2012 年第三届全国大学生数学竞赛决赛
(非数学专业）参考答案

一、计算题

1.【参考解析】：$\lim\limits_{x\to 0}\dfrac{\sin^2 x-x^2\cos^2 x}{x^2\sin^2 x}=\lim\limits_{x\to 0}\dfrac{\sin^2 x-x^2\cos^2 x}{x^4}$

$$=\lim_{x\to 0}\frac{\left(\sin x-x\cos x\right)\left(\sin x+x\cos x\right)}{x^4}$$

$$=\lim_{x\to 0}\frac{\sin x+x\cos x}{x}\lim_{x\to 0}\frac{\sin x-x\cos x}{x^3}$$

$$=2\lim_{x\to 0}\frac{x\sin x}{3x^2}=\frac{2}{3}\lim_{x\to 0}\frac{\sin x}{x}=\frac{2}{3}.$$

2.【参考解析】：令$t=\dfrac{1}{x}$，则

原式$=\lim\limits_{t\to 0^+}\dfrac{1}{t^3}\left[\left(1+\dfrac{t^2}{2}-t^3\tan t\right)e^t-\sqrt{1+t^6}\right]$

$$=\lim_{t\to 0^+}\frac{1}{t^3}\left[\left(1+\frac{t^2}{2}\right)e^t-1\right]=\lim_{t\to 0^+}\frac{2+2t+t^2}{6t^2}\lim_{t\to 0^+}e^t=+\infty.$$

3.【参考解析】：对$z=f(x,y)$两边对x求两次偏导，分别得

$$0=f_x+f_y\frac{\partial y}{\partial x},0=f_{xx}+2f_{xy}\frac{\partial y}{\partial x}+f_{yy}\left(\frac{\partial y}{\partial x}\right)^2+f_y\frac{\partial^2 y}{\partial x^2}.$$

由前面的式子解出$\dfrac{\partial y}{\partial x}=-\dfrac{f_x}{f_y}$，代入第二个式子并求解，得

$$f_y\frac{\partial^2 y}{\partial x^2}=0\text{，即}\frac{\partial^2 y}{\partial x^2}=0.$$

4.【参考解析】：$I=\displaystyle\int e^{x+\frac{1}{x}}\,\mathrm{d}x+\int x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,\mathrm{d}x$

$$=\int e^{x+\frac{1}{x}}\,\mathrm{d}x+xe^{x+\frac{1}{x}}-\int e^{x+\frac{1}{x}}\,\mathrm{d}x=xe^{x+\frac{1}{x}}+C.$$

5.【参考解析】：联立两个曲面方程，解得交线所在平面$z=a$（$z=4a$舍去)，它将表面积分为S_1,S_2两部分，它们在xOy面上的投影为$x^2+y^2\le a^2$.

$$S=\iint\limits_{x^2+y^2\le a^2.}\left(\sqrt{1+\frac{4x^2}{a^2}+\frac{4y^2}{a^2}}+\sqrt{2}\right)\mathrm{d}\,x\,\mathrm{d}\,y$$
$$=\int_0^{2\pi}\mathrm{d}\,\theta\int_0^a\frac{\sqrt{a^2+4r^2}}{a}r\,\mathrm{d}\,r+\sqrt{2}\pi a^2=\pi a^2\left(\frac{5\sqrt{5}-1}{6}+\sqrt{2}\right).$$

二、【参考证明】：记 $f(x)=\dfrac{x}{\cos^2 x+x^\alpha\sin^2 x}$.

若 $\alpha\le 0, f(x)\ge\dfrac{x}{2}\left(\forall x>1\right)$；

若 $0<\alpha\le 2,\alpha-1\le 1, f(x)\ge\dfrac{x^{1-\alpha}}{2}\left(\forall x>1\right)$；

所以积分发散.

若 $\alpha>2, a_n=\int_{n\pi}^{(n+1)\pi}f(x)\,\mathrm{d}\,x$，考虑级数 $\sum_{n=1}^{\infty}a_n$ 的收敛性即可.

