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Procedia IUTAM 26 (2018) 92–101

Procedia IUTAM

www.elsevier.com/locate/procedia

IUTAM Symposium Wind Waves, 4-8 September 2017, London, UK

Generation of Wave Groups

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Abstract

The well-known linear stability theory of wind-wave generation is revisited with a focus on the generation of wave groups. As well as recovering the usual temporal instability, the analysis has the outcome that the wave group must move with a real-valued group velocity. This has the consequence that both the wave frequency and the wavenumber should be complex-valued. In the frame of reference moving with the group velocity, the growth rate is enhanced above that for just a temporally growing monochromatic sinusoidal wave. The analysis is extended to the weakly nonlinear regime where a nonlinear Schrödinger equation with a linear growth term is discussed.

Keywords: Wave group, group velocity, wind wave generation;

1. Introduction

Various mechanisms have been proposed to describe the generation of waves by wind. But despite decades of theoretical research, observations and more recently, detailed numerical simulations, the nature of these mechanisms and their practical applicability remains controversial, see the recent assessments by [12], [16] and [14], and the comprehensive reviews by [1] and [7].

Two main mechanisms have been developed. One is a classical shear flow instability mechanism developed by [10] and subsequently adapted for routine use in wave forecasting models, see the review by [7]. In this theory, turbulence in the wind is used only to determine a logarithmic profile for the mean wind shear $u_0(z)$. Then, a sinusoidal wave field is assumed, with a real-valued wavenumber k and a complex-valued phase speed, $c = c_r + ic_i$ so that the waves may have a growth rate kc_i. In the limit $c_i/u_* \to 0$ where u_* is the wind friction velocity, there is significant transfer of energy from the wind to the waves at the critical level z_c where $u_0(z_c) = c_r$. Pertinent to the context of this paper, we note that that this was extended to allow for spatial growth instead of temporal growth by [18].

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2210-9838
10.1016/j.piutam.2018.03.009

The other is a steady-state theory, developed originally by [8] for separated flow over large amplitude waves, and later adapted for non-separated flow over low-amplitude waves, see the reviews by [1] and most recently by [16]. Here the wind turbulence is taken into account through an eddy viscosity term in an inner region near the wave surface. Asymmetry in this inner region then allows for an energy flux to the waves.

Neither theory alone has been found completely satisfactory, and in particular, both fail to take account of wave transience and the tendency of waves to develop into wave groups. This is the issue we propose to address in this article. The strategy we employ is again based on linear shear flow instability theory, but to incorporate at the outset that the waves will have a wave group structure with both temporal and spatial dependence. In fluid flows this was initiated by [4, 5], see the summary by [3] and the reviews by [6, 15]. The essential feature that we exploit is that the wave group moves with a real-valued group velocity $c_g = d\omega/dk$ even when for unstable flows the frequency $\omega = kc$ and the wavenumber k may be complex-valued.

2. Formulation

It is useful to begin with a linear stability theory for a general stratified shear flow, and then develop the theory for the air-water system as a special case. The basic state is the density profile $\rho_0(z)$ and the horizontal shear flow $u_0(z)$. Then the linearized equations are

$$\rho_0(Du + wu_{0z}) + p_x = 0\,, \tag{1}$$
$$\rho_0 Dw + p_z + g\rho = 0\,, \tag{2}$$
$$D\rho + \rho_{0z} w = 0\,, \tag{3}$$
$$u_x + w_z = 0\,. \tag{4}$$
$$\text{where} \quad D = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\,. \tag{5}$$

Here, the terms (u, w) are the perturbation velocity components in the (x, z) directions, ρ is the perturbation density, and p is the perturbation pressure. Equations (1 -4) represent conservation of horizontal and vertical momentum, conservation of mass, and incompressibility respectively in this linearised setting. It is useful to introduce the vertical particle displacement ζ defined in this linearised formulation by

$$D\zeta = w\,. \tag{6}$$

Then the density field is given by integrating equation (3) to get

$$\rho = -\rho_{0z}\zeta\,. \tag{7}$$

Substituting (6, 7) into the remaining equations (1, 2, 4) yields

$$\rho_0 D\tilde{u} + p_x = 0\,, \tag{8}$$
$$\rho_0 D^2\zeta + p_z - g\rho_{0z}\zeta = 0\,, \tag{9}$$
$$\tilde{u}_x + D\zeta_z = 0\,,\,, \tag{10}$$
$$\text{where} \quad \tilde{u} = u + u_{0z}\zeta\,.$$

Finally eliminating $\tilde{u}, p$ yields a single equation for ζ,

$$\{\rho_0 D^2\zeta_z\}_z + \{\rho_0 D^2\zeta\}_{xx} - g\rho_{0z}\zeta_{xx} = 0\,. \tag{11}$$

This equation, together with the boundary conditions that $\zeta = 0$ at $z = -H$ (the bottom of the ocean) and as $z \to \infty$ (the top of the atmosphere) is the basic equation to be studied here.

