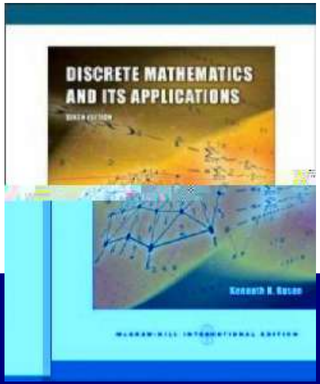
Discrete Mathematics



Basic Structures : Sets, Functions
Relations, Sequences, and Sums

Sets

- Def 1 : A set is an unordered collection of objects.
- Def 2 : The objects in a set are called the elements, or members of the set.
- Example 1 : common Important collections
  - N = { 0,1,2,3,…} , the set of natural number
  - Z = { …,-2,-1,0,1,2,…} , the set of integers
  - Z+ = { 1,2,3,…} , the set of positive integers
  - Q = { $p / q \mid p \in Z , q \in Z , q \neq 0$ } , the set of rational numbers
  - R = the set of real numbers

- Def 4 : Two sets A and B are equal iff
  $\forall x\ (x \in A \leftrightarrow x \in B)$
- Def 4 : $A \subseteq B$ iff $\forall x,\ x \in A \rightarrow x \in B$
  Supplement : $A \subset B$ means $A \subseteq B$ But $A \neq B$
- Def 5 : S : a finite set

  The cardinality of S , denoted by $|S|$, is the number of elements in S.
- Def 7 : S : a set

  The power set of S , denoted by $P(S)$, is the set of all subsets of S.
- Example 1 : $S = \{0,1,2\}$

  $P(S) = \{\{\ \}, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
- Def 8 : A , B : sets The Cartesian Product of A and B, denoted by $A \times B$, is the set
  $A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$

- Note. $|A \times B| = |A| . |B|$
- **Example 2 :**

$A = \{1,2\}$, $B = \{a, b, c\}$

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

Exercise

- For each of the following sets, determine whether 2 is an element of that set.
  - **a)** $\{x \in \mathbf{R} \mid x \text{ is an integer greater than } 1\}$
  - **b)** $\{x \in \mathbf{R} \mid x \text{ is the square of an integer}\}$
  - **c)** $\{2,\{2\}\}$ **d)** $\{\{2\},\{\{2\}\}\}$
  - **e)** $\{\{2\},\{2,\{2\}\}\}$ **f)** $\{\{\{2\}\}\}$
- Determine whether each of these statements is true or false.
  - **a)** $0 \in \emptyset$ **b)** $\emptyset \in \{0\}$
  - **c)** $\{0\} \subset \emptyset$ **d)** $\emptyset \subset \{0\}$
  - **e)** $\{0\} \in \{0\}$ **f)** $\{0\} \subset \{0\}$
  - **g)** $\{\emptyset\} \subseteq \{\emptyset\}$
- Determine whether these statements are true or false.
  - **a)** $\emptyset \in \{\emptyset\}$ **b)** $\emptyset \in \{\emptyset, \{\emptyset\}\}$
  - **c)** $\{\emptyset\} \in \{\emptyset\}$ **d)** $\{\emptyset\} \in \{\{\emptyset\}\}$
  - **e)** $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ **f)** $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
  - **g)** $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Exercise Contd

- What is the cardinality of each of these sets?
  - **a)** $\{a\}$ **b)** $\{\{a\}\}$
  - **c)** $\{a, \{a\}\}$ **d)** $\{a, \{a\}, \{a, \{a\}\}\}$

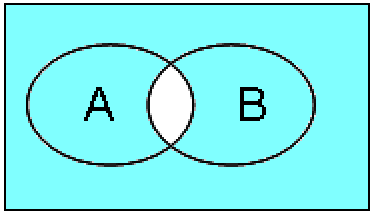
- How many elements does each of these sets have where *a* and *b* are distinct elements?
  - **a)** $P(\{a, b, \{a, b\}\})$
  - **b)** $P(\{\emptyset, a, \{a\}, \{\{a\}\}\})$
  - **c)** $P(P(\emptyset))$
- Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find
  - **a)** $A \times B$. **b)** $B \times A$.

Set Operations

- Def 1,2,4 : A,B : sets
  - $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$ (union)
  - $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$ (intersection)
  - $A - B = \{ x \mid x \in A \text{ and } x \notin B \}$ (Is often written as $A \setminus B$)
- Def 3 : Two sets A,B are disjoint if $A \cap B = \varnothing$
- Def 5 : Let U be the universal set.

  The complement of the set A, denoted by $\overline{A}$, is the set $U - A$.
- Example 3 : Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- pf :

  Known as the Venn Diagram



- **Def 6 :** $\boldsymbol{A_1}$ **,** $\boldsymbol{A_2}$ **, … ,** $\boldsymbol{A_n}$: sets

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

Let I = {1,3,5} , $\bigcup_{i \in I} A_i = A_1 \cup A_3 \cup A_5$

- **Def :** (p.131) A,B : sets

The symmetric difference of A and B, denoted by A⊕B, is the set

{ x | x ∈ A – B or x ∈ B – A } = (A∪B) – (A ∩B)

- ※Inclusion – Exclusion Principle

|A ∪ B| = |A| + |B| – |A ∩ B|

Exercise

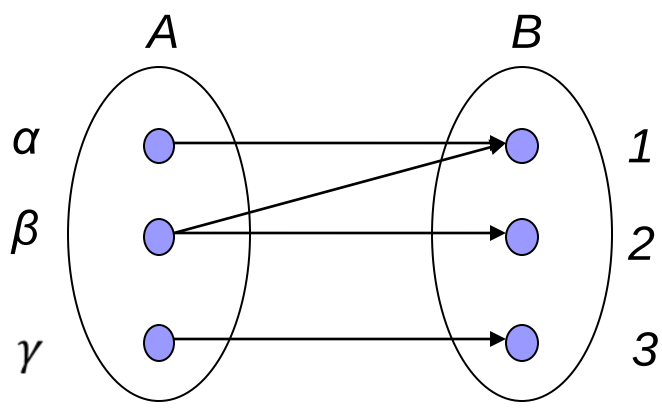
- Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, *and* $C = \{4, 5, 6, 7, 8, 9, 10\}$. *Find*
  - a) $A \cap B \cap C$. b) $A \cup B \cup C$.
  - c) $(A \cup B) \cap C$. d) $(A \cap B) \cup C$.
- Draw the Venn diagrams for each of these combinations of the sets *A, B, and C.*
- a) $A \cap (B \cup C)$ b) $\overline{A} \cap \overline{B} \cap \overline{C}$
- c) $(A - B) \cup (A - C) \cup (B - C)$

Functions

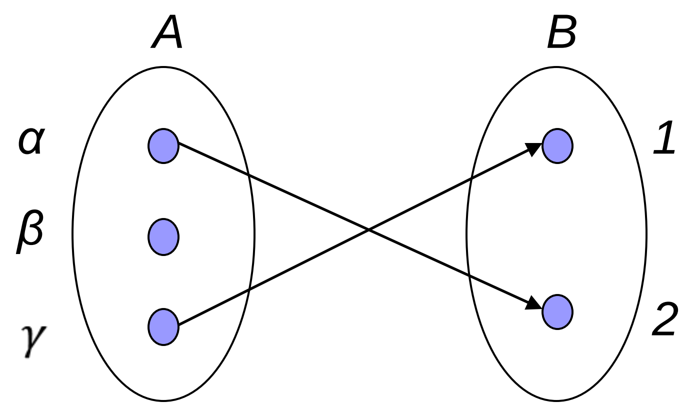
- Def 1 : A,B : sets

  A function $f : A \to B$ is an assignment of exactly one element of B to each element of A.
  We write $f(a) = b$ if b is the unique element of B assigned by f to $a \in A$.

- eg.

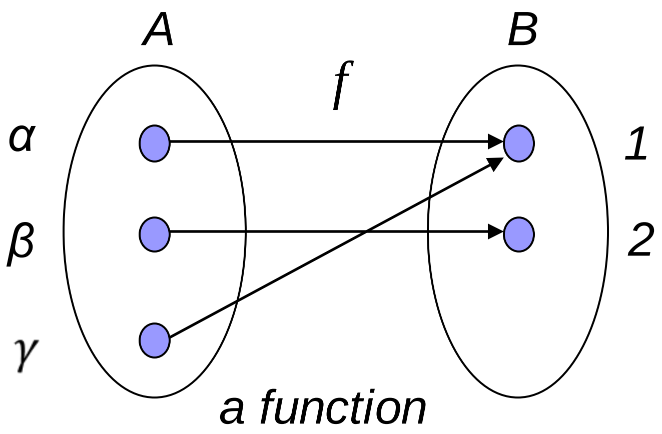


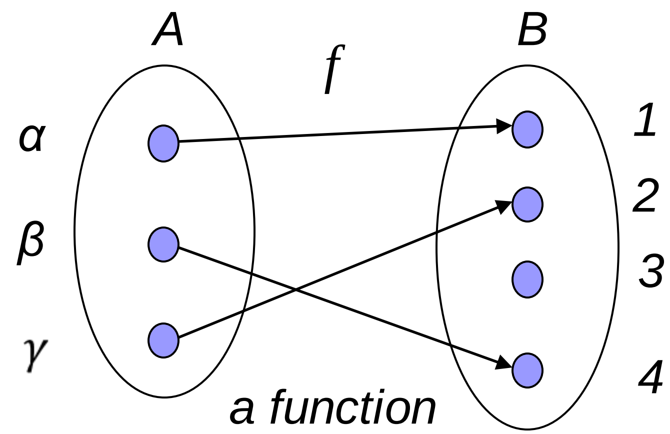
Not a function



Not a function

- If *f* is a function from *A* to *B*, we say that *A* is the *domain* of *f* and *B* is the *codomain* of *f*. If *f (a)* = *b*, we say that *b* is the *image* of *a* and *a* is a *preimage* of *b*. The *range*, or *image*, of *f* is the set of all images of elements of *A*. Also, if *f* is a function from *A* to *B*, we say that *f maps A* to *B*





- Def 2 : (f : A→B) considering the case on the right

$f(\alpha) = 1$, $f(\beta) = 4$, $f(\gamma) = 2$

1 is the image α (unique) , α is the pre-image of 1(not unique)

A : domain of f , B : codomain of f

range of $f = \{f(a) \mid a \in A\} = f(A) = \{1,2,4\}$ (may not be=B)

Example 1 : $f : Z \rightarrow Z$, $f(x) = x^2$, find the domain, codomain and range of f ?

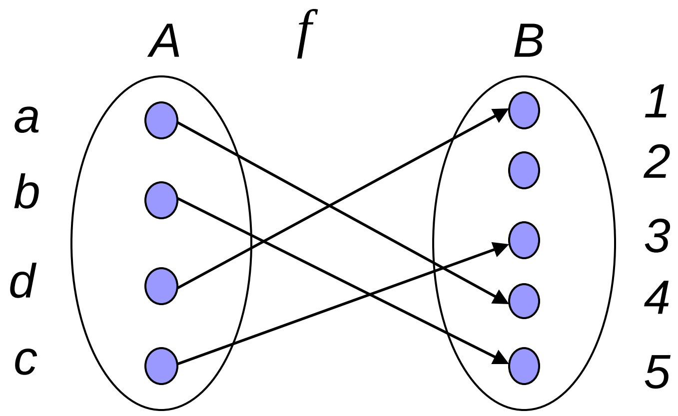
- Def 3: Let f_1 and f_2 be functions from A to **R**. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to **R** defined for all $x \in A$ by
  - $(f_1 + f_2)(x) = f_1(x) + f_2(x)$,
  - $(f_1 f_2)(x) = f_1(x) \cdot f_2(x)$

- **Example 2 :** Let $f_1 : R \to R$ and $f_2 : R \to R$ s.t. $f_1(x) = x^2$, $f_2(x) = x - x^2$, What are the function $f_1 + f_2$ and $f_1 f_2$?
- **Sol :**

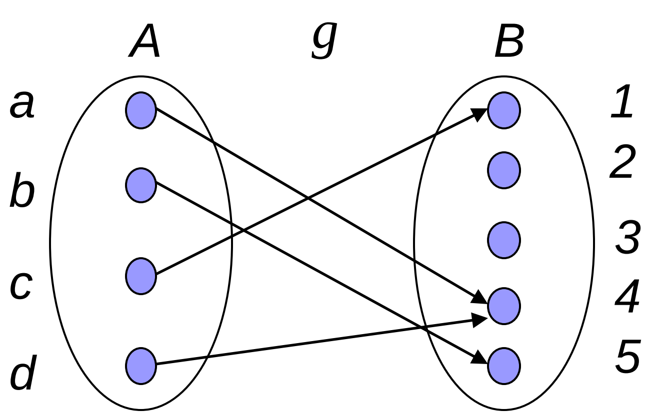
$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

$$(f_1 f_2)(x) = f_1(x) \cdot f_2(x) = x^2(x - x^2) = x^3 - x^4$$

- **Def 5:** A function f is said to be one-to-one, or injective, iff $f(x) \neq f(y)$ whenever $x \neq y$.
- **Example 3 :**



is 1-1

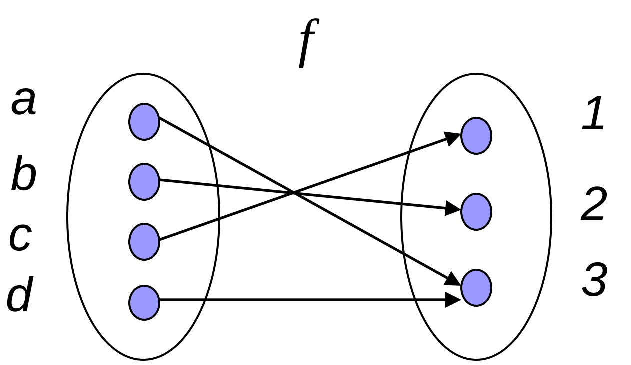


not 1-1 because $g(a) = g(d) = 4$

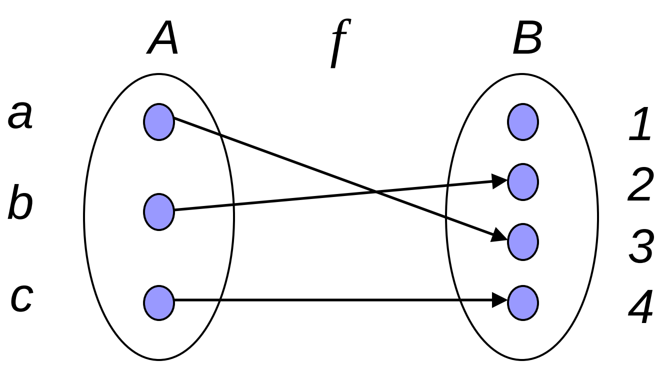
- Example 4: Determine whether the function $f(x) = x + 1$ is one-to-one ?
- Sol : when $x \neq y$ then $x + 1 \neq y + 1$

  $f(x) \neq f(y)$

  $\therefore f$ is 1-1
- Def 7 : A function $f : A \rightarrow B$ is called onto, or surjective, iff for every element $b \in B$, $\exists a \in A$ with $f(a) = b$. (That is, all the elements of b corresponds to f)
- Example 5 :



onto

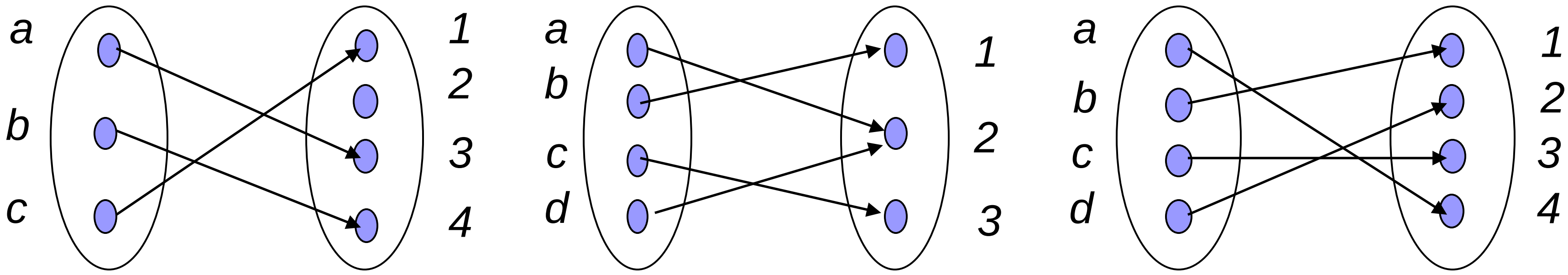


not onto

Note :
when $|A| < |B|$ then, function is not onto.

Def 8 : The function f is a one-to-one correspondence, or a bijection, if it is both 1-1 and onto.

Examples in Fig 5



1-1 , not onto *not 1-1 , onto* *1-1 and onto*

※Suppose that: f : A →B

(1) If f is 1-1 , then |A| ≤ |B|

(2) If f is onto , then |A| ≥ |B|

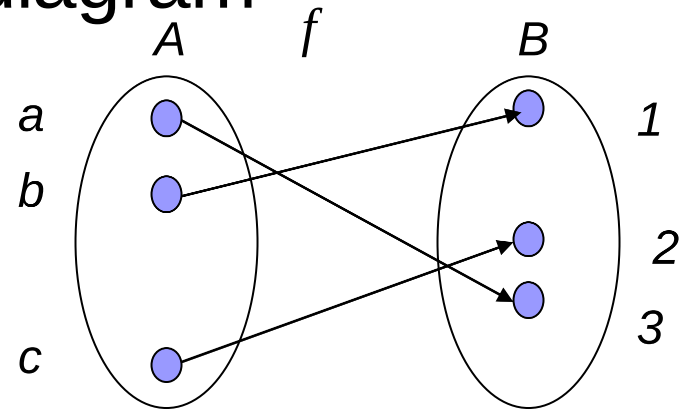
(3) if f is 1-1 and onto , then |A| = |B|.

Inverse Functions

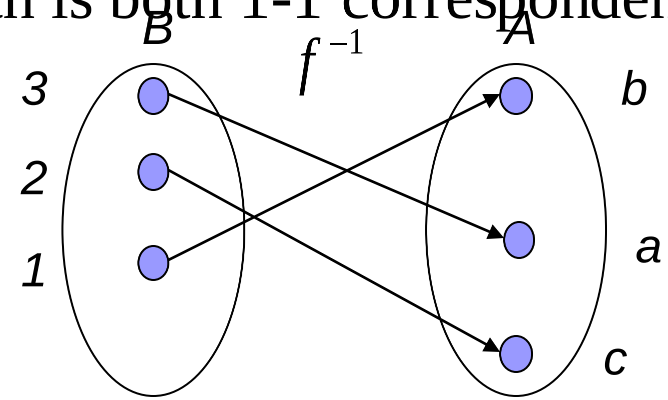
- Let *f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A* such that *f (a) = b. The inverse function of f is denoted by* f^{-1}. Hence, f^{-1}*(b) = a when f (a) = b.*
  - *Remark: Be sure not to confuse the function* f^{-1} with the function 1/*f , which is the function* that assigns to each *x in the domain the value 1/f (x). Notice that the latter makes sense only* when *f (x) is a non-zero real number.*

- Example 6: Let *f be the function from {a, b, c} to {1, 2, 3} such that f (a) = 2, f (b) = 3, and f (c) = 1.* Is *f invertible, and if it is, what is its inverse?*
- *Solution: The function f is invertible because it is a one-to-one correspondence. The inverse* function f^{-1} reverses the correspondence given by *f , so* f^{-1}*(1) = c,* f^{-1}*(2) = a, and* f^{-1}*(3) = b.*

- Example 7: Let A={a, b, c} and B={1, 2, 3}. Suppose f : A →B be defined in the following diagram



- Find the inverse of f
- Sol : f is both is both 1-1 correspondence, hence



- Example 8: Let f : R →R be defined by f(x)=3x-1. Find the inverse of f
- Sol : f is 1-1 and onto and hence f has an inverse

$$Now\ let\ y = 3x - 1$$

$$\Rightarrow x = \frac{y+1}{3}$$

$$\therefore f^{-1}(y) = \frac{y+1}{3}$$

$$\therefore f^{-1}(x) = \frac{x+1}{3}$$

f^{-1} *is also 1-1 and onto*

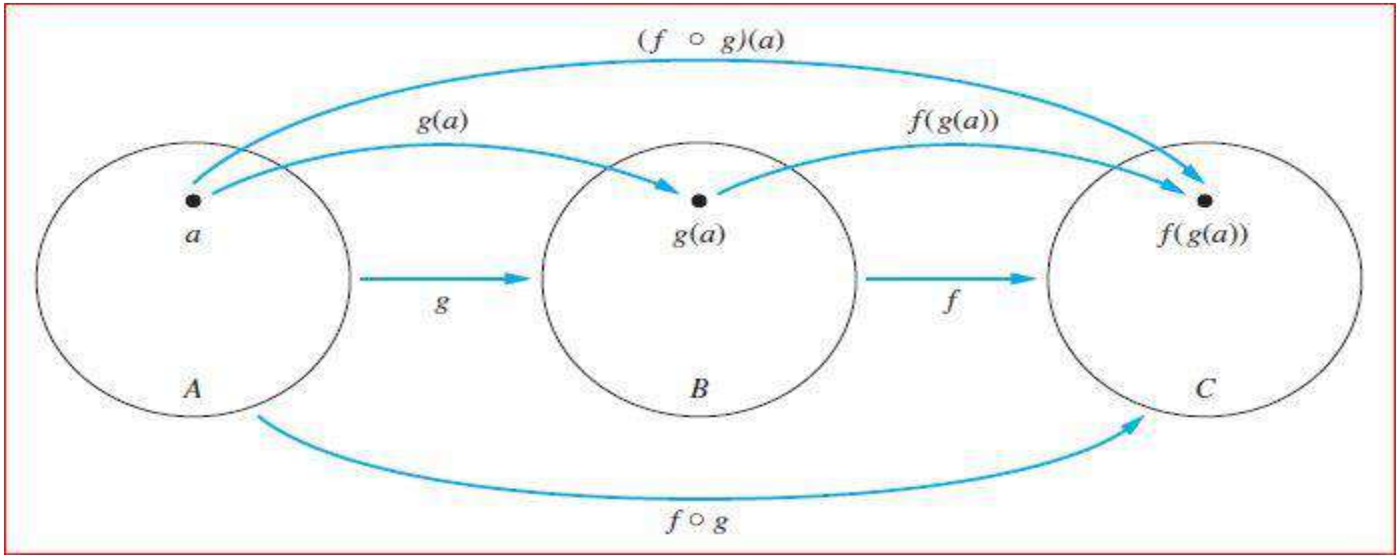
Alternative method of finding inverse

- Replace x with $f^{-1}(x)$ in both sides
- Substitute in $f(f^{-1}(x)) = x$
- Solve for $f^{-1}(x)$ in terms of x

- Example 9: find the inverse of $f(x) = (x+4)^2 + 5$
- Example 10: find the inverse of $f(x) = \frac{x-1}{x-2}$

Composition of Functions

- Let *g be a function from the set A to the set B and let f be a function from the set B to the set* C. *The composition of the functions f and g, denoted for all a ∈ A by f ∘ g, is defined* by $(f \circ g)(a) = f(g(a))$.



- Example :Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

- *Solution:* Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover, $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$ and $(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$.

    - ***Remark:*** $f \circ g$ and $g \circ f$ are not equal. In other words, the commutative law does not hold for the composition of functions.

- Some important functions
- **Def 12 :**
  - floor function : $\lfloor x \rfloor$: $\leq x$ also known as the largest integer, that is,[x]
  - ceiling function : $\lceil x \rceil$: $\geq x$ The smallest integer.
- **Example 11 :**

$\lceil ½ \rceil =$ $\lceil -½ \rceil =$ $\lceil 7 \rceil =$

$\lfloor ½ \rfloor =$ $\lfloor -½ \rfloor =$ $\lfloor 7 \rfloor =$

- **Example 12 :**
  - factorial function :

$f : N \rightarrow Z+ ,\ f(n) = n! = 1 \times 2 \times \ldots \times n$

Exercise

- Why is *f* not a function from **R** to **R** if
  - **a)** $f(x) = 1/x$?
  - **b)** $f(x) = \sqrt{x}$?
  - **c)** $f(x) = \pm\sqrt{(x2 + 1)}$?
- Determine whether each of these functions from **Z to Z** is one-to-one.
  - **a)** $f(n) = n - 1$
  - **b)** $f(n) = n^2 + 1$
  - **c)** $f(n) = n^3$
  - **d)** $f(n) = \lfloor n/2 \rfloor$
- Give an explicit formula for a function from the set of integers to the set of positive integers that is
  - **a)** one-to-one, but not onto.
  - **b)** onto, but not one-to-one.
  - **c)** one-to-one and onto.
  - **d)** neither one-to-one nor onto.

Exercise Contd

- Determine whether each of these functions is a bijection from **R** to **R**.
  - **a)** $f(x) = 2x + 1$
  - **b)** $f(x) = x^2 + 1$
  - **c)** $f(x) = x^3$
  - **d)** $f(x) = (x^2 + 1)/(x^2 + 2)$
- Find these values.
  - **a)** $\lfloor 1.1 \rfloor$ ***b)*** $\lceil 1.1 \rceil$ **c)** $\lfloor -0.1 \rfloor$ ***d)*** $\lceil -0.1 \rceil$ **e)** $\lceil 2.99 \rceil$ ***f)*** $\lceil -2.99 \rceil$
  - **g)** $\left\lfloor \frac{1}{2} + \left\lceil \frac{1}{2} \right\rceil \right\rfloor$ **h)** $\left\lceil \left\lfloor \frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil + \frac{1}{2} \right\rceil$
- Let f be the function from **R** to **R** defined by $f(x) = x^2$. Find
  - **a)** $f^{-1}(\{1\})$ **b)** $f^{-1}(\{x \mid 0 < x < 1\})$
  - **c)** $f^{-1}(\{x \mid x > 4\})$

Outline

- Relations and their properties
- Representing Relations

Relations and their properties.

The most direct way to express a relationship between elements of two sets is to use ordered pairs.

For this reason, sets of ordered pairs are called **binary relations**.

Def 1

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be sets. A binary relation from* $\mathbf{A}$ *to* $\mathbf{B}$ *is a subset* $\mathbf{R}$ *of* $\mathbf{A} \times \mathbf{B} = \{ (a, b) : a \in A, b \in B \}$.

Example 1.

$\mathbf{A}$: *the set of students in your school.*

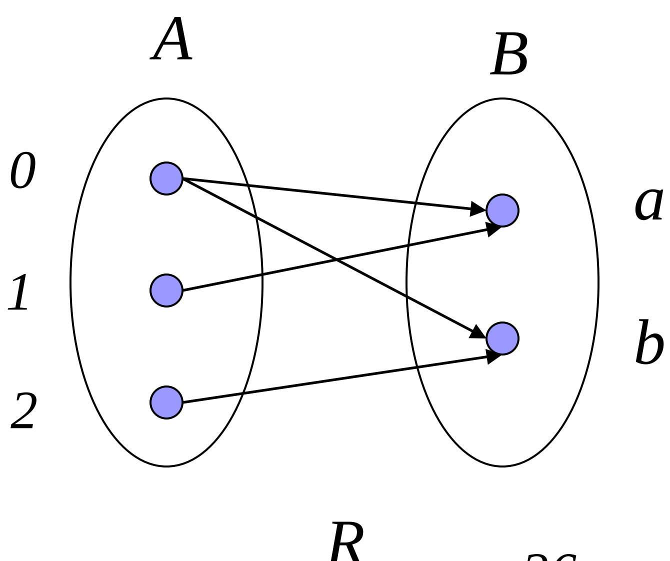
$\mathbf{B}$: *the set of courses.*

$\mathbf{R} = \{ (a, b) : a \in A, b \in B, a \text{ is enrolled in course } b \}$

Def 1'. We use the notation aRb to denote that $(a, b)\in R$, and $a \not R b$ to denote that $(a,b)\notin R$.

Moreover, a is said to be related to b by R if aRb.

Example 2. *Let* $A=\{0, 1, 2\}$ *and* $B=\{a, b\}$, *then* $\{(0,a),(0,b),(1,a),(2,b)\}$ *is a relation* R *from* A *to* B. *This means, for instance, that* $0Ra$, *but that* $1 \not R b$.



$$R \subseteq A\times B = \{ (0,a) , (0,b) , (1,a) , \underset{\notin R}{\underline{(1,b)}} , \underset{\notin R}{\underline{(2,a)}} , (2,b)\}$$

Example 3

- Let *A be the set of cities in the Nigeria, and let B be the set of the 37 states in the Nigeria.* Define the relation *R by specifying that (a, b) belongs to R if a city with name a is in* the state *b. For instance, (Zaria, Kaduna), (Minna, Niger), (Ikeja, Lagos),* (Owerri, Imo), (Ogbomaso, Oyo), (Bama, Borno), and (Otukpo, Benue) are in *R.*

Note. Relations vs. Functions

A relation can be used to express a 1-to-many relationship between the elements of the sets ***A*** and ***B***.

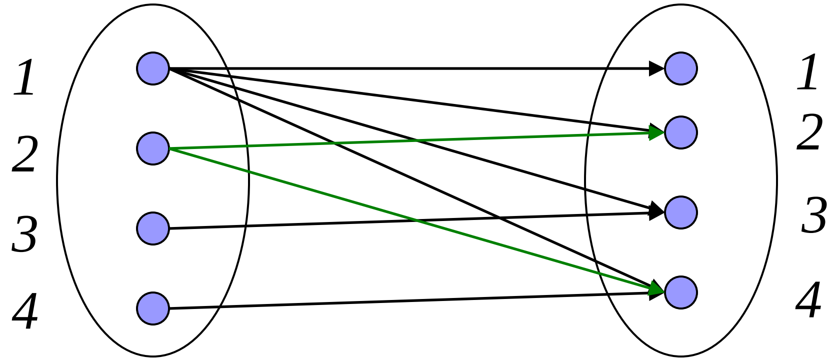
A function represents a relation where exactly one element of *B* is related to each element of *A*

Def 2. *A relation on the set **A** is a subset of* $\boldsymbol{A} \times \boldsymbol{A}$ *(i.e., a relation from **A** to **A**).*

Example 4.

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{ (a, b) | \ a \text{ divides } b \}$?

Sol :



$R = \{ (1,1), (1,2), (1,3), (1,4),$

$(2,2), (2,4),$

$(3,3),$

$(4,4) \}$

Example 5. Consider the following relations on **Z**.

$R_1 = \{ (a, b) \mid a \leq b \}$

$R_2 = \{ (a, b) \mid a > b \}$

$R_3 = \{ (a, b) \mid a = b \text{ or } a = -b \}$

$R_4 = \{ (a, b) \mid a = b \}$

$R_5 = \{ (a, b) \mid a = b+1 \}$

$R_6 = \{ (a, b) \mid a + b \leq 3 \}$

Which of these relations contain each of the pairs (1,1), (1,2), (2,1), (1,–1), and (2,2)?

Sol :

	(1,1)	(1,2)	(2,1)	(1,–1)	(2,2)
R_1	•	•			•
R_2			•	•	
R_3	•			•	•
R_4	•				•
R_5			•		
R_6	•	•	•	•	

Properties of Relations

Def 3. A relation R on a set A is called reflexive if $(a,a)\in R$ for every $a\in A$.

Example 6. Consider the following relations on $\{1, 2, 3, 4\}$:

$R_2 = \{ (1,1), (1,2), (2,1) \}$

$R_3 = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$R_4 = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

which of them are reflexive ?

Sol :

R_3

Example 7. Which of the relations from Example 5 are reflexive?

$R_1 = \{ (a, b) \mid a \le b \}$

$R_2 = \{ (a, b) \mid a > b \}$

$R_3 = \{ (a, b) \mid a = b \text{ or } a = -b \}$

$R_4 = \{ (a, b) \mid a = b \}$

$R_5 = \{ (a, b) \mid a = b+1 \}$

$R_6 = \{ (a, b) \mid a + b \le 3 \}$

Sol : R_1, R_3 and R_4

Example 8. *Is the "divides" relation on the set of positive integers reflexive?*

Sol : *Yes.*

Def 4.

(1) A relation R on a set A is called symmetric

if for $a, b \in A$,

$$(a, b) \in R \Rightarrow (b, a) \in R.$$

(2) A relation R on a set A is called antisymmetric if for $a, b \in A$,

$$(a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b.$$

Example 9. *Which of the relations from Example 7 are symmetric or antisymmetric ?*

$\boldsymbol{R_2} = \{ (1,1), (1,2), (2,1) \}$

$\boldsymbol{R_3} = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$\boldsymbol{R_4} = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

Sol :

$\boldsymbol{R_2}$, $\boldsymbol{R_3}$ *are symmetric*

$\boldsymbol{R_4}$ *are antisymmetric.*

Example 10. *Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?*

Sol : *It is not symmetric since* $1|2$ *but* $2 \nmid 1$.

It is antisymmetric since $a|b$ *and* $b|a$ *implies* $a=b$.

Def 5. A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called transitive if for $\boldsymbol{a, b, c \in A}$, $\boldsymbol{(a, b) \in R}$ and $\boldsymbol{(b, c) \in R \Rightarrow (a, c) \in R}$.

Example 11. *Is the "divides" relation on the set of positive integers transitive?*

$$\textbf{Sol :}\ \textit{Suppose } a|b \textit{ and } b|c$$
$$\Rightarrow a|c$$
$$\Rightarrow \textit{transitive}$$

Example 12. *Which of the relations in Example 7 are transitive ?*

$\boldsymbol{R_2} = \{ (1,1), (1,2), (2,1) \}$

$\boldsymbol{R_3} = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$\boldsymbol{R_4} = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

Sol :

$\boldsymbol{R_2}$ *is not transitive since*

$(2,1) \in \boldsymbol{R_2}$ and $(1,2) \in \boldsymbol{R_2}$ but $(2,2) \notin \boldsymbol{R_2}$.

$\boldsymbol{R_3}$ *is not transitive since*

$(2,1) \in \boldsymbol{R_3}$ and $(1,4) \in \boldsymbol{R_3}$ but $(2,4) \notin \boldsymbol{R_3}$.

$\boldsymbol{R_4}$ *is transitive.*

Exercise

- Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if
  - **a)** $x != y$. **b)** $xy \geq 1$. **c)** $x = y + 1$ or $x = y - 1$. **d)** $x \equiv y \pmod 7$.
  - **e)** x is a multiple of y. **f)** x and y are both negative or both nonnegative.
  - **g)** $x = y2$. **h)** $x \geq y2$.

- For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.
  - **a)** $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
  - **b)** $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
  - **c)** $\{(2, 4), (4, 2)\}$ **d)** $\{(1, 2), (2, 3), (3, 4)\}$
  - **e)** $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$ **f)** $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$
-
- Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if
  - **a)** a is taller than b. **b)** a and b were born on the same day.
  - **c)** a has the same first name as b. **d)** a and b have a common grandparent.

Combining Relations

Example 13. Let $\boldsymbol{A} = \{1, 2, 3\}$ and $\boldsymbol{B} = \{1, 2, 3, 4\}$.

The relation $\boldsymbol{R_1} = \{(1,1), (2,2), (3,3)\}$

and $\boldsymbol{R}_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain

$\boldsymbol{R_1} \cup \boldsymbol{R_2} = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$

$\boldsymbol{R_1} \cap \boldsymbol{R_2} = \{(1,1)\}$

$\boldsymbol{R_1} - \boldsymbol{R_2} = \{(2,2), (3,3)\}$

$\boldsymbol{R_2} - \boldsymbol{R_1} = \{(1,2), (1,3), (1,4)\}$

$\boldsymbol{R_1} \oplus \boldsymbol{R_2} = \{(2,2), (3,3), (1,2), (1,3), (1,4)\}$

symmetric difference, That is, $(\mathbf{R_1} \cup \mathbf{R_2}) - (\mathbf{R_1} \cap \mathbf{R_2})$

- **Def 6.** Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by $S \circ R$.

Example 14. *What is the composite of relations* **R** *and* **S**, *where* **R** *is the relation from* $\{1, 2, 3\}$ *to* $\{1, 2, 3, 4\}$ *with* **R** $= \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ *and* **S** *is the relation from* $\{1, 2, 3, 4\}$ *to* $\{0, 1, 2\}$ *with* **S** $= \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Sol. $S \circ R$ *is the relation from* $\{1, 2, 3\}$ *to* $\{0, 1, 2\}$ *with*
$S \circ R = \{(1, 0), (1,1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

n-ary Relations and Their Applications

Def 1. *Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and n is called its degree*

- **Example 1.** Let *R be the relation on* $N \times N \times N$ *consisting of triples (a, b, c), where a, b, and c are integers* with $a < b < c$.
  - *Then* $(1, 2, 3) \in R$, *but* $(2, 4, 3) \notin R$. *The degree of this relation is 3. Its domains* are all equal to the set of natural numbers.

Databases and Relations

- A database consists of **records,** which are *n-tuples, made up of fields.*
- *The fields are the* entries of the *n-tuples. For instance, a database of student records may be made up of fields* containing the name, student number, major, and grade point average of the student. The relational data model represents a database of records as an *n-ary relation.*

Databases and Relations

- A database consists of **records,** which are *n-tuples, made up of fields.*
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TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

- *Thus, student records* are represented as 4-tuples of the form

  (*Student_name, ID_number, Major, GPA).*
- *A sample* database of six such records is
  - (Ackermann, 231455, Computer Science, 3.88)
  - (Adams, 888323, Physics, 3.45)
  - (Chou, 102147, Computer Science, 3.49)
  - (Goodfriend, 453876, Mathematics, 3.45)
  - (Rao, 678543, Mathematics, 3.90)
  - (Stevens, 786576, Psychology, 2.99).

Representing Relations

Representing Relations using Matrices

Suppose that R is a relation from $A=\{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.

The relation R can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

Example 1. Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$

Let $R = \{(a, b) \mid a > b, a \in A, b \in B\}$.

What is the matrix M_R representing R?

Sol :

$$R = \{(2, 1), (3, 1), (3, 2)\}$$

$$\begin{array}{ccc} & & B \\ & & \begin{array}{cc} 1 & 2 \end{array} \\ A & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

$$\therefore M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Let $A=\{a_1, a_2, \ldots, a_n\}$.

A relation R on A is reflexive iff $(a_i,a_i)\in R, \forall i$.

i.e.,

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{array} \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & \cdots & & \cdots & a_n \end{array} \\ \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & & \\ & & & & & 1 \end{bmatrix} \end{array}$$

Elements on the diagonal = 1

- ※ The relation R is symmetric iff $(a_i,a_j) \in R \Rightarrow (a_j,a_i) \in R$.
  This means $m_{ij} = m_{ji}$ (M_R is the symmetry matrices).

$$M_R = \begin{bmatrix} & & 1 & & \\ & & & & \\ 1 & & & & 0 \\ & & & & \\ & & 0 & & \end{bmatrix} = (M_R)^t$$

The relation R is antisymmetric iff

$$(a_i,a_j)\in R \text{ and } i \neq j \Rightarrow (a_j,a_i)\notin R.$$

This means that if $m_{ij}=1$ with $i\neq j$, then $m_{ji}=0$.

i.e.,

$$M_R = \begin{bmatrix} & 1 & & \\ 0 & & 0 & \\ & 1 & & 0 \\ & & 0 & \end{bmatrix}$$

(Transitive nature of the matrix is not easy to judge directly, need to do some operations)

Example 2. Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric ?

Sol :

reflexive

symmetric

not antisymmetric

eg. Suppose that $S=\{0, 1, 2, 3\}$. Let R be a relation containing (a, b) if $a \le b$, where $a \in S$ and $b \in S$.
Is R reflexive, irreflexive, symmetric, antisymmetric ?

Sol :

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\therefore$ *R is reflexive and antisymmetric, not symmetric.*

Def. irreflexive: $(a,a) \notin R, \ \forall a \in A$

Example 3. Suppose the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ *and* $R_1 \cap R_2$*?*

Sol :

$$M_{R_1 \cup R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R_1 \cap R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 4. Find the matrix representing the relation $S \circ R$, where the matrices representing R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Sol :

$$M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$M_R \times M_S$ (Matrix multiplication) after if >1 figure change to 1)

Example 5. Find the matrix representing the relation R^2, where the matrix representing R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Sol :

$$M_{R^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing Relations using Digraphs

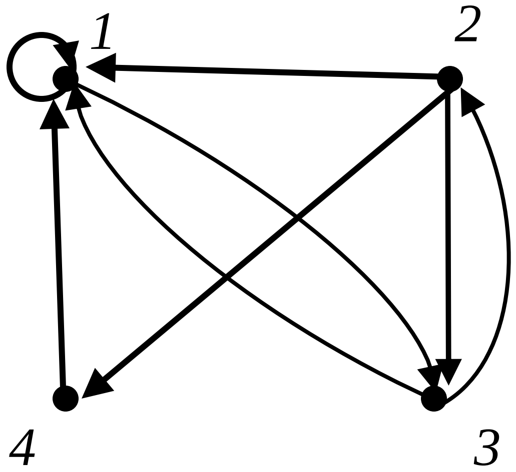
Def 1. *A directed graph (digraph) consists of a set* **V** *of vertices (or nodes) together with a set* **E** *of ordered pairs of elements of* **V** *called edges (or arcs).*

Example 6. Show the digraph of the relation
$R=\{(1,1),(1,3),(2,1),(2,3),(2,4),\ (3,1),(3,2),(4,1)\}$
on the set $\{1,2,3,4\}$.

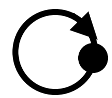
Sol :

vertex : 1, 2, 3, 4

edge : (1,1), (1,3),
(2,1), (2,3), (2,4),
(3,1), (3,2),
(4,1)

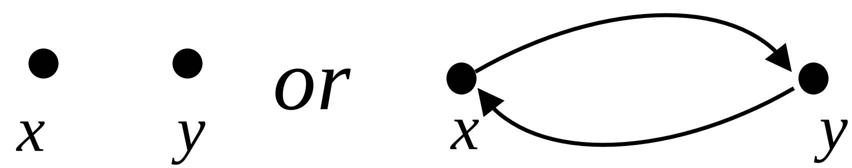


The relation R is reflexive iff
for every vertex,



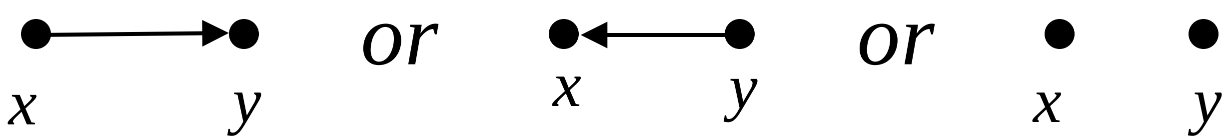
loop

The relation R is *symmetric* *iff for any vertices* $x \neq y$, *either*



two edges in the opposite direction between distinct vertices

The relation R is *antisymmetric* *iff for any* $x \neq y$,



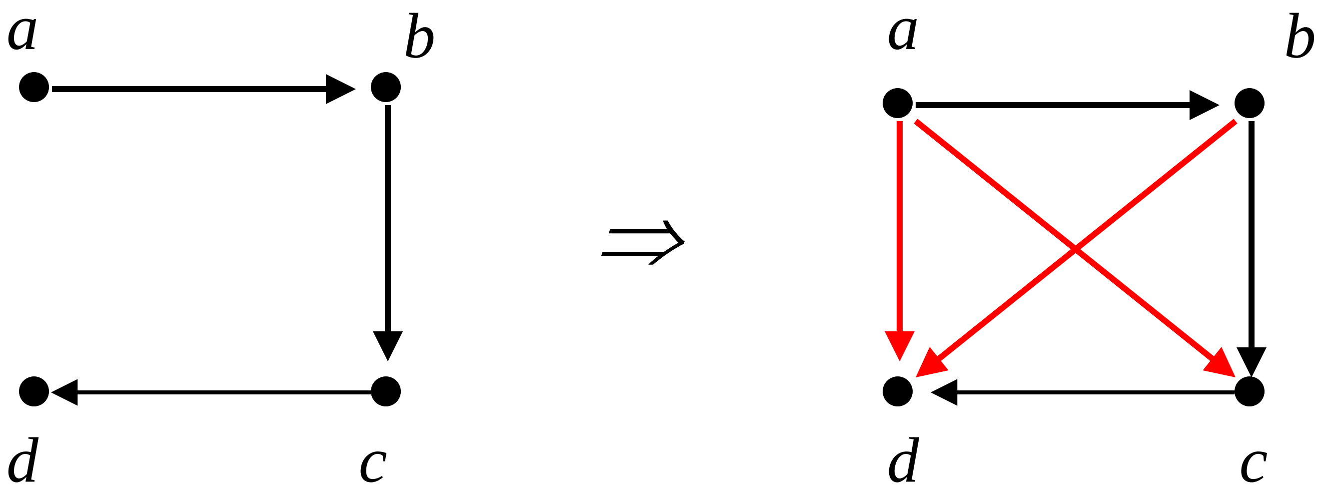
No two edges in opposite directions between distinct vertices

The relation R is transitive iff

for $\boldsymbol{a, b, c} \in \boldsymbol{A}$,

$(\boldsymbol{a, b}) \in \boldsymbol{R}$ and $(\boldsymbol{b, c}) \in \boldsymbol{R} \Rightarrow (\boldsymbol{a, c}) \in \boldsymbol{R}$.

