ALGEBRA 2: COURSE NOTES

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> Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.
>
> Sir Michael Atiyah

Contents

Part 1. Preliminaries

1. Binary Operations

The whole study of mathematics is about Objects and Relationships between the Objects.

The most basic type of object is a number and the most basic operations on numbers are operations we've gotten to be familiar with since primary school: addition (+), subtraction (−), multiplication (×) and division (÷). These operations are what we call ***binary operations*** since these are instructions on *how to combine* ***two*** *objects*.

If we have x and y, then $x + y$ is formed from taking x and then combining it with y and treating the combined group as it's own group.

If we were asked to combine $x = 3$ apples and $y = 5$ apples, then $x + y = 8$ apples and we treat the group of 8 apples as it's own thing. To communicate that the combined group is to be treated as it's own individual thing, we place ( ) around $(x + y)$, but this is frequently dropped because it's tedious to keep the ( ) around. We will keep the ( ) if in the case someone might get confused, but we drop the ( ) when there's no risk of confusion.

1.1. Order of Operations.

While the operations $+, -, \times, \div$ tell us how to combine numbers, we should also be familiar with how the operations act *with each other*.

To help clarify the confusion and the symbolic mess that can occur, we use the convention PEMDAS to inform us which operations happen first:

(1) **P**arantheses
(2) **E**xponents
(3) **M**ultiplication
(4) **D**ivision
(5) **A**ddition
(6) **S**ubtraction

So we simplify all of the **P**arantheses first (going from Left to Right) and then move onto simplifying all of the material at the level of **E**xponents (going from Left to Right), and then all of the **M**ultiplication (going from Left to Right), and so on.

Example 1. Simplify the expression

$$10 - 2 \cdot 7 + (11 - 5)^2 \div 3 + 2(10 - 3)^{4^2 - 45 \div 3}.$$

To start, we take it slowly and do the Parentheses first, going from left to right:

$$\begin{aligned} 10 - 2 \cdot 7 + (11 - 5)^2 \div 3 + 2(10 - 3)^{4^2 - 45 \div 3} &= 10 - 2 \cdot 7 + (6)^2 \div 3 + 2(7)^{4^2 - 45 \div 3} \\ &= 10 - 2 \cdot 7 + (6)^2 + 2(7)^{16 - 45 \div 3} \\ &= 10 - 2 \cdot 7 + (6)^2 + 2(7)^{16 - 15} \end{aligned}$$

$$
\begin{aligned}
&= 10 - 2 \cdot 7 + 6^2 + 2(7)^1 \\
&= 10 - 2 \cdot 7 + 36 + 2(7) \\
&= 10 - 14 + 36 + 14 \\
&= -4 + 36 + 14 \\
&= 32 + 14 \\
&= 46
\end{aligned}
$$

Question 1. Simplify and compute the expression

$$10 - (\frac{6}{3} - 4 \times 2) - 8 \times 3 + 1.$$

1.2. **Commutative Operations.** We want to be able to come up with some shortcuts that help us accurately simplifies big expressions.

Considering addition $(+)$, we have that $3 + 5 = 5 + 3$ and in general, we can always take $x + y = y + x$ whenever x, y are replaced with numbers. The importance of this is that we see

$$\textsc{First Object} + \textsc{Second Object} = \textsc{Second Object} + \textsc{First Object}$$

as in we can change the order between two objects under addition! Whenever this occurs, we have a ***commutative*** operation.

We cannot always change the order we see the objects since the operation we work with may not be commutative. For example, since we have

$$5 - 3 \neq 3 - 5$$

and so subtraction is not commutative.

Question 2. Would the mixing of MILK and CHEER-E-O's be a commutative operation?

A few shortcuts that is useful for simplifying an expression are

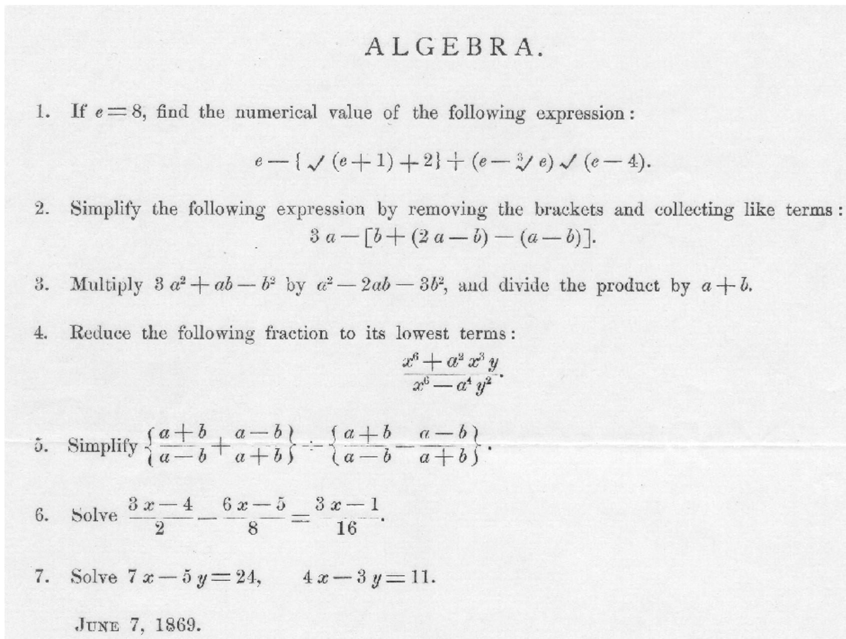
(1) $+$ and $\times$ are commutative
(2) $1 \times a = a = a \times 1$ (multiplication by 1 does nothing)
(3) $0 + a = a = a + 0$ (addition by 0 does nothing)
(4) $b - a = b + (-a) = -a + b$
(5) $-(a - b) = -a + b = b - a$

Question 3. Compute the following:

$$
\begin{aligned}
(a) &\quad 5\big((2 + 7) \div 3\big) \\
(b) &\quad \frac{8 + 2 \times 3}{12}
\end{aligned}
$$

2. Characterizations and Algebraic Equations

2.1. Letters in Algebra. Your first encounter with mathematics when you were younger might have been memorizing addition and multiplication tables. However, this is in fact only *arithmetic*[1] (no letters or variables) while *algebra*, the arithmetic with letters, was considered some of the more difficult material traditionally. For example, the picture below is the 1869 MIT admissions algebra exam.



