## 2016 年第七届全国大学生数学竞赛决赛 (非数学专业)参考答案

## 一、填空题

(1) 【参考解答】:令y'=p,则 $y''=p'=p^3$ ,这是可分离变量的微分方程,有 $\dfrac{\mathrm{d}\,p}{p^3}=\mathrm{d}\,x$ ,积分得

到 
$$-rac{1}{2}\,p^{-2}=x-C_1$$
,即  $p=y'=rac{\pm 1}{\sqrt{2ig(C_1-xig)}}$ ,积分得  $y=C_2\pm\sqrt{2(C_1-x)}$  .

(2)【参考解答】:利用对称性和极坐标,有

$$I = 4e^4 \int_0^{rac{\pi}{2}} \mathrm{d}\, heta \int_1^2 r^2 \sin^2 heta e^{-r^2} r \, \mathrm{d}\, r = rac{\pi}{2} e^4 \int_1^4 u e^{-u} \, \mathrm{d}\, u = rac{\pi}{2} (2e^3 - 5).$$

(3)【参考解答】:  $\mathrm{d}\,x = fig(tig)\mathrm{d}\,t,\,\mathrm{d}\,y = f'ig(tig)\mathrm{d}\,t$ ,所以

$$rac{\mathrm{d}\,y}{\mathrm{d}\,x} = rac{f^{\prime}ig(tig)}{fig(tig)}, \;\; rac{\mathrm{d}^2\,y}{\mathrm{d}\,x^2} = rac{\mathrm{d}}{\mathrm{d}\,t}igg(rac{f^{\prime}ig(tig)}{fig(tig)}igg)rac{\mathrm{d}\,t}{\mathrm{d}\,x} = rac{fig(tig)f^{\prime\prime}ig(tig)-f^{\prime}ig(tig)^2}{f^3ig(tig)}.$$

(4) 【参考解答】:  $\left|f\left(A\right)\right|=f\left(\lambda_1\right)f\left(\lambda_2\right)\cdots f\left(\lambda_n\right)$ .

(5) [参考解答]: 
$$\pi n! e = \pi n! \left[ 2 + \frac{1}{2!} + \cdots \frac{1}{n!} + \frac{1}{(n+1)!} + o\left(\frac{1}{(n+1)!}\right) \right]$$

$$= \pi a_n + \frac{\pi}{n+1} + o\left(\frac{1}{n+1}\right)$$

其中 $a_n$ 为整数,并且当n=2k时

$$egin{align} fig(2kig) &= 2\cdotig(2kig)! + rac{ig(2kig)!}{2!} + rac{ig(2kig)!}{3!} + \cdots + rac{ig(2kig)!}{ig(2kig)!} \ &= 2\cdotig(2kig)! + ig(2kig)ig(2k-1ig)\cdot 3 + \cdots + ig(2kig) + 1 \ \end{array}$$

为奇数

$$fig(2k+1ig) = 2\cdotig(2k+1ig)! + rac{ig(2k+1ig)!}{2!} + rac{ig(2k+1ig)!}{3!} + \cdots + rac{ig(2k+1ig)!}{ig(2k+1ig)!} = 2\cdotig(2k+1ig)! + ig(2k+1ig)ig(2kig)\cdot 3 + \cdots + ig(2k+1ig) + 1$$

为偶数. 所以

$$\lim_{n o\infty} n \sin(\pi n! e) = \pm \lim_{n o\infty} \left[ n \sin\left(rac{\pi}{n+1} + o(rac{1}{n+1})
ight) 
ight] = \pm \pi.$$

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即极限不存在,如果加上绝对值则极限存在等于 $\pi$ .

二、【参考证明】: 记
$$F\left(x,y,z\right)=f\left(\dfrac{x-a}{z-c},\dfrac{y-b}{z-c}\right)$$
,则

$$\left(F_x,F_y,F_z
ight) = \left(rac{f_1}{z-c},rac{f_2}{z-c},rac{-ig(x-aig)f_1-ig(y-big)f_2}{ig(z-cig)^2}
ight)$$

取曲面的法向量

$$\vec{n} = ((z-c)f_1, (z-c)f_2, -(x-a)f_1 - (y-b)f_2).$$

记 $\left(x,y,z\right)$ 为曲面上的点, $\left(X,Y,Z\right)$ 为切面上的点,则曲面上过点 $\left(x,y,z\right)$ 的切平面方程为

$$\begin{split} & \big[ \big(z-c\big)f_1 \big] \big(X-c\big) + \big[ \big(z-c\big)f_2 \big] \big(Y-y\big) \\ & - \big[ \big(x-a\big)f_1 + \big(y-b\big)f_2 \big] \big(Z-z\big) = 0 \end{split}$$

容易验证,对任意 $\left(x,y,z\right)\left(z\neq c\right),\left(X,Y,Z\right)=\left(a,b,c\right)$ 都满足上述切平面方程

三、【参考证明】:由f(x)在[a,b]上连续,知f(x)在[a,b]上可积.

