Part I
References

Dynamics and Control

Part II
Dynamic Modeling

Linear Time Invariant (LTI) System

State Space Form $\leftrightarrow$ Differential Equation $\leftrightarrow$ Transfer Function

1 Steps in Dynamic Modeling

1. Identify objective for the simulation
2. Draw a schematic diagram, labeling process variables
3. Determine spatial dependence
   yes = Partial Differential Equation (PDE)
   no = Ordinary Differential Equation (ODE)
4. Write dynamic balances (mass, species, energy)
5. Other relations (thermo, reactions, geometry, etc.)
6. Degrees of freedom, does number of equations = number of unknowns?

   $$DOE = N_V - N_E = \text{total variables} - \text{state (constrained) (output) variables} = \text{controllable (input) variables}$$

7. Classify inputs as
   Fixed values
   Disturbances
   Manipulated variables
8. Classify outputs as
   States
   Controlled variables
9. Simplify balance equations based on assumptions
10. Simulate steady state conditions (if possible)
11. Simulate the output with an input step

2 Balance Equations

$$\text{Accumulation} = \text{In} - \text{Out} + \text{Generation} - \text{Consumption}$$

2.1 Mass Balance

$$\frac{dm}{dt} = \frac{d(\rho V)}{dt} = \sum \dot{m}_{in} - \sum \dot{m}_{out}$$

2.2 Species Balance

$$\frac{dn_A}{dt} = \sum \dot{n}_{A_{in}} - \sum \dot{n}_{A_{out}} + \sum \dot{n}_{A_{gen}} - \sum \dot{n}_{A_{cons}}$$

$$\frac{dc_A V}{dt} = \sum c_{A_{in}} \dot{V}_{in} - \sum c_{A_{out}} \dot{V}_{out} + r_A V$$

2.3 Momentum Balance

$$\frac{d(m\,v)}{dt} = \sum F$$

2.4 Energy Balance

$$\frac{dE}{dt} = \frac{d(U+K+P)}{dt} = \sum \dot{m}_{in}\left(\hat{h}_{in} + \frac{v_{in}^2}{2g_c} + \frac{z_{in}g_{in}}{g_c}\right) - \sum \dot{m}_{out}\left(\hat{h}_{out} + \frac{v_{out}^2}{2g_c} + \frac{z_{out}g_{out}}{g_c}\right) + Q + W_s$$

$$\frac{dh}{dt} = \sum \dot{m}_{in}\hat{h}_{in} - \sum \dot{m}_{out}\hat{h}_{out} + Q + W_s$$

$$m\,c_p\frac{dT}{dt} = \sum \dot{m}_{in}c_p\left(T_{in} - T_{ref}\right) - \sum \dot{m}_{out}c_p\left(T_{out} - T_{ref}\right) + Q + W_s$$

$$\frac{\mathrm{d}(V\rho\int C_{\mathrm{d}}T)}{\mathrm{d}t} = (\phi\rho\int C_{\mathrm{d}}T)_{\mathrm{in}} - (\phi\rho\int C_{\mathrm{d}}T)_{\mathrm{out}} + rate\cdot V\cdot\Delta H$$

3 Linearization of Differential Equations (Linear Function in Deviation Variable Form)

Linearization is the process of taking the *gradient* of a nonlinear function with respect to all variables and creating a linear representation at that point. (*input* u, *output* y, *disturbance variable* d)

$$\frac{dy}{dt} = f(y,u)$$

linearized by a Taylor series expansion, using only the first two terms:

$$\frac{dy}{dt} = f(y,u) \approx f\left(\bar{y},\bar{u}\right) + \left.\frac{\partial f}{\partial y}\right|_{\bar{y},\bar{u}}(y-\bar{y}) + \left.\frac{\partial f}{\partial u}\right|_{\bar{y},\bar{u}}(u-\bar{u})$$

$$\frac{dy'}{dt} = \alpha y' + \beta u' + \gamma d'$$

$$y' = y - \overline{y} \quad u' = u - \overline{u} \quad d' = d - \overline{d}$$

$$\alpha = \left.\frac{\partial f}{\partial y}\right|_{\overline{y},\overline{u},\overline{d}} \quad \beta = \left.\frac{\partial f}{\partial u}\right|_{\overline{y},\overline{u},\overline{d}} \quad \gamma = \left.\frac{\partial f}{\partial d}\right|_{\overline{y},\overline{u},\overline{d}}$$

