A HITCHHIKERS GUIDE TO RIEMANN SURFACE THEORY

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Contents

The purpose of this talk is to give a quick view into the study of algebraic curves and Riemann surfaces, somewhat accessible to students who have seen vector calculus. Due to time constraints, we won't be proving a lot of the statements.

1. Covering Spaces of the Circle in $\mathbb{C}$

Consider the complex valued function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$. If we then took a particle that lived on the unit circle, i.e., let $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$, and suppose we track the amount of time it takes to travel around the unit circle. Clearly this is 2π (radians) units since $z \in \mathbb{C}$ has multiple polar representations as the angle can be represented by any $\theta + 2\pi\mathbb{Z}$.

Now what happens if we considered the speed of the particle of $f(z)$, the output? Well it follows that θ will be periodic for every $2\pi\mathbb{Z}$ and so we compute

$$f(z) = z^2 = (e^{i(\theta+2\pi\mathbb{Z})})^2 = e^{i(2\theta+2\pi\mathbb{Z})}$$

and so it becomes more clear that for every loop that z makes around the unit circle, the particle $f(z)$ will have gone around the same loop *twice* in the exact same time period.

A few quick computations convey this pretty quickly:

$$\theta = 0 \Longrightarrow z = 1, \quad z^2 = 1 \qquad\qquad \theta = \pi/2 \Longrightarrow z = i \quad z^2 = -1$$

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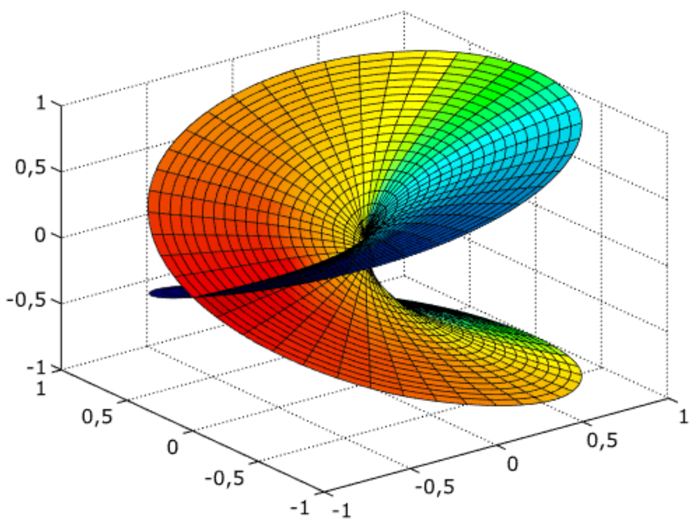
$$\theta = \pi \Longrightarrow z = -1, \quad z^2 = 1 \qquad\qquad \theta = 2\pi \Longrightarrow z = 1, \quad z^2 = 1$$

Going on with this for the rest of time, we can imagine two sets of infinite loops, one generated by the particle z and the other by $f(z)$. To go from one loop to another, we almost imagine a splitting of each loop into two; so what would happen if we were to try to reverse this process? how should each loop interact with each other?

Trying to reverse this process obviously gives us the squareroot function, which is clear that it is not well defined since every poing of $f(z)$ (except zero) has *two* distinct points mapping into it. In order for us to consider the squareroot function as an (partial) inverse to the square function, we are forced to take a *branch cut*, restricting the domain on which we consider to be the inverse of the function f.

We are left with a weird distorted surface as the map for the squareroot:



So we ask: what happens if we *don't* take a branch cut? Instead, of working with functions, we can simply work on the preimage of f. The work may not be as straightforward but this leads us to have coordinate-free results about the inverse of f and we can learn more about what the function f does to our domain space. This leads us the notion of covering spaces (see figure below)

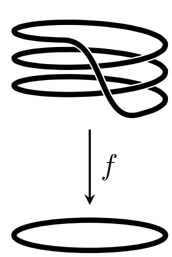


Figure 1. A covering space of a 3-to-1 map

and clearly these are strange looking shapes and figures. Can we learn more about these? Well, it turns out the slightly more general theory of Riemann surfaces will help us out with studying what these domain spaces might look like!

