Schwarz-Christoffel Transformation and Elliptic Functions

University of Colorado at Boulder

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Complex Variables and Applications

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Abstract Riemann's Mapping Theorem (RMT) claims that any simply-connected domain of the z-plane, but not the entire z-plane nor the extended z-plane, can be mapped onto the unit disk. However, the proof is not constructive. Schwarz-Christoffel Transformations (SCT) explicitly constructs a bijective mapping between the upper-half plane and an arbitrary polygon. This project presents the theoretical background of SCT, including conformal mapping and a heuristic derivation of SCT. Additionally, SCT is applied in the derivation of elliptic functions and their properties. Finally, other cases of mapping to regular polygons are explored.

1. Introduction. We first walk through various types of transformations as background before delving into elliptic functions and integrals. Simple transformations are first introduced by example, where a straight line contour is mapped from the z-plane to the w-plane. Then, we present multiple graphs throughout the discussion to help the visual reader better understand the transformations. Despite the first example being simple, each step of the mathematical derivation is provided to help the reader understand how to analytically calculate a contour transformed by a function. Afterwards, a slightly more difficult example is presented in the same fashion.

After building upon the general idea of transformations, we define and analyze conformal maps. In addition, we explore critical points and their effects on whether a transformation is conformal or not. Once again, graphs are provided throughout the discussion to support the main ideas.

Once transformations and conformal maps have been sufficiently analyzed, we present univalent transformations, the Riemann Mapping Theorem, and the Schwarz-Christoffel Transformation (SCT). We also guide the reader through a heuristic derivation of SCT to assist in understanding the derivation.

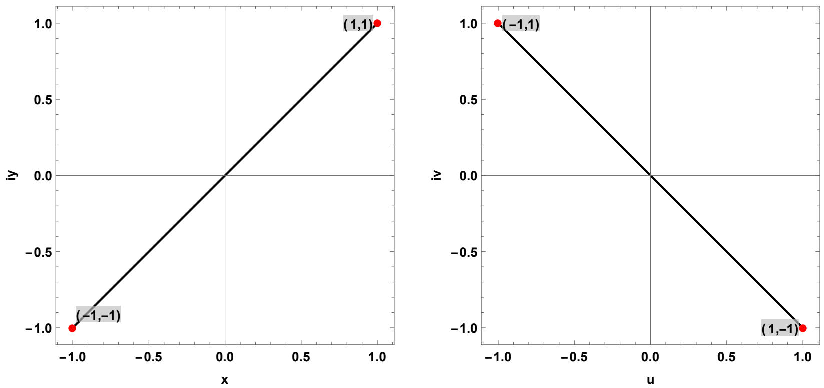
After deriving the SCT, we consider the special case of mapping the upper-half plane (UHP) to a rectangle. Remarkably, this SCT naturally gives rise to elliptic functions and allows us to understand their properties, including double-periodicity and single-valuedness, in an intuitive way.

The double periodicity of the Jacobi elliptic functions comes from the fact that rectangles completely tiles a two-dimensional plane. This naturally leads us to investigate similar functions determined by regular polygons that tile a two dimensional-plane and the function's properties.

2. Project Description.

2.1. Transformations. We assume that the reader already has a background in complex math and complex functions. We can now state that a function (from here on understood to be a complex function) $w = f(z)$ defined on a domain D is a *mapping* from the complex z-plane to the complex w-plane. Often the terms *map*, *function*, and *transformation*, are used interchangeably and we will do the same here. As a result, The function $f(z)$ is a *mapping* as well as a *transformation*. We will be working with the complex z and w-planes in our examples. Please note that we let $z = x + iy$ and $w = u + iv$ where x,y,u, and v are real while z and w are complex.

Consider the contour $z(s) = s + is$ for $s \in (-1, 1)$ and the function $w = f(z) = \overline{z}$ which transforms the contour from the z-plane to the w-plane.



z-plane $\quad\quad\quad\quad$ w-plane

FIG. 2.1. *The contour $z(s) = s + is$ for $s \in (-1, 1)$ in the complex z and in the w-plane transformed by $f(z) = \overline{z}$*

We can figure out how the contour in the z-plane is mapped to the w-plane by hand. We do so by solving for the real and imaginary parts of w as functions of x and y.

$$\begin{aligned} f(z) &= \overline{z} \\ f(x, y) &= \overline{x + iy} \\ &= x + i(-y) \\ &= u(x, y) + iv(x, y) \\ &\text{Where } u(x, y) = x, v(x, y) = -y \end{aligned} \tag{2.1}$$

Next, we see that the contour defined by $z(s)$ is $y = x$ in the complex z-plane by utilizing the equations $x = s$ and $y = s$. We then solve for a relation between u and v from the equations $y = x$, $u = x$, and $v = -y$ to get the contour in the w-plane:

$$u = x \tag{2.2}$$

$$\begin{aligned} v &= -y \\ &= -x \\ -v &= x \end{aligned} \tag{2.3}$$

$$\begin{aligned} u &= x = -v \\ v &= -u \end{aligned} \tag{2.4}$$

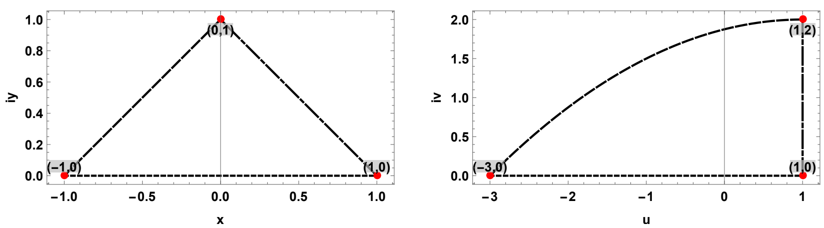
The final step is to see for which end-points we have the contour defined by $v = -u$ in the w-plane:

$$\begin{aligned} f(x, y) &= x - iy \\ f(-1, -1) &= -1 + i \end{aligned} \tag{2.5}$$

$$f(x, y) = x - iy \tag{2.6}$$
$$f(1, 1) = 1 - i$$

The final result is the same as shown above (Figure 2.1). We have a contour defined by $v = -u$ in the complex w-plane which is a line with a slope of -1 starting at the point $(-1, 1)$ and ending at the point $(1, -1)$.

Next, we consider the closed contour defined by the three parametric functions $z_1(s) = s - 1 + is$, $z_2(s) = 1 - s + is$, and $z_3(s) = 2s - 1$ for $s \in (0, 1)$ and the function $w = f(z) = 2z - z^2$ which transforms the closed contour from the z-plane to the w-plane.



z-plane w-plane

FIG. 2.2. *The closed contour defined by the three parametric functions $z_1(s) = s - 1 + is$, $z_2(s) = 1 - s + is$, and $z_3(s) = 2s - 1$ for $s \in (0, 1)$ and the function $w = f(z) = 2z - z^2$ which transforms the closed contour from the z-plane to the w-plane.*

We proceed by deriving the closed contour in the w-plane by hand. First, we recognize that the three parametric functions z_1, z_2, and z_3 are the lines $y = x + 1$, $y = 1 - x$, and $y = 0$ respectively. z_1 begins at $(-1, 0)$ and ends at $(0, 1)$, z_2 begins at $(0, 1)$ and ends at $(1, 0)$, and z_3 begins at $(-1, 0)$ and ends at $(1, 0)$.

