Applications of Polynomials: Secret Sharing and Erasure Codes

CS70 Summer 2016 - Lecture 7D

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UC Berkeley

Today

Counting polynomials

Shamir's Secret Sharing

Erasure Codes

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How many polynomials are there that pass through k points that I give you (assuming $k \leq d+1$)? m^{d+1-k}. Why? Polynomial fully determined by $d+1$ points. We have k. How we set the remaining $d+1-k$ fully specifies the polynomial.

Secret Sharing (1/2)

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Shamir's secret sharing scheme: a way to distribute a secret (e.g. nuclear launch codes) such that:

1. A group of sufficient size can recover the secret without all of them needing to be present.
2. No group that is too small to recover the entire secret can recover any information about the secret without the cooperation of more people.

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2. Pick a degree-$k-1$ polynomial P such that $P(0) = s$, i.e. $P(x) = s + a_1 x + a_2 x^2 + \ldots + a_{k-1} x^{k-1}$, where $a_1, \ldots, a_{k-1}$ are chosen randomly.

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What happens when fewer that k officials go rogue and try to order a nuclear strike? They have less than k points so they can't gain iny information about what $P(0)$ is!To see this: what happens if $k-1$ officials try to get P? There are q polynomials passing through their points, one for every possible value of $P(0)$. No new information gained!

Live Demo

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Packets dropped $\rightarrow$ dead officials.

Packets you receive $\rightarrow$ live officials.

You want to recover the original message if you receive enough information!

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Want to send n packets over a lossy channel (each one some number over $GF(q)$, q prime); call the packets $m_1, m_2, \ldots, m_n$. Say the channel drops d packets (although we don't know which).

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Note: does require that $q \geq n+d$, but finding big primes is easy so it's not normally a problem.

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