Process Algebra (2IMF10)
Replication

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The goal is to present a solution to Exercise 7.6.9 in [1]. Since we can conveniently use solutions to Exercise 7.4.8(a) and Exercises 7.4.9(a),(c) and (d) we will present those too.

Solution to Exercise 7.4.8(a)

The following derivation proves that $(\mathrm{BCP} + \mathrm{FMA})(A, \emptyset) \vdash (x \mid y) \mathbin{⌊\!⌊} z = 0$:

$$\begin{aligned}
(x \mid y) \mathbin{⌊\!⌊} z &= (x \mid y) \mathbin{⌊\!⌊} z + 0 && \text{(by A6)}\\
&= (x \mid y) \mathbin{⌊\!⌊} z + 1 \mathbin{⌊\!⌊} z && \text{(by LM2)}\\
&= (x \mid y + 1) \mathbin{⌊\!⌊} z && \text{(by LM4)}\\
&= 1 \mathbin{⌊\!⌊} z && \text{(by FMA)}\\
&= 0 && \text{(by LM2)} \ .
\end{aligned}$$

Solution to Exercises 7.4.9(a), (c), and (d)

(a) We prove that for all closed $(\mathrm{BCP} + \mathrm{FMA})(A, \emptyset)$-terms p and q and for every action a we have $(\mathrm{BCP} + \mathrm{FMA})(A, \emptyset) \vdash a.p \mid q = 0$. By the Elimination Theorem for $\mathrm{BCP}(A, \gamma)$ (Theorem 7.4.3 in [1]) we may assume without loss of generality that q is a closed $\mathrm{BSP}(A)$-term; we proceed by induction on the structure of q.

- If $q \equiv 0$, then $a.p \mid q \equiv a.p \mid 0 = 0 \mid a.p = 0$ by SC1 and CM1.
- If $q \equiv 1$, then $a.p \mid q \equiv a.p \mid 1 = 0$ by CM4.
- If $q \equiv b.q'$, then $a.p \mid q \equiv a.p \mid b.q' = 0$ by CM6 (use that $\gamma = \emptyset$!).
- Let $q \equiv q_1 + q_2$ and suppose that

$$\begin{aligned}
&(\mathrm{BCP} + \mathrm{FMA})(A, \emptyset) \vdash a.p \mid q_1 = 0 \ , \text{ and}\\
&(\mathrm{BCP} + \mathrm{FMA})(A, \emptyset) \vdash a.p \mid q_2 = 0 \ .
\end{aligned} \tag{IH}$$

  Then $a.p \mid q \equiv a.p \mid (q_1 + q_2) = (q_1 + q_2) \mid a.p = q_1 \mid a.p + q_2 \mid a.p = a.p \mid q_1 + a.p \mid q_2 = 0 + 0 = 0$ by SC1, CM2, SC1, IH and A3.

(c) We prove that $(\mathrm{BCP} + \mathrm{FMA})(A, \emptyset) \vdash p + p \mid q = p$ for all closed $(\mathrm{BCP} + \mathrm{FMA})(A, \emptyset)$-terms p and q. By the Elimination Theorem for $\mathrm{BCP}(A, \gamma)$ (Theorem 7.4.3 in [1]) we may assume without loss of generality that p is a closed $\mathrm{BSP}(A)$-term; we proceed by induction on the structure of p.

- If $p \equiv 0$, then $p + p \mid q \equiv p + 0 \mid q = p + 0 = p$ by CM1 and A6.
- If $p \equiv 1$, then $p + p \mid q \equiv 1 + p \mid q = p \mid q + 1 = 1$ by A1 and FMA.

- If $p \equiv a.p'$, then $p + p \mid q \equiv p + a.p' \mid q = p + 0 = p$ by (a) and A6.
- Let $p \equiv p_1 + p_2$ and suppose that

$$\begin{array}{l} (\mathrm{BCP} + \mathrm{FMA})(A, \emptyset) \vdash p_1 + p_1 \mid q = p_1 \ \text{, and} \\ (\mathrm{BCP} + \mathrm{FMA})(A, \emptyset) \vdash p_2 + p_2 \mid q = p_2 \ . \end{array} \tag{IH}$$

  Then $p + p \mid q \equiv p + (p_1 + p_2) \mid q \equiv p + (p_1 + p_2) \mid q = p + p_1 \mid q + p_2 \mid q = p_1 + p_1 \mid q + p_2 + p_2 \mid q = p_1 + p_2 \equiv p$ by CM2, A1, A2 and IH.

