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Magnetic satellite detumbling: the b-dot algorithm revisited

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***Abstract*— The first task a spacecraft attitude control system must perform after separation from the launcher is to detumble the spacecraft, *i.e.*, to bring it to a final condition with a sufficiently small angular momentum. Magnetic control has been used for decades to fulfill this task, typically using the so-called b-dot control law (in all its variants developed and flown through the years). In this paper the b-dot law is reviewed, a novel convergence analysis based on tools from averaging theory and periodic Lyapunov theory is presented and a simulation study is used to illustrate the results.**

I. INTRODUCTION

Magnetic coils have been used since the beginning of the space age as a simple and efficient torque generation mechanism for attitude and momentum control [18]. Their operation is based on the interaction between the magnetic field generated by the coils and the magnetic field of the Earth and therefore provides a simple solution to the problem of generating torques on board a satellite. More precisely, magnetic torquers can be used either as main actuators for attitude control in momentum biased or gravity gradient architectures or as secondary actuators for momentum management tasks in zero momentum reaction wheel based configurations. The main issue associated with magnetic coils is the fact that magnetic torques are constrained to lie in the plane orthogonal to the local magnetic field vector, so that the angular dynamics of a magnetically actuated spacecraft is completely controllable only in the time-varying sense (due to the variability of the geomagnetic field). This leads to a number of difficulties in the design of the attitude control law, which have been studied extensively in the last few years. In particular, as far as the design of nominal attitude control laws is concerned the recent work has focused on the application of optimal and robust periodic control theory to a number of attitude control system (ACS) configurations, namely gravity gradient spacecraft [24], [16], momentum biased spacecraft [15], [13], [9], [18], [4], or satellites exploiting passive aerodynamic properties [25]. The global formulation of the problem has also been studied extensively, both for Earth pointing (see, *e.g.*, [23], [3], [2], [20]) and inertially pointing (see [10], [11]) spacecraft. Limited attention, with the partial exception of the recent papers [5], [1], has been devoted to the detumbling problem, which is the first task the ACS must perform after separation of the spacecraft from the launcher. Detumbling consists in bringing the spacecraft from an initial condition possibly characterised by a large angular momentum, to a final one which, while remaining essentially arbitrary in terms of direction of the angular rate vector, is compatible with the operation of subsequent ACS modes capable of achieving also (coarse) pointing requirements. Detumbling must be performed by the spacecraft in a fully autonomous way. Therefore, in order to minimise the risk of failure in the course of this operation, tight requirements on the reliability of the involved actuators and sensors and on the simplicity of the adopted control law are usually driving the design of the detumbling mode. Magnetic control is an excellent candidate for this task (see, *e.g.*, [19]) as it allows to detumble a spacecraft using only magnetic torquers as actuators and magnetometers as angular rate sensors. Also, magnetic torquers are extremely reliable, as they do not involve moving parts and are therefore less sensitive to the large vibratory loads during launch than, *e.g.*, inertial actuators. Similarly, magnetometers are reliable, relatively inexpensive equipment, which can be easily redunded if necessary. The idea enabling magnetic detumbling is the celebrated b-dot control law (in all its variants developed and flown through the years), which will be presented and analysed in the following. More precisely, the convergence properties of the b-dot law are investigated first using an averaging-based approach, along the lines of [11], and then using results from the stability theory for non autonomous, time-periodic systems, under the assumption of a time-periodic evolution of the geomagnetic field along the spacecraft orbit. The paper is organised as follows. In Section II the mathematical model for the system under study is constructed, while the b-dot control law is presented and discussed in Section III. Sections IV and V then provide the main results dealing with the converging properties of the angular rate of the spacecraft under b-dot feedback, focusing on the angular-rate dependent part of the b-dot law only. In Section VI, instead, a complete analysis, taking also into account the "residual" part of the b-dot law is presented. Finally, some simulation results are presented in Section VII.

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II. SPACECRAFT MODEL

The model of a rigid spacecraft with magnetic actuation can be described in various reference frames [22]. For the purpose of the present analysis, the following reference systems are adopted. Earth Centered Inertial reference axes (ECI). The origin of these axes is in the Earth's centre. The X-axis is parallel to the line of nodes, that is, the intersection between the Earth's equatorial plane and the plane of the ecliptic, and is positive in the Vernal equinox direction (Aries point). The Z-axis is defined as being parallel to the Earth's geographic north-south axis and pointing north. The Y-axis completes the right-handed orthogonal triad. Satellite body axes. The origin of these axes is in the satellite centre of mass; the axes are assumed to coincide with the body's principal inertia axes.

The attitude dynamics can be expressed by Euler's equations ([22])

$$J_0\dot{\omega}_b = S(\omega_b)J_0\omega_b + T_{coils} + T_{dist} \tag{1}$$

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where $\omega_b = \begin{bmatrix}\omega_{bx} & \omega_{by} & \omega_{bz}\end{bmatrix}^T \in \mathbb{R}^3$ is the vector of spacecraft angular rates, expressed in body frame, $J_0 \in \mathbb{R}^{3\times 3}$ is the inertia matrix, $S(\omega_b)$ is given by

$$S(\omega_b) = \begin{bmatrix} 0 & \omega_{bz} & -\omega_{by} \\ -\omega_{bz} & 0 & \omega_{bx} \\ \omega_{by} & -\omega_{bx} & 0 \end{bmatrix}, \tag{2}$$

$T_{coils} \in \mathbb{R}^3$ is the vector of magnetic torques and $T_{dist} \in \mathbb{R}^3$ is the vector of disturbance torques. The latter will be neglected in the following, as they play a marginal role during detumbling.

In turn, the attitude kinematics can be described by means of the four Euler parameters (or quaternions), which lead to the following representation

$$\dot{\boldsymbol{q}} = W(\boldsymbol{q})\omega_b \tag{3}$$

where $\boldsymbol{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4\end{bmatrix}^T = \begin{bmatrix}\mathbf{q}^T & q_4\end{bmatrix}^T$ and

$$W(\boldsymbol{q}) = \frac{1}{2}\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}. \tag{4}$$

Note that the attitude of inertially pointing spacecraft is usually referred to the ECI reference frame.

Magnetic coils, which are assumed to be aligned with the spacecraft principal inertia axes, generate torques according to the law

$$T_{coils} = m \times \tilde{b}(t) = S(\tilde{b}(t))m, \tag{5}$$

where $\times$ denotes the vector cross product, $m \in \mathbb{R}^3$ is the vector of magnetic dipoles for the three coils, $\tilde{b}(t) \in \mathbb{R}^3$ is the vector formed with the components of the Earth's magnetic field in the body frame of reference. Note that the vector $\tilde{b}(t)$ can be expressed in terms of the attitude matrix $A(\boldsymbol{q})$ [22] and of the magnetic field vector expressed in the inertial coordinates, namely $\tilde{b}_0(t)$, as

$$\tilde{b}(t) = A(\boldsymbol{q})\tilde{b}_0(t), \tag{6}$$

and that the orthogonality of $A(\boldsymbol{q})$ implies $\|\tilde{b}(t)\| = \|\tilde{b}_0(t)\|$. Since $S(\tilde{b}(t))$ is structurally singular, as mentioned in the Introduction, magnetic actuators do not provide full controllability of the system at each time instant. In particular, it is easy to see that rank $\left(S(\tilde{b}(t))\right) = 2$ (since $\|\tilde{b}_0(t)\| \neq 0$ along all orbits of practical interest for magnetic control) and that the kernel of $S(\tilde{b}(t))$ is given by the vector $\tilde{b}(t)$ itself, *i.e.*, at each time instant it is *not* possible to apply a control torque along the direction of $\tilde{b}(t)$.

If a preliminary feedback of the form

$$m = \frac{1}{\|\tilde{b}_0(t)\|^2} S^T(\tilde{b}(t))v \tag{7}$$

is applied to the system, where $u \in \mathbb{R}^3$ is a new control vector, the overall dynamics can be written as

$$\begin{aligned} \dot{\boldsymbol{q}} &= W(\boldsymbol{q})\omega_b \\ J_0\dot{\omega}_b &= S(\omega_b)J_0\omega_b + \Gamma(t)v \end{aligned} \tag{8}$$

where $\Gamma(t) = S(b(t))S^T(b(t)) \geq 0$ and $b(t) = \frac{1}{\|\tilde{b}_0(t)\|}\tilde{b}(t) = \frac{1}{\|\tilde{b}(t)\|}\tilde{b}(t)$. Similarly, let $\Gamma_0(t) = S(b_0(t))S^T(b_0(t)) \geq 0$ and $b_0(t) = \frac{1}{\|\tilde{b}_0(t)\|}\tilde{b}_0(t)$. Note, also, that $\Gamma(t)$ can be written as $\Gamma(t) = \mathcal{I}_3 - b(t)b(t)^T$, where $\mathcal{I}_3$ is the 3×3 identity matrix.

