Twisted Reed-Solomon Codes

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Abstract—In this article, we present a new construction of evaluation codes in the Hamming metric, which we call twisted Reed—Solomon codes. Whereas Reed—Solomon (RS) codes are MDS codes, this need not be the case for twisted RS codes. Nonetheless, we show that our construction yields several families of MDS codes. Further, for a large subclass of (MDS) twisted RS codes, we show that the new codes are not generalized RS codes. To achieve this, we use properties of Schur squares of codes as well as an explicit description of the dual of a large subclass of our codes. We conclude the paper with a description of a decoder, that performs very well in practice as shown by extensive simulation results.

Index Terms—MDS Codes, Reed-Solomon Codes, Evaluation Codes, Decoding, Code Equivalence, Dual Codes

I. INTRODUCTION

Maximum distance separable (MDS) codes are error-correcting codes with a particularly large minimum distance. More precisely, they are linear [n,k,d] codes over a finite field \mathbb{F}_q where d=n-k+1, i.e., meeting the Singleton bound. The well known family of generalized Reed–Solomon (GRS) codes are MDS codes, thus giving examples of MDS codes of length up to q+1. Other known MDS codes have been constructed from n-arcs in projective geometry [3], circulant matrices [4], or Hankel matrices [4]. In this paper, we consolidate and extend the study of twisted Reed–Solomon (twisted RS) codes initiated in the conference papers [1], [2]. This new code family is inspired by Sheekey's twisted Gabidulin codes [5], a class of rank-metric codes. The class of twisted RS codes contains several subfamilies of long MDS codes.

More precisely, after giving some needed preliminaries in the second section, we introduce the class of twisted RS codes in Section III. After this, we study several special cases in the fourth section that give rise to MDS codes. In Section V, we present results on the duals of twisted RS-codes. We compare

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twisted RS codes with GRS codes in the sixth section. We are able to give various families of MDS twisted RS codes that are not monomially equivalent to GRS codes. Our main tool for this will be the Schur square of a code, which has low dimension for a GRS code, but can have a large dimension for a twisted RS code. In the last section, we discuss decoding of twisted RS codes and indicate a decoder that works very well in practice.

While working on this paper, related papers on twisted RS codes have begun to appear. In [6], a construction was presented that can give slightly longer MDS twisted RS codes by modifying two of our special classes of MDS twisted RS codes. Further, one-twisted RS codes were used for obtaining LCD MDS codes. New non-GRS LCD MDS codes based on twisted RS codes were also presented in [7]. In [8], self-dual MDS and near MDS codes were constructed using twisted RS codes for t = 1 and h = k - 1. For this twist and hook, a parity-check matrix was given as well. An AG variant of twisted RS codes was investigated in [9] using codes coming from the Hermitian curve. Further results on twisted Reed-Solomon codes can also be found in the dissertation of the second author [10]. In [11], the construction of twisted RS codes with multiple twists was used to further generalize the class of twisted Gabidulin codes [5] in the rank metric. Recently, a twisted variant of linearized Reed-Solomon codes (a mix of Reed-Solomon and Gabidulin codes, considered in the sumrank metric) was proposed in [12].

II. PRELIMINARIES

For $\alpha = [\alpha_1, \dots, \alpha_n] \in \mathbb{F}_q^n$, we define the evaluation map

$$\operatorname{ev}_{\alpha}(\cdot) : \mathbb{F}_{q}[X] \to \mathbb{F}_{q}^{n},$$

$$f \mapsto [f(\alpha_{1}), \dots, f(\alpha_{n})].$$

Note that $\operatorname{ev}_{\alpha}(\cdot)$ is an \mathbb{F}_q -linear map. If the α_i are distinct, then the restriction of $\operatorname{ev}_{\alpha}(\cdot)$ to polynomials of degree < n, i.e., $\operatorname{ev}_{\alpha}(\cdot) \big|_{\mathbb{F}_q[X]_{< n}}$, is bijective.

For distinct evaluation points $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$ and arbitrary column multipliers $v_1, \ldots, v_n \in \mathbb{F}_q^*$, the corresponding *generalized Reed–Solomon (GRS) code* of dimension k is defined by

$$C_{\mathsf{GRS}} = \mathrm{ev}_{\alpha}(\mathbb{F}_q[X]_{\leq k}) \cdot \mathrm{diag}(v_1, \dots, v_n).$$

Two linear codes are called (monomially) equivalent if one code can be obtained from the other by permutation of codeword positions and entry-wise multiplication with non-zero field elements. In particular, any GRS code is equivalent to an RS code.

III. TWISTED REED-SOLOMON CODES

In this section, we define twisted Reed–Solomon codes and show some of their properties.

Definition 1 Let $n, k, \ell \in \mathbb{N}$ be positive integers with k < n. We call ℓ the *number of twists*. Futhermore, choose three vectors

- $t = [t_1, \dots, t_\ell] \in \{1, \dots, n-k\}^\ell$ (called twist vector),
- $h = [h_1, ..., h_\ell] \in \{0, ..., k-1\}^{\ell}$ (called *hook vector*),
- $\eta = [\eta_1, \dots, \eta_\ell] \in \mathbb{F}_q^\ell$ (called *coefficient vector*)

such that the tuples $[h_i, t_i]$ for $i = 1, ..., \ell$ are distinct. We define the set of $[k, t, h, \eta]$ -twisted polynomials by

$$\mathcal{P}_{\mathbf{t},h,\eta}^{n,k} = \left\{ f = \sum_{i=0}^{k-1} f_i X^i + \sum_{j=1}^{\ell} \eta_j f_{h_j} X^{k-1+t_j} : f_i \in \mathbb{F}_q \right\}.$$

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$ be distinct and write $\alpha = [\alpha_1, \ldots, \alpha_n]$. The corresponding $[\alpha, t, h, \eta]$ -twisted Reed-Solomon code is defined by

$$\mathcal{C}^{n,k}_{\boldsymbol{lpha},\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}} := \operatorname{ev}_{\boldsymbol{lpha}} \Big(\mathcal{P}^{n,k}_{\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}} \Big) \subseteq \mathbb{F}_q^n.$$

For brevity, we often say *twisted polynomials* and *twisted RS codes*, respectively.

Lemma 1 The set of $[k, t, h, \eta]$ -twisted polynomials $\mathcal{P}_{t,h,\eta}^{n,k}$ is a k-dimensional subspace of $\mathbb{F}_q[X]$. A basis of $\mathcal{P}_{t,h,\eta}^{n,k}$ is given by

$$g_i := X^i + \sum_{\substack{j=1\\h_i = i}}^{\ell} \eta_j X^{k-1+t_j} \tag{1}$$

for i = 0, ..., k - 1.

Proof: For any $f = \sum_{i=0}^{k-1} f_i X^i + \sum_{j=1}^\ell \eta_j f_{h_j} X^{k-1+t_j} \in \mathcal{P}^{n,k}_{\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}}$, we can write $f = \sum_{i=0}^{k-1} f_i g_i$, where $f_i \in \mathbb{F}_q$. Furthermore, $g_i \in \mathcal{P}^{n,k}_{\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}}$ and the g_i are linearly independent since the monomial X^i appears in g_i only for each $i=0,\ldots,k-1$ (note that $k-1+t_j>k-1$).

Proposition 2 A $[\alpha, t, h, \eta]$ -twisted Reed–Solomon code $\mathcal{C}^{n,k}_{\alpha,t,h,\eta}$ is a linear [n,k] code. With $g_0,\ldots,g_{k-1}\in\mathbb{F}_q[X]$ as in (1), the matrix

$$G := \begin{bmatrix} \operatorname{ev}_{\alpha}(g_0) \\ \vdots \\ \operatorname{ev}_{\alpha}(g_{k-1}) \end{bmatrix} \in \mathbb{F}_q^{k \times n}$$
 (2)

is a generator matrix of $\mathcal{C}^{n,k}_{\alpha,t,h,\eta}$.

Proof: Since $\operatorname{ev}_{\alpha}(\cdot)$ is \mathbb{F}_q -linear and $\mathcal{P}^{n,k}_{t,h,\eta}$ is an \mathbb{F}_q -vector space, the code $\mathcal{C}^{n,k}_{\alpha,t,h,\eta}$ is linear. Furthermore, we have $\deg f < n$ for all $f \in \mathcal{P}^{n,k}_{t,h,\eta}$ due to $t_i \leq n-k$. Hence, $\operatorname{ev}_{\alpha}(\cdot)$ is injective on the evaluation polynomials, which implies $\dim_{\mathbb{F}_q}\left(\mathcal{C}^{n,k}_{\alpha,t,h,\eta}\right) = \dim_{\mathbb{F}_q}\left(\mathcal{P}^{n,k}_{t,h,\eta}\right) = k$. The same argument implies that the $\operatorname{ev}_{\alpha}(g_i)$ are a basis of $\mathcal{C}^{n,k}_{\alpha,t,h,\eta}$, i.e., G is in fact a generator matrix.

Remark 3 Some remarks about Definition 1.

- Twisted RS codes are not related to twisted BCH codes as defined in [13]. The name is inspired by Sheekey's twisted Gabidulin codes [5], which are related to generalized twisted fields.
- The condition that the tuples $[h_i, t_i]$ are distinct is no restriction in general. Assume that $[h_i, t_i] = [h_j, t_j]$ for some $i \neq j$. Then we obtain the same code by removing h_j, t_j, η_j from the twist, hook and coefficient vector (note that the number of twists decreases), respectively, and replacing η_i by $\eta_i + \eta_j$. We can repeat this process until all tuples are distinct.
- Setting $\eta_i \neq 0$ for all i is in principle no restriction if we are interested in codes that are not obviously RS codes. However, we allow the η_i to be 0 such that the family of twisted codes includes RS codes in a natural way.
- The restriction $t_i \leq n-k$ is not necessary for $\operatorname{ev}_{\alpha}(\cdot)$ to be injective on $\mathcal{P}^{n,k}_{t,h,\eta}$. Hence, it might be possible to relax this condition. A necessary and sufficient condition is that the polynomials

$$g_i \mod \prod_{j=1}^n (X - \alpha_j)$$

for $i=0,\ldots,k-1$ with g_i as in (1) are linearly independent. This condition is obviously fulfilled if $\deg g_i < n$ since $\deg \prod_{j=1}^n (X-\alpha_j) = n$, but it would require rather technical conditions on α , t, h, and η to guarantee it for for $\deg g_i \geq n$.

Example 4 We give three example generator matrices. For easier notation, we write $\alpha^i := [\alpha_1^i, \alpha_2^i, \dots, \alpha_n^i]$.

For $\eta = 0$, we obtain a Reed–Solomon code since the basis of $\mathcal{P}^{n,k}_{t,h,\eta}$ given in Lemma 1 is $g_i = X^i$ and, hence, the generator matrix in Proposition 2 is a Vandermonde matrix

$$oldsymbol{G} = egin{bmatrix} oldsymbol{lpha}^0 \ oldsymbol{lpha}^1 \ dots \ oldsymbol{lpha}^{k-1} \end{bmatrix} = egin{bmatrix} lpha_1^0 & \dots & lpha_n^0 \ lpha_1^1 & \dots & lpha_n^1 \ dots & \ddots & dots \ lpha_1^{k-1} & \dots & lpha_n^{k-1} \end{bmatrix}.$$

For q = n = 9, k = 5, $\ell = 1$, $h_1 = 2$, $t_1 = 2$, and η_1 a non-square of \mathbb{F}_9 , we obtain a punctured Glynn's code [14] (by evaluating "at infinity" (cf. Remark 16) in addition, we get exactly Glynn's code). Glynn's code is the first-known MDS code with odd field size, length n = q + 1, and dimension $3 \le k \le q - 1$ that is not a Generalized Reed–Solomon code. The generator matrix in Proposition 2 is given by

$$oldsymbol{G} = egin{bmatrix} oldsymbol{lpha}^0 & oldsymbol{lpha}^1 \ oldsymbol{lpha}^2 + \eta_1 oldsymbol{lpha}^6 \ oldsymbol{lpha}^3 \ oldsymbol{lpha}^4 \end{bmatrix}$$

¹This means that the t and h vectors may have repeated entries, just not in the same coordinates

For $q \ge n > 7$, k = 5, t = [1, 3, 3], and h = [4, 4, 2], the generator matrix in Proposition 2 is of the form

$$m{G} = egin{bmatrix} m{lpha}^0 \ m{lpha}^1 \ m{lpha}^2 + \eta_3 m{lpha}^7 \ m{lpha}^3 \ m{lpha}^4 + \eta_1 m{lpha}^5 + \eta_2 m{lpha}^7 \end{bmatrix}.$$

IV. MDS TWISTED RS CODES

Not all twisted RS codes are MDS. In this section, we give several families of MDS twisted RS codes.

A. A General MDS Condition

Definition 2 Let $\mathbb{F}_q/\mathbb{F}_{q_0}$ be a field extension. A vector $\eta \in \mathbb{F}_q^\ell$ is called \mathbb{F}_{q_0} -sum-product free if

$$\sum_{\substack{\mathcal{S} \subseteq \{1, \dots, \ell\} \\ \mathcal{S} \neq \emptyset}} a_{\mathcal{S}} \prod_{i \in \mathcal{S}} \eta_i \notin \mathbb{F}_{q_0}^* \quad \forall \, a_{\mathcal{S}} \in \mathbb{F}_{q_0}.$$

Equivalently, a vector η is \mathbb{F}_{q_0} -sum-product free exactly when there is no polynomial $f \in \mathbb{F}_{q_0}[X_1,\ldots,X_\ell]$ with non-zero constant coefficient and of degree at most 1 in each X_i such that $f(\eta_1,\ldots,\eta_\ell)=0$.

Proposition 5 Let $\mathbb{F}_q/\mathbb{F}_{q_0}$ be an extension of finite fields. Let $k < n \leq q_0$ and let $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_{q_0}$ be distinct. For any t, h and η chosen as in Definition 1 and such that η is \mathbb{F}_{q_0} -sumproduct free, then the twisted RS codes $\mathcal{C}_{\alpha,t,h,\eta}^{n,k}$ is MDS.

