MATH 11B INTEGRATION TECHNIQUES AND APPLICATIONS

Contents

1. Change of Variables

For this technique, let u be a variable that stands for something inside your integral. If you can replace all of the other variables in the integral with u and scalar multiples of its differential $\mathrm{d}u$, you are done!

Example 1. Integrate $\displaystyle\int x^3(2x^4+4)^4\,\mathrm{d}x$.

Picking $u = 2x^4+4$, we compute $\mathrm{d}u = 8x^3\,\mathrm{d}x$ and we can rewrite the integral to be

$$\begin{aligned}\int x^3(2x^4+4)^4\,\mathrm{d}x &= \int (2x^4+4)^4 x^3\,\mathrm{d}x \\ &= \int u^4\cdot\frac{1}{8}\,\mathrm{d}u \\ &= \frac{1}{8}\int u^4\,\mathrm{d}u\end{aligned}$$

now integrating with respect to the variable u, we have

$$\begin{aligned}\frac{1}{8}\int u^4\,\mathrm{d}u &= \frac{1}{8}\cdot\frac{u^5}{5}+C && \text{for some constant } C \\ &= \frac{(2x^4+4)^5}{40}+C && \text{resubbing in our variable}\end{aligned}$$

Thus, we conclude that

$$\boxed{\int x^3(2x^4+4)^4\,\mathrm{d}x = \frac{(2x^4+4)^5}{40}+C}$$

The general formula for this technique is

$$\boxed{\int f(g(x))\cdot g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u\,.}$$

We will include a table of integrals that result from this below.

Tips and warnings on substitution:

- Look for a product sign hiding in the integral (this also includes the division sign!!) and ask yourself "*is anything the derivative of another thing?*" (up to constants)
- If you end up with
$$\int \frac{f(u)}{g(u)\,\mathrm{d}u}$$
you made a mistake in the choice of u (don't have your $\mathrm{d}u$ in the denominator). Try again!

We include a table of a few integrals that result from the chain rule.

Table of Integrals

Let k be a constant number. The $+C$ denotes the integration constant.

$$\int \big(f(x)\big)^k \cdot f'(x)\,\mathrm{d}x = \begin{cases} \dfrac{1}{k+1}\big(f(x)\big)^{k+1} + C & \text{if } k \neq -1 \\ \ln|f(x)| + C & \text{if } k = -1 \end{cases}$$

$$\int e^{f(x)} \cdot f'(x)\,\mathrm{d}x = e^{f(x)} + C$$

$$\int k^{f(x)} f'(x)\,\mathrm{d}x = k^{f(x)} + C$$

$$\int \sin\big(f(x)\big) \cdot f'(x)\,\mathrm{d}x = -\cos\big(f(x)\big) + C$$

$$\int \cos\big(f(x)\big) \cdot f'(x)\,\mathrm{d}x = \sin\big(f(x)\big) + C$$

$$\int \sec^2\big(f(x)\big) \cdot f'(x)\,\mathrm{d}x = \tan\big(f(x)\big) + C$$

$$\int \csc^2\big(f(x)\big) \cdot f'(x)\,\mathrm{d}x = -\cot\big(f(x)\big) + C$$

$$\int \sec\big(f(x)\big) \cdot \tan\big(f(x)\big) \cdot f'(x)\,\mathrm{d}x = \sec\big(f(x)\big) + C$$

$$\int \csc\big(f(x)\big) \cdot \cot\big(f(x)\big) \cdot f'(x)\,\mathrm{d}x = -\csc\big(f(x)\big) + C$$

Example 2. Integrate $\displaystyle\int xe^{-x^2}\,\mathrm{d}x$.

