Lecture 8

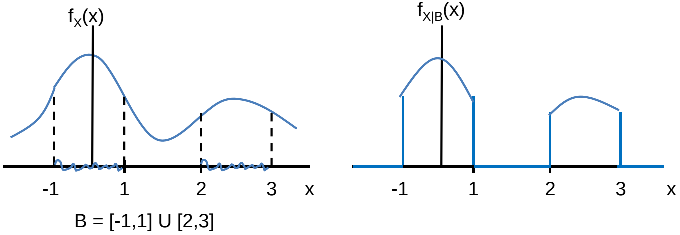
- **Read:** Chapter 3.8, 4.1, 4.4-4.10.
- Continuous Random Variables
  - Conditioning a Continuous Random Variable
- Multiple Continuous Random Variables
  - Joint Cumulative Distribution Function
  - Joint Probability Density Function
  - Marginal Probability Density Function
  - Functions of Two Random Variables
  - Expected Values
  - Conditioning by an Event/Conditioning by a Random Variable
  - Independent Random Variables

Conditioning a Continuous Random Variable

- Suppose that X has PDF $f_X(x)$ and let B be an event (i.e., a subset of $\mathbb{R}$, with $P[B] > 0$).
- **Definition:** The conditional PDF of X given B is given by

$$f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{P[B]} & , x \in B \\ 0 & , \text{otherwise} \end{cases}$$

- **Interpretation:** Having observed B, we know that X must lie in this set, so the new PDF is the same as the old one, but renormalized by $P[B]$.



Conditioning a Continuous Random Variable: Conditional Expectations

$$E[X|B] = \int_{-\infty}^{+\infty} x f_{X|B}(x) dx$$

$$E[g(X)|B] = \int_{-\infty}^{+\infty} g(x) f_{X|B}(x) dx$$

Conditioning a Continuous Random Variable: Example

- Suppose that the holding time (duration) in minutes, T, of a telephone call is known to have an exponential distribution.
- $T \sim \exp(1/3)$ or

$$f_T(t) = \begin{cases} \frac{1}{3} e^{-1/3t} & , \ t \geq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

- Let $B = \{T > 2\}$. Find $f_{T|B}(t)$.

. . . .

$$\begin{aligned} P[B] = \int_2^{+\infty} f_T(t)dt &= 1 - P[T \leq 2] \\ &= 1 - (1 - e^{-2/3}) \\ &= e^{-2/3} \end{aligned}$$

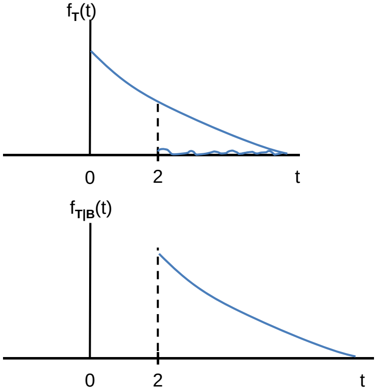
$$f_{T|B}(t) = \begin{cases} \frac{f_T(t)}{P[B]} = \frac{\frac{1}{3} e^{-1/3t}}{e^{-2/3}} = \frac{1}{3} e^{-\frac{1}{3}(t-2)} & , \ t > 2 \\ 0 & , \text{ otherwise} \end{cases}$$

Conditioning a Continuous Random Variable: Example (cont.)

- Let $B = \{T > 2\}$. Find $E[T|B]$.

