sine-Gordon, the nonlinear Schrödinger, the massive Thirring model and presumably many other equations. The important idea here is that the three ways to construct single valued, multiperiodic, meromorphic functions from theta functions, as specified in the above Baker-Mumford theorem, reappear directly in the construction of multiperiodic Fourier series solutions of the integrable wave equations. Table 1 gives a number of results.

Table 1. Integrable equations and solutions in terms of theta functions as derived from the Fay trisecant identity [24].

Name of equation	The equation	Solution in terms of theta functions
Korteweg-deVries	$u_t + 6uu_x + u_{xxx} = 0$	$u(x,t) = 2\partial_{xx}\ln\theta(x,t)$
Kadomtsev-Petviashvili	$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0$	$u(x,y,t) = 2\partial_{xx}\ln\theta(x,y,t)$
Nonlinear Schrödinger	$iu_t + u_{xx} + 2\|u\|^2 u = 0$	$u(x,t) = \theta(x,t\mid\mathbf{B},\phi^-)/\theta(x,t\mid\mathbf{B},\phi^+)$
Sine-Gordon	$u_{xx} - u_{tt} = \sin u$	$u(x,t) = 2i\ln[\theta^*(x,t)/\theta(x,t)]$
Gardner	$u_t + 6uu_x + u_{xxx} + u^2 u_x = 0$	$u(x,t) = 2\partial_x \ln[\theta(x,t\mid\mathbf{B},\phi^-)/\theta(x,t\mid\mathbf{B},\phi^+)]$

It is clear from the Baker-Mumford theorem that it is quite natural to find the solutions in Table 1 are single valued, quasiperiodic, meromorphic Fourier series. Construction of these series from the right hand column of Table 1 and the Riemann spectrum of FGT is a goal of this paper.

2.3. Finite Gap Theory of Nonlinear Wave Equations

The *periodic inverse scattering transform*, or *finite gap theory*, solves nonlinear wave equations starting with knowledge of the Lax pair [4, 22]. The spatial part of the Lax pair is an eigenvalue problem that provides a set of eigenvalues from which particular algebro-geometric loop integrals determine the parameters of the Riemann theta functions. The phase information provides a solution to the Cauchy problem. Random phases allow for the study of nonlinear stochastic systems. The full FGT solution of an integrable equation is contained in the eigenvalue problem, loop integrals and theta functions, together with the time dependant part of the *Lax pair*. The beauty of finite gap theory is that all the information about the solution can be computed *explicitly*.

It is worthwhile noting that the *Its-Matveev formula* for the KdV equation

$$u(x,t) = 2\partial_{xx}\ln\theta(x,t) \tag{15}$$

is a *tour de force* in soliton theory. The original periodic solutions found were related to the hyperelliptic functions, but Its and Matveev brought the power of Riemann theta functions to the forefront of the field. The Its-Matveev formula was hinted at in the work of Baker and Hirota (see also Mumford [24], Table 1), but the real breakthrough came with the derivation of the theta function solutions of the KdV equation using FGT. For those interested in finite gap theory and a historical review by one of the main practitioners, see Matveev [22]. The formula (15) is not very useful for computing the correlation function for random function solutions of KdV. To compute the correlation function of such solutions we use (11) from the Baker-Mumford theorem: To do this we require a method to compute the coefficients $u_{\mathbf{n}}$ from the Its-Matveev formula. This provides us with the multiperiodic Fourier series solution of KdV. The approach is given in Section 4.1 below.

3. Random Phase Approximation for the Solutions of Nonlinear, Integrable Wave Equations

3.1. The Nonlinear Random Phase Approximation

Osborne [26] has suggested that a kind of *nonlinear random phase approximation* be applied to nonlinear evolution equations. Thus, when the phase vector $\boldsymbol{\phi}$ appears in a theta function or in a multiperiodic series such as (1), (2) or (4), we will often find it is convenient to assume that *the components of the vector* $\boldsymbol{\phi}$ *are uniformly distributed random numbers*. This idea parallels the use of random phases in the Fourier transform, which is often assumed in many applications of wind waves. For example, when we analyse a time series, we typically find that the Fourier analysis gives us random phases as a function of frequency. Thus it seems natural to take the vector phases $\boldsymbol{\phi}$ as random numbers in multiperiodic Fourier series for *nonlinear* systems. However, see eq. (21) below for further perspective.