Proof: Let G be the generator matrix of $\mathcal{C}_{\alpha,t,h,\eta}^{n,k}$ given in (2). Since each $\alpha_i \in \mathbb{F}_{q_0}$, we can consider the entries of G to be in $\mathbb{F}_{q_0}[\eta_1,\ldots,\eta_\ell]$. The code $\mathcal{C}_{\alpha,t,h,\eta}^{n,k}$ is MDS if and only if every $k\times k$ minor of G is non-zero. Observe that each η_i appears in exactly one row of G, and hence as a polynomial in η_1,\ldots,η_ℓ , any such $k\times k$ minor has degree at most one in each variable. Moreover, its constant term is non-zero since setting $\eta_1=\ldots=\eta_\ell=0$ yields an RS code which is MDS, and hence has non-zero $k\times k$ minors. Since η is \mathbb{F}_{q_0} -sum-product free, then every such expression is non-zero.

The following gives two constructions of \mathbb{F}_{q_0} -sum-product free sets:

Proposition 6 Let $\mathbb{F}_{q_0} \subsetneq \mathbb{F}_{q_1} \subsetneq \ldots \subsetneq \mathbb{F}_{q_\ell} = \mathbb{F}_q$ be a proper chain of subfields. Let $\eta_1, \ldots, \eta_\ell$ be chosen with the condition that $\eta_i \in \mathbb{F}_{q_i} \backslash \mathbb{F}_{q_{i-1}}$. Then $\boldsymbol{\eta} := [\eta_1, \ldots, \eta_\ell]$ is \mathbb{F}_{q_0} -sum-product free.

Proof: We prove the claim by induction on ℓ . If $\ell=1$, we have $a\eta_1\notin \mathbb{F}_{q_0}^*$ for any $a\in \mathbb{F}_{q_0}$ since $\eta_1\notin \mathbb{F}_{q_0}$. For the inductive step, we can split any sum product

$$A := \sum_{\substack{\mathcal{S} \subseteq \{1,\dots,\ell\} \\ \mathcal{S} \neq \emptyset}} a_{\mathcal{S}} \prod_{i \in \mathcal{S}} \eta_i = a(\eta_1,\dots,\eta_{\ell-1}) + \eta_{\ell} b(\eta_1,\dots,\eta_{\ell-1}),$$

where $a,b \in \mathbb{F}_{q_0}[X_1,\ldots,X_{\ell-1}]$ are polynomials with degree at most one in each variable X_i and a has zero constant term (i.e., $a(\eta_1,\ldots,\eta_{\ell-1})$ is a sum-product of $\eta_1,\ldots,\eta_{\ell-1}$). By the inductive step and the choice of the η_i , we have $a(\eta_1,\ldots,\eta_{\ell-1}) \in \mathbb{F}_{q_{\ell-1}} \setminus \mathbb{F}_{q_0}^*$ and $b(\eta_1,\ldots,\eta_{\ell-1}) \in \mathbb{F}_{q_{\ell-1}}$. If

 $\begin{array}{l} b(\eta_1,\ldots,\eta_{\ell-1})=0\text{, then we have }A=a(\eta_1,\ldots,\eta_{\ell-1})\notin\mathbb{F}_{q_0}^*.\\ \text{Else, we have }A\notin\mathbb{F}_{q_{\ell-1}}\text{ since otherwise, }\eta_\ell=\frac{A-a(\eta_1,\ldots,\eta_{\ell-1})}{b(\eta_1,\ldots,\eta_{\ell-1})}\\ \text{would be in }\mathbb{F}_{q_{\ell-1}}.\text{ In particular, we have }A\notin\mathbb{F}_{q_0}.\end{array}$

Proposition 7 Let $\mathbb{F}_q/\mathbb{F}_{q_0}$ be an extension of finite fields of degree at least $\ell+1\geq 2$, and let $1,\psi,\ldots,\psi^{[F_q:\mathbb{F}_{q_0}]-1}\in\mathbb{F}_q$ be a power basis of the extension. Then any $\pmb{\eta}:=[a_1\psi,\ldots,a_\ell\psi]$ with $a_i\in\mathbb{F}_{q_0}\setminus\{0\}$ is \mathbb{F}_{q_0} -sum-product free.

Proof: For any non-empty $\mathcal{I}\subseteq\{1,\ldots,\ell\}$, we have $\prod_{i\in\mathcal{I}}\eta_i=b\psi^{|\mathcal{I}|}$ for some $b\in\mathbb{F}_{q_0}\setminus\{0\}$. Hence, an \mathbb{F}_{q_0} -linear combination of such terms must be of the form $b_1\psi+b_2\psi^2+\ldots+b_\ell\psi^\ell$ with $b_i\in\mathbb{F}_{q_0}$, and this will never be 0 as \mathbb{F}_q has degree at least $\ell+1$ over \mathbb{F}_{q_0} .

Remark 8 The propositions above provide two constructions of MDS twisted RS codes. Both methods require that the evaluation points are chosen from a proper subfield of the code's base field. Since $n \leq q_0$ and the smallest prime number q_0 greater or equal to n satisfies $q_0 < 2n$ by Bertrand's postulate, the smallest overall field sizes q for the constructions fulfill

$$n^{2^{\ell}} \le q = q_0^{2^{\ell}} < (2n)^{2^{\ell}},$$
 (Proposition 6), $n^{\ell+1} \le q = q_0^{\ell+1} < (2n)^{\ell+1},$ (Proposition 7).

For these smallest-possible field sizes, we have

$$\begin{array}{ll} \frac{1}{2}q^{2^{-\ell}} < n \leq q^{2^{-\ell}} & \textit{(Proposition 6),} \\ \frac{1}{2}q^{\frac{1}{\ell+1}} < n \leq q^{\frac{1}{\ell+1}} & \textit{(Proposition 7).} \end{array}$$

It can be seen that, compared to Proposition 6, the construction in Proposition 7 is able to provide smaller field sizes for any given code length. We include Proposition 6 for the sake of having a second construction that might prove useful for other purposes than merely minimizing the field size. For instance, there is an analog of Proposition 6 (based on a conference version of this paper) for constructing twisted Gabidulin codes in the rank metric [11], but there is no rank-metric analog of Proposition 7, yet, and it is not obvious how to adapt it.

B. (*)-Twisted RS Codes

The \mathbb{F}_{q_0} -sum-product free property as in the previous subsection is a rather strong restriction on the η_i and yields relatively short MDS codes. In this and the following subsection, we will obtain longer MDS codes for two specific choices of t and t.

We first consider twisted RS codes with one twist $\ell=1$, hook h=0, and twist t=1. In this case, we have the following necessary and sufficient MDS condition on the α_i and η_1 .

Lemma 9 Let $\ell=1$ and n,k,α,η be chosen as in Definition 1. The code $\mathcal{C}^{n,k}_{\alpha,1,0,\eta}$ is MDS if and only if

$$\eta_1(-1)^k \prod_{i \in \mathcal{I}} \alpha_i \neq 1 \quad \forall \mathcal{I} \subseteq \{1, \dots, n\} \text{ s.t. } |\mathcal{I}| = k.$$
 (3)

Proof: All evaluation polynomials are of the form $f = \sum_{i=0}^{k-1} f_i x^i + \eta_1 f_0 x^k$. If $f_0 = 0$, the weight of the corresponding codeword is either 0 or at least n-k+1 since $\deg f < k$. Otherwise, f corresponds to a codeword of weight < n-k+1 if and only if f has exactly k roots among the α_i , i.e., there is a subset $\mathcal{I} \subseteq \{1,\ldots,n\}$ with $|\mathcal{I}| = k$ such that

 $f = \eta_1 f_0 \prod_{i \in \mathcal{I}} (x - \alpha_i)$. The constant term of f is $f_0 = f(0) = \eta_1 f_0 \prod_{i \in \mathcal{I}} (-\alpha_i)$. Due to $f_0 \neq 0$, we have

$$\eta_1(-1)^k \prod_{i \in \mathcal{I}} \alpha_i = 1.$$

Hence, all non-zero evaluation polynomials have at most k-1 roots among the α_i if and only if (3) is satisfied.

For $\eta_1 \neq 0$, a sufficient condition for (3) to be fulfilled is that $(-1)^k \eta_1^{-1}$ is not contained in the multiplicative group generated by the α_i 's. This motivates the following definition.

Definition 3 Let G be a proper subgroup of (\mathbb{F}_q^*, \cdot) , $\alpha_i \in G \cup \{0\}$ for all i, and $(-1)^k \eta_1^{-1} \in \mathbb{F}_q^* \setminus G$. Then, we call $\mathcal{C}_{\boldsymbol{\alpha},1,0,\boldsymbol{\eta}}^{n,k}$ a (*)-twisted code.

Theorem 10 Any (*)-twisted code is MDS.

Proof: For any $\mathcal{I} \subseteq \{1,\ldots,n\}$, we have $\prod_{i\in\mathcal{I}}\alpha_i\in G\cup\{0\}$. Since $(-1)^k\eta_1^{-1}\notin G\cup\{0\}$, Condition (3) is fulfilled, and the code is MDS by Lemma 9.

For any divisor a>1 of q-1, there is a proper subgroup G of \mathbb{F}_q^* of cardinality (q-1)/a. This means that (*)-twisted codes can have length $n=\frac{q+1}{a}$ and can be rather long compared to the constructions in the previous subsection. In particular, if q is odd, then $2\mid q-1$ and $n=\frac{q+1}{2}$ is possible.

For even q, there is no multiplicative subgroup of this cardinality. However, if we allow arbitrary evaluation points and $\eta \in \mathbb{F}_q^*$, MDS twisted RS codes with t=1, h=0, and length $n\approx q/2$ may exist for even q: our computer search (cf. [1] shows, e.g., for q=16, there are many such codes of length n=9 for k=3,4,5.

Choosing the evaluation points from a multiplicative group appears to be rather restrictive. However, the following analysis shows that for odd q, (*)-twisted codes have maximal length among all MDS twisted RS codes with t=1 and h=0. To show this, we use the notion of k-sum generators in finite abelian groups, which was introduced in [15], [16] and originally used to construct non-RS MDS codes.

Definition 4 ([16]) Let (A, \oplus) be a finite abelian group and $k \in \mathbb{N}$. A k-sum generator of A is a subset $S \subseteq A$ such that for any $a \in A$, there are distinct $s_1, \ldots, s_k \in S$ with $a = \bigoplus_{i=1}^k s_i$. The smallest integer such that any $S \subset A$ with |S| > M(k, A) is a k-sum generator of A is denoted by M(k, A).

Lemma 11 ([16]) Let A be a finite abelian group of order |A| = 2r for some $r \ge 6$. For any k with $3 \le k \le r - 2$, we have

$$M(k,A) = \begin{cases} r+1, & \textit{if } A \in \{\mathbb{Z}_2^m, \mathbb{Z}_4 \times \mathbb{Z}_2^{m-1}\} \textit{ for some} \\ & m>1 \textit{ and } k \in \{3,r-2\}, \\ r, & \textit{else}. \end{cases}$$

Lemma 12 Let k, n, α, η be chosen as in Definition 1 such that $S := \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{F}_q^*$ is a k-sum generator of (\mathbb{F}_q^*, \cdot) and $\eta \in \mathbb{F}_q^*$. Then, the code $\mathcal{C}_{\alpha,1,0,\eta}^{n,k}$ is not MDS.

Proof: Since S is a k-sum generator of (\mathbb{F}_q^*,\cdot) and $(-1)^k\eta_1^{-1}\neq 0$, there is an index set $\mathcal{I}\subseteq\{1,\ldots,n\}$ with $|\mathcal{I}|=k$ such that $\prod_{i\in\mathcal{I}}\alpha_i=(-1)^k\eta_1^{-1}$. Lemma 9 then implies the claim.

Theorem 13 Let q be odd and $3 \le k \le \frac{q-1}{2} - 2$. If $n > \frac{q+1}{2}$, then $C^{n,k}_{\alpha,1,0,\eta}$ is not MDS for any choice of α and $\eta \ne 0$ as in Definition 1.

Proof: The abelian group (\mathbb{F}_q^*,\cdot) is cyclic and of even order $|\mathbb{F}_q^*|=q-1$ since q is odd. Thus, Lemma 11 implies that the maximal cardinality of a subset of \mathbb{F}_q^* that is not a k-sum generator is $M(k,\mathbb{F}_q^*)=\frac{q-1}{2}$. Since, for $S:=\{\alpha_1,\ldots,\alpha_n\}$, we have $|S\setminus\{0\}|\geq n-1>M(k,\mathbb{F}_q^*)$, the set S is therefore a k-sum generator of \mathbb{F}_q^* . By Lemma 12, the code $\mathcal{C}_{\pmb{\alpha},1,0,\eta}^{n,k}$ is not MDS.

C. (+)-Twisted Reed-Solomon Codes

We consider twisted RS codes with one twist $\ell=1$, hook h=k-1, and twist t=1. In this case, we can also give a necessary and sufficient MDS condition, which can be seen as the additive analog of Lemma 9. It gives rise to a similar construction as the (*)-twisted codes using additive instead of multiplicative subgroups of \mathbb{F}_a .

Lemma 14 Let $\ell=1$ and n,k,α,η be chosen as in Definition 1. The code $\mathcal{C}^{n,k}_{\alpha,1,k-1,\eta}$ is MDS if and only if

$$\eta \sum_{i \in \mathcal{I}} \alpha_i \neq -1 \quad \forall \mathcal{I} \subseteq \{1, \dots, n\} \text{ s.t. } |\mathcal{I}| = k.$$
(4)

Proof: The code is MDS if and only if any non-zero evaluation polynomial has at most k-1 zeros among the α_i .

