MATH 11A WEEK 4
LIMITS AND CONTINUOUS FUNCTIONS FUNSIES

Contents

1. Limits of Functions

Recall that a sequence (a_n) converges to a limit L if after some index N, all the terms $a_{N+1}, a_{N+2}, \ldots$ are arbitrarily close to the constant L. Remember that we often denote that (a_n) converges to L by writing $a_n \to L$ or $\lim_{n\to\infty} a_n = L$.

So suppose we have a random sequence of real numbers $(x_n) \to a$ and let f be a function where its domain includes all of the numbers in our sequence (x_n). We can then create a new sequence

$$y_n \stackrel{\text{def}}{=} f(x_n).$$

Now, what the essential question that we are interested in is: since we are assuming that $x_n \to a$, what is the relation between $\lim_{n\to\infty} f(x_n)$ and $f(\lim_{n\to\infty} x_n)$?

Suppose we take a sequence $(a_n) \to a^-$, where the $-$ in the superscript denotes that every term in the sequence $a_i \leq a$ for every i (the entire sequence approaches a from below). We are able to consider a (new) sequence given by $f(a_n)$. Now if we have that $\lim_{n\to\infty} f(a_n)$ converges to a limit ℓ, we call ℓ the **left hand limit** of $f(x)$ as x approaches a. This is often written as $\lim_{x\to a^-} f(x) = \ell$.

In a similar manner, define another sequence $(b_n) \to a^+$ where the $+$ in the superscript denotes that $b_i \geq a$ (the entire sequences approaches a from above). If we have the sequence given by $\lim_{n\to\infty} f(b_i) = r$, then r is the **right-hand limit** of $f(x)$ as x approaches a. This is often denoted with $\lim_{x\to a^+} f(x) = r$.

Now in the special case where the right-hand limit of f and the left-hand limit of f actually are equal to each other, i.e.,

$$\lim_{x\to a^-} f(x) = \ell = r = \lim_{x\to a^+} f(x),$$

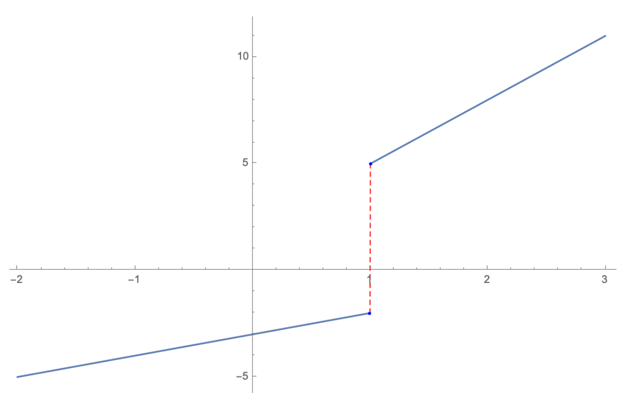
then we say that the value $\ell = r$ is the limit of $f(x)$ as $x \to a$. We should highlight that the sequences were arbitrarily given and so if we have that $\lim_{x\to a} f(x) = L$, then we must get that $f(x_n) \to L$ *for every sequence* such that $x_n \to a$. If we can find a single sequence that breaks that conclusion, we get that the limit of $f(x)$ does not exist at the point a.

Let's look at an example:

Example 1. Consider a function f given by

$$f(x) = \begin{cases} x - 3 & x < 1 \\ 3x + 2 & x > 1 \\ 1 & x = 1 \end{cases}$$

The graph of this function is given by[1]



The point of interest is what happens when $x = 1$. So we come up with any sequence $(a_n) \to 1^-$ and we plug in every term in the sequence into f to get a new sequence $f(a_n)$. Since $a_n \to 1^-$, we see that $f(a_n)$ approaches -2 and so -2 is the left-hand limit of f as x approaches 1. That is, $\lim_{x\to 1^-} f(x) = -2$.

[1]actually given by Mathematica

Now coming up with any sequence $(b_n) \to 1^+$, plug in every term in the sequence into f to get a new sequence $f(b_n)$. Since $b_n \to 1^+$, we see that $f(b_n)$ approaches 5 and so 5 is the right-hand limit of f as x approaches 1. That is, $\lim\limits_{x \to a^+} f(x) = 5$.

Since the left hand limit was -2 and the right hand limit is 5, and since $-2 \neq 5$, we get that this function does not have a limit at $x = 1$.[2] We note that $f(1) = 1$ by how f was defined (picture is slightly misleading); the limits are independent of how functions behave at a specified points!

