Exponents and Logarithms

In the last three chapters, we dealt exclusively with relationships expressed by polynomial functions or by quotients of polynomial functions. However, in many economics models, the function which naturally models the growth of a given economic or financial variable over time has the independent variable t appearing as an *exponent*; for example, $f(t) = 2^t$. These exponential functions occur naturally, for example, as models for the amount of money in an interest-paying savings account or for the amount of debt in a fixed-rate mortgage account after t years.

This chapter focuses on exponential functions and their derivatives. It also describes the inverse of the exponential function — the logarithm, which can turn multiplicative relationships between economic variables into additive relationships that are easier to work with. This chapter closes with applications of exponentials and logarithms to problems of present value, annuities, and optimal holding time.

5.1 EXPONENTIAL FUNCTIONS

When first studying calculus, one works with a rather limited collection of functional forms: polynomials and rational functions and their generalizations to fractional and negative exponents — all functions constructed by applying the usual arithmetic operations to the monomials ax^k. We now enlarge the class of functions under study by including those functions in which the variable x appears as an *exponent*. These functions are naturally called **exponential functions**.

A simple example is $f(x) = 2^x$, a function whose domain is all the real numbers. Recall that:

(1) if x is a positive integer, 2^x means "multiply 2 by itself x times";
(2) if $x = 0$, $2^0 = 1$, by definition;
(3) if $x = 1/n$, $2^{1/n} = \sqrt[n]{2}$, the nth root of 2;
(4) if $x = m/n$, $2^{m/n} = (\sqrt[n]{2})^m$, the mth power of the nth root of 2; and
(5) if x is a negative number, 2^x means $1/2^{|x|}$, the reciprocal of $2^{|x|}$

In these cases, the number 2 is called the **base** of the exponential function.

To understand this exponential function better, let's draw its graph. Since we do not know how to take the derivative of 2^x yet — $(2^x)'$ is certainly not $x2^{x-1}$ —

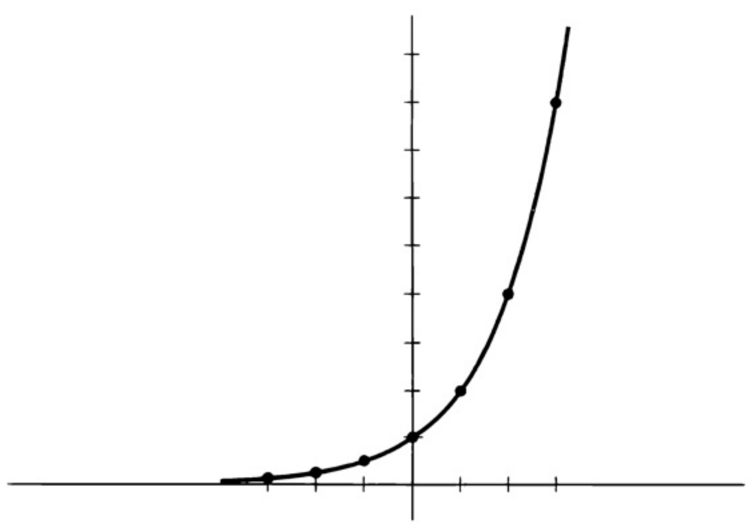
x	2^x
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

we will have to plot points. We compute values of 2^x in Table 5.1 and draw the corresponding graph in Figure 5.1.

Note that the graph has the negative x-axis as a horizontal asymptote, but unlike any rational function, the graph approaches this asymptote in only one direction. In the other direction, the graph increases very steeply. In fact, it increases more rapidly than *any* polynomial — "exponentially fast."

In Figure 5.2, the graphs of $f_1(x) = 2^x$, $f_2(x) = 3^x$, and $f_3(x) = 10^x$ are sketched. Note that the graphs are rather similar; the larger the base, the more quickly the graph becomes asymptotic to the x-axis in one direction and steep in the other direction.

The three bases in Figure 5.2 are greater than 1. The graph of $y = b^x$ is a bit different if the base b lies between 0 and 1. Consider $h(x) = (1/2)^x$ as an example. Table 5.2 presents a list of values of (x, y) in the graph of h for small integers x. Note that the entries in the y-column of Table 5.2 are the same as the entries in the y-column of Table 5.1, but in reverse order, because $(1/2)^x = 2^{-x}$. This means that the graph of $h(x) = (1/2)^x$ is simply the reflection of the graph of $f(x) = 2^x$ in the y-axis, as pictured in Figure 5.3. The graphs of $(1/3)^x$ and $(1/10)^x$ look similar to that of $(1/2)^x$.



The graph of $y = 2^x$.