3.2. Explicit Forms for the Power Spectrum and Other Properties of Random Function Solutions

The power spectrum is usually obtained from a time series by taking the Fourier transform and then graphing the squared-amplitudes as a function of frequency. Let us see how this plays out for a system that uses multiperiodic, meromorphic functions, for which we also assume *spatially periodic boundary conditions*. We use (3) to compute the correlation function

$$C(\mathcal{L}) = \langle u(x,t)u(x+\mathcal{L},t)\rangle , \qquad \langle . \rangle = \frac{1}{L}\int_0^L . \, dx \tag{16}$$

where the *spatial average* has the symbol $\langle . \rangle$. We find

$$C(\mathcal{L},t) = \langle u(x,t)u(x+\mathcal{L},t)\rangle = \sum_{n=-\infty}^{\infty} u_n(t)u_n^*(t)e^{ik_n\mathcal{L}} = |u_o(t)|^2 + \frac{1}{2}\sum_{n=1}^{\infty}|u_n(t)|^2\cos(k_n\mathcal{L}) \tag{17}$$

Now address the Fourier coefficients in terms of their modulus $A_n(t)$ and phases $\Phi_n(t)$ from (4)

$$u_n(t) = x_n(t) + iy_n(t) = A_n(t)e^{i\Phi_n(t)} \tag{18}$$

where

$$x_n(t) = \sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ I(\mathbf{n}\cdot\mathbf{k})=n\}} u_{\mathbf{n}}\cos(\mathbf{n}\cdot\boldsymbol{\omega}t - \mathbf{n}\cdot\boldsymbol{\phi}), \qquad y_n(t) = -i\sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ I(\mathbf{n}\cdot\mathbf{k})=n\}} u_{\mathbf{n}}\sin(\mathbf{n}\cdot\boldsymbol{\omega}t - \mathbf{n}\cdot\boldsymbol{\phi}) \tag{19}$$

The *spectral moduli* are given by:

$$\begin{aligned} A_n(t) &= |u_n(t)| = \sqrt{x_n^2(t) + y_n^2(t)} = \\ &= \left\{\left[\sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ I(\mathbf{n}\cdot\mathbf{k})=n\}} u_{\mathbf{n}}\cos(\mathbf{n}\cdot\boldsymbol{\omega}t - \mathbf{n}\cdot\boldsymbol{\phi})\right]^2 + \left[\sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ I(\mathbf{n}\cdot\mathbf{k})=n\}} u_{\mathbf{n}}\sin(\mathbf{n}\cdot\boldsymbol{\omega}t - \mathbf{n}\cdot\boldsymbol{\phi})\right]^2\right\}^{1/2} \end{aligned} \tag{20}$$

and the time evolution of the *linear Fourier phases* $\Phi_n(t)$ in terms of the *FGT phases* $\phi = [\phi_1, \phi_2 .. \phi_N]$:

$$\tan\Phi_n(t) = \frac{y_n(t)}{x_n(t)} = -\frac{\displaystyle\sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ I(\mathbf{n}\cdot\mathbf{k})=n\}} u_{\mathbf{n}}\sin(\mathbf{n}\cdot\boldsymbol{\omega}t - \mathbf{n}\cdot\boldsymbol{\phi})}{\displaystyle\sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ I(\mathbf{n}\cdot\mathbf{k})=n\}} u_{\mathbf{n}}\cos(\mathbf{n}\cdot\boldsymbol{\omega}t - \mathbf{n}\cdot\boldsymbol{\phi})} \tag{21}$$

Hence the time evolution of the *power spectrum* is given by:

$$P(k_n,t) = |u_n(t)|^2 = \left[\sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ I(\mathbf{n}\cdot\mathbf{k})=n\}} u_{\mathbf{n}}\cos(\mathbf{n}\cdot\boldsymbol{\omega}t - \mathbf{n}\cdot\boldsymbol{\phi})\right]^2 + \left[\sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ I(\mathbf{n}\cdot\mathbf{k})=n\}} u_{\mathbf{n}}\sin(\mathbf{n}\cdot\boldsymbol{\omega}t - \mathbf{n}\cdot\boldsymbol{\phi})\right]^2 \tag{22}$$

One can compute triple correlation functions in a similar way for integrable soliton equations. This is the analytical form of the time varying power spectrum for a multiperiodic, meromorphic Fourier series solution to an integrable, nonlinear partial differential equation for spatially periodic boundary conditions. We have *collapsed* the FGT spectrum $u_{\mathbf{n}}$ (see (32) below) onto the time varying linear Fourier spectrum $u_n(t)$ (see (3, 4) above).

