# MATH 11A WEEK 10
# OPTIMISATION

## Contents



## 1. The Story

The entire semester had been building up to this point. To summarise everything we've seen, recall:

(1) We initially defined functions as a way to relate different numbers together (namely through inputs and outputs)
   (a) Domains, ranges, composition of functions
   (b) One-to-one functions, inverses (one-to-one functions are the only functions with inverses)
(2) We then started talking about *continuous functions* and *limits of sequences*
   (a) Convergence, divergence of sequences
   (b) Left and right handed limits for functions
   (c) Continuous functions, which require limits and actual outputs to be equal
   (d) Continuous functions and limits of sequences are related since continuous functions are the functions that allow limits to pass through, i.e., if $\lim_{n\to\infty} x_n = x$, and if $f$ is a continuous function then

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)) = f(x)$$

(3) We then looked a sub-class of continuous functions, namely *differentiable functions*
   (a) Derivatives are defined in terms of limits
   (b) Derivatives output slopes of tangent lines (they tell you what the rate of change at a point is)
   (c) Derivative rules (sum rule, product rule, quotient rule, chain rule, etc.)
   (d) L'Hopital's rule helps determine indeterminate limits
(4) Now we are at the whole purpose of derivatives: optimisation.
   (a) For this final part, we will use derivatives to tell us what the maximum or minimum of something is.

## 2. The Four-Step Programme

Our strategy to optimise will be given by a 4 step process:

**Your Four Step Guide for Optimisation**

In the case you are asked to maximise or minimise something, just break the problem up into 4 parts.

(1) Define/find a function that models what you are trying to measure
   - don't try to maximise/minimise anything yet. Just find an expression that will measure what you are trying to optimise
   - Examples: if you're asked to minimise area, just write area equation.

(2) Take the derivative of your function

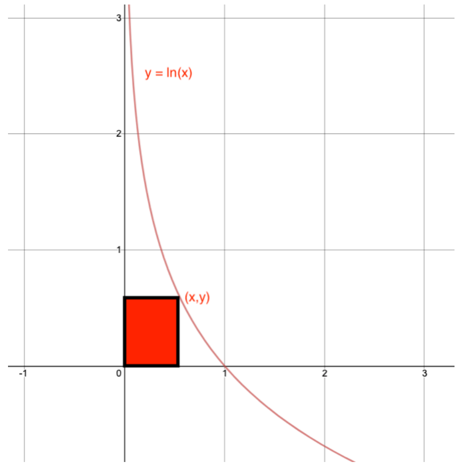
(3) Find the critical points of your function

(4) Double check that you are at a min or a max
   - Use the first derivative test (look for a sign change, going from $+$ to $-$ is a maximum, going from $-$ to $+$ is a minimum
   - For the first derivative test, create a table similar to the solution of Exercise 1 in this writeup but use $f'$ instead $f''$ and critical points instead of inflexion points.
   - Use the second derivative test (if $a$ is your critical point, and if $f''(a) < 0$, you are at a maximum but if $f''(a) > 0$, you are at a minimum
   - Use a graph to confirm
   - It does not matter which test you use, they are all equivalent.

We will see it on an example:

**Example 1.** *Suppose we have a rectangle bounded by the $x$-axis, the $y$-axis, and by the curve $y = -\ln(x)$. What is the largest possible area?*

The picture is given by




We follow the 4-step programme:

(1) Since we are trying to optimise the area of the rectangle, we write the area equation for the rectangle:

$$Area = (width) \times (length)$$

Since the width of the rectangle is determined by where we are on the $x$-axis, the width is determined simply by $x$. Since the rectangle is bounded above by the curve $y = -\ln(x)$, the length of the rectangle is $-\ln(x)$ if the width is $x$. Thus, we define the area function to be

$$A(x) = (width) \times (length) = -x\ln(x)$$

(2) We need to take the derivative:

$$\begin{aligned} A'(x) &= -\ln(x)\frac{\mathrm{d}}{\mathrm{d}x}[x] + x\frac{\mathrm{d}}{\mathrm{d}x}[-\ln(x)] \\ &= -\ln(x)\cdot 1 - x\cdot\frac{1}{x} \\ &= -\ln(x) - 1 \end{aligned}$$

(3) We need to find the critical points: To find the critical points, we need to find where the derivative is zero or undefined and so:

$$\begin{aligned} A'(x) = 0 &\implies -\ln(x) - 1 = 0 \\ &\implies \ln(x) = -1 \\ &\implies x = e^{-1}. \end{aligned}$$

(4) We should double check we are at a maximum. We can do this either by first derivative test, second derivative test, or a graph of $A(x)$. We'll do all 3.

