2014 年第五届全国大学生数学竞赛决赛
(非数学类）参考答案

一、解答下列各题

1.【参考解答】:【解法一】:

$$原式=\int_0^{2\pi}\frac{\sin^2 t}{t^2}\,\mathrm{d}t\int_0^t x\,\mathrm{d}x=\frac{1}{2}\int_0^{2\pi}\sin^2 t\,\mathrm{d}t=2\int_0^{\frac{\pi}{2}}\sin^2 t\,\mathrm{d}t=\frac{\pi}{2}.$$

【解法二】: 令 $f(x)=\int_x^{2\pi}\frac{\sin^2 t}{t^2}\,\mathrm{d}t$，则 $f'(x)=-\frac{\sin^2 x}{x^2}$ 且 $f\left(2\pi\right)=0.$

$$原式=\int_0^{2\pi}xf(x)\,\mathrm{d}x=\left[\frac{1}{2}x^2f(x)\right]_0^{2\pi}-\frac{1}{2}\int_0^{2\pi}x^2f'(x)\,\mathrm{d}x$$

$$=\frac{1}{2}\int_0^{2\pi}x^2\frac{\sin^2 x}{x^2}\,\mathrm{d}x=\frac{1}{2}\int_0^{2\pi}\sin^2 x\,\mathrm{d}x=\frac{\pi}{2}.$$

2.【参考解答】: $I=\int_0^1 f(x)\,\mathrm{d}x=\int_0^1 f(x)\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\,\mathrm{d}x$

$$\le\left(\int_0^1\left(1+x^2\right)f^2(x)\,\mathrm{d}x\right)^{1/2}\left(\int_0^1\frac{1}{1+x^2}\,\mathrm{d}x\right)^{1/2}=\left(\int_0^1\left(1+x^2\right)f^2(x)\,\mathrm{d}x\right)^{1/2}\left(\frac{\pi}{4}\right)^{1/2}$$

$\int_0^1\left(1+x^2\right)f^2(x)\,\mathrm{d}x\ge\frac{4}{\pi}$，取 $f(x)=\frac{4}{\pi\left(1+x^2\right)}$ 即可.

3.【参考解答】: 由两方程定义的曲面在 $P_0\left(x_0,y_0,z_0\right)$ 的切面分别为

$$F_x\left(P_0\right)(x-x_0)+F_y\left(P_0\right)(y-y_0)+F_z\left(P_0\right)(z-z_0)=0,$$
$$G_x\left(P_0\right)(x-x_0)+G_y\left(P_0\right)(y-y_0)+G_z\left(P_0\right)(z-z_0)=0.$$

上述两切面的交线就是 Γ 在 P_0 点的切线，该切线在 xOy 面上的投影就是 S 过 $\left(x_0,y_0\right)$ 的切线. 消去 $z-z_0$，有

$$\left(F_xG_z-G_xF_z\right)_{P_0}(x-x_0)+\left(F_yG_z-G_yF_z\right)_{P_0}(y-y_0)=0.$$

这里 $x-x_0$ 的系数 $\frac{\partial\left(F,G\right)}{\partial\left(x,z\right)}\ne 0$，故上式是一条直线的方程，就是所求的切线.

4.【参考解答】: 由关系式

$$AB=A-B+E\Rightarrow\left(A+E\right)\left(B-E\right)=0.$$

$$\Rightarrow rank(A+B)\le rank(A+E)+rank\left(B-E\right)\le 3$$

因为 $rank\left(A+B\right)=3,$ 所以

$$rank(A+E)+rank\left(B-E\right)=3.$$

又$rank(A+E)\geq 2$,考虑到B非单位，所以$rank\left(B-E\right)\geq 1$，只有$rank(A+E)=2$.

$$A+E=\begin{pmatrix}2&2&1\\3&5&a\\1&2&3\end{pmatrix}\to\begin{pmatrix}0&-2&-5\\0&-1&a-9\\1&2&3\end{pmatrix}\to\begin{pmatrix}0&0&13-2a\\0&-1&a-9\\1&2&3\end{pmatrix}$$

从而$a=\dfrac{13}{2}$.

二、【参考证明】：由泰勒公式

$$f\left(x+h\right)=f\left(x\right)+f'(x)h+\frac{f''\left(x\right)}{2}h^2+\frac{f'''\left(x\right)}{6}h^3+\frac{f^{(4)}\left(\xi\right)}{24}h^4 \qquad (1)$$

$$f''\left(x+\theta h\right)=f''\left(x\right)+f'''\left(x\right)\theta h+\frac{f^{(4)}\left(\eta\right)}{2}\theta^2h^2 \qquad (2)$$

其中ξ介于x与$x+h$之间，η介于x与$x+\theta h$之间，由(1)(2)式和已知条件

$$f\left(x+h\right)=f\left(x\right)+f'(x)h+\frac{f''\left(x+\theta h\right)}{2}h^2$$

可得$4\left(1-3\theta\right)f'''\left(x\right)=\left[6f^{(4)}\left(\eta\right)\theta^2-f^{(4)}\left(\xi\right)\right]h$.