Now, what if we wanted to look at what happens if we want to approach $x = 0$? Then we can come up with a sequence $(x_n) \to 0^-$ and another sequence $(y_n) \to 0^+$ and we would see that $f(x_n) \to -3$ and $f(y_n) \to -3$ as well. Thus, -3 is the left-hand limit of f as x approaches 0 and also is the right-hand limit of f as x approaches 0, and so we say that -3 is the limit of f as x goes to 0. $\square$

Just as we had a bunch of rules given for finding the limits of convergent sequences and combinations of convergent sequences via addition, subtraction, multiplication, and division, we have a very similar set of rules:

Rules about limits of functions

Fix a to be some real number and let $f(x)$ and $g(x)$ be functions such that $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$. Then we have the following rules:

(1) Constant and Identity function limits:
 (a) If k is a fixed number (not necessarily different from a), then $\lim\limits_{x\to a} k = k$.
 (b) Remember that the identity function $\mathrm{id}(x) = x$. It follows that
$$\lim_{x\to a} \mathrm{id}(x) = \lim_{x\to a} x = a.$$
(2) Limit of sums is Sums of limits:
$$\lim_{x\to a} \big(f(x) + g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$$
(3) Limit of differences is Differences of limits:
$$\lim_{x\to a} \big(f(x) - g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$$
(4) Limit of products is Products of limits:
$$\lim_{x\to a} \big(f(x) \cdot g(x)\big) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right) = L + M$$
(5) Limit of quotients is Quotients of limits:
$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{L}{M} \qquad \text{(assuming } M \neq 0\text{)}$$

[2] or as some people in one particular movie might say, "the limit does not exist"

Example 2

Evaluate the following limits:

(1) $\lim_{x\to 1} \frac{x^2+2x-5}{x^2+3x-5}$

For this, we apply the rules above:

$$\begin{aligned} \lim_{x\to 1} \frac{x^2+2x-5}{x^2+3x-5} &= \frac{\lim_{x\to 1} x^2+2x-5}{\lim_{x\to 1} x^2+3x-5} && \text{lim of quotients} \\ &= \frac{1^2+2-5}{1^2+3(1)-5} && \text{evaluate limits} \\ &= \frac{-2}{-1} = \boxed{2} \end{aligned}$$

(2) $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$.

For this one, we multiply by the "conjugate" and so

$$\begin{aligned} \lim_{x\to 0} \left(\frac{\sqrt{4+x}-2}{x}\right)\left(\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}\right) &= \lim_{x\to 0} \frac{4+x-4}{x\left(\sqrt{4+x}+2\right)} && \text{FOILing} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{4+x}+2} && \text{cancel } x \\ &= \frac{1}{\sqrt{4+0}+2} && \text{evaluate limit} \\ &= \boxed{1/4} \end{aligned}$$

(3) $\lim_{x\to 0} |x| = 0$.

We can rewrite $|x|$ as a piecewise function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

and so we can take the left hand limit and the right hand limits:

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0} -x = 0$$
$$\lim_{x\to 0^+} |x| = \lim_{x\to 0} x = 0$$

where we picked the functions that were directly left and right of zero respectively.

2. Continuous Functions

We saw that limits of a function f at a point a is independent of how f actually is defined. That is, $f(a)$ may not even exist but we can still evaluate limits concerning what happens near $f(a)$. Even if $\lim_{x\to a} f(x)$ exists, we can find examples where

$$\lim_{x\to a} f(x) \neq f(\lim_{x\to a} x) = f(a)$$

(see the first example).

So we must come with a class of functions for which we do achieve the equality

$$\lim_{x\to a} f(x) = f(a)$$

and these are the **continuous functions**.

So fixing a real number a, we say that $f(x)$ is **continuous at the point** a if for any sequence $x_n \to a$ we get that

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(a).$$

If this holds true for all real numbers a in the domain of the function f, then we say that f is a **continuous function**. In summary, continuous functions are precisely the functions where limits behave nicely, i.e., *all limits of a continuous function that should be there, actually are there.*

Theorem 1 (Intermediate Value Theorem)**.** *Let the closed interval $[a, b]$ be the domain of a continuous function f. Then for any constant k such that $f(a) \leq k \leq f(b)$ or $f(b) \leq k \leq f(a)$, then there exists a value c in the domain of f such that $a \leq c \leq b$ with $f(c) = k$.*

The consequence of this is that we can picture continuous functions on $\mathbb{R}$ to be the functions where there are no jump discontinuities in its graph. For reference, see the following figure:[3]

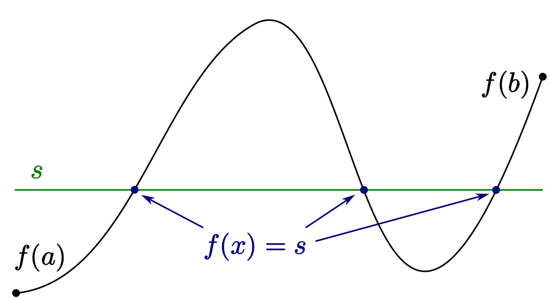


Figure 1. There are points c such that $f(c) = s$ for any s in between $f(a)$ and $f(b)$

[3]Figure is from the Wikipedia article: `https://en.wikipedia.org/wiki/Intermediate_value_theorem`

Theorem 2 (Extreme Value Theorem). *Let the closed interval $[a, b]$ be the domain of a continuous function f. Then f attains an absolute minimum and an absolute maximum, i.e., there are points c, d such that $a \leq c, d \leq b$ with $\underbrace{f(c)}_{\text{the min value}} \leq f(x) \leq \underbrace{f(d)}_{\text{the max value}}$ for all x in $[a, b]$.*

The extreme value theorem says that there will always be a minimum value and a maximum value that the function outputs, but it doesn't tell us when. One of the goals in calculus is to identify when these extreme values are attained and the extreme value theorem helps tells this story.