(a) For the first derivative test, we have to divide up the domain into intervals, splitting everything up on the critical points. Since the domain of the $A(x)$ is positive numbers, create the table:

| intervals | $(0, e^{-1})$ | $(e^{-1}, \infty)$ |
|---|---|---|
| point in interval | $a = e^{-2}$ | $a = 1$ |
| $A'(a)$ | $A'(e^{-2}) = 1$ | $A'(1) = -1$ |
| sign | $++++++$ | $------$ |

and since we changed signs from $+$ to $-$, we have $x = e^{-1}$ is a maximum of $A(x)$.

(b) The second derivative test, compute the second derivative to be

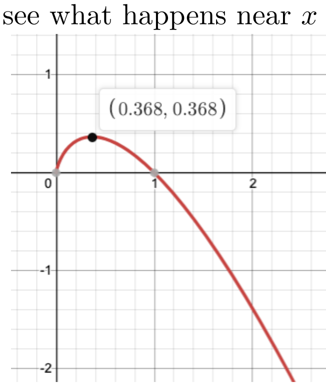
$$A''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[-\ln(x) - 1\right] = -\frac{1}{x}.$$

Now plugging in the critical point $x = e^{-1}$, see that

$$A''(e^{-1}) = -\frac{1}{e^{-1}} = -e < 0$$

and since $A''(critical point) < 0$, we have that the critical point is a maximum since $A$ is concave down at the critical point.

(c) Another alternative, which depends on using a graphing calculator, is to graph the area function $A(x)$ and see what happens near $x = e^{-1}$:




We're going to use the 4-step programme on an exercise we saw in Week 09.

**Example 2.** *Suppose we have two numbers $x$ and $y$ such that $x + y = 20$. What is the largest possible value the product $x^3y$ equal to?*

(1) For this, we need to maximise $x^3y$, but this has two variables as opposed to one. However, from the constraining condition, we have

$$x + y = 20 \implies y = 20 - x$$

and so we can now construct a function

$$f(x) = x^3(20 - x) = 20x^3 - x^4.$$

(2) We need to take the derivative:

$$f'(x) = 60x^2 - 4x^3.$$

(3) We need to find critical points:

$$f'(x) = 0 \Longrightarrow 60x^2 - 4x^3 = 0 \Longrightarrow 60x^2 = 4x^3 \Longrightarrow x = 15$$

(4) We have to confirm that this is a maximum. Confirm it via second derivative test (just because)

$$f''(x) = 120x - 12x^2 \Longrightarrow f''(15) = 120(15) - 12(15)^2 = 12(15)(-5) < 0$$

and since $f''(15) < 0$, $f$ is concave down at this critical point, giving us that $x = 15$ is a maximum of $f$.

Thus, the largest possible product is

$$f(15) = (15)^3(5).$$

## 3. The Exercises

**Question 1.** *What is the equation of the tangent line to the curve $y = x^3 - 3x^2 + 4x$ at the curve's inflexion point?*

**Question 2.** *Find the two numbers whose sum is 20 and whose product is a maximum.*

**Question 3.** *Let $R$ be the rectangle in the first quadrant bounded by the $x$-axis, $y$-axis, and the unit circle. What is the largest possible area of $R$? What is the area of the largest possible rectangle that can be inscribed inside the unit circle?*

## 4. Solutions to the Exercises

**Question 1.** *What is the equation of the tangent line to the curve* $y = x^3 - 3x^2 + 4x$ *at the curve's inflexion point?*

Solution. Let $f(x) = x^3 - 3x^2 + 4x$. To find the equation of the tangent line at the inflexion point, we need:

(1) to know where the inflexion point $x = a$ is (therefore the second derivative is necessary to compute)
(2) the slope of the tangent line at the point $a$ (therefore the first derivative is necessary to compute)
(3) the $y$-intercept of the tangent line.

Compute the first and second derivative of $f$ to be:

$$\begin{aligned} f(x) &= x^3 - 3x^2 + 4x \\ f'(x) &= 3x^2 - 6x + 4 \\ f''(x) &= 6x - 6 \end{aligned}$$

(1) To find the inflexion point, we set its second derivative to zero, i.e.,

$$f''(x) = 0 \Longrightarrow 6x - 6 = 0 \Longrightarrow 6x = 6 \Longrightarrow x = 1.$$

Check that this is an inflexion point by seeing if there is a sign change in $f''(x)$ at $x = 1$:

| intervals | $(-\infty, 1)$ | $(1, \infty)$ |
|---|---|---|
| point in interval | $a = 0$ | $a = 2$ |
| $f''(a)$ | $f''(0) = -6$ | $f''(2) = 6$ |
| sign | $------$ | $++++++$ |

and so $x = 1$ is an inflexion point of $f(x)$.

