Process Algebra (2IMF10)
Expansion

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We prove a useful Expansion Lemma, which differs from the Expansion Law presented in [1].

Lemma (Expansion)**.** *Let* $t \equiv \sum_{i \in I} a_i.x_i$ $[+\ 1]$, *let* $u \equiv \sum_{j \in J} b_j.y_j$ $[+\ 1]$, *and let*

$$\Gamma = \{(i,j) \in I \times J \mid \gamma(a_i, b_j) \textit{ is defined}\} \enspace .$$

Then

$$\mathrm{BCP}(A,\gamma) \vdash t \parallel u = \sum_{i \in I} a_i.(x_i \parallel u) + \sum_{j \in J} b_j.(t \parallel y_j) + \sum_{(i,j) \in \Gamma} \gamma(a_i, b_j).(x_i \parallel y_j) \quad [+\ 1] \enspace ,$$

where $t \parallel u$ *has the* 1*-summand if, and only if,* t *and* u *both have it.*

Proof. First, note that

$$t \parallel u = (t \mathbin{⫽} u + u \mathbin{⫽} t) + t \mid u \qquad \text{(by M)} \tag{1}$$

Next, note that

$$\begin{aligned}
t \mathbin{⫽} u &\equiv \left(\sum_{i \in I} a_i.x_i\ [+1]\right) \mathbin{⫽} u \\
&= \sum_{i \in I} (a_i.x_i \mathbin{⫽} u) \ \ [+\ 1 \mathbin{⫽} u] && \text{(by LM4)} \\
&= \sum_{i \in I} a_i.\,(x_i \parallel u) \ \ [+\ 1 \mathbin{⫽} u] && \text{(by LM3)} \\
&= \sum_{i \in I} a_i.\,(x_i \parallel u) \ \ [+\ 0] && \text{(by LM1)} \\
&= \sum_{i \in I} a_i.\,(x_i \parallel u) && \text{(by A6)}
\end{aligned}$$

Analogously, we can derive that

$$u \mathbin{⫽} t = \sum_{j \in J} b_j.\,(y_j \parallel t) \enspace ,$$

and hence, using that $\mathrm{BCP}(A,\gamma) \vdash x \parallel y = y \parallel x$ (see Exercise 7.4.3),

$$u \mathbin{⫽} t = \sum_{j \in J} b_j.\,(t \parallel y_j) \enspace ,$$

It remains to derive that

$$t \mid u = \sum_{(i,j)\in\Gamma} \gamma(a_i, b_j).(x_i \parallel y_j) \quad [+\ 1] \;;$$

we proceed as follows:

$$\begin{aligned}
t \mid u &\equiv \left(\sum_{i\in I} a_i.x_i \ [+1]\right) \mid u \\
&= \sum_{i\in I} (a_i.x_i \mid u) \ \ [+\ 1 \mid u] && \text{(by CM2)} \\
&= \sum_{i\in I} (u \mid a_i.x_i) \ \ [+\ u \mid 1] && \text{(by SC1)} \\
&\equiv \sum_{i\in I} \left(\left(\sum_{j\in J} b_j.y_j \ [+\ 1]\right) \mid a_i.x_i\right) \ \ \left[+ \ \left(\sum_{j\in J} b_j.y_j \ [+\ 1]\right) \mid 1\right] \\
&= \sum_{i\in I}\sum_{j\in J} (b_j.y_j \mid a_i.x_i \ [+\ 1 \mid a_i.x_i]) \ \ \left[+ \ \sum_{j\in J} b_j.y_j \mid 1 \ [+\ 1 \mid 1]\right] && \text{(by CM2)} \\
&= \sum_{i\in I}\sum_{j\in J} (a_i.x_i \mid b_j.y_j \ [+\ a_i.x_i \mid 1]) \ \ \left[+ \ \sum_{j\in J} b_j.y_j \mid 1 \ [+\ 1 \mid 1]\right] && \text{(by SC1)} \\
&= \sum_{i\in I}\sum_{j\in J} (a_i.x_i \mid b_j.y_j \ [+\ 0]) \ \ \left[+ \ \sum_{j\in J} 0 \ [+\ 1 \mid 1]\right] && \text{(by CM4)} \\
&= \sum_{i\in I}\sum_{j\in J} (a_i.x_i \mid b_j.y_j) \ \ [+\ 1 \mid 1] && \text{(by A6)} \\
&= \sum_{i\in I}\sum_{j\in J} (a_i.x_i \mid b_j.y_j) \ \ [+\ 1] && \text{(by CM3)} \\
&= \sum_{(i,j)\in\Gamma} (a_i.x_i \mid b_j.y_j) + \sum_{(i,j)\notin\Gamma} (a_i.x_i \mid b_j.y_j) \ \ [+\ 1] && \text{(by A1, A2)} \\
&= \sum_{(i,j)\in\Gamma} \gamma(a_i, b_j).(x_i \parallel y_j) + \sum_{(i,j)\notin\Gamma} 0 \ \ [+\ 1] && \text{(by CM5, CM6)} \\
&= \sum_{(i,j)\in\Gamma} \gamma(a_i, b_j).(x_i \parallel y_j) \ \ [+\ 1] && \text{(by A6)}
\end{aligned}$$

Note that the optional 1-summand stems from $1 \mid 1$, and so it is only present if both t and u have the optional 1-summand. □

References

[1] J. C. M. Baeten, T. Basten, and M. A. Reniers. *Process Algebra (Equational Theories of Communicating Processes)*. Cambridge University Press, 2010.