4

Basic process theory

4.1 Introduction

Chapter 3 has introduced the notion of transition systems both as an abstract operational model of reactive systems and as a means to give operational semantics to equational theories. The latter has been illustrated by giving an operational interpretation of the equational theory of natural numbers. The aim of this book, however, is to develop equational theories for reasoning about reactive systems. This chapter provides the first simple examples of such theories. Not surprisingly, the semantics of these theories is defined in terms of the operational framework of the previous chapter. For clarity, equational theories tailored towards reasoning about reactive systems are referred to as *process theories*. The objects being described by a process theory are referred to as *processes*. The next two sections introduce a *minimal* process theory with its semantics. This minimal theory is mainly illustrative from a conceptual point of view. Sections 4.4 and 4.5 provide some elementary extensions of the minimal theory, to illustrate the issues involved in extending process theories. Incremental development of process theories is crucial in tailoring a process theory to the specifics of a given design or analysis problem. The resulting framework is flexible and it allows designers to include precisely those aspects in a process theory that are relevant and useful for the problem at hand. Incremental design also simplifies many of the proofs needed for the development of a rich process theory. The remainder of this book builds upon the simple theory introduced in Sections 4.2 through 4.4. Sections 4.5 and 4.6 present extensions of this basic process theory that provide a basis for the next chapter on recursion. Recursion is essential in any practically useful theory. The final section of this chapter puts everything into the perspective of the relevant literature, and provides pointers for further reading.

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4.2 The process theory MPT

This chapter starts from a very simple process theory MPT(A), where MPT is an abbreviation for Minimal Process Theory and where A is a set of actions (sometimes also referred to as *atomic* actions). The set of actions is a parameter of the theory. The idea is that terms over the signature of theory MPT(A) specify processes. The signature has one constant, one binary function, and a set of unary functions. Note that it is common practice to refer to functions in the signature of a process theory as *operators*.

(i) The constant in the signature of MPT(A) is the constant 0, denoting *inaction*. Process 0 cannot execute any action; in other words, 0 can only deadlock.

(ii) The signature also has a binary operator $+$, denoting *alternative composition* or *choice*. If x and y are terms over the signature of MPT(A), the process corresponding to term $x + y$ behaves either as x or as y, but not as both. A choice is resolved upon execution of the first action. The inaction constant 0 is the identity element (also called unit element or neutral element) of alternative composition.

(iii) Finally, the signature of theory MPT(A) has, for each action $a \in A$, a unary operator $a.\_$, denoting *action prefix*. If x is a term, the process corresponding to $a.x$ executes action a and then proceeds as x. Substituting the inaction constant 0 for x yields the basic process $a.0$ that can execute an a and then deadlocks.

From these descriptions, it is already clear that the theory MPT(A) allows for the specification of simple sequential processes with branching behavior. The latter is obtained via the choice operator. It remains to give the axioms of the equational theory MPT(A). The process theory is axiomatized by four axioms, namely A1, A2, A3, and A6, given in Table 4.1. The naming is not consecutive for historical reasons (see (Baeten & Weijland, 1990)). The table adheres to the format introduced in Section 2.2. Notation $(a.\_)_{a \in A}$ in the signature of theory MPT(A) means that MPT(A) has a function $a.\_$ for every $a \in A$.

Axioms A1, A2, and A3 express properties of alternative composition. The fact that a choice between x and y is the same as a choice between y and x is reflected by Axiom A1. Axiom A2 expresses that a choice between x and choosing between y and z is the same as a choice between choosing between x and y, and z. In both cases, a choice is made between three alternatives. If one has to choose between two identical alternatives, this is the same as not choosing at all; this is expressed by Axiom A3. The three properties of alternative composition reflected by Axioms A1, A2, and A3, are often referred to as *commutativity*, *associativity*, and *idempotency*, respectively. Axiom A6 expresses

MPT(A)		
constant: 0;	unary: $(a.\_)_{a \in A}$;	binary: $\_ + \_$;
x, y, z;		
	$x + y = y + x$	A1
	$(x + y) + z = x + (y + z)$	A2
	$x + x = x$	A3
	$x + 0 = x$	A6

Table 4.1. The process theory MPT(A).

that in the context of a choice deadlock is avoided as long as possible. It is not allowed to choose for inaction if there is still an alternative. Stated mathematically, the inaction constant is an *identity element* for alternative composition.

To improve readability, terms over the signature of some process theory T are simply referred to as T-terms. Furthermore, binding priorities are introduced. Action-prefix operators always bind stronger than other operators; the binary + always binds weaker than other operators. Hence, the term $a.x + y$, with x and y arbitrary MPT(A)-terms and a an action in A, is in fact the term $(a.x) + y$; similarly, $a.a.x + a.y$ represents $(a.(a.x)) + (a.y)$.

Using the axioms of the theory MPT(A) and the rules of equational logic presented in Chapter 2, equalities between process terms can be derived.

Example 4.2.1 (Proofs in a process theory) The following derivation is a compact proof of the equality $a.x + (b.y + a.x) = a.x + b.y$, where a and b are actions in A and x and y are arbitrary MPT(A)-terms:

$$\begin{aligned} \mathrm{MPT}(A) \vdash \; & a.x + (b.y + a.x) = a.x + (a.x + b.y) \\ & = (a.x + a.x) + b.y = a.x + b.y. \end{aligned}$$

The above example shows how a process theory can be used to reason about the equivalence of processes. This is one of the main objectives of a process theory. Another major objective of a process theory is that it serves as a means to specify processes.

Example 4.2.2 (The lady or the tiger?) Recall Example 3.1.9 from the previous chapter, describing a situation where a prisoner is confronted with two closed doors, one hiding a dangerous tiger and the other hiding a beautiful lady. The described situation can be modeled by a process term using the three actions introduced in Example 3.1.9, namely the term

$$open.eat.0 + open.marry.0. \tag{4.2.1}$$

This term describes that after opening a door the prisoner is confronted with

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either the tiger or the lady. He does not have a choice; in fact, the choice is made as soon as he chooses a door to open, which is the desired situation. In the previous chapter, such a choice was called a non-deterministic choice.

In Example 3.1.9, the described situation has been captured in the leftmost transition system of Figure 3.3. Term (4.2.1) corresponds to this transition system, except for the termination behavior. In Example 3.1.9, a distinction was made between successful and unsuccessful termination, expressed through a terminating state and a deadlock state, respectively. The process theory MPT(A) does not have the possibility to distinguish between these two kinds of termination behavior. Termination is specified by the inaction constant 0. In the next section, where a model of MPT(A) is given, it becomes clear that 0 corresponds to unsuccessful termination. In Section 4.4, theory MPT(A) is extended with a new constant that allows the specification of successful termination.

In Example 3.1.9, a second transition system is discussed, namely the rightmost transition system of Figure 3.3. Except for the termination behavior, also this transition system can be specified as an MPT(A)-term:

$$\mathit{open}.(\mathit{eat}.0 + \mathit{marry}.0). \tag{4.2.2}$$

For the same reasons as those given in Example 3.1.9, terms (4.2.1) and (4.2.2) describe different situations. Thus, it is desirable that the two terms are not derivably equal.

Generalizing the last observation in the above example means that terms of the form $a.(x+y)$ and $a.x+a.y$, where a is an action and x and y are arbitrary but different terms, should not be derivably equal (unless x and y are derivably equal). It turns out that the general equality $a.(x + y) = a.x + a.y$ is indeed not derivable from theory MPT(A) (see Exercise 4.3.7). In other words, action prefix does not distribute over alternative composition.

In term $a.x + a.y$, the choice between the eventual execution of x or y is made upon the execution of a, whereas in $a.(x+y)$ this choice is made after the execution of a. In the terminology of Section 3.1, the two terms specify processes with different branching structure. Thus, (at least some) processes with different branching structure cannot be proven equal in theory MPT(A). In the next section, a model of MPT(A) is given in terms of the semantic domain introduced in the previous chapter. It turns out that derivability in MPT(A) coincides with bisimilarity in the semantic domain.

In Section 3.1, it has already been mentioned that it is not always necessary to distinguish between processes that only differ in branching structure. In the semantic domain, this translates to choosing another equivalence on transition systems. In the framework of the process theory MPT(A), it is possible to add the distributivity of action prefix over choice as an axiom to the theory. In general, one has to design a process theory in such a way that derivability in the theory coincides with the desired semantic equivalence. In Chapter 3, bisimilarity has been chosen as the basic semantic equivalence. As a consequence, distributivity of action prefix over choice is not included as an axiom in theory MPT(A). Note that this conforms to the goal of this section that aims at developing a *minimal* process theory. By omitting distributivity of action prefix over choice from the minimal theory MPT(A), the freedom of adding it when necessary or desirable is retained. Of course, doing so has consequences for the semantic domain in the sense that a different equivalence on transition systems is needed in order to construct a model of the extended theory. Chapter 12 investigates a number of interesting axioms that lead to different semantic equivalences when they are included in a process theory.

Exercises

4.2.1 Prove that the three axioms A1, A2, and A3 of MPT(A) are equivalent to the following *two* axioms:

$$(x + y) + z = (y + z) + x \qquad \text{A2}'$$
$$x + x = x \qquad \text{A3}$$

(with x, y, z MPT(A)-terms).
(Hint: first show that commutativity (A1) is derivable from A2′ and A3.)

4.2.2 Prove that the three axioms A1, A2, and A3 of MPT(A) are equivalent to the following two axioms:

$$(x + y) + z = y + (z + x) \qquad \text{A2}''$$
$$x + x = x \qquad \text{A3}$$

(with x, y, z MPT(A)-terms).
(Hint: as in the previous exercise, show commutativity first. Use this several times in order to show associativity.)

4.2.3 For any (open) MPT(A)-terms x and y, define $x \leq y$ if and only if MPT(A) $\vdash x + y = y$; if $x \leq y$, it is said that x is a *summand* of y. The idea is that summand x describes a (not necessarily strict) subset of the alternatives of term y, and can thus be added to y without essentially changing the process.

(a) Prove that $x \leq y$, for terms x and y, if and only if there is a term z such that MPT(A) $\vdash x + z = y$.
(b) Prove that $\leq$ is a partial ordering, i.e., that

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1. $\leq$ is reflexive (for all terms x, $x \leq x$),
2. $\leq$ is anti-symmetric (for all x, y, $x \leq y$ and $y \leq x$ implies $\mathrm{MPT}(A) \vdash x = y$),
3. $\leq$ is transitive (for all x, y, z, $x \leq y$ and $y \leq z$ implies $x \leq z$).

(c) Give an example of two closed MPT(A)-terms p and q such that not $p \leq q$ and not $q \leq p$.

(d) Prove that for any terms x, y, z, $x \leq y$ implies $x + z \leq y + z$.

(e) Give an example of two closed MPT(A)-terms p and q such that $p \leq q$ but not $a.p \leq a.q$ (with $a \in A$).

4.3 The term model

In the previous section, the process theory MPT(A), with A a set of actions, has been introduced. This section presents a model of this equational theory, following the concepts explained in Section 2.3. The model is built using the semantic framework of Chapter 3.

The starting point of the construction of this model is a so-called *term algebra*. The term algebra for the theory MPT(A) has as its universe the set of all closed MPT(A)-terms. The constants and functions of the term algebra are the constants and functions of MPT(A). The identity on the domain of the algebra is the syntactical equivalence of terms (denoted $\equiv$; see Definition 2.2.3 (Terms)). The term algebra itself is not a model of MPT(A). The reason is that syntactical equivalence of terms does not coincide with derivability as defined by the axioms of MPT(A). The term algebra can be turned into a model by transforming its domain, the closed MPT(A)-terms, into a transition-system space, as defined in Definition 3.1.1. As a consequence, bisimilarity, as defined in Definition 3.1.10, becomes an equivalence relation on closed MPT(A)-terms. As already suggested before, it turns out that bisimilarity is a suitable equivalence that matches derivability in MPT(A). It is shown that bisimilarity is a congruence on the term algebra. The term algebra modulo bisimilarity is shown to be a model of MPT(A), referred to as the term model. Finally, it is shown that the process theory MPT(A) is a ground-complete axiomatization of the term model.

The remainder often uses the notation $\mathcal{C}(T)$ to denote the set of closed terms over the signature of a process theory T. (Note that this slightly generalizes the notation for closed terms introduced in Definition 2.2.3 (Terms).)

Definition 4.3.1 (Term algebra) The algebra $\mathbb{P}(\mathrm{MPT}(A)) = (\mathcal{C}(\mathrm{MPT}(A)), +, (a.\_)_{a \in A}, 0)$ is called the *term algebra* for theory MPT(A). Recall from

Definition 2.2.3 (Terms) that syntactical equivalence, $\equiv$, is the identity on the domain $\mathcal{C}(\mathrm{MPT}(A))$ of this algebra.

The term algebra is not a model of the process theory $\mathrm{MPT}(A)$, assuming the identity function as the interpretation of the signature of $\mathrm{MPT}(A)$ into $\mathbb{P}(\mathrm{MPT}(A))$. The term algebra does not capture the equalities defined by the axioms of $\mathrm{MPT}(A)$. This can be easily seen as follows. Consider the equality $a.0 + a.0 = a.0$, with $a \in A$. Clearly, because of Axiom A3, $\mathrm{MPT}(A) \vdash a.0 + a.0 = a.0$; however, $a.0 + a.0 \not\equiv a.0$. According to Definition 2.3.6 (Validity), this means that Axiom A3 is not valid in the term algebra $\mathbb{P}(\mathrm{MPT}(A))$. As a direct consequence, the term algebra is not a model of $\mathrm{MPT}(A)$ (see Definition 2.3.8 (Model)).