This means:

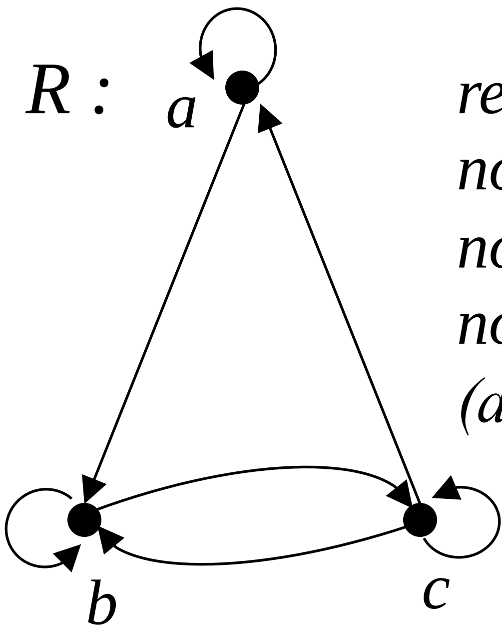


an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z, there is an edge from x to z

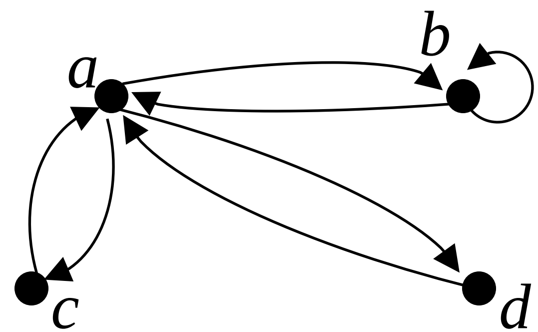
Example 7. Determine whether the relations R and S are reflexive, symmetric, antisymmetric, and/or transitive

Sol :

R :



reflexive,
not symmetric,
not antisymmetric,
not transitive
$(a \to b,\ b \to c,\ a \not\to c)$



S

not reflexive,
symmetric
not antisymmetric
not transitive
$(b \to a,\ a \to c,\ b \not\to c)$

Exercise

- Represent each of these relations on {1, *2, 3, 4} with a* matrix (with the elements of this set listed in increasing order).
  - a) ***{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}***
  - b) ***{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}***
  - c) ***{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2),*** *(3, 4), (4, 1), (4, 2), (4, 3)}*
  - d) ***{(2, 4), (3, 1), (3, 2), (3, 4)}***
- List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

$$a)\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}\quad \text{b)}\begin{bmatrix}0&1&0\\0&1&0\\0&1&0\end{bmatrix}\quad \text{c)}\begin{bmatrix}1&1&1\\1&0&1\\1&1&1\end{bmatrix}$$

Exercise Contd

- Let *R*1 and *R*2 be relations on a set *A* represented by the matrices

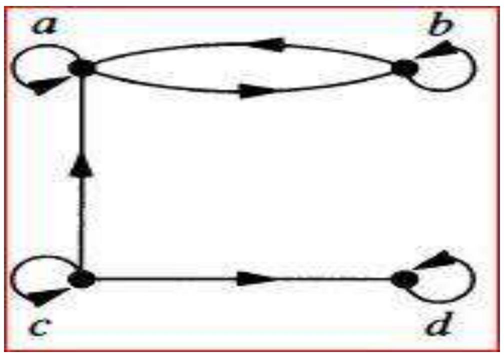
$$M_{R1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } M_{R2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

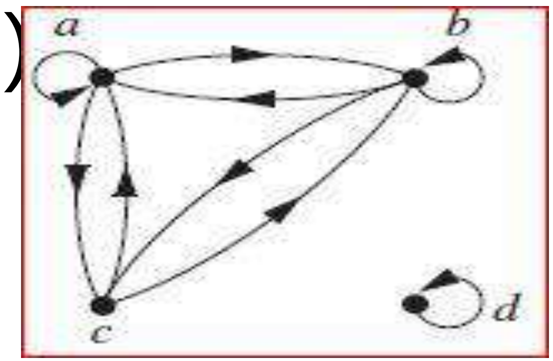
  - Find the matrices that represent

$$\text{a) } R_1 \cup R_2 \quad \text{b) } R_1 \cap R_2 \quad \text{c) } R_2 \circ R_1 \quad \text{d) } R_1 \circ R_1 \quad \text{e) } R_1 \oplus R_2$$

- Draw the directed graph that represents the relation
  - {*(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)*}

Exercise Contd

- List the ordered pairs in the relations represented by the directed graphs below
- a) 

b) 

- Determine whether the relations represented by the directed graphs above are reflexive, symmetric, antisymmetric, and/or transitive.

Sequences and Summations

Sequence (Series)

Def 1. A sequence is a function f from $A \subseteq \mathbf{Z}+$ (or $A \subseteq N$) to a set S. We use $\boldsymbol{a_n}$ to denote $f(n)$, and call $\boldsymbol{a_n}$ a <u>term</u> (item) of the sequence.

Example 1. $\{a_n\}$, where $a_n = 1/n$, $n \in \mathbf{Z}+$

$\Rightarrow a_1 = 1, a_2 = 1/2 , a_3 = 1/3, \ldots$

Example 2. $\{b_n\}$, where $b_n = (-1)^n$, $n \in \mathbf{N}$

$\Rightarrow b_0 = 1, b_1 = -1 , b_2 = 1, \ldots$

- An *arithmetic progression (AP)is a sequence of the form a, a + d, a + 2d, . . . , a + nd, . . .*
  - where the *initial term a and the common difference d are real numbers.*
- *Sum of AP:* $s_n = \frac{n}{2}(2a + (n-1)d)$

- A *geometric progression (GP) is a sequence of the form* $a, ar, ar^2, \ldots, ar^n, \ldots$
  - where the *initial term a and the common ratio r are real numbers.*
- *Sum of GP:* $S_n = \frac{a(r^{n+1}-1)}{r-1} \quad r \neq 1$

  $S_n = (n+1)a \quad r = 1$

Example 3. How can we produce the terms of a sequence if the first 10 terms are

5, 11, 17, 23, 29, 35,41, 47, 53, 59?

Sol :

$$a_1 = 5$$

$$a_2 = 11 = 5 + 6$$

$$a_3 = 17 = 11 + 6 = 5 + 6 \times 2$$

$$\vdots$$

$$\vdots$$

$$\therefore\ a_n = 5 + 6 \times (n-1) = 6n-1$$

Example 4. Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are

$1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047$?

Sol :

Apparently non-arithmetic progression terms are previously values close to 3

$\Rightarrow$ Guess number as $3^n \pm \ldots$

Compare :

$$\{3^n\} : \ 3, 9, 27, 81, 243, 729, 2187, \ldots$$
$$\{a_n\} : \ 1, 7, 25, 79, 241, 727, 2185, \ldots$$
$$\therefore \ a_n = 3^n - 2 \ \ , n \geq 1$$

※ Summations

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n$$

Here, the variable j is call the index of summation, m is the lower limit, and n is the upper limit.

Example 5. $\sum_{j=1}^{5} j^2 = 1 + 4 + 9 + 16 + 25 = 55$

Example 6.

$$\sum_{i=1}^{n} i = 1+2+3+4+\ldots+n = \frac{n(n+1)}{2}$$

Example 7.

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2} \qquad \sum_{i=1}^{n-1} i = \frac{(n-1)(n)}{2}$$

Rules of manipulating sums

1) $\sum_{i=1}^{n} ki = k\sum_{1=1}^{n} i$

2) $\sum_{i=1}^{n} (f(i)+g(i)) = \sum_{1=1}^{n} f(i) + \sum_{1=1}^{n} g(i)$

3) $\sum_{i=1}^{n} k = k\sum_{1=1}^{n} = kn$

$Example$: find the value of $\sum_{i=1}^{n}(2i+3)$

$$sol: \sum_{i=1}^{n}(2i+3) = \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 3$$

$$= 2\sum_{i=1}^{n} i + 3\sum_{i=1}^{n} \quad = 2\left[\frac{n(n+1)}{2}\right] + 3n$$

$$= n^2 + n + 3n = n^2 + 4n$$

Double/Multiple summation

- **These arise when analyzing nested loops in computer science programs. Example of nested loops**

$$\sum_{i=1}^{20} \sum_{j=1}^{5} ij\,, \qquad \sum_{i=1}^{3} \sum_{j=0}^{2} \sum_{k=1}^{2} ijk$$

- **To evaluate double/multiple summation, we first expand the inner summation and then continue with the outer summation**
- **Example 8.**

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i+2i+3i) = \sum_{i=1}^{4} 6i = 6(1+2+3+4) = 60$$

Example 9.

$$\sum_{S\in\{0,2,4\}} S = 0+2+4=6$$

Table 2. Some useful summation formulae

$$(1)\quad \sum_{k=0}^{n} ar^k = \frac{a(r^{n+1}-1)}{r-1},\quad r\neq 1$$

$$(2)\quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$(3)\quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

- Given the sum

$$\sum_{i=1}^{7} 2i$$

- Assuming we know the index of summation to be in the range 0 to 6 instead of 1 to 7. then we let j=i-1 and substitute it in the summation

$$\sum_{i=1}^{7} 2i = \sum_{i=0}^{6} 2(j+1) \qquad \text{since } j = i-1 \Rightarrow i = j+1$$

- **Example 9.** Find $\sum_{i=5}^{10} i$
- **Sol.** $\sum_{i=1}^{10} i = \sum_{i=1}^{4} i + \sum_{i=5}^{10} i$

$$\sum_{i=5}^{10} i = \sum_{i=1}^{10} i - \sum_{i=1}^{4} i \qquad \text{from } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\frac{10(10+1)}{2} - \frac{4(4+1)}{2} = 55 - 10 = 45$$

Exercise

- What is the value of

$$(1)\ \sum_{i=1}^{7} 8i-7 \quad (2)\ \sum_{m=1}^{10} 6m+5 \quad (3)\ \sum_{i=1}^{5} i(3+2i)$$

- Compute the following

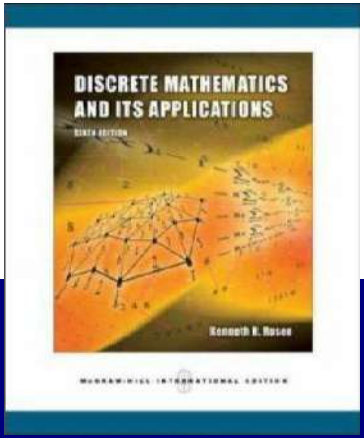
$$(1)\ \sum_{i=1}^{3}\sum_{j=1}^{2}(i-j) \quad (2)\ \sum_{i=1}^{3}\sum_{j=0}^{2} j \quad (3)\ \sum_{i=0}^{3}\sum_{j=0}^{2}(3i+2j) \quad (4)\ \sum_{i=0}^{3}\sum_{j=0}^{2} i^2 j^3$$

- Find the 1st, 2nd, 3rd and 4th of the sequence if the nth term is

$$(1)\ 2^n+1 \quad (2)\ \frac{n}{2} \quad (3)\ 5 \quad (4)\ 7+4^n$$

- What is the value of

$$(1)\ \sum_{i=3}^{10}\frac{i}{3} \quad (2)\ \sum_{k=2}^{n} 20k \quad (3)\ \sum_{k=100}^{200} k \quad (4)\ \sum_{k=99}^{200} k^3$$



Discrete Mathematics

Counting

The Basics of counting

※ A counting problem:

Each user on a computer system has a password, which is six to eight characters long, where each characters is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

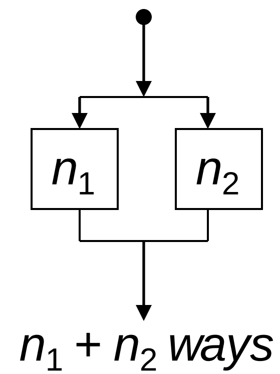
※ This section introduces

- a variety of other counting problems
- the basic techniques of counting.

Basic counting principles

※ **The sum rule:**

If a first task can be done in n_1 ways and a second task in n_2 ways, and if these tasks cannot be done at the same time. then there are n_1+n_2 ways to do either task.



Example 1

- Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Sol: There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student.

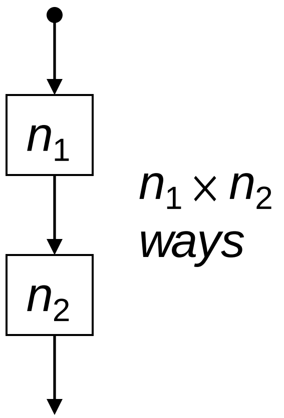
By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick this representative.

Example 2 A student can choose a computer project from one of three lists. The three lists contain 23, 15 and 19 possible projects respectively.
How many possible projects are there to choose from?

Sol: $23+15+19=57$ projects.

※ **The product rule:**

Suppose that a procedure can be broken down into two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.



Example 3 The chair of an auditorium (The great Hall) is to be labeled with <u>a letter</u> and <u>a positive integer not exceeding 100.</u> What is the largest number of chairs that can be labeled differently?

Sol: $\underbrace{26}_{\text{letter}} \times \underbrace{100}_{\boxed{1 \le x \le 100,\ x \in \mathbf{N}}} = 2600$ ways to label chairs.

Example 4 How many different bit strings are there of length seven?

Sol:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \square & \square & \square & \square & \square & \square & \square \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 0,1 & 0,1 & 0,1 & & \ldots\ldots & & 0,1 \end{array} \quad \rightarrow 2^7 \text{ ways}$$

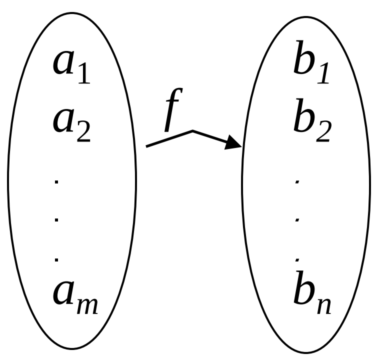
Example 4

How many different <u>license plates</u> are available if each plate contains a sequence of <u>3 letters followed by 3 digits</u> ?

Sol: $\underbrace{\square\ \square\ \square}_{\text{letter}} \quad \underbrace{\square\ \square\ \square}_{\text{digit}} \rightarrow 26^3 \,.\, 10^3$

Example 6 How many functions are there from a set with m elements to one with n elements?

Sol:



$f(a_1)=?$ Can map to $b_1, b_2, \ldots b_n$, Totalling ***n*** ways

$f(a_2)=?$ Can map to $b_1, b_2, \ldots b_n$, Totalling ***n*** ways

$\vdots$

$f(a_m)=?$ Can map to $b_1, b_2, \ldots b_n$, Totalling ***n*** ways

$\therefore\ n^m$

Example 5 How many one-to-one functions are there from a set with m elements to one with n element? $(m \le n)$

Sol: $f(a_1) = ?$ Can map to $b_1, b_2, \ldots b_n$, a total of n functions

$f(a_2) = ?$ Can map to $b_1, b_2, \ldots b_n$, but not $= f(a_1)$, a total of $n-1$ functions

$f(a_3) = ?$ Can map to $b_1, b_2, \ldots b_n$, but not $= f(a_1)$, or $= f(a_2)$, a total of $n-2$ functions

⋮

⋮

$f(a_m) = ?$ Not $= f(a_1), f(a_2), \ldots, f(a_{m-1})$, a total of $n-(m-1)$ functions

∴there are a total of $n \times (n-1) \times (n-2) \times \ldots \times (n-m+1)$ 1-1 functions #

Example 6 Each user on a computer system has a password which is 6 to 8 characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit.
How many possible passwords are there?

Sol: P_i : # of possible passwords of length i , i=6,7,8

$P_6 = 36^6 - 26^6$

$P_7 = 36^7 - 26^7$

$P_8 = 36^8 - 26^8$

$\therefore P_6 + P_7 + P_8 = 36^6 + 36^7 + 36^8 - 26^6 - 26^7 - 26^8$ possible passwords

Example 7

In a version of Basic, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of Basic?

Sol:

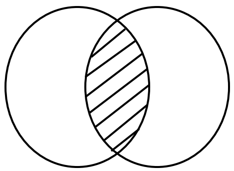
Let V_i be the number of variable names of length i.

$$V_1 = 26$$

$$V_2 = 26 \cdot 36 - 5$$

$\therefore 26 + 26 \cdot 36 - 5$ different names.

※ The Inclusion-Exclusion Principle



A B

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 8 How many bit strings of length eight either start with a 1 bit or end with the two bits 00 ?

Sol:

1 2 3 4 5 6 7 8

□ □ □ □ □ □ □ □

↑ ↑ . . . .

① **1** 0,1 0,1 $\rightarrow$ A total of 2^7 bit strings

② . . . . 0 0 $\rightarrow$ A total of 2^6 bit strings

③ **1** . . . . 0 0 $\rightarrow$ A total of 2^5 bit strings

$\therefore\ 2^7 + 2^6 - 2^5$ bit strings

Example 9 A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Sol:

34 of the applicants majored neither in computer science nor in business

- The subtraction rule, or the principle of inclusion–exclusion, can be generalized to find the number of ways to do one of *n different tasks or, equivalently, to find the number of elements* in the union of *n sets, whenever n is a positive integer.*
- For 3 sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example 10

A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Sol:

$|S \cap F \cap R| = 7.$

The Principle of Inclusion–Exclusion

- Let $A_1, A_2, \ldots, A_n$ *be finite sets.* Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

- The inclusion–exclusion principle gives a formula for the number of elements in the union of *n sets for every positive integer n. There are terms in this formula for the number of elements* in the intersection of every nonempty subset of the collection of the *n sets is. Hence, there* are $2^n - 1$ *terms in this formula.*

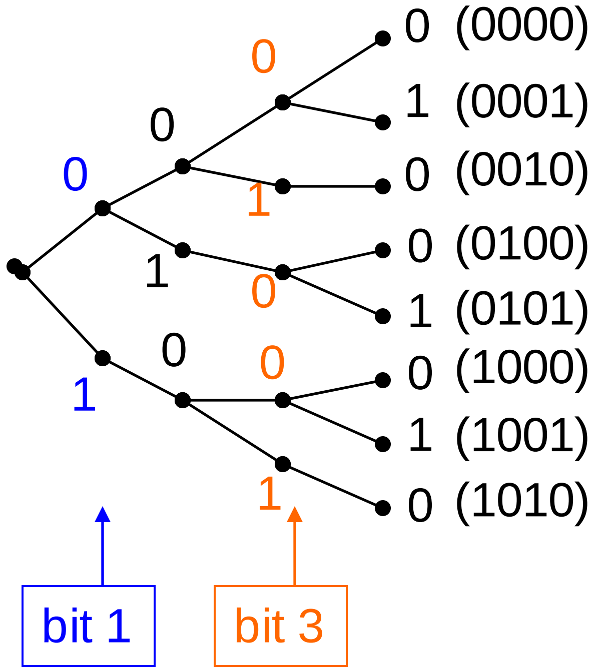
Example 11

Give a formula for the number of elements in the union of four sets.

※ Tree Diagrams

Example10 How many bit strings of length four do not have two consecutive 1s ?

Sol:



∴ 8 bit strings

Exercise

- How many bit strings of length ten both begin and end with a 1?
- How many strings of five ASCII characters contain the character @ (“at” sign) at least once? [*Note:* There are 128 different ASCII characters.
- How many strings of three decimal digits
  a) do not contain the same digit three times?
  b) begin with an odd digit?
  c) have exactly two digits that are 4s?
- How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?
- How many subsets of a set with 100 elements have more than one element?
- A **palindrome** is a string whose reversal is identical to the string. How many bit strings of length *n* are palindromes?
- How many positive integers not exceeding 100 are divisible either by 4 or by 6?
- How many ways are there to arrange the letters a, b, c, and d such that a is not followed immediately by b?

Ex 40. How many subsets of a set with 100 elements have more than one element ?

Sol: ?

Ex 41. A palindrome is a string whose reversal is identical to the string. How many bit strings of length n are palindromes ?

(abcdcba is a palindrome, abcd is not)

Sol: ?

The Pigeonhole Principle

Theorem 1 (The Pigeonhole Principle)

If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof

Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction.

Example 1. Among any 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example 2 In any group of 27 English words, there must be at least two that begin with the same letter.

Example 3 How many students must be in a class to guarantee that at least two students receive the same score on the final exam ? (0~100 points)

Sol: 102. (101+1)

Theorem 2. (The generalized pigeon hole principle)
If $\underline{N}$ objects are placed into $\underline{k}$ boxes, then there is at least one box containing at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

e.g. 21 objects, 10 boxes $\Rightarrow$ there must be one box containing at least $\left\lceil \frac{21}{10} \right\rceil = 3$ objects.

Example 4 Among 100 people there are at least $\left\lceil \frac{100}{12} \right\rceil = 9$ who were born in the same month. (100 objects, 12 boxes)

Def. Suppose that $a_1, a_2, \ldots, a_N$ is a sequence of numbers. A subsequence of this sequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ where $1 \le i_1 < i_2 < \ldots < i_m \le N$

(*i.e.*, A subsequence is a sequence obtained from the original sequence)

e.g. sequence: 8, 11, 9, 1, 4, 6, 12, 10, 5, 7

subsequence: 8, 9, 12 (✓)

9, 11, 4, 6 (✗)

Def. A sequence is called increasing if $a_i \le a_{i+1}$

A sequence is called decreasing if $a_i \ge a_{i+1}$

A sequence is called strictly increasing if $a_i < a_{i+1}$

A sequence is called strictly decreasing if $a_i > a_{i+1}$

Theorem 3. Every sequence of $\underline{n^2+1}$ distinct real numbers contains a subsequence of length $\underline{n+1}$ that is either strictly increasing or strictly decreasing.

Example 6. The sequence 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contains $10=3^2+1$ terms (i.e., n=3).
There is a strictly increasing subsequence of length four, namely, 1, 4, 5, 7. There is also a decreasing subsequence of length 4, namely, 11, 9, 6, 5.

Exercise 7 Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of 5 terms.

Sol: $$\| \; 4,3,2,1 \; \| \; 8,7,6,5 \; \| \; 12,11,10,9 \; \| \; 16,15,14,13$$

Exercise

- Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.
-
    a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.
    b) Is the conclusion in part (a) true if four integers are selected rather than five?
- How many numbers must be selected from the set { I , 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add up to 7?
- There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

Permutations and Combinations

Example 1. In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Sol: There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5 \times 4 \times 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways to arrange all five students in a line for a picture.

This example illustrates how ordered arrangements of distinct objects can be counted. This leads to some terminology.

Permutations and Combinations

Def. A <u>permutation</u> of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an <u>r-permutation</u>.

Example 1. Let $S = \{1, 2, 3\}$.

The arrangement <u>3,1,2</u> is a permutation of S.

The arrangement <u>3,2</u> is a 2-permutation of S.

Theorem 1. The number of r-permutation of a set with n distinct elements is

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

position : 1 2 3 … r

□ □ □ … □

insert : n $n-1$ $n-2$ ... $n-r+1$

Example 2. How many different ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest ?

Sol:

$$P(100,3) = 100 \times 99 \times 98$$

Example 3. Suppose that a saleswoman has to visit 8 different cities. She must begin her trip in a specified city, but she can visit the other cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities ?

Sol:

$$7! = 5040$$

Def. An *r-combination* of elements of a set is an unordered selection of r elements from the set.

Example 4: Let S be the set $\{1, 2, 3, 4\}$.
Then $\{1, 3, 4\}$ is a 3-combination from S.

Theorem 2 The number of r-combinations of a set with n elements, where n is a positive integer and r is an integer with $0 \le r \le n$, equals

$$C_r^n = C(n,r) = \binom{n}{r} = \frac{p(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

Known as the **binomial coefficient**

pf **:**

$$P(n,r) = C(n,r) \times r!$$

Example 10. We see that $C(4,2)=6$, since the 2-combinations of $\{a,b,c,d\}$ are the six subsets $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$ and $\{c,d\}$

Corollary 2. Let n and r be nonnegative integers with $r \le n$. Then $C(n,r) = C(n,n-r)$

pf **:** From Thm 2.

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = C(n,n-r)$$

Meaning: Selecting r elements from n elements, is equivalent to taking out r elements leaving n-r elements remaining

Example 6. How many ways are there to select 5 players from a 10-member tennis team to make a trip to a match at another school ?

Sol: $C(10,5)=252$

Example 7. Suppose there are 9 faculty members in the math department and 11 in the computer science department. How many ways are there to select a committee if the committee is to consist of 3 faculty members from the math department and 4 from the computer science department?

Sol: $C(9,3)\times C(11,4)$

Exercise

- How many permutations of {a , b, e, d, e , t, g} end with a?
- How many bit strings of length 10 contain
  a) exactly four 1s?
  b) at most four 1s?
  c) at least four 1s?
  d) an equal number of 0s and 1s?
- A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?
- How many permutations of the letters ABCDEFG contain

  a) the string BCD? **b)** the string CFGA?

  c) the strings BA and GF? **d)** the strings ABC and DE?

  e) the strings ABC and CDE? **f)** the strings CBA and BED?
- Suppose that a department contains 1 0 men and 1 5 women. How many ways are there to form a committee with six members if it must have the same number of men and women?
- Suppose that a department contains 1 0 men and 1 5 women. How many ways are there to form a committee with six members if it must have more women than men?

Binomial Coefficients

Example 1.

$$(x+y)^3=(x+y)(x+y)(x+y)=?\,x^3+?\,x^2y+?\,xy^2+y^3$$

To obtain a term of the form xy^2,

a y must be chosen from two of the brackets, and an x must be chosen from one of the brackets

(Note : x and y in the same bracket cannot be multiplied)

∴There are different ways to get $xy^2 \Rightarrow$ xy^2 The coefficient $=\binom{3}{2}$

$$\therefore\ (x+y)^3=\binom{3}{0}x^3+\binom{3}{1}x^2y+\binom{3}{2}xy^2+\binom{3}{3}y^3$$

Theorem 1. (The Binomial Theorem)

Let x,y be variables, and let n be a positive integer, then

$$(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\ldots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

Example 2. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Sol: $(2x-3y)^{25} = (2x+(-3y))^{25}$

$$\therefore \quad \binom{25}{13} \times 2^{12} \times (-3)^{13}$$

Cor 1. Let n be a positive integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

pf **:** By Thm 1, let $x = y = 1$

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

Cor 2. Let n be a positive integer. Then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

pf **:** by Thm 1. $(1-1)^n = 0$

Theorem 2. (Pascal's identity)

Let n and k be positive integers with $n \geq k$

Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

PASCAL's triangle

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
 & & \vdots & & & \binom{4}{3} &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

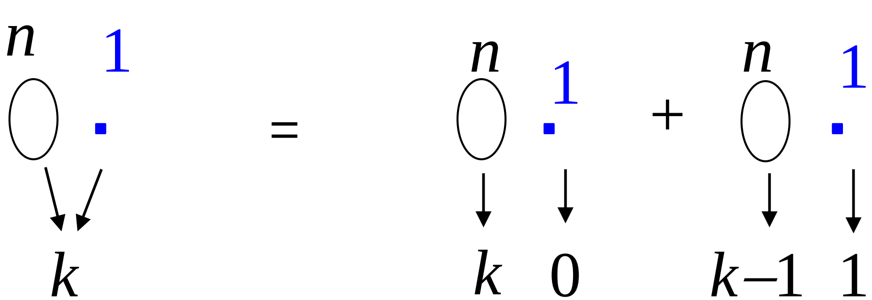
pf : ① (algebraic proof)

$$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!}$$

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{k \times n!}{k!(n-k+1)!} + \frac{(n-k+1)\times n!}{k!(n-k+1)!} = \frac{(n+1)\times n!}{k!(n-k+1)!}$$

② (combinatorial proof):



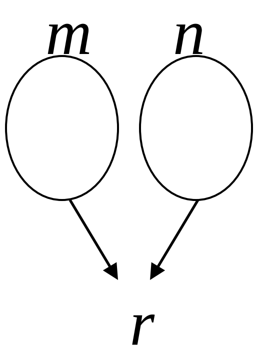
$$\Rightarrow \binom{n+1}{k} = \binom{n}{k-1}\binom{1}{0} + \binom{n}{k}\binom{1}{1}$$

Theorem 3. (Vandermode's Identity)

$$m, n, r \in \mathrm{Z}^+, \quad 0 \le r \le m, n$$

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k) \cdot C(n, k)$$

pf :

$C(m+n, r) =$ 

$$= \begin{array}{cc} m & n \\ \downarrow & \downarrow \\ r, & 0 \end{array} + \begin{array}{cc} m & n \\ \downarrow & \downarrow \\ r-1, & 0 \end{array} + \ldots + \begin{array}{cc} m & n \\ \downarrow & \downarrow \\ 0, & r \end{array}$$

$$= \binom{m}{r}\binom{n}{0} + \binom{m}{r-1}\binom{n}{1} + \ldots + \binom{m}{0}\binom{n}{r}$$

$$= \sum_{k=0}^{r} C(m, r-k) \cdot C(n, k)$$

Exercise

- What is the coefficient of x^9 in $(2 - x)^{19}$?
- Prove that if n and k are integers with $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$
  a) using a combinatorial proof. [Hint: Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]
  b) using an algebraic proof based on the formula for $\binom{n}{k}$ given in Theorem 2.
- Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$
  a) using a combinatorial argument.

Generalized Permutations and Combinations

Permutations with Repetition

Example 1 How many strings of length r can be formed from the English alphabet?

Sol. 26^r

Thm 1. The number of r-permutations of a set of n objects with repetition allowed is n^r.

Thm 2. There are $C(r+n-1, r)$ r-combinations from a set with n elements when repetition of elements is allowed.

pf : (Each r-combination of a set with n elements when repetition is allowed can be represented by a list of $n - 1$ bars and r stars. The $n - 1$ bars are used to mark off n different cells, with the ith cell containing a star for each time the ith element of the set occurs in the combination)

Example : $n = 4$, set is $\{a_1, a_2, a_3, a_4\}$, $r = 6$

A_2 appear 1 time a_3 does not appear

$$\underbrace{*\ *}_{a_1 \text{ appears 2 times}} \mid * \mid \mid \underbrace{*\ *\ *}_{a_4 \text{ appears 3 times}} \Longrightarrow \{a_1, a_1, a_2, a_4, a_4, a_4\}$$

$\therefore \binom{r+n-1}{r}$ combinations #

Example 3. Suppose that a cookie shop has 4 different kinds of cookies. How many different ways can 6 cookies be chosen?

Sol: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. From Theorem 2 this equals $C(4 + 6 - 1, 6) = C(9, 6)$ *ways*

Example 4. How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, x_3 are nonnegative integers?'

Sol: 11 stars to be inserted between 2 bars

By theorem 2 ⇒ $\binom{11+2}{11}$ solutions

※ since x_1, x_2, x_3 are nonnegative integers, $x_1 \ge 1, x_2 \ge 2, x_3 \ge 3,$

Original equation is $x_1 + x_2 + x_3 = 11$

can be changed to $(x_1 - 1) + (x_2 - 2) + (x_3 - 3) = 11 - 1 - 2 - 3 = 5$

At this point $y_1 + y_2 + y_3 = 5$

Where $y_1 = x_1 - 1,\ y_2 = x_2 - 2, y_3 = x_3 - 3$ and $y_1,\ y_2, y_3 \in \boldsymbol{N}$

∴5 stars to be inserted between 2 bars $\Rightarrow \binom{5+2}{5}$ solutions

(Note：case $y_1 = 1, y_2 = 3, y_3 = 1$ is equal to $x_1 = 2, x_2 = 5, x_3 = 4$)

※if the problem is $1 \le x_1 \le 3, x_2 \ge 2, x_3 \ge 3,$ you need to exclude $x_1 > 3$ case

(i.e $x_1 \ge 4$ case)

because $(x_1 - 4) + (x_2 - 2) + (x_3 - 3) = 11 - 9 = 2$

$\therefore$ A total of $\binom{5+2}{5} - \binom{2+2}{2}$ combinations

Exercise

- How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 21$, where x_i , i = 1 , 2, 3, 4, 5, is a nonnegative integer such that
  a) $\mathrm{x}_1 \geq 1$?
  b) $\mathrm{x}_1 \geq 2$ for $\mathrm{i} = 1,2,3,4,5$?
  c) $0 \leq x_1 \leq 10$?
  d) $0 \leq x_1 \leq 3, 1 \leq x_2 < 4, and\ x_3 \geq 15$?
- How many solutions are there to the inequality $x_1 + x_2 + x_3 \leq 11$, where $x_1\ x_2,\ and\ x_3$ are nonnegative integers? [*Hint:* Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$].
- How many positive integers less than 1,000,000 have the

※ Permutations with indistinguishable objects

Example 5. How many different strings can be made by reordering the letters of the word SUCCESS ?

Sol:

There are 3 *S*s, 2 Cs, 1 U and 1 E,

The three *S*s can be placed among the seven positions in **C(7,3)** different ways leaving four positions free.

Then the two Cs can be placed in **C(4, 2)** ways, leaving two free positions.

Then the two Cs can be placed in **C(2, 1)** ways, leaving two free positions.

Hence *E* can be placed in **C(1, 1)** way
Consequently, from the product rule, the number of different strings that can be made is

C(7, 3)C(4, 2)C(2, 1)C(1, 1) = 420

Thm 3. The number of different permutations of n objects, where

$$\begin{cases} \text{type } 1 : & n_1 \text{ ways} \\ \text{type } 2 : & n_2 \text{ ways} \\ \vdots & \\ \text{type } k : & n_k \text{ ways} \end{cases}$$

is $\dfrac{n!}{n_1!n_2!\cdots n_k!}$

pf :

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}=\frac{n!}{n_1!n_2!\cdots n_k!}$$

Exercise

- Solve the following
  a) How many ways are there to deal hands of five cards to six players from a standard 52-card deck?
  b) How many ways are there to distribute ***n*** distinguishable objects into ***k*** distinguishable boxes so that n_i objects are placed in box ***i***?
- How many ways are there to choose a dozen apples from a bushel containing 20 indistinguishable Delicious apples, 20 indistinguishable Macintosh apples, and 20 indistinguishable Granny Smith apples, if at least three of each kind must be chosen?
- How many ways are there to pack eight identical DVDs into five indistinguishable boxes so that each box contains at least one DVD?
- How many ways are there to distribute six indistinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

Distributing objects into Boxes

Distinguishable Objects and distinguishable boxes

Example 8. How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Sol: player 1 : $\binom{52}{5}$ ways
player 2 : from the remaining 5 $\Rightarrow \binom{47}{5}$

$$\therefore \binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5} = \frac{52!}{5!\,47!}\cdot\frac{47!}{5!\,42!}\cdot\frac{42!}{5!\,37!}\cdot\frac{37!}{5!\,32!} = \frac{52!}{5!\,5!\,5!\,5!\,32!}$$

Note: the above question is equivalent to 52 cards in 5 different box method,
that is box 1 for player 1,
box 2 for player 2
box 3 for player 3
box 4 for player 4
while box 5 is for the cards left.

Exercise

- How many ways can ***n*** books be placed on ***k*** distinguishable shelves
  a) if the books are indistinguishable copies of the same title?
  b) if no two books are the same, and the positions of the books on the shelves matter?

Thm 4. The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i=1, 2, \ldots, k$, equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$ (the same as Thm 3)

Indistinguishable Objects and distinguishable boxes

Example 9. How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?

Sol: $C(8+10-1, 10)$

$\Rightarrow$There are $C(n+r-1, n-1)$ ways to place r indistinguishable objects into n distinguishable boxes.

Distinguishable Objects and indistinguishable boxes

Example 6. How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Sol: employees: A, B, C, D

4 people in the same office: {{A, B, C, D}} $\Rightarrow$ 1 way

3 people in the same office, 1 person in another office
: {{A, B, C}, {D}}, {{A, B, D}, {C}},… $\Rightarrow$ 4 ways

2 people in the same office, 2 people in another office
: {{A, B}, {C, D}}, {{A, D}, {B, C}},… $\Rightarrow$ 3 ways

2 people in the same office, The other 2 people in the remaining two Office:
{{A, B}, {C}, {D}}, {{A, D}, {B}, {C}}, … $\Rightarrow$ 6 ways

$\therefore$ Total: 14 ways

Note. There is no simple closed formula. You may refer to Stirling numbers of the second kind. (p. 378)

Indistinguishable Objects and indistinguishable boxes

Example 7. How many ways are there to pack six copies of the same book into four identical boxes, where a box can obtain as many as six books?

Sol:

6	3, 3	2, 2, 1, 1
5, 1	3, 2, 1	$\therefore$ Total 9
4, 2	3, 1, 1, 1	
4, 1, 1	2, 2, 2	

Note. This problem is the same as writing n as the sum of at most k positive integers in nonincreasing order.

That is, $a_1+a_2+\ldots+a_j = n$, where $a_1, a_2, \ldots, a_j$ are positive integers with $a_1 \geq a_2 \geq \ldots \geq a_j$ and $j \leq k$.

No simple closed formula exists.

Discrete Mathematics

Advanced Counting Techniques

Outline

- Recurrence Relations
- Solving Linear Recurrence Relations
- Generating Functions

Recurrence Relations

- We specified sequences by providing explicit formulas for their terms. There are many other ways to specify a sequence. For example, another way to specify a sequence is to provide one or more initial terms together with a rule for determining subsequent terms from those that precede them
- A rule of the latter sort is known as **Recurrence Relation**

DEFINITION 1: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to *recursively define* a sequence.)

Recurrence Relations Contd

- **EXAMPLE 1:** Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$ and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Sol: 5, 8, 11

EXAMPLE 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots,$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Sol: 2, -3

The **initial conditions** for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect. For instance, the initial condition in Example 1 is $a_0 = 2$, *and the initial conditions in Example 2 are* $a_0 = 3$ *and* $a_1 = 5$.

Fibonacci sequence

DEFINITION 2: The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ is defined by the initial conditions $f_0 = 0$, $f_1 = 1$, and the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$.

EXAMPLE 3: Find the Fibonacci numbers $f_2, f_3, f_4, f_5,$ *and* f_6.

Sol: 1, 2, 3, 5, 8

We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula,** for the terms of the sequence.

EXAMPLE 4: Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$. Answer the same question where $a_n = 2n$ and where $a_n = 5$.

EXAMPLE 5: Solve the recurrence relation and initial condition in Example 1.

i. starting with the initial condition $a_1 = 2$, *and working upward until we reach* a_n *to deduce a closed formula for* the sequence.

ii. starting with the term a_n *and working downward until we reach the initial condition* $a_1 = 2$ *to deduce this same* formula.

The technique used in Example 5 is called **iteration.** We have iterated, or repeatedly used, the recurrence relation.

•The first approach is called **forward substitution –** we found successive terms beginning with the initial condition and ending with a_n.

•*The second approach is called* **backward substitution**, because we began with a_n *and iterated to express it in terms of falling* terms of the sequence until we found it in terms of a_1.

EXAMPLE 6: Compound Interest

Suppose that a person deposits $10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Sol: $228,*922.97.*

EXAMPLE 6

- **a)** Find a recurrence relation for the number of ways to climb *n* stairs if the person climbing the stairs can take one stair or two stairs at a time.
- **b)** What are the initial conditions?
- **c)** In how many ways can this person climb a flight of eight stairs?

- **EXAMPLE 7:** A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the nth month.

  a) Set up a recurrence relation for the number of cars produced in the first n months by this factory.

  b) How many cars are produced in the first year?

  c) Find an explicit formula for the number of cars produced in the first n months by this factory

Exercises

- Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

  a) $a_n = 6a_{n-1}$, $a_0 = 2$

  b) $a_n = a^2_{n-1}$, $a_1 = 2$

  c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

- Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

  a) $a_n = -2a_{n-1}$, $a_0 = -1$

  b) $a_n = a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$

  c) $a_n = 3a^2_{n-1}$, $a_0 = 1$

  d) $a_n = na_{n-1} + a^2_{n-2}$, $a_0 = -1$, $a_1 = 0$

- Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

  a) $a_n = 0$. **b)** $a_n = 1$.

  c) $a_n = (-4)^n$. **d)** $a_n = 2(-4)^n + 3$.

Exercises Contd

- Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if

  a) $a_n = 0$? **b)** $a_n = 1$?

  c) $a_n = 2^n$? **d)** $a_n = 4^n$?

  e) $a_n = n4^n$? **f)** $a_n = 2.4^n + 3n4^n$?

- A person deposits $1000 in an account that yields 9% interest compounded annually.

  a) Set up a recurrence relation for the amount in the account at the end of *n* years.

  b) Find an explicit formula for the amount in the account at the end of *n* years.

  c) How much money will the account contain after 100 years?

- Suppose that the number of bacteria in a colony triples every hour.

  a) Set up a recurrence relation for the number of bacteria after *n* hours have elapsed.

  b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

Exercises Contd

- Assume that the population of the world in 2010 was 6.9 billion and is growing at the rate of 1.1% a year.

  a) Set up a recurrence relation for the population of the world *n* years after 2010.

  b) Find an explicit formula for the population of the world *n* years after 2010.

  c) What will the population of the world be in 2030?

- A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with *n* cars made in the *n*th month.

  a) Set up a recurrence relation for the number of cars produced in the first *n* months by this factory.

  b) How many cars are produced in the first year?

  c) Find an explicit formula for the number of cars produced in the first *n* months by this factory.

- An employee joined a company in 2009 with a starting salary of $50,000. Every year this employee receives a raise of $1000 plus 5% of the salary of the previous year.

  a) Set up a recurrence relation for the salary of this employee *n* years after 2009.

  b) What will the salary of this employee be in 2017?

  c) Find an explicit formula for the salary of this employee *n* years after 2009.

Recurrence Relations

- **Example 1.** Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n=a_{n-1}-a_{n-2}$ for $n=2,3,\ldots,$ and suppose that $a_0=3$,and $a_1=5$.

  Here $a_0=3$ and $a_1=5$ are the initial conditions.

  By the recurrence relation,

$$a_2 = a_1-a_0 = 2$$
$$a_3 = a_2-a_1 = -3$$
$$a_4 = a_3-a_2 = -5$$
$$\vdots$$

Q1: Applications ?

Q2: Are there better ways for computing the terms of $\{a_n\}$?

Applications of Recurrence Relations

- We will show that recurrence relations can be used to study and to solve counting problems.

EXAMPLE 2: For example, suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in *n* hours?

Let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer and the initial condition $a_0 = 5$, uniquely determines *an* for all nonnegative integers n.

We can find a formula for *an* using the iterative approach $a_n = 5 \cdot 2^n$ for all nonnegative integers n

※Modeling with Recurrence Relations

We can use recurrence relations to model (describe) a wide variety of problems.

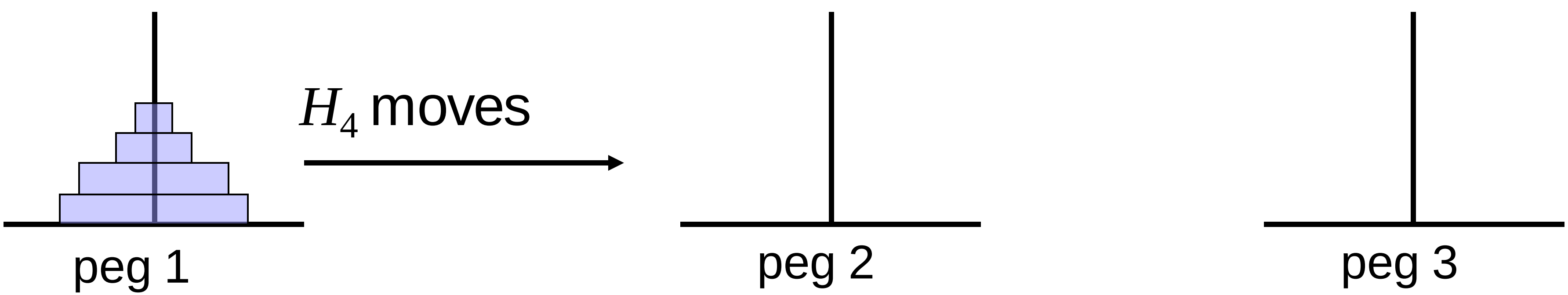
Such as finding:

- ❑compound interest
- ❑counting rabbits on an island
- ❑determining the number of moves in the Tower of Hanoi puzzle and
- ❑counting bit strings with certain properties.

