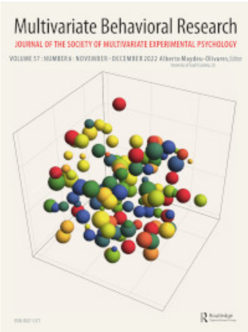

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A Sequential Response Model for Analyzing Process Data on Technology-Based Problem-Solving Tasks

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ABSTRACT
Students' response sequences to a technology-based problem-solving task can be treated as a discrete time stochastic process with a conditional Markov property—after conditioning on the students' abilities of problem solving, the next state only depends on the current state. This article proposes a sequential response model (SRM) with a Bayesian approach for parameter estimation that incorporates comprehensive information from the response process to infer problem-solving ability more effectively. A Monte Carlo simulation study showed that parameters were well-recovered. An illustrated example is provided to showcase additional gains using our model for understanding the response process with a real-world interactive assessment item "Tickets" in the programme for international student assessment (PISA) 2012.

KEYWORDS
Sequential response model; technology-based assessment; process data; response sequence

Introduction

To evaluate the students' complex problem-solving abilities in more realistic situations, an emerging number of technology-based interactive evaluation systems have been developed. For example, internationally, the program for international student assessment (PISA) of the organization for economic cooperation and development (OECD) launched a computer-based problem-solving assessment in 2012, and added a human-computer interactive collaborative problem-solving test in 2015 (OECD, 2014, 2016). OECD also organized the programme for international assessment of adult competencies (PIAAC) to assess the skills of literacy, numeracy, and problem-solving in technology-rich environments in 2012 (OECD, 2012; Schleicher, 2008). Recently in the United States, the National Assessment of Education Progress (NAEP) implemented technology and engineering literacy assessments based on technology-enhanced learning environments (TEL, 2013). Compared with traditional paper-pencil tests, these new technology-based tests have substantial changes in building virtual simulation scenarios which often includes multiple operations to reach the final answer in a real-world setting. In addition, one of the most promising features is that it can record process data, which often comes with keystrokes, click flow, text, and other forms of data while students complete the task. Therefore, process data can provide more enriched information about the students' cognitive and response processes beyond the traditional right or wrong simplification.

Most of the existing technology-based problem-solving questions are well-defined, which have a clearly initial state, target state, and intermediate operations (Mayer & Wittrock, 2006). Therefore, the students' operations can be extracted from process data in a labeled form with time-stamped information, for which we refer to as response sequences in the current study, and this kind of new response type has become increasingly useful in understanding the students' behaviors in the problem-solving process. In the literature, three types of strategies have been proposed to utilize the ordered and time-stamped response sequence for describing sequential behavioral characteristics: natural language processing (NLP) techniques, traditional psychometric models on the recoded data, and dynamic Bayesian networks (DBNs; Hao et al., 2015; He et al., 2020; He & von Davier, 2016; Tang et al., 2020; Tang et al., 2021). Response sequences can be analogized to strings in natural languages and analyzed by NLP techniques. He and von Davier (2016) utilized N-gram, and term frequency-inverse document frequency (TF-IDF; Manning & Schutze, 1999) to detect key action

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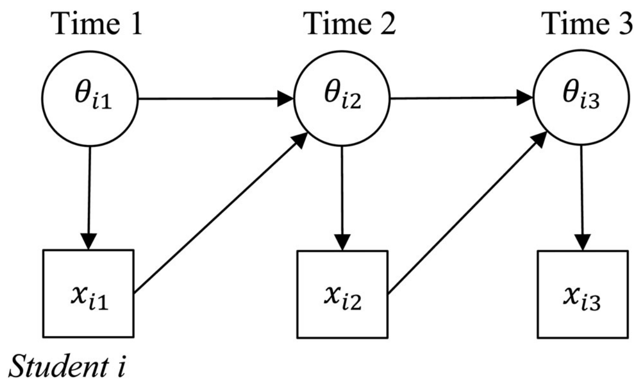

Figure 1. Graph for a dynamic Bayesian network (Levy & Mislevy, 2016, p. 384).

sequences that are associated with success or failure on a problem-solving item. A sequence-to-sequence auto-encoder and multidimensional scaling (MDS) were applied to extract latent variables from response sequences (Tang et al., 2020; 2021), but it is difficult to interpret the standard numerical vectors due to a mismatch between the number of dimensions of the extracted data representation and that of the ability. The edit distance and the longest common subsequence (LCS) were used to compare the distance or similarity for the response sequences and the optimal sequence, which can be considered as a new feature variable that encodes information about the students' proficiency (Hao et al., 2015; He et al., 2020). However, using NLP techniques cannot obtain continuous ability estimates as in the traditional psychometric models.

Besides NLP, some studies treated the operation or state transitions in response sequences as indicators then recoded them according to their frequency. The scored indictors could fit in a traditional psychometric model to estimate the students' ability (Han & Wilson, 2020; Shu et al., 2017). This approach allows us to avoid much information loss by involving the whole response sequence. However, it needs to bucketize the sequential response process into disordered operation or states, which leads to the loss of sequence information to a certain extent.

Another potential approach for modeling response sequence in problem-solving tasks is to use dynamic Bayesian networks (DBNs; Käser et al., 2017; Reichenberg, 2018; Reye, 2004; Rowe & Lester, 2010; VanLehn & Learning, 2008). DBNs are extensions of the original Bayesian network to model state transition incorporating temporal information. Figure 1 illustrates a path diagram representing the two essential parts of a DBN. Variables in circles and rectangles correspond to latent abilities and observed responses, respectively; and paths represent the dependence structure among the variables across time (Levy & Mislevy, 2016). It is noteworthy that the latent variables in DBNs are often categorical, indicating whether the student has mastered a certain knowledge skill, or representing different levels of mastery of knowledge, rather than specifying a continuous underlying latent variable as is done in tradtional psychometric models.

DBNs have been applied to assessment and learning analytics. For example, Reye (2004) demonstrated how preceding lines of work on modeling longitudinal patterns could be framed as DBNs, paving the way for their applications in tutoring systems (Reye, 2004; VanLehn & Learning, 2008) and game-based assessments of learning or change (Iseli et al., 2010; Levy, 2019). Levy (2019) constructed a DBN for *Save Patch*, an educational game targeting rational number addition (Chung et al., 2010). Bergner et al. (2017) explored peer tutoring collaborations in a computer-based adaptive peer tutoring assistant setting (APTA; Walker et al., 2009a, 2009b, 2011), and demonstrated the use of a special case of DBNs, specifically, hidden Markov Models (HMM). Arieli-Attali et al. (2019) extracted process data in an adaptive testing setting under different target conditions and applied HMM to model the students' choice behaviors. DBNs have a unique advantage of utilizing the information from different patterns of a response sequence and maintaining the sequential structure of the response sequence. However, the current categorical nature and dynamic characteristics of the latent variable hinders its application to sequential response data in the modern assessment setting.

Although the new sequential nature of the modern assessment data has the potential in understanding the students' answering process and inferring their academic achievement, existing solutions fail to fully utilize this rich information: NLP techniques can be used to extract the features of response sequences effectively, but these features cannot infer the students' abilities. Traditional psychometric models can be used to obtain the students' ability estimates, but this approach can only applicable after we recode the state transitions according to their frequency which loses the ordering information. DBNs can analyze the response sequence directly, but the latent variables obtained are categorical and dynamic, rather than continuous and static abilities. Thus, we hope to propose a hybrid model that joins item response models with DBNs, by extending the DBN model to estimate a continuous and static latent variable which is often the case in traditional psychometric models, such as item response theory (IRT) models (Lord, 1980) or nominal response models (NRM; Bock, 1972). The purpose of this study is to develop a sequential response model (SRM) that combines DBNs with traditional psychometric models to be able to infer a continuous latent variable from a sequence of categorical variables. The proposed measurement model allows for analyzing the students' whole response sequences in technology-based problem-solving tasks.

The next section describes the formulation of the proposed SRM as well as its parameter estimation. This is followed by a simulation study to determine how well the estimation procedure recovers model parameters. Next, we illustrate the use of the proposed model with an empirical example of analyzing process data in a problem-solving task. The paper concludes with a discussion and a summary for future methodological research in this area.

Sequential response model

Theoretical foundations of SRM

To solve a problem (i.e., to move from the initial state to the final state), certain steps, so-called solution operations, must be taken (Anderson et al., 2007; Dörner, 1976). The process of human problem-solving is to achieve the goal through a series of psychological calculations or external operations (Newell & Simon, 1972). In this process, the student encounters various problem scenarios, and the sum of these problem scenarios constitutes the problem states. The problem states can follow three categories: *initial state*, *target state*, and a series of *intermediate states* in between. For students with different levels of knowledge and intelligence, the available operations may be different, and the transition sequence of the problem state can be different as well. Therefore, the problem state sequence available in process data of the technology-based task can be treated as an external manifestation of latent ability, which can reflect the mental process of the student in the process of problem solving. We define the state transition as the change caused by the students' operation and all the transitions moving closer to the target state as correct and all the others as incorrect, which will be demostrated below using an example.[1] Therefore, the objective of modeling is to infer the latent ability from the observed state sequence of problem solving. In our study, the response sequence also refers to the problem state sequence, the two terms will be used interchangeably in the current paper.

For example, assuming that there is a task with only three states — A, B, and C, where A is the *initial state* and C is the *target state*. To move from the initial state A to the target state C, the student must go through the intermediate state B. We denote A, B, and C as the states of the problem and A → B → C as the optimal sequence to solve the problem, with the arrows representing the operations that causes the state changes. Since the consequences of the same operation under different problem states may be different, we use the transition of the problem states (e.g., A → B or AB in short) to characterize the response process. In practice, few students take the optimal sequence to solve problems. They may make mistakes in the process of solving problems — jumping off the optimal sequence or struggling between the intermediate states. Suppose that there are several possible state transitions in this simple problem scenario as summarized in Table 1.