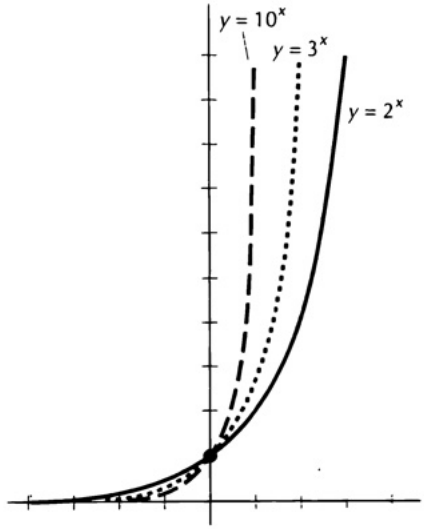


Figure 5.2 *The graphs of $f_1(x) = 2^x$, $f_2(x) = 3^x$, and $f_3(x) = 10^x$.*

Table 5.2

x	$(1/2)^x$
−3	8
−2	4
−1	2
0	1
1	1/2
2	1/4
3	1/8

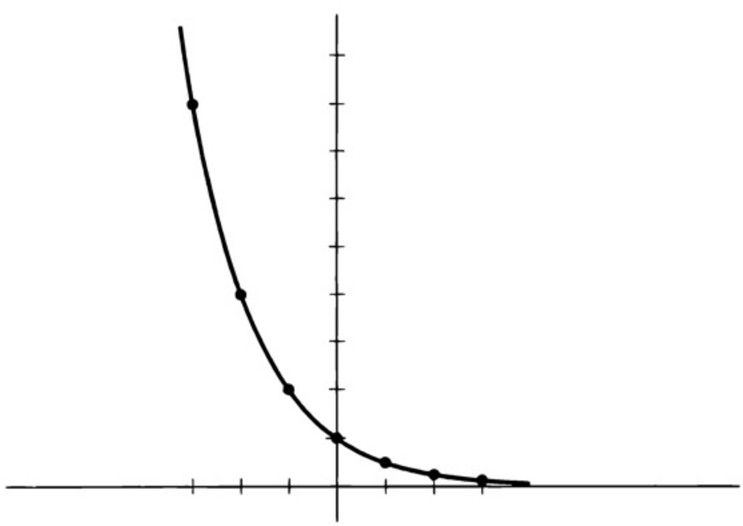


Figure 5.3 *The graph of $y = (1/2)^x$.*

Negative bases are not allowed for the exponential function. For example, the function $k(x) = (-2)^x$ would take on positive values for x an even integer and negative values for x an odd integer; yet it is never zero in between. Furthermore, since you cannot take the square root of a negative number, the function $(-2)^x$ is not even defined for $x = 1/2$ or, more generally, whenever x is a fraction p/q and q is an even integer. So, we can only work with exponential functions a^x, where a is a number greater than 0.

EXERCISES

5.1 Evaluate each of the following:

$$2^3, \quad 2^{-3}, \quad 8^{1/3}, \quad 8^{2/3}, \quad 8^{-2/3}, \quad \pi^0, \quad 64^{-5/6}, \quad 625^{3/4}, \quad 25^{-5/2}.$$

5.2 Sketch the graph of: *a)* $y = 5^x$; *b)* $y = .2^x$; *c)* $y = 3(5^x)$; *d)* $y = 1^x$.

5.2 THE NUMBER e

Figure 5.2 presented graphs of exponential functions with bases 2, 3, and 10, respectively. We now introduce a number which is the most important base for an exponential function, the irrational number e. To motivate the definition of e, consider the most basic economic situation — the growth of the investment in a savings account. Suppose that at the beginning of the year, we deposit \$$A$ into a savings account which pays interest at a simple annual interest rate r. If we will let the account grow without deposits or withdrawals, after one year the account will grow to $A + rA = A(1 + r)$ dollars. Similarly, the amount in the account in any one year is $(1 + r)$ times the previous year's amount. After two years, there will be

$$A(1 + r)(1 + r) = A(1 + r)^2$$

dollars in the account. After t years, there will be $A(1 + r)^t$ dollars in the account.

Next, suppose that the bank compounds interest four times a year; at the end of each quarter, it pays interest at $r/4$ times the current principal. After one quarter of a year, the account contains $A + \frac{r}{4}A$ dollars. After one year, that is, after four compoundings, there will be $A\left(1 + \frac{r}{4}\right)^4$ dollars in the account. After t years, the account will grow to $A\left(1 + \frac{r}{4}\right)^{4t}$ dollars.

More generally, if interest is compounded n times a year, there will be $A\left(1 + \frac{r}{n}\right)$ dollars in the account after the first compounding period, $A\left(1 + \frac{r}{n}\right)^n$ dollars in the account after the first year, and $A\left(1 + \frac{r}{n}\right)^{nt}$ dollars in the account after t years.

Many banks compound interest daily; others advertise that they compound interest *continuously*. By what factor does money in the bank grow in one year at interest rate r if interest is compounded so frequently, that is, if n is very large? Mathematically, we are asking, "What is the limit of $(1 + \frac{r}{n})^n$ as $n \to \infty$?" To simplify this calculation, let's begin with a 100 percent annual interest rate; that is, $r = 1$. Some countries, like Israel, Argentina, and Russia, have experienced interest rates of 100 percent and higher in recent years.

We compute $(1 + \frac{1}{n})^n$ with a calculator for various values of n and list the results in Table 5.3.

Table 5.3

n	$\left(1 + \dfrac{1}{n}\right)^n$
1	2.0
2	2.25
4	2.4414
10	2.59374
100	2.704814
1,000	2.7169239
10,000	2.7181459
100,000	2.71826824
10,000,000	2.718281693

One sees in Table 5 3 that the sequence $(1 + \frac{1}{n})^n$ is an increasing sequence in n and converges to a number a little bigger than 2.7. The limit turns out to be an irrational number, in that it cannot be written as a fraction or as a repeating decimal. The letter e is reserved to denote this number; formally,

$$e \equiv \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n. \tag{1}$$

To seven decimal places, $e = 2.7182818 \cdots$.

This number e plays the same fundamental role in finance and in economics that the number π plays in geometry. In particular, the function $f(x) = e^x$ is called *the* **exponential function** and is frequently written as $\exp(x)$. Since $2 < e < 3$, the graph of $\exp(x) = e^x$ is shaped like the graphs in Figure 5.2.

Next, we reconsider the general interest rate r and ask: What is the limit of the sequence

$$\left(1 + \frac{r}{n}\right)^n$$

in terms of e? A simple change of variables answers this question. Fix $r > 0$ for the rest of this discussion. Let $m \equiv n/r$; so $n = mr$. As n gets larger and goes to infinity, so does m. (Remember r is fixed.) Since $r/n = 1/m$,

$$\left(1 + \frac{r}{n}\right)^n = \left(1 + \frac{1}{m}\right)^{mr} = \left(\left(1 + \frac{1}{m}\right)^m\right)^r$$

by straightforward substitution. Letting $n \longrightarrow \infty$, we find

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n} &= \lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^{m}\right)^{r} \\ &= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right)^{r} \\ &= e^{r}.\end{aligned}$$

In the second step, we used the fact that x^r is a continuous function of x, so that if $\{x_m\}_{m=1}^{\infty}$ is a sequence of numbers which converges to x_0, then the sequence of powers $\{x_m^r\}$ converges to x_0^r; that is

$$\left(\lim_{m\to\infty} x_m\right)^{r} = \lim_{m\to\infty}\left(x_m^r\right).$$

If we let the account grow for t years, then

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} &= \lim_{n\to\infty}\left(\left(1+\frac{r}{n}\right)^{n}\right)^{t} \\ &= \left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right)^{t} \\ &= (e^r)^t = e^{rt}.\end{aligned}$$

The following theorem summarizes these simple limit computations.