Example 3. (The Tower of Hanoi)

The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

Target: ***n*** Generic classifier **disk** From **peg 1** Move to **peg 2**



Sol : $H_n = 2H_{n-1} + 1$, $H_1 = 1$

($n-1$ Generic classifier disk From peg $1 \rightarrow$ peg 3, Subsection n Generic classifier disk From peg $1 \rightarrow$ peg 2, $n-1$ Generic classifier disk From peg $3 \rightarrow$ peg 2)

In the previous example: $H_n=2H_{n-1}+1, \quad H_1=1$

Sol: 2^n-1

Example 4. (Bit Counting)

Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length 5 ?

Sol :

Let a_n be the number of bit strings of length n that do not have two consecutive 0s.

$\boxed{1}$ $\boxed{2}$ $\ldots$ $\boxed{n\text{-}3}$ $\boxed{n\text{-}2}$ $\boxed{n\text{-}1}$ $\boxed{n}$

$\boldsymbol{a_{n-1}}$ ways ______ **1**

$\boldsymbol{a_{n-2}}$ ways ______ **1** **0**

$$\therefore a_n = a_{n-1} + a_{n-2},\ \ n \geq 3$$

$$a_1 = 2 \text{ (string : 0,1)}$$

$$a_2 = 3 \text{ (string : 01,10,11)}$$

$$\therefore a_3 = a_2 + a_1 = 5,\ a_4 = 8,\ a_5 = 13$$

Example 5. (Codeword enumeration)

A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. Let a_n be the number of valid n-digit codewords.

Find a recurrence relation for a_n.

Sol :

1	2	3	…	n-1	n

a_{n-1} ways — $1\sim 9$

$10^{n-1} - a_{n-1}$ ways — 0

$$\therefore a_n = 9a_{n-1} + (10^{n-1} - a_{n-1})$$
$$= 8a_{n-1} + 10^{n-1},\ n \geq 2$$
$$a_1 = 9$$

Exercises

- **a)** Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.
- **b)** What are the initial conditions?
- **c)** How many bit strings of length seven contain two consecutive 0s?

- **a)** Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 0s.
- **b)** What are the initial conditions?
- **c)** How many bit strings of length seven do not contain three consecutive 0s?

- **a)** Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.
- **b)** What are the initial conditions?
- **c)** In how many ways can this person climb a flight of eight stairs?

- **a)** Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.
- **b)** What are the initial conditions?
- **c)** In many ways can this person climb a flight of eight stairs?

- **a)** Find a recurrence relation for the number of ternary strings of length n that do not contain two consecutive 0s.
- **b)** What are the initial conditions?
- **c)** How many ternary strings of length six do not contain two consecutive 0s?

Solving Recurrence Relations

Def 1. A linear homogeneous recurrence relation of degree k (i.e., k terms) with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_i \in \mathbf{R}$ and $c_k \neq 0$

Example 1 and 2.

$f_n = f_{n-1} + f_{n-2}$	(True, deg=2)
$a_n = a_{n-5}$	(True, deg=5)
$a_n = a_{n-1} + a_{n-2}{}^2$	(False, not linear)
$a_n = n a_{n-1}$	(False , not linear)
$H_n = 2H_{n-1} + 1$	(False, not homogeneous)

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

Theorem 1.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a recurrence relation with $c_1, c_2 \in \mathbf{R}$.

If $r^2 - c_1 r - c_2 = 0$ (Known as the characteristic equation) has two distinct roots r_1 and r_2.

Then the solution of a_n is $a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$,

for $n=0,1,2,\ldots,$ where α_1, α_2 are constants.

(α_1, α_2 Available a_0, a_1 Work out)

Example 3.

What's the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with a_0=2 and a_1=7 ?

Sol :

$$a_n = 3\times 2^n - (-1)^n.$$

Example 4. Find an explicit formula for the Fibonacci numbers.

Sol :

$$f_n = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Thm 2.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a recurrence relation with $c_1, c_2 \in \mathbf{R}$.

If $r^2 - c_1 r - c_2 = 0$ has only one root r_0.

Then the solution of a_n is

$$a_n = \alpha_1 \cdot r_0^n + \alpha_2 \cdot n \cdot r_0^n$$

for $n=0,1,2,\ldots,$ where α_1 and α_2 are constants.

Example 5.

What's the solution of $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0=1$ and $a_1=6$?

Sol : $a_n = 3^n + n \times 3^n$

Thm 3.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a recurrence relation with $c_1, c_2, \ldots, c_k \in \mathbf{R}$.

If $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$.

Then the solution of a_n is

$a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n} + \ldots + \alpha_k r_k^{\,n}$, for $n = 0, 1, 2, \ldots$

where $\alpha_1, \alpha_2, \ldots \alpha_k$ are constants.

Example 6 (k = 3)

Find the solution of $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial conditions $a_0=2$, $a_1=5$ and $a_2=15$.

Sol : $a_n = 1 - 2^n + 2 \times 3^n$

Thm 4.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a recurrence relation with $c_1, c_2, \ldots, c_k \in \mathbf{R}$.

If $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c^k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, where $m_i \geq 1, \forall i$, and $m_1 + m_2 + \ldots + m_t = k$,

then

$$
\begin{aligned}
a_n &= (\alpha_{1,0} + \alpha_{1,1} \cdot n + \ldots + \alpha_{1,m_1-1} \cdot n^{m_1-1}) r_1^{\,n} \\
&\quad + (\alpha_{2,0} + \alpha_{2,1} \cdot n + \ldots + \alpha_{2,m_2-1} \cdot n^{m_2-1}) \cdot r_2^{\,n} \\
&\quad + \ldots + (\alpha_{t,0} + \alpha_{t,1} \cdot n + \ldots + \alpha_{t,m_t-1} \cdot n^{m_t-1}) \cdot r_t^{\,n}
\end{aligned}
$$

where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

For example :

If an equation has the roots: 1 (appearing two times),
-2 (appearing three times),
3 (appearing 1 time)

Then, the general solution of the above example using the theorem in the previous slide is :

$$a_n = (\alpha_{1,1} + \alpha_{1,2} \cdot n) \cdot 1^n + (\alpha_{2,1} + \alpha_{2,2} \cdot n + \alpha_{2,3} \cdot n^2) \cdot (-2)^n + \alpha_{3,1} \cdot 3^n$$

(Variable α Subscripts can be renamed, as along as you do not repeat a subscript)

$$a_n = (\alpha_1 + \alpha_2 \cdot n) \cdot 1^n + (\alpha_3 + \alpha_4 \cdot n + \alpha_5 \cdot n^2) \cdot (-2)^n + \alpha_6 \cdot 3^n$$

- **Example 7.** Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Sol : $a_n = (1+3n-2n^2) \times (-1)^n$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Example: $a_n = 3a_{n-1} + 2n$

A recurrence relation of the form

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)},$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}}$$

is called the associated homogeneous recurrence relation.

Example 8:

$a_n = a_{n-1} + 2^n,$ associated h.r.r $\Rightarrow a_n = a_{n-1}$

$a_n = a_{n-1} + a_{n-2} + n^2+1,$ associated h.r.r $\Rightarrow a_n = a_{n-1} + a_{n-2}$

$a_n = 3a_{n-1} + n3^n,$ associated h.r.r $\Rightarrow a_n = 3a_{n-1}$

$a_n = a_{n-1} + a_{n-3} + n!,$ associated h.r.r $\Rightarrow a_n = a_{n-1} + a_{n-3}$

Theorem 5. If $\{a_n^{(p)}\}$ is a particular solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Proof. If $\{a_n^{(p)}\}$ and $\{b_n\}$ are both solutions of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

then $a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \ldots + c_k a_{n-k}^{(p)} + F(n)$,

and $b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + F(n)$.

$$\Rightarrow a_n^{(p)} - b_n = c_1(a_{n-1} - b_{n-1}) + c_2(a_{n-2} - b_{n-2}) + \ldots + c_k(a_{n-k} - b_{n-k})$$

$\Rightarrow \{a_n^{(p)} - b_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

$$\Rightarrow b_n = a_n^{(p)} + a_n^{(h)}$$

Example 9. Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1=3$?

Sol : $a_n = (11/6) \times 3^n - n - 3/2$

Example 10. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Sol : $a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 \times 3^n + \alpha_2 \times 2^n + (49/20) \cdot 7^n$

Theorem 6.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

where $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$.

When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n.$$

When s is a root of the characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n.$$

Example 11. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2+1)3^n$.

Sol :

The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$.

characteristic equation: $r^2 - 6r + 9 = 0 \Rightarrow r = 3$ (Multiple root)

$F(n) = 3^n$, and 3 is a root $\Rightarrow a_n^{(p)} = p_0 n^2 3^n$

$F(n) = n3^n$, and 3 is a root $\Rightarrow a_n^{(p)} = n^2(p_1 n + p_0)\, 3^n$

$F(n) = n^2 2^n$, and 2 is not a root $\Rightarrow a_n^{(p)} = (p_2 n^2 + p_1 n + p_0) 2^n$

$F(n) = (n^2+1)3^n$, and 3 is a root

$\Rightarrow a_n^{(p)} = n^2\, (p_2 n^2 + p_1 n + p_0)\, 3^n$

Example 12. Find the solutions of the recurrence relation $a_n = a_{n-1} + n$ with $a_1=1$.

Sol : $a_n = a_n^{(p)} + a_n^{(h)} = (n^2+n)/2$

Exercises

- Solve these recurrence relations together with the initial conditions given.

  a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$

  b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$

  c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

  d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

  e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$

  f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$

  g) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

- Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

- Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$, and $a_3 = 8$.

- Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

- Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, and $a_2 = 15$.

Exercises Contd

- Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

  a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.

  b) Use Theorem 5 to find all solutions of this recurrence relation.

  c) Find the solution with $a_0 = 1$.

- What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation

  $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if

  a) $F(n) = n^3$?
  b) $F(n) = (-2)^n$?
  c) $F(n) = n2^n$?
  d) $F(n) = n^2 4^n$?
  e) $F(n) = (n^2 - 2)(-2)^n$?
  f) $F(n) = n^4 2^n$?
  g) $F(n) = 2$?

- **a)** Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.
- **b)** Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.

Generating Functions.

Def 1. The generating function for the sequence $a_0, a_1, a_2, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \ldots + a_n x^n + \ldots$$

$$= \sum_{k=0}^{\infty} a_k x^k$$

(If the series $\{a_n\}$ Is a finite, can be seen as Infinite, but the latter is equal to 0)

Example 1. Find the generating functions for the sequences $\{a_k\}$ with

(1) $a_k = 3$

(2) $a_k = k+1$

(3) $a_k = 2^k$

Sol :

(1) $G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 3x^k$

(2) $G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (k+1)x^k$

(3) $G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 2^k x^k$

Example 2. What is the generating function for the sequence 1,1,1,1,1,1 ?

Sol :

$a_0\ a_1\ a_2\ a_3\ a_4\ a_5 \quad a_6=0$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= 1 + x + x^2 + \ldots + x^5$$ (expansion)

$$= \frac{x^6 - 1}{x - 1}$$ (closed form)

Example 3.

Let $m \in \mathbf{Z}^+$ and $a_k = \binom{m}{k}$, for $k = 0, 1, \ldots, m$.

What is the generating function for the sequence $a_0, a_1, \ldots, a_m$?

Sol :

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$$

$$= \binom{m}{0} + \binom{m}{1} x + \binom{m}{2} x^2 + \cdots + \binom{m}{m} x^m$$

$$= (1+x)^m$$

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Useful Facts About Power Series

Example 4.

The function $f(x) = \dfrac{1}{1-x}$ is the generating function of the sequence $1, 1, 1, \ldots,$ because

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots = \frac{1}{1-x} \text{ when } |x| < 1.$$

Example 5.

The function $f(x) = \dfrac{1}{1-ax}$ is the generating function of the sequence $1, a, a^2, \ldots,$ because

$$\sum_{k=0}^{\infty} (ax)^k = 1 + ax + a^2x^2 + \ldots = \frac{1}{1-ax} \text{ when } |ax| < 1 \text{ for } a \neq 0.$$

Theorem 1.

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$.

Then $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$.

$$f(x)\, g(x) = (a_0 + a_1 x + a_2 x^2 + \ldots)(b_0 + b_1 x + b_2 x^2 + \ldots)$$
$$= (a_0 b_0) + (a_0 b_1 + a_1 b_0)\, x + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, x^2 + \ldots$$
$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

Example 6.

Let $f(x) = f(x) = \frac{1}{(1-x)^2}$. Use Example 4 to find the coefficients $a_0, a_1, a_2, \ldots$ in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

Sol : $a_k = k+1$

Def 2.

Let $u \in \mathbf{R}$ and $k \in \mathbf{N}$. Then the extended binomial coefficient $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)(u-2)\ldots(u-k+1)/k!, & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

Example 7. Find $\binom{-2}{3}$ and $\binom{1/2}{3}$

Sol : $\dfrac{1}{16}$

Example 8

When the top parameter is a negative integer, the extended binomial coefficient can be expressed in terms of an ordinary binomial coefficient.

$$\binom{-n}{r} = \frac{(-n)(-n-1)\ldots(-n-r+1)}{r!}$$

$$= \frac{(-1)^r (n)(n+1)\ldots(n+r-1)}{r!}$$

$$= \frac{(-1)^r (n+r-1)!}{r!(n-1)!}$$

$$= (-1)^r \binom{n+r-1}{r}$$

Thm 2. (The Extended Binomial Theorem)

Let $x \in \mathbf{R}$ with $|x|<1$ and let $u \in \mathbf{R}$, then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Example 9.

Find the generating functions for $(1+x)^{-n}$ and $(1-x)^{-n}$ where $n \in \mathbf{Z}^+$

Sol : By the Extended Binomial Theorem,

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{k=0}^{\infty} \frac{1}{k!}(-n)(-n-1)\ldots(-n-k+1)x^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(n)(n+1)\ldots(n+k-1)x^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k$$

$$\boxed{\text{Note}: \binom{-n}{k} = (-1)^k \binom{n+k-1}{k}}$$

By replacing x by $-x$ we have

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

Counting Problems and Generating Functions

Generating functions can be used to count the number of combinations of various types.

Example 10.

Find the number of solutions of $e_1 + e_2 + e_3 = 17$, where e_1, e_2, e_3 are integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

Sol : $(4, 6, 7), (5, 5, 7), (5, 6, 6)$ A total of 3

Example 11.

In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

Sol : The number of solutions is the coefficient of x^8 in the expansion of

$$(x^2 + x^3 + x^4)^3$$

$$\therefore (c_1, c_2, c_3) = (2, 2, 4), (2, 3, 3), (2, 4, 2), (3, 2, 3), (3, 3, 2), (4, 2, 2)$$

※Using Generating Functions to solve Recurrence Relations.

Example 12.

Solving the recurrence relation $a_k = 3a_{k-1}$ for k=1,2,3,… and initial condition $a_0 = 2$.

Sol :

Another method : (From Thm 1)

$$r - 3 = 0 \Rightarrow r = 3 \Rightarrow a_n = \alpha \cdot 3^n$$

$$\because a_0 = 2 = \alpha$$

$$\therefore a_n = 2 \cdot 3^n$$

Alternatively, use generating functions

Example 13.

Solving the recurrence relation

$a_n = 5a_{n-1}$ - $4a_{n-2}$ for $n \geq 0$, and initial condition $a_0 = 1$, $a_1 = 2$.

Sol :

$$a_n = \frac{4^n + 2}{3}$$

Example 14

Solving $a_k = 8a_{k-1} + 10^{k-1}$ for $k = 1,2,3,\ldots$ and initial condition $a_1 = 9$.

Sol : $a_k = (10^k + 8^k)/2$

Exercises

- Find the generating function for the finite sequence 1, 4, 16, 64, 256.

- Find a closed form for the generating function for the sequence $\{a_n\}$, where

  a) $a_n = 5$ for all $n = 0, 1, 2, \ldots$.

  b) $a_n = 3^n$ for all $n = 0, 1, 2, \ldots$.

- Find the coefficient of x^{10} in the power series of each of these functions.

  a) $1/(1 - 2x)$ **b)** $1/(1 + x)^2$

  c) $1/(1 - x)^3$ **d)** $1/(1 + 2x)^4$ **e)** $x4/(1 - 3x)^3$

- **a)** What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1 + x_2 + x_3 = k$ when x_1, x_2, and x_3 are integers with $x_1 \geq 2$, $0 \leq x_2 \leq 3$, and $2 \leq x_3 \leq 5$?

  b) Use your answer to part (a) to find a_6.

Exercises Contd

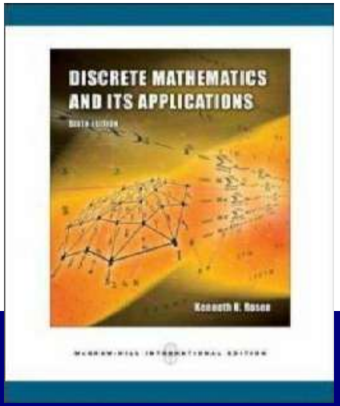
- Find a closed form for the generating function for the sequence $\{a_n\}$, where
- **a)** $a_n = 5$ for all $n = 0, 1, 2, \ldots$.
- **b)** $a_n = 3^n$ for all $n = 0, 1, 2, \ldots$.
- **c)** $a_n = 2$ for $n = 3, 4, 5, \ldots$ and $a_0 = a_1 = a_2 = 0$.
- **d)** $a_n = 2n + 3$ for all $n = 0, 1, 2, \ldots$.

- For each of these generating functions, provide a closed formula for the sequence it determines.

  a) $(3x - 4)3$ **b)** $(x^3 + 1)^3$

  c) $1/(1 - 5x)$ **d)** $x^3/(1 + 3x)$

  e) $x^2 + 3x + 7 + (1/(1 - x^2))$ **f)** $(x^4/(1 - x^4)) - x^3 - x^2 - x - 1$
- **g)** $x^2/(1 - x)^2$ **h)** $2e^{2x}$
- Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.



Discrete Mathematics

Graphs

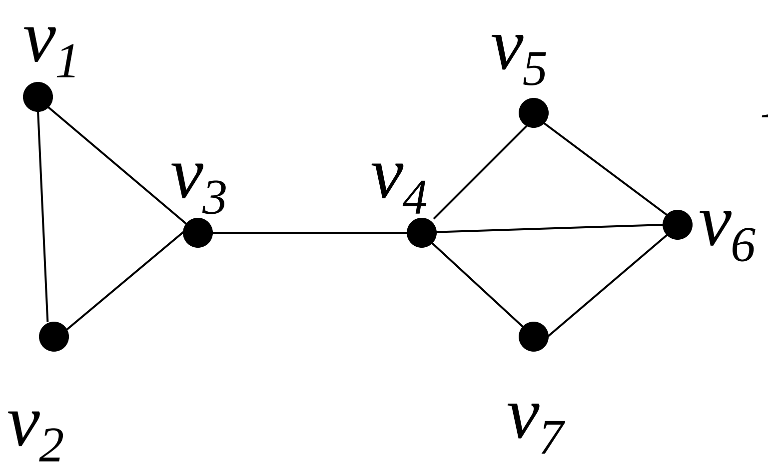
Outline

- Graph and Graph Models
- Graph Terminology and Special Types of Graphs
- Representing Graphs and Graph Isomorphism
- Connectivity
- Euler and Hamiltonian Paths
- Shortest-Path Problems
- Planar Graphs
- Graph Coloring

Introduction to Graphs

Def 1. A graph $\boldsymbol{G = (V, E)}$ consists of $\boldsymbol{V}$, a nonempty set of vertices (or nodes), and $\boldsymbol{E}$, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

eg.



$G=(V, E)$, *where*

$V=\{v_1,v_2, \ldots, v_7\}$

$E=\{\{v_1,v_2\}, \{v_1,v_3\}, \{v_2,v_3\}$
$\{v_3,v_4\}, \{v_4,v_5\}, \{v_4,v_6\}$
$\{v_4,v_7\}, \{v_5,v_6\}, \{v_6,v_7\}\}$

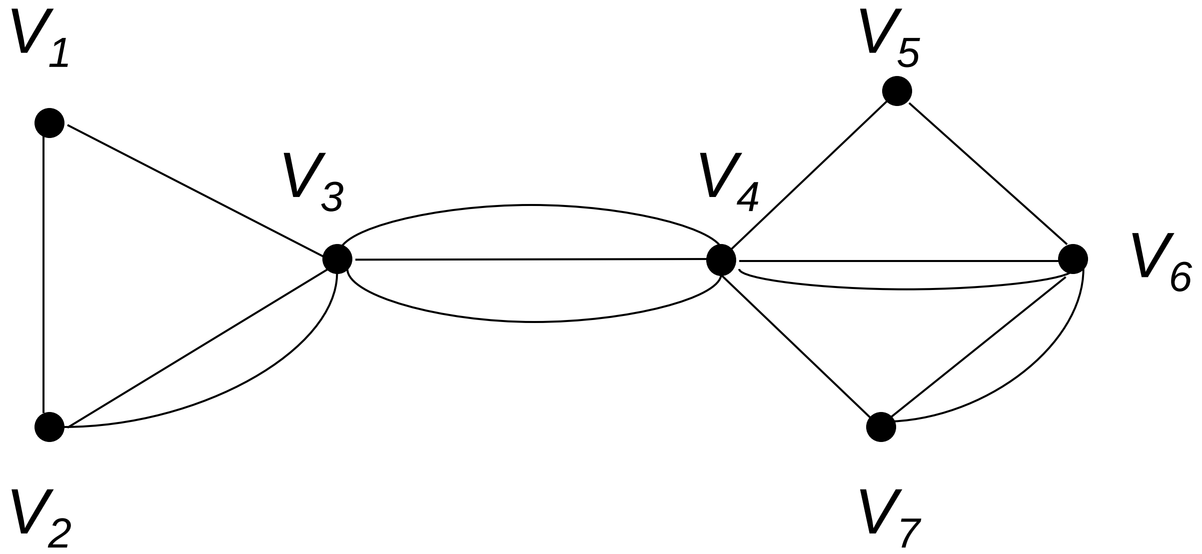
Def *A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.*

Def Multigraph:

simple graph + multiple edges (multiedges)

(Between two points to allow multiple edges)

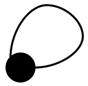
eg.



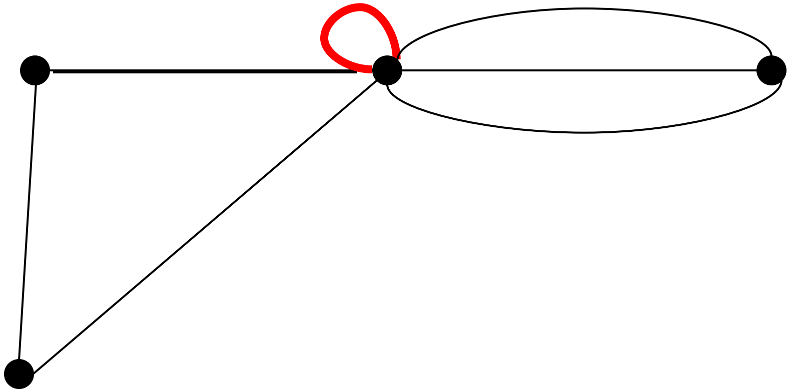
Def. Pseudograph:

simple graph + multiedge

+ loop

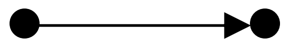
(a loop:  *)*

eg.



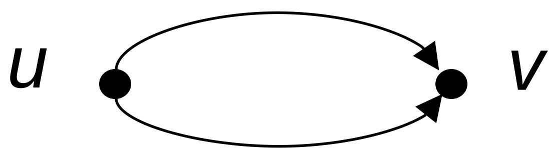
Def 2. Directed graph (digraph):

simple graph with each edge directed

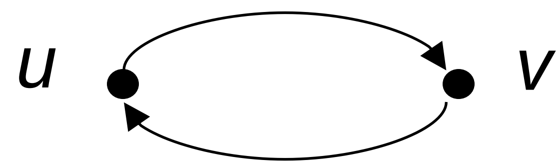


Note:  *is allowed in a directed graph*

Note:



The two edges (u,v),(u,v) are multiedges.



The two edges (u,v), (v,u) are not multiedges.

Def. Directed multigraph: digraph+multiedges

Table 1. Graph Terminology

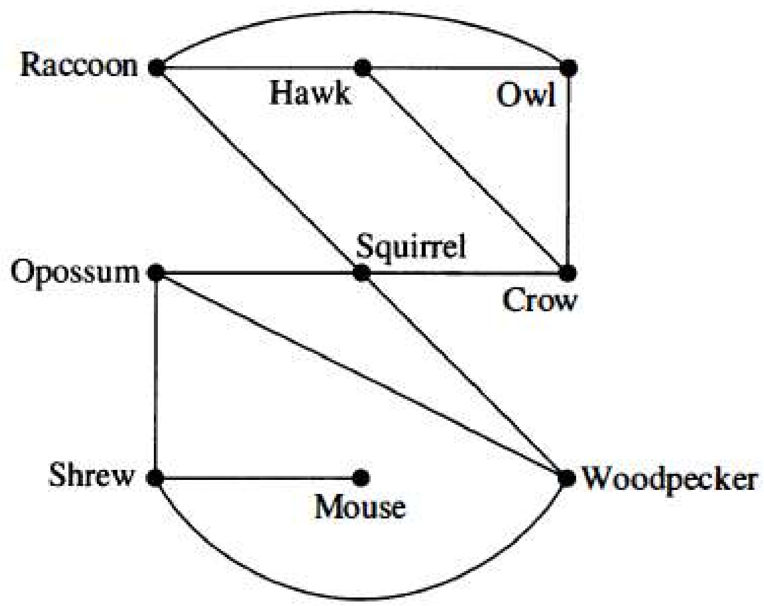
Type	Edges	Multiple Edges	Loops
(simple) graph	undirected edge: {u,v}	✗	✗
Multigraph	undirected edge: {u,v}	✓	✗
Pseudograph	undirected edge: {u,v}	✓	✓
Directed graph	directed edge: (u,v)	✗	✓
Directed multigraph	directed edge: (u,v)	✓	✓

Graph Models

Example 1. *(Niche Overlap graph)*

We can use a simple graph to represent interaction of different species of animals. Each animal is represented by a vertex. An undirected edge connects two vertices if the two species represented by these vertices compete.

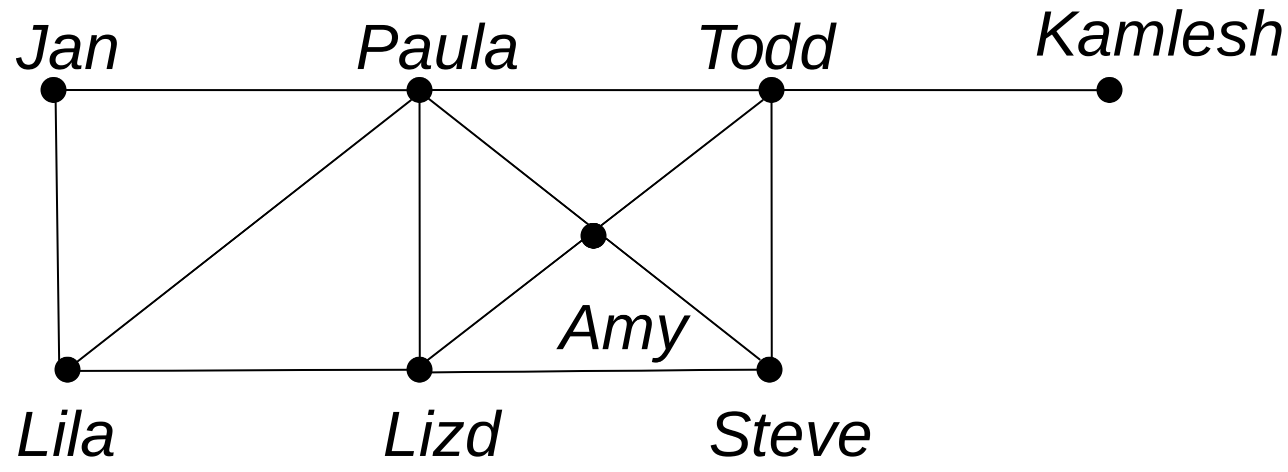
eg



Example 2. *(Acquaintanceship graphs)*

We can use a simple graph to represent whether two people know each other. Each person is represented by a vertex. An undirected edge is used to connect two people when these people know each other.

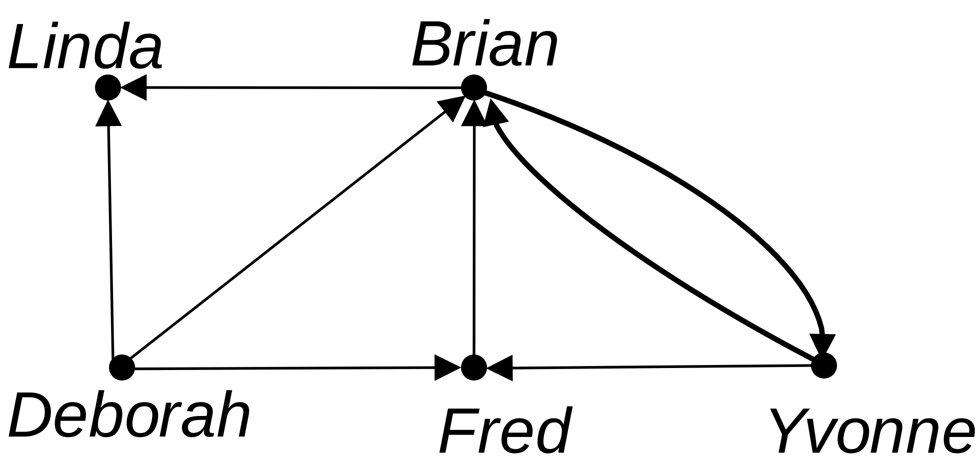
eg



Example 3. *(Influence graphs)*

In studies of group behavior it is observed that certain people can influence the thinking of others. Simple digraph $\Rightarrow$ *Each person of the group is represented by a vertex. There is a directed edge from vertex* a *to vertex* b *when the person* a *influences the person* b.

eg



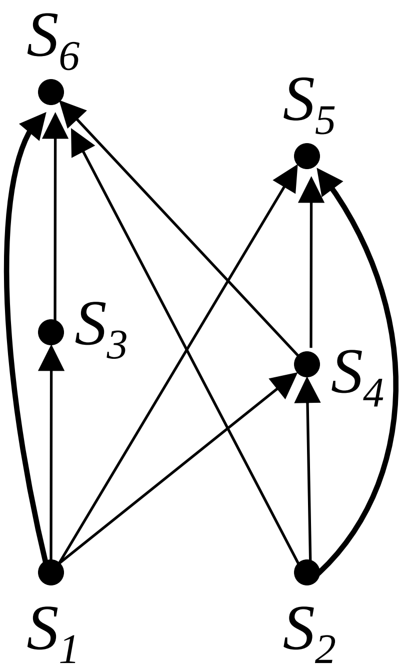
Example 9. *(Precedence graphs and concurrent processing)*

Computer programs can be executed more rapidly by executing certain statements concurrently. It is important not to execute a statement that requires results of statements not yet executed.

Simple digraph $\Rightarrow$ *Each statement is represented by a vertex, and there is an edge from* a *to* b *if the statement of* b *cannot be executed before the statement of* a.

eg

S_1: $a{:=}0$
S_2: $b{:=}1$
S_3: $c{:=}a+1$
S_4: $d{:=}b+a$
S_5: $e{:=}d+1$
S_6: $e{:=}c+d$



Graph Terminology

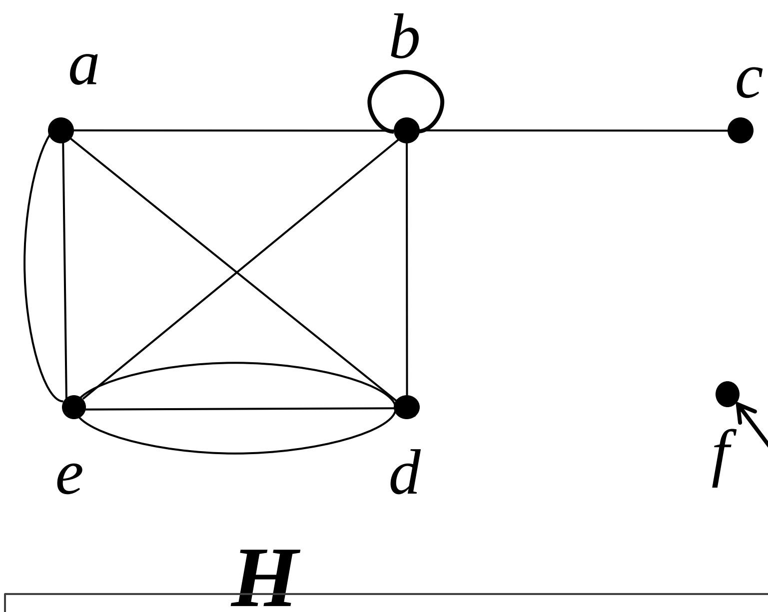
Def 1. Two vertices u and v in a undirected graph G are called adjacent (or neighbors) in G if $\{u, v\}$ is an edge of G.

Note : adjacent: a vertex connected to a vertex
incident: a vertex connected to an edge

Def 2. The degree of a vertex v, denoted by deg(v), in an undirected graph is the number of edges incident with it.

(Note : A loop adds 2 to the degree.)

Example 1. What are the degrees of the vertices in the graph H ?



Sol :

$deg(a)=4$

$deg(b)=6$

$deg(c)=1$

$deg(d)=5$

$deg(e)=6$

$deg(f)=0$

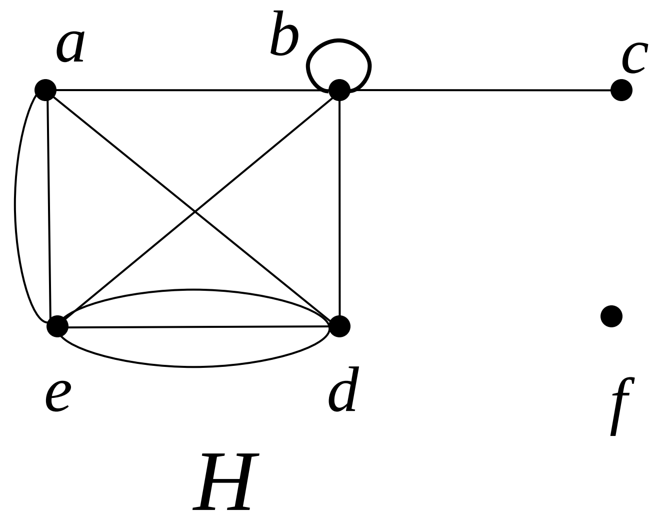
Def. *A vertex of degree 0 is called isolated.*

Def. *A vertex is **pendant if and only** if it has degree one.*

Thm 1. *(The Handshaking Theorem)*
Let $\boldsymbol{G} = \boldsymbol{(V, E)}$ *be an undirected graph with* e *edges (i.e.,* $|\boldsymbol{E}| = \boldsymbol{e}$*). Then*

$$\sum_{v \in V} \deg(v) = 2e$$

eg.



The graph H has 11 edges, and

$$\sum_{v \in V} \deg(v) = 22$$

Example 3. *How many edges are there in a graph with 10 vertices each of degree six?*

Sol :

$$10 \cdot 6 = 2e \quad \Rightarrow \quad e = 30$$

Thm 2. An undirected graph $G = (V, E)$ has an even number of vertices of odd degree.

Pf : *Let* $V_1=\{v \in V \mid deg(v) \text{ is even}\}$,

$V_2=\{v \in V \mid deg(v) \text{ is odd}\}$.

$$2e = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v) \quad \Rightarrow \quad \sum_{v \in V_2} \deg(v) \ \textit{is even.}$$

Def 3. *$G = (V, E)$: directed graph,*

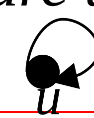
*$e = (u, v) \in E$: u is **adjacent to** v*

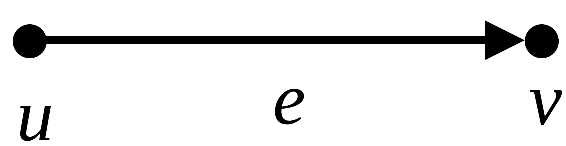
*v is **adjacent from** u*

*u : **initial vertex** of e*

*v : **terminal (end) vertex** of e*

The initial vertex and terminal vertex of a loop are the same





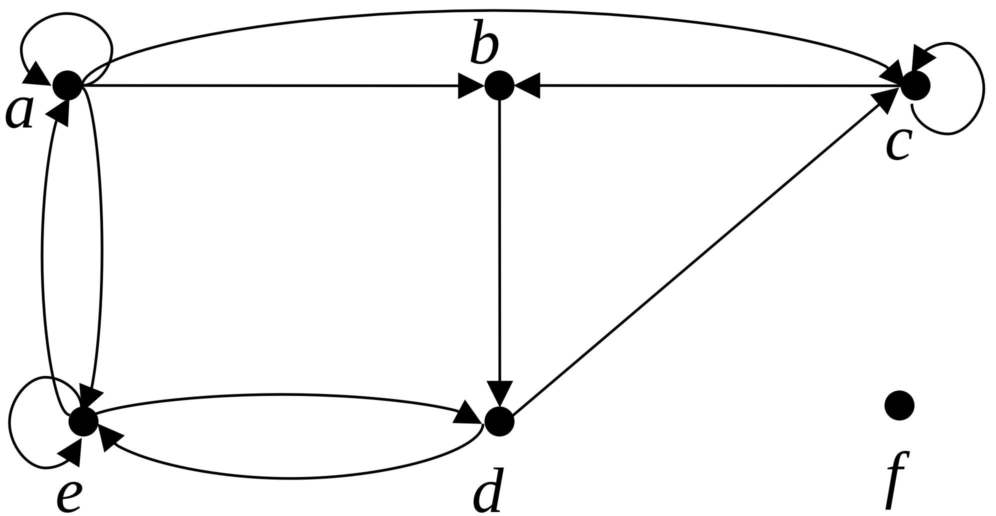
Def 4.

$G = (V, E)$: directed graph, $v \in V$

$\deg^-(v)$: # of edges with v as a terminal. (in-degree)

$\deg^+(v)$: # of edges with v as a initial vertex (out-degree)

Example 4.



$deg^-(a)=2, deg^+(a)=4$

$deg^-(b)=2, deg^+(b)=1$

$deg^-(c)=3, deg^+(c)=2$

$deg^-(d)=2, deg^+(d)=2$

$deg^-(e)=3, deg^+(e)=3$

$deg^-(f)=0, deg^+(f)=0$

Thm 3. Let $G = (V, E)$ be a digraph. Then

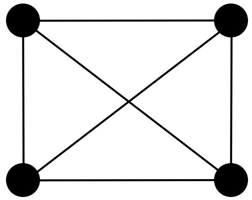
$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

Regular Graph

A simple graph $G=(V, E)$ is called regular if every vertex of this graph has the same degree. A regular graph is called n-regular if $\deg(v)=n$, $\forall v \in V$.

eg.

K_4 :

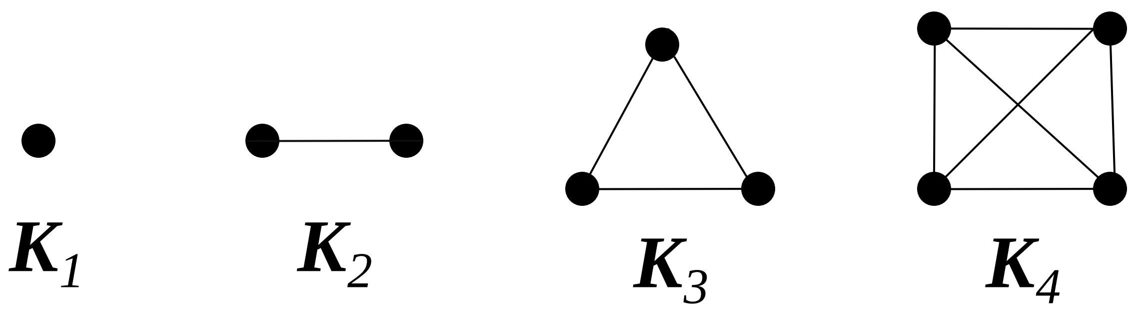


is 3-regular.

Some Special Simple Graphs

Example 5.

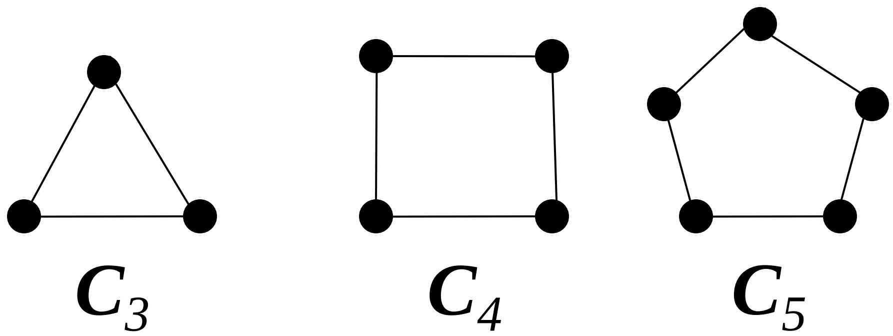
The complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



Note. K_n is $(n-1)$-regular, $|V(K_n)|=n$,

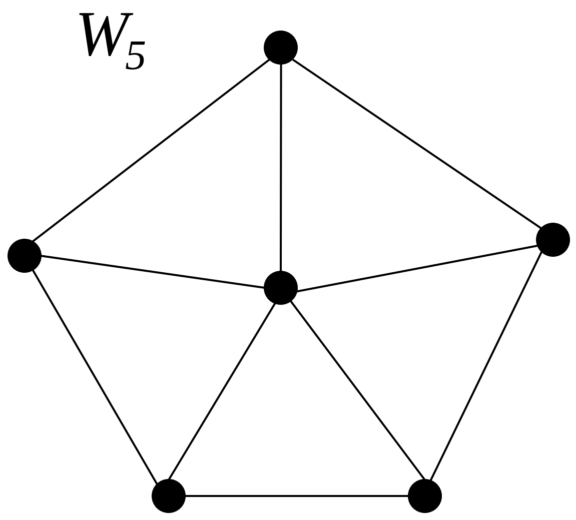
$$|E(K_n)| = \binom{n}{2}$$

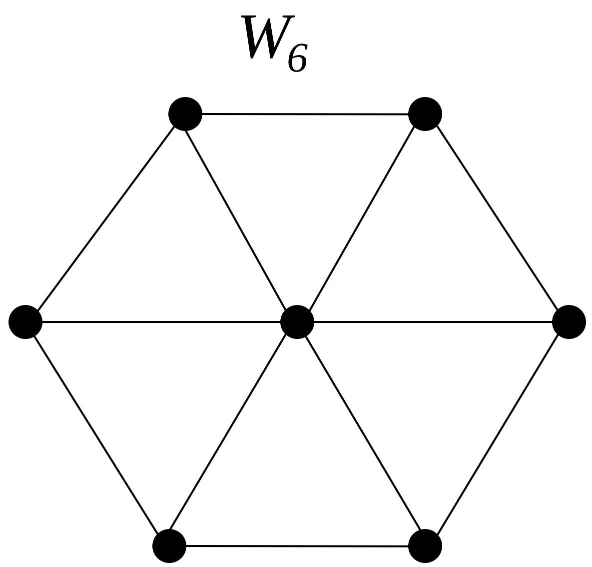
Example 6. The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1,v_2\}$, $\{v_2,v_3\}, \ldots, \{v_{n-1},v_n\}, \{v_n,v_1\}$.



