Extend the intuitionistic logic - Nature Deduction, with Kolmogorov double negation

Yantian Yin, Yifei Che

Motivation: Extending Intuitionistic ND to Classical Logic

- **Missing Excluded Middle:**
  Intuitionistic logic does not assume that every proposition satisfies A ∨ ¬A.
- **Our Aim:**
- Introduce a rule (via double negation) to bridge this gap.
- Extend Natural Deduction (ND) to a classical system (KND).
- Demonstrate that classical logic and intuitionistic logic are “the same” by Proving that the excluded middle is admissible through soundness and completeness of KND relative to ND.

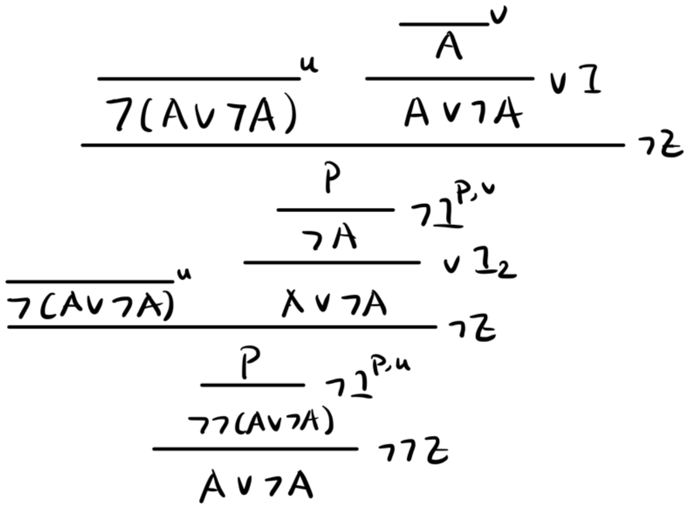
Introduction Rules	Elimination Rules
$\dfrac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\wedge\mathrm{I}$	$\dfrac{\vdash A \wedge B}{\vdash A}\wedge\mathrm{E_L} \qquad \dfrac{A \wedge B}{B}\wedge\mathrm{E_R}$
$\dfrac{\begin{array}{c}\overline{\vdash A}^{\,u} \\ \vdots \\ \vdash B\end{array}}{\vdash A \supset B}\supset\mathrm{I}^u$	$\dfrac{\vdash A \supset B \qquad \vdash A}{\vdash B}\supset\mathrm{E}$
$\dfrac{\vdash A}{\vdash A \vee B}\vee\mathrm{I_L} \qquad \dfrac{\vdash B}{\vdash A \vee B}\vee\mathrm{I_R}$	$\dfrac{\vdash A \vee B \qquad \begin{array}{c}\overline{\vdash A}^{\,u_1} \\ \vdots \\ \vdash C\end{array} \qquad \begin{array}{c}\overline{\vdash B}^{\,u_2} \\ \vdots \\ \vdash C\end{array}}{\vdash C}\vee\mathrm{E}^{u_1,u_2}$
$\dfrac{\begin{array}{c}\overline{\vdash A}^{\,u} \\ \vdots \\ \vdash p\end{array}}{\vdash \neg A}\neg\mathrm{I}^{p,u}$	$\dfrac{\vdash A \qquad \vdash \neg A}{\vdash C}\neg\mathrm{E}$
$\dfrac{}{\vdash \top}\top\mathrm{I}$	
	$\dfrac{\vdash \bot}{\vdash C}\bot\mathrm{E}$

No exclude middle.
Can't prove A ∨ ¬A directly

Objective

- Extend ND to KND and make KND classical.
- Define the ktrans that translates context between ND and KND
- Add inference rule to help with our prove
  - ND:¬¬X, ¬¬¬R. (could be ¬¬I and ¬¬¬E but we stick with the notation of the paper by Chad E. Brown 1998)
  - KND: ¬¬k
- Prove soundness
- Prove decomposability
- Prove completeness

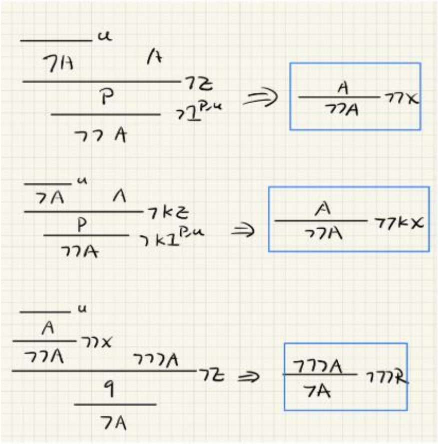
How we achieve our objectives



- ND->KND
  - by adding the Kolmogorov double negation elimination rule to embed the exclude middle property.
- Ktrans
  - We define the translation as the right side and apply recursively