The second step in the construction of a model of the theory $\mathrm{MPT}(A)$ is to turn the set of process terms $\mathcal{C}(\mathrm{MPT}(A))$ into a transition-system space as defined in Definition 3.1.1. The sets of states and labels are the sets of closed terms $\mathcal{C}(\mathrm{MPT}(A))$ and actions A, respectively. A direct consequence of this choice is that closed $\mathrm{MPT}(A)$-terms are turned into transition systems. The termination predicate $\_\downarrow$ and the (ternary) transition relation $\_ \xrightarrow{\_} \_$ are defined through a term deduction system. Recall that deduction systems are discussed in Section 3.2. The transition relation is constructed from a family of binary transition relations $\_ \xrightarrow{a} \_$ for each atomic action $a \in A$. If p and p' are closed terms in $\mathcal{C}(\mathrm{MPT}(A))$, intuitively, $p \xrightarrow{a} p'$ holds if and only if p can execute an action a and thereby transforms into p'. The deduction system defining the termination predicate and the transition relation is given in Table 4.2. The tabular presentation of the deduction system is similar to the presentation of equational theories as outlined in Chapter 2. The term deduction system corresponding to an equational theory T is referred to as $TDS(T)$. The first entry of Table 4.2 gives the signature of the term deduction system. The second entry contains the deduction rules. The termination predicate and the ternary transition relation of the transition-system space under construction are defined as the smallest set and relation satisfying these deduction rules, as explained in Definition 3.2.3 (Transition-system space induced by a deduction system).

The first deduction rule (which actually has an empty set of premises and is therefore also called an axiom, as explained in Section 3.2) states that any process term $a.p$, with a an action and p another process term, can execute the atomic action a and thereby transforms into process term p. The other two deduction rules explain that a process term $p + q$ can execute either an action from p or an action from q. As a consequence, the alternative disappears and can no longer be executed. Since the term deduction system for theory $\mathrm{MPT}(A)$ does not have any rules for termination, the termination predicate

$TDS(\mathrm{MPT}(A))$		
constant: 0;	unary: $(a.\_)_{a \in A}$;	binary: $\_ + \_$;
x, x', y, y';		
$a.x \xrightarrow{a} x$	$\dfrac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'}$	$\dfrac{y \xrightarrow{a} y'}{x + y \xrightarrow{a} y'}$

Table 4.2. Term deduction system for MPT(A) (with $a \in A$).

becomes the empty set. This means that successful termination is not possible, which corresponds to the intuitive explanations given earlier.

Example 4.3.2 (Transition systems for closed MPT(A)**-terms)** As an example, Figure 4.1 shows the transition systems induced by the process terms $a.b.0$ and $a.b.0 + a.(b.0 + b.0)$. Note that the two terms are derivably equal, whereas the two transition systems are obviously not the same. However, it is straightforward to show that the two transition systems are bisimilar (see Section 3.1). Notice that any transition system induced by a closed MPT(A)-term is a regular transition system (see Definition 3.1.15), that does not contain any cycles (i.e., from any state, that state is not reachable by a non-empty sequence of steps). The fact that closed MPT(A)-terms correspond to regular transition systems is an immediate consequence of Theorem 5.8.2 proven in Section 5.8. The fact that these transition systems cannot contain cycles follows from Theorem 4.5.4 proven later in this chapter.

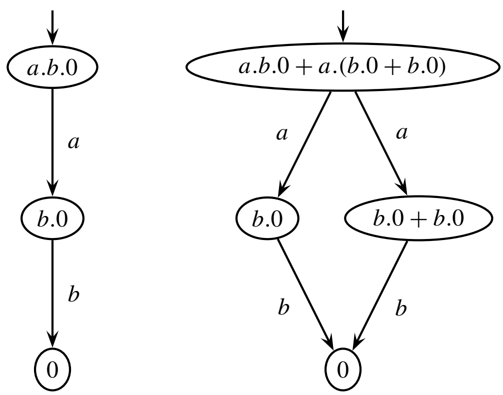

Fig. 4.1. Transition systems $a.b.0$ and $a.b.0 + a.(b.0 + b.0)$.

At this point, the term algebra $\mathbb{P}(\mathrm{MPT}(A))$ of Definition 4.3.1 has been

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turned into a transition-system space. In the remainder, a term algebra such as $\mathbb{P}(\mathrm{MPT}(A))$ is also referred to as an algebra of transition systems. This implicitly assumes that the term algebra is accompanied by a term deduction system defining a transition-system space. It is always clear from the context which term deduction system is meant.

A consequence of the fact that term algebra $\mathbb{P}(\mathrm{MPT}(A))$ has been turned into a transition-system space is that bisimilarity is inherited as an equivalence relation on closed terms. It has already been suggested a number of times that derivability in $\mathrm{MPT}(A)$ captures the same notion of equivalence as bisimilarity. At this point, it is possible to be more precise. It can be shown that two closed terms are derivably equal if and only if the induced transition systems are bisimilar. Let us examine the relation between bisimilarity and derivability in some more detail.

Recall the context rule from Definition 2.2.8 (Derivability). This rule plays a crucial role in equational derivations. It implies that terms can be replaced by derivably equal terms in arbitrary contexts. If bisimilarity and derivability are supposed to be the 'same' notion of equivalence, the context rule from equational logic must in some sense also be valid for bisimilarity. It turns out that the notion of a *congruence relation*, as defined in Definition 2.3.16, is the desired equivalent of the context rule.

Theorem 4.3.3 (Congruence) Bisimilarity is a congruence on the algebra of transition systems $\mathbb{P}(\mathrm{MPT}(A))$.

Proof Using Theorem 3.2.7 (Congruence theorem), the proof is trivial because it is easy to see that the deduction system of Table 4.2 is in path format, as defined in Definition 3.2.5.

Unfortunately, deduction systems are not always in path format, which means that it sometimes may be necessary to prove a congruence result directly from the definition of a congruence relation, namely Definition 2.3.16. To illustrate such a proof, it is given here as well. Definition 2.3.16 states two requirements on a relation, bisimilarity in this case. The first requirement is that bisimilarity is an equivalence. From Theorem 3.1.13, it is already known that bisimilarity is an equivalence. Thus, in order to prove that bisimilarity is a congruence on $\mathbb{P}(\mathrm{MPT}(A))$, it suffices to show that for each n-ary ($n \geq 1$) function f of $\mathbb{P}(\mathrm{MPT}(A))$ and for all $p_1, \ldots, p_n, q_1, \ldots, q_n \in \mathcal{C}(\mathrm{MPT}(A))$, $p_1 \leftrightarrows q_1, \cdots, p_n \leftrightarrows q_n$ implies that $f(p_1, \ldots, p_n) \leftrightarrows f(q_1, \ldots, q_n)$.

Consider the binary function $+$. Assume that $p_1 \leftrightarrows q_1$ and $p_2 \leftrightarrows q_2$, with $p_1, p_2, q_1, q_2 \in \mathcal{C}(\mathrm{MPT}(A))$. By definition, this means that there exist bisimulation relations R_1 and R_2 such that $(p_1, q_1) \in R_1$ and $(p_2, q_2) \in R_2$. Define relation R as follows: $R = R_1 \cup R_2 \cup \{(p_1 + p_2, q_1 + q_2)\}$. It can be shown that relation R is a bisimulation relation. It needs to be shown that R satisfies for each pair of process terms in the relation the transfer conditions of Definition 3.1.10 (Bisimilarity). The pairs of process terms in R that are elements of R_1 or R_2 obviously satisfy the transfer conditions because R_1 and R_2 are bisimulation relations that satisfy those conditions. Thus, it remains to prove that the pair of process terms $(p_1 + p_2, q_1 + q_2)$ satisfies the transfer conditions. Only condition (i) is proven; condition (ii) follows from the symmetry in Definition 3.1.10 (Bisimilarity). Since terms in $\mathcal{C}(\mathrm{MPT}(A))$ cannot terminate successfully, transfer conditions (iii) and (iv) are trivially satisfied.

(i) Suppose that $p_1 + p_2 \stackrel{a}{\to} p'$ for some $a \in A$ and $p' \in \mathcal{C}(\mathrm{MPT}(A))$. It must be shown that there is a $q' \in \mathcal{C}(\mathrm{MPT}(A))$ such that $q_1 + q_2 \stackrel{a}{\to} q'$ and $(p', q') \in R$. Inspection of the deduction rules in Table 4.2 reveals that $p_1 + p_2 \stackrel{a}{\to} p'$ must follow from the fact that $p_1 \stackrel{a}{\to} p'$ or $p_2 \stackrel{a}{\to} p'$. In the first case, the fact that R_1 is a bisimulation relation that relates the terms p_1 and q_1 yields the existence of a term $q' \in \mathcal{C}(\mathrm{MPT}(A))$ such that $q_1 \stackrel{a}{\to} q'$ and $(p', q') \in R_1$; in the second case, for similar reasons, $q_2 \stackrel{a}{\to} q'$ for some q' such that $(p', q') \in R_2$. Thus, in both cases, $q_1 + q_2 \stackrel{a}{\to} q'$ for some $q' \in \mathcal{C}(\mathrm{MPT}(A))$ such that $(p', q') \in R$.

The proof that bisimilarity is a congruence with respect to the action-prefix operators is left as an exercise (Exercise 4.3.3). □

The above theorem implies that bisimilarity on closed terms respects the rules of equational logic. It provides the basis for the definition of the term model of the process theory MPT(A) of the previous section. The term model is in fact a quotient algebra, namely the term algebra modulo bisimilarity (see Definition 2.3.18 (Quotient algebra)).

Definition 4.3.4 (Term model) The term model of process theory MPT(A) is the quotient algebra $\mathbb{P}(\mathrm{MPT}(A))_{/\leftrightarrow}$, where $\mathbb{P}(\mathrm{MPT}(A))$ is the term algebra defined in Definition 4.3.1.

The elements of the universe of the algebra $\mathbb{P}(\mathrm{MPT}(A))_{/\leftrightarrow}$ are called *processes*. In the term model, a process is an equivalence class of closed terms under bisimilarity. It can be shown that there is a strong correspondence between bisimilarity and the equalities that can be obtained from the equations of the process theory MPT(A): (1) any two derivably equal MPT(A)-terms fall into the same equivalence class when interpreted in the term model, and (2) any two closed MPT(A)-terms that are bisimilar are also derivably equal. In Chapter 2, the first notion has been introduced as soundness (see Definition 2.3.8

(Model)), whereas the second property has been called ground-completeness (Definition 2.3.23). In order to prove soundness and ground-completeness of MPT(A) for the term model, an interpretation of constants and functions of MPT(A) into those of the term model is needed. The construction of the model allows a very straightforward interpretation. The constant 0 is mapped onto the equivalence class $[0]_{\leftrightarrows}$, whereas each of the operators is mapped onto the corresponding function in $\mathbb{P}(\mathrm{MPT}(A))_{/\leftrightarrows}$ following Definition 2.3.18 (Quotient algebra). In the remainder, it is always assumed that this standard interpretation is used; thus, it is omitted in the various notations whenever possible.

Theorem 4.3.5 (Soundness) The process theory MPT(A) is a sound axiomatization of the algebra $\mathbb{P}(\mathrm{MPT}(A))_{/\leftrightarrows}$, i.e., $\mathbb{P}(\mathrm{MPT}(A))_{/\leftrightarrows} \models \mathrm{MPT}(A)$.

Proof According to Definition 2.3.8 (Model), it must be shown that, for each axiom $s = t$ of MPT(A), $\mathbb{P}(\mathrm{MPT}(A))_{/\leftrightarrows} \models s = t$. Below, the proof is given for Axiom A6; the proofs for the other axioms are left as an exercise (Exercise 4.3.4).

Recall the notations concerning quotient algebras introduced in Section 2.3. Let p be a closed term in $\mathcal{C}(\mathrm{MPT}(A))$. It follows from Definition 2.3.6 (Validity) that it must be shown that $[p]_{\leftrightarrows} +_{\leftrightarrows} [0]_{\leftrightarrows} = [p]_{\leftrightarrows}$. Definition 2.3.18 (Quotient algebra) implies that it must be shown that $[p + 0]_{\leftrightarrows} = [p]_{\leftrightarrows}$ and, thus, that $p + 0 \leftrightarrows p$.

It suffices to give a bisimulation relation on process terms that contains all the pairs $(p + 0, p)$ for closed MPT(A)-term p. Let $R = \{(p + 0, p) \mid p \in \mathcal{C}(\mathrm{MPT}(A))\} \cup \{(p, p) \mid p \in \mathcal{C}(\mathrm{MPT}(A))\}$. It must be shown that all elements of R satisfy the transfer conditions of Definition 3.1.10 (Bisimilarity). The transfer conditions trivially hold for all pairs (p, p) with p a closed term. Hence, only elements of R of the form $(p + 0, p)$ with $p \in \mathcal{C}(\mathrm{MPT}(A))$ need to be considered. Let $(p + 0, p)$ be such an element. Transfer conditions (iii) and (iv) are trivially satisfied because closed MPT(A)-terms cannot terminate successfully.