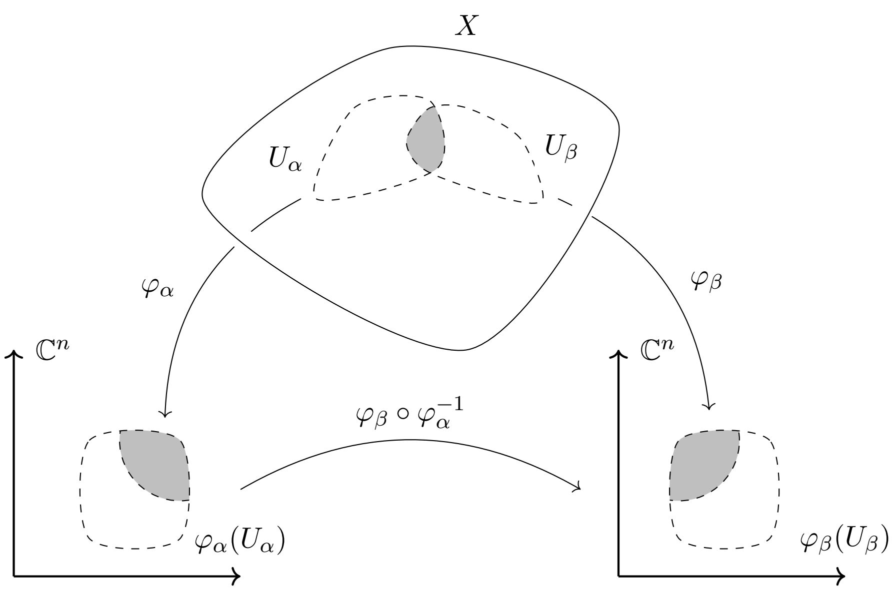
2. First Definitions

Definition. A *Riemann surface* is a connected complex manifold of dimension one.

For clarity, we include the definition of a manifold:

Definition. A *complex manifold* X of dimension n is a topological space X with the following properties:

(1) X is Hausdorff
(2) X is second countable (has a countable basis)
(3) we have open covering: $X = \bigcup_\alpha U_\alpha$, where each $U_\alpha \subseteq X$ is an open set, we call these *coordinate neighbourhoods*
(4) for each U_α, we have $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}^n$ where each φ_α is a homeomorphism. These maps are called *local charts*. We also require all of the transition maps (the maps such as $\varphi_\beta \circ \varphi_\alpha^{-1}$ below) to be holomorphic (see image below)



Example 1. A few examples of complex manifolds are

(1) complex projective space $\mathbb{CP}^n = \{\text{lines of } \mathbb{C}^{n+1} \text{ through origin}\} = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$
(2) finite dimensional vector spaces over $\mathbb{C}$
(3) The group $GL(n, \mathbb{C})$ can be viewed as a complex manifold when viewed as a complex Lie group
(4) all Riemann surfaces including

- elliptic curves
- the Riemann sphere
- *smooth plane affine curves*

The audience most likely has heard of elliptic curves and the Riemann sphere, but not necessarily *smooth plane affine curves*; these are the algebraic curves that motivate our talk:

Definition. Let $f \in \mathbb{C}[z, w]$ and let $X = \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0\}$. Assume at a point $p \in X$ that at least one of the partials does not vanish at the point.

If this condition holds, then the set $X \subseteq \mathbb{C}^2$ is called a smooth plane affine curve.

For example, consider the two functions:

- $f(z, w) = z^2 + w^2 - 1$, the condition holds
- $f(z, w) = w^2 - z^3$, the condition does not hold

We will focus our study towards compact, connected Riemann surfaces especially those associated with algebraic equations, including these smooth plane affine curves.

Now the tool that will allow us to study these mysterious objects by bringing them into more familiar spaces is the *implicit function theorem*!

Theorem 1 (Implicit Function Theorem)**.** *Let $X \subseteq \mathbb{C}^2$ be a smooth plane affine curve and let $p = (z_0, w_0) \in X$ such that $\frac{\partial f}{\partial z} \neq 0$. Then there exists neighbourhoods $U, V \subseteq \mathbb{C}$ such that $z_0 \in U$ and $w_0 \in V$ and there exists an analytic function $\varphi : V \to \mathbb{C}$ such that*

$$(z, w) \in X \cap (U \times V) \iff w \in V, \text{ and } z = \varphi(w).$$

That is the coordinate neighbourhood is $X \cap (U \times V)$ and the local chart is given by the projection map $(z, w) \mapsto w$.