Next we seek a solution describing a wave group,

$$\zeta = \{A(X, T)\phi(z) + \phi^{(2)}(X, T, z) + \cdots\}\exp(-ik(x - ct)) + \text{c.c.}\,, \quad \text{where} \quad X = \epsilon x\,,\ T = \epsilon t\,. \tag{12}$$

Here c.c. denotes the complex conjugate, and $\epsilon \ll 1$ is a small parameter describing the slow variation of the amplitude $A(X, T)$ relative to the carrier wave. The frequency $\omega = kc$, the phase speed c and the wavenumber k may be complex-valued, and then the imaginary part of the frequency $\mathrm{Im}[kc]$ is the temporal growth rate of an unstable wave. At leading order we obtain the modal equation, well-known as the Taylor-Goldstein equation,

$$(\rho_0 W^2 \phi_z)_z - (g\rho_{0z} + k^2 W^2)\phi = 0\,, \quad W = c - u_0\,. \tag{13}$$

This defines the modal functions and the dispersion relation specifying $\omega = \omega(k)$ where $\omega = kc$ is the frequency. At the next order in ϵ we obtain the equations determining the wave envelope, and these will be presented later. It is useful to note the integral identity

$$P(c, k) \equiv \int_{-H}^{\infty} \rho_0 W^2(|\phi_z|^2 + k^2|\phi|^2) + g\rho_{0z}|\phi^2|\, dz = 0\,. \tag{14}$$

This can be regarded as defining the dispersion relation, expressed here as $c = c(k)$ instead of the more usual $\omega = \omega(k) = kc(k)$. The group velocity can now be defined as

$$c_g = \frac{d\omega}{dk} = c + k\frac{dc}{dk}\,. \tag{15}$$

Differentiation of (14) with respect to k yields

$$k\frac{dc}{dk} = \frac{J}{I}\,, \quad J = -\int_{-H}^{\infty} \rho_0 k^2 W^2 |\phi|^2\, dz\,, \quad I = \int_{-H}^{\infty} \rho_0 W(|\phi_z|^2 + k^2|\phi|^2)\, dz\,. \tag{16}$$

This enables us to obtain a useful expression for the group velocity.

2.1. Air-water system

For an air-water system, we first write

$$\rho_0(z) = \rho_a H(z) + \rho_w H(-z)\,, \quad \rho_{0z} = (\rho_a - \rho_w)\delta(z)\,. \tag{17}$$

Here ρ_a, ρ_w are the constant air and water density respectively, the undisturbed air-water interface is at $z = 0$, $H(z)$ is the Heaviside function and $\delta(z)$ is the Dirac delta function. Further we suppose that $u_0(z)$ is continuous for all z. The water is bounded below at $z = -H$, and the air is unbounded above. Continuity of ζ at the interface $z = 0$ implies that ϕ is continuous across $z = 0$. Since the modal equation (13) is homogeneous, without loss of generality we can set $\phi(z = 0) = 1$. Then in the air ($z > 0$ and water ($z < 0$) the modal equation (13) collapses to the Rayleigh equation

$$(W^2\phi_z)_z - k^2 W^2 \phi = 0\,. \tag{18}$$

The dynamical boundary condition at $z = 0$ is found by integrating (13) across $z = 0$, with the outcome

$$[\rho_0 W^2 \phi_z]_-^+ = g(\rho_a - \rho_w)\phi(0)\,. \tag{19}$$

The system (18, 19) is supplemented by the boundary conditions that $\phi = 0$ at $z = -H$ and that $\phi \to 0$ as $z \to \infty$, where we assume that in $z > 0$ the background shear flow (wind) is such that $u_{0z}/W \to 0, z \to \infty$. This completes the formulation of the eigenvalue problem for the determination of $c(k)$.