Note *C_n is 2-regular, $|V(C_n)| = n$, $|E(C_n)| = n$*

Example 7. We obtained the wheel W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

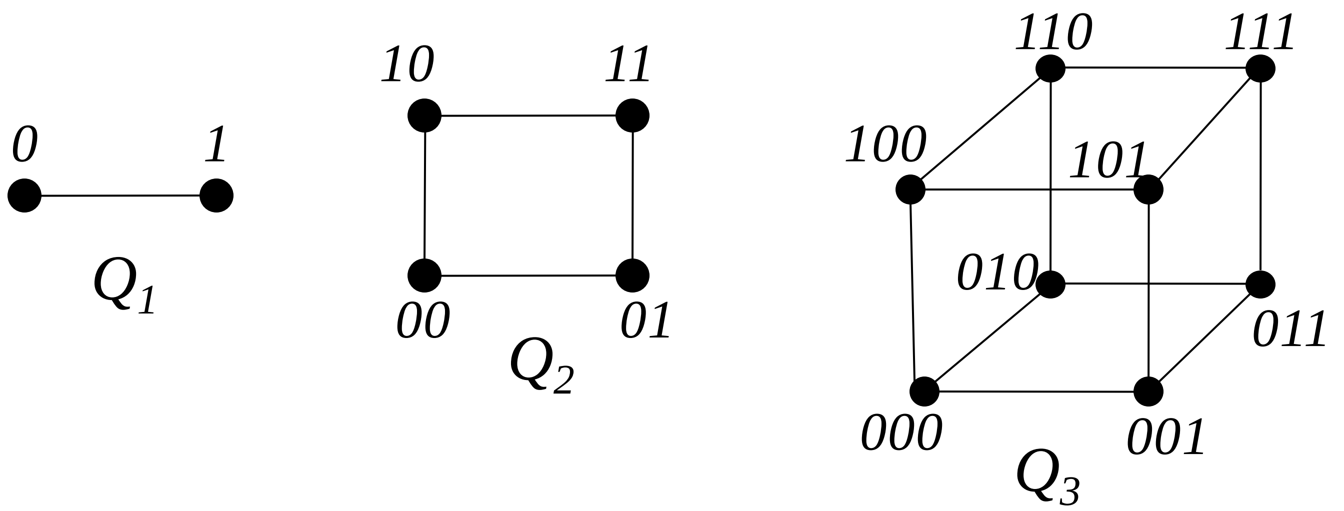




Note. $|V(W_n)| = n + 1$, $|E(W_n)| = 2n$,

W_n *is not a regular graph if* $n \neq 3$.

Example 8. The n-dimensional hypercube, or n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

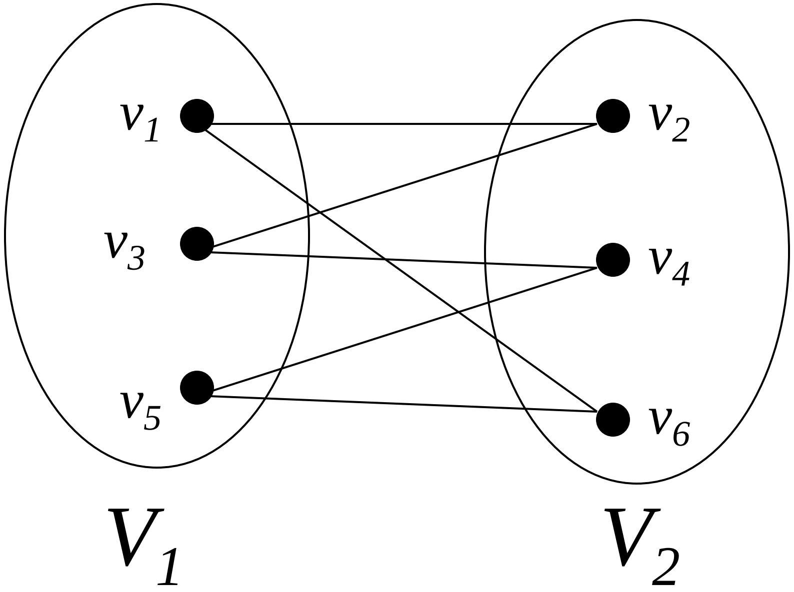


Note. Q_n *is n-regular,* $|V(Q_n)| = 2^n$, $|E(Q_n)| = (2^n n)/2 = 2^{n-1}n$

Some Special Simple Graphs

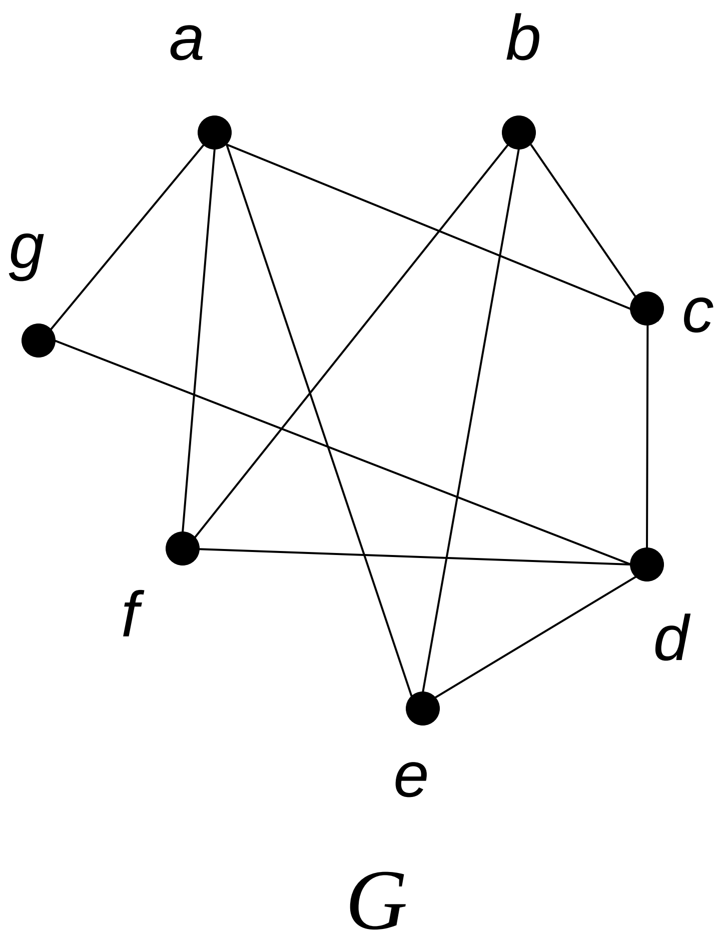
Def 5. A simple graph $G=(V,E)$ is called bipartite if V can be partitioned into V_1 and V_2, $V_1 \cap V_2 = \varnothing$, such that every edge in the graph connect a vertex in V_1 and a vertex in V_2.

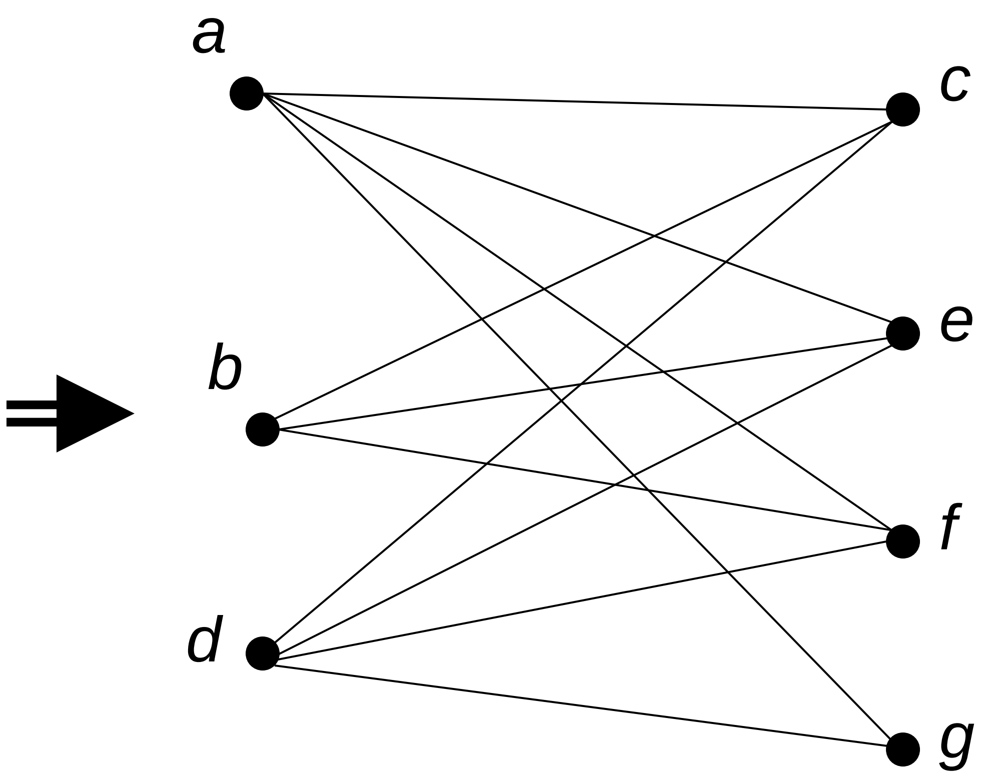
Example 9.



$\therefore$ *C_6 is bipartite.*

Example 10. Is the graph G bipartite ?

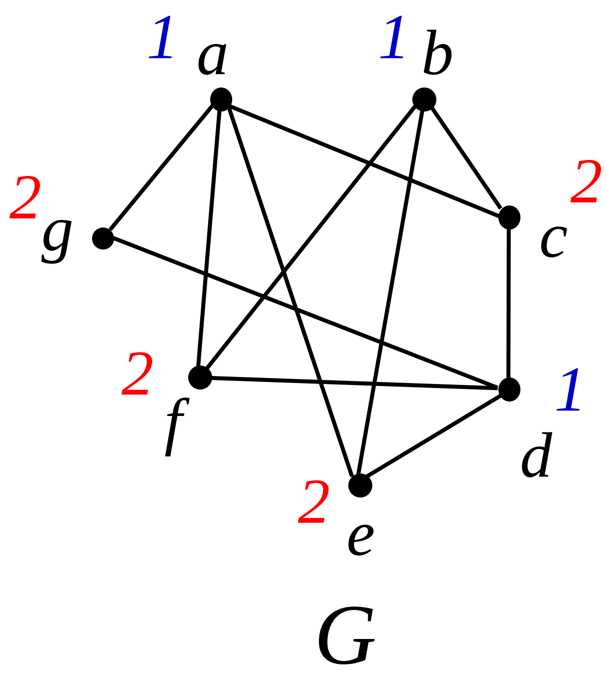




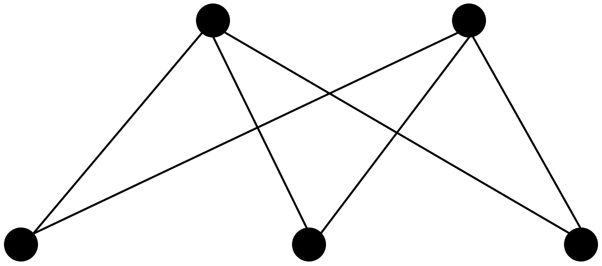
Yes !

Thm 4. *A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.*

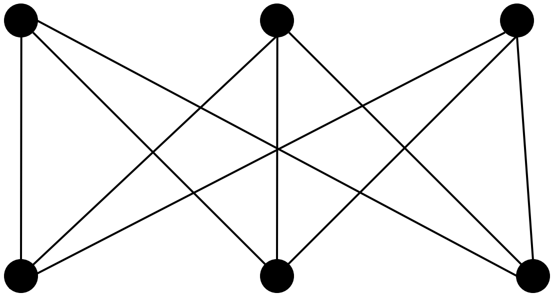
Example 12. *Use Thm 4 to show that* G *is bipartite.*



- Example 11. Complete Bipartite graphs ($K_{m,n}$)



$K_{2,3}$



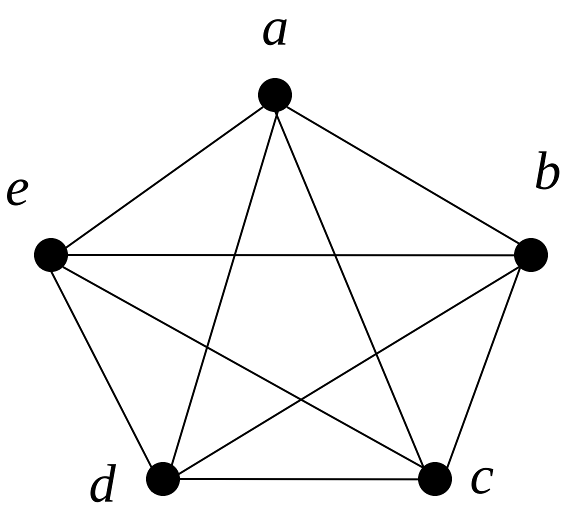
$K_{3,3}$

Note. $|V(K_{m,n})| = m+n$, $|E(K_{m,n})| = mn$,
$K_{m,n}$ *is regular if and only if* $m=n$.

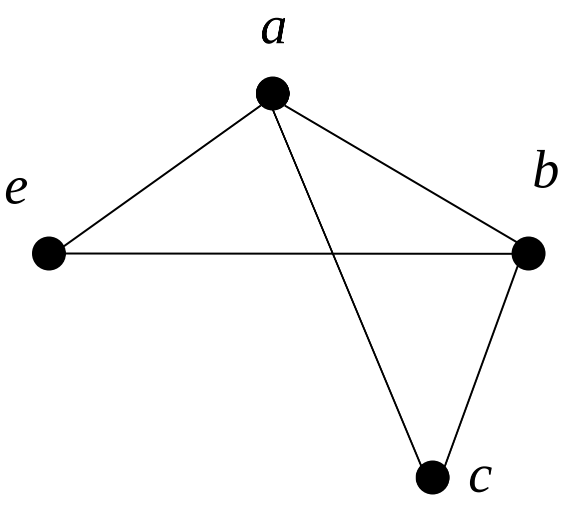
New Graphs from Old

Def 6. *A* subgraph *of a graph* $G=(V, E)$ *is a graph* $H=(W, F)$ *where* $W \subseteq V$ *and* $F \subseteq E$. *(Notice the f point w to connect)*

Example 14. A subgraph of K_5



K_5

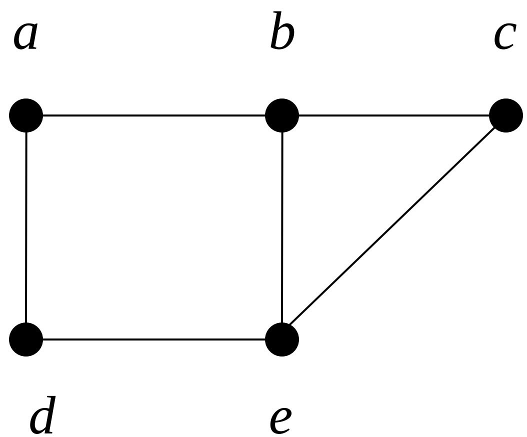


subgraph of K_5

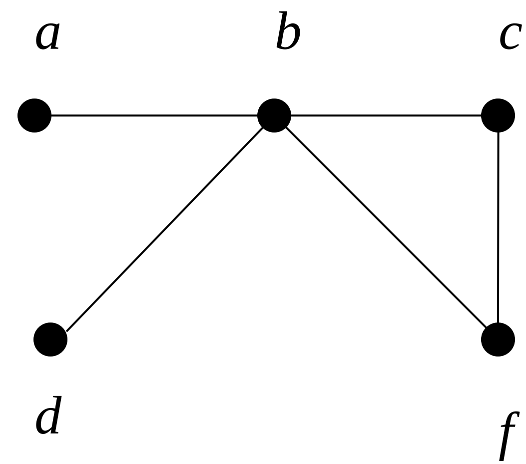
Def 7. The union of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is the simple graph $G_1 \cup G_2=(V_1 \cup V_2, E_1 \cup E_2)$

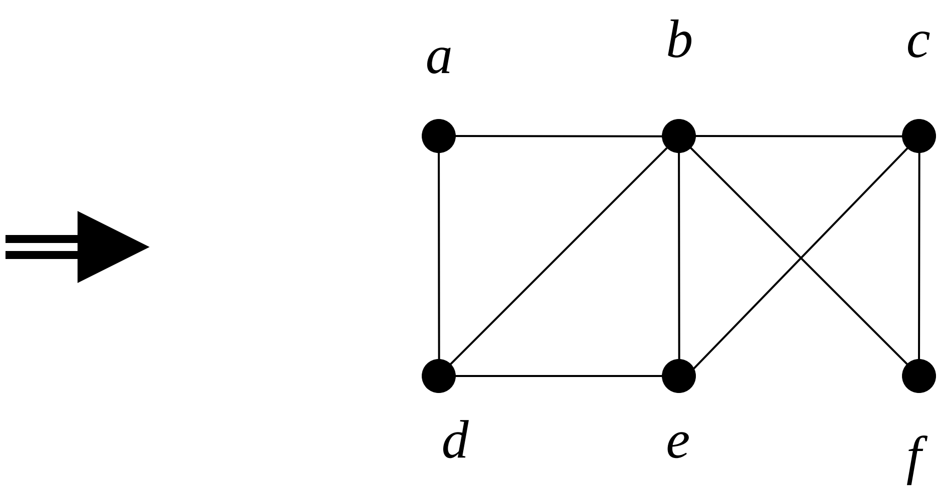
Example 15.

G_1



G_2



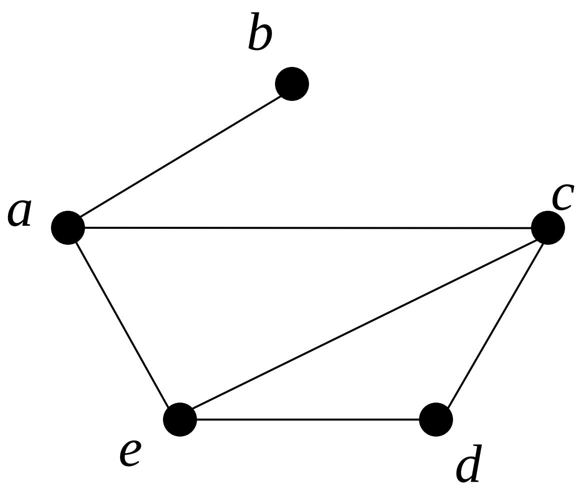


$G_1 \cup G_2$

Representing Graphs and Graph Isomorphism

※Adjacency list

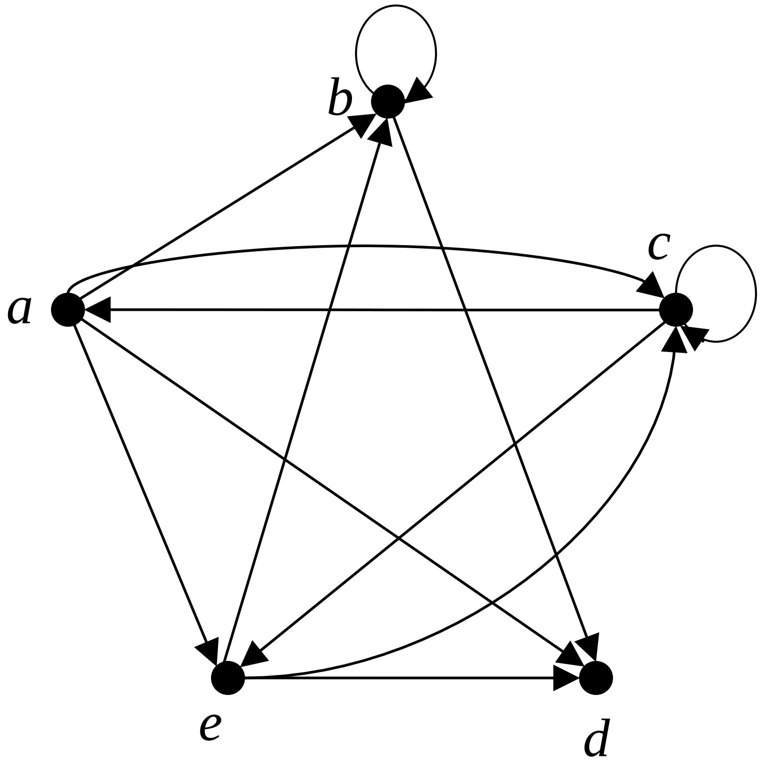
Example 1. Use adjacency lists to describe the simple graph given below.



Sol :

Vertex	Adjacent Vertices
a	b,c,e
b	a
c	a,d,e
d	c,e
e	a,c,d

Example 2. (digraph)



Initial vertex	Terminal vertices
a	b,c,d,e
b	b,d
c	a,c,e
d	
e	b,c,d

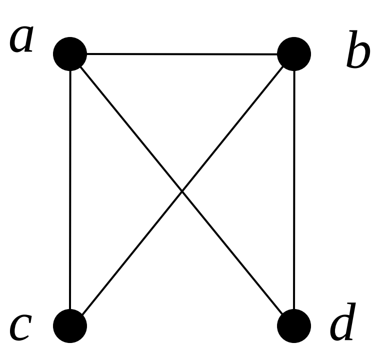
※Adjacency Matrices

Def. $G=(V, E)$: simple graph, $V=\{v_1,v_2,\ldots,v_n\}$.

A matrix A is called the adjacency matrix of G

if $A=[a_{ij}]_{n\times n}$, where $a_{ij}=\begin{cases} 1, \text{ if } \{v_i,v_j\}\in E, \\ 0, \text{ otherwise.} \end{cases}$

Example 3.



$$A_1 = \begin{array}{c c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c c} & \begin{matrix} b & d & c & a \end{matrix} \\ \begin{matrix} b \\ d \\ c \\ a \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

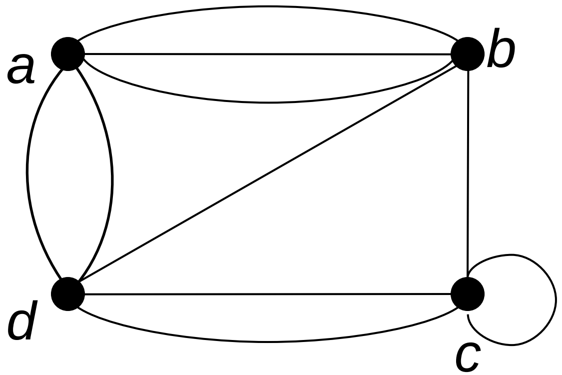
Note:

1. There are $n!$ different adjacency matrices for a graph with n vertices.

2. The adjacency matrix of an undirected graph is symmetric.

3. $a_{ii}=0$ (simple matrix has no loop)

Example 5. (Pseudograph) (Matrix may not be 0,1 matrix.)



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Def. *If $A=[a_{ij}]$ is the adjacency matrix for the directed graph, then*

$$a_{ij} = \begin{cases} 1 & \text{, if } v_i \longrightarrow v_j \\ 0 & \text{, otherwise} \end{cases}$$

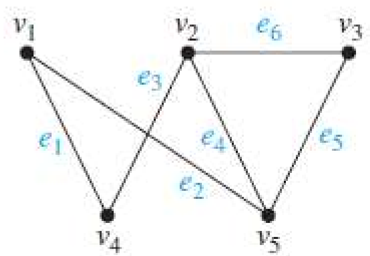
(So the matrix is not necessarily symmetrical)

※Incidence Matrices

Def. Let $G=(V, E)$: be an undirected graph. Suppose that $v_1,v_2,\ldots,v_n$ are the vertices and $e_1,e_2,\ldots,e_n$ are the edges of G . Then the incidence matrix with respect to this ordering of V and E is the n x m matrix $M=[m_{ij}]$, *where*

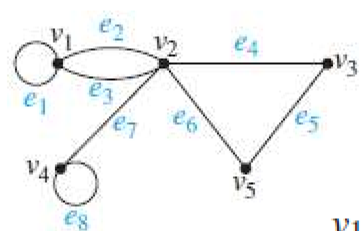
$$m_{i,j} = \begin{cases} 1 \text{ when edge } e_j \text{ is incident with } v_i, \\ 0 \text{ otherwise.} \end{cases}$$

Example 6.



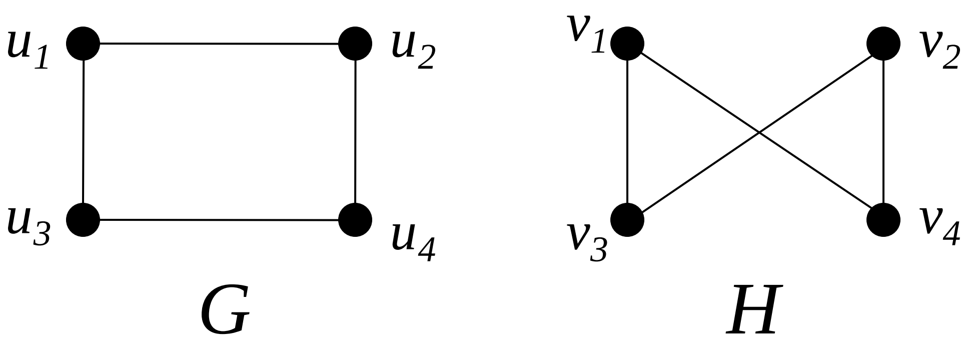
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Example 7.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

※*Isomorphism of Graphs*



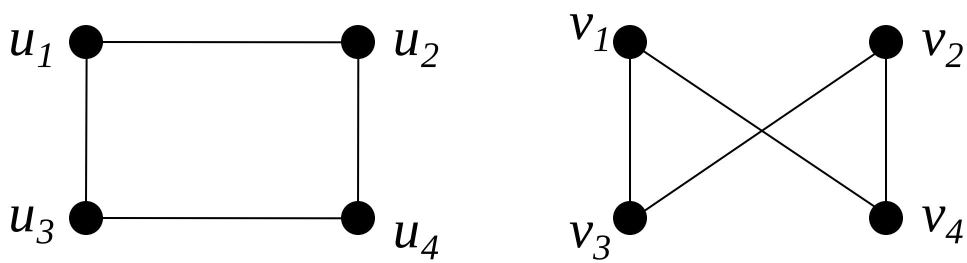
G is isomorphic to H

Def 1.

The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is an one-to-one and onto function f from V_1 to V_2 with the property that $a \sim b$ in G_1 iff $f(a) \sim f(b)$ in G_2, $\forall a,b \in V_1$

f is called an isomorphism.

Example 8. Show that G and H are isomorphic.



G H

Sol.

The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, *and* $f(u_4) = v_2$ *is a one-to-one correspondence between* $V(G)$ *and* $V(H)$.

※*Isomorphism graphs there will be:*

(1) *The same number of points (vertices)*

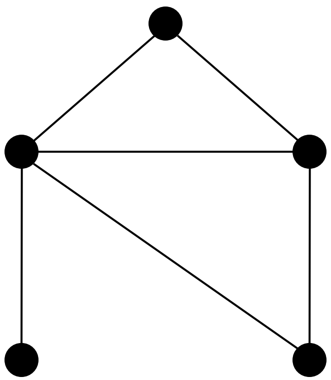
(2) *The same number of edges*

(3) *The same number of degree*

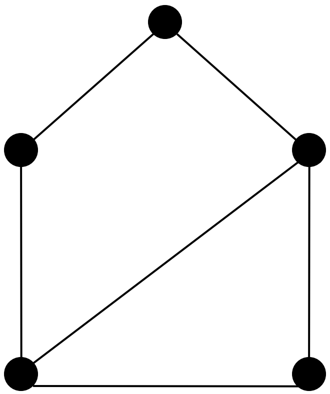
※ Given figures, judging whether they are isomorphic in general is not an easy task.

Example 9. Show that G and H are not isomorphic.

G



H

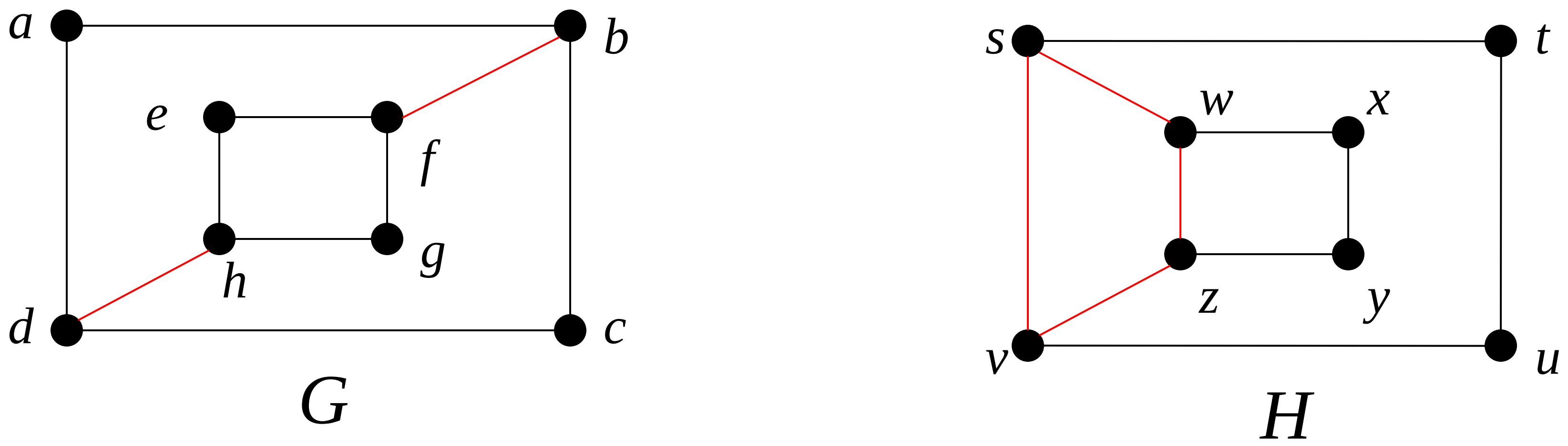


Sol :

G has a vertex of degree = 1 , H don't

Example 10.

Determine whether G and H are isomorphic.



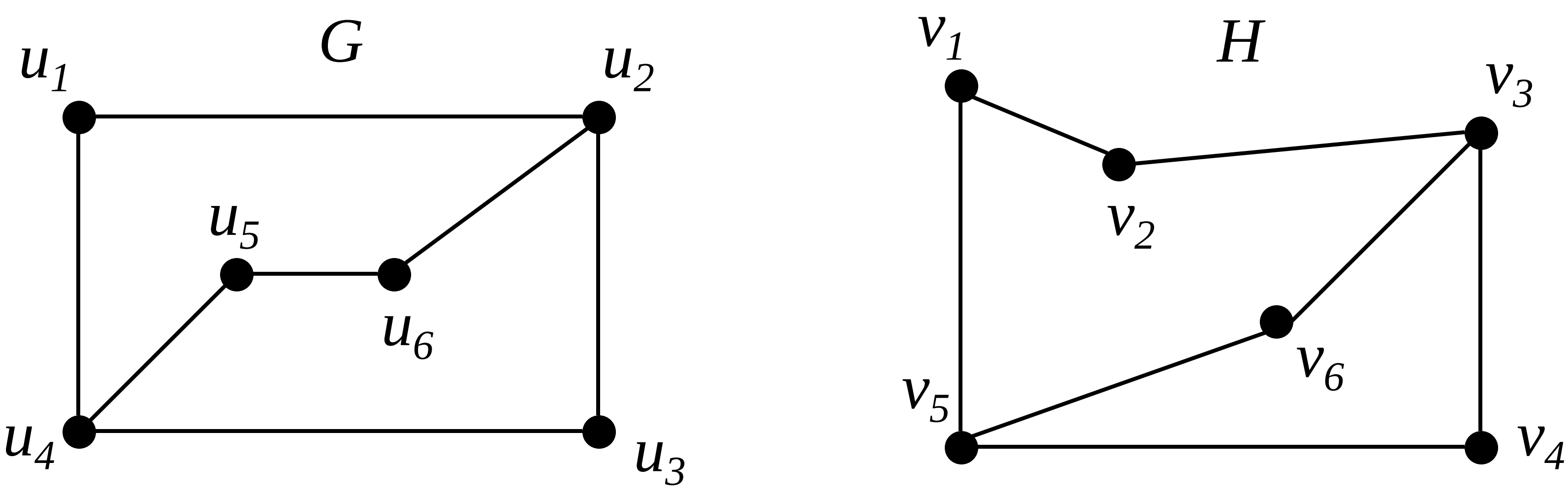
Sol : ∵ *In G, deg(a)=2, which must correspond to either t, u, x, or y in H degree*

Each of these four vertices in H is adjacent to another vertex of degree two in H,

which is not true for a in G

∴ *G and H are not isomorphic.*

Example 11. Determine whether the graphs G and H are isomorphic.



Sol:

$f(u_1)=v_6,\ f(u_2)=v_3,\ f(u_3)=v_4,\ \ f(u_4)=v_5,\ f(u_5)=v_1,\ f(u_6)=v_2$

$\Rightarrow$ Yes

Connectivity

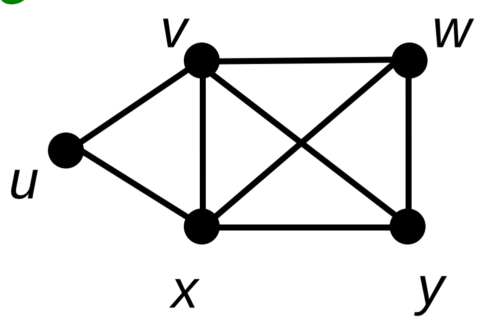
Def. 1 :

In an undirected graph, a path of length n from u to v is a sequence of $n+1$ adjacent vertices going from vertex u to vertex v.
(e.g., P: $u=x_0, x_1, x_2, \ldots, x_n=v$.) (P has n edges.)

Def. 2:

path: Points and edges in unrepeatable
trail: Allows duplicate path (not repeatable)
walk: Allows point and duplicate path

Example



path: u, v, y
trail: u, v, w, y, v, x, y
walk: u, v, w, v, x, v, y

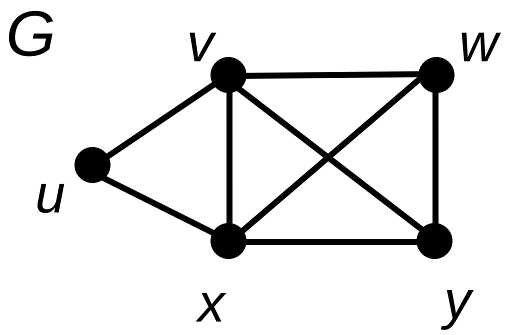
Def:

cycle: *path with u=v*

circuit: *trail with u=v*

closed walk: *walk with u=v*

Example



cycle: u, v, y, x, u

trail: u, v, w, y, v, x, u

walk: u, v, w, v, x, v, y, x, u

Paths in Directed Graphs

The same as in undirected graphs, but the path must go in the direction of the arrows.

Connectedness in Undirected Graphs

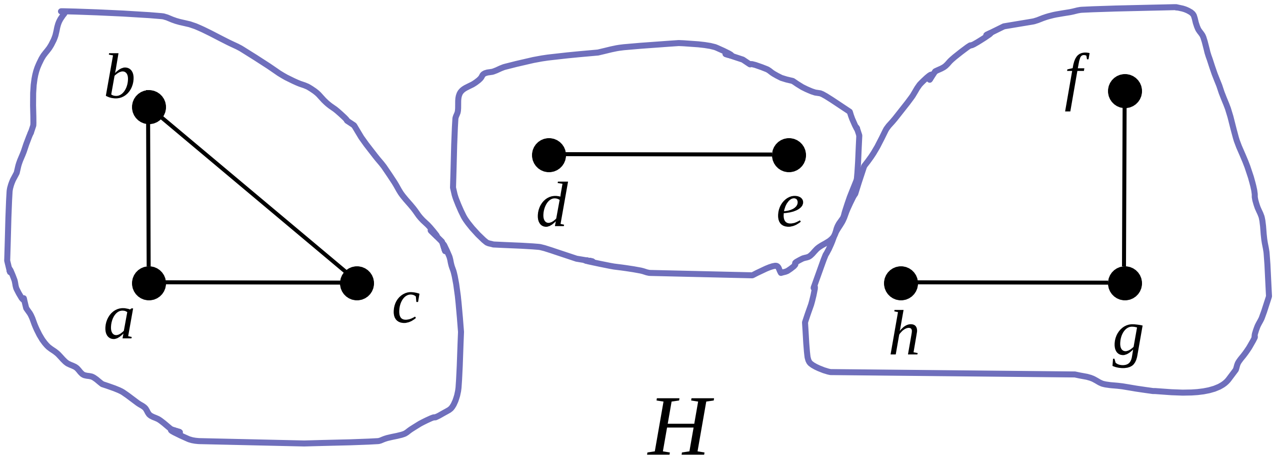
Def. 3:

An undirected graph is connected if there is a path between every pair of distinct vertices in the graph.

Def:

Connected component: maximal connected subgraph. (An unconnected graph will have several component)

Example 6 *What are the connected components of the graph H?*

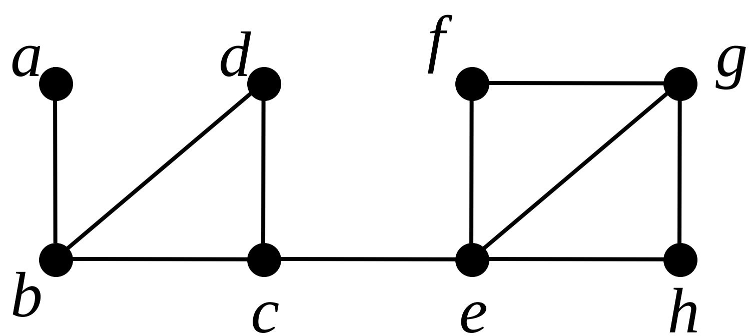


Def:

A *cut vertex* separates one connected component into several components if it is removed.
A *cut edge* separates one connected component into two components if it is removed.

Example 8. *Find the cut vertices and cut edges in the graph G.*

G



Sol:

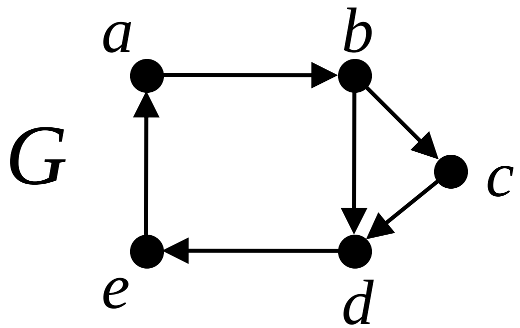
cut vertices: b, c, e

cut edges:
$\{a, b\}, \{c, e\}$

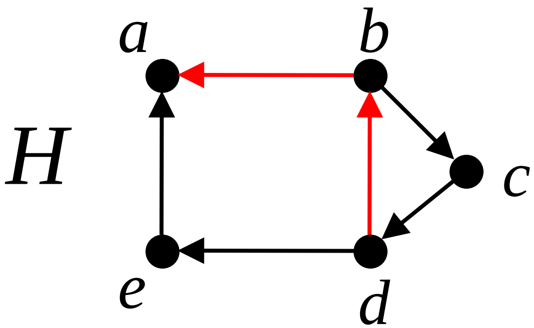
Connectedness in Directed Graphs

Def. 4: A directed graph is *strongly connected* if there is a path from a to b for any two vertices a, b. A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graphs.

Example 9 *Are the directed graphs G and H strongly connected or weakly connected?*



strongly connected

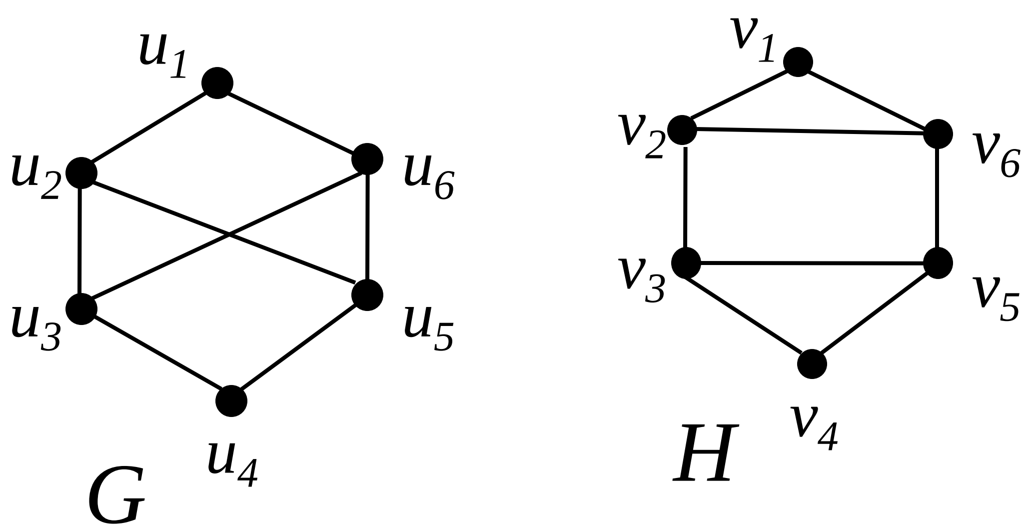


weakly connected

Paths and Isomorphism

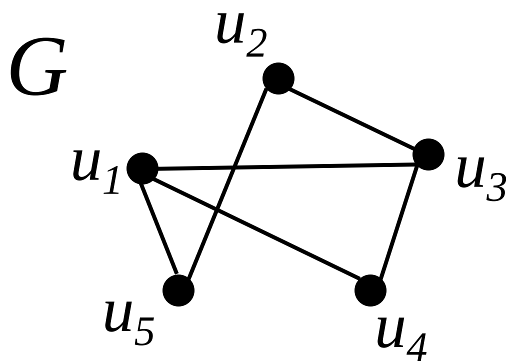
Note that connectedness, and the existence of a circuit or simple circuit of length k are graph invariants with respect to isomorphism.

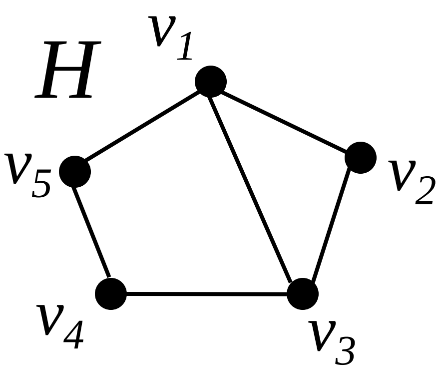
Example 12. *Determine whether the graphs G and H are isomorphic.*



Sol: *No, Because G has no simple circuit of length three, but H does*

Example 13. *Determine whether the graphs G and H are isomorphic.*





Sol.

Both G and H have 5 vertices, 6 edges, two vertices of deg 3, three vertices of deg 2, a 3-cycle, a 4-cycle, and a 5-cycle. $\Rightarrow$ G and H may be isomorphic.

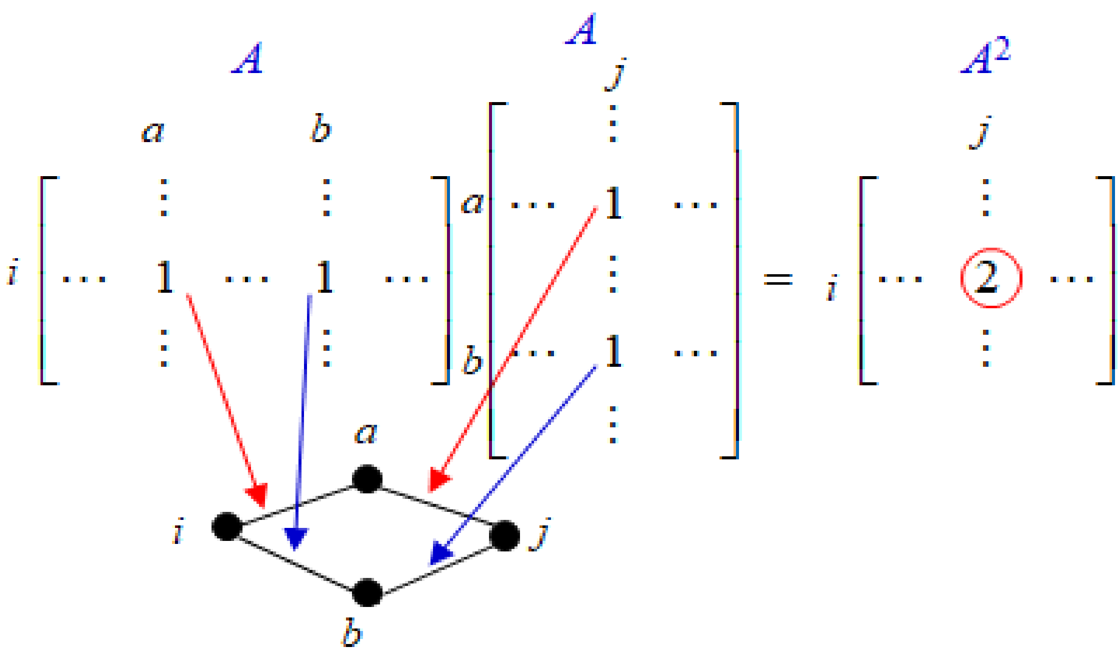
The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$ and $f(u_5) = v_5$ is a one-to-one correspondence between $V(G)$ and $V(H)$. $\Rightarrow$ G and H are isomorphic.

