MATH 11B: WHAT IS INTEGRAL CALCULUS?

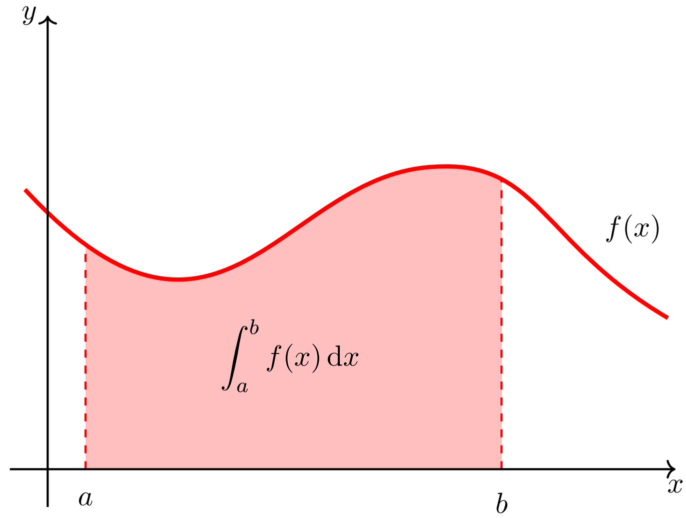
Contents

In this writeup, we try to address how the area problem is related with differential calculus, which was covered in Math 11A (limits and derivatives).

1. The Area Problem

Suppose we have a function $f(x)$. What is the area of the region bounded by $f(x)$, the x-axis, and the vertical segments $x = a$ and $x = b$? (see picture below).

These types of problems come up all the time; for example in physics, these questions come up in trying to determine displacement from velocity. In microeconomics, these questions come up in trying to determine consumer and producer surplus. In biology, the Poiseuille's law of fluids, which says flux is proportionate to the radius of the blood vessel, is also a type of area problem.



2. Riemann Sums are approximations

Looking at the picture again, denote the total area of the region shaded by $\displaystyle\int_a^b f(x)\,\mathrm{d}x$. Thus the question becomes what is the value of $\displaystyle\int_a^b f(x)\,\mathrm{d}x$?

We don't know how to take the area immediately, but we can approximate $\int_a^b f(x)\,\mathrm{d}x$ by drawing geometric shapes that we *do know how to calculate the area of*, namely rectangles, trapezoids, triangles, and parallelograms. This is the idea of the the *Riemann sum*.

In Math 11A, you couldn't calculate the slope of the tangent line of $f(x)$ at a point $x = a$ immediately...we had to first use the slope formula on two points on f, namely $f(a)$ and $f(a+h)$ to give us the slope of a secant line:

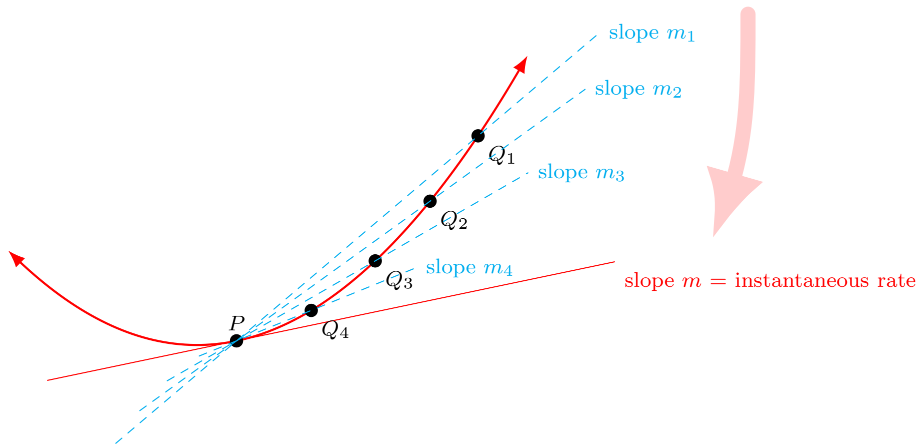
$$\big(\text{slope of tangent line of } f(x) \text{ at } x = a\big) \approx \underbrace{\frac{f(a+h) - f(a)}{h}}_{\text{slope of a secant line passing through } x=a}$$

and only when we finally took the **limit** we were able to get the slope of the tangent line:

$$\big(\text{slope of tangent line of } f(x) \text{ at } x = a\big) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The picture below shows the limiting process for approximating the tangent line from secant lines[1]:

[1]TeX code from `https://tex.stackexchange.com/questions/460632/tikz-and-secant-line-diagram`.