4 State Space Models

4.1 Linear Time Invariant (LTI) State Space Models

Linear Time Invariant (LTI) state space models are a linear representation of a dynamic system in either discrete or continuous time.

input u to state x

$$\dot{x} = Ax + Bu$$

state x to (measured) output y

$$y = Cx + Du$$

Stability The linear state space model is stable if all eigenvalues of A are negative real numbers or have negative real parts to complex number eigenvalues. If all real parts of the eigenvalues are negative then the system is stable, meaning that any initial condition converges exponentially to a stable attracting point. If any real parts are zero then the system will not converge to a point and if the eigenvalues are positive the system is unstable and will exponentially diverge.

4.2 General Nonlinear System

$$\frac{dx}{dt} = f(x,u)$$
$$y = g(x)$$

Model linearization

$$\frac{dx}{dt} = f(x,u) \approx f(\bar{x},\bar{u}) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}} (x-\bar{x}) + \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u}} (u-\bar{u})$$

$$y - \bar{y} = g(x) \approx g(\bar{x}) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} (x-\bar{x})$$

State space

$$\frac{dx'}{dt} = \alpha\, y' + \beta\, u'$$
$$y' = \gamma\, x'$$
$$\alpha = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}} \qquad \beta = \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u}} \qquad \gamma = \left.\frac{\partial g}{\partial x}\right|_{\bar{x}}$$

5 Common Empirical Description - First Order Plus Dead Time (FOPDT)

A first-order linear system with time delay is a common empirical description of many stable dynamic processes. The First Order Plus Dead Time (FOPDT) model is used to obtain initial controller tuning constants.

$$\tau_p \frac{dy(t)}{dt} = -y(t) + K_p u\left(t - \theta_p\right)$$

5.1 Process Gain K_p - magnitude of the response

The process gain is the change in the output y induced by a unit change in the input u.

$$K_p = \frac{\Delta y}{\Delta u} = \frac{y_{ss_2} - y_{ss_1}}{u_{ss_2} - u_{ss_1}}$$

5.2 Process Time Constant τ_p - speed of response

Given a change in $u(t) = \Delta u$,

$$y(t) = \left(e^{-t/\tau_p}\right) y(0) + \left(1 - e^{-t/\tau_p}\right) K_p \Delta u$$

$$y_j = e^{\frac{-\Delta t}{\tau_p}} \left(y_{j-1} - y_0\right) + \left(1 - e^{\frac{-\Delta t}{\tau_p}}\right) K_p \left(u_{j-\theta_p-1} - u_0\right) + y_0$$

If the initial condition $y(0) = 0$ and at $t = \tau_p$, the solution is simplified to the following.

$$y\left(\tau_p\right) = \left(1 - e^{-\tau_p/\tau_p}\right) K_p \Delta u = \left(1 - e^{-1}\right) K_p \Delta u = 0.632 K_p \Delta u$$

5.3 Process Time Delay θ_p - response time delay

The effect of θ_p is to delay the effect of $u(t)$.

$$u\left(t-\theta_p\right)$$

$$y(t<\theta_p)=y(0)$$

$$y(t\geq\theta_p)=\left(e^{-(t-\theta_p)/\tau_p}\right)y(0)+\left(1-e^{-(t-\theta_p)/\tau_p}\right)K_p\Delta u$$

5.4 Transfer Function

$$G(s)=\frac{Y(s)}{U(s)}=\frac{K_p e^{-\theta_p s}}{\tau_p s+1}$$

Without time delay θ_p:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{K_p}{\tau_p s+1}$$

5.5 FOPDT to Step Test

Fitting the parameters K_p, τ_p, θ_p to a step response.

$$\text{experimental data}\rightarrow\text{model parameters}$$

1. Find Δy and Δu from step response (step test experimental data), then calculate $K_p=\frac{\Delta y}{\Delta u}$.
2. Find θ_p apparent dead time, from step response.
3. Find $t_{0.632}$ for $y(t_{0.632})=0.623\Delta y$ from step response, calculate $\tau_p=t_0.632-\theta_p$ (step start at $t=0$).

6 Laplace Transform

$$F(s)=\mathcal{L}\left(f(t)\right)=\int_0^\infty f(t)e^{-s\,t}dt$$

6.1 Final Value Theorem (FVT)

$$y_\infty=\lim_{s\to 0}s\,Y(s)$$

6.2 Initial Value Theorem (IVT)

$$y_0=\lim_{s\to\infty}s\,Y(s)$$

7 Transfer Function

Transfer functions are input to output representations of dynamic systems.

$$G(s)=\frac{Y(s)}{U(s)}$$

Transfer Function Matrix form State Space Model

$$G(s)=C\cdot(SI-A)^{-1}B$$

From transfer function and input to get output:

$$Y(s)=G(s)U(s)$$

$$U(s)=\mathcal{L}(U(s))$$

$$Y(t)=\mathcal{L}^{-1}(Y(s))=\mathcal{L}^{-1}(G(s)U(s))$$

Properties

- Relate *single input* to *single output*.
- Represent an *algebraic* relationship in s domain.
- Can be *combined* to give the total system behavior.
- Convenient to use with *block diagram*.
- Require initial condition to be *0*, (use *deviation* variables)
- Additive property (*parallel* process)
- Multiplicative property (*series* process)
- $\lim_{s\to 0} G(s) = K_p$

7.1 Transfer Function Gain

$$K_p = \lim_{s\to 0} G(s)$$

$$K_p = \frac{\Delta y}{\Delta u} = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{Y(s)}{U(s)} = \lim_{s\to 0} \frac{Y(s)}{\frac{1}{s}}$$

corresponding to FVT. Also in deviation variable form,

$$\frac{y_\infty - 0}{1-0} = y_\infty = \lim_{s\to 0} \frac{Y(s)}{\frac{1}{s}} = \lim_{s\to 0} sY(s)$$

7.2 PID Equation in Laplace Domain

$$u(t) = u_{bias} + K_c\, e(t) + \frac{K_c}{\tau_I} \int_0^t e(t)dt - K_c \tau_D \frac{d(PV)}{dt}$$

$$u'(t) = u(t) - u_{bias}$$

$$U(s) = K_c\, E(s) + \frac{K_c}{\tau_I\, s} E(s) - K_c\, \tau_D PV(s)$$

If $\tau_D = 0$ for a PI controller:

$$G_c(s) = \frac{U(s)}{E(s)} = K_c + \frac{K_c}{\tau_I\, s} = K_c \frac{(\tau_I s + 1)}{\tau_I s}$$

8 Second Order Systems (SOPDT)

A second-order linear system is a common description of many dynamic processes. The response depends on whether it is an overdamped, critically damped, or underdamped second order system.

$$\tau_s^2 \frac{d^2 y}{dt^2} + 2\zeta\tau_s \frac{dy}{dt} + y = K_p\, u\,(t-\theta_p)$$

Damping factor ζ

Laplace Domain, Transfer Function

$$\frac{Y(s)}{U(s)} = \frac{K_p}{\tau_s^2 s^2 + 2\zeta\tau_s s + 1} e^{-\theta_p s}$$

State Space Form

$$\tau_s^2 \frac{dx_2}{dt} = -2\zeta\tau_s x_2 - x_1 + K_p u\,(t-\theta_p)$$

$$\frac{dx_1}{dt} = x_2$$

output x_1

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{1}{\tau_s^2} & -\frac{2\zeta}{\tau_s}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{K_p}{\tau_s^2}\end{bmatrix} u\left(t-\theta_p\right)$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\end{bmatrix} u$$

Process Gain K_p

$$K_p = \frac{\Delta y}{\Delta u} = \frac{y_{ss_2} - y_{ss_1}}{u_{ss_2} - u_{ss_1}}$$

Damping Factor ζ

- Overdampled $\zeta > 1$
- Critically damped $\zeta = 1$
- Underdamped $\zeta < 1$

Second Order Time Constant τ_s The second order process time constant is the speed that the output response reaches a new steady state condition. An overdamped second order system may be the combination of two first order systems.

$$\tau_{p1}\frac{dx}{dt} = -x + K_p u \qquad \frac{X(s)}{U(s)} = \frac{K_p}{\tau_{p1}\,s+1}$$

$$\tau_{p2}\frac{dy}{dt} = -y + x \qquad \frac{Y(s)}{X(s)} = \frac{1}{\tau_{p2}\,s+1}$$

he combination of these two first order systems becomes

$$\frac{Y(s)}{X(s)}\frac{X(s)}{U(s)} = \frac{Y(s)}{U(s)} = \left(\frac{K_p}{\tau_{p1}\,s+1}\right)\left(\frac{1}{\tau_{p2}\,s+1}\right) = \frac{K_p}{\tau_{p1}\tau_{p2}\,s^2 + \left(\tau_{p1}+\tau_{p2}\right)s+1}$$

With $\tau_{p1}\tau_{p2} = \tau_s^2$ and $\tau_{p1}+\tau_{p2} = 2\zeta\tau_s$ in second order form.

Process Time Delay θ_p A time delay adds $e^{-\theta_p,s}$ to the second order transfer function; and in the time domain, the output response is multiplied by the step function $S(t-\theta_p)$.

Part III
Control Design

Proportional Integral Derivative (PID) control automatically adjusts a control output based on the difference between a set point (SP) and a measured process variable (PV). The value of the controller output $u(t)$ is transferred as the system input.

$$u(t) = u_{bias} + K_c\,e(t) + \frac{K_c}{\tau_I}\int_0^t e(t)dt - K_c\tau_D\frac{d(PV)}{dt}$$

The error from the set point is the difference between the SP and PV and is defined as:

$$e(t) = (SP - PV)$$

Set Point (SP) The target value.

Process Variable (PV) The measured value that may deviate from the desired value.

Integrating Processes Opening a valve to fill a tank. (P-only control)

Non-Integrating Processes Meaning any process that eventually returns to the same output given the same set of inputs and disturbances. (PI control)

9 Proportional-only Control

$$u(t) = u_{bias} + K_c \, e(t)$$

- $u(t)$ controller outputs
- u_{bias} controller bias: (baseline) the needed output of controller to achieve the staady state from manual to auto. This gives *"bumpless" transfer* if the error is zero when the controller is turned on.
- $e(t)$ controller error
- K_c controller gain: a multiplier on the proportional error and integral term.
  units $= \frac{\text{units of controller}}{\text{units of measured vatiable}}$ (K_p units $= \frac{\text{units of measured vatiable (output)}}{\text{units of controller (controlled variable) (input)}}$)
  reverese acting $K_p > 0$
  direct acting $K_p < 0$

Controller Offset Offset occurs under P-Only control when the set point and/or disturbances are at values other than that used as the design level of operation (that used to determine u_{bias}) Integral action is typically used to remove offset.

Tuning Correlations The parameters K_p, τ_p and θ_p are obtained by fitting dynamic input and output data to a first-order plus dead-time (FOPDT) model.

ITAE (Integral of Time-weighted Absolute Error) Method

disturbance rejection (regulatory control)

$$K_c = \frac{0.50}{K_p}\left(\frac{\tau_p}{\theta_p}\right)^{1.08}$$

set point tracking (servo control)

$$K_c = \frac{0.20}{K_p}\left(\frac{\tau_p}{\theta_p}\right)^{1.22}$$

10 Proportional Integral (PI) Control

$$u(t) = u_{bias} + K_c \, e(t) + \frac{K_c}{\tau_I}\int_0^t e(t)dt$$

- controller gain K_c
- integral time constant τ_I

Discrete PI Controller

$$u(t) = u_{bias} + K_c \, e(t) + \frac{K_c}{\tau_I}\sum_{i=1}^{n_t} e_i(t)\Delta t$$

"I" Term - Moving Bias Integral action is used to remove offset and can be thought of as an adjustable u_{bias}. The PI controller stops computing changes in controlled output when $e(t)$ equals zero for a sustained period. At that point, the proportional term equals zero, and the integral term may have a residual value. It eliminates *offset*.

Increaseing the oscillatory or rolling behavior

Tuning Correlations The parameters K_p, τ_p and θ_p are obtained by fitting dynamic input and output data to a first-order plus dead-time (FOPDT) model. The parameters K_p, τ_p and θ_p are obtained by fitting dynamic input and output data to a first-order plus dead-time (FOPDT) model.

IMC (Internal Model Control) An extension of λ tuning by accounting for time delay.

$$\text{Aggressive Tuning}: \quad \tau_c = \max\left(0.1\tau_p, 0.8\theta_p\right)$$

$$\text{Moderate Tuning}: \quad \tau_c = \max\left(1.0\tau_p, 8.0\theta_p\right)$$

$$\text{Conservative Tuning}: \quad \tau_c = \max\left(10.0\tau_p, 80.0\theta_p\right)$$

$$K_c = \frac{1}{K_p}\frac{\tau_p}{(\theta_p+\tau_c)} \qquad \tau_I = \tau_p$$

Note that with moderate tuning and negligible dead-time ($\theta_p \to 0\, then\, \tau_c = 1.0\tau_p$), , IMC reduces to simple tuning correlations that are easy to recall without a reference book.

$$K_c = \frac{1}{K_p} \qquad \tau_I = \tau_p \qquad \text{Simple tuning correlations}$$

ITAE (Integral of Time-weighted Absolute Error) Method

$$K_c = \frac{0.586}{K_p}\left(\frac{\theta_p}{\tau_p}\right)^{-0.916} \qquad \tau_I = \frac{\tau_p}{1.03 - 0.165\left(\theta_p/\tau_p\right)} \quad \text{Set point tracking}$$

$$K_c = \frac{0.859}{K_p}\left(\frac{\theta_p}{\tau_p}\right)^{-0.977} \qquad \tau_I = \frac{\tau_p}{0.674}\left(\frac{\theta_p}{\tau_p}\right)^{0.680} \quad \text{Disturbance rejection}$$

Anti-Reset Windup The integral term does not accumulate if the controller output is saturated at an upper or lower limit.

11 Proportional Integral Derivative (PID)

$$u(t) = u_{bias} + K_c\, e(t) + \frac{K_c}{\tau_I}\int_0^t e(t)dt + K_c\tau_D\frac{e(t))}{dt}$$

$$u(t) = u_{bias} + K_c\, e(t) + \frac{K_c}{\tau_I}\int_0^t e(t)dt - K_c\tau_D\frac{d(PV)}{dt}$$

- derivative time constant τ_D

Discrete PID Controller

$$u(t) = u_{bias} + K_c\, e(t) + \frac{K_c}{\tau_I}\sum_{i=1}^{n_t} e_i(t)\Delta t - K_c\tau_D\frac{PV_{n_t} - PV_{n_t-1}}{\Delta t}$$