So in this case, we see that the expression inside the integral is (find where the product is)

$$x \cdot e^{x^2}$$

and x is the derivative of $-x^2$ just differing by multiplication by a scalar. Set $u = -x^2$ and compute that $\mathrm{d}u = -2x\,\mathrm{d}x$ and so we have $x\,\mathrm{d}x = \frac{-1}{2}\,\mathrm{d}u$. We need to divide by the -2 since we don't have a factor of -2 in the original integral, but it's okay since it's a scalar and not a variable like x. We can replace all of the x variables with u variables, i.e.,

$$\begin{aligned} \int xe^{-x^2}\,\mathrm{d}x &= \int e^{-x^2} \cdot x\,\mathrm{d}x \\ &= \int e^u \cdot \frac{-1}{2}\,\mathrm{d}u && \text{setting } u = -x^2 \\ &= -\frac{1}{2}\int e^u\,\mathrm{d}u \\ &= -\frac{1}{2}e^u + C && \text{taking integral with resp. to } u \\ &= -\frac{1}{2}e^{-x^2} + C && \text{re-introducing original variable} \end{aligned}$$

2. Integration by Parts

This should be your second go-to technique after trying to look for u-sub (change of variables) when faced with an integral that is a product. For this technique, the formula is

$$\boxed{\int \underbrace{f(x)}_{\text{call } u} \cdot \underbrace{g(x)\,\mathrm{d}x}_{\text{call } \mathrm{d}v} = uv - \int v\,\mathrm{d}u}$$

and so when faced with this, the big question is *how do we know which is u and which is $\mathrm{d}v$?* For this, we have to remember the mnemonic LIATE (or DETAIL, just flip the table below upside down), which is

$$\begin{array}{cl} \uparrow & L = \text{Logarithmic} \\ u & I = \text{Inverse trig e.g. } \arcsin(x), \arccos(2x), \text{ etc.} \\ & A = \text{Algebraic e.g. } x, 3x^2, x^5 + 2x, \text{ etc. (polynomials)} \\ \downarrow & T = \text{Trigonometric e.g., } \sin(x), \cos(2x), \text{ etc.} \\ \mathrm{d}v & E = \text{Exponential} \end{array}$$

so compare your f and g and pick your u to be whichever one is higher/comes first in the LIATE hierarchy.

Example 3. Find the integral of $\displaystyle\int 2x\cos(x)\,\mathrm{d}x$.

We use the LIATE hierarchy and identify that we should pick $u = 2x$ and $\mathrm{d}v = \cos(x)$ since $2x$ is algebraic and $\cos(x)$ is a trig function.

$$u = 2x \qquad\qquad \mathrm{d}v = \cos(x)$$

$$\mathrm{d}u = 2\,\mathrm{d}x \qquad\qquad v = \int \mathrm{d}v = \sin(x)$$

and using the boxed formula above, we have

$$\begin{aligned} \int 2x\cos(x)\,\mathrm{d}x &= uv - \int v\,\mathrm{d}u \\ &= 2x\sin(x) - \int sin(x)\cdot 2\,\mathrm{d}x \\ &= 2x\sin(x) - 2\int \sin(x)\,\mathrm{d}x \\ &= 2x\sin(x) + 2\cos(x) + C && \text{taking integral of } \sin(x) \text{ and considering sign change} \end{aligned}$$

2.1. **Tabular Method.**

We note that we might have to use integration by parts repeatedly since the integration by parts formula spits out another integral, which we might have to use integration by parts, and again...and again...

Instead of stopping at every moment, and if you aren't looking at a Trig-Trig or Trig-Exp combination, you can simplify it all by following the recipe:

(1) Pick your u and your $\mathrm{d}v$
(2) Repeatedly take derivatives of u until you get zero
(3) Repeatedly integrate your $\mathrm{d}v$ the same number of times you took derivative of u
(4) Draw diagonal lines, changing signs every line
(5) Multiply terms that are connected by the diagonal line and by the sign
(6) Add everything together

This is a lot, so it's probably best to look at an example.

Example 4. Find integral of $\displaystyle\int x^4 \cos(x)\,\mathrm{d}x$.