Next suppose that in the water, there is no background current, so that $u_0(z) = 0$, $-H \leq z \leq 0$, and in particular $u_0(0) = 0$. Then in the water

$$\phi = \frac{\sinh(k(z + H))}{\sinh(kH)}\,, \tag{20}$$

noting that $\phi(0) = 1$. The boundary condition (19) then reduces to

$$s\phi_z(+0) = \{k \coth(kH) - (1 - s)\frac{g}{c^2}\}\,, \quad s = \frac{\rho_a}{\rho_w}\,. \tag{21}$$

If there is no air ($s = 0$), then this reduces to the usual water wave dispersion relation Note that $s \ll 1$ which we will exploit as the analysis develops.

$$c^2 = c_0^2 = \frac{g}{k} \tanh(kH) . \tag{22}$$

It is then convenient to rewrite (21) in the form

$$s\phi_z(0+) = g\{\frac{1}{c_0^2} - (1 - s)\frac{1}{c^2}\} . \tag{23}$$

Without loss of generality, we assume that $\mathrm{Re}[c_0] > 0$. In deep water $kH \to \infty$, $c_0^2 \to g/|k|$. In general, this finite depth formulation has the same form as the deep water limit, and only c_0 is changed.

2.2. *Growth rate*

Noting that the growth rate is zero when $s = 0$, we put $\mathrm{Im}[kc] = s\gamma$. It is then useful to write

$$\phi_z(0+) = \frac{g\sigma}{c_0^2} , \quad (1 - s)\frac{1}{c^2} = \frac{1}{c_0^2}(1 - s\sigma) , \tag{24}$$

In the limit $s \to 0$, $c_0 > 0$ is real-valued and then

$$\mathrm{Im}[\phi_z(0+)] \approx \frac{g\mathrm{Im}(\sigma)}{c_0^2} , \quad \mathrm{Im}[\sigma] \approx \frac{2\gamma}{|k|c_0} , \tag{25}$$

The task now is to solve the modal equation (18) in $z > 0$ with the boundary conditions (24) at $z = 0$ and $\phi \to 0$ as $z \to \infty$. When $c_i \neq 0$, the modal equation is regular and has just one solution. However, in the limit $s \to 0$, $W = u_0 - c \to u_0 - c_0$ is real-valued, and the modal equation (18) is singular at a critical level z_c where $u_0(z_c) = c_0$. We assume henceforth that the wind shear profile is monotonically increasing from a zero value at $z = 0$, so that $u_{0z} > 0$ for $z \geq 0$. We further assume that u_{0z} decreases with height. Then there can be only one critical level. In this limit, the modal equation has real-valued coefficients. But (25) shows that we are seeking complex-valued solutions, so that then there are two linearly independent solutions ϕ, ϕ^* (* indicates the complex conjugate), whose Wronskian

$$\mathcal{W} = -iW^2(\phi_z\phi^* - \phi\phi_z^*) , \tag{26}$$

is a constant in any region where the solutions are regular. Evaluating the Wronskian at $z = 0$ yields

$$\mathcal{W} = 2c_0^2\mathrm{Im}[\phi_z(0+)] = 2g\mathrm{Im}[\sigma] = \frac{4g\gamma}{|k|c_0} , \quad \text{as} \quad s \to 0 . \tag{27}$$

Essentially, in this limit the instability arises from a coalescence of two modes. If there is no critical level where $u_0(z_c) = c_0$, then evaluating the solutions as $z \to \infty$ yields $\mathcal{W} = 0$ and there is no instability. Hence, there must be a critical level, and we assume as above that there is only one. Using Frobenius expansions near the critical level, it can be shown that the Wronskian has a jump discontinuity there, and that

$$\mathcal{W}(z = z_c-) = -2\pi|K|^2 u_{0zz}(z_c)|u_{0z}(z_c)| , \tag{28}$$

$$\text{where as} \quad z \to z_c , \quad \phi \sim K\{\frac{1}{z - z_c} + \frac{u_{0zz}(z_c)}{u_{0z}(z_c)} \log(z - z_c) + \cdots\} . \tag{29}$$

Here the branch of the logarithm when $z - z_c < 0$ must be chosen corresponding to the requirement that the growth rate $\gamma > 0$, that is

$$\log(z - z_c) = \log|z - z_c| - i\pi\mathrm{sign}[u_{0z}(z_c)] , \quad \text{when} \quad z < z_c .$$

Hence, combining (24, 27, 28) yields

$$\gamma = -\frac{kc_0\pi|K|^2 u_{0zz}(z_c)|u_{0z}(z_c)|}{2g}\,. \tag{30}$$

$$\text{so that} \quad \Gamma = \frac{\gamma}{kc_0} = -\frac{\pi|K|^2 u_{0zz}(z_c)|u_{0z}(z_c)|}{2g}\,, \tag{31}$$

is the non-dimensional growth rate. The expression (30) can also be obtained directly from the dispersion relation (14) taking the limit $s \to 0$ and using the Frobenius expansion (29). In the deep water limit $kH \to \infty$ these agree with the corresponding expressions of [10] and many other authors. Instability occurs only when $u_{0zz}(z_c) < 0$.

