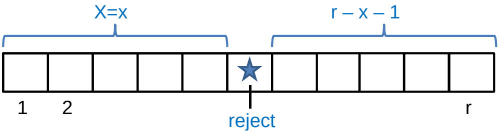
Lecture 13

Review

Number of Rejects



- r experiments, probability of accept is p
- Let $X = \#$ of accepts before first reject
- Let $N = \#$ of rejects
- Find $p_{N,X}(n,x)$

. . . .

$$p_{N,X}(n,x) = \underbrace{(\quad\checkmark\quad)}_{p_X(x)}\underbrace{(\quad\checkmark\quad)}_{p_{N|X}(n|x)}$$

$$= p^x \cdot \left[\binom{r-x-1}{k-1} p^{r-x-1-k+1}(1-p)^k\right]$$

Axioms and Properties of Probability

- Sample space - S
- Events - subsets of S, e.g., $A \subset S$
- Probability measure: P: events $\to$ [0,1]

$$A \mapsto P[A]$$

1. $P[A] \geq 0$
2. $P[S] = 1$
3. If A_1, A_2, ..., A_n are disjoint, then

$$P\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} P[A_i]$$

Conditional Probability

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

- **Independence:** A and B are independent events

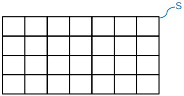
$$\text{if and only if } P[A \cap B] = P[A] \cdot P[B]$$
$$\text{or iff } P[A|B] = P[A]$$
$$\text{or iff } P[B|A] = P[B]$$

- **Note:** Disjoint events are not necessarily independent

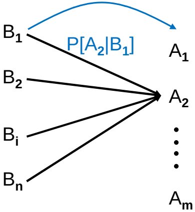
$$P[A \cap A^c] = 0 \stackrel{?}{=} P[A]P[A^c]$$

Total Probability and Bayes' Rule

- Suppose B_1, B_2, ..., B_n are an event space or a partition of S.

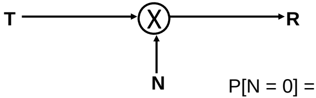


- Then, given
  1. the priors $P[B_i]$, $i = 1, ..., n$
  2. the transition probabilities $P[A_j|B_i]$, $i = 1, ..., n$, $j = 1, ..., m$



- **Theorem of Total Probability:** $P[A_j] = \sum_{i=1}^{n} P[A_j|B_i]P[B_i]$
- **Bayes' Rule:** $P[B_i|A_j] = \frac{P[A_j|B_i]P[B_i]}{P[A_j]}$

Channel Problem



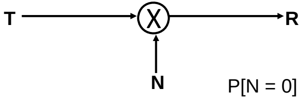
Priors: P[T = 0] = P[T = 1] = 1/2

- A noisy binary communication channel
- $T =$ input or transmitted bit
- $R =$ output or received bit
- Let N be an independent (noise) Bernoulli random variable where

$$P[N = 0] = 0.1 \text{ and } P[N = 1] = 0.9$$

- The received bit is corrupted in a multiplicative fashion, that is, $R = N \times T$.

Channel Problem



P[N = 0] = 0.1 P[N = 1] = 0.9

Priors: P[T = 0] = P[T = 1] = 1/2

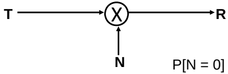
- $\mathrm{P[T{=}0]} = \mathrm{P[T{=}1]} = 1/2$
- **Question:** $\mathrm{P[T{=}0|R{=}1]}$? maximum likelihood detection
  $\mathrm{P[T{=}1|R{=}1]}$?

$$P[T=0|R=1] = \frac{P[R=1|T=0]P[T=0]}{P[R=1]}$$

$$P[R=1|T=0] = P[T \times N = 1|T=0] = 0$$

$$P[T=1|R=1] = \frac{P[R=1|T=1]\overbrace{P[T=1]}^{1/2}}{P[R=1]}$$

Channel Problem (cont.)



P[N = 0] = 0.1 P[N = 1] = 0.9

Priors: P[T = 0] = P[T = 1] = 1/2

- P[T=0]= P[T=1]= 1/2
- **Question:** P[T=0|R=1]? maximum likelihood detection
  P[T=1|R=1]?

