Math for Political Scientists Workshop

Day Four: Linear Algebra II

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Review Time

Linear (Matrix) Algebra:

- Scalar
- Vector
- Matrix
- Square Matrix
- Identity Matrix
- Matrix Addition (Subtraction)
- A scalar $\times$ a matrix
- A matrix $\times$ a matrix
- Matrix transposition
- Matrix inversion

Linear Algebra II: Determinants

An $n \times n$ square matrix A is called invertible if there exists an $n \times n$ matrix B such that $AB = BA = I_n$.

- Only a square matrix has an inverse.

However, not any square matrix has an inverse.

- If an matrix does not have an inverse, we say it is noninvertible.
- A square matrix that is not invertible is called a singular matrix.

Find Matrix Inverse in R

Compute and verify the inverse of a matrix in R in three steps

1. Create a 3x3 matrix

```r
# Example: Create a 3x3 matrix
A <- matrix(c(4, 7, 2,
              3, 6, 1,
              2, 5, 3), nrow=3, byrow=TRUE)

# Print the matrix
print(A)
```

```
##      [,1] [,2] [,3]
## [1,]    4    7    2
## [2,]    3    6    1
## [3,]    2    5    3
```

Find Matrix Inverse in R

2. Compute the inverse of the matrix in R

```r
# Find the inverse of the matrix
A_inverse <- solve(A)

# Print the inverse
print(A_inverse)
```

```
##            [,1]       [,2]       [,3]
## [1,]  1.4444444 -1.2222222 -0.5555556
## [2,] -0.7777778  0.8888889  0.2222222
## [3,]  0.3333333 -0.6666667  0.3333333
```

Find Matrix Inverse in R

3. Verify the Result

```r
# Verify by multiplying the original matrix by its inverse
identity_matrix1 <- round(A %*% A_inverse, digits=0)
# Verify by multiplying the inverse by the original matrix
identity_matrix2 <- round(A_inverse %*% A, digits=0)

# Print the identity matrices
print(identity_matrix1)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

```r
print(identity_matrix2)
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

Find Matrix Inverse in R

Example: Create a 2x2 singular matrix

```
# Example: Create a 2x2 singular matrix
B <- matrix(c(2, 4,
              1, 2), nrow=2, byrow=TRUE)

# Print the matrix
print(B)
```

```
##      [,1] [,2]
## [1,]    2    4
## [2,]    1    2
```

```
# Print the inverse
verif_inverse <- try({solve(B)})
```

```
## Error in solve.default(B) :
##   Lapack routine dgesv: system is exactly singular: U[2,2] = 0
```

```
verif_inverse
```

```
## [1] "Error in solve.default(B) : \n  Lapack routine dgesv: system is exa
## attr(,"class")
```

Linear Dependence and Singular Matrix (cont.)

If any two (or more) rows/columns of a square matrix are linearly dependent, the matrix is singular, meaning it does not have an inverse.

- Rows (or columns) of a matrix are said to be linearly dependent if one row (or one column) can be written as a linear combination of the others.
- For example, one row/column is a multiple of another or the sum of multiples of other rows/columns.

Linear Dependence and Singular Matrix

Linear combination/dependence by row

```r
# Example: Create a 4x4 singular matrix
C <- matrix(c(1, 2, 3, 4,
              2, 4, 6, 8,
              5, 7, 9, 10,
              1, 1, 1, 1), nrow=4, byrow=TRUE)

# Print the matrix
print(C)
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    2    3    4
## [2,]    2    4    6    8
## [3,]    5    7    9   10
## [4,]    1    1    1    1
```

```r
# Print the inverse
verif_inverse <- try({solve(C)})
```

```
## Error in solve.default(C) :
##   Lapack routine dgesv: system is exactly singular: U[4,4] = 0
```

Linear Dependence and Singular Matrix

Linear combination/dependence by column

```r
# Example: Create a 4x4 singular matrix
C <- matrix(c(1, 2, 3, 4,
              2, 4, 6, 8,
              5, 7, 9, 10,
              1, 1, 1, 1), nrow=4, byrow=FALSE)

# Print the matrix
print(C)
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    2    5    1
## [2,]    2    4    7    1
## [3,]    3    6    9    1
## [4,]    4    8   10    1
```

```r
# Print the inverse
verif_inverse <- try({solve(C)})
```

```
## Error in solve.default(C) :
##   Lapack routine dgesv: system is exactly singular: U[2,2] = 0
```

Determinants

Since not all square matrices are nonsingular (invertible), we need a test to *determine* whether a given matrix is nonsingular or not.

- We define a number called the determinant,with the property that the square matrix is nonsingular if and only if its determinant is not zero.

Why we learn determinant?

The determinant of a matrix is a single numerical value which is used:

- when calculating the inverse
- when solving systems of linear equations.

Defining the Determinant (1)

For a 1×1 matrix, it is just a scalar,a.

- Since the inverse of a, $\frac{1}{a}$ exists if and only if a is nonzero.
- We define the determinant of such matrix to be det(a) or $|a|$ =a.

