MATH 11A WEEK 6
DERIVATIVES RULE(S)!

1. Computing Derivatives

The $P \implies Q$ is read "if P then Q", e.g. you're apple $\implies$ you're fruit is read "if you're an apple, then you're a fruit."

Derivatives of Elementary Functions

(1) Derivative of Constants: if k is any real number, and $f(x) = k$ is constant, then $f'(x) = 0$. In the logical symbols using the arrow, this is written as

$$f(x) = k, \quad k \text{ is constant} \implies f'(x) = 0$$

(2) Derivative of Monomial: The **power rule** says: If $f(x) = x^n$ where n is any real number, then $f'(x) = nx^{n-1}$. In symbols and more generally,

$$f(x) = x^n \implies f'(x) = nx^{n-1}$$

(3) Derivative of Exponential: If $f(x) = e^x$, then $f'(x) = e^x$. More generally, if $a \neq 0$ is a constant number,

$$f(x) = a^x \implies f'(x) = \ln(a) \cdot a^x$$

(4) Derivative of Logarithm: If $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$. More generally, if $a \neq 0$ is a constant number,

$$f(x) = \log_a(x) \implies f'(x) = \frac{1}{x \log a}$$

(5) Derivatives of Trig Functions:

$$f(x) = \sin(x) \implies f'(x) = \cos(x)$$
$$f(x) = \cos(x) \implies f'(x) = -\sin(x)$$
$$f(x) = \tan(x) \implies f'(x) = \sec^2(x)$$
$$f(x) = \csc(x) \implies f'(x) = -\csc(x)\cot(x)$$
$$f(x) = \sec(x) \implies f'(x) = \sec(x)\tan(x)$$
$$f(x) = \cot(x) \implies f'(x) = -\csc^2(x)$$

These derivatives are all computed by applying each individual case to the definition of the derivative, which was

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

From this definition of the derivative, we also get more abstract rules which we now list:

Rules of Derivatives

(1) Derivative of sum: The derivative of sums is the sums of derivatives:

$$(f+g)'(x) = f'(x) + g'(x).$$

(2) Derivatives of scalar multiple: For any real number k, we have

$$(kf)'(x) = k \cdot f'(x)$$

(3) Derivative of products: "share the prime" or "FIG GIF" (reading $'$ as i)[a]

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

(4) Derivative of quotient: "low d high minus high d low all over low squared"

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

(5) Derivative of composition: the **chain rule** says "d of out times d of in"

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

[a] Credit to Lauren Song

Using these two tables, you should be able to compute almost any derivative!! These calculations can get a little complicated, but nevertheless we have rules that allow us to break any problem down.

The most important of these rules is the chain rule since that is the rule that makes everything else work wonderfully; *if you see a complicated problem, make variable substitutions and use the chain rule.*

That is, if we want a derivative of $f(x)$, then using the $\frac{\mathrm{d}f}{\mathrm{d}x}$ notation, the derivative of $f(x)$ can be computed by

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

It may be tempting to cancel out the terms in the fraction, but we can't do this because these are operations, not numbers! What this is saying is we can make a substitution of variables, and change $f(x)$ into a function $f(u)$, but if we do that, then we must compute how the variable u changes with respect to x, which is precisely given by $u'(x)$, which is also written as $\mathrm{d}u/\mathrm{d}x$.

We try this idea out in the following example:

Example 1. Find the derivatives of the following functions:

$$f(x) = \sqrt{\sin(3x)}, \quad g(x) = \ln\big(\ln(x)\big) \quad h(x) = \frac{\ln\big(\sin(2x)\big)}{\cos x}$$

Since $f(x) = \sqrt{\sin(3x)} = \big(\sin(3x)\big)^{1/2}$, we need to apply the chain rule a couple times. Following the pattern *make variable substitutions and use the chain rule* gives us the recipe:

(1) Make a substitution $u(x) = \sin(3x)$. Then $f(u) = \sqrt{u}$ and so by the chain rule, $f'(u) = \frac{1}{2}u^{-1/2} \cdot u'(x)$.
(2) Since $u(x) = \sin(3x)$, we only need to find what $u'(x)$ is. Make a substitution $v(x) = 3x$ so $u(v) = \sin(v)$
(3) We know from the chain rule that $u'(v) = \cos(v) \cdot v'(x)$.
(4) We can compute directly that $v'(x) = 3$ since $v(x) = 3x$.
(5) Put everything together:

$$f'(u) = \frac{1}{2}u^{-1/2} \cdot \cos(v) \cdot v'(x) = \frac{1}{2}(\sin(3x))^{-1/2} \cdot \cos(3x) \cdot 3.$$

Now without any words, this can rewritten as:

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(3x)\big)^{1/2} & \\
&= \frac{1}{2}\big(\sin(3x)\big)^{-1/2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin(3x) && \text{chain rule} \\
&= \frac{1}{2\sqrt{\sin(3x)}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin(3x) && \text{rewriting } (\ )^{-1/2} = \frac{1}{\sqrt{\ }} \\
&= \frac{1}{2\sqrt{\sin(3x)}} \cdot \cos(3x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(3x) && \text{chain rule} \\
&= \frac{3\cos(3x)}{2\sqrt{\sin(3x)}} && \text{simplifying}
\end{aligned}$$

and so

$$f'(x) = \frac{3\cos(3x)}{2\sqrt{(3x)}}.$$

To find $g'(x)$, where $g(x) = \ln\big(\ln(x)\big)$, we give it a very similar treatment.

(1) Let $u = \ln(x)$ and as a result of the chain rule, if we make this substitution, we better compute what $u'(x)$ is.
(2) We know that the derivative of $\ln(x)$ is given by $1/x$ and so $u'(x) = 1/x$.
(3) So we have $g(x) = \ln\big(u(x)\big)$ and so

$$g'(x) = \left(\frac{\mathrm{d}}{\mathrm{d}u}\ln(u)\right) \cdot u'$$

(4) Putting everything together,

$$g'(x) = \frac{1/x}{\ln(x)} = \frac{1}{x\ln(x)}$$

Now we come to $h(x) = \dfrac{\ln\big(\sin(2x)\big)}{\cos x}$. We're going to have to do a mix of both quotient rule and a chain rule.

(1) Since this is first and foremost a quotient, apply the quotient rule.
(2) To find the derivative of the numerator, we need to apply the chain rule where $u = \sin(2x)$ and then another $v = 2x$ and we can compute that $u' = \cos(2x)v' = \cos(2x) \cdot 2$.
(3) Put everything together!

Now to see it in action:

$$\begin{aligned}
h'(x) &= \frac{\cos(x)\dfrac{\mathrm{d}}{\mathrm{d}x}\Big[\ln\big(\sin(2x)\big)\Big] - \ln\big(\sin(2x)\big)\dfrac{\mathrm{d}}{\mathrm{d}x}\big[\cos(x)\big]}{(\cos(x))^2} && \text{by quotient rule} \\
&= \frac{\cos(x) \cdot \dfrac{1}{\sin(2x)} \cdot 2\cos(2x) - \ln\big(\sin(2x)\big) \cdot -\sin(x)}{(\cos(x))^2} && \text{by chain rule} \\
&= \frac{2\cos(x)\cos(2x) + \sin(x)\sin(2x)\ln\big(\sin(2x)\big)}{\sin(2x) \cdot \cos^2(x)} && \text{simplifying}
\end{aligned}$$

2. Exercises

Question 1. *Given a differentiable function $f(x)$, what can you say about the behaviour of the function $f(x)$ at a point $x = a$ if*

(1) $f'(a) > 0$*?*
(2) $f'(a) = 0$*?*
(3) $f'(a) < 0$*?*

Question 2. *Compute the derivatives of*

$$f(x) = 2x^3 + 5x, \quad g(x) = 3x\sin(2x), \quad h(x) = 10e^{2x}, \quad \ell(x) = \sqrt{2x}$$

Question 3. *Find the equation of the line that is tangent to the function*

$$f(x) = (x^2 + \sin(2\pi x))^{2020}$$

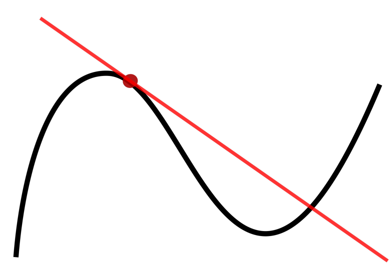
at the point $(1, 1)$

3. Solutions to Exercises

Question 1. *Given a differentiable function $f(x)$, what can you say about the behaviour of the function $f(x)$ at a point $x = a$ if*