We now solve for $u(x, y)$ and $v(x, y)$ from the transformation $f(z)$:

$$\begin{aligned} f(z) &= 2z - z^2 \\ f(x, y) &= 2(x + iy) - (x + iy)^2 \\ &= 2x + i2y - (x^2 + i2xy - y^2) \\ &= 2x - x^2 + y^2 + i(2y - 2xy) \\ &= u(x, y) + iv(x, y) \end{aligned} \tag{2.7}$$
$$\text{Where } u(x, y) = 2x - x^2 + y^2 \text{ and } v(x, y) = 2y - 2xy$$

As before, we'll use the functions $u(x, y)$ and $v(x, y)$ along with the curves $y = x + 1$, $y = 1 - x$, and $y = 0$ in the z-plane to calculate the curves in the w-plane:

$$\begin{aligned} &y = 0 \text{ from } (-1, 0) \text{ to } (1, 0): \\ u(x, y) &= 2x - x^2 + y^2 \\ u(x, 0) &= 2x - x^2 \\ v(x, y) &= 0 \\ u(-1, 0) &= -3 \\ u(1, 0) &= 1 \end{aligned} \tag{2.8}$$

Since we now know the mapping of $y = 0$ is also a line segment in the w-plane (along $v = 0$), we must also check for any maxima/minima of $u(x, y)$ since we can have a case where the end-points of $u(x, y)$ are not the maxima/minima. Not doing so can lead to an inaccurate representation of the transformation.[1]

$\frac{d}{dx}u(x, y) = 2-2x$ has a local maximum at $x = 1$, which we have already accounted for earlier. Therefore, the line $y = 0$ from $(-1, 0)$ to $(1, 0)$ in the z-plane is mapped to the line $u = 0$ from $(-3, 0)$ to $(1, 0)$ in the w-plane.

$$
\begin{aligned}
y &= x + 1 \text{ from } (-1, 0) \text{ to } (0, 1) : \\
u(x, y) &= 2x - x^2 + y^2 \\
u(x, x + 1) &= 2x - x^2 + x^2 + 2x + 1 \\
&= 4x + 1 \\
x &= \frac{u(x, y) - 1}{4} \\
v(x, y) &= 2y - 2xy \\
v(x, x + 1) &= 2x + 2 - 2x^2 - 2x \\
&= 2 - 2x^2 \\
&= 2 - 2\left(\frac{u(x, y) - 1}{4}\right)^2
\end{aligned}
\tag{2.9}
$$

We now check the start and endpoints of the mapped curve in the w-plane.

$$
\begin{aligned}
u(-1, 0) &= -3 \\
v(-1, 0) &= 0 \\
u(0, 1) &= 1 \\
v(0, 1) &= 2
\end{aligned}
\tag{2.10}
$$

The line $y = x + 1$ from $(-1, 0)$ to $(0, 1)$ in the z-plane is mapped to the line $v = 2 - 2\left(\frac{u-1}{4}\right)^2$ from $(-3, 0)$ to $(1, 2)$ in the w-plane.

$$
\begin{aligned}
y &= 1 - x \text{ from } (0, 1) \text{ to } (1, 0) : \\
u(x, y) &= 2x - x^2 + y^2 \\
u(x, x + 1) &= 2x - x^2 + x^2 - 2x + 1 \\
&= 1 \\
v(x, y) &= 2y - 2xy \\
v(x, x + 1) &= 2 - 2x - 2x + 2x^2 \\
&= 2 - 4x + 2x^2 \\
v(0, 1) &= 2 \\
v(1, 0) &= 0
\end{aligned}
\tag{2.11}
$$

Since we now know the mapping of $y = 1 - x$ is also a line segment in the w-plane, we must also check for any maxima/minima of $v(x, y)$ since we can have a case

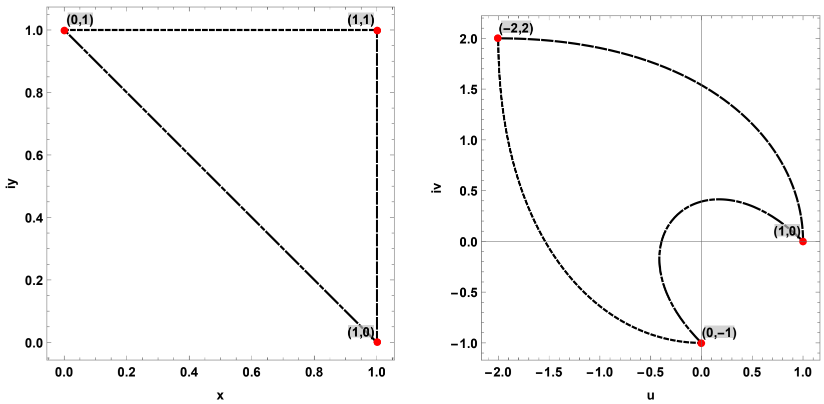
[1] In the previous example the maxima were not checked since we were dealing with linear functions.

where the end-points of $u(x, y)$ are not the maxima/minima. Not doing so can lead to an inaccurate representation of the transformation. $\frac{d}{dx}v(x, y) = -4 + 4x$ has a local maximum at $x = 1$, which we already accounted for earlier. Therefore, the line $y = x + 1$ from $(0, 1)$ to $(1, 0)$ in the z-plane is mapped to the line $u = 1$ from $(1, 2)$ to $(1, 0)$ in the w-plane.

2.2. Conformal maps.

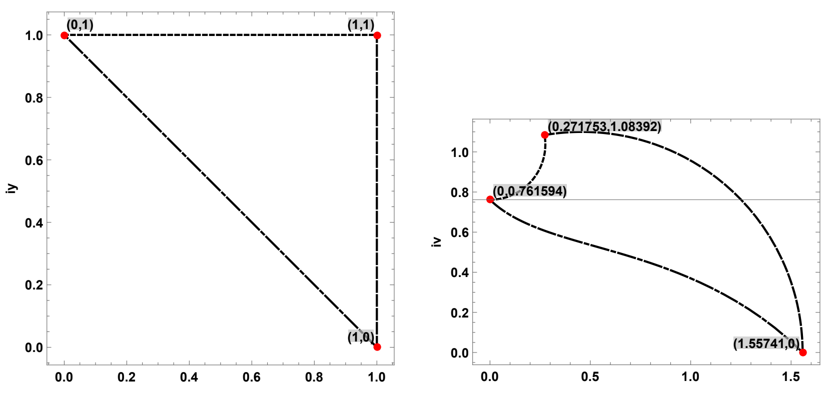
DEFINITION 2.1. A **conformal** map is a function that preserves the angles of intersection of intersecting curves.

THEOREM 2.2. *Assume that $f(z)$ is analytic and not constant in a domain D of the complex z plane. For any point $z \in D$ for which $f'(z) \neq 0$, this mapping is conformal, that is, it preserves the angle between two differentiable arcs.*



z-plane $\qquad$ w-plane

FIG. 2.3. *A conformal transformation of a right isosceles triangle in the z-plane to the w-plane by $w = f(z) = z^3$*



z-plane $\qquad$ w-plane

FIG. 2.4. *A conformal transformation of a right isosceles triangle in the z-plane to the w-plane by $w = f(z) = tan(z)$*

Figures 2.3 and 2.4 show conformal transformations since each angle between any two curves is preserved from the z-plane to the w-plane. This is because the transformations $f : \mathbb{C} \to \mathbb{C}$ ($f(z) = z^3$ and $f(z) = \tan(z)$) are analytic and not constant in the domain defined by the interior of the closed contour of the z-plane and have a first derivative that doesn't evaluate to zero at the points of intersection.