Pick your $u = x^4$ and $\mathrm{d}v = \cos(x)\,\mathrm{d}x$ since x^4 is algebraic and $\cos(x)$ is trig. We now create the table and draw lines and signs (steps 2-4):

$$\begin{array}{ccc}
u & & \mathrm{d}v \\
 & & \\
x^4 & & \cos(x) \\
 & \overset{+}{\searrow} & \\
4x^3 & & \sin(x) \\
 & \overset{-}{\searrow} & \\
12x^2 & & -\cos(x) \\
 & \overset{+}{\searrow} & \\
24x & & -\sin(x) \\
 & \overset{-}{\searrow} & \\
24 & & \cos(x) \\
 & \overset{+}{\searrow} & \\
0 & & \sin(x)
\end{array}$$

and so we get that the integral is equal to

$$\begin{aligned}
\int x^4 \cos(x) &= x^4 \sin(x) - 4x^3 \cdot (-\cos(x)) + 12x^2(-\sin(x)) - 24x\cos(x) + 24\sin(x) + C \\
&= \boxed{x^4 \sin(x) + 4x^3 \cos(x) - 12x^2 \sin(x) - 24x\cos(x) + 24\sin(x) + C}
\end{aligned}$$

2.2. **Entering the abyss (infinite loops).** See Exercise solutions for a method on how to exit the infinite loop.

3. Partial Fraction Decomposition

Definition. A *polynomial* is a function $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ where n is some (finite) positive integer and each a_i is a real number. The number (the highest exponent) n is called the *degree* of the polynomial.

So for example, all of the following are polynomials

$$\begin{aligned} f(x) &= x^2 + 1 & \deg(f) &= 2 \\ g(x) &= 4x^3 + \pi x^2 + 2 & \deg(g) &= 3 \\ h(x) &= 2x^3 & \deg(h) &= 3 \end{aligned}$$

while functions like $e^x, \ln(x), \sin(x)$ are not polynomials.

Partial fraction decomposition (uses a lot of high school algebra) is a method for simplifying integrals $\displaystyle\int \frac{p(x)}{q(x)}\,\mathrm{d}x$ where both $p(x), q(x)$ are polynomials, but also $\deg(q) > \deg(p)$.

Since this technique is only useful when the degree of denominator is greater than degree of numerator, we provide a conversion table that is completely dependent on what the denominator looks like.

Partial Fraction Conversion Table

The a, b, c are real numbers that you would find from the polynomial on the bottom.

$$\begin{aligned} \frac{p(x)}{ax+b} &= \frac{A}{ax+b} \\ \frac{p(x)}{(ax+b)^k} &= \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_{k-1}}{(ax+b)^{k-1}} + \frac{A_k}{(ax+b)^k} \\ \frac{p(x)}{ax^2+bx+c} &= \frac{Ax+B}{ax^2+bx+c} \\ \frac{p(x)}{(ax^2+bx+c)^k} &= \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k} \end{aligned}$$

When converting to the right hand side, just write letters A, B and you will have to find what these A's and B's are, using any algebraic technique you can.

We demonstrate how to use partial fraction decomposition by laying out the steps and following an example at each step. These steps are the general method on how to solve it.

Example 5. Using partial fraction decomposition, find the integral of

$$\int \frac{2x^3 + 5x^2 + 6x + 12}{x^4 + 3x^3 + 6x^2}\,\mathrm{d}x\,.$$

(1) First factor the denominator into product of simpler terms (simplify as much as possible):

$$x^4 + ex^3 + 6x^2 = x^2(x^2 + 3x + 6)$$

(2) Use conversion table for each factor and add the result:

$$\frac{2x^3 + 5x^2 + 6x + 12}{x^2(x^2 + 3x + 6)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 3x + 6}$$

where A, B, C, D are some constants. Our job now is to compute these constants.

(3) Simplify the right hand side by expanding to make a common denominator:

$$\begin{aligned}\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} &= \frac{A \cdot x(x^2 + 3x + 6) + B(x^2 + 3x + 6) + (Cx + D)(x^2)}{x^2(x^2 + 3x + 6)} \\ &= \frac{Ax^3 + 3Ax^2 + 6Ax + Bx^2 + 3Bx + 6B + Cx^3 + Dx^2}{x^2(x^2 + 3x + 6)} \\ &= \frac{(A + C)x^3 + (3A + B + D)x^2 + (6A + 3B)x + 6B}{x^2 + 3x + 6}\end{aligned}$$

(4) Compare this with original fraction. Using degrees, compare the coefficients to create a system of linear equations. That is we know that

$$2x^3 + 5x^2 + 6x + 12 = (A + C)x^3 + (3A + B + D)x^2 + (6A + 3B)x + 6B$$

and this gives us the system

$$\begin{aligned} x^3 : &\quad A + C = 2 \\ x^2 : &\quad 3A + B + D = 5 \\ x : &\quad 6A + 3B = 6 \\ constant : &\quad 6B = 12 \end{aligned}$$

See that $B = 2$ from the constant equation (just try to find the simplest one first), and so puzzling around with the different equations, we get that $A = 0, C = 2$ and $D = 3$.