(i) Suppose that $p + 0 \xrightarrow{a} p'$ for some $a \in A$ and $p' \in \mathcal{C}(\mathrm{MPT}(A))$. By inspection of the deduction rules in Table 4.2, it easily follows that necessarily $p \xrightarrow{a} p'$. The fact that $(p', p') \in R$ proves this case.

(ii) Suppose that $p \xrightarrow{a} p'$ for some $a \in A$ and $p' \in \mathcal{C}(\mathrm{MPT}(A))$. The application of the bottom left deduction rule of Table 4.2 yields $p + 0 \xrightarrow{a} p'$. As $(p', p') \in R$, also this transfer condition is satisfied.

□

The above theorem has two immediate corollaries.

Corollary 4.3.6 (Soundness) Let s and t be MPT(A)-terms. If MPT$(A) \vdash s = t$, then $\mathbb{P}(\text{MPT}(A))_{/\leftrightarrows} \models s = t$.

Corollary 4.3.7 (Soundness) Let p and q be two closed MPT(A)-terms. If MPT$(A) \vdash p = q$, then $p \leftrightarrows q$.

This last corollary states that by using the process theory to reason about closed terms only bisimilar terms can be proved equivalent. Under the assumption that the model captures the appropriate intuitions about processes (such as, for example, operational intuitions concerning the execution of actions), this property guarantees that using the equational theory for deriving equalities cannot lead to mistakes. The converse of this property, ground-completeness, assures that the process theory is strong enough to derive all bisimilarities between closed terms. That is, assuming two closed MPT(A)-terms p and q that are bisimilar, it can be shown that p and q are also derivably equal, i.e., MPT$(A) \vdash p = q$. The idea behind the ground-completeness proof, given below, is to show MPT$(A) \vdash p = q$ via the intermediate results MPT$(A) \vdash p = p + q$ and MPT$(A) \vdash p + q = q$. Clearly, these intermediate results imply MPT$(A) \vdash p = p + q = q$. To understand the intuition behind this idea, recall the notion of summands as introduced in Exercise 4.2.3. The crucial idea of the ground-completeness proof is that, if p and q are bisimilar, it must be the case that every summand of p is a summand of q and vice versa. This implies that p as a whole can be seen as a summand of q and q as a whole as a summand of p, which corresponds to the above mentioned intermediate results.

The following two lemmas are needed in the ground-completeness proof.

Lemma 4.3.8 (Towards completeness) Let p be a closed MPT(A)-term. If $p \xrightarrow{a} p'$ for some closed term p' and $a \in A$, then MPT$(A) \vdash p = a.p' + p$.

Proof The property is proven by induction on the structure of p.

(i) $p \equiv 0$. This case cannot occur as $0 \not\xrightarrow{a}$.

(ii) $p \equiv b.p''$ for some $b \in A$ and closed term p''. The assumption that $p \xrightarrow{a} p'$ implies that $b \equiv a$ and $p'' \equiv p'$. Thus, MPT$(A) \vdash p = p + p = b.p'' + p = a.p' + p$.

(iii) $p \equiv p_1 + p_2$ for some closed terms p_1 and p_2. From $p \xrightarrow{a} p'$, it follows that $p_1 \xrightarrow{a} p'$ or $p_2 \xrightarrow{a} p'$. By induction, (1) $p_1 \xrightarrow{a} p'$ implies MPT$(A) \vdash p_1 = a.p' + p_1$, and (2) $p_2 \xrightarrow{a} p'$ implies MPT$(A) \vdash p_2 = a.p' + p_2$. If $p_1 \xrightarrow{a} p'$, then MPT$(A) \vdash p = p_1 + p_2 = (a.p' + p_1) + p_2 = a.p' + (p_1 + p_2) = a.p' + p$. If $p_2 \xrightarrow{a} p'$, then

$\text{MPT}(A) \vdash p = p_1 + p_2 = p_1 + (a.p' + p_2) = a.p' + (p_1 + p_2) = a.p' + p.$ □

Lemma 4.3.9 Let p, q, and r be closed MPT(A)-terms. If $(p + q) + r \leftrightarrow r$, then $p + r \leftrightarrow r$ and $q + r \leftrightarrow r$.

Proof See Exercise 4.3.5. □

Theorem 4.3.10 (Ground-completeness) Theory MPT(A) is a ground-complete axiomatization of the term model $\mathbb{P}(\text{MPT}(A))_{/\leftrightarrow}$, i.e., for any closed MPT(A)-terms p and q, $\mathbb{P}(\text{MPT}(A))_{/\leftrightarrow} \models p = q$ implies $\text{MPT}(A) \vdash p = q$.

Proof Suppose that $\mathbb{P}(\text{MPT}(A))_{/\leftrightarrow} \models p = q$, i.e., $p \leftrightarrow q$. It must be shown that $\text{MPT}(A) \vdash p = q$. It suffices to prove that, for all closed MPT(A)-terms p and q,

$$p + q \leftrightarrow q \quad \text{implies} \quad \text{MPT}(A) \vdash p + q = q \tag{4.3.1}$$

and that, for all closed MPT(A)-terms p and q,

$$p \leftrightarrow p + q \quad \text{implies} \quad \text{MPT}(A) \vdash p = p + q. \tag{4.3.2}$$

That these properties are sufficient to prove the theorem can be seen as follows. Suppose that properties (4.3.1) and (4.3.2) hold. If $p \leftrightarrow q$, then, by the fact that bisimilarity is reflexive (i.e., $p \leftrightarrow p$ and $q \leftrightarrow q$) and the fact that bisimilarity is a congruence on $\mathbb{P}(\text{MPT}(A))$, $p + p \leftrightarrow p + q$ and $p + q \leftrightarrow q + q$. The soundness of MPT($A$) for $\mathbb{P}(\text{MPT}(A))_{/\leftrightarrow}$, more in particular the validity of Axiom A3, implies that $p + p \leftrightarrow p$ and $q + q \leftrightarrow q$. Using symmetry and transitivity of bisimilarity, $p \leftrightarrow p + q$ and $p + q \leftrightarrow q$ are obtained. Thus, properties (4.3.2) and (4.3.1), respectively, yield that $\text{MPT}(A) \vdash p = p + q$ and $\text{MPT}(A) \vdash p + q = q$. These last results can be combined to show that $\text{MPT}(A) \vdash p = p + q = q$.

Property (4.3.1), namely, for all closed MPT(A)-terms p and q, $p + q \leftrightarrow q$ implies $\text{MPT}(A) \vdash p + q = q$, is proven by induction on the total number of symbols (counting constants and action-prefix operators) in closed terms p and q. The proof of property (4.3.2) is similar and therefore omitted.

The induction proof goes as follows. Assume $p + q \leftrightarrow q$ for some closed MPT(A)-terms p and q. The base case of the induction corresponds to the case that the total number of symbols in p and q equals two, namely when p and q are both equal to 0. Using Axiom A3, it trivially follows that $\text{MPT}(A) \vdash 0 + 0 = 0$, proving the base case. The proof of the inductive step consists of a case analysis based on the structure of term p.

(i) Assume $p \equiv 0$. It follows directly from the axioms in Table 4.1 that $\text{MPT}(A) \vdash p + q = 0 + q = q + 0 = q$.

(ii) Assume $p \equiv a.p'$ for some $a \in A$ and closed term p'. Then, $p \xrightarrow{a} p'$ and thus $p + q \xrightarrow{a} p'$. As $p + q \leftrightarrow q$, also $q \xrightarrow{a} q'$ for some closed term q' such that $p' \leftrightarrow q'$. By Lemma 4.3.8 (Towards completeness), $\text{MPT}(A) \vdash q = a.q' + q$. From $p' \leftrightarrow q'$, as in the first part of the proof of this theorem, it follows that $p' + q' \leftrightarrow q'$ and $q' + p' \leftrightarrow p'$ and, hence, by induction, $\text{MPT}(A) \vdash p' + q' = q'$ and $\text{MPT}(A) \vdash q' + p' = p'$. Combining these last two results gives $\text{MPT}(A) \vdash p' = q' + p' = p' + q' = q'$. Finally, $\text{MPT}(A) \vdash p + q = a.p' + q = a.q' + q = q$.

(iii) Assume $p \equiv p_1 + p_2$ for some closed terms p_1 and p_2. As $(p_1 + p_2) + q \leftrightarrow q$, by Lemma 4.3.9, $p_1 + q \leftrightarrow q$ and $p_2 + q \leftrightarrow q$. Thus, by induction, $\text{MPT}(A) \vdash p_1 + q = q$ and $\text{MPT}(A) \vdash p_2 + q = q$. Combining these results gives $\text{MPT}(A) \vdash p + q = (p_1 + p_2) + q = p_1 + (p_2 + q) = p_1 + q = q$, which completes the proof. □

Corollary 4.3.11 (Ground-completeness) Let p and q be arbitrary closed $\text{MPT}(A)$-terms. If $p \leftrightarrow q$, then $\text{MPT}(A) \vdash p = q$.

An interesting observation at this point is that the term model constructed in this section is isomorphic (as defined in Definition 2.3.26) to the initial algebra of equational theory $\text{MPT}(A)$ (as defined in Definition 2.3.19). Exercise 4.3.6 asks to prove this fact.

Exercises

4.3.1 Consider the terms $a.(b.(c.0 + d.0))$, $a.(b.c.0 + b.d.0)$, and $a.b.c.0 + a.b.d.0$. Give the transition systems for these terms, following the example of Figure 4.1.

4.3.2 Consider the terms $a.c.0 + b.d.0$ and $a.c.0 + b.c.0$. Give the transition systems for these terms.

4.3.3 Prove that bisimilarity is a congruence with respect to the action-prefix operators $a.\_$ $(a \in A)$ on the term algebra $\mathbb{P}(\text{MPT}(A))$ defined in Definition 4.3.1, i.e., complete the second proof of Theorem 4.3.3 (Congruence).

4.3.4 Complete the proof of Theorem 4.3.5 (Soundness) by proving the validity of Axioms A1 through A3.

4.3.5 Prove Lemma 4.3.9.

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4.3.6 Give the initial algebra (see Definition 2.3.19) of the process theory MPT(A). Show that this algebra is isomorphic to the term model given in Definition 4.3.4.

4.3.7 Consider the process theory MPT(A) of Table 4.1. Show that *not* MPT(A) $\vdash a.x + a.y = a.(x + y)$ (with $a \in A$ and x, y terms). (Hint: construct a model of theory MPT(A) that does not validate the equality.)

4.3.8 Show that the transition systems of the terms $a.(0 + 0) + a.0$ and $a.0 + a.0$ are not isomorphic (as defined in Exercise 3.1.10), whereas $a.(0 + 0)$ and $a.0$ are isomorphic. This implies that isomorphism is not a congruence relation on algebra $\mathbb{P}(\text{MPT}(A))$. Isomorphism can be turned into a congruence relation by allowing that states from a transition system are replaced by isomorphic states in the transition-system space. Show that turning isomorphism into a congruence in this way gives exactly bisimilarity, i.e., two transition systems are bisimilar if and only if they become isomorphic by replacing some of their reachable states by isomorphic states.

4.4 The empty process

The process theory MPT(A) is a *minimal* theory; not much can be expressed in it. One aspect that cannot be addressed is successful termination. The semantic framework of Chapter 3 distinguishes between terminating states and deadlock states; this distinction cannot be made in MPT(A). This issue was already addressed in Examples 3.1.9 and 4.2.2 (The lady or the tiger?), illustrating that the distinction is meaningful from a specification point of view. The distinction between successful and unsuccessful termination also turns out to be essential when sequential composition is introduced in Chapter 6.

To ease the description of processes, it is quite common to take an existing process theory and add a constant or an operator plus axioms. There are two types of extensions. First, there are extensions that only allow to describe certain processes more conveniently than before. Second, extensions may allow processes to be described that could not be described in the original theory. In Section 4.5, an extension of the first type is discussed; in this section, an extension of the second type is discussed, namely an extension that allows to distinguish successful and unsuccessful termination.

In order to express successful termination, the new constant 1, referred to as the empty process or the termination constant, is introduced. The extension of the process theory MPT(A) with the empty process 1 results in process

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theory BSP(A), the theory of Basic Sequential Processes. This section gives the equational theory as well as its term model.

Table 4.3 defines process theory BSP(A). The only difference between the signature of MPT(A) and the signature of BSP(A) is the constant 1. The axioms of BSP(A) are exactly the axioms of MPT(A), given in Table 4.1. The layout of Table 4.3 follows the rules set out in Chapter 2. The first entry formally states that BSP(A) is an extension of MPT(A). The second entry introduces the new constant, whereas the main part of the table gives additional axioms, none in this case.

BSP(A)
MPT(A);
constant: 1;
–

Table 4.3. The process theory BSP(A).

The extension of the process theory MPT(A) with the empty process resulting in the process theory BSP(A) has enlarged the set of terms that can be used to specify processes. The terms specifying processes are the closed terms in a process theory. Clearly, the set of closed BSP(A)-terms is strictly larger than the set of closed MPT(A)-terms. A question that needs answering is whether this extension has really enlarged the set of processes that can be specified in the process theory. If this is the case, the expressiveness of the process theory has increased. If this is not the case, the same set of processes can be specified, but in more syntactically different ways. The latter can thus be considered an extension for convenience.