For a 2×2 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- A is nonsingular if and only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

We define the determinant of a 2×2 matrix:

For a 2×2 matrix,

$$det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

det(A) is the product of the two diagonal entries minus the product of the two off-diagonal entries.

Defining the Determinant (2)

Since for a 1×1 matrix, det(a)=a.

Then, for a 2×2 matrix, we can rewrite it like:

$$det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}det(a_{22}) - a_{12}det(a_{21})$$

Defining the Determinant (3)

To the same token,

the determinant of a 3×3 matrix can be calculated by breaking it down into smaller 2×2 matrices, as follows:

$$det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Formal Definition of the Determinant

For a general $n \times n$ matrix A:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The determinant of A, denoted as $\det(A)$, is defined recursively by the formula:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \cdot \det(A_{1j})$$

Where:

- a_{1j} are the elements of the first row of the matrix.
- A_{1j} is the $(n-1) \times (n-1)$ matrix obtained by removing the first row and the j-th column of A, named as cofactor matrices.
- The term $(-1)^{1+j}$ is known as the **cofactor sign** and accounts for alternating positive and negative signs. Odd (even) columns go with positive (negative) signs.

Theorem

The determinant of the transpose of a square matrix is equal to the determinant of the matrix, that is, $det(A) = det(A^T)$

Example 1

Find the determinant of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 3 & 1 & 4 \end{pmatrix}$.

$$|\mathbf{A}| = 1\begin{vmatrix} 3 & 4 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 3 \\ 3 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 1(3 \times 4 - 4 \times 1) - 2(0 \times 4 - 4 \times 3) + 1(0 \times 1 - 3 \times 3) \\ &= 1(12 - 4) - 2(0 - 12) + 1(0 - 9) \\ &= 8 + 24 - 9 \\ &= 23 \end{aligned}$$

Example: finding the determinant in R

Verify in R:

```r
A <- matrix(c(1, 2, 1,
              0, 3, 4,
              3, 1, 4), nrow=3, byrow=TRUE) # by row

# Calculate the determinant
det_A <- det(A)
cat("Determinant of A:\n")
```

```
## Determinant of A:
```

```r
print(det_A)
```

```
## [1] 23
```

Example: finding the determinant in R

Verify $det(A) = det(A^T)$in R:

```
At <- matrix(c(1, 2, 1,
               0, 3, 4,
               3, 1, 4), nrow=3, byrow=FALSE) # by column

# Calculate the determinant
det_At <- det(At)
cat("Determinant of At:\n")
```

```
## Determinant of At:
```

```
print(det_At)
```

```
## [1] 23
```

Example 2

Find the determinant of the matrix $\begin{pmatrix} 1 & 0 & 3 \\ -1 & -1 & -3 \\ 0 & 0 & 6 \end{pmatrix}$.

Example: finding the determinant

Practice Solution

Example 2

Find the determinant of the matrix $\begin{pmatrix} 1 & 0 & 3 \\ -1 & -1 & -3 \\ 0 & 0 & 6 \end{pmatrix}$.

$$|\mathbf{A}| = 1\begin{vmatrix} -1 & -3 \\ 0 & 6 \end{vmatrix} - 0\begin{vmatrix} -1 & -3 \\ 0 & 6 \end{vmatrix} + 3\begin{vmatrix} -1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$\begin{aligned}
&= 1\big((-1)\times 6 - (-3)\times 0\big) - 0(\dots) + 3\big((-1)\times 0 - (-1)\times 0\big) \\
&= 1\big((-6) - 0\big) - 0 + 3(0 - 0) \\
&= -6 + 0 \\
&= -6
\end{aligned}$$

Use Determinants and Adjoint Matrix to Find Matrix Inverse (by hand).

(This is not required. Just FYI!)

Definition

For any $n \times n$ matrix A, let C_{ij} denote the (i, j)th cofactor of A, that is, $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting row i and column j from A. The $n \times n$ matrix whose (i, j)th entry is C_{ji}, the (j, i)th cofactor of A (note the switch in indices), is called the **adjoint/adjugate matrix** of A and is written adj A.[1]

Theorem

Let A be a nonsingular matrix. Then,

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj } A,$$

[1] The adjoint matrix is the transpose of the matrix of the cofactors

Example (FYI)

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{8}$$

$$\begin{aligned}
C_{11} &= +\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3, & C_{12} &= -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0, & C_{13} &= +\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -3, \\
C_{21} &= -\begin{vmatrix} 4 & 5 \\ 0 & 1 \end{vmatrix} = -4, & C_{22} &= +\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = -3, & C_{23} &= -\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 4, \\
C_{31} &= +\begin{vmatrix} 4 & 5 \\ 3 & 0 \end{vmatrix} = -15, & C_{32} &= -\begin{vmatrix} 2 & 5 \\ 0 & 0 \end{vmatrix} = 0, & C_{33} &= +\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} = 6,
\end{aligned}$$

$$\det A = -9,$$

$$\operatorname{adj} A = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 3 & -4 & -15 \\ 0 & -3 & 0 \\ -3 & 4 & 6 \end{pmatrix}$$

So,

$$A^{-1} = -\frac{1}{9}\begin{pmatrix} 3 & -4 & -15 \\ 0 & -3 & 0 \\ -3 & 4 & 6 \end{pmatrix}. \tag{9}$$

Linear Algebra II: Eigenvalue and Eigenvector

Review: Matrix Multiplication

Let

$$A = \begin{pmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{pmatrix}$$

Compute the product AX for

$$X = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}, \quad X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

What do you notice about AX in each of these products?

