Lecture 7

- **Read:** Chapter 3.0-3.7

Continuous Random Variables

- Probability Density Function
- Cumulative Distribution Function
- Expected Values
- Some Common Continuous Random Variables
  - Uniform, Exponential
- Gaussian Random Variables
- Mixed Random Variables
  - Delta function, Unit step function
- Probability Models of Derived Random Variables

Continuous Random Variable

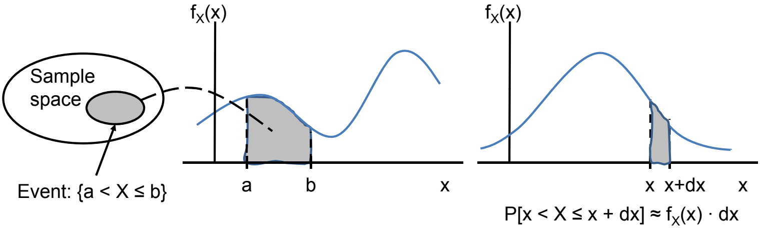
- **Definition: (Continuous Random Variable)** An RV X is said to be continuous if its probability law can be described in terms of a nonnegative function $f_X(x)$ called its probability density function (PDF), such that

$$P[X \in B] = \int_B f_X(u)du$$

  for every subset B of the real line.

- **Example:** The velocity of a randomly selected car measured by an analog speedometer.

Probability Density Function: Properties



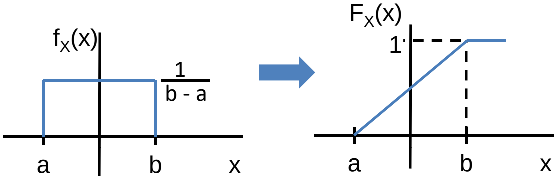
1. $P[a < X \leq b] = \int_a^b f_X(x)dx$
2. $P[x < X \leq x + \delta] = f_X(x) \cdot \delta$
   Interpretation: $f_X(x)$ as the "probability per unit length" at x
3. $P[X = x] = 0$, for all x
4. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
   Note: $f_X(x) \geq 0$, can exceed 1, but must integrate to 1!

Cumulative Distribution Function (CDF)

- **Definition:** The CDF of a RV X is defined as $F_X(x) = P[X \leq x]$. In particular for every x, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

- **Example:**



Properties of CDF

1. $F_X(x)$ is monotonically increasing, i.e., if $x \leq y$, then $F_X(x) \leq F_X(y)$.
2. $F_X(x)$ tends to 0 as $x \to -\infty$ and tends to 1 as $x \to \infty$.
3. $F_X(x)$ is a continuously differentiable function if X is continuous.
4. If X is continuous, the PDF and CDF are related as follows:

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

5. $P[a < X \leq b] = F_X(b) - F_X(a) = \int_a^b f_X(x)dx$

Expected Value

- **Definition: (Expected Value)** The expected value of a continuous random variable X is
$$\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Properties of expected value
$$E[X - \mu_X] = 0$$
$$E[1] = 1$$
$$E[aX + b] = aE[X] + b$$

- Expected value of $g(X)$
$$E[g(X)] = E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Variance

- **Definition: (Variance)**

$$\sigma_X^2 = Var[X] = E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx$$

- Some useful properties

1. $Var[X] = E[X^2] - (E[X])^2$
2. $Var[aX] = a^2 Var[X]$
3. $Var[X + a] = Var[X]$
4. If X always takes the value a, then $Var[X] = 0$.

Some Common Continuous Distributions

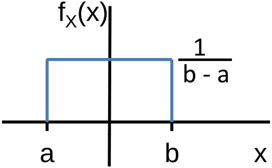
Some Common Continuous Distributions

Uniform Distribution

- **Definition: (Uniform random variable)** X is a uniform random variable if the PDF of X is

$$f_X(x) = \begin{cases} 1/(b-a) & , a \le x < b \\ 0 & , \text{otherwise} \end{cases}$$

where the two parameters are $b > a$.



X ~ uniform([a,b])

- Expected value: $\mu_X = (a+b)/2$
- Variance: $\sigma_X^2 = E[X^2] - \mu_X^2 = (b-a)^2/12$
- **Example:** Find the mean and variance of $Z = 3X + 10$.