In this case, it is not difficult to see that the expressiveness has increased. A term that contains an occurrence of 1 can never be equal to a term without a 1; inspection of the axioms (of MPT(A)) shows this easily. Thus, any term over the signature of BSP(A) that is not a term over the signature of MPT(A) cannot be derivably equal to a term over the signature of MPT(A). Observe that closed terms that are derivably equal must refer to the same process in some given model of an equational theory. Closed terms that are not derivably equal may be used to specify different processes. In fact, it is always possible to construct a model in which closed terms that are not derivably equal specify different processes (e.g., the initial algebra as defined in Definition 2.3.19). Thus, since any closed MPT(A)-term is a closed BSP(A)-term and since there are closed BSP(A)-terms that are not derivably equal to closed

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MPT(A)-terms, the expressiveness of BSP(A) is strictly larger than the expressiveness of MPT(A). This fact is confirmed below by the fact that the term model of BSP(A) contains strictly more processes than the term model of MPT(A) given in the previous section.

Recall the notion of elimination as introduced in Section 2.2 in Example 2.2.13 and Proposition 2.2.20. An immediate consequence of the above discussion is that process theory BSP(A) does not have the elimination property for MPT(A); that is, it is not possible to eliminate the empty-process constant from BSP(A)-terms to obtain MPT(A)-terms. Nevertheless, it can be shown that the theory BSP(A) is a conservative ground-extension of the theory MPT(A), as defined in Definition 2.2.19. This assures that the theory BSP(A) can always be used to derive equalities between closed MPT(A)-terms.

Theorem 4.4.1 (Conservative ground-extension) Process theory BSP(A) is a conservative ground-extension of process theory MPT(A).

Proof Recall Definition 2.2.19 that states two proof obligations. By definition, the signature of BSP(A) includes the elements of the signature of MPT(A), and the axioms of BSP(A) are the axioms of MPT(A). This proves that BSP(A) is an extension of MPT(A), as defined in Definition 2.2.14, implying that it is also a ground-extension as defined in Definition 2.2.18. This satisfies the first proof obligation of Definition 2.2.19. Thus, it remains to be shown that for all closed MPT(A)-terms p and q, BSP(A) $\vdash p = q$ implies that MPT(A) $\vdash p = q$. Suppose that a proof of BSP(A) $\vdash p = q$ is given. Because neither p nor q contain any occurrences of 1 and because the axioms of BSP(A) are such that any term with an occurrence of 1 can only be equal to a term also containing an occurrence of 1, no term that occurs in the proof of BSP(A) $\vdash p = q$ contains an occurrence of 1. In other words, all terms that occur in the proof of BSP(A) $\vdash p = q$ are MPT(A)-terms. This means that the proof of BSP(A) $\vdash p = q$ is also a proof of MPT(A) $\vdash p = q$, completing the proof. □

Figure 4.2 visualizes the fact that theory BSP(A) is a conservative ground-extension of MPT(A). At this point, the figure is very simple and may be considered redundant. However, in the remainder, this figure is incrementally extended to show the relationships between the various process theories introduced in this book.

It remains to provide a model for BSP(A). In the same way as for the process theory MPT(A), a term model can be defined. Starting from the term algebra, the meaning of the constants and operators is defined using a term deduction

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$$\begin{tikzcd} \boxed{\mathrm{MPT}(A)} \arrow[d] \\ \boxed{\mathrm{BSP}(A)} \end{tikzcd}$$

Fig. 4.2. BSP(A) is a conservative ground-extension of MPT(A).

system. This term deduction system is an extension of the term deduction system for MPT(A) given in Section 4.3. The term model is the quotient algebra of the term algebra modulo bisimilarity. The first step is to define the term algebra of closed BSP(A)-terms.

Definition 4.4.2 (Term algebra) The term algebra for BSP(A) is the algebra $\mathbb{P}(\mathrm{BSP}(A)) = (\mathcal{C}(\mathrm{BSP}(A)), +, (a.\_)_{a\in A}, 0, 1)$.

The next step in the construction of a model is to turn the set of closed terms into a transition-system space (see Definition 3.1.1). The sets of states and labels are the sets of closed terms $\mathcal{C}(\mathrm{BSP}(A))$ and actions A, respectively. Both the ternary transition relation $\rightarrow \subseteq \mathcal{C}(\mathrm{BSP}(A)) \times A \times \mathcal{C}(\mathrm{BSP}(A))$ and the termination predicate $\downarrow \subseteq \mathcal{C}(\mathrm{BSP}(A))$ are defined through the term deduction system in Table 4.4 that extends the term deduction system in Table 4.2. In other words, the transition relation and the termination predicate are the smallest relation and predicate satisfying the deduction rules in Tables 4.2 and 4.4. Note that the domain over which the variables range in Table 4.2 has (implicitly) been extended from the closed MPT(A)-terms to the set of closed BSP(A)-terms.

$TDS(\mathrm{BSP}(A))$		
$TDS(\mathrm{MPT}(A))$;		
constant: 1;		
x, y;		
$1\downarrow$	$\dfrac{x\downarrow}{(x+y)\downarrow}$	$\dfrac{y\downarrow}{(x+y)\downarrow}$

Table 4.4. Term deduction system for BSP(A).

The term deduction system for BSP(A) has three extra deduction rules when compared to the term deduction system for MPT(A). The first rule is an axiom: it says that 1 can terminate. The other two rules state that an alternative-

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composition term has a termination option, as soon as one of the components has this option.

Example 4.4.3 (Transition systems of BSP(A)**-terms)** For any given closed term, a transition system can be obtained from the deduction system in Table 4.4 in the same way as before. The transition systems associated with the process terms $a.1 + b.0$ and $a.(b.0 + 1)$ are given in Figure 4.3. Recall that terminating states (such as the ones represented by the process terms 1 and $b.0 + 1$) are labeled by a small outgoing arrow.

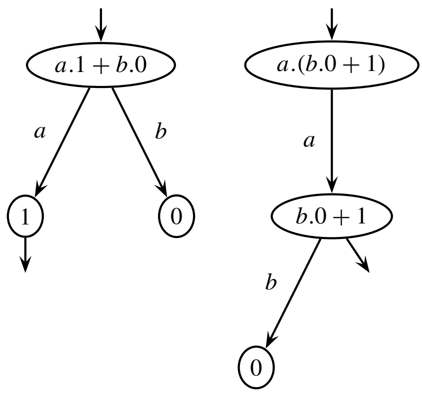

Fig. 4.3. Transition systems associated with $a.1 + b.0$ and $a.(b.0 + 1)$.

Example 4.4.4 (Transition systems) In drawing transition systems, it is often natural or convenient to abstract from the precise terms corresponding to a state. Usually, it is known from the context what terms are meant. Note that transition systems with identical graph structure are no longer distinguishable when terms are omitted from the nodes. For example, the transition systems associated with the processes $a.1$, $a.1 + 0$, and $a.1 + a.1$ are indistinguishable if the process terms are omitted, as illustrated in Figure 4.4. Note that transition systems with identical graph structure but different terms in the nodes are always bisimilar.

As in Section 4.3, bisimilarity is a congruence on the term algebra. This result forms the basis for the term model of BSP(A), which is the term algebra modulo bisimilarity. Note that the definition of bisimilarity (Definition 3.1.10) takes into account the termination relation. It is not difficult to see that transition systems that have termination options cannot be bisimilar to transition systems that do not have such options. Thus, transition systems with and

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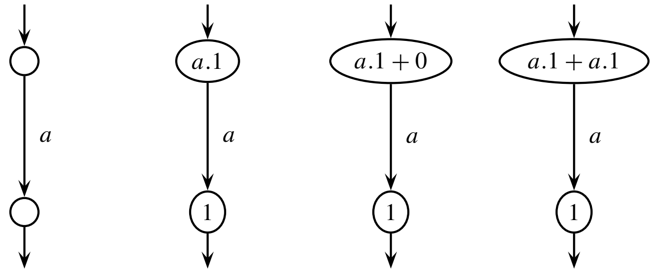

Fig. 4.4. Transition systems with identical graph structure.

without termination options end up in different equivalence classes in the term algebra modulo bisimilarity, i.e., they form different processes. Since all processes that can be specified in the theory MPT(A) of the previous section can also be specified in BSP(A), and since MPT(A) cannot be used to specify processes with termination options, the examples of Figure 4.3 confirm the earlier observation that BSP(A) is more expressive than MPT(A).

Proposition 4.4.5 (Congruence) Bisimilarity is a congruence on the term algebra $\mathbb{P}(\mathrm{BSP}(A))$.

Proof The property follows immediately from the format of the deduction rules in Tables 4.2 and 4.4 and Theorem 3.2.7 (Congruence theorem). □

As mentioned, the resulting term model of BSP(A) is the term algebra of Definition 4.4.2 modulo bisimilarity (see Definition 2.3.18 (Quotient algebra)).

Definition 4.4.6 (Term model of BSP(A)**)** The term model of BSP(A) is the quotient algebra $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrow}$.

Theorem 4.4.7 (Soundness) The process theory BSP(A) is a sound axiomatization of the algebra $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrow}$, i.e., $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrow} \models \mathrm{BSP}(A)$.

Proof The proof of this theorem follows the same lines as the proof of the soundness of theory MPT(A) with respect to the algebra $\mathbb{P}(\mathrm{MPT}(A))_{/\leftrightarrow}$ (Theorem 4.3.5). It must be shown that, for each axiom $s = t$ of BSP(A), $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrow} \models s = t$. As the axioms of MPT($A$) and the axioms of BSP($A$) are identical, it may seem that there is nothing new to prove. This is not true, however, as deduction rules defining the termination predicate have been added to the term deduction system for MPT(A), and hence transfer conditions (iii) and (iv) of Definition 3.1.10 (Bisimilarity) that were trivially true in the proof of Theorem 4.3.5 are no longer satisfied trivially. The remainder gives the proof of Theorem 4.4.7 for Axiom A6; the proofs for the other axioms are left as Exercise 4.4.4.

Recall that it suffices to give a bisimulation that relates all pairs of left-hand- and right-hand sides of closed instantiations of Axiom A6. Let $R = \{(p + 0, p) \mid p \in \mathcal{C}(\mathrm{BSP}(A))\} \cup \{(p, p) \mid p \in \mathcal{C}(\mathrm{BSP}(A))\}$. It must be shown that all elements of R satisfy the transfer conditions of Definition 3.1.10 (Bisimilarity). Since the transfer conditions trivially hold for all pairs (p, p) with p a closed term, only elements of R of the form $(p + 0, p)$ need to be considered. Let $(p + 0, p)$ be such an element. For transfer conditions (i) and (ii), the proofs as given in Theorem 4.3.5 can be copied as no deduction rules involving the transition relation have been added.

(iii) Suppose that $(p + 0)\downarrow$. By inspection of the deduction rules in Table 4.4, it easily follows that necessarily $p\downarrow$, as it is impossible to derive $0\downarrow$. This immediately proves this case.

(iv) Suppose that $p\downarrow$. The application of the middle deduction rule of Table 4.4 yields $(p + 0)\downarrow$, which proves also this case.

□

Corollary 4.4.8 (Soundness) Let s and t be two BSP(A)-terms. If $\mathrm{BSP}(A) \vdash s = t$, then $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrows} \models s = t$.

Corollary 4.4.9 (Soundness) Let p and q be two closed BSP(A)-terms. If $\mathrm{BSP}(A) \vdash p = q$, then $p \leftrightarrows q$.

The soundness result shows that equational theory BSP(A) is a sound axiomatization of the term model $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrows}$. It remains to see whether or not the axiomatization is also ground-complete. It turns out that this is the case. In order to prove this result, a similar approach is followed as the one leading to the ground-completeness result for MPT(A). The following lemmas provide steps towards the ground-completeness theorem.

Lemma 4.4.10 (Towards completeness) Let p be a closed BSP(A)-term. If $p \xrightarrow{a} p'$ for some closed term p' and $a \in A$, then $\mathrm{BSP}(A) \vdash p = a.p' + p$. If $p\downarrow$, then $\mathrm{BSP}(A) \vdash p = 1 + p$.

Proof The proof of the first property is similar to the proof of Lemma 4.3.8 (Towards completeness). There is one additional case in the induction.

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This is the case that $p \equiv 1$. As $1 \overset{a}{\nrightarrow}$, for any $a \in A$, the property is trivially satisfied in this case. The proofs of the other cases can be copied as no deduction rules involving the binary transition relations $\_ \xrightarrow{a} \_$ have been added to the term deduction system.

The second property is proven by induction on the structure of p.

(i) $p \equiv 0$. The property is satisfied as $0 \not\downarrow$.

(ii) $p \equiv 1$. Using Axiom A3, obviously, $\mathrm{BSP}(A) \vdash p = 1 = 1 + 1 = 1 + p$, which completes this case.

(iii) $p \equiv a.p'$ for some $a \in A$ and closed term p'. This case is satisfied as $a.p' \not\downarrow$.

