第十一届全国大学生数学竞赛决赛
（非数学类）试题与参考答案

一、填空题（本题满分 30 分，每小题 6 分）

1、极限 $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{(1-\sqrt{\sin x})(1-\sqrt[3]{\sin x})\cdots(1-\sqrt[n]{\sin x})}{(1-\sin x)^{n-1}} =$______.

【参考答案】：由等价无穷小，

$$\lim_{x \to \frac{\pi}{2}} \frac{1-\sqrt[k]{\sin x}}{1-\sin x} = \lim_{x \to \frac{\pi}{2}} \frac{1-\sqrt[k]{1+\left(\sin x - 1\right)}}{1-\sin x}$$

$$= -\lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{k}\left(\sin x - 1\right)}{1-\sin x} = \frac{1}{k}$$

故由极限的乘法法则，得

$$\text{原式} = \lim_{x \to \frac{\pi}{2}} \frac{1-\sqrt{\sin x}}{1-\sin x} \frac{1-\sqrt[3]{\sin x}}{1-\sin x} \cdot \cdots \cdot \frac{1-\sqrt[n]{\sin x}}{1-\sin x}$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \cdots \cdot \frac{1}{n} = \frac{1}{n!}$$

2、设函数 $y = f\left(x\right)$ 由方程 $3x - y = 2\arctan(y-2x)$ 所确定，则曲线 $y = f\left(x\right)$ 在点 $P\left(1+\frac{\pi}{2}, 3+\pi\right)$ 处的切线方程为______________.

【参考答案】：对方程 $3x - y = 2\arctan(y-2x)$ 两边求导，得

$$3 - y' = 2\frac{y'-2}{1+(y-2x)^2}$$

将点 P 的坐标代入，得曲线 $y = f(x)$ 在 P 点的切线斜率为 $y' = \frac{5}{2}$. 因此，切线方程为 $y-(3+\pi) = \frac{5}{2}\left(x - 1 - \frac{\pi}{2}\right)$，即 $y = \frac{5}{2}x + \frac{1}{2} - \frac{\pi}{4}$.

3、设平面曲线 L 的方程为 $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$，且通过五个点 $P_1(-1,0), P_2(0,-1), P_3(0,1), P_4(2,-1)$ 和 $P_5(2,1)$，则 L 上任意两点之间的直线距离最大值为________.

【参考答案】：将所给点的坐标代入方程得

$$\begin{cases} A - D + F = 0 \\ B - E + F = 0 \\ B + E + F = 0 \\ 4A + B - 2C + 2D - E + F = 0 \\ 4A + B + 2C + 2D + E + F = 0 \end{cases}$$

解得曲线L的方程为$x^2 + 3y^2 - 2x - 3 = 0$，其标准型为$\dfrac{(x-1)^2}{4} + \dfrac{y^2}{4/3} = 1$. 因此曲线$L$上两点间的最长直线距离为 4.

4、设$f(x) = \left(x^2 + 2x - 3\right)^n \arctan^2 \dfrac{x}{3}$， 其中$n$为正整数，则$f^{(n)}(-3) =$________.

【参考答案】: 记$g(x) = (x-1)^n \arctan^2 \dfrac{x}{3}$，则$f(x) = (x+3)^n g(x)$. 利用莱布尼兹法则，可得

$$f^{(n)}(x) = n!g(x) + \sum_{k=0}^{n-1} C_n^k \left[(x+3)^n\right]^{(k)} g^{(n-k)}(x)$$

所以 $f^{(n)}(-3) = n!g(-3) = (-1)^n 4^{n-2} n! \pi^2$.

5、设函数$f(x)$的导数$f'(x)$在$[0,1]$上连续，$f(0) = f(1) = 0$， 且满足

$$\int_0^1 \left[f'(x)\right]^2 \mathrm{d}x - 8\int_0^1 f(x)\mathrm{d}x + \frac{4}{3} = 0$$

则$f(x) =$____________.

