J. Zhou 23-Feb-2015

Derivation of Serdes Channel Simulation (Y)

differential ~~mode~~ sources only

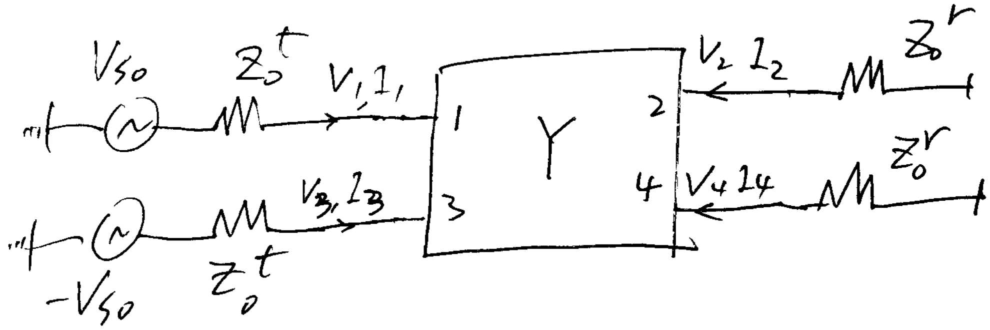

Fig. 1

excitations are purely differential

p/n impedances are balanced (equal)

$$I = [I_1 \ \ I_2 \ \ I_3 \ \ I_4]^T \tag{1a}$$

$$V = [V_1 \ \ V_2 \ \ V_3 \ \ V_4]^T \tag{1b}$$

$$I = YV \tag{1c}$$

Note: this derivation is for differential sources only. however, even if the sources are diff-only, there could still be common-mode response, unless the diff-to-common mode conversion is zero, i.e. $Y_{dc} = Y_{cd} = 0$.

case 11	$0<\lvert Z_s\rvert<\infty$	$0<\lvert Z_L\rvert<\infty$	pg 2
case 01	$\lvert Z_s\rvert=0$	$0<\lvert Z_L\rvert<\infty$	pg 9
case 1i	$0<\lvert Z_s\rvert<\infty$	$\lvert Z_L\rvert=\infty$	pg 22
case 0i	$\lvert Z_s\rvert=0$	$\lvert Z_L\rvert=\infty$	pg 14

	$0<\lvert Z_L\rvert<\infty$	$\lvert Z_L\rvert=\infty$
$\lvert Z_s\rvert=0$	case 01	case 0i
$0<\lvert Z_s\rvert<\infty$	case 11	case 1i

$\boxed{\text{Case 11}}$ $0 < Z_0^t < \infty$

$0 < Z_0^r < \infty$

Terminal equations:

$$V_2 = -Z_0^r I_2 \quad (2a)$$

$$V_4 = -Z_0^r I_4 \quad (2b)$$

$$V_{s0} - V_1 = Z_0^t I_1 \quad (3a)$$

$$-V_{s0} - V_3 = Z_0^t I_3 \quad (3b)$$

mixed mode voltages and currents:

$$V_{d1} = V_1 - V_3, \quad V_{c1} = (V_1 + V_3)/2 \quad (4a, 4b)$$

$$V_{d2} = V_2 - V_4, \quad V_{c2} = (V_2 + V_4)/2 \quad (5a, 5b)$$

$$I_{d1} = (I_1 - I_3)/2, \quad I_{c1} = I_1 + I_3 \quad (6a, 6b)$$

$$I_{d2} = (I_2 - I_4)/2, \quad I_{c2} = I_2 + I_4 \quad (7a, 7b)$$

single-ended to mixed mode mapping

$$V_m = \begin{bmatrix} V_{d1} \\ V_{d2} \\ V_{c1} \\ V_{c2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = M_V \cdot V \qquad (mV)$$

$$I_m = \begin{bmatrix} I_{d1} \\ I_{d2} \\ I_{c1} \\ I_{c2} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = M_I \cdot I \qquad (mI)$$

