2012 年第三届全国大学生数学竞赛决赛 (非数学专业)参考答案

一、计算题

1. [参考解析]:
$$\lim_{x\to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x} = \lim_{x\to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^4}$$
$$= \lim_{x\to 0} \frac{\left(\sin x - x \cos x\right) \left(\sin x + x \cos x\right)}{x^4}$$
$$= \lim_{x\to 0} \frac{\sin x + x \cos x}{x} \lim_{x\to 0} \frac{\sin x - x \cos x}{x^3}$$
$$= 2 \lim_{x\to 0} \frac{x \sin x}{3x^2} = \frac{2}{3} \lim_{x\to 0} \frac{\sin x}{x} = \frac{2}{3}.$$

原式 =
$$\lim_{t \to 0^+} \frac{1}{t^3} \left[\left(1 + \frac{t^2}{2} - t^3 \tan t \right) e^t - \sqrt{1 + t^6} \right]$$

= $\lim_{t \to 0^+} \frac{1}{t^3} \left[\left(1 + \frac{t^2}{2} \right) e^t - 1 \right] = \lim_{t \to 0^+} \frac{2 + 2t + t^2}{6t^2} \lim_{t \to 0^+} e^t = +\infty.$

3. 【参考解析】: 对z = f(x,y) 两边对x 求两次偏导,分别得

$$0=f_x+f_yrac{\partial y}{\partial x}, 0=f_{xx}+2f_{xy}rac{\partial y}{\partial x}+f_{yy}iggl(rac{\partial y}{\partial x}iggr)^2+f_yrac{\partial^2 y}{\partial x^2}.$$

由前面的式子解出 $\dfrac{\partial y}{\partial x}=-\dfrac{f_x}{f_y}$,代入第二个式子并求解,得

$$f_yrac{\partial^2 y}{\partial x^2}=0$$
,即 $rac{\partial^2 y}{\partial x^2}=0$.

4. [参考解析]:
$$I = \int e^{x+\frac{1}{x}} dx + \int x \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}} dx$$
$$= \int e^{x+\frac{1}{x}} dx + x e^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}} dx = x e^{x+\frac{1}{x}} + C.$$

5.【参考解析】: 联立两个曲面方程,解得交线所在平面 z=a (z=4a 舍去),它将表面积分为 S_1,S_2 两部分,它们在 xOy 面上的投影为 $x^2+y^2\leq a^2$.

$$egin{aligned} S &= \int \int \limits_{x^2+y^2 \leq a^2.} \Biggl(\sqrt{1 + rac{4x^2}{a^2} + rac{4y^2}{a^2}} + \sqrt{2} \Biggr) \mathrm{d}\,x\,\mathrm{d}\,y \ &= \int \limits_0^{2\pi} \mathrm{d}\, heta \int \limits_0^a rac{\sqrt{a^2 + 4r^2}}{a} \, r\,\mathrm{d}\,r + \sqrt{2}\pi a^2 = \pi a^2 \Biggl(rac{5\sqrt{5} - 1}{6} + \sqrt{2} \Biggr). \end{aligned}$$

二、【参考证明】: 记 $f(x) = \frac{x}{\cos^2 x + x^\alpha \sin^2 x}$.

若
$$lpha \leq 0, f(x) \geq \frac{x}{2} ig(orall x > 1 ig)$$
 ;

若
$$01ig)$$
 ;

所以积分发散.

若 $lpha>2, a_n=\int_{n\pi}^{(n+1)\pi}f(x)\,\mathrm{d}\,x$, 考虑级数 $\sum_{n=1}^{\infty}a_n$ 的收敛性即可.