What happens if a transformation $f(z)$ is analytic but has a first-derivative that evaluates to zero at a point of intersection? The point z_0 where $f'(z_0) = 0$ is called a *critical point*.

THEOREM 2.3. *Assume that $f(z)$ is analytic and not constant in a domain D of the complex z plane. Suppose that $f'(z_0) = f''(z_0) = \cdots = f^{(n-1)}(z_0) = 0$, while $f^{(n)}(z_0) \neq 0$, $z_0 \in D$. Then the mapping $z \to f(z)$ magnifies n times the angle between two intersecting differentiable arcs that meet at z_0.*

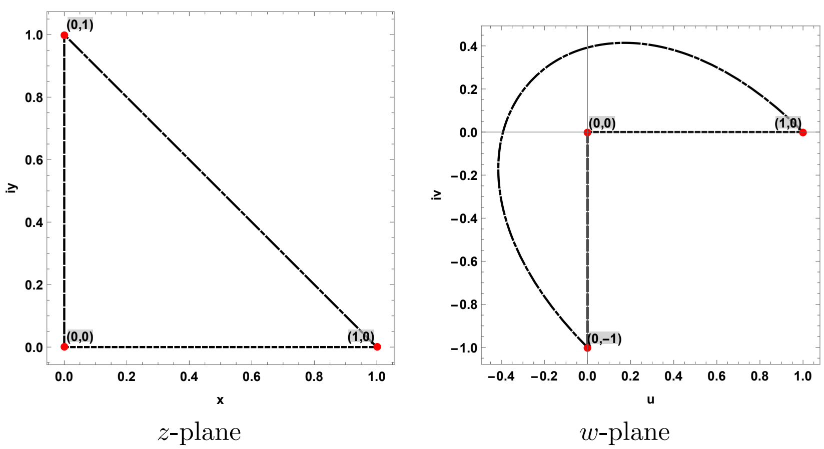
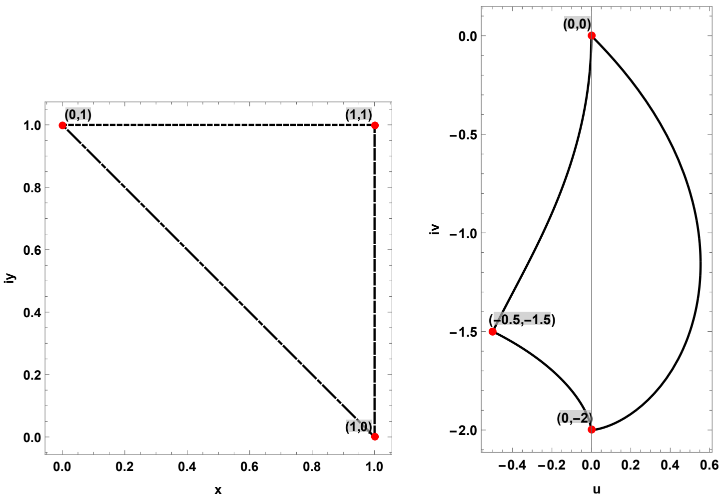


FIG. 2.5. *A conformal transformation of a right isosceles triangle in the z-plane to the w-plane by $w = f(z) = z^3$*

Note that the transformation for Figure 2.5 is $f(z) = z^3$, which evaluates to 0 at $z = 0$ for the first 2 derivatives. By Theorem 2.3, $n = 3$ meaning the angle in the z-plane at $(0, 0)$ of 90 degrees is magnified 3 times, resulting in an angle of 270 degrees in the w-plane, which is at the point $(0, 0)$. All other points of intersection in the z-plane are non-zero for $f'(z)$ and therefore their angle is preserved. Recall that Figure 2.3 has the same transformation except that the right isosceles triangle in the z-plane has its right angle at $(1, 1)$, not at $(0, 0)$, preserving all of its angles in the w-plane.



z-plane $\quad$ w-plane

FIG. 2.6. *A conformal transformation of a right isosceles triangle in the z-plane to the w-plane by $w = f(z) = \frac{1}{z} - z$*

For Figure 2.6 the transformation is $f(z) = \frac{1}{z} - z$. The first derivative of the transformation evaluates to 0 at $(0, 1)$ but not the second derivative derivative. By Theorem 2.3, $n = 2$ and the angle at $(0, 1)$ of 45 degrees in the z-plane is mapped to an angle of 90 degrees in the w-plane which occurs at the point $(0, -2)$. Note that there are two 90 degree angles in the w plane and only one 90 degree angle in the z plane.

2.3. Schwarz-Christoffel Transformation.

DEFINITION 2.4. An analytic transformation is **univalent** in a domain D if it obtains a unique value for every value $z \in D$.

REMARK 2.5. We note that an analytic univalent function always has a unique analytic inverse, and the proof can be found in Chapter 5.5 of [1].

The Schwarz-Christoffel Transformation (SCT) is a specific case of the Riemann Mapping Theorem.

THEOREM 2.6 (Riemann Mapping Theorem). *Let D be a simply connected domain in the z plane, which is neither the z plane or the extended z plane. Then there exists a univalent function $f(z)$, such that $w = f(z)$ maps D onto the disk $|w| < 1$.*

It is important to note that the bilinear transformation

$$w = i\left(\frac{1-z}{1+z}\right) \tag{2.12}$$

maps the interior of the unit circle centered at the origin of the z plane to the upper half of the w plane. Thus, the Riemann Mapping Theorem can be extended to conclude that if there's a univalent function $f(z)$ that maps D onto the disk $|w| < 1$, then there's a univalent function $f(z)$ that maps D onto the upper half w plane.

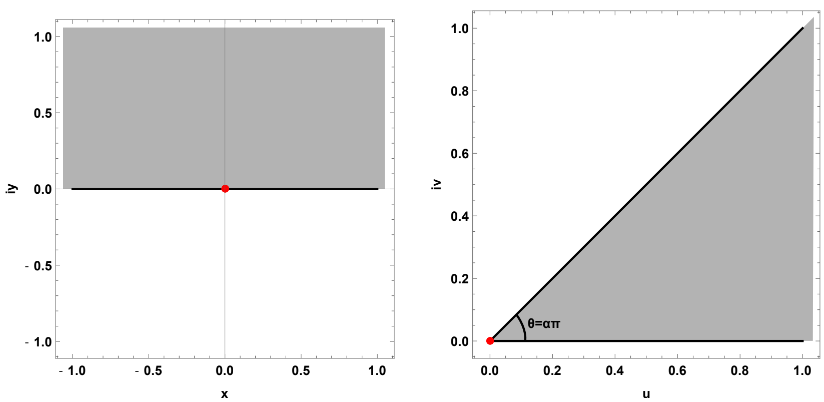
The Riemann Mapping Theorem, unfortunately, is non-constructive; however, if the simply connected domain in the z plane D is the interior of a polygon, then the

Schwarz-Christoffel Transformation provides a constructive formula for mapping the domain D to the upper half w plane. If a unit disk is desired, then the bilinear transformation (also known

$$z = i\left(\frac{1+\zeta}{1-\zeta}\right), \quad \zeta = \frac{z-i}{z+i}, \quad \frac{dz}{d\zeta} = \frac{2i}{(1-\zeta)^2} \tag{2.13}$$

can be used to map the upper half z plane to the disk $|\zeta| < 1$.