(d) We prove that $(\mathrm{BCP} + \mathrm{FMA})(A, \emptyset) \vdash p \mid p \mid q = p$ for all closed $(\mathrm{BCP} + \mathrm{FMA})(A, \emptyset)$-terms p and q. By the Elimination Theorem for $\mathrm{BCP}(A, \gamma)$ (Theorem 7.4.3 in [1]) we may assume without loss of generality that p is a closed $\mathrm{BSP}(A)$-term; we proceed by induction on the structure of p.

- If $p \equiv 0$, then $p \mid p \mid q \equiv 0 \mid 0 \mid q = 0 \mid q \equiv p \mid q$ by CM1.
- If $p \equiv 1$, then $p \mid p \mid q \equiv 1 \mid 1 \mid q = 1 \mid q \equiv p \mid q$ by CM3.
- If $p \equiv a.p'$, then $p \mid p \mid q \equiv a.p' \mid a.p' \mid q = 0 = a.p' \mid q \equiv p \mid q$ by (a).
- Let $p \equiv p_1 + p_2$ and suppose that

$$\begin{array}{l} (\mathrm{BCP} + \mathrm{FMA})(A, \emptyset) \vdash p_1 \mid p_1 \mid q = p_1 \mid q \ \text{, and} \\ (\mathrm{BCP} + \mathrm{FMA})(A, \emptyset) \vdash p_2 \mid p_2 \mid q = p_2 \mid q \ . \end{array} \tag{IH}$$

  Then

$$\begin{aligned} p \mid p \mid q &= p_1 \mid p_1 \mid q + p_1 \mid p_2 \mid q + p_2 \mid p_1 \mid q + p_2 \mid p_2 \mid q && \text{(by SC1 and CM2)} \\ &= p_1 \mid q + p_1 \mid p_2 \mid q + p_2 \mid p_1 \mid q + p_2 \mid q && \text{(by IH)} \\ &= (p_1 + p_1 \mid p_2) \mid q + (p_2 + p_2 \mid p_1) \mid q && \text{(by CM2)} \\ &= p_1 \mid q + p_2 \mid q && \text{(by (c))} \\ &= (p_1 + p_2) \mid q && \text{(by CM2)} \\ &\equiv p \mid q \end{aligned}$$

Solution to Exercise 7.6.9

Replication $!\,\_$ is a well-known unary operation that is included in some process calculi to express infinite behaviour. You may think of $!\,p$ as denoting the process that spawns an unbounded number of parallel copies of p. The exercise is about finitely defining, for an arbitrary closed $\mathrm{BSP}(A)$-term p, the process $!\,p$ in $\mathrm{BCP}(A, \emptyset)$.

The process $!\,p$ should satisfy

$$!\,p = p \parallel !\,p \ . \tag{1}$$

So, you might think that we could 'define' $!\,p$ as the solution of the equation

$$X = p \parallel X \ . \tag{2}$$

The problem is, however, that this recursive specification is not guarded and does not have a *unique* solution. (To see this, note that if q denotes a solution for Equation (2), then q satisfies

$$q = p \parallel q$$

and hence we have, for every r, that

$$q \parallel r = (p \parallel q) \parallel r = p \parallel (q \parallel r)$$

showing that then also $q \parallel r$ is a solution for Equation (1). Since r can be any process, infinitely many solutions can be constructed by suitably choosing r.)

The exercise proposes to define $!\,p$, instead, as the unique solution of the guarded recursive specification

$$E = \{X = p \mathbin{⌊\!⌊} X + p \mid 1\} \ . \tag{3}$$

To see that this recursive specification is indeed guarded, note that, by the elimnation theorem for BCP (Theorem 7.4.3 in [1]), for every closed BCP(A, γ)-term p there exist actions a_i and closed BSP(A)-terms p_i such that (modulo the axioms of BSP(A))

$$p = \sum_{i \in I} a_i.p_i \ [+1]$$

(the [+1] indicates that p may have an optional summand 1). Hence, by the axioms of BCP$(A, \emptyset)$, we have

$$p \mathbin{⌊\!⌊} X + p \mid 1 = \sum_{i \in I} a_i.(p_i \parallel X) + p \mid 1$$

