第十届全国大学生数学竞赛试卷

(非数学类，2018 年 10 月)

一、填空题（本题满分 24 分，共 4 小题，每小题 6 分）

(1) 设$\alpha \in (0,1)$，则 $\lim\limits_{n\to+\infty}\left[(n+1)^{\alpha}-n^{\alpha}\right]=$________.

【参考解析】:**【思路一】**因为 $\left(1+\dfrac{1}{n}\right)^{\alpha}<1+\dfrac{1}{n}$，所以

$$0<(n+1)^{\alpha}-n^{\alpha}=n^{\alpha}\left[\left(1+\frac{1}{n}\right)^{\alpha}-1\right]<n^{\alpha}\left(1+\frac{1}{n}-1\right)=\frac{1}{n^{1-\alpha}}\to 0(n\to 0)$$

所以由夹逼准则可得 $\lim\limits_{n\to+\infty}\left[(n+1)^{\alpha}-n^{\alpha}\right]=0.$

【思路二】 $\lim\limits_{n\to+\infty}\left[(n+1)^{\alpha}-n^{\alpha}\right]=\lim\limits_{x\to0^+}\dfrac{(1+x)^{\alpha}-1}{x^{\alpha}}$

$$=\lim_{x\to0^+}\frac{\alpha x}{x^{\alpha}}=\alpha\lim_{x\to0^+}x^{1-\alpha}=0.$$

(2) 若曲线$y=y(x)$由$\begin{cases}x=t+\cos t\\ e^{y}+ty+\sin t=1\end{cases}$确定，则此曲线在$t=0$对应点处的切线方程为___________.

【参考解析】: 当$t=0$时，$x=1$且$e^{y}=1$，即$y=0$，即求点$(1,0)$处曲线$y=y(x)$的切线方程. 在方程组两端对t求导，得

$$\begin{cases}x'(t)=1-\sin t\\ e^{y}\cdot y'(t)+y+ty'(t)+\cos t=0\end{cases}$$

将$t=0$，$y=0$代入方程，得$x'(0)=1,y'(0)=-1$，所以$\left.\dfrac{dy}{dx}\right|_{x=0}=\dfrac{y'(0)}{x'(0)}=-1$，所以切线方程为$y-0=(-1)(x-1)$，即$y=-x+1.$

(3) $\displaystyle\int\frac{\ln\left(x+\sqrt{1+x^{2}}\right)}{\left(1+x^{2}\right)^{3/2}}dx=$___________.

【参考解析】:**【思路一】**典型三角代换结构$\sqrt{1+x^{2}}$，令$x=\tan t,dx=\sec^{2}t\,dt$，所以

$$F(x)=\int\frac{\ln\left(x+\sqrt{1+x^{2}}\right)}{\left(1+x^{2}\right)^{3/2}}dx=\int\frac{\ln(\sec t+\tan t)}{\sec t}dt$$

$$=\int \ln\left(\sec t+\tan t\right)\mathrm{d}\left(\sin t\right)=\sin t\ln\left(\sec t+\tan t\right)-\int\frac{\sin t}{\cos t}\mathrm{d}\,t$$

$$=\sin t\ln\left(\sec t+\tan t\right)+\ln|\cos t|+C$$

由于$\tan t=\dfrac{x}{1}$，所以 $\cos t=\dfrac{1}{\sqrt{1+x^2}},\sin t=\dfrac{x}{\sqrt{1+x^2}}$， $\sec t=\sqrt{1+x^2}$，代入得原积分为

$$F\left(x\right)=\frac{x}{\sqrt{1+x^2}}\ln\left(\sqrt{1+x^2}+x\right)+\ln\frac{1}{\sqrt{1+x^2}}+C$$

$$\text{或}\frac{x}{\sqrt{1+x^2}}\ln\left(\sqrt{1+x^2}+x\right)-\frac{1}{2}\ln\left(1+x^2\right)+C$$

【思路二】 $F\left(x\right)=\int\ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}\dfrac{x}{\sqrt{1+x^2}}$

$$=\frac{x}{\sqrt{1+x^2}}\ln\left(x+\sqrt{1+x^2}\right)-\int\frac{x}{x^2+1}\mathrm{d}\,x$$

$$=\frac{x}{\sqrt{1+x^2}}\ln\left(x+\sqrt{1+x^2}\right)-\frac{1}{2}\ln\left(x^2+1\right)+C$$

(4) $\lim\limits_{x\to0}\dfrac{1-\cos x\sqrt{\cos 2x}\sqrt[3]{\cos 3x}}{x^2}=$__________.