4. The Korteweg-deVries Equation

4.1. Construction of Multiperiodic Meromorphic Fourier Series Solutions of the KdV and exKdV Equations

The above results are possible because theta functions have remarkable properties and these allow one to discuss *an algebra of theta functions,* their derivatives, integrals. Furthermore theta functions can be *added, subtracted, multiplied and divided* as exploited below. As an example for the construction of a multiperiodic Fourier series, I now discuss the KdV and extended KdV (exKdV) equations.

The next equation above KdV in the Whitham hierarchy (exKdV) [15, 36] is:

$$u_t + 6uu_x + u_{xxx} = \lambda_1 u_{xxxxx} + \lambda_2 u_x u_{xx} + \lambda_3 u u_{xxx} + \lambda_4 u^2 u_x \tag{23}$$

Here the coefficients are given by $\lambda_1 = 1$, $\lambda_2 = 100/19$, $\lambda_3 = 230/19$ and $\lambda_4 = -60/19$. The *asymptotic solution* to (23) is given by the *near identity transformation* (Lie-Kodama transform [15, 18, 19, 27]):

$$u = 2Z_x + \varepsilon\left[4\lambda_1 Z_x^2 + 2\lambda_2 Z_{xxx} + 4\lambda_3 Z Z_{xx}\right] \tag{24}$$

where $Z(x,t) = \partial_x \ln\theta(x,t)$.

What follows will be based upon the multidimensional Fourier series for Z:

$$Z(x,t) = \partial_x \ln\theta(x,t) = \frac{\theta_x(x,t)}{\theta(x,t)} = \theta_x(x,t)\theta^{-1}(x,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} Z_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{k}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}} \tag{25}$$

Here $\theta^{-1}(x,t)$ means the series

$$Q(x,t) = \frac{1}{\theta(x,t)} = \sum_{\mathbf{n}\in\mathbb{Z}^N} Q_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{k}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}}, \qquad Q_{\mathbf{n}} = \theta_{\mathbf{n}}^{-1} = \sum_{\mathbf{m}\in\mathbb{Z}^N} \left\{e^{-(\mathbf{n}-\mathbf{m})\cdot\tilde{\mathbf{B}}(\mathbf{n}-\mathbf{m})/2}\right\}^{-1} \delta_{\mathbf{m}} \tag{26}$$

where $\theta_{\mathbf{n}}^{-1}$ are the coefficients of the series (26). The "-1" serves to remind us that they are coefficients of the series for $\theta^{-1}(x,t)$; $Q_{\mathbf{n}} = \theta_{\mathbf{n}}^{-1}$ does *not* mean $1/\theta_{\mathbf{n}}$, but is given by the right equation of (26), a result that I have determined by a method of Zygmund [40] for ordinary Fourier series, but which is here applied to multiperiodic Fourier series. To get the result (26) set $Q(x,t) = 1/\theta(x,t)$ so that $\theta(x,t)Q(x,t) = 1$, implying a *convolution* for the coefficients of the product $\theta(x,t)Q(x,t)$:

$$\sum_{\mathbf{m}\in\mathbb{Z}^N} \theta_{\mathbf{n}-\mathbf{m}} Q_{\mathbf{m}} = \boldsymbol{\delta}_{\mathbf{n}}, \qquad \boldsymbol{\delta} = \{\delta_{\mathbf{n}}\} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

and in matrix-vector notation we have $\tilde{\boldsymbol{\theta}}\mathbf{Q} = \boldsymbol{\delta}$ where $\tilde{\boldsymbol{\theta}} = \{\theta_{\mathbf{n}-\mathbf{m}}\}$ is an infinite dimensional matrix formed from the coefficients $\theta_{\mathbf{n}} = \exp(-\mathbf{n}\cdot\mathbf{B}\mathbf{n}/2)$ of the theta function and $\mathbf{Q} = \{Q_{\mathbf{n}}\}$ is a vector we seek. This leads to the coefficients $\mathbf{Q} = \tilde{\boldsymbol{\theta}}^{-1}\boldsymbol{\delta}$ (or $\{Q_{\mathbf{n}}\} = \{\theta_{\mathbf{n}-\mathbf{m}}\}^{-1}\{\boldsymbol{\delta}_{\mathbf{m}}\}$) of the inverse series (26). Here $\boldsymbol{\delta}$ is an infinite length column vector whose elements are all zero except for that at the central position which is 1, i.e. for which $\mathbf{n} = 0$. The convergence of the series (25, 26) occurs because of the extraordinary properties of theta functions [1-4, 22, 23]. In this way one is led to the conclusion that the matrix $\tilde{\boldsymbol{\theta}} = \{\theta_{\mathbf{n}-\mathbf{m}}\}$ is invertible (see Zygmund [40] for arguments relating to the inversion of ordinary trigonometric series in this way). Furthermore:

$$\theta_x(x,t) = i\sum_{\mathbf{n}\in\mathbb{Z}^N} (\mathbf{n}\cdot\mathbf{k})\theta_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{k}x - i\mathbf{n}\cdot\boldsymbol{\omega}t + i\mathbf{n}\cdot\boldsymbol{\phi}} \tag{27}$$

To compute the multiperiodic Fourier series (25) we compute the product of the two series $\theta_x(x,t)$ and $\theta^{-1}(x,t)$, so that the coefficients of the product $\theta_x(x,t)\theta^{-1}(x,t)$ are a convolution of the coefficients

$$Z_{\mathbf{n}} = i \sum_{\mathbf{m}\in\mathbb{Z}^N} (\mathbf{m}\bullet\mathbf{k})\theta_{\mathbf{m}}\theta_{\mathbf{n}-\mathbf{m}}^{-1} = i \sum_{\mathbf{m}\in\mathbb{Z}^N} (\mathbf{n}-\mathbf{m})\mathbf{m}\bullet\mathbf{k}\,\theta_{\mathbf{n}-\mathbf{m}}\theta_{\mathbf{m}}^{-1} \tag{28a}$$

$$\theta_{\mathbf{n}-\mathbf{m}} = \exp(-(\mathbf{n}-\mathbf{m})\bullet\mathbf{B}(\mathbf{n}-\mathbf{m})/2), \quad Q_{\mathbf{m}} = \theta_{\mathbf{m}}^{-1} = \sum_{\mathbf{n}\in\mathbb{Z}^N} \left\{e^{-(\mathbf{m}-\mathbf{n})\bullet\tilde{\mathbf{B}}(\mathbf{m}-\mathbf{n})/2}\right\}^{-1}\delta_{\mathbf{n}} \tag{28b}$$

We see that the $Z_{\mathbf{n}}$ are derived entirely from the coefficients $\theta_{\mathbf{n}}$ of the theta function (10).

It is now easy to see that the coefficients of the KdV equation solution (1) are given by

$$u(x,t) = 2Z_x = 2i \sum_{\mathbf{n}\in\mathbb{Z}^N} (\mathbf{n}\bullet\mathbf{k})Z_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} \tag{29}$$

which means that (see (28a) for other forms for $Z_{\mathbf{n}}$):

$$u_{\mathbf{n}} = 2i(\mathbf{n}\bullet\mathbf{k})Z_{\mathbf{n}} = -2(\mathbf{n}\bullet\mathbf{k}) \sum_{\mathbf{m}\in\mathbb{Z}^N} (\mathbf{m}\bullet\mathbf{k})\theta_{\mathbf{m}}\theta^{-1}{}_{\mathbf{n}-\mathbf{m}} \tag{30}$$

It is interesting to consider the *physical meaning* for (29, 30). To do this, write the convolution (30) in the following way

$$c_{\mathbf{n}} = \sum_{\mathbf{m}\in\mathbb{Z}^N} a_{\mathbf{m}} b_{\mathbf{n}-\mathbf{m}} = \sum_{\mathbf{m}+\mathbf{l}=\mathbf{n}\in\mathbb{Z}^N} a_{\mathbf{m}} b_{\mathbf{l}} \tag{31}$$

Hence, the *convolution* between the coefficients $a_{\mathbf{n}}$, $b_{\mathbf{n}}$ of the product of two multidimensional Fourier series can also be written as a *three-wave interaction coefficient*. Therefore (30), the *coefficients of the multidimensional Fourier series solutions of the KdV equation,* are given by two expressions:

$$u_{\mathbf{n}} = 2i(\mathbf{n}\bullet\mathbf{k}) \sum_{\mathbf{m}\in\mathbb{Z}^N} (\mathbf{m}\bullet\mathbf{k})\theta_{\mathbf{m}}\theta_{\mathbf{n}-\mathbf{m}}^{-1} = 2i(\mathbf{n}\bullet\mathbf{k}) \sum_{\mathbf{m}+\mathbf{l}=\mathbf{n}\in\mathbb{Z}^N} (\mathbf{m}\bullet\mathbf{k})\theta_{\mathbf{m}}\theta_{\mathbf{l}}^{-1} \tag{32}$$