Suppose that there is a polynomial $f \in \mathcal{P}_{t,h,\eta}^{n,k} \setminus \{0\}$ with k roots among the α_i . Then, we have $f_{k-1} \neq 0$ and there is a set $\mathcal{I} \subseteq \{1,\ldots,n\}$ with $|\mathcal{I}|=k$ and $f=f_k\prod_{i\in\mathcal{I}}(x-\alpha_i)$, i.e., $f_{k-1}=f_k\sum_{i\in\mathcal{I}}(-\alpha_i)$. Due to the choice of t and h, we have $f_k=\eta f_{k-1}$, so $\eta \sum_{i\in\mathcal{I}}\alpha_i=-1$ for this \mathcal{I} .

Conversely, assume that there is such a set \mathcal{I} with $\eta \sum_{i \in \mathcal{I}} \alpha_i = -1$. Then, $f = \eta \prod_{i \in \mathcal{I}} (x - \alpha_i)$ is an evaluation polynomial and has k roots among the evaluation points.

Analog to the multiplicative case, a sufficient condition for (4) to be fulfilled is to choose the evaluation points from a proper subgroup of $(\mathbb{F}_q,+)$ and $-\eta^{-1}$ not in this subgroup. This gives the following class of MDS twisted RS codes.

Definition 5 Let V be a proper subgroup of $(\mathbb{F}_q, +)$, $\eta^{-1} \in \mathbb{F}_q \setminus V$, and α consist of n distinct elements of V. Then, $\mathcal{C}^{n,k}_{\alpha,1,k-1,\eta}$ is called a (+)-twisted code.

Theorem 15 Any (+)-twisted code is MDS.

Proof: This follows immediately from Lemma 14. If p is the characteristic of \mathbb{F}_q , then there is a proper subgroup V of $(\mathbb{F}_q, +)$ with order q/p. Hence, a (+)-twisted code can have length up to $n = \frac{q}{p}$. In particular, for even q, we get codes of length $n = \frac{q}{2}$.

Remark 16 For $\ell = 1$ and general h_1 and t_1 , we define the evaluation at infinity as $f(\infty) := f_{k-1+t_1}$ (note that $k-1+t_1$ is the maximal degree of a polynomial in $\mathcal{P}^{n,k}_{t,h,\eta}$). Due to $(\alpha f + \beta g)(\infty) = \alpha f(\infty) + \beta g(\infty)$ for all $f, g \in \mathcal{P}^{n,k}_{t,h,\eta}$ and $\alpha, \beta \in \mathbb{F}_q$, adding ∞ to the evaluation point set gives a linear code. For $h_1 = k-1$, we have $f(\infty) = 0$ if and only if $\deg(f) < k-1$. Hence, if a twisted RS code with these parameters and $\alpha \in \mathbb{F}_q^n$ is MDS, then the "extended" code with the same k, t, h, η

and evaluation points $\alpha' := [\alpha_1, \dots, \alpha_n, \infty]$ is also MDS. By extending a (+)-twisted code, we get an MDS code of length up to $n = \frac{q}{2} + 1$ over a field of characteristic 2.

As in the (*)-twisted case, we study the maximal length of a twisted RS code with t = 1 and h = k - 1, and arbitrary α and $\eta \neq 0$. The proofs of the following two statements are similar to those of Lemma 12 and Theorem 13, respectively, and are therefore omitted.

Lemma 17 Let k, n, α, η be chosen as in Definition 1 such that $S:=\{\alpha_1,\ldots,\alpha_n\}\in\mathbb{F}_q$ is a k-sum generator of $(\mathbb{F}_q,+)$. Then, the code $\mathcal{C}^{n,k}_{\boldsymbol{\alpha},1,k-1,\boldsymbol{\eta}}$ is not MDS.

Theorem 18 Let q be even and $3 \le k \le \frac{q}{2} - 2$. If the code length satisfies

$$n > \begin{cases} \frac{q}{2}, & \text{if } 3 < k < \frac{q}{2} - 3, \\ \frac{q}{2} + 1, & \text{if } k \in \{3, \frac{q}{2} - 2\}, \end{cases}$$

then the twisted code $\mathcal{C}^{n,k}_{\alpha,1,k-1,\eta}$ is not MDS for any choice of η as in Definition 1.

V. DUALS OF TWISTED RS CODES

In this section, we show that the family of twisted RS codes whose evaluation points form a multiplicative group are closed under duality. We use the following auxiliary matrices.

Definition 6 Let $r \in \mathbb{Z}_{>0}$ and $\alpha \in \mathbb{F}_q^r$. i) The *reversal matrix* $\boldsymbol{J}_r \in \mathbb{F}_q^{r \times r}$ is the square matrix

$$oldsymbol{J}_r = \left[egin{array}{ccc} & 1 \ 1 \end{array}
ight].$$

ii) The Vandermonde matrix of α is denoted by

$$oldsymbol{V}_r(oldsymbol{lpha}) = egin{bmatrix} lpha_1^0 & lpha_2^0 & \dots & lpha_r^0 \ lpha_1^1 & lpha_2^1 & \dots & lpha_r^1 \ dots & \ddots & dots \ lpha_1^{r-1} & lpha_2^{r-1} & \dots & lpha_r^{r-1} \end{bmatrix}.$$

For a matrix $A \in \mathbb{F}_q^{r \times r'}$, then $J_r A$ is A with the rows in reverse order. Similarly, $AJ_{r'}$ is A with the columns in reverse order. And $B:=J_rAJ_{r'}\in \mathbb{F}_q^{r imes r'}$ is A "rotated", i.e., $B_{i,j} = A_{r-i+1,r'-j+1}$. If the α_i form a multiplicative group, we can give the inverse of the Vandermonde matrix $\boldsymbol{V}_n(\boldsymbol{\alpha})$ with the help of the reversal matrix as follows. This is a reformulation of a result in [17].

Lemma 19 Let $\alpha \in \mathbb{F}_q^n$ such that the α_i are distinct and form a multiplicative subgroup of \mathbb{F}_q^* . Then,

$$(\boldsymbol{V}_n(\boldsymbol{\alpha})^\top)^{-1} = \boldsymbol{J}_n \cdot \boldsymbol{V}_n(\boldsymbol{\alpha}) \cdot \operatorname{diag}(\boldsymbol{\alpha}/n).$$

Proof: Since the entries of α are a multiplicative group, we have $\prod_{i=1}^{n} (x - \alpha_i) = x^n - 1$ and

$$(\boldsymbol{V}_n(\boldsymbol{\alpha})^{\top})^{-1} = \frac{1}{n} \begin{bmatrix} 1 & 1 & \dots & 1 \\ \alpha_1^{-1} & \alpha_2^{-1} & \dots & \alpha_n^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{-(n-1)} & \alpha_2^{-(n-1)} & \dots & \alpha_n^{-(n-1)} \end{bmatrix}$$

$$= \boldsymbol{J}_n \cdot \boldsymbol{V}_n(\boldsymbol{\alpha}) \cdot \operatorname{diag}(\boldsymbol{\alpha}/n),$$

where the first equality follows by [17].

Lemma 19 enables us to describe the duals of the following class of codes, which includes the family of twisted RS codes with evaluation points forming a multiplicative group.

Lemma 20 Let C[n,k] be a linear code with a generator matrix of the form

$$G = [I \mid L] \cdot V_n(\alpha),$$

where $I \in \mathbb{F}_q^{k \times k}$ is the identity matrix, $L \in \mathbb{F}_q^{k \times n - k}$, and the entries of $\alpha \in \mathbb{F}_q^n$ are distinct and form a multiplicative subgroup of \mathbb{F}_q^* . Then, the following is a generator matrix of

$$\boldsymbol{H} = [\boldsymbol{I} \mid -\boldsymbol{J}_{n-k} \boldsymbol{L}^{\top} \boldsymbol{J}_{k}] \cdot \boldsymbol{V}_{n}(\boldsymbol{\alpha}) \cdot \operatorname{diag}(\boldsymbol{\alpha}/n).$$

Proof: By construction, \boldsymbol{H} has full rank n-k and fulfills $oldsymbol{G} \cdot oldsymbol{H}^{ op}$

$$\begin{split} &= [\boldsymbol{I} \mid \boldsymbol{L}] \boldsymbol{V}_n(\boldsymbol{\alpha}) \cdot \left([\boldsymbol{I} \mid -\boldsymbol{J}_{n-k} \boldsymbol{L}^\top \boldsymbol{J}_k] \boldsymbol{V}_n(\boldsymbol{\alpha}) \operatorname{diag}(\boldsymbol{\alpha}/n) \right)^\top \\ &= [\boldsymbol{I} \mid \boldsymbol{L}] \boldsymbol{V}_n(\boldsymbol{\alpha}) \cdot \left(\boldsymbol{J}_{n-k} [-\boldsymbol{L}^\top \mid \boldsymbol{I}] \underbrace{\boldsymbol{J}_n \boldsymbol{V}_n(\boldsymbol{\alpha}) \operatorname{diag}(\boldsymbol{\alpha}/n)}_{= (\boldsymbol{V}_n(\boldsymbol{\alpha})^{-1})^\top \text{ (Lemma 19)}} \right)^\top \\ &= [\boldsymbol{I} \mid \boldsymbol{L}] \begin{bmatrix} -\boldsymbol{L} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{J}_{n-k} = \mathbf{0} \end{split}$$

so it is a parity-check matrix of C, and thus, a generator matrix of the dual code.

Lemma 20 implies the following duality statement for twisted RS codes with evaluation points forming a multiplicative group.

Theorem 21 Let n, k, α, t, h, η be chosen as in Definition 1 such that the entries of α form a multiplicative subgroup of \mathbb{F}_a^* . Then, the dual of $C_{\alpha,k,h,\eta}^{n,k}$ is equivalent to $C_{\alpha,k-h,n-k-t,-\eta}^{n,n-k}$, where $k-\mathbf{h}:=[k-h_1,\ldots,k-h_\ell]$ and $n-k-\mathbf{t}$ is defined analogously.

Proof: The canonical generator matrix (as in (2)) of any twisted RS code $C_{\alpha,t,h,\eta}^{n,k}$ can be written as

$$G = [I \mid L] \cdot V_n(\alpha),$$

where the entries of $oldsymbol{L} \in \mathbb{F}_q^{k imes n-k}$ are of the form

$$L_{i,j} = \begin{cases} \eta_{\mu}, & \text{if } [i,j] = [h_{\mu} + 1, t_{\mu}], \\ 0, & \text{else.} \end{cases}$$

Since we assume that the α_i form a multiplicative group, we can apply Lemma 20 and obtain the following generator matrix of the dual code:

$$\boldsymbol{H} = [\boldsymbol{I} \mid -\boldsymbol{J}_{n-k} \boldsymbol{L}^{\top} \boldsymbol{J}_{k}] \cdot \boldsymbol{V}_{n}(\boldsymbol{\alpha}) \cdot \operatorname{diag}(\boldsymbol{\alpha}/n)$$

Hence, the dual of $\mathcal{C}^{n,k}_{\alpha,t,h,\eta}$ is equivalent to a code \mathcal{C}' with generator matrix $[I_{\perp}] - J_{n-k} L^{\top} J_k] \cdot V_n(\alpha)$. Since the entries of $\boldsymbol{B} := -\boldsymbol{J}_{n-k} \boldsymbol{L}^{\top} \boldsymbol{J}_k$ are of the form

$$B_{i,j} = \begin{cases} -\eta_{\mu}, & \text{if } [i,j] = [n-k-t_{\mu}+1,k-h_{\mu}] \\ 0, & \text{else}, \end{cases}$$

we have $\mathcal{C}' = \mathcal{C}^{n,n-k}_{\boldsymbol{\alpha},k-\boldsymbol{h},n-k-\boldsymbol{t},-\boldsymbol{\eta}}$, which proves the claim. \blacksquare A suitable example of twisted RS codes with evaluation

points forming a multiplicative group are the (*)-twisted codes described in Section IV-B.

Corollary 22 Let G be a proper subgroup of (\mathbb{F}_q^*, \cdot) and $\ell = 1, k, n, \alpha, \eta$ be chosen as in Definition I such that $G = \{\alpha_1, \dots, \alpha_n\}$ and $(-1)^{n-k+1}\eta_1^{-1} \notin G \cup \{0\}$. Then, the code $\mathcal{C}_{\alpha,n-k,k-1,\eta}^{n,k}$ is equivalent to the dual of the (*)-twisted code $\mathcal{C}_{\alpha,1,0,-\eta}^{n,n-k}$. In particular, it is MDS.

Remark 23 Theorem 21 could be generalized if the inverse of $V_n(\alpha)$ could be described similarly as in Lemma 19 for a wider class of evaluation points α . It is not true, however, that the dual of any twisted RS code is equivalent to a twisted RS code with the same number of twists: by computer search, we found twisted RS codes over \mathbb{F}_{11} of length n=8 and with one twist $(\ell=1)$ whose dual codes are not equivalent to any twisted RS code with one twist.

Remark 24 We can generalize Lemma 20 and Theorem 21 to allow also $\alpha_i = 0$, if we in addition assume $t_i \neq n - k$ for all i or $h_i \neq 0$ for all i. The proof idea is as follows. Let $\alpha = [\alpha_1, \ldots, \alpha_n, 0]$ and $\mathbf{L} \in \mathbb{F}_q^{k \times n + 1 - k}$ be a matrix whose first row is of the form $[\mathbf{l}_1 \mid 0]$ with $\mathbf{l}_1 \in \mathbb{F}_q^{n - k}$. Then,

$$egin{aligned} oldsymbol{H} &= egin{bmatrix} oldsymbol{I} &- egin{bmatrix} 1 \ oldsymbol{l}_1^{ op} & oldsymbol{I} \end{bmatrix} \cdot oldsymbol{J}_{n+1} oldsymbol{L}^{ op} oldsymbol{J}_k \end{bmatrix} \ \cdot oldsymbol{V}_{n+1}(oldsymbol{lpha}) \cdot \mathrm{diag}(1/n, \ldots, 1/n, -1) \end{aligned}$$

is a valid parity-check matrix for the code with generator matrix $G = [I \mid L] \cdot V_n(\alpha) \in \mathbb{F}_q^{k,n+1}$. If the first row $(t_i \neq n-k \ \forall i)$ or the last column $(h_i \neq 0 \ \forall i)$ of L is zero, then we have

$$-\begin{bmatrix} 1 \ l_1^{ op} & I \end{bmatrix} \cdot \boldsymbol{J}_{n+1-k} \boldsymbol{L}^{ op} \boldsymbol{J}_k = -\boldsymbol{J}_{n+1-k} \boldsymbol{L}^{ op} \boldsymbol{J}_k.$$

VI. RELATION TO GRS CODES

Using two different techniques, we show that many twisted RS codes are not GRS codes. Section VI-A uses the Schur square of a code to distinguish a large class of low-rate (and special high-rate) twisted codes from GRS codes. In Section VI-B, we derive a combinatorial statement, which states that if all code parameters are fixed except for η , either all η for which the code is MDS give a GRS code, or only a few of them result in GRS codes.