(1) $f'(a) > 0$*?*
(2) $f'(a) = 0$*?*
(3) $f'(a) < 0$*?*

Solution. As we saw before, the $f'(a)$ gives the slope of the line that is tangent to $f(x)$ at the point $x = a$. For reference, see the picture from Wikipedia (which can be accessed at the link `https://en.wikipedia.org/wiki/Tangent`):



So if $f'(a) > 0$, our slope of the tangent line is positive and the function is **increasing**. If $f'(a) < 0$, then the slope of the tangent line is negative (just like in the picture above) and we see that the function is **decreasing**. If $f'(a) = 0$, the slope of the tangent line is 0 and we are therefore neither increasing nor are we decreasing (oftentimes this is a local maximum or minimum, but we can't say what this point $x = a$ is just yet without further work).

Question 2. *Compute the derivatives of*

$$f(x) = 2x^3 + 5x, \quad g(x) = 3x\sin(2x), \quad h(x) = 10e^{2x}, \quad \ell(x) = \sqrt{2x}$$

Solution. Jumping right in,

$$\begin{aligned}
f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2x^3 + 5x\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(2x^3\right) + \frac{\mathrm{d}}{\mathrm{d}x}(5x) && \text{addition rule} \\
&= 3 \cdot 2x^2 + 5 && \text{power rule} \\
&= 6x^2 + 5 && \text{simplify}
\end{aligned}$$

Looking at $g(x) = 3x\sin(2x)$, we see that this is a product of the functions $3x$ and $\sin(2x)$. So computing the derivative,

$$\begin{aligned}
g'(x) = \frac{\mathrm{d}g}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\big(3x\sin(2x)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(3x)\cdot\sin(2x) + 3x\cdot\frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(2x)\big) && \text{product rule} \\
&= 3\sin(2x) + 3x\cdot\frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(2x)\big) && \text{power rule}
\end{aligned}$$

we are now left with computing the derivative of $\sin(2x)$. Let $u = 2x$ and by the chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(2x)\big) = \frac{\mathrm{d}}{\mathrm{d}u}\big(\sin(u)\big)\frac{\mathrm{d}u}{\mathrm{d}x}$$

We see that since $u = 2x$, then $u' = 2$ and so

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(2x)\big) &= \frac{\mathrm{d}}{\mathrm{d}u}\big(\sin(u)\big)\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\big(\sin(u)\big)\,2 && \text{since } u' = \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= 2\cos(u) && \text{trig derivative!} \\
&= 2\cos(2x) && \text{since } u = 2x
\end{aligned}$$

Thus,

$$\begin{aligned}
g'(x) &= 3\sin(2x) + 3x\cdot\frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(2x)\big) && \text{our work above} \\
&= 3\sin(2x) + 3x\cdot\big(2\cos(2x)\big) \\
&= 3\sin(2x) + 6x\cos(2x).
\end{aligned}$$

Now consider $h(x) = 10e^{2x}$. Compute that

$$\begin{aligned}
h'(x) = \frac{\mathrm{d}h}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(10e^{2x}\right) \\
&= 10\frac{\mathrm{d}}{\mathrm{d}x}e^{2x} && \text{scalar rule}
\end{aligned}$$

To find the derivative of e^{2x}, we make a substitution $u = 2x$, and compute that $u' = 2$. Now using the chain rule, we see

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}e^{2x} &= \frac{\mathrm{d}}{\mathrm{d}u}e^{u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= 2\frac{\mathrm{d}}{\mathrm{d}u}e^{u} && \text{since } u' = 2 \\
&= 2e^{u} && \text{since deriv. of exp is itself} \\
&= 2e^{2x} && \text{since } u = 2x
\end{aligned}$$