2.1. **The set up.** The implicit function theorem will be our guiding tool for studying the slightly more general Riemann surfaces associated with algebraic equations: let $f \in \mathbb{C}[z, w]$ where f is irreducible polynomial with $\deg(f) > 1$. We can then write

$$f(z, w) = \sum p_i(z) w^i, \quad p_i \in \mathbb{C}[z]$$

Define the two sets

$$S = \{z \in \mathbb{C} : p_n(z) = 0, \ \text{ and } f(z, 0) \ \text{ has multiple roots}\}$$
$$X_0 = \Big\{(z, w) \in \mathbb{C}^2 : f(z, w) = 0, \text{ but } (z, w) \notin S\Big\}$$

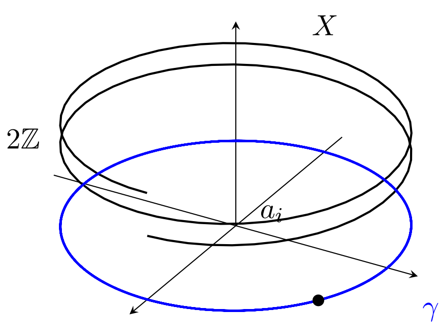
Example 2. Guiding example Let $F(z, w) = w^2 - (z - a_1)(z - a_2)(z - a_3)$ where each $a_i \in \mathbb{C}$ is distinct. We then have multiple roots in the variable w but p_2 is 1. So $S = \{a_1, a_2, a_3\}$.

Let's consider the projection map $\pi : X \to \mathbb{C} \setminus S$ given by $(z, w) \mapsto z$. By the implicit function theorem and by the multiple roots condition, we get that $\pi : X \to \mathbb{C} \setminus S$ is a covering.

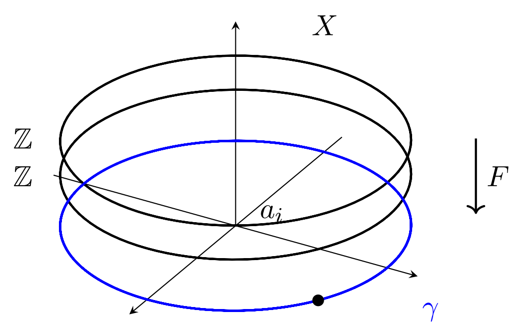
Remark. X_0 is (in general) connected space and we can add finitely many points to compactify X_0. This construction does rely on the Riemann sphere and so in our work, we will have to analyse each of the points in the set S and an extra point at infinity.

Returning to our guiding example, with $F(z, w) = w^2 - (z - a_1)(z - a_2)(z - a_3)$ and $S = \{a_1, a_2, a_3\}$. We want to have a better idea what the curve $X = \left\{(z, w) \in \mathbb{C}^2 : F(z, w) = 0\right\}$ will look like both locally and globally. To do this, we need to analyse what happens at each a_i locally and also analyse what happens at the point at infinity as we know that everywhere else away from these strange points that X will look very much like $\mathbb{C}$.

Around each a_i, we are left with two possible covers: connected and not connected. To detect whether we have a connection or not, we have to take a small contour around the point a_i and compute the period of the contour to detect the branches (see figures below).



connected



not connected

At the specific point $a_i \in \mathbb{C}$, we add the point and also add the point(s) $F^{-1}(a_i) := \overline{a_i}$ such that $t^2 = z - a_i$ where the local coordinate is t with $t^2 = z - a_i$ and we see that this has period $2\mathbb{Z}$, so we safely conclude that our final result is that X is connected at each of the a_i.

We also need to analyse the point at infinity, and we use local coordinates $t = 1/z$ and send $t \to 0$. We eventually find that the period is also $2\mathbb{Z}$.

3. Riemann-Hurwitz Formula

A common way of studying mathematical objects is through relations and mappings.