More generally, you may have to work a bit more complicated methods such as substitution or elimination.

(5) Put it back into the integral! That is, we've found that

$$\frac{2x^3 + 5x^2 + 5x + 12}{x^2(x^2 + 3x + 6)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 3x + 6} = \frac{2}{x^2} + \frac{2x + 3}{x^2 + 3x + 6} \qquad \text{step 4}$$

and so we have

$$\begin{aligned}\int \frac{2x^3+5x^2+5x+12}{x^2(x^2+3x+6)}\,\mathrm{d}x &= \int \frac{2}{x^2}\,\mathrm{d}x + \int \frac{2x+3}{x^2+3x+6}\,\mathrm{d}x \\ &= -\frac{2}{x} + \ln|x^2+3x+6| + C \qquad \text{using } u\text{-sub}\end{aligned}$$

4. Improper Integrals

Most of the definite integrals that we've seen are on bounded intervals $[a, b]$.

Definition. An *improper integral* is an integral that has unbounded intervals or is an integral. Another type of improper integral is an integral where the integrand becomes infinite at one or both of the points of integration or a point between them.

If you encounter these kinds of problems, convert it into a limit of a definite integral. That is,

$$\boxed{\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x\,.}$$

You get to use whatever technique possible to work out the definite integral; improper integrals are just definite integral problems with just one extra step.

Since we are now playing with infinite things, we need to consider notions of convergence/divergence. If the limit on the RHS of the boxed equation is finite and is equal to a constant number, then the integral converges to that constant. Otherwise the integral diverges.

Example 6. What is the value of the following integral

$$\int_0^\infty xe^{-x^2}\,\mathrm{d}x$$

This is actually an example that we've already looked at (indefinite integral); and we note that x only differs by a scalar multiple from $-2x$ which is the derivative of $-x^2$ and so we should use a u-sub by setting $u = -x^2$. This gives us

$$\begin{aligned}\int_{x=0}^\infty xe^{-x^2} &= -\frac{1}{2}\lim_{b\to\infty}\int_{x=0}^b e^u\,\mathrm{d}u \\ &= \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_{x=0}^b \\ &= -\frac{1}{2}\lim_{b\to\infty}\left(e^{-b^2}-e^0\right)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}(0-1) \\
&= \frac{1}{2}
\end{aligned}
$$

and so the integral converges to 1/2.

5. Exercises

5.1. **Week 3 Exercises.** Week 3 exercises are about u-sub and integration by parts.

Question 1. *Perform a substitution to find the indefinite integral*

$$\int \tan(5x)\,\mathrm{d}x$$

Question 2. *Evaluate the following integral*

$$\int \sin(2x)\cos(2x)\,\mathrm{d}x$$

Question 3. *Evaluate the indefinite integral*

$$\int \frac{\mathrm{d}x}{x\ln(x)}$$

Question 4. *Use integration by parts to evaluate the indefinite integral*

$$\int x\ln(x)\,\mathrm{d}x$$

Question 5. *Evaluate the following integral*

$$\int e^x \sin(2x)\,\mathrm{d}x$$

5.2. **Week 4 Exercises.** Week 4 exercises are about partial fraction decomposition and improper integrals.