Counting Paths between Vertices

Theorem 2:

Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$. The number of walks of length r from v_i to v_j is equal to $(A^r)_{i,j}$.

Proof *(Only simple examples)*

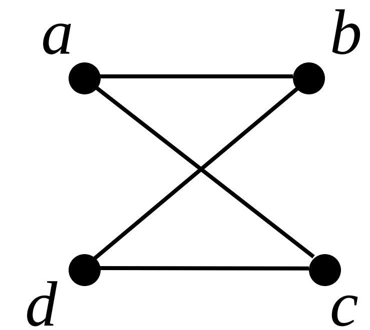


Example 14. *How many walks of length 4 are there from a to d in the graph G?*

Sol.

The adjacency matrix of G (ordering as a, b, c, d) is

G



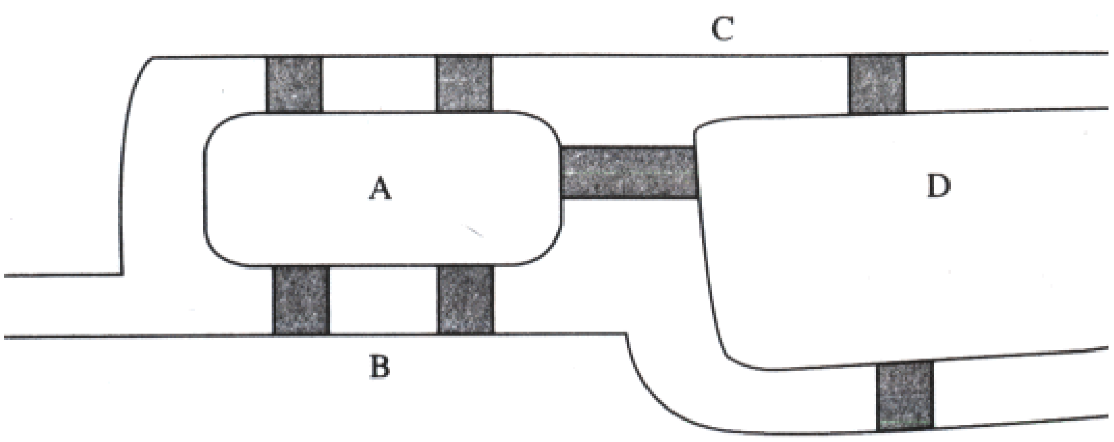
$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \Rightarrow A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix} \Rightarrow 8$$

a-b-a-b-d, a-b-a-c-d, a-c-a-b-d, a-c-a-c-d, a-b-d-b-d, a-b-d-c-d, a-c-d-b-d, a-c-d-c-d

Euler & Hamilton Paths

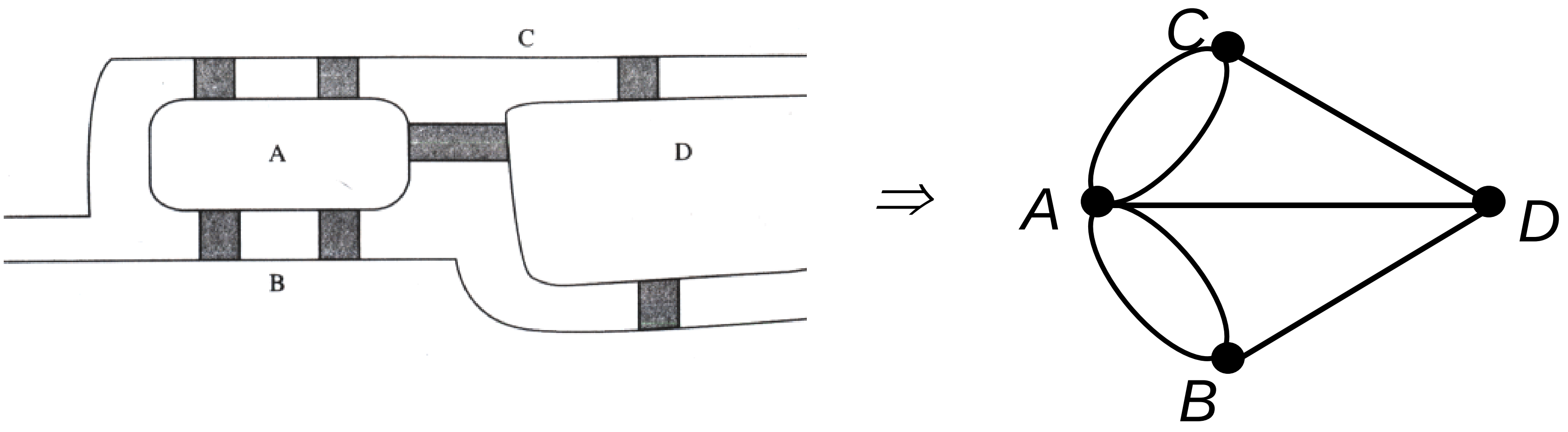
Graph Theory

- 1736, Euler solved the Königsberg Bridge Problem (Seven bridges problem)



Q:Is there a way can each bridge once, and return to the starting point?

Königsberg Bridge Problem



Q: Is there a way, you can walk down each side, and back to the starting point?

Ans: (Because each time a point is required from one side to the point, then the other side out, so after each time you want to use a pair of side.

- *connection must be an even number of sides on each point*
- *the move does not exist*

Def 1:

An *Euler circuit* in a graph *G* is a simple circuit containing every edge of *G*.
An *Euler path* in *G* is a simple path containing every edge of *G*.

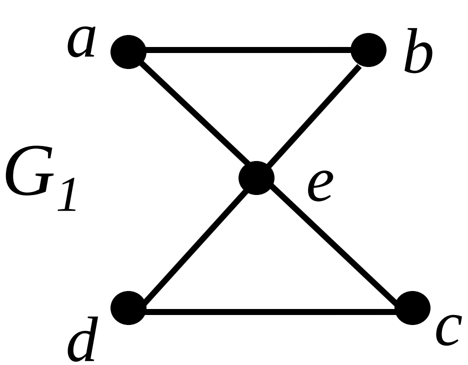
Thm. 1:

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

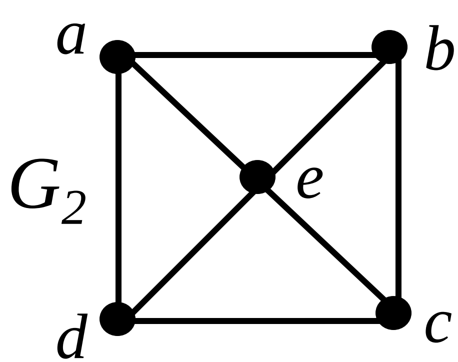
Thm. 2:

A connected multigraph has an Euler path (but not an Euler circuit) if and only if it has exactly 2 vertices of odd degree.

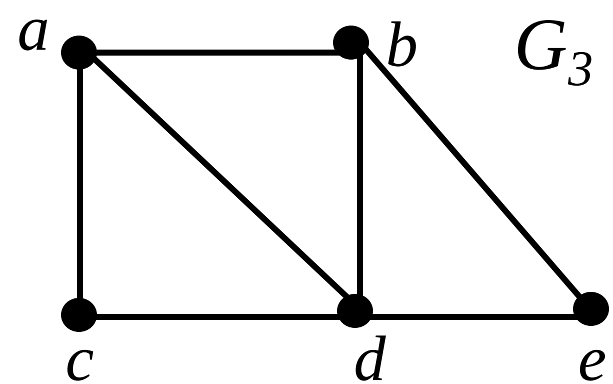
Example 1. *Which of the following graphs have an Euler circuit or an Euler path?*



Euler circuit

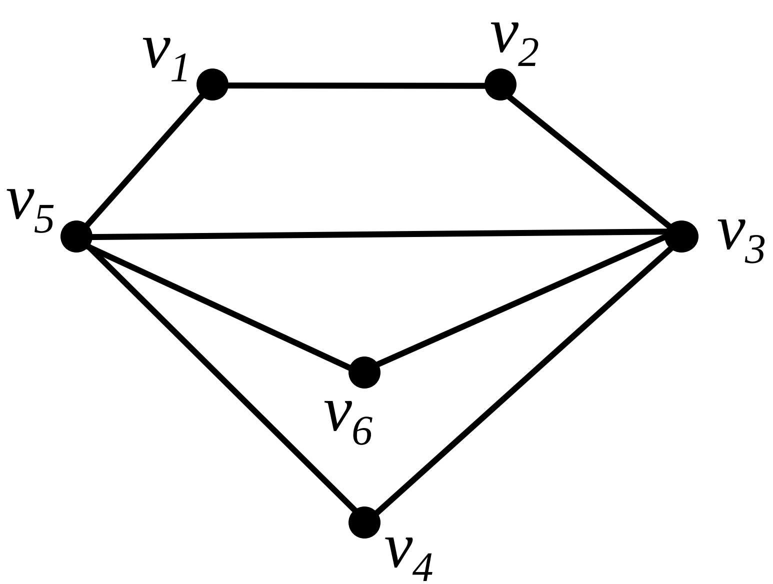


none



Euler path

Example



Step 1: find the 1^{st} circuit

C: $v_1, v_2, v_3, v_4, v_5, v_1$

Step 2: $H = G - C \neq \varnothing$, *find subcircuit*

SC: v_3, v_5, v_6, v_3

Step 3:

$C = C \cup SC$,

$H = G - C = \varnothing$, *stop*

C: $v_1, v_2, \underbrace{v_3, v_5, v_6, v_3}_{SC\ embedded}, v_4, v_5, v_1$

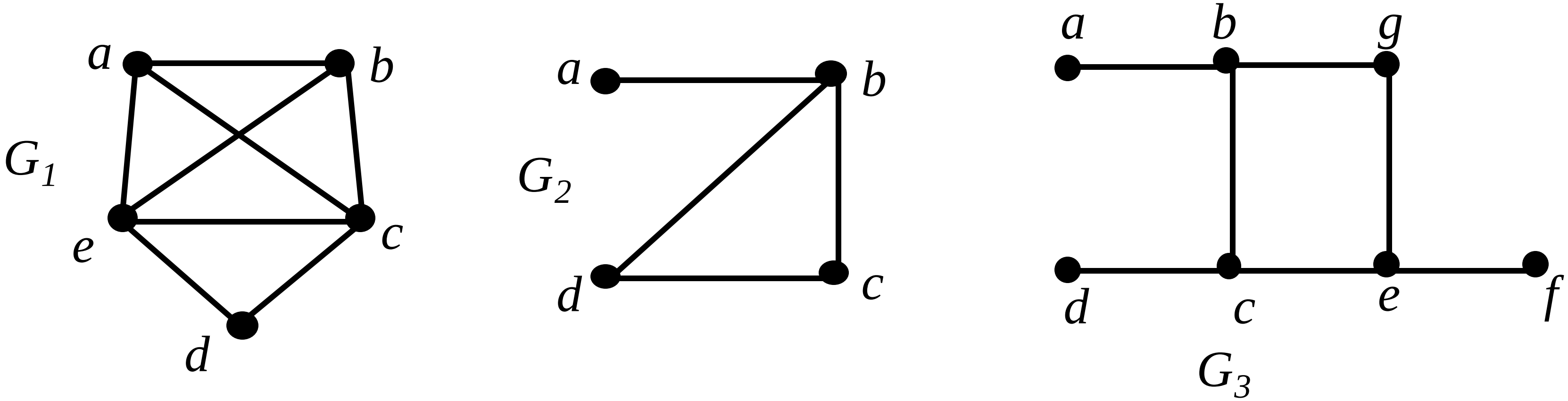
APPLICATIONS OF EULER PATHS AND CIRCUITS

- Euler paths and circuits can be used to solve many practical problems
  - traversing each street in a neighborhood
  - each road in a transportation network
  - each connection in a utility grid, or
  - each link in a communications network exactly once
- Among the other areas where Euler circuits and paths are applied is in
  - the layout of circuits,
  - in network multicasting, and
  - in molecular biology, where Euler paths are used in the sequencing of DNA

Hamilton Paths and Circuits

Def. 2: A *Hamilton path* is a path that traverses each vertex in a graph G exactly once.
A *Hamilton circuit* is a circuit that traverses each vertex in G exactly once.

Example 1. *Which of the following graphs have a Hamilton circuit or a Hamilton path?*

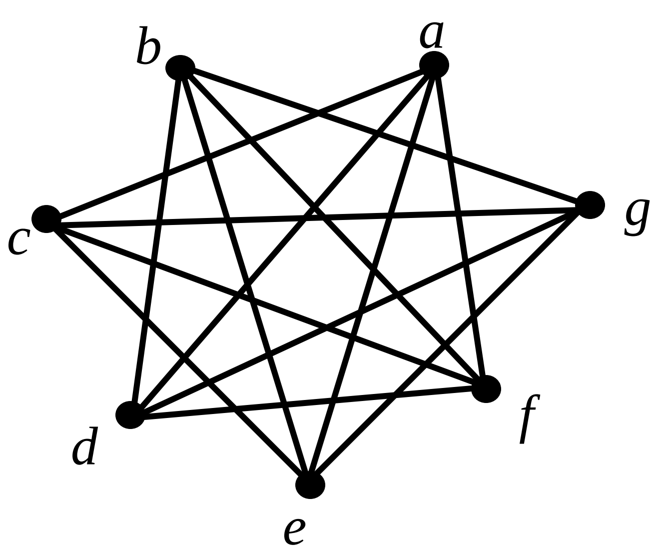


Hamilton circuit: G_1 *Hamilton path:* G_1, G_2

Thm. 3 (Dirac's Thm.):

If (but not only if) G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Example



each vertex has deg $\geq n/2 = 3.5$

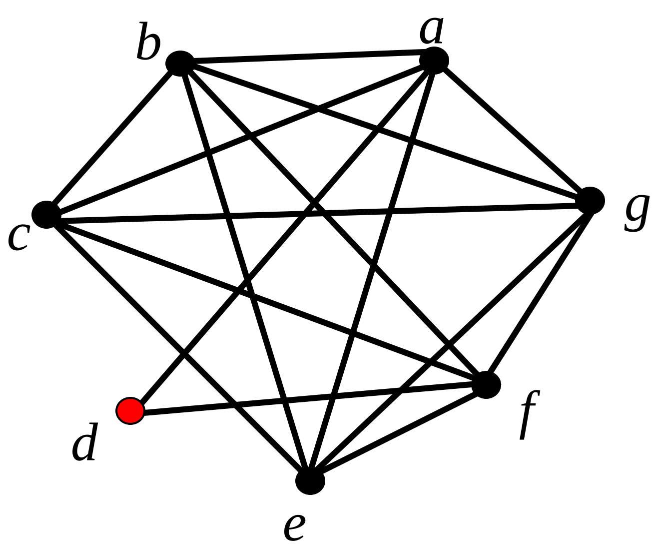
$\Rightarrow$Hamilton circuit exists

Such as: a, c, e, g, b, d, f, a

Thm. 4 (Ore's Thm.):

If G is a simple graph with $n \geq 3$ vertices such that $\deg(u)+\deg(v) \geq n$ for every pair of nonadjacent vertices u and v, then G has a Hamilton circuit.

Example



each nonadjacent vertex pair has deg sum $\geq n = 7$

$\Rightarrow$Hamilton circuit exists

Such as: a, d, f, e, c, b, g, a

Applications of Hamilton Circuits

- The famous **traveling salesperson problem** or **TSP** (also known in older literature as the **traveling salesman problem**)

Shortest-Path Problems

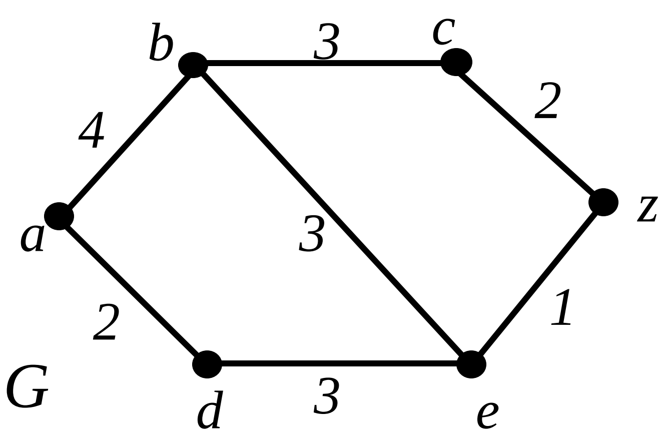
Def:

1. Graphs that have a number assigned to each edge are called *weighted graphs*.
2. The length of a path in a weighted graph is the sum of the weights of the edges of this path.

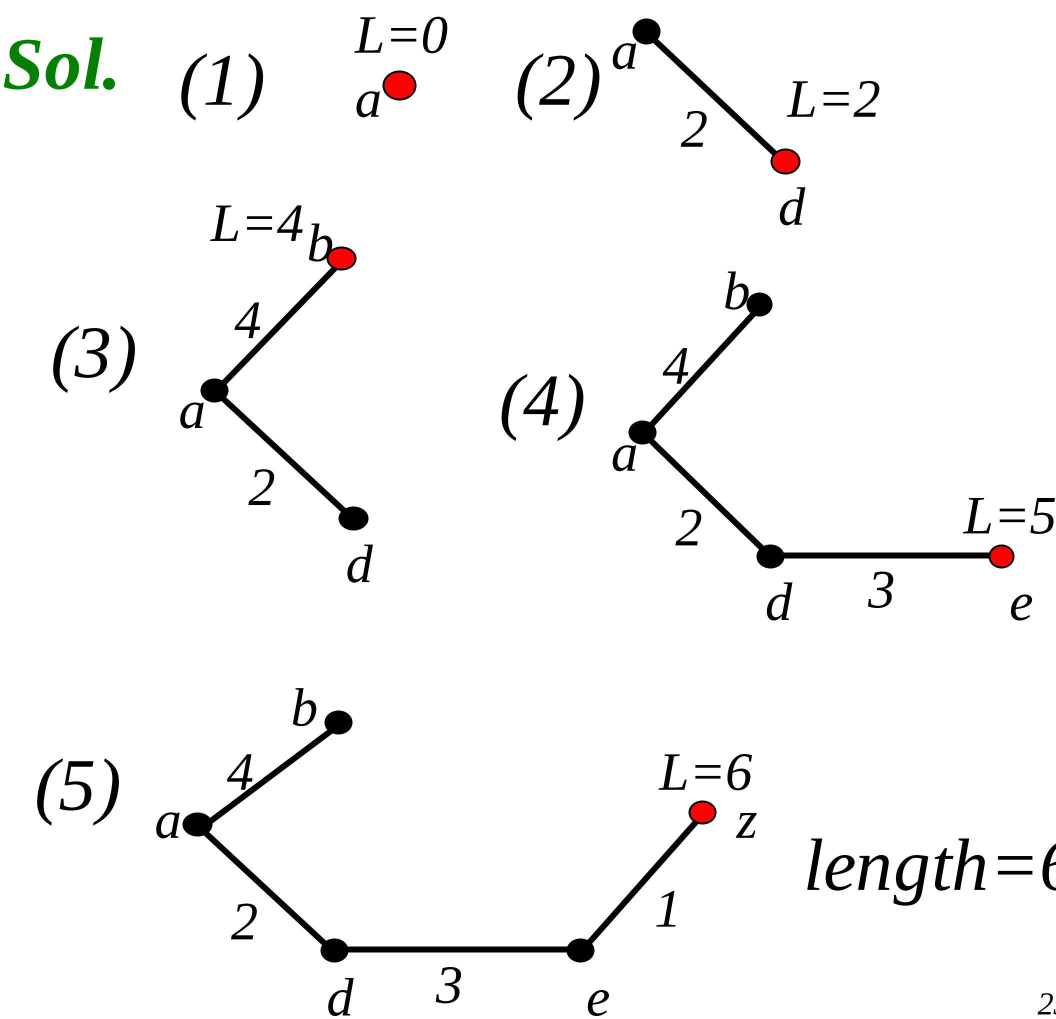
Shortest path Problem:

Determining the path of least sum of the weights between two vertices in a weighted graph.

Example 1. *What is the length of a shortest path between a and z in the weighted graph G?*



Sol.



length=6

Dijkstra's Algorithm *(find the length of a shortest path from a to z)*

Procedure *Dijkstra*(G: weighted connected simple graph, with all weights positive)

{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and weights $w(v_i, v_j)$ where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}

for $i := 1$ **to** n

  $L(v_i) := \infty$

$L(a) := 0$

$S := \emptyset$

while $z \notin S$

begin

  $u :=$ a vertex not in S with $L(u)$ minimal

  $S := S \cup \{u\}$

  for all vertices v not in S

    if $L(\mathrm{u}) + w(u, v) < L(v)$ **then** $L(v) := L(\mathrm{u}) + w(u, v)$

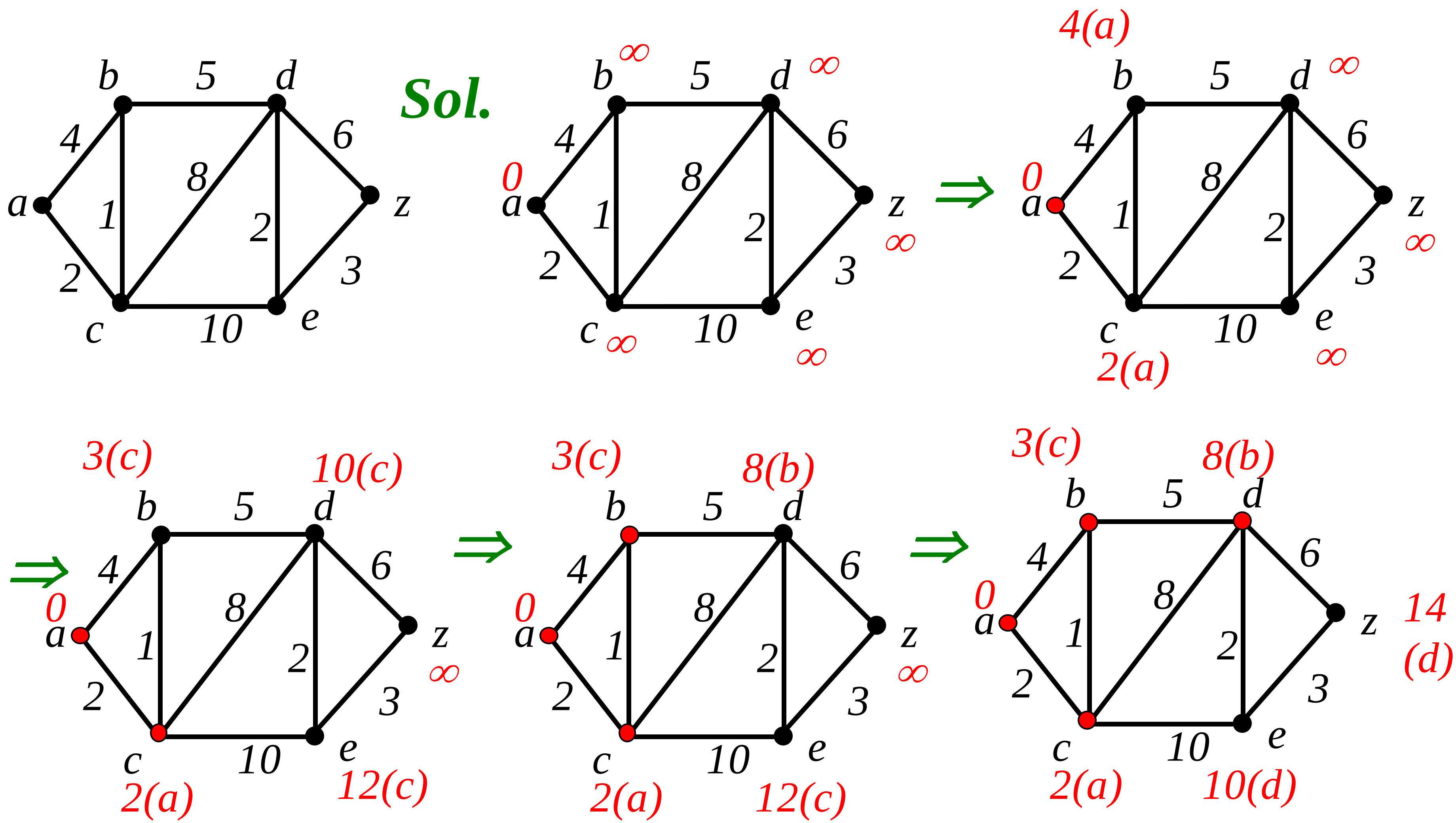
end $\{L(z) =$ length of a shortest path from a to $z\}$

This algorithm can be extended to construct a shortest path.

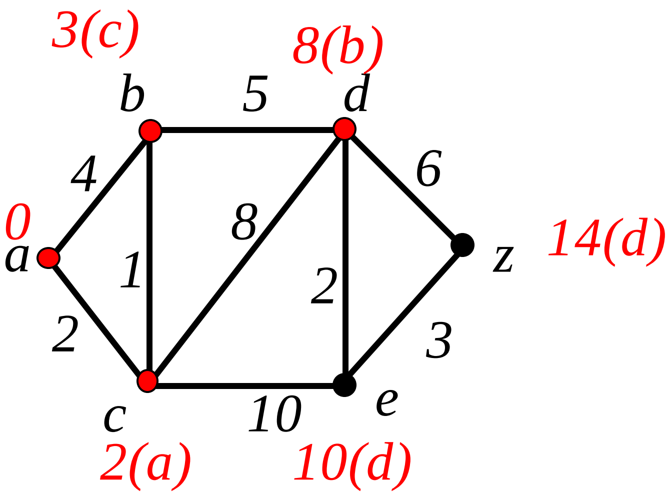
trace(We add a variable record thing is u before v previous (v): Finally, going on from z = u algorithm trace)

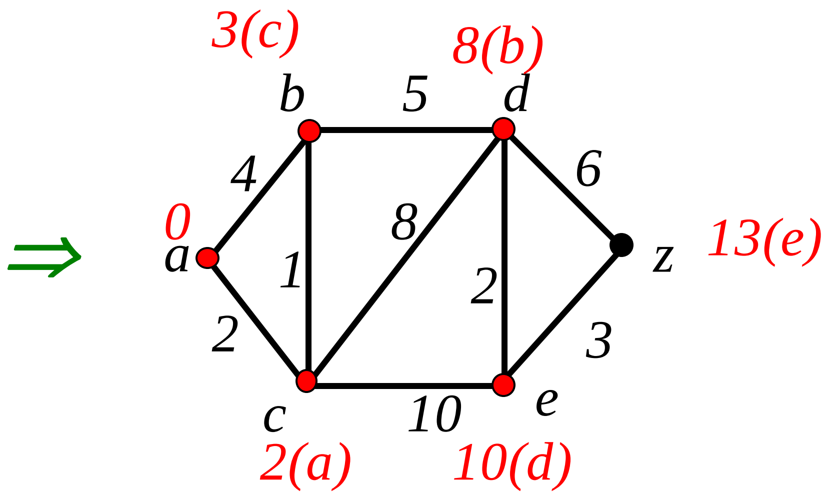
Example 2. *Use Dijkstra's algorithm to find the length of a shortest path between a and z in the weighted graph.*

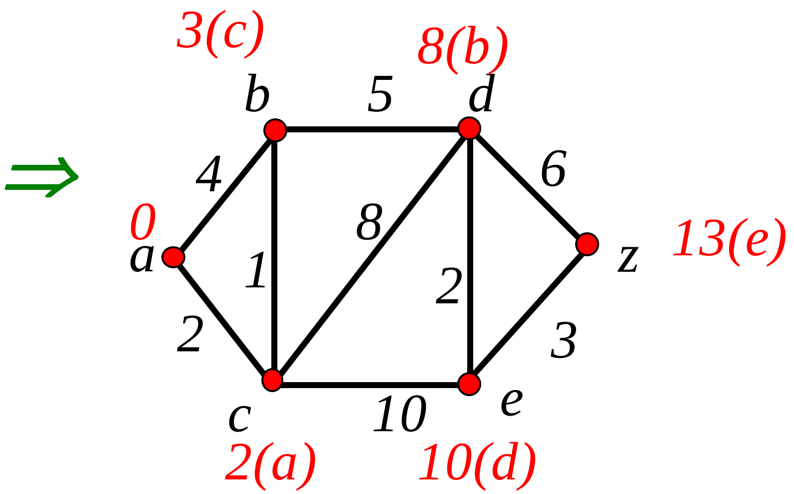
Sol.



Contd







$\Rightarrow$ path: a, c, b, d, e, z
length: 13

Thm. 1

Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.

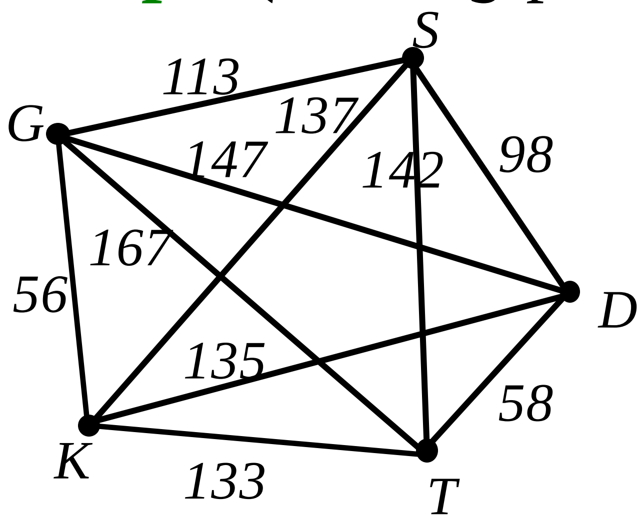
Thm. 2

Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with n vertices.

The Traveling Salesman Problem:

A traveling salesman wants to visit each of n cities exactly once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?

Example *(starting point D)*



$D \to T \to K \to G \to S \to D$: 458

$D \to T \to S \to G \to K \to D$: 504

$D \to T \to S \to K \to G \to D$: 540

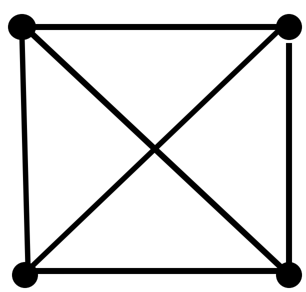
…

Planar Graphs

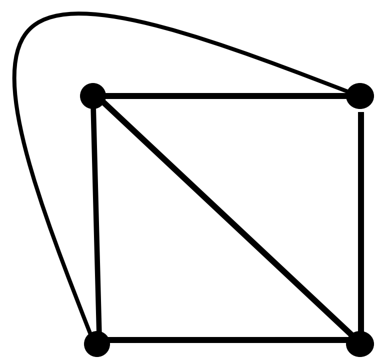
Def 1.

A graph is called *planar* if it can be drawn in the plane without any edge crossing. Such a drawing is called a *planar representation* of the graph.

Example 1: *Is K_4 planar?*



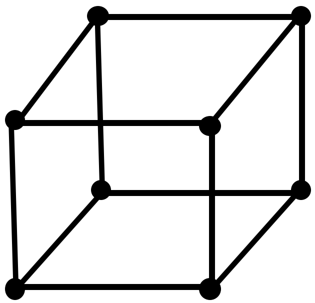
K_4



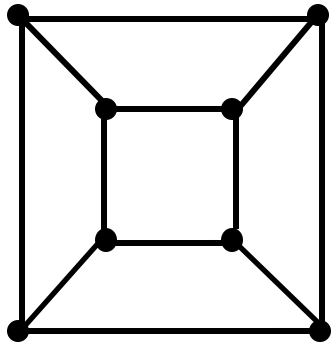
K_4 drawn with no crossings

$\therefore K_4$ *is planar*

Example 2: *Is Q_3 planar?*



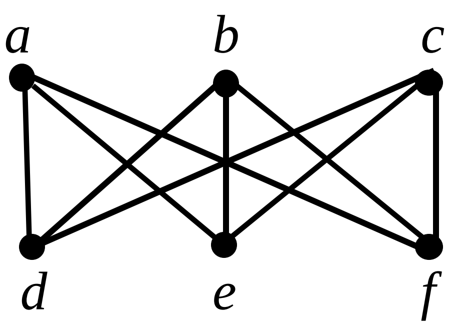
Q_3



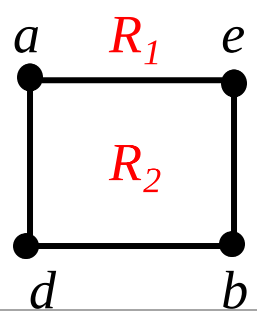
Q_3 drawn with no crossings

$\therefore Q_3$ *is planar*

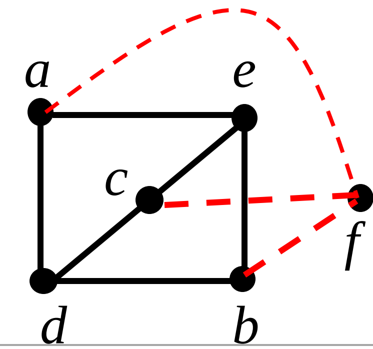
Example 3: *Show that $K_{3,3}$ is nonplanar.*



Sol.



In any drawing, aebd is cycle, and will cut the plane into two region

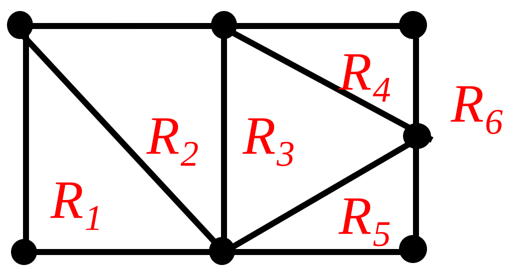


Regardless of which region c, could no longer put the f in that side staggered

Euler's Formula

A planar representation of a graph splits the plane into regions, including an unbounded region.

Example : *How many regions are there in the following graph?*



Sol. 6

Thm 1 (Euler's Formula)

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Example 4: *Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?*

Sol.

$$v = 20,\ 2e = 3\times 20 = 60,\ e = 30$$

$$r = e-v+2 = 30-20+2 = 12$$

Corollary 1

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Example 5: *Show that K_5 is nonplanar.*

Sol.

$$v = 5,\ e = 10,\ \text{but } 3v - 6 = 9.$$

Corollary 2

If G is a connected planar simple graph, then G has a vertex of degree ≤ 5.

pf: Let G be a planar graph of v vertices and e edges.

If $deg(v) \geq 6$ for every $v \in V(G)$

$$\Rightarrow \sum_{v \in V(G)} \deg(v) \geq 6v$$

$$\Rightarrow 2e \geq 6v \qquad \rightarrow\leftarrow (e \leq 3v - 6)$$

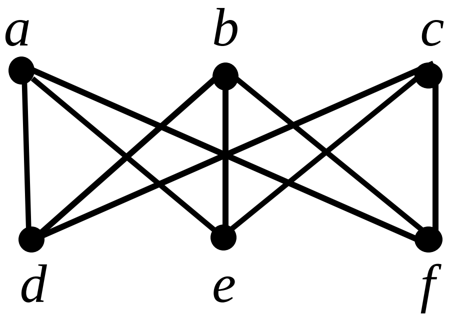
Corollary 3

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Example 6: *Show that $K_{3,3}$ is nonplanar by Cor. 3.*

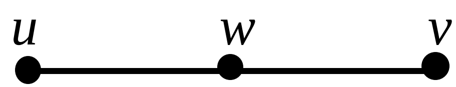
Sol.

Because $K_{3,3}$ has no circuits of length three, and $v = 6$, $e = 9$, but $e = 9 > 2v - 4$.



Kuratowski's Theorem

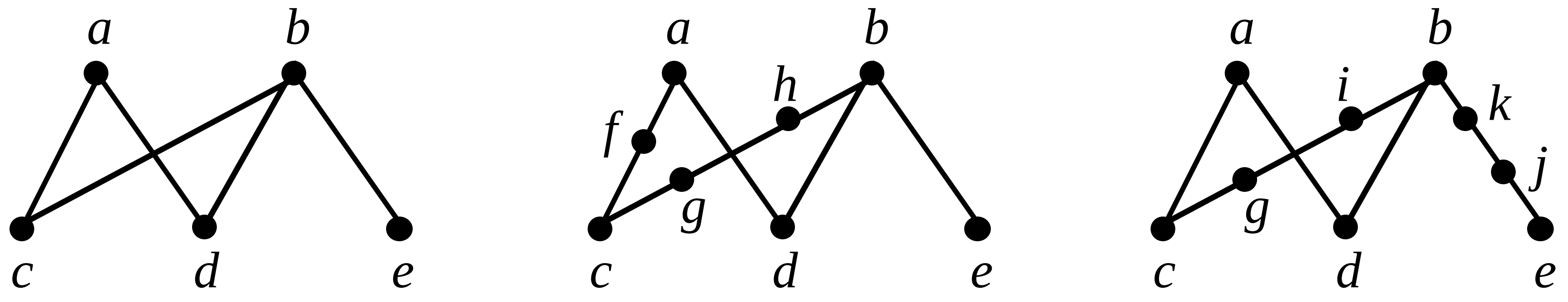
If a graph is planar, so will be any graph obtained by removing an edge {u, v} and adding a new vertex w together with edges {u, w} and {v, w}.



Such an operation is called an elementary subdivision.

Two graphs $G_1 = (V_1, E_1)$, $G_2=(V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisons.

Example 7: *Show that the graphs G_1, G_2, and G_3 are all homeomorphic.*

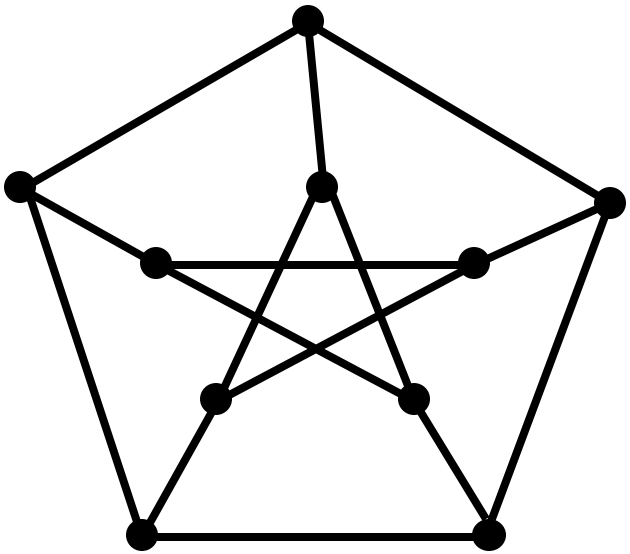


Sol: *all three can be obtained from G_1*

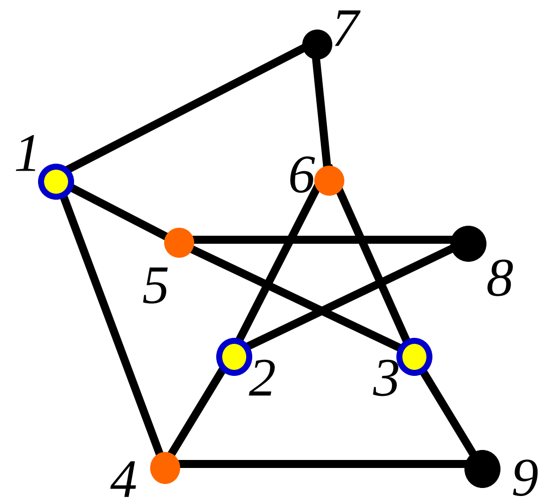
Thm 2. (Kuratowski Theorem)

A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

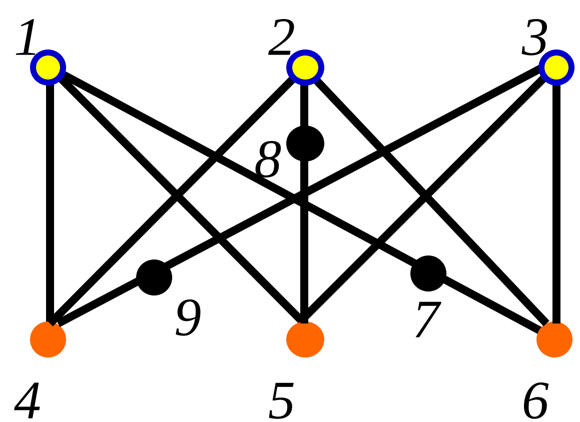
Example 9: *Show that the Petersen graph is not planar.*



Sol:



It is homeomorphic to $K_{3,3}$.

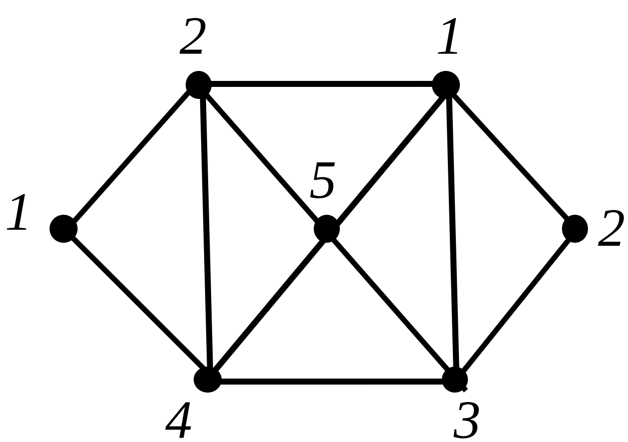


Graph Coloring

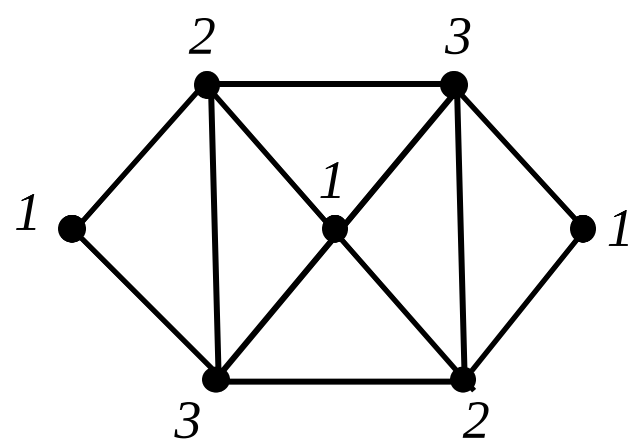
Def. 1:

A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Example:



5-coloring



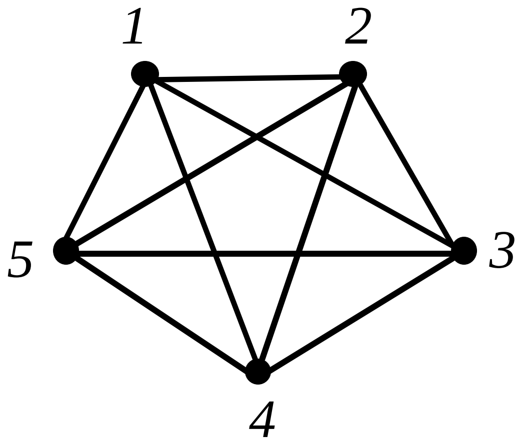
3-coloring

Less the number of colors, the better

Def. 2:

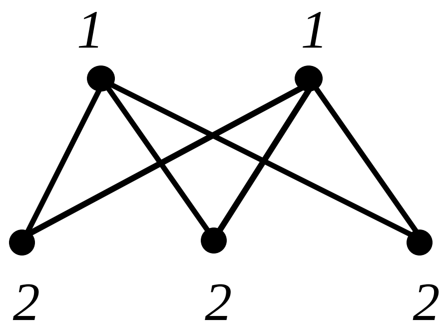
The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph. (denoted by $\chi(G)$)

Example 2: $\chi(K_5)=5$



Note: $\chi(K_n)=n$

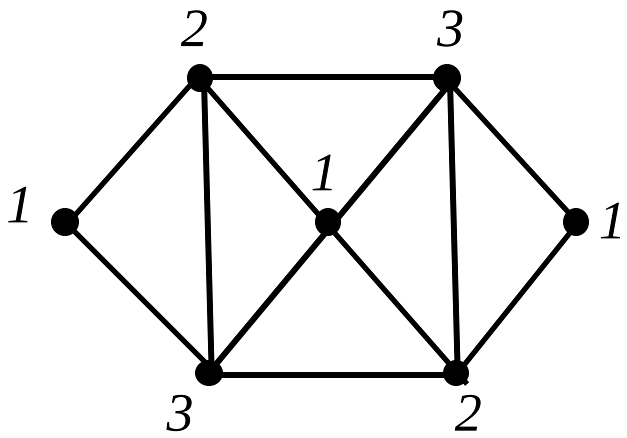
Example: $\chi(K_{2,3}) = 2$.



