VHF Repeater Placement

Control # 12114

February 14, 2011

Summary Statement

Our task is to develop a model that determines how to minimally and adequately place VHF repeaters so that low-power users, such as mobile stations, can communicate with one another in situations where direct user-to-user contact would not be possible. We assume that we are given a population of users within a circular region of radius 40 miles, and we attempt to find the minimal number of repeaters necessary to provide adequate coverage while minimizing the number of radio bands in use and keeping interference negligible. Although we assume that there must be a direct line of sight between a repeater and a user, our model allows for terrains of complex geometry including mountains and valleys. Throughout most of our analysis, we assume that repeater antennas are 18 feet above the ground, which corresponds to a coverage radius of 10 miles in a perfectly flat region.

Initially, we employ a greedy algorithm to successively place repeaters in the locations that provide coverage to the most additional users. In subsequent incarnations of the model, we consider goals other than maximal coverage in selecting the optimal positions. To assign frequency channels to these repeaters we form a graph whose vertices represent the repeaters, and we consider vertices to be adjacent if the repeaters are sufficiently close to one another. This graph is then colored greedily.

To test our model, we generate population distributions using both uniformly random placement of users and a preferential attachment model that more closely matches real world population distributions. We find that, using realistic population distributions, we were able to provide adequate coverage to 1000 simultaneous users with about 15 repeaters and 10000 users with about 20 repeaters; however by relaxing these conditions and requiring that only 95% of users be covered, we could lower these numbers to 10 and 15 respectively. We also showed that 21 repeaters should always be enough to provide coverage to a region with a radius of 40 miles.

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1 Background

1.1 Very High Frequency (VHF)

In most countries, the VHF frequencies 144-146 MHz are allocated for amateur radio. In the Americas, this band is 144-148 MHz. These frequencies are commonly referred to as the 2-meter amateur radio. The transmissions are local and generally require more compact machinery than operating on other frequencies. Because of their reliability, this band is also used for communicating emergencies via mobile and hand-held transmission devices. Because they provide back-up emergency communication, transmitting on these frequencies usually requires a license. Transmission time may be limited to a set time, such as 30 seconds, and many people send *courtesy tones* to remind people to leave gaps between transmissions; these breaks accommodate accident or emergency reporting as well as prevent one person from hogging the frequency.

To prevent interference and to control the usage of transmitting, government agencies manage the frequency allocations. In the United States, the Federal Communications Commission "is charged with regulating interstate and international communications by radio, television, wire, satellite and cable."[4] Originally, the spacing between 2-meter channels was 30kHz. When the bands became overloaded with users. the states split in their spacing method: some divided the stations in half such that the spacing between them became 15kHz; and others adopted a 20kHz spacing. When frequencies from the transmitters overlap with only 15kHz difference, there can still be interference. With a spacing of 20kHz, there is virtually no interference [2].

1.2 Repeaters

In order to extend the distance a transmission can reach, repeaters receive input on a prescribed frequency and output the amplified signal at the original frequency ±600kHz. This pair of frequencies for a given repeater is commonly referred to as a *channel*. Over an antenna, repeaters transmit the signal usually as far as the observable horizon. For mobile and hand-held devices, this line-of-sight provides a very limited range of only a few miles. Due to the fact that repeaters are typically placed on high locations such as mountains, the observable horizon of the repeater spans much further than it would otherwise. Tropospheric bending extends this *Radio Horizon* even more from

$$r \approx 1.23\sqrt{h}$$

to

$$r \approx \sqrt{2h} \approx 1.41\sqrt{h},$$

where r is the radius or distance from the repeater antenna to the Radio Horizon and h is the height of the antenna in feet [9]. Note that at the edges of the coverage of a repeater

– the *fringe* – the signal becomes weak and may drop out. Repeaters can also broadcast their received signal over the internet through applications such as `EchoLink`, which allows the transmissions to reach across the globe.

As listed on the National Association for Amateur Radio's website, the band plan for 144-148MHz specifies the following table[1]:

Frequency (MHz)	Allocated
146.01-146.37	Repeater inputs
146.40-146.58	Simplex†
146.52	National Simplex Calling Frequency
146.61-146.97	Repeater outputs
147.00-147.39	Repeater outputs
147.42-147.57	Simplex
147.60-147.99	Repeater inputs

UK Repeater Frequencies

As an example of actual repeater frequencies, we look at those in the UK.

The UK Amateur Radio Repeater Resource website lists all of the "existing and licensed analogue and dual-mode 145MHz amateur repeaters" in the UK. Of the 127 operational repeaters listed, there are only 15 distinct frequencies used. For a given frequency, the regions reached by the repeaters do not overlap and some repeaters may even have the same PL code; however, these regions are a significant distance apart (~200 miles). Furthermore, the frequencies used for repeaters are in a contiguous band. [Note that for context, the UK is 94,060 square miles, whereas our region is approximately 5,027 square miles.]

Channel	Output	#
RV48	145.6000	10
RV49	145.6125	5
RV50	145.6250	10
RV51	145.6375	4
RV52	145.6500	8
RV53	145.6625	8
RV54	145.6750	7
RV55	145.6875	7
RV56	145.7000	10
RV57	145.7125	6
RV58	145.7250	11
RV59	145.7375	6
RV60	145.7500	9
RV61	145.7625	5
RV62	145.7750	13
RV63	145.7875	8

Figure 1: The number of repeaters which output at given frequencies in the UK.[10]

[1] Note that we only show frequencies 146.01-147.99 and that, previous to this, there are blocks of other miscellaneous standards.

2 Problem Interpretation

Given a population of users within a circular region of radius 40 miles, we attempt to minimally and adequately place repeaters to assist with communication via VHF transmitters. Because VHF frequencies can be in high demand, we also minimize the number of frequencies used by our repeaters and analyze the benefit of using PL codes. We look at the cases where there are $1,000$ and $10,000$ simultaneous users with two types of population distributions: uniformly random and preferentially attached. The preferential distribution simulates the tendency of users to be found in clusters. In this way, we look at different types of regions which encapsulates sparse rural regions, towns, suburbs, and cities.

Because signals travel primarily by line-of-sight, we also analyze optimal placement of the repeaters when mountains are present in the given terrain and the effect this has on the number of the repeaters. Further, we discuss the effects of various attenuation factors on the signal.

3 Assumptions

There are three instances possible for when we would need to lay repeaters:

1. People reside within the region and no repeaters exist.
2. People reside within the region and inadequate repeaters exist.
3. The distributions of people and repeaters are unknown or not present.

In the third case, in order to adequately and minimally cover all of the simultaneous users, we must cover the entire region. See Circle Covering (Section 4) for repeater placement.

In the following analysis, we assume the first case, which could also easily be extended to the second case with our model by laying down a framework of already-existing repeaters and checking which users are already covered.

Repeaters

Usually repeaters are placed on high buildings to maximize their coverage; however, with a strong repeater and a reasonably tall building, this range could easily be more than 40 miles. If we used such repeaters, the minimal repeater placement would be directly in the center of the region.

To avoid looking at such a trivial case, we assume an antenna height of 18 feet and consider the range within 10 miles of the repeater, a range which does not impede on the "fringe" area and hence allows us to assume that all people within this circle are adequately covered. We discard the "fringe" range as unsatisfactory because is unreliable and signals may drop. Though the 18 feet and 10 mile range are for the most part arbitrary, any smaller of a range would be unrealistically small and any larger converges to the trivial case.

We also assume that any person within the range specified of the repeater can communicate with any other user who is within range of a, possibly distinct, repeater through an application such as `EchoLink`.

Also, we assume that repeaters fail independently.

Discretized Space

For ease of modeling, we assume a grid on which we place people, repeaters, and do calculations. We generally use a fineness of 1/4 mile for this grid. We also ensure that the people, repeaters, and calculations are within the circle centered at the center of the grid.

When placing repeaters, we assume that the circular coverage never escapes the surrounding grid boundary.

Usage Patterns

We assume that one circle covering any number of people is sufficient coverage. We address this issue further with *utility* (Section 6.2).

Obstructions

In our basic model, we assume that reflections, diffraction, and obstacles such as buildings have a neglectably small effect on the range of the repeater, which generally holds true in practice. We do discuss obstructions such as forests and consider the effect that mountains have on the line-of-sight, including the effects of diffraction (Sections 4.1, 7.2).

Interference

Interference can occur if two repeaters with the same frequency have overlapping coverages, even if their PL codes differ. We keep this as an assumption in our model and choose frequencies such that they are more than 25 miles apart and are not adjacent. Because we want to minimize the number of frequencies used, as mentioned in Section 2, we also attempt to reuse frequencies in our model and to keep these frequencies in a block as in Figure 1. Also, we assume that given a sufficiently small number of frequencies required, it is trivial to assign the channels frequencies without collision due to the 600kHz shifting of input and output of repeaters.

4 Circle Covering

One method of ensuring all users have coverage is to cover the entire region. We first look analytically at how to minimally cover the entire region with repeaters.

Let us assume that each repeater has a circular range with radius 10. Here we analyze the number of repeaters, n, needed to cover the region of radius 40.

Simple lower bound

We first find a lower bound on the number of repeaters needed to achieve a full covering. Using the fact that a circle with radius r has area πr^2, we note that the region to cover has area

$$A_{region} = \pi(40)^2 = 1600\pi$$

and each repeater has a coverage of area

$$A_{repeater} = \pi(10)^2 = 100\pi.$$

Because

$$\frac{A_{region}}{A_{repeater}} = 16,$$

we see that the area covered by 16 repeaters is equivalent to the area we intend to cover. This does not, however, imply that we only need 16 repeaters, because placing the circles would require overlap of their coverages. Hence, we find for a lower bound

$$16 \leq n.$$

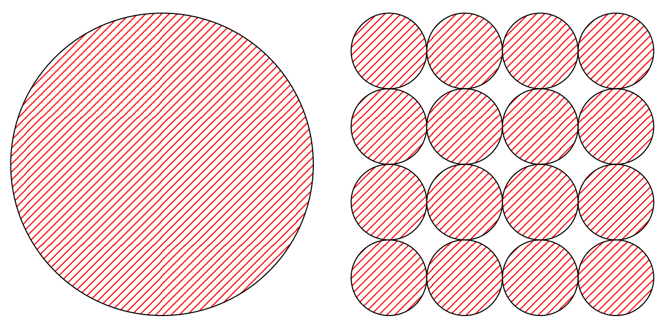


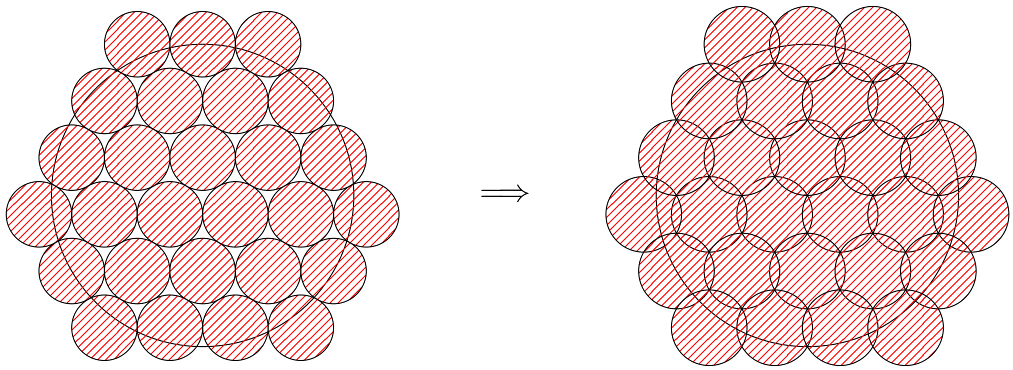
Figure 2: Circles of radii 40 and 10.