The model takes a remarkably simple form if the attitude dynamics is represented with respect to the inertial frame rather than with respect to the body frame. With this choice the dynamics of the magnetically controlled spacecraft can be written as

$$\begin{aligned} \dot{\boldsymbol{q}} &= \tilde{W}(\boldsymbol{q})\omega \\ \dot{\widehat{J\omega}} &= A(\boldsymbol{q})^T\Gamma(t)v \end{aligned} \tag{9}$$

where $\omega \in \mathbb{R}^3$ is the vector of spacecraft angular rates, expressed in the inertial frame, $J = A(\boldsymbol{q})^T J_0 A(\boldsymbol{q})$ and $\tilde{W}(\boldsymbol{q})$ is given by [17]

$$\tilde{W}(\boldsymbol{q}) = \frac{1}{2}\begin{bmatrix} q_4 & q_3 & -q_2 \\ -q_3 & q_4 & q_1 \\ q_2 & -q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}. \tag{10}$$

Let now $v = A(\boldsymbol{q})u$ and note that $\Gamma(t) = A(\boldsymbol{q})\Gamma_o(t)A(\boldsymbol{q})^T$. Therefore, (9) can be written as

$$\begin{aligned} \dot{\boldsymbol{q}} &= \tilde{W}(\boldsymbol{q})\omega \\ \dot{\widehat{J\omega}} &= \Gamma_0(t)u. \end{aligned} \tag{11}$$

III. MAGNETIC DETUMBLING: THE B-DOT PRINCIPLE

Consider the problem of designing a feedback law for the dynamics of a rigid body subject to magnetic actuation

$$\begin{aligned} \dot{\boldsymbol{q}} &= \tilde{W}(\boldsymbol{q})\omega \\ J_0\dot{\omega}_b &= S(\omega_b)J_0\omega_b + S(b)m, \end{aligned} \tag{12}$$

with the objective of reducing the angular rate of the spacecraft from the initial value $\omega(0) = \omega_o$ to, ideally, zero, using only feedback provided by a triaxial magnetometer:

$$b = A(\boldsymbol{q})b_0. \tag{13}$$

Clearly, magnetic field measurements can provide attitude information (in the form of a scalar vector measurement), but cannot provide rate information to a feedback controller. Magnetometer output, however, becomes relevant to the solution of the above stated problem if one considers the time derivative of the measured geomagnetic field, as computed from the above measurement equation:

$$\dot{b} = \dot{\widehat{Ab_0}} = \dot{A}b_0 + A\dot{b}_0, \tag{14}$$

which, recalling that

$$\dot{A} = S(\omega)A, \tag{15}$$

can be equivalently written as

$$\dot{b} = S(\omega)Ab_0 + A\dot{b}_0 = S(\omega)b + A\dot{b}_0. \tag{16}$$

Finally, by commuting the factors in the cross product appearing at the right-hand side of (16) and recalling that $S^T(\cdot) = -S(\cdot)$, one gets

$$\dot{b} = S^T(b)\omega + A\dot{b}_0. \tag{17}$$

With the derivative of the measured geomagnetic field ($\dot{b}$, hence the name b-dot for the magnetic detumbling controller) written in this form, it is apparent that the first term is proportional to the body angular rate of the spacecraft, through a (singular) matrix gain ($S^T(b)$) which is a function of the measured magnetic field itself. Furthermore, the presence of the second term, which is proportional to the rate of variation of the geomagnetic field in the orbital frame (as seen from the body frame) can be interpreted as a limit to the precision of the angular rate information that $\dot{b}$ can provide. So $\dot{b}$ can be considered a meaningful measure of the spacecraft angular rate provided that its magnitude is significantly larger than the orbital angular rate ω_0. As this is typically the case for the initial condition of a detumbling mode, $\dot{b}$ can be considered a valuable source of rate feedback for magnetic detumbling.

