Assignment 5

MATH 381 A - Winter 2022

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Introduction

Suppose we are playing a dice rolling game. We start with a score of zero, and have a goal score of M. On each turn, we will roll a fair six-sided die and have the following rules:

- If our score plus the roll is greater or equal to M, the game ends.
- Otherwise:
  - If our score plus the roll is not a prime number, our score plus the roll becomes our new score.
  - If our score plus the roll is a prime number, our score minus the roll becomes our new score.
    - If our new score becomes negative, it will be converted to zero automatically.

We continue rolling and updating our score until the game ends.

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This game can be modeled with a Markov Chain with $M+1$ states: $0, ..., M-1, M$. The frist M states indicate our current score, and the state M indicates that the game has ended. We start in state 0 with a current score of zero. State M is an absorbing state since we never leave it once entered.
Let A be the transition matrix for this chain. A_{ij} gives the probability of transitioning from state i to state j in one step. Here is the Sage code to generate the transition matrix with $M = 15$:

```
# generate an 16 by 16 transition matrix for this dice rolling game
# We have states {0,1,2,...,14} corresponding to our scores and a state 15
   corresponding to a score of 15 or more.
M = 15;
A = zero_matrix(QQ,M+1);

A[M,M]=1;
```

```python
for i in range(0,M):
    for j in range(6):
        roll=j+1
        attemptScore = i+roll
        if (attemptScore >= M):
            A[i,M]+= 1/6
        else:
            if (is_prime(attemptScore)):
                newScore = i-roll
            else:
                newScore = i+roll
            if (newScore > 0):
                A[i,newScore]+= 1/6
            else:
                A[i,0]+= 1/6
```

Then we can get a transition matrix with $M = 15$:

$$A = \begin{pmatrix}
\frac{1}{2} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{2}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Put this transition matrix into canonical form:

$$P = \left(\begin{array}{c|c} Q & R \\ \hline O & J \end{array}\right)$$

where J is an identity matrix and O is a matrix of all zeros. Q and R are non-negative matrices that arise from the transition probabilities between non-absorbing states. The matrix N is defined as

$$N = (I - Q)^{-1}$$

with the following theorem:

- The ij-th entry of N is the expected number of times that the chain will be in state j after starting in state i.
- The sum of the i-th row of N gives the mean number of steps until absorbtion when the chain is started in state i.

According to the canonical form we have

$$Q = \begin{pmatrix}
\frac{1}{2} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and get