Hexagonal approximation upper bound

To get an upper bound on the number of repeaters needed, we look to hexagonal packing. The problem we are solving is easily relatable to the *Disk Covering Problem*, where one finds the smallest radius r for which n disks of radius r can cover the unit disk. This is an unsolved, $\mathcal{NP}$-complete problem with analytic answers for small n and approximations up to $n = 10$ [5]; however, we are looking at $n \geq 16$. Our problem is also relatable to the *Circle Packing Problem*, where circles are packed into a region without overlap. Using a circle packing of a smaller radius than that of the repeater coverage, we can find a circle packing and then expand the radius to obtain overlapping circles which applies to our problem. On the euclidean plane, hexagonal packing provides the optimal lattice solution for circle packing [6]; however, when the boundary is finite, this is not usually the case. Because our problem remains unsolved, we look to hexagonal packing for an approximation by placing a hexagonal lattice of smaller circles and extending their boundaries.

Because we can find a hexagonal scheme which uses 21 circles of radius 10 to cover our region of radius 40 (Figure 3), we know that *at most* we will need 21 repeaters. Hence, our new bounds on n are

$$16 \leq n \leq 21.$$



Hexagonal lattice packing with circle radii $5\sqrt{3}$. Hexagonal lattice covering with circle radii 10.

Figure 3: Upper bound of 21 derived from hexagonal packing.

Thus, optimally we can hope to obtain results from our model which require at least 16 and at most 21 repeaters.

4.1 Radio Wave Propogation

In free space, the power denisty of all electromagnetic radiation follows the inverse square law $\rho \propto \frac{1}{r^2}$, so that a doubling of distance results in one-fourth the power density. However propogation of radio waves near the surface of the earth can be significantly more complicated than this. Lower frequency radio waves often propogate by sticking to the surface of the earth, while higher frequency radio waves need a direct line of sight, and radio waves of certain middle range frequencies can propogate by refraction through the atmosphere. We are considering only VHF radio waves which propogate mostly by direct line of sight, however other factors such as refraction, reflection, and diffraction have an effect.

Refraction

Radio waves in the HF range can travel very great distances through a process called skywave propogation. In this process, waves are bent downard by refraction in the ionosphere, as if they were reflected off a mirror. When the waves reach the earth, they may reflect upwards again. This process may occur several times, allowing the wave to travel accross continents. While refraction of VHF waves is not strong enough to induce skywave propogation, it still slightly bends the path of the waves back towards the earth. The result is that the waves are able to propogate farther than if they were traveling in a straight line. Without the refractive effect, VHF waves would only be able to travel as far as the visible horizon before being absorbed into the Earth. However, this bending allows the waves to propogate beyond the visible horizon to something often called the Radio Horizon. The distance in miles to the Radio Horizon is approximately $\sqrt{2A}$ where A is the antenna hight in feet [3].

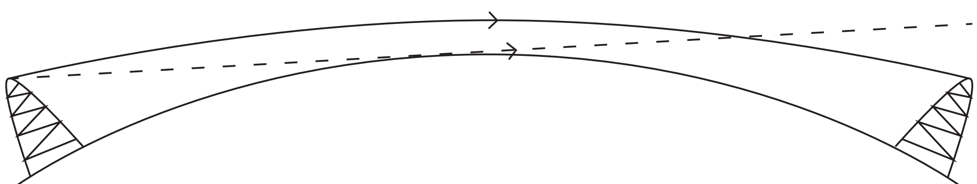


Figure 4: Signal range extends due to tropospheric interference.

Reflection

Ground plane reflections creat a second path for radio wave. When this wave recombines with the direct line of sight wave, there can be either constructive or deconstructive interference. However, ordinarily the ground plane reflection reduce the strength of the signal considerably, effectively changing the power density relationship to an inverse fourth law $\rho \propto \frac{1}{r^4}$ rather than the inverse square law [3].

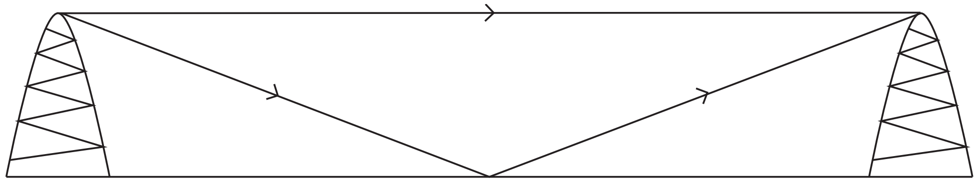


Figure 5: Reflections of the signal can interfere or strengthen the direct wave.

Diffraction

When VHF radio waves encounter large obstacles such a mountains, the wave is generally absorbed or reflected by the object. However in some cases, if the radio wave propogates near the top of the object, diffraction can redirect the wave towards the target. In this way the two radios can communicate even when their line of sight is blocked, however the signal strength is reduced by the diffraction process. In other cases, when the path goes directly above an object, diffraction can attenuate the signal even though there was a direct line of sight between te two antennas. In particular, obstructions inside the fresnel zone contribute to interference by diffraction.

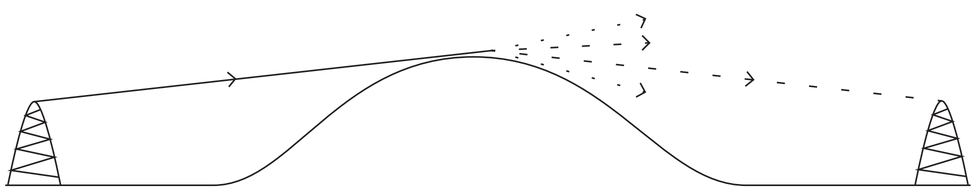


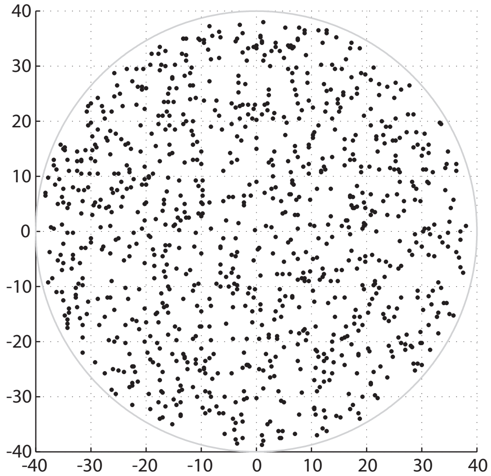
Figure 6: Diffraction over surfaces can cause indirect paths to connect antennas.