Many choices for the design of a magnetic detumbling controller based on $\dot{b}$ have been considered in the literature. The simplest one is given by static linear feedback of the form

$$m = -k\frac{1}{\|\tilde{b}\|^2}\dot{\tilde{b}}, \tag{18}$$

where $k > 0$ is a scalar control gain. Now, substituting (18) in (12), the closed-loop dynamics of the magnetically controlled spacecraft can be written as

$$\begin{array}{rcl} \dot{\boldsymbol{q}} & = & \tilde{W}(\boldsymbol{q})\omega \\ J_0\dot{\omega}_b & = & S(\omega_b)J_0\omega_b - kS(b)S^T(b)\omega - kA\dot{b}_0. \end{array} \tag{19}$$

As can be seen from equation (19), up to the effect of the "residual" term $-kA\dot{b}_0$, the b-dot law introduces an angular rate feedback with a gain given by $-kS(b)S^T(b)$, which is negative semi-definite in view of the fact that $S(b)S^T(b) \geq 0$ for all t and $\boldsymbol{q}$. In other words, letting $T(\omega_b) = \frac{1}{2}\omega_b^T J_0\omega_b$ the angular kinetic energy of the spacecraft, the computation of the derivative along the trajectories of the closed-loop system (19) of $T(\omega_b)$ gives $\dot{T}(\omega_b) = -k\omega_b^T S(b)S^T(b)\omega_b$, and since $\dot{T}(\omega_b) \leq 0$, from an engineering perspective it can be claimed that the b-dot law can effectively remove angular momentum from the spacecraft whenever ω_b is not parallel to b, while leaving it unchanged when $\omega_b^T b = 0$.

IV. Averaging-based analysis of the b-dot law

The dynamics of the rigid body spacecraft under feedback from the b-dot law can be analysed in a number of different ways. For analysis purposes, however, it is convenient to express the closed-loop dynamics in the inertial frame rather than in the body frame, so as to allow the time-variability of the closed-loop dynamics to appear as an explicit function of the geomagnetic field vector expressed in the inertial frame, *i.e.*, as a pure function of time. With this choice the dynamics of the magnetically controlled spacecraft given by equation (19) can be written as

$$\begin{array}{rcl} \dot{\boldsymbol{q}} & = & \tilde{W}(\boldsymbol{q})J^{-1}h \\ \dot{h} & = & -kA(\boldsymbol{q})^T\Gamma_0(t)\omega - kA(\boldsymbol{q})^T S(b)A(\boldsymbol{q})\dot{b}_0, \end{array} \tag{20}$$

where $h \in \mathbb{R}^3$ is the vector of spacecraft angular momentum expressed in the inertial frame. Therefore, (20) can be written as

$$\begin{array}{rcl} \dot{\boldsymbol{q}} & = & \tilde{W}(\boldsymbol{q})J^{-1}h \\ \dot{h} & = & -k\Gamma_0(t)J^{-1}h - kS(b)A(\boldsymbol{q})\dot{b}_0(t). \end{array} \tag{21}$$

In order to apply averaging theory to the closed-loop model (21), the following assumption on matrix $\Gamma_0(t)$ is needed.

Assumption 1: The considered orbit for the spacecraft satisfies the condition

$$\bar{\Gamma}_0 = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Gamma_0(t)dt = \\ = \lim_{T\to\infty}\frac{1}{T}\int_0^T S(b_0(t))S^T(b_0(t))dt > 0. \tag{22}$$

Note that Assumption 1 is only mildly restrictive as it can be easily verified (see, *e.g.*, [12]) that it holds for most orbits of practical interest for LEO spacecraft. Note also that the assumption is formulated in terms of so-called generalised averaging, so that the elements of $\Gamma_0(t)$ are not assumed to be periodic functions of time. Finally, as the aim of this analysis is to characterise the detumbling transient starting from an initial condition with a large magnitude of the angular momentum, in the following the term $-kS(b)A(\boldsymbol{q})\dot{b}_0(t)$ will be neglected, so the closed-loop model

$$\begin{array}{rcl} \dot{\boldsymbol{q}} & = & \tilde{W}(\boldsymbol{q})J^{-1}h \\ \dot{h} & = & -k\Gamma_0(t)J^{-1}h \end{array} \tag{23}$$

will be considered.

Proposition 1: Consider the closed-loop magnetically actuated spacecraft described by (23). Then, under Assumption 1, there exists $\varepsilon^\star > 0$ such that for any $0 < \varepsilon < \varepsilon^\star$ all trajectories of (11) are such that $\omega \to 0$.