(iv) $p \equiv p_1 + p_2$ for some closed terms p_1 and p_2. From $(p_1 + p_2)\downarrow$, it follows that $p_1\downarrow$ or $p_2\downarrow$. By induction, (1) $p_1\downarrow$ implies $\mathrm{BSP}(A) \vdash p_1 = 1+p_1$, and (2) $p_2\downarrow$ implies $\mathrm{BSP}(A) \vdash p_2 = 1+p_2$. If $p_1\downarrow$, then $\mathrm{BSP}(A) \vdash p = p_1+p_2 = (1+p_1)+p_2 = 1+(p_1+p_2) = 1+p$. If $p_2\downarrow$, then $\mathrm{BSP}(A) \vdash p = p_1+p_2 = p_1+(1+p_2) = (p_1+1)+p_2 = (1 + p_1) + p_2 = 1 + (p_1 + p_2) = 1 + p$. □

Lemma 4.4.11 Let p, q, and r be closed $\mathrm{BSP}(A)$-terms. If $(p + q) + r \leftrightarrow r$, then $p + r \leftrightarrow r$ and $q + r \leftrightarrow r$.

Proof See Exercise 4.4.5. □

Theorem 4.4.12 (Ground-completeness) Theory $\mathrm{BSP}(A)$ is a ground-complete axiomatization of the term model $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrow}$, i.e., for any closed $\mathrm{BSP}(A)$-terms p and q, $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrow} \models p = q$ implies $\mathrm{BSP}(A) \vdash p = q$.

Proof Suppose that $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrow} \models p = q$, i.e., $p \leftrightarrow q$. It must be shown that $\mathrm{BSP}(A) \vdash p = q$. Since bisimilarity is a congruence on $\mathbb{P}(\mathrm{BSP}(A))$ (Proposition 4.4.5) and $\mathrm{BSP}(A)$ is sound for $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrow}$ (Theorem 4.4.7), as in the proof of Theorem 4.3.10, it suffices to prove that, for all closed $\mathrm{BSP}(A)$-terms p and q, $p + q \leftrightarrow q$ implies $\mathrm{BSP}(A) \vdash p + q = q$ and $p \leftrightarrow p + q$ implies $\mathrm{BSP}(A) \vdash p = p + q$.

Only the first property is proven, by induction on the total number of symbols in closed terms p and q. Assume $p + q \leftrightarrow q$ for some closed $\mathrm{BSP}(A)$-terms p and q. Three of the four cases in the induction proving that $\mathrm{BSP}(A) \vdash p = p + q$ can be copied directly from the proof of Theorem 4.3.10. Thus, only the remaining case, when $p \equiv 1$, is detailed. If $p \equiv 1$, $p\downarrow$ and thus $(p + q)\downarrow$. As $p + q \leftrightarrow q$, also $q\downarrow$. By Lemma 4.4.10 (Towards completeness), $\mathrm{BSP}(A) \vdash q = 1 + q$. Hence, since $p \equiv 1$, $\mathrm{BSP}(A) \vdash p + q = 1 + q = q$. □

Corollary 4.4.13 (Ground-completeness) Let p and q be arbitrary closed BSP(A)-terms. If $p \leftrightarroweq q$, then BSP(A) $\vdash p = q$.

Theory BSP(A) with alternative composition, prefixing and constants for successful and unsuccessful termination as presented in this section is the basic theory for the rest of this book. Almost all theories presented further on are extensions of BSP(A). This chapter started out from a still simpler theory, the theory MPT(A), in order to present a process theory that is as simple as possible, and also to facilitate comparisons to the literature, in which often only one termination constant is present.

The term model for BSP(A) constructed in this section is a quotient algebra, namely the term algebra of Definition 4.4.2 modulo bisimilarity. In Chapter 3 also language equivalence was introduced as an equivalence relation on transition systems, see Definition 3.1.7. It is interesting to note that the term algebra modulo language equivalence is also a model of theory BSP(A). However, BSP(A) is not a ground-complete axiomatization of this model, because action prefix does not distribute over choice (see the discussion in Section 4.2), but also because language equivalence identifies all sequences of actions that end in a non-terminating state (see Example 3.1.8 (Language equivalence)). The axioms of BSP(A) do not capture these equivalences. Note that language equivalence is not very meaningful in the context of theory MPT(A), because it only has the inaction constant and not the empty process. This means that the model of closed terms modulo language equivalence is a so-called one-point model, which has only one process. Chapter 12 considers different semantic equivalences in more detail.

To conclude this section, let us briefly reconsider the notion of deadlocks in processes. In Chapter 3, Definition 3.1.14 (Deadlock), it has been defined when a transition system has a deadlock, and when it is deadlock free. Since theory BSP(A) allows to distinguish successful and unsuccessful termination, it is possible to define when a process term is deadlock free and when it has a deadlock.

Definition 4.4.14 (Deadlocks in BSP(A)**-terms)** Let p be a closed BSP(A)-term. Term p is deadlock free if and only if there is a closed BSP(A)-term q without an occurrence of the inaction constant 0 such that BSP(A) $\vdash p = q$. Term p has a deadlock if and only if it is not deadlock free.

It is possible to prove that the model-independent notion of deadlocks for closed BSP(A)-terms introduced in the above definition is consistent with the notion of deadlocks for such terms that can be derived from the term model of theory BSP(A) of Definition 4.4.6. (Recall that the term model is based on an algebra of transition systems, namely the term algebra of Definition 4.4.2, in which each closed BSP(A)-term induces a transition system.)

Proposition 4.4.15 (Deadlocks) Let p be some closed BSP(A)-term. The transition system of p obtained from the deduction system in Table 4.4 is deadlock free if and only if p is deadlock free according to Definition 4.4.14.

Proof The proof goes via induction on the structure of p.

(i) $p \equiv 0$. It follows immediately from the deduction system in Table 4.4 and Definition 3.1.14 (Deadlock) that the transition system of p consists of a single deadlock state, and, hence, is not deadlock free. It remains to prove that p is not deadlock free according to Definition 4.4.14, i.e., that 0 is not derivably equal to a closed term without occurrence of 0. Assume towards a contradiction that there is a closed term q without occurrence of 0 such that BSP(A) $\vdash 0 = q$. It follows from Corollary 4.4.9 that $0 \leftrightarrow q$. Definition 3.1.10 (Bisimilarity) and the fact that the transition system of 0 is a single deadlock state then imply that also the transition system of q must be a single deadlock state. However, the only closed BSP(A)-terms resulting in such a transition system are 0, $0 + 0$, $0 + 0 + 0$, et cetera, i.e., any summation of inaction constants. Clearly this means that q always has a 0 occurrence, which yields a contradiction, and thus proves that term p is not deadlock free according to Definition 4.4.14.

(ii) $p \equiv 1$. It follows immediately from the deduction system in Table 4.4 that the transition system of p is deadlock free. Since $p \equiv 1$, also BSP(A) $\vdash p = 1$, which means that p is also deadlock free according to Definition 4.4.14.

(iii) $p \equiv a.p'$ for some $a \in A$ and closed term p'. The transition system of p is deadlock free if and only if the transition system of p' is deadlock free. By induction, it follows that there is a 0-free closed term q such that BSP(A) $\vdash p' = q$. Thus, $a.q$ does not contain a 0 occurrence and BSP(A) $\vdash p = a.p' = a.q$, which shows that p is deadlock free according to Definition 4.4.14, completing also this case.

(iv) $p \equiv p_1 + p_2$ for some closed terms p_1 and p_2. The transition system of p is deadlock free if and only if (a) the transition systems of both p_1 and p_2 are deadlock free or if (b) one of the two is deadlock free and the other one is the transition system consisting of a single deadlock state. Assume case (a). By induction, it follows that there are two 0-free closed terms q_1 and q_2 such that BSP(A)$\vdash p_1 = q_1$ and BSP(A)$\vdash$

$p_2 = q_2$. Hence, $q_1 + q_2$ is 0-free and BSP$(A) \vdash p = p_1 + p_2 = q_1 + q_2$, which shows that p is deadlock free according to Definition 4.4.14. Assume case (b), and assume without loss of generality that the transition system of p_1 is deadlock free. By induction, it follows that there is a 0-free closed term q_1 such that BSP$(A) \vdash p_1 = q_1$. Following the reasoning in the first item of this induction proof, it furthermore follows that p_2 is some arbitrary summation of inaction constants. Thus, based on Axiom A6, it follows that BSP$(A) \vdash p = p_1 + p_2 = q_1$, showing also in this final case that p is deadlock free according to Definition 4.4.14. □

The following corollary immediately follows from the above result and the fact that both in Definition 3.1.14 (Deadlock) and in Definition 4.4.14 (Deadlocks in BSP(A)-terms) the presence of deadlocks and deadlock freeness are defined as opposites.

Corollary 4.4.16 (Deadlocks) Let p be some closed BSP(A)-term. The transition system of p obtained from the deduction system in Table 4.4 has a deadlock if and only if p has a deadlock according to Definition 4.4.14.

Exercises

4.4.1 Draw transition systems for the following closed BSP(A)-terms:

(a) $a.(1 + 0)$
(b) $a.1 + a.0$
(c) $1 + a.1$
(d) $1 + a.0$

Show that there does not exist a bisimulation relation between any pair of these transition systems.

4.4.2 Prove that the term algebra of BSP(A) of Definition 4.4.2 modulo language equivalence, Definition 3.1.7, is a model of theory BSP(A).

4.4.3 Recall Definition 3.1.14 (Deadlock) and Definition 4.4.14 (Deadlocks in BSP(A)-terms). Let p and q be closed BSP(A)-terms, and let a be an action in A.

(a) Which of the terms of Exercise 4.4.1 has a deadlock?
(b) Show that, if p has a deadlock and BSP$(A) \vdash p = q$, then q has a deadlock.
(c) Show that, if p has a deadlock, then $a.p$ has a deadlock.

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(d) Give an example showing that $p + q$ may be deadlock free even if one of the two terms p or q has a deadlock.

(e) Show that, if p has a deadlock, then $a.p + q$ has a deadlock.

(f) Give an inductive definition defining when a closed BSP(A)-term has a deadlock in a model-independent way, that is consistent with the earlier two definitions concerning deadlocks (Definitions 3.1.14 and 4.4.14).

(g) Give an inductive definition defining deadlock freeness in a model-independent way.

(h) Prove that a closed term has a deadlock according to the inductive definition of item (f) if and only if it is not deadlock free according to the inductive definition of item (g).

4.4.4 Complete the proof of Theorem 4.4.7 (Soundness) by proving the validity of Axioms A1 through A3.

4.4.5 Prove Lemma 4.4.11.

4.5 Projection

This section considers an extension of the process theory BSP(A). Contrary to the situation in the previous section, this extension does not allow to describe more processes, but rather allows to describe already present processes in more ways than before. Thus, it is an extension for convenience.

Theory BSP(A) can be extended with a family of so-called projection operators π_n, for each $n \in \mathbf{N}$. There are several reasons for introducing such operators. First, they are interesting in their own right. Second, they are a good candidate to explain what concepts play a role in extending a process theory with additional syntax and axioms. (Recall that the simple extension of MPT(A) to BSP(A) discussed in the previous section does not introduce any new axioms.) Third, the projection operators turn out to be useful in reasoning about iterative and recursive processes; a simple form of iteration is introduced later in this chapter, and recursion is the topic of the next chapter. A more general form of iteration is briefly discussed in Chapter 6.

Let x be some term in some process theory. Intuitively speaking, the term $\pi_n(x)$, for some $n \in \mathbf{N}$, corresponds to the behavior of x up to depth n, where the depth is measured in terms of the number of actions that have been executed. Table 4.5 gives process theory $(\mathrm{BSP} + \mathrm{PR})(A)$, the extension of theory BSP(A) with projection operators.

Table 4.5 expresses that $(\mathrm{BSP}{+}\mathrm{PR})(A)$ is an extension of BSP(A), and gives the additional syntax and axioms. Note that the five axioms are in fact *axiom*

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(BSP + PR)(A)	
BSP(A);	
unary: $(\pi_n)_{n \in \mathbf{N}}$;	
x, y;	
$\pi_n(1) = 1$	PR1
$\pi_n(0) = 0$	PR2
$\pi_0(a.x) = 0$	PR3
$\pi_{n+1}(a.x) = a.\pi_n(x)$	PR4
$\pi_n(x + y) = \pi_n(x) + \pi_n(y)$	PR5

Table 4.5. The process theory (BSP + PR)(A) (with $a \in A$).

schemes. Consider PR1, PR2 and PR5, for example. There is one instance of PR1, PR2 and PR5 for each of the projection operators π_n, with $n \in \mathbf{N}$. For PR3, there is one instance for each action-prefix operator $a.\_$, with $a \in A$, and PR4 has an instance for each combination of a projection operator and an action-prefix operator. Despite the fact that PR1 through PR5 are in fact axiom schemes, they are usually referred to as simply axioms.

An interesting observation is that the five axioms follow the structure of BSP(A)-terms. This is often the case when extending a process theory for the ease of specification; it guarantees that the extension, in this case projection, is defined for all possible process terms in the theory that is being extended.

Axiom PR1 says that the behavior of the termination constant up to some arbitrary depth n equals termination, or, in other words, the empty process; this obviously corresponds to the fact that the empty process cannot execute any actions. Likewise, Axiom PR2 says that the behavior of the inaction constant up to arbitrary depth equals inaction. The third axiom, Axiom PR3, states that the behavior of a process up to depth 0, i.e., the behavior corresponding to the execution of no actions, simply equals inaction for processes that have no immediate successful termination option. The fourth axiom is the most interesting one; Axiom PR4 states that the behavior of a process with a unique initial action up to some positive depth m equals the initial action of this process followed by the behavior of the remainder up to depth $m - 1$. The final axiom simply says that projection operators distribute over choice.