Theorem 5.1 As $n \longrightarrow \infty$, the sequence $\left(1+\frac{1}{n}\right)^n$ converges to a limit denoted by the symbol e. Furthermore,

$$\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^{n} = e^{k}.$$

If one deposits A dollars in an account which pays annual interest at rate r compounded continuously, then after t years the account will grow to Ae^{rt} dollars.

Note the advantages of frequent compounding. At $r = 1$, that is, at a 100 percent interest rate, A dollars will double to $2A$ dollars in a year with no compounding. However, if interest is compounded continuously, then the A dollars will grow to eA dollars with $e > 2.7$; the account nearly triples in size.

5.3 LOGARITHMS

Consider a general exponential function, $y = a^x$, with base $a > 1$. Such an exponential function is a strictly increasing function:

$$x_1 > x_2 \implies a^{x_1} > a^{x_2}.$$

In words, the more times you multiply a by itself, the bigger it gets. As we pointed out in Theorem 4.1, strictly increasing functions have natural inverses. Recall that the inverse of the function $y = f(x)$ is the function obtained by solving $y = f(x)$ for x in terms of y. For example, for $a > 0$, the inverse of the increasing linear function $f(x) = ax + b$ is the linear function $g(y) = (1/a)(y - b)$, which is computed by solving the equation $y = ax + b$ for x in terms of y:

$$y = ax + b \iff x = \frac{1}{a}(y - b). \tag{2}$$

In a sense, the inverse g of f undoes the operation of f, so that

$$g(f(x)) = x.$$

See Section 4.2 for a detailed discussion of the inverse of a function.

We cannot compute the inverse of the increasing exponential function $f(x) = a^x$ explicitly because we can't solve $y = a^x$ for x in terms of y, as we did in (2). However, this inverse function is important enough that we give it a name. We call it the **base a logarithm** and write

$$y = \log_a(z) \iff a^y = z.$$

The **logarithm** of z, by definition, is the power to which one must raise a to yield z. It follows immediately from this definition that

$$a^{\log_a(z)} = z \qquad \text{and} \qquad \log_a(a^z) = z. \tag{3}$$

We often write $\log_a(z)$ without parentheses, as $\log_a z$.

Base 10 Logarithms

Let's first work with base $a = 10$. The logarithmic function for base 10 is such a commonly used logarithm that it is usually written as $y = \operatorname{Log} x$ with an uppercase L:

$$y = \operatorname{Log} z \iff 10^y = z.$$

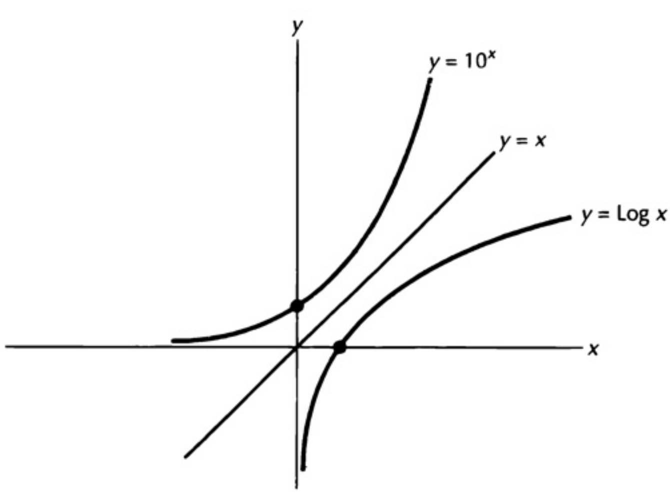
Example 5.1 For example, the Log of 1000 is that power of 10 which yields 1000. Since $10^3 = 1000$, $\text{Log}\,1000 = 3$. The Log of 0.01 is -2, since $10^{-2} = 0.01$. Here are a few more values of $\text{Log}\,z$:

$$\begin{aligned}
\text{Log}\,10 &= 1 && \text{since} \quad 10^1 &&= 10,\\
\text{Log}\,100{,}000 &= 5 && \text{since} \quad 10^5 &&= 100{,}000,\\
\text{Log}\,1 &= 0 && \text{since} \quad 10^0 &&= 1,\\
\text{Log}\,625 &= 2.79588\cdots && \text{since} \quad 10^{2.79588\cdots} &&= 625.
\end{aligned}$$

For most values of z, you'll have to use a calculator or table of logarithms to evaluate $\text{Log}\,z$.

One forms the graph of the inverse function f^{-1} by reversing the roles of the horizontal and vertical axes in the graph of f. In other words, the graph of the inverse of a function $y = f(x)$ is the reflection of the graph of f across the diagonal $\{x = y\}$, because (y, z) is a point on the graph of f^{-1} if and only if (z, y) is a point on the graph of f. In Figure 5.4, we have drawn the graph of $y = 10^x$ and reflected it across the diagonal $\{x = y\}$ to draw the graph of $y = \text{Log}\,x$.

Since the negative "x-axis" is a horizontal asymptote for the graph of $y = 10^x$, the negative "y-axis" is a vertical asymptote for the graph of $y = \text{Log}\,x$. Since 10^x grows very quickly, $\text{Log}\,x$ grows very slowly. At $x = 1000$, $\text{Log}\,x$ is just at $y = 3$; at x equals a million, $\text{Log}\,x$ has just climbed to $y = 6$. Finally, since for *every* x, 10^x is a positive number, $\text{Log}\,x$ is only defined for $x > 0$. Its domain is $\mathbf{R}_{++}$, the set of strictly positive numbers.