$$N = \begin{pmatrix}
\frac{37736035}{10834313} & \frac{544807}{570227} & 0 & \frac{451186}{1547759} & \frac{1220135}{1547759} & \frac{519799}{1547759} & \frac{1510124}{1547759} & \frac{300264}{1547759} & \frac{666918}{1547759} & \frac{941752}{1547759} & \frac{859832}{1547759} & \frac{122254}{1547759} & \frac{733524}{1547759} & 0 & \frac{1662140}{4643277} \\
\frac{28449481}{10834313} & \frac{1033573}{570227} & 0 & \frac{451186}{1547759} & \frac{1220135}{1547759} & \frac{519799}{1547759} & \frac{1510124}{1547759} & \frac{300264}{1547759} & \frac{666918}{1547759} & \frac{941752}{1547759} & \frac{859832}{1547759} & \frac{122254}{1547759} & \frac{733524}{1547759} & 0 & \frac{1662140}{4643277} \\
\frac{86320429}{43337252} & \frac{5723993}{6842724} & 1 & \frac{1369456}{4643277} & \frac{4254553}{6191036} & \frac{2071053}{6191036} & \frac{7963231}{9286554} & \frac{290316}{1547759} & \frac{1734389}{3095518} & \frac{5473079}{9286554} & \frac{4978297}{9286554} & \frac{362281}{4643277} & \frac{724562}{1547759} & 0 & \frac{20726477}{55719324} \\
\frac{31929847}{21668626} & \frac{806469}{1140454} & 0 & \frac{1964140}{1547759} & \frac{1779779}{3095518} & \frac{999991}{3095518} & \frac{1121537}{1547759} & \frac{310692}{1547759} & \frac{797709}{1547759} & \frac{1080727}{1547759} & \frac{783425}{1547759} & \frac{116974}{1547759} & \frac{701844}{1547759} & 0 & \frac{3480679}{9286554} \\
\frac{1147149}{1140454} & \frac{669673}{1140454} & 0 & \frac{31108}{81461} & \frac{216549}{162922} & \frac{46749}{162922} & \frac{48755}{81461} & \frac{16926}{81461} & \frac{38073}{81461} & \frac{53419}{81461} & \frac{48137}{81461} & \frac{5866}{81461} & \frac{35196}{81461} & 0 & \frac{180691}{488766} \\
\frac{187153}{162922} & \frac{72393}{162922} & 0 & \frac{31108}{81461} & \frac{53627}{162922} & \frac{209671}{162922} & \frac{48755}{81461} & \frac{16926}{81461} & \frac{38073}{81461} & \frac{53419}{81461} & \frac{48137}{81461} & \frac{5866}{81461} & \frac{35196}{81461} & 0 & \frac{180691}{488766} \\
\frac{17709661}{21668626} & \frac{532023}{1140454} & 0 & \frac{311320}{1547759} & \frac{766109}{3095518} & \frac{1190965}{3095518} & \frac{2093903}{1547759} & \frac{278934}{1547759} & \frac{610449}{1547759} & \frac{868543}{1547759} & \frac{805061}{1547759} & \frac{133054}{1547759} & \frac{798324}{1547759} & 0 & \frac{3215431}{9286554} \\
\frac{9441372}{10834313} & \frac{290988}{570227} & 0 & \frac{540168}{1547759} & \frac{446460}{1547759} & \frac{346836}{1547759} & \frac{578676}{1547759} & \frac{1824876}{1547759} & \frac{622836}{1547759} & \frac{879504}{1547759} & \frac{783198}{1547759} & \frac{133974}{1547759} & \frac{803844}{1547759} & 0 & \frac{537226}{1547759} \\
\frac{11612693}{21668626} & \frac{275923}{1140454} & 0 & \frac{482982}{1547759} & \frac{562309}{3095518} & \frac{1015169}{3095518} & \frac{412611}{1547759} & \frac{240576}{1547759} & \frac{1868577}{1547759} & \frac{764319}{1547759} & \frac{679137}{1547759} & \frac{113292}{1547759} & \frac{679752}{1547759} & 0 & \frac{1368359}{3095518} \\
\frac{1155695}{3095518} & \frac{29205}{162922} & 0 & \frac{210560}{1547759} & \frac{355285}{3095518} & \frac{826385}{3095518} & \frac{276115}{1547759} & \frac{422520}{1547759} & \frac{250005}{1547759} & \frac{1953035}{1547759} & \frac{582085}{1547759} & \frac{99536}{1547759} & \frac{597216}{1547759} & 0 & \frac{3481877}{9286554} \\
\frac{2266979}{10834313} & \frac{66021}{570227} & 0 & \frac{126124}{1547759} & \frac{104863}{1547759} & \frac{128639}{1547759} & \frac{143780}{1547759} & \frac{376578}{1547759} & \frac{146664}{1547759} & \frac{481390}{1547759} & \frac{1778078}{1547759} & \frac{83614}{1547759} & \frac{501684}{1547759} & 0 & \frac{1495715}{4643277} \\
\frac{693417}{10834313} & \frac{17523}{570227} & 0 & \frac{36096}{1547759} & \frac{30453}{1547759} & \frac{70833}{1547759} & \frac{47334}{1547759} & \frac{72432}{1547759} & \frac{42858}{1547759} & \frac{334806}{1547759} & \frac{99786}{1547759} & \frac{1609044}{1547759} & \frac{367710}{1547759} & 0 & \frac{409034}{1547759} \\
\frac{231139}{21668626} & \frac{5841}{1140454} & 0 & \frac{6016}{1547759} & \frac{10151}{3095518} & \frac{23611}{3095518} & \frac{7889}{1547759} & \frac{12072}{1547759} & \frac{7143}{1547759} & \frac{55801}{1547759} & \frac{16631}{1547759} & \frac{268174}{1547759} & \frac{1609044}{1547759} & 0 & \frac{1956793}{9286554} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Summing the first row of N yields the expected value of the number of rolls made before the game ends:

$$\frac{44248376}{4643277}$$

which approximately equals to

$$9.52955768092233$$

2

Now we want to make a simulation of the game to support the expected value for M = 15. We will do the simulation 10 times. In each simulation, we will simulate 100000 games. Here is the Sage code:

```
# simulation on the game
import random

for i in range(10):
    totalStep = 0
    for j in range(100000):
        currentScore = 0
        step = 0
        while (currentScore < M):
            roll = random.randint(1,6)
            attScore = currentScore+roll
            step+= 1
            if (attScore < M):
                if (is_prime(attScore)):
                    currentScore = currentScore-roll
                    if (currentScore < 0):
                        currentScore = 0
                else:
                    currentScore = currentScore+roll
            else:
                currentScore = currentScore+roll
        totalStep+= step
    meanStep = totalStep/100000*1.
    print("Simulation #",i+1," ",meanStep)
```