【参考答案】: 因为 $\int_0^1 f(x)\mathrm{d}x = -\int_0^1 xf'(x)\mathrm{d}x$，$\int_0^1 f'(x)\mathrm{d}x = 0$ 且

$$\int_0^1 \left(4x^2 - 4x + 1\right)\mathrm{d}x = \frac{1}{3}$$

所以

$$\begin{aligned} &\int_0^1 f'^2(x)\mathrm{d}x - 8\int_0^1 f(x)\mathrm{d}x + \frac{4}{3} \\ =& \int_0^1 \left[f'^2(x) + 8xf'(x) - 4f'(x) + \left(16x^2 - 16x + 4\right)\right]\mathrm{d}x \\ =& \int_0^1 \left[f'(x) + 4x - 2\right]^2 \mathrm{d}x = 0 \end{aligned}$$

因此 $f'(x) = 2 - 4x$，$f(x) = 2x - 2x^2 + C$. 由 $f(0) = 0$ 得 $C = 0$. 因此 $f(x) = 2x - 2x^2$.

二、（12 分） 求极限 $\lim\limits_{n\to\infty} \sqrt{n}\left(1 - \sum\limits_{k=1}^{n} \dfrac{1}{n + \sqrt{k}}\right)$.

【参考解答】: 记$a_n = \sqrt{n}\left(1 - \sum\limits_{k=1}^{n} \dfrac{1}{n + \sqrt{k}}\right)$，则

$$a_n = \sqrt{n}\sum_{k=1}^{n}\left(\frac{1}{n}-\frac{1}{n+\sqrt{k}}\right)=\sum_{k=1}^{n}\frac{\sqrt{k}}{\sqrt{n}(n+\sqrt{k})}\le\frac{1}{n\sqrt{n}}\sum_{k=1}^{n}\sqrt{k}$$

因为$\sum_{k=1}^{n}\sqrt{k}\le\sum_{k=1}^{n}\int_{k}^{k+1}\sqrt{x}\mathrm{d}x=\int_{1}^{n+1}\sqrt{x}\mathrm{d}x=\frac{2}{3}((n+1)\sqrt{n+1}-1)$，所以

$$a_n<\frac{2}{3}\cdot\frac{(n+1)\sqrt{n+1}}{n\sqrt{n}}=\frac{2}{3}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{1}{n}}$$

又$\sum_{k=1}^{n}\sqrt{k}\ge\sum_{k=1}^{n}\int_{k-1}^{k}\sqrt{x}\mathrm{d}x=\int_{0}^{n}\sqrt{x}\mathrm{d}x=\frac{2}{3}n\sqrt{n}$，得

$$a_n\ge\frac{1}{\sqrt{n}(n+\sqrt{n})}\sum_{k=1}^{n}\sqrt{k}\ge\frac{2}{3}\cdot\frac{n}{n+\sqrt{n}}$$

于是可得

$$\frac{2}{3}\cdot\frac{n}{n+\sqrt{n}}\le a_n<\frac{2}{3}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{1}{n}}$$

故由夹逼准则，得

$$\lim_{n\to\infty}\sqrt{n}\left(1-\sum_{k=1}^{n}\frac{1}{n+\sqrt{k}}\right)=\lim_{n\to\infty}a_n=\frac{2}{3}$$

三、(12 分)设$F(x_1,x_2,x_3)=\int_0^{2\pi}f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)\mathrm{d}\varphi$,其中$f(u,v)$具有二阶连续偏导数. 已知$\frac{\partial F}{\partial x_i}=\int_0^{2\pi}\frac{\partial}{\partial x_i}\left[f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)\right]\mathrm{d}\varphi$，

$$\frac{\partial^2 F}{\partial x_i^2}=\int_0^{2\pi}\frac{\partial^2}{\partial x_i^2}\left[f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)\right]\mathrm{d}\varphi,\quad i=1,2,3$$

试求$x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}$并要求化简.