A. Inequivalence Based on Schur Squares

Schur squares of codes have become an increasingly studied object in coding theory in the last years due to several applications [18]–[20].

Definition 7 Let C[n, k] be a linear code. The *Schur square* of C is defined as

$$\mathcal{C}^2 := \left\langle \left\{ \boldsymbol{c} \star \boldsymbol{c}' \, : \, \boldsymbol{c}, \, \boldsymbol{c}' \in \mathcal{C} \right\} \right\rangle,$$

where $c \star c' = [c_1 c'_1, \dots, c_n c'_n]$ is the Schur product of two vectors.

The dimension of the Schur product of a code is an invariant under code equivalence and satisfies

$$\dim(\mathcal{C}^2) \le \min\{n, \frac{1}{2}k(k+1)\}.$$

A random linear code attains this upper bound with high probability, cf. [21]. An MDS code has Schur square dimension

at least $\dim(\mathcal{C}^2) \geq \min\{n, 2k-1\}$ [20] and GRS codes attain this lower bound. We will make use of these properties in this section by showing that a large family of twisted RS codes of rate less than 1/2 has Schur square dimension at least 2k, and thus is non-GRS.

We start with a generic lower bound on the Schur square dimension of an evaluation code.

Definition 8 Let $\mathcal{P} \subseteq \mathbb{F}_q[x]_{\leq n}$ be an \mathbb{F}_q -subspace and α consist of n distinct elements α_i of \mathbb{F}_q . We define

$$\begin{split} & \mathrm{D}(\mathcal{P})_{< n} := \{ \deg(f \cdot g) \, : \, f, g \in \mathcal{P}, \, \deg(f \cdot g) < n \} \ \text{ and } \\ & \overline{\mathrm{D}}(\mathcal{P}, \pmb{\alpha}) := \{ \deg(\overline{f \cdot g}) \, : \, f, g \in \mathcal{P} \} \, , \end{split}$$

where
$$\overline{f} := (f \mod \prod_{i=1}^n (X - \alpha_i))$$
 for any $f \in \mathbb{F}_q[X]$.

Lemma 25 Let $\alpha \in \mathbb{F}_q^n$ with distinct entries, $\mathcal{P} \subseteq \mathbb{F}_q[x]_{< n}$ be an \mathbb{F}_q -subspace, and $\mathcal{C} = \operatorname{ev}_{\alpha}(\mathcal{P})$ be the evaluation code of \mathcal{P} at the evaluation points α . Then,

$$\mathcal{C}^2 = \operatorname{ev}_{\alpha}(\langle fg : f, g \in \mathcal{P} \rangle)$$
 and $\dim (\mathcal{C}^2) \ge |\overline{\mathbb{D}}(\mathcal{P}, \alpha)| \ge |D(\mathcal{P})_{< n}|$.

Proof: The first part of the statement follows directly from $f(\alpha) \cdot g(\alpha) = (f \cdot g)(\alpha)$ for $f, g \in \mathbb{F}_q[X]$ and $\alpha \in \mathbb{F}_q$. Since

$$\operatorname{ev}_{\alpha}(\langle fg : f, g \in \mathcal{P} \rangle) = \operatorname{ev}_{\alpha}(\langle \overline{fg} : f, g \in \mathcal{P} \rangle)$$

and the evaluation $\operatorname{ev}_{\alpha}(\cdot)$ is a bijection between $\mathbb{F}_q[X]_{< n}$ and \mathbb{F}_q^n , the Schur square dimension $\dim\left(\mathcal{C}^2\right)$ is greater or equal to the dimension of $\langle \overline{f\cdot g}:f,g\in\mathcal{P}\rangle$, which in turn is lower-bounded by $|\overline{\mathbb{D}}(\mathcal{P},\alpha)|$. Note also that $\operatorname{D}(\mathcal{P})_{< n}\subseteq\overline{\mathbb{D}}(\mathcal{P},\alpha)$.

Using Lemma 25, we get the following lower bound on the Schur square dimension of twisted RS codes.

Proposition 26 Let α , t, h, and η be as in Definition 1. Denote by $g_0, \ldots, g_{k-1} \in \mathcal{P}^{n,k}_{t,h,\eta}$ the basis of $\mathcal{P}^{n,k}_{t,h,\eta}$ given in Lemma 1 and define $S^{n,k}_{t,h,\eta} := \{\deg(g_1), \ldots, \deg(g_{k-1})\}$. Then,

$$S_{\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}}^{n,k} = \left(\{0,\dots,k-1\} \setminus \{h_j : \eta_j \neq 0\} \right) \cup \left\{ k-1 + \max\{t_j : h_j = i, \eta_j \neq 0\} : i \in \{h_j : \eta_j \neq 0\} \right\}.$$

Thus, the dimension of the Schur square satisfies

$$\dim \left(C_{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}^{n,k} \right) \ge \left| \left\{ d_1 + d_2 : d_1, d_2 \in S_{\boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}^{n,k}, d_1 + d_2 < n \right\} \right|.$$

Proof: Recall $g_i = X^i + \sum_{j=1, h_j=i}^{\ell} \eta_j X^{k-1+t_j}$ from (1). Hence, for $i \notin \{h_j : \eta_j \neq 0\}$, we have $g_i = X^i$ and otherwise, its degree is determined by the term $\eta_j X^{k-1+t_j}$ with largest t_j among those j with $h_j = i$ and $\eta_j \neq 0$. The second part follows directly from Lemma 25.

Lemma 25 and Proposition 26 imply the following three inequivalence statements for (*)-twisted and (+)-twisted codes.

Corollary 27 Let $3 \leq k < \frac{n}{2}$. Then, any (*)-twisted code is non-GRS. If $k = \frac{n}{2}$ and $\eta_1^2 \prod_{i=1}^n \alpha_i \neq 1$, then any (*)-twisted code with such η and α is non-GRS.

Proof: For a (*)-twisted code, we have $S^{n,k}_{t,h,\eta}=\{1,\ldots,k\}$, so the set $A:=\langle fg:f,g\in\mathcal{P}\rangle$ contains polynomials of degrees $\{2,\ldots,2k\}$. Furthermore, A contains a polynomial of degree 1 since $X^1\cdot(\eta_1X^k+X^0)-\eta_1X^{k-1}\cdot X^2=X^1$

(here we need $k \geq 3$). In fact, we can choose as a basis of A the polynomials $X^1, X^2, \ldots, X^{2k-1}, (1+\eta_1 X^k)^2$ since $X^i = X^{i_1} X^{i_2}$ for some $1 \leq i_1, i_2 \leq k-1$ for any $i = 2, \ldots, 2k-2$, and $X^{2k-1} = \eta_1^{-1} \cdot X^{k-1} \cdot (\eta_1 X^k + X^0) - \eta_1^{-1} \cdot X^1 \cdot X^{k-2}$ (here we need $k \geq 3$).

If 2k < n, the set A hence contains 2k polynomials of distinct degrees less than n, and by Lemma 25, we have $\dim \mathcal{C}^2 \geq 2k$. In particular, \mathcal{C} is non-GRS.

If 2k=n, then we must reduce the basis polynomial $(1+\eta_1X^k)^2$ modulo $\prod_{i=1}^n(X-\alpha_i)$ in order to determine the Schur square dimension. As the monomials X^1,X^2,\ldots,X^{n-1} are in A, the Schur square has dimension n if and only if the constant term of

$$\overline{(1+\eta_1 X^k)^2} = (1+\eta_1 X^k)^2 - \eta_1^2 \prod_{i=1}^n (X-\alpha_i)$$

is non-zero.

Corollary 28 Let $3 \le k < \frac{n}{2}$ Then, any (+)-twisted code C is non-GRS.

Proof: We have $S_{t,h,\eta}^{n,k}=\{0,1,\ldots,k-2,k\}$ and thus $\{d_1+d_2:d_1,d_2\in S_{t,h,\eta}^{n,k},d_1+d_2< n\}=\{0,\ldots,2k-2,2k\}$ (here we use $k\geq 3$ and 2k< n). By Proposition 26, we have $\mathcal{C}^2>2k$ and the claim follows.

For (*)-twisted codes with evaluation points forming a multiplicative group, we can use the duality statements of Section V and show that also high-rate codes are non-GRS.

Corollary 29 Let $\frac{n}{2} < k \le n-3$ and suppose the α form a proper subgroup of (\mathbb{F}_q^*, \cdot) . Then, any (*)-twisted code with evaluation points α is non-GRS.

Proof: By Theorem 21, the dual code of the (*)-twisted code is equivalent to a low-rate twisted RS code $\mathcal{C}[n,n-k]$ with $\ell=1,\,t_1=k,\,h_1=n-k-1$, and the same evaluation points. Hence, the evaluation polynomial set \mathcal{P} of \mathcal{C} is spanned by the polynomials $X^0,\ldots,X^{n-k-2},\eta'X^{n-1}+X^{n-k-1}$ for some $\eta'\neq 0$. As \mathcal{C} is a low-rate code, it suffices to show that $\dim\mathcal{C}^2\geq 2(n-k)$. We show this by finding 2(n-k) polynomials of distinct degrees in $B:=\langle \overline{fg}:f,g\in\mathcal{P}\rangle$ and applying Lemma 25.

First note that by combining the basis elements, B obviously contains elements of degrees $0,\ldots,2(n-k)-4$ and n-1. We construct two more elements of B with different degrees using the structure of α . Since $\prod_{i=1}^n (X-\alpha_i)=X^n-1$, the set B contains a polynomial of degree 2(n-k)-3 as

$$\overline{X^{n-k-2}\left(\eta'X^{n-1} + X^{n-k-1}\right)} = \eta'X^{n-k-3} + X^{2(n-k)-3}$$

(we use that $n-k-2 \ge 1$ due to $k \le n-3$) and a polynomial of degree n-2 as

$$\overline{(\eta' X^{n-1} + X^{n-k-1})^2}$$

$$= {\eta'}^2 X^{n-2} + 2\eta' X^{n-k-2} + X^{2(n-k)-2}.$$

Note that in the latter polynomial, X^{n-2} is indeed the leading term due to $\frac{n}{2} < k$. For the same reason, we have 2(n-k)-3 < n-2. This concludes the proof.

The following theorem shows that many twisted RS codes of rate smaller than 1/2 are not GRS codes. The only restriction

is a mild technical condition on the hook vector h, which we require to not contain the two smallest or the two largest possible values, or contain consecutive elements.

Theorem 30 Let $k < \frac{n}{2}$ and choose α, h, t, η as in Definition I with the additional requirements $\eta_i \neq 0$, $1 < h_i < k - 2$, and either $h_i = h_j$ or $|h_i - h_j| > 1$ for all $i \neq j$. Then, the code $\mathcal{C} := \mathcal{C}^{n,k}_{\alpha,t,h,\eta}$ has Schur square dimension $\dim(\mathcal{C}^2) \geq 2k$. In particular, it is not a GRS code.

Proof: By Proposition 26, the set of evaluation polynomial degrees is given by $S_{t,h,n}^{n,k} = A \cup B$, where

$$A = \{0, \dots, k-1\} \setminus \{h_1, \dots, h_{\ell}\}\$$

 $\emptyset \neq B \subseteq \{k-1+t_i : i=1, \dots, \ell\}.$

By the restrictions on h_i , we have

$$\{0, 1, k - 2, k - 1\} \subseteq A$$
 and $\{h_i - 1, h_i + 1\} \subseteq A$ $\forall i = 1, \dots, \ell.$

We show that $\{0,\ldots,2k-2,\mu\}\subseteq \mathrm{D}(\mathcal{P}^{n,k}_{\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}})$ for some $\mu\in\{2k-1,\ldots,n-1\}$. Let $0\leq j\leq k-1$. Then, j can be written as the sum of two elements in A as follows:

$$j = \begin{cases} j+0, & \text{if } j \in A \text{ (i.e., } j \neq h_i \text{ for all } i), \\ (h_i-1)+1, & \text{if } j=h_i. \end{cases}$$

Hence, $j\in \mathrm{D}(\mathcal{P}^{n,k}_{\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}}).$ We used $0,1,h_i-1\in A.$ Let $k\leq k-1+j\leq 2k-2.$ Then,

$$k-1+j = \begin{cases} (k-1)+j, & \text{if } j \in A, \\ (k-2)+(h_1+1), & \text{if } j=h_i, \end{cases}$$

i.e., $k-1+j \in D(\mathcal{P}_{t,h,\eta}^{n,k})$. We used $k-1,k-2,h_i+1 \in A$.