ALGEBRA.

1. If $e=8$, find the numerical value of the following expression:

$$e-\{\sqrt{(e+1)}+2\}+(e-\sqrt[3]{e})\sqrt{(e-4)}.$$

2. Simplify the following expression by removing the brackets and collecting like terms:

$$3a-[b+(2a-b)-(a-b)].$$

3. Multiply $3a^2+ab-b^2$ by $a^2-2ab-3b^2$, and divide the product by $a+b$.

4. Reduce the following fraction to its lowest terms:

$$\frac{x^6+a^2x^3y}{x^6-a^4y^2}.$$

5. Simplify $\left\{\frac{a+b}{a-b}+\frac{a-b}{a+b}\right\}\div\left\{\frac{a+b}{a-b}-\frac{a-b}{a+b}\right\}$.

6. Solve $\frac{3x-4}{2}-\frac{6x-5}{8}=\frac{3x-1}{16}$.

7. Solve $7x-5y=24, \quad 4x-3y=11.$

June 7, 1869.

A question that many students ask themselves when studying algebra is "why do we use letters in place of numbers in algebra?" The idea is for us to generalize our observations and emphasize key ideas. Moreover, it helps make the expressions shorter.

To solve problems involving letters in math, we think of these letters as either abbreviations or nicknames. For example, the *mass-energy equivalence equation* states that

$$\text{Energy of Particle} = \text{Mass of Particle} \cdot (\text{speed of light})^2$$

but this is popularly written as

$$E = mc^2, \quad c = \text{speed of light}$$

because it doesn't take too much work to explain that E represents "energy" and m represents "mass" while the formula is more compact.

An ***equation*** takes *two* (possibly different) objects and says that these two objects *are actually the same object.*

[1]This was later renamed as "mathematics" as colloquially known today

For example, suppose you were trying to explain to a child what it would mean to finish with a silver medal at the Olympics. We implicitly equate silver with finishing second place-but what exactly does "second" mean in this context? In English, the word "two" and "second" (or 2nd) both express the same idea, referring to the smallest positive even number 2. So you might explain that "the silver medal means achieving the second best result in the competition- with one person doing better than second place". So in a way, in terms of placement in a competition, we equate "gold = 1st = 1", "silver = 2nd = 2", "bronze = 3rd = 3" and so on. We should note that "two" and "second" are *not* identical- they're spelt differently from each other! However, these differences in spelling do not matter in the context of trying to talk about placement in an ordering, and so we draw the equality between 2nd with 2 and similarly, equate silver medal placement in sports as 2nd (or 2).[2]

When presented with an equation, you are being asked two things about a statement:

(1) is the equation (as a statement) `True` or is it `False`?
(2) what conditions are needed to make that statement `True` (or `False`)?

You should ask yourself these two questions whenever you try to write down algebraic expressions that captures the idea of what you are trying to model.

Example 1. Consider the equation below:

$$x + y = y + x - z.$$

Here, the letters x, y, and z all represent *any* (fixed) object.

Question: under what conditions, if any, is the equality true?

For us to tackle this, we set the x, y, z equal to certain numbers, then we evaluate the left side and then evaluate the right side.

If we set $x = 1, y = -3, z = 6$, then we get

$$\begin{aligned}\underbrace{x+y}_{\text{left side}} = \underbrace{y+x-z}_{\text{right side}} &\Longrightarrow 1 + -3 = -3 + 1 - 6 \\ &\Longrightarrow -2 = -8 \Longrightarrow \texttt{False}\end{aligned}$$

$\Longrightarrow x = 1, y = -3, z = 6$ does not work.

Testing a whole bunch of numbers is a very poor approach, so we use algebra to help reduce the work:

$$\begin{aligned}\underbrace{x+y}_{\text{left side}} = \underbrace{y+x-z}_{\text{right side}} &\Longrightarrow x + y + (-y - x) = y + x - z + (-y - x) \\ &\Longrightarrow 0 = -z \\ &\Longrightarrow z = 0\end{aligned}$$

so we get that equality will be true if $z = 0$ while x and y are free.

[2]In other words, context matters!

We confirm our conclusion with a test: setting $x = 23, y = -3$ and $z = 0$, we get

$$\underbrace{x+y}_{\text{left side}} = \underbrace{y+x-z}_{\text{right side}} \Longrightarrow 23 + -3 = -3 + 23 - 0$$

$$\Longrightarrow 20 = 20 \Longrightarrow \texttt{True}$$

Question 4. Solve for all values of x that will make the following equation true:

$$2x + 3 - 4 = 3x - 5 - x$$

Question 5. Solve for all values of x that will make the following equation true:

$$5 - \frac{x}{7} = 19$$

Question 6. Solve for x in the equation $2x - 5 = -3(x - 5)$.

Question 7. Think of a number. Add 3 to it, then multiply the result by 2. Then subtract your number. Subtract 4. Then subtract your number once more.

You get 2, don't you?

Explain how the trick works.

Question 8. Evaluate each expression for the given values of the variables.

$$\begin{array}{lll} (a) & 4a + 7b + 3b - 2b + 2a & \text{where } a = -5, b = 3 \\ (b) & -k^2 - (3k - 5n) + 5n & \text{where } k = -1, n = -2 \\ (c) & -5(x + 2y) + 15(x + 2y) & \text{where } x = 7, y = -7 \end{array}$$

Question 9. Write an expression that is simpler version of $4m + 3(m + n)$.

Question 10. Consider the two following quantities:

Option A	**Option B**
$4(\frac{1}{2}x + 2y)$	$2x + 8y$

Which of the following is `True`?

(a) Option A is larger than Option B.
(b) Option B is larger than Option A.
(c) Option A and B are equal in value.
(d) Not enough information was given.

2.2. **Distributive Law.**

Question 11. Let x, y be numbers. Using the distributive law, expand out

$$(x + y)^2$$

in red level rigor. (Hint: you will need to use commutative property).

Question 12. Let x, y be numbers.

(a) Prove that
$$(x+y)(x-y) = x^2 - y^2.$$
(b) Compute $(105) \cdot (95)$ using the formula above (try to pattern match; what should we set x and y to?)