Theorem 2. *Let X, Y be compact connected Riemann surfaces. Then any holomorphic nonconstant map $f : X \to Y$ is surjective.*

Proof. Since X is compact, $f(X)$ is compact and since f is holomorphic, $f(X)$ is open since f is an open map and since X is connected, $f(X) = Y$ or $f(X) = \emptyset$. $\square$

Definition. Let $f : X \to Y$ be a holomorphic map. A point $a \in X$ is a *ramification point* if $f'(a) = 0$.

Returning to our example, of X being the Riemann surface corresponding to the equation $w^2 = (z - a_1)(z_{a2})(z - a_3)$, let's compute the ramification points. To do this, it might be a good idea on how we define $f'(a) = 0$:

Suppose f is ramified at a. Since f is analytic,

$$\begin{aligned} f(z) &= f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots \\ &= f(a) + c_k(z-a)^k + c_{k+1}(z-a)^{k+1} + \cdots \end{aligned}$$

where $k > 1 \iff f'(a) = 0$. So in a punctured neighbourhood of $a \in X$, if f is k-to-1, then the ramification index is k.

Applying this to our example, $w^2 = (z - a_1)(z - a_2)(z - a_3)$, we see that the ramification index of each of the $\overline{a_i} \in X$ is 2.

Let's list a few boring topology technicalities that we won't prove:

Proposition 3. *Let X be compact, connected Riemann surface. If $f : X \to Y$ is a nonconstant holomorphic mapping.*

(1) The number of ramification points is finite
(2) All fibres of f are finite
(3) f is a covering map

Definition. The cardinality of all fibres of

$$X \setminus f^{-1}(B) \longrightarrow Y \setminus B$$

is called the *degree* of f.

Another fun result we won't prove:

Theorem 4. *If a Riemann surface is compact and connected, then it is homeomorphic to a sphere with handles. The number of handles is called the genus which we will denote with $g(X)$.*

We now have enough to write the interesting result helps classify Riemann surfaces:

Theorem 5 (Riemann-Hurwitz)**.** *If X, Y are conncected compact Riemann surfaces, with $f : X \to Y$ a holomorphic nonconstant map, then*

$$2 - 2g(X) = \deg(f)\left(2 - 2g(Y)\right) - \sum_{j=1}^{m}\left(e(p_j) - 1\right)$$

where the points $p_1, \ldots, p_m$ are the ramification points of f and $e(p_j)$ is the ramification index of p_j.

We won't prove this, and instead we are going to find the genus of our Riemann surface X associated with the algebraic equation $w^2 = (z-a_1)(z-a_2)(z-a_3)$ with $a_1, a_2, a_3 \in \mathbb{C}$ distinct:

Recall from our previous calculation of the a_i, ∞ that they were ramification points with ramification index 2. Using this and noting that the projection map $z : X \to \overline{\mathbb{C}}$ the projection map has degree 2, we can plug this into the formula to give us:

$$2 - 2g(X) = 2(2-0) - 4(2-1) \implies g(X) = 1.$$

Example 1. Let's compute the genus of the Riemann surface associated to the algebraic equation $w^2 = (z-a_1)^2(a-a_2)$ with $a_1 \neq a_2$. We note that the point $z = a_1$ is *not* a ramification point since its derivative is nonzero. Using the same projection map $z : X \to \overline{\mathbb{C}}$, we get

$$2 - 2g(X) = 2(2-0) - 2(2-1) \implies g(X) = 0.$$

3.1. **Construction of a Riemann Surface of arbitrary genus.**

Consider the equation

$$w^2 = \prod_{i=1}^{2k}(z-a_i)$$

and let X be the associated Riemann surface. Using the Riemann-Hurwitz formula and the projection map $z : X \to \overline{\mathbb{C}}$, we get the computation that

$$2 - 2g(X) = 2 \cdot 2 - 2k \implies \boxed{g(X) = k-1}.$$

The Riemann Hurwitz formula gives us our first interesting result on maps:

Theorem 6. *Let $f : X \to Y$ holomorphic map. If $g(X) < g(Y)$, then f must be constant.*

Proof. The proof comes from $g(X) \geq 0$ and the Riemann Hurwitz formula, which will give us the silly result $g(X) < 0$ from the assumption $g(X) < g(Y)$. □

4. Divisors

For the rest of the talk and writeup, we assume all of the Riemann surfaces are compact and connected.

