Electrodynamics I

Fall Semester 2023

Prof. Dr. Martin Spinrath
TA in charge: Jean Nelson
(jeannelson247@gmail.com)

Exercise sheet 7 (bonus)
Due date: 11.01.2024, 1pm online (google classroom)
Please write your name and student number on every page of the solution.

Problem 15: (20 bonus points) Dynamics of Particles in Electromagnetic Fields
There is many applications where one wants to control the motion of a charged particle. This can be achieved using electromagnetic fields and is the subject of this exercise. The equations of motion for a particle with charge q and mass m are

$$m \frac{\mathrm{d}\,\vec{v}}{\mathrm{d}\,t} = q\,\vec{E} + q\,\vec{v} \times \vec{B}\;, \tag{1}$$

$$\frac{\mathrm{d}\,E}{\mathrm{d}\,t} = q\,\vec{v} \cdot \vec{E}\;, \tag{2}$$

with $\vec{v}$ the velocity of the particle, $\vec{p} = m\,\vec{v}$ its momentum and E its energy. The electric field is $\vec{E}$ and the magnetic field is $\vec{B}$. Note that we use SI units here. Please use SI units in your solution as well.

1. Consider a constant magnetic field $\vec{B} = B_0\,\hat{e}_z$ and constant electric field $\vec{E} = E_0\,\hat{e}_y$. Calculate and plot the position of the particle using the boundary conditions

$$\vec{v}(t=0) = (E_0/B_0, -v_0, 0) \text{ and } \vec{x}(t=0) = (v_0/\Omega, 0, 0) \tag{3}$$

with $\Omega \equiv q\,B_0/m$. What is the drift (linear) velocity $\vec{v}_D$ of the particle? If you consider a charge neutral plasma, would you generate a net current applying this field configuration to the plasma?

2. Consider now a time-dependent magnetic field $\vec{B} = B_0(t)\,\hat{e}_z$ and $E_0 = 0$. What are the equations of motions in this case? Note that you still need to respect Maxwell's equations. We can define a perpendicular energy increase

$$\Delta W_\perp = \oint q\vec{E} \cdot \mathrm{d}\,\vec{l} \tag{4}$$

where the path is a circular orbit around the z-axis. Show that approximately

$$\frac{\mathrm{d}\,\mu}{\mathrm{d}\,t} = 0\;, \tag{5}$$

with $\mu \equiv W_\perp/B_0$. This quantity is an adiabatic invariant.

Plot $y(x)$ and $\mathrm{d}\,\mu(t)/\mathrm{d}\,t$ for $B_0(t) = 1 + 0.1\,t$, $m = q = 1$, $\vec{x}(0) = (1, 0, 0)$ and $\vec{v}(0) = (0, -1, 0)$ in a time interval $t \in [0, 100]$. You can solve the equations of motion numerically.

3. What happens when the perpendicular energy, $W_\perp$, is equal to the total energy, i.e., $E = W_\perp$ and the parallel energy $W_\| = 0$?

   As a concrete example consider the magnetic field

$$\vec{B} = (B_0 + \alpha\, z)\, \hat{e}_z - \frac{\alpha}{2} \rho\, \hat{e}_\rho \tag{6}$$

   in cylindrical coordinates $\{\rho, \varphi, z\}$. Plot $x(z)$ and $x(y)$ for the initial conditions $x(0) = 1$, $y(0) = 0$, $z(0) = 0$, $v_x(0) = 0$, $v_y(0) = -1$, $v_z(0) = 1$ in the time range $t \in [0, 40]$ and take $q = m = B_0 = 1$ and $\alpha = 0.1$. The point where $E = W_\perp$ corresponds to which point in this plots? Note that the magnetic field is not time-dependent here.