Question 13. Expand out $(x+5)(x+5)$.

Question 14. Expand out $(x-2)(x-2)$.

Question 15. Expand out $(x+9)(x+9)$.

Question 16. Expand out $(x+9)(x-9)$.

Question 17. Expand out $(x+2)(x-2)$.

Question 18. Expand out $(x+\sqrt{2})(x-\sqrt{2})$

Question 19. Expand out $(x+3)(x-2)$

Question 20. Expand out $(x^2+y^2)(x^2-y^2)$.

2.3. Solving for x.

Question 21. Indicate which of the integers below that have all of the following properties:

(1) it is prime
(2) when it is squared, it is going to be greater than 20.

Question 22. Indicate which of the integers below that have all of the following properties:

(1) the list of primes in its prime factorization, when added together, is going to be greater 20.
(2) the number is less than 10 and is the product of two primes.

2.4. Prime Factorization.

Recall that a number x is an ***integer*** if it is has no decimal parts. So for example, -2 and 2023 are integers, but the number 2.5 and π are not integers.

Definition. A positive integer x is ***prime*** if only has two different factors: 1 and itself.

- Primes are strictly positive, so -3 and 0 are not prime even though they are integers (called *composite numbers*).
- 1 is not prime since the definition requires two *different* factors.
- 2 is the only prime even integer.
- There are an infinite number of primes! This was proved by Euclid in 300 B.C.
- The first few primes are:
$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots$$

While all positive integers are either *prime* or *non-prime*, which is commonly called composite, Euclid proved that there was a stronger relationship between primes and composites- namely he proved that every composite is made up of smaller primes and so we can start viewing prime numbers as the core building blocks for all other numbers!

Theorem 1 (Fundamental Theorem of Arithmetic, Euclid)**.** *Every positive integer can be written as the product of primes. That is, every positive integer $x > 1$ can be written as*

$$x = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \cdots \times p_r^{e_r}$$

where the $p_1, p_2, p_3, \ldots, p_r$ are primes. Moreover, this factorization is unique, up to reordering, so any prime factorization of x is called the prime factorization.

The upshot about the Fundamental Theorem of Arithmetic:

- Tweaking any of the primes p_i will completely change the value of x. Similarly, tweaking any of the details about the exponents e_i by either swapping them to a different prime or by changing its values will also completely change the value of x.
- We can assign an "identity" to each positive integer where that identity is going to be unique and only made up of primes.
- The only changes that is allowed in a prime factorization is the order in which we do the multiplication. So

$$45 = 3^2 \times 5^1$$

is going to be the same as

$$45 = 5^1 \times 3^2$$

since we only changed the order of multiplication, but *kept all the other details the exact same*!

We summarize the process of finding prime factorization using a factoring tree.

Example 2. Compute the prime factorization of $112, 612, 715$ and 1224.

112612

7151224

Question 23. Compute the prime factorisation of 2020 and 2022. Is it true that we can decompose *any* positive integer into primes? If so, is the factorisation unique?

2.5. **GCD and LCM.**

2.6. Inequalities.

Question 24. Sketch on a number line each of the following:

(a) $x \in \mathbb{R}$ such that $-2 \leq x < 3$
(b) $x \in \mathbb{Z}$ such that $-2 \leq x < 3$
(c) $x \in \mathbb{R}$ such that $x \geq 2$

2.7. Intervals.

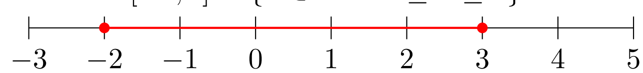
Assuming we know how to compare any two objects in $\mathbb{R}$ (or $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or irrationals) we can place all of the objects on a number line.

An ***interval*** is a set of the form:

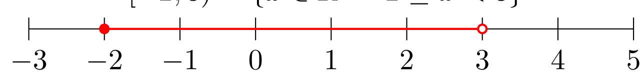
$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$
$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$
$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

For example we let $a = -2$ and $b = 3$. Note the endpoints: if it is a hard bracket [ ] then you include that point and have a filled in circle. if it is () then you don't include that point and have an open circle:

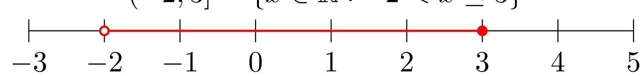
$$[-2, 3] = \{x \in \mathbb{R} : -2 \leq x \leq 3\}$$



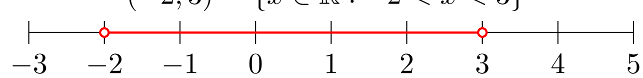
$$[-2, 3) = \{x \in \mathbb{R} : -2 \leq x < 3\}$$



$$(-2, 3] = \{x \in \mathbb{R} : -2 < x \leq 3\}$$

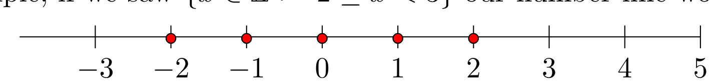


$$(-2, 3) = \{x \in \mathbb{R} : -2 < x < 3\}$$



The "$x \in \mathbb{R}$" is specifying that we are working on the number line and the : is a "such that", which is followed by the inequalities.

If we changed the setting from $x \in \mathbb{R}$ to $x \in \mathbb{Z}$, then we only want to highlight the integers on the line. For example, if we saw $\{x \in \mathbb{Z} : -2 \leq x < 3\}$ our number line would look like:



(note that 3 is not touched due the fact it was $x < 3$ and $x \in \mathbb{Z}$; compare this to the second number line on the previous page).

Question 25. Sketch on a number line each of the following sets:

(a) $\{x \in \mathbb{N} : 4 \leq x < 10\}$
(b) $\{x \in \mathbb{Z} : -2 \leq x < 4\}$
(c) $\{x \in \mathbb{R} : -2 \leq x < 4\}$
(d) $\{x \in \mathbb{Z} : x \geq 2\}$
(e) $\{x \in \mathbb{R} : x > 0\}$

2.8. **Combining Intervals.**

Question 26. Let $A = [0, 1]$ and $B = (-1, 1)$. Sketch on a number line each of the following sets:

(a) $A \cup B$
(b) $A \cap B$

Part 2. **Sequences**

3. Patterns and Equations

One of the most important skills to have in problem solving is being able to recognize patterns and communicate these observations to others. One of the most important and accessible concepts in math is the idea of a ***sequence***, which is an ordered list of numbers.