We will get 10 number of mean roll to end the game for each simulation. Sort them from the smallest to the largest, we get a interval of the mean roll:

$$(9.50616, 9.55122)$$

The expected value we computed lies in between this interval, which give us confidence on our computation.

3

To investigate the distribution of the number of rolls until the game ends when M = 15, we will run the following Sage code to get the probability that we are in state M where the game ends after different number of rolls:

```
# calculate the probability that the game ends after differnt number of turn
import numpy as np
prob = np.zeros(50)
for i in range(50):
    prob[i] = (A^i)[0][M]
import matplotlib.pyplot as plt
i = np.arange(50)
plt.plot(i,prob,'o')
plt.xlabel('number of turn')
plt.ylabel('probability to end the game after that turn')
```

and get a plot of the probability:

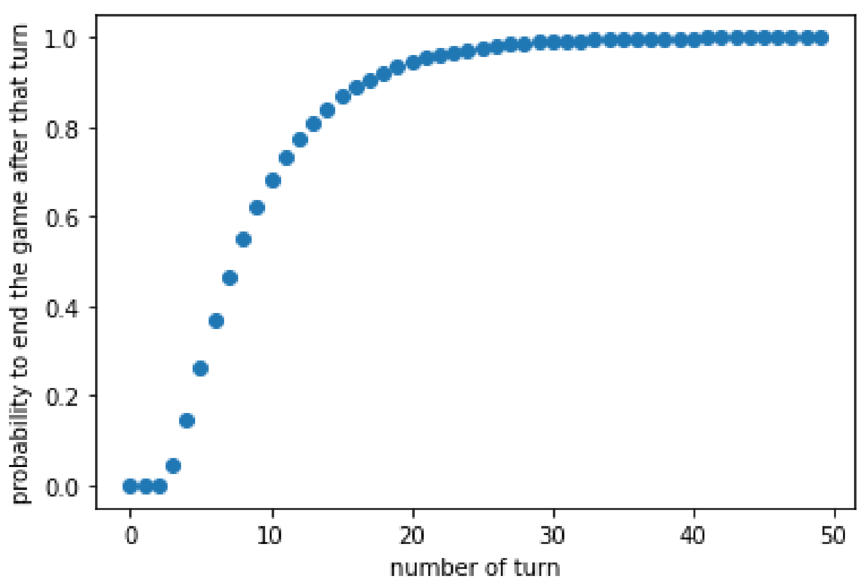


Figure 1: Probability to end the game after different number of turn

Here is the Sage code to generate the probability that we end the game on each different number of turn:

```
# calculate the probability that the game ends after differnt number of turn
prob2 = np.zeros(50)
prob2[0] = prob[0]
for i in range(1,50):
    prob2[i] = prob[i]-prob[i-1]
i = np.arange(50)
plt.plot(i,prob2,'o')
```

```
plt.xlabel('number of turn')
plt.ylabel('probability to end the game on that turn')
```

We get a plot of the probability:

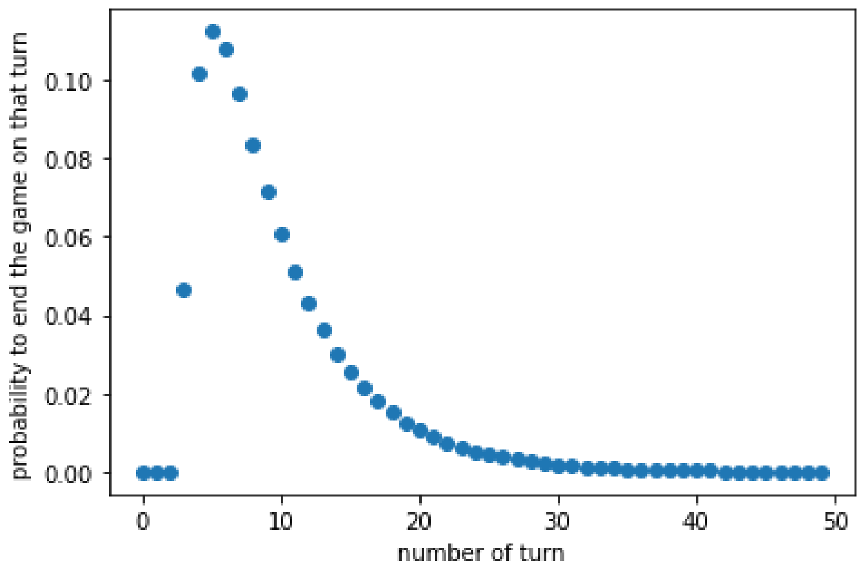


Figure 2: Probability to end the game on each different number of turn

From the probability that we are in state M where the game ends after different number of rolls, we can get that the median game length, where the probability to end the game is 50%, is between 7 and 8, since after the 7^{th} roll the probability to end the game is 0.46500986 and after the 8^{th} roll the probability to end the game is 0.5487927. After the 21 roll, the probability to end the game is 0.95350975. So 95% of all games end on or before the 21^{th} turn. After the 30 roll, the probability to end the game is 0.990398. So 99% of all games end on or before the 30^{th} turn.

4

Lastly we want to investigate how the expected number of turns in the game changes with the value of M. Let $f(M)$ be the expected value for M. We will use the following Sage code to calculate $f(M)$ for M from 1 to 300:

```
# to generate the transition matrix A
def getA(M):
    A = zero_matrix(QQ,M+1);
    A[M,M]=1;
    for i in range(0,M):
        for j in range(6):
            roll=j+1
            attemptScore = i+roll
            if (attemptScore >= M):
```

```
                A[i,M]+= 1/6
            else:
                if (is_prime(attemptScore)):
                    newScore = i-roll
                else:
                    newScore = i+roll
                if (newScore > 0):
                    A[i,newScore]+= 1/6
                else:
                    A[i,0]+= 1/6
    return A
# generate the expected value for different values of M
f = np.zeros(300)
for M in range(1,301):
    A = getA(M)
    Q = A[:M,:M]
    N=(matrix.identity(M)-Q).inverse()
    f[M-1] = sum(N[0][i] for i in range(M))
plt.figure(figsize=(24,16))
plt.plot(np.arange(1,301),f,'.')
```

The reason we choose 300 to be the maximum value of M is that when changed the maximum value to be 500, it took Sage for more than an hour to compute, so we investigate $f(M)$ with $M \in \{1, 300\}$. We get the plot:

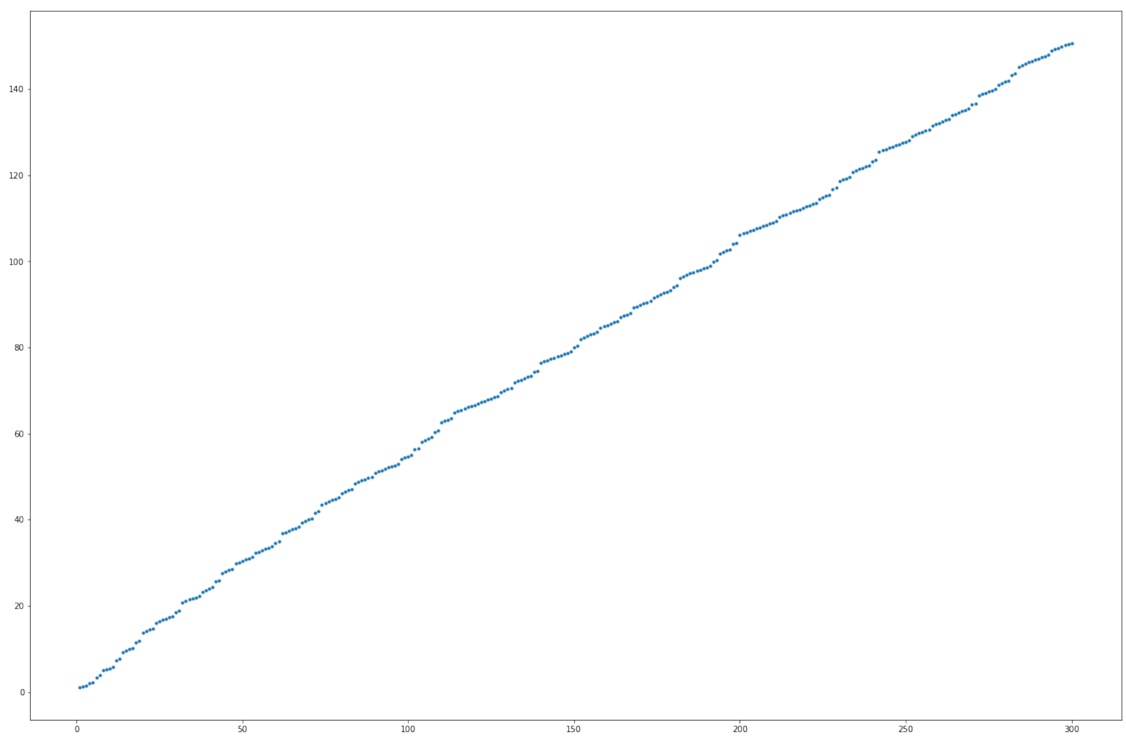


Figure 3: $f(M)$ vs M

We can see that relation between $f(M)$ and M is not linear. There are many jumps of $f(M)$ while the M is increasing. To investigate the these jumps, we will plot $f(M+1)-f(M)$ vs M:

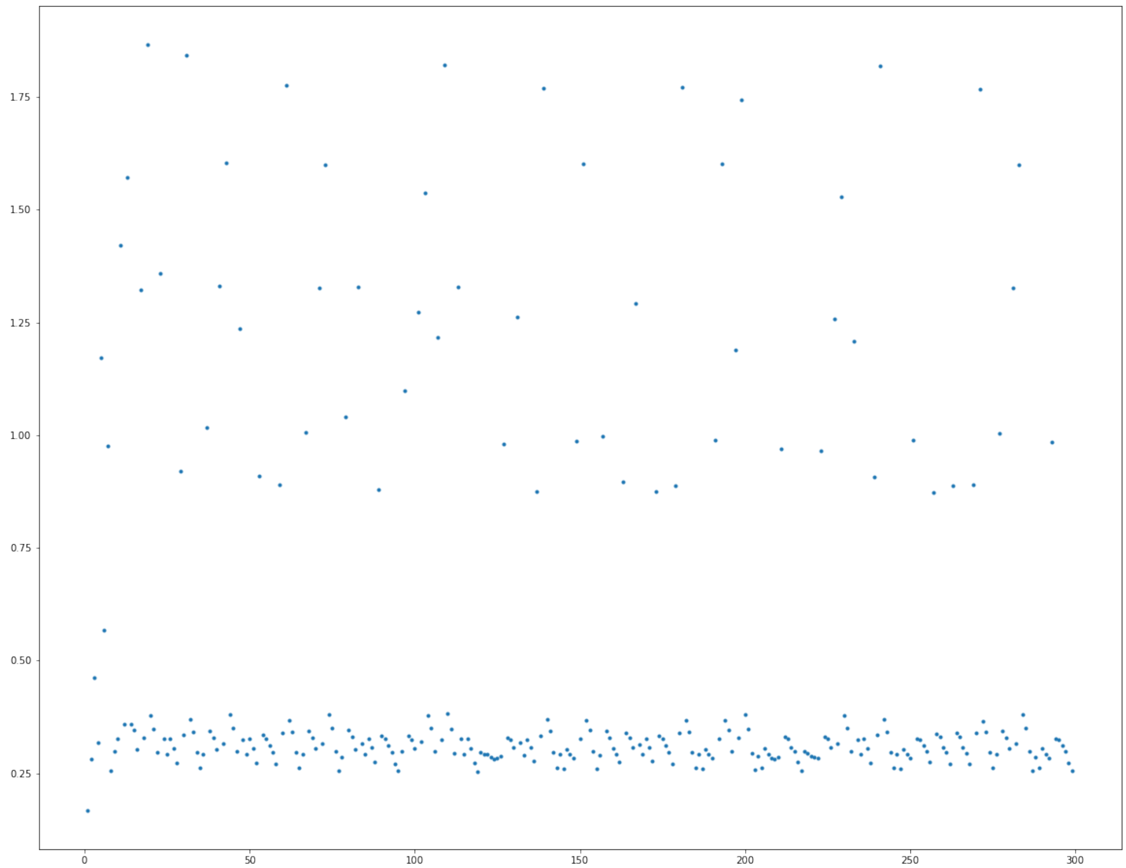
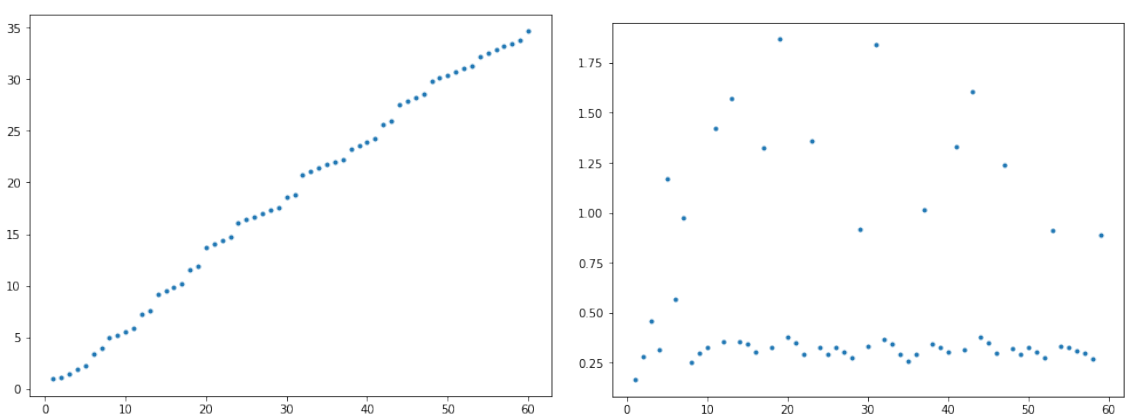