Exponential Distribution

- **Definition: (Exponential Distribution)** X is an exponential RV with parameter $\lambda > 0$ iff $f_X(x) = \lambda e^{-\lambda x}$.
  - $P[X \geq x] = e^{-\lambda x}$
- Good model for the the amount of time until a part breaks, e.g., light bulb burns out, or an accident occurs.
  - The larger $\lambda > 0$ is, the sooner it breaks, i.e., has a higher failure rate.

Normal (Gaussian) Distribution

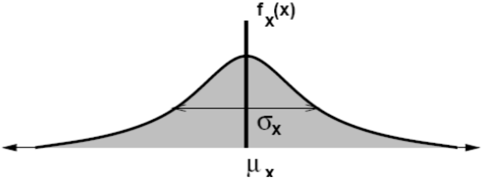
- **Definition:** X is a normally distributed RV with mean μ_X and variance σ_X^2 if it has

$$\text{PDF: } f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-(x-\mu_X)^2/2\sigma_X^2}, \text{ for } -\infty < x < \infty$$

or

$$\text{CDF: } F_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{x} e^{-(x-\mu_X)^2/2\sigma_X^2}$$

Note: $\sigma_X > 0$ and $-\infty < \mu_X < \infty$.



$X \sim N(\mu_X, \sigma_X^2)$

- We say Z is a standard normal RV if $Z \sim N(0, 1)$.

Standard Gaussian

- **Definition:** A standard Gaussian RV $Z \sim N(0,1)$ has PDF

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \text{ for } -\infty < z < \infty$$

- The CDF of a standard Gaussian is usually denoted by

$$\Phi(z) = F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du$$

Linear Transformations

Fact: Linear transformation of Gaussian RV is another Gaussian RV.

1. Suppose $X \sim N(\mu_X, \sigma_X^2)$ and $Y = \frac{X - \mu_X}{\sigma_X}$ (renormalized RV). Then,

$$Y \sim N(0,1) \Rightarrow \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \text{ for } -\infty < y < \infty$$

2. Alternately, suppose $Z \sim N(0,1)$ and let $Y = aZ + b$. Then,

$$Y \sim N(b, a^2)$$

Linear Transformations: Proof of 1

$$
\begin{aligned}
F_Y(y) = P[Y \le y] = P\left[\frac{X-\mu}{\sigma} \le y\right] &\\
= P[X - \mu \le y\sigma] &\\
= P[X \le \mu + y\sigma] &\\
= F_X[\mu + y\sigma] &\\
= \int_{-\infty}^{\mu+y\sigma} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(u-\mu)^2/2\sigma^2} du &
\end{aligned}
$$

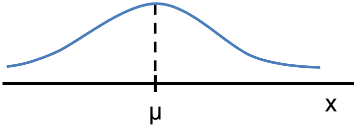
change of variables:
$$
\begin{aligned}
v &= \frac{u-\mu}{\sigma}\\
u &= \mu + \sigma v\\
du &= \sigma dv
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}\sigma} e^{-v^2/2} \sigma dv\\
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv
\end{aligned}
$$

Linear Transformations: Intuition

Intuition: You can think of Gaussian RV $X \sim N(\mu, \sigma^2)$ as

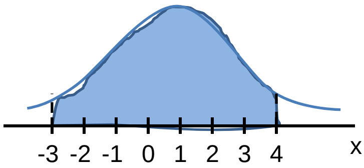
$$X = \underbrace{\mu}_{\text{a constant: the mean}} + Z \underbrace{\sigma}_{\text{fluctuation}}, \text{ where } Z \sim N(0,1)$$



Using Fact and Tables for CDF: Example

- Suppose $X \sim N(1, 16)$. Find $P[-3 < X < 4]$.

. . . .



Using Fact and Tables for CDF: Example (cont.)

- Suppose $X \sim N(1, 16)$. Find $P[-3 < X < 4]$.

. . . .