5 Population Distribution

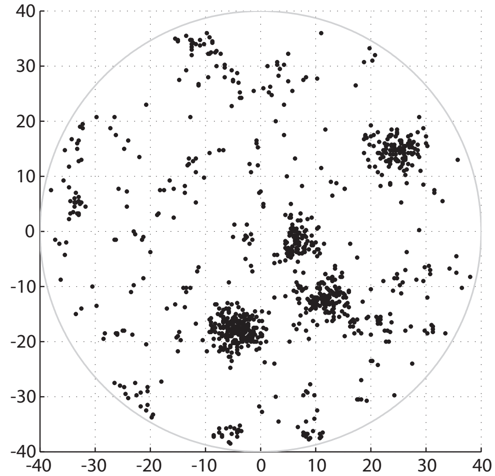
In order to acquire test data on which to execute our model, we need to generate population distributions to recreate user distribution scenarios similar to those that could be expected in real life. We consider two different algorithms to generate reasonable distributions within a circle with a 40-mile radius.

5.1 Uniform Algorithm

For each inhabitant, the uniform algorithm naïvely selects a point within the circle uniformly at random as their place of residence. This covers the circle reasonably evenly, without yielding the unnatural regularity attained by placing the points with equal spacing. As we will see in Section 7, this distribution is oftentimes the hardest to deal with as there are few patterns or clusters to be taken advantage of.



(a) Uniform



(b) Preferential

Figure 7: The two types of considered user distributions with 1000 users.

5.2 Preferential Attachment Algorithm

An alternative algorithm, based on the model of the preferential attachment suggested by Barabási and Albert [1], distributes the users in a slightly more organized way. Initially, some small number of users are distributed uniformly at random throughout the region to act as a seed for the algorithm. Afterwards, new users either select a location uniformly at random with some small fixed probability or select an existing user uniformly at random and located themselves within a small distance of that user.

This method ends up forming a population distribution approximating a preferrential attachment network, with clusters of users benefiting from the rich-get-richer effect in bringing in additional users. This is a highly desirable quality for our model, as population distributions tend to roughly follow the scale-free design formed by preferential attachment [7].

For clarity, pseudocode for this algorithm is given in Program 1.

6 Our Approach

Here we begin discussing the core of our model and algorithms.

6.1 Basic Coverage

Two similar goals can be sought in trying to attain basic coverage. We would like to be able to both:

Program 1 Preferrential Attachment Algorithm

init_users is the number of initial users distributed uniformly at random
total_users is the number of total users
r is the maximum radius between two "attached" users
connect_chance is the chance of a new user being attached to a random vertex
`greedy_color` refers to the common greedy coloring algorithm

Function `preferential_attachment`

Let *pop* be an empty list of users

/* *Place the initial set of users* */
loop *init_users* times:
    append to *pop* a user located uniformly at random in the region of interest

/* *Obtain a coloring of the generated graph* */
coloring = `greedy_color`(G)

For each vertex v in G:
    freqmap$[v]$ = *frequency_set*[*coloring*[v]]

return *freqmap*

1. Minimize the number of repeaters necessary to cover some fraction p of the population.
2. Maximize the number of people covered by some fixed number, n, of repeaters.

Since the decision problems corresponding to the above two goals are both known to be $\mathcal{NP}$-hard [11], we employ a greedy algorithm to try approximating the solution by repeatedly taking the locally-optimal move (that is, repeatedly covering users with maximally-covering circles). This greedy approach was shown to be a $2-$approximation of the optimal solution in a more general setting, but has been demonstrated to approximate the optimal solution rather well in simulations [11]. Interestingly, the monotonic reasoning of the greedy algorithm allows us to simultaneously calculate our success at attaining the two goals mentioned above: while greedily computing the number of repeaters needed to cover a certain fraction of the population, we can record the total coverage at each intermediate step in order to compute the effectiveness of greedily placing n repeaters.

Because this method is a specific case of the utility method introduced in Section 6.2, we delay the introduction of pseudocode for the time-being. More thorough analysis of the code's design and implications will be presented with the pseudocode.

6.1.1 Algorithmic Runtime

Letting k denote the maximum number of circles of radius r needed to cover the 40-mile-radius circle, the 2-approximation factor discussed in the previous section implies that the greedy algorithm will terminate within *at most* $2k$ iterations. To find the optimal circle in each iteration, the body of the loop iterates over $\mathcal{O}(s^2)$ grid points (for s representing the fineness of the grid), each of which requires $\mathcal{O}((rs)^2)$ calculations, leading to an overall runtime of $\mathcal{O}(r^2s^4)$. Because k is $\mathcal{O}(r^{-2})$, the overall runtime of this algorithm is

$$\mathcal{O}(r^{-2}r^2s^4) = \mathcal{O}(s^4),$$

or quartic in the fineness of the grid.

6.2 Additional Layers: Maximizing Utility

Although the algorithm described in Section 6.1 can successfully cover any given percentage of users, it often employs a number of repeaters which cover only a relatively small fraction of the population. Instead, a better use of these repeaters might be in the establishment of secondary, backup networks from which the high-population districts can benefit.

The reasoning is as follows: if an average repeater is malfunctioning with some fraction p of the time, the expected loss of utility (ignoring regions of overlap) is pn, where n is the number of people covered by the repeater's signal. Thus, placing a repeater to get rid of most of this downtime in a high-population region might be more beneficial than putting one somewhere that covers a small number of users. If we assume that the probabilities of each of two repeaters experiencing faults are independent, the probability of all repeaters in an area covered by q repeaters being down simultaneously is np^q. Alternatively, we can treat the first repeater covering a specific individual as having p utility, the next having p^2 utility, and so on, forming a geometric series that converges to $1/(1-p)$ as the supremum of attainable utilities. As this value forms a multiplicative constant for any chosen set of parameters, the fact that the maximum attainable utility per person is more than 1 does not affect the calculation of the placement of repeaters in any scale-independent covering algorithm, including the greedy algorithm we employ.

To clarify the workings of our algorithm, we present pseudocode describing its pre-optimization behavior in Program 2.

6.2.1 Basic Coverage as a Special Case

In addition to maximizing the total utility of the covering, the above pseudocode can actually be used to satisfy the two goals mentioned in Section 6.1. To do this, one simply needs to set u to 0, reducing the utility-maximization problem to coverage maximization, which in turn solves both tasks outlined in Section 6.1. This minimizes the amount of code reuse needed, and simplifies making later changes to the model.