The remaining task is to determine the constant K, which has the dimensions of a length. It is obtained by solving the modal equation (18) with the boundary condition (24) and matching this with the Frobenius expansion (29). This either requires a specification of the wind profile $u_0(z)$ which allows for an exact solution to be obtained, see [13] for instance, or an asymptotic analysis as performed by [10, 11] and [7] amongst several others using the well-known logarithmic profile, see [10, 11] and [7],

$$u_0(z) = \frac{u_*}{\kappa}\log\left(1 + \frac{z}{z_*}\right), \quad z_* = \frac{\alpha_C u_*^2}{g}\,. \tag{32}$$

Here u_* is the friction velocity, z_* is a roughness length scale, $\kappa = 0.4$ is von Karman's constant, and $\alpha_C \approx 0.01$ is the Charnock parameter. However, the specific value of K is not our main concern here, and we proceed on the basis that it has been determined. A slight variation of the well-known asymptotic determinations of K are described in the Appendix.

3. Wave groups

To find the equation determining the evolution of the amplitude A we return to the expression (12). After substitution into (11) we find that the $O(\epsilon)$ term yields the equation

$$(\rho_0 W^2 \phi_z^{(2)})_z - (g\rho_{0z} + k^2W^2)\phi^{(2)} = -\frac{2iF^{(1)}}{k}\,, \tag{33}$$

$$F^{(1)} = \{\rho_0 W\phi_z \mathcal{D}A\}_z + \rho_0 k^2\phi(-W\mathcal{D}A + W^2 A_X) - g\rho_{0z}\phi A_X\,, \tag{34}$$

$$\text{and} \quad \mathcal{D}A = A_T + u_0 A_X\,. \tag{35}$$

The left-hand side is just the modal equation operator, and it is readily shown that the forced modal equation (18) has a solution if and only if a certain compatibility condition is satisfied. This is the requirement that $F^{(1)}$ be orthogonal to the complex conjugate of the modal function ϕ,

$$\int_{-H}^{\infty} F^{(1)}\phi^*\,dz = 0\,. \tag{36}$$

Substituting the expression (34) into (36) leads to the desired equation for the amplitude A,

$$A_T + c_g A_X = 0\,, \quad \text{where} \quad c_g = c + k\frac{dc}{dk}\,, \tag{37}$$

where we have used the dispersion relation (14) and the expressions (15, 16).

For stable waves when ω, k are real-valued, c_g is also real-valued and this is the well-known result that the wave group moves with the group velocity. The result obtained here that it also holds for unstable waves when ω, k may be complex-valued is not so well-known. Importantly, causality considerations now demand that nevertheless the group velocity must be real-valued, see the reviews by [6] and [15] for instance. For unstable waves when the frequency ω is complex-valued this leads to the necessity that the wavenumber k also be complex-valued. In the limit as $kc_i \to 0$, we can let $k = k_r + ik_i, |k_i| \ll |k_r|$, and then determine k_i so that c_g is real-valued, that is $\mathrm{Re}[c_{gk}(k_r)]k_i \approx -\mathrm{Im}[c_g(k_r)]$.

The outcome is that, to leading order, the wave group amplitude A propagates with a real-valued group velocity, c_{g0} evaluated at $k = k_r$, with a localised shape determined by the initial conditions.

It is natural here to assume that c_i, k_i are $O(\epsilon)$, and we can write $k_i = \epsilon\delta, kc_i = \epsilon\gamma$. Then we see from the phase expression in (12) that the amplitude is multiplied by the exponential factor

$$E = \exp(-\delta(X - c_0 T) + \gamma T) = \exp(-\delta(X - c_{g0}T) + \Sigma T)\,, \tag{38}$$

$$\text{where} \quad \Sigma = \gamma + \delta(c_0 - c_{g0})\,. \tag{39}$$

Thus Σ is the growth rate in the frame of reference moving with the group velocity. We then put

$$\xi = X - c_{g0}T\,, \quad B(\xi, T) = EA\,, \quad \text{so that} \quad B_T - \Sigma B = 0\,. \tag{40}$$

Note then the expression (12) for a wave packet becomes, at leading order,

$$\zeta = \{B(X, T)\phi(z) + \cdots\}\exp(i\Theta) + \text{c.c.}\,, \qquad \text{where} \quad \Theta = k_r(x - c_r t)\,, \tag{41}$$

for the real-valued phase Θ. Although $\gamma > 0$, Σ could be positive or negative, and correspondingly the instability is absolute or convective in the wave packet reference frame. Note that since $c_0 > c_{g0}$ for water waves, the instability is always absolute if $\delta > 0$ and then the exponential factor E spatially enhances the waves in the rear of the packet.

