4 Neural Networks

1. Give brief definitions of the following terms:

- **neuron**

  a simple information processing unit.

- **action potential**

  the signal outputted by a biological neuron.

- **firing rate**

  the number of action potentials emitted during a defined time-period.

- **synapse**

  the connection between two neurons.

- **an artificial neural network**

  a parallel architecture composed of many simple processing elements interconnected to achieve certain collective computational capabilities

2. A neuron has a transfer function which is a linear weighted sum of its inputs and an activation function that is the Heaviside function. If the weights are $\mathbf{w} = [0.1, -0.5, 0.4]$ what is the output of this neuron when the input is: $\mathbf{x}_1 = [0.1, -0.5, 0.4]^t$ and $\mathbf{x}_2 = [0.1, 0.5, 0.4]^t$?

Output of neuron is defined as:

$$y = H(\mathbf{wx})$$

$$y_1 = H(\mathbf{wx}_1) = H((0.1 \times 0.1) + (-0.5 \times -0.5) + (0.4 \times 0.4)) = H(0.42) = 1$$

$$y_2 = H(\mathbf{wx}_2) = H((0.1 \times 0.1) + (-0.5 \times 0.5) + (0.4 \times 0.4)) = H(-0.08) = 0$$

3. A Linear Threshold Unit has one input, x_1, that can take binary values. Apply the sequential Delta learning rule so that the output of this neuron, y, is equal to $\bar{x}_1$, *i.e.*, such that:

x_1	y
0	1
1	0

Assume initial values of $\theta = 1.5$ and $w_1 = 2$, and use a learning rate of 1.

Using Augmented notation, $y = H(\mathbf{wx})$ where $\mathbf{w} = [-\theta, w_1]$, and $\mathbf{x} = [1, x_1]^T$.
For the Delta rule, weights are updated such that: $\mathbf{w} \leftarrow \mathbf{w} + \eta(t - y)\mathbf{x}^t$
Initial $\mathbf{w} = [-1.5, 2]$

$\mathbf{x}^t$	t	$y = H(\mathbf{wx})$	$t - y$	$\eta(t-y)\mathbf{x}^t$	$\mathbf{w}$
$(1,0)$	1	$H(-1.5 \times 1 + 2 \times 0) = 0$	1	$[1,0]$	$[-0.5,2]$
$(1,1)$	0	$H(-0.5 \times 1 + 2 \times 1) = 1$	-1	$[-1,-1]$	$[-1.5,1]$
$(1,0)$	1	$H(-1.5 \times 1 + 1 \times 0) = 0$	1	$[1,0]$	$[-0.5,1]$
$(1,1)$	0	$H(-0.5 \times 1 + 1 \times 1) = 1$	-1	$[-1,-1]$	$[-1.5,0]$
$(1,0)$	1	$H(-1.5 \times 1 + 0 \times 0) = 0$	1	$[1,0]$	$[-0.5,0]$
$(1,1)$	0	$H(-0.5 \times 1 + 0 \times 1) = 0$	0	$[0,0]$	$[-0.5,0]$
$(1,0)$	1	$H(-0.5 \times 1 + 0 \times 0) = 0$	1	$[1,0]$	$[0.5,0]$
$(1,1)$	0	$H(0.5 \times 1 + 0 \times 1) = 1$	-1	$[-1,-1]$	$[-0.5,-1]$
$(1,0)$	1	$H(-0.5 \times 1 - 1 \times 0) = 0$	1	$[1,0]$	$[0.5,-1]$
$(1,1)$	0	$H(0.5 \times 1 + -1 \times 1) = 0$	0	$[0,0]$	$[0.5,-1]$
$(1,0)$	1	$H(0.5 \times 1 - 1 \times 0) = 1$	0	$[0,0]$	$[0.5,-1]$

Learning has converged, so required weights are $\mathbf{w} = [0.5, -1]$, *or equivalently* $\theta = -0.5$, $w_1 = -1$.