令 
$$F\left(x
ight)=\int\limits_{x}^{b}f\left(t
ight)\mathrm{d}\,t$$
 ,则  $F'\left(x
ight)=-f\left(x
ight)$ .由此, 
$$2\int\limits_{a}^{b}f\left(x
ight)\!\!\left(\int\limits_{x}^{b}f\left(t
ight)\mathrm{d}\,t\right)\!\mathrm{d}\,x=2\int\limits_{a}^{b}f\left(x
ight)\!F\left(x
ight)\!\mathrm{d}\,x=-2\int\limits_{a}^{b}F\left(x
ight)\!F'\left(x
ight)\mathrm{d}\,x=-2\int\limits_{a}^{b}F\left(x
ight)\mathrm{d}\,F\left(x
ight)$$
 
$$=-F^{2}\left(x
ight)\!\!\left|_{a}^{b}\!=\!F^{2}\left(b
ight)\!-\!F^{2}\left(a
ight)\!=\!F^{2}\left(a
ight)\!=\!\left(\int\limits_{a}^{b}f\left(x
ight)\!\!\right|^{2}$$

四、【参考证明】: 要证明不等式成立, 即要证明

$$\begin{split} R\left(AB\right) + R\left(BC\right) &\leq R\left(B\right) + R\left(ABC\right) = R \begin{bmatrix} ABC & O \\ O & B \end{bmatrix} \\ & \oplus \mp \begin{bmatrix} E_m & A \\ O & E_n \end{bmatrix} \begin{bmatrix} ABC & O \\ O & B \end{bmatrix} \begin{bmatrix} E_q & O \\ -C & E_p \end{bmatrix} = \begin{bmatrix} O & AB \\ -BC & B \end{bmatrix} \\ & \begin{bmatrix} O & AB \\ E_p & O \end{bmatrix} = \begin{bmatrix} AB & O \\ B & BC \end{bmatrix} \\ & \oplus \begin{bmatrix} E_m & A \\ O & E_n \end{bmatrix}, \begin{bmatrix} E_q & O \\ -C & E_p \end{bmatrix}, \begin{bmatrix} O & -E_q \\ E_p & O \end{bmatrix}$$
 可逆,所以 
$$R \begin{bmatrix} ABC & O \\ O & B \end{bmatrix} = R \begin{bmatrix} AB & O \\ B & BC \end{bmatrix} \geq R\left(AB\right) + R\left(BC\right). \end{split}$$

五【参考解答】: (1) 
$$I_n + I_{n-2} = \int\limits_0^{\pi/4} \tan^n x \mathrm{d}x + \int\limits_0^{\pi/4} \tan^{n-2} x \mathrm{d}x$$

$$=\int\limits_{0}^{\pi/4} an^{n-2}\,x\,\mathrm{d}ig( an \,xig) = rac{1}{n-1} an^{n-1}\,xigg|_{0}^{\pi/4} = rac{1}{n-1}$$

(2) 由于 $0 < x < \frac{\pi}{4}$ ,所以0 < an x < 1, $an^{n+2} x < an^n x < an^{n-2} x$ .从而

$$I_{n+2} < I_n < I_{n-2}, \\$$

于是 $I_{n+2} + I_n < 2I_{\mathrm{n}} < I_{n-2} + I_n$  ,

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}, \left(\frac{1}{2(n+1)}\right)^p < I_n^p < \left(\frac{1}{2(n-1)}\right)^p.$$

当
$$p>1$$
时, $\left|\left(-1\right)^pI_n^p
ight|\leq I_n^p<rac{1}{2^p\left(n-1\right)^p}, (n\geq 2)$ .由 $\sum_{n=2}^\inftyrac{1}{\left(n-1\right)^p}$ 收敛,所以 $\sum_{n=1}^\infty\left(-1\right)^nI_n^p$ 绝

对收敛.