$$V_m = M_V V \qquad (mVb)$$

$$I_m = M_I I \qquad (mIb)$$

$$I_m = M_I Y V = M_I Y M_V^{-1} V_m \qquad (Im) \; \cancel{(Vm)}$$

$$Y_m = M_I Y M_V^{-1} \qquad (Yma)$$

$$= \begin{bmatrix} Y_{dd} & Y_{dc} \\ Y_{cd} & Y_{cc} \end{bmatrix} \qquad (Ymb)$$

$$\begin{bmatrix} I_{d1} \\ I_{d2} \\ I_{c1} \\ I_{c2} \end{bmatrix} = \begin{bmatrix} Y_{dd} & Y_{dc} \\ Y_{cd} & Y_{cc} \end{bmatrix} \begin{bmatrix} V_{d1} \\ V_{d2} \\ V_{c1} \\ V_{c2} \end{bmatrix} \qquad (I_m Y V_m)$$

to solve for I_m, V_m from 3a 3b we have

$$V_1 - V_3 = 2V_{s0} - Z_0^t (I_1 - I_3) \qquad (10)$$

$$V_1 + V_3 = -Z_0^t (I_1 + I_3) \qquad (11)$$

let $V_s^d = 2V_{s0}$ (12a)

$V_s^c = V_{s0} - V_{s0} = 0$ (12b)

from 4a, 4b, 6a, 6b we have

$$V_{d1} = V_s^d - 2Z_0^t \cdot I_{d1}$$

$$V_{c1} = -\frac{Z_0^t}{2} I_{c1}$$

similarly $V_{d2} = -Z_0^r (I_2 - I_4)$ (14a)

$$= -2Z_0^r I_{d2} \qquad (14a)$$

$$V_{c2} = -Z_0^r \frac{I_2 + I_4}{2} = -Z_0^r \frac{I_{c2}}{2} \qquad (14b)$$

$$\begin{bmatrix} V_{d1} \\ V_{d2} \\ V_{c1} \\ V_{c2} \end{bmatrix} = \underbrace{\begin{bmatrix} -2Z_0^t & 0 & 0 & 0 \\ 0 & -2Z_0^r & 0 & 0 \\ 0 & 0 & -\frac{Z_0^t}{2} & 0 \\ 0 & 0 & 0 & -\frac{Z_0^r}{2} \end{bmatrix}}_{\tilde{Z}^{tm}} \begin{bmatrix} I_{d1} \\ I_{d2} \\ I_{c1} \\ I_{c2} \end{bmatrix} + \begin{bmatrix} V_s^d \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (15*)$$

$$V_m = \tilde{Z}^{tm} I_m + V_s \qquad (15b)$$

substitute eqn (Z_m) from pg 3 into above

$$V_m = \tilde{Z}^{tm} M_I Y M_V^{-1} V_m + V_s \qquad (16)$$

$$V_m = \left(U - \tilde{Z}^{tm} M_I Y M_V^{-1}\right)^{-1} V_s \qquad (17)$$

$$I_m = M_I Y M_V^{-1} V_m \qquad (18)$$

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to further simplify the solution in eqn ($I_m Y V_m$) on pg 3, assume

$$Y_{dc} = Y_{cd} = \cancel{Y_{cc}} = 0$$

we have

$$\begin{pmatrix} I_{d1} \\ I_{d2} \end{pmatrix} = \left(Y_{dd} \right) \begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} \qquad (101)$$

$$\begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} = \underbrace{\begin{bmatrix} -2 Z_0^t & 0 \\ 0 & -2 Z_0^r \end{bmatrix}}_{\tilde{\tilde{Z}}^t} \begin{pmatrix} I_{d1} \\ I_{d2} \end{pmatrix} + \begin{bmatrix} V_s^d \\ 0 \end{bmatrix} \qquad (102)$$

$$\begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} = \tilde{\tilde{Z}}^t \cdot Y_{dd} \begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} + \begin{pmatrix} V_s^d \\ 0 \end{pmatrix} \qquad (103)$$

$$\begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} = \left(U - \tilde{\tilde{Z}}^t \cdot Y_{dd} \right)^{-1} \begin{bmatrix} V_s^d \\ 0 \end{bmatrix} \qquad (104)$$

$$\begin{pmatrix} I_{d1} \\ I_{d2} \end{pmatrix} = \left[Y_{dd} \right] \begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} \qquad (105)$$