In Table 1, S_t represents the current state, and S_{t+1} represents the possible next states, with the elements in the table representing the possible state transitions, and the blank cells representing the impossible state transitions in this situation. For example, if the current state is A, the possible state transitions are A# and A → B. A# means that the student stops without reaching the target state, that is, the task is ended without any effective operation. A → B means that the student has reached the intermediate state B, which is one step closer to the target state. Transition A → C is not possible because we assume that students have to go step by step following the optimal sequence, so it has been left blank. When the current state is B, the possible state transitions are B → A and B → C. B → A means that the student has returned to the initial state A (i.e., moving away from the target state), which is considered as an incorrect operation per our definition, while B → C means that the student has correctly reached the target state C, which is considered correct. And we assume that in state B, it is impossible for the task to terminate prematurely, so we left transition B# blank.

As defined above, according to whether the state transition is on the optimal sequence, we can divide the state transitions into correct transitions (e.g, A → B and B → C) and incorrect transitions (e.g., A# and B → A). The state sequences of students are different

[1] It is worth mentioning that the correctness of a certain state transition depends on the specific situation. For simple cases where all states are included in the optimal sequence, we can divide the state transitions into correct transitions and incorrect transitions based on whether the state transition is on the optimal solution sequence or not. In complex situations, we still follow the same principle. For example, it is considered incorrect to deviate from the optimal sequence and reach an incorrect state, or to transition from one incorrect state to another incorrect state. However, we denote the transitions as correct when they return to the optimal sequence.

Table 1. Possible state transitions in a simple three-states problem.

	S_{t+1}			
S_t	A	B	C	*End*
A		**AB**		**A#**
B	BA		**BC**	

Note: This table contain two corecct transition AB and BC, two incorrect transition A# and BA. A# represnts that the student stops without reaching the target state. The blank cells represent the impossible state transitions in this situation. C# is not included in the table for reaching the target state means the task is terminated.

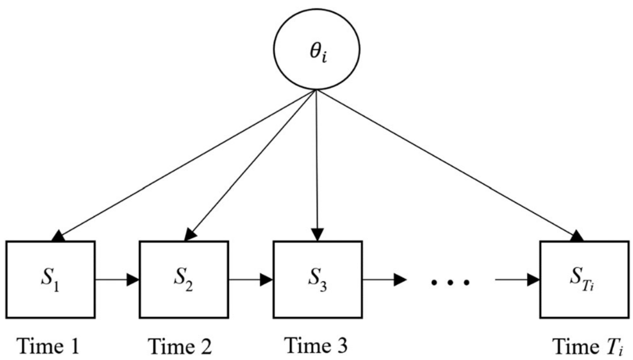

Figure 2. Schematic diagram of sequential response model.

in length, and may have infinite theoretical possibilities. For example, the state sequence can be A, A → B → C, A → B → A, A → B → A → B → A, or even A → B → A → B → A → B → A → B → A → B → A and so on. In a well structured problem with finite problem states, infinite response sequences can be represented by finite problem states, and the finite state transitions can be divided into correct and incorrect. We expect that different latent ability values should result in different response sequences. We will propose a modeling solution to the senario described above in the next subsection.

Specification of SRM

Suppose that a task has R discrete states, and the set of all states can be expressed as $\boldsymbol{x} = \{x_1, x_2, x_3, ..., x_R\}$. $S_{i,t} = x_j$ indicates that student i has selected the state x_j $(x_j \in \boldsymbol{x})$ at time t $(t = 1, 2, ..., T_i)$, where T_i represents the length of state sequence for student i. To facilitate understanding, Figure 2 shows a visual representation of our proposed model. The observed data is a state sequence with T_i time points, where S_t represents the state at time t and θ_i represents the latent ability value for student i. Notice that the subscript i of $S_{i,t}$ is omitted for brevity.

We assume that during the problem-solving process, the next state S_{t+1} only depends on the current state S_t and the latent ability θ_i, which is equivalent to assuming that the response sequence can be treated as a discrete time stochastic process with a conditional Markov property. It should be noted that latent variables in our model are continuous, while observed variables (i.e., problem states) are nominal and arranged in order according to time.

Suppose the state at time t is x_j, $\boldsymbol{M}_{x_j}$ are used to represent the set of all possible states at time $t + 1$. And all the possible transitions can be classified as correct and incorrect. Use $I^+_{x_j,x_k}$ to indicate whether the state transition $x_j \longrightarrow x_k$ $(x_k \in \boldsymbol{M}_{x_j})$ is correct. The value of $I^+_{x_j,x_k}$ is 1, when the transition is correct, otherwise it is -1. Before analysis, the limited problem states contained in the task and the transition rules between states need to be pre-set. The pre-set rules of the task itself are represented by $\mathcal{R}$.

The SRM describes how the current state S_t and the latent ability θ_i affect the next state S_{t+1}, which can be expressed by the following equation:

$$P\left(S_{i,t+1} = x_k | S_{i,t} = x_j, \theta_i, \boldsymbol{\lambda}, \mathcal{R}\right) = \frac{\exp\left(\lambda_{x_j,x_k} + I^+_{x_j,x_k} \cdot \theta_i\right)}{\sum_{x_h \in \boldsymbol{M}_{x_j}} \exp\left(\lambda_{x_j,x_h} + I^+_{x_j,x_h} \cdot \theta_i\right)}, \quad x_j \in \boldsymbol{x}, x_k \in \boldsymbol{M}_{x_j} \tag{1}$$

Here, λ_{x_j,x_k} represents the tendency of transition from state x_j to state x_k, reflecting the possibility of the transition $x_j \longrightarrow x_k$. For each possible state transition in the task, there is a transition parameter to describe its possibility of occurrence and $\boldsymbol{\lambda}$ represents a vector of the tendency parameters for all possible state transitions. In the same state, the correct transition parameter is greater than that of the incorrect ones, indicates that it is more likely to make the right choice under this state; whereas the transition parameter of an incorrect transition is greater than that of correct transition indicates that the incorrect transition is highly error-prone—it is more likely to make the wrong choice in the next step. The range of the transition parameter λ is similar to that of the ability parameter, given that both are defined on the same scale.

The transition parameter λ reflects the characteristics of the task itself, and is independent of the student ability parameter. The transition parameter and the latent ability jointly determine the possibility of each state transition, as shown in Equation (1). When $x_j \longrightarrow x_k$ is a correct state transition, the numerator in Equation (1) is simplified as $\exp\left(\lambda_{x_j,x_k} + \theta_i\right)$, that is, higher-ability students are more likely to make the correct state transition; whereas, when $x_j \longrightarrow x_k$ is an incorrect state

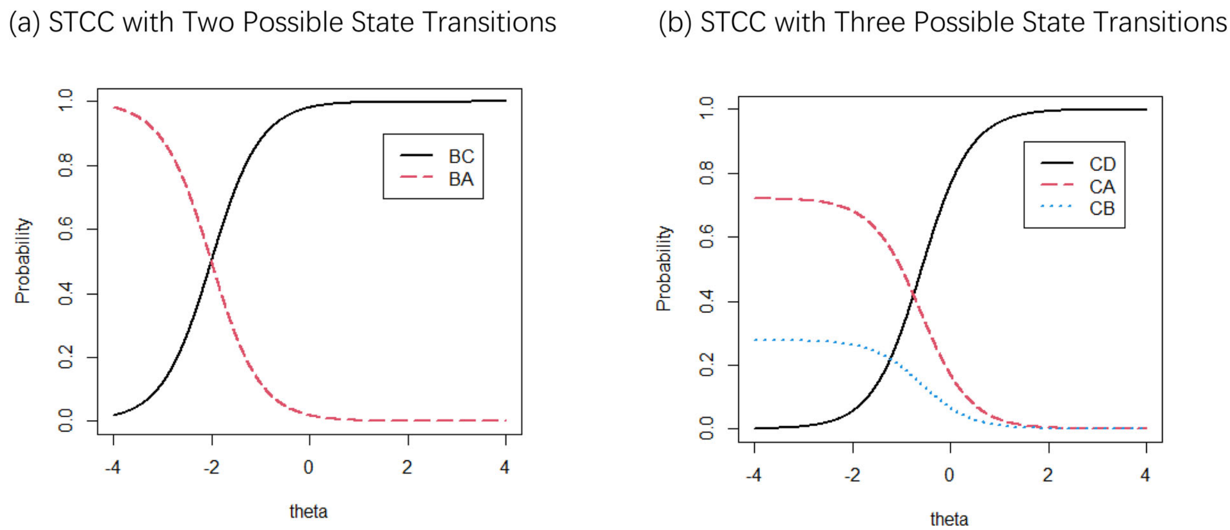

Figure 3. State transition characteristic curves.
Note: in Figure 3(a), the solid line represents the probability of B $\rightarrow$ C and the dash line represents the probability of B $\rightarrow$ A. In Figure 3(b), the solid line represents the probability of selecting C $\rightarrow$ D. The dash line represents the probability of selecting C $\rightarrow$ A, and the dotted line represents the probability of selecting C $\rightarrow$ B.

transition, the numerator is $\exp\left(\lambda_{x_j,x_k} - \theta_i\right)$, indicating that students with higher ability are less likely to make the incorrect state transition. The denominator of Equation (1) is to add all possible state transitions under the condition of state x_j, which normalizes the probability.

To sum up, we assume that higher ability leads to a greater probability of taking the correct transition and lower ability leads to a greater probability of taking the incorrect transition. And the sum of probabilities of all state transitions given a certain state is 1.

