Factorial Function

UNIVERSITY OF COLORADO AT BOULDER

APPM 4450

UNDERGRADUATE APPLIED ANALYSIS 2

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1. Mathematical Preliminaries. The contents of this exploration deal heavily with functions of 2 variables on rectangular domains. Let's begin with a working definition of these two-variable functions and some familiar notions we'll use throughout the project.

We first define the domain of functions of two-variables to be in $\mathbf{R}^2$.

DEFINITION 1.1 (Rectangle Domain of a 2D Function). Let $f(x, t)$ be a function of two variables, defined for all $a \le x \le b$ and $c \le t \le d$. The domain of f is then called a *rectangle* D in $\mathbf{R}^2$.

We then define a notion of distance between two points in $\mathbf{R}^2$.

DEFINITION 1.2 (Euclidean Distance Formula). The *distance* between two points (x_0, t_0) and (x, t) in $\mathbf{R}^2$ is defined as:

$$||(x,t) - (x_0, t_0)|| = \sqrt{(x - x_0)^2 + (t - t_0)^2}$$

We use this to establish a definition of continuity and uniform continuity:

DEFINITION 1.3 (Continuity of a 2-Variable Function (Definition 8.4.4 in Abbott [1])). A function $f : D \to \mathbf{R}$ is continuous at (x_0, t_0) if for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $||(x, t) - (x_0, t_0)|| < \delta$ it follows that

$$\left| f(x,t) - f(x_0, t_0) \right| < \epsilon$$

If there exists such a delta for each $(x_0, t_0) \in D$, then we say f is continuous on D.

DEFINITION 1.4 (Uniform Continuity of a 2-Variable Function). A function $f : D \to \mathbf{R}$ is called *uniformly continuous* on D if for each $(x, t), (y, s) \in D$ and for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $||(x, t) - (y, s)|| < \delta$ it follows that

$$\left| f(x,t) - f(y, s) \right| < \epsilon$$

THEOREM 1.5 (Uniform Continuity on Compact Sets 2D (extension of Theorem 4.4.7 in Abbott [1])). *A function $f : D \to \boldsymbol{R}$ that is continuous on a compact set D is uniformly continuous on D.*

Proof. This proof is nearly identical to the one done in class and is omitted here. Please review Theorem 4.4.7 in Abbott [1] and the proof we did in class. ∎

We then show that if we fix one of the variables of a 2D function, the resulting function inherits properties of continuity as a function of one variable.

THEOREM 1.6 (Continuity from 2D to 1D). *If a function $f : D \to \boldsymbol{R}$ is continuous on the rectangle $D = \{(x, t) : a \le x \le b, c \le t \le d\}$, then for every $x_0 \in [a, b]$ the function $f(x_0, t)$ is continuous over $t \in [c, d]$. Similarly, for every $t_0 \in [c, d]$ the function $f(x, t_0)$ is continuous over $x \in [a, b]$.*

Proof. Let $f(x, t)$ be continuous over a rectangle $D = \{(x, t) : a \le x \le b, c \le t \le d\}$. Fix $x_0 \in [a, b]$ and $t_0 \in [c, d]$. Let $\epsilon > 0$ be fixed but arbitrary. Then, by Definition 1.3 there exists a $\delta > 0$ such that when $||(x, t) - (x_0, t_0)|| < \delta$ $(x, x_0, t, t_0 \in D)$ it follows that $|f(x, t) - f(x_0, t_0)| < \epsilon$.

First I show $f(x_0, t)$ is continuous over $t \in [c, d]$. Let $k \in [c, d]$. Note that for the same δ, $||(x_0, t) - (x_0, k)|| = \sqrt{(x_0 - x_0)^2 + (t - k)^2} = \sqrt{(t - k)^2} = |t - k| < \delta$. It immediately follows that $|f(x_0, t) - f(x_0, k)| < \epsilon$. Thus, $f(x_0, t)$ is continuous over $t \in [c, d]$. Note also that our choice of δ does not depend on what x_0 is.

Next I show $f(x, t_0)$ is continuous over $x \in [a, b]$. Let $k \in [a, b]$. Note that for the same δ, $||(x, t_0) - (k, t_0)|| = \sqrt{(x - k)^2 + (t_0 - t_0)^2} = \sqrt{(x - k)^2} = |x - k| < \delta$. It immediately follows that $|f(x, t_0) - f(k, t_0)| < \epsilon$. Thus, $f(x, t_0)$ is continuous over $x \in [a, b]$. Note also that our choice of δ does not depend on what t_0 is. □

Theorem 1.7 (Uniform Continuity from 2D to 1D). *If a function $f(x, t)$ is uniformly continuous (in the 2-variable sense) over a rectangle $D = \{(x, t) : a \leq x \leq b, c \leq t \leq d\}$, then for every x_0 in D the function $f(x_0, t)$ is uniformly continuous (in the 1-variable sense) over $t \in [c, d]$. Similarly, for every t_0 in D the function $f(x, t_0)$ is uniformly continuous over $x \in [a, b]$.*

Proof. Let $f(x, t)$ be uniformly continuous over a rectangle $D = \{(x, t) : a \leq x \leq b, c \leq t \leq d\}$. Fix $x_0 \in [a, b]$ and $t_0 \in [c, d]$. Note that by Theorem 1.6 $f(x, t_0)$ and $f(x_0, t)$ are continuous. But $f(x, t_0)$ and $f(x_0, t)$ are continuous on compact sets, so they are uniformly continuous by Theorem 4.4.7 in Abbott [1]. □

In our study of the factorial function, we'll be working with functions of one variable defined by integrating a 2-variable function.

Exercise 1.8 (Exercise 8.4.12 in Abbott [1]). Assume the function $f(x, t)$ is continuous on the rectangle $D = \{(x, t) : a \leq x \leq b, c \leq t \leq d\}$. Explain why the function

$$F(x) = \int_c^d f(x, t)dt$$

is properly defined for all $x \in [a, b]$.

Proof. Let $f(x, t)$ be continuous on the rectangle $D = \{(x, t) : a \leq x \leq b, c \leq t \leq d\}$. Define $F(x) = \int_c^d f(x, t)dt$. Let $x_0 \in [a, b]$ be fixed but arbitrary. Then, $f(x_0, t)$ is a continuous function of t by Theorem 1.6. By Theorem 7.2.9 in Abbott [1], $\int_c^d f(x_0, t)dt$ is well defined. Thus, $F(x_0) = \int_c^d f(x_0, t)dt$ is well defined. Since x_0 was an arbitrary element of the interval $[a, b]$, $F(x) = \int_c^d f(x, t)dt$ is properly defined for all $x \in [a, b]$. □

Like other functions of one variable, these functions can be uniformly continuous.

Theorem 1.9 (Uniform Continuity (Theorem 8.4.5 in Abbott [1])). *If f(x,t) is continuous on D, then $F(x) = \int_c^d f(x, t)dt$ is uniformly continuous on $[a, b]$.*

Proof. Let $F(x) = \int_c^d f(x, t)dt$ and assume $f(x, t)$ is continuous on the rectangle $D = \{(x, t) : a \leq x \leq b, c \leq t \leq d\}$. By Theorem 1.9, $f(x, t)$ is uniformly continuous on D. So, there exists a $\delta > 0$ such that when $||(x, t) - (y, s)|| < \delta$ it follows $|f(x, t) - f(y, s)| < \frac{\epsilon}{2(d-c)}$ for all $x, y, t, s \in D$.
Note:

$$
\begin{aligned}
|F(x)-F(y)| &= \left|\int_c^d f(x,t)dt - \int_c^d f(y,t)dt\right| \\
&= \left|\int_c^d f(x,t)-f(y,t)dt\right| \\
&\leq \int_c^d |f(x,t)-f(y,t)|dt && \text{(By Theorem 7.4.2 (v) in Abbott [1])} \\
&\leq \int_c^d \frac{\epsilon}{2(d-c)}dt && \text{(By Theorem 7.4.2 (iv) in Abbott [1])} \\
&= \frac{\epsilon}{2} && \text{(By Theorem 7.4.2 (iii) in Abbott [1])} \\
&< \epsilon
\end{aligned}
$$

Thus, $F(x)$ is uniformly continuous on $[a,b]$. ◻

We will also be working with partial derivatives of functions of 2-variables.

Definition 1.10. For each fixed value of t in $[c,d]$, the function $f(x,t)$ is a differentiable function of x if

$$f_x(x,t) = \lim_{z\to x} \frac{f(z,t)-f(x,t)}{z-x}$$

exists for all $(x,t) \in D$.

We can relate the derivative of $F(x)$ to the derivative of $f(x,t)$ in an intuitive way from our definition of $F(x)$ seen in 1.8.