For the present air-water system, the expressions (16) can be reduced using the density profile (17) and the solution (20) for the modal functions in the water. The small parameter ϵ is now chosen so that $\epsilon = s$ for an optimal balance. Then we get that

$$c = c_0 + sc_1 + \cdots\,, \quad c_g = c_{g0} + sc_{g1} + \cdots\,, \tag{42}$$

$$c_{g1} = c_1 - k\frac{dc_0}{dk}\frac{I_1}{I_0} + \frac{J_1}{I_0} + \cdots\,, \quad I_0 = \frac{g}{c_0}\,, \tag{43}$$

$$J_1 = -\int_0^\infty k^2 W_0^2|\phi|^2\,dz\,, \quad I_1 = \int_0^\infty W_0(|\phi_z|^2 + k^2|\phi|^2)\,dz\,, \tag{44}$$

Here c_{g0}, is the group velocity for water waves alone, that is $c_{g0} = c_0 + kc_0/dk$, and is real-valued for all real-valued wavenumbers k. In the limit $s \to 0$, at leading order the amplitude propagates with the real-valued group velocity c_{g0} without change of form. At the next order c_{g1} is complex-valued when the wavenumber k is real-valued, with contributions coming from both c_1 where $k_r\text{Im}[c_1] = \gamma$, and I_1, while J_1 is real-valued. We can estimate the imaginary part of I_1 from (43) and the Frobenius expansion (29), expanded here to

$$\phi_z = K\{-\frac{u_{0z}^2(z_c)}{W_0^2} + \frac{k^2}{2} + \cdots\}\,, \tag{45}$$

Then, using (30) we find that

$$\text{Im}[c_{g1}] = \frac{\gamma}{k} - \frac{\pi|K|^2 c_0}{2g}\{\frac{3u_{0zz}^2(z_c)}{|u_{0z}(z_c)|} - \text{sign}[u_{0z}(z_c)]u_{0zzz}(z_c)\}kc_{0k}\,, \tag{46}$$

and recalling that then $\text{Re}[c_{gk}(k_r)]\delta \approx -\text{Im}[c_{g1}(k_r)]$, from (39) we get that

$$\Sigma = \gamma\{1 + \frac{c_{0k}}{c_{g0k}} + [\frac{3u_{0zz}(z_c)}{u_{0z}(z_c)^2} - \frac{u_{0zzz}(z_c)}{u_{0zz}(z_c)u_{0z}(z_c)}]\frac{kc_{0k}^2}{c_{g0k}}\}\,. \tag{47}$$

For water waves, $c_{0k} < 0, c_{g0k} < 0$, and $u_{0zz}(z_c) < 0$ for an instability. Hence, if also $u_{0z}(z_c)u_{0zzz}(z_c) < 3u_{0zz}^2(z_c)^2$, then $\Sigma > 0$ and there is an absolute instability. This is only a necessary condition, but as the first three terms in Σ are positive, it seems very likely that $\Sigma > 0$ for all typical wind profiles. For the well-known logarithmic profile (32),

$$u_0(z) = \frac{u_*}{\kappa}\log(1 + \frac{z}{z_*})\,, \quad [\frac{3u_{0zz}(z_c)}{u_{0z}(z_c)^2} - \frac{u_{0zzz}(z_c)}{u_{0zz}(z_c)u_{0z}(z_c)}] = -\frac{\kappa}{u_*} < 0\,, \tag{48}$$

and so $\Sigma > 0$. In the deep water limit, the growth rate Σ is enhanced over the temporal growth rate γ by a factor $3 + (\kappa c_0/u^*)$. The latter term indicates that the amplification enhancement is larger for old wave groups than for young wave groups.