The joint probability of the whole state sequence $\boldsymbol{S_i} = (S_{i,1}, S_{i,2}, \ldots, S_{i,T_i})$ is modeled as the product of the probabilities of each state transition:

$$L(\boldsymbol{S_i}|\theta_i, \boldsymbol{\lambda}, \mathcal{R}) = \prod_{t=1}^{T_i-1} P(S_{i,t+1}|S_{i,t}, \theta_i, \boldsymbol{\lambda}, \mathcal{R}). \tag{2}$$

Consider once again the simple three states A, B, and C scenario and the state transitions in Table 1 as an example. If the current state is B, then the next state has two possibilities: state A or state C. We assume that state transition B $\rightarrow$ C is correct, while state transition B $\rightarrow$ A is incorrect. We use $\lambda_{B,C}$ and $\lambda_{B,A}$ to represent their tendency to for these transitions to occur, respectively. Therefore, for student i with ability θ_i, given the current state B, the probability of selecting state C in the next state is

$$P(S_{i,t+1} = C|S_{i,t} = B, \theta_i, \boldsymbol{\lambda}, \mathcal{R}) = \frac{\exp(\lambda_{B,C} + \theta_i)}{\exp(\lambda_{B,C} + \theta_i) + \exp(\lambda_{B,A} - \theta_i)} \tag{3}$$

and the probability of selecting state A in the next state is

$$P(S_{i,t+1} = A|S_{i,t} = B, \theta_i, \boldsymbol{\lambda}, \mathcal{R}) = \frac{\exp(\lambda_{B,A} - \theta_i)}{\exp(\lambda_{B,C} + \theta_i) + \exp(\lambda_{B,A} - \theta_i)} \tag{4}$$

where

$$P(S_{i,t+1} = C|S_{i,t} = B, \theta_i, \boldsymbol{\lambda}, \mathcal{R}) + P(S_{i,t+1} = A|S_{i,t} = B, \theta_i, \boldsymbol{\lambda}, \mathcal{R}) = 1.$$

If the response sequence of student i is A $\rightarrow$ B $\rightarrow$ C, that is, $T_i = 3$, and $S_{i,1} = A$, $S_{i,2} = B$, $S_{i,3} = C$, denoted as $\boldsymbol{S_i} =$ A, B, C, where A $\rightarrow$ B and B $\rightarrow$ C are all considered as correct transitions, then the probability of its occurrence is

$$L(\boldsymbol{S_i}|\theta_i, \boldsymbol{\lambda}) = P(B|A, \theta_i, \boldsymbol{\lambda}, \mathcal{R}) \cdot P(C|B, \theta_i, \boldsymbol{\lambda}, \mathcal{R}) = \frac{\exp(\lambda_{A,B} + \theta_i)}{\exp(\lambda_{A,B} + \theta_i) + \exp\left(\lambda_{A,\#} - \theta_i\right)} \cdot \frac{\exp(\lambda_{B,C} + \theta_i)}{\exp(\lambda_{B,C} + \theta_i) + \exp(\lambda_{B,A} - \theta_i)} \tag{5}$$

State transition characteristic curve (STCC)

In SRM, given the latent ability and state transition parameters, the probability of each state transition being selected can be plotted as a STCC. The STCC in SRM is similar to the item characteristic curve (ICC) in IRT models. Consider once again the three states example, and further assume that $\lambda_{B,A} = -2$

and $\lambda_{B,C} = 2$. Figure 3(a) shows the probability of selecting B → A (represented by the dashed line) or B → C (represented by the solid line) by students with different abilities.

As shown in Figure 3(a), the probability of choosing the correct transition B → C increases with the increase of the students' ability, while the probability of choosing the incorrect transition B → A decreases accordingly. The intersection point of the correct transition probability curve and the wrong transition probability curve is at $\theta = \frac{\lambda_{B,A} - \lambda_{B,C}}{2} = \frac{2\lambda_{B,A}}{2} = \lambda_{B,A} = -2$, where the probability of correct transition $P(C|B, \theta, \mathcal{R}, \lambda_{B,C} = 2)$ equals the incorrect transition $P(A|B, \theta, \mathcal{R}, \lambda_{B,A} = -2)$.

Taking another slightly more complicated example that the current state is C, and there are two possible incorrect transitions C→A and C→B, and a correct transition C→D. Assuming further that $\lambda_{C,A} = -2$, $\lambda_{C,B} = -1$, and $\lambda_{C,D} = 3$, Figure 3(b) shows the STCCs when there are three possible state transitions. In Figure 3(b), the solid line represents the probability of the correct transition $P(D|C, \theta, \mathcal{R}, \lambda_{C,D} = 3)$, which increases as θ increases. The dashed line represents $P(A|C, \theta, \mathcal{R}, \lambda_{C,A} = -2)$, and the dotted line represents $P(B|C, \theta, \mathcal{R}, \lambda_{C,B} = -1)$, which both decrease as θ increases.

With STCC, we can see that the performance of the probabilistic model is consistent with our assumption stated before: the higher the abilities of the students are, the more likely they choose the correct state transition, and the less likely they choose the incorrect state transition. Note that there may be multiple possible incorrect transitions under the same state. In this case, the tendency parameter indicates the propensity to move to the corresponding state. The tendency parameter λ of incorrect transitions represents the degree of being error-prone, and the higher the tendency is, the harder to avoid taking the transition. Similarly, higher tendency parameter of the correct transition indicates the greater the probability of making a correct choice.

Bayesian estimation

We choose to use a Bayesian approach to estimate the proposed model to obtain the estimates of the latent ability and state transition parameters for its flexibility and convenience in implementation. Letting $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ denote the collection of latent variables for n examinees and letting $\boldsymbol{S}$ denote the full collection of n response sequences. The posterior probability of the model parameters is as follows:

$$p(\boldsymbol{\theta}, \boldsymbol{\lambda}|\boldsymbol{S}, \mathcal{R}) \propto p(\boldsymbol{S}|\boldsymbol{\theta}, \boldsymbol{\lambda}, \mathcal{R}) p(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \prod_{i=1}^{n} \prod_{t=1}^{T_i - 1} p(S_{i,t+1}|S_{i,t}, \theta_i, \boldsymbol{\lambda}, \mathcal{R}) p(\theta_i) p(\boldsymbol{\lambda}). \tag{6}$$

In Equation (6), $p(\theta_i)$ and $p(\boldsymbol{\lambda})$ are the prior distributions of the latent ability and state transition tendency parameter vector, respectively. They are assumed to be independent of each other. The prior distributions for the latent variables are assumed to be standard normal distributions for simplicity in Bayesian estimation. In addition, note that any constant to all of the transition parameters in Equation (1) yields different parameter sets but the same values of $P(S_{i,t+1}|S_{i,t}, \theta_i, \boldsymbol{\lambda}, \mathcal{R})$. For the purpose of model identification, the sum of all possible transition parameters, given state x_j is zero. That is, we constrain $\sum_{x_k \in \boldsymbol{M}_{x_j}} \lambda_{x_j, x_k} = 0$. Metropolis-Hastings-within-Gibbs sampling approach was used to implement the Markov chain Monte Carlo (MCMC) estimation to empirically approximate the joint posterior distribution (Patz & Junker, 1999a, 1999b). The detailed sampling procedures are given in the appendix.

In Bayesian estimation, convergence and data model fitting need to be evaluated. Convergence is defined when the chains have 'forgotten' their initial values, and the posterior draws from all chains are indistinguishable. The potential scale reduction factor (PSRF; Gelman & Rubin, 1992) is used to monitor convergence of MCMC. Generally, approximate convergence is diagnosed when R is close to 1. For each replication and given parameter, the standard deviation of the sample means from the chains, which can be called Monte Carlo error (MCE; Koehler et al., 2009), was also used as an evidence of convergence. In addition, the trace plots of MCMC and the standard deviations of the estimated parameters of multiple chains were also used to check the convergence.

Posterior predictive checking (PPC) using the test statistics approach can be used to check model-data fit (Gelman et al., 1996; 2014; Guttman, 1967; Rubin, 1984). In our study, model fit was evaluated by comparing the frequencies of state transitions in the observed data to the frequencies obtained in the posterior predicted data.

Simulation study

In this section, a Monte Carlo simulation study was carried out to demonstrate the rationale of the proposed sequential response model in dealing with

Table 2. All possible state transitions in the simulation study.

S_t	S_{t+1} A	B	C	D	E	F	G	End
A		**AB**						A#
B	BA		**BC**					B#
C	CA	CB		**CD**				
D		DB	DC		**DE**			D#
E			EC			**EF**		E#
F			FC	FD			**FG**	

Note: Each cell represents a possible state transition, totaling 18 possible state transitions. The 6 state transitions in bold are correct while the other 12 state transitions are incorrect. A#, B#, D#, and E# stand for early termination of the task. Blank cells indicate that there are no such transitions.

Table 3. True values of transition parameters for the simulation study.

Transition parameters	Short sequence	Long sequence
$\lambda_{A\#}$	−3.016	−3.016
λ_{AB}	3.016	3.016
λ_{BA}	−1.527	−0.627
$\lambda_{B\#}$	−1.469	−1.969
λ_{BC}	2.996	2.596
λ_{CA}	−0.166	−0.316
λ_{CB}	−0.138	0.012
λ_{CD}	0.304	0.304
λ_{DB}	0.197	0.097
λ_{DC}	−0.060	0.640
$\lambda_{D\#}$	−0.128	−0.828
λ_{DE}	−0.009	0.091
λ_{EC}	0.235	0.435
$\lambda_{E\#}$	−0.076	−0.776
λ_{EF}	−0.159	0.341
λ_{FC}	0.704	0.704
λ_{FD}	1.229	1.229
λ_{FG}	−1.933	−1.933

response sequences and its parameter recovery with the Bayesian estimation procedure. The effects of different prior distributions, sample sizes, and response sequence length on the parameter recovery of SRM were evaluated.

3.1. Simulation design

In the simulation study, we hypothesize a scenario with seven finite problem states: A, B, C, D, E, F, and G, where A is the initial state and G is the target state with an optimal sequence being A → B → C → D → E → F → G. All possible state transitions are summarized in Table 2.