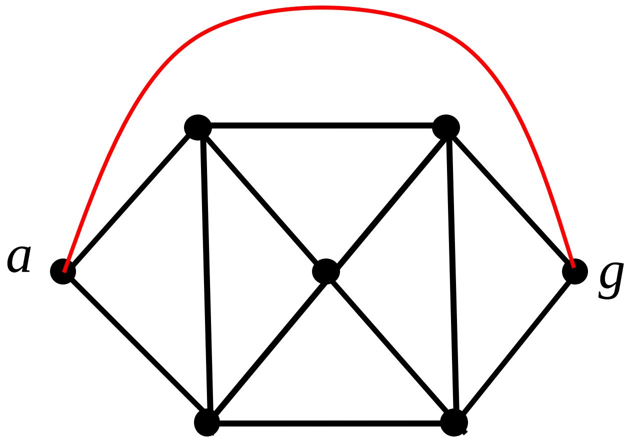
Note: $\chi(K_{m,n}) = 2$

Note: *If G is a bipartite graph, $\chi(G) = 2$.*

Example 1: *What are the chromatic numbers of the graphs G and H?*



G



H

Sol: *G has a 3-cycle*

$\Rightarrow \chi(G) \geq 3$

G has a 3-coloring

$\Rightarrow \chi(G) \leq 3$

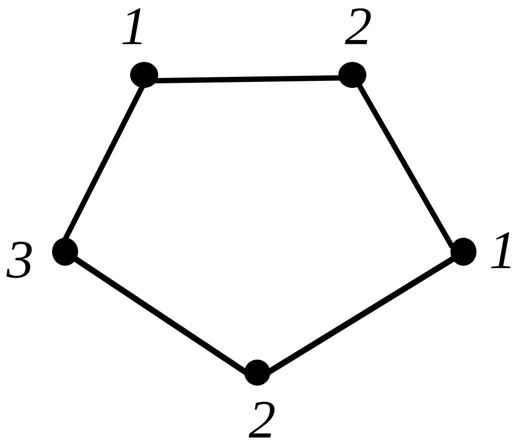
$\Rightarrow \chi(G) = 3$

Sol: *any 3-coloring for $H - \{(a,g)\}$ gives the same color to a and g*

$\Rightarrow \chi(H) > 3$

4-coloring exists $\Rightarrow \chi(H) = 4$

Example 4: $\chi(C_n) = \begin{cases} 2 \text{ if } n \text{ is even,} \\ 3 \text{ if } n \text{ is odd.} \end{cases}$



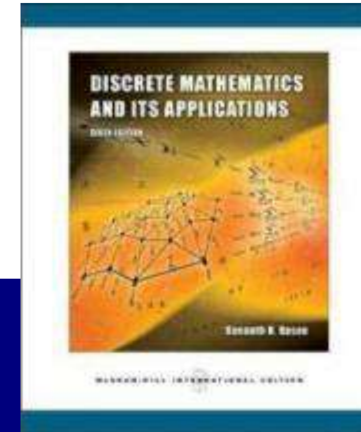
C_n is bipartite when n is even.

Thm 1. (The Four Color Theorem)

The chromatic number of a planar graph is no greater than four.

Corollary

Any graph with chromatic number >4 is nonplanar.



Discrete Mathematics

Trees

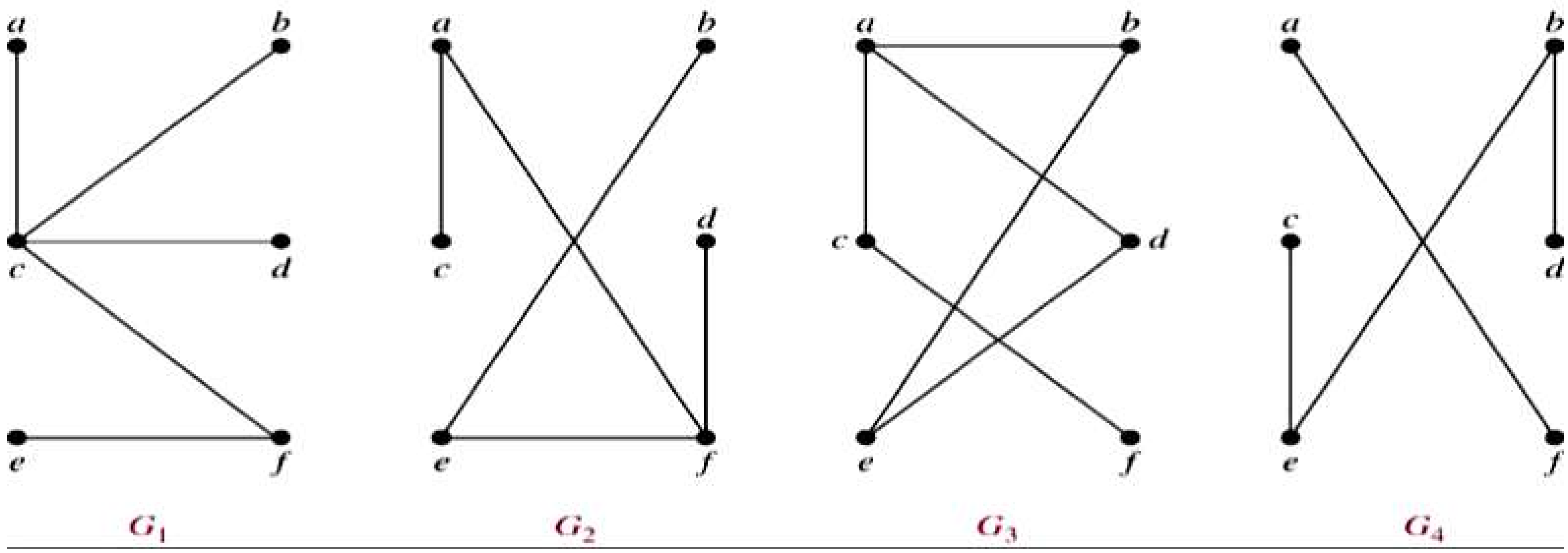
Outline

- 10.1 Introduction to Trees
- 10.2 Applications of Trees
- 10.3 Tree Traversal
- 10.4 Spanning Trees
- 10.5 Minimal Spanning Trees

10.1 Introduction to Trees

Def 1 A tree is a connected undirected graph with no simple circuits.

Example 1. Which of the graphs are trees?



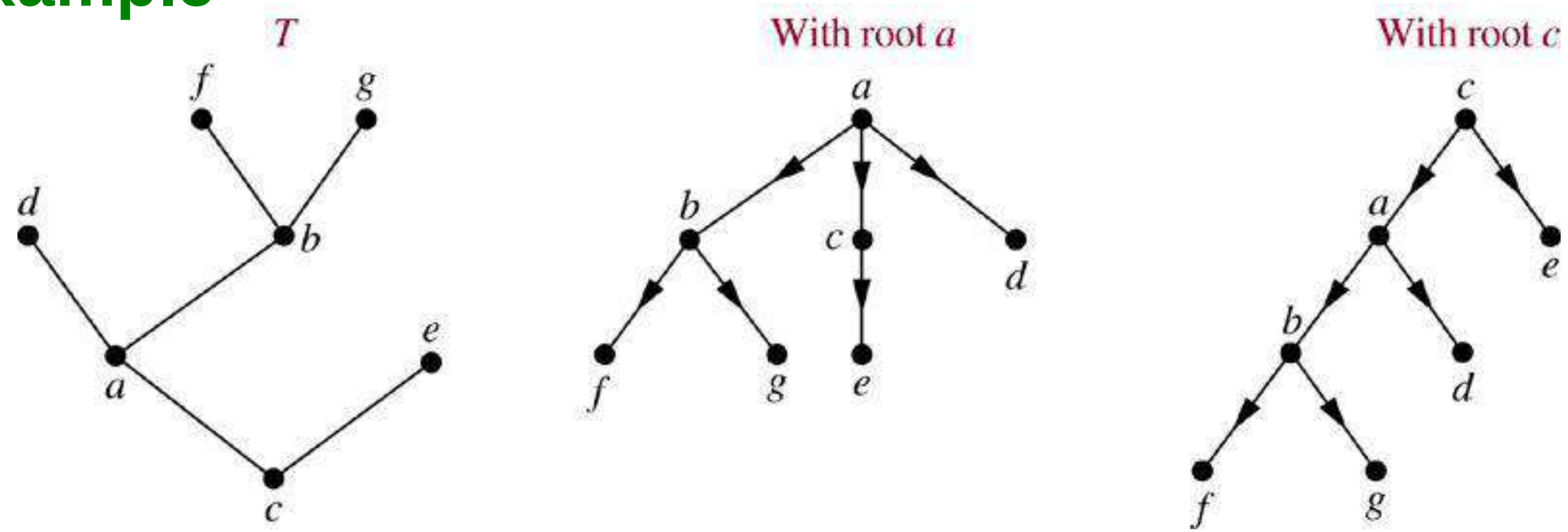
Sol: G_1, G_2

Note. If connected condition is removed, it becomes forest

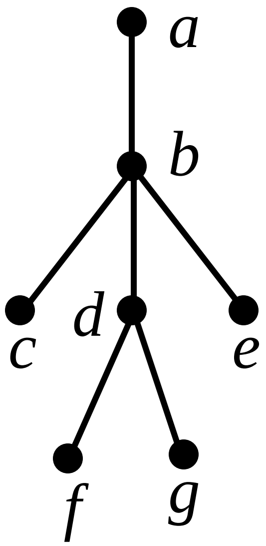
Thm 1. Any undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Def 2. A rooted tree is a tree in which one vertex has been designed as the root and every edge is directed away from the root. (Arrow to disappear)

Example



Def:



a is the parent of b, b is the child of a,

c, d, e are siblings,

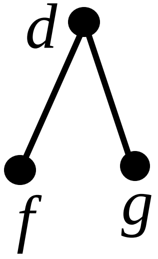
a, b, d are ancestors of f

c, d, e, f, g are descendants of b

c, e, f, g are leaves of the tree (deg=1)

a, b, d are internal vertices of the tree (at least one child)

subtree with d as its root:

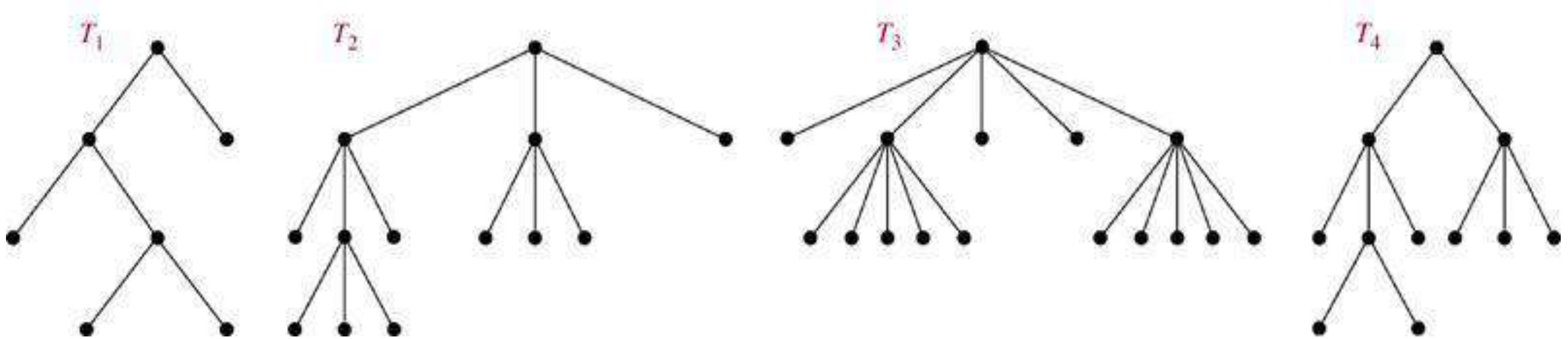


Def:

Vertices that have children are called **internal vertices**

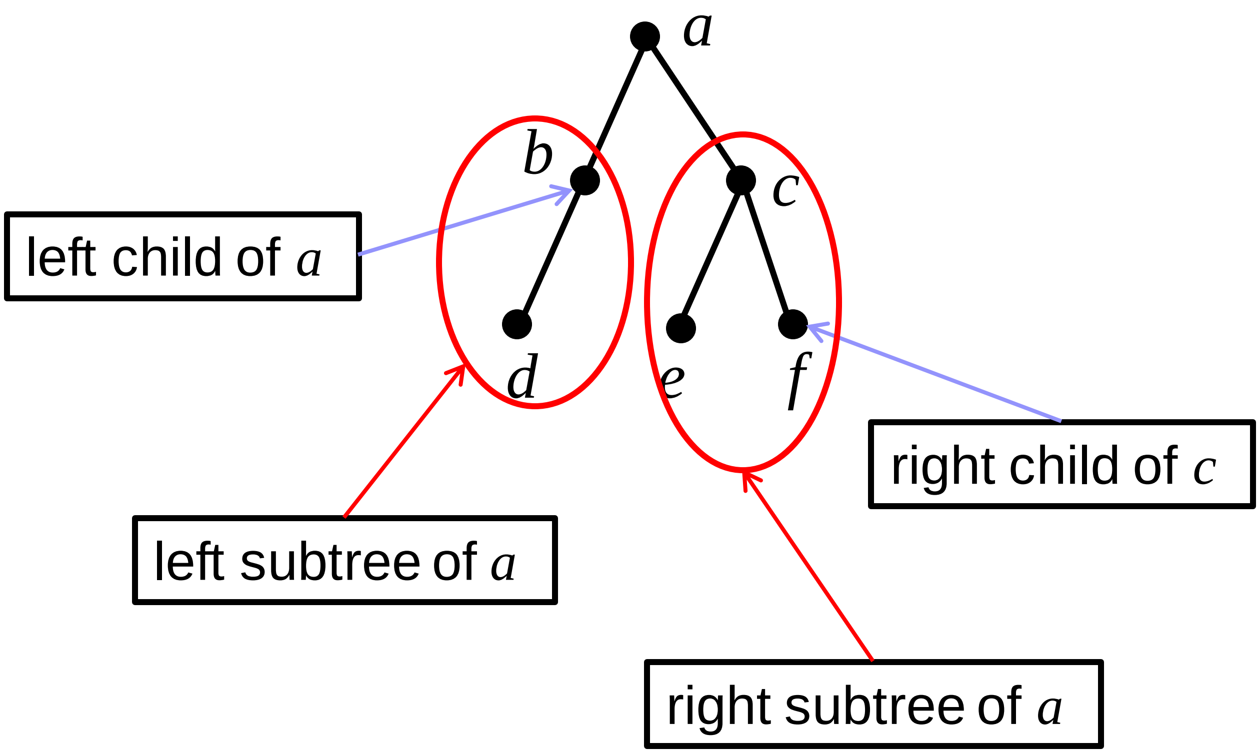
Def 3 A rooted tree is called an m-ary tree if every internal vetex has no more than m children. The tree is called a full m-ary tree if every internal vertex has exactly m children. An m-ary tree with $m=2$ is called a binary tree.

Example 3



full binary tree | full 3-ary tree | full 5-ary tree | not full 3-ary tree

Def:



a
b
c
left child of a
d
e
f
right child of c
left subtree of a
right subtree of a

Properties of Trees

Thm 2. A tree with n vertices has $n-1$ edges.

Thm 3. A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.

Pf. Every vertex, except the root, is the child of an internal vertex.

Each internal vertex has m children.

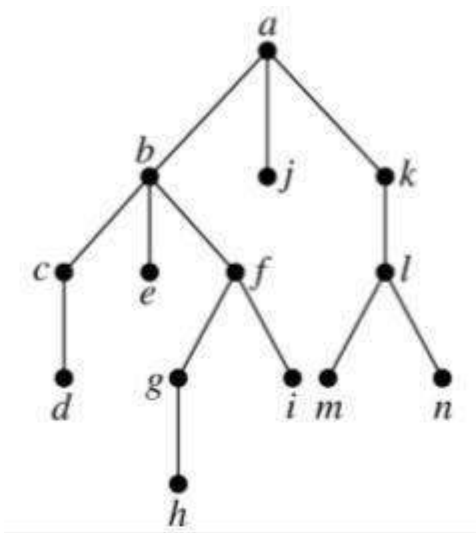
$\Rightarrow$ there are $mi + 1$ vertices in the tree

Cor. A full m-ary tree with n vertices contains $(n-1)/m$ internal vertices, and hence $n - (n-1)/m = ((m-1)n+1)/m$ leaves.

Def: The level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex. The level of the root is defined to be zero.

The height of a rooted tree is the maximum of the levels of vertices.

Example 10.

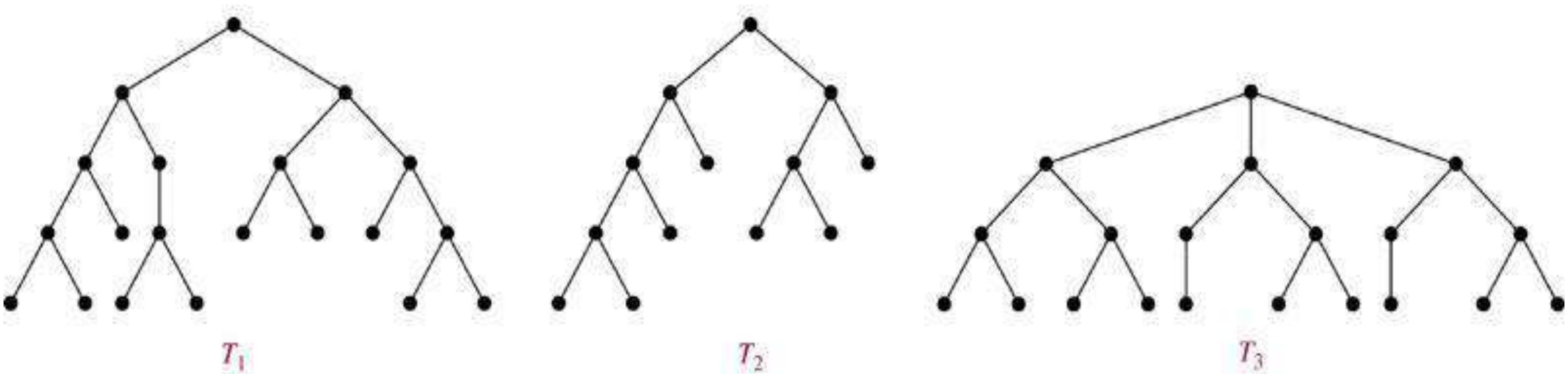


level
0
1
2
3
4

height = 4

Def: A rooted m-ary tree of height h is balanced if all leaves are at levels h or $h-1$.

Example 11 Which of the rooted trees shown below are balanced?



Sol. T_1, T_3

Thm 5. There are at most m^h leaves in an m-ary tree of height h.

Def: A complete m-ary tree is a full m-ary tree, where every leaf is at the same level.

Ex 28 How many vertices and how many leaves does a complete m-ary tree of height h have?

Sol.

# of vertices = $1+m+m^2+\ldots+m^h = (m^{h+1}-1)/(m-1)$

# of leaves = m^h

10.2 Applications of Trees

- **Binary Search Trees**
- **Decision Trees**
- **Prefix Codes**
- **Game Trees**

Binary Search Trees

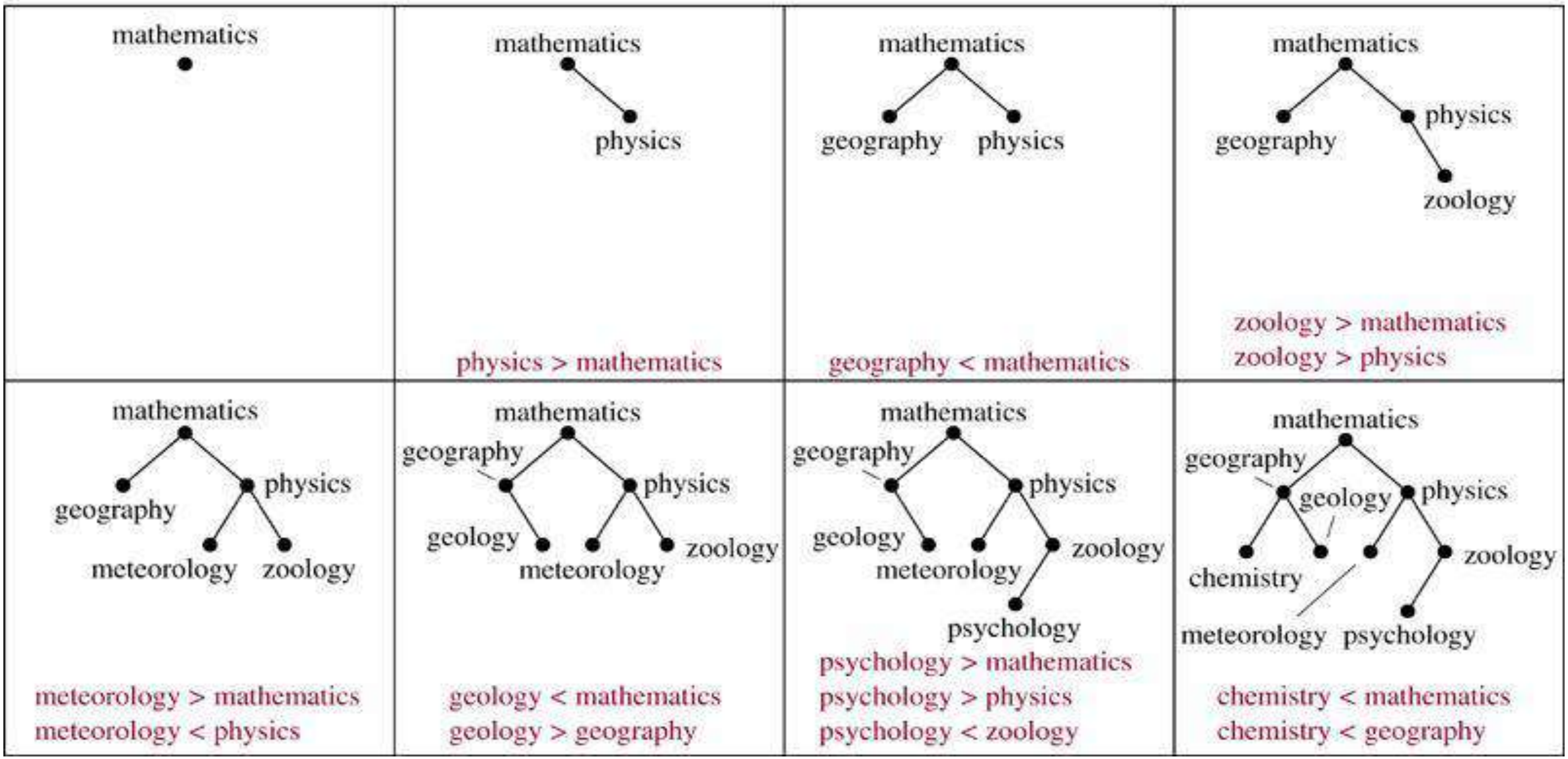
Goal: Implement a searching algorithm that finds items efficiently when the items are totally ordered.

Binary Search Tree: Binary tree + each child of a vertex is designed as a right or left child, and each vertex v is labeled with a key $label(v)$, which is one of the items.

Note: $label(v) > label(w)$ if w is in the left subtree of v and $label(v) < label(w)$ if w is in the right subtree of v

Example 1 Form a binary search tree for the words *mathematics*, *physics*, *geography*, *zoology*, *meteorology*, *geology*, *psychology*, and *chemistry* (using alphabetical order).

Sol.



Algorithm 1 (Locating and Adding Items to a Binary Search Tree.)

```
Procedure insertion(T: binary search tree, x: item)
v := root of T
{a vertex not present in T has the value null}
while v ≠ null and label(v) ≠ x
begin
    if x < label(v) then
        if  left child of v ≠ null then v:=left child of v
        else add new vertex as a left child of v and set v := null
    else
        if  right child of v ≠ null then v:= right child of v
        else add new vertex as a right child of v and set v := null
end
if  root of T = null then add a vertex v to the tree and label it with x
else if v is null or label(v) ≠ x then label new vertex with x and
                                       let v be this new vertex
{v = location of x}
```

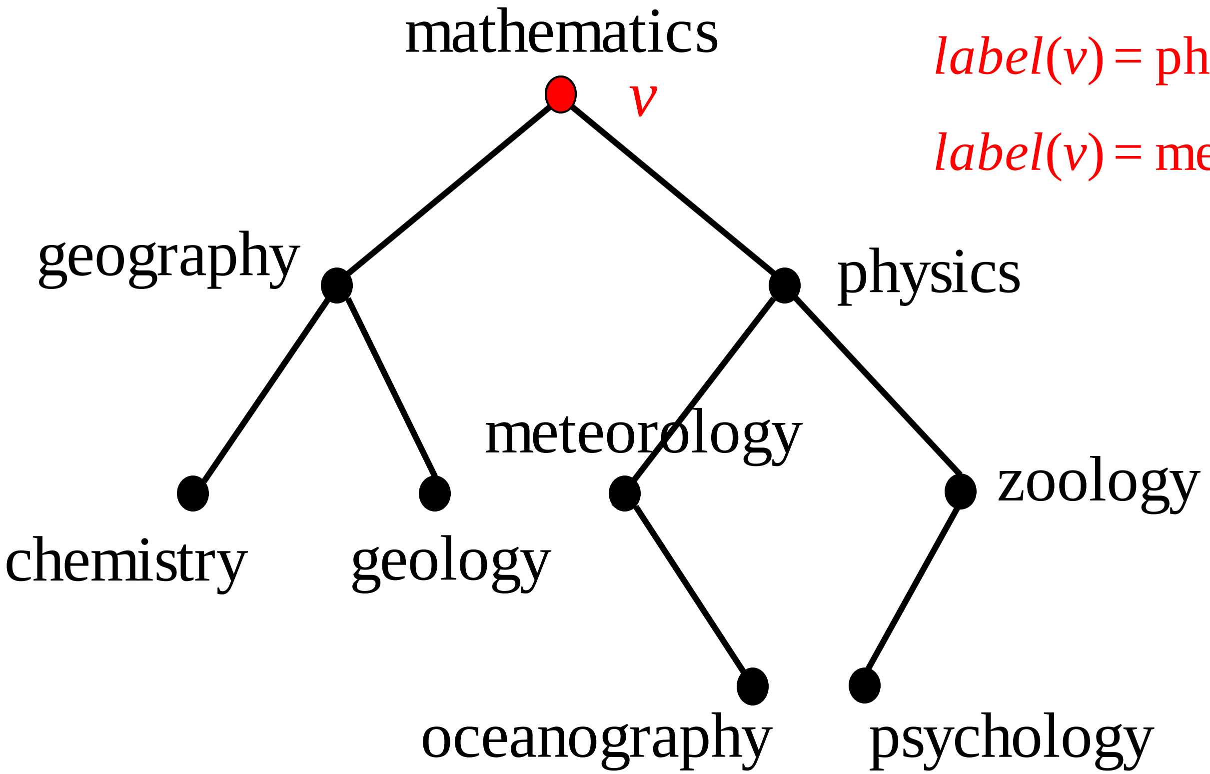
Example 2 Use Algorithm 1 to insert the word *oceanography* into the binary search tree in Example 1.

Sol.

$label(v) = \text{mathematics} < oceanography$

$label(v) = \text{physics} > oceanography$

$label(v) = \text{meteorology} < oceanography$



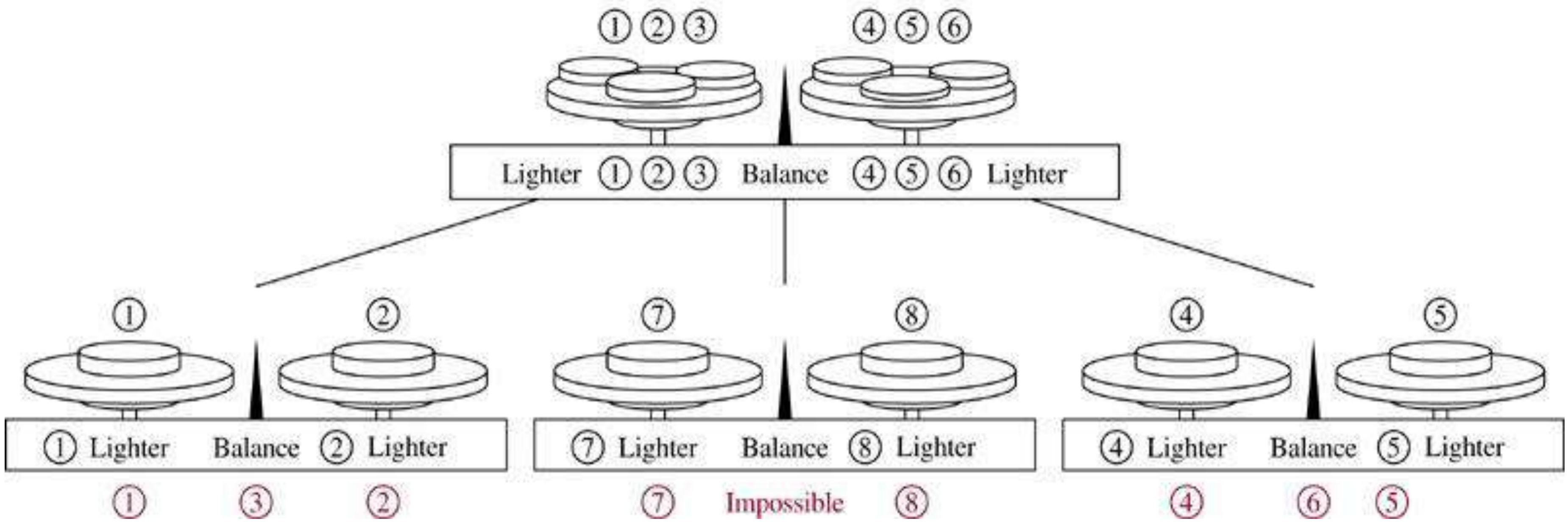
Decision Trees

A rooted tree in which each internal vertex corresponds to a decision, with a subtree at these vertices for each possible outcome of the decision, is called a decision tree.

Example 3 Suppose there are seven coins, all with the same weight, and a counterfeit coin that weights less than the others. How many weighings are necessary using a balance scale to determine which of the eight coins is the counterfeit one? Give an algorithm for finding this counterfeit coin.

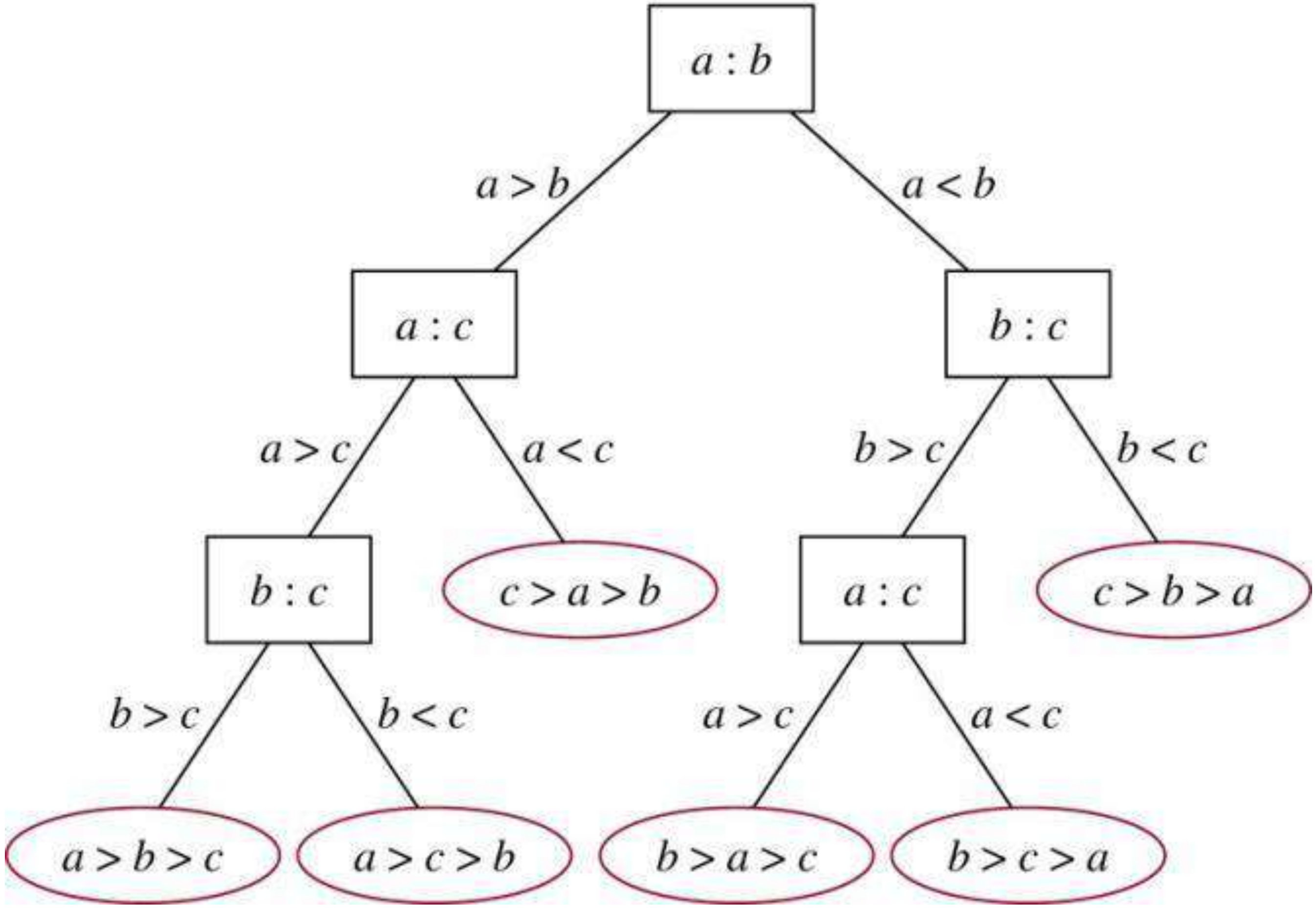
Sol. When weighing, it is either the two pans can have equal weight or the first pan can be heavier, or the second pan can be heavier $\Rightarrow$ 3-ary tree

Need 8 leaves $\Rightarrow$ Need to weigh at least twice



Example 4 A decision tree that orders the elements of the list a, b, c.

Sol.



Prefix Codes

Problem: Using bit strings to encode the letter of the English alphabet (Not case sensitive)

$\Rightarrow$ each letter needs a bit string of length 5 (Because $2^4 < 26 < 2^5$)

$\Rightarrow$ Is it possible to find a coding scheme of these letter such that when data are coded, fewer bits are used?

$\Rightarrow$ Encode letters using varying numbers of bits.

$\Rightarrow$ Some methods must be used to determine where the bits for each character start and end.

$\Rightarrow$ Prefix codes: Codes with the property that the bit string for a letter never occurs as the first part of the bit string for another letter.

Example: (not prefix code)

e : 0, a : 1, t : 01

The string 0101 could correspond to eat, tea, $eaea$, or tt.

Example: (prefix code)

e : 0, a : 10, t : 11

The string 10110 is the encoding of ate.

A prefix code can be represented using a binary tree.

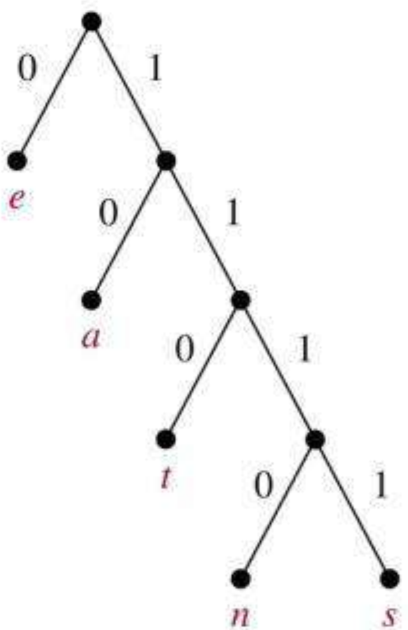
character: the label of the leaf

edge label: left child $\rightarrow$ 0, right child $\rightarrow$ 1

The bit string used to encode a character is the sequence of labels of the edges in the unique path from the root to the leaf that has this character as its label.

Example:

Walking from the root to the leaf repeatedly



encode

$e : 0$

$a : 10$

$t : 110$

$n : 1110$

$s : 1111$

decode

$$\underbrace{1111}_{s}\underbrace{10}_{a}\underbrace{1110}_{n}\underbrace{0}_{e}$$

$$\Rightarrow \; sane$$

Huffman Coding (data compression)

Input the frequencies of symbols in a string and output a prefix code that encodes the string using the fewest possible bits, among all possible binary prefix codes for these symbols.

Start with a lot of isolated points and label is the symbol,
Use at least two symbol into a subtree,
Repeat this concept
Using at least two of the subtrees into a subtree, ...

Algorithm 2 (Huffman Coding)

Procedure *Huffman*(C: symbols a_i with frequencies w_i, $i = 1, \ldots, n$)

F := forest of n rooted trees, each consisting of the single vertex a_i
and assigned weighted w_i

while F is not a tree

begin

Replace the rooted trees T and T' of least weights from F with $w(T) \geq w(T')$ with a tree having a new root that has T as its left subtree and T' as its right subtree. Label the new edge to T with 0 and the new edge to T' with 1.

Assign $w(T)+w(T')$ as the weight of the new tree.

end

Example 5 Use Huffman coding to encode the following symbols with the frequencies listed:
A: 0.08, B: 0.10, C: 0.12, D: 0.15, E: 0.20, F: 0.35.
What is the average number of bits used to encode a character?

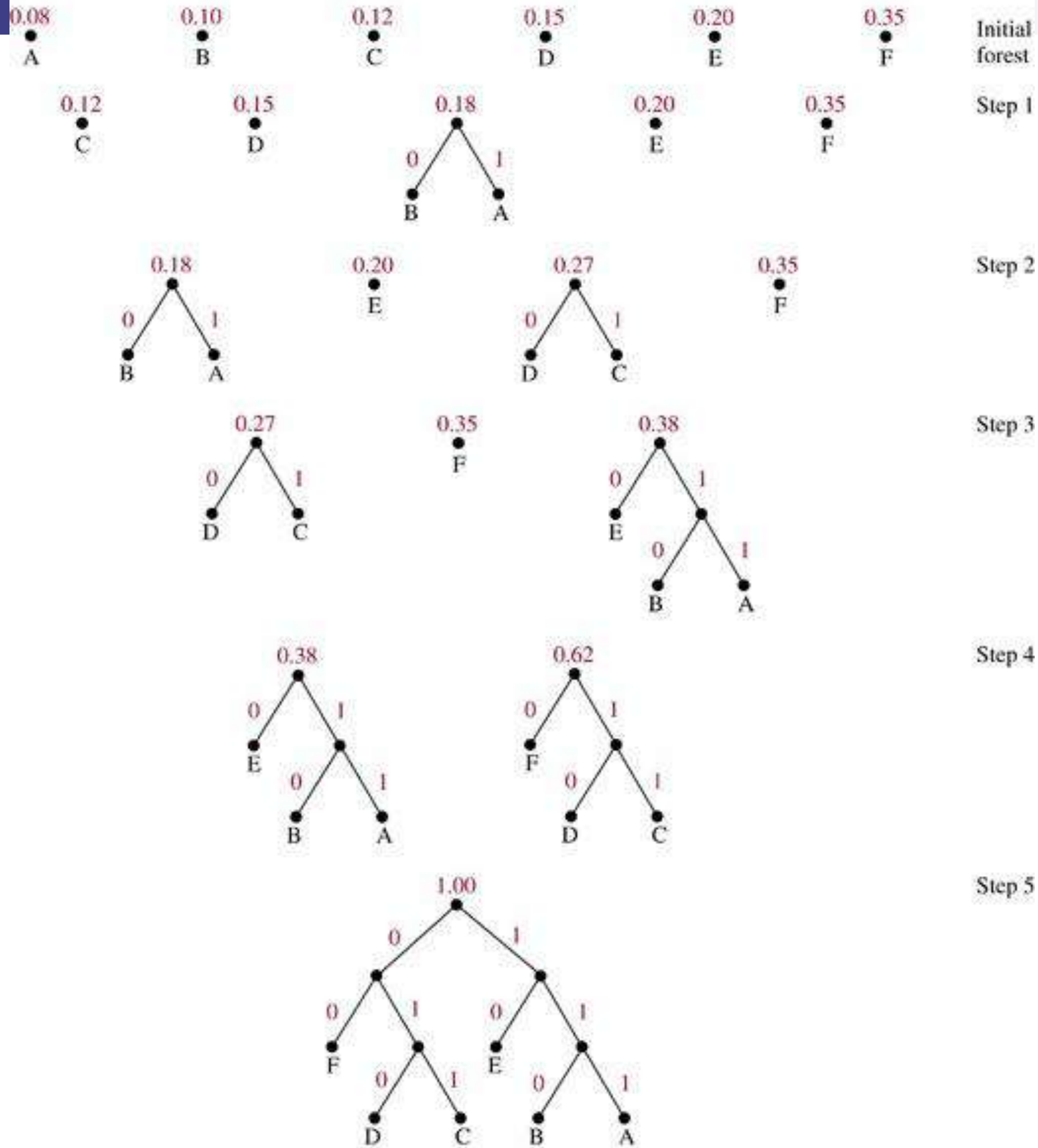
Sol:

1. Next map
2. The average number of bits is:

   Each symbol total length $\times$ frequency

$$= 3\times0.08+3\times0.10+3\times0.12+3\times0.15+2\times0.20+2\times0.35$$

$$=2.45$$


0.08 A
0.10 B
0.12 C
0.15 D
0.20 E
0.35 F
Initial forest
Step 1
Step 2
Step 3
Step 4
Step 5
0.18
0.27
0.38
0.62
1.00
0
1

10.3 Tree Traversal

We need procedures for visiting each vertex of an ordered rooted tree to access data.

Universal Address Systems

Label vertices:

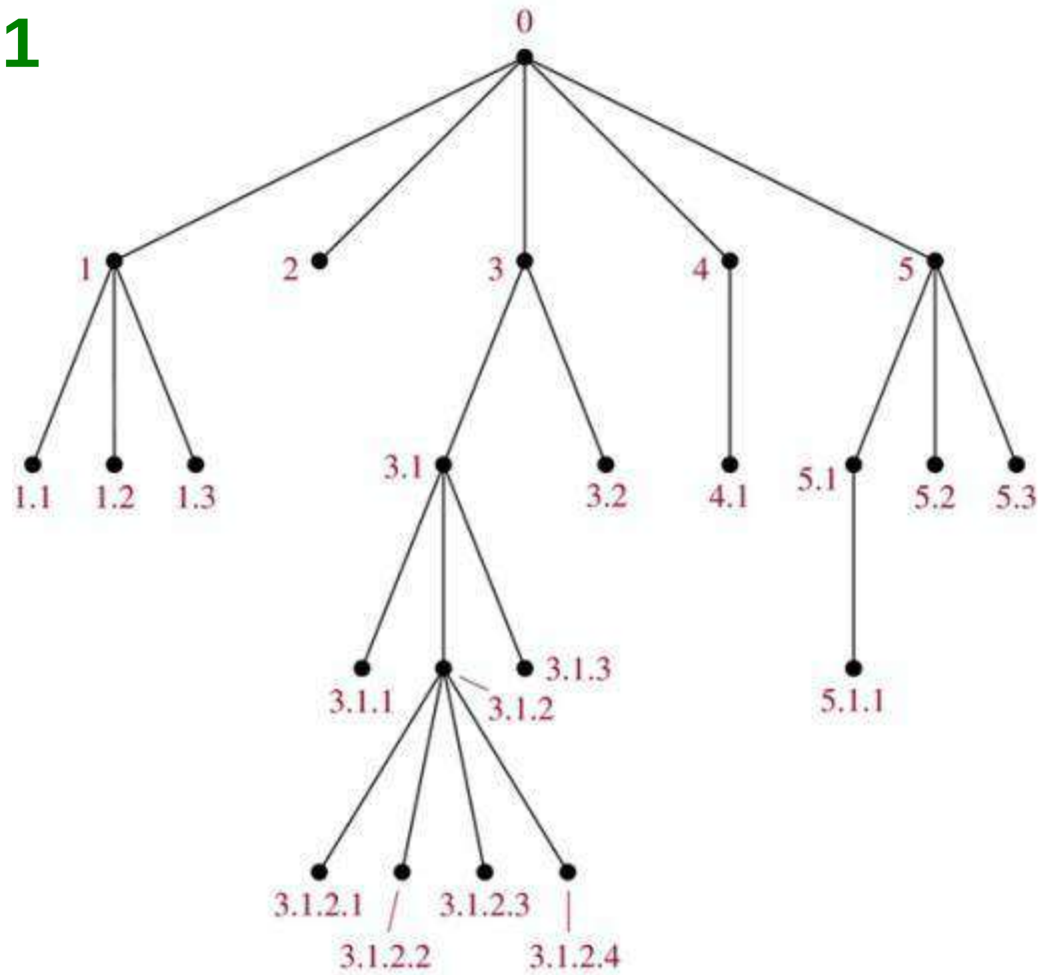
1. root $\rightarrow$ 0, its k children $\rightarrow 1, 2, \ldots, k$ (from left to right)
2. For each vertex v at level n with label A,
its r children $\rightarrow A.1, A.2, \ldots, A.r$ (from left to right).