当 $n\pi \le x \le (n+1)\pi$ 时,

$$rac{n\pi}{1+ig(n+1ig)^lpha\,\pi^lpha\sin^2x} \leq f(x) \leq rac{(n+1)\pi}{1+n^lpha\pi^lpha\sin^2x}$$
 ,

对任何b>0,有

$$\int_{n\pi}^{(n+1)\pi} \frac{\mathrm{d}\,x}{1+b\sin^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\,x}{1+b\sin^2 x}$$
$$= 2\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\cot x}{b+\csc^2 x} = 2\int_0^{+\infty} \frac{\mathrm{d}\,t}{b+1+t^2} = \frac{\pi}{\sqrt{b+1}}$$

这样,存在 $0 < A_1 \le A_2$,使得 $\dfrac{A_1}{n^{\alpha/2-1}} \le a_n \le \dfrac{A_2}{n^{\alpha/2-1}}$,从而可知,当 $\alpha > 4$ 时,所讨论的积分收敛;否则发散.

三、【参考证明】: 因为 f(x) 在 $\left(-\infty, +\infty\right)$ 上无穷次可微,且满足: $\left|f^{(k)}(x)\right| \leq M\left(k=1,2,\cdots\right)$,所以

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$
 (*)

由
$$figg(rac{1}{2^n}igg) = 0, ig(n=1,2,\cdotsig)$$
得 $f(0) = \lim_{n o\infty} figg(rac{1}{2^n}igg) = 0$. 于是 $f'(0) = \lim_{n o\infty} rac{figg(rac{1}{2^n}igg) - f(0)}{rac{1}{2^n}} = 0$. 由

洛尔定理,对于任意自然数 n ,在 $\left[rac{1}{2^{n+1}},rac{1}{2^n}
ight]$ 上,存在 $\xi_n^{(1)}\in\left(rac{1}{2^{n+1}},rac{1}{2^n}
ight)$,使得 $f'\left(\xi_n^{(1)}
ight)=0$,从而

$$\xi_n^{(1)} o 0ig(n o\inftyig)$$
,这里

$$\xi_1^{(1)} > \xi_2^{(1)} > \xi_3^{(1)} > \dots > \xi_n^{(1)} > \xi_{n+1}^{(1)} \cdots$$

在 $\left[\xi_{n+1}^{(1)},\xi_n^{(1)}\right]$ 上, 对 f'(x) 由 洛 尔 定 理, 存 在 $\xi_n^{(2)}\in\left(\xi_{n+1}^{(1)},\xi_n^{(1)}\right)$, 使 得 $f''\left(\xi_n^{(2)}\right)=0$ 且 $\xi_n^{(2)}\to 0$ $\left(n\to\infty\right)$,于是

$$f''(0) = \lim_{n o \infty} rac{f'\Big(\xi_n^{(2)}\Big) - f'(0)}{\xi_n^{(2)}} = 0$$
 .

类似地,对于任意的n,有 $f^{(n)}(0)=0$.由(*),有

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \equiv 0.$$

四、【参考解析】: (1)
$$J = \iint\limits_{D} \left[\left(c+x
ight)^2 + y^2
ight]
ho \,\mathrm{d}\,x\,\mathrm{d}\,y$$

$$=2\rho\int_0^\pi\mathrm{d}\,\varphi\int_0^1\!\!\left(c^2+2act\cos\varphi+a^2t^2\cos^2\varphi+b^2t^2\sin^2\varphi\right)\!abt\,\mathrm{d}\,t=\frac{ab\pi}{4}\!\!\left(5a^2-3b^2\right)\!\rho.$$

$$\int_0^\pi \mathrm{d}\,arphi \int_0^1 c^2 a b t \, \mathrm{d}\, t = a b c^2 rac{\pi}{2} = a b \Big(a^2 - b^2\Big) rac{\pi}{2};$$

$$\int_0^{\pi} \mathrm{d}\varphi \int_0^1 2act \cos\varphi abt \, \mathrm{d}t = 0;$$

$$\int_0^\pi \mathrm{d}\,arphi \int_0^1 a^2 t^2 \cos^2arphi abt \, \mathrm{d}\,t = rac{a^3 b}{8} \int_0^\pi igl(1 + \cos 2arphiigr) \, \mathrm{d}\,arphi = rac{\pi a^3 b}{8};$$

$$\int_0^\pi \mathrm{d}\,arphi \int_0^1 b^2 t^2 \sin^2arphi abt\,\mathrm{d}\,t = rac{ab^3}{8} \int_0^\pi igl(1-\cos 2arphiigr) \mathrm{d}\,arphi = rac{\pi ab^3}{8}.$$