6.2.2 Runtime Analysis and Optimizations

As the overall structure of the main loop's body has not changed significantly since previous runtime analysis, the runtime of each iteration is again $\mathcal{O}(r^2s^4)$. The number of iterations, however, is now also dependent on the rate of convergence of the utility to the desired amount, namely,

$$k \sim \frac{\log_u(1-W)}{r^2}$$

meaning that the runtime of the algorithm as a whole is

$$\mathcal{O}\left(s^4 \log_u(1-W)\right)$$

whenever $u > 0$ and $\mathcal{O}(s^4)$ otherwise.

In order to successfully deal with this approximately-quartic runtime, various optimizations had to be made. Firstly, instead of recalculating the populations of the various repeater regions of effect in each iteration, we can precompute each of these populations beforehand. This can be done efficiently (in $\mathcal{O}(rn^2)$ time) using standard dynamic programming approaches. With these precomputed values and a maintained sorted list of circle populations, we can reduce the $\mathcal{O}(r^2s^4)$ portion of the algorithm to $\mathcal{O}(rs^2\log(rs))$ average-case runtime, drastically reducing the number of computations necessary. Further, although not implemented in our model, the search portion of this step can be sped up

Program 2 Utility-Based Greedy Cover

r is the effective radius of a repeater's signal
u is the expected proportion of downtime of the repeaters
W is the utility ratio at which the program is to stop

```
Function utility_based_greedy_cover
covering = empty list
current_utility = 0

/*Precompute maximum possible utility using infinite geometric series formula*/
max_utility = population/(1-u)

set the utility of all users to 1
loop until broken:
    let greedy_value = -1, best_point = (0, 0)
    for each grid point p:
        compute total utility u of all users within r miles of p
        if u > greedy_value:
            best_point = p
            greedy_value = u

    /*Store result*/
    append best_point to covering

for each user x within r miles of best_point:
    current_utility += x.utility
    x.utility *= u

    /*Check stopping condition*/
    if current_utility/max_utility < W:
        break

return covering
```

further by parallelizing the search over multiple cores, reducing the total runtime by a small amount.

6.2.3 Additional Notes

This algorithm can also handle instances in which the search area surrounding the various potential repeaters is non-circular, although the various sequential optimizations (i.e. all those besides the parallelization) noted in 6.2.2 are eliminated, and the tight runtime is much more difficult to calculate (though it can easily be shown to be no worse than $\mathcal{O}(\frac{s^4 \max(\mathbf{a})}{\min(\mathbf{a})})$ where $\mathbf{a}$ is the array of areas of the various regions covered by different locations).

6.3 Frequency Selection

To assign frequencies to the various repeaters, we form a graph whose vertices represent the repeaters and whose edges represent locality between the two repeaters at its endpoints. After this graph is created, a reasonable vertex coloring is found (greedily or otherwise), with the different colors representing distinct frequencies of the repeaters. Pseudocode for this idea is given in Program 3. Note that because our model allows for only a small number of frequencies, we assume that assigning these colors to frequencies is trivial and so do not assign specific numbers.

Note that although we can simply define two repeaters to be adjacent if there's overlap in the regions covered by the pair, our current algorithm maintains an extra half-radius separating the regions to ensure minimal interference.

7 Results

This section will discuss the results from our model. We will look at the results from the basic model with both uniform and preferential distributions. This includes looking at both potential populations of users, and will look at coverage and utility. In each of the plots, each distinct frequency is a distinct color.

7.1 Basic Model

Covering

Using the *Greedy Covering Algorithm*, described in Program 2, we not only look at the minimum number of circles needed to cover all users but also what happens if a smaller

---

Program 3 Frequency Allocation Algorithm

---

repeater_list is the list containing the locations of repeaters
r is the effective radius of a repeater's signal
frequency_set is a list of allowable, non-overlapping frequencies

---

Function `greedy_frequency_allocator`

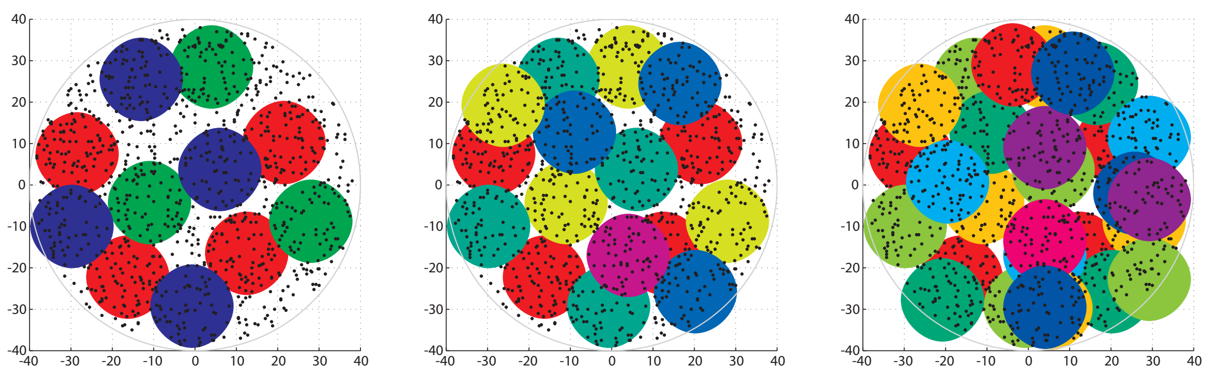
Let G be an edgeless undirected graph with one vertex per repeater in *repeater_list*
Let *freqmap* be a mapping from repeaters into frequencies

for each repeater pair (i, j) with $i \neq j$:
  if $\texttt{dist}(i, j) \leq \frac{5}{2}r$:
    Add an edge in G connecting the vertices representing i and j

/* *Obtain a coloring of G, represented as a vertex → integer map* */
$coloring = \texttt{greedy\_color}(G)$