当 $n\pi\le x\le\left(n+1\right)\pi$ 时，

$$\frac{n\pi}{1+\left(n+1\right)^\alpha\pi^\alpha\sin^2 x}\le f(x)\le\frac{(n+1)\pi}{1+n^\alpha\pi^\alpha\sin^2 x},$$

对任何 $b>0$，有

$$\int_{n\pi}^{(n+1)\pi}\frac{\mathrm{d}\,x}{1+b\sin^2 x}=2\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\,x}{1+b\sin^2 x}$$
$$=2\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\cot x}{b+\csc^2 x}=2\int_0^{+\infty}\frac{\mathrm{d}\,t}{b+1+t^2}=\frac{\pi}{\sqrt{b+1}}$$

这样，存在 $0<A_1\le A_2$，使得 $\dfrac{A_1}{n^{\alpha/2-1}}\le a_n\le\dfrac{A_2}{n^{\alpha/2-1}}$，从而可知，当 $\alpha>4$ 时，所讨论的积分收敛；否则发散.

三、【参考证明】：因为 $f(x)$ 在 $\left(-\infty,+\infty\right)$ 上无穷次可微，且满足：$\left|f^{(k)}(x)\right|\le M\left(k=1,2,\cdots\right)$，所以

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n. \qquad (*)$$

由 $f\left(\dfrac{1}{2^n}\right)=0,\left(n=1,2,\cdots\right)$ 得 $f(0)=\lim\limits_{n\to\infty}f\left(\dfrac{1}{2^n}\right)=0$. 于是 $f'(0)=\lim\limits_{n\to\infty}\dfrac{f\left(\dfrac{1}{2^n}\right)-f(0)}{\dfrac{1}{2^n}}=0$. 由

洛尔定理，对于任意自然数 n，在 $\left[\dfrac{1}{2^{n+1}},\dfrac{1}{2^n}\right]$ 上，存在 $\xi_n^{(1)}\in\left(\dfrac{1}{2^{n+1}},\dfrac{1}{2^n}\right)$，使得 $f'\left(\xi_n^{(1)}\right)=0$，从而

$\xi_n^{(1)} \to 0\left(n \to \infty\right)$，这里

$$\xi_1^{(1)} > \xi_2^{(1)} > \xi_3^{(1)} > \cdots > \xi_n^{(1)} > \xi_{n+1}^{(1)} \cdots.$$

在 $\left[\xi_{n+1}^{(1)}, \xi_n^{(1)}\right]$ 上，对 $f'(x)$ 由洛尔定理，存在 $\xi_n^{(2)} \in \left(\xi_{n+1}^{(1)}, \xi_n^{(1)}\right)$，使得 $f''\left(\xi_n^{(2)}\right) = 0$ 且 $\xi_n^{(2)} \to 0\left(n \to \infty\right)$，于是

$$f''(0) = \lim_{n \to \infty} \frac{f'\left(\xi_n^{(2)}\right) - f'(0)}{\xi_n^{(2)}} = 0 .$$

类似地，对于任意的 n，有 $f^{(n)}(0) = 0$. 由(*)，有

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \equiv 0.$$

四、【参考解析】：(1) $J = \iint\limits_D \left(\left(c + x\right)^2 + y^2\right) \rho \,\mathrm{d}x\,\mathrm{d}y$

$$= 2\rho \int_0^{\pi} \mathrm{d}\varphi \int_0^1 \left(c^2 + 2act\cos\varphi + a^2t^2\cos^2\varphi + b^2t^2\sin^2\varphi\right) abt\,\mathrm{d}t = \frac{ab\pi}{4}\left(5a^2 - 3b^2\right)\rho.$$

$$\int_0^{\pi} \mathrm{d}\varphi \int_0^1 c^2 abt\,\mathrm{d}t = abc^2 \frac{\pi}{2} = ab\left(a^2 - b^2\right)\frac{\pi}{2};$$

$$\int_0^{\pi} \mathrm{d}\varphi \int_0^1 2act\cos\varphi\, abt\,\mathrm{d}t = 0;$$