We unfortunately arrive at a dictionary dump:

Definition. A *divisor* on a Riemann surface X is a formal finite linear combination of poitns of X with integer coefficients. The *degree* of the divisor is the sum of the integer coefficients.

Two divisors D_1, D_2 are said to be *equivalent* if $D_1 - D_2 = (f)$ for some holomorphic function f.

A divisor is called *effective* if all coefficients are non-negative.

Definition. Let D be a divisor on a Riemann surface X. Then

$$\mathcal{L}(D) = \{0\} \cup \left\{f - \text{meromorphic on } X : (f) + D \geq 0\right\}$$

This condition $(f) + D$ is equivalent to saying if $D = \sum a_i p_i$

- f has no poles outside of $\{p_1, \ldots, p_n\}$ (this set of points is called the *support* of the divisor)
- if $a_i \geq 0$, then
$$\operatorname{ord}_{p_i}(f) + a_i \geq 0 \iff \operatorname{ord}_{p_i}(f) \geq -a_i$$
this is saying f is allowed to have a pole at $p_i \in X$ but the order of the pole is *at most* a_i
- if $a_i < 0$, this is saying f **must** have a zero at $p_i \in X$, and the order of the zero must be *at least* a_i.

Example 1. Let's look at an example. Suppose $X = \overline{\mathbb{C}}$ (the Riemann sphere) and let $D = -2(1) + 4(\infty)$.

This is saying that $\mathcal{L}(D)$ is comprised of the meromorphic functions such that it has a zero of order 2 at $z = 1$ and a pole of order at most 4 at ∞...noting that $(z-1)^2$ will give a pole at ∞ of order 2, we might think that

$$\mathcal{L}(D) = \left\{(z-1)^2 Q(z) : \deg(Q) \leq 2\right\}$$

It is clear that the degree of this space is 3, and in other words, $\mathcal{L}(D)$ is a complex vector space of dimension 3, something we're quite familiar with!

We also denote $\dim_{\mathbb{C}}(\mathcal{L}(D)) = \ell(D)$.

We will prove the following statment sincce it does give us a bit more insight of this vector space $\mathcal{L}(D)$ on a Riemann surface X:

Theorem 7. *Let D be a divisor on a Riemann surface X. Then $\ell(D) < \infty$.*

Proof. We're going to provide an upper bound:

Claim: $\ell(D) \leq 1 + a_i \leq \deg(D) + 1$.
We can prove this by considering the map

$$\mathcal{L}(D) \longrightarrow \mathbb{C}^{a_1} \oplus \mathbb{C}^{a_2} \oplus \cdots \oplus \mathbb{C}^{a_n}$$

where

$$\begin{aligned}\mathbb{C}^{a_i} &= \{\text{principal parts at } a_i \text{ such that no. of negative terms is at most } n_i\} \\ &= \left\{a_1 z_1^{-1} + \cdots + a_{n_i} z^{n_i}\right\}\end{aligned}$$

where we send the map

$$f \longmapsto (\text{principal part at } p_1, \ldots, \text{principal part at } p_i).$$

The kernel of this map is $\{\text{holomorphic} = \text{constants}\} = \mathbb{C}$ and so we get that $\dim_{\mathbb{C}}(\mathcal{L}(D)) \leq 1 + \deg(D)$. $\square$

Recall that two divisors D_1, D_2 are equivalent, which we denote with $D_1 \sim D_2$ if $D_1 - D_2 = (f)$. It isn't too surprising that $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$. To see this, let f be a map such that $(f) + D_1 > 0$ and $(f) + D_2 > 0$. Noting that $(fg) = (f) + (g)$, let $D_1 - D_2 = (\varphi)$ and defining a map

$$\begin{aligned} \psi : \mathcal{L}(D_1) &\longrightarrow \mathcal{L}(D_2) \\ f &\longmapsto f \cdot \varphi \end{aligned}$$

will be an isomorphism.