. . . .

$$
\begin{aligned}
E[T|B] &= \int_{-\infty}^{+\infty} t f_{T|B}(t) dt \\
&= \int_{2}^{+\infty} t \frac{1}{3} e^{-\frac{1}{3}(t-2)} dt \\
&= 5 \text{ minutes}
\end{aligned}
$$

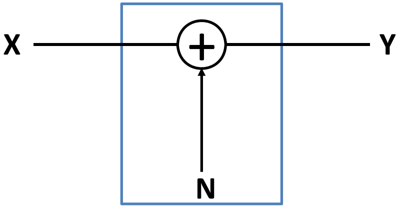


Reminder on integration by parts:

$$
\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx
$$

Multiple Continuous Random Variables

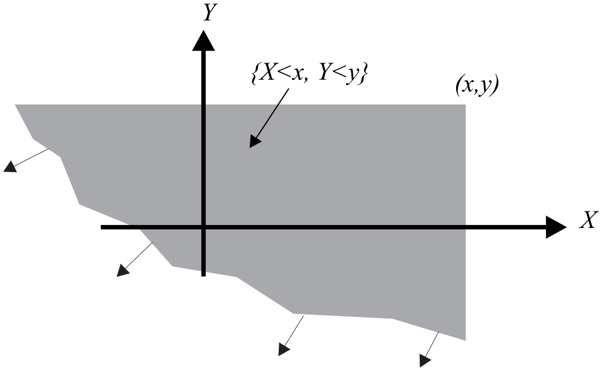
- **Example:** We would like to consider pairs of continuous RVs, e.g., (X, Y). Experiment produces at least two continuous RVs.



Joint CDF

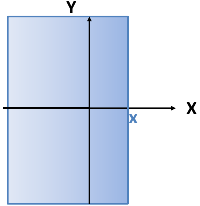
- **Definition: (Joint CDF)** The joint CDF of X and Y is given by

$$F_{X,Y}(x,y) = P[X \leq x, Y \leq y]$$



Multiple Continuous RVs: Joint CDF Properties

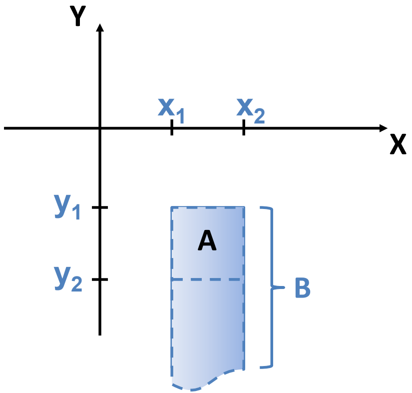
- $0 \leq F_{X,Y}(x, y) \leq 1$
- $$\begin{aligned} F_{X,Y}(x, +\infty) &= P[X \leq x, Y \leq +\infty] \\ &= P[X \leq x] \\ &= F_X(x) \end{aligned}$$



- $F_Y(y) = F_{X,Y}(+\infty, y)$
- $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$
- If $x_1 \geq x$ and $y_1 \geq y$, then
  $$F_{X,Y}(x_1, y_1) \geq F_{X,Y}(x, y).$$

Multiple Continuous RVs: Joint CDF and Rectangles

- We can use the joint CDF to compute the probability associated with rectangles as follows:



- $P[B] = F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_1)$
- $P[A] = P[B] - (F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2))$

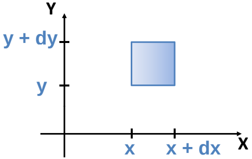
Joint Probability Density Function (PDF)

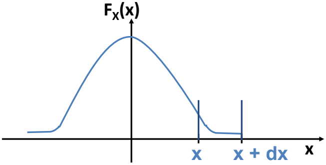
- **Definition: (Joint PDF)** The joint PDF of (X, Y) is $f_{X,Y}(x, y)$ satisfying

$$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)dvdu$$

$$\text{equivalently, } f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$$

- **Interpretation:** $f_{X,Y}$ as the probability per unit area around (x, y). It can exceed 1, but must be such that $f_{X,Y} \geq 0$.
$P[x \leq X \leq x + dx, y \leq Y \leq y + dy] \approx f_{X,Y}(x, y)dxdy$





$P[x \leq X \leq x + dx] \approx f_X(x)dx$

Joint PDF Properties

- $f_{X,Y}(x, y) \geq 0$ (for all (x, y))
- $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x, y) dxdy = 1.$
- For any event $A \subset \mathbb{R}^2$ (i.e., subset of the x-y plane)

$$P[A] = \int_A \int f_{X,Y}(x, y) dxdy$$

Marginal PDF

- **Definition: (Marginal PDF)** Experiment produces continuous RVs X and Y, with joint PDF $f_{X,Y}(x,y)$, marginal PDFs are given by

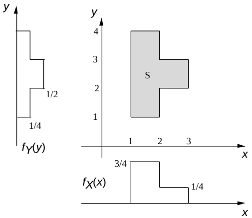
$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy$$
$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx$$

- **Proof:** Write $F_X(x)$ as an integral, take the derivative.

