MA2104

AY24/25 SEM 1

yyccbb

Vectors

- **a** · **b** $= \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$
- $\text{comp}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$
- The vector projection: $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{a}}\mathbf{b} \times \frac{\mathbf{a}}{\|\mathbf{a}\|}$
- $\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$
- $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$
- $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = volume of parallelepiped.
- Representations of a line: vector/parametric.

Vector-valued Function

- $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$
- **r**(t) is a parametrization of curve *C*
- $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}(t)\times\mathbf{s}(t)) = \mathbf{r}'(t)\times\mathbf{s}(t) + \mathbf{r}(t)\times\mathbf{s}'(t)$
- **Arc length**:
  - Assumption: continuous, traversing exactly once.
  - $S = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|\,dt$

Cylinders and Quadric Surfaces

- **Cylinder** $\rightarrow$ ∃plane P such that all the planes parallel to P interset the surface in the same curve.
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$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$	elliptic paraboloid
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$	hyperbolic paraboloid
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	ellipsoid
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	elliptic cone
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	hyperboloid of one sheet
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$	hyperboloid of two sheets

Limits and Continuity

- **Continuous** $\rightarrow$ f is continuous at (a,b) if $\lim_{(x,y)\to(a,b)}$ exists and is equal to f(a,b).
- A limit at (a,b) does not exist if the limits of f when approached from different paths do not agree.
- **How to prove a limit exists?** go polar!
  - 2D: $x = r\cos\theta + a, y = r\sin\theta + b, r\to 0^+$ when approaching (a,b)
  - 3D: $x = p\sin\phi\cos\theta, y = p\sin\phi\sin\theta, z = p\cos\phi, p\to 0^+$
  - Squeeze Theorem:
  - Since $\frac{x^4}{x^4+y^4} \le 1, 0 \le \left|\frac{x^4y}{x^4+y^4}\right| = \left|\frac{x^4}{x^4+y^4}y\right| \le |y|$

Partial Derivatives

PDE Examples

- Wave equation: $u_{tt}(x,t) = a^2u_{xx}(x,t)$
- Transport equation: $u_t(x,t) = u_x(x,t)$
- Heat diffusion: $u_t(x,t) = u_{xx}(x,t)$
- Laplace equation: $u_{xx}(x,y) + u_{yy}(x,y) = 0$

Tangent Planes and Linear Approximations

- Equation of tangent plane at (a,b): $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
- **Linearisation** of f at (a,b): L(x,y) = z
- f is **differentiable** at (a,b) if $f(x,y)\approx L(x,y)$
- Alternatively, f is differentiable if f_x and f_y exist near (a,b) and are continuous at (a,b).
- Total differential: $dz = f_x(x,y)dx + f_y(x,y)dy$

The Chain Rule

- $\frac{\mathrm{d}}{\mathrm{d}t}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t))\cdot\mathbf{r}'(t)$
- $\frac{\partial}{\partial t}f(\mathbf{r}(s,t)) = \nabla f(\mathbf{r}(s,t))\cdot\frac{\partial}{\partial t}\mathbf{r}(s,t)$

Implicit Differentiation

- Idea: $G(x,y) = F(x(x,y), y(x,y), z(x,y))$, $G_x/G_y = 0$
- $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$
- Alternatively, apply $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x}$.

Directional Derivatives

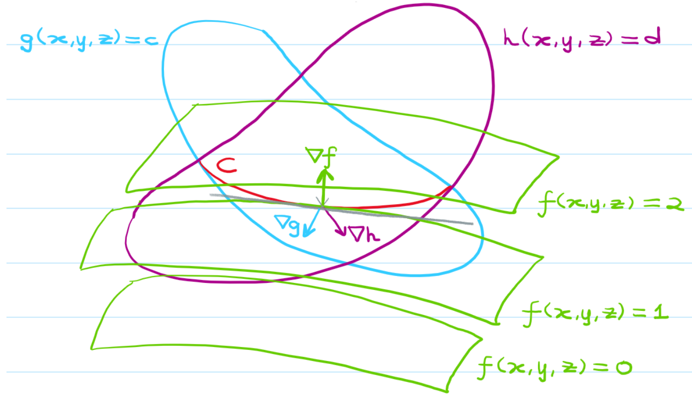
- $\nabla f(x_0,y_0)$ is normal to the level curve at (x_0,y_0).
- For **unit vector** $\vec{u}$, $D_{\vec{u}}f(x,y) = \nabla f(x,y)\cdot\vec{u}$.
- $\vec{u}, D_{\vec{u}}f(x,y) = \nabla f\cdot\vec{u} = |\nabla f|\cos\theta$

Extrema and Saddle Points, Lagrange Multipliers

Extrema and Saddle Points

- A point (a,b) is a **critical point** of f(x,y) if $\nabla f(a,b) = \vec{0}$ or if one of the partial derivatives does not exist.
- A **saddle point** is **NOT** a local extremum.
- (2nd Derivative Test) $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$
- $D < 0$: saddle point.
- $D > 0, f_{xx} > 0$: minimum.
- $D > 0, f_{xx} < 0$: maximum.
- $D = 0$: no information.

Lagrange Multipliers

- Scenario: optimise f(x,y) restricted to constraint g(x,y)=c
- $\nabla f \parallel \nabla g, \nabla f = \lambda\nabla g$
- System of equations: $\nabla f = \lambda\nabla g, g(x,y) = c$
- **Case of two constraints**
  - Idea: optimise along a constraint curve in 4D
  - 

  - $\nabla f = \lambda\nabla g + \mu\nabla h$ (because $\nabla f, \nabla g, \nabla h$ are coplanar).