Example 2. Another example computation. Let D be a divisor on X such that $\deg(D) < 0$. Then $\mathcal{L}(D) = \{0\}$.

Proof. Suppose $f \in \mathcal{L}(D)$ is nonzero. Then we can consider another divisor D' with $D' = (f) + D$; since $f \in \mathcal{L}(D)$, and D' being effective, we get that the degree is positive. However, since $\deg(f) = 0$, we have $\deg(E) = \deg(F) < 0$, giving us a contradiction. $\square$

Definition. Let D be a divisor on X. The *complete linear system of* D, which is denoted with $|D|$, is the set of all effective divisors equivalent to D. That is,

$$|D| = \left\{ D' \in \operatorname{Div}(X) : D' \sim D, \quad D' \geq 0 \right\}.$$

Combining this with the previous example result , we get that $|D| = \emptyset$ if $\deg(D) < 0$.

Now slightly more generally, there is an algebraic-geometric struccture to $|D|$ related to the vector space $\mathcal{L}(D)$:

Definition. If V is a vector space, we define the *projectivization* $\mathbb{P}(V)$ to be the set of 1-dimensional subspaces of V. If $\dim_{\mathbb{C}}(V) = n + 1$, then $\mathbb{P}(V)$ is in bijection with $\mathbb{P}^n$.

Let's try to study what might the space $\mathbb{P}(\mathcal{L}(D))$ might look like. If we define a function

$$\begin{aligned} \mathcal{F} : \mathbb{P}(\mathcal{L}(D) &\longrightarrow |D| \\ \operatorname{span}(f) \in \mathcal{L}(D) &\longmapsto (f) + D \end{aligned}$$

we get that $\mathcal{F}$ is a bijection: if $D' \in |D|$, then there must exist a meromorphic function f on X such that $D' = (f) + D$ since $D' \sim D$. Moreover, since D' is effective, the function $f \in \mathcal{L}(D)$ and so we can conclude that $\mathcal{F}f = D'$. This proves $\mathcal{F}$ is surjective. To show that $\mathcal{F}$ is injective, suppose $\mathcal{F}f = \mathcal{F}g$. Staring at this equation long enough, one will find that $(f) = (g)$ and so we get that $(f) - (g) = (f/g) = 0$; that is, the function f/g has no poles or zeros on X. Since X is compact, we conclude that f/g is a nonzero constant map and so f and g will have the same span in the space $\mathcal{L}(D)$, giving us our desired result.

Definition. The *dimension* of a linear system is the dimension of the space $|D|$ considered as a projective space. A linear system of dimension one is a *pencil*, a linear system of dimension two is a *net*, and a linear system of dimension three is a *web*.

Remark. With a few more computations such as the one we did before, we can prove the following results:

Theorem 8. *let D be a divisor on the Riemann sphere. Then*

$$\ell(D) = \dim_{\mathbb{C}}(\mathcal{L}(D)) = \begin{cases} 0 & \textit{if } \deg(D) < 0 \\ 1 + \deg(D) & \textit{else} \end{cases}$$

Theorem 9. *Let $X = \mathbb{C}/\Lambda$ be an elliptic curve (or a complex torus) for some lattice Λ and D be a divisor on X. Then*

- *if* $\deg(D) < 0$, $\mathcal{L}(D) = \{0\}$
- *if* $\deg(D) = 0$
  - *if $D \sim 0$, then $\ell(D) = 1$.*
  - *if $D \not\sim 0$, then $\mathcal{L}(D) = \{0\}$*
- *if* $\deg(D) > 0$, *then* $\ell(D) = \deg(D)$.