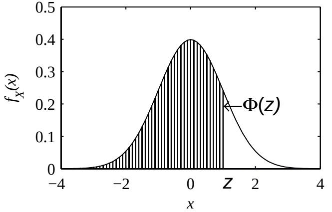
- We need to compute $P[-3 < X < 4] = F_X(4) - F_X(-3)$.
- Instead, using previous fact: $X \sim 4Z + 1$, where $Z \sim N(0, 1)$.
- $P[-3 < X < 4] = P[-3 < 4Z + 1 < 4]$ because X has the same distribution as $1 + 4Z$. So,

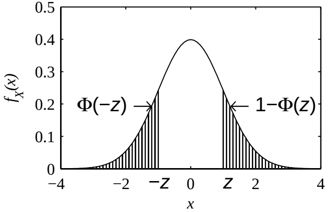
$$\begin{aligned} P[-3 < 4Z + 1 < 4] = P\left[-1 < Z < \frac{3}{4}\right] &= F_Z\left(\frac{3}{4}\right) - F_Z(-1) \\ &= \Phi\left(\frac{3}{4}\right) - \Phi(-1) \end{aligned}$$

Using Fact and Tables for CDF: Example (cont.)

- We look up these values in tables for $\Phi(z)$ and $Q(z) = P[Z > z] = 1 - \Phi(z)$ (Tables 3.1 and 3.2 on p. 123 and p. 124 of our textbook)
- Note that by symmetry: $\Phi(-z) = 1 - \Phi(z)$, so we need to only tabulate positive values

Using Fact and Tables for CDF: Example (cont.)





- How do we deal with negative values?

$$\Phi(-1) = 1 - \Phi(1)$$

$$P[-3 < X < 4] = \Phi\left(\frac{3}{4}\right) - 1 + \Phi(1)$$

- In general, if $X \sim N(\mu, \sigma^2)$, then

$$F_X(x) = P[X \le x] = P[\mu + \sigma Z \le x] = P\left[Z \le \frac{x-\mu}{\sigma}\right]$$
$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Mixed RVs

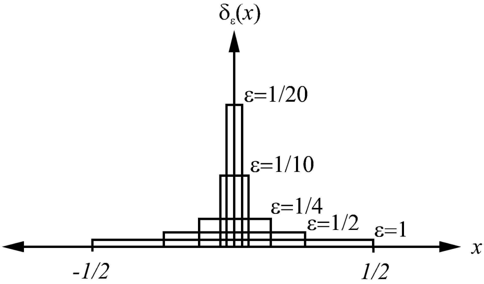
Mixed RVs

Unit Impulse/Delta Function

- **Definition:** The unit impulse or delta function, δ, is defined as

$$\delta(x) = \lim_{\epsilon \to 0} d_\epsilon(x)$$

$$d_\epsilon(x) = \begin{cases} +\frac{1}{\epsilon} & , -\frac{\epsilon}{2} \leq x \leq \frac{\epsilon}{2} \\ 0 & , \text{otherwise} \end{cases}$$

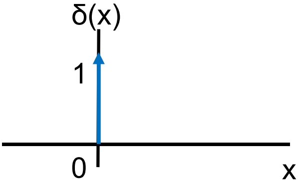


- As $\epsilon \to 0$, $d_\epsilon(x)$ approaches the delta function $\delta(x)$.
- For each ϵ, the area under the curve of $d_\epsilon(x)$ equals 1.

Unit Impulse/Delta Function Properties

1. $\int_{-\infty}^{+\infty} \delta(x)dx = 1$
2. For any continuous function g,

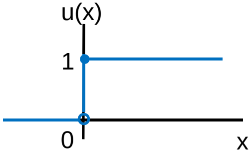
$$\int_{-\infty}^{+\infty} g(x)\delta(x - x_0)dx = g(x_0) \qquad \text{"sifting property"}$$



Unit Step Function

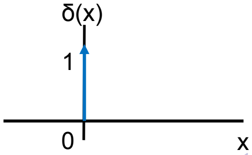
- **Definition:** The unit step function, $u(x)$,

$$u(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$



- **Properties:**
  1. $\int_{-\infty}^{x} \delta(u)du = u(x)$

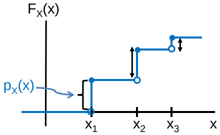
     Equivalently, we think of $\frac{\partial u(x)}{\partial x} = \delta(x)$



PDFs for Discrete Random Variables

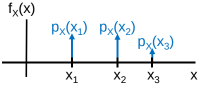
- Consider a discrete RV X with P.M.F. $p_x(x)$, $x \in S_X$.
- We can write the CDF of X as

$$F_X(x) = \sum_{x_i \in S_x} p_X(x_i) u(x - x_i)$$



- Can we define the P.D.F. of a discrete RV?

$$f_X(x) = \frac{\partial}{\partial x} F_X(x) = \sum_{x_i \in S_x} p_X(x_i)\delta(x - x_i)$$



PDFs for Discrete Random Variables (cont.)