- there does **not** need to be a pattern, though we like to study sequences that do exhibit some observable pattern
- sequences can be finite or infinite in length

Example 1. Each of the following is a sequence- there does not need to be pattern!

(a) $1, 2, 3, 4, 5, \ldots$
(b) $-1, 0, -4, 2, 2, 4, -2022, \frac{\pi}{2}, \sqrt{5}, \sqrt{\pi^3}, \sqrt{5}, 1, \ldots$
(c) $1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1, \ldots$

Notation. The convention is to label a sequence of numbers $a_1, a_2, a_3, \ldots$ as (a_n). That is, the (a_n) is the "list name" while the a_1 refers to the first item in (a_n), a_2 refers to the second item in (a_n), while a_k refers to the kth item in (a_n), where k is free to be any positive integer. For example, if we consider the sequence $(2023, 7, 0, -1, 5, 5, 8, 1, 12)$, then

$$\underbrace{2023}_{a_1}, \underbrace{7}_{a_2}, \underbrace{0}_{a_3}, \underbrace{-1}_{a_4}, \underbrace{5}_{a_5}, \underbrace{5}_{a_6}, \underbrace{8}_{a_7}, \ldots$$

and we say a_1 is "first term", a_2 is "second term", a_3 is "third term", etc.[3].

When talking about sequences, there's two sets of data we need to keep track of: (1) the objects on the list and (2) where the objects are within the list. To help keep track of both, we let a_n to be the object that's on the list while n keeps track of where the object is placed. The n is a *variable* since n can be any natural number.

Example 2. Let (a_n) be the sequence given below:

$$(3, 5, 1, -3, 7, \pi, 5^{23}, 0, 2023)$$

For each given n, identify a_n.

(a) $n = 1$
(b) $n = 4$
(c) $n = 2$
(d) $n = 7$

If we set $n = 1$, then we get that $a_{n=1} = a_1 = 3$ since 3 is the first term. If we set $n = 4$, then we get that $a_4 = -3$ since -3 is the fourth term. If we set $n = 2$, then we get $a_2 = 4$ since 4 is the second term. If we set $n = 7$, then we get $a_7 = 5^{23}$ since 5^{23} is the seventh term.

Question 27. Let (a_n) and (b_n) be the sequences given below:

n	1	2	3	4	5	6	7
(a_n)	1	4	6	8	10	12	14
(b_n)	1	4	9	16	25	36	49
(c_n)	3	3	3	-3	-3	-3	3

(1) Identify the 5th object in each sequence (which of the above are sequences?)
(2) Define a new sequence by the equation

$$(d_n) \overset{\text{def}}{=} a_n + b_n + c_n$$

What are the values of d_1, d_2, d_3, d_4, and d_5 going to be equal to?

4. Recursive Formulas

We're mostly interested in sequences that exhibit some sort of pattern, and we can "automate" the process of creating sequences.

Consider a sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots \qquad (\textit{called Fibonacci sequence})$$

- what are the next 3 numbers?

[3]It is also very common to start sequences off at a_0, the zeroth term. There is no need to start at a non-negative number, we can start at a_{-5}, for example (any non-fraction works). Can you explain why?

- how did you get those 3 numbers?

You most likely used a ***recursive formula***:

$$\begin{aligned} 2+3&=5 & 3+5&=8 \\ 5+8&=13 & 8+13&=21 \\ 13+21&=34 & 21+34&=55 \end{aligned}$$

if F_n denotes the nth Fibonacci number, then you followed the recipe

$$\underbrace{F_{n+1}}_{\text{next term}} = \underbrace{F_{n-1}}_{\text{previous term}} + \underbrace{F_n}_{\text{current term}} \quad \text{for } n \geq 2, \quad \text{with } \underbrace{F_1 = 1, F_2 = 1}_{\text{initial conditions}}$$

This is a recursive formula since it tells you how to generate the next term from previously known terms. A ***recursive formula*** *will tell you how to generate the next term in the sequence from the previous terms.* In the case of the Fibonacci sequence, we see that it if we follow the recursive formula defined above, then we get

$$\begin{aligned} &F_1 = 1, && \text{initial conditions} \\ &F_2 = 1, && \text{initial conditions} \\ &F_3 = F_1 + F_2 \Longrightarrow F_3 = 2, && \text{when } n = 2 \\ &F_4 = F_2 + F_3 \Longrightarrow F_4 = 3, && \text{when } n = 3 \\ &F_5 = F_3 + F_4 \Longrightarrow F_5 = 5 && \text{when } n = 4 \\ &F_6 = F_4 + F_5 \Longrightarrow F_6 = 8 && \text{when } n = 5 \end{aligned}$$

Even though we write down the pattern we observe amongst the numbers as a recursive formula, we need to include the initial conditions since your results will vary wildly even if we follow the same recursive formula. To see this, consider the sequence with the same recursive formula as the Fibonacci sequence but with slightly different conditions:

$$f_{n+1} \stackrel{\text{def}}{=} f_{n-1} + f_n \quad \text{for } n \geq 2, \quad \text{with } f_1 = 5, f_2 = -1$$

We then see that

$$\begin{aligned} &f_1 = 5, && \text{initial conditions} \\ &f_2 = -1, && \text{initial conditions} \\ &f_3 = f_1 + f_2 \Longrightarrow f_3 = 4, && \text{when } n = 2 \\ &f_4 = f_2 + f_3 \Longrightarrow f_4 = 3, && \text{when } n = 3 \\ &f_5 = f_3 + f_4 \Longrightarrow f_5 = 7 && \text{when } n = 4 \\ &f_6 = f_4 + f_5 \Longrightarrow f_6 = 10 && \text{when } n = 5 \end{aligned}$$

Example 1. Check that the following recursive formulas match the sequence of numbers below:

Sequence	Recursive Form
$(a_n) = (1, 4, 7, 10, 13, \ldots)$	$a_{n+1} = a_n + 3, \quad a_1 = 1$
$(b_n) = (128, 64, 32, 16, 8, 4, 2, 1, \frac{1}{2}, \cdots)$	$b_{n+1} = b_n \cdot \dfrac{1}{2}, \quad b_1 = 128$
$(c_n) = (0, 3, 9, 21, 45, 93, \ldots)$	$c_{n+1} = 2c_n + 3, \quad c_1 = 0$

Example 2. Let (x_n) be a sequence defined by $x_{n+1} = 2x_n - 3n$, for $n \geq 1$ with $x_1 = 2$. Let's find the next 4 terms of the sequence x_2, x_3, x_4 and x_5.