The graph of $y = \text{Log}\,x$ is the reflection of the graph of $y = 10^x$ across the diagonal $\{y = x\}$. **Figure 5.4**

Base e Logarithms

Since *the* exponential function $\exp(x) = e^x$ has all the properties that 10^x has, it also has an inverse. Its inverse works the same way that $\operatorname{Log} x$ does. Mirroring the fundamental role that e plays in applications, the inverse of e^x is called the **natural logarithm** function and is written as $\ln x$. Formally,

$$\ln x = y \iff e^y = x;$$

$\ln x$ is the power to which one must raise e to get x. As we saw in general in (3), this definition can also be summarized by the equations

$$e^{\ln x} = x \quad \text{and} \quad \ln e^x = x. \tag{4}$$

The graph of e^x and its reflection across the diagonal, the graph of $\ln x$, are similar to the graphs of 10^x and $\operatorname{Log} x$ in Figure 5.4.

Example 5.2 Let's work out some examples. The natural log of 10 is the power of e that gives 10. Since e is a little less than 3 and $3^2 = 9$, e^2 will be a bit less than 9. We have to raise e to a power bigger than 2 to obtain 10. Since $3^3 = 27$, e^3 will be a little less than 27. Thus, we would expect that ln 10 to lie between 2 and 3 and somewhat closer to 2. Using a calculator, we find that the answer to four decimal places is $\ln 10 = 2.3026$.

We list a few more examples. Cover the right-hand side of this table and try to estimate these natural logarithms.

$$\begin{aligned}
\ln e &= 1 && \text{since} \quad e^1 &&= e;\\
\ln 1 &= 0 && \text{since} \quad e^0 &&= 1;\\
\ln 0.1 &= -2.3025\cdots && \text{since} \quad e^{-2.3025\cdots} &&= 0.1;\\
\ln 40 &= 3.688\cdots && \text{since} \quad e^{3.688\cdots} &&= 40;\\
\ln 2 &= 0.6931\cdots && \text{since} \quad e^{0.6931\cdots} &&= 2.
\end{aligned}$$

EXERCISES

5.3 First estimate the following logarithms without a calculator. Then, use your calculator to compute an answer correct to four decimal places:

a) $\operatorname{Log} 500$, *b*) $\operatorname{Log} 5$, *c*) $\operatorname{Log} 1234$, *d*) $\operatorname{Log} e$,
e) $\ln 30$, *f*) $\ln 100$, *g*) $\ln 3$, *h*) $\ln \pi$.

5.4 Give the exact values of the following logarithms without using a calculator:

a) Log 10,	*b*) Log 0.001,	*c*) Log(billion),
d) $\log_2 8$,	*e*) $\log_6 36$,	*f*) $\log_5 0.2$,
g) $\ln(e^2)$,	*h*) $\ln\sqrt{e}$,	*i*) ln 1.

5.4 PROPERTIES OF EXP AND LOG

Exponential functions have the following five basic properties:

(1) $a^r \cdot a^s = a^{r+s}$,
(2) $a^{-r} = 1/a^r$,
(3) $a^r/a^s = a^{r-s}$,
(4) $(a^r)^s = a^{rs}$, and
(5) $a^0 = 1$.

Properties 1, 3, and 4 are straightforward when r and s are positive integers. The definitions that $a^{-n} = 1/a^n$, $a^0 = 1$, $a^{1/n}$ is the nth root of a, and $a^{m/n} = (a^{1/n})^m$ are all specifically designed so that the above five rules would hold for *all real* numbers r and s.

These five properties of exponential functions are mirrored by five corresponding properties of the logarithmic functions:

(1) $\log(r \cdot s) = \log r + \log s$,
(2) $\log(1/s) = -\log s$,
(3) $\log(r/s) = \log r - \log s$,
(4) $\log r^s = s \log r$, and
(5) $\log 1 = 0$.

The fifth property of logs follows directly from the fifth property of a^x and the fact that a^x and $\log_a$ are inverses of each other. To prove the other four properties, let $u = \log_a r$ and $v = \log_a s$, so that $r = a^u$ and $s = a^v$. Then, using the fact that $\log_a(a^x) = x$, we find:

$$\begin{aligned}
&(1)\ \log(r \cdot s) = \log(a^u \cdot a^v) = \log(a^{u+v}) = u + v = \log r + \log s,\\
&(2)\ \log(1/s) = \log(1/a^v) = \log(a^{-v}) = -v = -\log s,\\
&(3)\ \log(r/s) = \log(a^u/a^v) = \log(a^{u-v}) = u - v = \log r - \log s,\\
&(4)\ \log r^s = \log(a^u)^s = \log a^{us} = us = s \cdot \log r.
\end{aligned}$$

Logarithms are especially useful in bringing a variable x that occurs as an exponent back down to the base line where it can be more easily manipulated.

Example 5.3 To solve the equation $2^{5x} = 10$ for x, we take the Log of both sides:

$$\text{Log}\, 2^{5x} = \text{Log}\, 10 \quad \text{or} \quad 5x \cdot \text{Log}\, 2 = 1.$$

It follows that

$$x = \frac{1}{5\ \text{Log}\, 2} \approx .6644.$$

We could have used ln instead of Log in this calculation.

Example 5.4 Suppose we want to find out how long it takes A dollars deposited in a saving account to double when the annual interest rate is r compounded continuously. We want to solve the equation

$$2A = Ae^{rt} \tag{5}$$

for the unknown t. We first divide both sides of (5) by A. This eliminates A from the calculation — a fact consistent with our intuition that the doubling time should be independent of the amount of money under consideration. To bring the variable t down to where we can work with it, take the natural log of both sides of the equation $2 = e^{rt}$:

$$\begin{aligned} \ln 2 &= \ln e^{rt} \\ &= rt, \end{aligned} \tag{6}$$

using (4). Solving (6) for t yields the fact that the doubling time is $t = (\ln 2)/r$.

