Math for Political Scientists Workshop

Day Two: Exponential and Logarithm Functions & Intro to Probability

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Review Time

- Calculus
- Differential Calculus
- Derivative
- How to get derivative?
- Integral
- Definite vs. indefinite integral
- How to calculate definite vs. indefinite integral?
- Second-order derivative
- Critical point
- Continuous vs. differentiable function

Exponential and Logarithm Functions

In mathematics, exponentiation is an operation involving two numbers: the **base** and the **exponent** or **power**.

- Exponentiation is written as b^n, where b is the base and n is the power; this is pronounced as “b (raised) to the (power of) n.”

The exponential function is a mathematical function denoted by

$$f(x) = a^x$$

(where a is a constant (typically a >0) and the variable x is written as an exponent).

Natural exponential function is denoted by

$$f(x) = exp(x) = e^x$$

Natural exponential functions commonly use e, Euler's number, for their base constant. Euler's number is an irrational number.[1]

[1] which means that it cannot be reduced to a simple fraction

Do not confuse a^x and x^a:

The function $y = 2^x$ is an exponential function, but the function $y = x^2$ is a polynomial function, specifically a quadratic function.

- Why?
- Because an exponential function is defined as a function in which the variable x appears in the exponent.

Exponential Function

Motivation

- The growth of the investment in a savings account.
- Suppose we deposit \$$A$ into an account, which pays interest n times a year with an annual interest rate r.
- With no deposits or withdrawals:

after the first compounding period:

$$A + A\frac{r}{n} = A(1 + \frac{r}{n})$$

after one year:

$$A(1 + \frac{r}{n})^n$$

after t years:

$$A(1 + \frac{r}{n})^{nt}$$

What is the limit of $(1 + \frac{r}{n})^n$ as $n \to \infty$?

Exponential Function

Suppose $r = 1$, then $(1 + \frac{1}{n})^n$

Table 1: Values of $\left(1 + \frac{1}{n}\right)^n$ for different n

n	$\left(1 + \frac{1}{n}\right)^n$
1	2.0
2	2.25
4	2.4414
10	2.59374
100	2.704814
1,000	2.7169239
10,000	2.7181459
100,000	2.71826824
10,000,000	2.718281693

The letter e is reserved to denote this number; formally:

$$e \equiv \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

Exponential Function

We know that

$$e \equiv \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

Suppose m=n/r, so n=mr,

$$\begin{aligned}
\lim_{n\to\infty} (1 + \frac{r}{n})^n &= \lim_{m\to\infty} (1 + \frac{r}{mr})^{mr} \\
&= \lim_{m\to\infty} (1 + \frac{1}{m})^{mr} \\
&= \lim_{m\to\infty} ((1 + \frac{1}{m})^m)^r \\
&= (\lim_{m\to\infty} (1 + \frac{1}{m})^m)^r \\
&= e^r
\end{aligned}$$

$$f(x) = exp(x) = e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

Logarithm Function

The logarithm is the inverse function to exponentiation.

$$y = log_b(x) \iff b^y = x$$

which means that the logarithm of a number x to the base b is the exponent to which b must be raised to produce x.

	Logarithm	=	**Exponent**
	$\log_a N = x$	$\longleftrightarrow$	$N = a^x$
(Common Log)	$\log N = x$	$\longleftrightarrow$	$N = 10^x$
(Natural Log)	$\ln N = x$	$\longleftrightarrow$	$N = e^x$

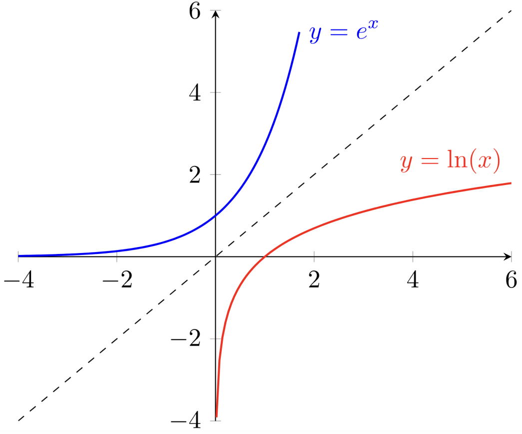
The inverse of e^x is called natural logarithm function and is written as $ln(x)$.Formally,

$$ln(x) = y \iff e^y = x$$

The log of a negative number is undefined.

The Graphs

The graphs of the natural exponential and logarithmic functions



Derivatives of Exponential and Logarithm Functions

Theorem The functions e^x and $\ln x$ are continuous functions on their domains and have continuous derivatives of every order.

- Their first derivatives are given by

(a) $(e^x)' = e^x$,

(b) $(\ln x)' = \frac{1}{x}$.

