CHAPTER 8

Matrix Algebra

Matrices were introduced in the previous chapter to organize our calculations for solving systems of linear equations. Matrices play an important role in many other areas of economics and applied mathematics. The *input-output matrix* of Example 2 in Chapter 6, and the *Markov matrix* of Example 3 are but two examples. Other examples include *payoff matrices* from the theory of games, *coefficient matrices* and *correlation matrices* from econometrics, *Slutsky* and *Antonelli matrices* from consumer theory, and the *Hessian* and *bordered Hessian* matrices that embody the second order conditions in multivariable optimization theory.

A **matrix** is simply a rectangular array of numbers. So, any table of data is a matrix. The size of a matrix is indicated by the number of its rows and the number of its columns. A matrix with k rows and n columns is called a $k \times n$ ("k by n") matrix. The number in row i and column j is called the (i, j)th entry, and is often written a_{ij}, as we did in Chapter 7. Two matrices are *equal* if they both have the same size and if the corresponding entries in the two matrices are equal.

Matrices are in a sense generalized numbers. When the sizes are right, two matrices can be added, subtracted, multiplied and even divided. Whenever an economic model uses matrices, we can learn a lot about the underlying model via these algebraic operations. In this chapter, we describe the algebra of matrices. This chapter is a bit more abstract than previous chapters since it focuses on algebraic operations and their properties. But we will use these operations throughout this book. We illustrate this use in Section 8.5 where we derive the basic property of Leontief input-output models.

8.1 MATRIX ALGEBRA

Addition

We begin with addition of matrices. One can add two matrices of the same size, which is to say, with the same number of rows and columns. Their sum is a new matrix of the same size as the two matrices being added. The (i, j)th entry of the sum matrix is simply the sum of the (i, j)th entries of the two matrices being added.

In symbols

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & b_{ij} & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & a_{ij} + b_{ij} & \vdots \\ a_{k1} + b_{k1} & \cdots & a_{kn} + b_{kn} \end{pmatrix}$$

For example,

$$\begin{pmatrix} 3 & 4 & 1 \\ 6 & 7 & 0 \\ -1 & 3 & 8 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 7 \\ 6 & 5 & 1 \\ -1 & 7 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 8 \\ 12 & 12 & 1 \\ -2 & 10 & 8 \end{pmatrix}$$

but
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$$

is not defined.

The matrix $\mathbf{0}$ whose entries are all zero is an *additive identity* since

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} + \begin{pmatrix} 0_{11} & \cdots & 0_{1n} \\ \vdots & 0_{ij} & \vdots \\ 0_{k1} & \cdots & 0_{kn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix},$$

that is, $A + \mathbf{0} = A$ for all matrices A.

Subtraction

Since $-A$ is what one adds to A to obtain $\mathbf{0}$,

$$-\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} = \begin{pmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & -a_{ij} & \vdots \\ -a_{k1} & \cdots & -a_{kn} \end{pmatrix}.$$

Since $A - B$ is just shorthand for $A + (-B)$, we *subtract* matrices of the same size simply by subtracting their corresponding entries:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} - \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & b_{ij} & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - b_{11} & \cdots & a_{1n} - b_{1n} \\ \vdots & a_{ij} - b_{ij} & \vdots \\ a_{k1} - b_{kn} & \cdots & a_{kn} - b_{kn} \end{pmatrix}$$

Scalar Multiplication

Matrices can be multiplied by ordinary numbers, which we also call **scalars**. This operation is called **scalar multiplication**. Implicitly we have already used this operation in defining $-A$, which is $(-1)A$. More generally, the product of the matrix A and the number r, denoted rA, is the matrix created by multiplying each entry of A by r.

$$r\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} = \begin{pmatrix} ra_{11} & \cdots & ra_{1n} \\ \vdots & ra_{ij} & \vdots \\ ra_{k1} & \cdots & ra_{kn} \end{pmatrix}.$$

In summary, within the class of $k \times n$ matrices, addition, subtraction, and scalar multiplication are all defined in the obvious way and act just as one would expect.

Matrix Multiplication

Just as two numbers can be multiplied together, so can two matrices. But at this point matrix algebra becomes a little bit more complicated than the algebra for real numbers. There are two differences: Not all pairs of matrices can be multiplied together, and the order in which matrices are multiplied can matter.

We can define the matrix product AB if and only if

number of columns of A = number of rows of B.

For the matrix product to exist, A must be $k \times m$ and B must be $m \times n$. To obtain the (i, j)th entry of AB, multiply the ith row of A and the jth column of B as follows:

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{im} \end{pmatrix} \cdot \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}.$$

In other words, the (i, j)th entry of the product AB is defined to be

$$\sum_{h=1}^{m} a_{ih}b_{hj}.$$

For example,

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \\ eA + fC & eB + fD \end{pmatrix}$$

Note that in this case, the product taken in reverse order,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix},$$

is not defined. See Exercise 8.2.

If A is $k \times m$ and B is $m \times n$, then the product AB will be $k \times n$. The product matrix AB inherits the number of its rows from A and the number of its columns from B:

$$\begin{aligned} \text{number of rows of } AB &= \text{number of rows of } A; \\ \text{number of columns of } AB &= \text{number of columns of } B; \\ (k \times m) \cdot (m \times n) &= (k \times n). \end{aligned}$$

The $n \times n$ matrix

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

with $a_{ii} = 1$ for all i and $a_{ij} = 0$ for all $i \neq j$, has the property that for any $m \times n$ matrix A,

$$AI = A,$$

and for any $n \times l$ matrix B,

$$IB = B.$$

The matrix I is called the $n \times n$ **identity matrix** because it is a multiplicative identity for matrices just as the number 1 is for real numbers.

Laws of Matrix Algebra

We can think of matrices as generalized numbers because matrix addition, subtraction and multiplication obey most of the same laws that numbers do.

Associative Laws: $\quad (A + B) + C = A + (B + C),$

$$(AB)C = A(BC),$$

Commutative Law for Addition: $\quad A + B = B + A,$

Distributive Laws:

$$A(B+C) = AB + AC,$$
$$(A+B)C = AC + BC.$$

The one important law which numbers satisfy but matrices do not, is the *commutative law for multiplication*. Although $ab = ba$ for all numbers a and b, it is not true that $AB = BA$ for matrices, *even when both products are defined*. We have already seen examples where only one product is defined. But notice that even if both products exist, they need not be the same size. For example, if A is 2×3 and B is 3×2, then AB is 2×2 while BA is 3×3. Even if AB and BA have the same size, AB need not equal BA. For example,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix},$$

while

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}.$$

Transpose

Finally, there is one other operation on matrices which we shall frequently use. The **transpose** of a $k \times n$ matrix A is the $n \times k$ matrix obtained by interchanging the rows and columns of A. This matrix is often written as A^T. The first row of A becomes the first column of A^T. The second row of A becomes the second column of A^T, and so on. Thus, the (i, j)th entry of A becomes the (j, i)th entry of A^T. For example,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix},$$

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}^T = (a_{11} \quad a_{21}).$$

The following rules are fairly straightforward to verify:

$$(A+B)^T = A^T + B^T,$$
$$(A-B)^T = A^T - B^T,$$
$$(A^T)^T = A,$$
$$(rA)^T = rA^T,$$

where A and B are $k \times n$ and r is a scalar. The following rule is not so obvious and takes a little work to prove:

$$(AB)^T = B^T A^T.$$

Note the change in the order of the matrix multiplication.

Theorem 8.1 Let A be a $k \times m$ matrix and B be an $m \times n$ matrix. Then, $(AB)^T = B^T A^T$.

Proof We will be working with six different matrices: $A, B, A^T, B^T, (AB)^T$, and $B^T A^T$. For notation's sake, if C is any of these matrices, we will write C_{ij} for the (i, j)th element of C. For example, $((AB)^T)_{ij}$ will denote the (i, j)th element of the matrix $(AB)^T$. Now,

$$\begin{aligned}
((AB)^T)_{ij} &= (AB)_{ji} && \text{(definition of transpose)} \\
&= \sum_h A_{jh} \cdot B_{hi} && \text{(definition of matrix multiplication)} \\
&= \sum_h (A^T)_{hj} \cdot (B^T)_{ih} && \text{(definition of transpose, twice)} \\
&= \sum_h (B^T)_{ih} \cdot (A^T)_{hj} && (a \cdot b = b \cdot a \text{ for scalars}) \\
&= (B^T A^T)_{ij} && \text{(definition of matrix multiplication.)}
\end{aligned}$$

Therefore, $(AB)^T = B^T A^T$ ■

Systems of Equations in Matrix Form

The algebra that we have developed so far is already very powerful. Consider the systems of linear equations from the previous chapter. The typical system looked like

$$\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad & \quad \vdots \\
a_{k1}x_1 + \cdots + a_{kn}x_n &= b_k.
\end{aligned}$$

This system can be expressed much more compactly using the notation suggested by matrix algebra. As before, let A denote the coefficient matrix of the system:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}$$

Also, let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}.$$

Both $\mathbf{x}$ and $\mathbf{b}$ are matrices, called column matrices. The $n \times 1$ matrix $\mathbf{x}$ contains variables, and the $k \times 1$ matrix $\mathbf{b}$ contains the parameters from the right-hand side of the system. Then, the system of equations can be written as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix},$$

or simply as

$$A\mathbf{x} = \mathbf{b},$$

where $A\mathbf{x}$ refers to the matrix product of the $k \times n$ matrix A with the $n \times 1$ matrix $\mathbf{x}$. This product is a $k \times 1$ matrix, which must be made equal to the $k \times 1$ matrix $\mathbf{b}$. Check that carrying out the matrix multiplication in $A\mathbf{x} = \mathbf{b}$ and applying the definition of equality of matrices gives back exactly the original system of linear equations. The matrix notation is much more compact than writing out arrays of coefficients, and, as we shall see, it suggests how to find the solution to the system by analogy with the one-variable case.

EXERCISES

8.1 Let

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix},$$

$$D = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad E = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

a) Compute each of the following matrices if it is defined:

$$\begin{array}{llllll} A+B, & A-D, & 3B, & DC, & B^T, & A^TC^T, \\ C+D, & B-A & AB, & CE, & -D, & (CE)^T, \\ B+C, & D-C, & CA, & EC, & (CA)^T, & E^TC^T. \end{array}$$

b) Verify that $(DA)^T = A^TD^T$.

c) Verify that $CD \neq DC$.

8.2 Check that

$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 5 & 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 2 & 3 \\ 10 & 21 \end{pmatrix}.$$

Note that the reverse product is not defined.

8.3 Show that if AB is defined, then B^TA^T is defined but A^TB^T need not be defined.

8.4 If you choose four numbers at random for the entries of a 2×2 matrix A, and four others for another 2×2 matrix B, AB will probably not equal BA. Carry out this procedure a few times.

