MATH 11A WEEK 8
MORE DERIVATIVES

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1. Chain Rule Formulas

Recall that the chain rule gives us

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

The real power of the chain rule is that it simplifies a lot of the calculations we would normally have to do. If $f(x)$ is a differentiable function, then by the chain rule, we have the following formulas:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{f(x)}\right) = \frac{f'(x)}{2\sqrt{f(x)}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right)^n = n\left(f(x)\right)^{n-1} f'(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\sin\big(f(x)\big)\Big) = f'(x)\cos\big(f(x)\big)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\ln\left(f(x)\right)\Big) = \frac{f'(x)}{f(x)}$$

2. Exercises

Question 1. *For the following derivatives, identify which of the derivatives is correct. If there is an incorrect derivative, please identify what rule was incorrectly applied and provide the correct derivative.*

$$(a)\ \frac{\mathrm{d}}{\mathrm{d}x} \ln\left(x^2\right) = \frac{2}{x}$$

$$(b)\ \frac{\mathrm{d}}{\mathrm{d}x} x^x = x \cdot x^{x-1}(x-1)$$

$$(c)\ \frac{\mathrm{d}}{\mathrm{d}x} e^{2x^3} = 2e^{2x^3}$$

$$(d)\ \frac{\mathrm{d}}{\mathrm{d}x} \pi^2 = 2\pi$$

$$(e)\ \frac{\mathrm{d}}{\mathrm{d}x} \sqrt{x^3} = 3\sqrt{x^2}$$

$$(f)\ \frac{\mathrm{d}}{\mathrm{d}x} x\sin(2x) = \sin(2x) + 2x^2\cos(2x)$$

Question 2. *Let $f(x) = x^k e^{-x}$, where k is some positive real constant. For $x > 0$, what is the maximum value attained by f? (You may use calculators/Desmos to help).*

$$(a)\ \left(\frac{e}{k}\right)^k \qquad (b)\ \sqrt[k]{\frac{e}{k^k}} \qquad (c)\ \frac{\ln(k)^k}{k} \qquad (d)\ \left(\frac{k}{e}\right)^k$$

3. Solutions to Exercises

Question 1. *For the following derivatives, identify which of the derivatives is correct. If there is an incorrect derivative, please identify what rule was incorrectly applied and provide the correct derivative.*

(*a*) $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(x^2\right) = \frac{2}{x}$ It is correct.

(*b*) $\frac{\mathrm{d}}{\mathrm{d}x}x^x = x \cdot x^{x-1}(x-1)$ It is not correct. The correct derivative would be

$$\frac{\mathrm{d}}{\mathrm{d}x}x^x = x^x\left(\ln(x)+1\right)$$

(*c*) $\frac{\mathrm{d}}{\mathrm{d}x}e^{2x^3} = 2e^{2x^3}$ Chain rule incorrectly applied.

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{2x^3} = 6x^2e^{2x^3}$$

(*d*) $\frac{\mathrm{d}}{\mathrm{d}x}\pi^2 = 2\pi$ It is not correct since π^2 is constant.

$$\frac{\mathrm{d}}{\mathrm{d}x}\pi^2 = 0$$

(*e*) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^3} = 3\sqrt{x^2}$ Chain/power rule(s) incorrectly applied.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^3} &= \frac{\mathrm{d}}{\mathrm{d}x}x^{3/2} = \frac{3}{2}\sqrt{x} && \text{(power rule)}\\ &= \frac{3x^2}{2\sqrt{x^3}} && \text{(chain rule)}\end{aligned}$$

(*f*) $\frac{\mathrm{d}}{\mathrm{d}x}x\sin(2x) = \sin(2x) + 2x^2\cos(2x)$ Chain rule not applied correctly.

$$\frac{\mathrm{d}}{\mathrm{d}x}x\sin(2x) = \sin(2x) + 2x\cos(2x)$$

Question 2. *Let $f(x) = x^k e^{-x}$, where k is some positive real constant. For $x > 0$, what is the maximum value attained by f? (You may use calculators/Desmos to help).*

(*a*) $\left(\frac{e}{k}\right)^k$ (*b*) $\sqrt[k]{\frac{e}{k^k}}$ (*c*) $\frac{\ln(k)^k}{k}$ (*d*) $\left(\frac{k}{e}\right)^k$

Recall that the maximum/minimum values occur whenever the derivative is zero. So calculate the derivative of f and set it equal to zero.

The derivative of f is found through the product rule:

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}x^k e^{-x} &= kx^{k-1}e^{-x} + x^k\left(-e^{-x}\right)\\ &= kx^{k-1}e^{-x} - x^k e^{-x}\end{aligned}$$

Thus setting this equal to zero, we have

$$\begin{aligned}f'(x) = 0 \iff kx^{k-1}e^{-x} - x^k e^{-x} = 0 &\iff kx^{k-1}e^{-x} = x^k e^{-x}\\ &\iff x = k\end{aligned}$$

That is, $x = k$ is the point when the derivative is zero, i.e., $f'(k) = 0$. Thus, $f(k)$ is either a local minimum or a local maximum (it turns out to be a maximum).

Evaluating $f(k)$ gives

$$f(k) = k^k e^{-k} = \frac{k^k}{e^k} = \left(\frac{k}{e}\right)^k$$

and so we get that the answer is d. A graph for $k = 1, 2, 3$ are included below: