Tripartite Entanglement of Matrix Multiplication

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1 From matrices to qudits

We consider the multiplication of two $N \times N$ matrices, $c_{ji} = \sum_{j=1}^{N} a_{jk}\, b_{ki}$. Let the uppercase indices represent pairs of lowercase indices, e.g., $I = (i_1, i_2)$. The matrix multiplication is described by a certain 3-tensor C_{IJK} as $c_I = \sum_{J,K=1}^{N^2} C_{IJK}\, a_J\, b_K$.

This double-index notation $I = (i_1, i_2)$ hides the special definition of the matrix multiplication tensor, which is shown in the following equation (1). To reveal this, we reinstate the row and column indices of each matrix $(a_{ii'}, b_{jj'}, c_{kk'})$, instead of combining them to a single index. Then, the matrix multiplication tensor becomes a 6-tensor, defined as

$$c_{i'i} = \sum_{i,i',j,j'=1}^{N} C_{ii'jj'kk'}\, a_{jj'}\, b_{kk'} \ , \qquad C_{ii'jj'kk'} = \delta_{i'j}\, \delta_{j'k}\, \delta_{k'i} \ . \tag{1}$$

We use this tensor to introduce a quantum state of six N-level qudits, or three N^2-level qudits,

$$\begin{aligned} |\mathrm{MM}_N\rangle &= \frac{1}{N^{3/2}} \sum_{i,i',j,j',k,k'=1}^{N} C_{ii'jj'kk'}\, |i\rangle \otimes |i'\rangle \otimes |j\rangle \otimes |j'\rangle \otimes |k\rangle \otimes |k'\rangle \\ &= \frac{1}{N^{3/2}} \sum_{i,i',j,j',k,k'=1}^{N} C_{ii'jj'kk'}\, |i, i'\rangle \otimes |j, j'\rangle \otimes |k, k'\rangle \ . \end{aligned} \tag{2}$$

For example, the 2×2 matrix multiplication is described by the 6-qubit state

$$
\begin{aligned}
|\mathrm{MM}_2\rangle = \tfrac{1}{\sqrt{8}} \big(& |00\rangle \otimes |00\rangle \otimes |00\rangle + |01\rangle \otimes |10\rangle \otimes |00\rangle \\
& + |00\rangle \otimes |01\rangle \otimes |10\rangle + |01\rangle \otimes |11\rangle \otimes |10\rangle \\
& + |10\rangle \otimes |00\rangle \otimes |01\rangle + |11\rangle \otimes |10\rangle \otimes |01\rangle \\
& + |10\rangle \otimes |01\rangle \otimes |11\rangle + |11\rangle \otimes |11\rangle \otimes |11\rangle \big) \,.
\end{aligned} \tag{3}
$$

Next, we are interested in finding the optimal algorithm for matrix multiplication that uses the minimum number of *products*. Each product corresponds to a separable (non-entangled) state between the three N^2-level qudits.

Consider Strassen's algorithm [1] as an example,

$$
\begin{aligned}
p_1 &= (a_{11} + a_{22})\,(b_{11} + b_{22}) \ , \\
p_2 &= (a_{21} + a_{22})\,b_{11} \ , \\
p_3 &= (a_{11} + a_{12})\,b_{22} \ , \\
p_4 &= a_{11}\,(b_{12} - b_{22}) \ , \\
p_5 &= a_{22}\,(b_{11} - b_{21}) \ , \\
p_6 &= (a_{12} - a_{22})\,(b_{21} + b_{22}) \ , \\
p_7 &= (a_{11} - a_{21})\,(b_{11} + b_{12}) \ ,
\end{aligned}
\qquad
\begin{aligned}
c_{11} &= p_1 - p_3 - p_5 + p_6 \ , \\
c_{21} &= p_2 - p_5 \ , \\
c_{12} &= p_3 + p_4 \ , \\
c_{22} &= p_1 - p_2 + p_4 - p_7 \ .
\end{aligned} \tag{4}
$$

We match the matrix entries (which are indeterminates) to the N^2-level qudits as follows: (note that c_{ji} is transposed due to (1))

$$
\begin{array}{ccccc}
\underline{\text{1st qudit (“Charlie”)}} & & \underline{\text{2nd qudit (“Alice”)}} & & \underline{\text{3rd qudit (“Bob”)}} \\
c_{11} \leftrightarrow |00\rangle & , & a_{11} \leftrightarrow |00\rangle & , & b_{11} \leftrightarrow |00\rangle \ , \\
c_{21} \leftrightarrow |01\rangle & , & a_{12} \leftrightarrow |01\rangle & , & b_{12} \leftrightarrow |01\rangle \ , \\
c_{12} \leftrightarrow |10\rangle & , & a_{21} \leftrightarrow |10\rangle & , & b_{21} \leftrightarrow |10\rangle \ , \\
c_{22} \leftrightarrow |11\rangle & , & a_{22} \leftrightarrow |11\rangle & , & b_{22} \leftrightarrow |11\rangle \ .
\end{array} \tag{5}
$$

Then, we can read off each product from Strassen's algorithm (4),

$$
\begin{aligned}
|p_1\rangle &= \tfrac{1}{2^{3/2}}\, (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \,, \\
|p_2\rangle &= \tfrac{1}{2}\, (|0\rangle - |1\rangle) \otimes |1\rangle \otimes |1\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle \otimes |0\rangle \ , \\
|p_3\rangle &= \tfrac{1}{2}\, (|0\rangle - |1\rangle) \otimes |0\rangle \otimes |0\rangle \otimes (|0\rangle + |1\rangle) \otimes |1\rangle \otimes |1\rangle \ , \\
|p_4\rangle &= \tfrac{1}{2}\, |1\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle \otimes |0\rangle \otimes (|0\rangle - |1\rangle) \otimes |1\rangle \ , \\
|p_5\rangle &= \tfrac{1}{2}\, |0\rangle \otimes (|0\rangle + |1\rangle) \otimes |1\rangle \otimes |1\rangle \otimes (|0\rangle - |1\rangle) \otimes |0\rangle \ , \\
|p_6\rangle &= \tfrac{1}{2}\, |0\rangle \otimes |0\rangle \otimes (|0\rangle - |1\rangle) \otimes |1\rangle \otimes |1\rangle \otimes (|0\rangle + |1\rangle) \,, \\
|p_7\rangle &= \tfrac{1}{2}\, |1\rangle \otimes |1\rangle \otimes (|0\rangle - |1\rangle) \otimes |0\rangle \otimes |0\rangle \otimes (|0\rangle + |1\rangle) \,.
\end{aligned} \tag{6}
$$

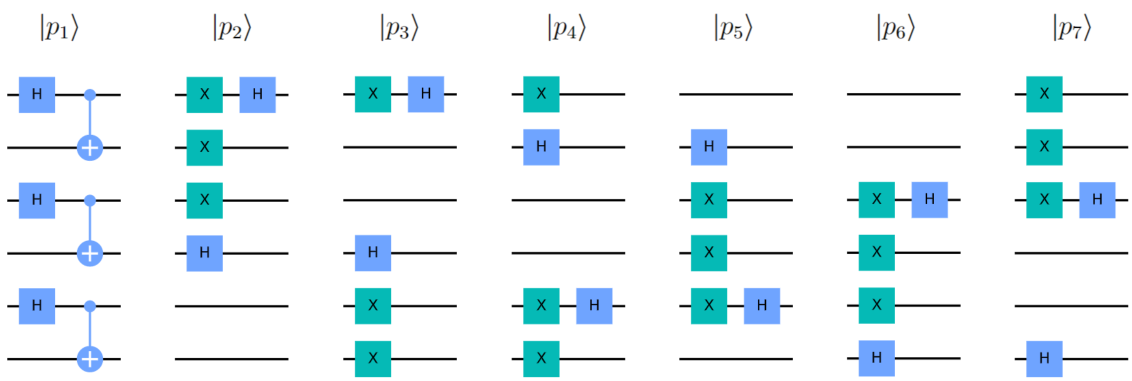
Strassen's algorithm is the statement that $|\mathrm{MM}_2\rangle$ is a superposition of these 7 vectors,

$$|\mathrm{MM}_2\rangle = |p_1\rangle + \tfrac{1}{\sqrt{2}}\,|p_2\rangle - \tfrac{1}{\sqrt{2}}\,|p_3\rangle + \tfrac{1}{\sqrt{2}}\,|p_4\rangle - \tfrac{1}{\sqrt{2}}\,|p_5\rangle + \tfrac{1}{\sqrt{2}}\,|p_6\rangle - \tfrac{1}{\sqrt{2}}\,|p_7\rangle\ . \tag{7}$$

Looking at the seven states in (6) closely, they appear as particularly symmetric: The red parts are Bell states, while the blue parts are X-eigenstates. Only for $|p_1\rangle$, all three of Alice, Bob, and Charlie have a (maximally entangled) Bell state, whereas for $|p_{2,\ldots,7}\rangle$, Alice, Bob, and Charlie each have separable states.

(I would like to suggest that the 'internal entanglement' between the two qubits in the hands of Alice, Bob, and Charlie is *"distilled"* into $|p_1\rangle$.)