We can totally order the vertices using the lexicographic ordering of their labels in the universal address system.

$x_1.x_2.\ldots.x_n < y_1.y_2.\ldots.y_m$

if there is an i, $0 \le i \le n$, with $x_1=y_1$, $x_2=y_2$, …, $x_{i-1}=y_{i-1}$, and $x_i<y_i$;
or if $n<m$ and $x_i=y_i$ for $i=1, 2, \ldots, n$.

Example 1

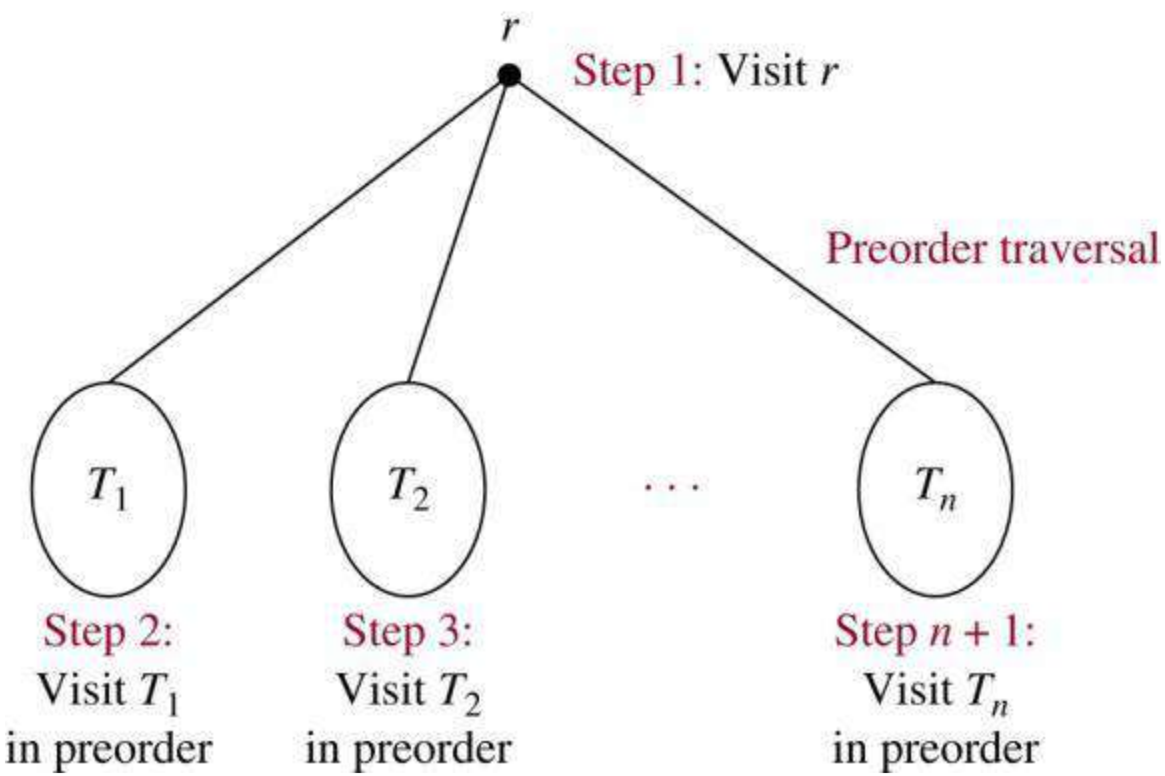


The lexicographic ordering is:

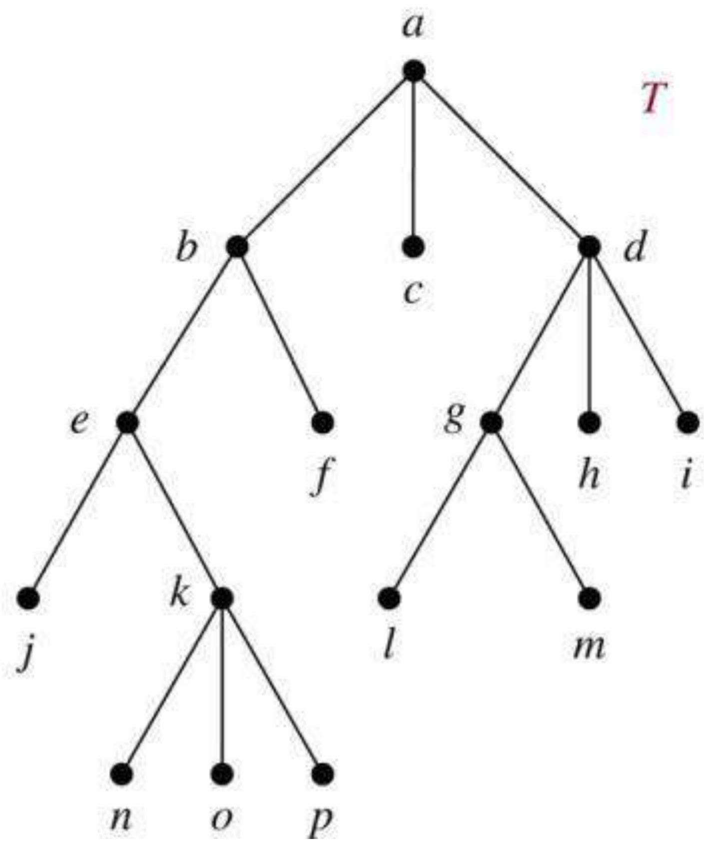
0 < 1<1.1 < 1.2 < 1.3 < 2 < 3 < 3.1 < 3.1.1 < 3.1.2 < 3.1.2.1 < 3.1.2.2 < 3.1.2.3 < 3.1.2.4 < 3.1.3 < 3.2 < 4 < 4.1 < 5 < 5.1 < 5.1.1 < 5.2 < 5.3

Traversal Algorithms

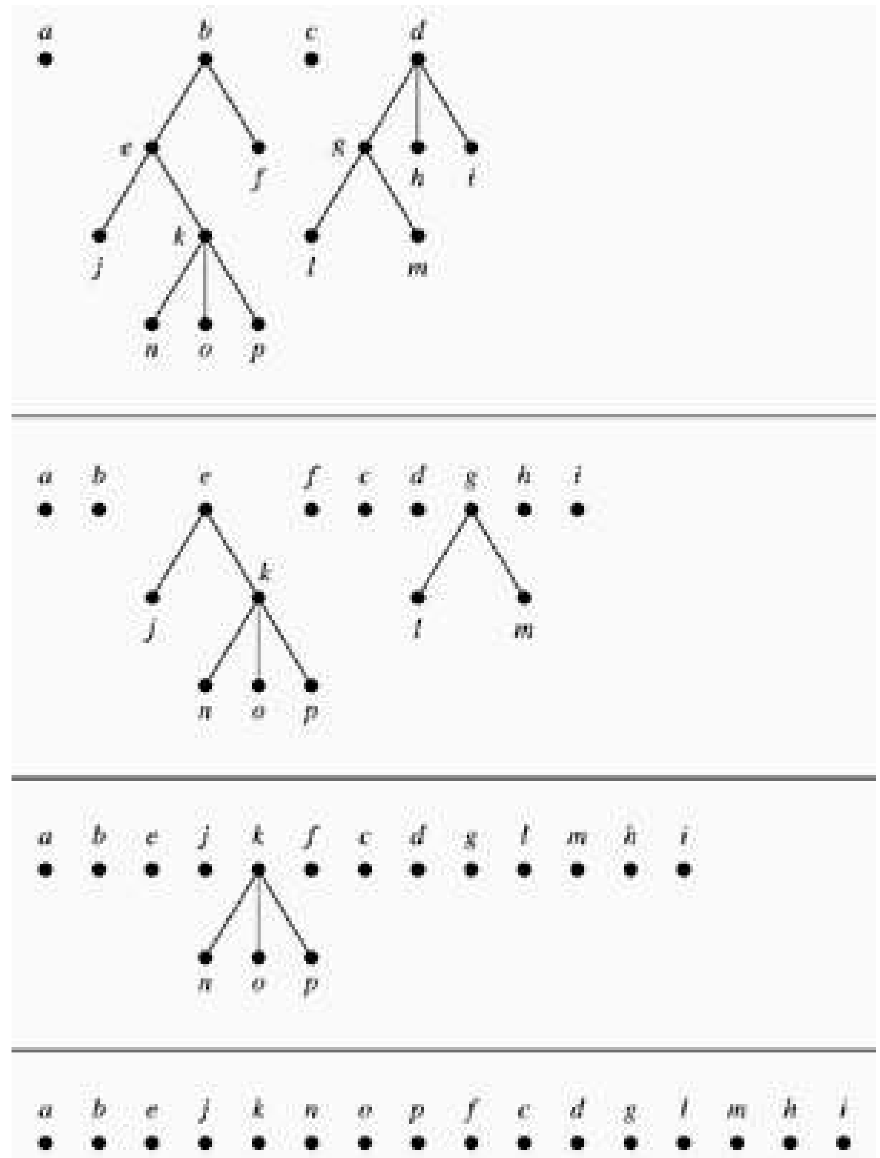
Preorder traversal (Preorder)



Example 2. In which order does a preorder traversal visit the vertices in the ordered rooted tree T shown below?



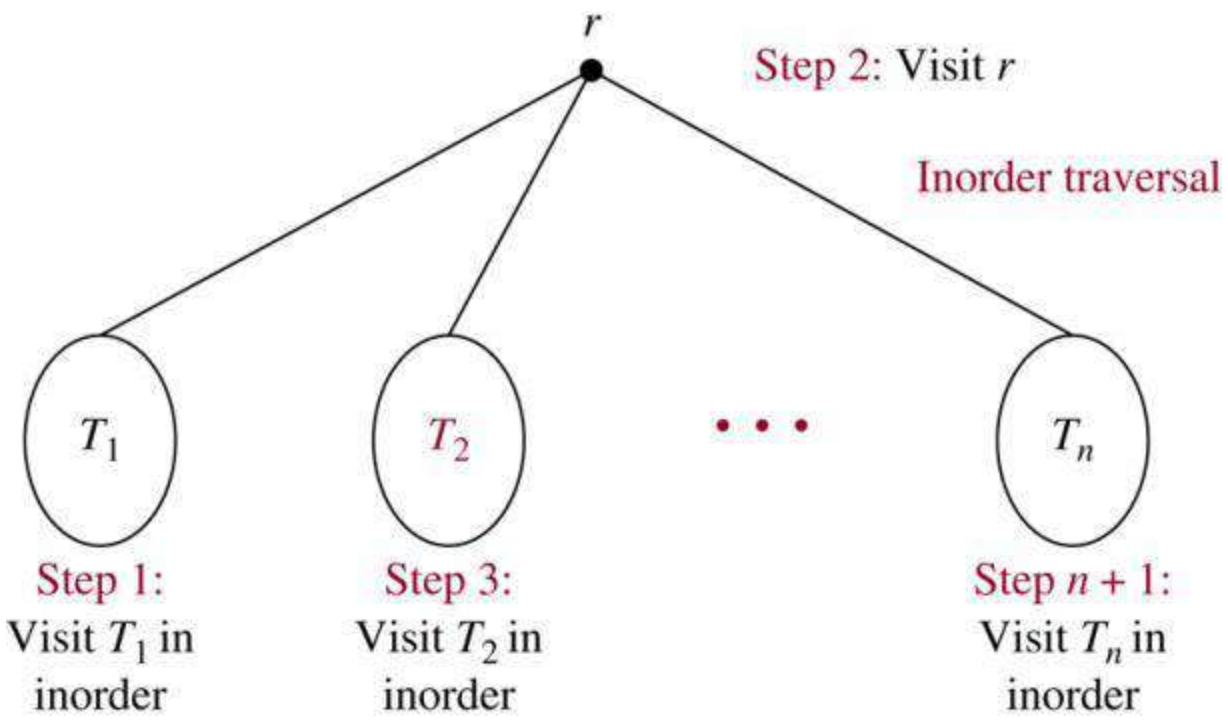
Sol:



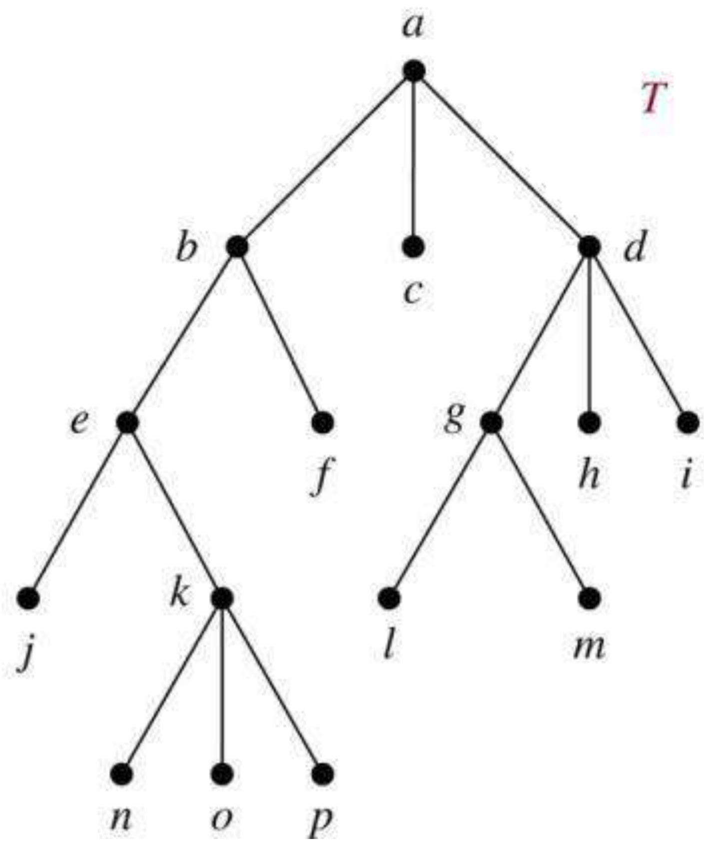
Algorithm 1 (Preorder Traversal)

Procedure *preorder*(T: ordered rooted tree)
r := root of T
list r
for each child c of r from left to right
begin
    $T(c)$:= subtree with c as its root
    preorder($T(c)$)
end

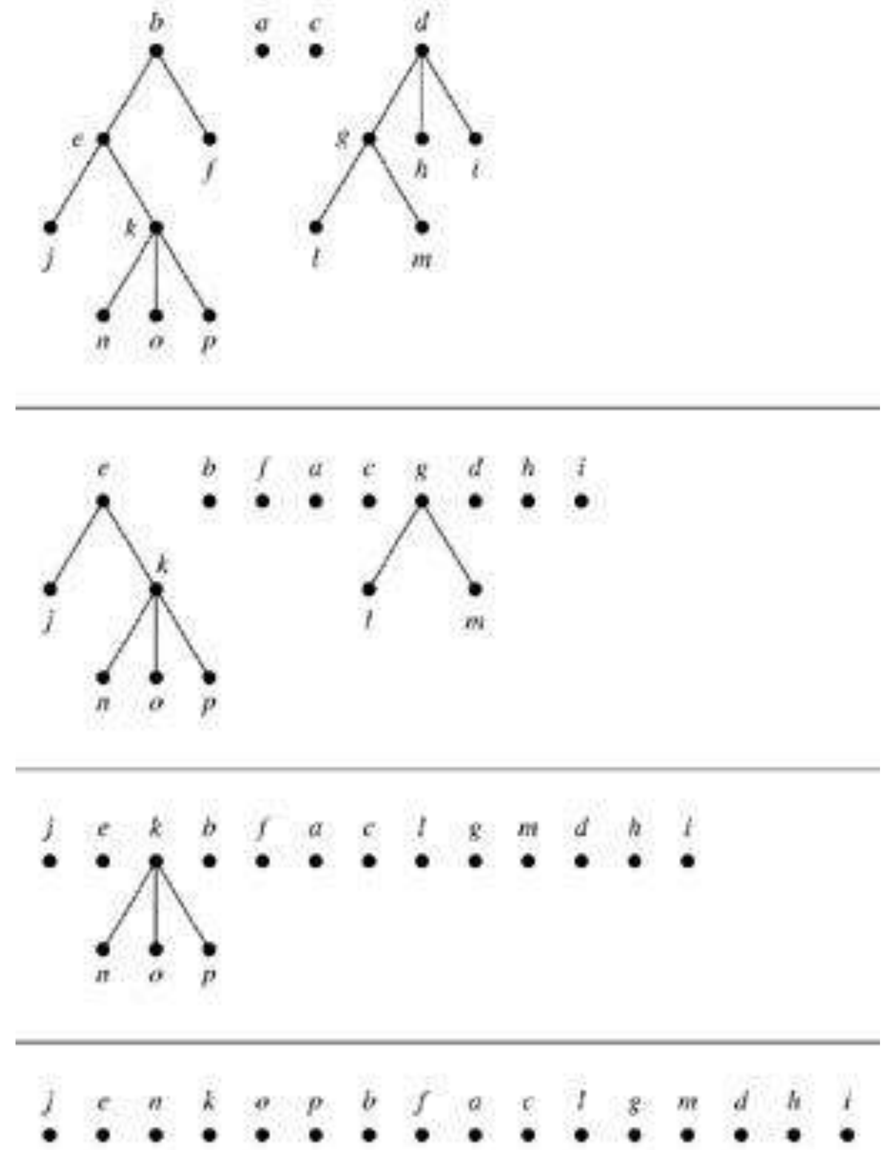
Inorder traversal(In the sequence)



Example 3. In which order does a preorder traversal visit the vertices in the ordered rooted tree T shown below?



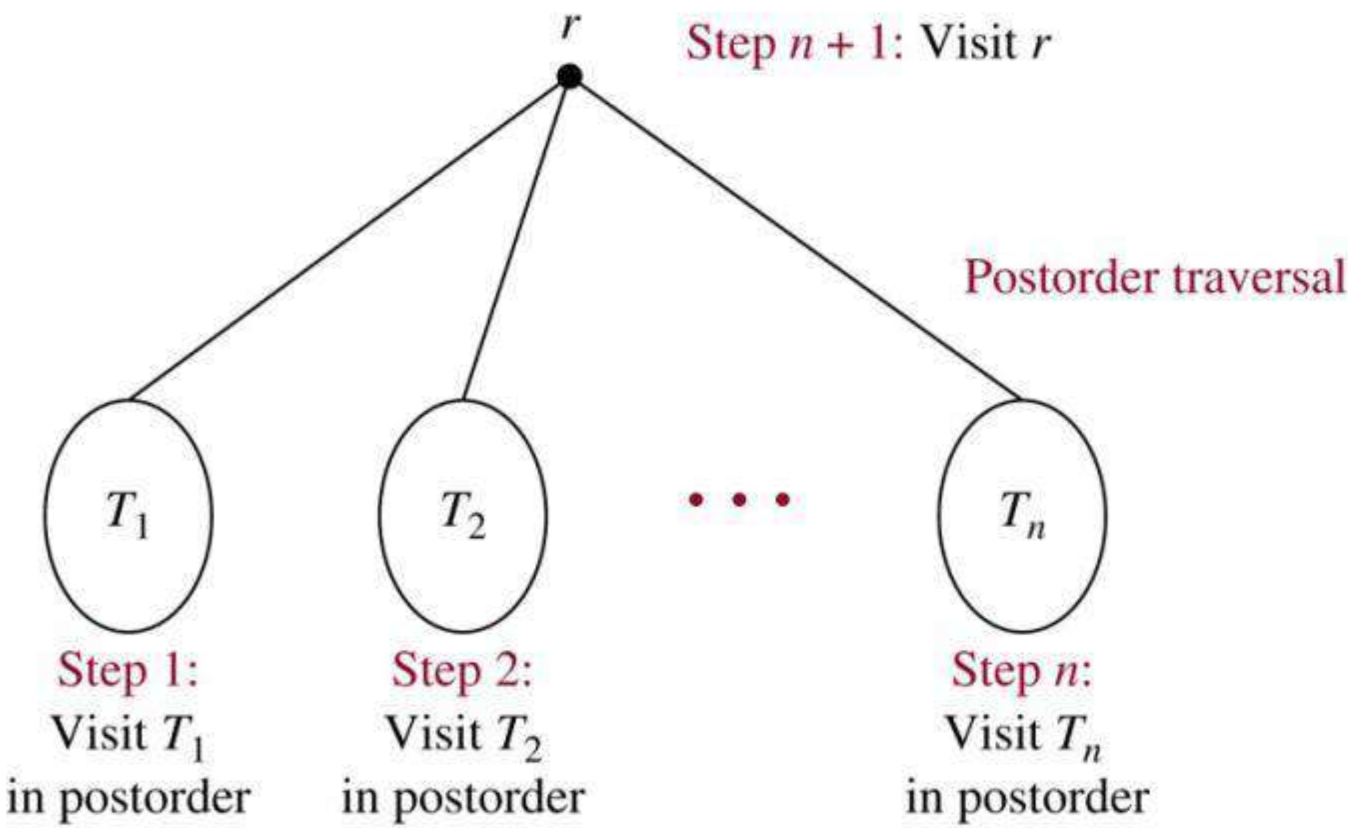
Sol:



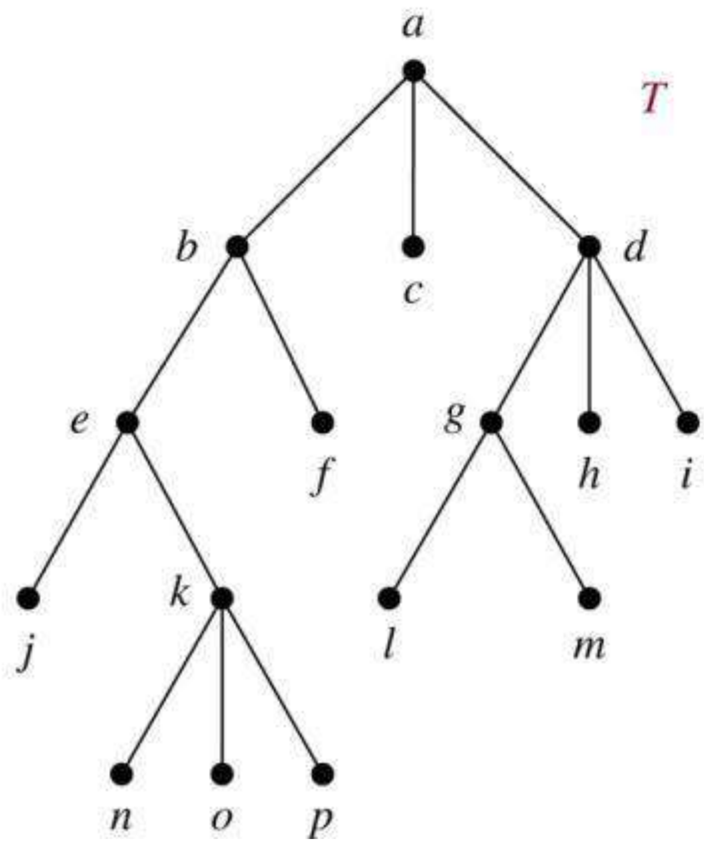
Algorithm 2 (Inorder Traversal)

Procedure *inorder*(T: ordered rooted tree)
r := root of T
If r is a leaf **then** list r
else
begin
    l := first child of r from left to right
    $T(l)$:= subtree with l as its root
    inorder($T(l)$)
    list r
    for each child c of r except for l from left to right
        $T(c)$:= subtree with c as its root
        inorder($T(c)$)
end

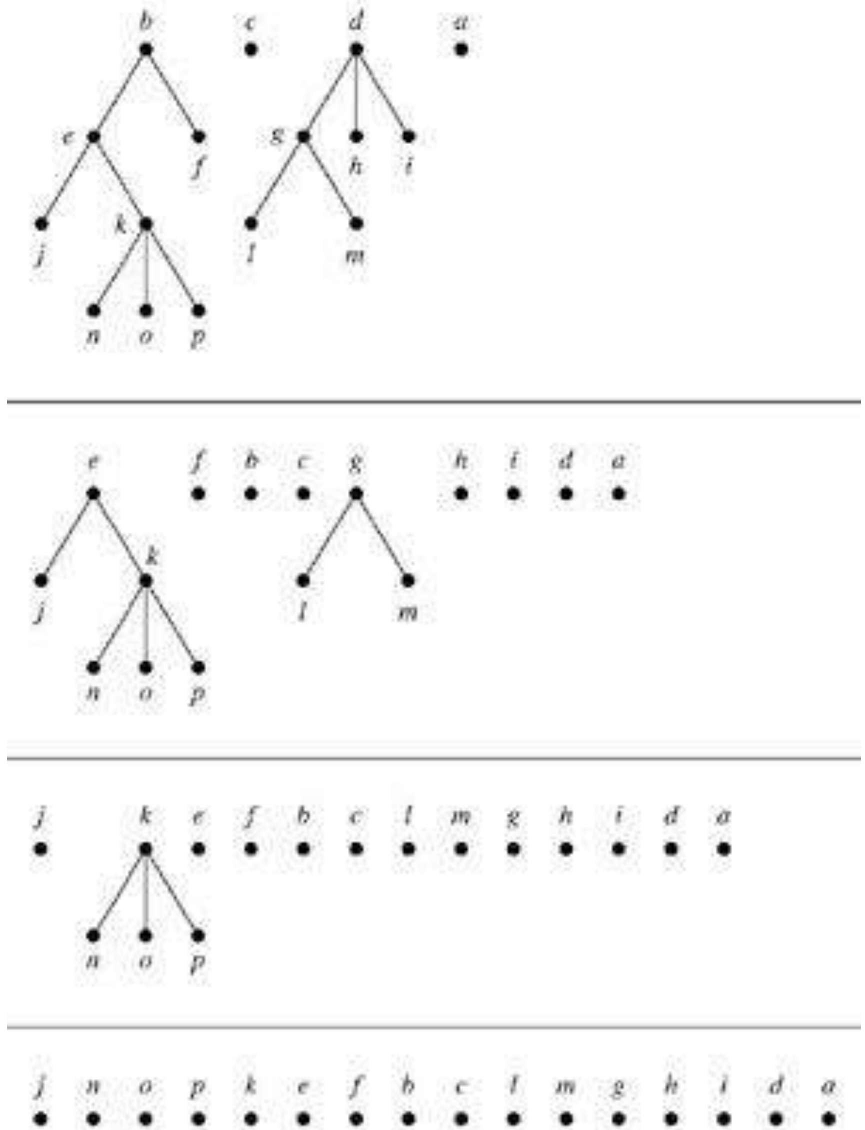
Postorder traversal



Example 4. In which order does a preorder traversal visit the vertices in the ordered rooted tree T shown below?



Sol:



Algorithm 3 (Postorder Traversal)

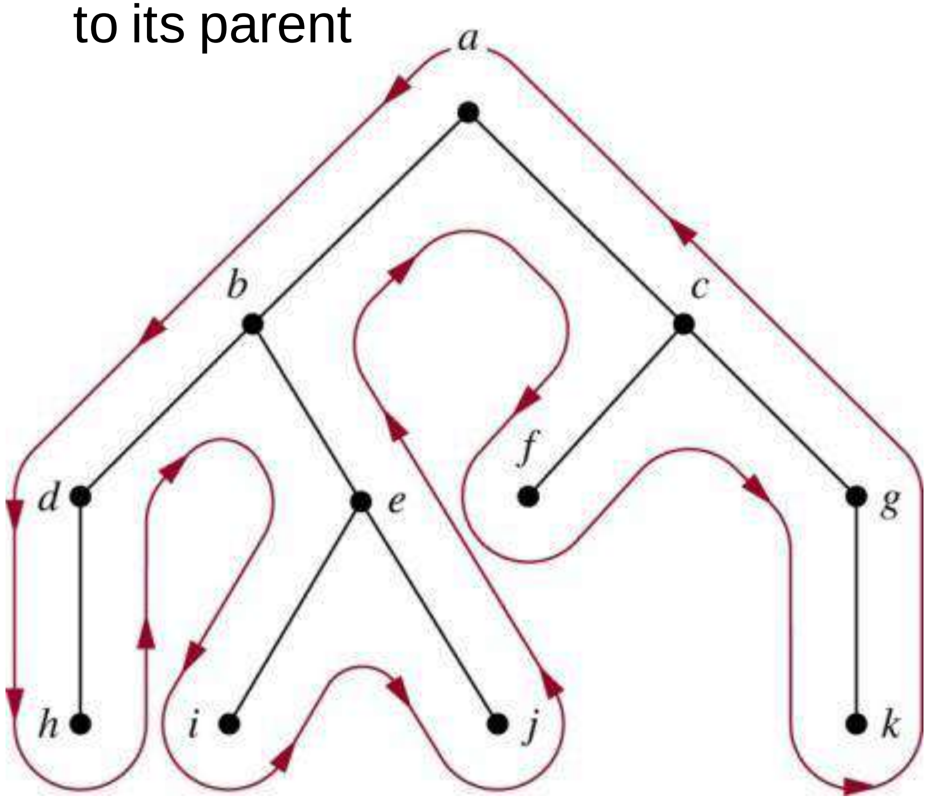
Procedure *postorder*(T: ordered rooted tree)
r := root of T
for each child c of r from left to right
begin
    $T(c)$:= subtree with c as its root
    postorder($T(c)$)
end
list r

Easy representation: draw a red line around the ordered rooted tree starting at the root, moving along the edges

Preorder: listing each vertex the first time this line passes it

Inorder: listing a leaf the first time the line passes it and listing each internal vertex the second time the line passes it

Postorder: listing a vertex the last time it is passed on the way back up to its parent



Preorder:

$a, b, d, h, e, i, j, c, f, g, k$

Inorder:

$h, d, b, i, e, j, a, f, c, k, g$

Postorder:

$h, d, i, j, e, b, f, k, g, c, a$

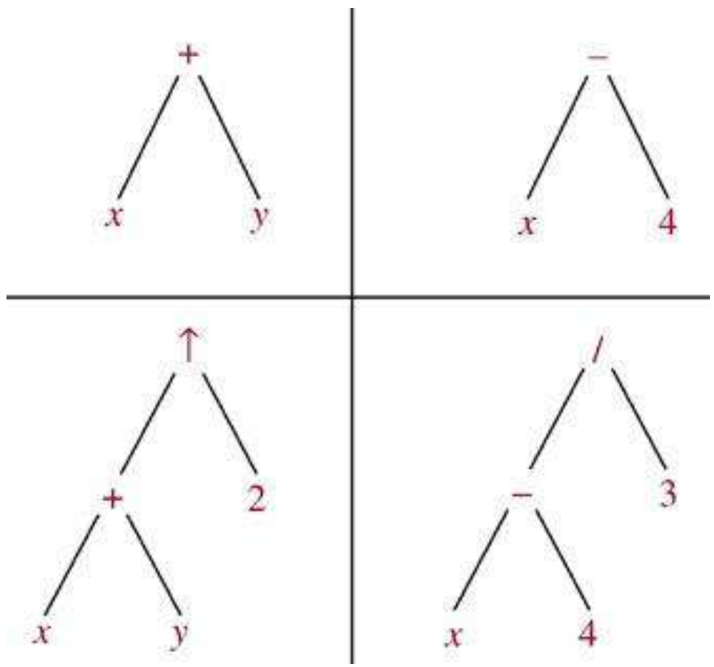
Infix, Prefix, and Postfix Notation

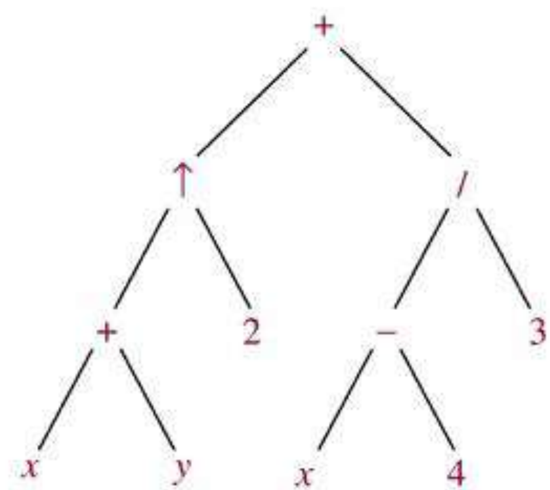
We can represent complicated expressions, such as compound propositions, combinations of sets, and arithmetic expressions using ordered rooted trees.

Example 1 Find the ordered rooted tree for
$((x+y)\uparrow 2)+((x-4)/3)$.

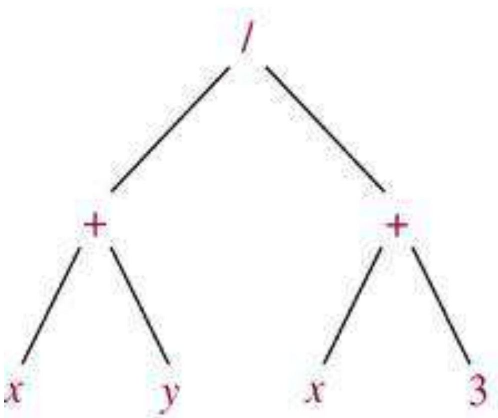
Sol.

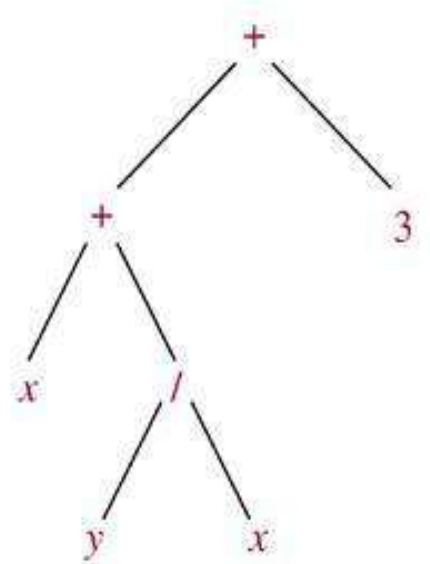
leaf:
variable
internal vertex:
operation on its left and right subtrees

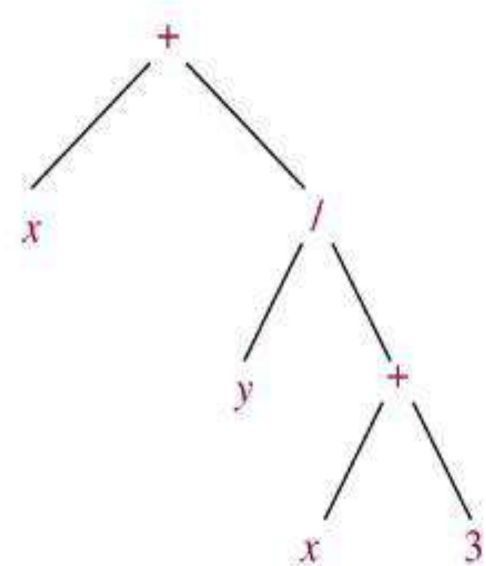




The following binary trees represent the expressions: (x+y)/(x+3), (x+(y/x))+3, x+(y/(x+3)).
All their inorder traversals lead to x+y/x+3 ⇒ ambiguous
⇒ need parentheses



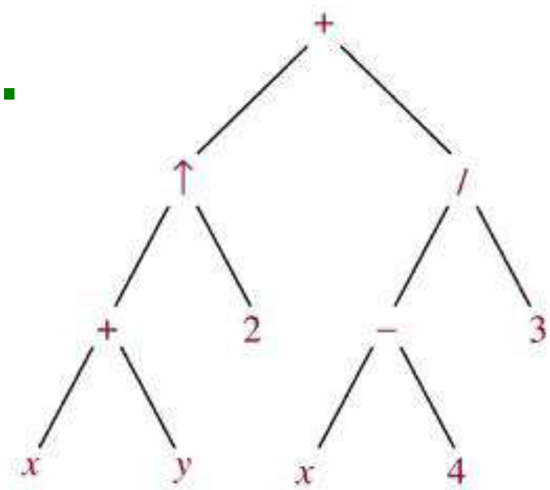




Infix form: An expression obtained when we traverse its rooted tree with <u>inorder</u>.
Prefix form: … … by <u>preorder</u>. (also named Polish notation)
Postfix form: … … by <u>postorder</u>. (reverse Polish notation)

Example 6 What is the prefix form for $((x+y)\uparrow 2)+((x-4)/3)$?

Sol.



$$+ \uparrow + \; x \; y \; 2 \; / - x \; 4 \; 3$$

Example 8 What is the postfix form of the expression $((x+y)\uparrow 2)+((x-4)/3)$?

Sol.

$$x \;\; y + 2 \uparrow x \; 4 - 3 \; / \; +$$

Note. An expression in prefix form or postfix form is unambiguous, so no parentheses are needed.

Example 7 What is the value of the prefix expression
$+ - * 2\ 3\ 5 / \uparrow 2\ 3\ 4$?

Sol.

Operation from right to left, First sign of operation (For example, $\uparrow$) The two figures on the right doing this operation, Results in original place, And so on.

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad / \quad \underbrace{\uparrow \quad 2 \quad 3}_{2 \uparrow 3 = 8} \quad 4$$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad \underbrace{/ \quad 8 \quad 4}_{8 / 4 = 2}$$

$$+ \quad - \quad \underbrace{* \quad 2 \quad 3}_{2 * 3 = 6} \quad 5 \quad 2$$

$$+ \quad \underbrace{- \quad 6 \quad 5}_{6 - 5 = 1} \quad 2$$

$$\underbrace{+ \quad 1 \quad 2}_{1 + 2 = 3}$$

Value of expression: 3

Example 9 What is the value of the postfix expression $7\ 2\ 3 * - 4 \uparrow 9\ 3 / +$?

Sol.

Operations from left to right, First sign of operation (For example, *) Two numbers on the left of this operation, Result replaces the original location, and so on.

$$7 \quad \underbrace{2 \quad 3 \quad *}_{2 * 3 = 6} \quad - \quad 4 \quad \uparrow \quad 9 \quad 3 \quad / \quad +$$

$$\underbrace{7 \quad 6 \quad -}_{7 - 6 = 1} \quad 4 \quad \uparrow \quad 9 \quad 3 \quad / \quad +$$

$$\underbrace{1 \quad 4 \quad \uparrow}_{1^4 = 1} \quad 9 \quad 3 \quad / \quad +$$

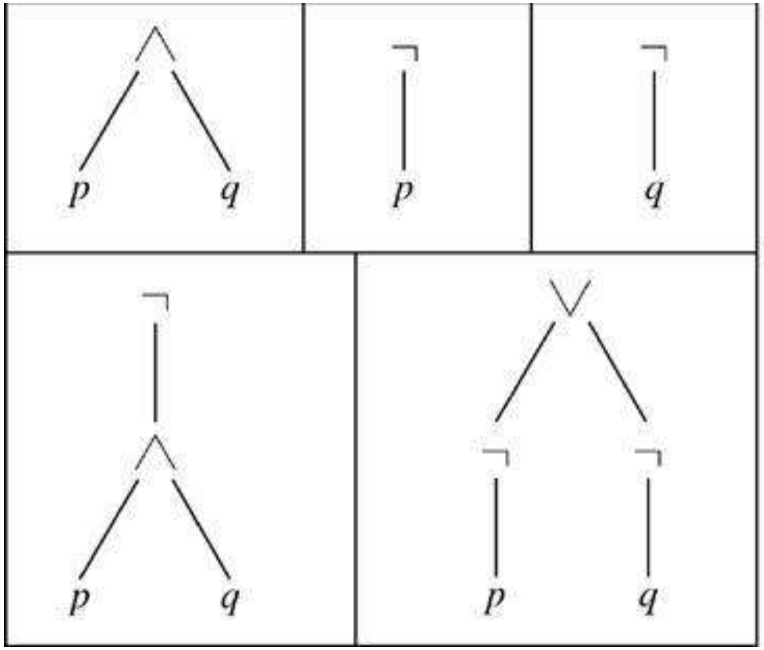
$$1 \quad \underbrace{9 \quad 3 \quad /}_{9 / 3 = 3} \quad +$$

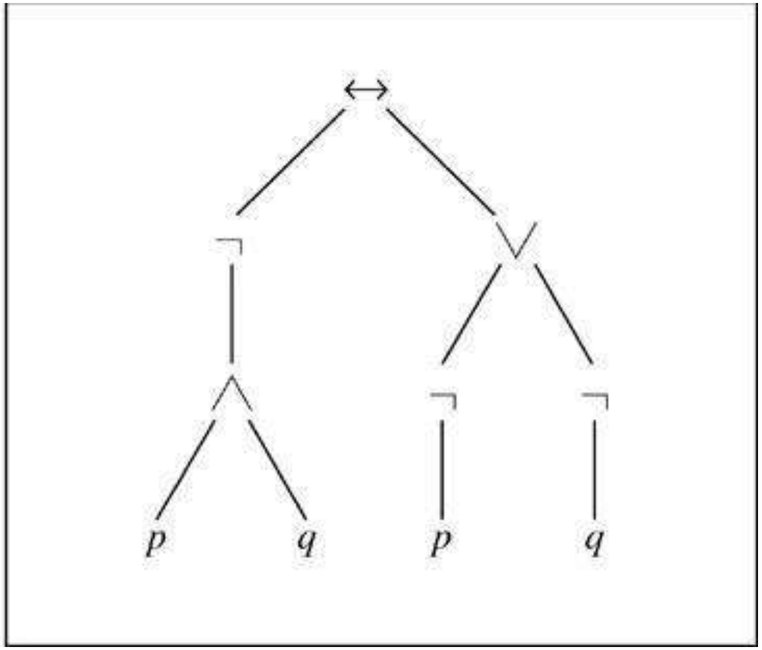
$$\underbrace{1 \quad 3 \quad +}_{1 + 3 = 4}$$

Value of expression: 4

Example 10 Find the ordered rooted tree representing the compound proposition $(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$. Then use this rooted tree to find the prefix, postfix, and infix forms of this expression.

