Math for Political Scientists Workshop

Day One: Intro to Calculus

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Welcome!

Introduction

- About Myself
- About You (name, major fields, interest, hobbies, and a fun fact)
- About the Workshop

Introduction

Why Math for Political Science?

- We need it! To master methods necessary for research!

" ...political science research has come to increasingly rely on quantitative and formal methods. Having a solid foundation in the underlying mathematical concepts will help you do better research in these areas. You will find the mathematical skills and intuition you gain ... useful to understand and conduct applied research in the discipline."

– ***Kosuke Imai***

Introduction

We have four main goals:

1. To enhance preparation for our methods sequence courses, specifically PLSC 502 and PLSC 503.
2. To improve comfort with comprehending all of the math concepts and techniques commonly used in the study of quantitative methodology in political science.
3. To provide useful resources that you can utilize for self-study in the future as needed.
4. To get familiar with your cohort.

Basic Calculus

What is Calculus?

Calculus is the mathematical study of **continuous change**.

It has two major branches:

- differential calculus [1]
- integral calculus [2]

[1] which concerns **instantaneous rates of change** (the slopes of curves).
[2] which concerns the accumulation of change.

Why should we care about it?

Broadly,

- As students of social sciences, we are interested in the relationships between variables in social sciences: **how a change in one variable affects/is associated with the other**.
- Optimization:finding (local) maximum/minimum values of a function.
  - Statistical Inference
  - Formal Modeling

Why should we care about it?

Specifically,

Differentiation:

- Ordinary Least Square Estimation (PLSC 502): finding the estimates of the regression coefficients that minimize the sum of squared residuals.
- Maximum Likelihood Estimation/Logistic Regression (PLSC 503): finding the parameters that maximize the likelihood function.

Integral:

- Probability: probability density function (PLSC 502).

Section 1: Differential Calculus

Essentially, we are interested in modeling and answering questions involving a rate of change: **what a change in X causes the change in Y?**

Let X be a set of discrete values such that x_1, $x_2 \in$ X and $x_1 \neq x_2$, and Y=f(x).

- One way is to calculate the **average rate of change**: $\frac{\Delta Y}{\Delta X} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$
- However, the average rate of change has *limitations* because it only can present the rate of change over two discrete values (a discrete interval).
- What if we want to know the rate of change at a specific value (**instantaneous rate of change**), rather than over an interval? What should we do?
- $\rightarrow$ Differential Calculus

Instantaneous Rate of Change: Derivative

Average rate of change: $m = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

Let $x_2 = x_1 + h$, where h is any real number.

$$m = \frac{f(x_1+h)-f(x_1)}{(x_1+h)-x_1} = \frac{f(x_1+h)-f(x_1)}{h}$$

Definition:
We take the limit as $h \to 0$, so we get the instantaneous rate of change (the derivative):

$$\lim_{h\to 0} \frac{f(x_1+h)-f(x_1)}{h}$$

Derivative in Geometry

Geometrically, the derivative is the slope of the tangent line to the graph of f at x_0. The tangent line is a limit of secant lines just as the derivative is a limit of difference quotients.

"https://en.wikipedia.org/wiki/Calculus#/media/File:Sec2tan.gif"

(Simon and Blume, 1994, page 24)

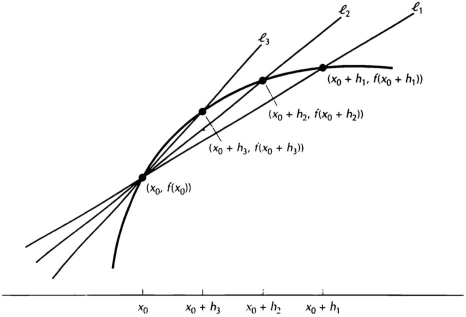


Figure 2.10 *Approximating the tangent line by a sequence of secant lines.*

Notations: Derivative

If $y = f(x)$, the derivative of f(x) or y:

Leibniz's Notation:

$$\frac{d}{dx}f(x) = \frac{dy}{dx}$$

which is read: the derivative of y with respect to x.

Lagrange's Notation:

$f'(x)$, which is read: f prime x.

Definition:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \frac{d}{dx}f(x) = \frac{dy}{dx}$$

Computing Derivative

Example 1.