5. Forms and the Riemann-Roch Theorem

Definition. A *meromorphic differential* ω on $X = \bigcup U_\alpha$ is a collection $\{f_\alpha \, \mathrm{d}z_\alpha\}$ where f_α is meromorphic on U_α such that on the overlaps

$$U_\alpha \cap U_\beta, \quad f_\alpha \, \mathrm{d}z_\alpha = f_\beta \, \mathrm{d}z_\beta \iff f_\alpha \frac{\mathrm{d}z_\alpha}{\mathrm{d}z_\beta} \, \mathrm{d}z_\beta - f_\beta \, \mathrm{d}z_\beta \iff f_\beta = f_\alpha \frac{\mathrm{d}z_\alpha}{\mathrm{d}z_\beta}$$

and so we have the relation

$$\mathrm{d}z_\alpha = \frac{\mathrm{d}z_\alpha}{\mathrm{d}z_\beta} \, \mathrm{d}z_\beta \, .$$

Example 1. Let f be a meromorphic function on X. Then the meromorphic differential

$$\mathrm{d}f = \left\{ \frac{\mathrm{d}f}{\mathrm{d}z_\alpha} \, \mathrm{d}z_\alpha \right\} .$$

Calling $f_\alpha = \frac{\mathrm{d}f}{\mathrm{d}z_\alpha}$, we have

$$f_\alpha \frac{\mathrm{d}z_\alpha}{\mathrm{d}z_\beta} = f_\beta \iff \frac{\mathrm{d}f}{\mathrm{d}z_\alpha} \frac{\mathrm{d}z_\alpha}{\mathrm{d}z_\beta} = \frac{\mathrm{d}f}{\mathrm{d}z_\beta}$$

which is the chain rule!

Remark. If ω_1, ω_2 are meromorphic differentials, then

$$\sum_{p \in X} \mathrm{ord}_p(\omega) p = \omega_1 \sim \omega_2 .$$

Proof. We get $\omega_1 = f\omega_2$ for some meromorphic function f and then get that $(\omega_1) = (f) + (\omega_2)$. □

Definition. The class of divisors of meromorphic forms is called the *canonical class*, which we denote K_X.

We also note that $\deg(K_X) = 2g - 2$

Let K_X be the canonical divisor. Then $\mathcal{L}(K_X) \cong \{\text{space of hol. forms on } X\}$, which we denote with $\Omega^1(X)$.

Proof. Let $K_X = (\omega)$ and $\mathcal{L}(K_X) = \left\{f : (f) + (\omega) \geq 0\right\}$ but $(f) + (\omega) = (f) \implies f\omega$ is a hol. form. □

Let $D = \sum n_i p_i$ be an effective divisor and let φ_j be the prinipal part of p_i. If ω is a holomorphic form and f is a meromorphic function with principal part φ_j at p_j, then

$$\sum_j \operatorname{Res}_{p_j}(f\omega) = \sum \operatorname{Res}_{p_j}(\varphi_j \omega) = 0$$

from the residue theorem of complex analysis. Thus, we have a linear functional

$$\begin{aligned} \operatorname{Res}_\omega : \Omega^1(X) &\longrightarrow \mathbb{C} \\ \omega &\longmapsto \sum \operatorname{Res}_{p_j}(\varphi_j \omega) \end{aligned}$$

Let's try to study the kernel: WLOG let's only consider φ_1 as the principal part. Suppose $\omega = h(z)\,\mathrm{d}z$ and we get that for all φ, $\operatorname{ord}_0(\varphi) \geq -n_1$. We conclude that $\operatorname{Res}_0(h\varphi_1) = 0 \iff \operatorname{ord}_0(h) \geq n_1$, giving us that

$$\ker(\omega) = I(D) := \left\{\omega \in \Omega^1(X) : (\omega) \geq D\right\}.$$

We are going to make a remark that we won't include the proof of, and state the most important result of the day: the *Riemann-Roch theorem.*

Remark. We will not prove this but: $I(D) \cong \mathcal{L}(K_X - D)$. This result leads us to the upper bound

$$\ell(D) \leq \deg(D) + 1 - (g(X) - \dim(I(D)).$$

Theorem 10 (Riemann-Roch)**.** *For any divisor D on a Riemann surface X, we have*

$$\ell(D) = \ell(K_X - D) + \deg(D) + 1 - g(X)$$

where K_X is the canonical divisor of X and $g(X)$ is the genus of X.

We won't prove it here, but let's get ready to use it!

Corollary 11. *If D is a divisor of degree at least $2g(X) - 1$, then $\deg(K_X - D) = 0$ and*

$$\ell(D) = \deg(D) + 1 - g(X).$$

Let's also do two computations.