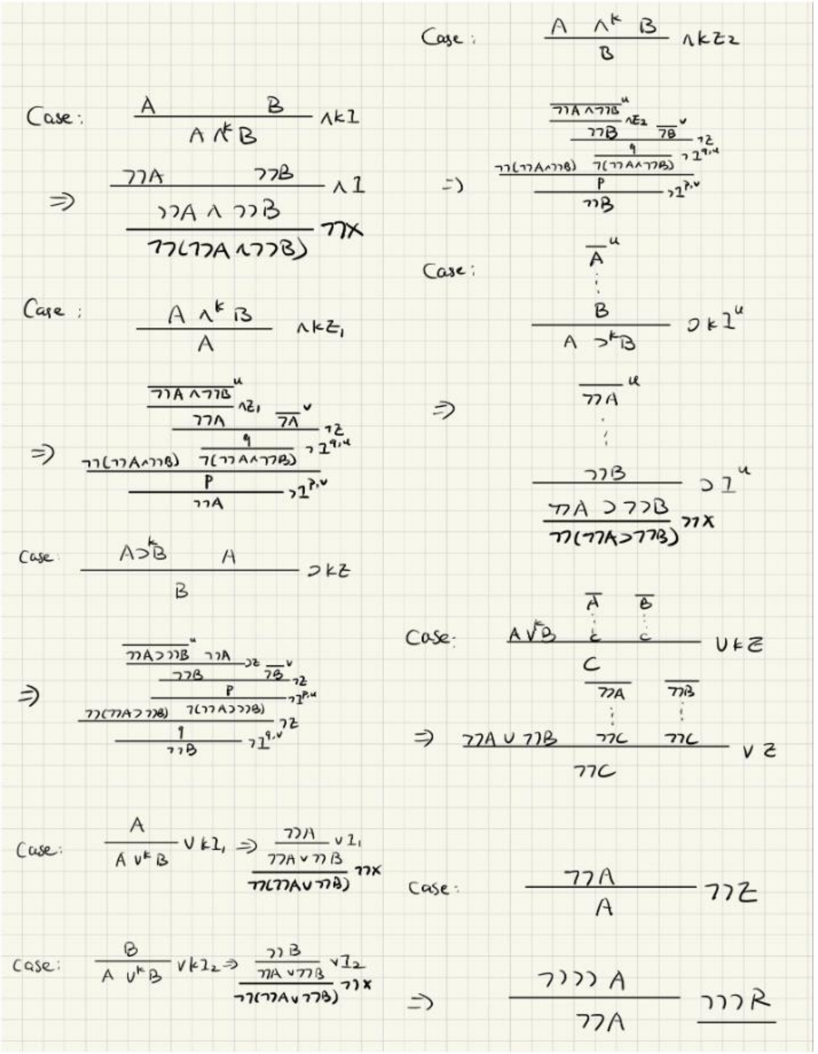
$$
\begin{aligned}
A^* &= nA \\
(A \wedge B)^* &= n(A^* \wedge B^*) \\
(A \supset B)^* &= n(A^* \supset B^*) \\
(A \vee B)^* &= n(A^* \vee B^*) \\
(\neg A)^* &= n(\neg A^*) \\
\top^* &= n\top \\
\bot^* &= n\bot
\end{aligned}
$$

- Inference rule
  - We prove each of them on paper.