4. Repeat the above question using the batch Delta learning rule.

Epoch 1, initial $\mathbf{w} = [-1.5, 2]$

$\mathbf{x}^t$	t	$y = H(\mathbf{wx})$	$t-y$	$\eta(t-y)\mathbf{x}^t$	$\mathbf{w}$
(1,0)	1	$H(-1.5 \times 1 + 2 \times 0) = 0$	1	[1,0]	
(1,1)	0	$H(-1.5 \times 1 + 2 \times 1) = 1$	−1	[−1,−1]	
			total weight change	[0,−1]	[−1.5,1]

Epoch 2, initial $\mathbf{w} = [-1.5, 1]$

$\mathbf{x}^t$	t	$y = H(\mathbf{wx})$	$t-y$	$\eta(t-y)\mathbf{x}^t$	$\mathbf{w}$
(1,0)	1	$H(-1.5 \times 1 + 1 \times 0) = 0$	1	[1,0]	
(1,1)	0	$H(-1.5 \times 1 + 1 \times 1) = 0$	0	[0,0]	
			total weight change	[1,0]	[−0.5,1]

Epoch 3, initial $\mathbf{w} = [-0.5, 1]$

$\mathbf{x}^t$	t	$y = H(\mathbf{wx})$	$t-y$	$\eta(t-y)\mathbf{x}^t$	$\mathbf{w}$
(1,0)	1	$H(-0.5 \times 1 + 1 \times 0) = 0$	1	[1,0]	
(1,1)	0	$H(-0.5 \times 1 + 1 \times 1) = 1$	−1	[−1,−1]	
			total weight change	[0,−1]	[−0.5,0]

Epoch 4, initial $\mathbf{w} = [-0.5, 0]$

$\mathbf{x}^t$	t	$y = H(\mathbf{wx})$	$t-y$	$\eta(t-y)\mathbf{x}^t$	$\mathbf{w}$
(1,0)	1	$H(-0.5 \times 1 + 0 \times 0) = 0$	1	[1,0]	
(1,1)	0	$H(-0.5 \times 1 + 0 \times 1) = 0$	0	[0,0]	
			total weight change	[1,0]	[0.5,0]

Epoch 5, initial $\mathbf{w} = [0.5, 0]$

$\mathbf{x}^t$	t	$y = H(\mathbf{wx})$	$t-y$	$\eta(t-y)\mathbf{x}^t$	$\mathbf{w}$
(1,0)	1	$H(0.5 \times 1 + 0 \times 0) = 1$	0	[0,0]	
(1,1)	0	$H(0.5 \times 1 + 0 \times 1) = 1$	−1	[−1,−1]	
			total weight change	[−1,−1]	[−0.5,−1]

Epoch 6, initial $\mathbf{w} = [-0.5, -1]$

$\mathbf{x}^t$	t	$y = H(\mathbf{wx})$	$t-y$	$\eta(t-y)\mathbf{x}^t$	$\mathbf{w}$
(1,0)	1	$H(-0.5 \times 1 - 1 \times 0) = 0$	1	[1,0]	
(1,1)	0	$H(-0.5 \times 1 - 1 \times 1) = 0$	0	[0,0]	
			total weight change	[1,0]	[0.5,−1]

Epoch 7, initial $\mathbf{w} = [0.5, -1]$

$\mathbf{x}^t$	t	$y = H(\mathbf{wx})$	$t-y$	$\eta(t-y)\mathbf{x}^t$	$\mathbf{w}$
(1,0)	1	$H(0.5 \times 1 - 1 \times 0) = 1$	0	[0,0]	
(1,1)	0	$H(0.5 \times 1 - 1 \times 1) = 0$	0	[0,0]	
			total weight change	[0,0]	[0.5,−1]

Learning has converged, so required weights are $\mathbf{w} = [0.5, -1]$, *or equivalently* $\theta = -0.5$, $w_1 = -1$.

5. A Linear Threshold Unit has two inputs, x_1 and x_2, that can take binary values. Apply the sequential Delta learning rule so that the output of this neuron, y, is equal to $x_1 AND x_2$, *i.e.*, such that:

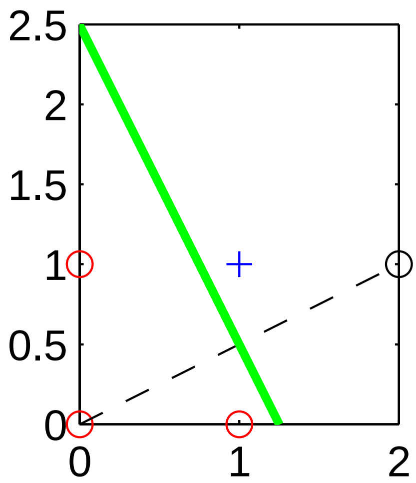
x_1	x_2	y
0	0	0
0	1	0
1	0	0
1	1	1

Assume initial values of $\theta = -0.5$, $w_1 = 1$ and $w_2 = 1$, and use a learning rate of 1.