Figure 4: $f(M+1) - f(M)$ vs M

To see the plot more clearly, let's zoom in these plots by focusing on $M \in \{1, 60\}$.



(a) $f(M)$ vs M

(b) $f(M+1) - f(M)$

Figure 5: $M \in \{1, 60\}$

We can now see that the jumps happen at when M is a prime number. To further explore the change in $f(M)$, we plot $f(M+50) - f(M)$ with $M \in \{1, 300\}$:

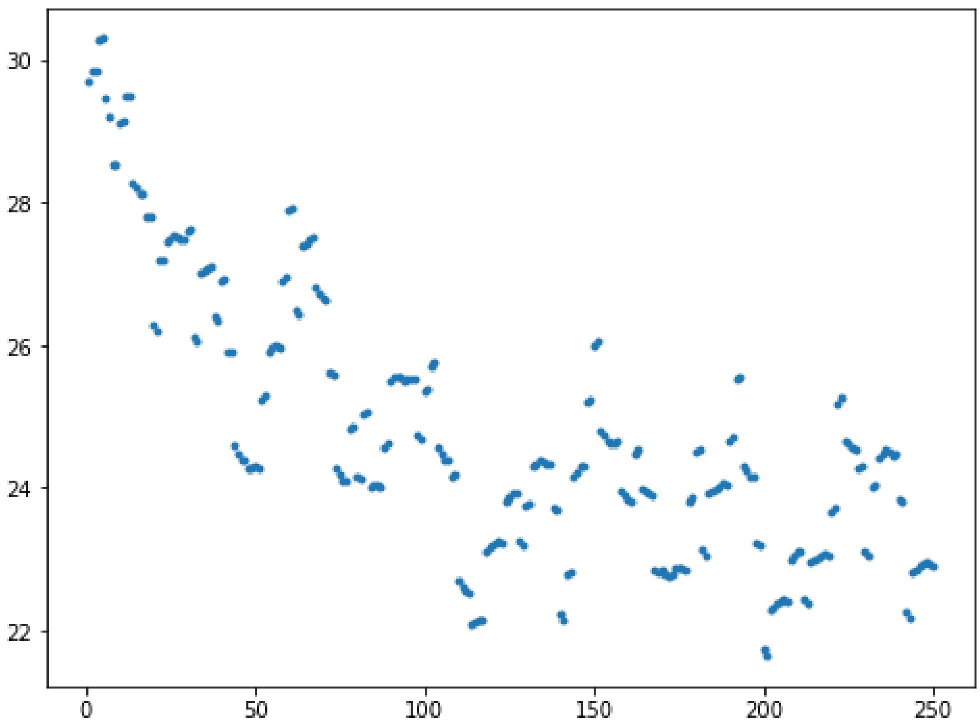


Figure 6: $f(M+50) - f(M)$ vs M

From the plot we can see that as M getting larger, $f(M+50) - f(M)$ getting smaller. So macroscopically $f(M)$ is concaving down as M growing.