Computing Pythagorean Triples

A **Pythagorean triple** (a, b, c) is a triple of positive integers that satisfy the equation $a^2 + b^2 = c^2$.
Goal: Describe all possible Pythagorean triples.

1. Give three examples of Pythagorean triples.

2. Show that if you have a Pythagorean triple (a, b, c), then (ka, kb, kc) is also a Pythagorean triple for $k \geq 1$.

Plato is usually known for philosophy, but philosophy also includes astronomy and mathematics! A formula ascribed to Plato for generating Pythagorean triples is the following:

$$x = n^2 - 1, \quad y = 2n, \quad z = n^2 + 1, \qquad \text{for } n \geq 2.$$

3. Show that every (x, y, z) produced by Plato's formula is a Pythagorean triple.

4. Can you find a Pythagorean triple that was not on your previous list?

5. Find a Pythagorean triple that does not follow this formula.

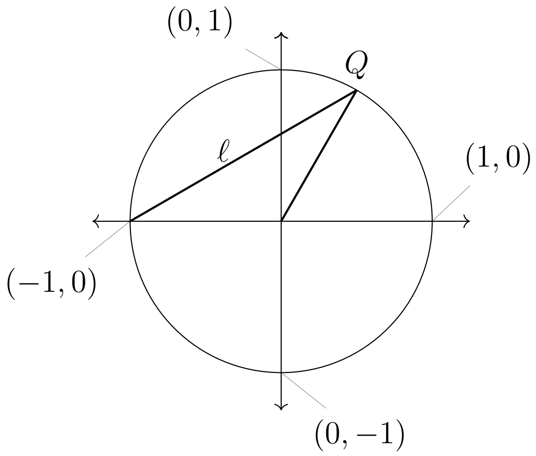
Since it's possible to retrieve a Pythagorean triple from a different Pythagorean triple by considering common factors, we start to focus on *primitive Pythagorean triples*.

A *primitive Pythagorean triple* is a triple (a, b, c) such that they have no common factors, e.g. $2, 3, 5$ is primitive, while $4, 6, 10$ is not.

Theorem 1 (Euclid's Theorem)**.** *Let a, b, c positive integers. The a, b, c is a primitive Pythagorean triple* **if and only if** there exist *relatively prime integers r, s such that $r \not\equiv s$ in $\mathbb{Z}/2\mathbb{Z}$ and $a = r^2 - s^2, \quad b = 2rs, \quad c = r^2 + s^2$.*

To establish why this works, we look to the *rational points* of the unit circle.

A *rational point* is a point (x, y) where both $x, y \in \mathbb{Q}$. Recall that the unit circle equation is $x^2 + y^2 = 1$.



6. (Triples to Rationals on Circle) Let a, b, c be a Pythagorean triple and let ℓ be a line from the origin to (a, b). Why must ℓ pass through a rational point on the unit circle?

7. (Rationals on Circle to Triples) Let $t \in \mathbb{Q}^+$ and let ℓ be the line passing through the point $(-1, 0)$ and $(0, t)$.

   (a) Explain why the line ℓ has the equation $\ell : y = t(x + 1)$.

   (b) Let $Q = (x_Q, y_Q)$ be a point on ℓ and on the unit circle. Show that the y-intercept of ℓ is at $(0, \dfrac{y_Q}{x_Q + 1})$.

   (c) Show that the x-coordinate of Q is given by $x_Q = \dfrac{1 - t^2}{1 + t^2}$.

   (d) Explain why the y-coordinate of Q is given by $y_Q = \dfrac{2t}{1 + t^2}$. We've found that the point Q is given by the equation

   $$Q = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right)$$

   (e) Writing $t = r/s$, show that $Q = (\dfrac{r^2 - s^2}{r^2 + s^2}, \dfrac{2rs}{r^2 + s^2})$ for $r, s \in \mathbb{Z}$.

   (f) Let $a = r^2 - s^2, b = 2rs$, and $c = r^2 + s^2$. Confirm that (a, b, c) is a Pythagorean triple.

Parametrization of the Circle

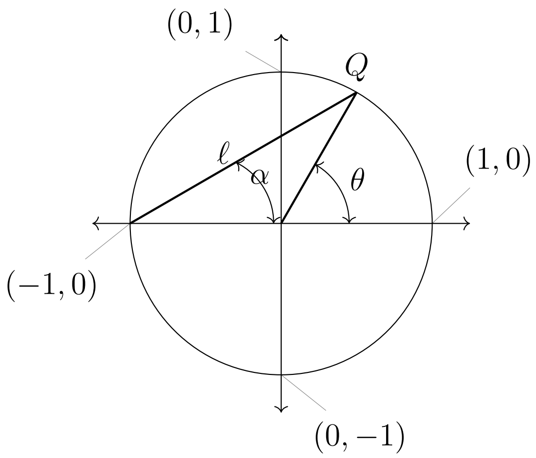
Exercises 6 and 7 is an example of a technique called *stereographic projection*, which gives us a parametrization of the circle that does not use trigonometry, and only uses the rational numbers $\mathbb{Q}$. We see that if we pick x_Q, y_Q to be rational points on the circle, then

$$x_Q = \frac{1-t^2}{1+t^2}, \quad y_Q = \frac{2t}{1+t^2}, \quad \text{and } t = \frac{y_Q}{x_Q+1}$$

so $t \in \mathbb{Q}$ if and only if both $x_Q, y_Q \in \mathbb{Q}$ as well.