Using the given recursive formula,

$$\begin{aligned}
&x_1 = 2 && \text{initial cond.}\\
&x_2 = x_{1+1} = 2x_1 - 3(1) \Rightarrow x_2 = 2(2) - 3 \Rightarrow x_2 = 1 && \text{when } n = 1\\
&x_3 = x_{2+1} = 2x_2 - 3(2) \Rightarrow x_3 = 2(1) - 6 \Rightarrow x_3 = -4 && \text{when } n = 2\\
&x_4 = x_{3+1} = 2x_3 - 3(3) \Rightarrow x_4 = 2(-4) - 9 \Rightarrow x_4 = -17 && \text{when } n = 3\\
&x_5 = x_{4+1} = 2x_4 - 3(4) \Rightarrow x_5 = 2(-17) - 12 \Rightarrow x_5 = -46 && \text{when } n = 4
\end{aligned}$$

Question 28. Find the next 3 terms in each sequence and then write a recursive formula for the sequence, including the initial conditions.

$$\begin{aligned}
(a_n) &= (80, 77, 74, 71, 68, \ldots)\\
(b_n) &= (4, 8, 16, 32, 64, \ldots)\\
(c_n) &= (\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \cdots)\\
(d_n) &= (1, 5, 14, 30, \ldots)
\end{aligned}$$

Example 3. Consider the *nested radical*

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$$

We can't solve for a numerical value of x just yet, but we can still model and the value that it can get really close to by using a sequence.

Suppose (x_n) is a sequence that satisfies the following recursive formula:

$$x_{n+1} = \sqrt{1 + x_n}, \quad x_1 = 1$$

As $n \to \infty$, we will get that nth term of the sequence (x_n) will become a better approximation for the true value of ϕ.

Question 29. A *continued fraction* x is a sequence of numbers $a_1, a_2, \ldots$ such that we can rewrite x as

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}$$

Let $a_1 = 1$ and let $a_{n+1} = 1 + \dfrac{1}{a_n}$ where $n \geq 1$.

Write out the expressions for a_2, a_3 and a_4 as continued fractions. Do not simplify.

Question 30. A *logistic map*, which models population given a reproduction rate r, is given by the recursive equation

$$x_{n+1} = rx_n(1 - x_n), n \geq 1, \quad 0 \leq x_1 \leq 1,\ r = \text{reproduction rate.}$$

John Von Neumann, a Hungarian-American mathematician who pioneered much of game theory, functional analysis, and quantum mechanics, thought up the logistic map $x_{n+1} = 4x_n(1 - x_n)$ as a random number generator (RNG).

(a) Using Von Neumann's RNG, what are the values of x_2, x_3, and x_4 if $x_1 = \dfrac{1}{2}$?
(b) If we use the initial condition $x_1 = 1$, what would be the values of x_2, x_3, and x_4 in the random number generator?

5. Closed Formulas

Look at the Fibonacci sequence $(1, 1, 2, 3, 5, 8, 13, 21, \ldots)$ and a second sequence $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$.

Question. What are the 100th terms of each sequence?

The 100th Fibonacci number would require computation of the 99th and 98th Fibonacci numbers, which those will require computations themselves, while the second sequence's 100th term is easily guessable from the pattern as "$\dfrac{1}{100}$".

This is the advantage of using a ***closed form***, which is a formula that automatically computes the kth term of a sequence for you, without needing any other information about the sequence.

- if you can give the 2023rd term of a sequence, you are most likely using a *closed formula*
- a closed formula for a given sequence might **not** be known!
- You can check `https://oeis.org/` to see if there's a known closed formula for a given sequence and even applications of a sequence!

Example 1. Suppose we have a closed form $a_n = \dfrac{(n+1)(n+2)}{n}$. Compute the first 4 terms of the sequence, i.e., compute a_1, a_2, a_3, a_4.

We get

$$\begin{aligned} a_1 &= \frac{(1+1)(1+2)}{1} = \frac{6}{1} \\ a_2 &= \frac{(2+1)(2+2)}{2} = \frac{12}{2} \\ a_3 &= \frac{(3+1)(3+2)}{3} = \frac{24}{3} \\ a_4 &= \frac{(4+1)(4+2)}{4} = \frac{30}{4} \end{aligned}$$

Example 2. Check that the following recursive formulas match the sequence of numbers below:

Sequence	Closed Form
$(a_n) = (1, 4, 7, 10, 13, \ldots)$	$a_n = 1 + 3(n-1)$
$(b_n) = (128, 64, 32, 16, 8, 4, 2, 1, \frac{1}{2}, \cdots)$	$b_n = \frac{128}{2^{n-1}}$
$(c_n) = (1, 4, 9, 16, 25, 36, \ldots)$	$c_n = n^2$

Question 31. Your friend claims that the sequence $(s_n) = (1, 4, 7, 10, \ldots)$ has the closed form $s_n = 3n + 1$. Is your friend right or wrong? What would be the correct closed form if you think they're incorrect?

Question 32. Find the 7th term of each of the following sequences:

$$a_n = 2n^2 - 7 \qquad b_n = \frac{3n-1}{n+2} \qquad c_n = 10(n-1) \qquad d_n = 10$$

Question 33. Consider the sequences a_n and b_n.

(a) Suppose we had formulas for the sequences a_n and b_n and we list them below:

$$\begin{aligned} a_n &= \frac{1}{n}, \quad n \geq 1 \\ b_n &= 2b_{n-1} + 1, \quad b_1 = 1,\ n \geq 2 \end{aligned}$$

Compute the first 3 terms of each sequence.

(b) Which sequence in the previous part was written in recursive form? Which one was in closed form?

Question 34. Consider the sequences a_n and b_n.