Since $\ln 2 \approx 0.69$, this rule says that to estimate the doubling time for interest rate r, just divide the interest rate into 69. For example, the doubling time at 10 percent interest is $69/10 = 6.9$ years; the doubling time at 8 percent interest is $69/8 = 8.625$ years. This calculation also tells us that it would take 8.625 years for the price level to double if the inflation rate stays constant at 8 percent.

As we discussed in Section 3.6, economists studying the relationship between the price p and the quantity q demanded of some good will often choose to work with the two-parameter family of **constant elasticity demand functions,** $q = kp^{\varepsilon}$, where k and ε are parameters which depend on the good under study. The parameter ε is the most interesting of the two since it equals the elasticity $(p/q)(dq/dp)$. Taking the log of both sides of $q = kp^{\varepsilon}$ yields:

$$\ln q = \ln kp^{\varepsilon} = \ln k + \varepsilon \ln p. \tag{7}$$

In logarithmic coordinates, demand is now a *linear* function whose slope is the elasticity ε.

EXERCISES

5.5 Solve the following equations for x:

$$a)\ 2e^{6x} = 18; \quad b) e^{x^2} = 1; \quad c) 2^x = e^5;$$
$$d)\ 2^{x-2} = 5; \quad e) \ln x^2 = 5; \quad f) \ln x^{5/2} - 0.5 \ln x = \ln 25.$$

5.6 Derive a formula for the amount of time that it takes money to triple in a bank account that pays interest at rate r compounded continuously.

5.7 How quickly will \$500 grow to \$600 if the interest rate is 5 percent compounded continuously?

5.5 DERIVATIVES OF EXP AND LOG

To work effectively with exponential and logarithmic functions, we need to compute and use their derivatives. The natural logarithmic and exponential functions have particularly simple derivatives, as the statement of the following theorem indicates.

Theorem 5.2 The functions e^x and $\ln x$ are continuous functions on their domains and have continuous derivatives of every order. Their first derivatives are given by

$$a)\quad (e^x)' = e^x,$$
$$b)\quad (\ln x)' = \frac{1}{x}.$$

If $u(x)$ is a differentiable function, then

$$c)\quad \left(e^{u(x)}\right)' = \left(e^{u(x)}\right) \cdot u'(x),$$
$$d)\quad (\ln u(x))' = \frac{u'(x)}{u(x)} \qquad \text{if } u(x) > 0.$$

We will prove this theorem in stages. That the exponential map is continuous should be intuitively clear from the graph in Figure 5.4; its graph has no jumps or discontinuities. Since the graph of $\ln x$ is just the reflection of the graph of e^x across the diagonal $\{x = y\}$, the graph of $\ln x$ has no discontinuities either, and so the function $\ln x$ is continuous for all x in the set $\mathbf{R}_{++}$ of positive numbers.

It turns out to be easier to compute the derivative of the natural logarithm first.

Lemma 5.1 Given that $y = \ln x$ is a continuous function on $\mathbf{R}_{++}$, it is also differentiable and its derivative is given by

$$(\ln x)' = \frac{1}{x}.$$

Proof We start, of course, with the difference quotient that defines the derivative, and we then simplify it using the basic properties of the logarithm. Fix $x > 0$.

$$\begin{aligned}\frac{\ln(x+h) - \ln x}{h} &= \frac{1}{h}\ln\left(\frac{x+h}{x}\right) = \ln\left(1 + \frac{h}{x}\right)^{\frac{1}{h}} \\ &= \ln\left(1 + \frac{1/x}{1/h}\right)^{\frac{1}{h}}.\end{aligned}$$

Now, let $m = 1/h$. As $h \to 0$, $m \to \infty$. Continuing our calculation with $m = 1/h$, we find

$$\begin{aligned}\lim_{h\to 0}\frac{\ln(x+h) - \ln x}{h} &= \lim_{m\to\infty}\ln\left(1 + \frac{1/x}{m}\right)^m \\ &= \ln\lim_{m\to\infty}\left(1 + \frac{1/x}{m}\right)^m \\ &= \ln e^{1/x} = \frac{1}{x}.\end{aligned}$$

Therefore, $(\ln x)' = 1/x$. The fact that we can interchange ln and lim in the above string of equalities follows from the fact that $y = \ln x$ is a continuous function: $x_m \to x_0$ implies that $\ln x_m \to \ln x_0$; or equivalently,

$$\lim_m(\ln x_m) = \ln\left(\lim_m x_m\right). \qquad \blacksquare$$

The other three conclusions of Theorem 5.2 follow immediately from the Chain Rule, as we now prove.

Lemma 5.2 If $h(x)$ is a differentiable and positive function, then

$$\frac{d}{dx}(\ln h(x)) = \frac{h'(x)}{h(x)}.$$

Proof We simply apply the Chain Rule to the composite function $f(x) = \ln h(x)$. The derivative of f is the derivative of the *outside* function $\ln y$—which is $1/y$—evaluated at the inside function $h(x)$—so it's $1/h(x)$—times the derivative $h'(x)$ of the inside function h:

$$(\ln h(x))' = \frac{1}{h(x)} \cdot h'(x) = \frac{h'(x)}{h(x)}. \quad \blacksquare$$

We can now easily evaluate the derivative of the exponential function $y = e^x$, using the fact that it is the inverse of $\ln x$.

Lemma 5.3 $(e^x)' = e^x$.