We have an exact equivalency between the *convolution operation* and the *three-wave interaction coefficients* for the solution of the KdV equation. Since the coefficients of the theta function Fourier series (10), $\theta_{\mathbf{n}} = \exp[-\mathbf{n}\bullet\tilde{\mathbf{B}}\mathbf{n}/2]$, contain the *solitons* as *diagonal elements of the period matrix* $\tilde{\mathbf{B}}$, we now have an *exact connection* between the *solitons* (characterized by the *Riemann spectrum*) and *three wave interactions* in the solutions of the KdV equation. Since we have $Z(x,t)$ in (25, 28) we can compute the asymptotic solution of the extended KdV equation via (24). This illustrates the advantage of combining multiperiodic, meromorphic Fourier series from FGT with Lie-Kodama transforms for computing perturbed solutions of nonlinear wave equations.

4.2. Computation of the Nonlinear Dispersion Relation for the KdV Equation

Consider the KdV equation in the usual dimensional form:

$$u_t + c_o u_x + \alpha u u_x + \beta u_{xxx} = 0 \tag{33}$$

The meromorphic series ansatz is

$$u(x,t) = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} \tag{34}$$

Remember that $u_{\mathbf{n}}$ can be expressed in terms of the theta functions (10). Then

$$u(x,t)u_x(x,t) = i\left(\sum_{\mathbf{m}\in\mathbb{Z}^N} u_{\mathbf{m}} e^{i\mathbf{m}\bullet\mathbf{k}x - i\mathbf{m}\bullet\boldsymbol{\omega}t + i\mathbf{m}\bullet\boldsymbol{\phi}}\right)\left(\sum_{\mathbf{n}\in\mathbb{Z}^N} (\mathbf{n}\bullet\mathbf{k})u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}}\right) \tag{35}$$

This results in a new series whose coefficients are a convolution of the coefficients of the series for $u(x,t)$ and $u_x(x,t)$:

$$u(x,t)u_x(x,t) = i\sum_{\mathbf{n}\in\mathbb{Z}^N} U_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\omega t + i\mathbf{n}\bullet\phi}, \quad U_{\mathbf{n}} = \sum_{\mathbf{m}\in\mathbb{Z}^N} (\mathbf{m}\bullet\mathbf{k})u_{\mathbf{m}}u_{\mathbf{n}-\mathbf{m}} = \sum_{\mathbf{m}+\mathbf{l}=\mathbf{n}\in\mathbb{Z}^N} (\mathbf{m}\bullet\mathbf{k})u_{\mathbf{m}}u_{\mathbf{l}} \quad (36)$$

Note that $U_{\mathbf{n}}$ is simultaneously a convolution and three-wave interaction in (36). KdV then becomes:

$$\begin{aligned} &-i\sum_{\mathbf{n}\in\mathbb{Z}^N} (\mathbf{n}\bullet\boldsymbol{\omega})u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} + ic_o\sum_{\mathbf{n}\in\mathbb{Z}^N} (\mathbf{n}\bullet\mathbf{k})u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} + \\ &+ i\alpha\sum_{\mathbf{n}\in\mathbb{Z}^N} U_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} - i\beta\sum_{\mathbf{n}\in\mathbb{Z}^N} (\mathbf{n}\bullet\mathbf{k})^3 u_{\mathbf{n}} e^{i\mathbf{n}\bullet\mathbf{k}x - i\mathbf{n}\bullet\boldsymbol{\omega}t + i\mathbf{n}\bullet\boldsymbol{\phi}} = 0 \end{aligned} \quad (37)$$

which means that

$$\mathbf{n}\bullet\boldsymbol{\omega} = c_o\mathbf{n}\bullet\mathbf{k} - \beta(\mathbf{n}\bullet\mathbf{k})^3 + \alpha U_{\mathbf{n}} / u_{\mathbf{n}} \quad (38)$$