8.5 It sometimes happens that $AB = BA$.

a) Check this for $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$.

b) Show that if B is a scalar multiple of the 2×2 identity matrix, then $AB = BA$ for *all* 2×2 matrices A.

8.2 SPECIAL KINDS OF MATRICES

Special problems use special kinds of matrices. In this section we describe some of the important classes of $k \times n$ matrices which arise in economic analysis.

Square Matrix. $k = n$, that is, equal number of rows and columns.

Column Matrix. $n = 1$, that is, one column. For example,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Row Matrix. $k = 1$, that is, one row. For example,

$$(2 \quad 1 \quad 0) \quad \text{and} \quad (2 \quad 3).$$

Diagonal Matrix. $k = n$ and $a_{ij} = 0$ for $i \neq j$, that is, a square matrix in which all nondiagonal entries are 0. For example,

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Upper-Triangular Matrix. $a_{ij} = 0$ if $i > j$, that is, a matrix (usually square) in which all entries below the diagonal are 0. For example,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

Lower-Triangular Matrix. $a_{ij} = 0$ if $i < j$, that is, a matrix (usually square) in which all entries above the diagonal are 0. For example,

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$$

Symmetric Matrix. $A^T = A$, that is, $a_{ij} = a_{ji}$ for all i, j. These matrices are necessarily square. For example,

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}.$$

Idempotent Matrix. A square matrix B for which $B \cdot B = B$, such as $B = I$ or

$$\begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}.$$

Permutation Matrix. A square matrix of 0s and 1s in which each row and each column contains exactly one 1. For example,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Nonsingular Matrix. A square matrix whose rank equals the number of its rows (or columns). When such a matrix arises as a coefficient matrix in a system of linear equations, the system has one and only one solution.

EXERCISES

8.6 Give an example with more than two rows or more than two columns of each of the above types of matrices.

8.7 Show that $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$ are idempotent.

8.8 Let D, U, L, and S denote, respectively, the sets of all diagonal, upper-triangular, lower-triangular, and symmetric matrices.

a) Show that D, U and L are each closed under matrix addition and multiplication, that is, that the sum or product of two matrices in one of the above sets is also a matrix in that set.

b) Show that $D \cap U = D$, $S \cap U = D$, and $D \subset S$.

c) Show that all matrices in D commute with each other. Is this true for matrices in U or S, too?

d) Show that S is closed under addition but not under multiplication.

8.9 How many $n \times n$ permutation matrices are there?

8.10 Is the set of $n \times n$ permutation matrices closed under addition or under matrix multiplication?

8.3 ELEMENTARY MATRICES

Another important class of matrices is the class of **elementary matrices**. Recall that the three elementary row operations that are used to bring a matrix to row echelon form are:

(1) interchanging rows,
(2) adding a multiple of one row to another, and
(3) multiplying a row by a nonzero scalar.

These operations can be performed on a matrix A by premultiplying A by certain special matrices called *elementary matrices*. For example, the following theorem illustrates how to interchange rows i and j of a given matrix A.

Theorem 8.2 Form the permutation matrix E_{ij} by interchanging the ith and jth rows of the identity matrix I. Left-multiplication of a given matrix A by E_{ij} has the effect of interchanging the ith and jth rows of A.

Proof To see this, let e_{hk} denote a generic element of E_{ij}:

$$\begin{aligned} e_{ij} &= e_{ji} = 1, \\ e_{ii} &= e_{jj} = 0, \\ e_{hh} &= 1 \qquad \text{if } h \neq i, j, \\ e_{hk} &= 0 \qquad \text{otherwise.} \end{aligned} \tag{1}$$

The element in row k and column n of $E_{ij}A$ is

$$\sum_m e_{km}a_{mn} = \begin{cases} a_{jn} & k = i, \\ a_{in} & k = j, \\ a_{kn} & k \neq i, j, \end{cases}$$

using (1). Therefore, $E_{ij}A$ is simply A with rows i and j interchanged. ∎

To carry out Row Operation 3, the multiplication of row i by the scalar $r \neq 0$, construct the matrix $E_i(r)$ by multiplying the ith row of the identity matrix I by the scalar r. The effect of premultiplication of A by $E_i(r)$ is to multiply each entry of the ith row of A by r. For example, in the case of the general 3×3 matrix A,

$$E_2(5) \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 5a_{21} & 5a_{22} & 5a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Finally, to perform Row Operation 2, the addition of r times the ith row of A to the jth row of A, form the matrix $E_{ij}(r)$ by adding r times row i to row j in the identity matrix I. In other words, replace the zero in column i and row j of I with r. Premultiplication of A by $E_{ij}(r)$ will add r times row i to row j in matrix A while leaving the entries in all other rows of A unchanged. For example, in the 3×3 case

$$\begin{aligned} E_{23}(5) \cdot A &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 5a_{21} + a_{31} & 5a_{22} + a_{32} & 5a_{23} + a_{33} \end{pmatrix}. \end{aligned}$$

Definition The matrices E_{ij}, $E_{ij}(r)$ and $E_i(r)$, which are obtained by performing an elementary row operation on the identity matrix, are called **elementary matrices**.

We summarize this discussion in the following theorem, whose proof is left as an exercise.

Theorem 8.3 Let E be an elementary $n \times n$ matrix obtained by performing a particular row operation on the $n \times n$ identity matrix. For any $n \times m$ matrix A, EA is the matrix obtained by performing that same row operation on A.

In Chapter 7, we showed that elementary row operations can be used to reduce any matrix to row echelon form. The matrix version of that fact is stated in the next theorem, whose proof is also left as an exercise.

Theorem 8.4 For any $k \times n$ matrix A there exist elementary matrices $E_1, E_2, \ldots, E_m$ such that the matrix product $E_m \cdot E_{m-1} \cdots E_1 \cdot A = U$ where U is in (reduced) row echelon form.

One can represent an elementary equation operation on the linear system $A\mathbf{x} = \mathbf{b}$ by multiplying both sides of the equation by the corresponding elementary matrix E to obtain the new system $EA\mathbf{x} = E\mathbf{b}$. This fact illustrates the convenience of matrix notation for representing systems of equations.

Example 8.1 Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}.$$

To bring $\mathbf{A}$ to row echelon form, we first add -12 times row 1 to row 2. This operation corresponds to the elementary matrix

$$E_{12}(-12) = \begin{pmatrix} 1 & 0 & 0 \\ -12 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then add -3 times row 1 to row 3 and finally $1/10$ times row 2 to row 3. These operations correspond to the elementary matrices

$$E_{13}(-3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

and

$$E_{23}(0.1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & .1 & 1 \end{pmatrix},$$

respectively. Check that the row echelon form of $\mathbf{A}$ is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -10 & -15 \\ 0 & 0 & -3.5 \end{pmatrix} = E_{23}(0.1) \cdot E_{13}(-3) \cdot E_{12}(-12) \cdot \mathbf{A}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -12 & 1 & 0 \\ -4.2 & .1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}.$$

EXERCISES

8.11 Carry out the matrix multiplication in Example 8.1.

8.12 Prove Theorem 8.3.

8.13 Prove Theorem 8.4.

8.14 *a*) Prove the following statement. If P is an $m \times m$ permutation matrix and A is $m \times n$, then PA is the matrix A with its rows permuted according to P. If $p_{ij} = 1$, then the ith row of PA will be the jth row of A.

b) State and prove a similar statement about the permutation of columns by the multiplication AP.

8.4 ALGEBRA OF SQUARE MATRICES

Within the class $\mathbf{M_n}$ of $n \times n$ (square) matrices, all the arithmetic operations defined so far can be used. The sum, difference and product of two $n \times n$ matrices is $n \times n$. Even transposes of matrices in $\mathbf{M_n}$ are $n \times n$. The $n \times n$ identity matrix I is a true multiplicative identity in $\mathbf{M_n}$ in that $AI = IA = A$ for all A in $\mathbf{M_n}$. The matrix I plays the role in $\mathbf{M_n}$ that the number 1 plays among the real numbers ($\mathbf{M_1}$). Recall, however, that if A and B are in $\mathbf{M_n}$, AB usually will not equal BA.

Since we can add, subtract, and multiply square matrices, it is reasonable to ask if we can divide square matrices too. For numbers, dividing by a is the same as multiplying by $1/a = a^{-1}$, and a^{-1} makes sense as long as $a \neq 0$. To carry out this program for matrices (if we can), we need to make sense of A^{-1} for matrices in $\mathbf{M_n}$. The number a^{-1} is defined to be that number b such that $ab = ba = 1$. The number b is called the inverse of the number a. We do the same for matrices in $\mathbf{M_n}$.

Definition Let A be a matrix in $\mathbf{M_n}$. The matrix B in $\mathbf{M_n}$ is an **inverse** for A if $AB = BA = I$.

If the matrix B exists, we say that A is **invertible**. Our definition has left open the possibility that a matrix A can have several inverses. This is not true for numbers, and neither is it true for matrices.

Theorem 8.5 An $n \times n$ matrix A can have at most one inverse.

Proof Suppose that B and C are both inverses of A. Then

$$C = CI = C(AB) = (CA)B = IB = B. \qquad \blacksquare$$

If an $n \times n$ matrix A is invertible, we write A^{-1} for its unique inverse matrix. Note that if A is 1×1, then $A^{-1} = 1/A$. So, multiplying by A^{-1} is the analog of dividing by the matrix A.

The only 1×1 matrix which is not invertible is 0. A main goal of this section is to identify exactly which $n \times n$ matrices are not invertible. We will see that a matrix is invertible if and only if it is nonsingular. In fact, the two properties reinforce each other. Recall that a square matrix $\mathbf{A}$ is called nonsingular if and only if the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for every right-hand side $\mathbf{b}$. Theorem 8.6 below states that if a square matrix has an inverse, then it is nonsingular. The proof of this theorem shows how to use the inverse of $\mathbf{A}$ to solve a general system $\mathbf{Ax} = \mathbf{b}$. Theorem 8.7 below is the converse statement: if a matrix is nonsingular, then it is invertible. Its proof shows how to use the fact that $\mathbf{A}$ is nonsingular to compute the inverse of $\mathbf{A}$. Before proving these theorems, we need two more definitions and a lemma.

Definition Let A be an $k \times n$ matrix. The $n \times k$ matrix B is a **right inverse** for A if $AB = I$. The $n \times k$ matrix C is a **left inverse** for A if $CA = I$.

Example 8.2 The matrix $\begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}$ is a right inverse for the matrix $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}$, but not a left inverse. On the other hand, the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ is a left inverse for $\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}$, but not a right inverse.

Lemma 8.1 If A has a right inverse B and a left inverse C, then A is invertible, and $B = C = A^{-1}$.