当$\theta\neq\dfrac{1}{3}$时，令$h\to 0$得$f'''(x)=0$，此时f是不超过二次的多项式;

当$\theta=\dfrac{1}{3}$时，有$\dfrac{2}{3}f^{(4)}\left(\eta\right)=f^{(4)}\left(\xi\right)$. 令$h\to 0$，注意到$\xi\to x,\eta\to x$，有$f^{(4)}(x)=0$，此时$f$是不超过三次的多项式.

三、【参考证明】：由题设可知$f'(0)=-1$，则所给方程可变形为$\left(1+x\right)f'(x)+\left(1+x\right)f(x)-\int_0^x f(t)\,\mathrm{d}\,t=0$，两端关于$x$求导并整理得

$$\left(1+x\right)f''(x)+\left(2+x\right)f'(x)=0$$

这是一个可降阶的二阶微分方程，可用分离变量法求得$f'(x)=\dfrac{Ce^{-x}}{1+x}$.

由$f'(0)=-1$得$C=-1$，即$f'(x)=-\dfrac{e^{-x}}{1+x}<0$.函数$f(x)$单调递减. 而$f(0)=1$，所以当$x\geq 0$，$f(x)\leq 1$.

对$f'(t)=-\dfrac{e^{-t}}{1+t}<0$在$\left[0,x\right]$上进行积分，得

$$f\left(x\right)=f\left(0\right)-\int_0^x\frac{e^{-t}}{1+t}\mathrm{d}\,t\geq 1-\int_0^x e^{-t}\,\mathrm{d}\,t=e^{-x}.$$

四、【参考证明】：$I=\int_0^1\mathrm{d}\,y\int_0^1 f(x,y)\,\mathrm{d}\,x=-\int_0^1\mathrm{d}\,y\int_0^1 f(x,y)\,\mathrm{d}\left(1-x\right)$

对于固定y，$\left(1-x\right)f\left(x,y\right)\Big|_{x=0}^{x=1}=0$，由分部积分法可得

$$\int_0^1 f(x,y)\,\mathrm{d}(1-x) = -\int_0^1 (1-x)\frac{\partial f(x,y)}{\partial x}\,\mathrm{d}x$$

交换积分次序后可得 $I = \int_0^1 (1-x)\,\mathrm{d}x \int_0^1 \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y$. 因为 $f(x,0)=0$，所以 $\frac{\partial f(x,y)}{\partial x}=0$；从而 $(1-y)\frac{\partial f(x,y)}{\partial x}\Big|_{y=0}^{y=1} = 0.$ 再由分部积分法得

$$\int_0^1 \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y = -\int_0^1 \frac{\partial f(x,y)}{\partial y}\,\mathrm{d}(1-y) = \int_0^1 (1-y)\frac{\partial^2 f}{\partial x\partial y}\,\mathrm{d}y.$$

$$I = \int_0^1 (1-x)\,\mathrm{d}x \int_0^1 (1-y)\frac{\partial^2 f}{\partial x\partial y}\,\mathrm{d}y = \iint_D (1-x)(1-y)\frac{\partial^2 f}{\partial x\partial y}\,\mathrm{d}x\,\mathrm{d}y$$

因为 $\frac{\partial^2 f}{\partial x\partial y} \le A$，且 $(1-x)(1-y)$ 在 D 上非负，故

$$I \le A\iint_D (1-x)(1-y)\,\mathrm{d}x\,\mathrm{d}y = \frac{A}{4}.$$

五、【参考解答】： 由高斯公式，有

$$I_t = \iiint_V \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}V = \iiint_V \left(2xz+2yz+x^2+y^2\right) f'\left(\left(x^2+y^2\right)z\right)\mathrm{d}V$$

由对称性，有 $\iiint_V (2xz+2yz) f'\left(\left(x^2+y^2\right)z\right)\mathrm{d}V = 0.$ 从而

$$I_t = \iiint_V \left(x^2+y^2\right) f'\left(\left(x^2+y^2\right)z\right)\mathrm{d}V = \int_0^1\left[\int_0^{2\pi}\mathrm{d}\theta\int_0^t f'\left(r^2 z\right) r^3\,\mathrm{d}r\right]\mathrm{d}z$$
$$= 2\pi\int_0^1\left[\int_0^t f'\left(r^2 z\right) r^3\,\mathrm{d}r\right]\mathrm{d}z$$

$$\lim_{t\to 0+}\frac{I_t}{t^4} = \lim_{t\to 0+}\frac{2\pi\int_0^1\left[\int_0^t f'\left(r^2 z\right) r^3\,\mathrm{d}r\right]\mathrm{d}z}{t^4} = \lim_{t\to 0+}\frac{2\pi\int_0^1 f'\left(t^2 z\right)t^3\,\mathrm{d}z}{4t^3}$$
$$= \lim_{t\to 0+}\frac{\pi}{2}\int_0^1 f'\left(t^2 z\right)\mathrm{d}z = \frac{\pi}{2}f'(0).$$