Example 4.5.1 (Projection) Let a, b, and c be actions in A. It is not difficult to verify that (BSP + PR)$(A) \vdash$

$$\begin{aligned} &\pi_0(a.b.1) = \pi_0(a.b.0) = 0, \\ &\pi_1(a.b.1) = \pi_1(a.b.0) = a.0, \\ &\pi_n(a.b.1) = a.b.1, \qquad\qquad \text{if } n \geq 2, \end{aligned}$$

$$\begin{aligned}
&\pi_n(a.b.0) = a.b.0, && \text{if } n \geq 2,\\
&\pi_n(a.0 + b.1) = a.0 + b.1, && \text{if } n \geq 1,\\
&\pi_0(a.0 + b.c.1) = 0,\\
&\pi_1(a.0 + b.c.1) = a.0 + b.0,\\
&\pi_n(a.0 + b.c.1) = a.0 + b.c.1, && \text{if } n \geq 2,\\
&\pi_1(a.(a.0 + b.c.1)) = a.0, && \text{and}\\
&\pi_2(a.(a.0 + b.c.1)) = a.(a.0 + b.0).
\end{aligned}$$

As a note aside, observe that it is also possible to extend the theory MPT(A) of Section 4.2 with projection operators, resulting in (MPT + PR)(A). This extension follows the lines of the extension of BSP(A) with projection; the only difference is that Axiom PR1 is omitted. As theory (MPT + PR)(A) is not needed in the remainder, its elaboration is left as Exercise 4.5.7.

The extension of the process theory BSP(A) with the projection operators to the process theory (BSP + PR)(A) has enlarged the set of closed terms that can be used for specifying processes. The question whether this extension has also enlarged the set of processes that can be specified in the process theory still needs to be addressed; stated in other words, it must be investigated whether the expressiveness of the process theory has increased. If this is not the case, the same set of processes can be specified, but in more syntactically different ways. This can thus be considered an extension for convenience. The following elimination theorem expresses that every closed (BSP + PR)(A)-term is derivably equal to a closed BSP(A)-term. The consequence of this theorem is that the addition of the projection operators has not enlarged the expressiveness of the process theory, which conforms to the claim made in the introduction to this section. Example 4.5.1 shows that it is often straightforward to eliminate projection operators from (BSP + PR)(A)-terms. Recall the concept of elimination from Example 2.2.13 and Proposition 2.2.20. Note that closed BSP(A)-terms play the role of basic terms. Proposition 2.2.20 has been proven via induction (Exercise 2.2.5). Another way of proving an elimination result is through rewrite theory. The proof given below is based on rewrite theory because it provides the most insight in the working of equational theory (BSP + PR)(A). However, it is also possible to prove Theorem 4.5.2 through induction (Exercise 4.5.4).

Theorem 4.5.2 (Elimination) For every closed (BSP + PR)(A)-term p, there exists a closed BSP(A)-term q such that (BSP + PR)$(A) \vdash p = q$.

Proof Recall Definition 2.4.1 (Term rewriting system). The first step of the proof is to turn equational theory (BSP+PR)(A) into a rewriting system.

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The signature of the rewriting system equals the signature of $(\mathrm{BSP} + \mathrm{PR})(A)$. The rewrite rules correspond to Axioms PR1–PR5 of Table 4.5 when read from left to right; for any $n \in \mathbf{N}$, $a \in A$, and $(\mathrm{BSP} + \mathrm{PR})(A)$-terms x, y:

$$
\begin{aligned}
&\pi_n(1) \to 1\\
&\pi_n(0) \to 0\\
&\pi_0(a.x) \to 0\\
&\pi_{n+1}(a.x) \to a.\pi_n(x)\\
&\pi_n(x + y) \to \pi_n(x) + \pi_n(y)
\end{aligned}
$$

The idea of the choice of rewrite rules is that for each rewrite rule $s \to t$ of the term rewriting system, $(\mathrm{BSP} + \mathrm{PR})(A) \vdash s = t$. This means that each rewrite step transforms a process term into a process term that is derivably equal.

The second step of the proof is to show that the above term rewriting system is strongly normalizing, i.e., that no process term allows an infinite reduction (see Definition 2.4.6). The details of this part of the proof are left as an exercise (Exercise 4.5.3) because they are not essential to the understanding of the proof of the elimination theorem. A consequence of the choice of the rewrite rules and the resulting strong-normalization property is that every $(\mathrm{BSP} + \mathrm{PR})(A)$-term has a normal form that is derivably equal.

The last part of the proof is to show that no *closed* normal form of the above term rewriting system contains a projection operator, i.e., the operators to be eliminated. Note that it is sufficient to prove the property for normal forms that are *closed terms* because the elimination theorem is formulated only for closed terms. (It is an interesting exercise for the reader to explain why the closed-term restriction is also necessary, implying that the elimination theorem cannot be extended to arbitrary open terms.)

Let closed $(\mathrm{BSP} + \mathrm{PR})(A)$-term u be a normal form of the above rewriting system. Suppose that u contains projection operators, i.e., it is not a (closed) $\mathrm{BSP}(A)$-term. Thus, u must contain at least one subterm of the form $\pi_n(v)$, for some natural number n and closed $(\mathrm{BSP} + \mathrm{PR})(A)$-term v. Clearly, this subterm can always be chosen to be minimal, such that v is a closed $\mathrm{BSP}(A)$-term containing no projection operators itself. It follows immediately from the structure of $\mathrm{BSP}(A)$-terms that precisely one of the rewrite rules of the above term rewriting system can be applied to $\pi_n(v)$. As a consequence, u is not a normal form. This contradiction implies that u must be a closed $\mathrm{BSP}(A)$-term.

Summarizing all the above, it has been shown that each closed $(\mathrm{BSP}+\mathrm{PR})(A)$-term p has a derivably equal normal form q that is a closed $\mathrm{BSP}(A)$-term, thus completing the proof. □

Now that it has been established that the additions to the basic process theory have not increased expressiveness, it is time to consider conservativity. Besides

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the addition of operators to the signature of the process theory, also axioms (involving these operators) have been added. Thus, the collection of proofs in the process theory has grown. Nevertheless, it can be shown that theory (BSP+PR)(A) is a conservative ground-extension of BSP(A). That is, it is not possible to obtain new equalities between BSP(A)-terms despite the addition of the extra axioms. Figure 4.5 extends Figure 4.2 to include this conservativity result; it also includes conservativity results for theory (MPT + PR)(A), which however are not explicitly formulated (see Exercise 4.5.7). Note that the combined elimination and conservativity results imply that the term models of BSP(A) and (BSP + PR)(A) have in fact the same number of processes.

$$\begin{tikzcd} \boxed{\mathrm{MPT}(A)} \arrow[r, "\text{PR2-PR5}"] \arrow[d] & \boxed{(\mathrm{MPT}+\mathrm{PR})(A)} \arrow[d, "\text{PR1}"] \\ \boxed{\mathrm{BSP}(A)} \arrow[r, "\text{PR1-PR5}"] & \boxed{(\mathrm{BSP}+\mathrm{PR})(A)} \end{tikzcd}$$

Fig. 4.5. Conservativity results for (MPT + PR)(A) and (BSP + PR)(A).

$$\begin{aligned} &x + x \to x \\ &x + 0 \to x \\ &\pi_n(1) \to 1 \\ &\pi_n(0) \to 0 \\ &\pi_0(a.x) \to 0 \\ &\pi_{n+1}(a.x) \to a.\pi_n(x) \\ &\pi_n(x + y) \to \pi_n(x) + \pi_n(y) \end{aligned}$$

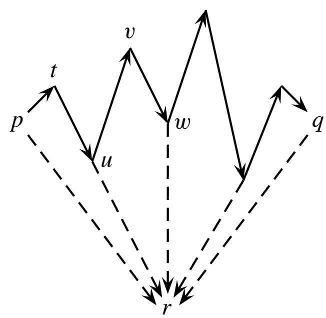

Fig. 4.6. Conservativity of the extension of BSP(A) with projection.

Theorem 4.5.3 (Conservative ground-extension) Theory (BSP + PR)(A) is a conservative ground-extension of process theory BSP(A).

Proof There are several ways to prove the above theorem. Two possibilities are outlined. The first possibility provides the most insight, whereas the second possibility is often technically more convenient.
Consider Definition 2.2.19 (Conservative ground-extension). Since the signature and axioms of theory (BSP + PR)(A) contain the signature and axioms

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of BSP(A), the ground-extension requirement is satisfied. Thus, it remains to prove that, for closed BSP(A)-terms p and q, $(\mathrm{BSP}+\mathrm{PR})(A) \vdash p = q$ implies $\mathrm{BSP}(A) \vdash p = q$. Assume that $(\mathrm{BSP} + \mathrm{PR})(A) \vdash p = q$; it must be shown that $\mathrm{BSP}(A) \vdash p = q$. In other words, a derivation in $(\mathrm{BSP}+\mathrm{PR})(A)$ must be transformed into a derivation in BSP(A).

The idea of the proof is visualized in Figure 4.6. The left part of this figure shows the rules of a term rewriting system, which correspond to the axioms of theory $(\mathrm{BSP} + \mathrm{PR})(A)$ read from left to right, excluding Axioms A1 (commutativity of choice) and A2 (associativity). The solid arrows in the right part of the figure visualize a derivation showing that p equals q. A downward arrow corresponds to a part of the derivation in which axioms are applied in the left-to-right direction, conform the rewrite rules in the left part of the figure, whereas an upward arrow corresponds to the application of axioms in the right-to-left direction. For the sake of simplicity, terms that differ only with respect to commutativity and associativity of choice are considered equal. Clearly, any derivation in $(\mathrm{BSP} + \mathrm{PR})(A)$ is of the form depicted in the figure.

Returning to the term rewriting system given in the figure, it can be shown that it is both strongly normalizing (see Definition 2.4.6) and confluent (see Definition 2.4.7), for the latter assuming that terms that differ only with respect to associativity and commutativity of choice may be considered equivalent. The proof of the confluence property uses the theory of term rewriting modulo a congruence relation. The reader interested in details is referred to (Dershowitz & Jouannaud, 1990). The confluence of the term rewriting system has some important consequences. Consider, for example, point t in Figure 4.6. It follows from the explanation of the meaning of the solid arrows given above that it is possible to rewrite t both into p and into u. Thus, the confluence property implies that it must be possible to rewrite both p and u into a single term r, as depicted by the dashed arrows in the figure. The strong-normalization property implies that r can be chosen to be a normal form. Next, consider term v. It is clear that it can be rewritten into r via u; it can also be rewritten into w. Thus, due to the confluence property, it must be possible to rewrite r and w into a common term. Since r is a normal form, this common term must be r, which explains the dashed arrow from w to r. Using similar reasoning, it can be shown that, in the end, also q can be rewritten into r.

It remains to show how this result can be used to turn the $(\mathrm{BSP} + \mathrm{PR})(A)$ derivation from p to q into a BSP(A) derivation. An important observation is that none of the rewrite rules introduces a projection operator and that both p and q are BSP(A)-terms and thus do not contain projection operators. An immediate consequence is that the dashed arrows from p to r and from q to r correspond to BSP(A) derivations. Clearly, the two derivations can be combined into one derivation showing that BSP$(A) \vdash p = q$ and completing the proof.

The second, technically more convenient way to prove the conservativity theorem builds upon the framework of Chapter 3. An interesting observation is that this proof uses soundness and ground-completeness results with respect to the term models for BSP(A) (see Section 4.3) and (BSP + PR)(A) (see below), whereas the above proof is independent of any given model.

The desired conservativity result is a direct consequence of the application of Theorem 3.2.21 (Conservativity). It uses the facts that BSP(A) is a ground-complete axiomatization of the term model $\mathbb{P}(\text{BSP}(A))_{/\underline{\leftrightarrow}}$ (Theorem 4.4.12), that (BSP + PR)(A) is a sound axiomatization of the term model $\mathbb{P}((\text{BSP} + \text{PR})(A))_{/\underline{\leftrightarrow}}$ (see Theorem 4.5.8 below), and that *TDS*((BSP + PR)(A)) (see Table 4.6 below) is an *operational* conservative extension of *TDS*(BSP(A)), as defined in Definition 3.2.16. This last fact follows immediately from the format of the deduction rules in Tables 4.2, 4.4 and 4.6, and Theorem 3.2.19 (Operational conservative extension). □

The following theorem shows an interesting application of projection operators. It proves that all closed (BSP + PR)(A)-terms have a finite depth, or in other words, that processes specified by closed (BSP + PR)(A)-terms cannot have unbounded executions. This implies among others that transition systems of closed (BSP + PR)(A)-terms cannot have cycles. Note that process theories (BSP + PR)(A) and BSP(A) allow the same set of processes to be specified (due to the elimination theorem), and that MPT(A) allows a subset of those processes to be specified. As a result, also in MPT(A) and BSP(A) only processes with bounded-depth executions can be specified.

Theorem 4.5.4 (Bounded depth) For each closed (BSP + PR)(A)-term p, there exists a natural number n such that for all $k \geq n$,

$$(\text{BSP} + \text{PR})(A) \vdash \pi_k(p) = p.$$

Proof Using Theorem 4.5.2 (Elimination), it suffices to prove the desired property for closed BSP(A)-terms; see also Chapter 2, Example 2.2.21. Thus, it suffices to show that, for each closed BSP(A)-term p_1, there exists a natural number n such that for all $k \geq n$, $(\text{BSP} + \text{PR})(A) \vdash \pi_k(p_1) = p_1$. The property is proven by induction on the structure of closed BSP(A)-term p_1.