Theorem 1.11 (Differentiation Under the Integral (Theorem 8.4.6 in Abbott [1])). *If $f(x,t)$ and $f_x(x,t)$ are continuous on D, then the function $F(x) = \int_c^d f(x,t)dt$ is differentiable and*

$$F'(x) = \int_c^d f_x(x,t)dt$$

Proof. Let $f(x,t)$ and $f_x(x,t)$ be continuous on D. Define $F(x) = \int_c^d f(x,t)dt$ (which we know exists by Exercise 1.8). Let $\epsilon > 0$ and fix $x_0 \in [a,b]$. I'll show there exists a $\delta > 0$ such that

$$\left|\frac{F(z)-F(x_0)}{z-x_0} - \int_c^d f_x(x_0,t)dt\right| < \epsilon$$

whenever $0 < |z-x_0| < \delta$.
Note:

$$\begin{aligned}
\left|\frac{F(z)-F(x_0)}{z-x_0}-\int_c^d f_x(x_0,t)dt\right| &= \left|\frac{\int_c^d f(z,t)dt-\int_c^d f(x_0,t)dt}{z-x_0}-\int_c^d f_x(x_0,t)dt\right| \\
&= \left|\int_c^d \frac{f(z,t)-f(x_0,t)}{z-x_0}dt-\int_c^d f_x(x_0,t)dt\right| \\
&= \left|\int_c^d \frac{f(z,t)-f(x_0,t)}{z-x_0}-f_x(x_0,t)dt\right| \\
&\leq \int_c^d \left|\frac{f(z,t)-f(x_0,t)}{z-x_0}-f_x(x_0,t)\right| dt
\end{aligned}$$

Note that since $f_x(x,t)$ exists on D, for the same ϵ there exists a $\delta > 0$ such that when $0 < |z-x_0| < \delta$ it follows $\left|\frac{f(z,t)-f(x_0,t)}{z-x_0}-f_x(x_0,t)\right| < \frac{\epsilon}{2(d-c)}$
So,

$$\begin{aligned}
\int_c^d \left|\frac{f(z,t)-f(x_0,t)}{z-x_0}-f_x(x_0,t)\right| dt &\leq \int_c^d \left|\frac{\epsilon}{2(d-c)}\right| dt \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

So, we have found a $\delta > 0$ such that

$$\left|\frac{F(z)-F(x_0)}{z-x_0}-\int_c^d f_x(x_0,t)dt\right| < \epsilon$$

whenever $0 < |z-x_0| < \delta$.
Since x_0 was an arbitrary x-value in D, $F'(x)$ exists for all x in D and is equal to $\int_c^d f_x(x,t)dt$. ∎

2. Improper Integration. In this section we develop the theory to improper integrals. This theory is central to our definition of the factorial function used later on.

DEFINITION 2.1 (Definition 8.4.3 in Abbott [1]). Suppose $f(x,t)$ is defined on $D = \{(x,t) : x \in A, a \leq t\}$, and $f(x,t)$ is integrable with respect to t on every interval of the form $[a,b]$ (where $a \leq b$). Then define $\int_a^\infty f(x,t)dt$ to be

$$\lim_{b\to\infty} \int_a^b f(x,t)dt$$

provided the limit exists. In this case we say the improper integral $\int_a^\infty f(x,t)$ *converges*.

When considering improper integrals of 2-variable functions we can define a notion of uniform convergence. This extends the idea of just "convergence" of improper integrals by allowing us to say an improper integral with respect to t of a 2-variable function converges for all x in the domain of f.

DEFINITION 2.2 (Definition 8.4.7 in Abbott [1]). Given $f(x,t)$ defined on $D = \{(x,t) : x \in A, c \leq t\}$, assume $F(x) = \int_c^\infty f(x,t)dt$ exists for all $x \in A$. We say the improper integral *converges uniformly* to $F(x)$ on A if for all $\epsilon > 0$, there exists $M > c$ such that

$$\left| F(x) - \int_c^d f(x,t)dt \right| < \epsilon$$

for all $d \geq M$ and all $x \in A$

EXERCISE 2.3 (Exercise 8.4.10 in Abbott [1]).

(a) Use the properties of e^t to show $\int_0^\infty e^{-t}dt = 1$.
(b) Show $\frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}dt$ for all $\alpha > 0$

SOLUTION 2.4.

(a) By Theorem 8.1.10 in Abbott [1], let $f(t) = e^t$ and $g(t) = -t$. Then, $(f \circ g)g' = -e^t$. Also, $g(b) = -b$ and $g(0) = 0$.
So,

$$\begin{aligned}
\int_0^\infty e^{-t}dt &= \lim_{b\to\infty} \int_0^b e^{-t}dt \\
&= \lim_{b\to\infty} \int_0^{-b} -e^u du \\
&= \lim_{b\to\infty} -\int_0^{-b} e^u du \\
&= \lim_{b\to\infty} \int_{-b}^0 e^u du
\end{aligned}$$

We proved e^x is differentiable and $\frac{d}{dt}e^x = e^x$. So, by FTC

$$\begin{aligned}
\lim_{b\to\infty} \int_{-b}^0 e^u du &= \lim_{b\to\infty} e^0 - e^{-b} \\
&= \lim_{b\to\infty} 1 - e^{-b}
\end{aligned}$$

I claim $\lim_{b\to\infty} e^{-b} = 0$. In HW 6 problem 8 (i), it was proven $e > 1$. So, $\frac{1}{e} < 1$. Thus, $\lim_{b\to\infty} e^{-b} = \frac{1}{e^b} = 0$.

Thus, $\lim_{b\to\infty} 1 - e^{-b} = 1$, proving $\int_0^\infty e^{-t}dt = 1$.

(b) Let $\alpha > 0$. By Theorem 8.1.10 in Abbott [1], let $f(t) = e^t$ and $g(t) = -\alpha t$. Then, $(f \circ g)g' = -\alpha e^{-\alpha t}$. Also, $g(b) = -b\alpha$ and $g(0) = 0$.

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}dt &= \lim_{b\to\infty} \int_0^b \frac{-\alpha}{-\alpha} e^{-\alpha t}dt \\
&= \lim_{b\to\infty} \int_0^{-b\alpha} \frac{-1}{\alpha} e^u du \\
&= \lim_{b\to\infty} \frac{-1}{\alpha} \int_0^{-b\alpha} e^u du \\
&= \lim_{b\to\infty} \frac{1}{\alpha} \int_{-b\alpha}^0 e^u du \\
&= \lim_{b\to\infty} \frac{1}{\alpha}(1) && \text{(This follows from part (a))} \\
&= \frac{1}{\alpha}
\end{aligned}$$

So, $\frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}dt$ for all $\alpha > 0$

There is also an analogy to the Cauchy Criterion, but for Improper Integrals.

Theorem 2.5 (Cauchy Criterion for Improper Integrals). *The improper integral $\int_a^\infty f(x,t)dt$ converges uniformly if and only if it is Cauchy for improper integrals, meaning for every $\epsilon > 0$ there exists an $M \in \boldsymbol{N}$ where $M > a$ such that whenever $d > c \geq M$ it follows*

$$\left| \int_c^d f(x,t)dt \right| < \epsilon$$

Proof.

$\Rightarrow$ First, we'll prove the forward direction.

Assume the improper integral $F(x) = \int_a^\infty f(x,t)dt$ converges. Let $\epsilon > 0$ be fixed but arbitrary. Then, there exists an $M \in \mathbf{N}$ $(M > a)$ such that when $c \geq M$ it follows

$$\left| \int_a^c f(x,t)dt - F(x) \right| < \frac{\epsilon}{2}$$

Let d satisfy $d > c$ and note:

$$\begin{aligned}
\left|\int_c^d f(x,t)dt\right| &= \left|F(x) - F(x) + \int_a^c f(x,t)dt - \int_a^d f(x,t)dt\right| \\
&\leq \left|F(x) - \int_a^d f(x,t)dt\right| + \left|F(x) - \int_a^c f(x,t)dt\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

$\Leftarrow$ Next, we prove the backward direction.
Let $f(x,t)$ be defined on $D = \{(x,t) : x \in A, a \leq t\}$ and integrable on every interval of $[a,b] \subset [a,\infty)$ with respect to t. Assume for every $\epsilon > 0$ there exists an $M \in \mathbf{N}$ where $M > a$ such that whenever $d > c \geq M$ it follows $\left|\int_c^d f(x,t)dt\right| < \epsilon$. Let $\epsilon > 0$ be fixed but arbitrary. Define

$$F_n(x) = \int_a^{a+n} f(x,t)dt$$

for every $n \in \mathbf{N}$.
I claim $F_n(x)$ is a Cauchy sequence of functions (Theorem 6.2.5 in Abbott [1]). For the same ϵ, there exists an $M_0 \in \mathbf{N}$ where $M_0 > a$ such that when $d > c \geq M_0$ it follows $\left|\int_c^d f(x,t)dt\right| < \epsilon$. By the Archimedean Property of real numbers, there exists an $M \in \mathbf{N}$ such that $M > M_0 - a$. Note that for all $n, m \in \mathbf{N}$ where $n > m \geq M$ it follows $\left|\int_{a+m}^{a+n} f(x,t)dt\right| < \epsilon$ since $n \geq M > M_0 - a$ implies $n + a \geq M_0$ and $m \geq M > M_0 - a$ implies $m + a \geq M_0$. It's clear that $n + a > m + a \geq M_0$.
Note:

$$\begin{aligned}
\left|\int_{a+m}^{a+n} f(x,t)dt\right| &= \left|\int_a^{a+n} f(x,t)dt - \int_a^{a+m} f(x,t)dt\right| \\
&= |F_n(x) - F_m(x)| \\
&< \epsilon
\end{aligned}$$

So $F_n(x)$ converges uniformly (by Abbott Theorem 6.2.5 [1]) to some function $F(x)$ since it's a Cauchy sequence of functions.

Note that the real number d can be represented as $d = a + n + \delta$ where $n \in \mathbf{N} \cup \{0\}$ and $\delta > 0$ when $d > a$.

