

Computing Pythagorean Triples

A **Pythagorean triple** (a, b, c) is a triple of positive integers that satisfy the equation $a^2 + b^2 = c^2$.

Goal: Describe all possible Pythagorean triples.

1. Give three examples of Pythagorean triples.
2. Show that if you have a Pythagorean triple (a, b, c) , then (ka, kb, kc) is also a Pythagorean triple for $k \geq 1$.

Plato is usually known for philosophy, but philosophy also includes astronomy and mathematics! A formula ascribed to Plato for generating Pythagorean triples is the following:

$$x = n^2 - 1, \quad y = 2n, \quad z = n^2 + 1, \quad \text{for } n \geq 2.$$

3. Show that every (x, y, z) produced by Plato's formula is a Pythagorean triple.

4. Can you find a Pythagorean triple that was not on your previous list?
5. Find a Pythagorean triple that does not follow this formula.

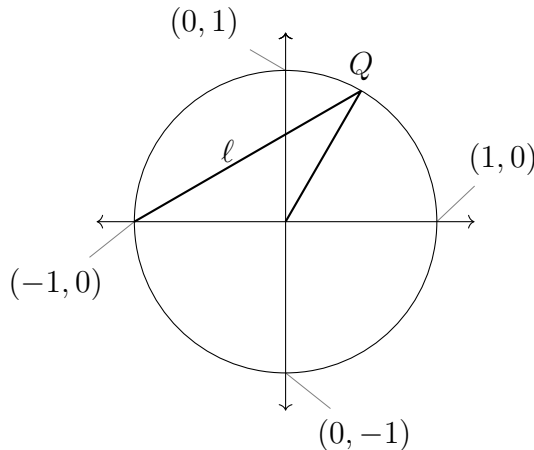
Since it's possible to retrieve a Pythagorean triple from a different Pythagorean triple by considering common factors, we start to focus on *primitive Pythagorean triples*.

A *primitive Pythagorean triple* is a triple (a, b, c) such that they have no common factors, e.g. $2, 3, 5$ is primitive, while $4, 6, 10$ is not.

Theorem 1 (Euclid's Theorem). *Let a, b, c positive integers. The a, b, c is a primitive Pythagorean triple if and only if there exist relatively prime integers r, s such that $r \not\equiv s \pmod{2}$ and $a = r^2 - s^2$, $b = 2rs$, $c = r^2 + s^2$.*

To establish why this works, we look to the *rational points* of the unit circle.

A *rational point* is a point (x, y) where both $x, y \in \mathbb{Q}$. Recall that the unit circle equation is $x^2 + y^2 = 1$.



6. (Triples to Rationals on Circle) Let a, b, c be a Pythagorean triple and let ℓ be a line from the origin to (a, b) . Why must ℓ pass through a rational point on the unit circle?
7. (Rationals on Circle to Triples) Let $t \in \mathbb{Q}^+$ and let ℓ be the line passing through the point $(-1, 0)$ and $(0, t)$.

- (a) Explain why the line ℓ has the equation $\ell : y = t(x + 1)$.
- (b) Let $Q = (x_Q, y_Q)$ be a point on ℓ and on the unit circle. Show that the y -intercept of ℓ is at $(0, \frac{y_Q}{x_Q + 1})$.

- (c) Show that the x -coordinate of Q is given by $x_Q = \frac{1 - t^2}{1 + t^2}$.

- (d) Explain why the y -coordinate of Q is given by $y_Q = \frac{2t}{1 + t^2}$. We've found that the point Q is given by the equation

$$Q = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right)$$

- (e) Writing $t = r/s$, show that $Q = (\frac{r^2 - s^2}{r^2 + s^2}, \frac{2rs}{r^2 + s^2})$ for $r, s \in \mathbb{Z}$.

- (f) Let $a = r^2 - s^2$, $b = 2rs$, and $c = r^2 + s^2$. Confirm that (a, b, c) is a Pythagorean triple.