- Using this notation, we can compute $E[X]$ as follows:

$$
\begin{aligned}
E[X] = \sum_{x_i \in S_x} x_i p_X(x_i) &= \int_{-\infty}^{+\infty} x f_X(x) dx \\
&= \int_{-\infty}^{+\infty} x \left(\sum_{x_i \in S_x} p_X(x_i) \delta(x - x_i) \right) dx \\
&= \sum_{x_i \in S_x} \left(\int_{-\infty}^{+\infty} x p_X(x_i) \delta(x - x_i) dx \right) \\
&= \sum_{x_i \in S_x} x_i p_X(x_i)
\end{aligned}
$$

Mixed RVs

- **Definition: (Mixed random variable)** X is a mixed random variable if and only if $f_X(x)$ contains both impulses and nonzero finite values.
- CDF $F_X(x)$ is piecewise continuous but has jumps at $x_1, x_2, \ldots$
- Jump at x_i is $P[X = x_i]$
- PDF has impulses at x_i weighted by $P[X = x_i]$

Mixed Random Variables: Example 1

- $X \sim \text{uniform}\{1,2,3\}$
- Find the CDF and PMF of X.

. . . .

- PMF: $p_X(1) = p_X(2) = p_X(3) = 1/3$
- CDF: $F_X(x) = \frac{1}{3}u(x-1) + \frac{1}{3}u(x-2) + \frac{1}{3}u(x-3)$
- PDF: $f_X(x) = \frac{1}{3}\delta(x-1) + \frac{1}{3}\delta(x-2) + \frac{1}{3}\delta(x-3)$

Mixed Random Variables: Example 2

- $W =$ time you wait at the ATM
  and

$$W = \begin{cases} 0 & \text{, with probability } p \text{ (no line)} \\ X & \text{, with probability } (1-p) \end{cases}$$

  and let $X \sim exp(a)$.

$$f_X(x) = \begin{cases} ae^{-ax} & , x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

- What is the CDF of W?

. . . .

- $F_W(w)$:
$$\begin{aligned} F_W(w) &= 0, & w &< 0 \\ F_W(w) &= p, & w &= 0 \\ F_W(w) &= p + (1-p)F_X(w), & w &> 0 \end{aligned}$$

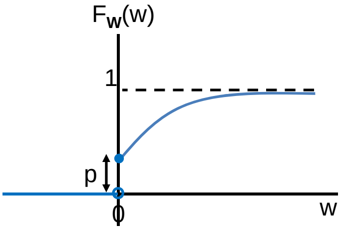
Mixed Random Variables: Example 2 (cont.)

For $w \geq 0$:

$$F_W(w) = P[W \leq w] = P[W = 0] + P[0 < W \leq w]$$
$$= p + (1-p)\underbrace{P[0 < X \leq w]}_{F_x(w)}$$

$$F_X(x) = \begin{cases} 1 - e^{-ax} & , x \geq 0 \\ 0 & , 0 \text{ otherwise} \end{cases}$$

$$F_W(w) = p + (1-p)(1 - e^{-aw}), w > 0$$



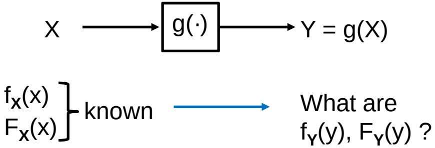
Mixed Random Variables: Example 2 (cont.)

- What is the PDF of W?

. . . .

$$f_W(w) = \begin{cases} 0 & , w < 0 \\ p\delta(w) + (1-p)ae^{-aw} & , w \geq 0 \end{cases}$$

Probability Models for Derived RVs



- **Recall:** If all we need is $E[Y]$, we do not need to compute $f_Y(y)$. Indeed,

$$E[Y] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

$$(E[Y] = \int_{-\infty}^{+\infty} y f_Y(y) dy)$$

Probability Models for Derived RVs: Example

- Let $Y = \alpha X + \beta \qquad (\beta > 0)$

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P[\alpha X + \beta \leq y] \\
&= P[\alpha X \leq y - \beta] \\
&= P\left[X \leq \frac{y-\beta}{\alpha}\right] \\
&= F_X\left(\frac{y-\beta}{\alpha}\right) \\
f_Y(y) &= \frac{\partial}{\partial y} F_Y(y) \\
&= \frac{1}{\alpha} f_X\left(\frac{y-\beta}{\alpha}\right)
\end{aligned}
$$

Probability Models for Derived RVs: 3-Step Procedure

1. Find CDF of Y ($F_Y(y) = P[Y \leq y] = P[g(X) \leq y]$) and express it in terms of $F_X(x)$.
2. Find $f_Y(y) = \frac{\partial}{\partial y} F_Y(y)$.
3. Determine the range of Y, S_Y.

