Network Parameter Analysis

Jianhua Zhou, Sheldon X.D. Tan

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1 Introduction

Network parameter matrices is an important and widely-adopted technique in the design and test of modern electronic systems. Large complex systems can be divided into smaller units with their characteristics represented by Z,Y,S,T or other matrices. The entire system can be analysed with much greater flexibility and efficiency without loss of accuracy. In this chapter we present concepts, principles and methods of network parameter analysis applicable to linear time-invariant (LTI) networks.

In lumped element circuit analysis, the governing laws of classical circuit theory (i.e. KVL, KCL and BCE) are only approximations of Maxwell equations under some rather restricting conditions. Once those conditions are violated, the classical circuit equations become increasingly inaccurate as frequency increases, regardless of how accurately they can be solved numerically. This is the fundamental reason why accurate high-frequency analysis cannot always be achieved by simply extending the classical circuit analysis to high frequencies with brute force.

Strictly speaking, accurate solutions at high frequencies can only be obtained by solving Maxwell equations. In practice this is called the full-wave solution. The only errors present in full-wave solutions are various numerical errors, which can be systematically reduced by setting lower thresholds of error in the simulator. It takes significant amount of memory and CPU time to obtain a full-wave solution for even a modest-sized structure at high frequencies.

Due to the significant amount of resources needed in solving the full-wave Maxwell equations, it is highly desirable to use classical circuit analysis methods in high-frequency simulations for improved efficiency and ease of use. As a result, it is very important to comprehend those situations when classical circuit analysis starts to deviate from their governing Maxwell equations.

1.1 Network and Port

This section discusses network and port in the context of general electromagnetic fields and field patterns. In order to emphasize the concepts in practical applications, the use of vector analysis notation is minimized.

The generalized definitions of network and port are given in the context of electromagnetic fields and guided waves theory in three-dimensional space.

Definition: A network is a structure enclosed in a 3D region designed for guiding electromagnetic waves. The structure may be consisted of various materials such as conductors or dielectrics. The surface area of the 3D region serves as the boundary of the network.

Based on electromagnetic principles, the electric and magnetic fields inside a 3D region are uniquely determined by the following conditions: (a) materials inside the region (b) boundary conditions on the enclosing surface (c) field excitations at ports (d) current sources inside the region

This principle has been rigorously proved in electromagnetic theory. For most readers it is sufficient to have intuitive interpretation of this important principle.

It is the subject of EMC/EMI to solve problems with external excitations on the boundary of the region (condition (b) in the list above). We only focus on problems that the excitations only exist at designated ports of transmission lines (condition (c) in the list above). Note that the lack of external excitations on the boundary does not preclude the fields radiating from the inside through the boundary. It is common to have fields radiating through the boundary surface from within a structure. This condition is commonly designated as radiation or absorbing boundary condition.

For readers who are familiar with lumped circuit analysis, at first sight it might appear rather restricting that transmission lines are the only means for signals to go in and out of a network. There seem to be numerous cases where ports of networks are only connected by lumped terminations and sources without any transmission line structures. In reality without exception, all port connections are some form of transmission line. Lumped element terminations are abstractions of real structures containing physical transmission lines.

A physical transmission line must always have a self-joint cross-sectional surface area which may have finite or infinite dimensions in the two-dimensional space. For example the cross sectional area of a coaxial transmission line is confined between the inner and outer conductors with a finite size. The cross sectional area of an ideal microtrip line contain the entire upper half space with unlimited size in both the horizontal and vertical directions. A physical transmission line is different from a ideal transmission line which may not have physical dimensions and are only characterized by RLGC parameters.

At this point we introduce the formal definition of a network port.

Definition: a port is the cross-sectional surface area of a transmission line connecting the network to the rest of the system. The surface area of the port is formed by intersecting the transmission line using a surface which is often called the reference plane. Although in most practical cases reference planes are planar, they are not required to be planar.

Port surfaces also serve as the boundary of the network, separating the transmission lines into the inside and outside region. To emphasize the importance of this phenomenon, we put it in the form of a observation:

Observation: from the location of a port going outward, a port is always extended by infinitely long uniform transmission lines having the same cross-section as the port.

Ports corresponding to condition (c) in the list above are often called wave-ports to illustrate the fact that they are excited by electromagnetic waves in the transmission lines extending outward from the ports.

Strictly speaking, current source excitations inside a region (condition (d)) have fundamentally different properties from electromagnetic fields in ports. In practice, these current sources are called lumped ports.

Definition: lumped ports are current sources inside a region.

One of the main characteristics of the lumped ports is that the current sources are known a priori, meaning that they are assumed to exist completely independently, where as the field patterns at wave ports must be solved according to Maxwell equations.

Another important difference between lumped and wave ports is that the current sources occupy the area or volume in which they are defined, whereas the fields in a wave port represent the travelling wave exist in the entire transmission line. This phenomenon imposes the important requirement on the wave ports that their locations must be sufficiently far away from any transmission line discontinuities such that the fields of other modes become negligible.

Rule: wave ports should be located sufficiently far away from transmission line discontinuities such that impact of higher order modes become negligible.

Some transmission lines are consisted of one or multiple conductors. Some transmission lines such as optical fibre are purely dielectric. For signal integrity applications we are mainly concerned with transmission lines formed by two or more conductors with one of them being the reference conductor (aka ground or return path).

1.2 Voltage and Current

In the previous section, the definitions of generalized network and their ports are given in terms of electromagnetic fields in arbitrary structures with transmission line ports of arbitrary cross sections.

In order to apply network analysis techniques the electric and magnetic field quantities at network port locations need to be mapped to voltages and currents.

In the most general cases, electric and magnetic fields in transmission lines cannot be uniquely mapped to voltage and current. This important fact is emphasized by the following observation.

Observation: in a uniform transmission line of arbitrary cross section, electric and magnetic fields can only be uniquely mapped to voltage and current if the are transverse electromagnetic (TEM).

It is important to note that this observation has nothing to do with the fact that voltage and current may be difficult to measure or simulate at high frequencies.

This observation has the same root with the following observation.

Observation: Voltage can only be defined for irrotational electric fields.

In irrotational fields the integration of electric field over any closed path is always zero.

$$V = \int Edl \tag{1}$$

Equivalently in irrotational fields the integration of electric field between any two points is path independent,

$$V = \int Edl \tag{2}$$

Irrotational fields can be intuitively understood as always having zero rotations. In order to be irrotational the fields lines must always start from a positive charge and end on an negative charge. Irrotational fields are also called conservative fields.

Now let's look at what type of structures or fields are irrotational.

Observation: the only two types of irrotational electromagnetic fields are, (a) static electric fields (b) electric fields in transverse electromagnetic waves in transmission lines.

The importance of TEM waves is fully demonstrated by this observation that they are the only type of fields in which voltage can be meaningfully defined when the frequency is greater than 0. For all other non-static fields, voltage is only an approximation of the static (i.e. DC) case and the error increases with frequency. These systematic errors cannot be reduced by increasing the accuracy of simulations.

Observation: for non-TEM waves, the systematic errors in voltages cannot be reduced by increasing the accuracy of the calculations or simulations.

TEM mode can only exist in multiple conductor transmission lines. With a zero cut-off frequency, TEM mode transmission lines work for all frequencies from DC to very high frequencies.

Voltage, current and characteristic impedance of TEM transmission lines can be uniquely defined by electromagnetic field distributions in the cross section. This is a very important property of TEM transmission lines in network analysis. This property allows the unique definition of port current, voltage and impedance which are the essential quantities in network analysis.

Definition: current on a TEM transmission line conductor equals the circular integral of magnetic field encircling that conductor.

Even though most transmission lines in high-speed serial applications are TEM (or quasi-TEM), other structures such as vias, BGAs and connectors are non-TEM. In 3DEM analysis, ports are often placed at locations of these non-TEM structures. Due to the importance of this issue it is worthwhile to further discuss this topic in more detail.

This important observation leads to the following rule,

Rule 1. The voltage between two points a and b can only be established in the following two cases, (a) a and b both lie on the same port of a TEM (or quasi-TEM) transmission line (b) the frequency is sufficiently low such that the electric field is practically conservative (quasi-static)

At this point we shall discuss the concept of global ground and voltage between two arbitrary nodes. We will also discuss the concept of KVL, KCL and BCE.

1.3 Voltage and Current Vectors

In frequency-domain the voltage and current at any given port k of a n-port network can be expressed as

$$V_k(\omega) = |V_k(\omega)|e^{j\theta_k^v(\omega)} \tag{3}$$

$$I_k(\omega) = |I_k(\omega)|e^{j\theta_k^i(\omega)} \tag{4}$$

Putting all variables into column vectors, we have

$$\mathbf{V}(\omega) = [V_1(\omega), V_2(\omega), \cdots, V_n(\omega)]^T \tag{5}$$

$$\mathbf{I}(\omega) = [I_1(\omega), I_2(\omega), \cdots, I_n(\omega)]^T \tag{6}$$

2 Network Parameter Matrices

A network parameter matrix is defined within the context of voltage, current of wave relationships at the ports of the network. The term parameter refers to the fact that it is a quantity representing such a relationship. The term matrix refers to the fact that such parameters always exist in the form of a matrix. There two terms are often used interchangeably in literatures.

For example, in Ohm's Law $V = RI$, R can be considered a parameter relating voltage and current as defined.

2.1 Impedance and Admittance Matrices

The impedance matrix $\mathbf{Z}(\omega)$ and admittance matrix $\mathbf{Y}(\omega)$ of the network are defined in terms of port voltage and current vectors,

$$\mathbf{V}(\omega) = \mathbf{Z}(\omega)\mathbf{I}(\omega) \tag{7}$$

$$\mathbf{I}(\omega) = \mathbf{Y}(\omega)\mathbf{V}(\omega) \tag{8}$$

where

$$\mathbf{Z}(\omega) = \begin{bmatrix} Z_{11}(\omega) & Z_{12}(\omega) & \cdots Z_{1n}(\omega) \\ Z_{21}(\omega) & Z_{22}(\omega) & \cdots Z_{2n}(\omega) \\ \vdots & \vdots & \ddots \\ Z_{n1}(\omega) & Z_{n2}(\omega) & \cdots Z_{nn}(\omega) \end{bmatrix} \tag{9}$$

$$\mathbf{Y}(\omega) = \begin{bmatrix} Y_{11}(\omega) & Y_{12}(\omega) & \cdots Y_{1n}(\omega) \\ Y_{21}(\omega) & Y_{22}(\omega) & \cdots Y_{2n}(\omega) \\ \vdots & \vdots & \ddots \\ Y_{n1}(\omega) & Y_{n2}(\omega) & \cdots Y_{nn}(\omega) \end{bmatrix} \tag{10}$$

and $\mathbf{Z}(\omega)$ and $\mathbf{Y}(\omega)$ are the reciprocal of each other,

$$\mathbf{Z}(\omega) = \mathbf{Y}(\omega)^{-1} \tag{11}$$

When it is not causing confusion between time and frequency domain, the ω symbol is often omitted for simplicity.

In (9) Z_{kk} is the self-impedance of port k when all other port currents are set to zero, i.e.

$$Z_{kk} = \left.\frac{V_k}{I_k}\right|_{I_i=0,\forall i\neq k} \tag{12}$$

Z_{pq} is the trans-impedance from port q to port p when all other port currents are set to zero,

$$Z_{pq} = \left.\frac{V_p}{I_q}\right|_{I_i=0,\forall i\neq q} \tag{13}$$

Similarly, Y_{kk} is the self-admittance of port k when all other port voltages are set to zero, i.e.

$$Y_{kk} = \left.\frac{I_k}{V_k}\right|_{V_i=0,\forall i\neq k} \tag{14}$$

and Y_{pq} is the trans-admittance from port q to port p when all other port voltages are set to zero,

$$Y_{pq} = \left.\frac{I_p}{V_q}\right|_{V_i=0,\forall i\neq q} \tag{15}$$

These parameters can also be viewed as system transfer functions in frequency domain,

$$V_p(\omega) = Z_{pq}(\omega)I_q(\omega)|_{I_i=0,\forall i\neq q} \tag{16}$$

$$I_p(\omega) = Y_{pq}(\omega)V_q(\omega)|_{V_i=0,\forall i\neq q} \tag{17}$$

In time domain, the above relationship can be expressed as,

$$v_p(t) = z_{pq}(t) * i_q(t)|_{i_j(t)=0,\forall j\neq q} \tag{18}$$

$$i_p(t) = y_{pq}(t) * v_q(t)|_{v_j(t)=0,\forall j\neq q} \tag{19}$$

where the time domain quantities are related to their frequency domain counterparts by inverse Fourier transform.

2.2 Scattering Parameters

Scattering parameter matrix, also known as S-parameter matrix, is defined in terms of incident and scattered waves at the ports. Incident waves travel from the outside to the inside region and, scattered waves travel from the inside to the outside region through the port interfaces. Incident and scattered waves co-exist at the ports.

The basic concept of the scattering-parameter approach is to express the scattered waves in terms of the incident waves, analogous to the phenomenon of electromagnetic scattering.

Due to the fact that incident and scattered waves are defined in terms of port voltages and currents rather than electric and magnetic fields, there are numerous variations of S-parameter definitions, each claim to have benefits over the others. It is important to understand the differences between various definitions of S-parameters and interpret their meanings properly.

In the following subsections we will discuss all types of S-parameters and examine the differences between them.

2.2.1 Ordinary S-Parameter

The so-called ordinary S-parameters maps incident voltage or current waves to scattered waves,

$$[scatteredwavevector] = [ordinaryS - parameters][incidentwavevector] \tag{20}$$

Voltage S-parameters are defined by incident and scattered voltages at port locations.

According to transmission line theory, the voltage and current waves in a TEM transmission line can be divided into forward and backward waves,

$$\tilde{V}(z) = \tilde{V}^{+}e^{-\gamma z} + \tilde{V}^{-}e^{\gamma z} \tag{21}$$

$$\tilde{I}(z) = \tilde{I}^{+}e^{-\gamma z} - \tilde{I}^{-}e^{\gamma z} \tag{22}$$

where $\tilde{V}^{+}, \tilde{V}^{-}, \tilde{I}^{+}$ and $\tilde{I}^{-}$ are complex coefficients of forward and backward waves.

The voltage and current wave coefficients are related by the characteristic impedance of the transmission line,

$$\tilde{I}^{+} = \frac{\tilde{V}^{+}}{Z_0} \tag{23}$$

$$\tilde{I}^{-} = \frac{\tilde{V}^{-}}{Z_0} \tag{24}$$

For lossy transmission lines, the characteristic impedance Z_0 is a complex constant,

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = |Z_0|\angle\theta \tag{25}$$

It is worth noting that for lossy transmission lines, Z_0 is not real and the voltage and current waves are not in phase to each other. This will cause significant impact on the definitions and interpretations of S-parameter which we will discuss in detail in subsequent sections.

At locations of ports, the z variables become have fixed values and equations (21) and (22) are reduced to,

$$V = V^{+} + V^{-} \tag{26}$$

$$I = I^{+} + I^{-} \tag{27}$$

where V^{+}, V^{-}, I^{+} and I^{-} are complex quantities representing the forward and backward waves at the locations of the ports.

2.2.2 Generalized S-Parameter

It is well-known from the transmission line theory that the voltage and current at any given port k ($k = 1, 2, ..., n$) is the superposition of the incident and scattered waves at the port,

$$V_k = V_k^{+} + V_k^{-} \tag{28}$$

$$I_k = I_k^{+} - I_k^{-} \tag{29}$$

where $+$ denotes the incident waves and, $-$ denotes the scattered waves.

The powers of the incident and scattered waves are

$$P_k^{+} = \frac{(V_k^{+})^2}{Z_{0k}} = (I_k^{+})^2 Z_{0k} \tag{30}$$

$$P_k^{-} = \frac{(V_k^{-})^2}{Z_{0k}} = (I_k^{-})^2 Z_{0k} \tag{31}$$

where Z_{0k} is the reference impedance of port k. The incident power wave a_k and scattered power wave b_k are defined as,

$$a_k = \frac{V_k^{+}}{\sqrt{Z}_{0k}} = I_k^{+}\sqrt{Z}_{0k} \tag{32}$$

$$b_k = \frac{V_k^-}{\sqrt{Z_{0k}}} = I_k^- \sqrt{Z_{0k}} \tag{33}$$

Eqs. (28) and (29) can now be expressed as

$$V_k = \sqrt{Z_{0k}}(a_k + b_k) \tag{34}$$

$$I_k = \sqrt{Y_{0k}}(a_k - b_k) \tag{35}$$

where Y_{0k} is the reference admittance of port k,

$$Y_{0k} = \frac{1}{Z_{0k}} \tag{36}$$

Putting these equations in the matrix form, we have

$$\mathbf{V} = \mathbf{Z}_0^R(\mathbf{a} + \mathbf{b}) \tag{37}$$

$$\mathbf{I} = \mathbf{Y}_0^R(\mathbf{a} - \mathbf{b}) \tag{38}$$

where $\mathbf{V}$ and $\mathbf{I}$ are column vectors as defined in Eqs.(5) and (6); $\mathbf{a}$ and $\mathbf{b}$ are the column vectors of a_k and b_k,

$$\mathbf{a} = [a_1, a_2, \cdots, a_n]^T \tag{39}$$

$$\mathbf{b} = [b_1, b_2, \cdots, a_n]^T \tag{40}$$

$\mathbf{Z}_0^R$, $\mathbf{Y}_0^R$ are the reciprocal diagonal matrices of $\sqrt{Z_{0k}}$ and $\sqrt{Y_{0k}}$,

$$\mathbf{Z}_0^R = \begin{bmatrix} \sqrt{Z_{01}} & & & \\ & \sqrt{Z_{02}} & & \\ & & \ddots & \\ & & & \sqrt{Z_{0n}} \end{bmatrix} \tag{41}$$

$$\mathbf{Y}_0^R = \begin{bmatrix} \sqrt{Y_{01}} & & & \\ & \sqrt{Y_{02}} & & \\ & & \ddots & \\ & & & \sqrt{Y_{0n}} \end{bmatrix} \tag{42}$$

$$\mathbf{Z}_0^R \mathbf{Y}_0^R = \mathbf{U} \tag{43}$$

where $\mathbf{U}$ is the identity matrix of size n.

Eqs. (37) and (38) can be rearranged to obtain

$$\mathbf{a} = \frac{1}{2}\left(\mathbf{Y}_0^R \mathbf{V} + \mathbf{Z}_0^R \mathbf{I}\right) \tag{44}$$

$$\mathbf{b} = \frac{1}{2}\left(\mathbf{Y}_0^R \mathbf{V} - \mathbf{Z}_0^R \mathbf{I}\right) \tag{45}$$

The scattering matrix of a n-port LTI network is defined in terms of the incident power wave $\mathbf{a}$ and scattered power wave $\mathbf{b}$,

$$\mathbf{b} = \mathbf{S}\mathbf{a} \tag{46}$$

where

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & \cdots S_{1n} \\ S_{21} & S_{22} & \cdots S_{2n} \\ \vdots & \vdots & \ddots \\ S_{n1} & S_{n2} & \cdots S_{nn} \end{bmatrix} \tag{47}$$

The element S_{ij} in the S-parameter matrix is a transfer function for input wave a_j and scattered output wave b_i.

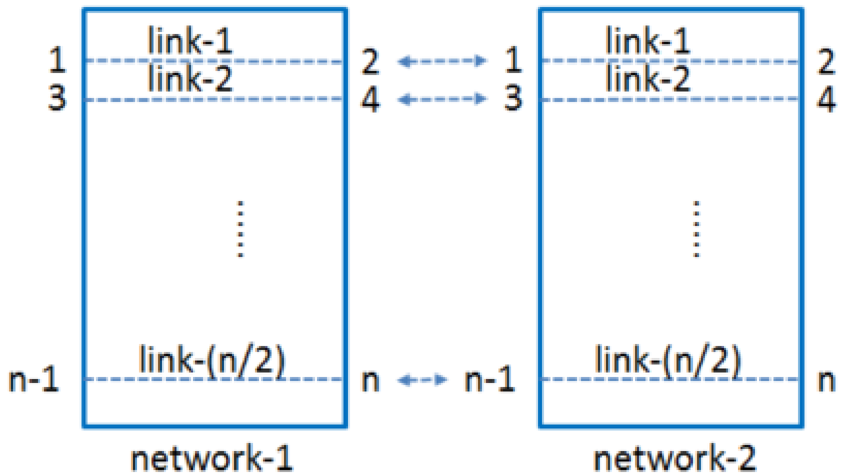

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Figure 1: Cascading of Networks

2.3 Transmission Parameters

Transmission matrix or T matrix is mainly used in the cascading of multiple networks in series as depicted in Fig.(1). All networks have the same number of input and output ports with the same port numbering scheme. There are many different ways to number the ports of such cascaded networks. Without loss of generality, we adopt the Z-convention of port numbering scheme as shown in Fig.(1). In this convention, the port numbers form a Z-shaped path in the network symbol when traversed sequentially in the order of $1, 2, 3, ..., n$, where n is always an even number.

2.3.1 Odd-even Permutated Block S Matrix

To facilitate the definition and derivation of $\mathbf{T}$ matrix, we considered the odd-even permuated block $\mathbf{S}$ matrix where the original $\mathbf{S}$ matrix and associated wave vectors $\mathbf{b}$ and $\mathbf{a}$ are resequenced into odd and even blocks by the permutation matrix $\mathbf{P}_{oe}$,

$$\mathbf{P}_{oe} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 \end{bmatrix} \tag{48}$$

$$\mathbf{P}_{oe} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_2 \\ b_4 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \mathbf{b}^o \\ \mathbf{b}^e \end{bmatrix} \tag{49}$$

$$\mathbf{P}_{oe}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{n-1} \\ a_2 \\ a_4 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}^o \\ \mathbf{a}^e \end{bmatrix} \tag{50}$$

$$\mathbf{P}_{oe}\mathbf{S}\mathbf{P}_{oe}^T = \begin{bmatrix} \mathbf{S}_{oo} & \mathbf{S}_{oe} \\ \mathbf{S}_{eo} & \mathbf{S}_{ee} \end{bmatrix} \tag{51}$$

where

$$\mathbf{S}_{oo} = \begin{bmatrix} S_{11} & S_{13} & \cdots & S_{1,n-1} \\ S_{31} & S_{33} & \cdots & S_{3,n-1} \\ \vdots & \vdots & \ddots & \\ S_{n-1,1} & S_{n-1,3} & \cdots & S_{n-1,n-1} \end{bmatrix} \tag{52}$$

$$\mathbf{S}_{oe} = \begin{bmatrix} S_{12} & S_{14} & \cdots & S_{1,n} \\ S_{32} & S_{34} & \cdots & S_{3,n} \\ \vdots & \vdots & \ddots & \\ S_{n-1,2} & S_{n-1,4} & \cdots & S_{n-1,n} \end{bmatrix} \tag{53}$$

$$\mathbf{S}_{eo} = \begin{bmatrix} S_{21} & S_{23} & \cdots & S_{2,n-1} \\ S_{41} & S_{43} & \cdots & S_{4,n-1} \\ \vdots & \vdots & \ddots & \\ S_{n,1} & S_{n,3} & \cdots & S_{n,n-1} \end{bmatrix} \tag{54}$$

$$\mathbf{S}_{ee} = \begin{bmatrix} S_{22} & S_{24} & \cdots & S_{2,n} \\ S_{42} & S_{44} & \cdots & S_{4,n} \\ \vdots & \vdots & \ddots & \\ S_{n,2} & S_{n,4} & \cdots & S_{n,n} \end{bmatrix} \tag{55}$$

The matrix form of the odd-even permutated **S** martrix is expressed as,

$$\begin{bmatrix} \mathbf{b}^o \\ \mathbf{b}^e \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{oo} & \mathbf{S}_{oe} \\ \mathbf{S}_{eo} & \mathbf{S}_{ee} \end{bmatrix} \begin{bmatrix} \mathbf{a}^o \\ \mathbf{a}^e \end{bmatrix} \tag{56}$$

2.3.2 Definition of T Matrix

T matrix can be defined either in terms of port voltages and currents or incident and scattered waves. Due to the fact that most network analysis problems are originated from S matrix data, we define the T matrix in terms of port incident and scattered waves.

Consider a transmission network with an even number of n ports where ports 1 and 2, 3 and 4, ..., $n-1$ and n are input/output pairs as depicted in Fig. (??). The T matrix of such a transmission network is defined as,

$$\begin{bmatrix} b_1 \\ b_3 \\ \vdots \\ b_{n-1} \\ a_1 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} \begin{bmatrix} a_2 \\ a_4 \\ \vdots \\ a_n \\ b_2 \\ b_4 \\ \vdots \\ b_n \end{bmatrix} \tag{57}$$

The matrix form of this definition is expressed in terms of odd and even wave vectors,

$$\begin{bmatrix} \mathbf{b}^o \\ \mathbf{a}^o \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{ba} & \mathbf{T}_{bb} \\ \mathbf{T}_{aa} & \mathbf{T}_{ab} \end{bmatrix} \begin{bmatrix} \mathbf{a}^e \\ \mathbf{b}^e \end{bmatrix} \tag{58}$$

This definition of T matrix satisfies the following cascading conditions of matrices $\mathbf{T}_1$ and $\mathbf{T}_2$,

$$\begin{bmatrix} \mathbf{a}_1^e \\ \mathbf{b}_1^e \end{bmatrix} = \begin{bmatrix} \mathbf{b}_2^o \\ \mathbf{a}_2^o \end{bmatrix} \tag{59}$$

allowing for direct concatenation of networks by matrix multiplication,

$$\begin{bmatrix} \mathbf{b}_1^o \\ \mathbf{a}_1^o \end{bmatrix} = \mathbf{T}_1 \begin{bmatrix} \mathbf{a}_1^e \\ \mathbf{b}_1^e \end{bmatrix} = \mathbf{T}_1 \begin{bmatrix} \mathbf{b}_2^o \\ \mathbf{a}_2^o \end{bmatrix} = \mathbf{T}_1\mathbf{T}_2 \begin{bmatrix} \mathbf{a}_2^e \\ \mathbf{b}_2^e \end{bmatrix} \tag{60}$$

3 Matrix Conversions

3.1 Z-Y Conversions

By definitions of $\mathbf{Z}$ and $\mathbf{Y}$ matrices in (7) and (8), $\mathbf{Z}$ and $\mathbf{Y}$ matrices are the inverse of each other,

$$\mathbf{Z} = \mathbf{Y}^{-1} \tag{61}$$

$$\mathbf{Y} = \mathbf{Z}^{-1} \tag{62}$$

3.2 S-Z Conversions

3.2.1 S to Z

To convert $\mathbf{S}$ matrix to $\mathbf{Z}$ matrix, substitute the expressions of $\mathbf{a}$ and $\mathbf{b}$ in (44) and (45) into the definition of $\mathbf{S}$ matrix (46) and rearrange the terms, we have

$$(\mathbf{U} - \mathbf{S})\mathbf{Y}_0^R\mathbf{V} = (\mathbf{U} + \mathbf{S})\mathbf{Z}_0^R\mathbf{I} \tag{63}$$

where $\mathbf{U}$ is the identity matrix.

In the case of $\mathbf{S} = \mathbf{U}$, the network is open circuit at all ports, and the $\mathbf{Z}$ matrix does not exist. Excluding this trivial case by assuming $\mathbf{S} \neq \mathbf{U}$, we have

$$\mathbf{V} = \mathbf{Z}_0^R(\mathbf{U} - \mathbf{S})^{-1}(\mathbf{U} + \mathbf{S})\mathbf{Z}_0^R\mathbf{I} \tag{64}$$

From the definition of $\mathbf{Z}$ matrix in (7), we readily obtain

$$\mathbf{Z} = \mathbf{Z}_0^R(\mathbf{U} - \mathbf{S})^{-1}(\mathbf{U} + \mathbf{S})\mathbf{Z}_0^R \tag{65}$$

where $\mathbf{U} - \mathbf{S} \neq 0$.

Alternatively, we could also substitute the definition of scattering matrix (46) into (37) and (38),

$$\mathbf{Y}_0^R\mathbf{V} = (\mathbf{U} + \mathbf{S})\mathbf{a} \tag{66}$$

$$\mathbf{Z}_0^R\mathbf{I} = (\mathbf{U} - \mathbf{S})\mathbf{a} \tag{67}$$

Solve for $\mathbf{a}$ from (67) and substitute into (66), we have

$$\mathbf{Z} = \mathbf{Z}_0^R(\mathbf{U} + \mathbf{S})(\mathbf{U} - \mathbf{S})^{-1}\mathbf{Z}_0^R \tag{68}$$

To prove that (65) and (68) are equivalent, consider the error term $\mathbf{E}$,

$$\mathbf{E} = (\mathbf{U} - \mathbf{S})^{-1}(\mathbf{U} + \mathbf{S}) - (\mathbf{U} + \mathbf{S})(\mathbf{U} - \mathbf{S})^{-1} \tag{69}$$

Multiply both sides of the equation by $\mathbf{U} - \mathbf{S}$ from the left and from the right, we obtain

$$(\mathbf{U} - \mathbf{S})\mathbf{E}(\mathbf{U} - \mathbf{S}) = (\mathbf{U} + \mathbf{S})(\mathbf{U} - \mathbf{S}) - (\mathbf{U} - \mathbf{S})(\mathbf{U} + \mathbf{S}) = 0 \tag{70}$$

Again by assuming $\mathbf{S} \neq \mathbf{U}$, we have $\mathbf{E} = 0$. The two expressions of $\mathbf{Z}$ matrix in (65) and (68) are equivalent.

For uniformly terminated networks where all ports having the same reference impedance,

$$Z_{01} = Z_{02} = \cdots = Z_{0n} = Z_0 \tag{71}$$

The expressions in (65) and (68) are reduced to

$$\mathbf{Z} = Z_0(\mathbf{U} - \mathbf{S})^{-1}(\mathbf{U} + \mathbf{S}) \tag{72}$$

$$\mathbf{Z} = Z_0(\mathbf{U} + \mathbf{S})(\mathbf{U} - \mathbf{S})^{-1} \tag{73}$$

3.2.2 Z to S

To convert $\mathbf{Z}$ matrix to $\mathbf{S}$ matrix, substitute the expressions of $\mathbf{V}$ and $\mathbf{I}$ in Eqs. (37) and (38) into the $\mathbf{Z}$ matrix definition (7), we have,

$$\mathbf{Z}_0^R(\mathbf{a} + \mathbf{b}) = \mathbf{Z}\mathbf{Y}_0^R(\mathbf{a} - \mathbf{b}) \tag{74}$$

Multiply both sides by $\mathbf{Z}_0^R$ and rearrange the terms, we obtain

$$\mathbf{b} = (\tilde{\mathbf{Z}} + \mathbf{Z}_0)^{-1}(\tilde{\mathbf{Z}} - \mathbf{Z}_0)\mathbf{a} \tag{75}$$

where

$$\begin{aligned}
\tilde{\mathbf{Z}} &= \mathbf{Z}_0^R\mathbf{Z}\mathbf{Y}_0^R \\
\mathbf{Z}_0 &= \mathbf{Z}_0^R\mathbf{Z}_0^R \\
&= diag(Z_{01}, Z_{02}, ..., Z_{0n}) \\
\tilde{\mathbf{Z}} &\neq -\mathbf{Z}_0
\end{aligned}$$

The $\mathbf{S}$ matrix is readily obtained by the definition of $\mathbf{b} = \mathbf{S}\mathbf{a}$,

$$\mathbf{S} = (\tilde{\mathbf{Z}} + \mathbf{Z}_0)^{-1}(\tilde{\mathbf{Z}} - \mathbf{Z}_0) \tag{76}$$

In case of uniformly terminated networks where all ports having the same reference impedance,

$$Z_{01} = Z_{02} = \cdots = Z_{0n} = Z_0 \tag{77}$$

$$\tilde{\mathbf{Z}} = \mathbf{Z}_0^R\mathbf{Z}\mathbf{Y}_0^R = \mathbf{Z} \tag{78}$$

$$\mathbf{S} = (\mathbf{Z} + \mathbf{Z}_0)^{-1}(\mathbf{Z} - \mathbf{Z}_0) \tag{79}$$

3.2.3 Z-S-Z

To verify the conversion between $\mathbf{S}$ and $\mathbf{Z}$ matrices, a known impedance matrix $\mathbf{Z}^{(1)}$ is first converted to $\mathbf{S}$ matrix $\mathbf{S}^{(1)}$ using (76),

$$\mathbf{S}^{(1)} = (\tilde{\mathbf{Z}}^{(1)} + \mathbf{Z}_0)^{-1}(\tilde{\mathbf{Z}}^{(1)} - \mathbf{Z}_0) \tag{80}$$

where

$$\tilde{\mathbf{Z}}^{(1)} = \mathbf{Z}_0^R\mathbf{Z}^{(1)}\mathbf{Y}_0^R \tag{81}$$

$\mathbf{S}^{(1)}$ is then converted back into $\mathbf{Z}^{(2)}$ matrix by substituting into (65),

$$\mathbf{Z}^{(2)} = \mathbf{Z}_0^R(\mathbf{U} - \mathbf{S}^{(1)})^{-1}(\mathbf{U} + \mathbf{S}^{(1)})\mathbf{Z}_0^R \tag{82}$$

$$(\mathbf{U} - \mathbf{S}^{(1)})\mathbf{Y}_0^R\mathbf{Z}^{(2)} = (\mathbf{U} + \mathbf{S}^{(1)})\mathbf{Z}_0^R \tag{83}$$

$$\mathbf{Y}_0^R\mathbf{Z}^{(2)} - \mathbf{Z}_0^R = \mathbf{S}^{(1)}(\mathbf{Y}_0^R\mathbf{Z}^{(2)} + \mathbf{Z}_0^R) \tag{84}$$

Substitute (80) into above equation, we have

$$\mathbf{Y}_0^R\mathbf{Z}^{(2)} - \mathbf{Z}_0^R = (\tilde{\mathbf{Z}}^{(1)} + \mathbf{Z}_0)^{-1}(\tilde{\mathbf{Z}}^{(1)} - \mathbf{Z}_0)(\mathbf{Y}_0^R\mathbf{Z}^{(2)} + \mathbf{Z}_0^R) \tag{85}$$

$$(\tilde{\mathbf{Z}}^{(1)} + \mathbf{Z}_0)(\mathbf{Y}_0^R\mathbf{Z}^{(2)} - \mathbf{Z}_0^R) = (\tilde{\mathbf{Z}}^{(1)} - \mathbf{Z}_0)(\mathbf{Y}_0^R\mathbf{Z}^{(2)} + \mathbf{Z}_0^R) \tag{86}$$

Expanding the above equation and eliminate identical terms from both sides, we obtain

$$\mathbf{Z}_0^R\mathbf{Z}^{(2)} = \mathbf{Z}_0^R\mathbf{Z}^{(1)} \tag{87}$$

Since $\mathbf{Z}_0^R$ is non-singular, we have proved that

$$\mathbf{Z}^{(2)} = \mathbf{Z}^{(1)} \tag{88}$$

3.3 S-Y Conversions

3.3.1 S to Y

To convert **S** matrix to **Y** matrix, substitute the expressions of **a** and **b** in Eqs.(44) and (45) into the definition of **S** in Eq.(46) and rearrange the terms, we obtain

$$(\mathbf{U}+\mathbf{S})\mathbf{Z_0}^R I = (\mathbf{U}-\mathbf{S})\mathbf{Y_0}^R\mathbf{V} \tag{89}$$

When $\mathbf{U}+\mathbf{S}=0$, all the ports of the network are short circuits. This would result in infinite currents due to finite voltage excitations. By excluding this trivial case, we have

$$\mathbf{I} = \mathbf{Y}_0^R(\mathbf{U}+\mathbf{S})^{-1}(\mathbf{U}-\mathbf{S})\mathbf{Y}_0^R\mathbf{V} \tag{90}$$

From the definition of Y matrix, we readily obtain the converted Y matrix from S matrix,

$$\mathbf{Y} = \mathbf{Y}_0^R(\mathbf{U}+\mathbf{S})^{-1}(\mathbf{U}-\mathbf{S})\mathbf{Y}_0^R \tag{91}$$

In the case of uniformly terminated networks where all ports having the same reference impedance, the converted **Y** matrix becomes,

$$\mathbf{Y} = \mathbf{Y}_0(\mathbf{U}+\mathbf{S})^{-1}(\mathbf{U}-\mathbf{S}) \tag{92}$$

3.3.2 Y to S

To convert **Y** matrix to **S** matrix, substitute the **V** and **I** expressions (37) and (38) into the definition of **Y** matrix (8) and rearrange the terms, we obtain,

$$(\tilde{\mathbf{Y}}+\mathbf{Y}_0)\mathbf{b} = -(\tilde{\mathbf{Y}}-\mathbf{Y}_0)\mathbf{a} \tag{93}$$

where

$$\tilde{\mathbf{Y}} = \mathbf{Y}_0^R\mathbf{Y}\mathbf{Z}_0^R \tag{94}$$

In the above equation, consider the case of

$$\tilde{\mathbf{Y}}+\mathbf{Y}_0 = 0 \tag{95}$$

Multiply both sides by $\mathbf{Z}_0^R$ from the left and, $\mathbf{Y}_0^R$ from the right and rearrange the terms, we have

$$\mathbf{Y} = -\mathbf{Y}_0 \tag{96}$$

This corresponds to the trivial case where the self-admittance of all ports equal to the negative value of their port characteristic admittance, respectively.

$$Y_{kk} = -Y_{0k}, k = 1, 2, ..., n \tag{97}$$

By excluding this trivial case from our consideration and assuming $\mathbf{Y} \neq \mathbf{Y}_0$, we obtain,

$$\mathbf{b} = -(\tilde{\mathbf{Y}}+\mathbf{Y}_0)^{-1}(\tilde{\mathbf{Y}}-\mathbf{Y}_0)\mathbf{a} \tag{98}$$

$$\mathbf{S} = -(\tilde{\mathbf{Y}}+\mathbf{Y}_0)^{-1}(\tilde{\mathbf{Y}}-\mathbf{Y}_0) \tag{99}$$

3.4 T-S Conversions

3.4.1 T to S

To obtain **S** matrix from **T** matrix, expand the matrix form of **T** definition in Eq.(58),

$$\mathbf{b}^o = \mathbf{T}_{ba}\mathbf{a}^e + \mathbf{T}_{bb}\mathbf{b}^e \tag{100}$$

$$\mathbf{a}^o = \mathbf{T}_{aa}\mathbf{a}^e + \mathbf{T}_{ab}\mathbf{b}^e \tag{101}$$

and rearrange the terms in Eq.(101) we have,

$$\mathbf{b}^e = \mathbf{T}_{ab}^{-1}\mathbf{a}^o - \mathbf{T}_{ab}^{-1}\mathbf{T}_{aa}\mathbf{a}^e \tag{102}$$

Substitute the expression of $\mathbf{b}^e$ into Eq.(100),

$$\mathbf{b}^o = \mathbf{T}_{bb}\mathbf{T}_{ab}^{-1}\mathbf{a}^o + (\mathbf{T}_{ba} - \mathbf{T}_{bb}\mathbf{T}_{ab}^{-1}\mathbf{T}_{aa})\mathbf{a}^e \tag{103}$$

We readily obtain the odd-even permutated $\mathbf{S}$ matrix equations,

$$\begin{bmatrix} \mathbf{b}^o \\ \mathbf{b}^e \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{bb}\mathbf{T}_{ab}^{-1} & (\mathbf{T}_{ba} - \mathbf{T}_{bb}\mathbf{T}_{ab}^{-1}\mathbf{T}_{aa}) \\ \mathbf{T}_{ab}^{-1} & -\mathbf{T}_{ab}^{-1}\mathbf{T}_{aa} \end{bmatrix} \begin{bmatrix} \mathbf{a}^o \\ \mathbf{a}^e \end{bmatrix} \tag{104}$$

and the expression of the permutated $\mathbf{S}$ matrix is ,

$$\begin{bmatrix} \mathbf{S}_{oo} & \mathbf{S}_{oe} \\ \mathbf{S}_{eo} & \mathbf{S}_{ee} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{bb}\mathbf{T}_{ab}^{-1} & (\mathbf{T}_{ba} - \mathbf{T}_{bb}\mathbf{T}_{ab}^{-1}\mathbf{T}_{aa}) \\ \mathbf{T}_{ab}^{-1} & -\mathbf{T}_{ab}^{-1}\mathbf{T}_{aa} \end{bmatrix} \tag{105}$$

The original $\mathbf{S}$ matrix is given by

$$\mathbf{S} = \mathbf{P}_{oe}^T \begin{bmatrix} \mathbf{S}_{oo} & \mathbf{S}_{oe} \\ \mathbf{S}_{eo} & \mathbf{S}_{ee} \end{bmatrix} \mathbf{P}_{oe}^T \tag{106}$$

3.4.2 S to T

To convert $\mathbf{S}$ matrix to $\mathbf{T}$ matrix, expand the permutated $\mathbf{S}$ matrix equation of Eq.(56),

$$\mathbf{b}^o = \mathbf{S}_{oo}\mathbf{a}^o + \mathbf{S}_{oe}\mathbf{a}^e \tag{107}$$
$$\mathbf{b}^e = \mathbf{S}_{eo}\mathbf{a}^o + \mathbf{S}_{ee}\mathbf{a}^e \tag{108}$$

and rearrange the terms in Eq.(108), we have

$$\mathbf{a}^o = -\mathbf{S}_{eo}^{-1}\mathbf{S}_{ee}\mathbf{a}^e + \mathbf{S}_{eo}^{-1}\mathbf{b}^e \tag{109}$$

Substitute this expression into Eq.(107), we have,

$$\mathbf{b}^o = (\mathbf{S}_{oe} - \mathbf{S}_{oo}\mathbf{S}_{eo}^{-1}\mathbf{S}_{ee})\mathbf{a}^e + \mathbf{S}_{oo}\mathbf{S}_{eo}^{-1}\mathbf{b}^e \tag{110}$$

From the $\mathbf{T}$ matrix definition in Eq.(57) we have,

$$\mathbf{T} = \begin{bmatrix} \mathbf{S}_{oe} - \mathbf{S}_{oo}\mathbf{S}_{eo}^{-1}\mathbf{S}_{ee} & \mathbf{S}_{oo}\mathbf{S}_{eo}^{-1} \\ -\mathbf{S}_{eo}^{-1}\mathbf{S}_{ee} & \mathbf{S}_{eo}^{-1} \end{bmatrix} \tag{111}$$

4 Port Translations

This section discusses port translations including termination, renormalization, resequence, merge and fold.

4.1 Terminate

A network port can be terminated by attaching an impedance load.

When port k of $\mathbf{S}$ matrix is terminated by a matched load which impedance equals the port reference impedance, the incident wave a_k becomes zero, the corresponding row and column k in the $\mathbf{S}$ matrix can be eliminated to obtain a new $\mathbf{S}$ matrix of reduced size. The port numbers of the new matrix should be reassigned accordingly to reflect this change.

Multiple ports can be terminated by eliminating all corresponding rows and columns in the S-parameter matrix.

When port k of $\mathbf{S}$ matrix is terminated by a mismatched load $Z_{0k'}$ which is not equal to Z_{0k}, the incident wave a_k is not zero.

(to be completed ...)

4.2 Open and Short

Open and short are special cases of port termination.

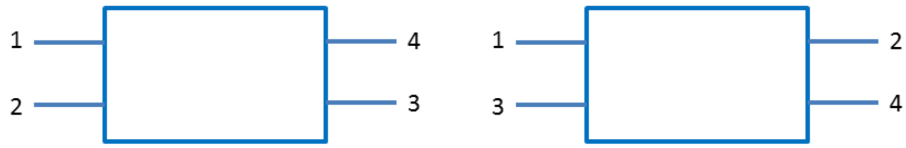

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Figure 2: Resequencing Port Numbers

4.3 Renormalize

To renormalize a S-parameter matrix from one set of reference impedance $\mathbf{Z}_{01}^R$ to another set of $\mathbf{Z}_{02}^R$, first convert it to $\mathbf{Z}$ matrix according to Eq.(65),

$$\mathbf{Z} = \mathbf{Z}_{01}^R(\mathbf{U} - \mathbf{S})^{-1}(\mathbf{U} + \mathbf{S})\mathbf{Z}_{01}^R \tag{112}$$

Then converting it back to $\mathbf{S}$ matrix using reference impedance $\mathbf{Z}_{02}^R$ according to Eq.(76),

$$\mathbf{S}' = (\tilde{\mathbf{Z}} + \mathbf{Z}_{02})^{-1}(\tilde{\mathbf{Z}} - \mathbf{Z}_{02}) \tag{113}$$

where

$$\tilde{\mathbf{Z}} = \mathbf{Z}_{02}^R\mathbf{Z}\mathbf{Z}_{02}^R \tag{114}$$

4.4 Resequence

Suppose the port numbers of a n-port network parameter needs to be resequenced according to the following 1:1 mapping,

$$\begin{aligned} 1 &\rightarrow N_1 \\ 2 &\rightarrow N_2 \\ &\vdots \\ n &\rightarrow N_n \end{aligned} \tag{115}$$

The permutation matrix $\mathbf{P}$ can be constructed by setting $P(N_i, i) = 1$ while keeping all other elements to 0. The resequenced $\mathbf{S}, \mathbf{Y}$ and $\mathbf{Z}$ network matrices are $\mathbf{PSP}^T$, $\mathbf{PYP}^T$ and $\mathbf{PZP}^T$

An example of a 4-port network port number resequence is shown in Fig.(2). The resequence mapping is,

$$\begin{aligned} 1 &\rightarrow 1 \\ 2 &\rightarrow 3 \\ 3 &\rightarrow 4 \\ 4 &\rightarrow 2 \end{aligned} \tag{116}$$

The permutation matrix is,

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{117}$$

Port number resequencing is achived by moving rows and columns according to the resequence mapping,

$$\begin{aligned} row(1) &\rightarrow row(N_1) \\ row(2) &\rightarrow row(N_2) \\ &\vdots \\ row(n) &\rightarrow row(N_n) \end{aligned} \tag{118}$$

$$\begin{aligned} col(1) &\rightarrow col(N_1) \\ col(2) &\rightarrow col(N_2) \\ &\vdots \\ col(n) &\rightarrow col(N_n) \end{aligned} \tag{119}$$

4.5 Merge

When two ports are merged, they are terminated by each other and both ports disappear from the resulting new network. Strictly speaking, only two ports having the exact same cross-section can be connected together without introducing a discontinuity at the interface. Such ports have exactly the same reference impedance. In practice, when two ports are connected together, the details about the port cross section may not be available. If in case the two ports to be merged have different reference impedances, they can be renormalized to the same impedance before merging.

First consider the merging of two ports on the same network. When two ports on the same network are merged, they terminate each other and both ports disappear from the network.

(details to be filled in here)

Now consider the merging of two ports from different networks. Assuming two S-matrices of dimensions n and m having k ports connected between them. The S-matrices are denoted as $\mathbf{S}^{(1)}_{n,n}$ and $\mathbf{S}^{(2)}_{m,m}$. The merged S-matrix has $n+m-2k$ ports.

To obtain the expression of the merged S-matrix, first rearrange the port numbers by putting all the connected ports to the end of a and b vectors. These connected ports will disappear from the merged network. The S-matrix can now be partitioned as the following,

$$\begin{bmatrix} \mathbf{b}^{(1)}_u \\ \mathbf{b}^{(1)}_k \end{bmatrix} = \begin{bmatrix} \mathbf{S}^{(1)}_{uu} & \mathbf{S}^{(1)}_{uk} \\ \mathbf{S}^{(1)}_{ku} & \mathbf{S}^{(1)}_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(1)}_u \\ \mathbf{a}^{(1)}_k \end{bmatrix} \tag{120}$$

$$\begin{bmatrix} \mathbf{b}^{(2)}_v \\ \mathbf{b}^{(2)}_k \end{bmatrix} = \begin{bmatrix} \mathbf{S}^{(2)}_{vv} & \mathbf{S}^{(2)}_{vk} \\ \mathbf{S}^{(2)}_{kv} & \mathbf{S}^{(2)}_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(2)}_v \\ \mathbf{a}^{(2)}_k \end{bmatrix} \tag{121}$$

In order to eliminate the k connected ports, these matrix equations are expanded into the following form,

$$\mathbf{b}^{(1)}_u = \mathbf{S}^{(1)}_{uu}\mathbf{a}^{(1)}_u + \mathbf{S}^{(1)}_{uk}\mathbf{a}^{(1)}_k \tag{122}$$

$$\mathbf{b}^{(1)}_k = \mathbf{S}^{(1)}_{ku}\mathbf{a}^{(1)}_u + \mathbf{S}^{(1)}_{kk}\mathbf{a}^{(1)}_k \tag{123}$$

$$\mathbf{b}^{(2)}_v = \mathbf{S}^{(2)}_{vv}\mathbf{a}^{(2)}_v + \mathbf{S}^{(2)}_{vk}\mathbf{a}^{(2)}_k \tag{124}$$

$$\mathbf{b}^{(2)}_k = \mathbf{S}^{(2)}_{kv}\mathbf{a}^{(2)}_v + \mathbf{S}^{(2)}_{kk}\mathbf{a}^{(2)}_k \tag{125}$$

For the connected ports, we have

$$\mathbf{a}^{(1)}_k = \mathbf{b}^{(2)}_k \tag{126}$$

$$\mathbf{a}^{(2)}_k = \mathbf{b}^{(1)}_k \tag{127}$$

Substitute Eqs.(126) and (127) into (123) and (125), we have

$$\mathbf{b}^{(1)}_k = \mathbf{S}^{(1)}_{ku}\mathbf{a}^{(1)}_u + \mathbf{S}^{(1)}_{kk}\mathbf{b}^{(2)}_k \tag{128}$$

$$\mathbf{b}^{(2)}_k = \mathbf{S}^{(2)}_{kv}\mathbf{a}^{(2)}_v + \mathbf{S}^{(2)}_{kk}\mathbf{b}^{(1)}_k \tag{129}$$

Now $\mathbf{b}^{(1)}_k$ and $\mathbf{b}^{(2)}_k$ can be readily eliminated from the right hand side,

$$\mathbf{b}^{(1)}_k = \mathcal{P}_{ku}\mathbf{a}^{(1)}_u + \mathcal{P}_{kv}\mathbf{a}^{(2)}_v \tag{130}$$

$$\mathbf{b}^{(2)}_k = \mathcal{Q}_{ku}\mathbf{a}^{(1)}_u + \mathcal{Q}_{kv}\mathbf{a}^{(2)}_v \tag{131}$$

where

$$\mathcal{P}_{ku} = \mathcal{E}_{kk}\mathbf{S}^{(1)}_{ku} \tag{132}$$

$$\mathcal{P}_{kv} = \mathcal{E}_{kk}\mathbf{S}^{(1)}_{kk}\mathbf{S}^{(2)}_{kv} \tag{133}$$

$$\mathcal{Q}_{ku} = \mathbf{S}^{(2)}_{kk}\mathcal{E}_{kk}\mathbf{S}^{(1)}_{ku} \tag{134}$$

$$\mathcal{Q}_{kv} = \mathbf{S}^{(2)}_{kv} + \mathbf{S}^{(2)}_{kk}\mathcal{E}_{kk}\mathbf{S}^{(1)}_{kk}\mathbf{S}^{(2)}_{kv} \tag{135}$$

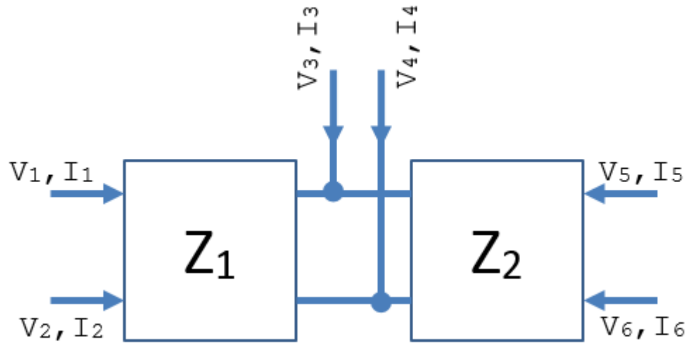

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Figure 3: Merge in Shunt

$$\mathcal{E}_{kk} = \left(\mathbf{U} - \mathbf{S}^{(1)}_{kk}\mathbf{S}^{(2)}_{kk}\right)^{-1} \tag{136}$$

The merged S-matrix can be obtained by substituting Eqs. (126), (127), (130) and (131) into (122) and (124),

$$\begin{bmatrix} \mathbf{b}^{(1)}_u \\ \mathbf{b}^{(2)}_v \end{bmatrix} = \begin{bmatrix} \mathbf{S}^{(1)}_{uu} + \mathbf{S}^{(1)}_{uk}\mathcal{Q}_{ku} & \mathbf{S}^{(1)}_{uk}\mathcal{Q}_{kv} \\ \mathbf{S}^{(2)}_{vk}\mathcal{P}_{ku} & \mathbf{S}^{(2)}_{vv} + \mathbf{S}^{(2)}_{vk}\mathcal{P}_{kv} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(1)}_u \\ \mathbf{a}^{(2)}_v \end{bmatrix} \tag{137}$$

4.6 Join

Two ports are joined to make a new port. Fig.(3) shows an example of the folding of two network ports. Networks Z_1 and Z_2 each has four ports. With two of their ports folded, the resulting network has a total of six ports, which are re-indexed from 1 to 6.

The Z-matrices of the two individual networks are given by,

$$\begin{bmatrix} \mathbf{V}^{(1)}_u \\ \mathbf{V}^{(1)}_k \end{bmatrix} = \begin{bmatrix} \mathbf{Z}^{(1)}_{uu} & \mathbf{Z}^{(1)}_{uk} \\ \mathbf{Z}^{(1)}_{ku} & \mathbf{Z}^{(1)}_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{I}^{(1)}_u \\ \mathbf{I}^{(1)}_k \end{bmatrix} \tag{138}$$

$$\begin{bmatrix} \mathbf{V}^{(2)}_v \\ \mathbf{V}^{(2)}_k \end{bmatrix} = \begin{bmatrix} \mathbf{Z}^{(2)}_{vv} & \mathbf{Z}^{(2)}_{vk} \\ \mathbf{Z}^{(2)}_{kv} & \mathbf{Z}^{(2)}_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{I}^{(2)}_v \\ \mathbf{I}^{(2)}_k \end{bmatrix} \tag{139}$$

In these expressions, the k-indexed ports are connected and folded into a new set of external ports. The port folding conditions are expressed as,

$$\begin{gathered} I_1 = I^{(1)}_1 + I^{(2)}_1 \\ I_2 = I^{(1)}_2 + I^{(2)}_2 \\ \vdots \\ I_K = I^{(1)}_K + I^{(2)}_K \end{gathered} \tag{140}$$

$$\begin{gathered} V_1 = V^{(1)}_1 = V^{(2)}_1 \\ V_2 = V^{(1)}_2 = V^{(2)}_2 \\ \vdots \\ V_K = V^{(1)}_K = V^{(2)}_K \end{gathered} \tag{141}$$

Expand the matrix form of the Z-parameters we have,

$$\mathbf{V}^{(1)}_u = \mathbf{Z}^{(1)}_{uu}\mathbf{I}^{(1)}_u + \mathbf{Z}^{(1)}_{uk}\mathbf{I}^{(1)}_k \tag{142}$$

$$\mathbf{V}^{(1)}_k = \mathbf{Z}^{(1)}_{ku}\mathbf{I}^{(1)}_u + \mathbf{Z}^{(1)}_{kk}\mathbf{I}^{(1)}_k \tag{143}$$

$$\mathbf{V}_v^{(2)} = \mathbf{Z}_{vv}^{(2)}\mathbf{I}_v^{(2)} + \mathbf{Z}_{vk}^{(2)}\mathbf{I}_k^{(2)} \tag{144}$$

$$\mathbf{V}_k^{(2)} = \mathbf{Z}_{kv}^{(2)}\mathbf{I}_v^{(2)} + \mathbf{Z}_{kk}^{(2)}\mathbf{I}_k^{(2)} \tag{145}$$

Left multiply Eqn.(143) with $\mathbf{Y}_{kk}^{(1)}$ and Eqn.(145) with $\mathbf{Y}_{kk}^{(2)}$ and rearrange the terms, we have

$$\mathbf{Y}_{kk}^{(1)}\mathbf{V}_k = \mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{ku}^{(1)}\mathbf{I}_u^{(1)} + \mathbf{I}_k^{(1)} \tag{146}$$

$$\mathbf{Y}_{kk}^{(2)}\mathbf{V}_k = \mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kv}^{(2)}\mathbf{I}_v^{(2)} + \mathbf{I}_k^{(2)} \tag{147}$$

where

$$\mathbf{Y}_{kk}^{(1)} = inv(\mathbf{Z}_{kk}^{(1)}) \tag{148}$$

$$\mathbf{Y}_{kk}^{(2)} = inv(\mathbf{Z}_{kk}^{(2)}) \tag{149}$$

$$\mathbf{V}_k = \mathbf{V}_k^{(1)} = \mathbf{V}_k^{(2)} \tag{150}$$

Adding Eqns.(146) and (147) together we obtain,

$$\mathbf{Y}_{kk}\mathbf{V}_k = \mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{ku}^{(1)}\mathbf{I}_u^{(1)} + \mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kv}^{(2)}\mathbf{I}_v^{(2)} + \mathbf{I}_k \tag{151}$$

$$\mathbf{V}_k = \mathbf{Z}_{kk}\mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{ku}^{(1)}\mathbf{I}_u^{(1)} + \mathbf{Z}_{kk}\mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kv}^{(2)}\mathbf{I}_v^{(2)} + \mathbf{Z}_{kk}\mathbf{I}_k \tag{152}$$

where

$$\mathbf{Y}_{kk} = \mathbf{Y}_{kk}^{(1)} + \mathbf{Y}_{kk}^{(2)} \tag{153}$$

$$\mathbf{Z}_{kk} = inv(\mathbf{Y}_{kk}) \tag{154}$$

$$\mathbf{I}_k = \mathbf{I}_k^{(1)} + \mathbf{I}_k^{(2)} \tag{155}$$

To eliminate $\mathbf{I}_k^{(1)}$ and $\mathbf{I}_k^{(2)}$ from Eqn.(142), (143), (144) and (145), rearrange the terms in Eq.(146), (147),

$$\mathbf{I}_k^{(1)} = \mathbf{Y}_{kk}^{(1)}\mathbf{V}_k - \mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{ku}^{(1)}\mathbf{I}_u^{(1)} \tag{156}$$

$$\mathbf{I}_k^{(2)} = \mathbf{Y}_{kk}^{(2)}\mathbf{V}_k - \mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kv}^{(2)}\mathbf{I}_v^{(2)} \tag{157}$$

Substitute the above expressions into Eqn.(142) and (144),

$$\mathbf{V}_u^{(1)} = (\mathbf{Z}_{uu}^{(1)} - \mathbf{Z}_{uk}^{(1)}\mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{ku}^{(1)})\mathbf{I}_u^{(1)} + \mathbf{Z}_{uk}^{(1)}\mathbf{Y}_{kk}^{(1)}\mathbf{V}_k \tag{158}$$

$$\mathbf{V}_v^{(2)} = (\mathbf{Z}_{vv}^{(2)} - \mathbf{Z}_{vk}^{(2)}\mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kv}^{(2)})\mathbf{I}_v^{(2)} + \mathbf{Z}_{vk}^{(2)}\mathbf{Y}_{kk}^{(2)}\mathbf{V}_k \tag{159}$$

Substitute Eq.(152) into the above equations and rearrange the terms, we obtain

$$\mathbf{V}_u^{(1)} = \tilde{\mathbf{Z}}_{uu}^{(11)}\mathbf{I}_u^{(1)} + \tilde{\mathbf{Z}}_{uv}^{(12)}\mathbf{I}_v^{(2)} + \tilde{\mathbf{Z}}_{uk}^{(1)}\mathbf{I}_k \tag{160}$$

where

$$\tilde{\mathbf{Z}}_{uk}^{(1)} = \mathbf{Z}_{uk}^{(1)}\mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{kk} \tag{161}$$

$$\tilde{\mathbf{Z}}_{uu}^{(11)} = \mathbf{Z}_{uu}^{(1)} + \mathbf{Z}_{uk}^{(1)}(\mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{kk} - \mathbf{U})\mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{ku}^{(1)} \tag{162}$$

$$= \mathbf{Z}_{uu}^{(1)} + (\tilde{\mathbf{Z}}_{uk}^{(1)} - \mathbf{Z}_{uk}^{(1)})\mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{ku}^{(1)} \tag{163}$$

$$\tilde{\mathbf{Z}}_{uv}^{(12)} = \mathbf{Z}_{uk}^{(1)}\mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kv}^{(2)} \tag{164}$$

$$\mathbf{V}_v^{(2)} = \tilde{\mathbf{Z}}_{vu}^{(21)}\mathbf{I}_u^{(1)} + \tilde{\mathbf{Z}}_{vv}^{(22)}\mathbf{I}_v^{(2)} + \tilde{\mathbf{Z}}_{vk}^{(2)}\mathbf{I}_k \tag{165}$$

where

$$\tilde{\mathbf{Z}}_{vk}^{(2)} = \mathbf{Z}_{vk}^{(2)}\mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kk} \tag{166}$$

$$\tilde{\mathbf{Z}}_{vu}^{(21)} = \mathbf{Z}_{vk}^{(2)}\mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{ku}^{(1)} \tag{167}$$

$$\tilde{\mathbf{Z}}_{vv}^{(22)} = \mathbf{Z}_{vv}^{(2)} + \mathbf{Z}_{vk}^{(2)}(\mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kk} - \mathbf{U})\mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kv}^{(2)} \tag{168}$$

$$= \mathbf{Z}_{vv}^{(2)} + (\tilde{\mathbf{Z}}_{vk}^{(2)} - \mathbf{Z}_{vk}^{(2)})\mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kv}^{(2)} \tag{169}$$

Eqn.(152) can be expressed in the following form,

$$\mathbf{V}_k = \tilde{\mathbf{Z}}_{ku}^{(1)}\mathbf{I}_u^{(1)} + \tilde{\mathbf{Z}}_{kv}^{(2)}\mathbf{I}_v^{(2)} + \mathbf{Z}_{kk}\mathbf{I}_k \tag{170}$$

where

$$\tilde{\mathbf{Z}}_{ku}^{(1)} = \mathbf{Z}_{kk}\mathbf{Y}_{kk}^{(1)}\mathbf{Z}_{ku}^{(1)} \tag{171}$$

$$\tilde{\mathbf{Z}}_{kv}^{(2)} = \mathbf{Z}_{kk}\mathbf{Y}_{kk}^{(2)}\mathbf{Z}_{kv}^{(2)} \tag{172}$$

Eqn.(160) (165) and (170) can be expressed in the matrix form,

$$\begin{bmatrix} \mathbf{V}_u^{(1)} \\ \mathbf{V}_v^{(2)} \\ \mathbf{V}_k \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{Z}}_{uu}^{(11)} & \tilde{\mathbf{Z}}_{uv}^{(12)} & \tilde{\mathbf{Z}}_{uk}^{(1)} \\ \tilde{\mathbf{Z}}_{vu}^{(21)} & \tilde{\mathbf{Z}}_{vv}^{(22)} & \tilde{\mathbf{Z}}_{vk}^{(2)} \\ \tilde{\mathbf{Z}}_{ku}^{(1)} & \tilde{\mathbf{Z}}_{kv}^{(2)} & \tilde{\mathbf{Z}}_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{I}_u^{(1)} \\ \mathbf{I}_v^{(2)} \\ \mathbf{I}_k^{(1)} \end{bmatrix} \tag{173}$$

4.7 Bridge

Two ports can be bridged together by a series impedance Z_b.

5 Mixed-Mode Parameters

Differential signals are widely used in modern electronic systems for their rejection of noise coupling which is mostly common-mode. A differential driver outputs two signals at opposite phase, relative to a common reference (aka ground). Such differential signals are usually transmitted by coupled transmission lines consisting of a pair of p(ositive) and n(egative) lines. Although most differential pair transmission lines use an explicit common reference (aka ground) as the return path, it is not necessary that such a return path has to exsit explicitly. For example in a twisted pair, there is a positive and negative wire but there is no explicit ground wire. The common reference for the twisted pair is the virtual ground. However the configuration of the return path does have an impact on common-mode characteristics.

For most practical applications, the p and n conductors of the coupled transmission line are symmetrical. Such transmission lines can support both the TEM (or quasi-TEM for microstrip) even- and odd-mode electromagnetic fields. The differential signal will excite the odd-mode field pattern while the common-mode signal will excite the even-mode field pattern. Unsymmetrical structures such as bends and skew will cause these two modes to be converted between each other. In some cases the terms of differential-mode and odd-mode are used interchangeably, same for common-mode and even-mode. It is important to understand that the terms even- and odd-mode refer to the electromagnetic field patterns in the physical structure and, differential and common-mode signals refer to the voltage waveform at a given point (such as the output of the driver or receiver).

Note that even though in real engineering design we are often more concerned about the conversion from the differential-mode to common-mode, the conversion from common-mode to differential-mode is equally important in the analysis.

From the perspective of network analysis, the two p and n terminals of the network are now treated as a differential port and a common-mode port. The total port numbers are still the same, albert they have different expressions for the port excitations.

5.1 Mixed-mode Z and Y Matrices

Expressions of mixed-mode voltage and current vectors are derived.

5.1.1 Four-port Network

We first look at an example of four-port mixed-mode network as depicted in Fig.(4). The singled-ended Z and Y matrices of the network are given by Eq.(7) and (8), which are repeated here,

$$\mathbf{V} = \mathbf{Z}\mathbf{I} \tag{174}$$

$$\mathbf{I} = \mathbf{Y}\mathbf{V} \tag{175}$$

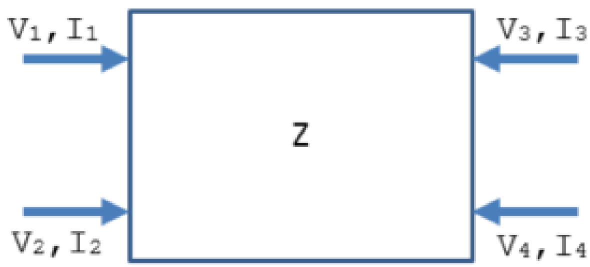

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Figure 4: Mixed-mode Four-Port Network

where the voltage and current vectors are expressed as,

$$\mathbf{V} = [V_1, V_2, ..., V_n] \tag{176}$$

$$\mathbf{I} = [I_1, I_2, ..., I_n] \tag{177}$$

The singled-ended ports 1 and 2 make up the positive and negative terminals for the first mixed-mode port. Ports 3 and 4 make up the positive and negative terminals of the second mixed-mode port. The differential voltages and currents of the two mixed-mode ports are defined as [5],

$$V_{d1} = V_1 - V_2 \tag{178}$$

$$I_{d1} = \frac{I_1 - I_2}{2} \tag{179}$$

$$V_{c1} = \frac{V_1 + V_2}{2} \tag{180}$$

$$I_{c1} = I_1 + I_2 \tag{181}$$

$$V_{d2} = V_3 - V_4 \tag{182}$$

$$I_{d2} = \frac{I_3 - I_4}{2} \tag{183}$$

$$V_{c2} = \frac{V_3 + V_4}{2} \tag{184}$$

$$I_{c2} = I_3 + I_4 \tag{185}$$

Putting these equations in matrix form, we have

$$\begin{bmatrix} V_{d1} \\ V_{c1} \\ V_{d2} \\ V_{c2} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} \tag{186}$$

$$\begin{bmatrix} I_{d1} \\ I_{c1} \\ I_{d2} \\ I_{c2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \tag{187}$$

By defining the mixed-mode voltage and current vectors as,

$$\mathbf{V}_m = [V_{d1}, V_{c1}, V_{d2}, V_{c2}]^T \tag{188}$$

$$\mathbf{I}_m = [I_{d1}, I_{c1}, I_{d2}, I_{c2}]^T \tag{189}$$

We obtain,

$$\mathbf{V}_m = \mathbf{M}_v \mathbf{V} \tag{190}$$

$$\mathbf{I}_m = \mathbf{M}_i \mathbf{I} \tag{191}$$

where $\mathbf{M}_v$ and $\mathbf{M}_i$ are the voltage and current mixed-mode conversion matrices as shown in Eqs.(186) and (187). Substuting Eq.(174) into the above equations, the relationship between the mixed-mode voltage and current vectors are readily obtained,

$$\mathbf{V}_m = \mathbf{M}_v \mathbf{Z} \mathbf{M}_i^{-1} \mathbf{I}_m \tag{192}$$

The mixed-mode impedance matrix $\mathbf{Z}_m$ is related to the singled-ended $\mathbf{Z}$ matrix by the following conversion,

$$\mathbf{Z}_m = \mathbf{M}_v \mathbf{Z} \mathbf{M}_i^{-1} \tag{193}$$

Similarly for the admittance matrix $\mathbf{Y}$,

$$\mathbf{I}_m = \mathbf{M}_i \mathbf{Y} \mathbf{M}_v^{-1} \mathbf{V}_m \tag{194}$$

$$\mathbf{Y}_m = \mathbf{M}_i \mathbf{Y} \mathbf{M}_v^{-1} \tag{195}$$

5.1.2 N-Port Network

Consider a generalized N-port network with some of its ports configured as mixed-mode ports while others remain as single-ended ports. The mixed-mode port indexing matrix is given by,

$$I_K = \begin{bmatrix} p_1 & n_1 \\ p_2 & n_2 \\ \vdots & \vdots \\ p_K & n_K \end{bmatrix} \tag{196}$$

Each row of the index matrix represents a pair of positive and negative ports of a mixed-mode port. The index of the mixed-mode port is the row number of the index matrix I_K.

For example, p_1 and n_1 are the positive and negative ports of mixed-mode port 1. There are a total of K mixed-mode ports and, p_K and n_K are the positive and negative ports for mixed-mode port K. The mixed-mode ports are indexed as $d1, c1, d2, c2, ..., dK, cK$, where d stands for differential and c stands for common-mode. The total number of mixed-mode ports and single-ended ports remain the same. The following matrix shows the mapping from single-ended ports to mixed-mode ports according to the port-mapping matrix in Eq.(240),

$$\begin{array}{ccc} p_1 & \mapsto & d1 \\ n_1 & \mapsto & c1 \\ p_2 & \mapsto & d2 \\ n_2 & \mapsto & c2 \\ \vdots & \vdots & \vdots \\ p_K & \mapsto & dK \\ n_K & \mapsto & cK \end{array} \tag{197}$$

Another way to look at this mapping is that differential port $d1$ replaces the singled-ended port p_1, common-mode port $c1$ replaces the singled-ended port c_1 in the matrix rows and columns. To put this in the matrix form, consider the k-th mixed-mode pair (p_k, c_k) in the index matrix I_K, the mixed-mode voltages and currents are given by,

$$V_{dk} = V_{p_k} - V_{n_k} \tag{198}$$

$$I_{dk} = \frac{I_{p_k} - I_{n_k}}{2} \tag{199}$$

$$V_{ck} = \frac{V_{p_k} + V_{n_k}}{2} \tag{200}$$

$$I_{ck} = I_{p_k} + I_{n_k} \tag{201}$$

The single-ended voltage vector is transformed into the mixed-mode voltage vector in the following matrix form,

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{dk} \\ \vdots \\ V_{ck} \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & \cdots & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{p_k} \\ \vdots \\ V_{n_k} \\ \vdots \\ V_N \end{bmatrix} \tag{202}$$

This transformation effectively replaces the single-ended voltages V_{p_k}, V_{n_k} with the mixed-mode voltages V_{dk}, V_{ck}. It is obvious that the completed transformation matrix $\mathbf{M}_v$ can be obtained by repeatedly stamping the four positions of $(p_k, p_k), (p_k, n_k), (n_k, p_k), (n_k, n_k)$ in the identity matrix with the values of $1, -1, \frac{1}{2}, \frac{1}{2}$, for all $k = 1, 2, ..., K$.

The mixed-mode voltage vector $\mathbf{V}_m$ can be derived from the singled-ended voltage vector $\mathbf{V}$ using the resulting transformation matrix $\mathbf{M}_v$,

$$\mathbf{V_m} = \mathbf{M}_v \mathbf{V} \tag{203}$$

In $\mathbf{V}_m$ all single-ended voltages corresponding to $V_{p_1}, V_{p_2}, ..., V_{p_K}$ are replaced in place by the mixed-mode voltages $V_{d1}, V_{c1}, V_{d2}, V_{c2}, ..., V_{dK}, V_{cK}$.

The expression for the mixed-mode current vector $\mathbf{I}_m$ can be obtained in a similar manner. The stamping of current pair $(I_{dk}, I_{ck}$ in the transformation matrix $\mathbf{M}_i$ is given by the following matrix form,

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{dk} \\ \vdots \\ I_{ck} \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & \cdots & -\frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{p_k} \\ \vdots \\ I_{n_k} \\ \vdots \\ I_N \end{bmatrix} \tag{204}$$

Note that the stamping values for current vectors are different from those for voltage vectors. The resulting final current tranformation matrix $\mathbf{M}_i$ converts the single-ended current vector $\mathbf{I}$ to its mixed-mode counterpart,

$$\mathbf{I_m} = \mathbf{M}_i \mathbf{I} \tag{205}$$

where in $\mathbf{I}_m$ all single-ended currents corresponding to $I_{p_1}, I_{p_2}, ..., I_{p_K}$ are replaced in place by the mixed-mode currents $I_{d1}, I_{c1}, I_{d2}, I_{c2}, ..., I_{dK}, I_{cK}$.

The conversion between mixed-mode and single-ended impedance and admittance matrices $\mathbf{Z}_m, \mathbf{Y}_m, \mathbf{Z}, \mathbf{Y}$ are readily obtained,

$$\mathbf{V}_m = \mathbf{M}_v \mathbf{Z} \mathbf{M}_i^{-1} \mathbf{I}_m \tag{206}$$

$$\mathbf{Z}_m = \mathbf{M}_v \mathbf{Z} \mathbf{M}_i^{-1} \tag{207}$$

$$\mathbf{Z} = \mathbf{M}_v^{-1} \mathbf{Z}_m \mathbf{M}_i \tag{208}$$

$$\mathbf{I}_m = \mathbf{M}_i \mathbf{Y} \mathbf{M}_v^{-1} \mathbf{V}_m \tag{209}$$

$$\mathbf{Y}_m = \mathbf{M}_i \mathbf{Y} \mathbf{M}_v^{-1} \tag{210}$$

$$\mathbf{Y} = \mathbf{M}_i^{-1} \mathbf{Y}_m \mathbf{M}_v \tag{211}$$

5.2 Mixed-mode S-Parameters

Mixed-mode S-parameters are used extensively in the characterization of differential serial links as well as clock signal distribution of parallel buses.

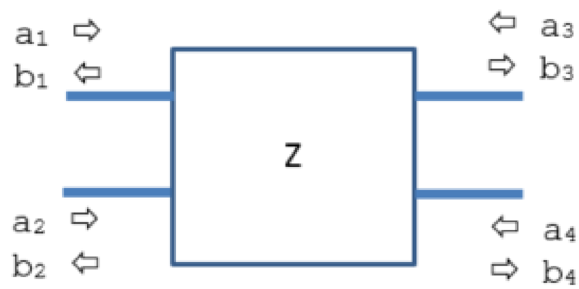

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Figure 5: Mixed-mode Four-Port Network

5.2.1 Four-port Network

Consider the four-port network depicted in Fig.(5). Assuming all ports are uncoupled at their respective reference planes, the differential and common-mode incident and scattered waves are defined as ([15]),

$$a_{d1} = \frac{1}{\sqrt{2}}(a_1 - a_2) \tag{212}$$

$$a_{c1} = \frac{1}{\sqrt{2}}(a_1 + a_2) \tag{213}$$

$$a_{d2} = \frac{1}{\sqrt{2}}(a_3 - a_4) \tag{214}$$

$$a_{c2} = \frac{1}{\sqrt{2}}(a_3 + a_4) \tag{215}$$

$$b_{d1} = \frac{1}{\sqrt{2}}(b_1 - b_2) \tag{216}$$

$$b_{c1} = \frac{1}{\sqrt{2}}(b_1 + b_2) \tag{217}$$

$$b_{d2} = \frac{1}{\sqrt{2}}(b_3 - b_4) \tag{218}$$

$$b_{c2} = \frac{1}{\sqrt{2}}(b_3 + b_4) \tag{219}$$

where the subscript d represents differential-mode, subscript c represents common-mode.

Putting these equations in the matrix form,

$$\begin{bmatrix} a_{d1} \\ a_{c1} \\ a_{d2} \\ a_{c2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{220}$$

$$\begin{bmatrix} b_{d1} \\ b_{c1} \\ b_{d2} \\ b_{c2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{221}$$

And the corresponding vector form,

$$\mathbf{a_m} = \mathbf{Ma} \tag{222}$$

$$\mathbf{b_m} = \mathbf{Mb} \tag{223}$$

where $\mathbf{a_m}$ and $\mathbf{b_m}$ are the mixed-mode incident and scattered wave vectors on the left hand side of Eqs.(220) and (221). $\mathbf{a}$ and $\mathbf{b}$ are the single-ended incident and scattered wave vectors.

The mode-conversion matrix M is,

$$\mathbf{M} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{224}$$

The mixed-mode S-matrix is defined as,

$$\mathbf{b_m} = \mathbf{S_m a_m} \tag{225}$$

Substituting Eqs.(222) and (223) into the definition of mixed-mode S-matrix and left-multiply both sides by M^{-1} we have,

$$\mathbf{b} = \mathbf{M^{-1} S_m M a} \tag{226}$$

The conversion between single-ended and mixed-mode S-matrices are readily obtained,

$$\mathbf{S} = \mathbf{M^{-1} S_m M} \tag{227}$$

$$\mathbf{S_m} = \mathbf{M S M^{-1}} \tag{228}$$

5.2.2 N-Port Network

For an arbitrary N-port network, some of its ports can be converted to mixed-mode ports while the rest of ports still remain as single-ended. The mixed-mode port indexing matrix is given below,

$$I_K = \begin{bmatrix} p_1 & n_1 \\ p_2 & n_2 \\ \vdots & \vdots \\ p_K & n_K \end{bmatrix} \tag{229}$$

The first column contains the single-ended port numbers of the positive ports, and second column the negative ports. For example, p_1 and n_1 are the positive and negative ports of mixed-mode port 1. There are a total of K mixed-mode ports and, p_K and n_K are the positive and negative ports for mixed-mode port K. The mixed-mode ports are indexed as $d1, c1, d2, c2, ..., dK, cK$, where d stands for differential and c stands for common-mode. The total number of mixed-mode ports and single-ended ports remain the same. The following matrix shows the mapping from single-ended ports to mixed-mode ports according to the port-mapping matrix in Eq.(240),

$$\begin{array}{ccc} p_1 & \mapsto & d1 \\ n_1 & \mapsto & c1 \\ p_2 & \mapsto & d2 \\ n_2 & \mapsto & c2 \\ \vdots & \vdots & \vdots \\ p_K & \mapsto & dK \\ n_K & \mapsto & cK \end{array} \tag{230}$$

The mixed-mode incident and scattered waves of port k are given by,

$$a_{dk} = \frac{1}{\sqrt{2}}(a_{p_k} - a_{n_k}) \tag{231}$$

$$a_{ck} = \frac{1}{\sqrt{2}}(a_{p_k} + a_{n_k}) \tag{232}$$

$$b_{dk} = \frac{1}{\sqrt{2}}(b_{p_k} - b_{n_k}) \tag{233}$$

$$b_{ck} = \frac{1}{\sqrt{2}}(b_{p_k} + b_{n_k}) \tag{234}$$

This transformation effectively replaces the single-ended waves $a_{p_k}, a_{n_k}, b_{p_k}, b_{n_k}$ with the mixed-modes waves $a_{dk}, a_{ck}, b_{dk}, b_{ck}$. The transformation can be viewed more clearly in its matrix form,

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{dk} \\ \vdots \\ a_{ck} \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{p_k} \\ \vdots \\ a_{n_k} \\ \vdots \\ a_N \end{bmatrix} \tag{235}$$

It is obvious that the completed transformation matrix $\mathbf{M}$ can be obtained by repeatedly stamping the four positions of $(p_k, p_k), (p_k, n_k), (n_k, p_k), (n_k, n_k)$ in the identity matrix with the values of $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$, for all $k = 1, 2, ..., K$.

The mixed-mode and single-ended wave vectors are related by the transformation matrix $\mathbf{M}$,

$$\mathbf{a_m} = \mathbf{Ma} \tag{236}$$

In the mixed-mode wave vector $\mathbf{a_m}$, all single-ended waves $a_{p_1}, a_{p_2}, ..., a_{p_K}$ have been replaced by the mixed-mode waves $a_{d1}, a_{c1}, a_{d2}, a_{c2}, ..., a_{dK}, a_{cK}$. Same is true for $\mathbf{b_m}$.

$$\mathbf{b_m} = \mathbf{Mb} = \mathbf{MSa} = \mathbf{MSM^{-1}a_m} \tag{237}$$

The conversion between the single-ended and mixed-mode S-matrices are given below,

$$\mathbf{S_m} = \mathbf{MSM^{-1}} \tag{238}$$

$$\mathbf{S} = \mathbf{M^{-1}S_mM} \tag{239}$$

5.3 Mixed-Mode T-Matrix

Consider the network and its associated T-matrix as shown in Fig.(6). Link-j and link-k form a mixed-mode pair of indices i and $i+1$, with link-j being the positive link and link-k the negative link, where the order of index j and k is completely arbitrary. The entry to the mixed-mode port indexing matrix is,

$$\begin{bmatrix} \cdots & \cdots \\ \vdots & \vdots \\ 2j-1 & 2k-1 \\ 2j & 2k \\ \vdots & \vdots \\ \cdots & \cdots \end{bmatrix} \tag{240}$$

where $2j-1, 2k-1$ are entered into row i and, $2j, 2k$ are entered into row $i+1$ with i and $i+1$ being the indices of mixed-mode ports.

The mixed-mode incident and scattered waves are given as,

$$a_{di} = \frac{1}{\sqrt{2}}(a_{2j-1} - a_{2k-1}) \tag{241}$$

$$a_{ci} = \frac{1}{\sqrt{2}}(a_{2j-1} + a_{2k-1}) \tag{242}$$

$$b_{di} = \frac{1}{\sqrt{2}}(b_{2j-1} - b_{2k-1}) \tag{243}$$

$$b_{ci} = \frac{1}{\sqrt{2}}(b_{2j-1} + b_{2k-1}) \tag{244}$$

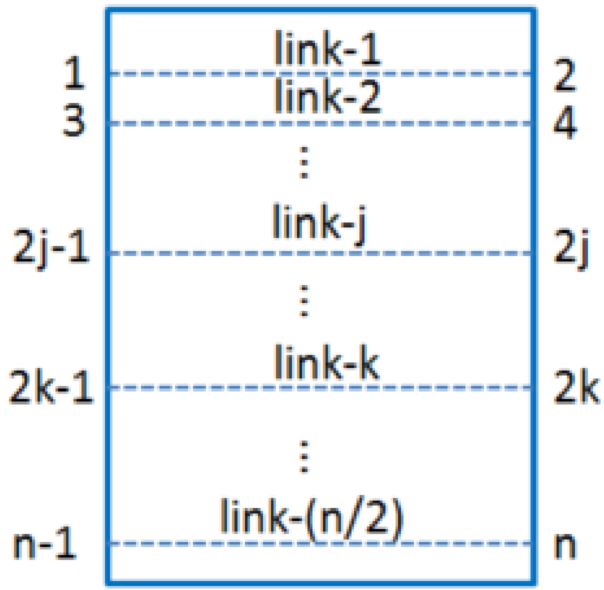

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Figure 6: Mixed-mode T-parameters

$$a_{d(i+1)} = \frac{1}{\sqrt{2}}(a_{2j} - a_{2k}) \tag{245}$$

$$a_{c(i+1)} = \frac{1}{\sqrt{2}}(a_{2j} + a_{2k}) \tag{246}$$

$$b_{d(i+1)} = \frac{1}{\sqrt{2}}(b_{2j} - b_{2k}) \tag{247}$$

$$b_{c(i+1)} = \frac{1}{\sqrt{2}}(b_{2j} + b_{2k}) \tag{248}$$

The odd mixed-mode wave vector $\mathbf{a^o}$ modified by port i can be obtained by stamping the mapping matrix $\mathbf{M}$ at the four positions corresponding to row and column indices $2j-1$ and $2k-1$,

$$\begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{di} \\ \vdots \\ a_{ci} \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{2j-1} \\ \vdots \\ a_{2k-1} \\ \vdots \\ a_{n-1} \end{bmatrix} \tag{249}$$

The final mapping matrix $\mathbf{M}$ is obtained by repeatedly stamping all positions according to the entries in the mixed-mode port indexing matrix. In matrix notation we obtain,

$$\begin{bmatrix} \mathbf{b^o_m} \\ \mathbf{a^o_m} \end{bmatrix} = \begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{b^o} \\ \mathbf{a^o} \end{bmatrix} = \tilde{\mathbf{M}} \begin{bmatrix} \mathbf{b^o} \\ \mathbf{a^o} \end{bmatrix} \tag{250}$$

Similarly, for the even wave vector,

$$\begin{bmatrix} \mathbf{a^e_m} \\ \mathbf{b^e_m} \end{bmatrix} = \begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{a^e} \\ \mathbf{b^e} \end{bmatrix} = \tilde{\mathbf{M}} \begin{bmatrix} \mathbf{a^e} \\ \mathbf{b^e} \end{bmatrix} \tag{251}$$

Substitute the definition of T-matrix Eq.(58) into Eq.(250),

$$\begin{bmatrix} \mathbf{b^o_m} \\ \mathbf{a^o_m} \end{bmatrix} = \tilde{\mathbf{M}} \begin{bmatrix} \mathbf{b^o} \\ \mathbf{a^o} \end{bmatrix} = \tilde{\mathbf{M}}\mathbf{T} \begin{bmatrix} \mathbf{a^e} \\ \mathbf{b^e} \end{bmatrix} = \tilde{\mathbf{M}}\mathbf{T}\tilde{\mathbf{M}}^{-1} \begin{bmatrix} \mathbf{a^e_m} \\ \mathbf{b^e_m} \end{bmatrix} \tag{252}$$

The conversion between single-ended and mixed-mode T-matrices are readily obtained,

$$\mathbf{T_m} = \tilde{\mathbf{M}}\mathbf{T}\tilde{\mathbf{M}}^{-\mathbf{1}} \tag{253}$$

$$\mathbf{T} = \tilde{\mathbf{M}}^{-\mathbf{1}}\mathbf{T_m}\tilde{\mathbf{M}} \tag{254}$$

6 Network Analysis Solutions

An interconnected network may contain multiple instances of network parameter matrices, lumped element circuit blocks, independent voltage and current sources, terminations and other components. This section provides the formal solutions for such interconnected networks.

Consider the passive LTI network as depicted in Fig.(7). The network is consisted of the following types of components,

1. parameterized networks (S, Y, Z, T parameters, shown as NPMat in Figure)
2. lumped element circuits (LEC)
3. independent voltage sources with series impedance and, independent current sources with shunt admittance
4. voltage and current probes

If the network contains one or multiple levels of hierarchy, it should be flattened to only contain the above four types of primitive components. The independent sources and probes are considered to be external and everything else are internal. Note that all controlled sources must reside inside the lumped element circuit blocks. The type of dependent sources supported by lumped element circuits include current-controlled current sources (CCCS), current-controlled voltage sources (CCVS), voltage-controlled current sources (VCCS) and voltage-controlled voltage sources (VCVS). The internal network is LTI due to the fact that it only contains passive components and controlled sources.

6.1 Independent Sources

At the top level of the network, all sources are independent. The equation of an independent voltage source with series impedance Z_{si} at port i is given by,

$$V_i = V_{si} - Z_{si} I_i \tag{255}$$

The equation of an independent current source with shunt admittance $Y_{si'}$ at port i' is given by,

$$V_{i'} = Z_{si'} I_{si'} - Z_{si'} I_{i'} \tag{256}$$

From Eqns.(255) and (256), we conclude that an independent current source with shunt admittance (I_s, Y_s) is equivalent to an independent voltage source with series impedance $(Z_s I_s, Z_s)$ where $Z_s Y_s = 1$. Due to this equivalency without loss of generality we will only consider independent voltage sources with series impedances in our discussion.

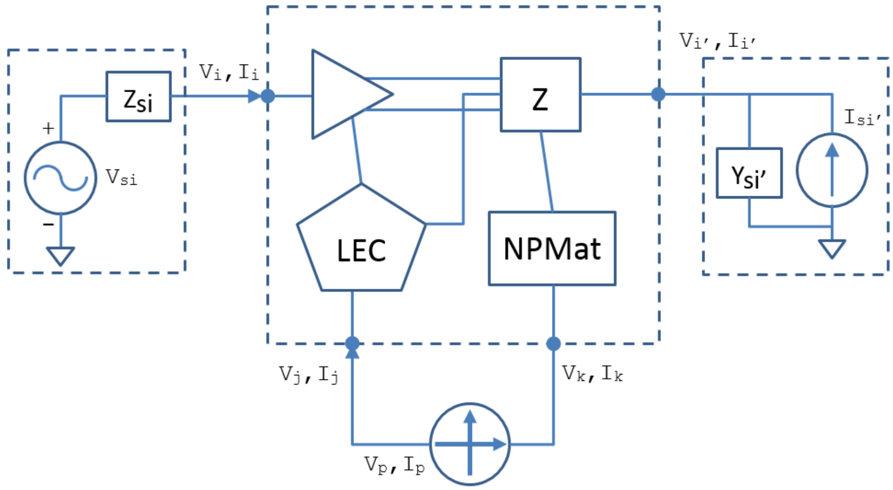

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Figure 7: Top-Level Network with Sources and Probes

6.2 Voltage and Current Probe

As depicted in Fig.(7), a voltage and current probe connected to ports j and k imposes the following conditions to the ports,

$$V_j = V_k = V_p \tag{257}$$

$$-I_j = I_k = I_p \tag{258}$$

where V_p and I_p are the voltage and current of the probe. Note that V_j, V_k and V_p reference the same ground.

6.3 Network Solution

The impedance equations of the internal network are,

$$\begin{bmatrix} \mathbf{V}_i \\ \mathbf{V}_j \\ \mathbf{V}_k \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{ii} & \mathbf{Z}_{ij} & \mathbf{Z}_{ik} \\ \mathbf{Z}_{ji} & \mathbf{Z}_{jj} & \mathbf{Z}_{jk} \\ \mathbf{Z}_{ki} & \mathbf{Z}_{kj} & \mathbf{Z}_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{I}_i \\ \mathbf{I}_j \\ \mathbf{I}_k \end{bmatrix} \tag{259}$$

$$\mathbf{V}_i = \mathbf{Z}_{ii}\mathbf{I}_i + \mathbf{Z}_{ij}\mathbf{I}_j + \mathbf{Z}_{ik}\mathbf{I}_k \tag{260}$$

$$\mathbf{V}_j = \mathbf{Z}_{ji}\mathbf{I}_i + \mathbf{Z}_{jj}\mathbf{I}_j + \mathbf{Z}_{jk}\mathbf{I}_k \tag{261}$$

$$\mathbf{V}_k = \mathbf{Z}_{ki}\mathbf{I}_i + \mathbf{Z}_{kj}\mathbf{I}_j + \mathbf{Z}_{kk}\mathbf{I}_k \tag{262}$$

The vector form of $\mathbf{V}_j$ represents the aggregation of all ports connected to the tail of voltage and current probes and, the vector form of $\mathbf{V}_k$ represents the aggregation of all ports connected to the head of voltage and current probes. It is obvious that $\mathbf{V}_j$ and $\mathbf{V}_k$ have the same dimension and, each quantity in $V_j(1), V_j(2), ..., V_j(N_j)$ is 1-to-1 mapped to $V_k(1), V_k(2), ..., V_k(N_k)$ corresponding to the voltage and current probes. $\mathbf{V}_i$ represents the aggregation of all ports connected to independent voltage sources with associated series source impedance.

To solve for probe voltages and currents $\mathbf{V}_j, \mathbf{V}_k, \mathbf{I}_j$ and $\mathbf{I}_k$, substitute Eqns.(257) and (258) into (261) and (262) to obtain,

$$(\mathbf{Z}_{ji} - \mathbf{Z}_{ki})\mathbf{I}_i = \mathbf{\Delta Z}_j^k \mathbf{I}_j \tag{263}$$

where

$$\mathbf{\Delta Z}_j^k = \mathbf{Z}_{kj} - \mathbf{Z}_{kk} - \mathbf{Z}_{jj} + \mathbf{Z}_{jk} \tag{264}$$

Substitute Eqns.(255) and (258) into Eqn.(260) and rearrange the terms, we have

$$(\mathbf{Z}_s + \mathbf{Z}_{ii})\mathbf{I}_i = \mathbf{V}_s - (\mathbf{Z}_{ij} - \mathbf{Z}_{ik})\mathbf{I}_j \tag{265}$$

From Eqns.(263) and (265) the expressions for $\mathbf{I}_i$ and $\mathbf{I}_j$ are readily obtained in the form of tran-admittance matrices,

$$\mathbf{I}_i = \mathbf{Y}_{is}\mathbf{V}_s \tag{266}$$
$$\mathbf{I}_j = \mathbf{Y}_{js}\mathbf{V}_s \tag{267}$$
$$\mathbf{I}_k = -\mathbf{I}_j = -\mathbf{Y}_{js}\mathbf{V}_s \tag{268}$$

where the trans-admittance matrices $\mathbf{Y}_{is}$ and $\mathbf{Y}_{js}$ are given by,

$$\mathbf{Y}_{is} = \{\mathbf{Z}_s + \mathbf{Z}_{ii} + (\mathbf{Z}_{ij} - \mathbf{Z}_{ik})(\boldsymbol{\Delta}\mathbf{Z}_j^k)^{-1}(\mathbf{Z}_{ji} - \mathbf{Z}_{ki})\}^{-1} \tag{269}$$
$$\mathbf{Y}_{js} = \{(\mathbf{Z}_{ji} - \mathbf{Z}_{ki})(\mathbf{Z}_s + \mathbf{Z}_{ii})^{-1}(\mathbf{Z}_{ij} - \mathbf{Z}_{ik}) + \boldsymbol{\Delta}\mathbf{Z}_j^k\}^{-1}(\mathbf{Z}_{ji} - \mathbf{Z}_{ki})(\mathbf{Z}_s + \mathbf{Z}_{ii})^{-1} \tag{270}$$

The voltages can be readily obtained from Eqns.(258), (260), (261) and (262).

$$\mathbf{V}_i = \mathbf{H}_{is}^v\mathbf{V}_s \tag{271}$$
$$\mathbf{V}_j = \mathbf{H}_{js}^v\mathbf{V}_s \tag{272}$$
$$\mathbf{V}_k = \mathbf{V}_j = \mathbf{H}_{js}^v\mathbf{V}_s \tag{273}$$
$$\mathbf{V}_k = \mathbf{H}_{ks}^v\mathbf{V}_s \tag{274}$$

where the voltage transfer matrices are given by,

$$\mathbf{H}_{is}^v = \mathbf{Z}_{ii}\mathbf{Y}_{is} + (\mathbf{Z}_{ij} - \mathbf{Z}_{ik})\mathbf{Y}_{js} \tag{275}$$
$$\mathbf{H}_{js}^v = \mathbf{Z}_{ji}\mathbf{Y}_{is} + (\mathbf{Z}_{jj} - \mathbf{Z}_{jk})\mathbf{Y}_{js} \tag{276}$$
$$\mathbf{H}_{ks}^v = \mathbf{Z}_{ki}\mathbf{Y}_{is} + (\mathbf{Z}_{kj} - \mathbf{Z}_{kk})\mathbf{Y}_{js} \tag{277}$$

The time-domain correspondence of these equations are obtained by iFFT and vector-matrix convolution,

$$\mathbf{i}_i(t) = \mathbf{y}_{is}(t) * \mathbf{v}_s(t) \tag{278}$$
$$\mathbf{i}_j(t) = \mathbf{y}_{js}(t) * \mathbf{v}_s(t) \tag{279}$$
$$\mathbf{i}_k(t) = -\mathbf{y}_{js}(t) * \mathbf{v}_s(t) \tag{280}$$
$$\mathbf{v}_i(t) = \mathbf{h}_{is}^v(t) * \mathbf{v}_s(t) \tag{281}$$
$$\mathbf{v}_j(t) = \mathbf{h}_{js}^v(t) * \mathbf{v}_s(t) \tag{282}$$
$$\mathbf{v}_k(t) = \mathbf{v}_j(t) = \mathbf{h}_{js}^v(t) * \mathbf{v}_s(t) \tag{283}$$
$$\mathbf{v}_k(t) = \mathbf{h}_{ks}^v(t) * \mathbf{v}_s(t) \tag{284}$$

7 Time Domain Analysis

7.1 Impulse Response

7.2 Impact of DC Value

7.3 TDR Impedance

8 Touchstone Data Processing

Touchstone files contain tabulated network parameters and associated network information such as reference impedance, data format (magnitude-phase, real-imaginary, dB-phase), etc. The syntax and rules of Touchstone 1.1 and 2.0 can be found in [1] and [2]. The Matlab TSNP object has been implemented to import Touchstone files and process its data. Touchstone files only allow one value of reference impedance for all the ports.

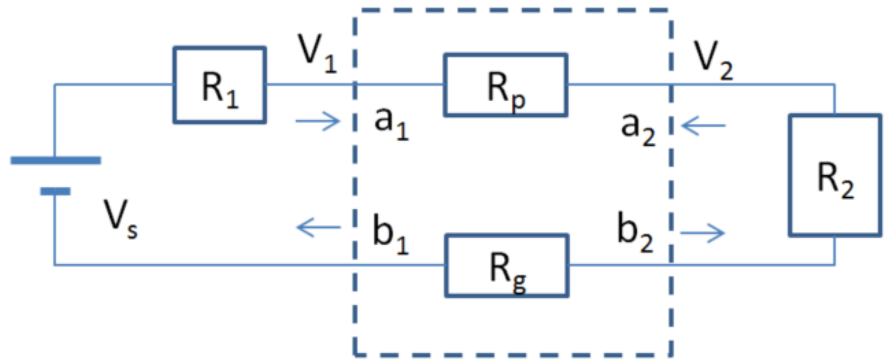

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Figure 8: DC Point of S-Parameter

9 Assignments

A Series Resistor Two-Port Network

The network of interest is shown inside the dashed box in Fig.(8). R_p and R_g are power and ground resistances at DC. For many structures used in engineering design, it is usually rather straightfoward to obtain the values of R_p and R_g from DC analysis. Due to the lack of shunt resistances between power and ground, the Z-parameter of such a network doesn't exist. For example, when port 1 is driven by a current source when port 2 is open, the voltage at port 2 would be infinite. Due to this reason, we derive the S-parameter of this network directly from incident and reflected waves at DC.

In the most general case, the reference impedances (resistances at DC) at port 1 and 2 are R_1 and R_2, respectively. The incident and reflected waves at ports 1 and 2 are denoted by a_1, a_2, b_1 and b_2. The voltages at ports 1 and 2 can also be decomposed into incident and reflected voltages,

$$V_1 = V_1^+ + V_1^- \tag{285}$$

$$V_2 = V_2^+ + V_2^- \tag{286}$$

According to the definition of incident and reflected waves,

$$a_1 = \frac{V_1^+}{\sqrt{R_1}} \tag{287}$$

$$b_1 = \frac{V_1^-}{\sqrt{R_1}} \tag{288}$$

$$a_2 = \frac{V_2^+}{\sqrt{R_2}} \tag{289}$$

$$b_2 = \frac{V_2^-}{\sqrt{R_2}} \tag{290}$$

The reflected wave at port 2 is non-exist, making $a_2 = 0$. From these equations, we obtain,

$$V_1 = (a_1 + b_1)\sqrt{R_1} \tag{291}$$

$$V_2 = (a_2 + b_2)\sqrt{R_2} = b_2\sqrt{R_2} \tag{292}$$

The loop current is,

$$I_L = \frac{V_s}{\Sigma R} \tag{293}$$

where $\Sigma R = R_1 + R_p + R_2 + R_g$. And the incident wave a_1 is solely determined by the source,

$$a_1 = \frac{V_s}{2\sqrt{R_1}} \tag{294}$$

From Ohms Law, we obtain,

$$V_1 = I_L(R_p + R_2 + R_g) \tag{295}$$

$$V_2 = I_L R_2 \tag{296}$$

$$V_1 - V_2 = \frac{R_p + R_g}{\Sigma R} V_s \tag{297}$$

From Eqs.(291) and (292), we also have,

$$V_1 - V_2 = a_1\sqrt{R_1} + b_1\sqrt{R_1} - b_2\sqrt{R_2} \tag{298}$$

From the above two equations, we obtain

$$a_1\sqrt{R_1} + b_1\sqrt{R_1} - b_2\sqrt{R_2} = \frac{R_p + R_g}{\Sigma R} V_s \tag{299}$$

Substitute V_s in Eq.(294) into the above and devide both sides by $a_1\sqrt{R_1}$ we have

$$1 + S_{11} - S_{21}\sqrt{\frac{R_2}{R_1}} = \frac{2(R_p + R_g}{\Sigma R} \tag{300}$$

S_{11} can be directly obtained from the reflection coefficient at port 1, when port 2 is terminated by its port impedance R_2,

$$S_{11} = \frac{(R_p + R_2 + R_g) - R_1}{\Sigma R} = 1 - \frac{2R_1}{\Sigma R} \tag{301}$$

Substitute S_{11} into Eq.(300) to obtain S_{21},

$$S_{21} = \frac{2\sqrt{R_1 R_2}}{\Sigma R} \tag{302}$$

Similarly we have

$$S_{22} = 1 - \frac{2R_2}{\Sigma R} \tag{303}$$

$$S_{12} = S_{21} = \frac{2\sqrt{R_1 R_2}}{\Sigma R} \tag{304}$$

The power conservation relationship is expressed as,

$$a_1^2 = b_1^2 + b_2^2 + I_L^2(R_p + R_g) \tag{305}$$

In this equation, a_1^2 is the incident power, b_1^2 and b_2^2 are the reflected powers, the last term is the power dissipated inside the network. The total sum of reflected and disspated power equals the incident power. Divide both sides by a_1^2, we obtain the power conservation relationship for this network,

$$1 = S_{11}^2 + S_{21}^2 + \frac{4(R_p + R_g)R_1}{\Sigma R^2} \tag{306}$$

Similarly,

$$1 = S_{22}^2 + S_{12}^2 + \frac{4(R_p + R_g)R_2}{\Sigma R^2} \tag{307}$$

When $R_1 = R_2 = R_0$, the above expressions can be simplified,

$$S_{11} = S_{22} = 1 - \frac{2R_0}{\Sigma R} \tag{308}$$

$$S_{12} = S_{21} = \frac{2R_0}{\Sigma R} \tag{309}$$

Note that in this case the reflected and transmitted waves satisfy the following relationship,

$$S_{11} + S_{21} = 1 \tag{310}$$

The power conservation relationships can be expressed as,

$$1 = S_{11}^2 + S_{21}^2 + \frac{4(R_p + R_g)R_0}{\Sigma R^2} \tag{311}$$

$$1 = S_{22}^2 + S_{12}^2 + \frac{4(R_p + R_g)R_0}{\Sigma R^2} \tag{312}$$

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