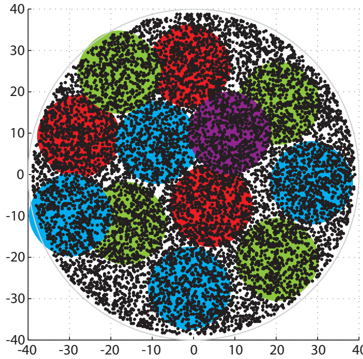
For each vertex v in G:
  $freqmap[v] = frequency\_set[coloring[\text{v}]]$

return *freqmap*

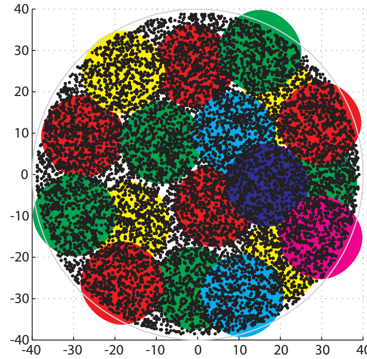


(a) 80% Coverage, 3 Colors, 11 Circles
(b) 92% Coverage, 5 Colors, 15 Circles
(c) 100% Coverage, 8 Colors, 28 Circles

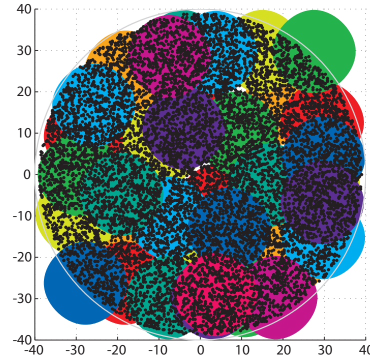
Figure 8: Uniform distribution of 1000 users with 80%, 92%, and 100% coverage. The number of colors corresponds to the number of frequencies used.



(a) 80% Coverage, 4 Colors, 12 Circles

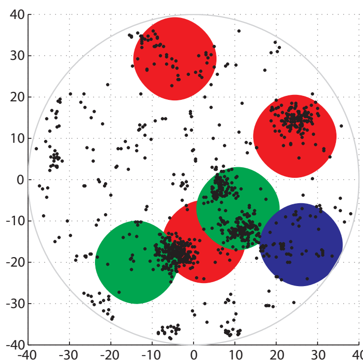


(b) 92% Coverage, 6 Colors, 18 Circles

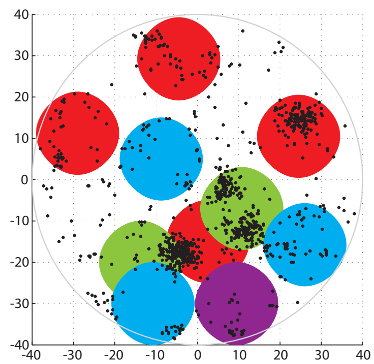


(c) 100% Coverage, 10 Colors, 35 Circles

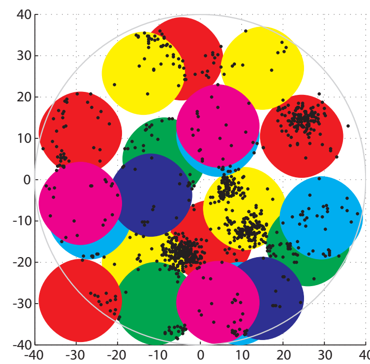
Figure 9: Uniform distribution of 10,000 users with 80%, 92%, and 100% coverage.



(a) 80% Coverage, 3 Colors, 6 Circles



(b) 92% Coverage, 4 Colors, 10 Circles



(c) 100% Coverage, 6 Colors, 21 Circles

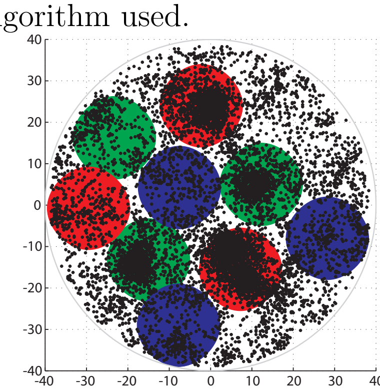
Figure 11: Preferential distribution of 1000 users with 80%, 92%, and 100% coverage.

percentage of users are required to be covered. For instance, if only 90% of users are required to be covered the number of repeaters required may be much smaller. This acts as a sensitivity analysis as well as offering insight into the algorithm used.

The first case we look at is that in Figure 8, which uses a uniform distribution for the population spread. We see that for 80% coverage of the users, only 11 repeaters are required. Again, for only 92% of users to be covered, only 15 are required; however, the last 8% require 13 extra repeaters, pushing the number of required repeaters to 28.

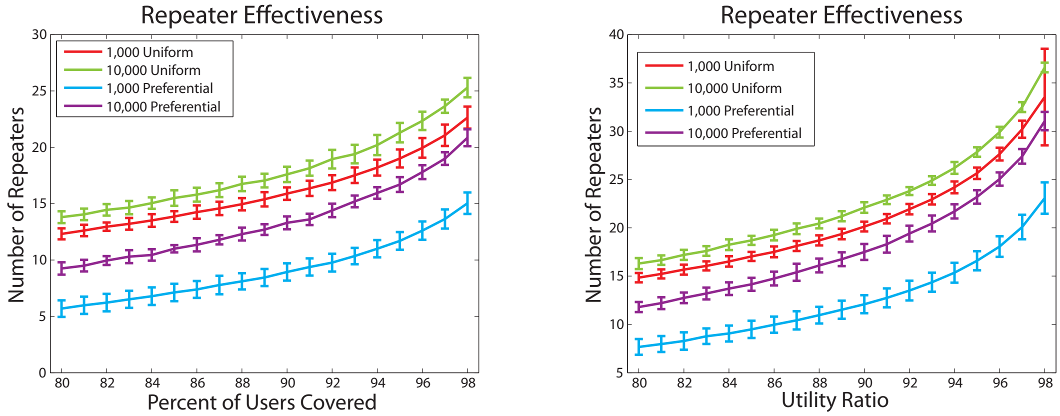
A similar phenomenon occurs in Figure 9 where the last 8% require 17 extra repeaters, pushing the number to 35.

In the preferential distribution case, we notice that covering 80% requires less repeaters. This is because a single



(a) 80% Coverage, 3 Colors, 10 Circles

Figure 10: Preferential distribution of 10,000 users with 80% coverage.