Using Augmented notation, $y = H(\mathbf{wx})$ where $\mathbf{w} = [-\theta, w_1, w_2]$, and $\mathbf{x} = [1, x_1, x_2]^t$.
For the Delta rule, weights are updated such that: $\mathbf{w} \leftarrow \mathbf{w} + \eta(t-y)\mathbf{x}^t$
Epoch 1, initial w=[0.5,1,1]

$\mathbf{x}^t$	t	$y = H(\mathbf{wx})$	$t-y$	$\eta(t-y)\mathbf{x}^t$	$\mathbf{w}$
$(1,0,0)$	0	$H(0.5\times1+1\times0+1\times0)=1$	-1	$[-1,0,0]$	$[-0.5,1,1]$
$(1,0,1)$	0	$H(-0.5\times1+1\times0+1\times1)=1$	-1	$[-1,0,-1]$	$[-1.5,1,0]$
$(1,1,0)$	0	$H(-1.5\times1+1\times1+0\times0)=0$	0	$[0,0,0]$	$[-1.5,1,0]$
$(1,1,1)$	1	$H(-1.5\times1+1\times1+0\times1)=0$	1	$[1,1,1]$	$[-0.5,2,1]$
$(1,0,0)$	0	$H(-0.5\times1+2\times0+1\times0)=0$	0	$[0,0,0]$	$[-0.5,2,1]$
$(1,0,1)$	0	$H(-0.5\times1+2\times0+1\times1)=1$	-1	$[-1,0,-1]$	$[-1.5,2,0]$
$(1,1,0)$	0	$H(-1.5\times1+2\times1+0\times0)=1$	-1	$[-1,-1,0]$	$[-2.5,1,0]$
$(1,1,1)$	1	$H(-2.5\times1+1\times1+0\times1)=0$	1	$[1,1,1]$	$[-1.5,2,1]$
$(1,0,0)$	0	$H(-1.5\times1+2\times0+1\times0)=0$	0	$[0,0,0]$	$[-1.5,2,1]$
$(1,0,1)$	0	$H(-1.5\times1+2\times0+1\times1)=0$	0	$[0,0,0]$	$[-1.5,2,1]$
$(1,1,0)$	0	$H(-1.5\times1+2\times1+1\times0)=1$	-1	$[-1,-1,0]$	$[-2.5,1,1]$
$(1,1,1)$	1	$H(-2.5\times1+1\times1+1\times1)=0$	1	$[1,1,1]$	$[-1.5,2,2]$
$(1,0,0)$	0	$H(-1.5\times1+2\times0+2\times0)=0$	0	$[0,0,0]$	$[-1.5,2,2]$
$(1,0,1)$	0	$H(-1.5\times1+2\times0+2\times1)=1$	-1	$[-1,0,-1]$	$[-2.5,2,1]$
$(1,1,0)$	0	$H(-2.5\times1+2\times1+1\times0)=0$	0	$[0,0,0]$	$[-2.5,2,1]$
$(1,1,1)$	1	$H(-2.5\times1+2\times1+1\times1)=1$	0	$[0,0,0]$	$[-2.5,2,1]$
$(1,0,0)$	0	$H(-2.5\times1+2\times0+1\times0)=0$	0	$[0,0,0]$	$[-2.5,2,1]$
$(1,0,1)$	0	$H(-2.5\times1+2\times0+1\times1)=0$	0	$[0,0,0]$	$[-2.5,2,1]$
$(1,1,0)$	0	$H(-2.5\times1+2\times1+1\times0)=0$	0	$[0,0,0]$	$[-2.5,2,1]$
$(1,1,1)$	1	$H(-2.5\times1+2\times1+1\times1)=1$	0	$[0,0,0]$	$[-2.5,2,1]$

Learning has converged, so required weights are $\mathbf{w} = [-2.5, 2, 1]$, or equivalently $\theta = 2.5$, $w_1 = 2$, $w_2 = 1$.
The decision surface looks like this:



6. Consider the following linearly separable data set.

$\mathbf{x}^t$	*class*
$(0,2)$	1
$(1,2)$	1
$(2,1)$	1
$(-3,1)$	0
$(-2,-1)$	0
$(-3,-2)$	0

Apply the Sequential Delta Learning Algorithm to find the parameters of a linear threshold neuron that will correctly classify this data. Assume initial values of $\theta = -1$, $w_1 = 0$ and $w_2 = 0$, and a learning rate of 1.