The Riemann sum will play an analogous role that the secant lines did in computing the slope of the tangent lines.

Example 1. Let's try to approximate the area of the region bounded by $x = 0, x = 10$ and the function $f(x) = x^2$. Say we want to use 10 rectangles and for simplicity, suppose all of the rectangles have the same base length. (It may help to draw this function out on paper and try to follow along by drawing rectangles on the paper!)

Since

$$\text{length of rectangle} = \frac{b-a}{\text{number of rectangles}},$$

we have length of each rectangle will be length 1. Starting our rectangles at 0, we generate the following table of values:

x	0	1	2	3	4	5	6	7	8	9	10
$f(x) = x^2$	0	1	4	9	16	25	36	49	64	81	100

Now suppose we want to use *left hand sum* (left Riemann sum) for each rectangle. To find the approximate area, we sum up the area of all of the individual rectangles. Since area of rectangle is (*base*) $\times$ (*height*), where base = length of rectangle, which in our case, we computed length = 1. So we now just need the height of the rectangles.

For the first rectangle (going left to right on the x-axis), sits on 0 and 1 and since we are using *left hand rule*, the height is determined by the $x = 0$. So the first rectangle's area is $0 \times 1 = 0$.

The second rectangle sitting right next to the first, which sits on 1 and 2, has height of 1 since 1 is the left-most number of this rectangle and $f(1) = 1$. Thus, area of this rectangle is 1.

The third rectangle is sitting on 2 and 3. The height is determined by $x = 2$ since we are considering left-hand rectangles (left Riemann sums) and so $f(2) = 4$ is the height of this rectangle. Thus the area of this third rectangle is $1 \times 4 = 4$.

Keep going onto this process until we get to the 10th rectangle, which sits on 9 and 10, has length 1 and height determined by $x = 9$. The height is $f(9) = 81$ and so the area of this rectangle is 81.

Summing up all of these 10 areas, we have

$$\begin{aligned}\int_0^{10} x^2 &\approx (1 \times 0) + (1 \times 1^2) + (1 \times 2^2) + (1 \times 3^2) + \cdots + (1 \times 9^2) \\ &= 0 + 1 + 4 + 9 + 16 + \cdots + 81 = 285.\end{aligned}$$

So when asked with to find a Riemann sum of a function $f(x)$ between $b - a$ and you're asked to use n-number of rectangles, the outline is:

(1) If all rectangles are of equal length, the length $= (b - a)/n$ where n is the number of rectangles
(2) Place all your rectangles on the interval from a to b
(3) Determine the height of each rectangle individually, starting from the leftmost rectangle, where height = the function's value at the leftmost point of the rectangle
    - If you're asked to do right hand rectangle (right hand Riemann sums), do the same, but change the height of the rectangle to be height = function's value at the rightmost point of each rectangle.
    - If you're asked to do midpoint rectangles, then do the same, but change the height so that the height = function's value at midpoint of each rectangle along the base
(4) Calculate the area of each individual rectangle
(5) Sum them all up!

3. The Fundamental Theorem of Calculus

If we recall from Math 11A, we never really used the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to talk about derivatives or calculate slope of tangent lines. They were only used in constructing the definition of the derivative; they are only helpers in creating the tool that we are interested in but the helpers themselves don't address the question we are genuinely interested in.

The Riemann sums, as we said in the previous section, are like the secant lines in differential calculus and they are only helpers in defining the integral. The **fundamental theorem of**

calculus allows us to finally ignore the definition of the integral (after it is defined in terms of a limit of a Riemann sum).

The fundamental theorem of calculus comes in two parts:

Fundamental Theorem of Calculus

Theorem 1. *Let f be a continuous function defined on a closed interval* $[a, b]$ (i.e., there are no discontinuities or asymptotes in $[a, b]$, including the endpoints).

(i) *Let c be a number such that $a \leq c \leq b$ and define another function*

$$F(x) = \int_c^x f(t)\,\mathrm{d}t \qquad \textit{for all } x \textit{ such that } a \leq x \leq b.$$

Then $F'(x) = f(x)$.

(ii) *Let F be any anti-derivative of the function $f(x)$* (with the conditions above). *Then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

What the fundamental theorem is basically saying is that the area problem can be solved by using anti-derivatives! The table below categorises the analogous roles the different characters in differential and integral calculus play:

	Differential Calculus	Integral Calculus
Problem	Finding rate of change of a function	Finding area under a function
Approximation tool	Secant lines	Riemann sums
Actual Tool	Derivative	Anti-derivatives
Result:	plug in point into derivative $f'(a)$	plug endpoints into anti-derivative

That is, if you were asked to find the slope of the tangent line of a function $f(x)$ at a point $x = a$, Math 11A skills tells us that we should

(1) calculate the derivative of $f(x)$ first (we get to use rules like chain rule, product, etc.)
(2) plug in $x = a$ into the derivative; the slope of the tangent line was equal to $f'(a)$.

In a similar capacity, if you are now asked to find the area under a function $f(x)$ between the points $x = a$ and $x = b$, the fundamental theorem says you should now

(1) calculate an anti-derivative $F(x)$, where $F(x)$ is a function (with the $+c$) such that $F'(x) = f(x)$,
(2) plug in the endpoints $x = b$ and $x = a$ into the antiderivative; the area of the region was $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.

We remark that in calculus, we have orientation involved and so it is possible to have negative areas, just as we could have negative velocities when viewing velocity as the derivative of displacement (speed is the absolute value of velocity and that's the reason why we can't have negative speed- there's no orientation).

We also make a remark that

$$\int f(x)\,\mathrm{d}x = \text{“indefinite integral”} = \textit{an} \text{ anti-derivative of } f(x)$$

$$\int_a^b f(x)\,\mathrm{d}x = \text{“definite integral”} = \text{area of region bounded by } x=a,\ x=b, \text{ and bounded by } f(x)$$

4. Integrals of Elementary Functions and Rules

Using the fundamental theorem of calculus, we are able to find integral of elementary functions. *If you ever want to double check your answer when finding an anti-derivative, just take the derivative of your result and you should get back the original function you are integrating.*

Integrals of Elementary Functions

Below, k denotes a constant number (it can be a negative number too). The $+C$ is necessary since all anti-derivatives of a given function will differ from each other by a constant.

$$\int k\,\mathrm{d}x = kx + C \qquad \int x^k\,\mathrm{d}x = \begin{cases} \dfrac{1}{k+1}x^{k+1} + C & \text{if } k \neq -1 \\ \ln|x| + C & \text{if } k = -1 \end{cases}$$

$$\int e^x\,\mathrm{d}x = e^x + C \qquad \int k^x\,\mathrm{d}x = \frac{1}{\ln(k)}k^x + C \text{ (assuming } k > 0)$$

$$\int \sin(x)\,\mathrm{d}x = -\cos(x) + C \qquad \int \cos(x)\,\mathrm{d}x = \sin(x) + C$$

$$\int \sec^2(x)\,\mathrm{d}x = \tan(x) + C \qquad \int \csc^2(x)\,\mathrm{d}x = -\cot(x) + C$$

$$\int \sec(x)\tan(x)\,\mathrm{d}x = \sec(x) + C \qquad \int \csc(x)\cot(x)\,\mathrm{d}x = -\csc(x) + C$$

Example 2. Try to evaluate $\int x^2\,\mathrm{d}x$ and $\int 2x\,\mathrm{d}x$.