Proof Exactly the same as the proof of Theorem 8.5. ■

Theorem 8.6 If an $n \times n$ matrix A is invertible, then it is nonsingular, and the unique solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof We want to show that if A is invertible, we can solve any system of equations $A\mathbf{x} = \mathbf{b}$. Multiply each side of this system by A^{-1} to solve for $\mathbf{x}$, as follows:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b}, \\ (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b}, \\ I\mathbf{x} &= A^{-1}\mathbf{b}, \\ \mathbf{x} &= A^{-1}\mathbf{b}. \end{aligned}$$

Make sure you can justify all the steps in this calculation. ■

Theorem 8.7 If an $n \times n$ matrix A is nonsingular, then it is invertible.

Proof Suppose that A is nonsingular. We shall prove that it has an inverse by showing how to compute this inverse. Let $\mathbf{e}_i$ denote the ith column of I. For example, when $n = 3$,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Write I with a focus on its columns as $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$. Since A is nonsingular, the equation $A\mathbf{x} = \mathbf{e}_i$ has a unique solution $\mathbf{x} = \mathbf{c}_i$. (Of course, $\mathbf{c}_i$ is an $n \times 1$ matrix.) Let C be the matrix whose n columns are the respective solutions $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$. Since one multiplies each row of A by the jth column of C to obtain the jth column of AC, we can write

$$\begin{aligned} AC &= A\,[\mathbf{c}_1, \ldots, \mathbf{c}_n] \\ &= [A\mathbf{c}_1, \ldots, A\mathbf{c}_n] \\ &= [\mathbf{e}_1, \ldots, \mathbf{e}_n] \\ &= I. \end{aligned} \tag{2}$$

So C is a right inverse of A.

To see that A has a left inverse too, use Theorem 8.4 to write $EA = U$ where E is a product of elementary matrices and U is the *reduced* row echelon form of A. Since A is nonsingular, U has no zero rows and each column contains exactly one 1. In other words, U is the identity matrix. Therefore, E is a left inverse of A. Since A has a right inverse and a left inverse, it is invertible. ■

Take time to study the calculation labeled (2) in the proof of Theorem 8.7, since we shall use it often. Once again, it follows from the fact that, to obtain the jth column of AC, one multiplies the rows of A by the jth column of C. No other column of C enters this calculation. In other words, if $\mathbf{c}_j$ is the jth column of C, then $A\mathbf{c}_j$ is the jth column of AC.

The proof of Theorem 8.7 actually shows how to compute the inverse of a nonsingular matrix. To find the ith column $\mathbf{c}_i$ of A^{-1}, we solve the system

$$A\mathrm{x} = \mathbf{e}_i$$

to find the solution $\mathbf{x} = \mathbf{c}_i$. Gauss-Jordan elimination can be used to solve this system for each i. In this case the augmented matrix is $[A \mid \mathbf{e}_i]$. The row operations which will reduce this depend only on the first n columns of the augmented matrix, in other words, only on the matrix A. One never uses the last column of an augmented matrix to determine which row operation to use on a system.

Therefore, the same row operations that reduce $[A \mid \mathbf{e}_i]$ to $[I \mid \mathbf{c}_i]$ will also reduce $[A \mid \mathbf{e}_j]$ to $[I \mid \mathbf{c}_j]$. We can be more efficient and pool all these data into a gigantic augmented matrix $[A \mid \mathbf{e}_1 \cdots \mathbf{e}_n] = [A \mid I]$ and perform Gauss-Jordan elimination only once rather than n times. In this process, the augmented matrix $[A \mid I]$ reduces to $[I \mid A^{-1}]$.

Example 8.3 We can apply this method to find the inverse of matrix A in Example 8.1:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}. \tag{3}$$

First, augment A with the identity matrix:

$$[A \mid I] = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 12 & 2 & -3 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right).$$

Then, perform the row operations on $[A \mid I]$ which reduce A to row echelon form. The first three such operations are described in Example 8.1 and result in the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -10 & -15 & -12 & 1 & 0 \\ 0 & 0 & -3.5 & -4.2 & 0.1 & 1 \end{array}\right).$$

Next reduce this matrix to *reduced row echelon form* using the operations described in Section 7.2:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.4 & \frac{3}{35} & -\frac{1}{7} \\ 0 & 1 & 0 & -0.6 & -\frac{2}{35} & \frac{3}{7} \\ 0 & 0 & 1 & 1.2 & -\frac{1}{35} & -\frac{2}{7} \end{array}\right).$$

As implied by the proof of Theorem 8.7, the right half of this augmented matrix,

$$\begin{pmatrix} 0.4 & \frac{3}{35} & -\frac{1}{7} \\ -0.6 & -\frac{2}{35} & \frac{3}{7} \\ 1.2 & -\frac{1}{35} & -\frac{2}{7} \end{pmatrix}, \tag{4}$$

is the inverse of A.

Example 8.4 We next apply this method to compute the inverse of an arbitrary 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5}$$

Begin by writing the augmented matrix

$$[A \mid I] = \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right).$$

If a and c are both $0, A$ will clearly be singular. Let us assume, then, that $a \neq 0$. First, add $-c/a$ times row 1 to row 2, to obtain the row echelon form

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \dfrac{ad-bc}{a} & -\dfrac{c}{a} & 1 \end{array}\right). \tag{6}$$

This short calculation tells us that when $a \neq 0, A$ is nonsingular (and therefore invertible) if and only if $ad - bc \neq 0$. Now we continue with Gauss-Jordan elimination to transform (6) to reduced row echelon form. Multiply the first row of (6) by $1/a$ and the second row of (6) by $a/(ad - bc)$ to obtain the matrix

$$\left(\begin{array}{cc|cc} 1 & \dfrac{b}{a} & \dfrac{1}{a} & 0 \\ 0 & 1 & -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{array}\right)$$

whose leading entries are both 1s. To complete the reduction, add $-b/a$ times row 2 to row 1. The final product is

$$\left(\begin{array}{cc|cc} 1 & 0 & \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ 0 & 1 & -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{array}\right).$$

Reading off the last half of the augmented matrix, we see that

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{7}$$

Note that if $ad - bc \neq 0$, a and c cannot both be 0. Thus, by Example 8.3 and Exercise 8.17, we have proven the following theorem on 2×2 matrices.

Theorem 8.8 The general 2×2 matrix given by (5) is nonsingular (and therefore invertible) if and only if $ad - bc \neq 0$. Its inverse is matrix (7).

The goal of the next chapter will be to generalize this convenient criterion to the case of arbitrary $n \times n$ matrices.

Putting together the facts about nonsingularity from Chapter 7 with what we have done here, we arrive at the following equivalencies.

Theorem 8.9 For any square matrix A, the following statements are equivalent:

(*a*) A is invertible.
(*b*) A has a right inverse.
(*c*) A has a left inverse.
(*d*) Every system $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$.
(*e*) Every system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$.
(*f*) A is nonsingular.
(*g*) A has maximal rank n.

Proof We saw the equivalence of statements *d*) through *g*) in Section 7.4. The statements and proofs of Theorems 8.6 and 8.7 indicate that statements *a*) through *d*) are equivalent. ∎

The following facts about the behavior of the inverse are easy to prove, and are left as an exercise.

Theorem 8.10 Let A and B be square invertible matrices. Then,

(*a*) $(A^{-1})^{-1} = A$,
(*b*) $(A^T)^{-1} = (A^{-1})^T$,
(*c*) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

The inverse for matrices works very much like the inverse for numbers. If A and B are invertible, $A + B$ need not be invertible, and even when it is, $(A + B)^{-1}$ is generally not $A^{-1} + B^{-1}$. Even for 1×1 matrices or scalars,

$$(3 + 2)^{-1} = \frac{1}{5}, \quad \text{but} \quad 3^{-1} + 2^{-1} = \frac{5}{6}.$$

If A is a square matrix, we can take integral powers of A. The matrix A^m is defined as the product $A \cdot A \cdots A$ (m times). For example, if

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

then
$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

If A is invertible, we can define negative powers of A as well:

$$A^{-m} = (A^{-1})^m = A^{-1} \cdot A^{-1} \cdots A^{-1} \qquad (m \text{ times}).$$

Taking powers of matrices follows most of the same basic rules as taking powers of scalars. This is summarized in the following theorem.

Theorem 8.11 If A is invertible:

(*a*) A^m is invertible for any integer m and $(A^m)^{-1} = (A^{-1})^m = A^{-m}$,
(*b*) for any integers r and s, $A^r A^s = A^{r+s}$, and
(*c*) for any scalar $r \neq 0$, rA is invertible and $(rA)^{-1} = (1/r)A^{-1}$.

Proof These easy computations are left as an exercise. ■

There are some differences between exponentiation of matrices and exponentiation of numbers, all due to the fact that matrix multiplication need not be commutative — that AB need not equal BA. These differences are explored in Exercise 8.27.

Example 8.5 Since each of the elementary row operations is reversible, each of the elementary matrices is invertible and has an elementary matrix for its inverse. For example, the inverse of the permutation matrix E_{ij} is E_{ji} $(= E_{ij})$, the inverse of $E_i(r)$ is $E_i(1/r)$, and the inverse of $E_{ij}(r)$ is $E_{ij}(-r)$.

Since each elementary matrix is invertible, any product of elementary matrices is also invertible by Theorem 8.10*c*. By inverting the elementary matrices in the statement of Theorem 8.4, we can write any matrix A as a product of elementary matrices times a reduced row echelon matrix U:

$$A = E_1^{-1} \cdot E_2^{-1} \cdots E_m^{-1} \cdot U.$$

Furthermore, if A is nonsingular, its reduced row echelon form is the identity matrix, as we saw in the proof of Theorem 8.7.

The foregoing discussion gives us a decomposition theorem for matrices which we will use in Chapter 26.

Theorem 8.12 Any matrix A can be written as a product

$$A = F_1 \cdots F_m \cdot U$$

where the F_i's are elementary matrices and U is in reduced row echelon form. When A is nonsingular, $U = I$ and $A = F_1 \cdots F_m$.

EXERCISES

8.15 Check that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} .5 & 0 & -.5 \\ .5 & 0 & .5 \\ -.5 & 1 & -.5 \end{pmatrix}.$$

8.16 Verify that matrix (4) is the inverse of matrix (3) by direct matrix multiplication.

8.17 Suppose that $a = 0$ but $c \neq 0$ in (5). Show that one obtains the same inverse (7) for A.

8.18 Show by simple matrix multiplication that, if $ad - bc \neq 0$,

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is both a left and a right inverse of A.