【参考解答】：令$u=x_1+x_3\cos\varphi,v=x_2+x_3\sin\varphi$，利用复合函数求偏导法则易知

$$\frac{\partial f}{\partial x_1}=\frac{\partial f}{\partial u},\quad\frac{\partial f}{\partial x_2}=\frac{\partial f}{\partial v},\quad\frac{\partial f}{\partial x_3}=\cos\varphi\frac{\partial f}{\partial u}+\sin\varphi\frac{\partial f}{\partial v},$$
$$\frac{\partial^2 f}{\partial x_1^2}=\frac{\partial^2 f}{\partial u^2},\quad\frac{\partial^2 f}{\partial x_2^2}=\frac{\partial^2 f}{\partial v^2},$$
$$\frac{\partial^2 f}{\partial x_3^2}=\frac{\partial^2 f}{\partial u^2}\cos^2\varphi+\frac{\partial^2 f}{\partial u\partial v}\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\sin^2\varphi$$

所以

$$
\begin{aligned}
&x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)\\
&=x_3\left[\begin{array}{l}\int_0^{2\pi}\frac{\partial^2 f}{\partial u^2}\mathrm{d}\varphi+\int_0^{2\pi}\frac{\partial^2 f}{\partial v^2}\mathrm{d}\varphi\\
-\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2}\cos^2\varphi+\frac{\partial^2 f}{\partial u\partial v}\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\sin^2\varphi\right)\mathrm{d}\varphi\end{array}\right]\\
&=x_3\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2}\sin^2\varphi-\frac{\partial^2 f}{\partial u\partial u}\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\cos^2\varphi\right)\mathrm{d}\varphi
\end{aligned}
$$

又由于$\frac{\partial F}{\partial x_3}=\int_0^{2\pi}\left(\cos\varphi\frac{\partial f}{\partial u}+\sin\varphi\frac{\partial f}{\partial v}\right)\mathrm{d}\varphi$，利用分部积分，得

$$
\begin{aligned}
\frac{\partial F}{\partial x_3}=&-\int_0^{2x}\sin\varphi\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial\varphi}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial v}{\partial\varphi}\right)\mathrm{d}\varphi\\
&+\int_0^{2\pi}\cos\varphi\left(\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial\varphi}+\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial\varphi}\right)\mathrm{d}\varphi\\
=&\,x_3\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2}\sin^2\varphi-\frac{1}{2}\sin 2\varphi\frac{\partial^2 f}{\partial u\partial v}\right)\mathrm{d}\varphi-x_3\int_0^{2n}\left(\frac{1}{2}\sin 2\varphi\frac{\partial^2 f}{\partial u\partial v}-\cos^2\varphi\frac{\partial^2 f}{\partial v^2}\right)\mathrm{d}\varphi\\
=&\,x_3\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2}\sin^2\varphi-\frac{\partial^2 f}{\partial u\partial v}\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\cos^2\varphi\right)\mathrm{d}\varphi
\end{aligned}
$$

所以$x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}=0$.

四、(10 分) 函数$f(x)$在$[0,1]$上具有连续导数，且

$$\int_0^1 f(x)\mathrm{d}x=\frac{5}{2},\int_0^1 xf(x)\mathrm{d}x=\frac{3}{2}$$

证明：存在 $\xi\in(0,1)$，使得$f'(\xi)=3$.

【参考解答】:**【思路一】** 考虑积分$\int_0^1 x(1-x)[3-f'(x)]\mathrm{d}x$. 利用分部积分及题设条件，得

$$
\begin{aligned}
&\int_0^1 x(1-x)[3-f'(x)]\mathrm{d}x\\
&=x(1-x)[3x-f(x)]_0^1-\int_0^1(1-2x)[3x-f(x)]\mathrm{d}x\\
&=\int_0^1 3x(2x-1)\mathrm{d}x+\int_0^1(1-2x)f(x)\mathrm{d}x\\
&=\left(2x^3-\frac{3}{2}x^2\right)\Big|_0^1+\int_0^1 f(x)\mathrm{d}x-2\int_0^1 xf(x)\,\mathrm{d}x\\
&=2-\frac{3}{2}+\frac{5}{2}-3=0
\end{aligned}
$$