Marginal PDF: Example

- Joint PDF which is uniform on region shown below.
- Find the constant c and marginals.

. . . .



- The area of the set S is equal to 4 and, therefore, $f_{X,Y}(x,y) = c = 1/4$, for $(x,y) \in S$.
- To find the marginal PDF $f_X(x)$ for some particular x, we integrate (with respect to y) the joint PDF over the vertical line corresponding to that x.
- We can compute $f_Y(y)$ similarly.

Marginal PDF: Example (cont.)

- For $1 \leq x \leq 2$, y ranges between 1 and 4:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_1^4 \frac{1}{4}dy = \left.\frac{y}{4}\right|_1^4 = 3/4$$

- For $2 \leq x \leq 3$, y ranges between 2 and 3:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_2^3 \frac{1}{4}dy = \left.\frac{y}{4}\right|_2^3 = 1/4$$

Marginal PDF: Example (cont.)

- For $1 \leq y \leq 2$, x ranges between 1 and 2:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_1^2 \frac{1}{4}dx = \frac{x}{4}\Big|_1^2 = 1/4$$

- For $2 \leq y \leq 3$, x ranges between 1 and 3:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_1^3 \frac{1}{4}dx = \frac{x}{4}\Big|_1^3 = 2/4 = 1/2$$

- For $3 \leq y \leq 4$, x ranges between 1 and 2:

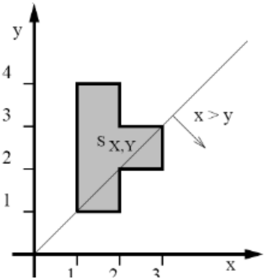
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_1^2 \frac{1}{4}dx = \frac{x}{4}\Big|_1^2 = 1/4$$

Marginal PDF: Example

- Joint PDF which is uniform on region shown on previous slide.
- Find $P[X \geq Y]$.

. . . .

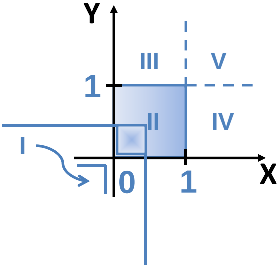
- Let $B = \{(x, y) | x \geq y\}$



$$P[X \geq Y] = P[(X, Y) \in B] = \int_B \int f_{X,Y}(x, y) dxdy$$
$$= \frac{1}{4} Area(B \cap S_{X,Y}) = \frac{1}{4}$$

Marginal PDF Example: Uniform Joint PDF

- Suppose (X, Y) is a randomly selected point out of the unit square.

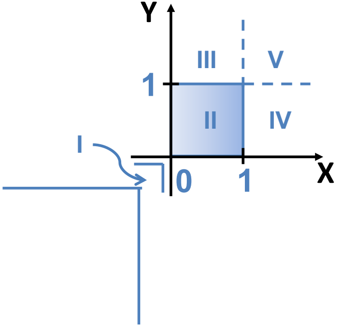


Then, $f_{X,Y}(x,y) = \begin{cases} 1 & , 0 \leq x, y \leq 1 \\ 0 & , \text{otherwise} \end{cases}$

I: $F_{X,Y}(x,y) = 0$
II: $F_{X,Y}(x,y) = x \cdot y \quad (x,y)$ are in region II: $0 \leq x \leq 1, 0 \leq y \leq 1$
III: $F_{X,Y}(x,y) = x$
IV: $F_{X,Y}(x,y) = y$
V: $F_{X,Y}(x,y) = 1$

Marginal PDF Example: Uniform Joint PDF, Region 1

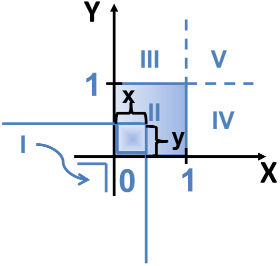
- Suppose (X, Y) lies in Region 1.