Before showing the theorem for SCT we introduce a simple example to give intuition into SCT. Consider an open triangle with one side along $\text{Re}(w) \geq 0$ and another intersecting the origin making an angle of $\theta = \alpha\pi$ from $\text{Re}(w) \geq 0$.



z-plane w-plane

FIG. 2.7. *A transformation of an open triangle in the w-plane to the upper half z-plane by $w = f(z) = z^\alpha$*

We show that the function $f(z) = z^\alpha$ $(0 < \alpha < 2)$ is a transformation from the upper half z-plane to the w-plane.

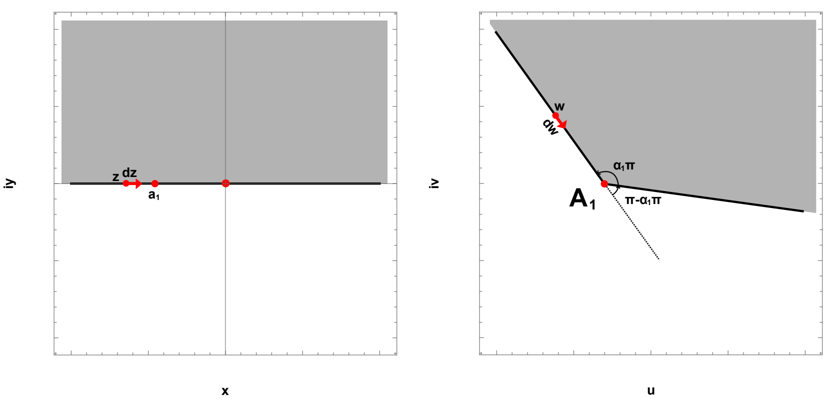
Let $z = re^{i\theta}$. We see that the sides of the triangle in the z-plane correspond to $z = r$ and $z = re^{i\pi}$ $(\forall r \geq 0)$. Applying the transformation $f(z) = z^\alpha$ to the z-plane results in $f(r) = r^\alpha$ (angle 0 in the w plane) and $f(re^{i\pi}) = r^\alpha e^{i\alpha\pi}$ (angle $\alpha\pi$ in the w plane). We see that $f(z)$ transforms $\text{Im}(z) \leq 0$ in the z-plane to the ray from $(0,0)$ at angle $\alpha\pi$ and $\text{Re}(z) > 0$ to $\text{Re}(w) > 0$.

To obtain a mapping from the w-plane to the z-plane we would take the inverse of $f(z)$ to get $g(w) = w^{\frac{1}{\alpha}}$. The restriction on $0 < \alpha < 2$ comes from the fact that we want to restrict the resulting mapping in the z-plane to not overlap with itself.

THEOREM 2.7 (Schwarz–Christoffel). *Let Γ be the piecewise linear boundary of a polygon in the w plane, and let the interior angles at successive vertices be $\alpha_1\pi, ..., \alpha_n\pi$. The transformation is defined by the equation*

$$\frac{dw}{dz} = \gamma(z-a_1)^{\alpha_1-1}(z-a_2)^{\alpha_2-1}\dots(z-a_n)^{\alpha_n-1}$$

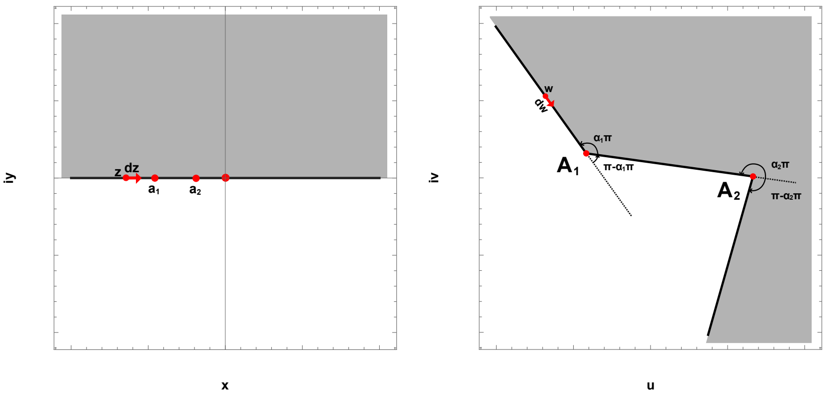
where γ is a complex number and $a_1, ..., a_n$ real numbers, maps Γ into the real axis of the z plane and the interior of the polygon to the upper half of the z plane. The *vertices of the polygon, $A_1, A_2, ..., A_n$, are mapped to the points $a_1, ..., a_n$ on the real axis. The map is an analytic one-to-one conformal transformation between the upper half z plane and the interior of the polygon.*



z-plane w-plane

FIG. 2.8. *A transformation of the upper half z-plane to an open polygon with one angle.*

We provide an informal derivation of SCT. Consider Figure 2.8, first observe the behavior of the vectors dz and dw on the z and w plane respectively. As dz crosses a_1, the phase of dw changes by $\pi - \alpha_1\pi$. We want to construct a function $w = f(z)$ that results in dw changing by $\pi - \alpha_1\pi$ as dz crosses a_1. It is easier if we analyze the behavior of $\frac{dw}{dz} = f'(z)$. Let $dw = \rho e^{i\phi}$ and $dz = re^{i\theta}$, then $f'(z) = \frac{\rho}{r}e^{i(\phi-\theta)}$. We see from Figure 2.8 that $\arg(dz) = \theta = 0$ always, resulting in $\arg(f'(z)) = \phi = \arg(dw)$. We can use this information to see that as we cross a_1 in the z plane, we want $\arg(w)$ to change by $\pi - \alpha_1\pi$. Letting $f'(z) = (z - a_1)^{\alpha_1 - 1}$ results in the aforementioned change occurring at a_1. To see this, let $z - a_1 = re^{i\theta}$, then $f'(z) = (z - a_1)^{\alpha_1-1} = (re^{i\theta})^{\alpha_1-1} = r^{\alpha_1-1}e^{i\theta(\alpha_1-1)}$ and $\arg(f'(z)) = \theta(\alpha_1 - 1)$. We also make use of the fact that before a_1 θ is π, and after a_1 θ is 0. Therefore, the change of $\arg(f'(z))$ as we cross a_1 in the z plane is $\pi - \alpha_1\pi$ which is exactly what we want. Next we consider an open polygon with two angles.



z-plane w-plane

FIG. 2.9. *A transformation of the upper half z-plane to an open polygon with two angles.*

Consider now Figure 2.9 and let $f'(z) = (z-a_1)^{\alpha_1-1}(z-a_2)^{\alpha_2-1}$, $z - a_1 = re^{i\theta}$, and $z - a_2 = \rho e^{i\phi}$. Then,

$$
\begin{aligned}
f'(z) &= (re^{i\theta})^{\alpha_1-1}(\rho e^{i\phi})^{\alpha_2-1} \\
&= r^{\alpha_1-1}e^{i\theta(\alpha_1-1)}\rho^{\alpha_2-1}e^{i\phi(\alpha_2-1)} \\
&= r^{\alpha_1-1}\rho^{\alpha_2-1}e^{i(\theta(\alpha_1-1)+\phi(\alpha_2-1))}
\end{aligned}
\tag{2.14}
$$

resulting in $\arg(f'(z)) = \theta(\alpha_1 - 1) + \phi(\alpha_2 - 1)$. When we cross aa_1 in the z plane, θ changes by $-\pi$ as before while ϕ does not change. This leads to a change in $\arg(f'(z))$ of $\pi - \alpha_1\pi$ just like the previous example. When we cross a_2 in the z plane, ϕ changes by $-\pi$ while θ does not change. This leads to a change in $\arg(f'(z))$ of $\pi - \alpha_2\pi$, which is what we want.