【参考解析】: **【思路一】** $A=\lim\limits_{x\to0}\dfrac{1-\cos x\sqrt{\cos 2x}\sqrt[3]{\cos 3x}}{x^2}$

$$=\lim_{x\to0}\frac{1-\cos x+\cos x\left(1-\sqrt{\cos 2x}\sqrt[3]{\cos 3x}\right)}{x^2}$$

$$=\frac{1}{2}+\lim_{x\to0}\frac{1-\sqrt{\cos 2x}\sqrt[3]{\cos 3x}}{x^2}$$

$$=\frac{1}{2}+\lim_{x\to0}\frac{1-\sqrt{\cos 2x}+\sqrt{\cos 2x}\left(1-\sqrt[3]{\cos 3x}\right)}{x^2}$$

$$=\frac{1}{2}+\lim_{x\to0}\left[\frac{1-\sqrt{\left(\cos 2x-1\right)+1}}{x^2}+\frac{1-\sqrt[3]{\left(\cos 3x-1\right)+1}}{x^2}\right]$$

$$=\frac{1}{2}+\lim_{x\to0}\frac{1-\cos 2x}{2x^2}+\lim_{x\to0}\frac{1-\cos 3x}{3x^2}$$

$$=\frac{1}{2}+1+\frac{3}{2}=3.$$

【思路二】带皮亚诺余项的麦克劳林公式，有

$$\cos x = 1 - \frac{x^2}{2} + o\left(x^2\right), \left(\cos 2x\right)^{\frac{1}{2}} = 1 - x^2 + o\left(x^2\right)$$

$$\left(\cos 3x\right)^{\frac{1}{3}} = 1 - \frac{3x^2}{2} + o\left(x^2\right)$$

所以$\cos x\left(\cos 2x\right)^{\frac{1}{2}}\left(\cos 3x\right)^{\frac{1}{3}} = 1 - 3x^2 + o\left(x^2\right)$，代入得

$$A = \lim_{x \to 0} \frac{3x^2 + o\left(x^2\right)}{x^2} = 3.$$

二(本题满分8分)设函数$f\left(t\right)$在$t \neq 0$时一阶连续可导,且$f\left(1\right) = 0$,求函数$f\left(x^2 - y^2\right)$,使得曲线积分$\int_L y\left[2 - f\left(x^2 - y^2\right)\right]\mathrm{d}x + xf\left(x^2 - y^2\right)\mathrm{d}y$与路径无关，其中$L$为任一不与直线$y = \pm x$相交的分段光滑曲线.

【参考解析】：令$P\left(x,y\right) = y\left[2 - f\left(x^2 - y^2\right)\right], Q\left(x,y\right) = xf\left(x^2 - y^2\right)$，于是

$$\begin{aligned}\frac{\partial P\left(x,y\right)}{\partial y} &= 2 - f\left(x^2 - y^2\right) + y\left[-f'\left(x^2 - y^2\right)\left(-2y\right)\right] \\ &= 2 - f\left(x^2 - y^2\right) + 2y^2 f'\left(x^2 - y^2\right)\end{aligned}$$

$$\frac{\partial Q\left(x,y\right)}{\partial x} = f\left(x^2 - y^2\right) + 2x^2 f'\left(x^2 - y^2\right)$$

由积分与路径无关的条件$\frac{\partial P\left(x,y\right)}{\partial y} = \frac{\partial Q\left(x,y\right)}{\partial x}$，代入结果整理得

$$\left(x^2 - y^2\right)f'\left(x^2 - y^2\right) + f\left(x^2 - y^2\right) - 1 = 0$$

令$x^2 - y^2 = u$，即$uf'\left(u\right) + f\left(u\right) - 1 = 0$，分离变量得$\frac{\mathrm{d}f\left(u\right)}{1 - f\left(u\right)} = \frac{1}{u}\mathrm{d}u$，由分离变量法，两端积分，得$\frac{1}{1 - f\left(u\right)} = C_1 u$，即$f\left(u\right) = 1 + \frac{C}{u}$，由$f\left(1\right) = 0$，得$C = -1$，即$f\left(x^2 - y^2\right) = 1 - \frac{1}{x^2 - y^2}$.

【注】其中微分方程$uf'\left(u\right) + f\left(u\right) - 1 = 0$的通解可以通过改写微分方程为$\left[uf\left(u\right)\right]' = 1$，得到通解为$uf\left(u\right) = u + C$.