Each of $|p_{1,\ldots,7}\rangle$ can be obtained from $|0\rangle^{\otimes 6}$ with a quantum circuit that only contains the X, Hadamard, and C-NOT gates:



However, this symmetric construction might be special to Strassen's 2×2 algorithm ($\omega = 2.807$) [1]. Pan's 70×70 exact multiplication algorithm with 143640 products ($\omega = 2.795$) [2], or Schönhage's 3×3 approximate multiplication algorithm with 21 products ($\omega = 2.771$) [3] do not appear to have a similar level of simplicity – at least not at a first glance.

2 Tripartite entanglement

Finding the optimal matrix multiplication algorithm is a special case of *tensor rank decomposition* (TRD) for certain 3-tensors. TRD can be done in polynomial time for 2-tensors, whereas it is an NP-hard problem for 3-tensors.

In the spirit of previous section, we reformulate the TRD problem as follows: *If we are given a generic* ***multi-partite*** *entangled state, how can we find the minimum number of separable states for which the given state is a superposition of them?*

Hence, part of the additional complexity of TRD for 3-tensors compared to 2-tensors can be attributed to the differences between bipartite and tripartite entanglement.

In the current discussion, we will use the *2-concurrence* (Tsallis-2 entropy) as our (bipartite) entanglement monotone, and denote it by $\mathcal{E}$. (Compare to [4] for other entanglement monotones.) Given a bipartite entangled pure state $|\phi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, the 2-concurrence $\mathcal{E}$ is defined as

$$\mathcal{E}(|\phi\rangle_{AB}) = 1 - \mathrm{Tr}\big(\rho_A^2\big) \ , \qquad \rho_A = \mathrm{Tr}_B(|\phi\rangle_{AB}\langle\phi|) \ . \tag{8}$$

There is no single quantity for tripartite entanglement because there are multiple ways in which tripartite entanglement can exist [5]: as biseparable states, or as genuine tripartite entanglement. Here, I like the approach from [4] to quantify tripartite entanglement as a *triplet* of bipartite entanglement monotones, which we will use in the following.

Given a pure state $|\phi\rangle_{ABC} \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ with tripartite entanglement, there are three ways to split it into two subsystems: $A|BC$, $B|CA$, and $C|AB$. We calculate the bipartite entanglement for each splitting using the 2-concurrence $\mathcal{E}$, and take the triplet $(\mathcal{E}^{A|BC}, \mathcal{E}^{B|CA}, \mathcal{E}^{C|AB})$ as measure for the tripartite entanglement. [4] shows that this triplet satisfies the triangle inequalities, and therefore it can be viewed as the side lengths of a triangle:

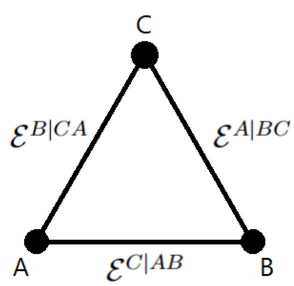


Figure 1: The triplet $(\mathcal{E}^{A|BC}, \mathcal{E}^{B|CA}, \mathcal{E}^{C|AB})$ satisfies the triangle inequalities, thus it forms a triangle. Each corner of this triangle represents an N^2-level qudit, i.e., one of the three matrices in the matrix multiplication.

The state $|\phi\rangle_{ABC}$ is completely separable if and only if $\mathcal{E}^{A|BC} = \mathcal{E}^{B|CA} = \mathcal{E}^{C|AB} = 0$. Furthermore, the state $|\phi\rangle_{ABC}$ is biseparable if and only if the area of this triangle is zero [4].

The area S of the triangle is given by Heron's formula,

$$\begin{aligned} S^2 = \tfrac{1}{16}\left(\mathcal{E}^{A|BC} + \mathcal{E}^{B|CA} + \mathcal{E}^{C|AB}\right)\left(-\mathcal{E}^{A|BC} + \mathcal{E}^{B|CA} + \mathcal{E}^{C|AB}\right) \\ \times\left(\mathcal{E}^{A|BC} - \mathcal{E}^{B|CA} + \mathcal{E}^{C|AB}\right)\left(\mathcal{E}^{A|BC} + \mathcal{E}^{B|CA} - \mathcal{E}^{C|AB}\right) \ . \end{aligned} \tag{9}$$

We can use this S as a measure for genuine tripartite entanglement, which is insensitive to the entanglement of biseparable states.

If we can reduce the $S(|\mathrm{MM}_N\rangle)$ to zero, then we will be reducing the TRD of a 3-tensor to the TRD of a 2-tensor, which is solvable in polynomial time.

