MATH 11A WEEK 3
LIMIT FUN

A **sequence** is simply an (ordered) list of numbers. There's nothing else to this definition- there does not need to be any sort of pattern, no restrictions on the length of the sequence, or anything else. It's just any list of numbers. However, we are most interested in infinitely long sequences that exhibit some sort of pattern, namely sequences whose terms eventually start to get closer and closer together.

So say we have a sequence of numbers

$$(a_n) = a_1, a_2, a_3, \ldots.$$

If the values in the sequence start to approach a constant number L, we say that the sequence (a_n) is **convergent**, and that the sequence **converges** to L. We call the number L the **limit** of the sequence (a_n). If there is no constant number L that terms are approaching, we say that the sequence is **divergent**.

For sequences, you may see different notation that mean the same thing. Both the notations $a_n \to L$ and $\lim_{n\to\infty} a_n = L$ mean that the sequence (a_n) converges to the limit L.

Example 1. Consider the sequence $a_n = 1/n$. Writing out terms of the sequence gives us

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$$

and we see that as $n \to \infty$ that $a_n \to 0$. We conclude that a_n is a convergent sequence and $L = 0$.

Consider the sequence $b_n = n$. Writing out this sequence gives

$$1, 2, 3, \ldots$$

and we see that as $n \to \infty$ that $b_n \to \infty$. Thus, the sequence b_n is divergent since there is no constant L that b_n approaches.

Consider the sequence $c_n = (-1)^n$. Writing out terms in this sequence gives us

$$-1, 1, -1, 1, \ldots$$

This sequence is divergent since there is no constant L that the terms c_n are getting closer to.

We say that a sequence is **monotonically increasing** if there is a starting index N such that for all $n \geq N$, $a_n \leq a_{n+1}$. Similarly, we say that a sequence is **monotonically decreasing** if there is a starting index N such that for all $n \geq N$ that $a_n \geq a_{n+1}$. If a sequence is either monotonically increasing or decreasing, we say that it is a **monotonic sequence**.[1]

Example 2. (1) The sequence $(a_n) = 1/n$ is a monotonically decreasing sequence.
(2) The sequence $(b_n) = n$ is a monotonically increasing sequence.
(3) The sequence $(c_n) = (-1)^n$ is not a monotonic sequence.

We now list off some of the most useful rules about convergent sequences:

Summary of Rules about convergent sequences

If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

- if k is a constant number, then $\lim_{n\to\infty} (ka_n) = k \lim_{n\to\infty} a_n = kA$.
- LIMIT OF SUMS IS SUM OF LIMITS
$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A + B$$
- LIMIT OF DIFFERENCES IS DIFFERENCE OF LIMITS:
$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = A - B$$
- LIMIT OF PRODUCTS IS PRODUCT OF LIMITS:
$$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = A \cdot B$$
- LIMIT OF QUOTIENTS IS QUOTIENT OF LIMITS: assuming $B \neq 0$,
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$$
Some caveats about limit of quotients:
  - if $B = 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n}$ does not exist.
  - if both $A = B = 0$, then the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ might exist
- If $k > 0$ is a fixed constant, then $\lim_{n\to\infty} \frac{1}{n^k} = 0$
- If $|k| > 1$, then $\lim_{n\to\infty} \frac{1}{k^n} = 0$.

Example 3. Show that the following sequences are convergent and find the limit:

- $a_n = \dfrac{4n^3 - n^2 + 5}{2n^3 + 6n^2 + n}$

[1] The for all $n \geq N$ is a technicality that translates to saying "ignore $a_1, a_2, \ldots a_N$ because we only care about the tail of the sequence".

  - Divide every term by n^3 and see that the limit is $4/2 = 2$.
- $b_n = \sqrt{n+1} - \sqrt{n}$
  - Multiply by the "conjugate" and get that the limit is 0.

Another handy trick to keep in mind is called *squeeze theorem* (aka sandwich theorem).

Theorem 1. *Let (a_n) and (c_n) be convergent sequences with the same limit L. If (b_n) is another sequence but*

$$a_n \leq b_n \leq c_n \quad \textit{for all } n \geq N, N \textit{ is some indexing number},$$

then $\lim_{n\to\infty} b_n = L$ as well.

Let's see this in action:

Example 4. Find the limit of the sequence (a_n) whose terms are given by the formula

$$a_n = \frac{\cos(n\pi) \cdot \sin^2(n)}{\sqrt{\log(n)}}.$$

We note that $-1 \leq \cos(n\pi) \leq 1$ and $-1 \leq \sin^2(n) \leq 1$ and so our sequence is "squeezed" in between the sequences

$$\frac{-1}{n} \leq a_n \leq \frac{1}{n}$$

and since we have that both $\pm(1/n) \to 0$, we get that $a_n \to 0$.

Before moving onto limits of functions, we go through one more trick via example:

Example 5. Let (a_n) be a sequence with $a_1 = 2$ and $a_n = \sqrt{2a_{n-1} + 15}$. Given that this sequence converges, what is the limit?