Tuning Correlations

IMC

$$\text{Aggressive Tuning}: \quad \tau_c = \max\left(0.1\tau_p, 0.8\theta_p\right)$$

$$\text{Moderate Tuning}: \quad \tau_c = \max\left(1.0\tau_p, 8.0\theta_p\right)$$

$$\text{Conservative Tuning}: \quad \tau_c = \max\left(10.0\tau_p, 80.0\theta_p\right)$$

$$K_c = \frac{1}{K_p}\frac{\tau_p + 0.5\theta_p}{(\tau_c + 0.5\theta_p)} \qquad \tau_I = \tau_p + 0.5\theta_p \qquad \tau_D = \frac{\tau_p\theta_p}{2\tau_p + \theta_p}$$

Optional Derivative Filter

$$\text{derivative filter contant } \alpha = \frac{\tau_c\left(\tau_p + 0.5\theta_p\right)}{\tau_p\left(\tau_c + \theta_p\right)}$$

$$u(t) = u_{bias} + K_c\, e(t) + \frac{K_c}{\tau_I}\int_0^t e(t)dt - K_c\tau_D\frac{d(PV)}{dt} - \alpha\tau_D\frac{du(t)}{dt}$$

Simple Tuning Rules Note that with moderate tuning and negligible dead-time ($\theta_p \rightarrow 0\, then\, \tau_c = 1.0\tau_p$), , IMC reduces to simple tuning correlations that are easy to recall without a reference book.

$$K_c = \frac{1}{K_p} \qquad \tau_I = \tau_p \qquad \text{Simple tuning correlations}$$

Anti-Reset Windup

Derivative Kick

$$\frac{de(t)}{dt} = \frac{d\left(SP - PV\right)}{dt} = \frac{d\left(SP\right)}{dt} - \frac{\left(PV\right)}{dt} = -\frac{\left(PV\right)}{dt}$$

12 Controller Stability Analysis (Laplace domain)

The purpose of controller stability analysis is to determine the range of controller gains that lead to a stable controller.

$$K_{cL} \leq K_c \leq K_{cU}$$

Zeros The roots of the transfer function numerator.

Poles The roots of the transfer function denominator. $\text{damp} + \text{i frequency}$

12.1 Root Locus

12.2 Converge or Diverge

Unstable System Any of the real parts of the *poles* are positive - mixed signs (+ or -) of denominator polynomial then there is at least one positive real root.

Routh-Hurwitz Stability Criterion

$$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

State Space Model A state space model is stable when the eigenvalues of the A matrix have negative real parts.

12.3 Oscillatory or Smooth

system oscillation Any of the roots of the denominator have an imaginary component. (Imaginary roots always come in pairs with the same positive and negative imaginary values.)

12.4 Final Value Theorem

12.5 Initial Value Theorem

13 Cascade Control - Rejecting Disturbance

Properties

- The secondary (inner or lower-level) control must respond much faster than the primary control.
- Cascade control has a single final control element (FCE).
- The secondary control manipulated the FCE.

14 Feedforward Control - Compensation

Key Characteristics

- An identifiable disturbance is affecting *significantly* the measured variable, in spite of the attempts of a feedback control system to regulate these effects.
- This disturbance*can be measured*, perhaps with the addition of instrumentation.

An ideal feedforward controller is the negative ratio of the disturbance transfer function divided by the process transfer function.

$$G_{ff} = -\frac{G_d}{G_p}$$

When the disturbance and controller output act on the process with similar dynamics $\tau_d \approx \tau_p$:

$$K_{ff} = -\frac{K_d}{K_p}$$

For PID controller (disturbance d):

$$u(t) = u_{bias} + K_c\, e(t) + \frac{K_c}{\tau_I}\int_0^t e(t)dt - K_c\tau_D\frac{d(PV)}{dt} + K_{ff}\, d$$