The result we have just shown for an open polygon with two angles is no different than an open polygon with n angles, resulting in the equation in Theorem 2.7. As before, if the interior of a unit circle is desired then a bilinear transformation to z can be applied.

2.4. Jacobi Elliptic Functions. As we increase the number of vertices of the polygon, the SCT can become surprisingly exotic. Even in the case of a rectangle, we can no longer use elementary functions to describe the transformation. It turns out that we can conveniently define an important family of functions called **Jacobi elliptical functions** using SCT. Before illustrating how, let's first motivate Jacobi elliptic functions.

2.4.1. Motivation. Recall that in classical mechanics, we used "small-angle approximation" to solve the pendulum problem as we were told that the seemingly innocuous differential equation $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$ has no closed-form solution. This is in fact a lie; all we need to do to make a function closed-form is to give it a name. It turns out that the "closed-form" functions that allow us to precisely describe the motion of pendulum are the Jacobi elliptic functions (or rather their inverses)! Moreover, SCT is a convenient tool we need to arrive at this conclusion. Let's start with a few definitions:

DEFINITION 2.8. The **incomplete elliptic integral of the first kind** is defined as

$$F(z;k) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}.$$

The parameter k is called the **modulus** of the elliptic integral. When $z = 1$, we call $F(1;k)$ the **complete elliptic integral of the first kind**, denoted as $K(k)$.

REMARK 2.9. Notice that $K(k)$ is usually a constant for a given problem. In our problem, k becomes a fixed parameter once the outer two points on the real line of the z-plane are determined (see Figure 2.10). Thus we use semicolon to remind readers that k is a fixed parameter. This constant will help us simplify some expressions in elliptic integrals.

For convenience, we define:

$$\begin{aligned} K'(k) &= \int_1^{1/k} \frac{d\xi}{\sqrt{(\xi^2-1)(1-k^2\xi^2)}} \\ &= \int_1^{1/k} \frac{d\xi}{\sqrt{-(1-\xi^2)(1-k^2\xi^2)}} \\ &= \int_1^{1/k} \frac{d\xi}{i\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \\ \implies iK'(k) &= \int_1^{1/k} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \end{aligned}$$

Notice

$$F\left(\frac{1}{k};k\right) = K(k) + iK'(k). \tag{2.15}$$

2.4.2. Derivation. Now we will show how SCT gives rise to Jacobi elliptic functions. Consider a rectangle with vertices $A_1(-1+is), A_2(-1), A_3(1), A_4(1+is)$ as shown in Figure 2.10. If we wish to map the interior of this rectangle to the entire upper-half plane, by Theorem 2.7 it suffices to map the four vertices carefully to four points on the real axis via SCT, since the boundary of the polygon must map onto the real line. In this case they map to $a_1(-1/k), a_2(-1), a_3(1), a_4(1/k)$ in that order. We kept A_2, A_3 fixed at $-1, 1$ since we can always perform an affine transformation on the rectangle to move to these simple coordinates for convenience.

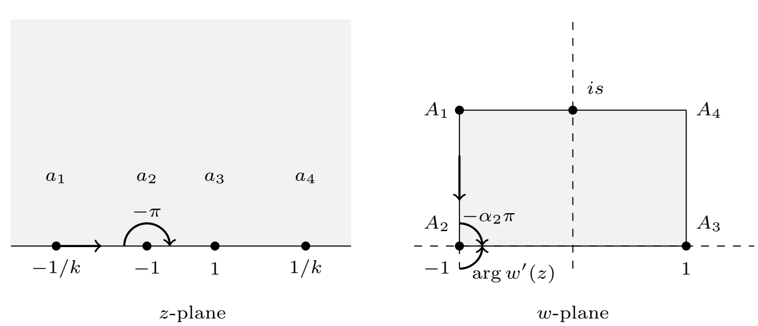


FIG. 2.10. *SCT from a rectangle to the upper-half plane. Here we assume $k \in (0,1)$.*

For the sake of practice, let's apply the heuristic argument again and consider what happens to the SCT when we traverse the vertex a_2 on the real axis of z-plane. In the z-plane, we see that the angle is changed by $-\pi$, yet in the w-plane the angle is only changed by $\arg w'(z) = -\alpha_2\pi - (-\pi) = -\pi(\alpha_2 - 1) = \frac{\pi}{2}$. Thus $\alpha_2 = \frac{1}{2}$. But of course we already know that $\alpha_i = \frac{1}{2}$ for all i from Theorem 2.7. Now let's directly apply Theorem 2.7 and the SCT is described by:

$$\frac{dw}{dz} = \gamma(z-a_1)^{\alpha_1-1}(z-a_2)^{\alpha_2-1}(z-a_3)^{\alpha_3-1}(z-a_4)^{\alpha_4-1} \tag{2.16}$$

$$\frac{dw}{dz} = \gamma(z-(-1/k))^{-\frac{1}{2}}(z-(-1))^{-\frac{1}{2}}(z-1)^{-\frac{1}{2}}(z-1/k)^{-\frac{1}{2}} \tag{2.17}$$

$$w(z;k) = \gamma\int_0^z \frac{d\zeta}{\sqrt{(\zeta^2-1)\left(\zeta^2-\frac{1}{k^2}\right)}} \tag{2.18}$$

$$= \tilde{\gamma}\int_0^z \frac{d\zeta}{\sqrt{(\zeta^2-1)(k^2\zeta^2-1)}} \tag{2.19}$$

$$= \tilde{\gamma}F(z;k) \tag{2.20}$$

We can solve for the constant $\tilde{\gamma}$ by plugging in $w(1;k) = 1$:

$$w(1;k) = \tilde{\gamma}F(1;k) = 1$$
$$\tilde{\gamma} = \frac{1}{K(k)}$$

It is easy to verify that this choice of $\tilde{\gamma}$ will simultaneously map a_2 to A_2 for us as well. Thus we obtain the function that maps the UHP to a generic rectangle with the bottom edge fixed at -1 and 1:

$$w(z;k) = \frac{F(z;k)}{K(k)} \tag{2.21}$$

It remains to map a_1, a_4 to A_1, A_4. Since SCT guarantees a rectangular output, the real components of A_1, A_4 are already fixed by A_2, A_3. Then to complete the mapping we need to find a relationship between k and s. Plug $w(1/k;k) = 1 + is$ into (2.21):

$$w\left(\frac{1}{k};k\right) = \frac{1}{K(k)}F\left(\frac{1}{k};k\right)$$
$$= \frac{1}{K(k)}(K(k) + iK'(k)) \qquad (2.15)$$
$$1 + is = 1 + i\frac{K'(k)}{K(k)}$$
$$\frac{K'(k)}{K(k)} = s$$

Hence, the function that completely describes the mapping from z-plane to w-plane illustrated by Figure 2.10 is

$$w(z;k) = \frac{F(z;k)}{K(k)}, \quad s = \frac{K'(k)}{K(k)} \tag{2.22}$$

Theorem 2.7 also tells us that this function is univalent. That is, it has an analytic inverse function that maps the rectangle to the UHP. It might be hard to imagine what exactly the function looks like, but since we know it exists, we can just give it a name:

DEFINITION 2.10 (elliptic sine). Let $w(z;k) = F(z;k)$ be the "normalized" function (scaled by $K(k)$) that maps the UHP to the rectangle in Figure 2.11. That is,

$$w(z;k) = F(z;k), \quad s = K'(k)$$

We define the **elliptic sine** as the inverse of the incomplete elliptic integral of the first kind. That is,

$$\mathrm{sn}(w;k) = F^{-1}(w;k) = z$$

Voilà! We obtain our first Jacobi elliptic function. This is quite remarkable because SCT not only guarantees the existence of the Jacobi elliptic function as the inverse of the elliptic integral but also allows us to directly visualize an application of the function on the complex plane. Although there are other ways to discover and define Jacobi elliptic functions, the derivation via SCT is perhaps the most convenient and illuminating.