I: $F_{X,Y}(x, y) = 0$

Marginal PDF Example: Uniform Joint PDF, Region 2

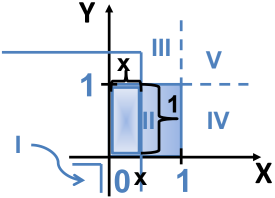
- Suppose (X, Y) lies in Region 2.



II: $F_{X,Y}(x, y) = x \cdot y \quad (x, y)$ are in region II: $0 \leq x \leq 1, 0 \leq y \leq 1$

Marginal PDF Example: Uniform Joint PDF, Region 3

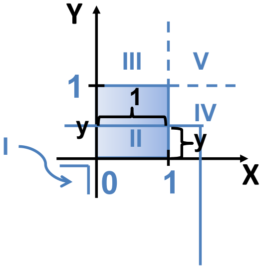
- Suppose (X, Y) lies in Region 3.



III: $F_{X,Y}(x, y) = x$

Marginal PDF Example: Uniform Joint PDF, Region 4

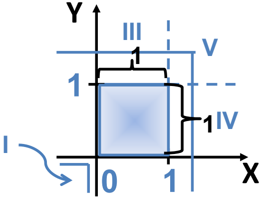
- Suppose (X, Y) lies in Region 4.



IV: $F_{X,Y}(x, y) = y$

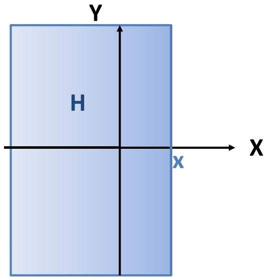
Marginal PDF Example: Uniform Joint PDF, Region 5

- Suppose (X, Y) lies in Region 5.



V: $F_{X,Y}(x, y) = 1$

Marginal CDF



$$
\begin{aligned}
F_X(x) &= P[X \leq x] \\
&= P[X \leq x, Y \leq \infty] \\
&= \int_H \int f_{X,Y}(\alpha, \beta) d\alpha d\beta \\
&= \int_{\alpha=-\infty}^{x} \int_{\beta=-\infty}^{\infty} f_{X,Y}(\alpha, \beta) d\beta d\alpha
\end{aligned}
$$

$$
f_X(x) = \int_{\beta=-\infty}^{\infty} f_{X,Y}(\alpha, \beta) d\beta
$$

Independent RVs

- X and Y are independent if $\forall x, y,\ F_{X,Y}(x,y) = F_X(x)F_Y(y)$ (equivalently, if $\forall x, y,\ f_{X,Y}(x,y) = f_X(x)f_Y(y)$).
- **Example:** Let X and Y be uniform on $[0,1] \times [0,1]$

$$f_{X,Y}(x,y) = \begin{cases} 1 & , 0 \le x \le 1, 0 \le y \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \begin{cases} 1 & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \begin{cases} 1 & , 0 \le y \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

Functions of Two Random Variables (I)

- **Example:** Receiver outputs X and Y from two antennas.

$$
\begin{aligned}
W_1 &= max(X, Y) \\
W_2 &= X + Y \\
W_3 &= aX + bY
\end{aligned}
$$

- What is the PDF of W_i?

. . . .