Conclusion:

(1) when common$\leftrightarrow$differential conversion is not zero (i.e. $\gamma_{dc} \neq 0$, $\gamma_{cd} \neq 0$)

~~use~~ eqns (17), (18) is the solution

(2) when common$\leftrightarrow$differential conversion is negligible (i.e. $\gamma_{dc} = \gamma_{cd} = 0$)

eqn (17) (18) can be simplified into eqns (104) (105), which is the solution of differential mode response only (the common-mode is non-exist)

(note: in either case, the excitation is always differential only).

case 01 $|Z_s| = 0$ $0 < |Z_L| < \infty$

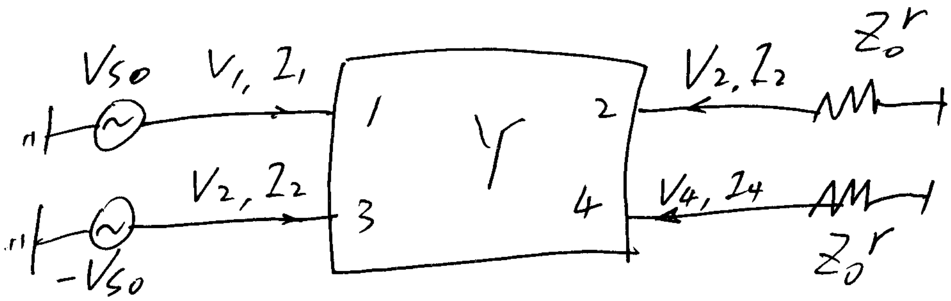


Fig. 2

$$I = YV \qquad (201c)$$

$$V_2 = -Z_0^r I_2 \qquad (202a)$$

$$V_4 = -Z_0^r I_4 \qquad (202b)$$

$$V_1 = V_{s0} \qquad (203a)$$

$$V_3 = -V_{s0} \qquad (203b)$$

mixed-mode voltages and currents:

$$V_{d1} = V_1 - V_3, \quad V_{c1} = (V_1 + V_3)/2 \qquad (204a, 204b)$$

$$V_{d2} = V_2 - V_4, \quad V_{c2} = (V_2 + V_4)/2 \qquad (205a, 205b)$$

$$I_{d1} = (I_1 - I_3)/2, \quad I_{c1} = I_1 + I_3 \qquad (206a, 206b)$$

$$I_{d2} = (I_2 - I_4)/2, \quad I_{c2} = I_2 + I_4 \qquad (207a, 207b)$$

$$V_m = M_V V \tag{208a}$$

$$I_m = M_I I \tag{208b}$$

$$I_m = M_I Y M_V^{-1} V_m \tag{208c}$$

$$Y_m = M_I Y M_V^{-1} = \begin{bmatrix} Y_{dd} & Y_{dc} \\ Y_{cd} & Y_{cc} \end{bmatrix} \tag{208d}$$

$$[I_{d1}, I_{d2}\ I_{c1}\ I_{c2}]^T = Y_m [V_{d1}\ V_{d2}\ V_{c1}\ V_{c2}]^T \tag{209}$$

from (203a, 203b) we have

$$V_1 - V_3 = 2V_{S0} \tag{210}$$

$$V_1 + V_3 = 0 \tag{211}$$

let $V_S^d = 2V_{S0}$ (212a)

$V_S^c = V_{S0} - V_{S0} = 0$ (212b)

from 204a, 204b, 206a, 206b, we have

$$V_{d1} = 2V_{S0} \triangleq V_S^d \tag{213a}$$

$$V_{c1} = 0 \tag{213b}$$

Similarly $V_{d2} = -Z_0^r (I_2 - I_4) = -2 Z_0^r I_{d2}$ (214a)

$$V_{c2} = -Z_0^r (I_2 + I_4)/2 = -Z_0^r I_{c2}/2 \tag{214b}$$

(eqn 215*)