Proof Use the definition of $\ln x$ in (4) to write $\ln e^x = x$. Taking the derivative of both sides of this equation and using the previous lemma, we compute

$$(\ln e^x)' = \frac{1}{e^x} \cdot (e^x)' = 1.$$

It follows that

$$(e^x)' = e^x. \quad \blacksquare$$

Finally, to prove part *c* of Theorem 5.2, we simply apply the Chain Rule to the composite function $y = e^{u(x)}$. The outside function is e^z, whose derivative is also e^z. Its derivative evaluated at the inside function is $e^{u(x)}$. Multiplying this by the derivative of the inside function $u(x)$, we conclude that

$$\left(e^{u(x)}\right)' = e^{u(x)}\, u'(x).$$

Example 5.5 Using Theorem 5.2, we compute the following derivatives:

a) $\left(e^{5x}\right)' = 5e^{5x}$, *b)* $\left(Ae^{kx}\right)' = Ake^{kx}$,

c) $\left(5e^{x^2}\right)' = 10xe^{x^2}$, *d)* $(e^x \ln x)' = e^x \ln x + \dfrac{e^x}{x}$,

e) $\left(\ln x^2\right)' = \dfrac{1}{x^2} \cdot 2x = \dfrac{2}{x}$, *f)* $\left((\ln x)^2\right)' = \dfrac{2 \ln x}{x}$,

g) $\left(xe^{3-x}\right)' = e^{3-x} - xe^{3-x} = (1-x)e^{3-x}$, *h)* $\left(\ln(x^2 + 3x + 1)\right)' = \dfrac{2x+3}{x^2 + 3x + 1}$.

Example 5.6 The density function for the standard **normal distribution** is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Let's use calculus to sketch the graph of its core function

$$g(x) = e^{-x^2/2}.$$

We first note that g is always positive, so its graph lies above the x-axis everywhere. Its first derivative is

$$g'(x) = -xe^{-x^2/2}.$$

Since $e^{-x^2/2}$ is always positive, $g'(x) = 0$ if and only if $x = 0$. Since $g(0) = 1$, the only candidate for max or min of g is the point $(0, 1)$. Furthermore, $g'(x) > 0$ if and only if $x < 0$, and $g'(x) < 0$ if and only if $x > 0$; so g is increasing for $x < 0$ and decreasing for $x > 0$. This tells us that the critical point $(0, 1)$ must be a max, in fact, a global max.

So far, we know that the graph of g stays above the x-axis all the time, increases until it reaches the point $(0, 1)$ on the y-axis, and then decreases to the right of the y-axis. Let's use the second derivative to fine-tune this picture:

$$g''(x) = \left(-xe^{-x^2/2}\right)' = x^2e^{-x^2/2} - e^{-x^2/2} = \left(x^2 - 1\right)e^{-x^2/2}.$$

Since $e^{-x^2/2} > 0$, $g''(x)$ has the same sign as $(x^2 - 1)$. In particular,

$$g''(0) < 0, \qquad \text{and} \qquad g''(x) = 0 \Longleftrightarrow x = \pm 1. \tag{8}$$

The first inequality in (8) verifies that the critical point $(0, 1)$ is indeed a local max of g. Using the second part of (8), we note that

$$\begin{aligned} -\infty < x < -1 \quad &\Longrightarrow \quad g''(x) > 0, \\ -1 < x < +1 \quad &\Longrightarrow \quad g''(x) < 0, \\ 1 < x < +\infty \quad &\Longrightarrow \quad g''(x) > 0; \end{aligned}$$

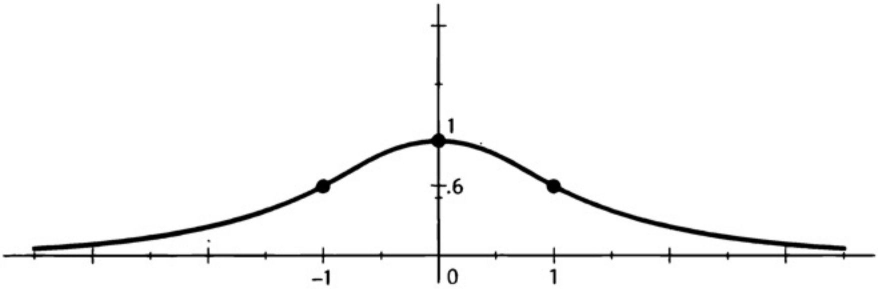
this implies that g is concave up on $(-\infty, -1)$ and on $(1, \infty)$ and concave down on $(-1, +1)$. The second order critical points occur at the points $(-1, e^{-1/2})$ and $(1, e^{-1/2})$. Putting all this information together, we sketch the graph of g in Figure 5.5.

The graph of g is the graph of the usual bell-shaped probability distribution. Since f is simply g times $(2\pi)^{-1/2} \approx .39$, the graph of f will be similar to the graph of g but closer to the x-axis.

We now use equation b in Example 5.5 to compute the derivative of the general exponential function $y = b^x$.

Theorem 5.3 For any fixed positive base b,

$$(b^x)' = (\ln b)(b^x). \tag{9}$$



The graph of $e^{-x^2/2}$.

Figure 5.5

Proof Since $b = e^{\ln b}$, then $b^x = \left(e^{\ln b}\right)^x = e^{(\ln b)x}$. By equation *b* in Example 5.5,

$$(b^x)' = \left(e^{(\ln b)\cdot x}\right)' = (\ln b)(e^{(\ln b)\cdot x}) = (\ln b)(b^x). \quad \blacksquare$$

Example 5.7 $(10^x)' = (\ln 10)(10^x)$.

Note that $(b^x)' = b^x$ if and only if $\ln b = 1$, that is, if and only if $b = e$. In fact, *the* exponential functions $y = ke^x$ are the only functions which are equal to their derivatives throughout their domains. This fact gives another justification for e being considered the ***natural*** base for exponential functions.

EXERCISES

5.8 Compute the first and second derivatives of each of the following functions:
a) xe^{3x}, *b)* e^{x^2+3x-2}, *c)* $\ln(x^4+2)^2$, *d)* $\dfrac{x}{e^x}$, *e)* $\dfrac{x}{\ln x}$, *f)* $\dfrac{\ln x}{x}$.