S_t and S_{t+1} represent the state at time t and $t+1$ respectively; the cells in the table are the pre-set possible state transitions. The six state transitions in bold (A → B, B → C, C → D, D → E, E → F, and F → G) on the optimal sequence A → B → C → D → E → F → G are regarded as correct state transitions. Transition A#, B#, D#, and E# are considered to be incorrect because they represent the early termination before the target is reached, and the other transitions are considered to be incorrect state transitions because they are the operations of deviating from the correct sequence.

Table 3 shows the true state transition parameters of the model. The lengths of observed sequences can be changed according to the setting of state transition parameters. Two sequence length conditions can be obtained by changing the values of transition parameters that represent return operations. For model identification, we constrained $\sum_{x_k \in \boldsymbol{M}_{x_j}} \lambda_{x_j, x_k} = 0$.

Two different configurations of prior distributions for the transition parameters were specified. First, we investigated the flat weakly informative (non-formative) distributions. The prior for each incorrect transition parameters had an expected value of 0 and a standard deviation of 10, $\boldsymbol{\lambda}_{\text{incorrect}} \sim \text{MVN}(0, \mathbf{I} \cdot 100)$. Where $\mathbf{I}$ is an identity matrix and $\boldsymbol{\lambda}_{\text{incorrect}} = (\lambda_{A\#,} \lambda_{BA}, \lambda_{B\#}, \lambda_{\mathrm{CA}}, \lambda_{\mathrm{CB}}, \lambda_{\mathrm{DB}}, \lambda_{\mathrm{DC}}, \lambda_{D\#}, \lambda_{EC}, \lambda_{E\#}, \lambda_{\mathrm{FC}}, \lambda_{\mathrm{FD}})'$ including all incorrect state transition parameters. We only set the prior distribution of incorrect state transition parameters because the correct state transition parameters are constrained to the opposite of the sum of all incorrect transition parameters under the same state. The second type of prior distribution we investigated was an informative distribution. The prior for each incorrect transition parameters followed a standard normal distribution with an expected value of 0 and a standard deviation of 1. Ability values followed a standard normal distribution.

The conditions of the simulation study consist of two sample size (1000, 2000 students); sequences length (short, long); and two levels of prior distributions (informative, non-informative) resulting in a $2 \times 2 \times 2 = 8$ fully crossed design.

One hundred samples were generated for each condition. State transition parameters remained constant across replications, and true person parameters were generated from the standard normal distribution for each replication. A self-generated Bayesian sampler in R (R core Team, 2018) was used for parameter estimation. The first 5,000 samples were used to select an appropriate variance of the proposal distribution. Then five chains were used and random initial values were assigned for each chain. Ten thousand iterations from each chain were obtained, and the first 3,000 iterations were discarded as burn-in, yielding 35,000 iterations that served to empirically approximate the posterior distribution. The R code for the Bayesian sampler is included in the supplementary material.

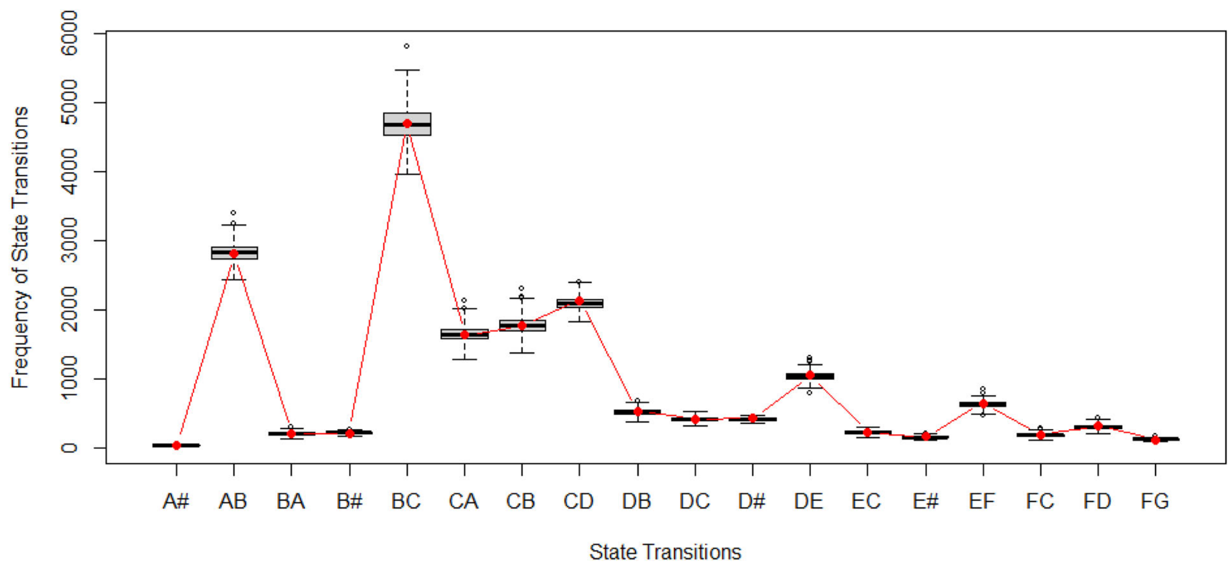

Figure 4. Observed and posterior predicted state transition distributions for the simulation study.
Note: At each transition, the box depicts the interquartile range, the notch in the middle depicts the median, and the whiskers depict the 2.5th and 97.5th percentiles. The points indicate the frequency in the observed data.

Results of the simulation study

Convergence and model fit

The PSRF values of all parameters in this study were between 1 and 1.1, which we surmised as being converged (Brooks & Gelman, 1998; Gelman & Rubin, 1992). In addition, the MCEs of ability and tendency parameters were less than 0.016 and 0.005 respectively. This means that the differences between the means of sampling chains for each parameter are very small, which further indicates the chains were converged. To check the convergence of MCMC in greater detail, we displayed the trace plot from the MCMC algorithm for the 18 state transition parameters in the cell that crossed the conditions of sample size of 1000, short sequence, and informative prior. These trace plots can be found in the supplementary material (see "Trace Plots for State Transition Parameters in the Simulation Study.pdf"), which showed that although the initial values were different, they converged to the same posterior distribution. The results of the other 99 replications and other conditions were similar to those of the first replication in this condition.

We used 2,500 iterations from MCMC (500 from each of five chains) of the first replication for the 1000 sample size, short sequence and informative prior condition to conduct PPC. In Figure 4, the red point indicates the frequency with that particular state transition in the observed data. Line segments connect the red points as a visual aid. And each boxplot presents the posterior predictive distribution of each state transition obtained by the model estimation.

It can be seen in Figure 4 that the empirical values from the observed data are all close to the median of the posterior predictive distributions, which provides evidence that the model fits the data well. Further, the results could be numerically summarized via the posterior predictive p-value (*ppp*) based on the *chi-square* of state transition distribution. The *ppp* is 0.53, which is close to 0.5 and indicates a good model fit (Gelman et al., 2014).

Estimation accuracy

The average absolute deviation between parameter estimates and true values (ABSE), root mean squared error (RMSE) were used to evaluate parameter recovery. We also calculated the correlation between the estimated values and the true values as a consistency index. Note that when evaluating the estimation accuracy of the students' abilities, we used the average ability of the same response pattern instead of a single ability. The average estimation accuracy of the 100 replications is shown in Table 4.

The estimation accuracy of state transition parameters presented in Table 4 is the average of 18 state transition parameters. In this study, the average sequence length of the short and long sequence condition was 18 and 31 respectively. As shown in Table 4, under all conditions, the correlations between the estimated values and the true values are relatively high, and all the correlation values are greater than 0.92. Parameter recovery was similar regardless of the prior configuration. As expected, the increase in sample size resulted in a more accurate estimation of the

Table 4. Summary of parameter estimation accuracy.

Simple Size	Sequence Length	Prior	Correlation θ	Correlation λ	ABSE θ	ABSE λ	RMSE θ	RMSE λ	Average Length
1000	Short	Non-Inf.	0.929	0.999	0.239	0.048	0.311	0.060	18.416
		Inf.	0.928	0.999	0.241	0.049	0.313	0.060	18.341
	Long	Non-Inf.	0.943	0.999	0.208	0.039	0.276	0.050	30.659
		Inf.	0.944	0.999	0.208	0.041	0.275	0.052	30.604
2000	Short	Non-Inf.	0.930	1.000	0.232	0.032	0.301	0.041	18.439
		Inf.	0.930	1.000	0.233	0.033	0.302	0.042	18.374
	Long	Non-Inf.	0.945	1.000	0.202	0.030	0.267	0.037	30.771
		Inf.	0.945	1.000	0.201	0.029	0.266	0.037	30.734

Note: Inf. and Non-Inf. are short for Informative and Non-informative, respectively.

transition parameters, reducing the ABSE and RMSE of the estimation. The sequence length was another key element to increase precision. The ABSE and RMSE of the latent abilities and transition parameters were all reduced as the sequence length increased.