2.5. Properties of the elliptic sine. It turns out that our understanding of complex analysis can help us gain a great deal of insights of the properties of the elliptic sine without too much work. Thus, we now turn to the two hallmark properties of elliptic functions: double-periodicity and single-valueness.

2.5.1. Double-periodicity. We wish to show that

$$\mathrm{sn}(w + n\omega_1 + im\omega_2; k) = \mathrm{sn}(w;k), \quad n, m \in \mathbb{Z} \tag{2.23}$$

The sketch of the proof requires an important concept called **Schwartz reflection principle**. Its most basic form claims the following

PROPOSITION 2.11 (Schwartz reflection principle). *Suppose that $f(z)$ is analytic in a domain D that lies in the upper half z-plane. Let $\tilde{D}$ be the domain obtained by reflecting D about the real axis to the lower half plane. Then then function $\tilde{f}(z) = \overline{f(\overline{z})}$ is analytic in $\tilde{D}$.*

Proof. Suppose $f(z) = u(x,y) + iv(x,y)$, then $\overline{f(\overline{z})} = u(x,-y) - iv(x,-y)$. The continuity of the partial derivatives of $\overline{f(\overline{z})}$ is immediately inherited from u, v and it's easy to check that the Cauchy-Riemann conditions are satisfied in $\tilde{D}$. Thus, $\overline{f(\overline{z})}$ is analytic in $\tilde{D}$. ∎

Let's come back to the rectangle. After normalization, the rectangle has been scaled to Figure 2.11.

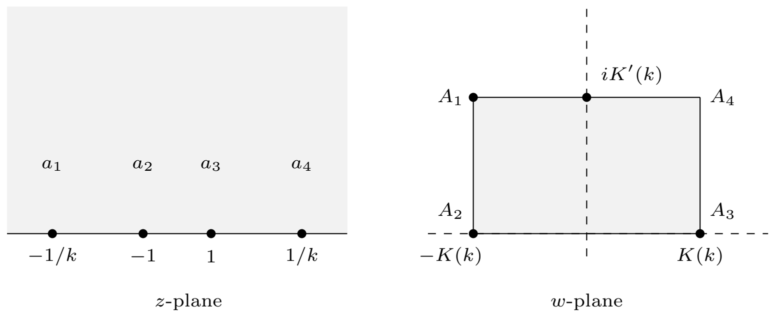


FIG. 2.11. *The function* sn *maps a rectangle with "normalized" coordinates in the w-plane to the upper-half z-plane.*

Let's name this principle rectangle R. As shown in Figure 2.12, if we reflect R about its edge A_3A_4, we obtain a rectangle R_1. This reflection in the w-plane corresponds via sn to the reflection of UHP onto the LHP in the z-plane. Applying the Schwartz reflection principle to $w(z;k)$, we obtain a function $\overline{w(\overline{z};k)}$ analytic in the LHP that maps LHP to R_1. Reflecting R_1 once more about its right edge onto R_2 corresponds to reflecting LHP back to UHP in the z-plane, yielding the same function outputs of sn from R. Thus, we show that $\mathrm{sn}(0 + 2(2K(k)) + i0) = \mathrm{sn}(0)$, and the periodicity of $4K(k)$ in the real component comes from the fact that it requires two reflections about the right edge to get back to the identity (UHP) in the z-plane. It's easy to see that the same idea holds for reflecting about the horizontal edges and gives us periodicity of $2K'(k)$ in the imaginary component. Taken together, we obtain double-periodicity:

$$\mathrm{sn}(w + n(4K(k)) + im(2K'(k)); k) = \mathrm{sn}(w; k), \quad n, m \in \mathbb{Z} \tag{2.24}$$

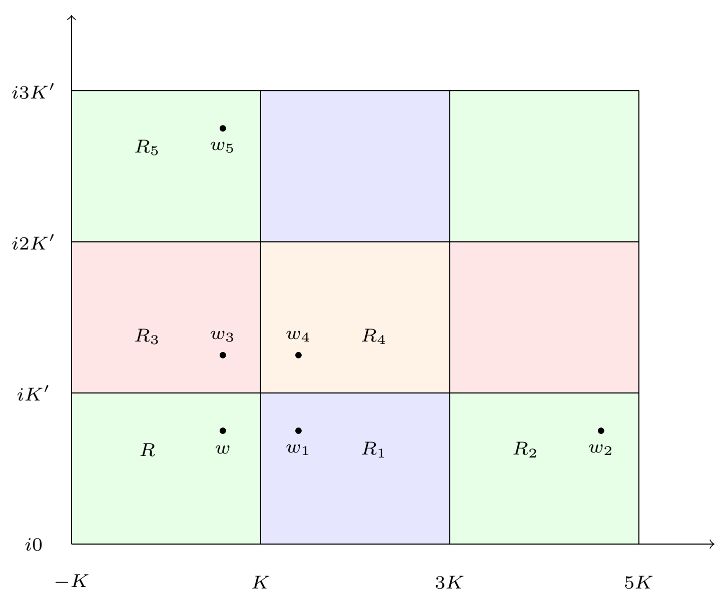


FIG. 2.12. *Double-periodicity of sn in the w-plane.*

What happens if we reflect once about a vertical and horizontal edges each? As shown in Figure 2.12, this gives us R_4. Clearly R_4 is not equivalent to R. This can also be understood via the image of sn in the z-plane. For example, we know that reflecting R about the right edge onto R_1 corresponds to reflecting UHP onto LHP about the real axis. Now the horizontal edges of R_1 are flipped. Thus when we reflect about the top edge of R_1, in the z-plane we reflect LHP back to UHP about the reversed real axis, so the UHP we obtain is also reversed in the real component. Thus, the rectangle R_4 corresponds to the UHP with the real part reversed.

By double-periodicity of sn, we can cover the entire w-plane with rectangles obtained from reflecting R over and over again. Via sn, an even number of reflections gives us rectangles that map to the UHP, and an odd number of reflections gives us rectangles that map to the LHP. Therefore, any *period rectangle* consisting of four rectangles meeting at a corner, such as R_1, R_2, R_3, R_4 in Figure 2.12, cover the z-plane twice via sn.