5.9 Use calculus to sketch the graph of each of the following functions:
a) xe^x, *b)* xe^{-x}, *c)* $\cosh(x) \equiv (e^x + e^{-x})/2$.

5.10 Use the equation $10^{\mathrm{Log}\,x} = x$, Example 5.7, and the method of the proof of Lemma 5.3 to derive a formula for the derivative of $y = \mathrm{Log}\,x$.

5.6 APPLICATIONS

Present Value

Many economic problems entail comparing amounts of money at different points of time in the same computation. For example, the benefit/cost analysis of the construction of a dam must compare in the same equation this year's cost of construction, future years' costs of maintaining the dam, and future years' monetary benefits from the use of the dam. The simplest way to deal with such comparisons is to use the concept of *present value* to bring all money figures back to the present.

If we put A dollars into an account which compounds interest continuously at rate r, then after t years there will be

$$B = Ae^{rt} \tag{10}$$

dollars in the account, by Theorem 5.1. Conversely, in order to generate B dollars t years from now in an account which compounds interest continuously at rate r, we would have to invest $A = Be^{-rt}$ dollars in the account now, solving (10) for A in terms of B. We call Be^{-rt} the **present value** (PV) of B dollars t years from now (at interest rate r).

Present value can also be defined using *annual* compounding instead of continuous compounding. In an account which compounds interest annually at rate r, a deposit of A dollars now will yield $B = A(1+r)^t$ dollars t years from now. Conversely, in this framework, the present value of B dollars t years from now is $B/(1+r)^t = B(1+r)^{-t}$ dollars. Strictly speaking, this latter framework only makes sense for integer t's. For this reason and because the exponential map e^{rt} is usually easier to work with than $(1+r)^t$, we will use the continuous compounding version of present value.

Present value can also be defined for *flows* of payments. At interest rate r, the present value of the flow — B_1 dollars t_1 years from now, B_2 dollars t_2 years from now, ..., B_n dollars t_n years from now — is

$$\text{PV} = B_1e^{-rt_1} + B_2e^{-rt_2} + \cdots + B_ne^{-rt_n}. \tag{11}$$

Annuities

An **annuity** is a sequence of equal payments at regular intervals over a specified period of time. The present value of an annuity that pays A dollars at the end of each of the next N years, assuming a constant interest rate r compounded continuously, is

$$\begin{aligned} \text{PV} &= Ae^{-r\cdot 1} + Ae^{-r\cdot 2} + \cdots + Ae^{-r\cdot N} \\ &= A\left(e^{-r} + e^{-r\cdot 2} + \cdots + e^{-r\cdot N}\right). \end{aligned} \tag{12}$$

Since $(a + \cdots + a^n)(1-a) = a - a^{n+1}$, as one can easily check,

$$a + \cdots + a^n = \frac{a(1-a^n)}{1-a}. \tag{13}$$

Substituting $a = e^{-r}$ and $n = N$ from (12) yields a present value for the annuity of

$$\text{PV} = A \cdot \frac{e^{-r}\left(1 - e^{-rN}\right)}{1 - e^{-r}} = \frac{A\left(1 - e^{-rN}\right)}{e^r - 1}. \tag{14}$$

To calculate the present value of an annuity which pays A dollars a year *forever*, we let $N \to \infty$ in (14):

$$\text{PV} = \frac{A}{e^r - 1}, \tag{15}$$

since $e^{-rN} \to 0$ as $N \to \infty$.

It is sometimes convenient to calculate the present value of an annuity using *annual* compounding instead of continuous compounding. In this case, equation (12) becomes

$$\text{PV} = \frac{A}{1+r} + \cdots + \frac{A}{(1+r)^N}.$$

Apply equation (13) with $a = 1/(1+r)$ and $n = N$:

$$\text{PV} = A \cdot \frac{1/(1+r)}{r/(1+r)} \left(1 - \left(\frac{1}{1+r}\right)^N\right) = \frac{A}{r}\left(1 - \left(\frac{1}{1+r}\right)^N\right) \tag{16}$$

To calculate the present value of an annuity which pays A dollars a year forever at interest rate r compounded annually, we let $N \to \infty$ in (16):

$$\text{PV} = \frac{A}{r}. \tag{17}$$

The intuition for (17) is straightforward; in order to generate a perpetual flow of A dollars a year from a savings account which pays interest annually at rate r, one must deposit A/r dollars into the account initially.

Optimal Holding Time

Suppose that you own some real estate the market value of which will be $V(t)$ dollars t years from now. If the interest rate remains constant at r over this period, the corresponding time stream of present values is $V(t)e^{-rt}$. Economic theory suggests that the optimal time t_0 to sell this property is at the maximum value of this time stream of present value. The first order conditions for this maximization problem are

$$(V(t)e^{-rt})' = V'(t)e^{-rt} - rV(t)e^{-rt} = 0,$$

or

$$\frac{V'(t)}{V(t)} = r \quad \text{at } t = \text{the optimal selling time } t_0. \tag{18}$$

Condition (18) is a natural condition for the **optimal holding time**. The left-hand side of (18) gives the rate of change of V divided by the amount of V — a quantity called the **percent rate of change** or simply the **growth rate**. The right-hand side gives the interest rate, which is the percent rate of change of money in the bank. As long as the value of the real estate is growing more rapidly than money in the bank, one should hold on to the real estate. As soon as money in the bank has a higher growth rate, one would do better by selling the property and banking the proceeds at interest rate r. The point at which this switch takes place is given by (18), where the percent rates of change are equal.

This principle of optimal holding time holds in a variety of circumstances, for example, when a wine dealer is trying to decide when to sell a case of wine that is appreciating in value or when a forestry company is trying to decide how long to let the trees grow before cutting them down for sale.