Note:

$$
\begin{aligned}
\left|F(x) - \int_a^d f(x,t)dt\right| &= \left|F(x) - \int_a^{a+n} f(x,t)dt - \int_{a+n}^{d} f(x,t)dt\right| \\
&= \left|F(x) - \int_a^{a+n} f(x,t)dt - \int_{a+n}^{a+n+\delta} f(x,t)dt\right| \\
&\leq \left|F(x) - \int_a^{a+n} f(x,t)dt\right| + \left|\int_{a+n}^{a+n+\delta} f(x,t)dt\right| \\
&= |F(x) - F_n(x)| + \left|\int_{a+n}^{a+n+\delta} f(x,t)dt\right|
\end{aligned}
$$

Since $F_n(x) \to F(x)$ there exists an M_1 such that when $n \geq M_1$ it follows $|F(x) - F_n(x)| < \frac{\epsilon}{3}$. By assumption there also exists an M_2 such that when $c > b \geq M_2 > a$ it follows $\left|\int_b^c f(x,t)dt\right| < \frac{\epsilon}{3}$.
By the Archimedean Property of the real numbers, there exists an $M_3 \in \mathbf{N}$ such that $M_3 > M_2 - a$, which means $a + M_3 + \delta > a + M_3 \geq M_2$.
Let $M = \max(M_1, M_3)$.
So, for all $n \geq M$ it follows that $\left|\int_{a+n}^{a+n+\delta} f(x,t)dt\right| < \frac{\epsilon}{3}$. Thus,

$$
|F(x) - F_n(x)| + \left|\int_{a+n}^{a+n+\delta} f(x,t)dt\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} < \epsilon \qquad \square
$$

Definition 2.6 (Definition 8.4.7 in Abbott [1]). Given $f(x,t)$ defined on $D = \{(x,t) : x \in A, c \leq t\}$, assume $F(x) = \int_c^\infty f(x,t)dt$ exists for all $x \in A$. We say the improper integral *converges uniformly* to $F(x)$ on A if for all $\epsilon > 0$, there exists $M > c$ such that

$$
\left|F(x) - \int_c^d f(x,t)dt\right| < \epsilon
$$

for all $d \geq M$ and all $x \in A$

Here is a useful theorem we will need later on:

Theorem 2.7 (Uniform Convergence of a Sequence of Integral Functions). *If the improper integral $\int_a^\infty f(x,t)$ converges uniformly then $F_n(x) = \int_a^{a+n} f(x,t)$ where $n \in \mathbf{N}$ also converges uniformly, and its limit is $\int_a^\infty f(x,t)$.*

Proof. Assume $\int_a^\infty f(x,t)$ converges. Let $F_n(x) = \int_a^{a+n} f(x,t)$ where $n \in \mathbf{N}$. Let $\epsilon > 0$. Since $\int_a^\infty f(x,t)$ converges, for this ϵ there exists an $M \in \mathbf{N}$ where for $b \geq M > a$ it follows $|\int_a^b f(x,t) - \int_a^\infty f(x,t)| < \epsilon$. By the Archimedean Property of real numbers, there is an $N \in \mathbf{N}$ where $N > M - a$. Note that it follows $N + a > M$.

For $n \geq N$ where $n \in \mathbf{N}$ it follows:

$$\left|\int_a^{n+a} f(x,t) - \int_a^{\infty} f(x,t)\right| < \epsilon$$
$$\left|F_n(x) - \int_a^{\infty} f(x,t)\right| < \epsilon$$

□

Like functions of 1-variable we've studied in class, the domains we consider affect whether or not we get uniform convergence of improper integrals.

EXERCISE 2.8 (Exercise 8.4.15 in Abbott [1]).

(a) Show that the improper integral $\int_0^{\infty} e^{-xt}dt$ converges uniformly to $1/x$ on the set $[1/2, \infty)$.

(b) Is the convergence uniform on $(0, \infty)$?

Proof. Let $x_0 \in (0, \infty)$ be fixed but arbitrary. Let $\epsilon > 0$ be fixed but arbitrary. Consider $x \in [x_0, \infty)$. By Theorem 8.1.10 in Abbott [1], let $f(t) = e^t$ and $g(t) = -xt$, then $(f \circ g)g' = -xe^{-xt}$. Note $g(b) = -bx$ and $g(0) = 0$.
Note:

$$\begin{aligned}
\int_0^b e^{-xt}dt &= \int_0^{-bx} e^u \frac{1}{-x} du \\
&= \frac{-1}{x} \int_0^{-bx} e^u du \\
&= \frac{-1}{x} \left[e^u \Big|_0^{-bx} \right] \\
&= \frac{-1}{x} \left[e^{-bx} - 1 \right] \\
&= \frac{1}{x} - \frac{e^{-bx}}{x}
\end{aligned}$$

Note for all $x \in [x_0, \infty)$, $\frac{-e^{-x}}{x}$ is an increasing function. This is since its derivative is positive at all values of $x \in [x_0, \infty)$ since $\frac{1}{x} > 0$ and $e^{-x} > 0$ for $x > 0$ (this shows it's increasing by a theorem proven in class):

$$\begin{aligned}
\frac{d}{dx} \frac{-e^{-x}}{x} &= \frac{d}{dx} \frac{1}{x} \times -e^{-x} \\
&= \frac{-1}{x^2}(-e^{-x}) + \left(\frac{1}{x} e^{-x}\right) \\
&= \frac{1}{x^2} e^{-x} + \frac{1}{x} e^{-x}
\end{aligned}$$

So, $\frac{e^{-x}}{x}$ is a decreasing function for $x \in [x_0, \infty)$.
Note:

$$\begin{aligned}
\left| \frac{1}{x} - \int_0^b e^{-xt}dt \right| &= \left| \frac{1}{x} - \frac{1}{x} + \frac{e^{-bx}}{x} \right| \\
&= \left| \frac{e^{-bx}}{x} \right|
\end{aligned}$$

Since $\frac{e^{-x}}{x}$ is a decreasing function for $x \in [x_0, \infty)$:

$$
\begin{aligned}
&= \frac{e^{-bx}}{x} \\
&\leq \frac{e^{-bx_0}}{x_0} \\
&= \frac{1}{x_0 e^{bx_0}}
\end{aligned}
$$

By the Archimedean Property of real numbers, there exists an $M \in \mathbf{N}$ such that $\frac{1}{x_0 e^{bx_0}} < \epsilon$ for $b \geq M$. This proves $\int_0^\infty e^{-xt}dt$ converges uniformly to $1/x$ on the set $[x_0, \infty)$ for every $x_0 \in (0, \infty)$.
Thus, part (a) is proven when choosing $x_0 = \frac{1}{2}$.

Part (b) is not true:
Let $\epsilon = 1$ and let $b \in (0, \infty)$ be fixed but arbitrary. Then,

$$
\left| \frac{1}{x} - \int_0^b e^{-xt}dt \right| = \frac{e^{-bx}}{x} < \epsilon
$$

becomes

$$
\frac{1}{xe^{bx}} < 1
$$

Let $x = \frac{1}{n}$, where $n \in \mathbf{N}$ (note $\frac{1}{n} \in (0, \infty)$ for $n \geq 1$).

$$
\frac{n}{e^{b/n}} < 1 \tag{2.1}
$$

In Homework 7 we proved e^x is differentiable, so it must be continuous. Thus,

$$
\lim_{n\to\infty} \frac{1}{e^{b \times \frac{1}{n}}} = \frac{1}{e^{\lim_{n\to\infty} b \cdot \frac{1}{n}}} = \frac{1}{e^0} = 1
$$

So, there exists an $N \in \mathbf{N}$ where for $n \geq N$ we get:

$$
\begin{aligned}
\left| \frac{1}{e^{b/n}} - 1 \right| &< \frac{1}{4} \\
\frac{-1}{4} < \frac{1}{e^{b/n}} - 1 &< \frac{1}{4} \\
\frac{3}{4} < \frac{1}{e^{b/n}} &< \frac{5}{4} \\
\frac{3n}{4} < \frac{n}{e^{b/n}} &< \frac{5n}{4}
\end{aligned} \tag{2.2}
$$

So, combining equations (2.1) and (2.2) we get:

$$
\frac{3n}{4} < \frac{n}{e^{b/n}} < 1
$$

Note we have a contradiction when $n = \max(N, 2)$, since $1 < \frac{3n}{4} < 1$
Thus, there is no $M \in \mathbf{N}$ where for $n \geq M$, $\frac{n}{e^{b/n}} < 1$, since eventually we have a contradiction. This proves the convergence of $\int_0^\infty e^{-xt}dt$ to $1/x$ is not uniform on $(0, \infty)$. $\square$

Like for functions of 1-variable, there is an analogue of the Weierstrass M-Test for showing uniform convergence of improper integrals.

Exercise 2.9 (Analogue of the Weierstrass M-Test for Improper Integrals (Exercise 8.4.16 in Abbott [1])). We will show if $f(x,t)$ satisfies $|f(x,t)| \leq g(t)$ for all $x \in A$ and $\int_a^\infty g(t)dt$ converges, then $\int_a^\infty f(x,t)dt$ converges uniformly on A.

Proof. Assume $|f(x,t)| \leq g(t)$ for all $x \in A$ and $\int_a^\infty g(t)dt$ converges. Let $\epsilon > 0$ be fixed but arbitrary. Since $\int_a^\infty g(t)dt$ converges, for this $\epsilon > 0$ there exists an $N \in \mathbf{N}$ where for $b \geq N$ it follows that:

$$\left|\int_a^\infty g(t)dt - \int_a^b g(t)dt\right| < \epsilon$$

Then:

$$\begin{aligned}
\left|\int_a^\infty f(x,t)dt - \int_a^b f(x,t)dt\right| &= \left|\int_b^\infty f(x,t)dt\right| \\
&\leq \int_b^\infty |f(x,t)|dt \\
&\leq \int_b^\infty g(t)dt \\
&= \int_a^\infty g(t)dt - \int_a^b g(t)dt \\
&= \left|\int_a^\infty g(t)dt - \int_a^b g(t)dt\right| \qquad \text{(since } g(t) > 0\text{)} \\
&< \epsilon
\end{aligned}$$

□

We can use the notion of uniform convergence of improper integrals to prove uniform continuity of the limit function $F(x)$.