(2) To find the slope of the tangent line at the point $x = 1$, plug $x = 1$ into the first derivative:

$$f'(1) = 3 \cdot 1^2 - 6 \cdot 1 + 4 = 1$$

and so the slope of the tangent line at $x = 1$ is one. That is, if $\ell(x)$ is the tangent line at $x = 1$, then

$$\ell(x) = x + b.$$

(3) We now need to find what $b$ (the $y$-intercept) is. To do this, we need a point on $\ell(x)$ and since we know that $\ell(1) = f(1)$ since they meet at that point, compute $f(1)$). Computing $f(1) = 1 - 3 + 4 = 2$, we have

$$\ell(1) = f(1) = 2 \Longrightarrow 2 = \ell(1) = 1 + b \Longrightarrow b = 1.$$

So the equation of the tangent line is $\ell(x) = x + 1$.

**Question 2.** *Find the two numbers whose sum is 20 and whose product is a maximum.*

Solution. Call the two numbers $x$ and $y$. The first condition saying their sum is 20 is equivalent to saying

$$x + y = 20.$$

We need to maximise the equation $xy$. Since we know $x + y = 20$, we can rewrite this equation to be $y = 20 - x$ and so substituting this into the equation $xy$, we have

$$f(x) = xy = x(20 - x) = 20x - x^2.$$

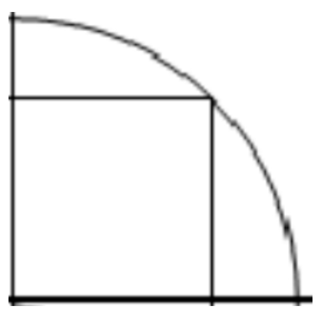
We now start the 4-step program! To find the maximum, we need to identify the critical points of $f(x)$ and so we compute the derivative and set it equal to zero:

$$f'(x) = 20 - 2x.$$

Setting $f'(x)$ equal to zero, we have $20 - 2x = 0 \Longrightarrow x = 10$ and so $x = 10$ is a critical point. It can be checked that this is a point where $f(x)$ achieves a maximum using either a graph, the first derivative test (create a table just like in exercise 1), or the second derivative test; it does not matter which test we use since these are all equivalent.
After getting that $x = 10$ gives a maximum for $f(x)$, we have found one of the numbers! Since $y = 20 - x$, and since $x = 10$, we have $y = 10$ as well.

**Question 3.** *Let $R$ be the rectangle in the first quadrant bounded by the $x$-axis, $y$-axis, and the unit circle. What is the largest possible area of $R$? What is the area of the largest possible rectangle that can be inscribed inside the unit circle?*

The picture in mind is:



Solution. We follow the 4-step program again. Since we need to maximise the area of the rectangle, with area given by $A = (length)(width)$. We note that a point on the unit circle $(x, y)$ determine both the length and the width of the rectangle. Recall that the equation of the unit circle is:

$$x^2 + y^2 = 1 \Longrightarrow y = \sqrt{1 - x^2}$$

and since the area is determined by $xy$, we have $A(x) = x(\sqrt{1 - x^2})$.

To determine the maximum, we find the critical points:

$$\begin{aligned} A'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}[x] \cdot \sqrt{1 - x^2} + x \cdot \frac{\mathrm{d}}{\mathrm{d}x}[\sqrt{1 - x^2}] \\ &= 1 \cdot \sqrt{1 - x^2} + x \frac{-2x}{2\sqrt{1 - x^2}} \end{aligned}$$

Setting this equal to zero, we have

$$\begin{aligned} A'(x) = 0 &\Longrightarrow \sqrt{1 - x^2} = \frac{x^2}{\sqrt{1 - x^2}} \\ &\Longrightarrow 1 - x^2 = x^2 && \text{multiply both sides by } (\sqrt{1 - x^2}) \\ &\Longrightarrow x^2 = 1/2 \end{aligned}$$

and so the critical point is at $x = 1/\sqrt{2}$. Checking that this is a maximum can be done with first derivative test, a graph, or the second derivative test.

Granting that $x = 1/\sqrt{2}$ is a maximum for $A(x)$, plug in that value to get the largest possible area:

$$\begin{aligned} A(\frac{1}{\sqrt{2}}) &= \frac{1}{\sqrt{2}} \left( \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} \right) \\ &= \frac{1}{\sqrt{2}} \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}. \end{aligned}$$

Thus the largest possible area is 1/2.

For the largest possible rectangle inside the unit circle (not bounded by the axes), we multiply by 4 since this is symmetric for every quarter of the circle. Thus the area of the largest possible rectangle that will fit inside the unit circle is 2.