根据积分中值定理，存在$\xi\in(0,1)$，使得 $\xi(1-\xi)[3-f'(\xi)]=0$ ，即

$$f'(\xi)=3.$$

【思路二】由定积分的分部积分法，有

$$\int_0^1 f(x)\,\mathrm{d}x = xf(x)\Big|_0^1 - \int_0^1 xf'(x)\,\mathrm{d}x = \frac{5}{2} \quad (*)$$

$$\int_0^1 xf(x)\,\mathrm{d}x = \frac{1}{2}f(x)\cdot x^2\Big|_0^1 - \frac{1}{2}\int_0^1 x^2 f'(x)\,\mathrm{d}x = \frac{3}{2} \quad (**)$$

用(*)$\times\dfrac{1}{2}-$(**)，得

$$\begin{aligned}&\frac{1}{2}\int_0^1 f(x)\,\mathrm{d}x - \int_0^1 xf(x)\,\mathrm{d}x \\ =& -\frac{1}{2}\int_0^1 xf'(x)\,\mathrm{d}x + \frac{1}{2}\int_0^1 x^2 f'(x)\,\mathrm{d}x = -\frac{1}{4}\end{aligned}$$

整理得

$$\int_0^1 f'(x)x(x-1)\,\mathrm{d}x = -\frac{1}{2}$$

由于$f'(x)$连续，而$x(x-1)$在$[0,1]$上不改变符号，故由第一积分中值定理知，存在$\xi\in(0,1)$，使得

$$f'(\xi)\int_0^1 (x^2-x)\,\mathrm{d}x = -\frac{1}{2}$$

其中$\displaystyle\int_0^1 (x^2-x)\,\mathrm{d}x = -\frac{1}{6}$，即$f'(\xi)=3$成立.

五、(12 分) 设 $B_1,B_2,\cdots,B_{2021}$ 为空间$\mathbf{R}^3$中半径不为零的 2021 个球，$A=(a_{ij})$为 2021 阶方阵，其(i,j)元a_{ij}为球B_i与B_j相交部分的体积. 证明：行列式$|E+A|>1$，其中E为单位矩阵.

【参考解答】：记Ω为以原点O为球心且包含$B_1,B_2,\cdots,B_{2021}$在内的球，考察二次型$f=\sum_{i=1}^{2021}\sum_{j=1}^{2011} a_{ij}z_iz_j$.注意到

$$a_{ij}=\iiint_{\Omega}\chi_i(t,u,v)\chi_j(t,u,v)\mathrm{d}t\mathrm{d}u\mathrm{d}v$$

其中$\chi_i(t,u,v)$的定义为$\chi_i(t,u,v)=\begin{cases}1, & (t,u,v)\in B_i\\ 0, & (t,u,v)\in\Omega\setminus B_i\end{cases}$ 于是有

$$\begin{aligned}f &= \sum_{i=1}^{2021}\sum_{j=1}^{201} a_{ij}z_iz_j \\ &= \sum_{i=1}^{2011}\sum_{j=1}^{2021}\iiint\left[\chi_i(t,u,v)z_i\right]\left[\chi_j(t,u,v)z_j\right]\mathrm{d}t\mathrm{d}u\mathrm{d}v \\ &= \iiint_{\Omega}\sum_{i=1}^{2021}\left[\chi_i(t,u,v)z_i\right]^2\mathrm{d}t\mathrm{d}u\mathrm{d}v \ge 0\end{aligned}$$

另一方面，存在正交变换$Z=PY$使得f化为

$$f=\lambda_1 y_1^2+\lambda_2 y_2^2+\cdots+\lambda_{2021}y_{2021}^2$$

其中$\lambda_1,\lambda_2,\cdots,\lambda_{2021}$为$A$的全部特征值. 因为二次型$f\geq 0$，所以$A$的特征值$\lambda_i\geq 0,(i=1,2,\cdots 2021)$. 于是

$$\begin{aligned}|E+A|&=\left|P^{-1}(E+A)P\right|\\&=(1+\lambda_1)(1+\lambda_2)\cdots(1+\lambda_{2021})\geq 1.\end{aligned}$$

注意到A不是零矩阵，所以至少有一个特征值$\lambda_i>0$，故$|E+A|>1$.