However, we are familiar with trigonometric parametrization of the circle: $x = \cos(\theta)$ and $y = \sin(\theta)$ so whenever these output rationals, we get the equations

$$\cos(\theta) = \frac{1-t^2}{1+t^2}, \quad \sin(\theta) = \frac{2t}{1+t^2}, \quad \text{and} \quad t = \frac{\sin(\theta)}{1+\cos(\theta)}.$$



8. Prove that $\alpha = \dfrac{\theta}{2}$ where θ is the angle from $(0,0)$ to $Q = (x_Q, y_Q)$ and α is the angle formed by the line from $(-1,0)$ to Q.

9. Explain why the slope t can be determined by the equation $t = \tan(\theta/2)$.

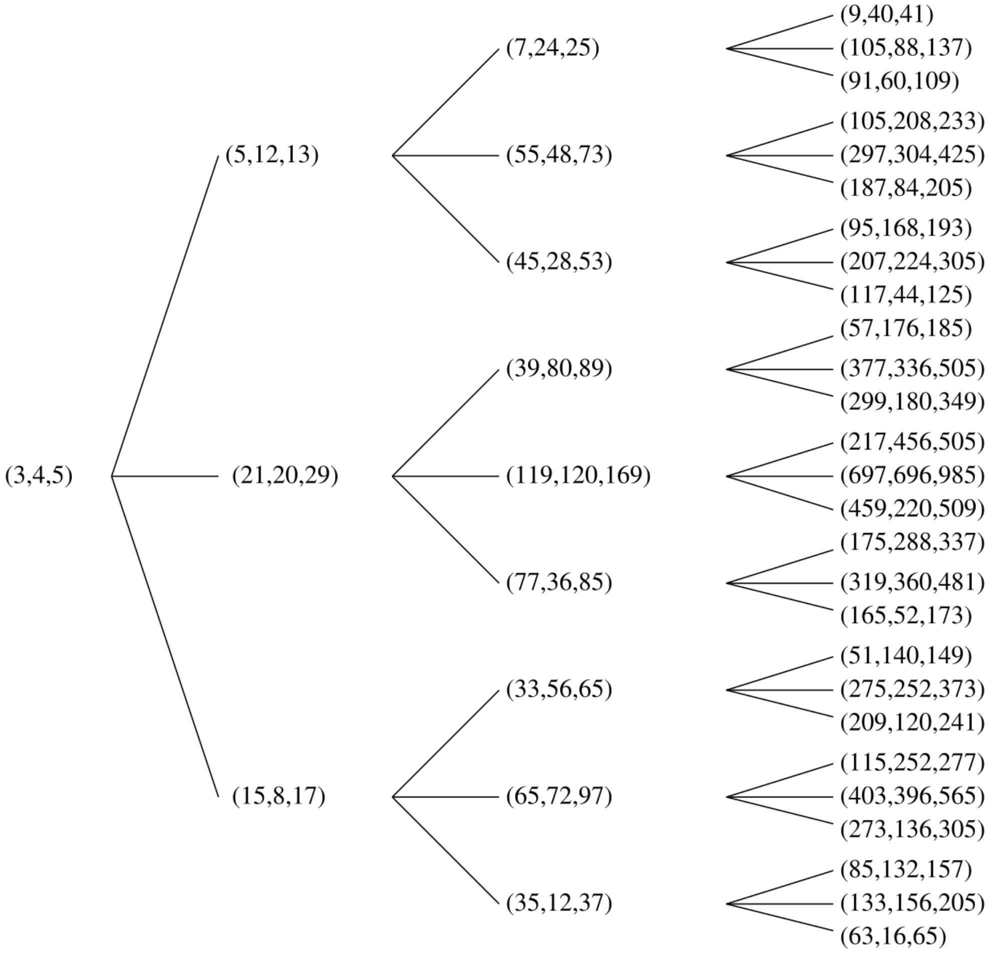
10. (Calculus required) Show that $\mathrm{d}\theta = \dfrac{2}{1+t^2}\,\mathrm{d}t$

This allows one to compute integrals of the form involving trigonometric functions into rational ones by substitution:

$$\int f(\cos(\theta), \sin(\theta))\,\mathrm{d}\theta = \int f\Big(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\Big)\frac{2}{1+t^2}\,\mathrm{d}t\,.$$

Tree of Pythagorean Triples

In 1934, the mathematician Berggren discovered that all of the primitive Pythagorean triples formed a rooted ternary tree (shown below):



The Dutch mathematician F.J.M. Barning proved in 1963 that if you consider the matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}, \quad v = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

where (x_0, y_0, z_0) are a Pythagorean triple, then the matrix products Av, Bv and Cv all produce Pythagorean triples. Moreover, if (x_0, y_0, z_0) are a primitive Pythagorean triple, then so are Av, Bv, Cv!