Using Augmented notation, $y = H(\mathbf{wx})$ where $\mathbf{w} = [-\theta, w_1, w_2]$, and $\mathbf{x} = [1, x_1, x_2]^T$. So initial weight values are

$\mathbf{w} = [1,0,0]$ *and the dataset is:*

$\mathbf{x}^t$	t
$(1,0,2)$	1
$(1,1,2)$	1
$(1,2,1)$	1
$(1,-3,1)$	0
$(1,-2,-1)$	0
$(1,-3,-2)$	0

For the Sequential Delta Learning Algorithm, weights are updated such that: $\mathbf{w} \leftarrow \mathbf{w} + \eta(t-y)\mathbf{x}^t$. *Here,* $\eta = 1$.

iteration	$\mathbf{w}$	$\mathbf{x}^t$	$y = H(\mathbf{wx})$	t	$\mathbf{w} \leftarrow \mathbf{w} + (t-y)\mathbf{x}^t$
1	$[1,0,0]$	$[1,0,2]$	1	1	$[1,0,0]$
2	$[1,0,0]$	$[1,1,2]$	1	1	$[1,0,0]$
3	$[1,0,0]$	$[1,2,1]$	1	1	$[1,0,0]$
4	$[1,0,0]$	$[1,-3,1]$	1	0	$[1,0,0]-[1,-3,1]=[0,3,-1]$
5	$[0,3,-1]$	$[1,-2,-1]$	0	0	$[0,3,-1]$
6	$[0,3,-1]$	$[1,-3,-2]$	0	0	$[0,3,-1]$
7	$[0,3,-1]$	$[1,0,2]$	0	1	$[0,3,-1]+[1,0,2]=[1,3,1]$
8	$[1,3,1]$	$[1,1,2]$	1	1	$[1,3,1]$
9	$[1,3,1]$	$[1,2,1]$	1	1	$[1,3,1]$
10	$[1,3,1]$	$[1,-3,1]$	0	0	$[1,3,1]$
11	$[1,3,1]$	$[1,-2,-1]$	0	0	$[1,3,1]$
12	$[1,3,1]$	$[1,-3,-2]$	0	0	$[1,3,1]$
13	$[1,3,1]$	$[1,0,2]$	1	1	$[1,3,1]$

Learning has converged (we have gone through all the data without needing to update the weights), so required parameters are $\mathbf{w} = (1,3,1)$.

7. A negative feedback network has three inputs and two output neurons, that are connected with weights $W = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$**. Determine the activation of the output neurons after 5 iterations when the input is** $\mathbf{x} = (1,1,0)^T$**, assuming that the output neurons are updated using parameter** $\alpha = 0.25$**, and the activations of the output neurons are initialised to be all zero.**

The activation of a negative feedback network is determined by iteratively evaluating the following equations:

$$\mathbf{e} = \mathbf{x} - \mathbf{W}^T\mathbf{y}$$

$$\mathbf{y} \leftarrow \mathbf{y} + \alpha\mathbf{We}$$

iteration	$\mathbf{e}^T$	$(\mathbf{We})^T$	$\mathbf{y}^T$	$(\mathbf{W}^T\mathbf{y})^T$
1	$(1,1,0)$	$(2,2)$	$(0.5,0.5)$	$(1,1,0.5)$
2	$(0,0,-0.5)$	$(0,-0.5)$	$(0.5,0.375)$	$(0.875,0.875,0.375)$
3	$(0.125,0.125,-0.375)$	$(0.25,-0.125)$	$(0.5625,0.34375)$	$(0.90625,0.90625,0.34375)$
4	$(0.09375,0.09375,-0.34375)$	$(0.1875,-0.15625)$	$(0.60938,0.30469)$	$(0.91406,0.91406,0.30469)$
5	$(0.085938,0.085938,-0.30469)$	$(0.17188,-0.13281)$	$(0.65234,0.27148)$	$(0.92383,0.92383,0.27148)$

So output is $\begin{pmatrix} 0.65234 \\ 0.27148 \end{pmatrix}$

Note, competition results in the first neuron increasing its output, while the output of the second neuron is suppressed. Also, note that the vector $\mathbf{W}^T\mathbf{y}$ *becomes similar to the input* $\mathbf{x}$. $\mathbf{W}^T\mathbf{y}$ *converges towards a reconstruction of the input.*