To compute these, we have

$$\int x^2\,\mathrm{d}x = \frac{x^3}{3} + C \qquad \text{since } \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^3}{3} + C\right] = x^2$$

and $$\int x \,\mathrm{d}x = \frac{x^2}{2} + C \qquad \text{since } \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2}{2} + C\right] = x$$

Rules for Integrals

(1) Integral of constant times f is constant times integral of f: if k is a number (including negative numbeers), then
$$\int_a^b k \cdot f(x) \,\mathrm{d}x = k \cdot \int_a^b f(x) \,\mathrm{d}x \,.$$
(2) Integral of Sums (resp. differences) is Sum (resp. difference) of integrals:
$$\int_a^b \big(f(x) + g(x)\big) \,\mathrm{d}x = \int_a^b f(x) \,\mathrm{d}x - \int_a^b g(x) \,\mathrm{d}x$$
$$\int_a^b \big(f(x) - g(x)\big) \,\mathrm{d}x = \int_a^b f(x) \,\mathrm{d}x - \int_a^b g(x) \,\mathrm{d}x$$
(3) Break up interval rule: Let c be a number such that f is still integrable from $[a, c]$ and $[c, b]$ (reverse the order in the interval if needed):
$$\int_a^b f(x) \,\mathrm{d}x = \int_a^c f(x) \,\mathrm{d}x + \int_c^b f(x) \,\mathrm{d}x$$
(4) Flipping endpoints costs a negative sign:
$$\int_b^a f(x) \,\mathrm{d}x = -\int_a^b f(x) \,\mathrm{d}x$$
(5) Monotonicity: If $f(x) \le g(x)$ for every single x in the interval $[a, b]$, (i.e., for every x satisfying $a \le x \le b$, we have $f(x) \le g(x)$), we have the inequality
$$\int_a^b f(x) \,\mathrm{d}x \le \int_a^b g(x) \,\mathrm{d}x \,.$$

5. Exercises

Question 1. *Evaluate the following integrals and also provide the geometric interpretation (i.e., in terms of areas) for each of the integrals:*

$$\text{(a)}\quad \int_0^{2\pi} \sin(x)\,\mathrm{d}x$$

$$\text{(b)}\quad \int_0^{2\pi} |\sin(x)|\,\mathrm{d}x$$

$$\text{(c)}\quad \left|\int_0^{2\pi} \sin(x)\,\mathrm{d}x\right|$$

Question 2. *Determine the value*

$$\int_{-2}^{20} f(x)\,\mathrm{d}x$$

given that $\displaystyle\int_0^{-2} f(x)\,\mathrm{d}x = 10,\ \int_0^{35} f(x)\,\mathrm{d}x = 42,\ \int_{10}^{20} f(x)\,\mathrm{d}x = 5,\ \int_{20}^{35} f(x)\,\mathrm{d}x = -10.$

Question 3. *Evaluate the following definite integral:*

$$\int_{\cos(x)}^{\sin(x)} t\,\mathrm{d}t\,.$$

Question 4. *Evaluate the following expression:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\cos(x)}^{\sin(x)} t\,\mathrm{d}t$$

6. Solutions to the Exercises

Question 1. *Evaluate the following integrals and also provide the geometric interpretation (i.e., in terms of areas) for each of the integrals:*

$$\text{(a)} \quad \int_0^{2\pi} \sin(x)\,\mathrm{d}x$$

$$\text{(b)} \quad \int_0^{2\pi} |\sin(x)|\,\mathrm{d}x$$

$$\text{(c)} \quad \left|\int_0^{2\pi} \sin(x)\,\mathrm{d}x\right|$$

Solution: (a) Using the fundamental theorem of calculus, we get that since $-\cos(x) + C$ is the anti-derivative of $\sin(x)$, we have that

$$\int_0^{2\pi} \sin(x)\,\mathrm{d}x = -\cos(x)\Big|_{x=0}^{2\pi} = -\cos(2\pi) - (-\cos(0)) = -1 + 1 = \boxed{0.}$$

For part (b), we note that $\sin(x)$ is negative for all x in the interval $(\pi, 2\pi)$ and positive from $(0, \pi)$, we see that

$$|\sin(x)| = \begin{cases} \sin(x) & \text{for all } x \text{ such that } 0 \le x \le \pi \\ -\sin(x) & \text{for all } x \text{ such that } \pi \le x \le 2\pi \end{cases}$$

