Introductory Kinematics

COSC 150 - Maxwell's Demon

These notes were derived from the Dynamics Reference written by Prof. Matthew West at the University of Illinois. The complete reference may be found at http://dynref.engr.illinois.edu.

I. Positions and Coordinates

The basic geometric objects used in kinematics (motions) are positions and vectors. Positions describe locations in space, while vectors describe direction and length (magnitude).

A *position* or *point* is a location in 2D (or 3D) space which may or may not be occupied by an object. To perform calculations, we must introduce *coordinates* for positions. A *coordinate chart* is a map relating positions in space to their coordinates. For example, the coordinates "37th and O Streets NW" label the intersection in front of Georgetown University according to the D.C. street map.

The most commonly-used coordinates are *Cartesian coordinates* (also called rectangular coordinates). These consist of set of perpendicular straight-line axes and corresponding coordinates to determine position along that axis.

It is also possible to express position as a vector, making use of the x and y coordinates as the corresponding *components* of the position vector. We use $\hat{\imath}$ to denote movement one pixel in the x-direction, and $\hat{\jmath}$ to denote movement one pixel in the y-direction, and can express the current position as

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \text{ px}$$

When a Java `Graphics` object is used to draw (e.g., within the `paint()` or `paintComponent()` methods) on a `JPanel`, a Cartesian coordinate grid is provided for you. The origin of this coordiante grid is the upper-left corner of the panel; increasing the x coordinate moves the object to the right, while increasing the y coordinate moves the object down.

II. Velocity

The *velocity* of a particle is expressed as a vector, which has both direction and magnitude. The *speed* of an object is exactly the magnitude of the velocity vector. The same velocity vector may be decomposed in different ways to perform different calculations.

When we are updating the position of an object over time, the velocity vector may be decomposed into two components: a component changing the x-coordinate of position called v_x, and a component changing the y-coordinate of position called v_y. When we do this, the velocity may be expressed as

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} \text{ px/s}$$

A. Updating Position from Velocity

When velocity is constant, and a fixed interval of time Δt elapses, the position of the object may be updated by the equation

$$\begin{aligned}\vec{r}_{t_1} &= \vec{r}_{t_0} + (\Delta t)\,\vec{v} \\ &= (x_{t_0}\hat{\imath} + y_{t_0}\hat{\jmath}) + (\Delta t)\,(v_x\hat{\imath} + v_y\hat{\jmath}) \\ &= (x_{t_0} + (\Delta t)\,v_x)\,\hat{\imath} + (y_{t_0} + (\Delta t)\,v_y)\,\hat{\jmath}\end{aligned}$$

This means that the new position coordinates can be computed from the old position coordinates and the velocity according to the update rules

$$\begin{aligned}x_{t_1} &= x_{t_0} + (\Delta t)\,v_x \\ y_{t_1} &= y_{t_0} + (\Delta t)\,v_y\end{aligned}$$

B. Velocity After a Reflection

When a particle in motion strikes a rigid surface (in what is called an elastic collision), the component of velocity parallel to the surface is unchanged, while the component perpendicular to the surface is reflected (that is, becomes the negation of its previous value). For example, a particle with velocity $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ striking a vertical wall would subsequently have velocity $\vec{v}_{new} = -v_x\hat{\imath} + v_y\hat{\jmath}$.

Reflections off of surface that are not purely horizontal or vertical require first decomposing the velocity into parallel and perpendicular components, and then reflecting the correct component in this new basis.

C. Velocity from Speed and Direction

Alternately, the velocity vector can be decomposed into its magnitude v (the speed) and a direction angle θ. If we measure θ clockwise, with zero corresponding to the positive $\hat{\imath}$ direction, then we can relate these values to the Cartesian decomposition via

$$v_x = v\cos(\theta)$$
$$v_y = v\sin(\theta)$$

or

$$v = \sqrt{v_x^2 + v_y^2}$$
$$\theta = \text{atan2}(v_y/v_x)$$

(Here the operator `atan2` computes the arctangent, but correctly accounts for the quadrant in which the vector $\vec{v}$ lies).

From these two relations, we can uniquely identify or generate a velocity vector either by choosing its horizontal and vertical components v_x and v_y, or by choosing a speed v and a direction θ. The velocity may be transformed between these two representations as needed to perform calculations.

III. Periodic Reflection in a Rectangular Area

As a particle moves in a rectangular area, it is possible that it's horizontal position and velocity will repeat periodically. For this to happen, the particle must 1) be in the same position x, and 2) have the same (signed) component of velocity v_x. When the particle is experiencing only elastic collisions, it can only have the same component of velocity after striking *both* side walls (negating its velocity each time); for this reason, a particle must cross the width of the container exactly twice in order to have the same position/velocity. The time required to do this will be

$$t_x = \frac{2w}{v_x}$$

where w is the width of the area.

By a similar argument, the particle will periodically repeat its vertical position and velocity in time

$$t_y = \frac{2h}{v_y}$$

where h is the height of the area. If $t_x = t_y$, then the particle will repeat its position (both x and y) and velocity (both v_x and v_y) periodically, and will follow a "diamond" trajectory in which it strikes each of the four walls of the area exactly once per cycle.

More generally, if there exist integers a and b such that

$$a\frac{2w}{v_x} = b\frac{2h}{v_y}$$

then the particle will once again follow a periodic trajectory. Over a single cycle, this particle will strike each vertical edge of the area a times, and each horizontal edge of the area b times. One can further argue that the distance between adjacent strikes is fixed.

If it is necessary that a particle strike a particular subsection of a particular surface at some point (e.g., a door in a wall), then it is necessary to ensure that the distance between strikes h/b is smaller than the door width d to guarantee that the ball will strike the door at least once per cycle.