(i) Assume $p_1 \equiv 1$. Choose $n = 0$ and observe that $(\text{BSP} + \text{PR})(A) \vdash \pi_k(1) = 1$ for all $k \geq 0$.
(ii) Assume $p_1 \equiv 0$. Choose $n = 0$ and observe that $(\text{BSP} + \text{PR})(A) \vdash \pi_k(0) = 0$ for all $k \geq 0$.

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(iii) Assume $p_1 \equiv a.q$, for some $a \in A$ and closed BSP(A)-term q. By induction, there exists an $m \in \mathbf{N}$ such that $(\mathrm{BSP} + \mathrm{PR})(A) \vdash \pi_l(q) = q$ for all $l \geq m$. Choose $n = m + 1$. It must be shown that $(\mathrm{BSP} + \mathrm{PR})(A) \vdash \pi_k(p_1) = p_1$ for all $k \geq n$. Observe that $n \geq 1$. Thus, for all $k - 1 \geq m$ and, hence, for all $k \geq n$, $(\mathrm{BSP} + \mathrm{PR})(A) \vdash \pi_k(p_1) = \pi_k(a.q) = a.\pi_{k-1}(q) = a.q = p_1$.

(iv) Assume $p_1 \equiv q_1 + q_2$ for some closed BSP(A)-terms q_1 and q_2. By induction, there exist m_1 and m_2 such that $(\mathrm{BSP} + \mathrm{PR})(A) \vdash \pi_{l_1}(q_1) = q_1$ and $(\mathrm{BSP} + \mathrm{PR})(A) \vdash \pi_{l_2}(q_2) = q_2$ for all $l_1 \geq m_1$ and $l_2 \geq m_2$. Choose n to be the maximum of m_1 and m_2: $n = \max(m_1, m_2)$. Then, for all $k \geq \max(m_1, m_2) = n$, $(\mathrm{BSP}+\mathrm{PR})(A) \vdash \pi_k(p) = \pi_k(q_1 + q_2) = \pi_k(q_1) + \pi_k(q_2) = q_1 + q_2 = p$. □

It remains to construct a model of theory $(\mathrm{BSP} + \mathrm{PR})(A)$. The model given in the remainder of this section is a term model, similar to the term model of BSP(A) given in Section 4.4. The first step is to define the term algebra of closed $(\mathrm{BSP} + \mathrm{PR})(A)$-terms.

Definition 4.5.5 (Term algebra) The *term algebra* for $(\mathrm{BSP} + \mathrm{PR})(A)$ is the algebra $\mathbb{P}((\mathrm{BSP}+\mathrm{PR})(A)) = (\mathcal{C}((\mathrm{BSP}+\mathrm{PR})(A)), +, (\pi_n)_{n\in\mathbf{N}}, (a.\_)_{a\in A}, 0, 1)$.

The next step in the construction of a model is to turn the set of closed terms $\mathcal{C}((\mathrm{BSP}+\mathrm{PR})(A))$ into a transition-system space. The sets of states and labels are the sets of closed terms $\mathcal{C}((\mathrm{BSP} + \mathrm{PR})(A))$ and actions A, respectively. The termination predicate and the ternary transition relation are defined via the term deduction system in Table 4.6, which extends the term deduction systems in Tables 4.2 and 4.4. In other words, the termination predicate and the transition relation are the smallest set and relation satisfying the deduction rules in Tables 4.2, 4.4 and 4.6.

$TDS((\mathrm{BSP} + \mathrm{PR})(A))$	
$TDS(\mathrm{BSP}(A))$;	
unary: $(\pi_n)_{n\in\mathbf{N}}$;	
x, x';	
$\dfrac{x\downarrow}{\pi_n(x)\downarrow}$	$\dfrac{x \xrightarrow{a} x'}{\pi_{n+1}(x) \xrightarrow{a} \pi_n(x')}$

Table 4.6. Term deduction system for $(\mathrm{BSP} + \mathrm{PR})(A)$ (with $a \in A$).

The term deduction system for $(\mathrm{BSP}+\mathrm{PR})(A)$ has two extra deduction rules

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when compared to the term deduction system for BSP(A). The first rule implies that any closed term $\pi_n(p)$ has a termination option exactly when closed term p has. This expresses the fact that projection does not influence termination. The second rule assumes that some closed term p can execute an action a thereby transforming into some other closed term p' as a result. It then states that the execution of an action by a closed term $\pi_m(p)$ with $m \in \mathbf{N}$ is allowed if, and only if, the subscript of the projection operator m is not zero. Since the execution of an action increases the actual depth of a process by one, the result of the execution is term $\pi_{m-1}(p')$, i.e., closed term p' up to depth *m minus one*. The resulting term model is the term algebra of Definition 4.5.5 modulo bisimilarity.

Proposition 4.5.6 (Congruence) Bisimilarity is a congruence on term algebra $\mathbb{P}((\mathrm{BSP} + \mathrm{PR})(A))$.

Proof The property follows immediately from the format of the deduction rules given in Tables 4.2, 4.4 and 4.6 (see Theorem 3.2.7 (Congruence theorem)). □

Definition 4.5.7 (Term model of $(\mathrm{BSP} + \mathrm{PR})(A)$) The term model of $(\mathrm{BSP}+\mathrm{PR})(A)$ is the quotient algebra $\mathbb{P}((\mathrm{BSP} + \mathrm{PR})(A))_{/\underline{\leftrightarrow}}$.

Theorem 4.5.8 (Soundness) Theory $(\mathrm{BSP} + \mathrm{PR})(A)$ is a sound axiomatization of the algebra $\mathbb{P}((\mathrm{BSP} + \mathrm{PR})(A))_{/\underline{\leftrightarrow}}$, i.e., $\mathbb{P}((\mathrm{BSP} + \mathrm{PR})(A))_{/\underline{\leftrightarrow}} \models (\mathrm{BSP} + \mathrm{PR})(A)$.

Proof According to Definition 2.3.8 (Model), it must be shown that, for each axiom $s = t$ of $(\mathrm{BSP} + \mathrm{PR})(A)$, $\mathbb{P}((\mathrm{BSP} + \mathrm{PR})(A))_{/\underline{\leftrightarrow}} \models s = t$. The proof for the axioms of BSP(A) carries over directly from the proof of Theorem 4.4.7 (Soundness of BSP(A)) and Exercise 4.4.4. The proof for the axioms that are new in $(\mathrm{BSP} + \mathrm{PR})(A)$ goes along the same lines and is left as Exercise 4.5.5. □

Having soundness, the question is addressed whether or not the axiomatization is also ground-complete. It turns out that this is the case. Informally, the ground-completeness of $(\mathrm{BSP} + \mathrm{PR})(A)$ follows from the fact that the addition of the projection operators to the basic theory BSP(A) does not increase the set of processes that can be expressed and the ground-completeness of BSP(A). There are two straightforward ways for proving the result formally. First, it is possible to apply Theorem 3.2.26 (Ground-completeness), which is a meta-result applicable to models fitting the operational framework

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of Chapter 3. However, the ground-completeness result also follows in a fairly straightforward way from the basic results developed in this chapter, namely the elimination theorem (Theorem 4.5.2), the conservativity theorem (Theorem 4.5.3), and the ground-completeness of BSP(A) (Theorem 4.4.12). For illustration purposes, both proofs are given below.

Theorem 4.5.9 (Ground-completeness) The process theory $(\mathrm{BSP}+\mathrm{PR})(A)$ is a ground-complete axiomatization of the term model $\mathbb{P}((\mathrm{BSP}+\mathrm{PR})(A))_{/\leftrightarrow}$, i.e., for any closed $(\mathrm{BSP}+\mathrm{PR})(A)$-terms p and q, $\mathbb{P}((\mathrm{BSP}+\mathrm{PR})(A))_{/\leftrightarrow} \models p = q$ implies $(\mathrm{BSP}+\mathrm{PR})(A) \vdash p = q$.

Proof The first way to prove the desired result is the application of Theorem 3.2.26 (Ground-completeness). In the proof of Theorem 4.5.3 (Conservative ground-extension), it has already been established that the two theories $(\mathrm{BSP}+\mathrm{PR})(A)$ and BSP(A) and their associated deduction systems and models satisfy the conditions of Theorem 3.2.21 (Conservativity). Thus, Theorem 4.5.2 (Elimination) immediately gives the desired result.

The second proof builds on the basic results for theories $(\mathrm{BSP}+\mathrm{PR})(A)$ and BSP(A) directly, without resorting to the meta-results of Chapter 3. Let p and q be closed $(\mathrm{BSP}+\mathrm{PR})(A)$-terms. Suppose that $\mathbb{P}((\mathrm{BSP}+\mathrm{PR})(A))_{/\leftrightarrow} \models p = q$. Then, by definition, $p \leftrightarrow q$. By Theorem 4.5.2 (Elimination), there exist closed BSP(A)-terms p_1 and q_1 such that $(\mathrm{BSP}+\mathrm{PR})(A) \vdash p = p_1$ and $(\mathrm{BSP}+\mathrm{PR})(A) \vdash q = q_1$. This implies $p \leftrightarrow p_1$ and $q \leftrightarrow q_1$. Since bisimilarity is an equivalence (Theorem 3.1.13), the fact that $p \leftrightarrow q$, $p \leftrightarrow p_1$, and $q \leftrightarrow q_1$ implies that $p_1 \leftrightarrow q_1$. From the facts that p_1 and q_1 are closed BSP(A)-terms and that BSP(A) is a ground-complete axiomatization of the term model $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrow}$, it follows that $\mathrm{BSP}(A) \vdash p_1 = q_1$ (Corollary 4.4.13). Since $(\mathrm{BSP}+\mathrm{PR})(A)$ is a conservative ground-extension of BSP(A) (Theorem 4.5.3) and since p_1 and q_1 are closed BSP(A)-terms, it follows that $(\mathrm{BSP}+\mathrm{PR})(A) \vdash p_1 = q_1$. Combining results yields that $(\mathrm{BSP}+\mathrm{PR})(A) \vdash p = p_1 = q_1 = q$, completing the proof. □

Exercises

4.5.1 Let p be the closed term $a.(b.1 + c.d.0) + a.b.1$. Calculate $\pi_n(p)$ (as a closed BSP(A)-term) for all $n \geq 0$.

4.5.2 Prove that, for all closed $(\mathrm{BSP}+\mathrm{PR})(A)$-terms p, it holds that $(\mathrm{BSP}+\mathrm{PR})(A) \vdash \pi_0(p) = 1$ if and only if $(\mathrm{BSP}+\mathrm{PR})(A) \vdash p = p + 1$ and that $(\mathrm{BSP}+\mathrm{PR})(A) \vdash \pi_0(p) = 0$ otherwise.

4.5.3 Prove that the term rewriting system that is given in the proof of Theorem 4.5.2 (Elimination) is strongly normalizing.
(Hint: use induction on the total number of symbols that occur in the term that is the operand of the projection operator.)

4.5.4 Give an inductive proof of Theorem 4.5.2 (Elimination).
(Hint: difficult.)

4.5.5 Complete the proof of Theorem 4.5.8 (Soundness) by proving the validity of Axioms PR1 through PR5 of Table 4.5.

4.5.6 Prove by structural induction that for all closed BSP(A)-terms p and all natural numbers $n, m \geq 0$ the following holds:

$$(\mathrm{BSP} + \mathrm{PR})(A) \vdash \pi_n(\pi_m(p)) = \pi_{\min(n,m)}(p).$$

4.5.7 Starting from theory MPT(A), develop the theory (MPT + PR)(A). That is, define the equational theory $(\mathrm{MPT} + \mathrm{PR})(A)$, prove an elimination and a conservativity result, and give a term model proving soundness and ground-completeness. Also, prove that theory $(\mathrm{BSP} + \mathrm{PR})(A)$ is a conservative ground-extension of $(\mathrm{MPT} + \mathrm{PR})(A)$.

4.6 Prefix iteration

Recall Theorem 4.5.4 (Bounded depth). A consequence of that theorem is the fact that none of the process theories considered so far allows us to specify behavior with unbounded depth. Most example processes that will be considered in the following chapters have unbounded depth. Thus, it is interesting to investigate extensions of basic process theory that remove this boundedness restriction. This section extends the process theory BSP(A) with prefix iteration. This will constitute a limited instantiation of much more general constructs to be discussed in the following chapter. For each action $a \in A$, a new unary operator $a^*\_$ called a *prefix-iteration* operator is added to the signature of BSP(A). The process a^*x can execute a any number of times before starting the execution of x. Note that this construction indeed allows unbounded-depth behavior; for instance, a^*0 will execute a an unbounded number of times.

The extension of BSP(A) with prefix-iteration operators is called BSP$^*(A)$. The axioms of BSP$^*(A)$ are the axioms of BSP(A) plus Axioms PI1 and PI2 given in Table 4.7. These axioms, and their names, are taken from (Fokkink, 1994). The first axiom explains that a process a^*x has a choice between executing a and repeating itself, and executing x. The second axiom expresses idempotency of prefix iteration, i.e., allowing two iterations before continuing with x is the same as allowing only one iteration.