当  $0 时,由于 <math>\left\{I_n^p\right\}$  单调减少,并趋近于 0,由莱布尼兹判别法,知  $\sum_{n=1}^{\infty} \left(-1\right)^n I_n^p$  收敛.

而 
$$I_n^p > \frac{1}{2^p \left(n+1\right)^p} \geq \frac{1}{2^p} \cdot \frac{1}{n+1}, \frac{1}{2^p} \sum_{n=1}^{\infty} \frac{1}{(n+1)}$$
 发散,所以  $\sum_{n=1}^{\infty} \left(-1\right)^n I_n^p$  是条件收敛的.

当  $p \leq 0$  时,则 $\left|I_n^p\right| \geq 1$ ,由级数收敛的必要条件,知 $\sum_{n=1}^{\infty} \left(-1\right)^n I_n^p$  发散.

六、【参考证明】:记上半球面S 的底平面为D,方向向下,S 和D 围成的区域记为 $\Omega$ ,由高斯公式得

$$\left(\iint\limits_{S} + \iint\limits_{D} \right) P \, \mathrm{d} \, y \, \mathrm{d} \, z + R \, \mathrm{d} \, x \, \mathrm{d} \, y = \iiint\limits_{\Omega} \left( \frac{\partial P}{\partial x} + \frac{\partial R}{\partial z} \right) \mathrm{d} \, V$$

由于  $\iint_D P \,\mathrm{d}\,y \,\mathrm{d}\,z + R \,\mathrm{d}\,x \,\mathrm{d}\,y = -\iint_D R \,\mathrm{d}\,\sigma$  和题设条件,其中  $\mathrm{d}\,\sigma$  是xOy 平面上的面积微元,则有

$$-\iint_{P} R \, \mathrm{d} \, \sigma = \iiint_{Q} \left[ \frac{\partial P}{\partial x} + \frac{\partial R}{\partial z} \right] \mathrm{d} V \tag{*}$$

注意到上式对任何 r>0 成立,由此证明  $R\left(x_{_{\!0}},y_{_{\!0}},z_{_{\!0}}\right)=0$  .

若不然,设 $R(x_0,y_0,z_0)\neq 0$ ,注意到

$$\iint\limits_{D}R\operatorname{d}\sigma=Rig(\xi,\zeta,z_{0}ig)\!\pi r^{2},$$
其中 $ig(\xi,\zeta,z_{0}ig)\in D$  ,

而当  $r \to 0^+, R\left(\xi,\zeta,z_0\right) \to R\left(x_0,y_0,z_0\right)$ , 故 (\*) 左端为一个二阶的无穷小. 类似地,当

$$\frac{\partial P(x_0, y_0, z_0)}{\partial x} + \frac{\partial R(x_0, y_0, z_0)}{\partial z} \neq 0,$$

$$\iiint_{\Omega} \left( \frac{\partial P}{\partial x} + \frac{\partial R}{\partial z} \right) dV$$
是一个 3 阶的无穷小;而当

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$$rac{\partial Pig(x_{_{\!0}},y_{_{\!0}},z_{_{\!0}}ig)}{\partial x}+rac{\partial Rig(x_{_{\!0}},y_{_{\!0}},z_{_{\!0}}ig)}{\partial z}=0$$
 ,

该积分趋于 0 的阶高于 3. 因此(\*) 式右端阶高于左端,从而当r 很小时,有

$$\left|\iint\limits_{D} R\,\mathrm{d}\,\sigma
ight| \geq \left|\iiint\limits_{\Omega} \left[rac{\partial P}{\partial x} + rac{\partial R}{\partial z}
ight] \mathrm{d}\,V
ight|$$
 ,

这与(\*)矛盾.

由于在任何点  $R\left(x_0,y_0,z_0
ight)=0$  ,故  $R\left(x,y,z
ight)\equiv 0$  .代入入 (\*) 式得到

$$\iiint\limits_{\Omega}\frac{\partial P\big(x,y,z\big)}{\partial x}\mathrm{d}\,V=0$$

重复前面的证明可知  $rac{\partial Pig(x_0,y_0,z_0ig)}{\partial x}=0$  . 由 $ig(x_0,y_0,z_0ig)$ 的任意性得 $rac{\partial P}{\partial x}\equiv 0$  .