(a) Suppose we had formulas for the sequences a_n and b_n and we list them below:

$$a_n = \frac{1}{2n+1}, \quad n \geq 1$$

$$b_{n+1} = -b_n + 2, \quad b_1 = 1, \ n \geq 2$$

Compute the first 3 terms of each sequence.

(b) Which sequence in the previous part was written in recursive form? Which one was in closed form?

Question 35. Let a_n be a sequence given by the recursive formula

$$a_{n+1} = 2a_n + n \cdot a_{n-1}, \quad \text{for } n \geq 2$$

where $a_1 = 1$ and $a_2 = 1$.

What are the next 3 terms?

6. Arithmetic Sequences

A sequence (a_n) is called an ***arithmetic sequence*** *if it has the recursive formula*

$$a_{n+1} = a_n + d, \quad \text{for } n \geq 1,$$

for *some fixed constant* d, which is oftentimes called the "common difference".

That is, an arithmetic sequence (x_n) is completely characterized by its initial condition x_1 and its common difference d.

Example 1. Some examples of arithmetic sequences are:

$$\begin{aligned}
(a_n) &= (1, 4, 7, 10, \ldots) && \text{where } d = 3, \text{initial cond. } a_1 = 1 \\
(b_n) &= (1, 2, 3, 4, 5, \ldots) && \text{where } d = 1, \text{initial cond. } b_1 = 1 \\
(c_n) &= (17, 13, 9, 5, \ldots) && \text{where } d = -4, \text{initial cond. } c_1 = 17 \\
(d_n) &= (\pi, \pi, \pi, \ldots) && \text{where } d = 0, \text{initial cond. } d_1 = \pi
\end{aligned}$$

The sequence $(1, 2, 4, 6, 8, 10, \ldots)$ is **not** an arithmetic sequence since the difference between terms changed from 1 to 2.

Question 36. Find the next three terms in the arithmetic sequence $5, -2, \ldots$.

Question 37. Find the 100th term in the pattern $2, 3, 4, 5, 6, \ldots$

Question 38. Find the 100th term in the pattern $2, 4, 6, 8, \ldots$

6.1. Closed Form.

Consider the sequence $1, 5, 9, 13, 17, \ldots$ What might be the 2023rd term be in the sequence? While we note that the sequence is arithmetic sequence, its recursive formula is not going to be very efficient for us to use.

Working more generally, suppose we have the arithmetic sequence (a_n). Listing out the terms in our sequence, and using the assumption that our sequence is arithmetic, we get that

$$a_1, \ a_2, \ a_3, \ \ldots \quad \text{where } \underbrace{a_{n+1} = a_n + d, \quad n \geq 1}_{\text{recursive formula for } (a_n)}.$$

Using the recursive formula, we can start rewriting the terms in the sequence:

$$\begin{aligned}
a_2 &= a_1 + d \\
a_3 &= a_2 + d \Longrightarrow a_3 = (a_1 + d) + d \Longrightarrow a_3 = a_1 + 2d \\
a_4 &= a_3 + d \Longrightarrow a_4 = (a_1 + 2d) + d \Longrightarrow a_4 = a_1 + 3d \\
a_5 &= a_4 + d \Longrightarrow a_5 = (a_1 + 3d) + d \Longrightarrow a_5 = a_1 + 4d \\
&\vdots \\
a_{100} &= a_{99} + d \Longrightarrow a_{100} = (a_1 + 98d) + d \Longrightarrow a_{100} = a_1 + 99d \\
&\vdots \\
a_n &= a_{n-1} + d \Longrightarrow a_n = (a_1 + (n-2)d) + d \Longrightarrow \boxed{a_n = a_1 + (n-1)d}
\end{aligned}$$

which is our desired closed formula for the arithmetic sequence.[4]

Example 2. Consider the sequence $1, 5, 9, 13, 17, \ldots$ What is the 2023rd term?

We can find this by first giving the closed form, then plugging in $n = 2023$.

Observe that the sequence is an arithmetic sequence with $d = 4$ so our closed form for

$$a_n = a_1 + d(n-1) \Longrightarrow a_n = 1 + 4(n-1)$$

so $a_{2023} = 1 + 4(2023 - 1) = 1 + 4(2022) = 8089$. **Remark.** In the example above, the

numbers should technically be written as $(1, 5, 9, 13, 17, \ldots)$ according to our notation. We will occasionally drop the () when there's no confusion between the *nth*-term of a sequence and the sequence name itself. This is because mathematicians are surprisingly lazy at times.

Example 3. Let (x_n) be an arithmetic sequence with $x_3 = 3$ and $x_6 = -3$.

(a) What is the common difference?
(b) What is the value of x_1?

Recall that arithmetic sequences are completely characterized by their common difference, which is the *constant* amount that's required to go between one term to the next, so starting with the information we're given:

$$\begin{aligned}
3 = x_3 \xrightarrow{+d} x_4 \xrightarrow{+d} x_5 \xrightarrow{+d} x_6 = -3 \\
&\Longrightarrow x_6 = -3 \text{and } x_6 = 3 + 3d \\
&\Longrightarrow -3 = 3 + 3d \\
&\Longrightarrow 6 = 3d \Longrightarrow 2 = d
\end{aligned}$$

so the common difference in this arithmetic sequence is $d = 2$.

[4] This technically needs to be proven rigorously by mathematical induction, but this suffices for now.

To find the value of x_1, we can use the closed formula:

$$\begin{aligned} x_3 = x_1 + (3-1)d &\Longrightarrow 3 = x_1 + 2(d) \\ &\Longrightarrow 3 = x_1 + 4 \Longrightarrow x_1 = -1 \end{aligned}$$

Question 39. Consider a sequence of numbers $2, 6, 10, 14, \ldots$

(a) Find the next three term in the sequence.
(b) Is this sequence arithmetic, geometric, or neither?
(c) Find a recursive form for this sequence.
(d) Find a closed form for this sequence (find the nth term).
(e) Find the 20th term in the sequence.

Question 40. Consider the sequence $(2, 9, 16, 23, 30, \ldots)$

(a) Show that the sequence is arithmetic by showing that there is a common difference.
(b) Find the closed formula for this sequence.
(c) Find the 100th term of the sequence.
(d) Prove that 828 is in the sequence by finding where it would be in the sequence.
(e) Is 2023 in the sequence?