The average frequency and estimation accuracy (RMSE) of each state transition parameter under each condition are further compared in Table 5. The RMSE of each state transition parameter reduced as the sample size and sequence length increased, which are the same as the conclusions from Table 4. Besides, the estimation accuracy of each state transition parameter showed some similar patterns across different conditions. For example, in the 1000 sample size, short sequence and informative prior condition, the RMSEs of $\lambda_{A\#}$, λ_{AB}, and λ_{FG} are 0.107, 0.107 and 0.089, respectively, which are larger than the average RMSE of the transition parameters under the same condition (0.060, see Table 4). Similar performance appeared in other conditions. The RMSEs of $\lambda_{A\#}$, λ_{AB}, and λ_{FG} are slightly larger than other state transition parameters across different conditions. Two reasons for this inconsistency are frequency and state transition true values. The lower the occurrence frequency of the state transition is, or the larger the discrepancy of frequency among transitions in the same state is, so that the lower the accuracy of the parameter estimation is. For example, the frequency of state transition A $\rightarrow$ # and F $\rightarrow$ G are relatively small under all conditions. Second, the true value of state transition parameters. If the true value of state transition parameter is more extreme (deviates further from 0), then its estimation accuracy is slightly lower. As shown in Table 3, the truth values of $\lambda_{A\#}$(-3.016) and λ_{AB}(3.016) are relatively extreme, and their estimation accuracy is relatively poor under various conditions. Table 5 also shows that the estimation accuracy of extreme values (e.g., $\lambda_{A\#}$ and λ_{AB}) under non-informative prior is slightly better than the informative condition. Whereas the informative prior is more favorable for the estimation of non-extreme values. Nevertheless, on average, there are no differences between the two prior configurations for the average RMSE of all transitions (see Table 4).

Empirical study

In this section, we illustrate the application of the proposed method using the process data of the *Tickets* task from PISA 2012.The proposed model is shown to effectively distinguish the different response patterns and give the corresponding ability estimates reasonably.

The tickets task and data

In the unit *Tickets*, students are asked to operate a virtual automated ticketing machine to buy a ticket per the requirements. This unit contains three items, and we take the first item in the following analysis (Question: Tickets CP038Q02). More details about the task unit can be found in the PISA 2012 technical report (OECD, 2014).

States and state transitions of the tickets task

In Question CP038Q02, students are invited to buy a full fare, country train ticket with two individual trips. The solution requires multiple steps — students have to first select the network (COUNTRY TRAINS), then the fare type (FULL FARE), next choose one for multiple individual trips, and finally indicate the number of trips (TWO) and press the BUY button. In addition to the best solution, students may make mistakes, such as choosing the wrong network, purchasing CONCESSION fare by mistake and so on. Before clicking the BUY button, students can click the CANCEL button on any interface to return to the initial interface. This makes the students' response sequences may have infinite theoretical possibilities. However, in this well-structured problem, the problem states and state transitions (i.e., operations) are limited.

The definition of problem states and state transitions is a key task in the pre-processing procedure to apply the SRM. In this item, in addition to the initial state without any operation and the termination state after clicking the BUY button, there are 22 possible intermediate states. The states were characterized by the students' choices of the four ordered variables: network, fare type, trip type and number of trips, as shown in columns 2 to 5 in Table 6. Students may make mistakes in each of the choices, which are shown in bold in Table 6. As long as students make

Table 5. The average frequency and estimation accuracy of transition parameters.

		Short sequence				Long sequence			
		Non-inf. prior		Inf. prior		Non-inf. prior		Inf. prior	
	Transition parameters	Freq.	RMSE	Freq.	RMSE	Freq.	RMSE	Freq.	RMSE
$N = 1000$	λ_A	42	0.096	43	0.107	62	0.075	61	0.085
	λ_{AB}	2954	0.096	2944	0.107	4410	0.075	4408	0.085
	λ_{BA}	212	0.062	213	0.058	980	0.035	977	0.039
	λ_B	225	0.054	226	0.052	256	0.047	256	0.052
	λ_{BC}	4930	0.064	4905	0.066	7505	0.051	7505	0.055
	λ_{CA}	1784	0.028	1774	0.032	2493	0.024	2493	0.024
	λ_{CB}	1835	0.031	1823	0.032	3453	0.024	3455	0.025
	λ_{CD}	2188	0.049	2184	0.055	3904	0.043	3875	0.042
	λ_{DB}	578	0.039	576	0.042	878	0.033	874	0.034
	λ_{DC}	448	0.046	450	0.045	1523	0.031	1501	0.034
	λ_D	421	0.044	420	0.047	349	0.048	349	0.048
	λ_{DE}	1069	0.065	1064	0.062	1806	0.059	1805	0.053
	λ_{EC}	236	0.055	233	0.056	437	0.049	434	0.054
	λ_E	171	0.067	174	0.058	133	0.058	132	0.066
	λ_{EF}	662	0.066	657	0.057	1236	0.059	1239	0.062
	λ_{FC}	194	0.062	193	0.071	385	0.052	383	0.050
	λ_{FD}	327	0.060	326	0.054	651	0.048	654	0.046
	λ_{FG}	141	0.092	138	0.089	200	0.079	202	0.073
		Short sequence				Long sequence			
		Non-inf. Prior		Inf. prior		Non-inf. prior		Inf. prior	
	Transition Parameters	Freq.	RMSE	Freq.	RMSE	Freq.	RMSE	Freq.	RMSE
$N = 2000$	λ_A	84	0.058	84	0.068	127	0.048	128	0.058
	λ_{AB}	5908	0.058	5898	0.068	8862	0.048	8813	0.058
	λ_{BA}	429	0.041	425	0.040	1977	0.029	1964	0.026
	λ_B	453	0.041	452	0.037	514	0.035	513	0.035
	λ_{BC}	9859	0.041	9827	0.045	15093	0.042	15003	0.044
	λ_{CA}	3563	0.024	3557	0.021	5012	0.021	4977	0.019
	λ_{CB}	3666	0.021	3652	0.024	6959	0.019	6923	0.020
	λ_{CD}	4392	0.039	4368	0.039	7802	0.035	7753	0.034
	λ_{DB}	1167	0.030	1154	0.028	1763	0.028	1744	0.025
	λ_{DC}	899	0.029	893	0.033	3031	0.023	3001	0.024
	λ_D	835	0.034	839	0.035	696	0.033	697	0.038
	λ_{DE}	2148	0.048	2138	0.048	3616	0.042	3614	0.039
	λ_{EC}	475	0.043	470	0.040	880	0.034	879	0.039
	λ_E	348	0.043	345	0.044	260	0.053	263	0.038
	λ_{EF}	1325	0.050	1323	0.045	2476	0.049	2472	0.044
	λ_{FC}	387	0.045	387	0.043	769	0.036	770	0.031
	λ_{FD}	657	0.038	656	0.038	1303	0.036	1302	0.035
	λ_{FG}	280	0.050	281	0.054	403	0.057	399	0.051

Note: Inf. and Non-Inf. are short for Informative and Non-informative respectively. Freq. is short for Frequency.

Table 6. The intermediate states of the *Tickets* task (Item CP038Q02).

No.	Network	Fare type	Trip type	Number of trips	States	State code
1	COUNTRY TRAINS	–	–	–	Correct network	B
2	COUNTRY TRAINS	FULL FARE	–	–	Correct fare type	C
3	COUNTRY TRAINS	FULL FARE	INDIVIDUAL	–	Correct trip type	D
4	COUNTRY TRAINS	FULL FARE	INDIVIDUAL	2	Correct trips	E
5	COUNTRY TRAINS	FULL FARE	INDIVIDUAL	**1,3,4,5**	Incorrect trips 1	F
6	**CITY SUBWAY**	–	–	–	Incorrect network	G
7	COUNTRY TRAINS	**CONCESSION**	–	–	Incorrect fare type	H
8	**CITY SUBWAY**	FULL FARE	–	–		
9	**CITY SUBWAY**	**CONCESSION**	–	–		
10	COUNTRY TRAINS	FULL FARE	**DAILY**	–	Incorrect trip type	I
11	COUNTRY TRAINS	**CONCESSION**	INDIVIDUAL	–		
12	COUNTRY TRAINS	**CONCESSION**	**DAILY**	–		
13	**CITY SUBWAY**	FULL FARE	INDIVIDUAL	–		
14	**CITY SUBWAY**	FULL FARE	**DAILY**	–		
15	**CITY SUBWAY**	**CONCESSION**	INDIVIDUAL	–		
16	**CITY SUBWAY**	**CONCESSION**	**DAILY**	–		
17	COUNTRY TRAINS	**CONCESSION**	INDIVIDUAL	2	Incorrect trips 2	J
18	COUNTRY TRAINS	**CONCESSION**	INDIVIDUAL	**1,3,4,5**		
19	**CITY SUBWAY**	FULL FARE	INDIVIDUAL	2		
20	**CITY SUBWAY**	FULL FARE	INDIVIDUAL	**1,3,4,5**		
21	**CITY SUBWAY**	**CONCESSION**	INDIVIDUAL	2		
22	**CITY SUBWAY**	**CONCESSION**	INDIVIDUAL	**1,3,4,5**		

Note: The contents in bold represent incorrect choices.