. . . .

$$
\begin{aligned}
P[R=1] &= P[R=1, T=1] + P[R=1, T=0] \\
&= P[\{R=1\} \cap \{T=1\}] + P[\{R=1\} \cap \{T=0\}] \\
&= P[R=1 | T=1] P[T=1] + P[R=1 | T=0] P[T=0]
\end{aligned}
$$

$$
\begin{aligned}
P[R=1 | T=1] &= P[T \times N = 1 | T=1] \\
&= P[N=1] = 0.9
\end{aligned}
$$

$$
P[T=1 | R=1] = \frac{0.9 \cdot \frac{1}{2}}{0.9 \cdot \frac{1}{2}} = 1
$$

Random Variables

- $X : S \to \mathbb{R}$
  $\quad s \mapsto X(s)$

- $S_X =$ set of possible values X can take
  $\quad =$ range of X

- If S_X is countable, then X is a discrete RV.

Discrete RVs

- PMF: $p_X(x) = P[X = x] = P[\overbrace{\{s \in S | X(s) = x\}}^{event}]$
- CDF: $F_X(x) = P[X \leq x] = P[\{s \in S | X(s) \leq x\}]$
- Given a set $B \subset \mathbb{R}$

$$P[B] = P[x \in B] = \sum_{x \in B} p_X(x)$$

Discrete RV Examples

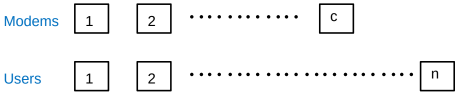
- **Uniform** (equal likelihood of occurrence for each outcome)
- **Bernoulli** (two possibilities: win (with probability p) or lose)
- **Binomial** $p_K(k) = \binom{n}{k} p^k (1-p)^{n-k}$ (probability of k wins out of n plays)
- **Geometric** (how long until the first win)
- **Pascal** (how long until we have seen k wins)
- **Poisson** (count arrivals)

Problem: Design a modem pool

- Campus population n
- At any given time, a user wants to connect to a modem pool with probability p
- How big should the modem pool be to ensure a low probability of a busy tone?

. . . .

- Let $c =$ capacity of modem pool



- Let $U = \#$ of users that want access to the pool

$$P[U > c] \le 0.1\% \Rightarrow \text{Binomial}$$

$$P[U > c] = \sum_{c+1}^{n} \binom{n}{j} p^j (1-p)^{n-j} \le 0.1\%$$

Expectation

$$E[X] = \sum_{x \in S_X} x p_X(x)$$

Function of an RV

$$X \longrightarrow \boxed{g(\cdot)} \longrightarrow Y = g(X)$$

- Two types of problems:
  1. Given $p_X(\mathrm{x})$ and $g()$, find $p_Y(y)$.
  2. Given $p_X(\mathrm{x})$ and $g()$, find $E[Y]$.

Function of an RV: Example

- $X \sim$ geometric(3/4)

$$p_X(x) = \frac{3}{4}\left(1 - \frac{3}{4}\right)^{x-1} \qquad , x = 1, 2, \ldots$$

- Let $Y = 2^X$. Find $p_Y(y)$.

....

- $S_Y = \{2, 4, 8, 16, \ldots\}$

$$p_Y(y) = \begin{cases} p_X(log_2 y) & , y = 2,\ 4,\ \ldots \\ 0 & , \text{otherwise} \end{cases}$$

- Suppose $y \in S_Y = \{2, 4, \ldots\}$

$$\begin{aligned} p_Y(y) &= P[Y = y] \\ &= P[2^X = y] \\ &= P[X = log_2 y] \\ &= p_X(log_2 y) = \frac{3}{4}\left(\frac{1}{4}\right)^{[log_2 y - 1]} \end{aligned}$$

Function of an RV: Example (cont.)