8.19 Use the technique of Example 8.3 to either invert each of the following matrices or prove that it is singular:

$$a)\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 4 & 5 \\ 2 & 4 \end{pmatrix}, \qquad c)\ \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix},$$

$$d)\ \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \qquad e)\ \begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix},$$

$$f)\ \begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 21 & 8 & 17 \\ 4 & 12 & -4 & 13 \\ 0 & -3 & -12 & 2 \end{pmatrix}.$$

8.20 Invert the coefficient matrix to solve the following systems of equations:

$$a)\ \begin{aligned} 2x_1 + x_2 &= 5 \\ x_1 + x_2 &= 3; \end{aligned} \qquad b)\ \begin{aligned} 2x_1 + x_2 \phantom{{}+6x_3} &= 4 \\ 6x_1 + 2x_2 + 6x_3 &= 20 \\ -4x_1 - 3x_2 + 9x_3 &= 3; \end{aligned}$$

$$\begin{aligned} &2x_1 + 4x_2 &&= 2 \\ c)\quad &4x_1 + 6x_2 + 3x_3 &&= 1 \\ &-6x_1 - 10x_2 &&= -6. \end{aligned}$$

8.21 Show that if A is $n \times n$ and $AB = BA$, then B is also $n \times n$.

8.22 For $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, compute A^3, A^4, and A^{-2}.

8.23 Verify the statements about the inverses of elementary matrices in the last sentence of Example 8.5.

8.24 *a*) Use Theorem 8.8 to prove that a 2×2 lower- or upper-triangular matrix is invertible if and only if each diagonal entry is nonzero.

b) Show that the inverse of a 2×2 lower triangular matrix is lower triangular.

c) Show that the inverse of a 2×2 upper triangular matrix is upper triangular.

8.25 *a*) Prove Theorem 8.10.

b) Generalize part *c* to the case of the product of k nonsingular matrices.

c) Show by example that if A and B are invertible, $A + B$ need not be invertible.

d) Show that, when it exists, $(A + B)^{-1}$ is generally not $A^{-1} + B^{-1}$.

8.26 Prove Theorem 8.11.

8.27 *a*) Prove that $(AB)^k = A^k B^k$ if $AB = BA$.

b) Show that $(AB)^k \neq A^k B^k$ in general.

c) Conclude that $(A + B)^2$ does not equal $A^2 + 2AB + B^2$ unless $AB = BA$.

8.28 What is the inverse of the $n \times n$ diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}?$$

8.29 Show that the inverse of a 2×2 symmetric matrix S is symmetric.

8.30 Show that the inverse of an $n \times n$ upper-triangular matrix U is upper-triangular. Can you find an easy argument to extend this result to lower-triangular matrices? [Hint: There are a number of ways to do the first part. You can use the inversion method described in the proof of Theorem 8.7, keeping track of the status of the 0s below the diagonal. Or, you can show by direct calculation that $BU = I$ implies that B has only 0s below the diagonal.]

8.31 Show that for any permutation matrix P, $P^{-1} = P^T$.

8.32 Use Gauss-Jordan elimination to derive a criterion for the invertibility of 3×3 matrices similar to the $ad - bc$ criterion for the 2×2 case. For simplicity, assume that no row interchanges are needed in the elimination process.

8.33 The definitions of left inverse and right inverse apply to nonsquare matrices. Use the ideas in the proof of Theorem 8.7 to prove the following statements for an $m \times n$ matrix A, where $m \neq n$.

a) A nonsquare matrix cannot have both a left and a right inverse.

b) If A has one left (right) inverse, it has infinitely many.

c) If $m < n$, A has a right inverse if and only if $\operatorname{rank} A = m$.

d) If $m > n$, A has a left inverse if and only if $\operatorname{rank} A = n$.

8.5 INPUT-OUTPUT MATRICES

The last section showed that solving a system $A\mathbf{x} = \mathbf{b}$ of n equations in n unknowns is closely related to inverting the matrix A since

$$\mathbf{x} = A^{-1}\mathbf{b}. \tag{8}$$

For a single fixed $\mathbf{b}$, it is usually quicker to solve $A\mathbf{x} = \mathbf{b}$ by Gaussian elimination (and back substitution). However, if one is going to work with many different right-hand sides $\mathbf{b}$ and the same A, it may be easier to invert A and use (8).

For example, consider the input-output example of Chapter 6. This is a model of an economy with n industries. Each industry produces a single output, using as inputs the products produced by the other industries. Write x_i for the gross output of product i, and let a_{ij} denote the amount of good i needed to produce one unit of good j. Let c_i denote consumer demand for product i. In Chapter 6, we saw that the market equilibrium condition that supply equal demand is given by the n equations

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + c_i,$$

for $i = 1, \ldots, n$. In matrix notation this system of equations becomes

$$\mathbf{x} = A\mathbf{x} + \mathbf{c},$$

which is more conveniently written as

$$(I - A)\mathbf{x} = \mathbf{c}. \tag{9}$$

(To keep all n-tuples nonnegative, we will ignore the labor sector described in Chapter 6.)

The matrix A of intermediate factor demands is sometimes called the **technology matrix**. We might expect this to remain relatively constant over long periods of time. The right-hand side of (9), $\mathbf{c}$, can be expected to vary more frequently. Thus it is convenient to study solutions to (9) by working with the inverse:

$$\mathbf{x} = (I - A)^{-1}\mathbf{c}.$$

Notice that in addition to requiring that $I - A$ be invertible, we also require that the solution to (9) be nonnegative whenever $\mathbf{c}$ is nonnegative. This corresponds to the requirement that any solution to our economic system produces nonnegative amounts of each commodity. For this to happen, all entries of the matrix $(I - A)^{-1}$ must be nonnegative. Furthermore, the study of this system is complicated by the fact that all the economic data in the model are contained in the matrix A. It is not enough simply to assume that $I - A$ has a nonnegative inverse. We must find assumptions on A which will imply the desired behavior of $I - A$.

Since the factors of production have different natural units, it is convenient to express them all in monetary terms, say in millions of dollars, in an input-output analysis. In this case, the (i, j)th entry a_{ij} of technology matrix A indicates how many millions of dollars of good i are needed to produce 1 million dollars of good j. The sum of the entries in each column of A gives the total cost of producing 1 million dollars of the product that column represents. Since we expect each industry to make a positive accounting profit, the sum of the entries in each column should be less than 1. This turns out to be one of the conditions on a technology matrix A which will guarantee that $I - A$ has a nonnegative inverse.

> **Theorem 8.13** Let A be an $n \times n$ matrix with the properties that each entry is nonnegative and the sum of the entries in each column is less than 1. Then, $(I - A)^{-1}$ exists and contains only nonnegative entries.

We will prove Theorem 8.13 at the end of this section. First, to make the preceding discussion concrete, consider a simple three-industry economy, with input-output matrix

$$A = \begin{pmatrix} 0.15 & 0.5 & 0.25 \\ 0.3 & 0.1 & 0.4 \\ 0.15 & 0.3 & 0.2 \end{pmatrix}.$$

Suppose that consumer demand fluctuates between

$$\mathbf{c} = \begin{pmatrix} 20 \\ 20 \\ 10 \end{pmatrix} \quad \text{and} \quad \mathbf{c}' = \begin{pmatrix} 10 \\ 20 \\ 20 \end{pmatrix}.$$

What will be the corresponding industry outputs?

First, compute $I - A$:

$$I - A = \begin{pmatrix} 0.85 & -0.5 & -0.25 \\ -0.3 & 0.9 & -0.4 \\ -0.15 & -0.3 & 0.8 \end{pmatrix}.$$

To invert $I - A$, write the augmented matrix

$$\left(\begin{array}{ccc|ccc} 0.85 & -0.5 & -0.25 & 1 & 0 & 0 \\ -0.3 & 0.9 & -0.4 & 0 & 1 & 0 \\ -0.15 & -0.3 & 0.8 & 0 & 0 & 1 \end{array}\right)$$

and use Gauss-Jordan elimination to reduce the first three columns to the identity matrix. The result, rounded to three decimal places, is

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1.975 & 1.564 & 1.399 \\ 0 & 1 & 0 & 0.988 & 2.115 & 1.366 \\ 0 & 0 & 1 & 0.741 & 1.086 & 2.025 \end{array}\right).$$

The last three columns are $(I - A)^{-1}$. Note that, as Theorem 8.13 predicts, all entries are positive.

When consumer demand is $\mathbf{c} = \begin{pmatrix} 20 \\ 20 \\ 10 \end{pmatrix}$, the total output should be

$$\mathbf{x} = (I - A)^{-1}\mathbf{c} = \begin{pmatrix} 1.975 & 1.564 & 1.399 \\ 0.988 & 2.115 & 1.366 \\ 0.741 & 1.086 & 2.025 \end{pmatrix} \begin{pmatrix} 20 \\ 20 \\ 10 \end{pmatrix} = \begin{pmatrix} 84.77 \\ 75.72 \\ 56.79 \end{pmatrix}.$$

When consumer demand is $\mathbf{c} = \begin{pmatrix} 10 \\ 20 \\ 20 \end{pmatrix}$, the total output should be

$$\mathbf{x} = (I - A)^{-1}\mathbf{c} = \begin{pmatrix} 1.975 & 1.564 & 1.399 \\ 0.988 & 2.115 & 1.366 \\ 0.741 & 1.086 & 2.025 \end{pmatrix} \begin{pmatrix} 10 \\ 20 \\ 20 \end{pmatrix} = \begin{pmatrix} 79.01 \\ 79.51 \\ 69.63 \end{pmatrix}.$$

Leontief used input-output analysis to study the 1958 U.S. economy. He divided the economy into 81 sectors and aggregated these sectors into six groups of related sectors. We will treat each of the six families as a separate industry in order to simplify our presentation. These six industries are listed in Table 8.1, and their intermediate factor demands are listed in Table 8.2. The units are millions of dollars. So the .173 in row 3 column 2 means that the production of $1 million worth of final metal products requiresthe expenditure of $173,000 on basic metal

	Sector	Examples
FN,	Final nonmetal	Leather goods, furniture, foods
FM,	Final metal	Construction mach'ry, household appliances
BM,	Basic metal	Mining, machine shop products
BN,	Basic nonmetal	Glass, wood, textile, and livestock products
E,	Energy	Coal, petroleum, electricity, gas
S,	Services	Govt. services, transportation, real estate

Table 8.1 *The Six Sectors*

	FN	FM	BM	BN	E	S
FN	0.170	0.004	0.000	0.029	0.000	0.008
FM	0.003	0.295	0.018	0.002	0.004	0.016
BM	0.025	0.173	0.460	0.007	0.011	0.007
BN	0.348	0.037	0.021	0.403	0.011	0.048
E	0.007	0.001	0.039	0.025	0.358	0.025
S	0.120	0.074	0.104	0.123	0.173	0.234

Table 8.2 *Internal demands for 1958 U.S. Economy*

FN	\$ 99,640
FM	75,548
BM	14,444
BN	33,501
E	23,527
S	263,985

External Demands for 1958 U.S. Economy (in millions of dollars) **Table 8.3**

goods. Table 8.3 lists Leontief's estimates of final demands in the 1958 U.S. economy. The problem is to determine how many units had to be produced in each of the six sectors in order to run the U.S. economy in 1958.