8. Repeat the previous question using a value of $\alpha = 0.5$**.**

$$\mathbf{e} = \mathbf{x} - \mathbf{W}^T\mathbf{y}$$

$$\mathbf{y} \leftarrow \mathbf{y} + \alpha\mathbf{We}$$

iteration	$\mathbf{e}^T$	$(\mathbf{We})^T$	$\mathbf{y}^T$	$(\mathbf{W}^T\mathbf{y})^T$
1	$(1,1,0)$	$(2,2)$	$(1,1)$	$(2,2,1)$
2	$(-1,-1,-1)$	$(-2,-3)$	$(0,-0.5)$	$(-0.5,-0.5,-0.5)$
3	$(1.5,1.5,0.5)$	$(3,3.5)$	$(1.5,1.25)$	$(2.75,2.75,1.25)$
4	$(-1.75,-1.75,-1.25)$	$(-3.5,-4.75)$	$(-0.25,-1.125)$	$(-1.375,-1.375,-1.125)$
5	$(2.375,2.375,1.125)$	$(4.75,5.875)$	$(2.125,1.8125)$	$(3.9375,3.9375,1.8125)$

So output is $\begin{pmatrix} 2.125 \\ 1.8125 \end{pmatrix}$

Note, competition results in oscillatory responses. If α is too large the network becomes unstable. Instability is a common problem with recurrent neural networks.

9. A more stable method of calculating the activations in a negative feedback network is to use the following update rules:

$$\mathbf{e} = \mathbf{x} \oslash \left[\mathbf{W}^T\mathbf{y}\right]_{\epsilon_2}$$

$$\mathbf{y} \leftarrow \left[\mathbf{y}\right]_{\epsilon_1} \odot \tilde{\mathbf{W}}\mathbf{e}$$

where $[v]_\epsilon = \mathtt{max}(\epsilon, v)$; ϵ_1 and ϵ_2 are parameters; $\tilde{\mathbf{W}}$ is equal to W but with each row normalised to sum to one; and $\oslash$ and $\odot$ indicate element-wise division and multiplication respectively.

This is called Regulatory Feedback or Divisive Input Modulation.

Determine the activation of the output neurons after 5 iterations when the input is $\mathbf{x} = (1,1,0)^T$, and $\mathbf{W} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, assuming that $\epsilon_1 = \epsilon_2 = 0.01$, and the activations of the output neurons are initialised to be all zero.

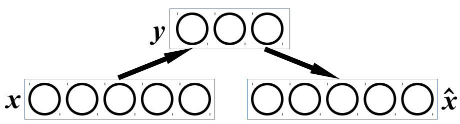
$$\tilde{\mathbf{W}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

iteration	$\mathbf{e}^T$	$(\tilde{\mathbf{W}}\mathbf{e})^T$	$\mathbf{y}^T$	$(\mathbf{W}^T\mathbf{y})^T$
1	$(100,100,0)$	$(100,66.66667)$	$(1,0.66667)$	$(1.6667,1.6667,0.66667)$
2	$(0.6,0.6,0)$	$(0.6,0.4)$	$(0.6,0.26667)$	$(0.86667,0.86667,0.26667)$
3	$(1.1538,1.1538,0)$	$(1.1538,0.76923)$	$(0.69231,0.20513)$	$(0.89744,0.89744,0.20513)$
4	$(1.1143,1.1143,0)$	$(1.1143,0.74286)$	$(0.77143,0.15238)$	$(0.92381,0.92381,0.15238)$
5	$(1.0825,1.0825,0)$	$(1.0825,0.72165)$	$(0.83505,0.10997)$	$(0.94502,0.94502,0.10997)$

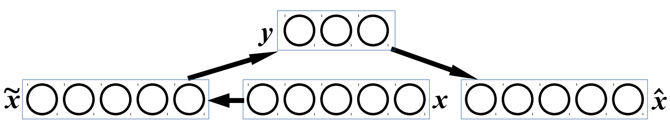
So output is $\begin{pmatrix} 0.83505 \\ 0.10997 \end{pmatrix}$

Note, competition results in the first neuron increasing its output, while the output of the second neuron is suppressed. Also, note that the vector $\mathbf{W}^T\mathbf{y}$ becomes similar to the input $\mathbf{x}$. $\mathbf{W}^T\mathbf{y}$ converges towards a reconstruction of the input.

10. The figure below shows an autoencoder neural network.



Draw a diagram of a de-noising autoencoder and briefly explain how a de-noising autoencoder is trained.



The network is trained so that the output, $\hat{x}$, reconstructs the input, x. However, before encoding is performed the input is corrupted with noise. This avoids overfitting.