- $X \sim \text{geometric}(3/4)$
$$p_X(x) = \frac{3}{4}\left(1 - \frac{3}{4}\right)^{x-1} \qquad , x = 1, 2, \ldots$$
- Let $Y = 2^X$. Find $E[Y]$.

$$E[Y] = E[2^X] = \sum_{x=1}^{\infty} 2^x p_X(x)$$
$$= \sum_{x=1}^{\infty} 2^x \frac{3}{4}\left(\frac{1}{4}\right)^{x-1}$$
$$= 2 \cdot \frac{3}{4} \sum_{x=1}^{\infty} 2^{x-1}\left(\frac{1}{4}\right)^{x-1}$$
$$= 2 \cdot \frac{3}{4} \sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^{x-1}$$
$$= \frac{6}{4} \sum_{x=0}^{\infty} \left(\frac{1}{2}\right)^{x} = \frac{6}{4} \cdot \frac{1}{1 - 1/2} = \frac{12}{4} = 3$$

Conditional RVs

- Given an event B, $P[B] > 0$,

$$p_{X|B}(x) = \frac{P[\{X = x\} \cap B]}{P[B]} = \begin{cases} \frac{p_X(x)}{P[B]} & , (x, y) \in B \\ 0 & , \text{otherwise} \end{cases}$$

$$E[X|B] = \sum_{x \in S_X} x p_{X|B}(x)$$

Conditional RVs: Example

- Let $U \sim$ uniform on $\{1,2,3,..., 10\}$

$$p_U(u) = \frac{1}{10} \quad , u = 1, 2, 3, ..., 10$$

- Let $A = \{\text{prime numbers in } S_U\} = \{1,2,3,5,7\}$
- Find $F_{U|A}$ and $E[U|A]$.

. . . .

$$p_{U|A}(u) = \frac{P[\{U = u\} \cap A]}{P[A]} = \begin{cases} \frac{1/10}{1/2} = \frac{1}{5} & , \ u = 1, 2, 3, 5, 7 \\ 0 & , \text{ otherwise} \end{cases}$$

$$E[U|A] = \sum_{u \in A} u p_{U|A}(u) = \frac{1}{5}(1 + 2 + 3 + 4 + 5 + 7) = 3.6$$

Multiple Discrete RVs

Joint PMF: $p_{X,Y}(x,y) = P[X = x, Y = y]$

marginal PMF

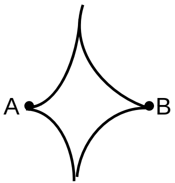
$p_X(x) = \sum_{y \in S_Y} p_{X,Y}(x,y)$

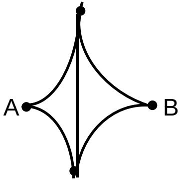
marginal PMF

$p_Y(y) = \sum_{x \in S_X} p_{X,Y}(x,y)$

Reliability

Find the reliability of a diamond network, and of a diamond network with a cross link on transit nodes shown below if the probability that a link is up is p.





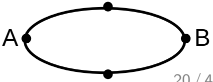
prob. of working link is p

Use conditioning!

prob. can get from
A to B = $1 - (1 - p^2)^2$

one in the middle working:

one in the middle not working:



Expectation

- **Definition:**

$$E[X] = \sum_x x p_X(x)$$

- Interpretations:
    - Center of gravity of PMF
    - Average in large number of repetitions of the experiment

- **Example:** Uniform on $\{0, 1, ..., n\}$

$$E[X] = 0 \times \frac{1}{n+1} + 1 \times \frac{1}{n+1} + ... + n \times \frac{1}{n+1} =$$

Properties of Expectations

- Let X be a R.V. and let $Y = \mathrm{g}(X)$
  - Hard: $E[Y] = \sum_y y p_Y(y)$
  - Easy: $E[Y] = \sum_x g(x) p_X(x)$
- "Second moment": $E[X^2]$
- Caution: In general, $E[g(X)] \neq g(E[X])$
- Variance:

$$Var[X] = E[(X - E[X])^2] = \sum_x (x - E[X])^2 p_X(x)$$

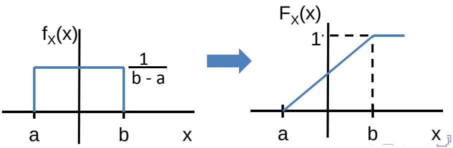
- If α is a constant: $E[\alpha] =$
- $E[\alpha X] =$
- $E[\alpha X + \beta] =$

Random Variables: Example

- $X \sim$ uniform[0,1]

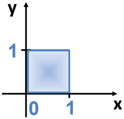
$$f_X(x) = \begin{cases} 1 & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(x)dx \\ &= \int_{0}^{x} f_X(x)dx \\ &= \begin{cases} 0 & , x < 0 \\ x & , 0 \le x \le 1 \\ 1 & , \text{otherwise} \end{cases} \end{aligned}$$