4. Nonlinear Schrödinger equation

This analysis is within the framework of the linearized equations, but we conjecture that a weakly nonlinear asymptotic analysis would lead to a nonlinear Schrödinger equation of the form

$$i(B_T - \Sigma B) - \lambda B_{\xi\xi} - \nu |B|^2 B = 0\,. \tag{49}$$

Here $\lambda = -c_{g0k}/2$ and ν is the well-known Stokes amplitude-dependent frequency correction for water waves. In deep water water $\lambda = c_0/8k$ and $\nu = c_0 k^3/2$. Formally, the derivation of (49) requires a re-scaling in which $T = \epsilon^2 t, \xi = \epsilon(x - c_{g0}t)$, the amplitude B is scaled with ϵ and we now put $s = \epsilon^2$. Since $s \approx 1.275 \times 10^{-3}$ this scaling implies that this model is restricted to waves with amplitudes of non-dimensional order 0.035. A nonlinear Schrödinger equation similar to (49) has been discussed by [9], [19], [12], [2] and [17] amongst others, but with the essential difference that they use the temporal growth rate γ instead of the spatial growth rate Σ expressed as here in the frame moving with the group velocity.

However, the nonlinear and dispersive terms in (49) are not sufficient to control the exponential growth of a localised wave packet, since

$$\frac{d}{dT}\int_{-\infty}^{\infty} |B|^2\, d\xi = 2\Sigma \int_{-\infty}^{\infty} |B|^2\, d\xi\,. \tag{50}$$

Further the modulational instability, present when $\nu\lambda > 0$ (as for deep water waves) in the absence of wind, is enhanced in the presence of wind, see [9]. To see this, first transform (49) into

$$B = \tilde{B}\exp(\Sigma T)\,, \quad \tau = \frac{\exp(2\Sigma T) - 1}{2\Sigma}\,, \quad i\tilde{B}_\tau - \lambda F \tilde{B}_{\xi\xi} - \nu|\tilde{B}|^2\tilde{B} = 0\,, \quad \text{where} \quad F = \frac{1}{1 + 2\Sigma\tau}\,. \tag{51}$$

This has the "plane wave" solution $\tilde{B} = B_0 \exp(-i\nu|B_0|^2\tau)$. Modulation instability is then found by putting $\tilde{B} = B_0(1 + b)\exp(-i\nu|B_0|^2\tau)$ into (51) and linearizing in b, so that

$$ib_\tau - \lambda F b_{\xi\xi} - \nu|B_0|^2(b + b^*) = 0\,. \tag{52}$$

Then we seek solutions of the form $b = (p(\tau) + iq(\tau))\cos(K\xi)$ where p, q are real-valued, and find that,

$$\{\frac{p_\tau}{F}\}_\tau + K^2\lambda(K^2\lambda F - 2\nu|B_0|^2)p = 0\,, \quad q = -\frac{p_\tau}{\lambda K^2 F}\,. \tag{53}$$

When $\Sigma = 0, F = 1$, and this yields the usual criterion for modulation instability, namely that $\lambda K^2(\lambda K^2 - \nu|B_0|^2) < 0$. When $\Sigma > 0$, F varies from 1 to 0 as τ increases from 0 to ∞. Then, as $T \to \infty, \tau \to \infty$ there is modulational instability provided only that $\lambda\nu > 0$, and so independent of $K, |B_0|$. The general solution of (53) can be expressed in terms of modified Bessel functions of imaginary order, $I_{in}(LF^{-1/2})$, where $n = \pm\lambda K^2/\Sigma$ and $L = |B_0|K(2\nu\lambda)^{1/2}/\Sigma$, see [9]. Note that as $\tau \to \infty$, $p \sim \tau^{-1/2}\exp(L(2\Sigma\tau)^{1/2})$. Even taking account of the cancellation of the factor $\tau^{-1/2}$ with the pre-factor $\exp(\Sigma T)$ in (51), we see that the growth rate is now super-exponential.

5. Summary and discussion

We have revisited the linear stability theory of wind-wave generation, initiated by [10] and subsequently developed by Miles and many others, see for instance [11, 7]. These theories were for a sinusoidal monochromatic wave, both in an inviscid context for laminar flow and in a context where wind turbulence is represented by an eddy viscosity term. In the former case the well-known logarithmic profile (32) is often invoked as a suitable model for the wind profile, although the theory can be developed for any wind profile for which the wind speed increases monotonically with height so that there is a single critical level.

