

A Random Walk through CS70, Pt. III: Number Theory, Polynomials, etc.

CS70 Summer 2016 - Lecture 8C

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UC Berkeley

Last lecture!

Fun with number theory and polynomials.

Modular Arithmetic

Covered in more detail in [M115](#).

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Division: multiplication by multiplicative inverse. How do we find MI? EGCD!

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1. If $y = 0$, return $(x, 1, 0)$: $x = 1x + 0y$.
2. Otherwise, let (d, a, b) be the return value of the extended GCD algorithm on $(y, x - y \lfloor x/y \rfloor)$.
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How do we find multiplicative inverse? Solve $ax + bm = 1$.

Exponentiation in Modular Arithmetic

Repeated squaring!

$$51^{43} \equiv 51^{32} \cdot 51^8 \cdot 51^2 \cdot 51^1 \equiv (60) * (53) * (60) * (51) \equiv 2 \pmod{77}.$$

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Immediate corollary: Fermat's little theorem. Suppose p is prime. Then $a^p \equiv a \pmod{p}$. Furthermore, if $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.

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Let $A_1, \dots, A_n, B_1, \dots, B_n$ be numbers in $\{0, \dots, p-1\}$ for some prime number p . At least one of them is not zero. We pick w_1, \dots, w_n , where each w_i is picked from the set $\{0, \dots, p-1\}$ uniformly at random. Let $\alpha = \sum_i w_i A_i$ and $\beta = \sum_i w_i B_i$.

1. **(11 points)** What is the probability that $\alpha = 0 \pmod{p}$?
2. **(11 points)** Give a strictly positive (non zero) lower bound to the probability that $\alpha \cdot \beta$ is not equal to zero. (Hint: union bound)

Dot Product over Finite Fields, Solution

Part 1:

- Case 1: *Two or more A_i 's are non-zero.* Look at the coefficient i of one of the non-zero ones. In order to make the sum non-zero, $w_i A_i$ must be equal to $S = \sum_{j \neq i} w_j A_j$. Therefore, we are asking for the probability that $w_i A_i = S$, which is $1/p$.

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$$\Pr[\alpha\beta = 0] = \Pr[\alpha = 0 \cup \beta = 0] \leq \Pr[\alpha = 0] + \Pr[\beta = 0] = \frac{2}{p}$$

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Pick random a . Check if $a^{p-1} \equiv 1 \pmod{p}$. No? then composite. Yes? Prime or Carmichael w.p. at least $1/2$.

Public Key Encryption, In General...

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Like this stuff? Want to learn more? [CS276](#).

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Expand to more congruences to get CRT! Let m_1, \dots, m_k be relatively prime numbers. Then the k equations $x \equiv a_1 \pmod{m_1}, \dots, x \equiv a_k \pmod{m_k}$ have a unique solution mod $m_1 m_2 \dots m_k$.

Euler's Criterion and Square Roots

Theorem (Euler's Criterion): Suppose p is an odd prime and a is some integer relatively prime to p . Then $a^{(p-1)/2}$ is $1 \pmod{p}$ if and only if there exists some integer x such that $a \equiv x^2 \pmod{p}$ and -1 otherwise.

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How to find the square root? If $p \equiv 3 \pmod{4}$, and the square roots exist, then square roots of $a \pmod{p}$ are given by $\pm a^{(p+1)/4}$.

Blum Coin Flipping

How to flip a coin over the phone?

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3. Alex, armed with knowledge of p, q , computes the square roots $\pm x, \pm y$ of a , mod n , and sends one to David.

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3. Alex, armed with knowledge of p, q , computes the square roots $\pm x, \pm y$ of a , mod n , and sends one to David.
4. If David got $\pm x$, then he says Alex guessed correctly. Otherwise, if he gets $\pm y$, he can factor n (since $pq \mid (x+y)(x-y)$) and use that to prove that he won.

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This material is covered in much greater depth in [M113](#).

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Points to coefficients? Lagrange interpolation:

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Or set up the Vandermonde matrix and solve.

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^d \\ 1 & x_2 & x_2^2 & \dots & x_2^d \\ 1 & x_3 & x_3^2 & \dots & x_3^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{d+1} & x_{d+1}^2 & \dots & x_{d+1}^d \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{d+1} \end{bmatrix}$$

Secret Sharing

1. Pick some prime $q > s, n$. We will operate in $GF(q)$.
2. Pick a degree- $k - 1$ polynomial P such that $P(0) = s$, i.e.
 $P(x) = s + a_1x + a_2x^2 + \dots + a_{k-1}x^{k-1}$, where a_1, \dots, a_{k-1} are chosen randomly.
3. Give $P(i)$ to the i th official.
4. To recover the secret: have k people get together and interpolate to find $P(0)$.

No information can be recovered with less than k people if done over a prime field!

Take original message $(1, m_1), (2, m_2), \dots, (n, m_n)$ in $GF(q)$ and then interpolate a polynomial.

Erasure Codes

Take original message $(1, m_1), (2, m_2), \dots, (n, m_n)$ in $GF(q)$ and then interpolate a polynomial.

Send k extra points. If k drop, it's ok! Just interpolate and evaluate.

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More on codes: [EE121](#), [EE229AB](#).

Theorem (Schwartz-Zippel Lemma) : Let $Q(x_1, \dots, x_n)$ be a *multivariate* polynomial of *total degree* d (i.e. the sum of the powers of all the variables in a term are at most d) over some field F .

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$Q(x_1, \dots, x_n) = \sum_{i=0}^k x_1^i Q_i(x_2, \dots, x_n)$ where k is the largest exponent of x_1 in Q , and each Q_i is nonzero.

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By inductive step, we know that $Q_k(r_2, \dots, r_n) = 0$ w.p. at most $(d - k)/|S|$ since total degree of Q_k is at most $d - k$.

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□

Conclusion

We hope you've enjoyed this semester and learned a lot.

Before CS70:



After CS70:



Thanks for taking CS70!