Jointly Distributed RVs: Example

$$f_{X,Y}(x,y) = \begin{cases} xy + \frac{3}{4} & , 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & , \text{otherwise} \end{cases}$$



- Find $f_X(x)$ and $F_X(x)$, the marginal distributions.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy \ ,\ 0 \le x \le 1 \\ &= \int_0^1 \left(xy + \frac{3}{4}\right) dy = \left[\frac{xy^2}{2} + \frac{3}{4}y\right]_{y=0}^{y=1} \\ &= \begin{cases} \frac{x}{2} + \frac{3}{4} & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases} \end{aligned}$$

Expectations

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

$$E[X] = \sum_{x_i \in S_X} x_i p_X(x_i) dx$$

Consider (X, Y) joint RVs and $Z = g(X, Y)$

$$E[Z] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) f_{X,Y}(x, y) dx dy$$

Expectations: Example

$$
\begin{aligned}
E[X] &= \int_0^1 x f_X(x) dx \\
&= \int_0^1 x \left(\frac{x}{2} + \frac{3}{4} \right) dx \\
&= \frac{1}{6} + \frac{3}{8}
\end{aligned}
$$

$$
\begin{aligned}
Var[X] &= E[(X - \mu_X)^2] \\
&= E[X^2] - \mu_X{}^2
\end{aligned}
$$

$$
Cov[X] = E[(X - \mu_X)(Y - \mu_Y)]
$$

Expectations: Key Fact

$$E\left[\sum_{i=1}^{n} \alpha_i X_i\right] = \sum_{i=1}^{n} \alpha_i E[X_i]$$

If X_i are independent:

$$Var\left[\sum_{i=1}^{n} \alpha_i X_i\right] = \sum_{i=1}^{n} {\alpha_i}^2 \sigma_{X_i}^2$$

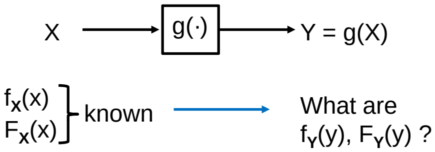
Expectations: Example

$$Y = \sum_{i=1}^{n} X_i$$

X_i are iid, $E[X_i] = 2,\ Var[X_i] = 3$

$$E[Y] = 2n$$
$$Var[Y] = 3n$$
$$E[Y/2] = n$$
$$Var[Y/2] = \frac{3}{4}n$$

Transformations of RVs



Procedure:

$$
\begin{aligned}
F_Y(y) &= P[Y \leq y] \\
&= P[g(X) \leq y] \qquad \text{write this in terms of } F_X(x)
\end{aligned}
$$

Transformations of RVs: Example

$$Y = X^2$$

X_i are iid, $E[X_i] = 2$, $Var[X_i] = 3$

$$f_X(x) = \frac{x}{2} + \frac{3}{4} \text{ , } 0 \leq x \leq 1$$
$$F_X(x) = \frac{x^2}{4} + \frac{3}{4}x \text{ , } 0 \leq x \leq 1$$

Transformations of RVs: Example (cont.)

$$
\begin{aligned}
F_Y(y) &= P[Y \leq y] \\
&= P[X^2 \leq y] \\
&= P[X \leq \sqrt{y}] \\
&= F_X(\sqrt{y}) \qquad \textcolor{blue}{\text{needs to be a function of } y}
\end{aligned}
$$

$$
F_Y(y) = \begin{cases} 0 & , y < 0 \\ \frac{y}{4} + \frac{3\sqrt{y}}{4} & , 0 \leq y \leq 1 \\ 0 & , y \geq 1 \end{cases}
$$

The PDF is obtained by taking the derivative of this.

Moment Generating Functions

$$\text{RV } X \qquad \Rightarrow \phi_X(s) = E\left[e^{sX}\right] = \int_{-\infty}^{+\infty} e^{sx} f_X(x)dx$$

$$E[X^n] = \left.\frac{d^n \phi_X(s)}{ds^n}\right|_{s=0}$$ should not have s in it; it is a number

Moment Generating Functions: Example

$$\phi_X(s) = exp\left[\mu s + \frac{s^2\sigma^2}{2}\right] \qquad \Rightarrow \text{Gaussian MGF}$$

$$\left.\frac{d\phi_X(s)}{ds}\right|_{s=0} = \left[(\mu + s\sigma^2)\phi_X(s)\right]_{s=0} = \mu$$

Find $E[X^4]$

$$Var[X] = E[X^2] - \mu_X^2$$

Gaussian RVs

$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Key Fact: $X \sim N(\mu, \sigma^2)$ can be expressed as $X = \mu + \sigma Z$, where $Z \sim N(0, 1)$.