This latter is the *N-dimensional nonlinear dispersion relation*. Note that the last term, which includes a convolution, is the *N*-dimensional nonlinear Stokes wave frequency correction. We could also write the above result in the following way:

$$\Omega_{\mathbf{n}} = c_o\mathbf{K}_{\mathbf{n}} - \beta\mathbf{K}_{\mathbf{n}}^3 + \alpha U_{\mathbf{n}} / u_{\mathbf{n}} \quad (39)$$

for $\Omega_{\mathbf{n}} = \mathbf{n}\bullet\boldsymbol{\omega}$ and $\mathbf{K}_{\mathbf{n}} = \mathbf{n}\bullet\mathbf{k}$. So all the frequencies of the discretuum are computed in this way. The Stokes wave for one degree of freedom agrees with that in Whitham for KdV [36].

4.3. Integrable Turbulence

Zakharov has studied integrable turbulence for the KdV equation [39] for a rarified soliton gas. He derived a soliton-gas kinetic equation for the KdV equation using the inverse scattering method. More recently, the kinetic equation for a *dense* soliton gas for KdV has been found by El and Kamchatnov [11] by taking the thermodynamic limit of the Whitham equations to obtain a nonlinear integrodifferential equation for the spectral measure. This result generalizes Zakharov's result for a rarified soliton gas. Experimentally Costa et al [8] have studied soliton turbulence in the surface wave field in Currituck Sound on the coast of North Carolina. Future work will emphasize the study of soliton gases using both the El-Kamchatnov method and finite gap theory, in theoretical, numerical and experimental contexts.

4.4. Fermi-Pasta-Ulam Recurrence

Fermi-Pasta-Ulam recurrence was discovered in the early 1950s and published in a Los Alamos report [14]. The problem was shown to be equivalent to using the Fourier transform to reduce the KdV equation to a set of ordinary differential equations (for spatially periodic boundary conditions) and then to study the time evolution of these equations [20]. Surprisingly, for a sine wave initial condition, the wave dynamics almost returned to the initial conditions after a *recurrence time* [25, 28]. From the perspective given herein the solution to KdV is given by (1) and the Fourier reduction of the equation to ordinary differential equations (for spatially periodic boundary conditions) is given by (2) and (3). The ordinary differential equations are solved by (3), a meromorphic Fourier series with incommensurable frequencies. This means that the dynamics of the ordinary differential equations (3) is quasiperiodic. Therefore FPU recurrence is approximate, not exact, due to the quasiperiodic nature of the ordinary differential equations. The study of the problem in the soliton limit is enlightening and is the topic of a future paper.

4.5. Wind Waves

The study of wind waves is normally based on the Hasselmann equation [16], a kinetic equation that is driven by wind forcing, by damping due to wave breaking and by four-wave nonlinear interactions. This is the basis of models such as WaveWatch III and WAM. The last important term is the full Boltzmann, nonlinear four-wave interactions represented by the Webb-Resio-Tracy (WRT) algorithm in WaveWatch III. A faster algorithm, but less representative of the physics, is the discrete interaction approximation (DIA). The documentation of the programs, and a full explanation of the physics and the requisite scientific papers, are easily found on the web. It is well known that the Hasselmann formulation equation does *not* reduce to a unidirectional spectrum because the nonlinear interactions reduce to zero for unidirectional wave motion. This of course suggests that the study of the modulational instability for unidirectional waves is "filtered out" in wind/wave models and that breather trains cannot therefore exist in the kinetic equation formulation. However, deterministic, nonlinear wave equations do not have this characteristic, so that unidirectional motion can be studied in a wave equation such as the one dimensional (1D) nonlinear Schrödinger equation, particularly with regard to the modulational instability, while the Hasselmann equation cannot be so studied. In order to fill this "hole" in the kinetic theory and in wind/wave models one might instead consider use of various deterministic equations such as the nonlinear Schrödinger equation and it's higher order versions such as the Dysthe [10] and the Zakharov [37] equations. By applying the methods given herein one could actually write the analytical form of the time varying power spectrum in order to include the coherent structures such as breathers in the wind wave formulation [30, 31]. This would allow the study of rogue waves in the context of wind waves. Using these deterministic wave equations one can formulate their multiperiodic, meromorphic Fourier series solutions, also for the driving terms, leads one to a "transfer function" that drives waves by wind, reduces them by dissipation and carries out the nonlinear four-wave interactions: This occurs because one connects the power spectrum of the waves to the wind, four-wave interactions and dissipation by a time-dependent, nonlinear, frequency domain transfer function. This approach may provide useful information about the behavior of the growth of rogue waves in wind/wave models. I will not discuss further details here.