Theorem 2.10 (Uniform Continuity of Improper Integrals (Theorem 8.4.8 in Abbott [1])). *If f(x,t) is continuous on $D = \{(x,t) : a \leq x \leq b, c \leq t\}$ then*

$$F(x) = \int_c^\infty f(x,t)dt$$

is uniformly continuous on $[a,b]$, provided the integral converges uniformly.

Proof. Let $f(x,t)$ be continuous on $D = \{(x,t) : a \leq x \leq b, c \leq t\}$. Let $F(x) \to \int_c^\infty f(x,t)dt$ uniformly. So, for every $\epsilon > 0$ there exists an $M_1 \in \mathbf{N}$ ($M_1 > c$) where for all $b \geq M_1$ it follows $\left|F(x) - \int_c^b f(x,t)dt\right| < \frac{\epsilon}{3}$ for all $x \in [a,b]$. Note $|f(x,t) - f(y,t)| < \frac{\epsilon}{3(M_1 - c)}$ for some $\delta > 0$ whenever $|x - y| < \delta$ by Theorem 1.7.

Note for the same $\delta > 0$, $|x-y| < \delta$, and M_1:

$$\begin{aligned}
|F(x) - F(y)| =& \left| F(x) - \int_c^{M_1} f(x,t)dt \right. \\
&+ \int_c^{M_1} f(x,t)dt - \int_c^{M_1} f(y,t)dt \\
&\left. + \int_c^{M_1} f(y,t)dt - F(y) \right| \\
\leq& \left| F(x) - \int_c^{M_1} f(x,t)dt \right| \\
&+ \left| \int_c^{M_1} f(x,t)dt - \int_c^{M_1} f(y,t)dt \right| \\
&+ \left| \int_c^{M_1} f(y,t)dt - F(y) \right| \\
\leq& \left| F(x) - \int_c^{M_1} f(x,t)dt \right| \\
&+ \int_c^{M_1} |f(x,t) - f(y,t)|dt \\
&+ \left| \int_c^{M_1} f(y,t)dt - F(y) \right| \\
<& \frac{\epsilon}{3} + (M_1 - c) \times \frac{\epsilon}{3(M_1 - c)} + \frac{\epsilon}{3} \\
=& \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
=& \epsilon
\end{aligned}$$

$\square$

3. The Factorial Function. In order to progress toward a definition for the factorial function, we'd like to differentiate both sides of the expression found in Exercise 2.3, namely:

$$\frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}\, dt \tag{3.1}$$

But first, we need to extend Theorem 1.11 to work for improper integrals.

THEOREM 3.1 (Differentiation Under Improper Integrals (Theorem 8.4.9 in Abbott [1])). *Suppose a function $f : D \to \mathbf{R}$ is continuous on $D = \{(x,t) : a \le x \le b, c \le t\}$, and $F(x) = \int_c^\infty f(x,t)\, dt$ exists for all $x \in [a,b]$. If $f_x(x,t)$ is continuous on D and $\int_c^\infty f_x(x,t)dt$ converges uniformly, then $F'(x)$ exists, and:*

$$F'(x) = \int_c^\infty f_x(x,t)\, dt$$

Proof. Let $f : D \to \mathbf{R}$ be a continuous function on $D = \{(x,t) : a \le x \le b, c \le t\}$. Assume $F(x) = \int_c^\infty f(x,t)\, dt$ exists for all $x \in [a,b]$, $f_x(x,t)$ is continuous on D, and $\int_c^\infty f_x(x,t)dt$ converges uniformly.

Let $F_n(x) = \int_c^{c+n} f(x,t)dt$, which exists by assumption for every $n \in \mathbf{N}$.
By Theorem 1.11, $F_n'(x) = \int_c^{c+n} f_x(x,t)dt$.

Note that we assumed $\int_c^\infty f_x(x,t)dt$ converges uniformly, so by Theorem 2.7 it follows that $F_n'(x) = \int_c^{c+n} f_x(x,t)dt$ converges uniformly to $\int_c^\infty f_x(x,t)dt$. By Theorem 6.4.3 in Abbott [1], since $F_n(x)$ converges for at least one $x \in [a,b]$ and $F_n'(x)$ converges uniformly to $\int_c^\infty f_x(x,t)dt$, it follows that $F'(x) = \int_c^\infty f_x(x,t)dt$. ∎

Before proceeding, we'll prove a small result that will be useful later.

THEOREM 3.2 (Uniform Convergence of Power-Exponential Integrals). *Given any $x \ge 0, \alpha > 0$, the integral:*

$$\int_0^\infty t^x e^{-\alpha t}\, dt \tag{3.2}$$

Is uniformly convergent.

Proof. Begin by noting that:

$$t^x e^{-\alpha t} = \left(e^{x\log(t)}\right) e^{\frac{-\alpha t}{2}} = e^{x\log(t) - \frac{\alpha t}{2}}$$

This motivates us to define and study $f : (0,\infty) \to \mathbf{R}$ given by:

$$f(t) = x\log(t) - \frac{\alpha t}{2}$$

Note that f is differentiable on its domain, since $\log(x)$ is known to be differentiable on that region. Further:

$$f'(t) = \frac{x}{t} - \frac{\alpha}{2}$$

Next, we note that for $t > t_0 \equiv 2x/\alpha$,

$$f'(t) < \frac{x}{\frac{2x}{\alpha}} - \frac{\alpha}{2} = 0$$

So, since $f'(t)$ is negative for all $t > t_0$, and f is continuous on its domain since it's differentiable, we conclude that for all $t > t_0$, we have the following inequality:

$$x \log(t) < \frac{\alpha t}{2} \tag{3.3}$$

And so, since $e^x : \mathbf{R} \to (0, \infty)$ is a strictly increasing increasing bijection, this implies the inequality:

$$e^{x \log(t)} = t^x < e^{\frac{\alpha t}{2}} \qquad\qquad t > t_0$$

Next, we split the integral (3.2) at $t_0 > 0$:

$$\int_0^\infty t^x e^{-\alpha t}\, dt = \int_0^{t_0} t^x e^{-\alpha t}\, dt + \int_{t_0}^\infty t^x e^{-\alpha t}\, dt$$

Note:

$$\int_{t_0}^\infty t^x e^{-\alpha t}\, dt < \int_{t_0}^\infty e^{\frac{-\alpha t}{2}}\, dt$$

By Exercise 2.9 we know $\int_{t_0}^\infty t^x e^{-\alpha t}\, dt$ converges uniformly. So, the integral:

$$\int_0^{t_0} t^x e^{-\alpha t}\, dt + \int_{t_0}^\infty t^x e^{-\alpha t}\, dt = \int_0^\infty t^x e^{-\alpha t}\, dt$$

must converge uniformly as well. Thus, (3.2) must also be uniformly convergent on any compact interval $x \in [0, x_0] \subset [0, \infty)$, and so converges uniformly for any fixed $x \in [0, \infty)$!□

Corollary 3.3 (Uniform Convergence of Log-Power-Exponential Integrals). *Given any $x > 0, \alpha > 0, m \in \mathbf{N} \cup \{0\}$, the integral:*

$$\int_0^\infty (\log(t))^m\, t^x e^{-\alpha t}\, dt \tag{3.4}$$

Is uniformly convergent.

Proof. We begin by noting that the integral can be split:

$$\int_0^\infty (\log(t))^m\, t^x e^{-\alpha t}\, dt = \int_0^1 (\log(t))^m\, t^x e^{-\alpha t}\, dt + \int_1^\infty (\log(t))^m\, t^x e^{-\alpha t}\, dt \tag{3.5}$$

Next, note that $\log(t) < t$ for all $t > 1$; this can be thought of as a statement of (3.3) with $x = 1, \alpha = 2$. Thus, because $\log(t), t$ are strictly non-negative on $[1, \infty)$, we can write:

$$\int_1^\infty (\log(t))^m\, t^x e^{-\alpha t}\, dt \leq \int_1^\infty t^{(x+m)} e^{-\alpha t}\, dt$$

And so the second integral can be bounded above by a known uniformly convergent improper integral by Theorem 3.2, and thus must also converge uniformly by Exercise 2.9.

Looking now at the first of the two integrals on the right of the equals-sign in (3.5), note that the integrand is continuously differentiable on $(0, 1)$ by the Algebraic Differentiability Theorem (Theorem 5.2.4 in Abbott [1])! This, combined with the requirement that $x > 0$, allows us to show that the left sided limit at zero exists:

$$\begin{aligned}
\lim_{t\to 0^+} (\log(t))^m t^x &= \lim_{t\to 0^+} \frac{(\log(t))^m}{t^{-x}} \\
&\overset{\text{L'H}}{=} \lim_{t\to 0^+} \frac{m(\log(t))^{m-1}}{-xt^{-x}} \\
&\overset{\text{L'H}}{=} \lim_{t\to 0^+} \frac{m(m-1)(\log(t))^{m-2}}{x^2 t^{-x}} \\
&\vdots \\
&\overset{\text{L'H}}{=} \lim_{t\to 0^+} \frac{m!}{(-x)^m t^{-x}} \\
&= \frac{m!}{(-x)^m} \lim_{t\to 0^+} \left(t^x\right) \\
&= 0
\end{aligned}$$

Since $t = 0$ was the only point at which any part of the integrand has an essential discontinuity, and the integrand is continuous everywhere else on $(0, 1)$, we conclude that the integrand is integrable on this region, and its integral exists

Thus, the original integral (3.4) must converge uniformly, since it is the sum of a uniformly convergent integral and a constant (in this case, a definite integral). □

With the new machinery developed in Theorem 3.1 and the assistance of Theorem 3.2, we can now explore an interesting question: What happens when we differentiate (3.1)?