三（本题满分 14 分）设$f\left(x\right)$在区间$\left[0,1\right]$上连续，且$1 \leq f\left(x\right) \leq 3$. 证明:

$$1 \leq \int_0^1 f\left(x\right)\mathrm{d}x \int_0^1 \frac{1}{f\left(x\right)}\mathrm{d}x \leq \frac{4}{3}.$$

【参考解析】：由柯西不等式，得

$$\int_0^1 f(x)\mathrm{d}x\int_0^1\frac{1}{f(x)}\mathrm{d}x\geq\left[\int_0^1\sqrt{f(x)}\sqrt{\frac{1}{f(x)}}\mathrm{d}x\right]^2=1$$

又由于$[f(x)-1][f(x)-3]\leq 0$，则$\dfrac{[f(x)-1][f(x)-3]}{f(x)}\leq 0$，即

$f(x)+\dfrac{3}{f(x)}\leq 4$，所以$\displaystyle\int_0^1\left[f(x)+\frac{3}{f(x)}\right]\mathrm{d}x\leq 4$. 由于

$$\int_0^1 f(x)\mathrm{d}x\int_0^1\frac{3}{f(x)}\mathrm{d}x\leq\frac{1}{4}\left[\int_0^1 f(x)\mathrm{d}x+\int_0^1\frac{3}{f(x)}\mathrm{d}x\right]^2\leq 4$$

所以$\displaystyle 1\leq\int_0^1 f(x)\mathrm{d}x\int_0^1\frac{1}{f(x)}\mathrm{d}x\leq\frac{4}{3}$.

四（本题满分 12 分）计算三重积分$\displaystyle\iiint\limits_{(V)}\left(x^2+y^2\right)\mathrm{d}V$，其中$(V)$是由

$$x^2+y^2+(z-2)^2\geq 4, x^2+y^2+(z-1)^2\leq 9$$

及$z\geq 0$所围成的空间图形.

【参考解析】: 画图（关键），考虑区域的特殊性，采用容易计算的整体减去容易计算的部分来完成计算，从而分成三个部分来讨论：

第一部分：整个大球(V_1)的积分：采用球坐标换元，令

$$x=r\sin\varphi\cos\theta, y=r\sin\varphi\sin\theta, z=1+r\cos\varphi$$
$$0\leq r\leq 3, 0\leq\varphi\leq\pi, 0\leq\theta\leq 2\pi.$$

于是有

$$\iiint\limits_{(V_1)}\left(x^2+y^2\right)\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^3 r^2\sin^2\varphi\cdot r^2\sin\varphi\,\mathrm{d}r=\frac{648\pi}{5}$$

第二部分：小球(V_2)的积分：采用球坐标换元，令

$$x=r\sin\varphi\cos\theta, y=r\sin\varphi\sin\theta, z=2+r\cos\varphi$$
$$0\leq r\leq 2, 0\leq\varphi\leq\pi, 0\leq\theta\leq 2\pi.$$

于是有

$$\iiint\limits_{(V_2)}\left(x^2+y^2\right)\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^2 r^2\sin^2\varphi\cdot r^2\sin\varphi\,\mathrm{d}r=\frac{256\pi}{15}$$

第三部分：大球$z=0$下部分的积分(V_3)，采用柱坐标：

$$x=r\cos\theta, y=r\sin\theta, 1-\sqrt{9-r^2}\leq z\leq 0$$
$$0\leq r\leq 2\sqrt{2}, 0\leq\theta\leq 2\pi$$

于是有$\displaystyle\iiint\limits_{(V_2)}\left(x^2+y^2\right)\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\sqrt{2}}r\,\mathrm{d}r\int_{1-\sqrt{9-r^2}}^0 r^2\,\mathrm{d}z=\frac{136\pi}{5}$

