MATH 11A WEEK 2
INVERSES, EXPONENTIATION, LOGS, AND SEQUENCE FUN

Contents

Announcements:

(1) Office hours
   - M 4:40 PM - 5:40 PM
   - Th 11:30 AM - 12:30 PM

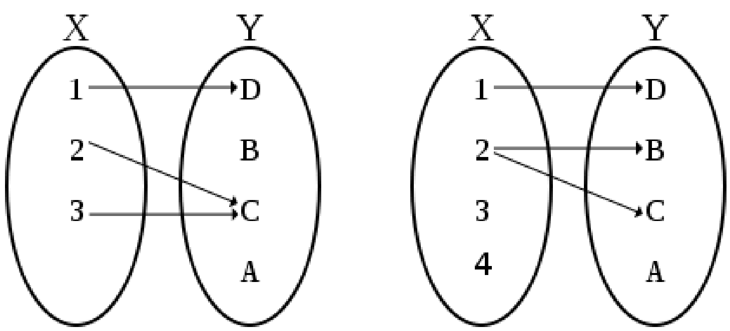
(2) Send an email to `ykim172@ucsc.edu` if these times do not work for you

1. Functions

Functions require

- domain set (collection of inputs to plug in)
- a range set (collection of outputs)
- a rule that assigns exactly one output (from the range set) for every input in the domain.

We can get the same output for different inputs, but never different outputs for the same input. For example, look at the pictures below[1].
We see that the picture on the left is a function (since every input has exactly one output) but the picture on the right is **not** a function since the input 2 has two different outputs.



The notion that there is exactly 1 output for every input is the reason why the vertical line test works in confirming whether a graph of a curve is a function or not.

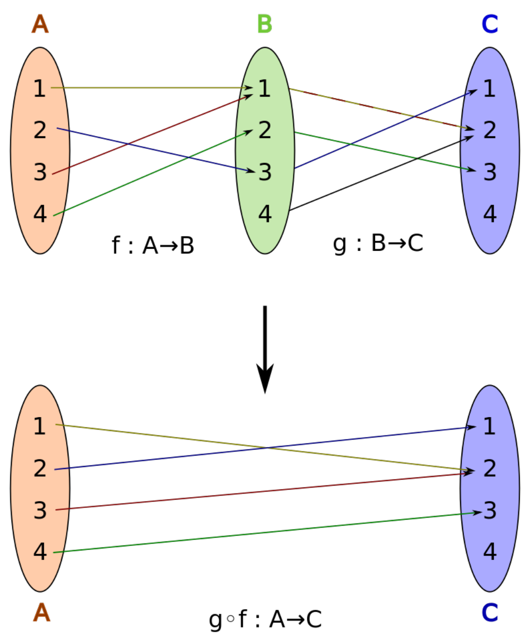
In the xy-coordinate plane, a point on the plane is given by (x, y). Given a function f, we consider the points in the xy-coordinate plane of the form $\big(x, f(x)\big)$. The collection of all the points of the form $(x, f(x))$ is called the **graph of f**.

2. Function Composition, Inverses

The definition of a function never says that the inputs have to be numbers- you can define a function on *any collection of objects*, such as fruits, lists, other functions, etc.

[1]Pictures are from Wikipedia, `https://en.wikipedia.org/wiki/Function_(mathematics)`.

We restrict to the case of other functions. If we have two function f and g where $Range(f) = Domain(g)$, then we can define a new function which takes inputs in $Domain(f)$ and sends it to $Range(g)$; consider the picture below[2]



Thus, we conclude that for any x in $Domain(f)$ that we define the function $g \circ f$ as

$$(g \circ f)(x) \stackrel{\text{def}}{=} g\left(f(x)\right)$$

We now come to the idea of **inverses** of functions. The idea is given a function f, can we “undo“ everything f did?

To talk undoing f, we have to come up with a **function** g which sends every $f(x)$ back to x. Thus, we have to come up with inputs, outputs, and a rule to define a function:

- $Domain(g) = Range(f)$
- $Range(g) = Domain(f)$
- rule: $f(x)$ gets sent to x

The last condition means that only a certain class of functions have an inverse. This leads us to the question:

[2]From Wikipedia, `https://en.wikipedia.org/wiki/Function_composition`

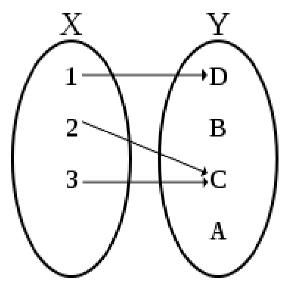
Which functions have an inverse?

We note that if g is the inverse function of f, then for every x in $Domain(f)$, we get

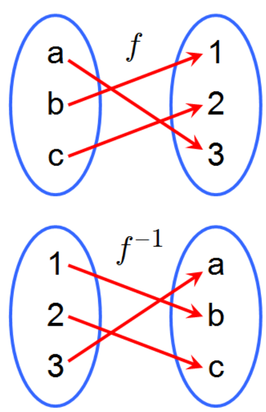
$$(g \circ f)(x) = g(f(x)) = x$$

where the last equality comes from the idea that g undoes f.