$$\underbrace{\begin{bmatrix} V_{d1} \\ V_{d2} \\ V_{c1} \\ V_{c2} \end{bmatrix}}_{V_m} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2Z_0^r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{Z_0^r}{2} \end{bmatrix}}_{\tilde{Z}^{tm}} \underbrace{\begin{bmatrix} I_{d1} \\ I_{d2} \\ I_{c1} \\ I_{c2} \end{bmatrix}}_{I_m} + \underbrace{\begin{bmatrix} V_s^d \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{V_s}$$

$$V_m = \tilde{Z}^{tm} I_m + V_s \qquad (215b)$$

substitute (208c) into above

$$V_m = \tilde{Z}^{tm} Y_m V_m + V_s \qquad (216)$$

$$V_m = \left(U - \tilde{Z}^{tm} Y_m \right)^{-1} V_s \qquad (217)$$

$$I_m = Y_m \left(U - \tilde{Z}^{tm} Y_m \right)^{-1} V_s \qquad (218)$$

To further simplify the solution by assuming $Y_{dc} = Y_{cd} = 0$ in eqn (208d)

we have
$$\begin{pmatrix} I_{d1} \\ I_{d2} \end{pmatrix} = \left[Y_{dd} \right]_{2\times 2} \begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} \qquad (2\text{-}101)$$

according to 213 a,

$$\begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & -2Z_0^r \end{pmatrix}}_{\tilde{Z}^t_{2\times 2}} \begin{pmatrix} I_{d1} \\ I_{d2} \end{pmatrix} + \underbrace{\begin{bmatrix} V_s^d \\ 0 \end{bmatrix}}_{V_s} \qquad (2\text{-}102)$$

$$\begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} = \tilde{Z}_0^t Y_{dd} \begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} + V_s \qquad (2\text{-}103)$$

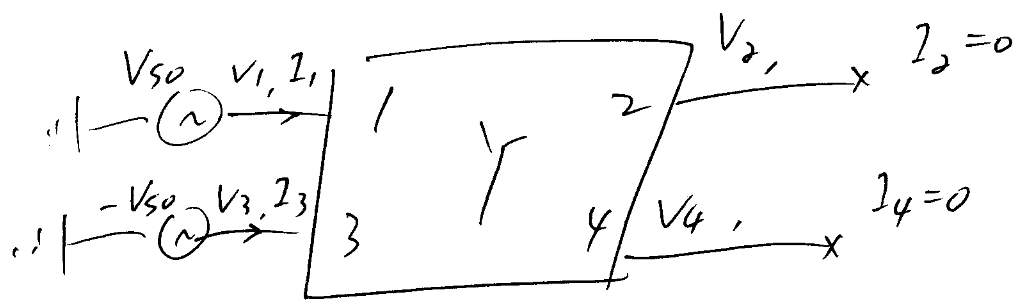
$$\begin{pmatrix} V_{d1} \\ V_{d2} \end{pmatrix} = \left(U - \tilde{Z}_0^t Y_{dd} \right)^{-1} V_s \qquad (2\text{-}104)$$

$$\begin{pmatrix} I_{d1} \\ I_{d2} \end{pmatrix} = Y_{dd\,2\times 2} \begin{bmatrix} V_{d1} \\ V_{d2} \end{bmatrix} \qquad (2\text{-}105)$$

Conclusion:

for case 01 where $|z_s| = 0$, $0 < |z_L| < \infty$
the solution has the same form as case 11,
as given by eqns (217, 218)
The only difference is that the rank of
$\tilde{Z}^{tm}$ is 2 for case 01. it is
full rank for case 11.
However, the matrix $(U - \tilde{Z}^{tm} Y_m)$
in eqn (217) should still be full
rank regardless of rank$(\tilde{Z}^{tm})$.

case 01 $|Z_s| = 0, \quad |Z_L| = \infty$



$I = YV$ eqn (301c)

$I_2 = 0, \quad I_4 = 0$ (202a, 202b)

$V_1 = V_{s0}$ (303a)

$V_3 = -V_{s0}$ (303b)

mixed mode voltages and currents

$V_{d1} = V_1 - V_3 \qquad V_{c1} = (V_1 + V_3)/2$ (304a, b)