- If $u(x)$ is a differentiable function, then

(c) $\left(e^{u(x)}\right)' = e^{u(x)} \cdot u'(x)$,

(d) $(\ln u(x))' = \frac{u'(x)}{u(x)}$ if $u(x) > 0$.

Practice

Let $y = ln(x^5 - 2x^2 - 12)$, then y'?

Answer

Let $y = ln(x^5 - 2x^2 = 12)$, then y'?

- by $(\ln u(x))' = \frac{u'(x)}{u(x)}$
- $y' = \frac{1}{x^5 - 2x^2 - 12} \cdot (5x^4 - 4x)$

Application in Political Methodology

- Exponential family of distribution
- Survival analysis and hazard functions
- Log-linear models
- Generalized linear models

Introduction to Probability

What is probability?

- Simply, probability is how likely something is to happen.
- It is a branch of mathematics that deals with the occurrence of a random event.

It is the fundamental to statistical inference. (PLSC 502)

Terms related to Probability

Random Event: A random event is one in which all the possible results are known in advance but none of them can be predicted with certainty.

- Eg. Flipping a coin

Outcome: The result of a random event is called an outcome.

- Eg. Head or Tail.

Sample Space: The set of all the possible outcomes of a random event is called Sample Space, and it is denoted by 'S'.

- Eg. $S_{flip-a-coin} \in \{\text{Head, Tail}\}$

With outcome, event, and sample space defined, we can define the classical probability of an event

$$Pr(e) = \frac{\text{No. of outcomes in event e}}{\text{No. of outcomes in the sample space}}$$

Simple Event and Compound Event

A simple event is when only one event can occur.

- Eg. if we roll a die, it gives only one outcome.

A compound event is the chance of two or more events occurring.

- Eg. rolling two or more dies together.

Independence, Mutual Exclusivity and Collective Exhaustivity

Independence

- Independent events are events which are not affected by the occurrence of other events.
- For example, if we roll a die twice, the outcome of the first roll and second roll have no effect on each other – they are independent.

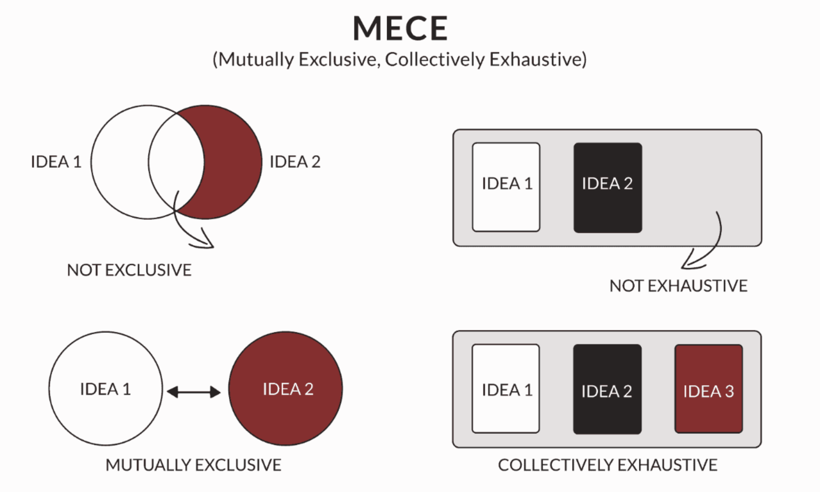
Mutually Exclusive

- means that two events cannot occur simultaneously.
- For example, if you roll a six-sided die, the outcomes of a six or a three are mutually exclusive.

Collectively Exhaustive

- means that the set of events includes all possible outcomes.
- For example, when throwing an unbiased six-sided die, the outcomes 1, 2, 3, 4, 5, and 6 are collectively exhaustive.

Mutually Exclusive and Collectively Exhaustive In Graph



Set Notations

Set Theory Symbols

Symbol	Name	Example	Explanation
$\{\ \}$	Set	$A = \{1,3\}$ $B = \{2,3,9\}$ $C = \{3,9\}$	Collection of objects
$\cap$	Intersect	$A \cap B = \{3\}$	Belong to both set A and set B
$\cup$	Union	$A \cup B = \{1,2,3,9\}$	Belong to set A or set B
$\subset$	Proper Subset	$\{1\} \subset A$ $C \subset B$	A set that is contained in another set
$\subseteq$	Subset	$\{1\} \subseteq A$ $\{1,3\} \subseteq A$	A set that is contained in or equal to another set
$\not\subset$	Not a Proper Subset	$\{1.3\} \not\subset A$	A set that is not contained in another set
$\supset$	Superset	$B \supset C$	Set B includes set C
$\in$	Is a member	$3 \in A$	3 is an element in set A
$\notin$	Is not a member	$4 \notin A$	4 is not an element in set A

We denote the probability that an event A occurs: $Pr(A)$ or $P(A)$.