(a) Covering (b) Utility

Figure 12: Uniform and preferential distributions of both $1,000$ and $10,000$ user population, looking at the covering and utility cases. In (a), the minimum number of repeaters, on the y-axis, needed to provide coverage to a percentage of people, on the x-axis, is plotted. In (b), the minimum number of repeaters to achieve the utility ratio is plotted. Each curve of 1000 users averages 100 runs; each of 10,000 users averages 20 runs. The error bars on each curve show one standard deviation away from the average.

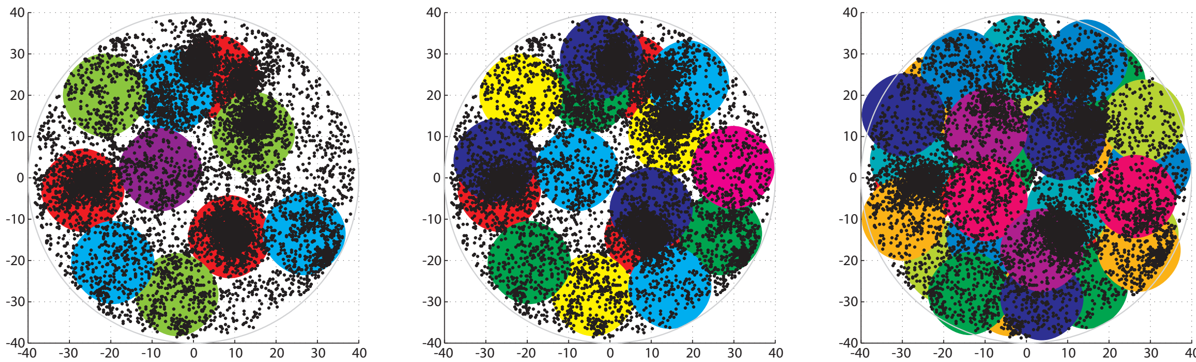
repeater can cover a larger percentage of the users due to them being clustered in location. The last 8% still requires a significant amount of repeaters.

Utility

Looking at Figures 13 and 14, we again see that as the percentage of users increases, the number of repeaters required increases. There again exists the shift perceived as before, and preferential distributions still require less repeaters than that for uniform distributions.

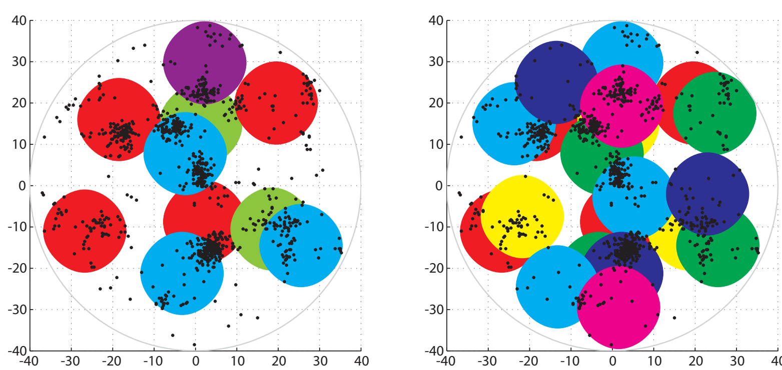
Overview

In Figure 12a, we average the minimum number of circles needed to cover some percentage of the population from 100 runs for each curve of 1000 users and 20 runs for each curve of 10,000 users. We do this by incrementally adding circles and checking how many circles pushes the percentage over that in the plot. We can see that either of the uniform populations requires more repeaters than either of the preferential populations. This is because the preferential clusters the points together, making them easier to cover with less repeaters. The uniform population, as the number of users increases, should converge to needing to cover the entire area. In fact, the preferential population should also cause this to happen; however, as we can see, the *Greedy Covering Algorithm* performs better in this case because the order of selection is close to optimal due to the extreme clusters. For



(a) 80% Coverage, 4 Colors (b) 90% Coverage, 6 Colors (c) 98% Coverage, 10 Colors

Figure 13: Preferential distribution of 10,000 users with 80%, 90%, and 98% utility ratios.



(a) 90% Coverage, 4 Colors (b) 98% Coverage, 6 Colors

Figure 14: Preferential distribution of 1000 users with 90% and 98% utility ratios.

smaller sizes, the greedy algorithm is optimal in the case where there exist only clusters with sufficient spacing. We can see the algorithm at work in the figures, selecting the high density areas first and then gathering the remainder population.

As the number of users increase, the curves are simply shifted up. This is because more area is covered by users as the number of users increases, and covering the same percentage of users should require more.

We perform the same operation to find the curves in Figure 12b. The behavior is very similar; however, the scale differs.

In each case, we can easily stay within the frequency band of 145-148MHz, and the number of colors rarely grow to above 12, meaning that we only need allocate 12 frequencies to the use of these repeaters.

7.2 Mountains and Complex Geometries

Up to this point we have only considered a completely flat surface for our reegion of interest. If we allow the elevation of the surface to deviate from the smooth curved surface of a sphere, the problem of placing repeaters becomes considerably more difficult. Previously the coverage of a repeater was independant of its location, however if we allow for complex elevation geometries, we must calculate the coverage of a repeater for any given location. This requires us to check for ever potential repater location at which locations would an antenna be in a direct line of sight path from the antenna of the repeater. To accomplish this we designed and implmented an algorithm that determines whether two points on an elevation map can be connected by an onubstructed line of sight path, and

Line of Sight

To determine whether or not one point is in the line of sight of another, we first find the elevation of the mesh points between the two radios, which we call the elevation profile curve. This is done using bilinear interpolation when accuracy is desired, or Bresenham's line algorithm when speed is desired over accuracy. Next we calculate the heights of the antennas places at the radio locations (generally assumed to be 18 feet). Then we form the straight line connecting the antennas and check to see if it intersects the elevation profile. If there is an intersection, then the line of sight is obstructed. In practice the curve and the line are calculated and compared simultaneously so that intersections can be found more quickly. This algorithm has a runtime of $\mathcal{O}(n)$ (although the expectation is constant for large n).