It is left to show that $D(\mathcal{P}^{n,k}_{t,h,\eta}) \cap \{2k-1,\ldots,n-1\}$ is non-empty. We distinguish three cases, of which at least one is true since B is non-empty and $k \leq b < n$ for all $b \in B$:

- 1) If there is a $b \in B$ with $b \ge 2k$, then $0 + b \in D(\mathcal{P}_{\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}}^{n,k})$ (recall that $0 \in A$) and the claim follows.
- 2) If $k \in B$, then $2k 1 = k + (k 1) \in D(\mathcal{P}_{t,h,\eta}^{n,k})$ (recall that $k 1 \in A$) and the claim follows.
- 3) If there is a $b \in B$ with k < b < 2k, then

$$D(\mathcal{P}_{t,h,\eta}^{n,k}) \cap \{2k-1,\ldots,n-1\}$$

$$\supseteq (b+A) \cap \{2k-1,\ldots,n-1\}$$

$$= \{ \max\{2k-1,b\},\ldots,\min\{n-1,b+k-1\} \} \setminus \{b+h_i : i=1,\ldots,\ell\}$$

$$=: B_1$$

Due to $\max\{2k-1,b\}=2k-1, 2k < n$, and b+k-1 > 2k-1, we have $\{2k-1,2k\}\subseteq B_1$. Since the h_i are nonconsecutive, we must have $2k-1 \notin B_2$ or $2k \notin B_2$. Hence, $B_1 \setminus B_2 \neq \emptyset$, which proves the claim.

Hence, $|D(\mathcal{P}_{t,h,\eta}^{n,k})_{< n}| \ge 2k$ and Lemma 25 implies the claim.

B. A Combinatorial Inequivalence Argument

In the following, we present combinatorial results on the inequivalence question. We rely on the following well-known characterization of GRS codes.

Lemma 31 ([4], [22]) Let \mathcal{C} be a linear code with a generator matrix of the form $G = [I \mid A]$. Then, \mathcal{C} is a GRS code if and only if, for $A' \in \mathbb{F}_q^{k \times n - k}$ with $A'_{ij} = A_{ij}^{-1}$,

- (i) all entries of A are non-zero,
- (ii) all 2×2 minors of A' are non-zero, and
- (iii) all 3×3 minors of \mathbf{A}' are zero.

An MDS code always has a systematic generator matrix $G = [I \mid A]$ and fulfills Conditions (i) and (ii). The crucial difference of a GRS and a non-GRS MDS code is hence Condition (iii). Note also that for $\min\{k,n-k\} < 3$, the matrix A' has no 3×3 minors, so any such MDS code is a GRS code. The following lemma states how the entries of A depend on η .

Lemma 32 Let α, t, h be chosen as in Definition 1. For these choices, let $\mathcal{H} \subseteq \mathbb{F}_q^\ell$ be a set of η 's such that $\mathcal{C}_{\alpha,t,h,\eta}^{n,k}$ is MDS. For any $\eta \in \mathcal{H}$, let $G^{(\mathrm{sys},\eta)} = [I \mid A^{(\eta)}]$ be the systematic generator matrix of $\mathcal{C}_{\alpha,t,h,\eta}^{n,k}$. Then, the entries of $A^{(\eta)} \in \mathbb{F}_q^{k \times n - k}$ can be written as

$$A_{i,j}^{(\boldsymbol{\eta})} = \frac{p^{(i,j)}(\eta_1,\ldots,\eta_\ell)}{p(\eta_1,\ldots,\eta_\ell)} \quad \forall \, \boldsymbol{\eta} = [\eta_1,\ldots,\eta_\ell] \in \mathcal{H},$$

where $p, p^{(i,j)} \in \mathbb{F}_q[X_1, \dots, X_\ell]$ are polynomials in ℓ variables of degree at most 1 in each variable that do not have a zero in \mathcal{H} and whose coefficients do not depend on η .

Proof: Consider the "canonical" generator matrix $G^{(\operatorname{can}, \eta)}$ in (2), i.e., the matrix in which the rows are the evaluations at α of the evaluation polynomial basis g_0, \ldots, g_{k-1} (cf. Lemma 1). By definition of the g_i , its entries are of the form

$$G_{i,j}^{(\operatorname{can},\boldsymbol{\eta})} = \alpha_j^{i-1} + \sum_{\substack{\kappa=1\\h_\kappa = i-1}}^{\ell} \eta_\kappa \alpha_j^{k-1+t_\kappa},\tag{5}$$

i.e., $G_{i,j}^{(\operatorname{can}, \boldsymbol{\eta})}$ is the evaluation at $\boldsymbol{\eta}$ of a polynomial in $\mathbb{F}_q[X_1, \dots, X_\ell]$ of total degree at most 1. Furthermore, for each variable X_i , there is only one row of $\boldsymbol{G}^{(\operatorname{can}, \boldsymbol{\eta})}$ for which these polynomials have non-zero degree in X_i (we abbreviate the latter property with X_i "appears in a polynomial" below).

We write $G^{(\operatorname{can},\eta)} = [B^{(\eta)} \mid D^{(\eta)}]$ with $B^{(\eta)} \in \mathbb{F}_q^{k \times k}$ and $D^{(\eta)} \in \mathbb{F}_q^{k \times (n-k)}$. Observe that since we only consider η for which the code is MDS, $B^{(\eta)}$ is invertible and we have

$$\boldsymbol{A^{(\eta)}} = \boldsymbol{B^{(\eta)}}^{-1} \boldsymbol{D^{(\eta)}} = \frac{\operatorname{adj}(\boldsymbol{B^{(\eta)}}) \boldsymbol{D^{(\eta)}}}{\operatorname{det}(\boldsymbol{B^{(\eta)}})},$$

where $adj(B^{(\eta)})$ is the adjunct matrix of $B^{(\eta)}$.

The determinant $\det \left(\boldsymbol{B}^{(\eta)} \right)$ is the evaluation at η of a polynomial $p \in \mathbb{F}_q[X_1,\ldots,X_\ell]$, where p has degree at most 1 in each variable. This follows inductively from the Laplace expansion of the determinant and the fact that each X_i appears only in the polynomials that correspond to one row of $\boldsymbol{G}^{(\operatorname{can},\eta)}$. Further, p has no zero in \mathcal{H} since the code is MDS for all $\eta \in \mathcal{H}$. This gives the sought polynomial p.

We study the entries of $A^{(\eta)} \cdot \det(B^{(\eta)})$, which are sums of products of one entry from the adjunct matrix and one entry from $D^{(\eta)}$ whose column and row index, respectively, coincide. By definition, the (i, j)-th entry of the adjunct matrix of $B^{(\eta)}$ is given by $(-1)^{i+j}$ times the determinant of the $(k-1) \times (k-1)$ submatrix of $B^{(\eta)}$ obtained by deleting its i-th column and j-th row. This means that it is the evaluation at η of a polynomial in $\mathbb{F}_q[X_1,\ldots,X_\ell]$ with degree at most 1 in each variable. Moreover, if κ satisfies $h_{\kappa} = j - 1$, the variable X_{κ} does not appear in those polynomials that correspond to the j-th column of the adjunct matrix. On the other hand, these X_{κ} are the only variables that appear in the polynomials corresponding to the j-th row of $D^{(\eta)}$ (cf. (5)). Hence, the (i, j)-th entry of $A^{(\eta)} \cdot \det(B^{(\eta)})$ can be written as evaluation at $\boldsymbol{\eta}$ of a polynomial $p^{(i,j)} \in \mathbb{F}_q[X_1,\ldots,X_\ell]$ of degree at most 1 in each X_i . Furthermore, each of the $p^{(i,j)}$ s does not have a zero in \mathcal{H} since otherwise $G^{(\text{sys},\eta)}$ would contain a row with k zeros, contradicting the MDS assumption. This gives the sought polynomials $p^{(i,j)}$.

Theorem 33 Let $\min\{k, n-k\} \geq 3$ and α , t, h be chosen as in Definition 1. Denote by $\mathcal{H} \subseteq \mathbb{F}_q^\ell$ the set of η such that $\mathcal{C}_{\alpha,t,h,\eta}^{n,k}$ is MDS and assume that there is an $\eta^* \in \mathcal{H}$ for which $\mathcal{C}_{\alpha,t,h,\eta^*}^{n,k}$ is non-GRS. Then there is a non-zero multivariate polynomial $P \in \mathbb{F}_q[X_1,\ldots,X_\ell]$ of degree at most 6 in each variable such that all $\eta \in \mathcal{H}$ for which $\mathcal{C}_{\alpha,t,h,\eta}^{n,k}$ is GRS are zeros of P.

Proof: Consider the systematic generator matrices $G^{(\mathrm{sys},\eta)} = [I \mid A^{(\eta)}]$ for all the codes indexed by η . By Lemma 31, the code $\mathcal{C}^{n,k}_{\alpha,t,h,\eta}$ is GRS if and only if all 3×3 minors of the element-wise inverse of $A^{(\eta)}$ vanish. Since there is an $\eta^* \in \mathcal{H}$ such that $\mathcal{C}^{n,k}_{\alpha,t,h,\eta^*}$ is non-GRS, there is at least one non-zero minor of the element-wise inverse of $A^{(\eta^*)}$. Fix this minor (i.e., the same 3×3 submatrix) for all η . We show that the η for which this minor is zero are zeros of a polynomial P as in the theorem statement.

By Lemma 32, the entries of the element-wise inverse of $A^{(\eta)}$ are evaluations at η of rational functions $\frac{p}{p^{(i,j)}} \in \mathbb{F}_q(X_1,\ldots,X_\ell)$, where $p,p^{(i,j)}$ are ℓ -variate polynomials of degree at most 1 in each variable which do not have a zero in \mathcal{H} . Hence, the fixed minor of the element-wise inverse of $A^{(\eta)}$ is the evaluation at η of a rational function $p^3\frac{P}{Q}$, where Q is the product of all nine $p^{(i,j)}$ in the 3×3 submatrix and P is a sum of products of six $p^{(i,j)}$'s each. Thus, P is a polynomial of degree at most 6 in each variable. As Q and p do not have a zero in \mathcal{H} , the minor can only vanish at zeros of P. Since $P(\eta^*) \neq 0$, the polynomial P is non-zero.

Theorem 33 states that for given α , t, and h, either all MDS twisted RS codes are GRS, or "many" are non-GRS. We will quantify what we mean by "many" in the following, but first we give an example for which the polynomial P in the proof of Theorem 33 vanishes, i.e., all MDS twisted RS codes of this α , t, and h are GRS.

 $^{^2}$ In the first conference paper about twisted RS codes [1] (case $\ell=1$), we mistakenly assumed that the polynomial P never vanishes. Hence, [1, Theorem 18] is not true in general, see Example 34.

Example 34 Consider a twisted RS code over a field \mathbb{F}_q with

$$[n, k] = [6, 3], \quad \ell = 1, \quad \mathbf{t} = 1, \quad \mathbf{h} = 2,$$

and evaluation points $\alpha = [\alpha_1, \ldots, \alpha_6]$. Let $\mathcal H$ be the set of all η such that the code $\mathcal C^{n,k}_{\boldsymbol{\alpha},t,h,\eta}$ is MDS. Then, using the notation as in the proof of Lemma 32, the determinant of the 3×3 matrix $\mathbf B^{(\eta)}$ is the evaluation at η of the polynomial

$$p(X) = -[1 + (\alpha_1 + \alpha_2 + \alpha_3) X] \prod_{\substack{i,j=1\\i \neq i}}^{3} (\alpha_i - \alpha_j).$$

The polynomials $p^{(i,j)}(X)$ $(i,j \in \{1,2,3\})$ as in Lemma 32 are given as

$$p^{(i,j)}(X) = (\alpha_{i_*} - \alpha_{j+3}) (\alpha_{i^*} - \alpha_{j+3}) (\alpha_{i_*} - \alpha_{i^*}) \cdot [1 + (\alpha_{i_*} + \alpha_{i^*} + \alpha_{j+3}) X]$$

where, or $i \in \{1, 2, 3\}$, we set $i_* := \min(\{1, 2, 3\} \setminus \{i\})$ and $i^* := \max(\{1,2,3\} \setminus \{i\})$. As shown in Lemma 32, these polynomials are all non-zero (since the α_i are distinct) and of degree at most 1.

Using the notation of the proof of Theorem 33, the determinant of the entry-wise inverse of $A^{(\eta)}$ (note that $A^{(\eta)}$ has only one 3×3 minor: the entire matrix) is the evaluation at η of the rational function $p^3 \frac{P}{Q}$, where

$$P = -X^{3} \left[1 + X \sum_{i=1}^{3} \alpha_{i} \right]^{2} \left[2 + X \sum_{i=1}^{6} \alpha_{i} \right] \prod_{\substack{i,j=4\\i < i}}^{6} (\alpha_{i} - \alpha_{j}),$$

$$Q = \prod_{\substack{i,j,\kappa=1\\i < j}}^{3} \left[1 + X \left(\alpha_i + \alpha_j + \alpha_{\kappa+3} \right) \right] \prod_{i,j=1}^{3} \left(\alpha_i - \alpha_{j+3} \right).$$

Observe that $\deg P \leq 6$ and $\deg Q = \binom{3}{2}3 = 9$. Furthermore, the polynomial Q has no zero in \mathcal{H} since each factor $[1 + X(\alpha_i + \alpha_j + \alpha_{\kappa+3})]$ is also a factor of the polynomial whose evaluation at η is the determinant of the 3×3 submatrix of $G^{(can,\eta)}$ consisting of the columns indexed by i, j, and $\kappa+3$ (which must be non-zero due to the MDS property).

By the same argument, the factor $1+X\sum_{i=1}^{3}\alpha_i$ of P cannot have a zero in \mathcal{H} . The factor X^3 has only $\eta=0$ as a zero, which obviously yields a GRS code. Hence, the code $C_{\alpha,t,h,\eta}^{n,k}$ with $\eta \in \mathcal{H} \setminus \{0\}$ is non-GRS if and only if η is a zero of $2 + X \sum_{i=1}^{6} \alpha_i$. In particular, P is the zero polynomial if and

- 1) \mathbb{F}_q has characteristic 2 and 2) $\sum_{i=1}^6 \alpha_i = 0$.