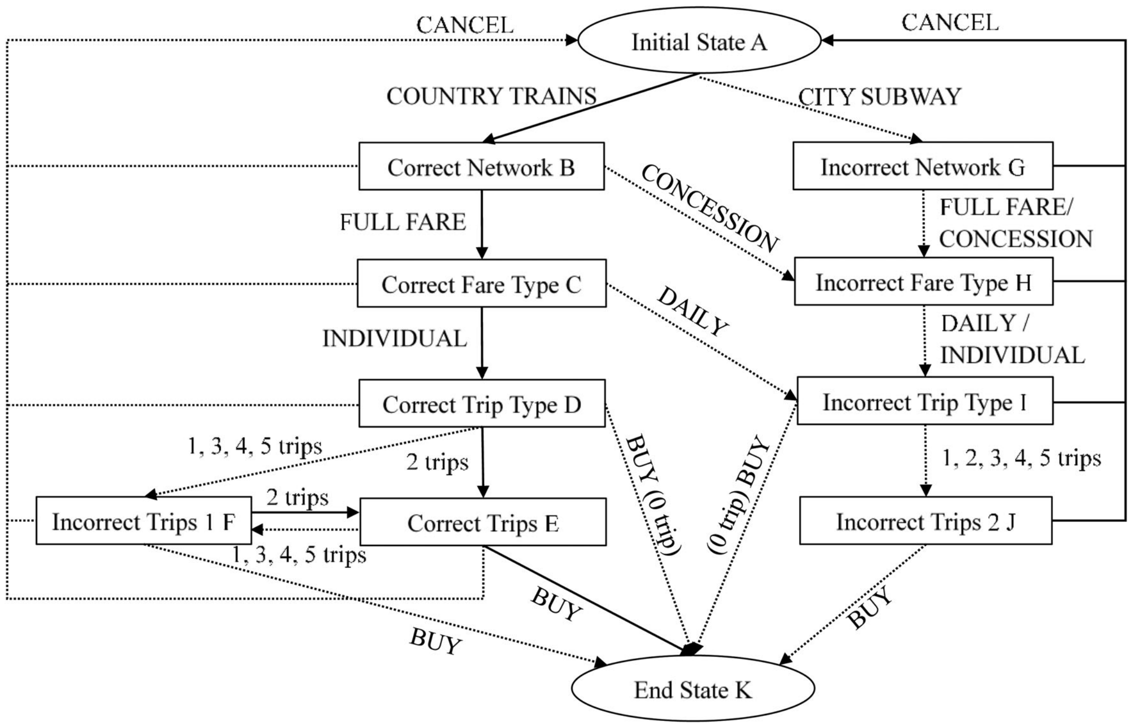

Figure 5. Diagram of all possible states and state transitions for the first item in the tickets task.
Note: the ellipses represent the start and end states, and the rectangles represent the intermediate states. The arrows represent the state transitions (i.e., operations): the solid arrows represent the correct state transitions, while the dotted arrows represent the incorrect state transitions.

mistakes in any of the choices, they will not be able to buy the right ticket. According to which variable is chosen and whether there are incorrect choices in the state, the 22 intermediate states can be categorized into nine states (see column 6 in Table 6). In the right-most column of Table 6, each of the nine states is given an alphabetic label for convenience of communication. In addition to the nine states summarized in Table 6, the initial state was coded as A and the termination state after clicking the BUY button was coded as K. Then the problem is decomposed into 11 possible states. All the possible operation sequences can be expressed as the transition path between these 11 states, as summarized in Figure 5.

In Figure 5, the ellipses represent the start and end states, and the rectangles represent the intermediate states. The arrows represent the state transitions (i.e., operations). The solid arrows represent the correct state transitions, while the dotted arrows represent the incorrect state transitions. The optimal sequence is Initial State (A) → Correct Network (B) → Correct Fare Type (C) → Correct Trip Type (D) → Correct Trips (E) → End State (K). We expected the highest ability estimates of the students whose response sequences perfectly match the optimal sequence. All transitions that move closer to by the right ticket are considered correct. Thus, the five state transitions on this optimal sequence (e.g., A → B, B → C, C → D, D → E, and E → K) and five transitions from incorrect state back to correct state or the initial state (e.g., G → A, H → A, I → A, J → A and F → E) are defined as the correct state transitions. All the other transitions are incorrect. A summary of the transitions is given in Table 7. In Table 7, the 10 correct transitions are in bold, and, the other 17 transitions are incorrect. There are 27 possible state transitions and 11 possible states in total. The definition of problem states and state transitions is relatively easy in this well-defined task, while it may not be the case in more complex scenarios. We further expand on this point subsequently.

Data description

After data cleaning, 28,838 students' response data were retained in this study. The state sequences were extracted from each the student's log file. Each sequence started with state A and ended with state K. The length of the response sequences was different. The shortest sequence consisted of 5 states, while the longest sequence consisted of 110 states, and the average sequence length of all students was about 7. There were 1,409 different response sequences. 15,379 students with 573 different patterns of response sequences finished the task correctly. Among them, 10,545 students completely followed the optimal path, and

Table 7. Summary of all state transitions of the first item in the *Tickets* task.

S_t		S_{t+1}										
Labels	States	A Initial State	B Correct Network	C Correct Fare Type	D Correct Trip Type	E Correct Trips	K End State	F Incorrect Trips 1	G Incorrect Network	H Incorrect Fare Type	I Incorrect Trip Type	J Incorrect Trips 2
A	Initial state		**COUNTRY TRAINS**						CITY SUBWAY			
B	Correct network	CANCEL		**FULL FARE**						CONCESSION		
C	Correct fare type	CANCEL			**INDIVIDUAL**						DAILY	
D	Correct trip type	CANCEL				**2 trips**	BUY	1, 3, 4, 5 trips				
E	Correct trips	CANCEL					**BUY**	1, 3, 4, 5 trips				
F	Incorrect trips 1	CANCEL				**2 trips**	BUY					
G	Incorrect network	**CANCEL**								FULL FARE/ CONCESSION		
H	Incorrect fare type	**CANCEL**									DAILY / INDIVIDUAL	
I	Incorrect trip type	**CANCEL**					BUY					1, 2, 3, 4, 5 trips
J	Incorrect trips 2	**CANCEL**					BUY					

Note: The contents in bold represent correct transitions.

the other 4,834 students made mistakes during their response processes, but then corrected the mistakes and finally completed the task correctly.

Parameter estimation

For this simple scenario, we assume that after conditioning on the students' abilities of problem solving, the choice of the next state depends on the current state, which means the response sequence can be treated as a discrete time stochastic process with a conditional Markov property. The response sequences were fitted using the proposed SRM in Equation (1), with the prior distributions specified as: $\theta \sim N(0, 1)$ and $\boldsymbol{\lambda} \sim \text{MVN}(0, \mathbf{I})$. The proposed Bayesian procedure was used to estimate the parameters with the same number of chains and draws in the simulation study. We used the frequency weighting for each response pattern to reduce the computation overload instead of the raw data. The detailed estimation procedure can be found in the appendix.

Results of the empirical study

Convergence and model fit

The model converged as the PSRF for all parameters were between 1 and 1.02 and the average MCE for ability and tendency parameters were 0.015 and 0.016, respectively. We also provided the posterior distribution plots of the state transition parameters in the supplementary file ("Posterior Distribution Plots of State Transition Parameters for the Tickets Task.pdf"). The posterior distributions from each chain overlapped well, which provides another piece of evidence of convergence.

We used 2,500 iterations from MCMC (500 from each of five chains) to conduct PPC. Figure 6 compares the frequencies of state transitions in the observed data to the frequencies obtained in the posterior predicted data, and the *ppp* value based on the *chi-square* of state transition distribution is 0.418, indicating a good model fit.

The marginal posterior distributions for the state transition parameters

The marginal posterior distributions were mostly unimodal and symmetric with several of them exhibiting some level of skewness. We report the median in addition to the mean as summaries of the central tendency for the state transition parameters in Table 8. The 95% highest posterior density (HPD) column is the interval that contains 95% of the space of the estimated parameter where the posterior density of the

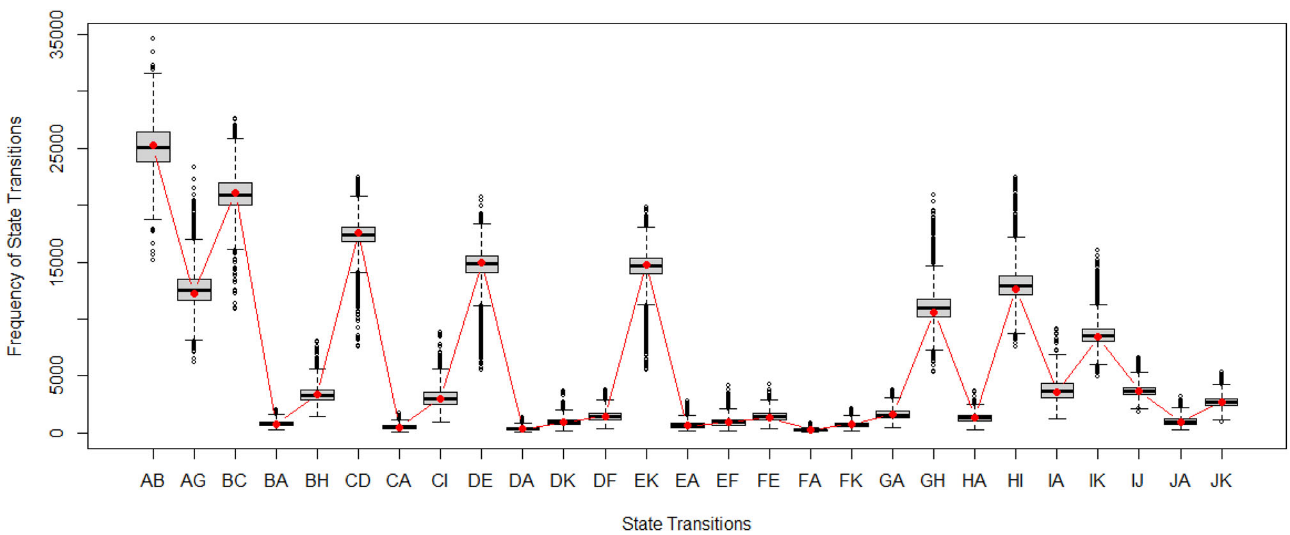

Figure 6. Observed and posterior predicted state transition distributions for the sequential response model of the tickets task data.
Note: At each transition, the box depicts the interquartile range, the vertical line that goes through the box in the middle depicts the median, and the whiskers depict the 2.5th and 97.5th percentiles. The points indicate the frequency in the observed data.

Table 8. Summary of the marginal posterior distributions for the state transition parameters on the *Tickets* task data.