Using the definition to calculate the derivative of $f(x) = 3x$:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h) - 3(x)}{h} \\
&= \lim_{h\to 0} \frac{3x + 3h - 3x}{h} = \lim_{h\to 0} \frac{3h}{h} \\
&\approx 3
\end{aligned}
$$

Computing Derivative

Example 2.

Using the definition to calculate the derivative of $f(x) = x^2$:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h} \\
&= \lim_{h \to 0} 2x + h \\
&\approx 2x
\end{aligned}
$$

The Rules of Differentiation

- Power Rule[3]

$$(x^n)' = nx^{n-1}$$

- Other Rules of Differentiation

Sum rule	$(f(x)+g(x))' = f'(x)+g'(x)$
Difference rule	$(f(x)-g(x))' = f'(x)-g'(x)$
Multiply by constant rule	$f'(ax) = af'(x)$
Product rule	$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$
Quotient rule	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain rule	$(g(f(x))' = g'(f(x))f'(x)$
Inverse function rule	$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$
Constant rule	$(a)' = 0$

[3] To Look for a formal proof, please read Simon and Blume (1994) on page 27.

The Rules of Differentiation

- Other Rules of Differentiation

Exponential rule 1	$(e^x)' = e^x$
Exponential rule 2	$(a^x)' = a^x(\ln(a))$
Logarithm rule 1	$(\ln(x))' = \frac{1}{x}$
Logarithm rule 2	$(\log_a(x))' = \frac{1}{x(\ln(a))}$

We will talk more about exponential and logarithm functions on Day 2!

Easy Exercise

- f(x)=$5x^2 + 5$
- f(x)=$3x^3 - 4x + 8$
- y=(x+5)(x+6)

A Little Bit Hard Exercise

Suppose $y = \frac{f(x)}{g(x)}$, $f(x) = 3x - 7$, and $g(x) = x^3 + 6$,

- Use the quotient rule to differentiate y.
- Alternatively, let $h(z) = z^{-1}$ and $z = g(x)$, so that $y = f(x) \cdot h(g(x))$. Use the product rule and chain rule to differentiate y.

Exercise Solutions

We know that $y = \frac{f(x)}{g(x)} = \frac{3x-7}{x^3+6}$, then $f'(x) = 3$ and $g'(x) = 3x^2$

By quotient rule,

$$\begin{aligned} y' &= \frac{d}{dx}(\frac{f(x)}{g(x)}) \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{3(x^3+6) - (3x-7)(3x^2)}{(x^3+6)^2} \\ &= \frac{3x^3 + 18 - (9x^3 - 21x^2)}{x^6 + 12x^3 + 36} \\ &= \frac{-6x^3 + 18 + 21x^2}{x^6 + 12x^3 + 36} \end{aligned}$$

Exercise Solutions

We know that $y = \frac{f(x)}{g(x)}$, $f(x) = 3x - 7$, and $g(x) = x^3 + 6$,

- Let $h(z) = z^{-1}$ and $z = g(x)$, so that $y = f(x) \cdot h(g(x))$.
- By product rule, $y' = f'(x) \cdot h(g(x)) + f(x) \cdot (h(g(x)))'$.
- Since $h(z) = z^{-1}$ and $z = g(x)$, we know that $h' = \frac{d}{dx}(g(x))^{-1}$.
- By chain rule, we know $(h(g(x)))' = h'(g(x))g'(x) = (-1)(x^3 + 6)^{-2}(3x^2)$

$$
\begin{aligned}
y' &= f'(x) \cdot h(g(x)) + f(x) \cdot (h(g(x)))' \\
&= 3(x^3 + 6)^{-1} + (3x - 7)(-1)(x^3 + 6)^{-2}(3x^2) \\
&= \frac{3(x^3 + 6)}{(x^3 + 6)^2} - \frac{(3x - 7)3x^2}{(x^3 + 6)^2} \\
&= \frac{-6x^3 + 18 + 21x^2}{x^6 + 12x^3 + 36}
\end{aligned}
$$

Compute Derivaties in R

- Compute Derivative of an Expression in R Programming

```
# Function Expression
y <- expression((3*x-7)/(x^3+6))

# Derivative
print(D(y, "x"))
```

```
## 3/(x^3 + 6) - (3 * x - 7) * (3 * x^2)/(x^3 + 6)^2
```

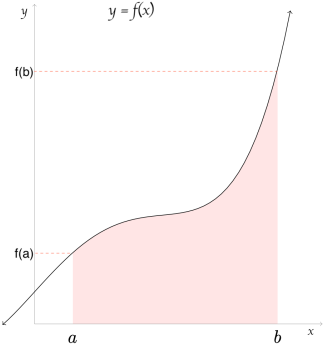
Section 2: Integral Calculus

Intuition

Why Integral Calculus?