To solve the problem, we turn Table 8.2 into the technology matrix A and Table 8.3 into the final demand column matrix $\mathbf{c}$. As before, the goal is to solve $(I - A)\mathbf{x} = \mathbf{c}$ for the output column matrix $\mathbf{x}$:

$$\mathbf{x} = (I - A)^{-1}\mathbf{c}.$$

First, we need to compute the net input-output matrix $I - A$.

$$I - A$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.170 & 0.004 & 0 & 0.029 & 0 & 0.008 \\ 0.003 & 0.295 & 0.018 & 0.002 & 0.004 & 0.016 \\ 0.025 & 0.173 & 0.460 & 0.007 & 0.011 & 0.007 \\ 0.348 & 0.037 & 0.021 & 0.403 & 0.011 & 0.048 \\ 0.007 & 0.001 & 0.039 & 0.025 & 0.358 & 0.025 \\ 0.120 & 0.074 & 0.104 & 0.123 & 0.173 & 0.234 \end{pmatrix}$$

$$= \begin{pmatrix} 0.830 & -0.004 & 0 & -0.029 & 0 & -0.008 \\ -0.003 & 0.705 & -0.018 & -0.002 & -0.004 & -0.016 \\ -0.025 & -0.173 & 0.540 & -0.007 & -0.011 & -0.007 \\ -0.348 & -0.037 & -0.021 & 0.597 & -0.011 & -0.048 \\ -0.007 & -0.001 & -0.039 & -0.025 & 0.642 & -0.025 \\ -0.120 & -0.074 & -0.104 & -0.123 & -0.173 & 0.766 \end{pmatrix}.$$

The inverse of this net input-output matrix can be computed by the methods of Section 8.4 and then used to compute the gross output column matrix.

$$\mathbf{x} = (I - A)^{-1}$$

$$= \begin{pmatrix} 1.234 & 0.014 & 0.006 & 0.064 & 0.007 & 0.018 \\ 0.017 & 1.436 & 0.057 & 0.012 & 0.020 & 0.032 \\ 0.071 & 0.465 & 1.877 & 0.019 & 0.045 & 0.031 \\ 0.751 & 0.134 & 0.100 & 1.740 & 0.066 & 0.124 \\ 0.060 & 0.045 & 0.130 & 0.082 & 1.578 & 0.059 \\ 0.339 & 0.236 & 0.307 & 0.312 & 0.376 & 1.349 \end{pmatrix} \begin{pmatrix} 99{,}640 \\ 75{,}548 \\ 14{,}444 \\ 33{,}501 \\ 23{,}527 \\ 263{,}985 \end{pmatrix}$$

$$= \begin{pmatrix} 131{,}161 \\ 120{,}324 \\ 79{,}194 \\ 178{,}936 \\ 66{,}703 \\ 426{,}542 \end{pmatrix}$$

We conclude, for example, that it requires \$131,161 million worth of final nonmetal products to meet both intermediate and final demands in the 1958 U.S. economy.

Proof of Theorem 8.13

We conclude this section by proving Theorem 8.13. Let A be a technology matrix that satisfies the hypotheses of Theorem 8.13: nonnegative entries and column sums less than 1. Then, $-A$ has all its entries and its column sums between 0 and -1 and $I - A$ satisfies the following three properties:

(a) each off-diagonal entry is ≤ 0,
(b) each diagonal entry is positive, and
(c) the sum of the entries in each column is positive.

Matrices which satisfy these three conditions are a special case of the class of **dominant diagonal matrices**. A more general definition of a dominant diagonal matrix requires that in each column the *absolute value* of the diagonal entry is at least as large as the sum of the absolute values of the other entries in that column. To prove Theorem 8.13, we need only prove the following result.

Theorem 8.14 Let B be a square matrix which satisfies conditions a, b, and c above. Then, all entries of B^{-1} are nonnegative.

Proof To keep better track of the signs and sizes of the entries of the matrix B, we write it as

$$B = \begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & b_{22} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & b_{nn} \end{pmatrix},$$

where $$\text{each } b_{ij} \geq 0 \quad \text{and} \quad 0 \leq \sum_{h \neq j} b_{hj} < b_{jj} \tag{10}$$

for all j. Let $\mathbf{c}$ be a vector with all positive entries and consider the system $B\mathbf{x} = \mathbf{c}$. To solve this system, we perform Gaussian elimination on the augmented matrix $[B \mid \mathbf{c}]$. Add b_{j1}/b_{11} times row 1 to row j for all $j > 1$. The result is the new augmented matrix

$$\begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} & | & c_1 \\ 0 & b_{22} - \frac{b_{21}}{b_{11}}b_{12} & \cdots & -b_{2n} - \frac{b_{21}}{b_{11}}b_{1n} & | & c_2 + \frac{b_{21}}{b_{11}}c_1 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & -b_{n2} - \frac{b_{n1}}{b_{11}}b_{12} & \cdots & b_{nn} - \frac{b_{n1}}{b_{11}}b_{1n} & | & c_n + \frac{b_{n1}}{b_{11}}c_1 \end{pmatrix}$$

$$\equiv \begin{pmatrix} b_{11} & * & | & c_1 \\ \mathbf{0} & \hat{B} & | & \hat{\mathbf{c}} \end{pmatrix}.$$

The $(n-1) \times (n-1)$ matrix $\hat{B}$ is still dominant diagonal, since its off-diagonal entries are still nonpositive and the sum of the entries in its $(j-1)$th column is

$$\begin{aligned} &\left(b_{jj} - \frac{b_{j1}}{b_{11}}b_{1j}\right) + \sum_{h \neq 1,j}\left(-b_{hj} - \frac{b_{h1}}{b_{11}}b_{1j}\right) \\ &\qquad = b_{jj} - \left(\sum_{h \neq 1,j} b_{hj}\right) - b_{1j}\frac{b_{21} + \cdots + b_{n1}}{b_{11}} \\ &\qquad > b_{jj} - \sum_{h \neq 1,j} b_{hj} - b_{1j} \\ &\qquad > 0, \qquad\qquad \text{(by (10) twice).} \end{aligned}$$

The new RHS $\hat{\mathbf{c}}$ has all entries positive. Continue applying Gaussian elimination; at each stage, the resulting submatrix still satisfies a, b, and c. We conclude that the row echelon form of $[B \mid \mathbf{c}]$ has the sign pattern

$$\begin{pmatrix} + & - & - & \cdots & - & | & + \\ 0 & + & - & \cdots & - & | & + \\ 0 & 0 & + & \cdots & - & | & + \\ \vdots & \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & 0 & 0 & \cdots & - & | & + \\ 0 & 0 & 0 & \cdots & + & | & + \end{pmatrix}$$

Back substitution from such a matrix yields a *positive* solution $\mathbf{x}$ to the system $B\mathbf{x} = \mathbf{c}$. If the nonzero right-hand side $\mathbf{c}$ had some zero entries and if A had some zero off-diagonal terms, the same argument yields a nonnegative solution of $B\mathbf{x} = \mathbf{c}$. Since the columns of B^{-1} are the solution vectors of $B\mathbf{x} = \mathbf{e}_i$ (Theorem 8.7), the entries of B^{-1} are all nonnegative numbers. ∎

EXERCISES

8.34 Let the technology matrix be given by $A = \begin{pmatrix} .7 & .2 & .2 \\ .1 & .6 & .1 \\ .1 & .1 & .6 \end{pmatrix}$. Find the gross output vectors when final demand is:

$$a)\ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad c)\ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

8.35 Let the general 2×2 technology matrix be given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove Theorem 8.13 directly for such a matrix using Theorem 8.8.

8.6 PARTITIONED MATRICES (optional)

Let A be an $m \times n$ matrix. A **submatrix** of A is a matrix formed by discarding some entire rows and/or columns of A. A **partitioned matrix** is a matrix which has been partitioned into submatrices by horizontal and/or vertical lines which extend along *entire* rows or columns of A. For example,

$$A = \left(\begin{array}{cc|c|ccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{array}\right), \tag{11}$$

which we can write as

$$A = \left(\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right).$$

Each submatrix A_{ij} is called a **block** of A. Augmented matrices are an example of partitioned matrices. They have been partitioned vertically into two blocks.

If A is a square matrix which has been partitioned as

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix} \tag{12}$$

where each A_{ii} is square and $A_{ij} = 0$ for $i \neq j$, then A is called a **block diagonal** matrix.

Suppose that A and B are two $m \times n$ matrices which are partitioned the same way; that is,

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}$$

where A_{11} and B_{11} have the same dimensions, A_{12} and B_{12} have the same dimensions, and so on. Then A and B can be added as if the blocks are scalar entries:

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} & A_{13} + B_{13} \\ A_{21} + B_{21} & A_{22} + B_{22} & A_{23} + B_{23} \end{pmatrix}.$$

Similarly, two partitioned matrices A and C can be multiplied, treating the blocks as scalars, if the blocks are all of a size such that the matrix multiplication of blocks can be done. For example, if

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{pmatrix}, \tag{13}$$

then

$$AC = \begin{pmatrix} A_{11}C_{11} + A_{12}C_{21} & A_{11}C_{12} + A_{12}C_{22} & A_{11}C_{13} + A_{12}C_{23} \\ A_{21}C_{11} + A_{22}C_{21} & A_{21}C_{12} + A_{22}C_{22} & A_{21}C_{13} + A_{22}C_{23} \end{pmatrix}$$

so long as the various matrix products $A_{ij}C_{jk}$ can be formed. For example, A_{11} must have as many columns as C_{11} has rows, and so on.

We used the block multiplication of partitioned matrices in Section 8.4 when we wrote the matrix product $AA^{-1} = I$ as

$$A\,(\mathbf{c}_1 \quad \cdots \quad \mathbf{c}_n) = (\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n),$$

where $\mathbf{c}_i$ is the ith column of A^{-1} and $\mathbf{e}_j$ is the jth column of the identity matrix. In this case, the jth block product yielded the equation $A\mathbf{c}_j = \mathbf{e}_j$ in (2).

One reason for partitioning matrices is that frequently inverses can be computed (or found not to exist) much more easily using the blocks than they can by direct computation. For example, the following result on partitions is useful for deriving propositions about how demand functions depend on the price level.

Theorem 8.15 Let A be a square matrix partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square submatrices. If both A_{22} and the matrix

$$D = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

are nonsingular, then A is nonsingular and

$$A^{-1} = \begin{pmatrix} D^{-1} & -D^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}D^{-1} & A_{22}^{-1}(I + A_{21}D^{-1}A_{12}A_{22}^{-1}) \end{pmatrix} \tag{14}$$

The proof of this theorem is left as an exercise.

EXERCISES

8.36 What must be true about the sizes of the various blocks A_{11}, A_{12}, C_{11}, and so on, in (13) in order for the block multiplications to make sense?

