Time-Frequency Transform of Band-limited Signals (2)

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Reference [1]
same title written on 2015-01-29

Intro:
This writing is to clarify some issues in [1]

Part 1. time and frequency sequence

first of all, DFT only involves indexes (indices) in time and frequency, meaning that there are no time or frequency units involved.

for example, the time-index is $n = 0, 1, 2, \cdots, N-1$
the frequency-index is $k = 0, 1, 2, \cdots, N-1$

these indices are translated into unitized time and frequency sequences according to the expressions on page 7 of [1]

Specifically, the time sequence is

$0, T_s, 2T_s, \cdots, (N-1)T_s$

the frequency sequence is

$0, \frac{f_s}{N}, \frac{2f_s}{N}, \cdots, \frac{(N-1)f_s}{N}$

time step $T_{step} = T_s$

frequency span $F_{span} = f_s = \frac{1}{T_s}$

time span $T_{span} = N T_s$

frequency step $F_{step} = \frac{f_s}{N} = \frac{1}{T_{span}}$

Note: for all practical purposes, time and frequency ~~all~~ always start from zero

$T_{start} = 0$

$F_{start} = 0.$

In summary, each ~~time-frequency~~ ~~pai~~ sequence pair has the following quantities that are always inter-related:

$T_{start} = 0$; start time

$T_{step} = T_s$; time step

$T_{stop} = (N-1)T_s$; the last time value

$T_{span} = N\,T_s$; the entire span of time interval

N ; # of samples

$F_{start} = 0$

$F_{step} = 1/T_{span} = \frac{1}{N\,T_s}$

$F_{stop} = (N-1)F_{step} = \frac{N-1}{N\,T_s}$

$F_{span} = N \cdot F_{step} = \frac{1}{T_s}$

The time and frequency domain samples always have the exact number of points N.

Part 2. Periodicity and Symmetry of discrete time and frequency sequences.

It is well documented in textbooks that the DFT time-frequency pair is periodic.

That is both the time and frequency sequences are periodic.

In practice, we almost always ignore the values beyond ~~out~~ the main sampling interval. the fact that the DFT time-frequency pair is periodic has little impact on the ~~repeat~~ results, provided the original signal is bandlimited to meet the Nyquist sampling Criteria $f_s \geq 2B$ where f_s is the sampling frequency

$f_s = \frac{1}{T_s}$ (aka fspan).

B is the max Bandwidth of the signal.

DFT transformed $X(f_k)$ satisfy the following symmetrical properties when $x[n]$ is real

(1) $Re[X(f_k)]$ is even function of f

(2) $Im[X(f_k)]$ is odd function of f

equivalently $\begin{cases} \text{Magnitude}[X(f_k)] \text{ is even} \\ \text{Angle}[X(f_k)] \text{ is odd} \end{cases}$

In order to satisfy this symmetry requirement, the frequency-sampled $\tilde{X}(f_{\tilde{k}})$ $\tilde{k}=0,1,2,\cdots,\tilde{N}$ must be "mirrored" to generate a symmetrical function $X(f_k)$ $k=0,1,2,\cdots,N$ before taking the IDFT.

The procedure of "mirror" is described below.