(2) 设
$$J$$
固定, $b(a)$ 是 $J=rac{ab
ho\pi}{4}\Big(5a^2-3b^2\Big)$ 确定的隐函数,则 $b'(a)=rac{3b^3-15a^2b}{5a^3-9ab^2}$.对

 $S = \pi a b(a)$ 求导,则有

$$S'(a) = \pi \Big[b \Big(a \Big) + ab'(a) \Big] = \pi \Bigg[b + \frac{3b^3 - 15a^2b}{5a^2 - 9b^2} \Bigg] = -2\pi b \Bigg(\frac{3b^2 + 5a^2}{5a^2 - 9b^2} \Bigg)$$

显然,当 $b=\frac{\sqrt{5}}{3}a$ 时,S'(a)不存在;当 $b<\frac{\sqrt{5}}{3}a$ 时,S'(a)<0;当 $\frac{\sqrt{5}}{3}a< b\leq a$ 时,S'(a)>0.

由
$$J=rac{ab
ho\pi}{4}igl(5a^2-3b^2igg)$$
,当 $b=a$ 时,

$$a = \left(\frac{2J}{
ho\pi}\right)^{1/4}, S = \left(\frac{2\pi J}{
ho}\right)^{1/2};$$

当
$$b=rac{\sqrt{5}}{3}a$$
时, $a=\left(rac{18J}{5\sqrt{5}
ho\pi}
ight)^{\!1/4},S=\left(rac{2\pi J}{\sqrt{5}
ho}
ight)^{\!1/2}$;

由
$$rac{\pi
ho}{2}a^3b \leq J = rac{ab
ho\pi}{4}igl(5a^2-3b^2igr)$$
可知,当 $a o +\infty$ 时, $b = Oigl(a^{-3}igr)$,所以 $\lim_{a o +\infty} S = 0$.由

此可知,椭圆的面积不存在最大值和最小值,且 $0 < S < \left(rac{2\pi J}{
ho}
ight)^{1/2}$.

五、【参考解析】:
$$\diamondsuit Q=xz^2+2yz, P=-\left(2xz+yz^2\right)$$
,则

$$rac{\partial Q}{\partial x} - rac{\partial P}{\partial y} = 2 \Big(xz + y \Big) rac{\partial z}{\partial x} + 2 \Big(x + yz \Big) rac{\partial z}{\partial y} + 2z^2 \, .$$

利用格林公式,有

$$I = 2 \int\!\!\!\int_{x^2+y^2 < 1} \!\! \left[\! \left(xz + y
ight) \! rac{\partial z}{\partial x} + \left(x + yz
ight) \! rac{\partial z}{\partial y} + 2z^2
ight] \! \mathrm{d}\, x \, \mathrm{d}\, y$$

方程F = 0对x求导,得到

$$\left(z+x\frac{\partial z}{\partial x}\right)F_u+\left(1-y\frac{\partial z}{\partial x}\right)F_v=0 \ \ \text{Im}\, \frac{\partial z}{\partial x}=-\frac{zF_u+F_v}{xF_u-yF_v}.$$

同样可得 $\dfrac{\partial z}{\partial y}=\dfrac{F_u+zF_v}{xF_u-yF_v}$. 于是可得

$$x rac{\partial z}{\partial x} + y rac{\partial z}{\partial y} = rac{yF_u - xF_v}{xF_u - yF_v} - z.$$
 $y rac{\partial z}{\partial x} + x rac{\partial z}{\partial y} = 1 - rac{z\left(yF_u - xF_v
ight)}{xF_u - yF_v}.$ $(xz + y) rac{\partial z}{\partial x} + (x + yz) rac{\partial z}{\partial y} = 1 - z^2.$ $I = \left(xz^2 + 2yz\right) dy - \left(2xz + yz^2\right) dx = 2$ $\int \int dx dy = 2\pi.$

 $I=\oint\limits_L \Big(xz^2+2yz\Big)\mathrm{d}\,y-\Big(2xz+yz^2\Big)\mathrm{d}\,x=2\iint\limits_{x^2+y^2\leq 1}\mathrm{d}\,x\,\mathrm{d}\,y=2\pi.$