8.37 Suppose that A is given by (11) and the matrix C is given by

$$C = \left(\begin{array}{c|cc|c} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ \hline c_{31} & c_{32} & c_{33} & c_{34} \\ \hline c_{41} & c_{42} & c_{43} & c_{44} \\ c_{51} & c_{52} & c_{53} & c_{54} \\ c_{61} & c_{62} & c_{63} & c_{64} \end{array}\right) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}.$$

a) Check that block multiplication can be carried out for the matrix product AC.

b) Compute the six block products $\Sigma A_{ij}C_{jk}$ for $i = 1, 2$ and $k = 1, 2, 3$.

c) Check that you reach the same answer for the matrix product whether you compute it with the block products or directly.

8.38 Show that the block diagonal matrix A in (12) is invertible if and only if each A_{ii} is invertible. Find A^{-1}.

8.39 Prove Theorem 8.15. First show that the matrix D exists. Then verify by block multiplication that matrix (14) is the inverse of A.

8.40 Replace the hypotheses on the matrix A of Theorem 8.15 by the hypothesis that both A_{11} and $A_{22} - A_{21}A_{11}^{-1}A_{12}$ are invertible. Prove that A is invertible and find its inverse.

8.41 Rewrite the invertibility conditions of Theorem 8.15 for the following cases.

a) $A_{21} = 0$;

b) A_{22} is 1×1 (a scalar);

c) A_{11} is the scalar 0, and $A_{21} = A_{12}^T = \mathbf{p}$ where $\mathbf{p}$ is a column vector.

8.7 DECOMPOSING MATRICES (optional)

This section demonstrates how most matrices can be written as a product of a lower-triangular matrix L and an upper-triangular matrix U. This **LU decomposition** leads to an efficient approach to solving systems of equations (Exercise 8.51 below). It is also the central technique in proving some important theorems about matrices (especially in Chapter 26). This decomposition is a direct consequence of Theorem 8.12 and the following lemma about the product of elementary matrices.

Lemma 8.2 Let L and M be two $n \times n$ lower-triangular matrices. Then, the matrix product LM is lower triangular. If L and M have only 1s on their diagonals, so does LM.

Proof The (i, j)th entry of the product LM is the product of the ith row of L and the jth column of M. Using the hypothesis that $l_{ik} = 0$ for $k > i$ and $m_{hj} = 0$ for $h < j$, we write this product as:

$$[LM]_{ij} = \begin{pmatrix} l_{i1} & \cdots & l_{i,i-1} & l_{ii} & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{jj} \\ m_{j+1,j} \\ \vdots \\ m_{nj} \end{pmatrix} \tag{15}$$

If $i < j$, each of the i nonzero entries at the beginning of the ith row of L will be multiplied by the i zero entries beginning the jth column of M. The result is a zero entry in LM. Therefore, LM is lower triangular.

It follows from (15) that the (i, i)th diagonal entry of LM is $l_{ii}m_{ii}$. If $l_{ii} = m_{ii} = 1$, then $l_{ii}m_{ii} = 1$. ∎

Now we can use our knowledge of elementary matrices to decompose matrices.

Theorem 8.16 Let A be a general $k \times n$ matrix, and suppose that no row interchanges are needed to reduce A to its row echelon form. Then A can be written as a product LU where L is an $k \times k$ lower-triangular matrix with only 1's on the diagonal, and U is an upper-triangular $k \times n$ matrix.

The U in Theorem 8.16 is the row echelon form of A. Although it is not necessarily a square matrix, we will call it upper triangular because its (i, j)th entries are all zero whenever $i > j$.

Proof Theorem 8.16 is a consequence of Theorems 8.4 and 8.12, which summarize the elementary matrix approach to Gaussian elimination. If no row interchanges are needed to reduce A to its row echelon form U, the only row operation required is the addition of a multiple of one row to a row which is farther down in the matrix. This operation is described by the elementary matrix $E_{ij}(r)$ where $i < j$. These elementary matrices are lower-triangular with 1s on the diagonal. Theorems 8.4 and 8.12 tell us that

$$A = E_1 \cdots E_m \cdot U \tag{16}$$

where E_1 is the inverse of the first elementary matrix used in the row reduction of A, E_2 is the inverse of the second elementary matrix used in the row reduction of A, and so on. In Example 8.5, we noted that the inverse of $E_{ij}(r)$ is $E_{ij}(-r)$. So the matrices $E_1, \ldots, E_m$ are all lower triangular with only 1's on their diagonals. Applying Lemma 8.2, we see that the product $E_1 \cdot E_2$ is lower triangular and has only 1's on the diagonal. Since the matrices E_3 and $E_1 \cdot E_2$ satisfy the hypotheses of the Lemma, $E_1 \cdot E_2 \cdot E_3$ is lower triangular and has 1s on the diagonal. Repeating this argument as many times as is necessary, we can see that the product $L = E_1 \cdots E_m$ is lower triangular and has only 1s on the diagonal. Consequently (16) can be rewritten as $A = LU$ where L is lower triangular with only 1s on the diagonal. ■

Example 8.6 To illustrate Theorem 8.16, let us return to Example 8.1, where we wrote the row echelon form U of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}$$

as

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -10 & -15 \\ 0 & 0 & -3.5 \end{pmatrix} = E_{23}(.1) \cdot E_{13}(-3) \cdot E_{12}(-12) \cdot A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -12 & 1 & 0 \\ -4.2 & .1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}.$$

Multiply the right-hand side by the inverses of the elementary matrices:

$$A = E_{12}(-12)^{-1} \cdot E_{13}(-3)^{-1} \cdot E_{23}(.1)^{-1} \cdot U$$

$$= E_{12}(12) \cdot E_{13}(3) \cdot E_{23}(-.1) \cdot U$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 3 & -.1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -10 & -15 \\ 0 & 0 & -3.5 \end{pmatrix}$$

$$= LU.$$

Notice that the negatives of the below-diagonal entries of L reflect the elementary row operations used to reduce A to U.

Mathematical Induction

The proof of Theorem 8.16 is not completely rigorous, because the statement "repeating this argument as many times as is necessary" is a bit vague. How do we know that we can really do this? How many times are necessary? There is a formal technique for making this argument, which is called the *principle of mathematical induction*. The principle of mathematical induction is described fully in the first Appendix of this book. Here we will only show how we would apply it in the proof of Theorem 8.16.

In the proof of Theorem 8.16 we want to show that, for all k, the matrix product of k lower-triangular matrices $L_1 \cdot L_2 \cdots L_k$ is lower triangular. This statement is clearly true when $k = 1$. Lemma 8.2 tells us that the statement is true for $k = 2$. For $k = 3$, we write $L_1 \cdot L_2 \cdot L_3$ as $(L_1 \cdot L_2) \cdot L_3$. Since the statement is true for $k = 2$, $(L_1 \cdot L_2)$ is lower triangular. Lemma 8.2 then assures us that the product $L_1 \cdot L_2 \cdot L_3$ is lower triangular, and so on.

To formalize this argument, we divide it into two steps:

(1) the product of two lower-triangular matrices is lower triangular, and
(2) *if* the product of k lower-triangular matrices is lower triangular, *then* the product of $k + 1$ lower-triangular matrices is lower triangular.

Statements 1 and 2 are true by Lemma 8.2. Taken together, statements 1 and 2 allow us to conclude that the product of an arbitrary number k of lower triangular matrices is lower triangular. First, let $k = 2$ in 2, then 1 and 2 imply that the statement is true for $k = 3$. Next, let $k = 3$ in 2 to conclude that the statement is true for $k = 4$, and so on. Statement 2 is called the **inductive hypothesis**. This proof by induction is a bootstrap method that is often used to prove propositions of the form: statement P(k) is true for every positive integer k.

Including Row Interchanges

In the hypothesis of Theorem 8.16 we assumed that no row interchanges were needed to reduce A to its row echelon form. Of course this is not always the case, and so we would like to know what happens to the conclusions of Theorem 8.16 when row interchanges are required. First consider the case of nonsingular A. The answer is very simple (although the proof is sufficiently tricky that we will only sketch it here). Row interchanges are required only because, at some stage in the reduction process, there arises a pivot whose value is 0. So, reduce A to its row echelon form, keeping track of the row interchanges that are required. Suppose now that these row interchanges were to be made *before* we began the reduction. Then all the pivots would be in the right places, and no 0 pivots would be encountered in the row reduction process. How do we swap the rows of A?

The row interchanges can be accomplished by premultiplying A by permutation matrices — E_{ij} matrices. The product of permutation matrices is a permutation matrix (Exercise 8.10), and so, to eliminate the need for row interchanges during the reduction process, we can just premultiply A by the appropriate permutation matrix P. Thus there exists a permutation matrix P, an upper-triangular matrix U, and a lower-triangular matrix L such that $PA = LU$.

When A is singular, the story is not much different. Here, when a 0 pivot is encountered, it may not be possible to replace it with a nonzero pivot using a row interchange. Everything below the pivot may also be 0. This presents no problem; just go on to the next column. Of course some 0 pivots may have nonzero elements below them, so row interchanges may still be required. Nonetheless, our conclusions are not altered. We summarize them in the following theorem:

Theorem 8.17 Let A be a general $k \times n$ matrix. Then one can write $PA = LU$ where P is a $k \times k$ permutation matrix, L is a $k \times k$ lower-triangular matrix with only 1s on the diagonal, and U is a $k \times n$ upper-triangular matrix.

EXERCISES

8.42 For each of the following matrices A, write down the string of elementary matrices which are needed to transform A to its row echelon form.

$$a)\quad \begin{pmatrix} 2 & 4 \\ -6 & -13 \end{pmatrix}, \qquad b)\quad \begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix},$$

$$c)\quad \begin{pmatrix} 2 & 4 & 0 & 1 \\ 4 & 6 & 3 & 3 \\ -6 & -10 & 0 & 4 \end{pmatrix}, \qquad d)\quad \begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 21 & 8 & 17 \\ 0 & -3 & -12 & 2 \\ 4 & 12 & -4 & 13 \end{pmatrix}.$$

8.43 Write down the LU decomposition of each matrix in Exercise 8.42.

8.44 Show that the LU decomposition of A is unique if A is square, invertible, and satisfies the hypotheses of Theorem 8.16.
[Hint: Write $A = L_1U_1 = L_2U_2$, where the L_i are invertible and lower triangular, with 1s on the diagonal. Show that the U_i are invertible and write $L_2^{-1}L_1 = U_2U_1^{-1}$. Conclude that both sides are diagonal and that the left side is in fact the identity matrix.]