Proof: Introduce the coordinates transformation

$$\boldsymbol{z}_1 = \boldsymbol{q} \qquad z_2 = \frac{h}{\varepsilon} \tag{24}$$

(so that $\mathbf{z}_1 = \mathbf{q}$ and $z_{14} = q_4$) in which the system (23) is described by the equations

$$\begin{array}{rcl} \dot{\boldsymbol{z}}_1 & = & \varepsilon\tilde{W}(\boldsymbol{z}_1)J^{-1}z_2 \\ \dot{z_2} & = & -\varepsilon k\Gamma_0(t)J^{-1}z_2. \end{array} \tag{25}$$

System (25) satisfies all the hypotheses for the applicability of the generalised averaging theory [6], which yields the averaged system

$$\begin{array}{rcl} \dot{\boldsymbol{z}}_1 & = & \varepsilon\tilde{W}(\boldsymbol{z}_1)J^{-1}z_2 \\ \dot{z_2} & = & -\varepsilon k\bar{\Gamma}_0 J^{-1}z_2. \end{array} \tag{26}$$

As a result, there exists $\varepsilon^\star > 0$ such that for any $0 < \varepsilon < \varepsilon^\star$ the trajectories of system (26) are close to the trajectories of system (25).

Consider now the Lyapunov function ([14], [21])

$$V_1(\boldsymbol{z}_1, z_2) = \frac{1}{2}(Jz_2)^T\bar{\Gamma}_0^{-1}(Jz_2). \tag{27}$$

Its time derivative

$$\dot{V}_1 = -\varepsilon k z_2^T z_2 \tag{28}$$

is negative semidefinite, therefore $z_2 \to 0$.

Finally, note that considering the linear approximation of system (26) around the generic equilibrium $(\bar{\boldsymbol{q}}, 0)$, which is given by

$$\begin{array}{rcl} \dot{\mathbf{z}}_1 & = & \frac{1}{2}\varepsilon z_2 \\ \widehat{Jz_2} & = & -\varepsilon\bar{\Gamma}_0 J^{-1}(kz_2), \end{array} \tag{29}$$

it is easy to verify that

$$V_L(z_2) = \frac{1}{2}(Jz_2)^T\bar{\Gamma}_0^{-1}(Jz_2) \tag{30}$$

is a Lyapunov function for the linear system given by the second equation in (29), so the convergence of z_2 is locally exponential. ■

V. Periodic analysis of the b-dot law

As discussed in the previous section, Assumption 1 on the time evolution of the geomagnetic field along the considered orbit allows to define conditions for the convergence of the angular rate and the angular momentum of the controlled spacecraft as functions of the scaling parameter ε. It is also interesting to note, however that as the duration of the detumbling procedure is usually not very long (typically 1 to 2 orbits), the effects which make periodicity assumptions on the geomagnetic field unrealistic (*i.e.*, slow orbital perturbation and the 24 hours period of Earth rotation) do not play a significant role, so that a periodic analysis of the closed-loop dynamics is, in this case, a viable approach. To this purpose, in this section the closed loop model

$$\begin{array}{rcl} \dot{\boldsymbol{q}} & = & \tilde{W}(\boldsymbol{q})J^{-1}h \\ \dot{h} & = & -k\Gamma_0(t)J^{-1}h \end{array} \quad (31)$$

will be studied under the following assumption on matrix $\Gamma_0(t)$.

Assumption 2: Along the considered orbit for the spacecraft the geomagnetic field satisfies for all t and for some $T > 0$ the condition

$$\Gamma_0(t+T) = \Gamma_0(t). \quad (32)$$

The main result is given in the following proposition.

Proposition 2: Consider the closed-loop magnetically actuated spacecraft described by (31). Then, under Assumption 2, all trajectories of (31) are such that $\omega \to 0$.

Proof: Consider the function ([14], [21])

$$V_2(h) = \frac{1}{2}h^T h. \quad (33)$$

Its time derivative is given by

$$\dot{V}_2 = -kh^T(\Gamma_0(t)J^{-1} + J^{-1}\Gamma_0(t))h. \quad (34)$$

It is easy to check that $\dot{V}_2$ is negative semidefinite. Indeed, recalling that $\Gamma_0(t)$ can be written as $\Gamma_0(t) = \mathcal{I}_3 - b_0(t)b_0^T(t)$, one has that $b_0(t)$ is a basis for the kernel of $\Gamma_0(t)$ for all t, so

$$b_0^T(t)(\Gamma_0(t)J^{-1} + J^{-1}\Gamma_0(t))b_0(t) = 0 \quad (35)$$

for all t. Similarly, for all $b_\perp$ such that $b_\perp \perp b_0(t)$ one has that

$$b_\perp^T(\Gamma_0(t)J^{-1} + J^{-1}\Gamma_0(t))b_\perp = 2b_\perp^T J^{-1}b_\perp > 0. \quad (36)$$

Therefore V_2 satisfies the assumptions for the application of Theorem 3 in [7], [8] (La Salle invariance principle for non autonomous, time-periodic systems), from which it follows that $h \to 0$ and $\omega \to 0$. ■

Remark 1: The analysis of the magnetic detumbling control law carried out in Sections IV and V can be modified to deal with saturation of the magnetic coils, along the lines of [11].