Question 41. Suppose (a_n) is an arithmetic sequence with $a_1 = 9$, $a_2 = 3k$ and $a_3 = k^2$. What is the value of k?

Question 42. How long is the sequence $(1, 4, 7, \ldots, 31, 34)$?

Question 43. How long is the sequence

$$(1, 3, 5, 7, \ldots, 995, 997, 999)?$$

7. Geometric Sequences

A sequence is called a **geometric sequence** *if it has a recursive form*

$$a_{n+1} = a_n \cdot r, \quad \textit{for } n \geq 1,$$

where r is some fixed constant, which we call a "common ratio".

That is, a geometric sequence (x_n) is completely characterized by its initial condition x_1 and its common ratio r.

Example 1. Some examples of geometric sequences are:

$$\begin{aligned} &(a_n) = (1, 2, 4, 8 \ldots), \text{ where } r = 2, \text{ initial cond. } a_1 = 1 \\ &(b_n) = (7, 21, 63, 189, \ldots), \text{ where } r = 3, \text{ initial cond. } b_1 = 7 \\ &(c_n) = (1, -1, 1, -1, 1, \ldots), \text{ where } r = -1, \text{ initial cond. } c_1 = 1 \\ &(d_n) = (100, 10, 1, \frac{1}{10}, \frac{1}{100}, \cdots), \text{ where } r = \frac{1}{10}, \text{ initial cond. } d_1 = 100 \end{aligned}$$

The sequence $1, 1, 1, 2, 4, 8, \ldots$ is *not* a geometric sequence since the ratio between terms changed from 1 to 2.

7.1. **Closed Form.** Consider the sequence $2024, 1012, 506, 253, \ldots$ What might be the 2023rd term be in the sequence? While we note that the sequence is geometric sequence, again we find that the recursive formula is not going to be very efficient for us to use.

So let's approach this more generally: suppose we have the arithmetic sequence (a_n). Listing out the terms in our sequence, and using the assumption that our sequence is geometric, we get that

$$a_1,\ a_2,\ a_3,\ \ldots \quad \text{where } \underbrace{a_{n+1} = a_n \cdot r, \quad n \geq 1}_{\text{recursive formula for } (a_n)}.$$

Using the recursive formula, we can start rewriting the terms in the sequence:

$$\begin{aligned}
a_2 &= a_1 r \\
a_3 &= a_2 r \Longrightarrow a_3 = (a_1 r)r \Longrightarrow a_3 = a_1 rr = a_1 r^2 \\
a_4 &= a_3 r^2 \Longrightarrow a_4 = (a_1 r^2)r \Longrightarrow a_4 = a_1 r^3 \\
a_5 &= a_4 r^3 \Longrightarrow a_5 = (a_1 r^3)r \Longrightarrow a_5 = a_1 r^4 \\
&\vdots \\
a_{100} &= a_{99} r \Longrightarrow a_{100} = (a_1 r^{98})r \Longrightarrow a_{100} = a_1 r^{99} \\
&\vdots \\
a_n &= a_{n-1} \cdot r \Longrightarrow (a_1 r^{n-2})r \Longrightarrow a_n = (a_1 \underbrace{rr \cdots r}_{(n-2)\text{-copies of } r} r) \Longrightarrow \boxed{a_n = a_1 r^{n-1}}
\end{aligned}$$

which is our desired closed formula for the geometric sequence.[5]

Example 2. Consider the sequence $2024, 1012, 506, 253, \ldots$ What is the 2023rd term?

We can find this by first giving the closed form, then plugging in $n = 2023$.

Observe that the sequence is an geometric sequence with $r = \dfrac{1}{2}$ so our closed formula for the sequence is

$$\begin{aligned}
a_n = a_1 r^{n-1} &\Longrightarrow a_{2023} = 2024 \cdot (\frac{1}{2})^{2023-1} \\
&\Longrightarrow a_{2023} = 2024 \cdot (\frac{1}{2})^{2022} \approx 4.202 \times 10^{-606}
\end{aligned}$$

Example 3. Let (x_n) be an geometric sequence of positive numbers, with $x_2 = 2$ and $x_6 = 1/13$.

[5]This technique is the idea behind mathematical induction!

(a) What is the common ratio?
(b) What is the value of x_1?

Since we're given that this is geometric, we have

$$2 = x_2 \xrightarrow{\times r} x_3 \xrightarrow{\times r} x_4 \xrightarrow{\times r} x_5 \xrightarrow{\times r} x_6 = 1/13$$
$$\implies x_6 = 1/13, \text{ and } x_6 = 2 \times r^4$$
$$\implies 1/13 = 2r^4 \implies \frac{1}{26} = r^4 \implies r = \sqrt[4]{\frac{1}{26}} \approx 0.442\ldots$$

so the common ratio in this geometric sequence is $r = \sqrt[4]{\frac{1}{26}}$.

To find the value of x_1, we can use the closed formula:

$$x_2 = x_1 \cdot r^{(2-1)} \implies 2 = x_1 r$$
$$\implies x_1 = \frac{2}{r} \implies x_1 = 2/\sqrt[4]{\frac{1}{26}} = 0.0384615$$

Question 44. Consider a sequence of numbers $3, -15, 45, \ldots$

(a) Find the next three term in the sequence.
(b) Is this sequence arithmetic, geometric, or neither?
(c) Find a recursive form for this sequence.
(d) Find a closed form for this sequence (find the nth term).
(e) Find the 20th term in the sequence.

Question 45. A geometric sequence has $a_4 = 24$ and $a_7 = 192$. Compute/give an expression for a_{100}. What is the first term in the sequence?
Hint: Use the closed formula and try to compute the common ratio r.

Question 46. What is the value of the first term of the geometric sequence (a_n) if the first three terms are $k, k + 8$, and $9k$?

Formula Page for Special Sequences

If you are working with a sequence $a_1, a_2, \ldots$, (you start indexing from $n = 1$), then

- the nth term of an **arithmetic sequence** is $a_n = a_1 + d(n-1)$ for $n \geq 1$
- the nth term of a **geometric sequence** is $a_n = a_1 r^{n-1}$ for $n \geq 1$
- the recursive formula of an **arithmetic sequence** is $a_{n+1} = a_n + d$ for $n \geq 1$
- the recursive formula of a **geometric sequence** is $a_{n+1} = a_n \cdot r$ for $n \geq 1$

Warning: just because you observe a pattern in a sequence does **not** make those sequences either arithmetic or geometric sequences! Arithmetic and geometric sequences have a very rigid definition to them.