8.45 Show that the LU decomposition of the $k \times n$ matrix A satisfying the hypothesis of Theorem 8.16 is unique if A has maximal rank.
[Hint: As in the previous exercise, write $L_2^{-1}L_1U_1 = U_2$. Check that U_1 and U_2 have no 0 rows and then show that the equation $L_2^{-1}L_1U_1 = U_2$ implies that $L_2^{-1}L_1$ is the identity matrix.]

8.46 Show by example that if A does not have maximal rank, then the LU decomposition of A need not be unique.

8.47 Prove the following proposition: If A is a square, nonsingular matrix and if row reduction of A requires no row interchanges, then A can be written uniquely as $A = LDU$ where L and U are lower- and upper-triangular matrices, respectively, with only 1s on their diagonals and D is a diagonal matrix. The diagonal entries of D are precisely the pivots of A.

[Hint: Start with the LU decomposition of A and decompose U into the product of two matrices, each with the desired properties.]

8.48 Find the LDU decomposition for the matrices in Exercise 8.42*a*, *b*, *d*.

8.49 The following two matrices require row interchanges to achieve their row echelon forms. For each matrix A:

a) Compute the row-echelon form.

b) Construct the permutation matrix P which corresponds to these row interchanges.

c) Compute the row echelon form of PA and compare your answer to that of part *a*.

d) Find the LU decomposition of PA.

$$i)\ \begin{pmatrix} 3 & 2 & 0 \\ 6 & 4 & 1 \\ -3 & 4 & 1 \end{pmatrix}, \qquad ii)\ \begin{pmatrix} 0 & 1 & 1 & 4 \\ 1 & 1 & 2 & 2 \\ -6 & -5 & -11 & -12 \\ 2 & 3 & -2 & 3 \end{pmatrix}$$

8.50 *a*) What must be true about the entries of the general 2×2 matrix if row interchanges are required for reduction to row-echelon form?

b) What about the general 3×3 case?

8.51 The LU decomposition provides an efficient way to solve a system of linear equations $A\mathbf{x} = \mathbf{b}$ for different values of $\mathbf{b}$. It requires many fewer arithmetic steps than matrix inversion, and it works when A is not square. Use the LU decomposition to rewrite the system of equations as $LU\mathbf{x} = \mathbf{b}$. Now the system can be solved by first letting $U\mathbf{x} = \mathbf{z}$, solving the system of equations $L\mathbf{z} = \mathbf{b}$ for $\mathbf{z}$, and then solving $U\mathbf{x} = \mathbf{z}$ for $\mathbf{x}$. Since both of these systems are triangular, only back substitution is required to solve them.

a) Verify that the solutions obtained this way are precisely the solutions to $A\mathbf{x} = \mathbf{b}$.

b) Solve the following systems using this technique:

$$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}; \quad \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix};$$

$$\begin{pmatrix} 5 & 3 & 1 \\ -5 & -4 & 1 \\ -10 & -9 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -10 \\ -24 \end{pmatrix}; \quad \begin{pmatrix} 5 & 3 & 1 \\ -5 & -4 & 1 \\ -10 & -9 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -14 \end{pmatrix}$$

NOTES

For an excellent summary of Leontief's study, see W. Leontief, "The structure of the U.S. economy," *Scientific American* 212 (April 1965). Our discussion of Leontief's 1958 model is adapted from the presentation in Stanley Grossman, *Applied Mathematics for the Management, Life, and Social Sciences* (Belmont, Calif.: Wadsworth, 1983). Our proof of Theorem 8.14 is adapted from Carl Simon, "Some Fine-Tuning for Dominant Diagonal Matrices," *Economic Letters* 30 (1989), 217–221.

Determinants: An Overview

The most important matrices in economic models are square matrices, in which the number of unknowns equals the number of equations. For example, all the matrices for economic analysis listed in the first paragraph of Chapter 8 are square matrices. The most important square matrices are the nonsingular ones. These are precisely the coefficient matrices A such that the system of n equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\quad &\ \ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

or in matrix notation $A\mathbf{x} = \mathbf{b}$, has one and only one solution for each right-hand side $\mathbf{b}$. As we saw in the last chapter, these are also the matrices which are invertible. Since not all square matrices are nonsingular, we will describe in this chapter a straightforward test to *determine* whether or not a given matrix is nonsingular. In particular, for any square matrix we will define a number called the *determinant*, with the property that the square matrix is nonsingular if and only if its determinant is not zero. Later we will use the determinant for other tasks, for example, for developing an *explicit formula* for the solution of (1) in terms of the a_{ij}'s and b_i's, for deriving a formula for the inverse of a matrix, and for classifying the behavior of quadratic functions.

Many mathematical models in economics center around constrained maximization or minimization problems. Determinants play a role here too, because the second order condition for such problems requires that one check the signs of determinants of certain matrices of second derivatives.

The determinant can be a fairly complex expression. For a general $n \times n$ matrix there are $n!$ terms, each the product of n different entries of the matrix. Some of the proofs of its properties are also fairly complex. Consequently, this chapter presents a comprehensive *overview* of the determinant: how to compute it and how to use it, with relatively little motivation and no complex proofs. Chapter 26 contains a complete analysis of the determinant, including proofs of its important properties and major uses. Depending on the amount of detail with which one wants to cover determinants, one can: 1) read this chapter and skip Chapter 26, at least for the time being; 2) read Chapter 26 now and skip this chapter; or 3) read this chapter as an overview on determinants, follow it with a careful reading of Chapter 26, and then return to Chapter 10.

9.1 THE DETERMINANT OF A MATRIX

Defining the Determinant

The determinant of a matrix is defined inductively. There is a natural definition for 1×1 matrices. Then, we use this definition to define the determinant of 2×2 matrices. Once we have defined the determinant for 2×2 matrices, we use this definition to define the determinant for 3×3 matrices, and so on.

A 1×1 matrix is just a scalar (a). Since the inverse of a, $1/a$, exists if and only if a is nonzero, it is natural to define the determinant of such a matrix to be just that scalar a:

$$\det(a) = a.$$

For a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

Theorem 8.8 states that A is nonsingular if and only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Therefore, we define the determinant of a 2×2 matrix A:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{2}$$

Notice that (2) is just the product of the two diagonal entries minus the product of the two off-diagonal entries. In order to motivate the general definition of a determinant, we write (2) as follows:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \det(a_{22}) - a_{12} \det(a_{21}). \tag{3}$$

The first term on the right-hand side of (3) is the $(1, 1)$th entry of A times the determinant of the submatrix obtained by deleting from A the row and column which contain that entry; the second term is the $(1, 2)$th entry times the determinant of the submatrix obtained by deleting from A the row and column which contain that entry. The terms alternate in sign; the term containing a_{11} receives a plus sign and the term containing a_{12} receives a minus sign.

The following definitions will simplify the task of defining the determinant of an $n \times n$ matrix.

Definition Let A be an $n \times n$ matrix. Let A_{ij} be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from A. Then, the scalar

$$M_{ij} \equiv \det A_{ij}$$

is called the (i, j)th **minor** of A and the scalar

$$C_{ij} \equiv (-1)^{i+j} M_{ij}$$

is called the (i, j)th **cofactor** of A. A cofactor is a signed minor. Note that $M_{ij} = C_{ij}$ if $(i+j)$ is even and $M_{ij} = -C_{ij}$ if $(i+j)$ is odd.

Formula (3) can be written as

$$\det A = a_{11}M_{11} - a_{12}M_{12} = a_{11}C_{11} + a_{12}C_{12}.$$

We use this expression as motivation for the definition of the determinant of a 3×3 matrix.

Definition The **determinant** of a 3×3 matrix is given by

$$\begin{aligned}
\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\
&= a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \\
&\qquad + a_{13} \cdot \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.
\end{aligned}$$

The jth term on the right-hand side of the definition is a_{1j} times the determinant of the submatrix obtained by deleting row 1 and column j from A. The term is preceeded by a plus sign if $1 + j$ is even and by a minus sign if $1 + j$ is odd.

Definition The **determinant** of an $n \times n$ matrix A is given by

$$\begin{aligned}
\det A &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\
&= a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{n+1}a_{1n}M_{1n}.
\end{aligned} \tag{4}$$

Notation In referring to the determinant of a $n \times n$ matrix A, one sometimes writes

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad \text{for} \qquad \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

and $|A|$ for $\det A$.

Computing the Determinant

Our definition of the determinant of a matrix involves expanding along its first row. There is nothing special about the first row. It turns out that one can use any row or column to compute the determinant of a matrix. For example, if one uses, say, the *second column* to compute the determinant of a 3×3 matrix, one computes

$$\det A = -a_{12} \cdot \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{22} \cdot \det \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} - a_{32} \cdot \det \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix},$$

or equivalently,

$$\det A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}. \tag{5}$$

The jth term on the right-hand side of (5) is a_{j2} times the determinant of the submatrix obtained by deleting the row and column of A which contains a_{j2}; it is preceeded by a plus sign if $(j + 2)$ is even and by a minus sign if $(j + 2)$ is odd.

In general, the determinant of an $n \times n$ matrix involves $n!$ terms, each a product of n entries. This can be a time-consuming computation. There are certain classes of matrices whose determinants are easy to compute, as the following theorem illustrates.

Theorem 9.1 The determinant of a lower-triangular, upper-triangular, or diagonal matrix is simply the product of its diagonal entries.

Example 9.1 For a lower- or upper-triangular 2×2 matrix A, $a_{12} = 0$ or $a_{21} = 0$. Therefore, by (2)

$$\det A = a_{11}a_{22} - 0 = a_{11}a_{22}.$$

For a lower-triangular 3×3 matrix, use the definition to compute

$$\det \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}C_{11} + 0 \cdot C_{12} + 0 \cdot C_{13}$$

$$= a_{11} \det \begin{pmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33}.$$

Theorem 9.1 along with the following theorem often leads to simpler calculations of $\det A$.

Theorem 9.2 Let A be an $n \times n$ matrix and let R be its row echelon form. Then

$$\det A = \pm \det R.$$

If no row interchanges are used to compute R from A, then $\det A = \det R$.

One can frequently combine the previous two theorems to compute $\det A$ more efficiently. First, convert A to its row echelon form R. Since R is an upper-triangular matrix, its determinant is simply the product of its diagonal entries.

Remark There is an easy-to-remember mnemonic device for computing the determinant of a 3×3 matrix A, that **works only for 3×3 matrices**. Form the partitioned matrix $\hat{A}$ by recopying the first and second rows of A right below A, as in Figure 9.1. Starting from a_{11} at the top left corner of $\hat{A}$, add together the three products along the three "diagonals" indicated by the solid lines in Figure 9.1:

$$a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}. \tag{6}$$

Then, starting from a_{21} at the bottom left corner of $\hat{A}$, subtract from (6) the three products along the three "counterdiagonals" indicated by the dotted lines in Figure 9.1:

$$-a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23} - a_{31}a_{22}a_{13}. \tag{7}$$

The result (6) + (7) is the determinant of A.