3. Exercises

Question 1. *Let*

$$f(x) = \begin{cases} \dfrac{|x-2|}{x-2} & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$$

Evaluate the following limits:

(a) $\lim_{x \to 2^+} f(x)$
(b) $\lim_{x \to 2^-} f(x)$
(c) *What is $f(2)$? Does the limit of $f(x)$ at $x = 2$ exist?*
(d) $\lim_{x \to 0^+} f(x)$
(e) $\lim_{x \to 0^-} f(x)$
(f) *What is $f(0)$? Is $f(x)$ continuous at $x = 0$? If so, why? If not, explain why it fails to be continuous.*

Question 2. *Recall that we have a special limit*

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Try to evaluate the limit

$$\lim_{\theta \to 0} \frac{\sin(3\theta)}{\theta}$$

Question 3. *What is the value of the limit*

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} = ?$$

Question 4. *Let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x *(*called the "floor function", and rounds down to the nearest whole number and never rounds up, for example $\lfloor 3 \rfloor = 3$, $\lfloor \pi \rfloor = 3$, $\lfloor 3.9 \rfloor = 3$.*) What is the value of the limit*

$$\lim_{x \to \ -4^-} \big(|x| - \lfloor x \rfloor\big) - \lim_{x \to 4^-} \big(|x| - \lfloor x \rfloor\big) = \ ?$$

Solutions to Exercises.

Question 1. *Let*

$$f(x) = \begin{cases} \dfrac{|x-2|}{x-2} & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$$

Evaluate the following limits:

(a) $\lim_{x\to 2^+} f(x) = \boxed{1}$
(b) $\lim_{x\to 2^-} f(x) = \boxed{-1}$
(c) *What is* $f(2)$*? Does the limit of* $f(x)$ *at* $x = 2$ *exist?*
$\boxed{f(2) = 0, \textit{No, limit does not exist.}}$
(d) $\lim_{x\to 0^+} f(x) = \boxed{-1}$
(e) $\lim_{x\to 0^-} f(x) = \boxed{-1}$
(f) *What is* $f(0)$*? Is* $f(x)$ *continuous at* $x = 0$*? If so, why? If not, explain why it fails to be continuous.* $\boxed{f(0) = -1, \textit{ is continuous at } x = 0.}$

Question 2. *Recall that we have a special limit*

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Try to evaluate the limit

$$\lim_{\theta\to 0} \frac{\sin(3\theta)}{\theta.}$$

Let $a = 3\theta$. Then $\theta = a/3$ and we get that if $\theta \to 0$, then $\theta = 3\theta \to 0$ remembering limits of products are products of limits. So we can rewrite the given equation as

$$\lim_{\theta\to 0} \frac{\sin(3\theta)}{\theta} = \lim_{a\to 0} \frac{\sin(a)}{a/3} = \lim_{a\to 0} \frac{3\sin(a)}{a} = 3 \lim_{a\to 0} \frac{\sin(a)}{a} = 3(1) = 3.$$

Question 3. *What is the value of the limit*

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = ?$$

Multiply by conjugate to get

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}+1}\right) = \lim_{x\to 1} \frac{(x-1)(\sqrt{x}+1)}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}+1}{1} = 2.$$

Question 4. *Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x (*called the "floor function", and rounds down to the nearest whole number and never rounds up, for example $\lfloor 3 \rfloor = 3, \lfloor \pi \rfloor = 3, \lfloor 3.9 \rfloor = 3.$*) What is the value of the limit*

$$\lim_{x \to \ -4^-} \left(|x| - \lfloor x \rfloor \right) - \lim_{x \to 4^-} \left(|x| - \lfloor x \rfloor \right) = \ ?$$

Let check and see what happens to the first term, and we see that

$$\begin{aligned}
&\lim_{x \to -4^-} |x| = 4 \\
&\lim_{x \to -4^-} \lfloor x \rfloor = -5 \qquad\qquad \text{since anything smaller than } -4 \text{ rounds down} \\
&\lim_{x \to 4^-} |x| = 4 \\
&\lim_{x \to 4^-} \lfloor x \rfloor = 3
\end{aligned}$$

and so we are left with

$$4 - (-5) - (4 - 3) = 9 - 1 = 8.$$