It is straightforward to extend the term model of theory BSP(A) to a model

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BSP*(A)	
BSP(A);	
unary: $(a^*\_)_{a\in A}$;	
x;	
$a.(a^*x) + x = a^*x$	PI1
$a^*(a^*x) = a^*x$	PI2

Table 4.7. The process theory BSP*(A).

of BSP*(A). The closed BSP*(A)-terms with the signature of theory BSP*(A) form the term algebra $\mathbb{P}$(BSP*(A)). This term algebra can be turned into a term model for BSP*(A) via the deduction system given in Table 4.8. Compared to the deduction system for theory BSP(A) (see Tables 4.2 and 4.4), it contains three new deduction rules, all three for prefix-iteration operators. The first prefix-iteration rule implies that a closed term a^*p can execute action a an arbitrary number of times. The other two rules state that closed term a^*p can mimic the behavior (action execution and termination) of closed term p.

TDS(BSP*(A))		
TDS(BSP(A));		
unary: $(a^*\_)_{a\in A}$;		
x, x';		
$a^*x \xrightarrow{a} a^*x$	$\dfrac{x \xrightarrow{b} x'}{a^*x \xrightarrow{b} x'}$	$\dfrac{x\downarrow}{a^*x\downarrow}$

Table 4.8. Term deduction system for BSP*(A) (with $a, b \in A$).

Example 4.6.1 (Transition systems) The transition systems that can be associated with terms a^*1, a^*0, and $a^*(b.1)$ are given in Figure 4.7. Note that transition systems induced by closed BSP*(A)-terms are still regular, but now, loops from a node to itself can occur. The relationship between (closed terms in) equational theories and regular transition systems is addressed in more detail in the next chapter. One of the results given in that chapter is an equational theory that precisely captures all regular transition systems.

Proposition 4.6.2 (Congruence) Bisimilarity is a congruence on term algebra $\mathbb{P}$(BSP*(A)).

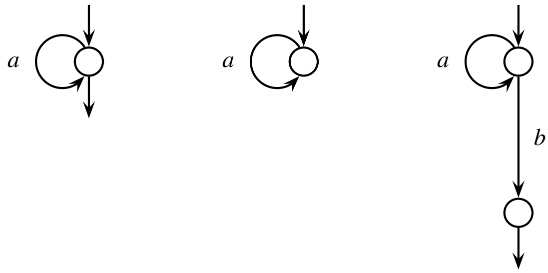

Fig. 4.7. Transition systems of a^*1, a^*0, and $a^*(b.1)$.

Proof Using Theorem 3.2.7 (Congruence theorem), the property follows immediately from the format of the deduction rules given in Tables 4.2, 4.4 and 4.8. □

Theorem 4.6.3 (Soundness) The process theory $\mathrm{BSP}^*(A)$ is a sound axiomatization of the algebra $\mathbb{P}(\mathrm{BSP}^*(A))_{/\leftrightarrow}$.

Proof Exercise 4.6.2. □

In contrast with the extension of the process theory $\mathrm{BSP}(A)$ with projection operators, the extension with prefix iteration actually allows to define more processes. It is not possible to prove an elimination result similar to Theorem 4.5.2 (Elimination). It is possible though to prove a conservativity result, which implies that in the extended theory $\mathrm{BSP}^*(A)$ it is not possible to prove any equalities between closed $\mathrm{BSP}(A)$-terms that cannot be proven in $\mathrm{BSP}(A)$.

Theorem 4.6.4 (Conservative ground-extension) Theory $\mathrm{BSP}^*(A)$ is a conservative ground-extension of process theory $\mathrm{BSP}(A)$.

Proof The result follows immediately from Theorem 3.2.21 (Conservativity) and the facts that $\mathrm{BSP}(A)$ is a ground-complete axiomatization of the term model $\mathbb{P}(\mathrm{BSP}(A))_{/\leftrightarrow}$ (Theorem 4.4.12), that $\mathrm{BSP}^*(A)$ is a sound axiomatization of $\mathbb{P}(\mathrm{BSP}^*(A))_{/\leftrightarrow}$, and that *TDS*($\mathrm{BSP}^*(A)$) is an operational conservative extension of *TDS*($\mathrm{BSP}(A)$) (Table 4.4). This last fact is a direct consequence of the format of the deduction rules in Tables 4.2, 4.4 and 4.8, and Theorem 3.2.19 (Operational conservative extension). □

Figure 4.8 visualizes the conservativity result proven by Theorem 4.6.4. For the sake of completeness, it also shows conservativity results regarding the

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$$
\begin{tikzcd}
\boxed{\mathrm{MPT}(A)} \arrow[r, "\mathrm{PI1,PI2}"] \arrow[d] & \boxed{\mathrm{MPT}^*(A)} \arrow[d] \\
\boxed{\mathrm{BSP}(A)} \arrow[r, "\mathrm{PI1,PI2}"] & \boxed{\mathrm{BSP}^*(A)}
\end{tikzcd}
$$

Fig. 4.8. Conservativity results for $\mathrm{MPT}^*(A)$ and $\mathrm{BSP}^*(A)$.

theory $\mathrm{MPT}^*(A)$, the minimal process theory of Section 4.2 extended with prefix iteration. The details of this theory and the conservativity results do not play a role in the remainder and are therefore omitted.

It still remains to support the claim, made in the introduction to this section, that the extension of BSP(A) with prefix iteration allows us to describe unbounded processes. Consider the extension of theory $\mathrm{BSP}^*(A)$ with projection operators, i.e., the theory $(\mathrm{BSP}^* + \mathrm{PR})(A)$. Details of this extension are straightforward and left to the reader; see Exercise 4.6.3. The extension allows us to introduce a notion of boundedness for process terms, following the idea of Theorem 4.5.4 (Bounded depth).

Definition 4.6.5 (Bounded depth) Let p be a closed $(\mathrm{BSP}^* + \mathrm{PR})(A)$-term. Term p has a bounded depth if and only if there exists a natural number n such that for all $k \geq n$,

$$(\mathrm{BSP}^* + \mathrm{PR})(A) \vdash \pi_k(p) = p.$$

Note that this definition introduces a notion of boundedness for all process theories introduced so far, because theory $(\mathrm{BSP}^* + \mathrm{PR})(A)$ contains all other theories.

Notation 4.6.6 (n-fold action prefix) For any atomic action $a \in A$, any natural number n, and any term x, the term $a^n x$ denotes the n-fold application of the action-prefix operator $a.\_$ to x. An inductive definition is as follows:

$$
\begin{aligned}
& a^0 x = x, \qquad \text{and, for all } n \in \mathbf{N}, \\
& a^{n+1} x = a.a^n x.
\end{aligned}
$$

Example 4.6.7 ((Un-)boundedness) Let a be an action in A. Consider the process term a^*0. It can easily be established that, for all natural numbers $n \in \mathbf{N}$,

$$(\mathrm{BSP}^* + \mathrm{PR})(A) \vdash \pi_n(a^*0) = a^n 0. \tag{4.6.1}$$

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The proof is left as an exercise (see Exercise 4.6.5).

Since all the finite projections of a^*0 are different, by Definition 4.6.5, it cannot have bounded depth. To see this, assume that there is a natural number n such that for all $k \geq n$, $(\mathrm{BSP}^* + \mathrm{PR})(A) \vdash \pi_k(a^*0) = a^*0$. It then follows from property (4.6.1) that, for all $k \geq n$, $(\mathrm{BSP}^* + \mathrm{PR})(A) \vdash a^k0 = a^*0$, which in turn implies that for all $k, l \geq n$ with $k \neq l$, $(\mathrm{BSP}^* + \mathrm{PR})(A) \vdash a^k0 = a^l0$. This contradicts the fact a^k0 and a^l0 correspond to different processes in the term model $\mathbb{P}(\mathrm{BSP}^*(A))_{/\leftrightarrow}$. Hence, a^*0 has unbounded depth.

Note that by Theorem 4.5.4 (Bounded depth) all closed $(\mathrm{BSP} + \mathrm{PR})(A)$-terms are bounded according to Definition 4.6.5. Thus, this example illustrates that the extension of $\mathrm{BSP}(A)$ with prefix iteration is a true extension that allows to describe unbounded-depth processes, as already claimed informally in the introduction to this section.

This section ends with a ground-completeness result.

Theorem 4.6.8 (Ground-completeness) Process theory $\mathrm{BSP}^*(A)$ is a ground-complete axiomatization of the term model $\mathbb{P}(\mathrm{BSP}^*(A))_{/\leftrightarrow}$.

Proof The proof is a straightforward adaptation of the proof given in (Fokkink, 1994) for a theory similar to BSP^* but without empty process. □

As a final remark, it is important to note that prefix iteration can be seen as a simple form of recursion, as process a^*p can be seen as the solution of the recursive equation $X = a.X + p$. The next chapter is fully devoted to recursion.

Exercises

4.6.1 Draw the transition system of the following closed $\mathrm{BSP}^*(A)$-terms:

(a) a^*b^*1,
(b) $a^*(b^*0 + c.1)$, and
(c) $a^*b.c^*0$

(with $a, b, c \in A$).

4.6.2 Prove Theorem 4.6.3 (Soundness).

4.6.3 Develop the theory $(\mathrm{BSP}^* + \mathrm{PR})(A)$. That is, define $(\mathrm{BSP}^* + \mathrm{PR})(A)$ by extending $\mathrm{BSP}^*(A)$ with the projection operators and axioms of Table 4.5, prove elimination and conservativity results, and give a term model proving soundness and ground-completeness.

4.6.4 Define by induction a series of closed BSP(A)-terms $(p_n)_{n\in\mathbf{N}}$ such that, for all $n \in \mathbf{N}$, $(\mathrm{BSP}^* + \mathrm{PR})(A) \vdash \pi_n(a^*1) = p_n$.

4.6.5 Recall Notation 4.6.6 (n-fold action prefix).

(a) Prove that, for all natural numbers $n \in \mathbf{N}$, $(\mathrm{BSP} + \mathrm{PR})(A) \vdash \pi_n(a^{n+1}0) = a^n 0$.

(b) Prove that, for all natural numbers $n \in \mathbf{N}$, $(\mathrm{BSP}^* + \mathrm{PR})(A) \vdash \pi_n(a^*0) = a^n 0$.

4.6.6 Consider the so-called *proper iteration* operators $a^{\oplus}\_$, with a an action in A. The process described by $a^{\oplus}x$, for some term x, is the process that executes a a number of times, but at least once, and then continues as the process described by x. Give deduction rules for the transition relations $\_ \xrightarrow{a} \_$ and the termination predicate $\_\downarrow$ that specify the expected operational semantics of proper iteration. Give also axioms for proper iteration without using prefix iteration. Finally, express proper iteration in terms of action prefix and prefix iteration, and express prefix iteration in terms of proper iteration.
(Hint: difficult.)

4.7 Bibliographical remarks

Minimal Process Theory is the same as basic CCS, see e.g. (Hennessy & Milner, 1980). The inaction constant is denoted 0 or *nil*. The present formulation of MPT is the theory MPA, minimal process algebra, from (Baeten, 2003). It is a subtheory of the theory BPA$_\delta$, Basic Process Algebra with inaction, of (Bergstra & Klop, 1984a). In both of these theories, the inaction constant is denoted δ. The system was called FINTREE in (Aceto *et al.*, 1994). The summand relation of Exercise 4.2.3 can be found in (Bergstra & Klop, 1985).

The material of Section 4.3 is based on (Van Glabbeek, 1987). Section 4.4 again follows (Baeten, 2003). The addition of the empty process 1 in ACP-style process algebra originates in (Koymans & Vrancken, 1985), and was continued in (Vrancken, 1997), (Baeten & Van Glabbeek, 1987). In all of these papers, the empty process is denoted ϵ. In CSP, the empty process is written SKIP; see (Hoare, 1985).

The theory BPA (Basic Process Algebra) of Bergstra and Klop (Bergstra & Klop, 1982) (later published as (Bergstra & Klop, 1992)), (Bergstra & Klop, 1984a; Bergstra & Klop, 1985) is related to BSP as follows: BPA has constants a for each atomic action a, which in BSP would be represented as $a.1$; furthermore, BPA has general sequential composition, introduced in the framework of this book in Chapter 6.

The operational semantics of BSP has an action relation and a termination predicate; there is no need for a mixed relation $\_ \xrightarrow{a} \surd$ as in (Baeten & Weijland, 1990) or (Baeten & Verhoef, 1995).

The material on presence and absence of deadlocks is based on (Baeten & Bergstra, 1988). Projection operators appear in many process algebras, see e.g. (De Bakker & Zucker, 1982b; Bergstra & Klop, 1982). Conservativity of the (ground-)extension is addressed e.g. in (Baeten & Verhoef, 1995).

Section 4.6 discusses an iteration operator. The origin of such an operator, and the star notation, dates back to (Kleene, 1956). The article (Copi *et al.*, 1958) introduced a unary variant of Kleene's operator, and later, (Bergstra *et al.*, 1994) went back to the binary variant (in the absence of the empty process 1). With the constants 0 and 1, there is no finite ground-complete axiomatization for iteration modulo bisimulation (Sewell, 1997), which led to the consideration of restricted iteration operators like the prefix iteration introduced in this chapter, see (Fokkink, 1994) and (Aceto *et al.*, 1998). A general (unary) iteration operator in the context of this book is discussed in Section 6.5. An overview of iteration concepts in a process-algebraic context is given in (Bergstra *et al.*, 2001).