$V_{d2} = V_2 - V_4 \qquad V_{c2} = (V_2 + V_4)/2$ (305a, b)

$I_{d1} = (I_1 - I_3)/2 \qquad I_{c1} = I_1 + I_3$ (306a, b)

$I_{d2} = (I_2 - I_4)/2 \qquad I_{c2} = I_2 + I_4$ (307a, b)

$$V_m = M_V V \tag{308a}$$

$$I_m = M_2 I \tag{308b}$$

$$I_m = M_2 Y M_V^{-1} V_m \tag{308c}$$

$$Y_m = M_2 Y M_V^{-1} = \begin{bmatrix} Y_{dd} & Y_{dc} \\ Y_{cd} & Y_{cc} \end{bmatrix}_{4\times4} \tag{308d}$$

$$[I_{d1}\ I_{d2}\ I_{c1}\ I_{c2}]^T = \underset{4\times4}{Y_m} [V_{d1}\ V_{d2}\ V_{c1}\ V_{c2}]^T \tag{309}$$

from (303 a. b) we have

$$V_1 - V_3 = 2V_{s0} \tag{310}$$

$$V_1 + V_3 = 0 \tag{311}$$

let $V_s^d = 2V_{s0}$ (312a)

$V_s^c = V_{s0} - V_{s0} = 0$ (312b)

from 304 a, b 306a, b we have

$$V_{d1} = 2V_{s0} \tag{313a}$$

$$V_{c1} = 0 \tag{313b}$$

$V_{d2} = V_2 - V_4 \qquad V_{c2} = (V_2 + V_4)/2$ (314a

$I_{d1} = (I_1 - I_3)/2 \qquad I_{c1} = I_1 + I_3$ (314b

$I_{d2} = (I_2 - I_4)/2 = 0, \quad I_{c2} = I_2 + I_4 = 0$ (314c

The formal solution for this case is different from the previous two cases.

The mixed-mode Y matrix is reorganized by ports:

$$\begin{array}{c} I_1^m \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ I_2^m \left\{ \begin{array}{c} \\ \\ \end{array} \right. \end{array} \begin{pmatrix} I_{d1} \\ I_{c1} \\ I_{d2} \\ I_{c2} \end{pmatrix} = \begin{pmatrix} Y_{11}^m & Y_{12}^m \\ Y_{21}^m & Y_{22}^m \end{pmatrix}_{4\times 4} \begin{bmatrix} V_{d1} \\ V_{c1} \\ V_{d2} \\ V_{c2} \end{bmatrix} \begin{array}{c} \left. \begin{array}{c} \\ \\ \end{array} \right\} V_1^m \\ \left. \begin{array}{c} \\ \\ \end{array} \right\} V_2^m \end{array} \qquad (315a$$

$$\begin{pmatrix} I_1^m \\ I_2^m \end{pmatrix} = \begin{pmatrix} Y_{11}^m & Y_{12}^m \\ Y_{21}^m & Y_{22}^m \end{pmatrix}_{4\times 4} \begin{bmatrix} V_1^m \\ V_2^m \end{bmatrix} \qquad (315b$$

The above Y matrix can be obtained from (308d) by swapping rows $2 \leftrightarrow 3$ and columns $2 \leftrightarrow 3$

from 315b we have

$$I_1^m = Y_{11}^m V_1^m + Y_{12}^m V_2^m \qquad (316a)$$

$$I_2^m = Y_{21}^m V_1^m + Y_{22}^m V_2^m \qquad (316b)$$

according to 314c, $I_2^m = [\, I_{d_2} \;\; I_{c_2} \,]^T = 0 \qquad (317)$

according to 313a $V_1^m = \begin{pmatrix} V_{d_1} \\ V_{c_1} \end{pmatrix} = \underbrace{\begin{pmatrix} V_s^d \\ 0 \end{pmatrix}}_{V_s} = V_s \qquad (318)$

we obtain

$$I_1^m = Y_{11}^m V_s + Y_{12}^m V_2^m \qquad (319a)$$

$$0 = Y_{21}^m V_s + Y_{22}^m V_2^m \qquad (319b)$$

$$V_2^m = -\left(Y_{22}^m\right)^{-1} Y_{21}^m V_s \qquad (320)$$

$$I_1^m = Y_{11}^m V_s - Y_{12}^m \left(Y_{22}^m\right)^{-1} Y_{21}^m V_s \qquad (321a)$$

$$I_1^m = \left(Y_{11}^m - Y_{12}^m \left(Y_{22}^m\right)^{-1} Y_{21}^m\right) V_s \qquad (321b)$$

eqn (320) is the solution for output voltage

eqn (321b) is the current at input port.