六、【参考证明】：（必要性）设 A,B 是两个 n 阶正定矩阵，从而为对称矩阵，即

$$(AB)^T = AB.$$

又 $A^T = A, B^T = B,$ 所以 $(AB)^T = B^T A^T = BA,$ 所以 $AB = BA.$

（充分性）因为 $AB = BA.$ 则 $(AB)^T = B^T A^T = BA = AB,$ 所以 AB 为实对称矩阵. 因为 A,B 是正定矩阵，存在可逆矩阵 P,Q，使得

$$A = P^T P, B = Q^T Q \Rightarrow AB = P^T P Q^T Q.$$

所以$\left(P^T\right)^{-1}ABP^T=PQ^TQP^T=\left(QP^T\right)^T\left(QP^T\right)$，即$\left(P^T\right)^{-1}ABP^T$是正定矩阵. 所以矩阵$\left(P^T\right)^{-1}ABP^T$的特征值 全为正实数, 而$AB$相似于$\left(P^T\right)^{-1}ABP^T$, 所以$AB$的特征值全为正实数, 所以$AB$为正定矩阵.

七、【参考证明】：由$\lim\limits_{n\to\infty} na_n=0$，知$\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=0}^{n}k\,|\,a_k\,|}{n}=0$，故对任意的$\varepsilon>0$，存在$N_1$，使得当$n>N_1$时，有

$$0\le\frac{\sum\limits_{k=0}^{n}k\,|\,a_k\,|}{n}<\frac{\varepsilon}{3},n\,|\,a_n\,|<\frac{\varepsilon}{3}.$$

又因为$\lim\limits_{x\to1^-}\sum\limits_{n=0}^{\infty}a_nx^n=A$. 所以存在$\delta>0$，当$1-\delta<x<1$时，

$$\left|\sum_{n=0}^{\infty}a_nx^n-A\right|<\frac{\varepsilon}{3}.$$

取N_2，使得当$n>N_2$时，$\dfrac{1}{n}<\delta$，从而$1-\delta<1-\dfrac{1}{n}$，取$x=1-\dfrac{1}{n}$，则

$$\left|\sum_{n=0}^{\infty}a_n\left(1-\frac{1}{n}\right)^n-A\right|<\frac{\varepsilon}{3}.$$

取$N=\max\left\{N_1,N_2\right\}$，当$n>N$时

$$\begin{aligned}\left|\sum_{k=0}^{n}a_k-A\right|&=\left|\sum_{k=0}^{n}a_k-\sum_{k=0}^{n}a_kx^k-\sum_{k=n+1}^{\infty}a_kx^k+\sum_{k=0}^{\infty}a_kx^k-A\right|\\&\le\left|\sum_{k=0}^{n}a_k\left(1-x^k\right)\right|+\left|\sum_{k=n+1}^{\infty}a_kx^k\right|+\left|\sum_{k=0}^{\infty}a_kx^k-A\right|\end{aligned}$$

取$x=1-\dfrac{1}{n}$，则有

$$\begin{aligned}\left|\sum_{k=0}^{n}a_k\left(1-x^k\right)\right|&=\left|\sum_{k=0}^{n}a_k\left(1-x\right)\left(1+x+x^2+\cdots+x^{k-1}\right)\right|\\&\le\sum_{k=0}^{n}\left|a_k\right|\left(1-x\right)k=\frac{\sum\limits_{k=0}^{n}k\left|a_k\right|}{n}<\frac{\varepsilon}{3}\end{aligned}$$

$$\left|\sum_{k=n+1}^{\infty}a_kx^k\right|\le\frac{1}{n}\sum_{k=n+1}^{\infty}k\,|\,a_k\,|\,x^k<\frac{\varepsilon}{3n}\sum_{k=n+1}^{\infty}x^k\le\frac{\varepsilon}{3n}\frac{1}{1-x}=\frac{\varepsilon}{3n\cdot\frac{1}{n}}=\frac{\varepsilon}{3}$$

又因为$\left|\sum_{k=0}^{\infty} a_k x^k - A\right| < \frac{\varepsilon}{3}$，则$\left|\sum_{k=0}^{n} a_k - A\right| < 3\frac{\varepsilon}{3} = \varepsilon.$