Application to Partial Differential Equations for Optimal Transport Problem

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1 Introduction

Interest in the optimal transportation problem has increased in recent years due to the discovery of a deep relationship between the optimal transportation problem and partial differential equations. The back-and-forth method (Matt Jacobs, Flavien Léger, 2020) is a new method for solving partial differential equations using the solution of the optimal transportation problem. This method is particularly effective for solving nonlinear partial differential equations and can solve a wider range of problems than previous methods because it is faster and does not require stability conditions. I am interested in the back-and-forth method and my ultimate goal is to create a program that solves partial differential equations using this method.

In this paper, I will explain the back-and-forth method, which efficiently solves the optimal transportation problem with strictly convex costs.

2 Background

2.1 Optimal transport problem

Optimal transform problem(Monge,1781). Find a method to minimize cost, which depends on weight and distance, for transporting sand from a sandpit with measure μ to a hole with the same volume and measure ν using a mapping T.

Definition 2.1 (pushforward measure)**.** Given a mapping T that transports from measure μ to measure ν ($T_\#\mu = \nu$), the pushforward measure is defined as:

$$\nu(A) = T_\#\mu(A) := \mu(T^{-1}(A)) \qquad A \subset \Omega.$$

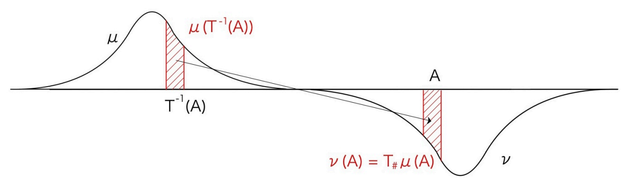

Figure 1: transport map

The optimal transportation problem can be formalized as follows.

Optimal transform problem. Let $\Omega \subset \mathbb{R}^d$ be a convex set, $c : \Omega \times \Omega \to \mathbb{R}$ be a cost function that represents the cost of transporting from x to y, $\mu, \nu : \Omega \to \mathbb{R}$ be probability measures on Ω, and $T : \Omega \to \Omega$ be a mapping such that $T_\#\mu = \nu$, meaning it transforms measure μ to measure ν.

The minimum cost to transport μ to ν is denoted by $C(\mu, \nu)$ and is given by:

$$C(\mu, \nu) = \inf_T \int_\Omega c(x, T(x)) d\mu(x).$$

The optimal cost C can be treated as a problem of maximizing the volume of sand transported, rather than fixing the transportation location.

Kantorovich dual problem. When ν and μ are probability measures and $c(x, y)$ represents the cost of transporting from x to y, the cost C can be expressed as maximizing the volume of sand transported as follows:

$$C(\mu, \nu) = \sup_{\phi,\psi} \int \phi d\nu + \int \psi d\mu$$

where $\phi(y)$ and $\psi(x)$ are the Kantorovich potentials and satisfy the inequality:

$$\phi(y) + \psi(x) \leq c(x, y).$$

If there exists an optimal map from μ to ν, the maximum value of the dual problem $\phi_(y), \psi_(x)$ can be restored. Then,

$$\phi_*(y) + \psi_*(x) = c(x, y).$$

2.2 c-transform

Definition 2.2 (c-transform)**.** Given a continuous function $\phi : \Omega \to \mathbb{R}$, we define its c-transform $\phi^c : \Omega \to \mathbb{R}$ as follows:

$$\phi^c(x) := \inf_{y \in \Omega}(c(x, y) - \phi(y))$$

A function ϕ is called a c-concave function if there exists a continuous function $\psi : \Omega \to \mathbb{R}$ such that $\phi = \psi^c$. Additionally, pair of functions (ϕ, ψ) is called c-conjugate if $\phi = \psi^c$ and $\psi = \phi^c$.

If (ϕ, ψ) is c-conjugate, then the maximum values ϕ_* and ψ_* are given by:

$$\phi_*(y) = \psi_*^c(y) = \inf_{x \in \Omega} c(x, y) - \psi_*(x),$$
$$\psi_*(x) = \phi_*^c(x) = \inf_{y \in \Omega} c(x, y) - \phi_*(y).$$

3 The back-and-forth method

The Kantorovich dual problem, given by

$$C = \sup_{\phi, \psi} \int \phi d\nu + \int \psi d\mu,$$

can be expressed as the sup of

$$J(\phi) = \int \phi d\nu + \int \phi^c d\mu,$$

and

$$I(\psi) = \int \psi^c d\nu + \int \psi d\mu.$$

using c-transformation. In other words, $c = \sup J = \sup I$.

The back-and-forth method solves the Kantorovich dual problem rapidly by finding the supremum of J and I using gradient ascent, and by alternating back and forth between J and I through c-transformations.