所以最终的积分为

$$\iiint\limits_{(V)}\left(x^2+y^2\right)\mathrm{d}V=\iiint\limits_{(V_1)}-\iiint\limits_{(V_2)}-\iiint\limits_{(V_3)}$$

$$=\frac{648}{5}\pi-\frac{256}{15}\pi-\frac{136}{5}\pi=\frac{256}{3}\pi.$$

五（本题满分 14 分） 设$f(x,y)$在区域D内可微，且$\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\le M$，$A(x_1,y_1),B(x_2,y_2)$是$D$内两点，线段$AB$包含在$D$内．证明：

$$\left|f(x_1,y_1)-f(x_2,y_2)\right|\le M\,|AB|,$$

其中$|AB|$表示线段AB的长度．

【参考解析】：作辅助函数$\varphi(t)=f\left[x_1+t(x_2-x_1),y_1+t(y_2-y_1)\right]$，显然函数$\varphi(t)$在$[0,1]$上可导．根据拉格朗日中值定理，存在$c\in(0,1)$，使得

$$\varphi(1)-\varphi(0)=\varphi'(c)=\frac{\partial f(u,v)}{\partial u}(x_2-x_1)+\frac{\partial f(u,v)}{\partial v}(y_2-y_1)$$

所以$\left|\varphi(1)-\varphi(0)\right|=\left|f(x_1,y_1)-f(x_2,y_2)\right|$

$$=\left|\frac{\partial f(u,v)}{\partial u}(x_2-x_1)+\frac{\partial f(u,v)}{\partial v}(y_2-y_1)\right|$$

$$\le\sqrt{\left[\frac{\partial f(u,v)}{\partial u}\right]^2+\left[\frac{\partial f(u,v)}{\partial v}\right]^2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\le M\,|AB|.$$

六（本题满分 14 分） 证明：对于连续函数$f(x)>0$，有

$$\ln\int_0^1 f(x)\mathrm{d}x\ge\int_0^1\ln f(x)\mathrm{d}x.$$

【参考解析】：由于$f(x)$在$[0,1]$上连续，所以

$$\int_0^1 f(x)\mathrm{d}x=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^n f(x_k),x_k\in\left[\frac{k-1}{n},\frac{k}{n}\right].$$

由算术几何不等式$\left[f(x_1)f(x_2)\cdots f(x_n)\right]^{\frac{1}{n}}\le\frac{1}{n}\sum_{k=1}^n f(x_k)$．于是有

$$\frac{1}{n}\sum_{k=1}^n\ln f(x_k)\le\ln\left[\frac{1}{n}\sum_{k=1}^n f(x_k)\right]$$

根据$\ln x$的连续性，两边取极限，得

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^n\ln f(x_k)\le\lim_{n\to+\infty}\ln\left[\frac{1}{n}\sum_{k=1}^n f(x_k)\right]$$

即$\ln\int_0^1 f(x)\mathrm{d}x\ge\int_0^1\ln f(x)\mathrm{d}x$.

七（本题满分 14 分） 已知$\{a_k\},\{b_k\}$是正数数列，且$b_{k+1}-b_k\ge\delta>0,k=1,2,\cdots$，$\delta$

为一常数. 证明：若级数 $\sum_{k=1}^{+\infty} a_k$ 收敛，则级数 $\sum_{k=1}^{+\infty} \frac{k\sqrt[k]{(a_1a_2\cdots a_k)(b_1b_2\cdots b_k)}}{b_{k+1}b_k}$ 收敛.

【参考解析】：令 $S_k = \sum_{i=1}^{k} a_ib_i, a_kb_k = S_k - S_{k-1}$，

$$S_0 = 0, a_k = \frac{S_k - S_{k-1}}{b_k}, k = 1, 2, \cdots$$

$$\sum_{k=1}^{N} a_k = \sum_{k=1}^{N} \frac{S_k - S_{k-1}}{b_k} = \sum_{k=1}^{N-1}\left(\frac{S_k}{b_k} - \frac{S_k}{b_{k+1}}\right) + \frac{S_N}{b_N}$$

$$= \sum_{k=1}^{N-1} \frac{b_{k+1} - b_k}{b_kb_{k+1}} S_k + \frac{S_N}{b_N} \geq \sum_{k=1}^{N-1} \frac{\delta}{b_kb_{k+1}} S_k$$

所以 $\sum_{k=1}^{+\infty} \frac{S_k}{b_kb_{k+1}}$ 收敛. 由不等式

$$\sqrt[k]{(a_1a_2\cdots a_k)(b_1b_2\cdots b_k)} \leq \frac{a_1b_1 + a_2b_2 + \cdots + a_kb_k}{k} = \frac{S_k}{k}$$

可知 $\sum_{k=1}^{+\infty} \frac{k\sqrt[k]{(a_1a_2\cdots a_k)(b_1b_2\cdots b_k)}}{b_{k+1}b_k} \leq \sum_{k=1}^{+\infty} \frac{S_k}{b_kb_{k+1}}$，故原不等式成立.