Example 9.2 Using this method, it is easy to see that

$$\begin{aligned}\det\begin{pmatrix}0 & 1 & 2\\ 3 & 4 & 5\\ 6 & 7 & 8\end{pmatrix} &= 0\cdot 4\cdot 8 + 3\cdot 7\cdot 2 + 6\cdot 1\cdot 5\\ &\quad - 3\cdot 1\cdot 8 - 0\cdot 7\cdot 5 - 6\cdot 4\cdot 2\\ &= 0 + 42 + 30 - 24 - 48\\ &= 0.\end{aligned}$$

Main Property of the Determinant

Finally, we put the above facts about determinants together to derive the main property of the determinant — the determinant *determines* whether or not a square matrix is nonsingular.

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Figure 9.1 *Computing the determinant of a* 3×3 *matrix.*

Theorem 9.3 A square matrix is nonsingular if and only if its determinant is nonzero.

Proof Sketch Recall that a square matrix A is nonsingular if and only if its row echelon form R has no all-zero rows. Since each row of the square matrix R has more leading zeros than the previous row, R has no all-zero rows if and only if the jth row of R has exactly $(j - 1)$ leading zeros. This occurs if and only if R has no zeros on its diagonal. Since $\det R$ is the product of its diagonal entries, A is nonsingular if and only if $\det R$ is nonzero. Since $\det R = \pm \det A$, A is nonsingular if and only if $\det A$ is nonzero. ■

Theorem 9.3 is obvious for 1×1 matrices, because the equation $ax = b$ has a unique solution, $x = b/a$, for every b if and only if $a \neq 0$. Theorem 8.8 demonstrates Theorem 9.3 for 2×2 matrices.

EXERCISES

9.1 Write out the complete expression for the determinant of a 3×3 matrix — six terms, each a product of three entries.

9.2 Write out the definition of the determinant of a 4×4 matrix in terms of the determinants of certain of its 3×3 submatrices. How many terms are there in the complete expansion of the determinant of a 4×4 matrix?

9.3 Compute out the expression on the right-hand side of (5). Show that it equals the expression calculated in Exercise 9.1.

9.4 Show that one obtains the same formula for the determinant of a 2×2 matrix, no matter which row or column one uses for the expansion.

9.5 Use a formula for the determinant to verify Theorem 9.1 for upper-triangular 3×3 matrices.

9.6 Verify the conclusion of Theorem 9.2 for 2×2 matrices by showing that the determinant of a general 2×2 matrix is not changed if one adds r times row 1 to row 2.

9.7 For each of the following matrices, compute the row echelon form and verify the conclusion of Theorem 9.2:

$$a)\ \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \qquad c)\ \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 7 & 8 \end{pmatrix}.$$

9.8 Use the observation following Theorem 9.2 to carry out a quick calculation of the determinant of each of the following matrices:

$$a)\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 1 & 9 & 6 \end{pmatrix}.$$

9.9 Use Theorem 9.3 to determine which of the matrices in Exercises 9.7 and 9.8 are nonsingular.

9.2 USES OF THE DETERMINANT

Since the determinant tells whether or not A^{-1} exists and whether or not $A\mathbf{x} = \mathbf{b}$ has a unique solution, it is not surprising that one can use the determinant to derive a formula for A^{-1} and a formula for the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$. First, we define the adjoint matrix of A as the transpose of the matrix of cofactors of A.

Definition For any $n \times n$ matrix A, let C_{ij} denote the (i, j)th cofactor of A, that is, $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting row i and column j from A. The $n \times n$ matrix whose (i, j)th entry is C_{ji}, the (j, i)th cofactor of A (note the switch in indices), is called the **adjoint** of A and is written **adj** A.

Theorem 9.4 Let A be a nonsingular matrix. Then,

(*a*) $A^{-1} = \dfrac{1}{\det A} \cdot \operatorname{adj} A$, and

(*b*) (**Cramer's rule**) the unique solution $\mathbf{x} = (x_1, \cdots, x_n)$ of the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ is

$$x_i = \frac{\det B_i}{\det A}, \qquad \text{for } i = 1, \ldots, n,$$

where B_i is the matrix A with the right-hand side $\mathbf{b}$ replacing the ith column of A.

For 3×3 systems,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

Cramer's rule states that

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

Example 9.3 Use Theorem 9.4 to invert the matrix

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{8}$$

$$C_{11} = +\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3, \quad C_{12} = -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0, \quad C_{13} = +\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -3,$$

$$C_{21} = -\begin{vmatrix} 4 & 5 \\ 0 & 1 \end{vmatrix} = -4, \quad C_{22} = +\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = -3, \quad C_{23} = -\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 4,$$

$$C_{31} = +\begin{vmatrix} 4 & 5 \\ 3 & 0 \end{vmatrix} = -15, \quad C_{32} = -\begin{vmatrix} 2 & 5 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{33} = +\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} = 6,$$

$$\det A = -9,$$

$$\operatorname{adj} A = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 3 & -4 & -15 \\ 0 & -3 & 0 \\ -3 & 4 & 6 \end{pmatrix}$$

So,

$$A^{-1} = -\frac{1}{9}\begin{pmatrix} 3 & -4 & -15 \\ 0 & -3 & 0 \\ -3 & 4 & 6 \end{pmatrix}. \tag{9}$$

Example 9.4 We can use Cramer's rule to calculate x_3 for the system in Example 7.1, which we write in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -4 \end{pmatrix}.$$

The determinant of the coefficient matrix A is 35. The determinant of

$$B_3 = \begin{pmatrix} 1 & 1 & 0 \\ 12 & 2 & 5 \\ 3 & 4 & -4 \end{pmatrix}$$

is also 35. Thus, $x_3 = |B_3|/|A| = 1$.

Finally, we note three algebraic properties of the determinant function which we will find important in our use of determinants.

Theorem 9.5 Let A be a square matrix. Then,

(*a*) $\det A^T = \det A$,
(*b*) $\det(A \cdot B) = (\det A)(\det B)$, and
(*c*) $\det(A + B) \neq \det A + \det B$, in general.

Gaussian elimination is a much more efficient method of solving a system of n equations in n unknowns than is Cramer's rule. Cramer's rule requires the evaluation of $(n+1)$ determinants. Each determinant is a sum of $n!$ terms and each term is a product of n entries. So, Cramer's rule requires $(n+1)!$ operations. On the other hand, the number of arithmetic operations required by Gaussian elimination for such a system is on the order of n^3. If $n = 6$ as in the Leontief model in Section 8.5, then $(n+1)!$ is 5040, while n^3 is only 216; the difference grows exponentially as n increases.

Nevertheless, Cramer's rule is particularly useful for small linear systems in which the coefficients a_{ij} are parameters and for which one wants to obtain a general formula for the endogenous variables (the x_i's) in terms of the parameters and the exogenous variables (the b_j's). One can then see more clearly how changes in the parameters affect the values of the endogenous variables.

EXERCISES

9.10 Verify directly that matrix (9) really is the inverse of matrix (8) in Example 9.3.

9.11 Use Theorem 9.4 to invert the following matrices:

$$a)\ \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}, \qquad c)\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

9.12 Use Cramer's rule to compute x_1 and x_2 in Example 9.4.

9.13 Use Cramer's rule to solve the following systems of equations:

$$a)\ \begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4; \end{aligned} \qquad b)\ \begin{aligned} 2x_1 - 3x_2 \phantom{{}+x_3} &= 2 \\ 4x_1 - 6x_2 + x_3 &= 7 \\ x_1 + 10x_2 \phantom{{}+x_3} &= 1. \end{aligned}$$

9.14 Verify the conclusions of Theorem 9.5 for the following pairs of matrices:

$$a)\ A = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix};$$

b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix};$

c) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$

9.3 IS-LM ANALYSIS VIA CRAMER'S RULE

As an illustrative example, consider the linear IS-LM national income model described in Chapter 6:

$$\begin{aligned} sY + ar &= I^o + G \\ mY - hr &= M_s - M^o \end{aligned} \tag{10}$$

where Y = net national product
r = interest rate
s = marginal propensity to save,
a = marginal efficiency of capital,
I = investment $(= I^o - ar)$,
m = money balances needed per dollar of transactions,
G = government spending,
M_s = money supply.

All the parameters are positive. Because the coefficients in this system are parameters instead of numbers, it is easiest to solve (10) using Cramer's rule:

$$Y = \frac{\begin{vmatrix} I^o + G & a \\ M_s - M^o & -h \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}} = \frac{(I^o + G)h + a(M_s - M^o)}{sh + am}$$

$$r = \frac{\begin{vmatrix} s & I^o + G \\ m & M_s - M^o \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}} = \frac{(I^o + G)m - s(M_s - M^o)}{sh + am}.$$

One can now use these expressions to see that, in this model, an increase in I^o, G, or M_s or a decrease in M^o or m will lead to an increase in the equilibrium net product Y. An increase in I^o or M^o or a decrease in M_s, h, or m will lead to an increase in equilibrium interest rate r.

EXERCISES

9.15 Verify the assertions in the last two sentences before these exercises.

9.16 If you are familiar with partial derivatives, compute

$$\frac{\partial Y}{\partial a} = \frac{-rh}{(sh + am)} \leq 0.$$

So an increase in the marginal efficiency of capital a will bring down the equilibrium Y and r. How will the equilibrium Y change if h increases? How will the equilibrium r change if m or s increases?

9.17 If we introduce tax rate t and let the consumption function depend on after-tax income, $C = b(Y - tY)$, then system (10) becomes

$$\begin{aligned} (1 - t)sY + ar &= I^o + G \\ mY - hr &= M_s - M^o. \end{aligned}$$

Use Cramer's rule to see how the equilibrium Y and r are affected by the tax rate t.

9.18 Consider the following more elaborate linear IS-LM.

$a)$ $Y = C + I + G$ $\qquad$ $b)$ $C = c_0 + c_1(Y - T) - c_2 r$

$c)$ $T = t_0 + t_1 Y$ $\qquad$ $d)$ $I = I^o + a_0 Y - ar$

$e)$ $M_s = mY + M^o - hr.$

Substitute c into b to obtain b'; then substitute b' and d into a to get the new IS-curve. Combine this with e and use Cramer's rule to solve this system for Y and r in terms of the exogenous variables. Show that an increase in G or a reduction of t_0 or t_1 will increase Y: in macroeconomic terms, Keynesian fiscal policy "works" in this model. Show that these changes also increase r. Regarding monetary policy, show that an increase in M_s increases Y and lowers r.

9.19 What is the effect of an increase in I^o, c_0, or m?

9.20 For Example 1 in Chapter 6, write out the linear system which corresponds to equation (1) in Chapter 6, but with a general before-tax profit P, a general contribution percentage c, and general state and federal tax rates r and f. Use Cramer's rule to compute C, S, and F in terms of P, c, s, and f.