六、(12 分) 设Ω是由光滑的简单封闭曲面Σ围成的有界闭区域，函数$f(x,y,z)$在Ω上具有连续二阶偏导数，且$f(x,y,z)\big|_{(x,y,z)\in\Sigma}=0$. 记$\nabla f$为$f(x,y,z)$的梯度，并令

$$\Delta f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.$$

证明：对任意常数$C>0$，恒有

$$C\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z+\frac{1}{C}\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\geq 2\iiint_\Omega|\nabla f|^2\ \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

【参考解答】： 首先利用 Gauss 公式，得

$$\iint_\Sigma f\frac{\partial f}{\partial x}\mathrm{d}y\mathrm{d}z+f\frac{\partial f}{\partial y}\mathrm{d}z\mathrm{d}x+f\frac{\partial f}{\partial z}\mathrm{d}x\mathrm{d}y=\iiint_\Omega\left(f\Delta f+|\nabla f|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

其中Σ取外侧. 因为$f(x,y,z)\big|_{(x,y,z)\in\Sigma}=0$，所以上式左端等于零. 利用 Cauchy 不等式，得

$$\begin{aligned}&\iiint_\Omega|\nabla f|^2\ \mathrm{d}x\mathrm{d}y\mathrm{d}z=-\iiint_\Omega(f\Delta f)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\&\leq\left(\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/2}\left(\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/2}\end{aligned}$$

故对任意常数$C>0$，恒有(利用均值不等式)

$$\begin{aligned}&C\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z+\frac{1}{C}\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\\&\geq 2\left(\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/2}\left(\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/2}\\&\geq 2\iiint_\Omega|\nabla f|^2\ \mathrm{d}x\mathrm{d}y\mathrm{d}z\end{aligned}$$

七、(12 分) 设$\{u_n\}$是正数列，满足$\dfrac{u_{n+1}}{u_n}=1-\dfrac{\alpha}{n}+O\left(\dfrac{1}{n^\beta}\right)$，其中常数$\alpha>0,\beta>1$.

(1) 对于$v_n=n^\alpha u_n$，判断级数$\displaystyle\sum_{n=1}^{\infty}\ln\frac{v_{n+1}}{v_n}$的敛散性;

(2) 讨论级数$\displaystyle\sum_{n=1}^{\infty}u_n$的敛散性.

[注：设数列$\{a_n\},\{b_n\}$满足$\lim\limits_{n\to\infty}a_n=0,\ \lim\limits_{n\to\infty}b_n=0$，则$a_n=O(b_n)\Leftrightarrow$存在常数 $M>0$ 及正整数N，使得$|a_n|\leq M|b_n|$对任意$n>N$成立.]

【参考解答】：(1) 注意到

$$\ln\frac{v_{n+1}}{v_n}=\alpha\ln\left(1+\frac{1}{n}\right)+\ln\frac{u_{n+1}}{u_n}$$
$$=\left(\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right)+\left(-\frac{\alpha}{n}+\frac{\alpha^2}{n^2}+O\left(\frac{1}{n^\beta}\right)\right)=O\left(\frac{1}{n^\gamma}\right)$$

其中$\gamma=\min\{2,\beta\}>1$，故存在常数$C>0$及正整数N，使得

$$\left|\ln\frac{v_{n+1}}{v_n}\right|\le C\left|\frac{1}{n^\gamma}\right|$$

对任意$n>N$成立，所以级数$\sum\limits_{n=1}^{\infty}\ln\frac{v_{n+1}}{v_n}$收敛.

(2) 因为$\sum\limits_{k=1}^{n}\ln\frac{v_{k+1}}{v_k}=\ln v_{n+1}-\ln v_1$，所以由(1)的结论可知，极限$\lim\limits_{n\to\infty}\ln v_n$存在. 令$\lim\limits_{n\to\infty}\ln v_n=a$，则$\lim\limits_{n\to\infty}v_n=e^a>0$，即$\lim\limits_{n\to\infty}\frac{u_n}{1/n^\alpha}=e^a>0$.根据正项级数的比较判别法，级数$\sum\limits_{n=1}^{\infty}u_n$当$\alpha>1$时收敛，当$\alpha\le 1$时发散.