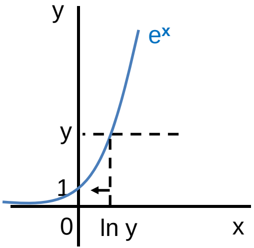
Probability Models for Derived RVs: 3-Step Procedure: Example 1

- Let $Y = e^X$.
- Find $f_Y(y)$ in terms of $f_X(x)$.

. . . .

$$F_Y(y) = P[Y \leq y] = P\left[e^X \leq y\right] = P[X \leq \ln y] = F_X(\ln y)$$

$$f_Y(y) = \frac{\partial}{\partial y} F_X(\ln y) = \frac{1}{y} f_X(\ln y)$$



Probability Models for Derived RVs: 3-Step Procedure: Example 1 (cont.)

- Let $Y = e^X$.
- Find $f_Y(y)$ in terms of $f_X(x)$.

. . . .

- In particular, suppose $X \sim N(\mu, \sigma^2)$ and $Y = e^X$. Then,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$f_Y(y) = \frac{1}{y} f_X(\ln y) = \begin{cases} \frac{1}{y}\frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(\ln y-\mu)^2}{2\sigma^2}\right) & , y > 0 \\ 0 & , \text{otherwise} \end{cases}$$

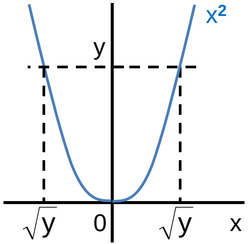
- **Note:** This is the lognormal PDF.

Probability Models for Derived RVs: 3-Step Procedure: Example 2

- Let $Y = X^2$.
- Find $f_Y(y)$ in terms of $f_X(x)$.

. . . .

$$
\begin{aligned}
F_Y(y) &= P[Y \le y] = P[X^2 \le y] \\
&= P[-\sqrt{y} \le X \le \sqrt{y}] \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \\
f_Y(y) &= \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})
\end{aligned}
$$



Probability Models for Derived RVs: 3-Step Procedure: Example 2 (cont.)

- Let $Y = X^2$.
- Find $f_Y(y)$ in terms of $f_X(x)$.

. . . .

In particular, let $X \sim$ uniform[-1,1] and $Y = X^2$. Then,

$$f_X(x) = \begin{cases} \frac{1}{2} & , -1 \leq x \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & , 0 < y \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

Application: Generating RVs on a Computer: Setup

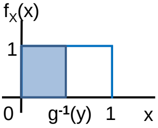
- Suppose your computer can generate $X \sim$ uniform[0,1] RVs (e.g., do a random() call).
- How do we generate some other random variable, say Y, with a given CDF, say $F(\cdot)$?

Application: Generating RVs on a Computer: Approach

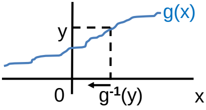
$X \sim \text{uniform}([0,1]) \longrightarrow \boxed{g(\cdot) = ?} \longrightarrow Y$ with $F_Y(y) = F(y)$

Suppose that g() is increasing.

$F_Y(y) = P[Y \leq y] = P[g(X) \leq y] = P[X \leq g^{-1}(y)]$



$P[X \leq g^{-1}(y)] = g^{-1}(y)$ when $0 \leq g^{-1}(y) \leq 1$



$y = g(x)$

$g^{-1}(y) = g^{-1}(g(x)) = x$

- Our goal is to make $F_Y(y) = g^{-1}(y) = \underbrace{F_Y(y)}_{\text{prespecified CDF}}$.
- Thus, $g^{-1}(y) = F(y)$
  and $\underbrace{g(g^{-1}(y))}_{y=g(F(y))} = g(F(y))$
- Thus, $g = F^{-1}$, the inverse of the specified CDF.

Application: Generating RVs on a Computer: Example

- How do we generate exponential RVs based on uniform RVs?