- All probabilities lie between zero and one, so $Pr(A) \in [0, 1]$.

$Pr(A|B)$ is the conditional probability of A on B.

- It is read "the probability of A given B.' '
- It means the probability that A occurs given that B has already occurred.
- If A and B are independent events, then $Pr(A|B)=Pr(A)$.
- If A and B are mutually exclusive, then $Pr(A|B)=Pr(B|A) = 0$.
- If A and B are dependent events, then $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$

Probability Notations

Joint Probability and Union Probability

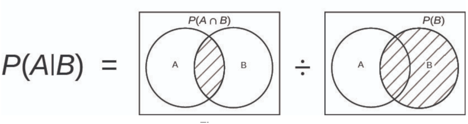
$Pr(A \cap B)$ is the joint probability of A and B.

- It is read "the joint probability of A and B."
- We read $A \cap B$ as "A and B."
- It means the probability of event B occurring at the same time that event A occurs.
- If A and B are dependent events, then $Pr(A \cap B) = Pr(A|B)Pr(B) = Pr(B|A)Pr(A)$.
- If A and B are independent events, then $Pr(A \cap B) = Pr(A)Pr(B)$

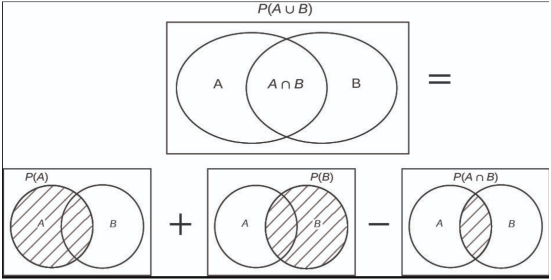
$Pr(A \cup B)$ is the union probability of A and B.

- It is read "the union probability of A and B."
- We read $A \cup B$ as "A or B."
- It means the probability that either event will happen, or that both will happen.
- If A and B are dependent events: $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$.
- If A and B are mutually exclusive, then $Pr(A \cup B) = Pr(A) + Pr(B)$.

Conditional Probability



Union Probability



n Factorial

In mathematics, the **factorial** of a non-negative integer n, denoted by $n!$, is the **product** of all positive integers less than or equal to n. The factorial of n also equals the product of n with the next smaller factorial:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 3 \times 2 \times 1 = n \times (n-1)!$$

For example,

$$5! = 5 \times 4! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

The value of 0! is 1, but why?

Combinations

Combination

- It is a way of choosing k objects from n objects when one does not care about the order in which one chooses the objects.
- Notation:

$$\binom{n}{k} \qquad \text{or} \qquad {}^{n}C_{k}$$

- It is read "n choose k."

For Example:

- A group of 3 lawn tennis players {S,T,U}. A team consisting of 2 players is to be formed. In how many ways can we do so?
- $\binom{3}{2}$ = ST or TS; SU or US; TU or UT.

Combination Formula

Choose k from n without considering the order:

$$^{n}C_k = \frac{n!}{k!(n-k)!}$$

where:

$$\begin{aligned} ^{n}C_k &= \text{number of combinations} \\ n &= \text{total number of objects in the set} \\ k &= \text{number of choosing objects from the set} \end{aligned}$$

Permutation

- It is a way of choosing k objects from n objects when one does care about the order in which one chooses the objects.
- Notation:

$$^{n}P_{k}$$

- It is read "n permute k.' '

For Example:

- You have 3 lawn tennis players $\{S, T, U\}$ in Team A. They need to play 3 matches against Team B, with each player playing one match. In how many different ways can the players from Team A be arranged to play in these matches?
- $^{3}P_{3} =$ STU, SUT, TUS, TSU, UST, and UTS.

Permutation Formula

Choose k from n with considering the order:

$$^{n}P_{k} = \frac{n!}{(n-k)!}$$

where:

$$\begin{aligned} ^{n}P_{k} &= \text{permutation} \\ n &= \text{total number of objects} \\ k &= \text{number of objects selected} \end{aligned}$$

There are a lot more about probability!

The left will be covered in PLSC 502.

- Random variables.
- Probability distribution.
- Sampling.
- Law of large numbers.
- Central limit theorem.
- ...
- ...

Wrap-up Time

Exponential and logarithm function

- Natural exponential function
- Natural logarithm function
- Their derivatives

Introduction to Probability

- Random event; outcome; sample space
- Simple event and compound event
- Independence; mutually exclusive; collectively exhaustive
- Conditional probability
- Joint probability; union probability
- Combination and permutation
- n factorial