Parameter	Frequency	Mean	Median	Standard Deviation	95% Highest Posterior Density Interval
$\boldsymbol{\lambda_{AB}}$	**26376**	**0.479**	**0.482**	**0.187**	**(0.086, 0.835)**
λ_{AG}	12889	−0.479	−0.482	0.187	(−0.835, −0.086)
$\boldsymbol{\lambda_{BC}}$	**22059**	**1.601**	**1.586**	**0.249**	**(1.101, 2.093)**
λ_{BA}	812	−1.544	−1.534	0.197	(−1.968, −1.184)
λ_{BH}	3505	−0.057	−0.048	0.165	(−0.391, 0.264)
$\boldsymbol{\lambda_{CD}}$	**18369**	**1.422**	**1.409**	**0.281**	**(0.877, 1.982)**
λ_{CA}	527	−1.627	−1.611	0.238	(−2.135, −1.196)
λ_{CI}	3163	0.204	0.209	0.194	(−0.167, 0.608)
$\boldsymbol{\lambda_{DE}}$	**15546**	**1.645**	**1.614**	**0.398**	**(0.895, 2.450)**
λ_{DA}	343	−1.425	−1.401	0.366	(−2.174, −0.725)
λ_{DK}	966	−0.340	−0.332	0.267	(−0.877, 0.185)
λ_{DF}	1514	0.120	0.130	0.253	(-0.421, 0.592)
$\boldsymbol{\lambda_{EK}}$	**15379**	**1.387**	**1.354**	**0.355**	**(0.722, 2.114)**
λ_{EA}	630	−0.904	−0.884	0.275	(−1.479, −0.401)
λ_{EF}	944	−0.483	−0.473	0.261	(−1.001, 0.036)
$\boldsymbol{\lambda_{FE}}$	**1407**	**0.748**	**0.740**	**0.232**	**(0.317, 1.212)**
λ_{FA}	251	−0.956	−0.952	0.165	(−1.286, −0.632)
λ_{FK}	800	0.208	0.209	0.147	(−0.089, 0.48)
$\boldsymbol{\lambda_{GA}}$	**1692**	**−0.652**	**−0.647**	**0.184**	**(−1.014, −0.300)**
λ_{GH}	11197	0.652	0.647	0.184	(0.300, 1.014)
$\boldsymbol{\lambda_{HA}}$	**1392**	**−0.832**	**−0.826**	**0.188**	**(−1.212, −0.468)**
λ_{HI}	13310	0.832	0.826	0.188	(0.468, 1.212)
$\boldsymbol{\lambda_{IA}}$	**3774**	**0.246**	**0.247**	**0.224**	**(−0.199, 0.696)**
λ_{IK}	8857	0.296	0.296	0.125	(0.041, 0.539)
λ_{IJ}	3842	−0.542	−0.543	0.126	(−0.788, −0.281)
$\boldsymbol{\lambda_{JA}}$	**1006**	**0.094**	**0.101**	**0.252**	**(−0.390, 0.609)**
λ_{JK}	2836	−0.094	−0.101	0.252	(−0.609, 0.390)

Note: The correct transitions are in bold.

estimated parameter is maximized. Each row in Table 8 represents a state transition, with the correct transitions are in bold.

It can be seen from Table 8 that the tendency parameter for the correct transition D → E — from taking the correct trip type state to choosing the correct number of trips state — has the highest estimated value (1.645). This implies that students who have correctly selected the trip type (INDIVIDUAL) are very likely to continue choosing the right number of trips. Furthermore, the tendency parameter for the correct transition B → C — from the correct network state to correct fare type state by selecting FULL FARE — has a high estimated value (1.601) as well, which means that students who have correctly selected the network (COUNTRY TRAINS) are likely to continue to choose FULL FARE correctly. In the incorrect network state G, the tendency parameter of continuing to select the fare type (i.e., incorrect transition G → H, 0.652) is greater than the tendency to return to the initial state (i.e., correct transition G → A, −0.652). Similarly, when the incorrect fare type state H is taken, the propensity to continue selecting trip type (i.e., incorrect transition H→I, 0.832) is higher than that of returning to the initial state (i.e., correct transition H→A, −0.832). This shows that in the middle interfaces of the task, students are unlikely to click CANCEL to return to the initial interface no matter whether the current choices are appropriate or not.

However, when arriving at the ticket purchase interface, the tendency to restart from the incorrect state is greater than that of direct purchase. In the state of wrong number of tickets (state J), the tendency of clicking the CANCEL button (J → A, 0.094) is larger than clicking the BUY button (J → K, −0.094). And the tendency parameter of the correct transition to click the CANCEL button in incorrect trip type state (I → A, 0.246) is similar to clicking the BUY button in incorrect trip type state (I → K, 0.296), which indicates that students who make

Table 9. Summary of the marginal posterior distributions for the ability estimates and its corresponding response sequences with frequency greater than 100 for the sequential response model of the *Tickets* task data.

Response pattern	Frequency	Finish	Mean	Median	Standard deviation	95% Highest posterior density interval
AGHIJK[1]	1609	No	−1.184	−1.142	0.653	(−2.449, 0.099)
AGHIK	4804	No	−1.067	−1.021	0.681	(−2.425, 0.244)
AGHIAGHIK	165	No	−0.764	−0.735	0.503	(−1.783, 0.175)
ABHIJK	604	No	−0.749	−0.733	0.521	(−1.798, 0.263)
ABHIK	855	No	−0.601	−0.592	0.559	(−1.689, 0.519)
ABCIK	1544	No	−0.298	−0.306	0.510	(−1.298, 0.708)
AGABCIK	105	No	−0.198	−0.202	0.449	(−1.132, 0.631)
ABCDFK	457	No	−0.178	−0.186	0.478	(−1.104, 0.78)
AGHIJABCDEK	177	Yes	0.069	0.065	0.376	(−0.635, 0.842)
ABCDK	708	No	0.096	0.072	0.560	(−0.96, 1.245)
AGHIABCDEK	661	Yes	0.236	0.231	0.411	(−0.577, 1.035)
ABCDFEK	329	Yes	0.371	0.348	0.515	(−0.637, 1.383)
ABABCDEK	172	Yes	0.374	0.339	0.511	(−0.562, 1.432)
ABHIABCDEK	152	Yes	0.386	0.370	0.437	(−0.444, 1.271)
AGHABCDEK	287	Yes	0.415	0.404	0.456	(−0.47, 1.305)
ABCDEFEK	259	Yes	0.427	0.400	0.507	(−0.566, 1.44)
ABCABCDEK	140	Yes	0.447	0.413	0.507	(−0.491, 1.501)
ABCDEABCDEK	209	Yes	0.542	0.510	0.478	(−0.395, 1.474)
AGABCDEK	543	Yes	0.591	0.563	0.511	(−0.362, 1.621)
ABCIABCDEK	239	Yes	0.594	0.561	0.488	(−0.321, 1.581)
ABHABCDEK	155	Yes	0.654	0.626	0.511	(−0.31, 1.689)
ABCDEK	10545	Yes	0.903	0.856	0.687	(−0.388, 2.27)

[1]AGHIJK is short for sequence A → G → H → I → J → K, with the arrowswere omitted for brevity. The same below.

mistakes in a certain intermediate state are more likely to restart (by clicking the CANCEL button) when they arrive at the ticket buying interface, rather than restart in a certain state in the middle.

The marginal posterior distributions for ability estimates

Table 9 summarizes the marginal posterior distributions for the latent abilities and the corresponding response sequences with frequency greater than 100, and they are sorted from smallest to largest according to the means of the ability estimates for different response patterns. The third column represents whether the task has been successfully solved, that is, to buy a full fare, country train ticket with two individual trips.

As expected, students who made more mistakes had lower estimated abilities. The sequence A → G → H → I → J → K in the first row indicates that students who made a mistake in selecting the type of network at the beginning and continued choosing incorrect transitions until they bought the ticket, thus these students obtained the lowest ability estimate. In contrast, students who made fewer mistakes had higher ability estimates. Those who took the optimal path A → B → C → D → E → K in the last row obtained the highest ability estimates. Next, students with sequence A → B → H → A → B → C → D → E → K in the second to last row, which represents the pattern that students made an error when selecting the far type for the first time but corrected it in the next step, also had a relatively high ability estimate 0.654. Note that students with the response sequence A → B → C → D → K in the middle part of Table 9 represents students who chose the right network, the fare type and the trip type but only missed the number of trips. Therefore, even if they did not complete the task exactly right, they were still in the middle of the ability range compared to the others. We can see that the output contains much useful information of understanding the students' response process.

In addition, the correlation between latent ability estimated from SRM and the outcome (correct/incorrect) of the single item (Tickets CP038Q02) was calculated. The correlation was positive and strong (0.901**), providing evidence that the latent ability was consistent with the scoring rubric defined by experts. However, the proposed model allows us to investigate the process-specific characteristics of problems-solving and give reasonable latent estimates based on the whole response sequence, which can provide additional information beyond outcome.

Comparing the latent ability estimates with the students' outcomes of multiple items

We further explored the explanatory power of the latent abilities from the SRM (denoted as θ_{SRM}) on the students' overall performance in PISA 2012 on

Table 10. The parameter estimation results of the regression model.

	Unstandardized coefficients				
Coefficients	Estimate	Std. error	Standardized coefficients	t	Sig.
(Constant)	0.088	0.007		12.374	0.000
θ_{SRM}	1.054	0.009	0.581	119.165	0.000

problem-solving tasks. There were 27,848 students left after merging the log and outcome dataset. And their overall performance (denoted as θ_{IRT}) was estimated from the outcomes of ten items[2] via "mirt" package in R (Chalmers, 2012) using Rasch models (Masters, 1982; Rasch, 1960). Specifically, we regressed the θ_{IRT} on θ_{SRM}. The regression model was significant, $F(1, 27846) = 14200.237$, $p < 0.001$, indicating that the θ_{SRM} extracted from the process data had a significantly effect on the overall performance score θ_{IRT}. The parameter estimation results are provided in Table 10. The slope parameter was positive, meaning that students with higher θ_{SRM} tended to have better overall performance in problem solving. In addition, the R^2 of the model was 0.338, showing that the explanation rate of θ_{SRM} to the total variance of overall performance score was about one third.