. . . .

- Recall Y is exponential(a) if

$$F_Y(y) = F(y) = \begin{cases} 1 - e^{-ay} & , y \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

- Since $g = F^{-1}$, if $x = 1 - e^{-ay}$,

$$x - 1 = -e^{-ay}$$

$$1 - x = e^{-ay}$$

$$\ln(1 - x) = -ay$$

$$\frac{\ln(1 - x)}{-a} = y$$

- So, $g(x) = \frac{\ln(1-x)}{-a}$.
- If $X \sim$ uniform[0,1], then $Y = \frac{\ln(1-X)}{-a} \sim$ exp(a).
- **Note:** $Y = \frac{\ln(X)}{-a}$ also works! (because if X is uniform on [0,1], then so is 1 - X)

Reminder: "Functions of Discrete RVs"

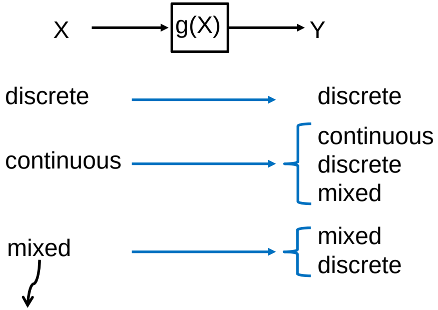
- Suppose X is a discrete RV with range S_X and PMF $p_X(x)$.
- Let $Y = g(X)$.
- Then, Y is also discrete with $S_Y = \{g(x) | x \in S_X\}$ and

$$p_Y(y) = \sum_{\substack{x:g(x)=y \\ x \in S_X}} p_X(x)$$

- **Example:** Suppose $X \sim$ uniform{-1,0,1,2} and $Y = X^2$.
  Then, $S_Y = \{0,1,4\}$ and

$$p_Y(0) = p_Y(4) = 1/4$$
$$p_Y(1) = 1/2$$

Summary: Possibilities for Derived Distributions



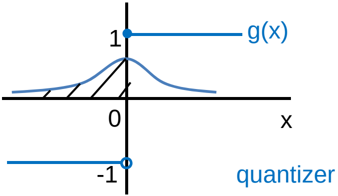
There is at least one value assumed with positive probability; cannot be continuous

Getting a Discrete RV from a Continuous RV

- **Example:** Let $X \sim N(0, 1)$ and let

$$g(x) = \begin{cases} 1 & , x \geq 0 \\ -1 & , x < 0 \end{cases}$$

Let $Y = g(X)$. What is the PDF/PMF of Y?



Note $S_Y = \{-1, 1\}$
and $P[Y = -1] = P[X < 0] = 1/2$
$P[Y = 1] = 1/2$

- **Remark:** In general, functions $g(\cdot)$ which have flat regions may lead to discrete/mixed RVs.

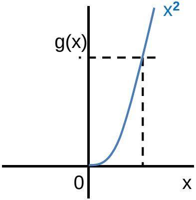
Derived Random Variables: Example

Let

$$g(x) = \begin{cases} x^2 & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

and define $Y = g(X)$. Find $f_Y(y)$, $F_Y(y)$ given $f_X(x)$, $F_X(x)$.

. . . .



$$\text{Find } F_Y(y) = \begin{cases} 0 & , \text{y} < 0 \\ F_X(0) & , \text{y} = 0 \\ ??? & , \text{y} > 0 \end{cases}$$

$$F_Y(0) = P[Y \leq 0] = P[Y = 0] = P[X \leq 0] = F_X(0)$$

$$F_Y(y) = P[Y \leq y] = P[g(X) \leq y]$$

$$= P[X \leq \sqrt{y}] = F_X(\sqrt{y})$$

Derived Random Variables: Example (cont.)

Let

$$g(x) = \begin{cases} x^2 & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

and define $Y = g(X)$. Find $f_Y(y)$, $F_Y(y)$ given $f_X(x)$, $F_X(x)$.

. . . .

- Suppose $X \sim N(0, 1)$. Then, $F_X(0) = 1/2$.

$$\text{Then, } F_Y(y) = \begin{cases} 0 & , y < 0 \\ 1/2 & , y = 0 \\ F_X(\sqrt{y}) & , y > 0 \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{1}{2}\delta(y) + \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) & , y \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$\text{where, } f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Summary: Possibilities for Derived Distributions