(verify this yourself!). It follows that E can be rewritten to the completely guarded recursive specification

$$E' = \{X = \sum_{i \in I} a_i.(p_i \parallel X) + p \mid 1\} \ .$$

So, according to RSP the guarded recursive specification E has *at most* one solution. Let us now denote the unique solution of E (in any model that satisfies both RDP and RSP, e.g., the term model $\mathbb{P}(\mathrm{BCP}(A, \emptyset))/\underline{\leftrightarrow}$) by $!\,p$. It remains to prove that $!\,p$ indeed satisfies Equation (1). For this we shall also use RSP.

On the one hand, we have:

$$\begin{aligned}
p \parallel !\,p &= p \mathbin{⌊\!⌊} !\,p + !\,p \mathbin{⌊\!⌊} p + p \mid !\,p && \text{(by M)}\\
&= p \mathbin{⌊\!⌊} !\,p + (p \mathbin{⌊\!⌊} !\,p + p \mid 1) \mathbin{⌊\!⌊} p + p \mid !\,p && \text{(by definition of } !\,p)\\
&= p \mathbin{⌊\!⌊} !\,p + p \mathbin{⌊\!⌊} (!\,p \parallel p) + (p \mid 1) \mathbin{⌊\!⌊} p + p \mid !\,p && \text{(by LM4, SC6)}\\
&= p \mathbin{⌊\!⌊} !\,p + p \mathbin{⌊\!⌊} (!\,p \parallel p) + p \mid !\,p && \text{(by Ex. 7.4.8(a) and A6)}\\
&= p \mathbin{⌊\!⌊} !\,p + p \mathbin{⌊\!⌊} (!\,p \parallel p) + p \mid (p \mathbin{⌊\!⌊} !\,p + p \mid 1) && \text{(by definition of } !\,p)\\
&= p \mathbin{⌊\!⌊} !\,p + p \mathbin{⌊\!⌊} (!\,p \parallel p) + p \mid (p \mathbin{⌊\!⌊} !\,p) + p \mid (p \mid 1) && \text{(by SC1, CM2)}\\
&= p \mathbin{⌊\!⌊} !\,p + p \mathbin{⌊\!⌊} (!\,p \parallel p) + (p \mid p) \mathbin{⌊\!⌊} !\,p + p \mid (p \mid 1) && \text{(by SC7)}\\
&= p \mathbin{⌊\!⌊} !\,p + p \mathbin{⌊\!⌊} (!\,p \parallel p) + p \mid (p \mid 1) && \text{(by Ex. 7.4.8(a) and A6)}\\
&= p \mathbin{⌊\!⌊} !\,p + p \mathbin{⌊\!⌊} (!\,p \parallel p) + p \mid 1 && \text{(by Ex. 7.4.9(d))}\\
&= p \mathbin{⌊\!⌊} (!\,p \parallel p) + p \mathbin{⌊\!⌊} !\,p + p \mid 1 && \text{(by A1 and A2)}\\
&= p \mathbin{⌊\!⌊} (!\,p \parallel p) + !\,p && \text{(by definition of } !\,p) \ .
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
!\,p &= p \mathbin{⌊\!⌊} !\,p + p \mid 1 && \text{(by definition of } !\,p)\\
&= p \mathbin{⌊\!⌊} !\,p + (p \mathbin{⌊\!⌊} !\,p + p \mid 1) && \text{(by A3)}\\
&= p \mathbin{⌊\!⌊} !\,p + !\,p && \text{(by definition of } !\,p) \ .
\end{aligned}$$

Hence, we may conclude that both $p \parallel !\,p$, $!\,p$ and $!\,p, !\,p$ are solutions for the guarded recursive specification

$$F = \left\{ \begin{array}{l} Y = p \mathbin{⌊\!⌊} Y + X, \\ X = p \mathbin{⌊\!⌊} X + p \mid 1 \end{array} \right\} \ .$$

Since F is clearly guarded, by RSP these two solutions must be equal, and hence it follows that $!\,p = p \parallel !\,p$.

References

[1] J. C. M. Baeten, T. Basten, and M. A. Reniers. *Process Algebra (Equational Theories of Communicating Processes)*. Cambridge University Press, 2010.