Question 6. *Use partial fraction decomposition to evaluate the indefinite integral*

$$\int \frac{6-x}{(x-3)(2x+5)}\,\mathrm{d}x\,.$$

Question 7. *Find the value of*

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

Question 8. *Which of the following converge?*

(i) $\displaystyle\int_1^\infty xe^{-x}\,\mathrm{d}x$

(ii) $\displaystyle\int_0^2 \frac{\mathrm{d}x}{(2-x)^2}$

(iii) $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x\ln(x)}$

6. Solutions to Exercises

6.1. **Week 3 Solutions.**

Question 1. *Perform a substitution to find the indefinite integral*

$$\int \tan(5x)\,\mathrm{d}x$$

Solution: Recalling that $\tan(\theta) = \sin(\theta)/\cos(\theta)$, we rewrite the integral to be

$$\int \tan(5x)\,\mathrm{d}x = \int \frac{\sin(5x)}{\cos(5x)}\,\mathrm{d}x$$

Setting $u = \cos(5x)$, take the derivative of u to get $\mathrm{d}u = -5\sin(5x)\,\mathrm{d}x$ and this tells us that $\sin(5x)\,\mathrm{d}x = -\frac{1}{5}\,\mathrm{d}u$ and so rewrite the integral to be

$$\begin{aligned}
\int \tan(5x)\,\mathrm{d}x &= \int \frac{\sin(5x)}{\cos(5x)}\,\mathrm{d}x \\
&= -\frac{1}{5}\int \frac{\mathrm{d}u}{u} && \text{where } u = \cos(5x) \\
&= -\frac{1}{5}\ln|u| + C && \text{where } C \text{ is a constant} \\
&= \boxed{-\frac{1}{5}\ln|\cos(5x)| + C} && \text{re-subbing}
\end{aligned}$$

Question 2. *Evaluate the following integral*

$$\int \sin(2x)\cos(2x)\,\mathrm{d}x$$

This is an interesting problem and depending on the method/substitutions you make, you will end up with different antiderivatives (something I only found out when writing these solutions...oops).

Solution 1: Recalling some fancy trig identities, $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, and so we can rewrite our integral to be

$$\int \sin(2x)\cos(2x)\,\mathrm{d}x = \int \frac{\sin(4x)}{2}\,\mathrm{d}x$$

and set $u = 4x$ and find that $\mathrm{d}u = 4\,\mathrm{d}x$ to get

$$\begin{aligned}
\int \sin(2x)\cos(2x)\,\mathrm{d}x &= \int \frac{\sin(4x)}{2}\,\mathrm{d}x \\
&= \frac{1}{2}\int \frac{\sin(u)}{4}\,\mathrm{d}u && \text{where } u = 4x
\end{aligned}$$

$$\begin{aligned} &= -\frac{1}{8}\cos(u) + C && \text{for some constant } C \\ &= \boxed{-\frac{1}{8}\cos(4x) + C} && \text{re-subbing} \end{aligned}$$

Solution 2: Setting $u = \sin(2x)$, compute the derivative to be $\mathrm{d}u = 2\cos(2x)\,\mathrm{d}x$ and rewrite the integral to be

$$\begin{aligned} \int \sin(2x)\cos(2x)\,\mathrm{d}x &= \frac{1}{2}\int u\,\mathrm{d}u && \text{where } u = \sin(2x) \\ &= \frac{1}{4}u^2 + C && \text{for some constant } C \\ &= \boxed{\frac{1}{4}\sin^2(2x) + C} && \text{re-subbing} \end{aligned}$$

Solution 3: Setting $u = \cos(2x)$, compute the derivative to be $\mathrm{d}u = -2\sin(2x)\,\mathrm{d}x$ and rewrite the integral to be

$$\begin{aligned} \int \sin(2x)\cos(2x)\,\mathrm{d}x &= -\frac{1}{2}\int u\,\mathrm{d}u && \text{where } u = \cos(2x) \\ &= -\frac{1}{4}u^2 + C && \text{for some constant } C \\ &= \boxed{-\frac{1}{4}\cos^2(2x) + C} && \text{re-subbing} \end{aligned}$$

We got *three different* anti-derivatives! They are all related by trig-identities and the derivative ignores what makes these anti-derivatives different, somewhat like how $x^2 + 1$ and x^2 have the same derivative.