VI. Taking the residual term into account

Up to now only the ω-dependent part of the b-dot law has been considered and the "residual" term has been neglected. This has enabled the derivation of convergence results for the controlled spacecraft which account for the initial part of the transient of the angular rate and of the angular momentum. The residual term, related to the rate of change of the geomagnetic field in the inertial frame, prevents the actual convergence to zero of the angular rate vector. In this section the role of this term is analysed and the true asymptotic behaviour of the controlled dynamics is investigated, on the basis of the following assumptions.

Assumption 3: Along the considered orbit for the spacecraft the geomagnetic field satisfies for all t the differential equation

$$\dot{b}_0 = S(\bar{\omega})b_0, \quad (37)$$

for some constant vector $\bar{\omega}$.

Assumption 4: Consistently with the analysis of gyro-based detumbling in [5], gyroscopic coupling terms are neglected.

Assumption 3 allows one to include the "residual" term $-kA\dot{b}_0$ in the analysis and actually characterise the true asymptotic behaviour of the controlled angular rate, along the following lines. First of all, note that in view of Assumption 3, equation (17) can be written as

$$\dot{b} = S^T(b)\omega_b + S^T(b)\bar{\omega}_b, \quad (38)$$

where $\bar{\omega}_b = A\bar{\omega}$. Then, substituting (38) in Euler's equation for the conservation of angular momentum and taking Assumption 4 into account one gets

$$J_0\dot{\omega}_b \quad = \quad -kS(b)S^T(b)\omega_b - kS(b)S^T(b)\bar{\omega}_b, \quad (39)$$

or, equivalently, in the inertial reference frame:

$$J\dot{\omega} \quad = \quad -kS(b_0)S^T(b_0)(\omega - \bar{\omega}). \quad (40)$$

Letting now $\tilde{\omega} = \omega - \bar{\omega}$, one has that

$$J\dot{\tilde{\omega}} \quad = \quad -kS(b_0)S^T(b_0)\tilde{\omega}, \quad (41)$$

so the complete system can be written as

$$\begin{array}{rcl} \dot{\boldsymbol{q}} & = & \tilde{W}(\boldsymbol{q})J^{-1}h \\ J\dot{\tilde{\omega}} & = & -kS(b_0)S^T(b_0)\tilde{\omega}, \end{array} \quad (42)$$

and the following result holds.

Proposition 3: Consider the closed-loop magnetically actuated spacecraft described by (42). Then, under Assumptions 3 and 4, all trajectories of (42) are such that $\tilde{\omega} \to 0$.

Proof: Consider the function

$$V_3(\tilde{\omega}) = \frac{1}{2}\tilde{\omega}^T\tilde{\omega}. \quad (43)$$

Its time derivative is given by

$$\dot{V}_3 = -k\tilde{\omega}^T(\Gamma_0(t)J^{-1} + J^{-1}\Gamma_0(t))\tilde{\omega}, \quad (44)$$

so the result follows along the same lines as for Proposition 2. ■

VII. SIMULATION EXAMPLE

A simulation study is presented, based on a mission scenario related to a typical LEO small satellite mission. The considered satellite operates along a near polar, circular orbit ($i = 89°$, $e = 0$) with an altitude of 450 km and a corresponding orbital period of 5614.8 seconds. For the b-dot detumbling algorithm a Monte Carlo study was carried out to define a robust tuning with respect to: uncertainty on the initial angular rate: $\pm 50\%$ of an initial reference value chosen as 50 times the orbital angular rate, per axis; uncertainty on the attitude: ± 1 on all components of the attitude quaternion; uncertainty on the moments of inertia: $\pm 10\%$ on each of the principal moments. More precisely, 50 simulations were carried out for different choices of the gain k of the b-dot law, to assess the settling time of the norm of the angular rate vector of the spacecraft with respect to a threshold equal to twice the orbital angular rate. As a result, the following statistics were computed: mean, standard deviation, maximum and minimum values. The numerical values are depicted in Figure 1, from which, in view of a conservative, worst-case tuning, a value of $k = 2 \times 10^{-3}$ is chosen, which guarantees a worst-case settling time of about 1.5 orbits. In Figure 2 a time-domain representation of the simulation results of the Monte Carlo study is provided.