So if we go back to the fist picture



we note that an inverse of this function does **not** exist since we cannot "undo" C since $g(C)$ would take on two outputs, namely 2 and 3. Thus, we get that the inverse of a function f only exists if **every input in** $Dom(f)$ **gets a unique output and every output can be written as** $f(x)$ **for some** x **in** $Dom(f)$**.** This idea is probably best understood using pictures[3]



we see that every input gets a unique output, $f(a) \neq f(b) \neq f(c)$, and every output has an input, i.e., $1 = f(b), 2 = f(c), 3 = f(a)$ and there is nothing else in the $Range(f)$. Again, we

[3]Picture is from `https://en.wikipedia.org/wiki/Inverse_function`

see that for every x in $Dom(f)$ and every y in $Range(f)$ that

$$(f^{-1} \circ f)(x) = x, \quad (f \circ f^{-1})(y) = y.$$

A common technique to solve for the inverse is if you're given an equation of the form $y = f(x)$, the way we can find f^{-1} is by interchanging x and y in the given equation, then solve for y.

We conclude that the functions which satisfy the condition the functions that satisfy the condition

$$x \neq y \Longrightarrow f(x) \neq f(y)$$

are the functions that have inverses. We call these functions **one-to-one** functions.

Example 1

(a) $f(x) = \sin(x)$. Since we have $y = f(x) = \sin(x)$ we apply our technique to get $\sin(y) = x$. Thus, the inverse of sin is given by $f^{-1}(x) = \arcsin(x)$, which is often also written as $\sin^{-1}(x)$ and is different from $\csc(x) = 1/(\sin(x))$.
(b) $g(x) = x^3$. Writing $y = x^3$, apply the technique to get $x = y^3$. Solving for y gives us $y = x^{1/3}$ and so $g^{-1}(x) = x^{1/3}$.
(c) $h(x) = x^2$. This cannot be inverted since $h(-1) = h(1) = 1$ and we see that two inputs get taken to the same output.

If the last example bothers you, where you want to say $h^{-1}(x) = \sqrt{x}$, we note that this works is only true if x is positive; if x is negative, then you have to take $h^{-1}(x) = -\sqrt{x}$.

If you're asked to find the value of $f^{-1}(k)$ (where k is an actual number), another useful technique is to simply set $f(x) = k$ and solve for x.

Example 2

If $f(x) = x^3 + x - 4$, and is invertible, what is $f^{-1}(-2)$?
Trying to solve for the general formula $f^{-1}(x)$ by interchanging x and y would lead to a horrible mess so it will be easier to simply solve for x in the equation

$$f(x) = x^3 + x - 4 = -2 \Longrightarrow x = 1.$$

Thus, $f^{-1}(-2) = 1$.

3. Exponents, Logarithms

Fix a to be some real number. The product $\underbrace{a \cdot a \cdots a}_{n\text{-times}}$ is denoted a^n, and we call the number n to be the **exponent** and the number a is called the **base**. From this, we get the following rules:

$$a^b \cdot a^c = a^{b+c} \qquad \frac{a^b}{a^c} = a^{b-c}$$

$$(a^b)^c = a^{b \cdot c} \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Rules of Exponents

- If you see exponents with equal bases and are being multiplied, you combine by adding powers.
  For example,
  $$2^3 \cdot 2^6 = 2^9$$
  More generally,
  $$a^b \cdot a^c = a^{b+c}.$$
  This holds for all numbers a, b, c (including negative numbers).
- When equal bases are divided, you subtract! (this is equivalent to above)
  $$\frac{2^3}{2^6} = 2^{-3}$$
  and more generally
  $$\frac{a^b}{a^c} = a^{b-c} = a^b \cdot a^{-c}$$
- When an exponent is raised to a power, multiply the powers.
  $$(2^3)^2 = 2^{2 \cdot 3} = 2^6$$
  $$(2^4)^{-5} = 2^{-20}$$
  $$(2^a)^b = 2^{ab}$$
  Of course, we don't need to work base 2, so in full generality, we have for all numbers
  $$(a^b)^c = a^{bc} = (a^c)^b$$
- Using the subtraction rule, if $a \neq 0$ and setting $b = c$, we get
  $$a^0 = 1 \quad \text{and} \quad a^{-b} = \frac{1}{a^b}$$

Fixing a to be some real number, we say f is an **exponential function** if $f(x) = a^x$.

We define the **logarithm** to be the inverse function of the exponential function. That is,

$$\log_a(a^x) = x \quad \text{and} \quad a^{\log_a(x)} = x.$$

It is impossible to define $\log(0)$ and $\log(x)$ if $x < 0$. (What is the domain of $\log(x)$ and why?)