Question 3. *Evaluate the indefinite integral*

$$\int \frac{\mathrm{d}x}{x\ln(x)}$$

Solution: Setting $u = \ln(x)$, compute the differential of u to be $\mathrm{d}u = 1/x\,\mathrm{d}x$, rewrite the integral to be

$$\begin{aligned} \int \frac{\mathrm{d}x}{x\ln(x)} &= \int \frac{\mathrm{d}u}{u} && \text{where } u = \ln(x) \\ &= \ln|u| + C && \text{for some constant } C \\ &= \boxed{\ln|\ln(x)| + C} && \text{re-subbing} \end{aligned}$$

Question 4. *Use integration by parts to evaluate the indefinite integral*

$$\int x \ln(x)\,\mathrm{d}x$$

Solution: Remembering LIATE, or dually, DETAIL, we see that we should pick $u = \ln(x)$ and $\mathrm{d}v = x$. With these, compute $\mathrm{d}u = 1/x\,\mathrm{d}x$ by taking derivative of u, and compute $v = x^2/2$ by taking integral of $\mathrm{d}v$. Now using the integration by parts formula which says

$$\begin{aligned}\int x \ln(x)\,\mathrm{d}x = \int \underbrace{\ln(x)}_{u}\underbrace{x\,\mathrm{d}x}_{\mathrm{d}v} &= uv - \int v\mathrm{d}u \\ &= \ln(x)\cdot\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,\mathrm{d}x \\ &= \frac{x^2\ln(x)}{2} - \frac{1}{2}\int x\,\mathrm{d}x \\ &= \boxed{\frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C} \qquad \text{for some constant } C\end{aligned}$$

Question 5. *Evaluate the following integral*

$$\int e^x \sin(2x)\,\mathrm{d}x$$

Solution: For this problem, we note that we should use integration by parts. However, since an $\int$ pops out in the integration by parts formula, we get that we will enter an infinite loop.

To get out of this loop, we will have to use integration by parts twice, and then observe that the integral that pops out after the second time is equal to the problem. It's probably easiest to learn by example (by this problem).

Using LIATE, we see $u = \sin(2x)$ and $\mathrm{d}v = e^x\,\mathrm{d}x$. Compute that $\mathrm{d}u = 2\cos(2x)\,\mathrm{d}x$ and $v = e^x$ to see that *first iteration* of integration by parts gives us

$$\int e^x \sin(2x)\,\mathrm{d}x = e^x\sin(2x) - 2\int e^x\cos(2x)\,\mathrm{d}x$$

and we now have to do integration by parts again. So now focusing on the integral portion of the right hand side of the equation, we set $u = \cos(2x)$ and $\mathrm{d}v = e^x$. Again compute that $\mathrm{d}u = -2\sin(2x)\,\mathrm{d}x$ and $v = e^x$ and so integration by parts gives us

$$\int e^x\cos(2x)\,\mathrm{d}x = e^x\cos(2x) - \int(-2\sin(2x))e^x\,\mathrm{d}x$$

$$= e^x \cos(2x) + 2 \int e^x \sin(2x)\, \mathrm{d}x \tag{1}$$

Now returning to the whole problem, we conclude that the *second iteration* of integration by parts gives us

$$\begin{aligned}
\underbrace{\int e^x \sin(2x)\, \mathrm{d}x}_{\text{our problem}} &= e^x \sin(2x) - 2 \int e^x \cos(2x)\, \mathrm{d}x && \text{first iteration} \\
&= e^x \sin(2x) - 2 \left(e^x \cos(2x) + 2 \int e^x \sin(2x)\, \mathrm{d}x \right) && \text{Equation (1)} \\
&= e^x \sin(2x) - 2e^x \cos(2x) - 4 \underbrace{\int e^x \sin(2x)\, \mathrm{d}x}_{\text{our problem!!!!}} && \text{simplify}
\end{aligned}$$

and so we essentially have an equation of the form

$$y = \text{our answer} - 4y.$$

With this note in mind, we add $\left[4 \int e^x \sin(2x)\, \mathrm{d}x\right]$ to both sides of the equation (the integral equation above with the "our problem" marked) and we see that

$$5 \underbrace{\int e^x \sin(2x)\, \mathrm{d}x}_{\text{our problem}} = e^x \sin(2x) - 2e^x \cos(2x)$$

and so to finish the problem off, divide by 5 on both sides. Thus, we have

$$\boxed{\int e^x \sin(2x)\, \mathrm{d}x = \frac{1}{5} \left(e^x \sin(2x) - 2e^x \cos(2x) \right) + C}$$

for some constant C.