Exercise 3.4 (8.4.19 in Abbot). Show that for all $\alpha > 0, n \in \mathbf{N}$,

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\, dt$$

Use this to derive an explicit formula for $n!$, namely:

$$n! = \int_0^\infty t^n e^{-t}\, dt$$

Solution 3.5. Note that, in Exercise 2.3 we showed:

$$\frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}\, dt$$

And further, we showed that the above integral converges uniformly for all $\alpha > 0$. We can now take the derivative of both sides of the above expression with respect to α, and note that uniform convergence of the integral allows us to apply Theorem 3.1:

$$\frac{d}{d\alpha}\left(\frac{1}{\alpha}\right) = \frac{d}{d\alpha}\int_0^\infty e^{-\alpha t}\, dt$$

$$
\begin{aligned}
\frac{-1}{\alpha^2} &= \int_0^\infty \frac{d}{d\alpha}\left(e^{-\alpha t}\right)\, dt \\
&= \int_0^\infty -te^{-\alpha t}\, dt \\
&= -\int_0^\infty te^{-\alpha t}\, dt
\end{aligned}
$$

Note that the above equation is well defined, since the resulting integral converges uniformly by Theorem 3.2. This is the last detail we need to apply Theorem 3.1, which informs us:

$$\frac{1!}{\alpha^2} = \int_0^\infty t^1 e^{-\alpha t}\, dt$$

We'll take this as a base case for induction on n. Let $S(n)$ be the given statement; namely, the following equality:

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\, dt$$

which is well defined, since the integral is uniformly convergent for each $n \in \mathbf{N}$ by Theorem 3.2. Suppose S(n) holds for some $n \in \mathbf{N}$. Then, we'll take a derivative with respect to α:

$$\frac{d}{d\alpha}\left(\frac{n!}{\alpha^{n+1}}\right) = \frac{d}{d\alpha}\int_0^\infty t^n e^{-\alpha t}\, dt$$

By Theorem 3.2 the necessary convergence is achieved to satisfy the hypothesis of Theorem 3.1. Then, differentiating under the integral is allowed, and we find:

$$
\begin{aligned}
\frac{-n!(n+1)}{\alpha^{n+2}} &= \int_0^\infty \frac{d}{d\alpha}\left(t^n e^{-\alpha t}\right)\, dt \\
\frac{(n+1)!}{\alpha^{n+2}} &= \int_0^\infty t^{n+1} e^{-\alpha t}\, dt
\end{aligned}
$$

Now, to verify our supposition, we note that the resulting integral is of the form described by Theorem 3.2 and so converges uniformly! Thus, the above equality holds, and $S(n+1)$ is satisfied; namely:

$$\frac{(n+1)!}{\alpha^{n+2}} = \int_0^\infty t^{n+1} e^{-\alpha t}\, dt$$

Therefore, by induction, $S(n)$ holds for all $n \in \mathbf{N}$.

The above equality also holds for all $\alpha \in (0, \infty)$. So, letting $\alpha = 1$ yields:

$$n! = \int_0^\infty t^n e^{-t}\, dt \tag{3.6}$$

Which demonstrates the desired result.

Equation (3.6) is the meaningful interpolation of the discrete factorial function we've been looking for! Or rather, it's one such possible function; we haven't proved uniqueness yet (but we will soon!). The above formula is well defined for any $x \geq 0$, which suggests a definition. But first, a small result we'll use later:

Exercise 3.6. Prove $\lim_{b\to\infty}\left(-\frac{b^{x+1}}{e^b}\right) = 0$:

Solution 3.7. Note:

$$\begin{aligned} -\frac{b^{x+1}}{e^b} &= -\frac{e^{(x+1)\log(b)}}{e^b} \\ &= -e^{(x+1)\log(b)-b} \end{aligned}$$

I claim $\lim_{b\to\infty} b - \log(b) = \infty$
Note:

$$\begin{aligned} b - \log(b) &= b - \int_1^b \frac{1}{x}dx \\ &= 1 + \int_1^b x - \int_1^b \frac{1}{x}dx \\ &= 1 + \int_1^b x - \frac{1}{x}dx \\ &= 1 + \int_1^b \frac{x^2-1}{x}dx \end{aligned}$$

I claim $\frac{x^2-1}{x}$ is a strictly increasing function.
Let $a > b$ and note:

$$\begin{aligned} \frac{a^2-1}{a} - \frac{b^2-1}{b} &= \frac{a^2b - b - b^2a + a}{ba} \\ &= \frac{a^2b - b^2a + (a-b)}{ba} \\ &> \frac{a^2b - b^2a}{ba} \quad \text{Since } a > b \text{ implies } a - b > 0 \\ &> \frac{0}{ba} \quad \text{Since } a^2b > b^2a \text{ implies } a^2b - b^2a > 0 \\ &> 0 \end{aligned}$$

So, $\int_1^b \frac{x^2-1}{x}dx$ is greater than or equal to the lower integral with partition $P = \{1, 2, ..., n\}$ where n is the largest integer less than $b-1$.
So,

$$\begin{aligned} \int_1^b \frac{x^2-1}{x}dx &\geq \sum_{m=1}^{n-1} \frac{m^2-1}{m} \\ &= \sum_{m=1}^{n-1} m - \frac{1}{m} \\ &\geq \sum_{m=1}^{n-1} m - 1 \quad \text{Since } m \geq 1 \text{ implies } \frac{1}{m} \leq 1 \end{aligned}$$

Note then, that as $b \to \infty$ it follows $m \to \infty$ and the integral diverges to ∞.
So, $\int_1^b \frac{x^2-1}{x} dx \to \infty$ as $b \to \infty$ means $1 + \int_1^b \frac{x^2-1}{x} dx = b - \log(b) \to \infty$.
Thus, $\log(b) - b \to -\infty$ and $(x+1)\log(b) - b \to -\infty$.
This proves that $-e^{(x+1)\log(b)-b} \to 0$ since $e^x \to 0$ as $x \to -\infty$, proven in Proof 9.

We now have everything we need to establish the Factorial Function!

DEFINITION 3.8 (The Factorial Function). For any $x \geq 0$, define the *Factorial Function*, $! : [0, \infty) \to \mathbf{R}$ by:

$$x! = \int_0^\infty t^x e^{-t}\, dt$$

Let's explore some key properties of this new function on its domain.

EXERCISE 3.9 (Convexity of the Factorial Function (8.4.20 in Abbot)).

(a) Show that $x!$ is an infinitely differentiable function on $(0,\infty)$ and produce a formula for the n^{th} derivative. In particular show that $x!$ is convex; namely $(x!)'' > 0$.

(b) Show that $x!$ satisfies the Functional Equation for the Factorial :

$$(x+1)! = (x+1)x! \tag{3.7}$$

SOLUTION 3.10 (Convexity of the Factorial Function).

(a) To prove this, we'll use induction.

$$\begin{aligned} \frac{d}{dx}\int_0^\infty t^x e^{-t}\, dt &= \int_0^\infty \frac{d}{dx}\left(t^x e^{-t}\right)\, dt \\ &= \int_0^\infty \frac{d}{dx} e^{x\log(t)} e^{-t}\, dt \\ &= \int_0^\infty \log(t) t^x e^{-t}\, dt \end{aligned}$$

To verify that our supposition was correct, we invoke Theorem 3.2 on the first integral, and Corollary 3.3 on the resulting integral. So, indeed, both of these integrals converge uniformly; Theorem 3.1 applies and we conclude:

$$\frac{d}{dx}\int_0^\infty t^x e^{-t}\, dt = \int_0^\infty \log(t) t^x e^{-t}\, dt$$

Proceeding now to the inductive case, let $S(n)$ be the statement:

$$\frac{d^n}{dx^n}\int_0^\infty t^x e^{-t}\, dt = \int_0^\infty (\log(t))^n t^x e^{-t}\, dt$$

And suppose that $S(n)$ holds for some $n \in \mathbf{N}$. Then, we can write:

$$\frac{d^{n+1}}{dx^{n+1}}\int_0^\infty t^x e^{-t}\, dt = \frac{d}{dx}\int_0^\infty (\log(t))^n t^x e^{-t}\, dt$$

Supposing we can bring the last derivative into the integral, we find:

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}\int_0^\infty t^x e^{-t}\ dt &= \int_0^\infty \frac{d}{dx}\Big((\log(t))^n t^x e^{-t}\Big)\ dt\\ &= \int_0^\infty (\log(t))^{n+1} t^x e^{-t}\ dt\end{aligned}$$

Corollary 3.3 guarantees the uniform convergence of all three of the above integrals, and so the above equation is well posed by Theorem 3.1. Thus, $S(n+1)$ holds.

By induction, we conclude that $x!$ is infinitely differentiable, with n^{th} derivative given by:

$$\frac{d^n}{dx^n}\int_0^\infty t^x e^{-t}\ dt = \int_0^\infty (\log(t))^n t^x e^{-t}\ dt$$

In particular, the second derivative $(x!)''$ is:

$$\begin{aligned}\frac{d^2}{dx^2}\int_0^\infty t^x e^{-t}\ dt &= \int_0^\infty (\log(t))^2 t^x e^{-t}\ dt\\ &= \int_0^\infty (\log(t))^2 e^{x\log(t)-t}\ dt\end{aligned} \tag{3.8}$$

As we've proven, $e^a > 0$ for any $a \in \mathbf{R}$. Of course, $(\log(t))^2 \geq 0$, and the integrand is everywhere non-negative. Further, the integrand is continuous by the Algebraic Continuity Theorem (Theorem 4.3.4 in Abbott [1]), since it is a product of continuous functions. Lastly, by (3.8) for all $x \geq 0$ and $t = e$:

$$(\log(e))^2\, e^{x\log(e)-e} = e^{x-e} > 0$$

And so since the integrand is continuous, everywhere non-negative, and nonzero at at least one point, the uniformly convergent integral given by:

$$\int_0^\infty (\log(t))^2 t^x e^{-t}\ dt$$

evaluates to a positive nonzero real number by HW5 Problem 8c. Since this is true for any $x \in [0,\infty)$, we conclude that $(x!)'' > 0$ for all such x, and so the factorial function $x!$ is convex on $[0,\infty)$ by definition.