Algorithm 1 The back-and-forth method

Given probability densities μ and ν, set $\phi_0 = 0, \psi_0 = 0$, and iterate:

$$\begin{aligned}
\phi_{n+\frac{1}{2}} &= \phi_n + \sigma \nabla_{\dot{H}^1} J(\phi_n), \\
\psi_{n+\frac{1}{2}} &= (\phi_{n+\frac{1}{2}})^c, \\
\psi_{n+1} &= \psi_{n+\frac{1}{2}} + \sigma \nabla_{\dot{H}^1} I(\psi_{n+\frac{1}{2}}), \\
\phi_{n+1} &= (\psi_{n+1})^c.
\end{aligned}$$

Where,

$$\nabla_{\dot{H}^1} J(\phi) = (-\Delta)^{-1}(\nu - T_{\phi\#}\mu),$$
$$\nabla_{\dot{H}^1} I(\psi) = (-\Delta)^{-1}(\mu - T_{\psi\#}\nu).$$

4 Results

An example of ν and ν optimal transport using the back-and-forth method is shown.

Example 4.1. Initial conditions

$$\mu = \begin{cases} 1 & 0.3 \leq x \leq 0.8 \\ 0 & otherwise \end{cases},$$
$$\nu = \begin{cases} 1 & -0.8 \leq x \leq -0.3 \\ 0 & otherwise \end{cases}.$$

Example 4.2. Initial conditions

$$\mu = \begin{cases} 0.5 & 0 \leq x \leq 0.5 \\ 0 & otherwise \end{cases}$$
$$\nu = \begin{cases} 1 & -0.5 \leq x \leq -0.25 \\ 0 & otherwise \end{cases}$$

5 Issues

Since the x-coordinates before and after transportation have a one-to-one correspondence, the mass

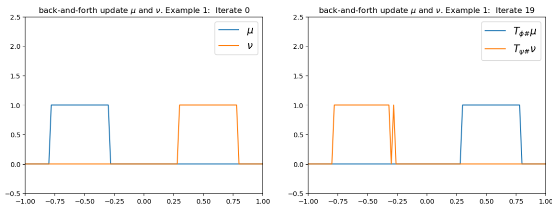

Figure 2: Example 4.1.

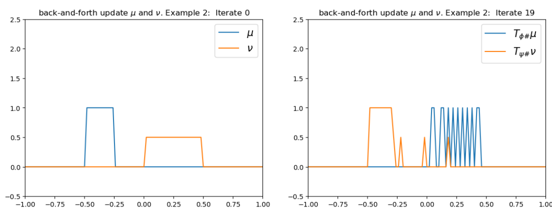

Figure 3: Example 4.2.

(value) of ν and μ at a certain x is reflected directly to the destination of the transportation.

Therefore, even though the calculation results in $T_{\phi\#}\mu = \nu, T_{\psi\#}\nu = \mu$, a large error occurs.

6 Next steps

1. Creation of a program for transport maps without a one-to-one correspondence (for improved accuracy)

2. Creation of a program that solves partial differential equations using the back-and-forth method

3. Development of a program to find new solutions for nonlinear equations

References

[1] Matt Jacobs, Flavien Léger. A fast approach to optimal transport: the back-and-forth method. Numerische Mathematik, 2020.

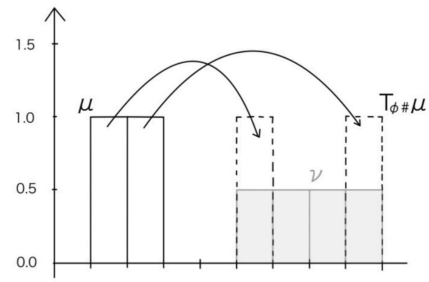

Figure 4: Transport map $T_{\phi\#}\mu$

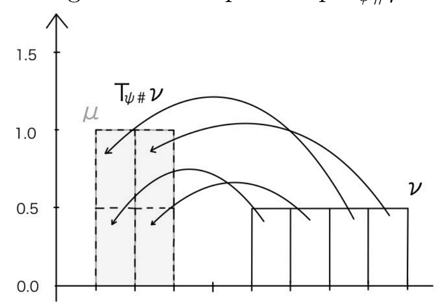

Figure 5: Transport map $T_{\psi\#}\nu$

[2] Shin-ichi OHTA. 最適輸送理論とその周辺. http://www4.math.sci.osaka-u.ac.jp/~sohta/jarts/kino09.pdf.

[3] Shinichiro KOBAYASHI. 距離コストに対する最適輸送問題について. https://www.math.sci.hokudai.ac.jp/~wakate/mcyr/2020/pdf/kobayashi_shinichiro.pdf.