In this work we have extended this approach to wave groups, using a well-known multi-scale asymptotic expansion to represent the group structure, where the amplitude $A(X, T)$ in the representation of the solution of the linearised stability problem depends on the slow variables $X = \epsilon x, T = \epsilon t$, see (12). At the leading order, discussed in section 2, we recover the familiar theory for the temporal growth of the amplitude of a sinusoidal monochromatic wave, and the main outcome is the temporal growth rate γ, expressed by (30, 31), in agreement with the expressions obtained by [10] and many others. Crucially in obtaining these expressions, we exploit the approximation $s = \rho_a/\rho_w \ll 1$ by formally taking the limit $s \to 0$.

The main new material is in section 3 where we show that at the next order in the asymptotic expansion the equation (37) determining the evolution of the amplitude is determined. It is no surprise that the amplitude evolves with the group velocity $c_g = \omega_k = c + kc_k$, well-known for stable waves but is also the case, as here, for unstable waves, see the reviews by [6] and [15]. It is an important consequence that the solution of (37) is that A is constant on the characteristics $dX/dT = c_g$ and consequently c_g must be real-valued. In turn, this implies that for unstable waves with a small, but non-zero, imaginary component to the frequency, there must also be a small, but non-zero, imaginary component to the wavenumber. This has the consequence, see (40, 41), that in the reference frame moving with the group velocity the wave group has a growth rate Σ, which differs from the temporal growth rate γ, see (39) for the general expression, and (47) for the present case of wind waves. We find further that for typical wind profiles such as the logarithmic profile (32), Σ is considerably larger than γ, indicating that wave group dynamics, even in this linearised formulation, acts to enhance the generation of wind waves. The implications of this finding for operational wind-wave models remain to be determined, and among the issues that will need to be addressed is how to move from this present deterministic model of a single wave packet, to a statistical ensemble of such wave packets, and in particular to extend this present analysis to two horizontal spatial dimensions. However, it would seem clear that the considerable difference in magnitude between Σ and γ, $\Sigma > \gamma$, will lead to a substantial modification of the growth term in current wind wave operational models.

Although the emphasis here has been on the linear stability problem for a wave group, in section 4 we indicate how an extension to the weakly nonlinear narrow band regime will lead to a nonlinear Schrödinger equation (49) with a forcing term. Because this forcing term is linear, the balance between weak nonlinearity and weak dispersion in this equation in not sufficient to control the instability, and the energy equation (51) shows, all solutions still grow. In particular we examine the effect of this forcing term on modulational instability, and find in agreement with others, such as [9, 17], that the modulational instability is enhanced with super-exponential growth.

Acknowledgement

RG was supported by the Leverhulme Trust through the award of a Leverhulme Emeritus Fellowship, EM-2015-37.

Appendix

The constant K in the Frobenius expansion (29) is formally obtained by solving the modal equation (18) with the boundary condition (24) and matching this with the Frobenius expansion (29). For most wind profiles, this cannot be done exactly, and a variety of asymptotic and approximate analyses have been performed, see [10, 11] and [7] amongst several others.

Here we make a further assumption that $u_0(z) \to U_0$ as $z \to \infty$, where $U_0 > c_0$, and use an approach similar to that of [7], which matches the solutions in two regimes. These are an inner layer including both $z = 0$ and $z \sim z_c$ and an outer layer where $z \gg z_c$. In the inner layer, the right-hand side of (18) is neglected, and then an approximate solution is,

$$\phi \approx \phi_{inner} = 1 + b \int_0^z \frac{dy}{W^2(y)}\,, \tag{54}$$

where the constant b is determined by matching with an outer solution. Formally, this is valid when $kz \ll 1$, and in particular, $kz_c \ll 1$. The second term in (54) is singular at $z = z_c$ and is evaluated by assuming as above that $c_i > 0$ and then taking the limit $c_i \to 0+$ This yields the Frobenius expansion (29,45)), and so

$$b = -Ku_{0z}^2(z_c)\,. \tag{55}$$

Matching with the outer solution will require that the inner solution (54) be expanded for $z > z_c$, yielding

$$\phi_{inner} = 1 + \{\mathcal{P} \int_0^z \frac{dy}{W^2(y)} - \frac{i\pi u_{0zz}(z_c)}{|u_{0z}(z_c)|^3}\}\,, \tag{56}$$

where $\mathcal{P}\int$ denotes the principal value integral.