8. Limits of Sequences

In many problem-solving oriented fields, you are given a “black box”, where you are tasked to describe and master how the box works without being able to open the box and see what’s inside.

To analyze this box, one approach is to give the box a bunch of different inputs and see what the outputs are to see if there’s any observable pattern amongst the outputs. We will be studying lots of different mathematical objects by using sequences and seeing if there’s observable patterns in the sequences. More specifically, the majority of functions that we will focus on will be *Cauchy-continuous functions* (see section 10). To prepare for this, we need to build the fundamental concept of *limits* of sequences.

8.1. Technical details.

Definition. A sequence (a_n) is ***monotonic increasing*** if $a_n \leq a_{n+1}$ for all n. Similarly, (a_n) is ***monotonic decreasing*** if $a_n \geq a_{n+1}$ for all n.

- For monotonic increasing sequences, all of the terms are getting *more positive* (or getting bigger).
- For monotonic decreasing sequences, all of the terms are getting *more negative* (or getting smaller).

Example 1. The following sequences are monotonic increasing:

(a) $-3, -1, 0, 1, 2, 3, \pi, 2023, 2023.1, 2023.2, 2024, \ldots$
(b) $-3, -3, -3, -3, 3, 3, 3, \ldots$
(c) $0, 1, 2, 3, 4, \ldots$

The sequence $0, 1, 1, 2, 2, 3, 3, 3, 1, 4, 5, 6, \ldots$ is **not** a monotonic increasing sequence.

Question 47. Let (a_n) be a monotonic increasing sequence. Define a sequence (b_n) where $b_n = -a_n$ for each n. Is (b_n) going to be a monotonic decreasing sequence?

Definition. A sequence (a_n) is ***bounded*** if there exists a number M such that $|a_n| \leq M$ for all n.

Question 48. Identify if the following sequences are monotonic increasing, monotonic decreasing, or bounded:

(a) $1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$
(b) $2, 2, 2, 2, 2, 2, 2, \ldots$
(c) $1, 2, 4, 8, 16, 32, \ldots$
(d) $1, -2, 4, -8, 16, -32, \ldots$
(e) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$
(f) $1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \ldots$

Definition. A sequence (a_n) ***converges*** (or *approaches*) a ***limit*** L if for all $\varepsilon > 0$, we can find an N such that for any $n > N$, we get the inequality

$$|a_n - L| < \varepsilon$$

When the sequence (a_n) converges to L, we write $a_n \to L$.

- The idea is that the $|a_{n+1} - a_n|$ is getting smaller and smaller as n gets bigger i.e. consecutive terms are getting closer together.
- The sequence is settling towards a "black hole", which is the number L, and eventually gets *sooo* close to L that you can't tell if the sequence becomes L or not.

8.2. **Computation of Limits.** Some properties that help simplify computations of limits:

Proposition 2. *Every convergent sequence is bounded.*

- The converse is not true: a bounded sequence is not necessarily convergent since $(a_n) = (-1)^n$ is bounded by $M = 1$ but is not convergent.

Proposition 3. *Any sequence that is both monotonic and bounded will also be convergent.*

Proposition 4. *Suppose (a_n) and (b_n) are convergent sequences with limits A and B, respectively. That is, assuming $a_n \to A$ and $b_n \to B$, then*

(1) $(a_n + b_n) \to A + B$ *(limit of sums is sum of limits)*
(2) $(a_n - b_n) \to A - B$ *(limit of differences is difference of limits)*
(3) $(a_n \cdot b_n) \to A \cdot B$ *(limit of products is product of limits)*
(4) $(\frac{a_n}{b_n}) \to \frac{A}{B}$ *(limit of quotients is quotient of limits)*

Question 49. For each sequence,

(1) Give the first three terms of each sequence.
(2) Identify if the formula given is *closed* or *recursive*.
(3) Compute the limit of the sequence as $n \to \infty$.

$$\begin{aligned}
&(a) \quad a_n = 2a_{n-1} + 3, \quad \text{where } a_1 = 3\\
&(b) \quad b_n = (n-5)(n+5)\\
&(c) \quad c_n = \frac{(-1)^n}{n}\\
&(d) \quad d_n = \frac{2n+1}{3n-1}\\
&(e) \quad a_{n+1} = \frac{1}{2}a_n + 2, \quad a_1 = \frac{1}{2}.
\end{aligned}$$

Question 50. Compute the limits of the following sequences (if they exist).

$$(a) \quad a_n = \frac{1}{n(n+1)}$$

$$(b) \qquad b_n = \frac{n}{n+1}$$

$$(c) \qquad c_{n+1} = \sqrt{1 + c_n}$$

Review Exercises. The following exercises will go only through §2.1 – §2.5.

Question 51. Consider the function $f(x) = x^3 - 3x + 1$. Let a_n be a sequence given by the recursive formula

$$a_{n+1} = a_n + f(a_n), \quad \text{for } n \geq 2$$

where $a_1 = 0$. What are the next 3 terms?

Question 52. Consider the sequence

$$\begin{array}{ll}
a_1 & \bullet \\
a_2 & \bullet\ \bullet\ \bullet \\
a_3 & \bullet\ \bullet\ \bullet\ \bullet\ \bullet \\
a_4 & \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet \\
a_5 & \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet
\end{array}$$

(a) What is the pattern observed?
(b) Find a formula that describes a_n, i.e., come up with a closed form.
(c) How many dots will be in a_{2023}?

Question 53. (a) Let a_n be an arithmetic sequence with $a_1 = 24$ and common difference of 16. What will be the value of a_{62}?
(b) Let b_n be a geometric sequence with $b_1 = 8$ and $b_4 = 216$. What is the common ratio?

Question 54. Try to come up with a sequence of numbers that meets the following criteria or justify why it cannot be done:

(1) it is arithmetic
(2) it is geometric
(3) it is not a constant sequence (cannot be a sequence of all the same numbers)