So to evaluate this integral, we should split the integral into two:

$$\begin{aligned} \int_0^{2\pi} |\sin(x)|\,\mathrm{d}x &= \int_0^{\pi} \sin(x)\,\mathrm{d}x + \int_{\pi}^{2\pi} -\sin(x)\,\mathrm{d}x \\ &= -\cos(x)\Big|_{x=0}^{\pi} - \left(-\cos(x)\Big|_{x=\pi}^{2\pi}\right) \\ &= \Big(-\cos(\pi) - (-\cos(0))\Big) - \Big(-\cos(2\pi) - (-\cos(\pi))\Big) \\ &= \boxed{4} \end{aligned}$$

and finally for (c), we simply need to look at (a) since we are just taking the absolute value of our result in (a). Thus

$$\left|\int_0^{2\pi} \sin(x)\,\mathrm{d}x\right| = \boxed{0}.$$

Question 2. *Determine the value*

$$\int_{-2}^{20} f(x)\,\mathrm{d}x$$

given that $\displaystyle\int_0^{-2} f(x)\,\mathrm{d}x = 10$, $\displaystyle\int_0^{35} f(x)\,\mathrm{d}x = 42$, $\displaystyle\int_{10}^{20} f(x)\,\mathrm{d}x = 5$, $\displaystyle\int_{20}^{35} f(x)\,\mathrm{d}x = -10$.

SOLUTION: We first notice that the first integral has its bounds flipped, so we flip the bounds, but at the cost of a negative sign. That is,

$$\int_{-2}^{0} f(x)\,\mathrm{d}x = \int_{0}^{-2} f(x)\,\mathrm{d}x = -10.$$

We can now use the broken interval rule

$$\int_{-2}^{35} f(x)\,\mathrm{d}x = \int_{-2}^{0} f(x)\,\mathrm{d}x + \int_{0}^{35} f(x)\,\mathrm{d}x = -10 + 42 = 32.$$

We note that we have another integral given going from 20 to 35 and so we can subtract that integral from our current integral, since

$$\underbrace{\int_{-2}^{35} f(x)\,\mathrm{d}x}_{\text{our current integral}} = \underbrace{\int_{-2}^{20} f(x)\,\mathrm{d}x}_{\text{what we want}} + \underbrace{\int_{20}^{35} f(x)\,\mathrm{d}x}_{\text{given value is } -10}$$

and so we have

$$\begin{aligned}\underbrace{\int_{-2}^{20} f(x)\,\mathrm{d}x}_{\text{what we want}} &= \underbrace{\int_{-2}^{35} f(x)\,\mathrm{d}x}_{\text{our current integral}} - \underbrace{\int_{20}^{35} f(x)\,\mathrm{d}x}_{\text{given value is } -10} \\ &= 32 - (-10) \\ &= \boxed{42.}\end{aligned}$$

Question 3. *Evaluate the following definite integral:*

$$\int_{\cos(x)}^{\sin(x)} t\,\mathrm{d}t\,.$$

SOLUTION: Using the fundamental theorem of calculus, we know that the anti-derivative (indefinite integral) of t is given by $\dfrac{t^2}{2} + C$ and so we just plug in our endpoints: thus we conclude that

$$\int_{\cos(x)}^{\sin(x)} t\,\mathrm{d}t = \frac{t^2}{2}\Big|_{t=\cos(x)}^{\sin(x)} = \frac{\sin^2(x)}{2} - \frac{\cos^2(x)}{2}.$$

Question 4. *Evaluate the following expression:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\cos(x)}^{\sin(x)} t\,\mathrm{d}t$$

SOLUTION: Let $F(x) = \displaystyle\int_{\cos(x)}^{\sin(x)} t\,\mathrm{d}t$. From the previous question, we compute that

$$F(x) = \frac{\sin^2(x)}{2} - \frac{\cos^2(x)}{2}.$$

Now the question is just asking what is the derivative of this? Remembering that $\sin^2(x) = (\sin(x))^2$ (just notation change, no fancy maths involved), we can use the *chain rule* and get

$$F'(x) = \sin(x)\cos(x) + \sin(x)\cos(x) = 2\sin(x)\cos(x).$$

Thus,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\cos(x)}^{\sin(x)} t\,\mathrm{d}t = 2\sin(x)\cos(x).}$$