2.5.2. Single-valuedness. We know that the elliptical integral transform and the Jacobi elliptic function sn are a pair of univalent mappings between the UHP in the z-plane and the principle rectangle R in the w-plane. Specifically, a w in R maps to a single z in the UHP. Double-periodicty allows us to extend this further to any reflected rectangle in the w-plane. That is, sn maps any w from any rectangle in the w-plane to a single z in the z-plane. Care needs to taken at the singular points of $w(z;k)$, *i.e.* the prevertices of R, because they are excluded from the univalent map. We can show that they are indeed single-valued by considering that in a small neighborhood near the isolated singular point $z = a_i$, the only term in $\frac{dw}{dz}$ that matters is $(z-a_i)^{-1/2}$. Intuitively, the other terms are too far away and contribute little to the change of w with respect to z. Hence, if we integrate $\frac{dw}{dz}$ by treating other terms as constant, then in this neighborhood $w(z;k)$ can be expressed by $(z-a_i)^{\frac{1}{2}}g(z)$, where $g(z)$ is analytic and thus has a Taylor series expansion about $z = a_i$. :

$$
\begin{aligned}
w - A_i &= (z-a_i)^{\frac{1}{2}} \sum_{k=0}^{\infty} c_i^{(k)} (z-a_i)^k \\
(z-a_i)^{\frac{1}{2}} &= \frac{w - A_i}{\sum_{k=0}^{\infty} c_i^{(k)} (z-a_i)^k} =: \sigma(w) \\
z - a_i &= \sigma(w)^2 \\
z &= \sigma(w)^2 + a_i
\end{aligned}
$$

Clearly for any given w, $\sigma(w)$ is single-valued, therefore so is z. This means that sn is a *single-valued function* for all w. This result is not obvious at all just by looking at its definition, but we arrive at it almost for free thanks to the power of complex analysis.

2.6. Doubly-periodic functions with other regular polygons. The double periodicity of the Jacobi elliptic functions comes from applying the Schwarz Reflection Principle to the "original" rectangle in the w-plane. Since a rectangle can be repeatedly reflected to cover the whole w-plane without gaps, the double periodicity is established. This can line of thought can be immediately generalized to other polygons which can cover the plane without gaps. The only polygons for which this is possible are the (regular) triangle and hexagon.

In deriving the Schwarz-Christoffel Transformation (SCT) it was convenient to use the upper-half plane (UHP) since the argument of the real axis is constant. However, when dealing with more than three vertices, it is impossible to know where to place the remaining pre-vertices unless there is a high degree of symmetry, as was leveraged in mapping to a rectangle. For our purposes, we can further increase the symmetry of the SCT if we instead map from a unit disk, rather than the UHP. This will allow us to choose the pre-vertices in such a way that no numerical computation is required. Furthermore, we will find an elegant solution for mapping to any regular polygon.

Let's begin by deriving the SCT for the unit disk. The bilinear transformation in (2.12) maps between the UHP in the z-plane and the unit disk in the ζ-plane. Using the chain rule,

$$\frac{dw}{d\zeta} = \frac{dz}{d\zeta}\frac{dw}{dz}$$

Direct substitution of (2.12) gives us,

$$\begin{aligned}\frac{dw}{d\zeta} &= \frac{2i}{(1-\zeta)^2}\gamma\prod_{k=1}^{n} i\left(\frac{1+\zeta}{1-\zeta} - \frac{1+\zeta_k}{1-\zeta_k}\right)^{\alpha_k-1} \\ &= \frac{\gamma}{(1-\zeta)^2}\prod_{k=1}^{n}\left(\left(\frac{1-\zeta_k}{2}\right)\frac{\zeta-\zeta_k}{1-\zeta}\right)^{\alpha_k-1}\end{aligned}$$

where constants are absorbed into γ. Notice that in the second line, the first term in parentheses is a constant. Then the SCT becomes,

$$\frac{dw}{d\zeta} = \frac{\gamma}{(1-\zeta)^2}\prod_{k=1}^{n}(1-\zeta)^{1-\alpha_k}(\zeta-\zeta_k)^{\alpha_k-1}$$

Recall that the sum of the exterior angles of a polygon must be 2π. That is, $\sum_{k=1}^{n}(1-\alpha_k) = 2$. So the SCT takes on a familiar form,

$$\frac{dw}{d\zeta} = \gamma\prod_{k=1}^{n}(\zeta-\zeta_k)^{\alpha_k-1}$$

The SCT for mapping from a disk is identical to the SCT for mapping from the upper half plane! Now we derive the SCT for mapping to any regular n-gon. The natural choice for the pre-vertices are the n^{th}-roots of unity. These points are of course also the vertices of the n-gon. This induces the high degree of symmetry that we need to avoid numerical computation. In particular, if we rotate the circle by $2\pi/n$ radians, every pre-vertex is taken to another pre-vertex. Additionally, the angle at every vertex has the same magnitude. Thus every pre-vertex is, in a sense, identical to the other pre-vertices; we only need to consider the action of the SCT on a single pre-vertex. The pre-vertices are the points,

$$\zeta_k = e^{i\frac{2\pi k}{n}}, k = 1, 2, ..., n$$

Every angle has the same magnitude and the sum of the interior angles, $\pi\alpha_k$, is $\pi(n-2)$ so,

$$\alpha_k = 1 - \frac{2}{n}$$

Substituting these numbers into the SCT,

$$\frac{dw}{d\zeta} = \gamma \prod_{k=1}^{n} \left(\zeta - e^{i\frac{2\pi k}{n}}\right)^{(1-\frac{2}{n})-1} \tag{2.25}$$

$$= \gamma \left(\prod_{k=1}^{n} \left(\zeta - e^{i\frac{2\pi k}{n}}\right)\right)^{-\frac{2}{n}} \tag{2.26}$$

Define the function $g(\zeta) = \zeta^n - 1$. By the Fundamental Theorem of Algebra, we can factor $g(\zeta)$ as the product of n terms with the form $(\zeta - r_k)$, where r_k is a root of $g(\zeta)$. But the roots of $g(\zeta)$ are just the n-roots of unity, so we can replace the product in the SCT with $g(\zeta)$, giving us the equation,

$$\frac{dw}{d\zeta} = \gamma(\zeta^n - 1)^{-\frac{2}{n}}$$

Finally, we integrate $dw/d\zeta$ from $\zeta = 0$ to ζ.

$$w(\zeta) = A + \gamma \int_0^{\zeta} \frac{d\epsilon}{(\epsilon^n - 1)^{\frac{2}{n}}}$$

where A and γ allow us to translate and scale the n-gon as necessary. This particular Schwarz-Christoffel Transformation is now complete; the unit circle can be mapped to any regular n-gon.

2.6.1. SCT of regular triangle and hexagon. As mentioned earlier, we are interested in mapping to triangles and hexagons since they are the only other regular polygons which can completely tile a two-dimensional plane.

Following the line of thinking from the "rectangular" SCT, we investigate the inverse function $\zeta(w) \equiv w^{-1}(\zeta)$. For the hexagonal SCT, we immediately encounter an issue when trying to analytically continue $\zeta(w)$ to the rest of the w-plane. When we tile the w-plane with hexagons, an odd number of hexagons (3) meet at each vertex.