5. The Nonlinear Schrödinger Equation

Traditionally, ocean waves are viewed as a *random process*: In terms of the Fourier transform this means that for a chosen spectrum, one selects a set of random phases to give a particular *realization* of the random process. For the extension of Fourier analysis to include the class of multiperiodic Fourier series, we once again take the phases to be random, i.e. the vector of phases $\boldsymbol{\phi}$. The procedure is based on (2, 3) where the *correlation function* is computed by $C(\mathcal{L}) = \langle u(x,t)u(x+\mathcal{L},t)\rangle$ and the spatial average has the form given above. In this Section I do *not* assume *spatially periodic boundary conditions*, but assume that the *wavenumbers are incommensurable*. We have the modulational dispersion relation $\Omega_n = \Omega_n(K_n)$ [28]. The nonlinear Schrödinger (NLS) equation has the form $iu_t + \mu u_{xx} + \nu|u|^2 u = 0$, where the solution is a ratio of Riemann theta functions $u(x,t) \sim \theta(x,t\,|\,\boldsymbol{\tau},\boldsymbol{\phi}^-)/\theta(x,t\,|\,\boldsymbol{\tau},\boldsymbol{\phi}^+) = \sum u_{\mathbf{n}} \exp[i\mathbf{n}\bullet\mathbf{K}x - i\mathbf{n}\bullet\boldsymbol{\Omega} + i\mathbf{n}\bullet\boldsymbol{\phi}]$. Assume that the $u_{\mathbf{n}}$ are already computed for the NLS equation as for KdV above and so we can compute the correlation function.

$$u(x,t)u(x+\mathcal{L},t) = \sum_{\mathbf{m}\in\mathbb{Z}^N} \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{m}}(t)u_{\mathbf{n}}(t)e^{i(\mathbf{m}+\mathbf{n})\bullet\mathbf{K}x}e^{i\mathbf{n}\bullet\mathbf{K}\mathcal{L}}$$

Now take the spatial average:

$$\langle u(x,t)u(x+\mathcal{L},t)\rangle = \underbrace{\lim}_{L\to\infty} \frac{1}{L} \sum_{\mathbf{m}\in\mathbb{Z}^N} \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{m}}(t)u_{\mathbf{n}}(t)e^{i\mathbf{n}\bullet\mathbf{K}\mathcal{L}} \int_{-\infty}^{\infty} e^{i(\mathbf{m}+\mathbf{n})\bullet\mathbf{K}x}dx$$

Then write the integral, a Kronecker delta function, in the form:

$$\delta_{\mathbf{m},-\mathbf{n}} = \underbrace{\lim}_{L\to\infty} \frac{1}{L} \int_{-L}^{L} e^{i(\mathbf{m}+\mathbf{n})\bullet\mathbf{K}x}\,dx = \underbrace{\lim}_{L\to\infty} f(K_{\mathbf{mn}}L) = \begin{cases} 1, & \text{if } \mathbf{m} = -\mathbf{n} \\ 0, & \text{Otherwise} \end{cases}$$

For

$$K_{\mathbf{mn}} = \tfrac{1}{2}(\mathbf{m}+\mathbf{n})\bullet\mathbf{K}, \qquad f\big(K_{\mathbf{mn}}L\big) = \frac{\sin(K_{\mathbf{mn}}L)}{K_{\mathbf{mn}}L}$$

The convergence to a Kronecker delta as $L \to \infty$ is seen in Fig. 1. The *correlation function for a multiperiodic Fourier series* (where we used $u_{-\mathbf{n}}(t) = u_{\mathbf{n}}^*(t)$) becomes:

$$C(\mathcal{L},t) = \langle u(x,t)u(x+\mathcal{L},t)\rangle = \sum_{\mathbf{n}\in\mathbb{Z}^N} u_{\mathbf{n}}^*(t)u_{\mathbf{n}}(t)e^{i\mathbf{n}\bullet\mathbf{K}\mathcal{L}} = \sum_{\mathbf{n}\bullet\mathbf{K}=0} |u_{\mathbf{n}}(t)|^2 + \frac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^N:\,\mathbf{n}\neq 0} |u_{\mathbf{n}}(t)|^2 \cos(\mathbf{n}\bullet\mathbf{K}\mathcal{L})$$

Finally, the *power spectrum* is then given by $P(\mathbf{n}\bullet\mathbf{K}) = |u_{\mathbf{n}}(t)|^2$. Note that for the multiperiodic Fourier series used here the wavenumbers $\mathbf{n}\bullet\mathbf{K}$ lie on a point set known as a *discretuum* in string theory [6]. There are hundreds of millions of these wavenumbers in a typical numerical simulation for ocean waves, in contrast to the thousands that occur in an ordinary Fourier series.