This implies a few interesting observations:

- Since the second condition can be satisfied for $q=2^m$ if and only if $m \geq 4$, this gives a family of twisted RS codes with non-trivial parameters that are GRS for all $\eta \in \mathcal{H}$.
- If $\sum_{i=1}^{6} \alpha_i = 0$, but the characteristic of \mathbb{F}_q is not 2, then any $\mathcal{C}_{\alpha,t,h,\eta}^{n,k}$ with $\eta \in \mathcal{H} \setminus \{0\}$ is non-GRS.
- If the characteristic is 2, but $\sum_{i=1}^{6} \alpha_i \neq 0$, then any $\mathcal{C}^{n,k}_{\boldsymbol{\alpha},t,h,\eta}$ with $\boldsymbol{\eta}\in\mathcal{H}\setminus\{0\}$ is non-GRS. • If the characteristic is not 2 and $\sum_{i=1}^6\alpha_i\neq 0$, then there
- is at most one $\eta \in \mathcal{H} \setminus \{0\}$ such that the code $\mathcal{C}^{n,k}_{\alpha,t,h,\eta}$ is GRS.

Theorem 33 can be interpreted as follows: for fixed n, k, t, h, and α , either all η corresponding to MDS codes are GRS, or only a number of them that is bounded by the number of roots of a non-zero ℓ-variable polynomial of degree at most 6 in each variable.

Lemma 35 Let $P \in \mathbb{F}_q[X_1,\ldots,X_\ell] \setminus \{0\}$ be a non-zero multivariate polynomial of degree at most 6 in each variable, and $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_\ell$, where $\mathcal{H}_i \subseteq \mathbb{F}_q$ with $|\mathcal{H}_i| > 6$ for all i. Then, P has at most $\prod_{i=1}^{\ell} |\mathcal{H}_i| - \prod_{i=1}^{\ell} (|\mathcal{H}_i| - 6)$ zeros

Proof: The evaluation of P at all elements of \mathcal{H} gives a codeword of an ℓ -fold product code of GRS codes of parameters $[n_i, k, d_i]$, where $n_i := |\mathcal{H}_i|$, k = 7, and $d_i = |\mathcal{H}_i| - 6$. It is well-known that such a code has length $n = \prod_{i=1}^{\ell} n_i$ and minimum distance $d = \prod_{i=1}^{\ell} d_i$, so any non-zero codeword has weight at least d. Hence, P has at most n-d zeros in \mathcal{H} , which gives the claim.

Theorem 33 and Lemma 35 imply the following corollary.

Corollary 36 Let n, k, t, h, and α be chosen as in Definition 1 such that there are sets $\mathcal{H}_i \subseteq \mathbb{F}_q$ with $|\mathcal{H}_i| > 6$ and $\mathcal{C}_{\alpha,t,h,\eta}^{n,k}$ is MDS for any $\eta \in \mathcal{H} := \mathcal{H}_1 \times \cdots \times \mathcal{H}_{\ell}$. Then, either

- all $C_{\alpha,t,h,\eta}^{n,k}$ with $\eta \in \mathcal{H}$ are GRS codes or $C_{\alpha,t,h,\eta}^{n,k}$ is a non-GRS MDS twisted RS code for at least a fraction $A := \prod_{i=1}^{\ell} \left(1 \frac{6}{|\mathcal{H}_i|}\right)$ of the elements η in \mathcal{H} . In particular, for the MDS constructions in Section IV-A:
 - Proposition 6: we have $\mathcal{H}_i := \mathbb{F}_{q_i} \setminus \mathbb{F}_{q_{i-1}}$, hence, for

$$A = \prod_{i=1}^{\ell} \left(1 - \frac{6}{q_i - q_{i-1}}\right) \ge \left(1 - \frac{6}{n(n-1)}\right)^{\ell}.$$

• Proposition 7: we have $\mathcal{H}_i := \{a\psi : a \in \mathbb{F}_{q_0}^*\}$, hence, for $q_0 \geq 8$,

$$A = \left(1 - \frac{6}{q_0 - 1}\right)^{\ell} \ge \left(1 - \frac{6}{n - 1}\right)^{\ell}.$$

The first conference version of this paper [1] contains several computer search results for twisted RS codes. Among others, we counted inequivalent MDS twisted RS codes and non-GRS twisted RS codes for small parameters ($q \le 13$). The results show that for these parameters, most MDS twisted RS codes are non-GRS and there are also several parameters resulting in mutually inequivalent twisted RS codes. We also compared twisted RS codes to Roth-Lempel codes [4], whose definition is similar to our (+)-twisted codes. The computer searches for small parameters show that the two code families are largely independent, i.e., only few of their equivalence classes intersect. More details and tables can be found in [1].

VII. DECODING

Twisted RS codes can be decoded using a simple but expensive strategy: Use brute force to determine the twist coefficients f_{h_i} for all $i=1,\ldots,\ell$, for each choice subtract the evaluation of $\sum_{j=1}^\ell \eta_j f_{h_j} X^{k-1+t_j}$ from the received word, and decode in the corresponding Reed-Solomon code. This way, we obtain a decoder with complexity q^{ℓ} times the complexity of the RS decoder, and decoding radius equal to the used RS

decoder. Note that the output list size is bounded by generic bounds on the list size, i.e., not necessarily exponential in ℓ .

In this section, we present a decoding strategy that is often faster than this brute-force decoder. This comes at the cost that we cannot rigorously prove that decoding works for any error vector up to the maximal decoding radius. However, we present a variety of numerical results that indicate that the new decoder, for large decoding parameter, is able to decode up to almost half the minimum distance with overwhelming probability.

A. Key Equations

We fix a decoding parameter $\zeta \in \mathbb{Z}_{\geq 0}$ and set up a system of key equations. For notational convenience, we define $\mathcal{I}_{\zeta} := \{ \boldsymbol{i} \in \mathbb{Z}_{\geq 0}^{\ell} : \sum_{\mu=1}^{\ell} i_{\mu} \leq \zeta \}$ and, for $\mu \in \{1, \dots, \ell\}$, $\boldsymbol{\delta}_{\mu} := [0, \dots, 0, 1, 0, \dots, 0]$ (μ -th unit vector). Note that

$$|\mathcal{I}_{\zeta}| = \begin{pmatrix} \ell + \zeta \\ \ell \end{pmatrix}$$
 and $|\mathcal{I}_{\zeta+1}| = \frac{\ell + \zeta + 1}{\zeta + 1} |\mathcal{I}_{\zeta}|$. (6)

We assume that we are given a received word $\boldsymbol{r}=\boldsymbol{c}+\boldsymbol{e}\in\mathbb{F}_q^n$, where $\boldsymbol{c}:=\operatorname{ev}_{\boldsymbol{\alpha}}(f)$, for $f\in\mathcal{P}_{\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}}^{n,k}$, is a codeword of a twisted RS code $\mathcal{C}_{\boldsymbol{\alpha},\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}}^{n,k}$, and $\boldsymbol{e}\in\mathbb{F}_q^n$ is an error of Hamming weight $\operatorname{wt}_H(\boldsymbol{e})=t$ and support $\mathcal{E}=\operatorname{supp}(\boldsymbol{e}):=\{i:e_i\neq 0\}$. We define the polynomials $\Lambda:=\prod_{i\in\mathcal{E}}(X-\alpha_i)$ (error locator polynomial), $g:=\sum_{i=0}^{k-1}f_iX^i$, where the f_i 's are the coefficients of $f,G:=\prod_{i=1}^n(X-\alpha_i)$, and R to be the unique polynomial of degree < n with $R(\alpha_i)=r_i$ for all $i=1,\ldots,n$ (interpolation polynomial of the received word). Define

$$egin{align} \Lambda_{m{i}} := \Lambda \prod_{\mu=1}^{\ell} f_{h_{\mu}}^{i_{\mu}} & \forall \, m{i} \in \mathcal{I}_{\zeta+1}, \ \Psi_{m{j}} := \Lambda_{m{j}} g & orall \, m{j} \in \mathcal{I}_{\zeta}. \ \end{split}$$

The following system of key equations relates the notions defined above.

Theorem 37 (Key Equations) Consider the setting and notation above. Then, we have for all $i \in \mathcal{I}_{\zeta}$

$$\Lambda_{\pmb{i}}R \equiv \Psi_{\pmb{i}} + \sum_{\mu=1}^\ell \Lambda_{(\pmb{i}+\pmb{\delta}_\mu)} \eta_\mu X^{k-1+t_\mu} \pmod{G}.$$

Furthermore, we have

$$\begin{split} \deg \Lambda_{\boldsymbol{i}} & \leq \deg \Lambda_{\boldsymbol{0}} & \forall \, \boldsymbol{i} \in \mathcal{I}_{\zeta+1}, \\ \deg \Psi_{\boldsymbol{j}} & \leq \deg \Lambda_{\boldsymbol{0}} + k - 1 & \forall \, \boldsymbol{j} \in \mathcal{I}_{\zeta}. \end{split}$$

Proof: We have $\Lambda R \equiv \Lambda f \pmod{G}$ since $[\Lambda(R-f)](\alpha_i) = 0$ for all $i = 1, \ldots, n$. By the structure of f, we thus have

$$\Lambda R \equiv \Lambda g + \sum_{\mu=1}^{\ell} \Lambda f_{h_{\mu}} \eta_{\mu} X^{k-1+t_{\mu}} \pmod{G}. \tag{7}$$

Multiplying (7) with $\prod_{\mu=1}^{\ell} f_{h_{\mu}}^{i_{\mu}}$ gives the result. The degree bounds follow immediately from the definition, $\deg g < k$, and the fact that the $f_{h_{\mu}}$ are scalars.

Solving the system of key equations in Theorem 37 for the unknowns Λ , g, and $f_{h_{\mu}}$ is a non-linear problem. To find a solution efficiently, we linearize the problem as follows.

Problem 38 Given t and η , and let r be a received word. Denote by R and G the polynomials defined above Theorem 37. Find polynomials $(\lambda_i)_{i \in \mathcal{I}_{\zeta+1}}$ and $(\psi_j)_{j \in \mathcal{I}_{\zeta}}$, not all zero, such that

$$\lambda_{i}R \equiv \psi_{i} + \sum_{\mu=1}^{\ell} \lambda_{(i+\delta_{\mu})} \eta_{\mu} X^{k-1+t_{\mu}} \pmod{G}, \quad (8)$$

$$\deg \lambda_i \le \deg \lambda_0, \tag{9}$$

$$\deg \psi_{i} \le \deg \lambda_{0} + k - 1,\tag{10}$$

for all $i \in \mathcal{I}_{\zeta}$.

For a solution $(\lambda_i)_{i \in \mathcal{I}_{\zeta+1}}$ and $(\psi_j)_{j \in \mathcal{I}_{\zeta}}$ of Problem 38, we call deg λ_0 the *degree* of the solution.

The problem is related to the decoding problem as follows: $(\lambda_i = \Lambda_i)_{i \in \mathcal{I}_{\zeta+1}}, \ (\psi_j = \Psi_j)_{j \in \mathcal{I}_{\zeta}}$ is a solution of Problem 38 of degree t, where $t = \operatorname{wt}_H(e)$ is the number of errors. As we want to find the error locator polynomial Λ_0 of minimal degree (i.e., the one corresponding to an error of smallest weight), we aim at finding a solution of Problem 38 of smallest-possible degree. If all goes well and there are no generic solutions of the problem of equal or smaller degree, then we indeed find the solution $(\lambda_i = \Lambda_i)_{i \in \mathcal{I}_{\zeta+1}}, \ (\psi_j = \Psi_j)_{j \in \mathcal{I}_{\zeta}}$ or a scalar multiple thereof. We can thus obtain g, i.e., the lowest k coefficients of the message polynomial, by division

$$g = \frac{\psi_{\mathbf{0}}}{\lambda_{\mathbf{0}}}.$$

Algorithm 1: Decoding Algorithm for Twisted RS Codes

Input: Received Word r, code $\mathcal{C}^{n,k}_{\alpha,t,h,\eta}$, and decoder parameter ζ

Output: A closest codeword $c \in \mathcal{C}^{n,k}_{\alpha,t,h,\eta}$ to r, or "decoding failure".

- 1 Compute R and G as defined above Theorem 37
- 2 $(\lambda_i)_{i \in \mathcal{I}_{\zeta+1}}$, $(\psi_j)_{j \in \mathcal{I}_{\zeta}} \leftarrow$ solution of minimal degree of Problem 38 with input R and G.
- 3 if λ_0 divides ψ_0 then

$$\begin{array}{c|c} \mathbf{4} & g \leftarrow \psi_{\mathbf{0}}/\lambda_{\mathbf{0}} \\ \mathbf{5} & c \leftarrow \operatorname{ev}_{\boldsymbol{\alpha}} \left(\sum_{i=0}^{k-1} g_{i} X^{i} + \sum_{j=1}^{\ell} \eta_{j} g_{h_{j}} X^{k-1+t_{j}} \right) \\ \mathbf{6} & \text{if } \operatorname{d}_{\mathbf{H}}(\boldsymbol{c}, \boldsymbol{r}) \leq \lfloor \frac{n-k}{2} \rfloor \text{ then} \\ \mathbf{7} & | \text{return } \boldsymbol{c} \end{array}$$

8 return "decoding failure"

The resulting decoder is summarized in Algorithm 1. Note that a minimal solution of Problem 38 can be found by solving the linear system of equations for any $\tau = \deg \lambda_0 = 0, 1, 2, \ldots$ (w.l.o.g. we choose λ_0 to be monic), implied by the congruence (8) and degree constraints (9) and (10), until a solution exists. In Appendix A, we show that the decoding algorithm can be implemented more efficiently, more precisely with complexity

$$O^{\sim}\left(\left(e^{\frac{\ell+\zeta+1}{\ell}}\right)^{\ell\omega}n\right) \tag{11}$$

operations in \mathbb{F}_q , where e is Euler's number.

B. Decoding Radius

The new decoder is a partial unique decoder, which means that for some error weights, some error patterns cannot be corrected, but if it works, then it returns a unique decoding solution. We informally call the maximal value τ up to which the decoder returns c from the input r = c + e for the majority of the error vectors e of weight τ the decoding radius of the new decoder, and denote it by τ_{max} .

Although we have no failure probability bound or the like for the new decoding algorithm, we present some heuristic arguments in this section that lead to expected upper and lower bounds of the decoding radius. Our numerical results in Section VII-C verify the expectation on various examples, with only very few exceptions.

