Process Algebra (2IMF10)

Bas Luttik

s.p.luttik@tue.nl

May 4, 2023

Solution to Exercise 4.3.5

Let p, q and r be closed MPT(A)-terms. We need to prove that if $(p+q)+r \leftrightarroweq r$, then $p+r \leftrightarroweq r$ and $q+r \leftrightarroweq r$. To this end, suppose that $(p+q)+r \leftrightarroweq r$; then there exists a bisimulation relation R such that $(p+q)+r\ R\ r$. We use R to define a relation R' by

$$R' = R \cup \{(p+r, r), (q+r, r)\} \cup \{(s, s) \mid s \text{ a closed MPT}(A)\text{-term}\}\ ,$$

and establish that R' is a bisimulation relation too; since $p+r\ R\ r$ and $q+r\ R\ r$, it then follows that $p+r \leftrightarroweq r$ and $q+r \leftrightarroweq r$.

To establish that R' is a bisimulation relation, we need to verify the conditions of bisimulation relations for all pairs in R'. Let us start with the pair $(p+r, r) \in R'$:

1. If $p+r \xrightarrow{a} s$ for some closed MPT(A)-term s, then there exists a formal derivation in accordance with the structural operational semantics of MPT(A) (i.e., the term deduction system for MPT(A)) with this derivation as conclusion. From the syntactic shape $p+q$ of the left-hand side of the transition, it follows that only two of the three rules can have been applied last in this derivation (viz. the two rules for $+$; note that $p+r$ has $+$ as main operator). We distinguish two cases according to which rule was applied last in the formal derivation:

   - If the last rule applied in the formal derivation is the first rule for $+$, then it follows that $p \xrightarrow{a} s$. From $p \xrightarrow{a} s$ it follows by the first rule for $+$ that $p+q \xrightarrow{a} s$, and hence, with another application of the same rule, that $(p+q)+r \xrightarrow{a} s$. Now, since $(p+q)+r\ R\ r$ and R is a bisimulation relation, it follows that there exists s' such that $r \xrightarrow{a} s'$ for some s' such that $s\ R\ s'$. Since $R \subseteq R'$, we now also have that $s\ R'\ s'$.
   - If the last rule applied in the formal derivation is the second rule for $+$, then it follows that $r \xrightarrow{a} s$. Note that $s\ R'\ s$ by the definition of R'.

2. If $r \xrightarrow{a} s$ for some closed MPT(A)-term s, then, by the second rule for $+$ we find that $p+r \xrightarrow{a} s$, and we have $s\ R'\ s$ by the definition of R'.

3. Since $p+r\downarrow$ does not hold, there is nothing to prove.

4. Since $r\downarrow$ does not hold, there is nothing to prove.

Checking the conditions for the pair $(q+r, r) \in R'$ proceeds analogously.

Since R is a bisimulation relation by assumption, all pairs in R satisfy the conditions of bisimulation relations.

The straightforward arguments showing that pairs $(s, s) \in R'$ satisfy the conditions of bisimulation relations are omitted.

We have argued that all pairs in R' satisfy the conditions of bisimulation relations, and thereby the proof is complete.