6. Possible Extensions to Nonintegrable Wave Equations

Just as it is natural to take the Fourier series coefficients to be a function of time in eq. (3) [9, 35, 37, 38], it is also natural to do similarly in (4) to take into account of nonintegrability:

$$u_n(t) = \sum_{\{\mathbf{n}\in\mathbb{Z}^N:\ k_n = \mathbf{n}\bullet\mathbf{k}\}} u_{\mathbf{n}}(t) e^{-i\mathbf{n}\bullet\boldsymbol{\omega}(t)t + i\mathbf{n}\bullet\boldsymbol{\phi}(t)}$$

Poincaré has suggested a similar idea [29]. Perturbations of integrable equations have been studied using FGT [13]. In this case one obtains nonlinear ordinary differential equations for $u_{\mathbf{n}}(t)$, $\boldsymbol{\omega}(t)$ and $\boldsymbol{\phi}(t)$. For example one might think of solving the higher order equations (beyond integrability) using the above series, but with additional adiabatically varying ordinary differential equations for $u_{\mathbf{n}}(t)$, $\boldsymbol{\omega}(t)$ and $\boldsymbol{\phi}(t)$. Poincaré [29] indicates that it may be possible to expand each of these ordinary differential equations in a multiperiodic Fourier series. Such approaches might make it possible to describe the nonintegrable adiabatic motions of time varying solitons or breather states. Calogero [7] has mentioned the technique of imbedding a chaotic (nonintegrable) system into a periodic/multiperiodic series.

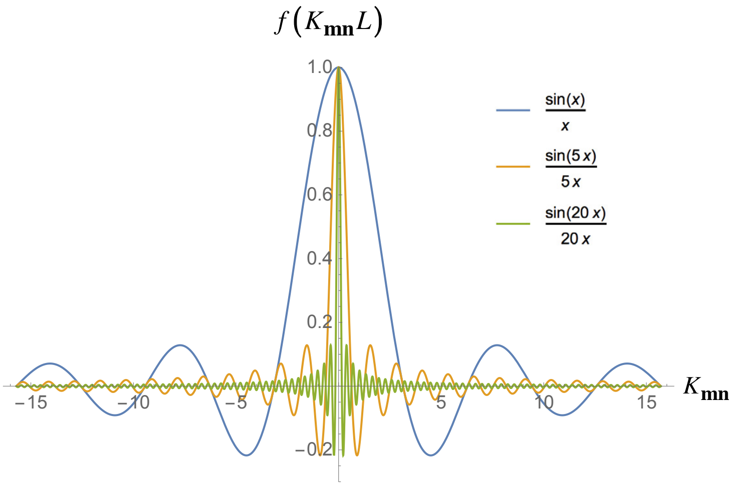

Fig. 1 Convergence of the function $f(K_{\mathbf{mn}}L)$ to a Kronecker delta function $\delta_{\mathbf{m},-\mathbf{n}}$ is shown for increasing values of L = 1, 5 and 20.

7. Conclusions and Projections

The developments here show that multiperiodic Fourier series solutions can be used to analytically compute the important physical properties of nonlinear wave equations with coherent structures such as Stokes waves, solitons, breathers and vortices. Additionally, nonlinear stochastic properties such as the correlation function, power spectrum, coherence functions, x etc. can also be computed. These nonlinear mathematical tools allow us to follow a path parallel to the linear Fourier method for stochastic systems that we have used since the 1930s [5]. The fact that the Baker-Mumford theorem constructs the most general multiperiodic meromorphic functions means that *singularities* can be included in a study of solutions for a number of equations for which "blow-ups" or "collapses" occur. It would seem that equations with blow-ups might be solved (or approximately solved) with multiperiodic Fourier series, thus providing a mathematical and physical description of this interesting phenomena and the types of coherent structures that may occur.

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