Recall that $(\lambda_i = \Lambda_i)_{i \in \mathcal{I}_{\zeta+1}}$, $(\psi_j = \Psi_j)_{j \in \mathcal{I}_{\zeta}}$ is a solution of Problem 38 of degree τ , where $\tau = \operatorname{wt}_H(e)$. Furthermore, λ_0 is monic and of degree τ . Decoding succeeds if $(\lambda_i = \Lambda_i)_{i \in \mathcal{I}_{\zeta+1}}$, $(\psi_j = \Psi_j)_{j \in \mathcal{I}_{\zeta}}$ is the only solution of Problem 38 of degree τ and monic λ_0 , and there is no solution of Problem 38 of smaller degree. Note that the other direction is not necessarily true.

All solutions of Problem 38 of degree exactly τ and monic λ_0 can be determined by an inhomogeneous linear system of equations, where the unknowns are the coefficients of the λ_i (except for the leading term of λ_0 , which is set to 1) and ψ_i (the number of coefficients, and thus unknowns is determined by the degree bounds (9) and (10)) and whose equations are given by the congruence relations (8). This means that the system has $NE = n|\mathcal{I}_{\zeta}|$ equations and $NV = |\mathcal{I}_{\zeta+1}|(\tau+1)+(\tau+k)|\mathcal{I}_{\zeta}|-1$ variables. The matrix of the linear system is of the form

$$\begin{bmatrix} R & I_{n\times(k+\tau)} & A \\ 0 & 0 & B \end{bmatrix}, \tag{12}$$

where $m{R} \in \mathbb{F}_q^{n imes \tau}$ (depends on the received word $m{r}$), $m{A} \in \mathbb{F}_q^{n imes (\mathrm{NV} - 2\tau - k)}$, $m{B} = \mathbb{F}_q^{(\mathrm{NE} - n) imes (\mathrm{NV} - 2\tau - k)}$ (depends on the received word $m{r}$), and $m{I}_{n imes (k + \tau)}$ is an $n imes (k + \tau)$ matrix with ones on the diagonal and zero otherwise. The columns of the submatrix $m{R}_0$ correspond to the coefficients $0, \dots, \tau - 1$ of λ_0 and the columns of $m{I}_{n imes (k + \tau)}$ correspond to the coefficients of ψ_0 .

The decoding radius corresponds to the maximal integer τ for which, for the majority of error vectors of weight τ , the linear system of equations has a unique solution and no solution for smaller values of τ .

The linear system has $NE = n|\mathcal{I}_{\zeta}|$ equations and $NV = |\mathcal{I}_{\zeta+1}|(\tau+1) + (\tau+k)|\mathcal{I}_{\zeta}| - 1$ variables. Hence, if τ is the number of errors, Problem 38 has more than one solution for

$$NE + 2 \leq NV$$

$$\Leftrightarrow n|\mathcal{I}_{\zeta}| + 3 \leq |\mathcal{I}_{\zeta+1}|(\tau+1) + (\tau+k)|\mathcal{I}_{\zeta}|$$

$$\Leftrightarrow \tau(|\mathcal{I}_{\zeta+1}| + |\mathcal{I}_{\zeta}|) \geq (n-k)|\mathcal{I}_{\zeta}| - |\mathcal{I}_{\zeta+1}| + 2$$

$$\Leftrightarrow \tau \geq \frac{|\mathcal{I}_{\zeta}|}{|\mathcal{I}_{\zeta+1}| + |\mathcal{I}_{\zeta}|}(n-k) - \frac{|\mathcal{I}_{\zeta+1}| - 3}{|\mathcal{I}_{\zeta+1}| + |\mathcal{I}_{\zeta}|}$$

$$= \frac{\zeta+1}{2(\zeta+1)+\ell}(n-k) - \frac{\zeta+\ell+1-3(\zeta+1)\binom{\ell+\zeta}{\ell}^{-1}}{2(\zeta+1)+\ell}, \quad (13)$$

where we use (6) to obtain the last line. Since the matrix \boldsymbol{R} and parts of the matrix \boldsymbol{B} in (12) depend on the received word \boldsymbol{r} and appear to behave somewhat like random matrices for random errors, we expect that the decoder behaves as follows: for the majority of error vectors of weight τ , for τ smaller than the right-hand side of (13), the linear system has only one solution, $(\lambda_i = \Lambda_i)_{i \in \mathcal{I}_{\zeta+1}}$, $(\psi_j = \Psi_j)_{j \in \mathcal{I}_{\zeta}}$, and no solution of smaller degree.

Based on these observations, we expect that the decoding radius is at least as large as

$$\tau_{\mathsf{LB}} := \left\lceil \frac{\zeta + 1}{2(\zeta + 1) + \ell} (n - k) - \frac{\zeta + \ell + 1 - 3(\zeta + 1) \binom{\ell + \zeta}{\ell}^{-1}}{2(\zeta + 1) + \ell} \right\rceil - 1 \\
\approx \frac{\zeta + 1}{2(\zeta + 1) + \ell} (n - k). \tag{14}$$

If we inspect the linear system closer, we observe that above the radius τ_{LB} , even if a solution λ_i , ψ_j is not unique, the polynomials λ_0 and ψ_0 may be the same for all solutions. Hence, in this case, Algorithm 1 is able to retrieve the correct error positions from any solution.

Consider again the system matrix in (12). If Problem 38 has multiple solutions, but λ_0 and ψ_0 are the same for all of them, then the rank of the entire matrix is less than NV (i.e., the number of columns), but we have rank $\begin{bmatrix} R & I_{n\times(k+\tau)} \\ 0 & 0 \end{bmatrix} = k+2\tau \text{ and the column spaces of } \begin{bmatrix} R \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} R & I_{n\times(k+\tau)} \\ 0 & 0 \end{bmatrix} \text{ do not intersect. It is quite involved to predict only from the code parameters } n, k, t, h, \eta, \alpha \text{ for which exact values of } \tau \text{ these properties are fulfilled with high probability, since the matrices } R \text{ and } B \text{ depend on the received word } r. \text{ However, it is clear that } rank \begin{bmatrix} R & I_{n\times(k+\tau)} \\ 0 & 0 \end{bmatrix} < k+2\tau \text{ for } \tau > \frac{n-k}{2}, \text{ which gives an upper bound on the decoding radius. In summary, we get the following conjecture.}$

Expectation The decoding radius τ_{max} of Algorithm 1 satisfies

$$au_{\mathsf{LB}} \leq au_{\mathsf{max}} \leq \lfloor rac{n-k}{2}
floor,$$

where τ_{LB} is defined as in (14).

Our numerical results in Section VII-C confirm this expectation for various parameters, with only very few exceptions. Furthermore, the numerical results show that for randomly chosen errors of a given weight τ , the success probability of the decoding is close to 1 for τ up to the decoding radius, and close to 0 above. Note that, for a given $\varepsilon>0$, we may choose $\zeta\geq \frac{1-\varepsilon}{2\varepsilon}\ell-1$ and get

$$\tau_{\mathsf{LB}} \geq (1-\varepsilon)\frac{n-k}{2}$$
.

Hence, τ_{LB} converges to $\lfloor \frac{n-k}{2} \rfloor$ for growing decoding parameter ζ .

Furthermore, for given ε , we can rewrite the decoding complexity expression of (11) into $O^{\sim}\left(\left(\frac{e}{2\varepsilon}\right)^{\ell\omega}n\right)$ operations in \mathbb{F}_q . For comparison, a brute-force decoder for correcting the same number of errors costs $O^{\sim}\left(q^{\ell}n\right)$ operations in \mathbb{F}_q . Hence, the new decoder is faster for

$$\left(\frac{e}{2\varepsilon}\right)^{\omega} \ll q$$
,

Note that the left-hand side does not depend on the code length n or the field size q.

C. Numerical Results

In the following, we present numerical results obtained through Monte-Carlo simulations, which verify the expectation on the decoding radius for a variety of code and decoder parameters.

1) Monte-Carlo Simulations: We consider the code parameters $q \in \{23, 64, 101\}, n = q - 1$, code rates $\approx 0.3, 0.5, 0.7$, and number of twists $\ell = 1, 2, 3$. For a fixed parameter tuple $[q, n, k, \ell]$, we selected 50 twisted RS codes at random in the following way:

- $\{\alpha_1, \ldots, \alpha_n\}$ is chosen uniformly at random from the set of subsets of \mathbb{F}_q^* of cardinality n.
- t, h is chosen uniformly at random from the set of valid twist/hook vectors with distinct entries, respectively.
- η is entry-wise chosen uniformly at random from \mathbb{F}_a^* .

Note that these twisted RS codes are not necessarily MDS codes. In total, we created 1350 random codes.

Then, for each such random code, we performed, for several decoding parameters $\zeta \in \{2,4,6\}$ and decoding radii $\tau \in \{\max\{0, \tau_{\mathsf{LB}} - 2\}, \dots, \lfloor \frac{n-k}{2} \rfloor\}$, the following Monte-Carlo

- Draw a codeword c of $\mathcal{C}^{n,k}_{\alpha,t,h,\eta}$ uniformly at random Draw an error e uniformly at random from the set of vectors of Hamming weight τ
- Decode with decoding parameter ζ
- If the decoder returns c, declare a success. Otherwise, declare a failure³.

We performed this simulation 1000 times for each parameters set and estimated the failure probability of the decoder for this code, ζ and radius $\tau.$ In total, we obtained $\approx 1.7 \cdot 10^7$ samples of the Monte-Carlo simulations.

- 2) Tables: Tables I, II, and III contain the following information extracted from these Monte-Carlo simulations:
 - For each code $\mathcal{C}^{n,k}_{\alpha,t,h,\eta}$ and each decoding parameter ζ , we determine the decoding radius as the maximal value of τ for which the estimated failure probability is < 0.2.
 - Each row of the table corresponds to a parameter set $[q, n, k, \ell]$ and decoding parameter ζ . We display the numbers of codes of the parameter set (out of 50) which have a certain decoding radius τ_{max} .
 - In each row of the table, the entry below the expected lower bound on the decoding radius, $\tau_{\text{max}} = \tau_{LB}$, is marked by a superscript L, and similarly the entry below the upper bound $\tau_{\text{max}} = \lceil \frac{n-k}{2} \rceil$ is marked by superscript U. The cells corresponding to the expected range of the decoding radius have gray background color.
 - The table also contains the following three probabilities:
 - $P_{\tau_{\max}-1}^{\max}$ is the maximal observed failure probability one below the decoding radius of a code, i.e., at $\tau_{max} - 1$, maximized over all 50 codes in this row.
 - $P_{ au_{\max}}^{\max}$ is the maximal observed failure probability at the decoding radius of a code, i.e., at τ_{max} , maximized over all 50 codes in this row.

³Note that this notion of failure includes the "decoding failure" declared (and noticed) by the decoder, as well as decoding errors (the decoder returns a valid codeword not equal to c, also called miscorrections).

- $P_{\tau_{\max}+1}^{\min}$ is the minimal observed failure probability one above the decoding radius of a code, i.e., at $\tau_{\text{max}} + 1$, minimized over all 50 codes in this row.

Note that we display only the "worst" probabilities (out of 50 codes) for each row, and that the three probabilities may correspond to different codes.

Table for q=23 and n=22. See Section VII-C for the DESCRIPTION.

Pa	ara	-		Nu	mber	of co	des (Observed										
me	etei	rs	İ		tha	at hav	e $ au_{max}$	ax =			Failure Rates							
k	ℓ	$ \zeta $	0	1	2	3	4	5	6	7	$\mathbf{P}^{\max}_{\tau_{\max}-1}$	$\mathbf{P}^{\max}_{\tau_{\max}}$	$\mathrm{P}^{min}_{ au_{max}+1}$					
7	1	2					0	0	43 ^L	7^{U}	0.000	0.047	0.905					
		4					0	0	43 ^L	7 ^U	0.000	0.004	0.915					
		6					0	0	43 ^L	7^{U}	0.000	0.005	0.905					
	2	2				0	0	42 ^L	8	$ 0^{0} $	0.000	0.073	0.890					
		4				0	0	29 ^L	21	0^{U}	0.001	0.057	0.878					
	3	2			0	0	36 ^L	13	1	0^{U}	0.000	0.089	0.861					
11	1	2			0	0	40 ^L	10 ^U			0.000	0.004	0.909					
		4			0	0	40 ^L	10 ^U			0.000	0.004	0.906					
		6			0	0	40 ^L	10 ^U			0.000	0.004	0.906					
	2	2		0	0	31 ^L	19	0_{Ω}			0.000	0.087	0.896					
		4			0	0	50 ^L	0_{Ω}			0.007	0.077	0.955					
	3	2		0	0	47 ^L	3	0_{Ω}			0.005	0.119	0.921					
15	1	2	0	0	36 ^L	$14^{\rm U}$					0.000	0.004	0.900					
		4	0	0	36 ^L	14^{U}					0.000	0.005	0.907					
		6	0	0	36 ^L	14^{U}					0.000	0.004	0.900					
	2	2	0	0	50 ^L	0_{Ω}					0.000	0.102	0.958					
		4	0	0	50 ^L	0^{U}					0.000	0.098	0.957					
	3	2	2	46 ^L	2	0^{U}					0.000	0.000	0.879					

3) Observations: It can be seen that for the vast majority of the codes, the decoding radius indeed lies between τ_{LB} and $\left|\frac{n-k}{2}\right|$. This confirms our expectation that we derived heuristically in the previous subsection. There are only very few exceptions: e.g., for

- $[q, n, k, \ell] = [23, 22, 15, 3]$ and $\zeta = 2$
- $[q, n, k, \ell] = [64, 63, 19, 2]$ and $\zeta = 2$
- $[q, n, k, \ell] = [64, 63, 32, 1]$ and $\zeta = 2$

there are 2, 1, and 6 codes, respectively, whose decoding radius is one below τ_{LB} . These are 9 exceptions out of in total 2700 code/decoding parameter pairs. Furthermore, in all exceptional cases, the decoding radius is only one below the expected smallest decoding radius.