- Find the CDF of W_i first.

$$
\begin{aligned}
F_{W_1}(w_1) &= P[W_1 \leq w_1] \\
&= P[max(X, Y) \leq w_1] \\
&= P[X \leq w_1, Y \leq w_1] \\
&= F_{X,Y}(w_1, w_1)
\end{aligned}
$$

Functions of Two Random Variables (II)

- If X and Y were independent, we could write $F_X(w_1)F_Y(w_1)$ instead of $F_{X,Y}(w_1, w_1)$:

$$
\begin{aligned}
F_{W_1}(w_1) &= F_X(w_1)F_Y(w_1) \\
f_{W_1}(w_1) &= f_X(w_1)F_Y(w_1) + F_X(w_1)f_Y(w_1)
\end{aligned}
$$

[the derivative of the product of two functions]

- If X and Y are not independent

$$
f_{W_1}(w_1) = \left.\frac{\partial F_{X,Y}(w_1, w_1)}{\partial x}\right|_{(w_1,w_1)} + \left.\frac{\partial F_{X,Y}(w_1, w_1)}{\partial y}\right|_{(w_1,w_1)}
$$

Functions of Two Random Variables (III)

- If X and Y were independent

$$
\begin{aligned}
F_{W_2}(w_2) &= P[W_2 \le w_2] \\
&= P[X + Y \le w_2] \\
&= \int_A \int f_{X,Y}(x,y)\,dx\,dy \\
&= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{w_2-y} f_{X,Y}(x,y)\,dx\,dy \\
&= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{w_2-y} f_X(x) f_Y(y)\,dx\,dy \\
&= \int_{y=-\infty}^{\infty} f_Y(y) F_X(w_2 - y)\,dy \\
f_{W_2}(w_2) &= \int_{y=-\infty}^{\infty} f_Y(y) f_X(w_2 - y)\,dy \\
f_X(h(w_2)) &= f_X(w_2 - y) \\
h(w_2) &= w_2 - y \\
f_{W_2} &= f_X * f_Y \text{ (Convolution)}
\end{aligned}
$$

Functions of Two Random Variables

- **Theorem:** For continuous random variables X and Y, the CDF of $W = g(X, Y)$ is

$$F_W(w) = P[W \leq w] = P[g(X, Y) \leq w] = \iint_{g(x,y) \leq w} f_{X,Y}(x, y) dxdy$$

$W = g(X, Y)$ Examples

- $W_1 = X + Y$
- $W_2 = max(X, Y)$
- $W_3 = XY$
- $W_4 = X/Y$

Finding the Expected Value $E[W]$

- We want to find the expectation of $W = g(X, Y)$. $(E[W] = E[g(X, Y)])$
- **Method 1:** Find the PDF of W, $f_W(w)$, then calculate

$$E[W] = \int_{-\infty}^{\infty} w f_W(w) dw$$

- **Method 2:** We can also compute the expected value of $W = g(X, Y)$ without going through the complicated process of deriving a probability model for W

$$E[W] = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dxdy$$

Expectation of Sums

- Expected value of $g(X, Y) = g_1(X, Y) + ... + g_n(X, Y)$ is

$$E[g(X, Y)] = E[g_1(X, Y)] + ... + E[g_n(X, Y)]$$

- Sums:

$$\begin{aligned} E[X + Y] &= E[X] + E[Y] \\ E[aX + bY + c] &= aE[X] + bE[Y] + c \text{ (Linear operator)} \\ Var[X + Y] &= Var[X] + Var[Y] + 2Cov[X, Y] \end{aligned}$$

- Covariance:

$$Cov[X, Y] = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X \mu_Y$$

Correlation Coefficient

- **Definition:** Correlation coefficient of two random variables X and Y is

$$\rho_{X,Y} = \frac{Cov[X, Y]}{\sqrt{Var[X]Var[Y]}}$$

- **Theorem:** $-1 \leq \rho_{X,Y} \leq 1$
- Same as for discrete random variables

Two Types of Conditioning

- By the occurrence of an event B of nonzero probability
  - Typically, this event B will be described in terms of a relationship between X and Y such as $X < Y$ or $X + Y \leq 100$.
  - Conditioning $f_{X,Y}(x, y)$ by an event is essentially the same as conditioning $f_X(x)$ by an event.
- By the observation that one of the random variables, say X, takes on the value x

Conditional Joint PDF

- When we learn that an event B occurs, we need to adjust our probability model for X and Y to reflect this knowledge.
- This modified probability model is the conditional joint PDF $f_{X,Y|B}(x,y)$.
- Given an event B with $P[B] > 0$, the conditional joint PDF of X and Y is

$$f_{X,Y|B}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P[B]} & ,\ (x,y) \in B \\ 0 & ,\ \text{otherwise} \end{cases}$$

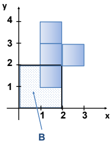
- Remove samples that do not belong to B and normalize.