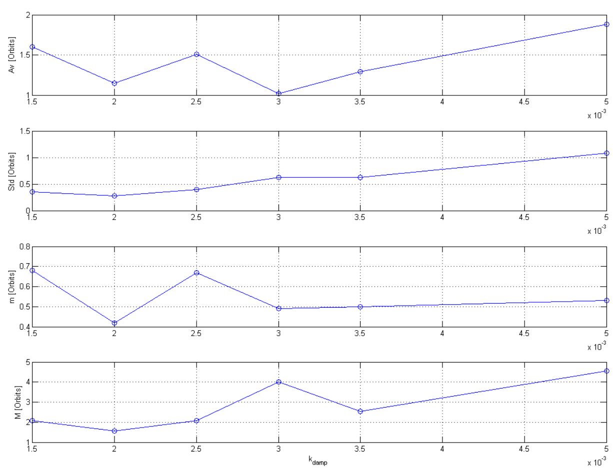

Fig. 1. Monte-Carlo controller tuning statistics.

Finally, the results of a simulation of the closed-loop dynamics in the inertial frame are presented, to support the conclusions of the previous sections. In Figures 3 and 4 the transient and steady state behaviour of the angular vector is illustrated. In particular, Figure 3 is representative of the convergent behaviour studied in Sections IV and V. Clearly, on the scale of this figure one can neglect the (nonzero) asymptotic value of the angular rate. On the contrary, Figure 4 shows that the asymptotic value of the angular rate is non zero, consistently with the analysis in Section VI. Similar conclusions can be reached by inspecting the transient and the steady state behaviour of the angular momentum, see Figures 5 and 6. Finally, the time histories of the magnetic dipoles are illustrated in Figure 7.

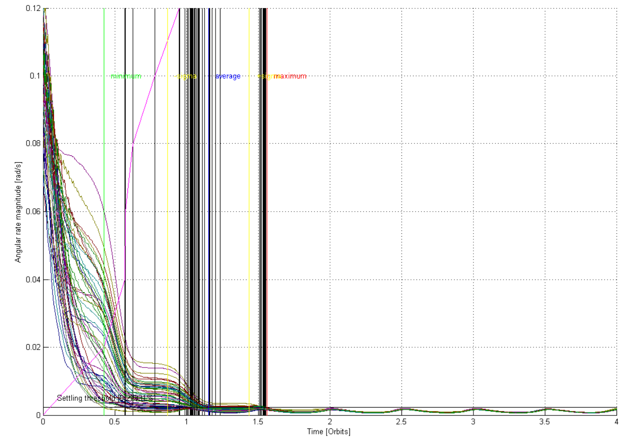

Fig. 2. Closed-loop evolution of angular rate magnitude for the Monte Carlo study.

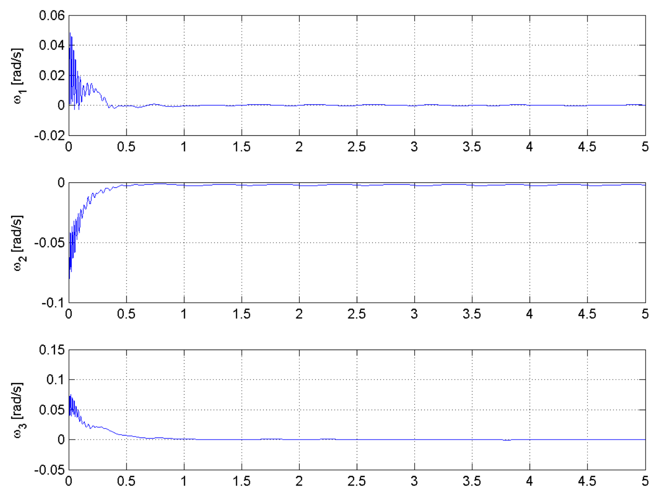

Fig. 3. Closed-loop evolution of the components of the angular rate in the inertial frame: transient.

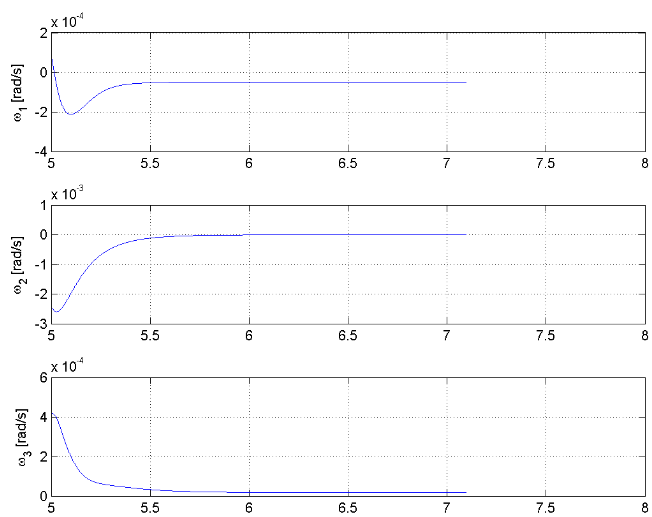

Fig. 4. Closed-loop evolution of the components of the angular rate in the inertial frame: steady-state.