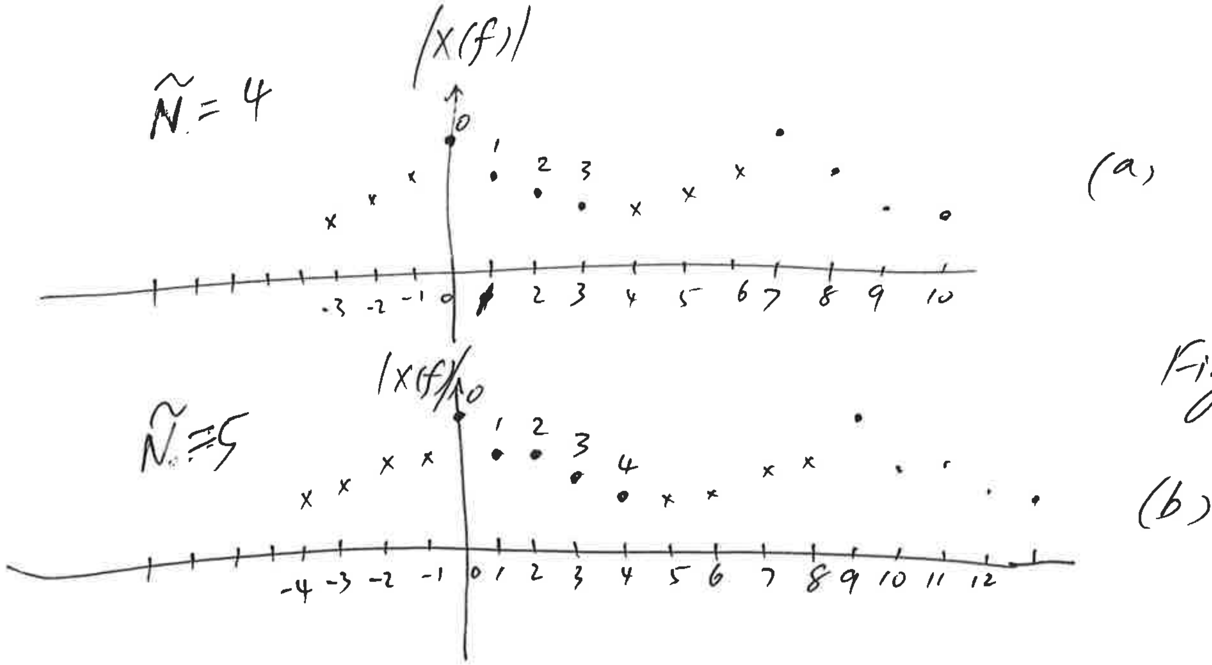

Fig. 100

As depicted in Fig. 100 (a)(b) on page 6.
the total number of frequency samples are
$\tilde{N} = 4$ and $\tilde{N} = 5$

Note this $\tilde{N}$ is not the same as the N
in the DFT and time-freq sequence
in Part 1

Now we apply the symmetry property
by padding (mirror) the samples using "*"
marked samples

for case (a) the complete sequence
has now becomes 0, 1, 2, 3, 4, 5, 6.
and this pattern is repeated to start from 7
its is obvious that the total number of
points $N = 2(\tilde{N}-1)+1 = 2\tilde{N}-1$

$$X(f_k) = \begin{cases} \tilde{X}(f_k) & 0 \le k \le \tilde{N}-1 \\ \tilde{X}(f_{2\tilde{N}-1-k}) & \tilde{N} \le k \le 2\tilde{N}-2 \end{cases}$$

for example, in Fig. 100 (a)

$\tilde{N} = 4$ (original freq samples inc. DC)

$X(f_0) = \tilde{X}(f_0)$ $\quad k=0$

$X(f_1) = \tilde{X}(f_1)$ $\quad k=1$

$X(f_2) = \tilde{X}(f_2)$ $\quad k=2$

$X(f_3) = \tilde{X}(f_3)$ $\quad k=3=\tilde{N}-1$

$X(f_4) = \tilde{X}(f_{2\tilde{N}-1-k})$

$= \tilde{X}(f_3)$ $\quad k=4=\tilde{N}$

$X(f_5) = \tilde{X}(f_{2\tilde{N}-1-k})$

$= \tilde{X}(f_2)$ $\quad k=5$

$X(f_6) = \tilde{X}(f_1)$ $\quad k=6=2\tilde{N}-2$

The total number of samples in $X(f_k)$ is $2\tilde{N}-1$

Note that the angle of $X(f)$ is odd symmetrical

$$\angle X(f_0) = \angle \tilde{X}(f_0), \quad k=0$$

$$\angle X(f_1) = \angle \tilde{X}(f_1), \quad k=1$$

$$\angle X(f_2) = \angle \tilde{X}(f_2), \quad k=2$$

$$\angle X(f_3) = \angle \tilde{X}(f_3), \quad k=3=\tilde{N}-1$$

$$\angle X(f_4) = (\text{minus}) \angle \tilde{X}(f_3)$$

$$\angle X(f_5) = (\text{minus}) \angle \tilde{X}(f_2)$$

$$\angle X(f_6) = (\text{minus}) \angle \tilde{X}(f_1)$$