(b) Using Definition 3.8 and applying the integration by parts formula which we proved in HW6 problem 2 (Exercise 7.5.6 in Abbott [1]) yields:

$$\begin{aligned}(x+1)! &= \int_0^\infty t^{(x+1)} e^{-t}\ dt\\ &= \int_0^\infty t^{(x+1)}\frac{d}{dt}\left(-e^{-t}\right)\ dt\\ &= \lim_{b\to\infty}\Big[-t^{(x+1)}e^{-t}\Big]_0^b + \int_0^\infty \frac{d}{dt}\Big(t^{(x+1)}\Big) - e^{-t}\ dt\end{aligned}$$

$$= \lim_{b\to\infty} \left(-\frac{b^{x+1}}{e^b} \right) + \int_0^\infty \frac{d}{dt} \left(t^{(x+1)} \right) e^{-t}\, dt$$

(The left limit goes to 0 by Exercise 3.6)

$$= \int_0^\infty (x+1) t^x e^{-t}\, dt$$

$$= (x+1) \int_0^\infty t^x e^{-t}\, dt$$

$$= (x+1)x!$$

Which proves the desired result.

The convexity of Definition 3.8 is a powerful result which allows us to apply many useful results from convex optimization. But it turns out that something more powerful is true: $\log(x!)$ is also a convex function! This proof requires a more tools than we currently have, and so is omitted here. A complete proof of this fact can be viewed in Artin [2]. This powerful fact has a notable corollary.

Theorem 3.11 (Bohr-Mollerup Theorem (8.4.11 in Abbott [1])). *There is exactly one positive continuous function $f : [0, \infty) \to \mathbf{R}$ satisfying:*

(i) $f(0) = 1$

(ii) $f(x+1) = (x+1)f(x)$

(iii) $\log(f(x))$ *is convex.*

Further, since the Factorial Function Definition 3.8 satisfies all three of these properties, it must be that $f(x) = x!$.

Proof. We begin by noting that, by virtue only of the fact that f satisfies properties (i) and (ii), it must be that $f(n) = n!$ for every $n \in \mathbf{N}$, by the recursive definition of the discrete factorial. Now, fix any such $n \in \mathbf{N}$, and $x \in (0, 1]$.

Next, we note that for any convex function $\phi : D \to \mathbf{R}$ and given any $a_0, a_1, b_0, b_1 \in D$ forming closed intervals of the form $[a_0, b_0], [a_1, b_1]$, and satisfying $a_0 \leq a_1, b_0 \leq b_1$, it follows (from Abbott's Theorem 8.4.11 proof [1]) that:

$$\frac{\phi(b_0) - \phi(a_0)}{b_0 - a_0} \leq \frac{\phi(b_1) - \phi(a_1)}{b_1 - a_1} \tag{3.9}$$

Since the logarithm of our mysterious function $\log(f(x))$ is assumed convex by property (iii), we can use (3.9) over the intervals $[n-1, n]$, and $[n, n+x]$ to write:

$$\frac{\log(f(n)) - \log(f(n-1))}{n - (n-1)} \leq \frac{\log(f(n+x)) - \log(f(n))}{(n+x) - n}$$

Similarly, applying (3.9) to the intervals $[n, n+x]$, and $[n, n+1]$ yields:

$$\frac{\log(f(n+x)) - \log(f(n))}{(n+x) - n} \leq \frac{\log(f(n+1)) - \log(f(n))}{(n+1) - n}$$

Combining these results using the shared central term and rearranging gives the joint inequality:

$$\begin{aligned}
\frac{\log(f(n)) - \log(f(n-1))}{n-(n-1)} &\le \frac{\log(f(n+x)) - \log(f(n))}{(n+x)-n} \le \frac{\log(f(n+1)) - \log(f(n))}{(n+1)-n} \\
x\log\left(\frac{f(n)}{f(n-1)}\right) &\le \log\left(f(n+x)\right) - \log\left(f(n)\right) \le x\log\left(\frac{f(n+1)}{f(n)}\right) \\
x\log\left(\frac{n!}{(n-1)!}\right) \;&\le\; \log\left(f(n+x)\right) - \log\left(n!\right) \;\le\; x\log\left(\frac{(n+1)!}{n!}\right) \\
x\log\left(n\right) \;&\le\; \log\left(f(n+x)\right) - \log\left(n!\right) \;\le\; x\log\left(n+1\right)
\end{aligned}$$

Next, note that we can apply property (ii) recursively to rewrite:

$$\begin{aligned}
f(x+n) &= (x+n)f(x+n-1) \\
&= (x+n)(x+n-1)f(x+n-2) \\
&\;\;\vdots \\
&= f(x)\prod_{i=1}^{n}(x+i) \qquad (3.10)
\end{aligned}$$

Applying a this result to the above joint inequality gives:

$$x\log\left(n\right) \le \log\left(f(x)\right) + \log\left(\prod_{i=1}^{n}(x+i)\right) - \log\left(n!\right) \le x\log\left(n+1\right)$$

Rearranging yields the result:

$$\begin{aligned}
0 &\le \log\left(f(x)\right) + \log\left(\prod_{i=1}^{n}(x+i)\right) - \log\left(n!\right) - x\log\left(n\right) \le x\log\left(\frac{n+1}{n}\right) \\
0 &\le \log\left(f(x)\right) - \log\left(\frac{n^x n!}{\prod_{i=1}^{n}(x+i)}\right) \le x\log\left(\frac{n+1}{n}\right)
\end{aligned}$$

This result holds for any $n \in \mathbf{N}$, so we can think of the above as sequences in n. Note that $\log : (0,\infty) \to \mathbf{R}$ is continuous, so:

$$\lim_{n\to\infty}\left(x\log\left(\frac{n+1}{n}\right)\right) = x\log\left(\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\right) = x\log(1) = 0$$

of course, $\{0\}_{n=1}^{\infty}$ converges to zero since it's a constant sequence. Thus, by the squeeze theorem for sequences, we conclude:

$$\lim_{n\to\infty}\left[\log\left(f(x)\right) - \log\left(\frac{n^x n!}{\prod_{i=1}^{n}(x+i)}\right)\right] = 0$$

Since $\log(f(x))$ doesn't depend on n, we know $\log(f(x)) \to \log(f(x))$ as $n \to \infty$. By the algebraic limit theorem (2.3.3 in Abbott [1]):

$$\lim_{n\to\infty}\left[-\log\left(f(x)\right)+\log\left(\frac{n^x n!}{\prod_{i=1}^n(x+i)}\right)\right]=0$$
$$\lim_{n\to\infty}\left[\log\left(f(x)\right)-\log\left(f(x)\right)+\log\left(\frac{n^x n!}{\prod_{i=1}^n(x+i)}\right)\right]=\log\left(f(x)\right)$$
$$\lim_{n\to\infty}\left[\log\left(\frac{n^x n!}{\prod_{i=1}^n(x+i)}\right)\right]=\log\left(f(x)\right)$$

Furthermore, since $\log(x)$ and its inverse function e^x are both continuous, we can bring the limit into or out of each function where appropriate. Doing so and rearranging gives:

$$\log\left(f(x)\right)=\log\left[\lim_{n\to\infty}\left(\frac{n^x n!}{\prod_{i=1}^n(x+i)}\right)\right]$$
$$f(x)=\lim_{n\to\infty}\left[\frac{n^x n!}{\prod_{i=1}^n(x+i)}\right] \tag{3.11}$$

The above sequence must converge for each $x\in(0,1]$, since x was arbitrary. Thus, we have a working definition for $f:[0,1]\to\mathbf{R}$, since $f(0)=1$ by property (i).

However, it's actually true that the above sequence converges for all $x\in[1,\infty)$ also. To show this, let $x\in[1,\infty)$ be arbitrary, and note that there exists an $m\in\mathbf{N}$ and a $q\in[0,1)$ such that $x=m+q$. Then, applying (3.10) and then (3.11) gives:

$$\begin{aligned}
f(x)&=f(q)\prod_{i=1}^m(q+i)\\
&=\lim_{n\to\infty}\left[\log\left(\frac{n^q n!}{\prod_{i=1}^n(q+i)}\right)\right]\prod_{i=1}^m(q+i)\\
&=\lim_{n\to\infty}\left[\frac{n^q n!}{\prod_{i=1}^n(q+i)}\prod_{i=1}^m(q+i)\right]\\
&=\lim_{n\to\infty}\left[\frac{n^q n!}{\prod_{i=m+1}^n(q+i)}\right]\\
&=\lim_{n\to\infty}\left[\frac{n^q n!}{\prod_{j=1}^{n-m}(q+j+m)}\cdot\frac{n^m}{n^m}\right]\\
&=\lim_{n\to\infty}\left[\frac{n^x n!}{\prod_{j=1}^n(x+j)}\cdot\frac{\prod_{j=n-m+1}^n(x+j)}{n^m}\right]
\end{aligned} \tag{3.12}$$

Let's take a closer look at the term on the right by studying the product:

$$\begin{aligned}
\frac{1}{n^m}\prod_{j=n-m+1}^{n}(x+j)&=\frac{1}{n^m}\prod_{k=1}^m(x-m+n+k)\\
&=\prod_{k=1}^m\frac{1}{n}(q+n+k)\\
&=\prod_{k=1}^m\left(1+\frac{q+k}{n}\right)
\end{aligned}$$

this is a finite product of convergent sequences in n; applying the Algebraic Limit Theorem gives:

$$\lim_{n\to\infty}\left[\prod_{k=1}^{m}\left(1+\frac{q+k}{n}\right)\right]=\prod_{k=1}^{m}\left[\lim_{n\to\infty}\left(1+\frac{q+k}{n}\right)\right]=\prod_{k=1}^{m}(1)=1$$

From this, it follows directly from Theorem 2.3.3 (iv) (where $a_n = 1$) in Abbott [1] that the reciprocal limit is also 1:

$$\lim_{n\to\infty}\left[\frac{n^m}{\prod_{j=n-m+1}^{n}(x+j)}\right]=1$$

Since nonvanishing products must necessarily have only nonzero elements; this is really a statement of the Algebraic Limit Theorem, since the product is finite.