6.2. **Week 4 Solutions.**

Question 6. *Use partial fraction decomposition to evaluate the indefinite integral*

$$\int \frac{6 - x}{(x-3)(2x+5)}\, \mathrm{d}x\,.$$

Solution: Using the conversion table, we have

$$\begin{aligned}
\frac{6 - x}{(x-3)(2x+5)} &= \frac{A}{x-3} + \frac{B}{2x+5} \\
&= \frac{A(2x+5) + B(x-3)}{(x-3)(2x+5)}
\end{aligned}$$

and this gives us that

$$6 - x = 2Ax + 5A + Bx - 3B$$

so we have the system of equations

$$\begin{aligned} x: &\quad 2A + B = -1 \\ const: &\quad 5A - 3B = 6 \end{aligned}$$

and so write $B = -1 - 2A$ and plug into the second equation, and compute that $A = 3/11$ and then replug that into $B = -1 - 2(3/11)$ to find that $B = -17/11$. So we have

$$\begin{aligned} \int \frac{6-x}{(x-3)(2x+5)}\,\mathrm{d}x &= \int \frac{3/11}{x-3}\,\mathrm{d}x + \int \frac{-17/11}{2x+5}\,\mathrm{d}x \\ &= \boxed{\frac{3}{11}\ln|x-3| - \frac{17}{22}\ln|2x+5| + C} && u\text{-sub} \end{aligned}$$

Question 7. *Find the value of*

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

Solution: We will have to use u-sub when evaluating the definite integral.

$$\begin{aligned} \int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x &= \lim_{b\to\infty}\int_0^b \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x \\ &= \lim_{b\to\infty}\int_0^b -2e^u\,\mathrm{d}u && \text{setting } u = -\sqrt{x} \\ &= -2\lim_{b\to\infty} e^{-\sqrt{x}}\Big|_{x=0}^{b} \\ &= -2\lim_{b\to\infty}\left(e^{-\sqrt{b}} - e^{-\sqrt{0}}\right) \\ &= -2(0-1) = \boxed{2} \end{aligned}$$

Question 8. *Which of the following converge?*

(i) $\displaystyle\int_1^\infty xe^{-x}\,\mathrm{d}x$

(ii) $\displaystyle\int_0^2 \frac{\mathrm{d}x}{(2-x)^2}$

(iii) $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x\ln(x)}$

SOLUTION: For the first one, use integration by parts to get

$$\begin{aligned}
\int_1^\infty xe^{-x}\,\mathrm{d}x &= \lim_{b\to\infty}\left(x(-e^{-x})\Big|_{x=1}^{b} - \int_1^b (-e^{-x})\,\mathrm{d}x\right)\\
&= \lim_{b\to\infty}\left(-xe^{-x} - e^{-x}\Big|_{x=1}^{b}\right)\\
&= \lim_{b\to\infty}\left(\frac{2}{e} - \frac{b+1}{e^b}\right)\\
&= \frac{2}{e}
\end{aligned}$$

So (i) converges to $2/e$.

For the second one, use a u-substitution to get

$$\begin{aligned}
\int_0^2 \frac{\mathrm{d}x}{(2-x)^2} = \lim_{b\to 2}\int_{x=0}^b \frac{-\,\mathrm{d}u}{u^2}\\
&= \lim_{b\to 2^-}\left(\frac{1}{2-x}\Big|_{x=0}^{b}\right)\\
&= \lim_{b\to 2^-}\left(\frac{1}{2-b} - \frac{1}{2}\right) = \infty
\end{aligned}$$

So (ii) does not converge.

For the final one, use u-substitution to get

$$\begin{aligned}
\int_1^\infty \frac{\mathrm{d}x}{x\ln(x)} &= \lim_{b\to\infty}\int_{x=1}^b \frac{\mathrm{d}u}{u} \qquad\qquad u=\ln(x)\\
&= \lim_{b\to\infty}\left(\ln\big(\ln(x)\big)\Big|_{x=1}^{b}\right)\\
&= \lim_{b\to\infty}\ln\ln(b)) = \infty
\end{aligned}$$

so (iii) does not converge.