conclusion: for case of, $|Z_S| = 0$, $|Z_L| = \infty$

the solution has a much simpler form of (320) and (321b) case 1i $\quad 0<|Z_s|<\infty, \quad |Z_L|=\infty$

terminal eqns:

$$I_2 = I_4 = 0 \qquad (402a,b)$$

$$V_{s0} - V_1 = Z_0^t I_1 \qquad (403a)$$

$$-V_{s0} - V_3 = Z_0^t I_3 \qquad (403b)$$

mixed mode voltages and currents

$$V_{d1} = V_1 - V_3, \quad V_{c1} = (V_1 + V_3)/2 \qquad (404a,b)$$

$$V_{d2} = V_2 - V_4, \quad V_{c2} = (V_2 + V_4)/2 \qquad (405a,b)$$

$$I_{d1} = (I_1 - I_3)/2, \quad I_{c1} = I_1 + I_3 \qquad (406a,b)$$

$$I_{d2} = (I_2 - I_4)/2, \quad I_{c2} = I_2 + I_4 \qquad (407a,b)$$

$$I_{d2} = 0, \quad I_{c2} = 0 \qquad \cdots \qquad (407d)$$

$$V_{d_1} = V_1 - V_3 = 2V_{s0} - Z_0^t (I_1 - I_3) \qquad (410a)$$

$$V_{d_1} = 2V_{s0} - 2Z_0^t I_{d_1} \qquad (410b)$$

$$V_{c_1} = \frac{-Z_0^t}{2} I_{c_1} \qquad (411)$$

$$V_{d_2} = V_2 - V_4 , \quad I_{d_2} = 0 \qquad (412a)$$

$$V_{c_2} = I_2 + I_4 = 0 , \quad I_{c_2} = 0 \qquad (412b)$$

$$\underbrace{\begin{pmatrix} V_{d_1} \\ V_{c_1} \end{pmatrix}}_{V_1^m} = \begin{bmatrix} -2Z_0^t & 0 \\ 0 & -\frac{Z_0^t}{2} \end{bmatrix} \underbrace{\begin{pmatrix} I_{d_1} \\ I_{c_1} \end{pmatrix}}_{I_1^m} + \underbrace{\begin{pmatrix} \overset{= 2V_{s0}}{V_s^d} \\ 0 \end{pmatrix}}_{V_s} \qquad (413a)$$

$$\left[V_1^m \right] \triangleq \begin{pmatrix} V_{d_1} \\ V_{c_1} \end{pmatrix} = \underbrace{\begin{pmatrix} -2Z_0^t & 0 \\ 0 & -\frac{Z_0^t}{2} \end{pmatrix}}_{\tilde{Z}^t} \left[I_1^m \right] + V_s \qquad (413b)$$

$$V_1^m = \tilde{Z}^t I_1^m + V_s \qquad (413c)$$

take similar approach as in case σ_i, to partition the Y matrix by ports, as in eqn (315b)

$$\begin{pmatrix} I_1^m \\ I_2^m \end{pmatrix} = \begin{pmatrix} Y_{11}^m & Y_{12}^m \\ Y_{21}^m & Y_{22}^m \end{pmatrix} \begin{pmatrix} V_1^m \\ V_2^m \end{pmatrix} \qquad (415b)$$

$$I_1^m = Y_{11}^m V_1^m + Y_{12}^m V_2^m$$

$$0 = I_2^m = Y_{21}^m V_1^m + Y_{22}^m V_2^m \qquad (416a,b)$$

now we can solve for V_1^m, V_2^m and I_1^m from the three eqns of (413c) and (416a,b), we have