Returning to the proof, note that the limit given in (3.12) must converge, since it was derived from a finite product multiplied by a known convergent infinite product (by (3.11)). By the Algebraic Limit Theorem, the product of two convergent sequences is equal to the limit of the product; hence:

$$\begin{aligned}
&\lim_{n\to\infty}\left[\frac{n^x n!}{\prod_{j=1}^{n}(x+j)}\cdot\frac{\prod_{j=n-m+1}^{n}(x+j)}{n^m}\right]\cdot(1)\\
&\quad=\lim_{n\to\infty}\left[\left(\frac{n^x n!}{\prod_{j=1}^{n}(x+j)}\cdot\frac{\prod_{j=n-m+1}^{n}(x+j)}{n^m}\right)\left(\frac{n^m}{\prod_{j=n-m+1}^{n}(x+j)}\right)\right]\\
&\quad=\lim_{n\to\infty}\left[\frac{n^x n!}{\prod_{j=1}^{n}(x+j)}\right]
\end{aligned}$$

And so, since $x \in [1, \infty]$ was arbitrary, we conclude:

$$f(x)=\lim_{n\to\infty}\left[\frac{n^x n!}{\prod_{i=1}^{n}(x+i)}\right] \tag{3.13}$$

holds for all $x \in [0, \infty)$. Thus, since we were able to derive an explicit definition for $f : [0, \infty)$ through use of only the properties (i), (ii), (iii), we conclude that $f(x)$ must be unique.

Since $x!$ satisfies all three properties, we therefore conclude that it must be the only such function which does so. $\square$

Of course, the identity (3.13) has much more utility than just a tool in the above proof. In fact, this expression is called the Gauss Product Formula, and by following the above steps, we've just proven an incredible equality.

$$x! = \int_0^\infty t^x e^{-t}\,dt=\lim_{n\to\infty}\left[\frac{n^x n!}{\prod_{i=1}^{n}(x+i)}\right] \tag{3.14}$$

This identity has a myriad of interesting corollaries! But before we explore them, we need to extend the Factorial Function Definition 3.8 once more, to include negative numbers also.

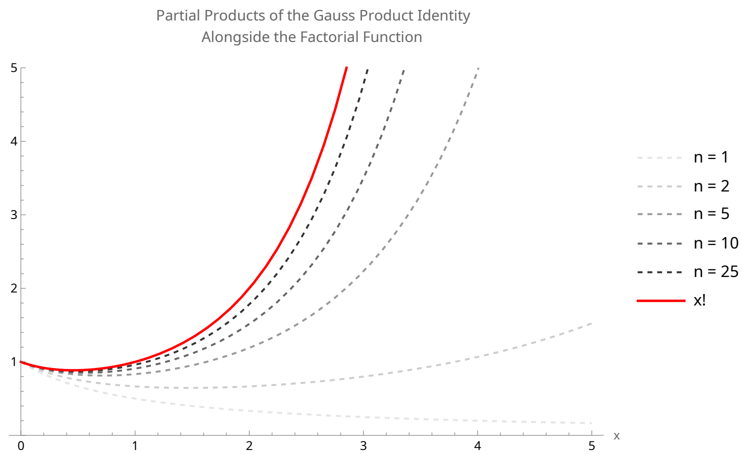

FIG. 3.1. *A comparison of some partial products of the Gauss Product Formula and the Factorial Function $x!$*

DEFINITION 3.12 (Fully Generalized Factorial Function). Using the Functional Equation (3.7), the Factorial Function Definition 3.8 can be extended to all $\mathbf{R}$, excluding negative integers $-1, -2, -3\ldots$. For $x < 0$ with $x \notin \mathbf{Z}$, note that the Archimedean Property guarantees that there exists an $M \in \mathbf{N}, M > 0$ with $M > -x$, so that $M + x > 0$. Applying the Functional Equation (3.7) recursively, we define:

$$x! = (M + x)! \prod_{i=1}^{M} \frac{1}{(x + i)}$$

The plot shown below in Figure 3.2 helps us get a sense for what this newly generalized factorial function looks like in much stranger negative domain.

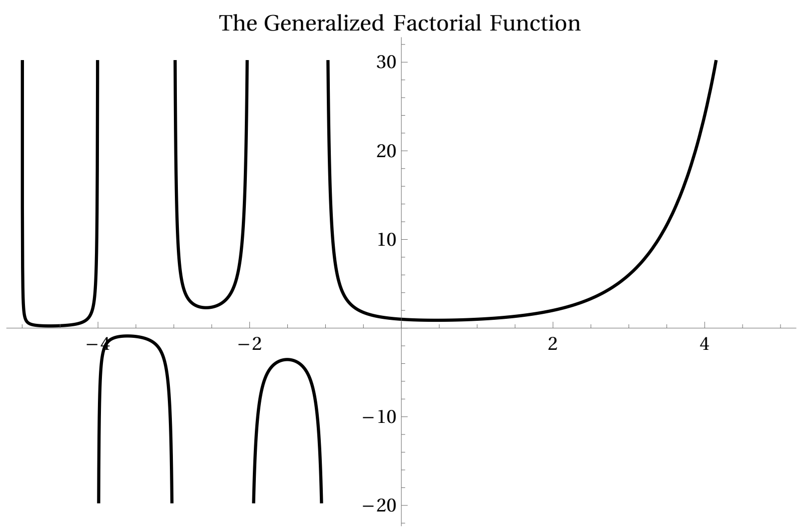

FIG. 3.2. *A plot of the Generalized Factorial Function $x!$ for $-5 \leq x \leq 5$.*

Armed with the fully generalized Definition 3.12, we can investigate some of the corollaries of the identity (3.14)!

THEOREM 3.13 (Wallis Product Formula for π). *The infinite product given by:*

$$2 \lim_{n\to\infty} \left[\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{2\cdot 2}{3\cdot 5}\right)\left(\frac{2\cdot 2}{5\cdot 7}\right)\cdots\left(\frac{(2n)\cdot(2n)}{(2n-1)\cdot(2n+1)}\right)\right]$$

converges to π.

Proof. To begin, we'll start with the integral identity:

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$$

This integrand is a well-studied function: the Gaussian distribution! The above identity is a known result from probability theory.[1] The integrand is even and well defined on all $\mathbf{R}$, so we can write:

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = 2\int_{0}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$$

Using the substitution $t = u^2$, we find:

$$\begin{aligned}\sqrt{\pi} &= 2\int_0^{\infty} e^{-t}\left(\frac{1}{2\sqrt{t}}\, dt\right)\\ &= \int_0^{\infty} t^{-1/2}e^{-t}\, dt\\ &= \left(\frac{-1}{2}\right)!\end{aligned}$$

Using the functional equation,

$$\left(\frac{1}{2}\right)! = \left(\frac{-1}{2}+1\right)! = \left(\frac{-1}{2}+1\right)\left(\frac{-1}{2}\right)! = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)!$$

And so, we find:

$$\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$$

Or, equivalently:

$$\pi = 4\big((1/2)!\big)^2 \tag{3.15}$$

Keeping (3.14) in mind, this suggestive equation motivates us to study a product of the form:

$$4\left[\frac{n^{1/2}n!}{\prod_{i=1}^{n}(1/2+i)}\right]^2$$

[1] To prove this, one can approach the problem with multivariable calculus, and use the Jacobian for a cylindrical coordinate system to cleverly evaluate the integral explicitly. We haven't yet developed these methods or their underlying theory in class, and so we'll take it as a given here.