Example 5.8 You own real estate the market value of which t years from now is given by the function $V(t) = 10{,}000e^{\sqrt{t}}$. Assuming that the interest rate for the foreseeable future will remain at 6 percent, the optimal selling time is given by maximizing the present value

$$F(t) = 10{,}000e^{\sqrt{t}}e^{-.06t} = 10{,}000e^{\sqrt{t}-.06t}.$$

The first order condition for this maximization problem is

$$0 = F'(t) = 10{,}000e^{\sqrt{t}-.06t}\left(\frac{1}{2\sqrt{t}} - .06\right),$$

which holds if and only if

$$\frac{1}{2\sqrt{t_0}} = .06 \quad \text{or} \quad t_0 = \left(\frac{1}{.12}\right)^2 \approx 69.44.$$

Since $F'(t)$ is positive for $0 < t < t_0$ and negative for $t > t_0$, $t_0 \approx 69.44$ is indeed the *global* max of the present value and is the optimal selling time of the real estate.

Logarithmic Derivative

Since the logarithmic operator turns exponentiation into multiplication, multiplication into addition, and division into subtraction, it can often simplify the computation of the derivative of a complex function, because, by Lemma 5.2,

$$(\ln u(x))' = \frac{u'(x)}{u(x)},$$

and therefore

$$u'(x) = (\ln u(x))' \cdot u(x). \tag{19}$$

If $\ln u(x)$ is easier to work with than $u(x)$ itself, one can compute u' more easily using (19) than by computing it directly.

Example 5.9 Let's use this idea to compute the derivative of

$$y = \frac{\sqrt[4]{x^2 - 1}}{x^2 + 1}. \tag{20}$$

The natural log of this function is

$$\ln\left(\frac{\sqrt[4]{x^2 - 1}}{x^2 + 1}\right) = \frac{1}{4}\ln(x^2 - 1) - \ln(x^2 + 1). \tag{21}$$

It is much simpler to compute the derivative of (21) than it is to compute the derivative of the quotient (20):

$$\begin{aligned}\frac{d}{dx}\ln\left(\frac{\sqrt[4]{x^2 - 1}}{x^2 + 1}\right) &= \frac{1}{4}\,\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1}\\ &= \frac{-3x^3 + 5x}{2(x^2 - 1)(x^2 + 1)}.\end{aligned}$$

Now, use (19) to compute y':

$$\begin{aligned}\left(\frac{\sqrt[4]{x^2 - 1}}{x^2 + 1}\right)' &= \frac{-3x^3 + 5x}{2(x^2 - 1)(x^2 + 1)} \cdot \frac{\sqrt[4]{x^2 - 1}}{x^2 + 1}\\ &= \frac{-3x^3 + 5x}{2(x^2 - 1)^{3/4}(x^2 + 1)^2}.\end{aligned}$$

Example 5.10 A favorite calculus problem, which can only be solved by this method, is the computation of the derivative of $g(x) = x^x$. Since

$$(\ln x^x)' = (x \ln x)' = \ln x + 1,$$

the derivative of x^x is $(\ln x + 1) \cdot x^x$, by (19).

Occasionally, scientists prefer to study a given function $y = f(x)$ by comparing $\ln y$ and $\ln x$, that is, by graphing f on log-log graph paper. See, for example, our discussion of constant elasticity demand functions in (7). In this case, they are working with the change of variables

$$Y = \ln y \quad \text{and} \quad X = \ln x.$$

Since $X = \ln x$, $x = e^X$ and $dx/dX = e^X = x$. In XY-coordinates, f becomes

$$Y = \ln f(x) = \ln f(e^X) \equiv F(X).$$

Now, the slope of the graph of $Y = F(X)$, that is, of the graph of f in log-log coordinates, is given by

$$\begin{aligned} \frac{dF(X)}{dX} &= \frac{dF(x(X))}{dx} \cdot \frac{dx}{dX} \qquad \text{(by the Chain Rule)} \\ &= \frac{d}{dx}(\ln f(x)) \cdot \frac{dx}{dX} = \frac{f'(x)}{f(x)} \cdot x. \end{aligned} \tag{22}$$

The difference approximation of the last term in (22) is

$$\frac{df(x)}{dx} \cdot \frac{x}{f(x)} \approx \frac{\Delta f}{\Delta x} \cdot \frac{x}{f(x)} = \frac{\Delta f}{f(x)} \Big/ \frac{\Delta x}{x},$$

the percent change of f relative to the percent change of x. This is the quotient we have been calling the (point) **elasticity** of f with respect to x, especially if f is a demand function and x represents price or income.

This discussion shows that the slope of the graph of f in log-log coordinates is the (point) elasticity of f:

$$\varepsilon = \frac{f'(x) \cdot x}{f(x)}.$$

In view of this discussion, economists sometimes write this elasticity as

$$\varepsilon = \frac{d(\ln f)}{d(\ln x)}.$$

EXERCISES

5.11 At 10 percent annual interest rate, which of the following has the largest present value:

a) \$215 two years from now,

b) \$100 after each of the next two years, or

c) \$100 now and \$95 two years from now?

5.12 Assuming a 10 percent interest rate compounded continuously, what is the present value of an annuity that pays \$500 a year *a*) for the next five years, *b*) forever?

5.13 Suppose that you own a rare book whose value at time t years from now will be $B(t) = 2^{\sqrt{t}}$ dollars. Assuming a constant interest rate of 5 percent, when is the best time to sell the book and invest the proceeds?

5.14 A wine dealer owns a case of fine wine that can be sold for $Ke^{\sqrt{t}}$ dollars t years from now. If there are no storage costs and the interest rate is r, when should the dealer sell the wine?

5.15 The value of a parcel of land bought for speculation is increasing according to the formula $V = 2000\,e^{t^{1/4}}$. If the interest rate is 10 percent, how long should the parcel be held to maximize present value?

5.16 Use the logarithmic derivative method to compute the derivative of each of the following functions: *a*) $\sqrt{(x^2 + 1)/(x^2 + 4)}$, *b*) $(x^2)^{x^2}$.

5.17 Use the above discussion to prove that the elasticity of the product of two functions is the sum of the elasticities.