$$\int_0^{\pi} \mathrm{d}\varphi \int_0^1 a^2t^2\cos^2\varphi\, abt\,\mathrm{d}t = \frac{a^3b}{8}\int_0^{\pi}\left(1 + \cos 2\varphi\right)\mathrm{d}\varphi = \frac{\pi a^3 b}{8};$$

$$\int_0^{\pi} \mathrm{d}\varphi \int_0^1 b^2t^2\sin^2\varphi\, abt\,\mathrm{d}t = \frac{ab^3}{8}\int_0^{\pi}\left(1 - \cos 2\varphi\right)\mathrm{d}\varphi = \frac{\pi ab^3}{8}.$$

(2) 设 J 固定，$b(a)$ 是 $J = \dfrac{ab\rho\pi}{4}\left(5a^2 - 3b^2\right)$ 确定的隐函数，则 $b'(a) = \dfrac{3b^3 - 15a^2b}{5a^3 - 9ab^2}$. 对 $S = \pi ab\left(a\right)$ 求导，则有

$$S'(a) = \pi\left[b\left(a\right) + ab'(a)\right] = \pi\left[b + \frac{3b^3 - 15a^2b}{5a^2 - 9b^2}\right] = -2\pi b\left(\frac{3b^2 + 5a^2}{5a^2 - 9b^2}\right)$$

显然，当 $b = \dfrac{\sqrt{5}}{3}a$ 时，$S'(a)$ 不存在；当 $b < \dfrac{\sqrt{5}}{3}a$ 时，$S'(a) < 0$；当 $\dfrac{\sqrt{5}}{3}a < b \le a$ 时，$S'(a) > 0$.

由 $J = \dfrac{ab\rho\pi}{4}\left(5a^2 - 3b^2\right)$，当 $b = a$ 时，

$$a = \left(\frac{2J}{\rho\pi}\right)^{1/4}, S = \left(\frac{2\pi J}{\rho}\right)^{1/2};$$

当 $b = \dfrac{\sqrt{5}}{3}a$ 时，$a = \left(\dfrac{18J}{5\sqrt{5}\rho\pi}\right)^{1/4}, S = \left(\dfrac{2\pi J}{\sqrt{5}\rho}\right)^{1/2}$；

由$\frac{\pi\rho}{2}a^3b \le J = \frac{ab\rho\pi}{4}\left(5a^2 - 3b^2\right)$可知，当$a \to +\infty$时，$b = O\left(a^{-3}\right)$，所以$\lim\limits_{a\to+\infty} S = 0$. 由此可知，椭圆的面积不存在最大值和最小值，且$0 < S < \left(\frac{2\pi J}{\rho}\right)^{1/2}$.

五、【参考解析】：令$Q = xz^2 + 2yz, P = -\left(2xz + yz^2\right)$，则

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2\left(xz + y\right)\frac{\partial z}{\partial x} + 2\left(x + yz\right)\frac{\partial z}{\partial y} + 2z^2.$$

利用格林公式，有

$$I = 2\iint\limits_{x^2+y^2\le 1}\left[\left(xz + y\right)\frac{\partial z}{\partial x} + \left(x + yz\right)\frac{\partial z}{\partial y} + 2z^2\right]\mathrm{d}x\,\mathrm{d}y$$

方程$F = 0$对x求导，得到

$$\left(z + x\frac{\partial z}{\partial x}\right)F_u + \left(1 - y\frac{\partial z}{\partial x}\right)F_v = 0 \text{ 即} \frac{\partial z}{\partial x} = -\frac{zF_u + F_v}{xF_u - yF_v}.$$

同样可得$\frac{\partial z}{\partial y} = \frac{F_u + zF_v}{xF_u - yF_v}$. 于是可得

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{yF_u - xF_v}{xF_u - yF_v} - z. \quad y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 1 - \frac{z\left(yF_u - xF_v\right)}{xF_u - yF_v}.$$

$$(xz + y)\frac{\partial z}{\partial x} + (x + yz)\frac{\partial z}{\partial y} = 1 - z^2.$$

$$I = \oint\limits_L \left(xz^2 + 2yz\right)\mathrm{d}y - \left(2xz + yz^2\right)\mathrm{d}x = 2\iint\limits_{x^2+y^2\le 1}\mathrm{d}x\,\mathrm{d}y = 2\pi.$$