Using the fact that

$$a^y = x \iff \log_a(x) = y$$

So if you see $\log_a(x) = y$, immediately convert it into the exponential form (the left side of the arrow above). Doing this, we get the following properties of logs:

$$\log_a(bc) = \log_a(b) + \log_a(c) \qquad \log_a\left(\frac{b}{c}\right) = \log_a(b) - \log_a(c)$$

$$\log_a(b^c) = c\log_a(b)$$

Summarising the properties of exponents and logarithms, we have

Rules of Exp and Log

$$a^b \cdot a^c = a^{b+c} \qquad \frac{a^b}{a^c} = a^{b-c}$$

$$(a^b)^c = a^{b\cdot c} \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$\log_a(bc) = \log_a(b) + \log_a(c) \qquad \log_a\left(\frac{b}{c}\right) = \log_a(b) - \log_a(c)$$

$$\log_a(b^c) = c\log_a(b)$$

4. Sequences

Another important concept is the idea of a sequence. A **sequence** is a (ordered) list of numbers $a_0, a_1, a_2, \ldots$. Sequences may be infinitely long. The n-th term of the sequence is a_n.

Examples of Sequences

- The collection of numbers $1, 2, 3, 4, 5, 6, ...$ is an infinite sequence where $a_n = n$.
- The collection of numbers $2, 7, 12, 17, \ldots$ is a sequence where $a_n = 2+5(n-1) = 5n-3$.

We say a sequence is **recursively defined** (or **recursive sequence**) if the n-th term of the sequence is generated by some formula involving previous terms.

Example of Recursive Sequence

The *Fibonacci sequence* is a very famous sequence $1, 1, 2, 3, 5, 8, 13, ...$ where $a_0 = 1, a_1 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for all $n \geq 1$.
This is a recursive sequence since we find what the next term in the sequence is by adding the previous two terms, which are also determined by previous values, which are....(keeps going on and on and on, hence the adjective "recursive".

5. Exercises

Solutions to the problems are included in the next page.

Question 1. *Solve for all x such that $9^x = 3^x + 4$.*
(Hint: write 9 as 3^2 and use quadratic formula.)

Question 2. *Let f be a function such that $f(n+1) = 1 - 2\left(f(n)\right)^2$ for all $n = 1, 2, \ldots$ (all positive whole numbers). Can we express $f(n+2)$ in terms of $f(n)$?*

6. Solutions

Question 1. *Solve for all x such that $9^x = 3^x + 4$.*

Since $9 = 3^2$, we have $9^x = (3^2)^x = 3^{2x} = (3^x)^2$. Let $a = 3^x$. This transforms the equation

$$(3^x)^2 = 3^x + 4 \xrightarrow{a=3^x \text{ substitution}} a^2 = a + 4$$

Thus $a^2 - a - 4 = 0$. Using the quadratic formula, we get that

$$a = \frac{1 \pm \sqrt{1^2 - 4(1)(-4)}}{2(1)} = \frac{1 \pm \sqrt{17}}{2}$$

However, we care about x, not a. However, since we know that $a = 3^x$, and using log base 3, we get that

$$\log_3(a) = \log_3(3^x) = x$$

and so our solution is

$$x = \log_3\left(\frac{1 \pm \sqrt{17}}{2}\right).$$

Before saying this is the answer, recall the domain of log, which is all positive real numbers (we cannot plug in 0 or negative numbers). Since $1 - \sqrt{17} < 0$, we don't consider this solution. Thus, we have

$$\boxed{x = \log_3\left(\frac{1 + \sqrt{17}}{2}\right)}$$

Question 2. *Let f be a function such that $f(n+1) = 1 - 2\left(f(n)\right)^2$ for all $n = 1, 2, \ldots$ (all positive whole numbers). Can we express $f(n+2)$ in terms of $f(n)$?*

Note that the relation $f(n+1) = 1 - 2\left(f(n)\right)^2$ holds for all positive integers, and so using the silly observation that $n + 2 = (n+1) + 1$, we get that

$$\begin{aligned} f(n+2) &= f\left((n+1)+1\right) \\ &= 1 - 2\left(f(n+1)\right)^2 && \text{using the relation} \end{aligned}$$

but since we know that $f(n+1) = 1 - 2\left(f(n)\right)^2$, we get that

$$\begin{aligned} (f(n+1))^2 &= \left(1 - 2\left(f(n)\right)^2\right)^2 \\ &= 1^2 + 2(1)\left(-2\left(f(n)\right)^2\right) + \left(-2\left(f(n)^2\right)\right)^2 \\ &= 1 - 4f(n)^2 + 4f(n)^4 \end{aligned}$$

Plugging this into the above equation, we get that

$$\begin{aligned}
f(n+2) &= 1 - 2\left(f(n+1)\right)^2 \\
&= 1 - 2\left(1 - 4f(n)^2 + 4f(n)^4\right) \\
&= 1 - 2 + 8f(n)^2 - 8f(n)^4 \\
&= -1 + 8f(n)^2 - 8f(n)^4
\end{aligned}$$

Thus, we have our solution

$$\boxed{f(n+2) = -1 + 8f(n)^2 - 8f(n)^4.}$$