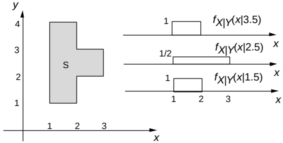
Conditional PDF of Y Given $X = x$

- For each fixed x, we consider the joint PDF along slice $X = x$ and renormalize (normalize it so that it integrates to 1).
- We can interpret the conditional PDF $f_{Y|X}(y|x)$ as:
$$f_{Y|X}(y|x)dy = P[y \leq Y \leq y + dy | x \leq X \leq x + dx]$$
- Using Bayes' Theorem,
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$
- $f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y)$
- When thinking about the conditional PDF, it is best to view x as a fixed number and consider $f_{Y|X}(y|x)$ as a function of the single variable y.
- As a function of y, the conditional PDF $f_{Y|X}(y|x)$ has the same shape as the joint PDF $f_{X,Y}(x,y)$ because the normalizing factor $f_X(x)$ does not depend on x.
- Note that the normalization ensures that $\int_{-\infty}^{\infty} f_{Y|X}(y|x)dy = 1$, so for any fixed y, $f_{Y|X}(y|x)$ is a legitimate PDF.

Conditional PDFs: Example

- Let X and Y have a joint PDF which is uniform on the set S and $B = [0,2] \times [0,2]$.
- What do $f_{X,Y|B}$, $f_{X|Y}$, and $f_{Y|X}$ look like?





$$f_{X,Y|B}(x,y) = \frac{f_{X,Y}(x,y)}{P[B]} = \frac{1/4}{1/4} = 1$$

$$f_{X|Y}(x|3.5) = \frac{f_{X,Y}(x,3.5)}{f_Y(3.5)} = \frac{1/4}{1/4} = 1$$

$$f_{X|Y}(x|2.5) = \frac{f_{X,Y}(x,2.5)}{f_Y(2.5)} = \frac{1/4}{1/2} = 1/2$$

$$f_{X|Y}(x|1.5) = \frac{f_{X,Y}(x,1.5)}{f_Y(1.5)} = \frac{1/4}{1/4} = 1$$

Conditional Expected Value

- **Definition: (Conditional Expected Value)** If $f_Y(y) > 0$, the conditional expected value of X given $Y = y$ is

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

- **Definition: (Conditional Expected Value of a Function)** For any y such that $f_Y(y) > 0$, the conditional expected value of $g(X, Y)$ given $Y = y$ is

$$E[g(X, Y)|Y = y] = \int_{-\infty}^{\infty} g(x, y) f_{X|Y}(x|y) dx$$

- Special case: conditional variance $Var[X|Y = y]$

$$Var[X|Y = y] = E[(X - E[X|Y = y])^2|Y = y]$$

Expected Value of Conditional Expected Value

- Note that the conditional expected value $E[g(X, Y)|Y = y]$ is a function of the observed value y of random variable Y.
- We can view the conditional expected value as a function of the random variable Y denoted $E[g(X, Y)|Y]$.
- Since $E[g(X, Y)|Y]$ is a function of Y, it is a random variable.
- We calculate the expected value of $E[g(X, Y)|Y]$ just as we would for any function $h(Y)$.
- **Theorem:**

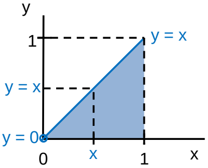
$$E[E[g(X, Y)|Y]] = \int_{-\infty}^{\infty} E[g(X, Y)|Y = y] f_Y(y) dy = E[g(X, Y)]$$

Expected Values: Example

- Let X and Y be random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6y & , 0 \le y \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

- Find the marginal PDF $f_X(x)$.