Label the limit L. Now since both $a_n \to L$ and so does $a_{n-1} \to L$ (we only care about tail of sequence which doesn't distinguish the two), we get that by taking the limit of the sequence on both sides of the equation we get another equation:

$$L = \sqrt{2L + 15} \implies L^2 = 2L + 15 \implies L^2 - 2L - 15 = 0$$

giving us that $L = -3$ and 5. Since every $a_i \geq 0$, we throw out the possibility that $L = -3$ and conclude that $L = 5$.

Similar tricks may be used when trying to find the limit of a function as its input go off to infinity, i.e., if $f(x)$ is a function, all of the tricks and rules for sequences work when trying to compute $\lim_{x\to\infty} f(x)$.

1. Exercises

Question 1. *What can you say about the following limits?*

- *(a)* $\lim_{x\to\infty} e^x$*?*
- *(b)* $\lim_{x\to\infty} e^{-x}$*?*
- *(c)* $\lim_{x\to\infty} \ln(x)$*?*

Question 2. *Find the limit of*

$$\lim_{x\to\infty} \frac{3x^2 - x + 5\sin(x)}{x^2 - 4}.$$

Question 3. *Suppose we have a recursively defined sequence*

$$a_n = \sqrt{a_{n-1} + a_{n-2}} \quad \textit{for } n \geq 3$$

with $a_1 = 1$*,* $a_2 = 1$*. What is the limit of this sequence?*

Question 4. *Find the limit of the sequence whose terms are given by*

$$a_n = e^{-n}\cos(n)$$

2. Solutions to Exercise

Question 1. *What can you say about the following limits?*

(a) $\lim_{x\to\infty} e^x = \infty$
(b) $\lim_{x\to\infty} e^{-x} = 0$
(c) $\lim_{x\to\infty} \ln(x) - \infty.$

Question 2. *Find the limit of*

$$\lim_{x\to\infty} \frac{3x^2 - x + 5\sin(x)}{x^2 - 4}.$$

Solution. Remembering that $-1 \leq \sin(x) \leq 1$, we get that $-5 \leq 5\sin(x) \leq 5$ and so we get that for all x that

$$\frac{3x^2 - x - 5}{x^2 - 4} \leq \frac{3x^2 - x + 5\sin(x)}{x^2 - 4} \leq \frac{3x^2 - x + 5}{x^2 - 4}$$

and so taking limits we get

$$\lim_{x\to\infty} \frac{3x^2 - x - 5}{x^2 - 4} \leq \lim_{x\to\infty} \frac{3x^2 - x + 5\sin(x)}{x^2 - 4} \leq \lim_{x\to\infty} \frac{3x^2 - x + 5}{x^2 - 4}$$

and we see that since $\lim_{x\to\infty} \dfrac{3x^2 - x - 5}{x^2 - 4} = 3$ (we can divide through by x^2 and then apply the summation/difference/quotient rules of limits) and using the same rules, we also get that $\lim_{x\to\infty} \dfrac{3x^2 - x + 5}{x^2 - 4} = 3$. By squeezing (theorem 1) we get that

$$\lim_{x\to\infty} \frac{3x^2 - x + 5\sin(x)}{x^2 - 4} = 3.$$

Question 3. *Suppose we have a recursively defined sequence*

$$a_n = \sqrt{a_{n-1} + a_{n-2}} \quad \textit{for } n \geq 3$$

with $a_1 = 1$, $a_2 = 1$. What is the limit of this sequence?
Solution. Write the limit as L, i.e., $\lim_{n\to\infty} a_n = L$. We note that because limits only care about the tail behaviour of the sequence that $a_{n-1} \to L$ and $a_{n-2} \to L$ as well. Thus, taking the limit of the equation gives

$$\begin{aligned}
\lim_{n\to\infty} a_n = L &= \lim_{n\to\infty} \sqrt{a_{n-1} + a_{n-2}} \\
&= \sqrt{\lim_{n\to\infty} (a_{n-1} + a_{n-2})} \\
&= \sqrt{\lim_{n\to\infty} a_{n-1} + \lim_{n\to\infty} a_{n-2}} \\
&= \sqrt{L + L}
\end{aligned}$$

$$= \sqrt{2L}$$

and so we get that $L^2 = 2L$ and so $L^2 - 2L = 0$, giving us that $L = 0, 2$. However, since the sequence (a_n) is a monotonically increasing sequence with $a_1, a_2 > 0$, we get that $L = 2$.

Question 4. *Find the limit of the sequence whose terms are given by*

$$a_n = e^{-n} \cos(n)$$

Solution. Note that $-1 \leq \cos(n) \leq 1$ for all n and so we get that

$$e^{-n}(-1) \leq e^{-n} \cos(n) \leq e^{-n}(1) \Longrightarrow -e^{-n} \leq e^{-n} \cos(n) \leq e^{-n}$$

but if we take the limit, we note that $\lim_{n\to\infty} e^{-n} = 0$ and so by squeezing (theorem 1), we get

$$0 \leq \lim_{n\to\infty} e^{-n} \cos(n) \leq 0 \Longrightarrow \lim_{n\to\infty} a_n = 0.$$