Example 2. Let X be a Riemann surface with genus $g(X) = 0$. We will prove that X is isomorphic to the Riemann sphere using the Riemann-Roch formula.

Let $p \in X$ and treating p as a divisor, e.g., a simple pole at p, we have

$$\ell(p) = \ell(K_X - p) + \deg(p) + 1 - g(X) \Longrightarrow \ell(p) = \ell(K_X - p) + 1$$

but since $\ell(K_X - p) = \ell(K_X) - 1$, and since we also see that the divisor $K_X - p$ will have degree -3, we have $\ell(K_X - p) = 0$ and so $\ell(p) = 2$.

Considering this, this would imply the existence of a nonconstant meromorphic function f with a simple pole at p and no other poles; the only map in this case would be a degree 1 map and therefore an isomorphism onto the Riemann sphere. Using the Riemann-Roch theorem, we also get a few other classifications:

Example 3. Let X be a Riemann surface with genus $g(X) = 1$. Then X is isomorphic to a smooth projective plan cubic curve, which is also a complex torus.

Example 4. Every Riemann surface of genus at least two is hyperelliptic.

Lemma 12. *Let D_1, D_2 be divisors on a Riemann surface X. Then*

(1) We have the inequality

$$\ell(D_1) + \ell(D_2) \leq \ell(\min\{D_1, D_2\}) + \dim(\{D_1, D_2\}).$$

(2) If $\ell(D) \geq 1$ and $\ell(K_X - D) \geq 1$, then $\ell(D) + \ell(K_X - D) \leq 1 + g(X)$.

Theorem 13 (Clifford's Theorem)**.** *Suppose D is a divisor on X such that $\ell(D)$ and $\ell(K_X - D)$ are both nonzero. Then*

$$2\ell(D) \leq \deg(D) + 2.$$

To conclude this talk, we will try to discuss what the Riemann-Roch means geometrically.

Definition. A *linear system* is a subset of a complete linear system $|D|$. A point $x \in X$ is said to be a base point for a linear system $Q \subseteq |D|$ if every divisor $D \in Q$ contains x. That is, every $D \in Q$ has $D \geq x$.

A linear system Q is *basepoint-free* if it contains no base points.

We also list a result we skip the proof of:

Proposition 14. *Let $Q \subseteq |D|$ be a basepoint-free linear system of ((projective) dimension n on a Riemann surface X. Then there exists a holomorphic map $\phi : X \to \mathbb{P}^n$ such that $Q = |\phi|$; this map is unique up to choice of coordinates.*

This result is saying that a basepoint-free system is characterized by the holomorphic maps $\phi : X \to \mathbb{P}^n$.

Now a question that is of interest to the theory of curves and Riemann surfaces is the following: If X is a Riemann surface and K_X is the canonical divisor on X, when is the associated holomorphic map $\phi_{|K_X|} : X \to \mathbb{P}^n$ injective? This map is called the *canonical map.*

Luckily for us, someone already proved the next result:

Theorem 15. *Let D be a divisor on a compact X and suppose $\ell(D) = n+1$, and $|D|$ contains no base point. Then*

$$\varphi_{|D|} \text{ is injective } \iff \ell(D-p-q) = \ell(D) - 2 \text{ for any two distinct } p, q \in X$$

Question 1.
(1) If X has genus 3, then either X is hyperelliptic or $\phi_{|K_X|}$ for K embeds X into $\mathbb{P}^2$ as a smooth plane curve defined by a quartic polynomial.
(2) If X has genus 4, then either X is hyperellipticc or $\phi_{|K_X|}$ embeds X into $\mathbb{P}^3$ as a smooth curve of degree 6 defined by a cubic and a quadratic.

This leads us to the geometric version of the Riemann-Roch theorem:

Theorem 16. *Let X be nonhyperelliptic curve of genus g, canonically embedded into $\mathbb{P}^n$ and let D be a divisor, then*

$$\ell(D) = \deg(D) - 1 - \dim_{\mathbb{C}}(\mathit{span}(D))$$

References

[1] Rick Miranda *Algebraic Curves and Riemann Surfaces*, Gradauate Studies in Mathematics, Volume 5, American Mathematical Society, Providence, Rhode Island. 1995