六、【参考解析】：(1) 微分方程为一阶线性微分方程，解方程的通解为

$$y = e^{\int x\,\mathrm{d}x}\left[\int xe^{x^2}e^{-\int x\,\mathrm{d}x}\,\mathrm{d}x + C\right] = e^{\frac{x^2}{2}}\left[\int xe^{x^2}e^{-\frac{x^2}{2}}\,\mathrm{d}x + C\right]$$

$$= e^{\frac{x^2}{2}}\left[\int xe^{\frac{x^2}{2}}\,\mathrm{d}x + C\right] = e^{\frac{x^2}{2}}\left[e^{\frac{x^2}{2}} + C\right] = e^{x^2} + Ce^{\frac{x^2}{2}}$$

由$y(0) = 1$，得$C = 0$，所以$y = e^{x^2}$.

(2) 证明：注意到

$$\lim_{n\to\infty}\int_0^1 \frac{n}{n^2x^2 + 1}\mathrm{d}x = \lim_{n\to\infty}\arctan n = \frac{\pi}{2},$$

$$\int_0^1 \frac{nf(x)}{n^2x^2 + 1}\mathrm{d}x = \int_0^1 \frac{ne^{x^2}}{n^2x^2 + 1}\mathrm{d}x$$

$$= \int_0^1 \frac{n}{n^2x^2 + 1}\left(e^{x^2} - 1\right)\mathrm{d}x + \int_0^1 \frac{n}{n^2x^2 + 1}\mathrm{d}x$$

$\forall\varepsilon>0$，由 $\lim\limits_{x\to0}\left(e^{x^2}-1\right)=0$ 知 $\exists\delta>0,\forall 0<x<\delta$ 时，有 $\left|e^{x^2}-1\right|<\varepsilon\dfrac{1}{\pi}$，因此有

$$\begin{aligned}&\int_0^1\frac{n}{n^2x^2+1}\left(e^{x^2}-1\right)\mathrm{d}x=\int_0^\delta\frac{n}{n^2x^2+1}\left(e^{x^2}-1\right)\mathrm{d}x+\int_\delta^1\frac{n}{n^2x^2+1}\left(e^{x^2}-1\right)\mathrm{d}x\\&\le\frac{\varepsilon}{\pi}\int_0^\delta\frac{n}{n^2x^2+1}\mathrm{d}x+\left(e-1\right)\int_\delta^1\frac{n}{n^2x^2+1}\mathrm{d}x\\&\le\frac{\varepsilon}{2}+\left(e-1\right)\frac{n}{n^2\delta^2+1}\left(1-\delta\right)\le\frac{\varepsilon}{2}+\left(e-1\right)\frac{n}{n^2\delta^2+1}\end{aligned}$$

$\exists N,\forall n>N$ 时，$\dfrac{n}{n^2\delta^2+1}\le\dfrac{\varepsilon}{2\left(e-1\right)}$，因此

$$\int_0^1\frac{n}{n^2x^2+1}\left(e^{x^2}-1\right)\mathrm{d}x<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

即 $\lim\limits_{n\to\infty}\displaystyle\int_0^1\frac{n}{n^2x^2+1}\left(e^{x^2}-1\right)\mathrm{d}x=0$. 所以

$$\lim_{n\to\infty}\int_0^1\frac{n}{n^2x^2+1}f(x)\,\mathrm{d}x=\lim_{n\to\infty}\int_0^1\frac{n}{n^2x^2+1}\left(e^{x^2}-1\right)\mathrm{d}x+\lim_{n\to\infty}\int_0^1\frac{n}{n^2x^2+1}\mathrm{d}x=\frac{\pi}{2}.$$