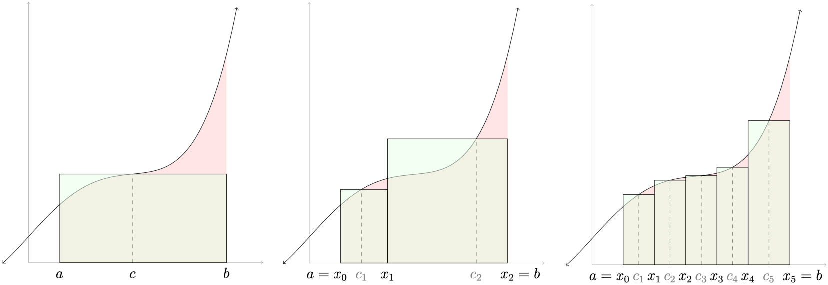
The derivative presents us with the instantaneous change in a continuous function at each point. But what if we care less about change than about the **net effect of change** (the sum of changes)?

Eg., below is the graph of a certain derivative function $y = f(x)$. What if we would like to know the net effect of change between x=a and x=b? (The area under the curve!)



The Definite Integral As A Limit of Sums

- We approximate the (red) shadow with a set of rectangles.



$$\begin{aligned} S_{approx} &= f(c_1)[x_1 - x_0] + f(c_2)[x_2 - x_1] + f(c_3)[x_3 - x_2] + f(c_4)[x_4 - x_3] + f(c_5)[x_5 - x_4] \\ &= f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4 + f(c_5)\Delta x_5 \\ &= \sum_{i=1}^{5} f(c_i)\Delta x_i \end{aligned}$$

Notations: Integral

And there we have our definition of integral: if

$$lim_{\|\Delta\|\to 0} \sum_{i}^{a\to b} f(c_i)\Delta x_i = L$$

then

$$\int_a^b f(x)dx = L$$

The integral sign $\int$, like an S, represents integration, which can remind you that it is in essence a sum. The symbol dx, called the differential of the variable x, indicates that the variable of integration is x. The function $f(x)$ is called the integrand, the points a and b are called the limits/bounds of integration. If the limits are specified, the integral is called a *definite integral*.

When the limits are omitted, as in, $\int f(x)dx = F(x)$, the integral is called an *indefinite integral*, which represents a class of functions (the antiderivative) whose derivative is the integrand.

Notations: From Difinite Integral to Indifinite Integral

- The **definite integral** concerns a *value*, the area under the curve in the plane that is bounded by the graph of a given function between two points in the real line.

$$\int_a^b f(x)dx = L$$

which is read "the integral of $f(x)$ with respect to dx, along the range from a to b."

- The **indefinite integral** concerns a *function* (an antiderivative[4]), whose derivative is the given function.Simply put, an antiderivative of a function f(x) is a function whose derivative is equal to f(x).

$$\int f(x)dx = F(x)$$

which is read "the indefinite integral of $f(x)$ with respect to dx."

[4] It can be thought of as the inverse operation for the derivative.

The Foundamental Theorem of Calculus

The foundamental theorem of calculus states that when the derivative of a function is integrated, the result is the orignal function evaluated at the bounds of the integration.

$$\int_a^b f(x)dx = F(b) - F(a)$$

In other words, the definite integral of a function from a to b is equal to the antiderivative of that function evaluated at b minus the same evaluated at a. Notation:

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\ |_a^b$$

where the vertical line means "evaluate the antiderivative F(x) at b and subtract the antiderivative evaluated at a."