Repeater Coverage

For each mesh point on our grid, we ascertain the area coverage of a potential repeater at that location. We do so by checking the line of sight visibility from that point to each other point on the grid. If the mesh is a square of size n, then ince there are n^2 points on the mesh, and for each of these points we must look at every other point, this algorithm calls the line of sight algorithm n^4 times, although exploiting the symmetry of the problem allows us to cut this in half (if point A can see B, then point B can see A). Thus determining the coverage of a repeater for every point on the mesh has an algorithmic complexity of $\mathcal{O}(n^5)$.

Constructing Terrain

To test our model we need realistic elevation data. While we can use real world elevation data, we also wanted a way to generate our own rough terrain for flexibility. We start with a smooth surface representing the curvature of the earth. This is given by

$$f(x, y) = \sqrt{R^2 - x^2 - y^2} + \sqrt{R^2 - r^2} - R$$

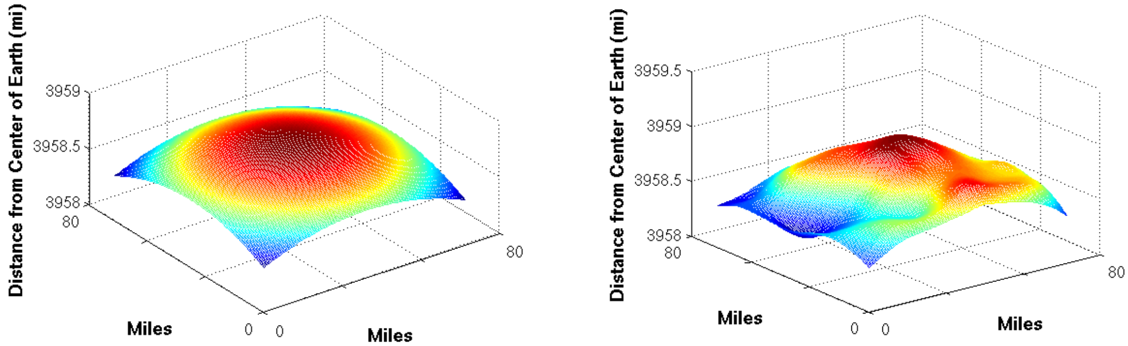
where R is the radius of the earth and r is the radius of the region. Next we can perturb this surface using either real world elevation data, or the Diamond-Square algorithm, to simulate mountains and other complex terrain.

Effects of Mountains

When we add mountains to our region, we find that some locations offer far more coverage for a repeater than other locations. Therefore, in general, our greedy algorithm has a tendency to pick these locations as they are more effective at both increasing coverage of users and utility ratio. In some scenarios, we find a location that offers an excepetionally great coverage, for instance at the top of a high peak. In these cases, generally fewer repeaters are needed to accomodate the users, since a few well placed ones are particularly effective. However, the mountains and valleys can also isolate users from the repeaters, requiring a repeater to be placed very close to them in order to provide service. These isolated repeaters only provide coverage to users in very close proximity, limiting there effectiveness. In these cases, more repeaters than usual may be necessary to provide adequate coverage for the region.

8 Conclusions

Although our analytic results in Section 4 suggest that an optimal covering of any collection of points within the circle should require no more than 21 repeaters, our greedy



(a) Smooth Curvature of Earth

(b) Perturbed by Mountains

Figure 15: Here we shows the curvature of the Earth in our model as well as the curvature perturbed by mountains.

approach consistently required more than this number. However, not only is this not surprising (given the simple nature of the algorithm), but it also allows for a simple solution: whenever the model requests more than 21 repeaters, simply place the repeaters according to the technique discussed in Section 4. Thus, although our model is weak when 100% coverage is sought, the fact that we know of an optimal solution to the problem (for most population distributions) makes this a moot point.

Instead, the strength of our model lies in its ability to present good solutions when fewer than 100% coverage is requested. As portrayed in Figures 8, 9, 11, and 12, relaxing the perfect covering requirement slightly allows for solutions involving much fewer than 21 repeaters, greatly improving the value one is attaining per placed repeater.

Interestingly, the same phenomenon holds when attempting to maximize the utility ratio. Figures 12, 13, and 14 all show that marginally weaker goals result in a substantially smaller number of required repeaters.

9 Future Considerations

9.1 Changes to the Model

9.1.1 Ad-hoc Networks and Game Theory

Currently, our model assumes that there is an external medium by which repeaters can communicate. It would be interesting to analyze what happens in the absense of this channel, where repeaters relay messages by spreading using their own transmitters (and thereby being bounded by some physical limitations).

Further, one can consider the question as to what kinds of game theoretic implications can arise. Can there be an analogue of selfish routing in the VHF wave networks?

9.1.2 One-to-One Communication

Whereas our model is built for one-to-many communication, there can also exist scenarios where the users wish to communicate between themselves. Although much more complex, creating a system that allows for any pair of users to converse could prove to be enlightening algorithmically.

9.1.3 Non-static and Introduced Users

As of now, our model assumes all users are completely stationary. Perhaps a more complete model would incorporate dynamic users in addition to static ones, allowing for mobile individuals to participate in the network as well. Further, how can our model incorporate the joining and leaving of additional users? Is it "stable" with respect to small perturbations in the population?

9.2 Model Correctness

9.2.1 Region Topography

A truly accurate model of repeater coordination would utilize terrain data to better predict just how the VHF radio waves will propogate throughout the region. Although our model crudely approximates this propogation with perfect circles, the various aspects of terrain data can greatly change the efficacy of various locations in housing repeater. The Longley-Rice Irregular Terrain Model[8], for example, demonstrates a simple way to model wave propogation on complex terrains.

9.3 Real World Data

Currently, all of our data is randomly generated. The best way to model real-world scenarios, however, is to acquire the actual population distributions from cities and find some topographic map of the regions in question. If chosen carefully, this data can much better reflect the intricacies of population spread and the abnormalities of terrain.

9.4 Theoretical Issues

9.4.1 Algorithmic Improvement

Currently, our analysis depends heavily on greedy algorithms. Although the optimal solution to these problems might be intractable, our model could still benefit from algorithms with stronger average-case behavior, many of which are readily available.

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