Distributing the exponent into the product and simplifying, we find:

$$\begin{aligned}
4\left[\frac{n^{1/2}n!}{\prod_{i=1}^{n}(1/2+i)}\right]^2 &= 4\left[\frac{n(n!)^2}{\prod_{i=1}^{n}(1/2+i)^2}\right] \\
&= 4n\prod_{i=1}^{n}\frac{i^2}{(1/2+i)^2} \\
&= 4n\prod_{i=1}^{n}\frac{4i^2}{(1+2i)^2} \\
&= 4n\left[\left(\frac{2\cdot 2}{3\cdot 3}\right)\left(\frac{4\cdot 4}{5\cdot 5}\right)\cdots\left(\frac{(2n)\cdot(2n)}{(2n+1)\cdot(2n+1)}\right)\right] \\
&= \frac{4n}{2n+1}\left[\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\cdots\left(\frac{(2n)\cdot(2n)}{(2n-1)\cdot(2n+1)}\right)\right] \\
&= \frac{4n}{2n+1}\prod_{i=1}^{n}\frac{(2i)\cdot(2i)}{(2i-1)\cdot(2i+1)}
\end{aligned}$$

This starting expression above is a finite sub-product of a known convergent infinite product, by Theorem 3.11. So, we can take the limit of both sides! Doing so gives the result:

$$\begin{aligned}
4\lim_{n\to\infty}\left[\frac{n^{1/2}n!}{\prod_{i=1}^{n}(1/2+i)}\right]^2 &= \lim_{n\to\infty}\left[\frac{4n}{2n+1}\prod_{i=1}^{n}\frac{(2i)\cdot(2i)}{(2i-1)\cdot(2i+1)}\right] \\
4\big((1/2)!\big)^2 &= 2\prod_{i=1}^{\infty}\frac{(2i)\cdot(2i)}{(2i-1)\cdot(2i+1)}
\end{aligned}$$

And so, by (3.15), we conclude

$$\pi = 2\lim_{n\to\infty}\left[\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{2\cdot 2}{3\cdot 5}\right)\left(\frac{2\cdot 2}{5\cdot 7}\right)\cdots\left(\frac{(2n)\cdot(2n)}{(2n-1)\cdot(2n+1)}\right)\right]$$

∎

Theorem 3.14 (Factorial Identity for Sine). *Considering the fully generalized factorial Definition 3.12 and filling removable discontinuities at every nonzero integer, we have the following equality for all $x \in \mathbf{R}$:*

$$\frac{\sin(\pi x)}{\pi} = \frac{x}{(x)!(-x)!}$$

Proof. We begin by noting noting that the Gauss Product Identity (3.14) holds for all real numbers, excluding negative integers. From Definition 3.12, given any non-integer $x < 0$, we define:

$$x! = (M+x)!\prod_{i=1}^{M}\frac{1}{(x+i)}$$

for any $M \in \mathbf{N}, M > -x$, which is guaranteed to exist by the Archimedian Property. Since $M + x > 0$, we can apply (3.14) directly to write:

$$
\begin{aligned}
x! &= \left[\prod_{i=1}^{M} \frac{1}{(x+i)}\right] \lim_{n\to\infty} \left[\frac{n^{(M+x)}n!}{\prod_{i=1}^{n}(M+x+i)}\right] \\
&= \left[\prod_{i=1-M}^{0} \frac{1}{(M+x+i)}\right] \lim_{n\to\infty} \left[\frac{n^{(M+x)}n!}{\prod_{i=1}^{n}(M+x+i)}\right] \\
&= \lim_{n\to\infty} \left[\frac{n^{(M+x)}n!}{\prod_{i=1-M}^{n}(M+x+i)}\right] \\
&= \lim_{n\to\infty} \left[\frac{n^{(M+x)}n!}{\prod_{j=1}^{n+M}(x+j)}\right] \\
&= \lim_{n\to\infty} \left[\left(\frac{n^M}{\prod_{j=n+1}^{n+M}(x+j)}\right)\left(\frac{n^x n!}{\prod_{j=1}^{n}(x+j)}\right)\right] \\
&= \lim_{n\to\infty} \left[\left(\prod_{j=1}^{M} \frac{n}{(x+j+n)}\right)\left(\frac{n^x n!}{\prod_{j=1}^{n}(x+j)}\right)\right]
\end{aligned}
$$

This expression was derived from a known convergent sequence of products, and so we can multiply the result by a few more known convergent sequences, namely:

$$1 = \lim_{n\to\infty} \prod_{j=1}^{M} \frac{x+j+n}{n}$$

Doing so, and noting that combining the two limits is justified by the Algebraic Limit Theorem, we find:

$$
\begin{aligned}
&\lim_{n\to\infty} \left[\left(\prod_{j=1}^{M} \frac{n}{(x+j+n)}\right)\left(\frac{n^x n!}{\prod_{j=1}^{n}(x+j)}\right)\right](1) \\
&= \lim_{n\to\infty} \left[\left(\prod_{j=1}^{M} \frac{n}{(x+j+n)}\right)\left(\frac{n^x n!}{\prod_{j=1}^{n}(x+j)}\right)\right]\left(\lim_{n\to\infty} \prod_{j=1}^{M} \frac{x+j+n}{n}\right) \\
&= \lim_{n\to\infty} \left[\left(\prod_{j=1}^{M} \frac{n}{(x+j+n)}\right)\left(\frac{n^x n!}{\prod_{j=1}^{n}(x+j)}\right)\left(\prod_{j=1}^{M} \frac{x+j+n}{n}\right)\right] \\
&= \lim_{n\to\infty} \left[\frac{n^x n!}{\prod_{j=1}^{n}(x+j)}\right]
\end{aligned}
$$

And so, we conclude that, even for $x < 0$, so long as $x \notin \mathbf{Z}$:

$$x! = \lim_{n\to\infty} \left[\frac{n^x n!}{\prod_{j=1}^{n}(x+j)}\right]$$

That is, the Gauss Product formula (3.14) also holds for negative factorials, as defined by (3.7)!

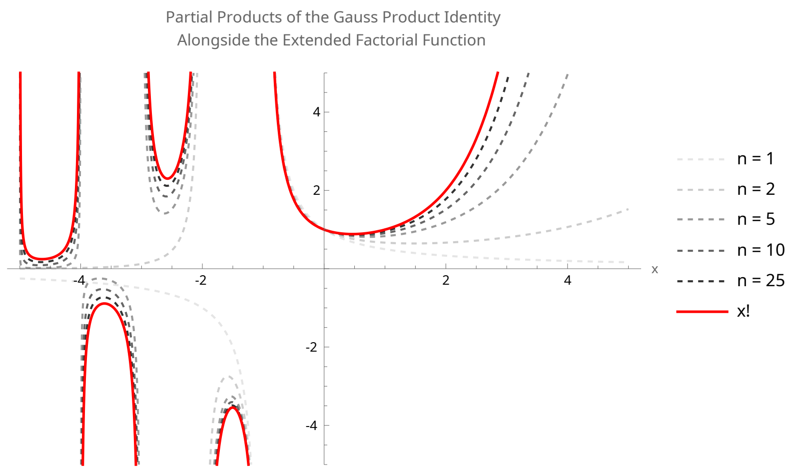

FIG. 3.3. *A comparison of some partial products of the Gauss Product Formula and the Fully Generalized Factorial Function* $x!$

Using this fact, we can begin to study an expression of the form $x!(-x)!$. Let $x \in \mathbf{R}, x \notin \mathbf{Z}$ be fixed. Then:

$$\begin{aligned} x!(-x)! &= \lim_{n\to\infty}\left[\left(\frac{n^x n!}{\prod_{i=1}^{n}(i+x)}\right)\left(\frac{n^{-x}n!}{\prod_{i=1}^{n}(i-x)}\right)\right] \\ &= \lim_{n\to\infty}\left[\frac{(n!)^2}{\prod_{i=1}^{n}(i^2-x^2)}\right] \\ &= \lim_{n\to\infty}\left[\prod_{i=1}^{n}\frac{i^2}{(i^2-x^2)}\right] \\ &= \lim_{n\to\infty}\left[\prod_{i=1}^{n}\frac{1}{\left(1-\left(\frac{x}{i}\right)^2\right)}\right] \end{aligned}$$

Using this, we can write:

$$\frac{x}{x!(-x)!} = x\prod_{i=1}^{\infty}\left(1-\left(\frac{x}{i}\right)^2\right)$$

This product defines a sine curve! From Abbot [1] Section 8.3 Equation (1),

$$\sin(x) = x\prod_{i=1}^{\infty}\left(1-\left(\frac{x}{\pi i}\right)^2\right)$$

Rewriting the above in a more suggestive form,

$$\frac{x}{x!(-x)!} = \frac{\pi x}{\pi}\prod_{i=1}^{\infty}\left(1-\left(\frac{\pi x}{\pi i}\right)^2\right) = \frac{\sin(\pi x)}{\pi} \tag{3.16}$$

□

To get a more intuitive sense for this incredible identity, some partial products can be viewed in Figure 3.4

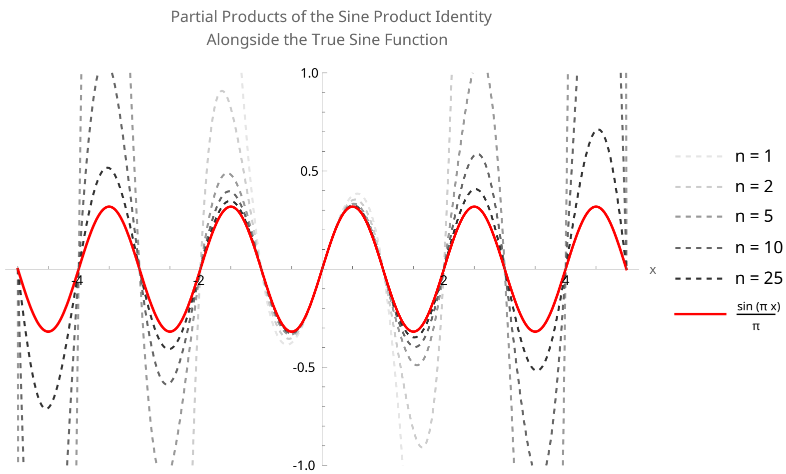

Fig. 3.4. *A comparison of some partial products of the Sine product identity to* $\frac{\sin(\pi x)}{\pi}$.

REFERENCES

[1] S. Abbott, *Understanding Analysis (Second Edition)*, Springer Science+Business Media, 2016.
[2] E. Artin, *The Gamma Function*, Dover Publications, 2015.