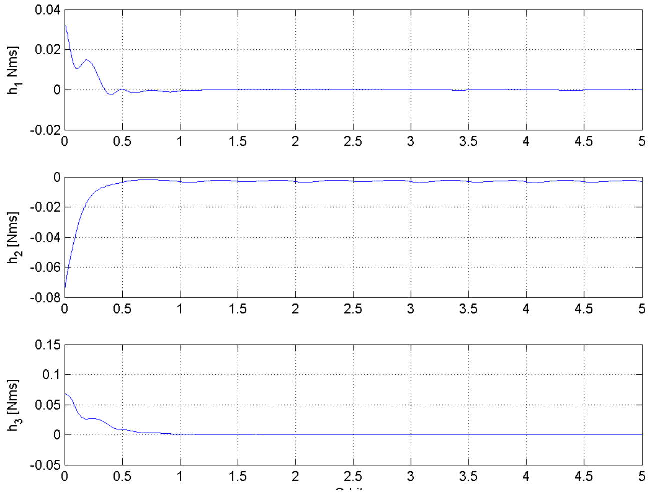

Fig. 5. Closed-loop evolution of the components of the angular momentum in the inertial frame: transient.

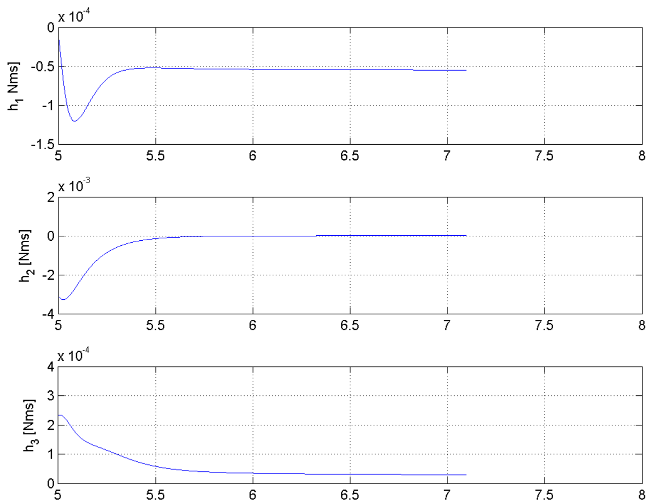

Fig. 6. Closed-loop evolution of the components of the angular momentum in the inertial frame: steady-state.

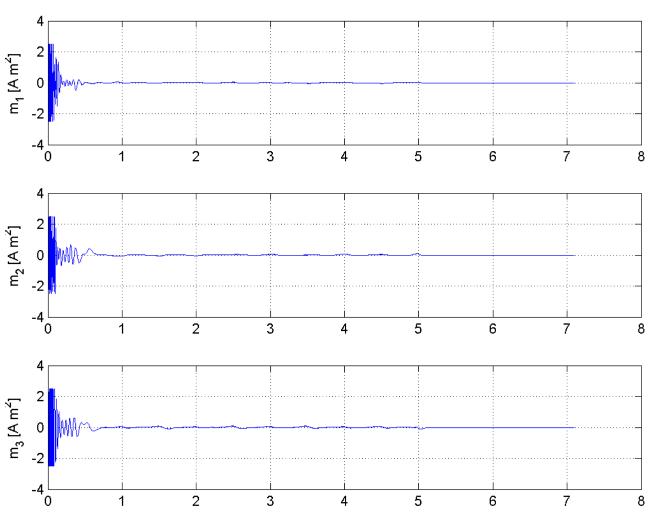

Fig. 7. Closed-loop evolution of the components of the dipole moments.

VIII. CONCLUDING REMARKS

The problem of magnetic detumbling of spacecraft attitude has been considered, with specific reference to the b-dot control law. Novel convergence analysis based on tools from averaging theory and periodic Lyapunov theory are presented and a simulation study is used to illustrate the results.

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