Rules of Integration

- List of Main Rules of Integration

Fundamental theorem of calculus	$\int_a^b f(x)dx = F(b) - F(a)$
Rules for bounds	$\int_a^b f(x)dx = -\int_b^a f(x)dx$ $\int_a^a f(x)dx = 0$ $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $c \in [a, b]$
Linear rule	$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$
Integration by substitution	$\int_a^b f(g(u))g'(u)du = \int_{g(a)}^{g(b)} f(x)dx$
Integration by parts	$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$
Power rule 1	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$
Power rule 2	$\int x^{-1} dx = \ln \lvert x \rvert + C$

Rules of Integration

- List of Other Rules of Integration

Exponential rule 1	$\int e^x dx = e^x + C$
Exponential rule 2	$\int a^x dx = \frac{a^x}{\ln(a)} + C$
Logarithm rule 1	$\int \ln(x) dx = x\ln(x) - x + C$
Logarithm rule 2	$\int \log_a(x) dx = \frac{x\ln(x) - x}{\ln(a)} + C$
Trigonometric rules	$\int \sin(x) dx = -\cos(x) + C$
	$\int \cos(x) dx = \sin(x) + C$
	$\int \tan(x) dx = -\ln(\lvert\cos(x)\rvert) + C$
Piecewise rules	Split definite integral into corresponding pieces

Examples of Evaluating Indefinite Integrals

Example 1: Simple Power Rule

Evaluate the indefinite integral:

$$\int (3x^2 + 2x + 5)\, dx$$

Solution:

Evaluate the indefinite integral:

$$\int (3x^2 + 2x + 5)\, dx$$

Solution:

Using the power rule, integrate each term individually:

$$\int 3x^2\, dx = 3 \cdot \frac{x^{2+1}}{2+1} = x^3$$

$$\int 2x\, dx = 2 \cdot \frac{x^{1+1}}{1+1} = x^2$$

$$\int 5\, dx = 5x$$

So, the solution is:

$$\int (3x^2 + 2x + 5)\, dx = x^3 + x^2 + 5x + C$$

Evaluate the indefinite integral:

$$\int (4x+1)^2 \, dx$$

Solution:

Solution:

Expand the integrand first:

$$\int (4x+1)^2 \, dx = \int (16x^2 + 8x + 1) \, dx$$

Now integrate each term:

$$\int 16x^2 \, dx = 16 \cdot \frac{x^{2+1}}{2+1} = \frac{16x^3}{3}$$

$$\int 8x \, dx = 8 \cdot \frac{x^{1+1}}{1+1} = 4x^2$$

$$\int 1 \, dx = x$$

So, the solution is:

$$\int (4x+1)^2 \, dx = \frac{16x^3}{3} + 4x^2 + x + C$$

Evaluate the indefinite integral:

$$\int 7(2x+3)\,dx$$

Solution:

First, distribute the constant:

$$\int 7(2x+3)\,dx = 7\int (2x+3)\,dx$$

Now, integrate term by term:

$$7\left(\frac{2x^2}{2}+3x\right)+C$$

Simplify:

$$= 7(x^2+3x)+C = 7x^2+21x+C$$

Compute Antiderivative in R

- Compute Antiderivative of an Expression in R Programming

```r
# Install the packages if they are not already installed
#install.packages("mosaic")
#install.packages("mosaicCalc")

# Load the necessary libraries
library(mosaic)
library(mosaicCalc)

# Find the antiderivative
antiD(a * x^2+2*x+5 ~ x)
```

```
## function (x, a, C = 0)
## (x^3 * a + 3 * x^2 + 15 * x)/3 + C
```

Wrap-up Time

- Calculus
- Differential Calculus
- Derivative
- How to get derivative?
- Integral
- Definite vs. indefinite integral
- How to calculate definite vs. indefinite integral?

Supplementary

Second-Order Derivative

- the derivative of the first derivative of the given function.
- denoted as $f''(x)$.

Critical Point

- A value x_0 in the domain of f where f is not differentiable or its derivative is 0 (i.e., $f'(x_0) = 0$ or f is not differentiable at x_0).[5]

If x_0 is a critical point of a function f, how can we use calculus to decide whether critical point x_0 is a max, a min, or neither? The answer to this question lies in the *second* derivative of f at x_0.

Theorem

(a) *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a max of f;*

(b) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a min of f; and*

(c) *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then x_0 can be a max, a min, or neither.*

[5] A continuous function is not necessarily differentiable, but a differentiable function is necessarily continuous (at every point where it is differentiable).

More Exercises