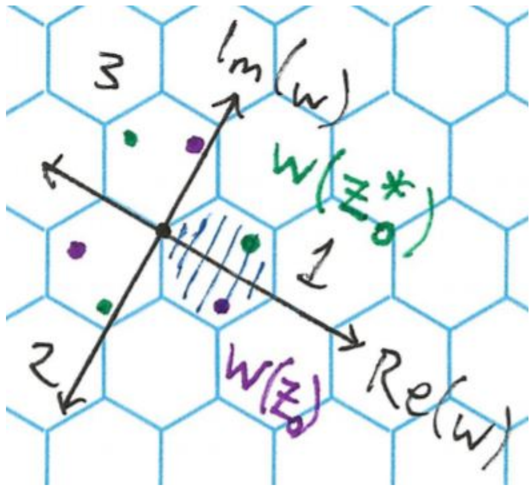


Fig. 2.13. *Hexagon Reflection in the w-plane*

Each time we reflect the hexagon in the w-plane we switch between the interior and exterior of the unit circle in the ζ-plane. In Figure 2.13, we begin in hexagon 1 with $w_0 = w(\zeta_0)$. We reflect around the origin, switching colors to represent the switch between the interior and exterior of the unit circle in the ζ-plane. When we return to hexagon 1, we find that the corresponding point is not w_0, but rather $w_0^* = w(\zeta_0^*)$, where ζ_0^* is the reflection of ζ_0 across the unit circle. However, each point in hexagon 1 is already associated with a point inside the ζ unit circle. Therefore, the inverse function, $\zeta(w)$, is multi-valued.

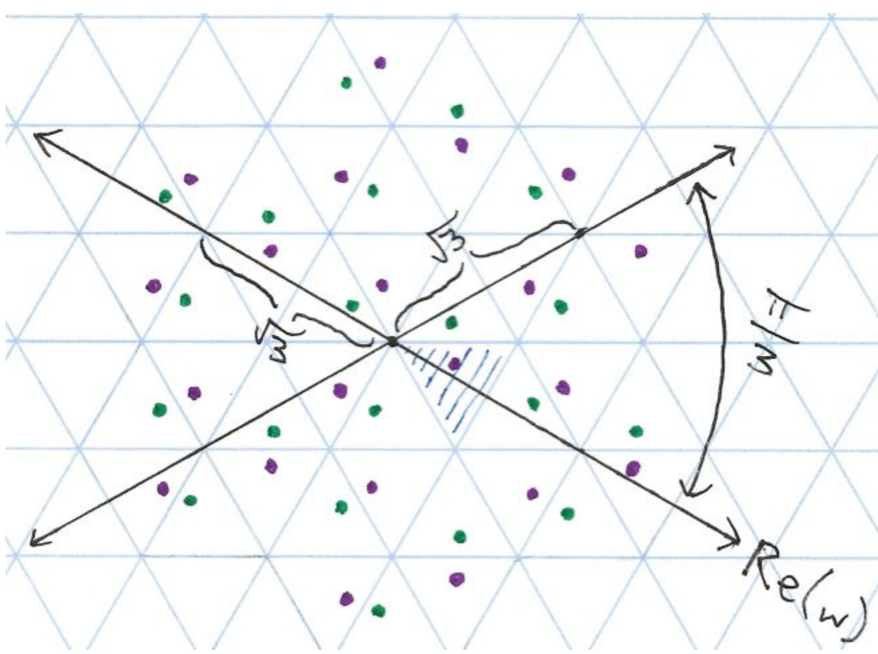


Fig. 2.14. *Triangle Reflection in the w-plane*

When analyzing the SCT for the triangle, we do not encounter this issue. For the triangular tiling of the w-plane, an even number of triangles (6) meet at each vertex. Thus, when we reflect the "original", shaded triangle around a vertex back onto itself, we do return to the interior of the unit circle in the ζ-plane. Hence the inverse function is single-valued. Furthermore, if we mark an arbitrary point in the "original" triangle and keep track of it as we tile the w-plane, we find that $\zeta(w)$ has interesting periodicity properties. For convenience, we choose A and γ such that there is a vertex at the origin in the w-plane and each side is of unit length.

From the diagram we see that we can translate the "original" point in two linearly independent directions, one along the real w-axis and the other along a line that makes a 60 degree angle with the real axis. Using the Pythagorean Theorem, we find that we must translate a distance of $\sqrt{3}$ in either direction. Thus the inverse function $\zeta(w)$ is doubly periodic,

$$\zeta(w) = \zeta(w + n\sqrt{3} + m\sqrt{3}e^{\frac{i\pi}{3}})$$

for integers m and n. However, these translations do not get us every reflected point. Note that this is different from the case in Figure 2.12, where w_4 corresponds to the original w, but does not sit in its respective rectangle with the same orientation as w. Instead, for the triangle, the "original" point can be rotated about the origin while maintaining proper orientation in its respective triangle. Thus, we have an additional symmetry, namely rotation. The angle of rotation is $e^{2\pi i/3}$. Taking into account this rotational symmetry,

$$\zeta(w) = \zeta(e^{p\frac{2\pi i}{3}}w + n\sqrt{3} + m\sqrt{3}e^{\frac{i\pi}{3}})$$

where m, n, and p are integers.

3. Conclusion. We learned a lot about transformations in complex variables through the creation of this report. A significant amount of time and care was put into learning about the topics explained in this report before putting pen to paper (or, fingers to keyboards in this case). As a result, we finish the semester strong having knowledge we gained from this complex variables course in addition to transformations (conformal, non-conformal and SCT), elliptic functions/integrals, and some of their applications.

We learned about transformations as a dense subject area with many different applications and directions. In this project, we first learned about transformations in the general sense, then we delved slightly deeper when talking about conformality. Conformality is an interesting result that gives insight into the behavior of different types of transformations. Finally, we reached SCT, which is one of the main subjects of this project. With SCT, we understood the heuristic derivation and the magnitude of its result complementary to the Riemann Mapping Theorem.

One of the biggest takeaways from this project (and this course) is that complex analysis is somewhat of a misnomer. Even though the subject itself is rich of deep results, it often greatly simplifies understanding of "complex" topics. We see that defining elliptic functions solely as the inverse of the elliptic integrals would seem perplexing and unmotivated. We might wonder why the inverse exists in the first place. However, complex analysis, particularly SCT, allows us to understand the univalent mapping between the elliptic integral and the elliptic sine through intuitive visualization. We can then understand the properties of elliptic functions just by reflecting rectangles around. How accessible is that?! The massive conceptual simplification of these challenging topics via complex analysis is remarkable and immensely satisfying.

Future directions certainly include the plethora of other applications that use the material covered here. It turns out that SCT of an arbitrary polygon with more than 4 vertices does not have a nice analytic solution and requires numerical methods to solve. Moreover, we can generalize SCT on a polygon to a curvilinear polygon and allow us to harness the power of SCT to a wider range of problems. For one, transformations can be studied further to model plane airfoils, which are curvilinear polygons. This transformation was critical in the analysis of lift in airplanes in the field of aerospace engineering. Other applications of elliptic integrals/functions would also be useful to explore, such as re-visiting the pendulum problem, finding the perimeter of an ellipse, or finding the perihelion of an orbiting body (ie. Mercury around the Sun).

Further analytic knowledge can be gained by exploring other types of transformations in complex variables and other types of elliptic functions. For one, the Mobius transformation can be analyzed and its applications to hyperbolic geometry. Also, Weierstrass elliptic functions would be important to consider in relation to physical phenomena like differential equations and movement of a spherical pendulum.

Another type of Jacobi elliptic function is the elliptic cosine, cn. These functions have been used to represent a solution to the Korteweg-de Vries equation, the so-called cnoidal waves. In particular one can physically see the double periodicity of cn in the cnoidal waves. The "hexagonal" and "triangular" SCT's could then perhaps arise in a solution to other differential equations, where rotational symmetry would have some physical significance.

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[1] M. J. Ablowitz, A. S. Fokas, and A. S. Fokas, *Complex variables: introduction and applications*, Cambridge University Press, 2003.