To sum up, the SRM takes all of the students' operations into account, which considers not only the correct operations but also the incorrect operations when inferring the latent abilities. Therefore, it utilizes more useful information than the traditional way of only considering the final correctness the task. Also from the above results, the latent ability estimates were consistent with the scoring rubric defined by experts, and the state transition parameters and the latent ability estimates obtained from the model can be reasonably explained.

Conclusion and discussion

Conclusion

In this study, we propose a sequential response model, which combines the model assumption of dynamic Bayesian models and the nature of latent variables as in traditional psychometric models. The SRM can infer the continuous latent ability from the observed nominal variable sequences. The proposed model can be used to analyze the whole response sequence of students in computer-based dynamic problem-solving tasks with information from correct and incorrect operations, therefore makes the analysis more comprehensive, diagnostic, and in-depth. The results of the simulation study and empirical study demonstrated the rationale of the model setup, the accuracy of the parameter estimation, the interpretability of the model parameters and the additional gain of understanding the students' response process. The proposed model fits for tasks with a clear problem-solving path. It can save the high cost of the expert-specified scoring rubrics with a focus only on the correct operations, and provides a clear story compared to a pure data-driven approach which loses interpretability. The parameters in the proposed model are highly interpretable, and it can be used across multiple tasks, in different context and scenarios to a certain extent. Generally speaking, the sequential response model provides a useful and effective framework for the analysis of process data in the new type of assessments based on technology.

Discussion and future directions

First, a data preprocessing procedure is needed in order to apply the model. The definition of problem states and state transitions is a key task in this process for appropriate use of the model. For the well-defined problems, such as the illustrated *Ticket* task, their initial state, target state and intermediate states are often clearly defined. We could enumerate all the possible states in the task scenario, and clarify the relations between states directly, that is, the possibility of the state transitions' occurrence and the correctness of the transitions. Fortunately, most of the current computer-based problem-solving tasks of this type with well-defined states. However, the use of current model with ill-defined problem scenarios is still an open problem, we are currently investigating into this issue and one possible solution is that the NLP techniques mentioned in the introduction could assist experts in exploring and defining the key states. Then the correctness of state transition can be determined by the distance toward the target state. It should be noticed that the definition of problem states and state transitions must be combined with the task design, and once the analysis results are not in line with our expectation, one must go back to the definition stage, think about whether the definition of critical states is

[2]The following ten items were used to estimate the overall performance on problem-solving, for that they were given to most of the students: CP018Q04T, CP018Q05, CP025Q01, CP025Q02, CP036Q01, CP036Q02, CP036Q03, CP038Q01, CP038Q02, and CP038Q03.

meaningful, and adjust the definition until the result is reasonable. Although it seems like a nontrivial and somewhat subjective procedure, the definition of state and state transitions is mostly easy and objective in our experience and can achieve a good balance between the full expert-driven and fully data-driven approach.

Second, although we introduced only the SRM for a single optimal path in this paper, the model can be extended to deal with the problem-solving tasks containing multiple optimal paths. When there are multiple optimal paths in the problem-solving task, the $I^{+}_{x_j, x_k}$ value in Equation (1) can be replaced by a weight between 0 and 1, with nonzero value indicating the correct transitions. Smaller positive weights indicating less efficiency correct transitions and larger positive weights indicating higher efficiency. Zero value indicating the incorrect transitions. The weights could be assigned manually or freely-estimated. We will leave it for the future research.

Third, the SRM proposed in this study is a basic model, which can be extended in different ways. First, we assume that the problem state sequences have the Markov property given the latent ability, which is a common assumption in the time series analysis literature and is reasonable for tasks such as *tickets* that are simple to operate and have little feedback. However, problem-solving behaviors may exhibit long-term dependence in more complex dynamic interaction scenarios. In addition to the application of high-order time series models, we can further add influence factors to account for lasting impacts to the basic model, to relax the condition independence. Second, we assumed a unidimensional ability in this paper. The propose model can be extended to deal with multidimensional data when measuring multidimensional constructs. Further, model fitting indices such as the deviance information criterion (DIC; Spiegelhalter et al., 2002) and pseudo Bayes factor (Gelfand, 1996; Levy & Mislevy, 2016) can be used in model comparison.

Finally, although the proposed model can use the whole response sequence to infer latent ability, there is still information in the process data can be utilized, such as response times. For students who selected the same response sequences, should their ability estimates vary with the time they spend? The method of utilizing response time to enhance the estimation of latent traits has been studied in the field of IRT (van der Linden, 2009). How to incorporate this unused information in our model is an issue worthy of our attention.

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Appendix. The Bayesian estimation procedure for SRM

Random initial values were assigned for all the components, yielding the collection of $\boldsymbol{\theta}^0$ and $\boldsymbol{\lambda}^0$. The superscript represents the sampling times. The sampling scheme generically for iteration $l+1$ works as follows ($l = 1, 2, 3, \ldots, L$, where L represents the length of the chain):

First, sample the abilities for n students or response patterns. For each student/response pattern $i = 1, 2, \ldots, n$, a candidate value θ_i^* is randomly selected from the proposal distribution dependent on the current value θ_i^l, $\theta_i^* \sim N(\theta_i^l, \sigma_\theta^2)$. Then sample a uniform variable $u \sim$ Uniform (0, 1), and set $\theta_i^{l+1} = \theta_i^*$ if $\alpha \geq u$, where

$$\alpha_i = \min\left\{1, \frac{p\left(\theta_i^* | \boldsymbol{\lambda}^l, \ \mathbf{S_i}, \mathcal{R}\right)}{p\left(\theta_i^l | \boldsymbol{\lambda}^l, \ \mathbf{S_i}, \mathcal{R}\right)}\right\} = \min\left\{1, \frac{p\left(\mathbf{S_i} | \theta_i^*, \boldsymbol{\lambda}^l, \mathcal{R}\right) p(\theta_i^*)}{p\left(\mathbf{S_i} | \theta_i^l, \boldsymbol{\lambda}^l, \mathcal{R}\right) p\left(\theta_i^l\right)}\right\} \tag{A1}$$

and set $\theta_i^{l+1} = \theta_i^l$ otherwise. Note the use of the values of the tendency parameters $\boldsymbol{\lambda}$ from the previous iteration l.

Second, sample the tendency parameters for state transitions. For each state $x_j \in \boldsymbol{x}$, a candidate vector for the tendency parameters of incorrect state transition(s) $\boldsymbol{\lambda}^*_{x_j,\text{incorrect}}$ is randomly selected from a proposal distribution dependent on the current value $\boldsymbol{\lambda}^l_{x_j,incorrect}$, $\boldsymbol{\lambda}^*_{x_j,\text{incorrect}} \sim MVN(\boldsymbol{\lambda}^l_{x_j,\text{incorrect}}, \mathbf{I} \cdot \boldsymbol{\sigma}^2_{x_j})$. And the candidate parameter of correct state transition $\lambda^*_{x_j,\text{correct}}$ is calculated by subtracting the sum of all the candidate incorrect transition parameters by zero,$\boldsymbol{\lambda}^*_{x_j,\text{correct}} = -\text{sum}(\boldsymbol{\lambda}^*_{x_j,\text{incorrect}})$. Therefore, $\boldsymbol{\lambda}_{x_j} = (\boldsymbol{\lambda}_{x_j,\text{incorrect}}, \lambda_{x_j,\text{correct}})$ is a vector, containing all state transition parameters with a given state x_j. Then sample a uniform variable $u \sim$ Uniform (0, 1), and set $\boldsymbol{\lambda}^{l+1}_{x_j} = \boldsymbol{\lambda}^*_{x_j}$ if the acceptance probability $\alpha \geq u$, where

$$\alpha_j = \min\left[1\ , \frac{p\left(\boldsymbol{\lambda}^*_{x_j} | \boldsymbol{\theta}^l, \mathbf{S}, \mathcal{R}\right)}{p\left(\boldsymbol{\lambda}^l_{x_j} | \boldsymbol{\theta}^l, \mathbf{S}, \mathcal{R}\right)}\right] = \min\left[1, \frac{p\left(\mathbf{S} | \boldsymbol{\theta}^l, \boldsymbol{\lambda}^*_{x_j}, \mathcal{R}\right) p\left(\boldsymbol{\lambda}^*_{x_j}\right)}{p\left(\mathbf{S} | \boldsymbol{\theta}^l, \boldsymbol{\lambda}^l_{x_j}, \mathcal{R}\right) p\left(\boldsymbol{\lambda}^l_{x_j}\right)}\right] \tag{A2}$$

and set $\boldsymbol{\lambda}^{l+1}_{x_j} = \boldsymbol{\lambda}^l_{x_j}$ otherwise. And $\boldsymbol{\lambda}^{l+1} = (\boldsymbol{\lambda}^{l+1}_{x_1}, \ldots, \boldsymbol{\lambda}^{l+1}_{x_R})$.

Note that when the latent abilities of response patterns instead of individuals were estimated, the likelihood of each state $p\left(\mathbf{S} | \boldsymbol{\theta}, \boldsymbol{\lambda}^*_{x_j}, \mathcal{R}\right)$ in Equation (A2) should be calculated according to the frequency of each response pattern as follows:

$$p(\mathbf{S} | \boldsymbol{\theta}, \boldsymbol{\lambda}_{x_j}) = \prod_{i=1}^{n} P(\mathbf{S_i} | \theta_i, \boldsymbol{\lambda}_{x_j}, \mathcal{R})^{F_i} \ . \tag{A3}$$

Where F_i represents the frequency of response pattern.

In the Bayesian estimation procedure, the hyper parameters σ_θ^2 and $\boldsymbol{\sigma}^2_{x_j}$ that govern the variability of the of proposal distribution are first tuned to control the acceptance rate of each parameter(s) between 20% and 40%. Then multiple chains with length L are used, and the initial iterations are discarded as burn-in.