Example:

$$\begin{aligned} P[2 < X < 3] &= P[2 \leq \mu + \sigma Z \leq 3] \\ &= P\left[\frac{2-\mu}{\sigma} \leq Z \leq \frac{3-\mu}{\sigma}\right] \\ &= F_Z\left(\frac{3-\mu}{\sigma}\right) - F_Z\left(\frac{2-\mu}{\sigma}\right) \end{aligned}$$

Table or necessary Gaussian values will be provided.

Bivariate Gaussian RVs

$$(X, Y) \sim N(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{X,Y})$$

$$f_{X,Y}(x, y) = \frac{exp\left[-\frac{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho)^2}\right]}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

Key Fact: Bivariate Gaussians are **independent** if and only if they are **uncorrelated** ($\rho = 0$).

Sums of RVs

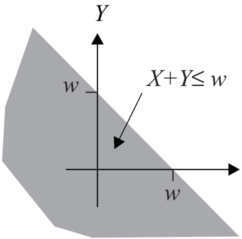
Z= X + Y

If X and Y are independent:

1. $f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x) dx$ (Use convolution)
2. $\phi_Z(s) = \phi_X(s) \cdot \phi_Y(s)$

If X and Y are **not** independent:

$$F_Z(z) = P[Z \le z] = P[X + Y \le z] = \int_{-\infty}^{+\infty} \int_{-\infty}^{z-x} f_{X,Y}(x,y) dy dx$$



Limit Theorems

Weak Law of Large Numbers (WLLN):

Suppose $X_1, X_2, ..., X_n$ are iid, then

$$\frac{X_1 + X_2 + ... + X_n}{n} \longrightarrow \mu_X \text{ "in probability"}$$

Central Limit Theorem (CLT):

Suppose $X_1, X_2, ..., X_n$ are iid, then

$$\frac{X_1 + X_2 + ... + X_n - n\mu_X}{\sqrt{n}\sigma_X} \overset{n\to\infty}{\longrightarrow} N(0,1) \text{ "in distribution"}$$

That is,

$$P\left[\frac{X_1 + X_2 + ... + X_n - n\mu_X}{\sqrt{n}\sigma_X} \leq z\right] = \Phi(z)$$

Conditional Expectation

Starting with a joint PDF $f_{X,Y}(x,y)$, the conditional PDF of X given Y

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Definition: $E[X|Y]$ is an RV, called **conditional expectation** defined by

$$E[X|Y] = g(Y)$$

where $g(y) = E[X|Y = y]$

Conditional Expectation: Example

- X, Y iid uniform[0,1].
- Let $Z = max(X, Y)$.
- Find $F_Z(z), E[Z|X]$.

. . . .

$$
\begin{aligned}
F_Z(z) = P[Z \leq z] &= P[max(X, Y) \leq z] \\
&= P[X \leq z, Y \leq z] \\
&= P[X \leq z, X \leq z] \ \color{blue}{(X, Y \text{ both iid uniform[0,1]})} \\
&= [F_X(x)]^2
\end{aligned}
$$

Conditional Expectation: Example (cont.)

- X, Y iid uniform[0,1].
- Let $Z = max(X, Y)$.
- Find $F_Z(z), E[Z|X]$.

....

Using the law of total probability,

$$\begin{aligned}
E[Z|X] &= E[max(X, Y)|X = x] \\
&= E[Z|Y \le x]P[Y \le x] + E[Z|Y > x]P[Y > x] \\
&= x \cdot x + \frac{(x+1)}{2}(1-x) \\
&= x^2 + (1-x)\frac{(x+1)}{2} = x^2 + \frac{(1+x)(1-x)}{2} \\
&= x^2 + \frac{(1-x^2)}{2} = \frac{(2x^2+1-x^2)}{2} = \frac{(x^2+1)}{2} \;, \; 0 \le x \le 1
\end{aligned}$$

So,

$$E[Z|X] = (1 + X^2)/2$$