六、【参考解析】: (1) 微分方程为一阶线性微分方程,解方程的通解为

$$y = e^{\int x \, dx} \left[\int x e^{x^2} e^{-\int x \, dx} \, dx + C \right] = e^{rac{x^2}{2}} \left[\int x e^{x^2} e^{-rac{x^2}{2}} \, dx + C \right]$$
 $= e^{rac{x^2}{2}} \left[\int x e^{rac{x^2}{2}} \, dx + C \right] = e^{rac{x^2}{2}} \left[e^{rac{x^2}{2}} + C \right] = e^{x^2} + Ce^{rac{x^2}{2}}$

由y(0)=1,得C=0,所以 $y=e^{x^2}$.

(2) 证明: 注意到

$$egin{split} &\lim_{n o\infty} \int_0^1 rac{n}{n^2 x^2 + 1} \, \mathrm{d}\, x = \lim_{n o\infty} rctan \, n = rac{\pi}{2}, \ &\int_0^1 rac{n f(x)}{n^2 x^2 + 1} \, \mathrm{d}\, x = \! \int_0^1 rac{n e^{x^2}}{n^2 x^2 + 1} \, \mathrm{d}\, x \ &= \! \int_0^1 \! rac{n}{n^2 x^2 + 1} \! \left(e^{x^2} - 1
ight) \! \mathrm{d}\, x + \! \int_0^1 \! rac{n}{n^2 x^2 + 1} \, \mathrm{d}\, x \end{split}$$

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$$\begin{split} \forall \varepsilon > 0 \;, \; & \text{由} \lim_{x \to 0} \left(e^{x^2} - 1 \right) = 0 \text{知} \exists \delta > 0, \forall 0 < x < \delta \, \text{时}, \; | a | e^{x^2} - 1 | < \varepsilon \frac{1}{\pi} \;, \; | a | b | \\ & \int_0^1 \frac{n}{n^2 x^2 + 1} \left(e^{x^2} - 1 \right) \mathrm{d} \, x = \int_0^\delta \frac{n}{n^2 x^2 + 1} \left(e^{x^2} - 1 \right) \mathrm{d} \, x + \int_\delta^1 \frac{n}{n^2 x^2 + 1} \left(e^{x^2} - 1 \right) \mathrm{d} \, x \\ & \leq \frac{\varepsilon}{\pi} \int_0^\delta \frac{n}{n^2 x^2 + 1} \mathrm{d} \, x + \left(e - 1 \right) \int_\delta^1 \frac{n}{n^2 x^2 + 1} \, \mathrm{d} \, x \\ & \leq \frac{\varepsilon}{2} + \left(e - 1 \right) \frac{n}{n^2 \delta^2 + 1} \left(1 - \delta \right) \leq \frac{\varepsilon}{2} + \left(e - 1 \right) \frac{n}{n^2 \delta^2 + 1} \\ & \exists N, \forall n > N \, \text{H}, \; \frac{n}{n^2 \delta^2 + 1} \leq \frac{\varepsilon}{2 \left(e - 1 \right)} \;, \; | a | b | b | \\ & \int_0^1 \frac{n}{n^2 x^2 + 1} \left(e^{x^2} - 1 \right) \mathrm{d} \, x < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{split}$$

$$\mathbb{P} \lim_{n \to \infty} \int_0^1 \frac{n}{n^2 x^2 + 1} \left(e^{x^2} - 1 \right) \mathrm{d} \, x = 0. \; | \mathbf{M} | \mathbf{M} | \mathbf{M}$$

$$\lim_{n \to \infty} \int_0^1 \frac{n}{n^2 x^2 + 1} f(x) \, \mathrm{d} \, x = \lim_{n \to \infty} \int_0^1 \frac{n}{n^2 x^2 + 1} \left(e^{x^2} - 1 \right) \mathrm{d} \, x + \lim_{n \to \infty} \int_0^1 \frac{n}{n^2 x^2 + 1} \, \mathrm{d} \, x = \frac{\pi}{2} \;. \end{split}$$