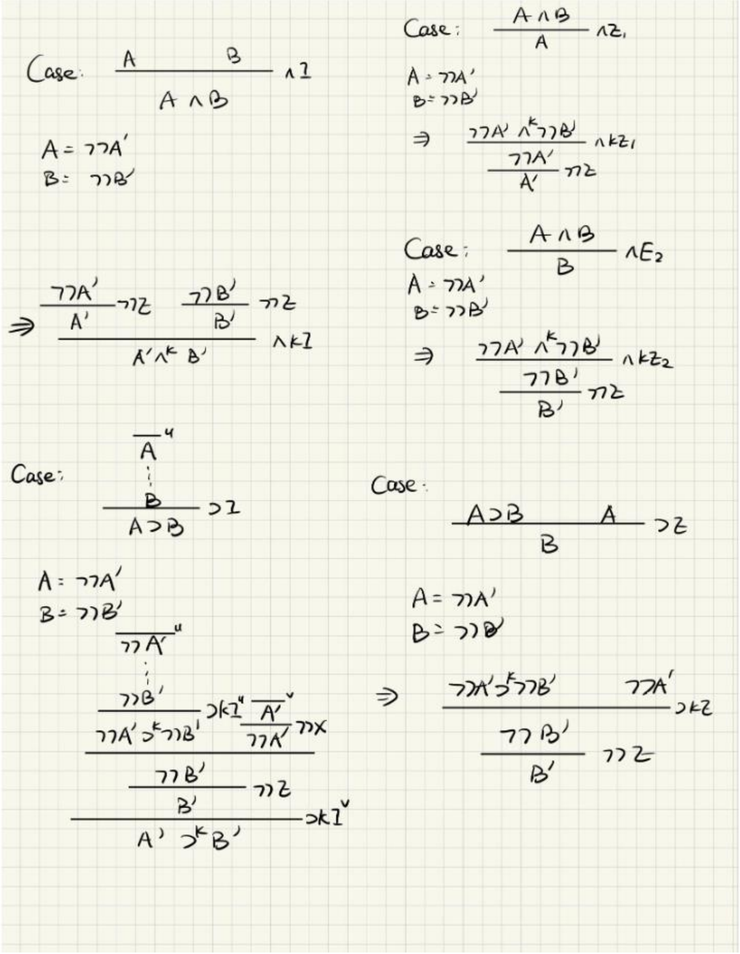
- Soundness

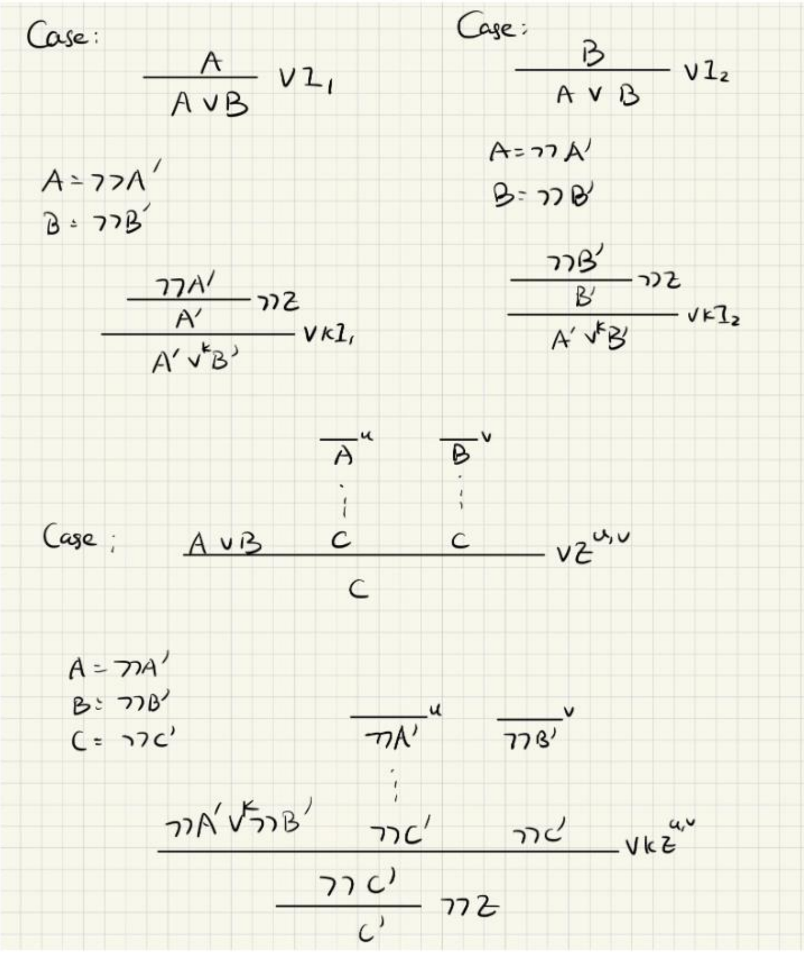
We are able to prove soundness with out any lemma directly



- Completeness

For completeness, we assume the decomposability





Currently working on

- Decomposability: for each A there exists a C such that A = ¬C
  - If proven, complete our completeness prove
- Excluded middle:
  - Show that for any formula A in KND we can prove (A or ¬A)* in ND
- Beluga implementations

Timeline and overall spilt up of work

- We are only a group of 2 so we do everything together.
- Yantian Yin:
  - On paper proof, documentation making and PPT making.
- Gufei Che:
  - Beluga code implementation.

Future timeline

- Week of March 31
  - Implement the decomposability and beluga code for ND, KND, Ktrans, and Decomp.
- Week of April 7
  - Implement soundness and completeness in Beluga
- Week of April 14
  - Prepare the anonymous YouTube video and the extended abstract pages