~~sub~~ we first eliminate V_2^m:

from (416b) $\quad V_2^m = -(Y_{22}^m)^{-1} Y_{21}^m V_1^m \qquad (417)$

substitute into (416a):

$$I_1^m = Y_{11}^m V_1^m - Y_{12}^m (Y_{22}^m)^{-1} Y_{21}^m V_1^m \qquad 418a$$

$$I_1^m = \underbrace{\left(Y_{11}^m - Y_{12}^m (Y_{22}^m)^{-1} Y_{21}^m\right)}_{\tilde{Y}_{11}^m} V_1^m \qquad 418b$$

$$I_1^m = \tilde{Y}_{11}^m V_1^m \qquad 418C$$

substitute into (413c) we obtain

$$V_1^m = \tilde{Z}^t \tilde{Y}_{11}^m V_1^m + V_s \qquad (419)$$

$$V_1^m = (U - \tilde{Z}^t \tilde{Y}_{11}^m)^{-1} V_s \qquad (420)$$

$$V_2^m = -(Y_{22}^m)^{-1} Y_{21}^m (U - \tilde{Z}^t \tilde{Y}_{11}^m)^{-1} V_s \quad (421)$$

$$I_1^m = \tilde{Y}_{11}^m (U - \tilde{Z}^t \tilde{Y}_{11}^m)^{-1} V_s \quad (422)$$

eqns 420, 421 and 422 are the solutions.

for the simplified case of zero mode conversion, eqn 415b becomes

$$\begin{pmatrix} I_1^d \\ I_2^d \end{pmatrix} = \underbrace{\begin{pmatrix} Y_{11}^d & Y_{12}^d \\ Y_{21}^d & Y_{22}^d \end{pmatrix}}_{2\times 2} \begin{pmatrix} V_1^d \\ V_2^d \end{pmatrix} \quad (415b\text{-}d)$$

$$I_1^d = Y_{11}^d V_1^d + Y_{12}^d V_2^d \quad 416a\text{-}d$$

$$0 = I_2^d = Y_{21}^d V_1^d + Y_{22}^d V_2^d \quad (416b\text{-}d)$$

and eqn 413 c becomes

$$V_1^d = -2 Z_0^t I_1^d + V_s^d \quad (413c\text{-}d)$$

$$(V_s^d \triangleq 2V_{s0})$$

The above three differential-only eqns can be solved to obtain I_1^d, V_2^d, V_1^d

from (416b-d): $V_2^d = -\frac{Y_{21}^d}{Y_{22}^d} V_1^d$ (417-d)

substitute into (416a-d):

$$I_1^d = Y_{11}^d V_1^d - Y_{12}^d \frac{Y_{21}^d}{Y_{22}^d} V_1^d \qquad (418a\text{-}d)$$

$$I_1^d = \underbrace{(Y_{11}^d - Y_{12}^d Y_{21}^d (Y_{22}^d)^{-1})}_{\stackrel{\Delta}{\equiv} \tilde{Y}_{11}^d} V_1^d \qquad (418b\text{-}d)$$

$$I_1^d = \tilde{Y}_{11}^d V_1^d \qquad (418c\text{-}d)$$

substitute into (413c-d):

$$V_1^d = -2 Z_0^t \tilde{Y}_{11}^d V_1^d + V_s^d \qquad (419\text{-}d)$$

$$V_1^d = \frac{V_s^d}{1 + 2 Z_0^t \tilde{Y}_{11}^d} \qquad (420\text{-}d)$$

$$V_2^d = -\frac{Y_{21}^d V_s^d}{Y_{22}^d (1 + 2 Z_0^t \tilde{Y}_{11}^d)} \qquad (421\text{-}d)$$

$$I_1^d = \frac{\tilde{Y}_{11}^d V_s^d}{1 + 2 Z_0^t Y_{11}^d} \qquad (422\text{-}d)$$

Conclusion:

The last case of $0<|Z_s|<\infty$, $|Z_L|=\infty$ is solved.

note: other cases are not allowed, such as $|Z_s|=\infty$ or $|Z_L|=0$.