Thus, we get that

$$h'(x) = 10\frac{\mathrm{d}}{\mathrm{d}x}e^{2x} = 10\cdot 2e^{2x} = 20e^{2x}$$

Now we come to the last one, $\ell(x) = \sqrt{2x}$. Since $\sqrt{a} = (a)^{1/2}$, write $\ell(x) = (2x)^{1/2}$. Now let $u = 2x$. Compute that $u' = 2$ and

$$\begin{aligned}
\ell'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left((2x)^{1/2}\right) &= \frac{\mathrm{d}}{\mathrm{d}u}(u)^{1/2}\frac{\mathrm{d}u}{\mathrm{d}x} && \text{chain rule}\\
&= \frac{1}{2}u^{-1/2}\cdot 2 && \text{power rule and since } u' = 2\\
&= \frac{1}{2}(2x)^{-1/2}\cdot 2 && \text{since } u = 2x\\
&= \frac{1}{\sqrt{2x}} && \text{simplifying}
\end{aligned}$$

Question 3. *Find the equation of the line that is tangent to the function*

$$f(x) = (x^2 + \sin(2\pi x))^{2020}$$

at the point $(1, 1)$

Recall that the equation of the line is given by $y = mx + b$, where m is the slope and b is the y-intercept. Since the derivative determines the slope, we must compute $f'(x)$ and evaluate at $x = 1$ (as $x = 1$ is the point of interest since we are looking at $(1, f(1))$).

Now to compute $f'(x)$, let $u = x^2 + \sin(2\pi x)$. Then we have $f(u) = u^{2020}$ and so by the chain rule

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left((x^2 + \sin(2\pi x))^{2020}\right)\\
&= \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} && \text{chain rule}\\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left(u^{2020}\right)\frac{\mathrm{d}u}{\mathrm{d}x}
\end{aligned}$$

Now to compute u', we have

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + \sin(2\pi x)\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(2\pi x)\right) && \text{addition law}\\
&= 2x + \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(2\pi x)\right)
\end{aligned}$$

and now to compute what the derivative of $\sin(2\pi x)$ is, let $v = 2\pi x$ and compute that $v' = 2\pi$. Thus, we have that

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x} &= 2x + \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(2\pi x)\right) && \text{from above}\\
&= 2x + \frac{\mathrm{d}}{\mathrm{d}v}\left(\sin(v)\right)\frac{\mathrm{d}v}{\mathrm{d}x} && \text{set } v = 2\pi x \text{ and use chain rule}\\
&= 2x + 2\pi\cos(v) && \text{since } v' = 2\pi
\end{aligned}$$

$$= 2x + 2\pi\cos(2\pi x) \qquad \text{since } v = 2\pi x$$

Now recall that from the first part of our work, we had

$$\begin{aligned} f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}u}\left(u^{2020}\right)\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 2020u^{2019}\frac{\mathrm{d}u}{\mathrm{d}v} && \text{power rule} \\ &= 2020u^{2019}\big(2x + 2\pi\cos(2\pi x)\big) \\ &= 2020\left(x^2 + \sin(2\pi x)\right)^{2019}\big(2x + 2\pi\cos(2\pi x)\big). \end{aligned}$$

Now we have found our general derivative and so because we want the slope of the line tangent to f at the point $x = 1$, plug in $x = 1$ into the derivative:

$$\begin{aligned} f'(1) &= 2020\left(1^2 + \sin(2\pi)\right)^{2020}\big(2 + 2\pi\cos(2\pi)\big) \\ &= 2020\,(1+0)^{2020}\,(2 + 2\pi) && \text{since } \sin(2\pi) = 0, \cos(2\pi) = 1 \\ &= 2020\,(2 + 2\pi). \end{aligned}$$

This is our slope!

Now since the point of interest is $(1, 1)$, and the equation of the line is $y = mx + b$, we only need to find what b is now. This gives us

$$1 = f'(1) \cdot 1 + b \Longrightarrow b = 1 - f'(1)$$

and so the $b = 1 - 2020\,(2 + 2\pi)$. Thus, we conclude that the equation of the tangent line to $f(x)$ at the point $(1, 1)$ is

$$y = \underbrace{2020(2 + 2\pi)}_{\text{this is } m}\,x + \underbrace{1 - 2020(2 + 2\pi)}_{\text{this is } b}.$$