From the values of $P_{\tau_{\max}-1}^{\max}$, $P_{\tau_{\max}}^{\max}$, and $P_{\tau_{\max}+1}^{\min}$, it can also be seen that the failure probability changes sharply around the decoding radius: for many parameters, the worst observed failure probability at the decoding radius is very small, e.g. 0.004 for some codes. One below the decoding radius, the observed failure probability is 0 for most parameters (recall that the number of samples is 1000, so we can say that it is $\lesssim 10^{-3}$ with some confidence). Above the decoding radius, the failure probability is always close to 1, as expected.

APPENDIX A EFFICIENT DECODING

In this appendix, we show how to implement the decoder in Section VII more efficiently than solving a linear system. The

Table II Table for q=64 and n=63. See Section VII-C for the description.

Par	am	eters		Number of codes (out of 50) that have $\tau_{\text{max}} =$															Observe	d Failu	re Rates				
\underline{k}	ℓ	ζ	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	${\rm P}_{ au_{ m max}-1}^{ m max}$	$\left \mathbf{P}_{\tau_{max}}^{max} \right $	${\rm P}_{ au_{ m max}+1}^{ m min}$
19	1	2														0	0	17 ^L	11	15	7	0_{Ω}	0.000	0.034	0.973
		4															0	0	17 ^L	20		0_{Ω}	0.001	0.034	0.976
		6																0	0	32 ^L	18	0^{U}	0.001	0.037	0.981
	2	2												0	1	12 ^L	15	11	10	1	0	0^{U}	0.000	0.021	0.971
		4													0	0	1^{L}	21	16	12	0	0^{U}	0.001	0.031	0.975
	3	2										0	0	12 ^L	18	10	4	5	1	0	0	0_{Ω}	0.000	0.022	0.950
32	1	2									0	6	17 ^L	24	3^{U}								0.000	0.023	0.979
		4									0	0	13 ^L	34	3^{U}								0.001	0.029	0.974
		6										0	0	47 ^L	3^{U}								0.001	0.033	0.972
	2	2							0	0	21 ^L	20	9	0	0_{Ω}								0.000	0.024	0.966
		4								0	0	21 ^L	28	1	0^{U}								0.001	0.035	0.960
	3	2					0	0	2^{L}	28	16	3	1	0	0_{D}								0.000	0.018	0.965
44	1	2			0	0	13 ^L	29	8^{0}														0.000	0.019	0.977
		4				0	0	42 ^L	8 ^U														0.001	0.024	0.973
		6				0	0	42^{L}	8^{U}														0.001	0.023	0.973
	2	2		0	0	14 ^L	32	4	0^{U}														0.001	0.054	0.957
		4			0	0	32 ^L	18	0^{U}														0.001	0.043	0.964
	3	2	0	0	6 ^L	38	6	0	0^{0}														0.001	0.030	0.948

Table III $\label{eq:table for q = 101 and n = 100. See Section VII-C for the description. }$

Paran	nete																Observe	d Failu	e Rates														
$k \mid \ell$	(,	$7 \mid 8$	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	$\mathbf{P}^{\max}_{\tau_{\max}-1}$	$\mathbf{P}^{\max}_{\tau_{\max}}$	$\mathrm{P}^{min}_{ au_{max}+1}$
30 1	2	2																				0	0	9^{L}	14	10	6	9	2	$ 0^{U} $	0.000	0.015	0.983
	4	1																						0	0	13 ^L	20	13	4	$ 0^{U} $	0.000	0.012	0.986
	1 6	3																							0	0	23^{L}	22	5	$ 0^{U} $	0.000	0.012	0.983
2	2	2																0	0	1^{L}	7	18	12	5	3	2	1	1		0^{U}	0.000	0.015	0.972
	4	1																			0	0	2^{L}	18	19	5	4	2	0	$ 0^{U} $	0.000	0.000	0.983
3	2	2													0	0	0_{Γ}	11	9	12	5	4	6	2	0	1	0	0	0	0^{U}	0.000	0.023	0.966
50 1	2	2												0	0	10 ^L	15	18	7	0_{Ω}											0.000	0.016	0.987
	4	1			İ										0	0	10 ^L	25	15	0^{U}											0.000	0.012	0.985
	1	3													0	0	3 ^L	28	19	0^{U}											0.000	0.015	0.985
2	2	2									0	0	6 ^L	22	9	10	1	2	0	0^{U}											0.000	0.014	0.973
	4	1											0	0	13^{L}	22	12	3	0	0_{Ω}											0.000	0.012	0.981
3	2	2							0	0	7^{L}	18	12	8	4	1	0	0	0	0_{II}											0.000	0.023	0.977
70 1	2	2			0	0	23^{L}	18	9	0^{U}																					0.000	0.016	0.983
	4	1				0	0	35 ^L	15	0^{U}																					0.000	0.012	0.984
	1	3				0	0	30^{L}	20	0_{Ω}																					0.000	0.015	0.986
2	2	2	(0	5 ^L	24	15	6	0	0^{U}																					0.000	0.021	0.974
	4	1		0	0	0_{Γ}	35	15	0	0^{U}																					0.000	0.008	0.984
3	2	2	0 (16 ^L	29	4	1	0	0	0^{U}																					0.000	0.013	0.965

bottleneck of Algorithm 1 is Line 2, which solves Problem 38. We first show how to solve it fast using row reduction.

The following theorem shows how a minimal solution of Problem 38 can be found and in which complexity. We need the following well-known notation. For a vector $\boldsymbol{m} \in \mathbb{F}_q[X]^r$, and a *shift vector* $\boldsymbol{s} \in \mathbb{Z}^r$, we define its *s-shifted degree* as $\deg_{\boldsymbol{s}} \boldsymbol{m} := \max_j \{\deg m_j + s_j\}$ and the *s-pivot of* \boldsymbol{m} to be the right-most index i such that $\deg m_i + s_i = \deg_{\boldsymbol{s}} \boldsymbol{m}$. A matrix $\mathcal{M} \in \mathbb{F}_q[X]^{r \times r}$ is in *s-shifted weak Popov form* if all its rows have distinct *s*-pivots. It is well-known that a matrix in *s*-shifted weak Popov form is *s*-row reduced, i.e., for all i, its row with *s*-pivot i has minimal *s*-shifted degree among all non-zero elements in the matrix' row space of *s*-pivot i. Furthermore, any full-rank square matrix $\boldsymbol{M} \in \mathbb{F}_q[X]^{r \times r}$ can be transformed (by preserving its row space) into *s*-shifted

weak Popov form using the Las-Vegas algorithm in [23] with complexity $O^{\sim}(r^{\omega} \deg M)$ operations in \mathbb{F}_q , where ω is the matrix multiplication exponent and $\deg M$ denotes the maximal degree of M. Note that the algorithm in [23] ouputs an *s-shifted Popov form*, which is in particular in *s-shifted weak Popov form*.

Theorem 40 Consider an instance of Problem 38. Let

$$m{M} := egin{bmatrix} m{I}_{|\mathcal{I}_{\zeta+1}| imes |\mathcal{I}_{\zeta+1}|} & m{A} \ G \cdot m{I}_{|\mathcal{I}_{\zeta}| imes |\mathcal{I}_{\zeta}|} \end{bmatrix},$$

where $\mathbf{A} \in \mathbb{F}_q[X]^{|\mathcal{I}_{\zeta+1}| \times |\mathcal{I}_{\zeta}|}$ is a matrix whose (i, j)-th entry for $i \in \mathcal{I}_{\zeta+1}$ and $j \in \mathcal{I}_{\zeta}$ (fix orders i_i and j_j of the elements

in $\mathcal{I}_{\zeta+1}$ and \mathcal{I}_{ζ} , respectively, both starting with $\mathbf{0}$) is

$$m{A_{i,j}} := egin{cases} R, & ext{if } m{i} = m{j}, \ -\eta_{\mu} X^{k-1+t_{\mu}}, & ext{if } m{i} = m{j} + m{\delta}_{\mu} ext{ for some } \mu, \ 0, & ext{else}. \end{cases}$$

Furthermore, let $s \in \mathbb{Z}^{|\mathcal{I}_{\zeta+1}|+|\mathcal{I}_{\zeta}|}$ be such that

$$s_i = \begin{cases} k, & \text{if } i = 1, \\ k - 1, & \text{if } 1 < i \le |\mathcal{I}_{\zeta+1}|, \\ 0, & \text{if } |\mathcal{I}_{\zeta}| < i \le |\mathcal{I}_{\zeta+1}| + |\mathcal{I}_{\zeta}|. \end{cases}$$

Let M' be a basis in s-shifted weak Popov form of the module spanned by the rows of M, and let m be the (unique) row of M' with s-pivot 1. Then,

$$[\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_{|\mathcal{I}_{\zeta+1}|}}, \psi_{j_1}, \dots, \psi_{j_{|\mathcal{I}_{\zeta}|}}] := m$$

is a solution of Problem 38 of minimal $\deg \lambda_0$.

The matrix M' can be computed using the Las-Vegas algorithm in [23] in

$$O^{\sim}((|\mathcal{I}_{\zeta+1}| + |\mathcal{I}_{\zeta}|)^{\omega}n) \subseteq O^{\sim}((e^{\ell+\zeta+1})^{\ell\omega}n)$$

operations over \mathbb{F}_q , where e is Euler's constant.

Proof: We first show that the rows of M form a basis of the module $\mathcal M$ of vectors

$$\boldsymbol{v} := [\lambda_{\boldsymbol{i}_1}, \lambda_{\boldsymbol{i}_2}, \dots, \lambda_{\boldsymbol{i}_{|\mathcal{I}_{\zeta+1}|}}, \psi_{\boldsymbol{j}_1}, \dots, \psi_{\boldsymbol{j}_{|\mathcal{I}_{\zeta}|}}] \in \mathbb{F}_q[X]^{|\mathcal{I}_{\zeta+1}| + |\mathcal{I}_{\zeta}|}$$

that satisfy the congruence relation in (9). Consider an element $v \in \mathcal{M}$. Then, there are polynomials $\chi_i \in \mathbb{F}_q[X]$, for $i \in \mathcal{I}_\zeta$, such that

$$\psi_{i} = \lambda_{i} R - \sum_{\mu=1}^{\ell} \lambda_{(i+\delta_{\mu})} \eta_{\mu} X^{k-1+t_{\mu}} + \chi_{i} G$$

for all $i \in \mathcal{I}_{\zeta}$. By the choice of M, we have

$$v = [\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_{|\mathcal{I}_{c+1}|}}, \chi_{j_1}, \dots, \chi_{j_{|\mathcal{I}_{c}|}}] \cdot M.$$

Hence, \mathcal{M} is contained in the row space of M. On the other hand, any row of M is in \mathcal{M} since:

• Index the first $|\mathcal{I}_{\zeta+1}|$ rows of M by i and the $|\mathcal{I}_{\zeta}|$ congruence relations in (9) by j. Then row i satisfies relation j since

$$egin{align} R &= R + \sum_{\mu=1}^{\ell} 0 \cdot \eta_{\mu} X^{k-1+t_{\mu}}, & ext{if } m{j} = m{i}, \ 0 &= -\eta_{\mu} X^{k-1+t_{\mu}} + \eta_{\mu} X^{k-1+t_{\mu}}, & ext{if } m{j} = m{i} - m{\delta}_{\mu}, \ 0 &= 0 + \sum_{\mu=1}^{\ell} 0 \cdot \eta_{\mu} X^{k-1+t_{\mu}}, & ext{else}. \ \end{pmatrix}$$

• The last $|\mathcal{I}_{\zeta}|$ rows of M satisfy (9) since

$$0 \equiv G + \sum_{\mu=1}^{\ell} 0 \cdot \eta_{\mu} X^{k-1+t_{\mu}} \mod G.$$

Also, the rows of M are linearly independent since the matrix is in upper triangular form with non-zero diagonal entries.

Since the row m of M' has s-pivot 1, the degree inequalities (9) and (10) are fulfilled. This is true since, by the definition of the s-pivot and the choice of the shift, we have

$$\deg \lambda_{\mathbf{0}} + k_1 = \deg \lambda_{i_1} + k > \deg \lambda_{i_i} + k - 1$$

$$\Leftrightarrow \deg \lambda_{i_i} \le \deg \lambda_{\mathbf{0}}$$

for all $i = 1, \ldots, |\mathcal{I}_{\zeta+1}|$, and

$$\deg \lambda_{\mathbf{0}} + k_1 = \deg \lambda_{i_1} + k > \deg \lambda_{j_j}$$

$$\Leftrightarrow \deg \lambda_{j_i} \leq \deg \lambda_{\mathbf{0}} + k - 1$$

for all $j=1,\ldots,|\mathcal{I}_{\zeta}|$. Hence, m is a solution of Problem 38. Moreover, it is also one of minimal degree since M' is srow reduced, i.e., m has minimal s-shifted degree among all non-zero vectors in the row space of M' with s-pivot 1.

As for the complexity, the matrix M has $|\mathcal{I}_{\zeta+1}| + |\mathcal{I}_{\zeta}|$ rows and columns, and maximal degree at most n. The complexity follows by the algorithm in [24], see complexity expression above the theorem.

The other operations of Algorithm 1 are standard polynomial operations, which all have complexity $O^{\sim}(n)$ operations in \mathbb{F}_q , see, e.g., [25]:

- R in Line 1 is obtained via Lagrange interpolation with n point tuples,
- G in Line 1 can be computed via a subproduct tree,
- Lines 3 and 4 can be implemented by a division with remainder,
- and Line 5 is a multi-point evaluation of a polynomial of degree < n at n points.

Hence, the bottleneck is Line 2, and Algorithm 1 can be implemented with complexity

$$O^{\sim}\left(\left(e^{\frac{\ell+\zeta+1}{\ell}}\right)^{\ell\omega}n\right)$$

operations in \mathbb{F}_q , where e is Euler's number.

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