- For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^x 6ydy = 3x^2$$

- So,

$$f_X(x) = \begin{cases} 3x^2 & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

Expected Values: Example (cont.)

- Let X and Y be random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6y & , 0 \le y \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

- Find the conditional PDF $f_{Y|X}(y|x)$. For what values of x is $f_{Y|X}(y|x)$ defined?

. . . .

- $f_{Y|X}(y|x)$ defined wherever $f_X(x) > 0$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} 2y/x^2 & , 0 \le y \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

Expected Values: Example (cont.)

- Let X and Y be random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6y & , 0 \leq y \leq x \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

- Find the conditional expected value $E[Y|X = x]$.

. . . .

$$E[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy \quad = \int_0^x y \frac{2y}{x^2} dy = \frac{2}{x^2} \left[\frac{y^3}{3} \right]_0^x = \frac{2}{3} x$$

Independent Continuous RVs

- **Definition: (Independence)** Continuous RVs X and Y are independent iff:

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

- If X and Y are independent,

$$f_{X|Y}(x|y) = f_X(x) \quad f_{Y|X}(y|x) = f_Y(y)$$

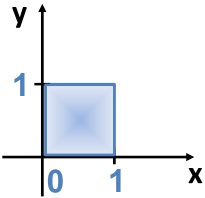
Independence: Example 1

- Are X and Y independent?

$$f_{X,Y}(x,y) = \begin{cases} 4xy & , 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

- Region of nonzero density is rectangular and

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$
$$= \int_0^1 4xydy = 2xy^2\Big|_{y=0}^{y=1}$$
$$= 2x - 0 = 2x$$



Independence: Example 1 (cont.)

- Are X and Y independent?

$$f_{X,Y}(x,y) = \begin{cases} 4xy & , 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

. . . .

- The marginal PDFs of X and Y are

$$f_X(x) = \begin{cases} 2x & , 0 \leq x \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y & , 0 \leq y \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

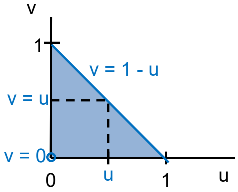
- Is $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all pairs (x,y)? Yes. X and Y are independent.

Independence: Example 2 (cont.)

- Are U and V independent?

$$f_{U,V}(u,v) = \begin{cases} 24uv & , \; u \geq 0, \; v \geq 0, \; u+v \leq 1 \\ 0 & , \; \text{otherwise} \end{cases}$$

. . . .



- Region of nonzero density is triangular and

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)dv \\ &= \int_{v=0}^{v=1-u} 24uvdv = 12uv^2 \Big|_{v=0}^{v=1-u} \\ &= 12(1-u)^2 - 0 = 12(1-u)^2 \end{aligned}$$

Independence: Example 2

- Are U and V independent?

$$f_{U,V}(u,v) = \begin{cases} 24uv & , u \geq 0,\ v \geq 0,\ u+v \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

. . . .

- Region of nonzero density is triangular and

$$f_U(u) = \begin{cases} 12u(1-u)^2 & , 0 \leq u \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$f_V(v) = \begin{cases} 12v(1-v)^2 & , 0 \leq v \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

- Is $f_{U,V}(u,v) = f_U(u)f_V(v)$? No. U and V are not independent!
- Learning the value of U changes our knowledge of V.
- For example, learning that $U = 1/2$ informs us that the event $P[V \leq 1/2] = 1$.

Independence: Example Summary

- In these two examples, we see that the region of nonzero probability plays a crucial role in determining whether random variables are independent.

Properties of Independent Continuous RVs

- **Theorem:** For independent random variables X and Y

$$
\begin{aligned}
E[g(X)h(Y)] &= E[g(X)]E[h(Y)] \\
Cov[X, Y] &= 0 \\
Var[X + Y] &= Var[X] + Var[Y]
\end{aligned}
$$