For the outer solution, we recall that as $z \to \infty$, $u_0 \to U_0, u_{0z} \to 0, z \to \infty$. In this limit an approximate solution of the modal equation (18) is found by setting

$$W\phi = \exp(-kz)\psi\,, \quad \text{so that} \quad \psi_{zz} - 2k\psi_z - \frac{u_{0zz}}{W}\psi = 0\,. \tag{57}$$

For our present purpose it is sufficient it assume that $\psi \sim \psi_{outer} = A$ as $z \to \infty$, where A is a constant. This outer solution is then matched to the inner solution. To this end, we expand the inner solution (56) as $z \to \infty$ so that

$$\phi_{inner} \sim 1 + b\{\frac{(z+z_0)}{W_0^2} - \frac{i\pi u_{0zz}(z_c)}{|u_{0z}(z_c)|^3}\}\,, \quad z_0 = \int_0^\infty \frac{W_0^2 - W^2}{W^2}\,dz\,, \quad W_0 = c_0 - U_0\,. \tag{58}$$

This is then matched to the inner limit of the outer expansion

$$\phi_{outer} \sim A\frac{(1-kz)}{W_0}\,. \tag{59}$$

The outcome, after eliminating the constant A, is

$$b = -\frac{kW_0^2}{\mu}\,, \quad \mu = 1 + kz_0 - \frac{i\pi k W_0^2 u_{0zz}(z_c)}{|u_{0z}(z_c)|^3}\,. \tag{60}$$

Then, using the expression (55) we finally obtain that

$$K = \frac{kW_0^2}{|u_{0z}(z_c)|^2\mu}\,. \tag{61}$$

Then the dimensionless growth rate, given by (31) becomes

$$\Gamma = -\frac{\pi k u_{0zz}(z_c) W_0^4 \tanh(kH)}{2c_0^2|u_{0z}(z_c)|^3|\mu|^2} = -\frac{\pi k^2 u_{0zz}(z_c) W_0^4}{2g|u_{0z}(z_c)|^3|\mu|^2}\,, \quad \text{where} \quad |\mu|^2 = (1+kz_0)^2 + B^2\,, \quad B = \frac{\pi k W_0^2 u_{0zz}(z_c)}{|u_{0z}(z_c)|^3}\,. \tag{62}$$

This depends on the parameters c_0, H and the parameters $u_{0z}(z_c), u_{0zz}(z_c), U_0, z_0$ contained in the shear flow profile $u_0(z)$. Note that the wavenumber k is subsumed into c_0. Note that $z_0 > 0$ when $U_0 > 2c_0$ and then $|\mu|^2 > 1$ for all parameter values. Typically B scales with kz_c, and then if $kz_0, kz_c \ll 1$, $|\mu|^2 \approx 1$.

As an illustration, consider the logarithmic wind profile (32) in the domain $0 < z < z_1$, with $u_0 = U_0, z > z_1$,

$$c_0 = \frac{u_*}{\kappa}\log(1 + \frac{z_c}{z_*})\,, \quad U_0 = \frac{u_*}{\kappa}\log(1 + \frac{z_1}{z_*})\,, \quad -\frac{u_{0zz}(z_c)}{|u_{0z}(z_c)|} = \frac{1}{z_* + z_c}\,, \tag{63}$$

$$B = \frac{\pi\kappa^2 k(z_c + z_*)W_0^2}{u_*^2} = \pi k(z_* + z_c)\log^2\{\frac{z_* + z_1}{z_* + z_c}\}\,, \tag{64}$$

$$z_0 = \int_0^{z_1} \frac{W_0^2 - W^2}{W^2}\,dz = \frac{\kappa z_*}{u_*}\int_{W_0}^{c_0} \frac{W_0^2 - W^2}{W^2}\,\exp(\kappa(c_0 - W)/u_*)\,dW\,, \tag{65}$$

$$\text{and} \quad \Gamma = \frac{\pi k(z_* + z_c)}{2\kappa^2|\mu|^2}\frac{ku_*^2}{g}\log^4\{\frac{z_* + z_1}{z_* + z_c}\}\,. \tag{66}$$

The Miles parameter β is defined by

$$\Gamma = \beta\frac{u_*^2}{c_0^2}\,, \quad \text{so that} \quad \beta = \frac{\pi k(z_* + z_c)}{2\kappa^2|\mu|^2}\frac{kc_0^2}{g}\log^4\{\frac{z_* + z_0}{z_* + z_c}\}\,. \tag{67}$$

In the deep water limit, this expression is similar to that obtained by [11], but differs in how the parameter μ is determined. For each fixed wavenumber k, Γ decreases as H decreases, albeit rather slowly, since z_c decreases as c_0 decreases with H for fixed k. Further, for each fixed c_0, and hence fixed z_c, Γ decreases as H decreases, since then k decreases. The expression (66) also contains a dependence on U_0 through z_0, and Γ increases as U_0 increases.

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