3 Entanglement on the quotient

Let $f : \mathcal{H} \to \mathbb{R}_{\geq 0}$ be a non-negative function on the Hilbert space $\mathcal{H}$. Let $\mathcal{G} \subset \mathcal{H}$ be a subspace of $\mathcal{H}$. Then, we define a new function $f_{\mathcal{H}/\mathcal{G}} : \mathcal{H} \to \mathbb{R}_{\geq 0}$ as

$$f_{\mathcal{H}/\mathcal{G}}(x) \equiv \min_{y\in\mathcal{G}} f(x+y) \ , \qquad x \in \mathcal{H} \ . \tag{10}$$

Now let's put everything together. We consider the state

$$\begin{aligned} &|\mathrm{MM}_N\rangle \equiv \frac{1}{N^{3/2}} \sum_{i,j,k=1}^{N} |i,j\rangle \otimes |j,k\rangle \otimes |k,i\rangle \ , \\ &|\mathrm{MM}_N\rangle \in \mathcal{H} \equiv \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \ , \end{aligned} \tag{11}$$

where $\mathcal{H}_{A,B,C}$ are N^2-dimensional Hilbert spaces. For any $|\alpha\rangle \in \mathcal{H}_A$, $|\beta\rangle \in \mathcal{H}_B$, $|\gamma\rangle \in \mathcal{H}_C$, we also consider the one-dimensional subspace,

$$\mathcal{G}_{(|\alpha\rangle,|\beta\rangle,|\gamma\rangle)} = \{\xi\,|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \in \mathcal{H} \mid \xi \in \mathbb{R}\} \subset \mathcal{H} \ , \tag{12}$$

and for any number of such triplets $(|\alpha_\mu\rangle, |\beta_\mu\rangle, |\gamma_\mu\rangle)$, we consider their linear span,

$$\mathcal{G} = \bigoplus_{\mu=1}^{M} \mathcal{G}_{(|\alpha_\mu\rangle,|\beta_\mu\rangle,|\gamma_\mu\rangle)} \ . \tag{13}$$

For any M, the quantity

$$\min_{\alpha_\mu,\beta_\mu,\gamma_\mu} S_{\mathcal{H}/\mathcal{G}}(|\mathrm{MM}_N\rangle) \tag{14}$$

tells how much we can possibly eliminate the genuine tripartite entanglement in the $N \times N$ matrix multiplication by dividing out M separable states.

Hence, we introduce a new cost function based on the tripartite entanglement as

$$\begin{aligned}&\mathcal{J}_3(|\alpha_\mu\rangle \in \mathcal{H}_A, |\beta_\mu\rangle \in \mathcal{H}_B, |\gamma_\mu\rangle \in \mathcal{H}_C, \xi_\mu \in \mathbb{R}) \\ &\quad = S\left(|\mathrm{MM}_N\rangle - \sum_{\mu=1}^{M} \xi_\mu \, |\alpha_\mu\rangle \otimes |\beta_\mu\rangle \otimes |\gamma_\mu\rangle\right) \ . \end{aligned} \tag{15}$$

Once we find a solution to $\mathcal{J}_3 = 0$, the argument of S in (15) will be a biseparable state, and we will only have a matrix left to decompose.

Comparison to the previous bipartite approach

The approach in this note preserves and emphasizes the trilinear nature of the matrix multiplication problem. In the previous "(bipartite) entanglement approach" [12/02], we considered Charlie as collapsing the tripartite state into an ensemble of N^2 bipartite states,

$$|\mathrm{MM}_N\rangle = \sum_{\ell=1}^{N^2} |\ell\rangle_C \otimes |\psi_\ell\rangle_{AB} \ . \tag{16}$$

Then, the cost function

$$\mathcal{J}_2 = \sum_{m=1}^{N^2} \mathcal{E}\left(\sum_{\ell=1}^{N^2} T_{m\ell} \left(|\psi_\ell\rangle_{AB} - \sum_{\mu=1}^{M} \xi_{\ell,\mu} \, |\alpha_\mu\rangle \otimes |\beta_\mu\rangle\right)\right) \ , \tag{17}$$

required optimizing not only over the qudits $|\alpha_\mu\rangle, |\beta_\mu\rangle$ and the coefficients $\xi_{\ell,\mu}$, but also over the matrix $T_{m\ell}$ which represents the different ways in which Charlie can choose a basis for his part to collapse the tripartite state.

In conclusion, we propose here the approach (15) as an alternative to (17). This exposition strengthens the connection between classical multiplication algorithms and quantum information that we are building in these reports.

References

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