Review: Matrix Multiplication

The first AX product is given by

$$AX = \begin{pmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{pmatrix} \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -50 \\ -40 \\ 30 \end{pmatrix} = 10 \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}$$

In this case, the product AX resulted in a vector which is equal to 10 times the vector X. In other words, $AX = 10X$.

The second product is given by

$$AX = \begin{pmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

In this case, the product AX resulted in a vector equal to 0 times the vector X, $AX = 0X$.

Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix and let $X \in \mathbb{C}^n$ be a **nonzero vector** for which

$$AX = \lambda X$$

for some scalar λ. Then λ is called an **eigenvalue** of the matrix A and X is called an **eigenvector** of A associated with λ, or a λ-eigenvector of A.

How to understand eigenvalues and eigenvectors

Eigenvalues: An eigenvalue of a square matrix is a scalar value that, when multiplied by a corresponding eigenvector, results in a new vector that points in the same direction as the original eigenvector. In other words, *the eigenvector only changes in magnitude but not in direction*.

Eigenvectors: An eigenvector is a non-zero vector that, when multiplied by a square matrix, results in a scalar multiple of itself. In other words, *the direction of the eigenvector remains the same even after the matrix transformation*.

Their Applications

PLSC 504 and PLSC 597 Machine Learning

- Principal Component Analysis (PCA): PCA is a technique used for dimensionality reduction in data analysis. It involves finding the eigenvalues and eigenvectors of the covariance matrix of a dataset. The eigenvectors with the highest eigenvalues represent the principal components of the data, which capture the most significant variations.

PLSC 597 Machine Learning

- Image and Signal Processing: Eigenvalues and eigenvectors are utilized in various image and signal processing techniques. For example, in image compression, eigenvalues are used to determine the most important image features for reconstruction. Eigenvectors are also employed in applications like image denoising, face recognition, and speech processing.

Find Eigenvalue and Eigenvectors in R

```r
# Create a matrix
A <- matrix(c(-5, 2,
              -7, 4), nrow=2, byrow=TRUE)

# Compute eigenvalues and eigenvectors
eigen_result <- eigen(A)
```

Print Eigenvalue and Eigenvectors in R

```r
# Output eigenvalues
print(eigen_result$values)
```

```
## [1] -3  2
```

```r
# Output eigenvectors
print(matrix(eigen_result$vectors[,1], ncol=1))
```

```
##            [,1]
## [1,] -0.7071068
## [2,] -0.7071068
```

```r
print(matrix(eigen_result$vectors[,2], ncol=1))
```

```
##            [,1]
## [1,] -0.2747211
## [2,] -0.9615239
```

Verify Eigenvalue and Eigenvectors in R

To verify your work, make sure that $AX = \lambda X$ for each λ and associated eigenvector X.

```
cat("AX:\n")
```

```
## AX:
```

```
print(A%*%eigen_result$vectors[,1])
```

```
##          [,1]
## [1,] 2.12132
## [2,] 2.12132
```

```
cat("lambda*X:\n")
```

```
## lambda*X:
```

```
print(matrix(-3*eigen_result$vectors[,1]), ncol=1)
```

```
##          [,1]
## [1,] 2.12132
## [2,] 2.12132
```

Verify Eigenvalue and Eigenvectors in R

To verify your work, make sure that $AX = \lambda X$ for each λ and associated eigenvector X.

```
cat("AX:\n")
```

```
## AX:
```

```
print(A%*%eigen_result$vectors[,2])
```

```
##            [,1]
## [1,] -0.5494423
## [2,] -1.9230479
```

```
cat("lambda*X:\n")
```

```
## lambda*X:
```

```
print(matrix(2*eigen_result$vectors[,2]),ncol=1)
```

```
##            [,1]
## [1,] -0.5494423
## [2,] -1.9230479
```

Find Eigenvalues and Eigenvectors by Hand (FYI)

By the definition:

$$AX - \lambda X = 0$$
$$(A - \lambda)X = 0$$
$$(A - \lambda I)X = 0$$

Suppose the matrix $(A - \lambda I)$ is invertible, so that $(A - \lambda I)^{-1}$ exists. Then the following equation would be true.

$$\begin{aligned} X &= IX \\ &= (A - \lambda I)^{-1}(A - \lambda I)X \\ &= (A - \lambda I)^{-1}((A - \lambda I)X) \\ &= (A - \lambda I)^{-1} \times 0 \\ &= 0 \end{aligned}$$

However, by definition, we know X is a non-zero vector, so that $(A - \lambda I)$ does not have an inverse. Therefore, $det(A - \lambda I) = 0$.

Wrap-up Time

- Singular Matrix
- Linear Dependence
- Determinant
- Eigenvalue & Eigenvector

We are done with linear algebra!