Sol.





prefix: $\leftrightarrow \neg \wedge p\ q \vee \neg\ p \neg\ q$

infix: $(\neg(p \wedge q)) \leftrightarrow ((\neg p) \vee (\neg q))$

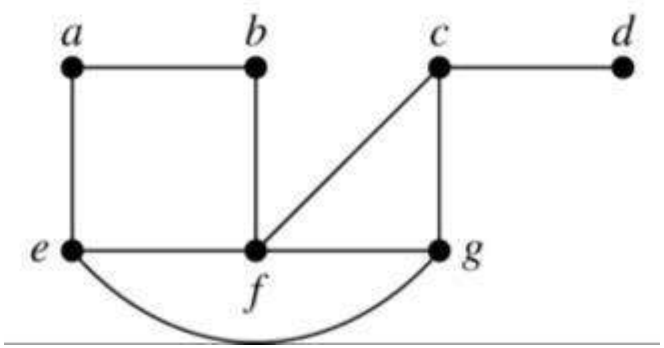
postfix: $p\ q \wedge \neg\ p \neg\ q \neg \vee \leftrightarrow$

10.4 Spanning Trees

Introduction

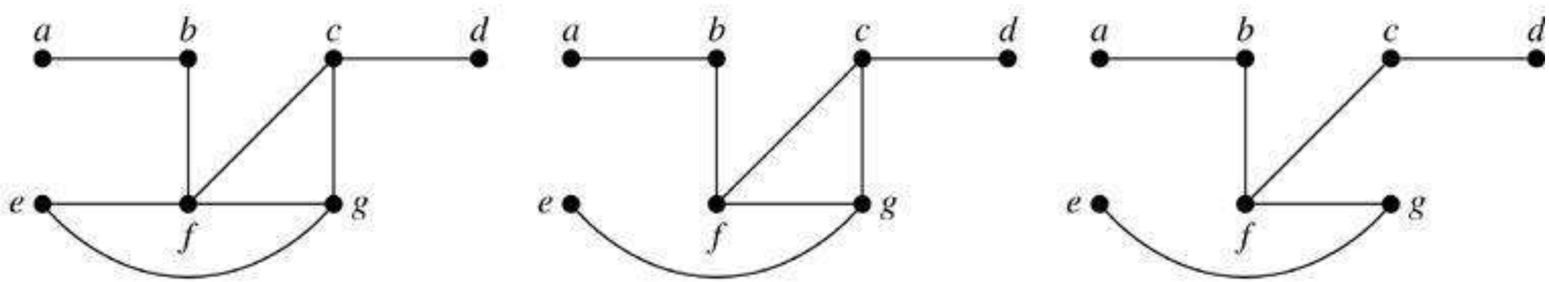
Def. Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

Example 1 Find a spanning tree of G.



Sol.

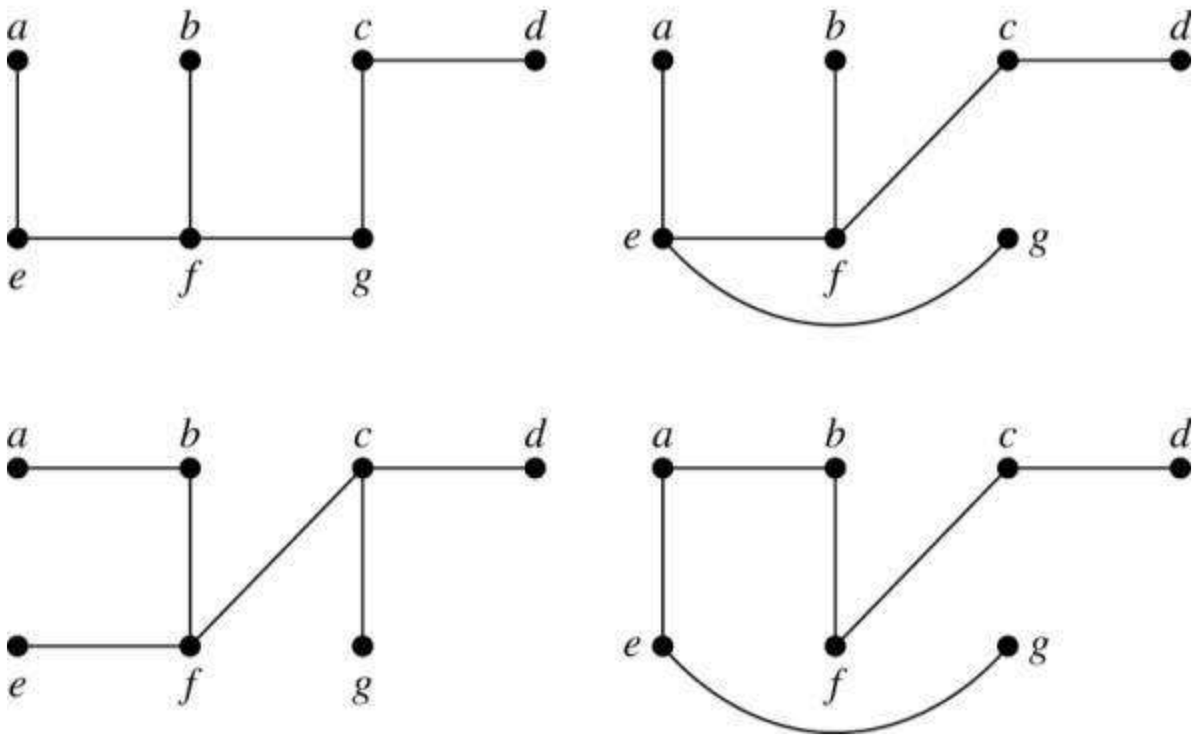
Remove an edge from any circuit.
(repeat until no circuit exists)



Edge removed: $\{a, e\}$ $\{e, f\}$ $\{c, g\}$

(a) (b) (c)

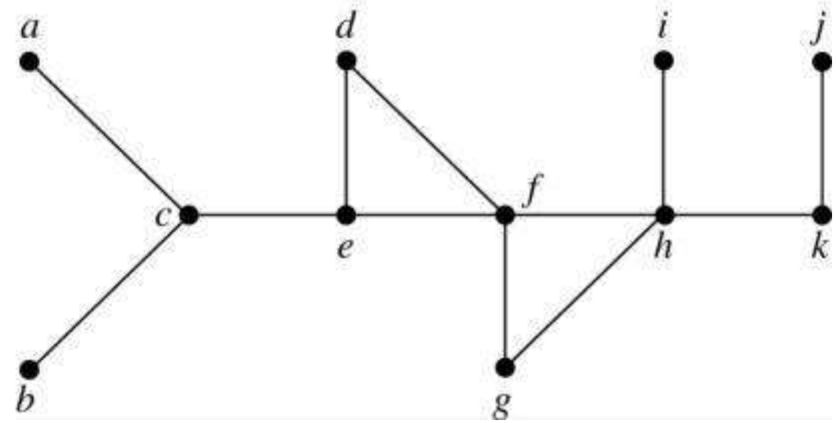
Four spanning trees of G:



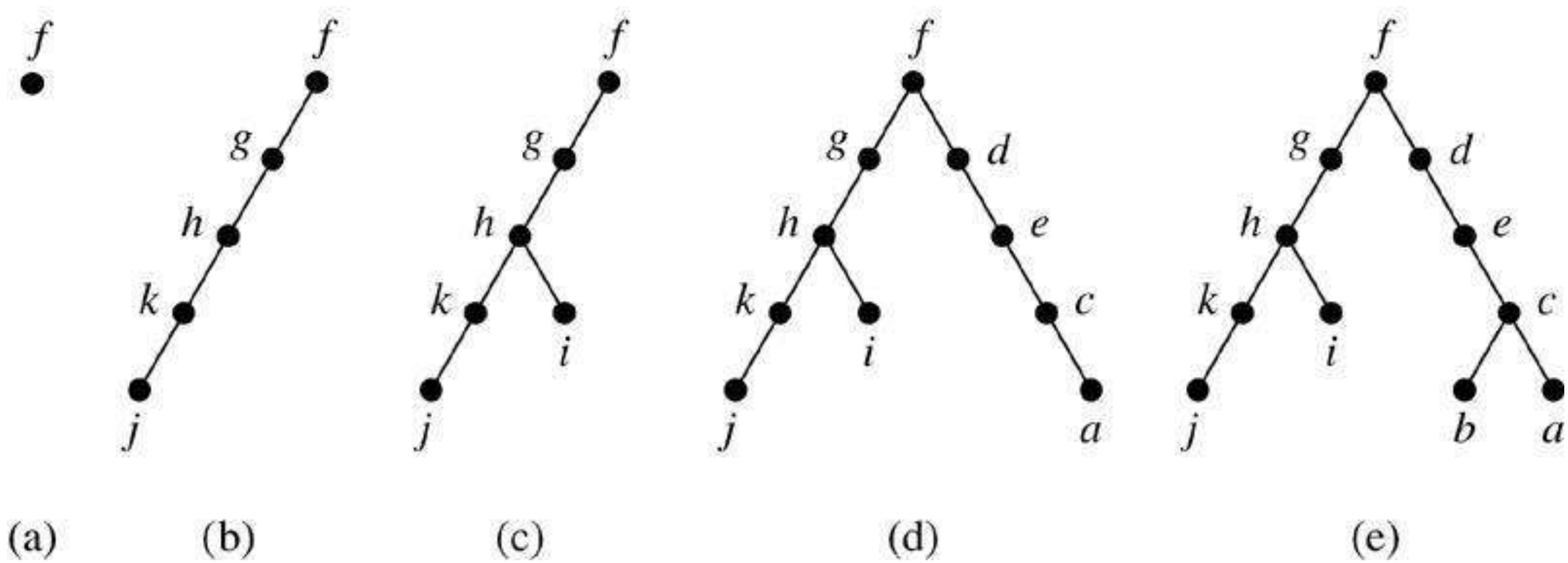
Thm 1 A simple graph is connected if and only if it has a spanning tree.

Depth-First Search (DFS)

Example 3 Use depth-first search to find a spanning tree for the graph.

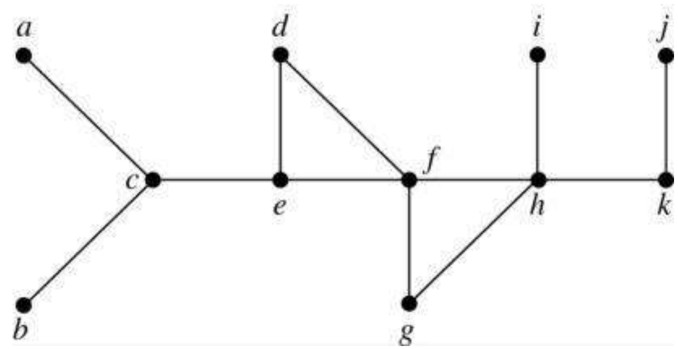


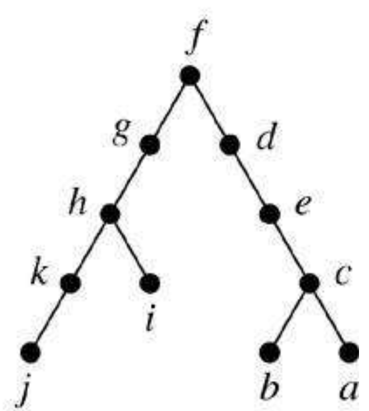
Sol. (arbitrarily start with the vertex *f*)



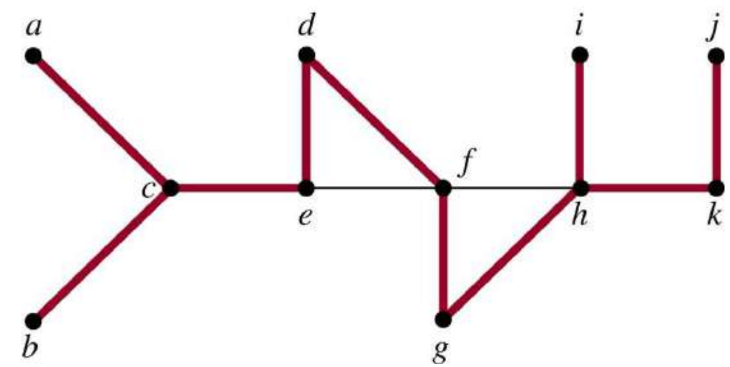
The edges selected by DFS of a graph are called tree edges. All other edges of the graph must connect a vertex to an ancestor or descendant of this vertex in the tree. These edges are called back edges.

Example 4 (Previous question)





⇒



The tree edges (red) and back edges (black)

Algorithm 1 (Depth-First Search)

Procedure $DFS(G$: connected graph with vertices $v_1, v_2, \ldots, v_n)$

T := tree consisting only of the vertex v_1

visit(v_1)

procedure *visit*(v: vertex of G)

for each vertex w adjacent to v and not yet in T

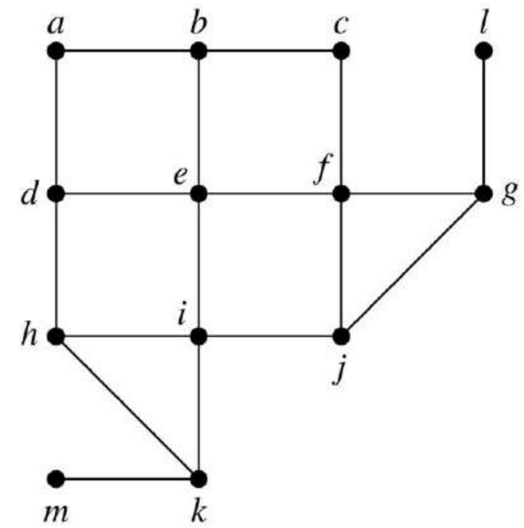
begin

    add vertex w and edge $\{v, w\}$ to T

    visit(w)

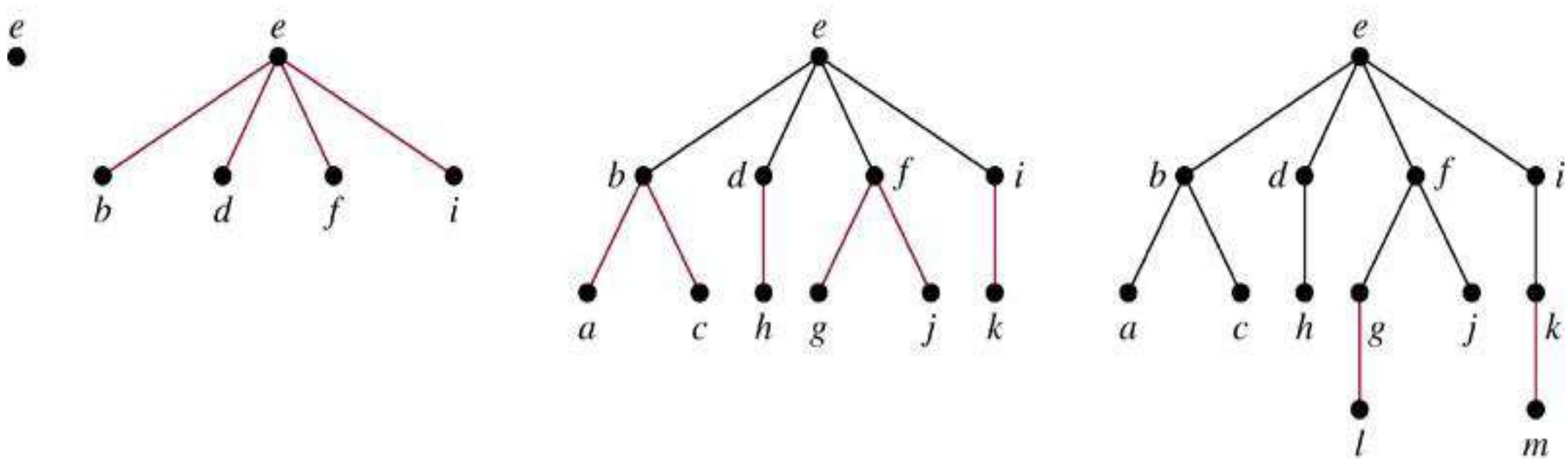
end

Breadth-First Search (BFS)



Example 5 Use breadth-first search to find a spanning tree for the graph.

Sol. (arbitrarily start with the vertex e)



Algorithm 2 (Breadth-First Search)

Procedure $BFS(G$: connected graph with vertices $v_1, v_2, \ldots, v_n)$

T := tree consisting only of vertex v_1

L := empty list

put v_1 in the list L of unprocessed vertices

while L is not empty

begin

  remove the first vertex v from L

  for each neighbor w of v

    if w is not in L and not in T **then**

    begin

      add w to the end of the list L

      add w and edge $\{v, w\}$ to T

    end

end

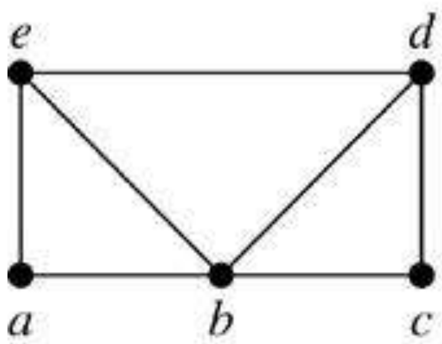
Backtracking Applications

There are problems that can be solved only by performing an exhaustive (Complete) search of all possible solutions.

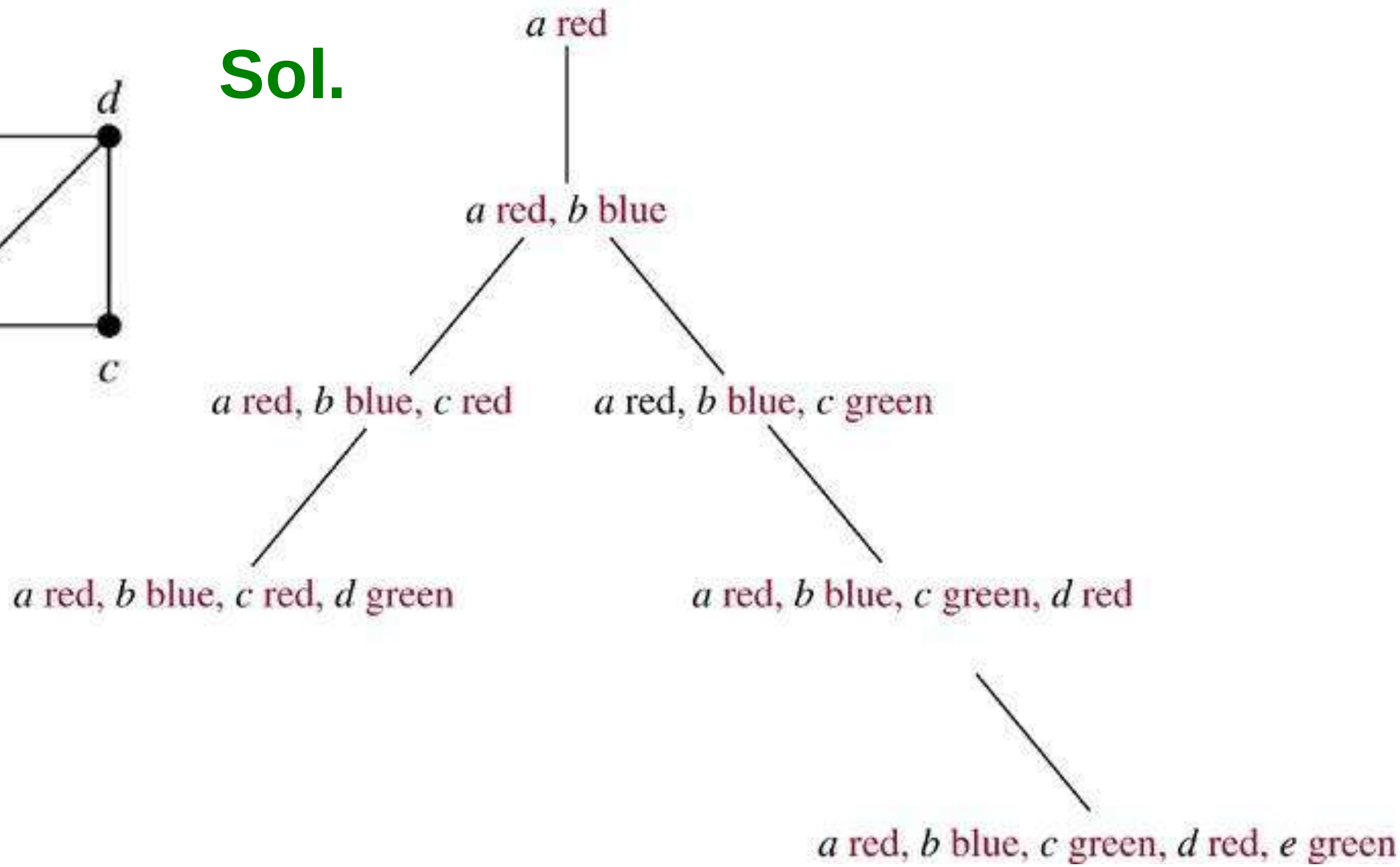
Decision tree: each internal vertex represents a decision, and each leaf is a possible solution.

To find a solution via backtracking: On the decision tree from the root to do a series of decision to the leaf, If the leaf is not the solution, or the entire subtree check solution is not found, Back to the top parent instead of looking for another tree.

Example 6 (Graph Colorings) How can backtracking be used to decide whether the following graph can be colored using 3 colors?



Sol.

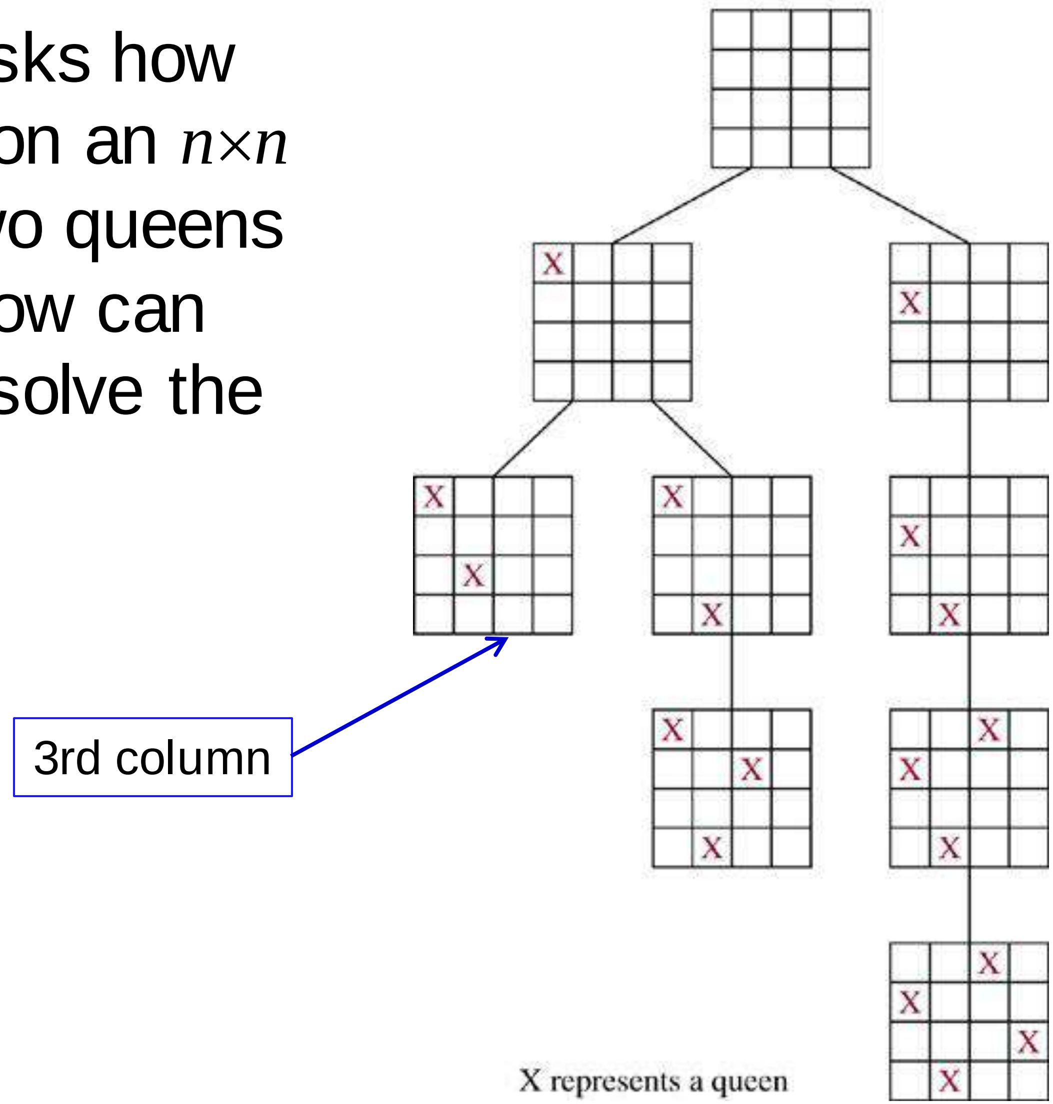


Example 7

(The n-Queens Problem)

The n-queens problem asks how n queens can be placed on an $n \times n$ chessboard so that no two queens can attack on another. How can backtracking be used to solve the n-queens problem.

Sol. A case study of n=4



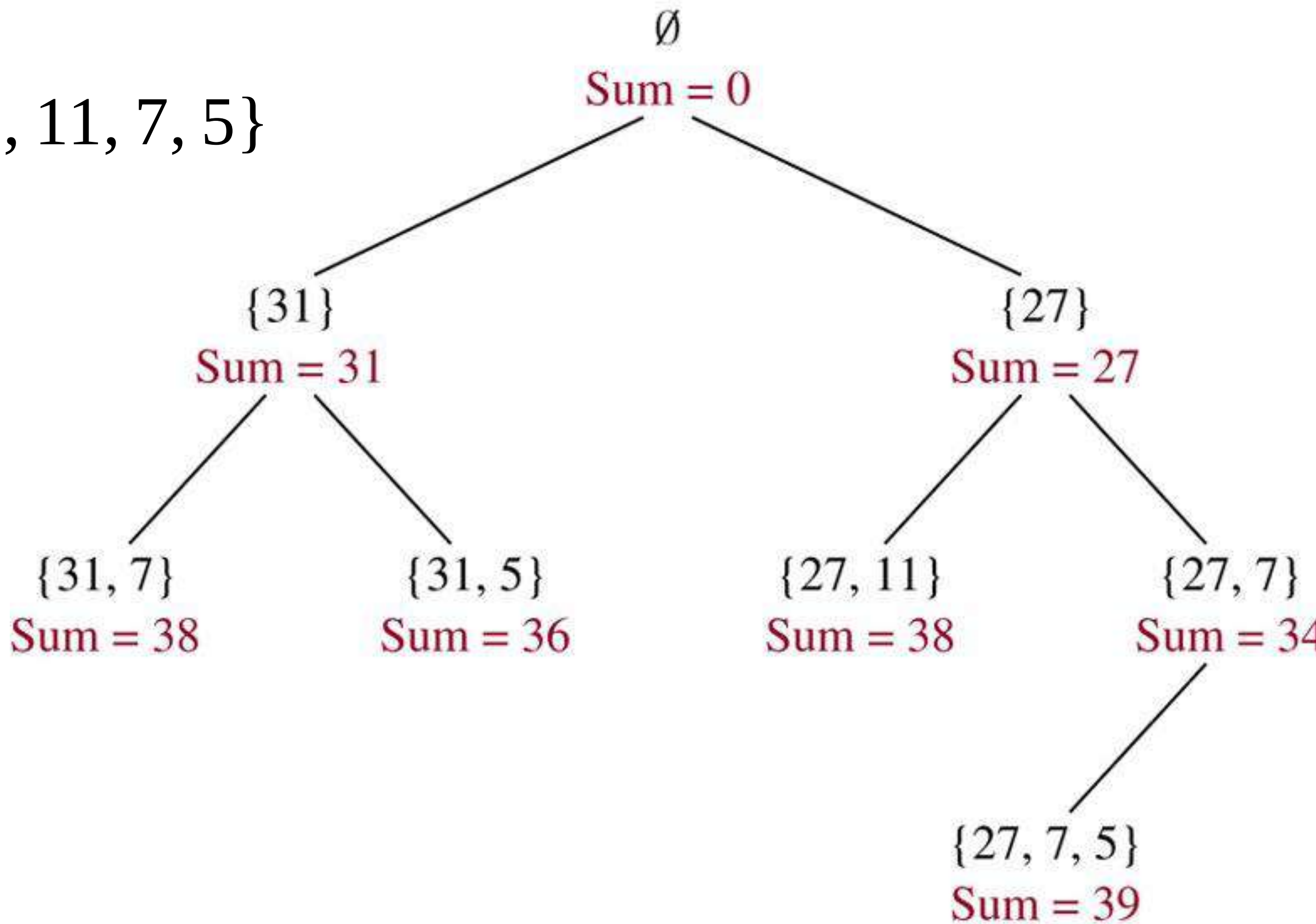
Example 8 (Sum of Subsets)

Give a set S of positive integers $x_1, x_2, \ldots, x_n$, find a subset of S that has M as its sum. How can backtracking be used to solve this problem.

Sol.

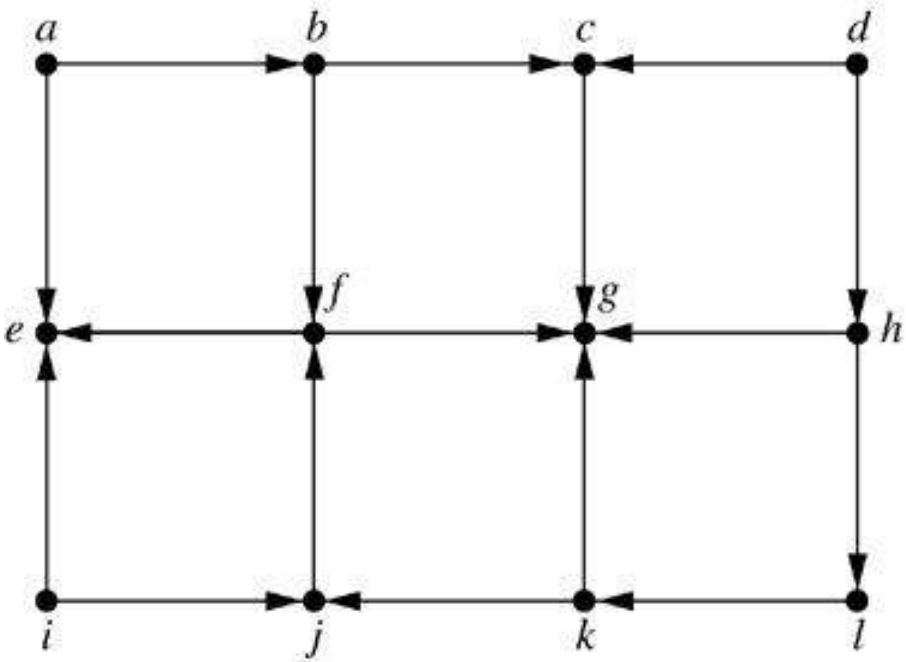
$S = \{31, 27, 15, 11, 7, 5\}$

$M = 39$



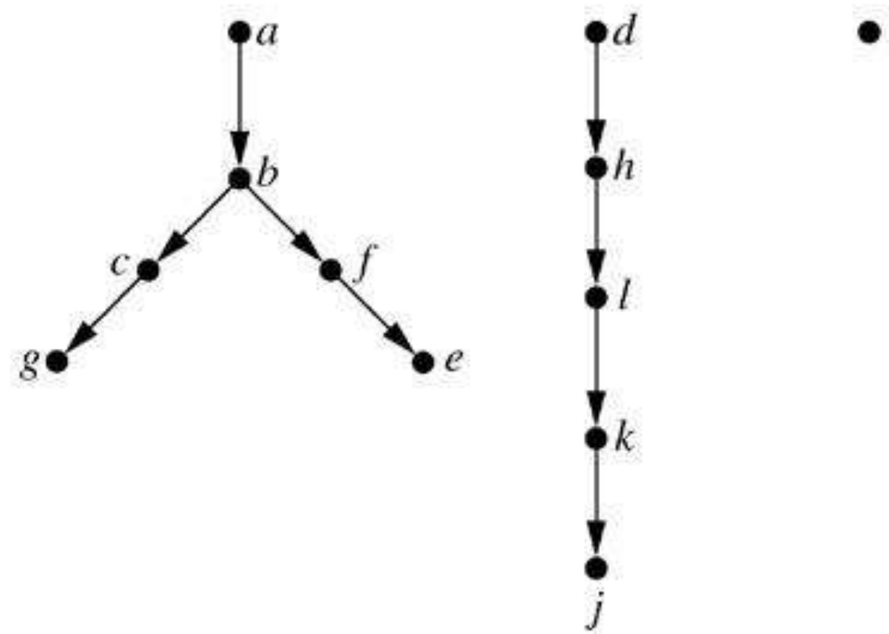
Depth-First Search in Directed Graphs

Example 9 What is the output of DFS given the graph G?



(a)

Sol.



(b)

10.5 Minimum Spanning Trees

G: connected weighted graph (each edge has an weight ≥ 0)

Def. minimum spanning tree of G: a spanning tree of G with smallest sum of weights of its edges.

Algorithms for Minimum Spanning Trees

Algorithm 1 (Prim's Algorithm)

Procedure *Prim*(G: connected weighted undirected graph with n vertices)

T := a minimum-weight edge

for i := 1 **to** n–2

begin

  e := an edge of minimum weight incident to a vertex in T and not forming a simple circuit in T if added to T

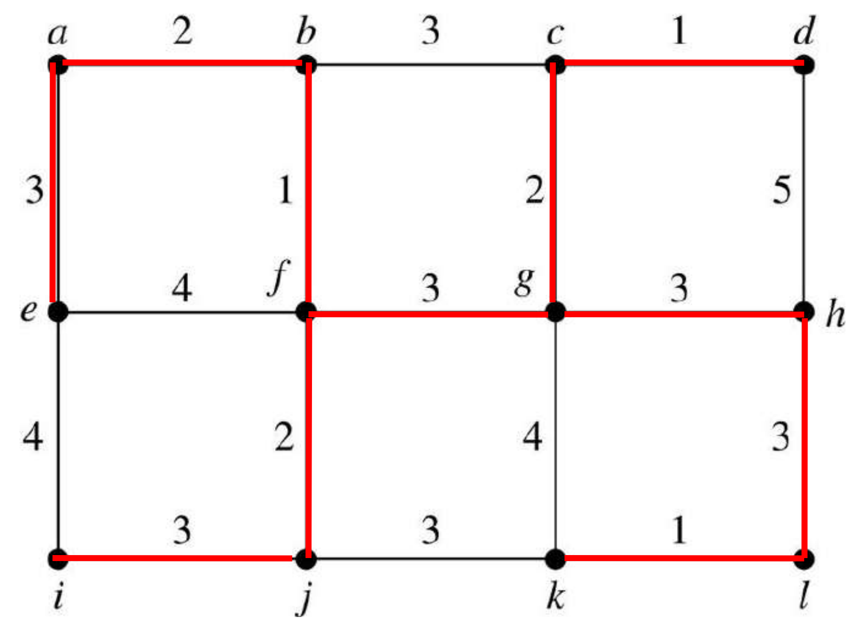
  T := T with e added

end {T is a minimum spanning tree of G}

Example 2 Use Prim's algorithm to find a minimum spanning tree of G.

Sol.

Choice	Edge	Weight
1	$\{b, f\}$	1
2	$\{a, b\}$	2
3	$\{f, j\}$	2
4	$\{a, e\}$	3
5	$\{i, j\}$	3
6	$\{f, g\}$	3
7	$\{c, g\}$	2
8	$\{c, d\}$	1
9	$\{g, h\}$	3
10	$\{h, l\}$	3
11	$\{k, l\}$	1
	Total:	24



(Maintain only one process tree)

Algorithm 2 (Kruskal Algorithm)

Procedure *Kruskal*(G: connected weighted undirected graph with n vertices)

T := empty graph

for $i := 1$ **to** $n-1$

begin

  e := any edge in G with smallest weight that does not form a simple circuit when added to T

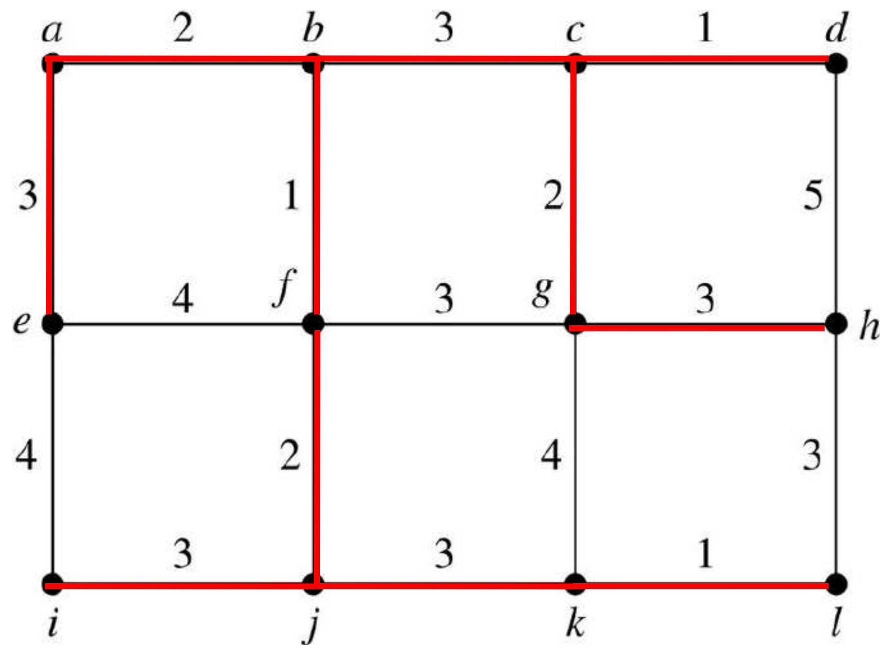
  $T := T$ with e added

end {T is a minimum spanning tree of G}

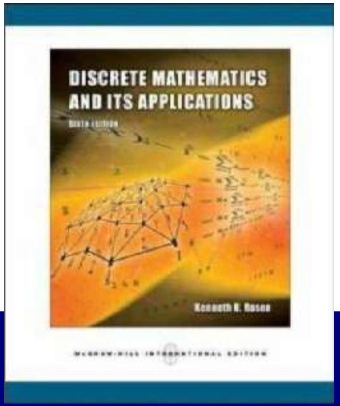
Example 3 Use Kruskal algorithm to find a minimum spanning tree of G.

Sol.

Choice	Edge	Weight
1	$\{c, d\}$	1
2	$\{k, l\}$	1
3	$\{b, f\}$	1
4	$\{c, g\}$	2
5	$\{a, b\}$	2
6	$\{f, j\}$	2
7	$\{b, c\}$	3
8	$\{j, k\}$	3
9	$\{g, h\}$	3
10	$\{i, j\}$	3
11	$\{a, e\}$	3
	Total:	24



Process tree will typically have several



Discrete Mathematics

Introduction to Discrete Probability

Outline

- Introduction to Probability
- Introduction to Discrete Probability
- Finite probability
- Probabilities of Complements and Unions
- Probability Theory

Introduction to Probability

- One of the most important course in computer science
  - Designing algorithm
  - Complexity analysis
  - Hash table
  - Code theory
  - Cryptography
  - Fault tolerance
  - Game theory or gambling

Introduction to Probability Contd

Can either be discrete or continuous

A discrete distribution is appropriate when the variable can only take on a fixed number of values. E.g. if you roll a normal die, you can get 1, 2, 3, 4, 5, or 6. You cannot get 1.2 or 0.1. If it is a fair die, the probability distribution will be 0.167 for the six possible outcomes

A continuous distribution is appropriate when the variable can take on an infinite number of values

Our concern will be on discrete probability

Introduction to Discrete Probability

- The words used in probability

An experiment is a procedure that yields one of a given set of possible outcomes

The sample space of the experiment is the set of possible outcomes.

An event is a subset of the sample space

The *event space is the power set of the* sample space

Introduction to Discrete Probability Contd

Example 1: Toss a coin.
Sample space: $U = \{H, T\}$
Event space: $P(U) = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$.

Remark: Often, some of the more interesting events have special names.

Example 2: Toss two coins.
Sample space: $U = \{HH, HT, TH, TT\}$
Event: subsets of U
Named events: *match* $= \{HH, TT\}$
at least one head $= \{HT, TH, HH\}$

Introduction to Discrete Probability Contd

- Finite Probability

If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the probability of E is

$$P(E) = \frac{|E|}{|S|}$$

The probability of an event is between 0 and 1

To see this, note that if *E is an event from a finite sample space S,* then

$0 \le |E| \le |S|$, *because* $E \subseteq S$.
Thus, $0 \le p(E) = |E|/|S| \le 1$.

Introduction to Discrete Probability Contd

- **Example 3:** An urn contains four blue balls and five red balls. What is the probability that a ball chosen at random from the urn is blue?
- **Example 4:** What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?
- **Example 5:** There are many lotteries now that award enormous prizes to people who correctly choose a set of six numbers out of the first *n positive integers, where n is usually between 30 and 60. What* is the probability that a person picks the correct six numbers out of 40?

Probabilities of Complements and Unions

- We can use counting techniques to find the probability of events derived from other events.

THEOREM 1 Let *E be an event in a sample space S. The probability of the event* $\overline{E} = S - E$*, the complementary* event of *E, is given by* $p(\overline{E}) = 1 - p(E)$.

EXAMPLE 6: A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

Sol: 1023/1024

Probabilities of Complements and Unions Contd

- We can also find the probability of the union of two events.

THEOREM 2: Let *$E1$ and $E2$ be events in the sample space S. Then $p(E1 \cup E2) = p(E1) + p(E2) - p(E1 \cap E2)$.*

EXAMPLE 7: What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

Sol: 3/5

Probabilistic Reasoning

- A common problem is determining which of two events is more likely. Analyzing the probabilities of such events can be tricky. Example 8 describes a problem of this type. It discusses a famous problem originating with the television game show ***Let's Make a Deal*** *and named after* the host of the show, Monty Hall.
  - EXAMPLE 8: The Monty Hall Three-Door Puzzle Suppose you are a game show contestant. You have a chance to win a large prize. You are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host, who knows what is behind each door, does the following. First, whether or not you selected the winning door, he opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors). Then he asks you whether you would like to switch doors. Which strategy should you use? Should you change doors or keep your original selection, or does it not matter?

Exercises

- What is the probability that a fair die comes up six when it is rolled?
- What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?
- What is the probability that a randomly selected day of a leap year (with 366 possible days) is in April?
- What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?
- What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?
- Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding
  - **a)** 30. **b)** 36. **c)** 42. **d)** 48

Probability Theory

- Probability theory can help us answer questions that involve uncertainty, such as determining whether we should reject an incoming mail message as spam based on the words that appear in the message

Recall that, we defined the probability of an event *E as*

$$P(E) = \frac{|E|}{|S|}$$

This definition assumes that all outcomes are equally likely

Probability Theory Contd

- However, many experiments have outcomes that are not equally likely.
  - For instance, a coin may be biased so that it comes up heads twice as often as tails.
  - Similarly, the likelihood that the input of a linear search is a particular element in a list, or is not in the list, depends on how the input is generated.
- How can we model the likelihood of events in such situations?
- In this section we will show how to define probabilities of outcomes to study probabilities of experiments where outcomes may not be equally likely.

Assigning Probabilities

- Let S *be the sample space of an experiment with a finite or countable number of outcomes. We* assign a probability *$p(s)$ to each outcome s. We require that two conditions be met:*

(i) $0 \leq p(s) \leq 1$ *for each* $s \in S$
and
(ii) $\sum_{s \in S} p(s) = 1$.

Condition (*i) states that the probability of each outcome is a nonnegative real number no greater* than 1.

Condition (*ii) states that the sum of the probabilities of all possible outcomes should* be 1;
that is, when the experiment is done, it is a certainty that one of these outcomes occurs.

Assigning Probabilities Contd

- Note that when there are *n possible outcomes*, $x_1, x_2, \ldots, x_n$, *the two conditions to be met* are:

(i) $0 \le p(x_i) \le 1$ for $i = 1, 2, \ldots n$
and

(ii) $\sum_{i=1}^{n} p(x_i) = 1$.

The function p from the set of all outcomes of the sample space S is called a **probability distribution.**

To model an experiment; the probability p(s) assigned to an outcome s should equal the limit of the number of times s occurs divided by the number of times the experiment is performed, as this number grows without bound.

Assigning Probabilities Contd

- We can model experiments in which outcomes are either equally likely or not equally likely by choosing the appropriate function *p(s)*

EXAMPLE 1: What probabilities should we assign to the outcomes *H (heads) and T (tails) when a fair coin* is flipped? What probabilities should be assigned to these outcomes when the coin is biased so that heads comes up twice as often as tails?

Sol: $p(H) = p(T) = 1/2$.

Sol: $p(T) = 1/3$ and $p(H) = 2/3$.

Assigning Probabilities Contd

- DEFINITION 1: Suppose that *S is a set with n elements. The uniform distribution assigns the probability 1/n* to each element of *S.*

The *probability of the event E is the sum of the probabilities of the outcomes in E. That is,*

$$p(E)=\sum_{s\in E} p(s)$$

(Note that when *E is an infinite set,* $\sum_{s\in E} p(s)$ *is a convergent infinite series.)*

Note that when there are *n outcomes in the event E, that is, if E =* $\{a_1, a_2, \ldots, a_n\}$, *then*

$$p(E)=\sum_{i=1}^{n} p(a_i)$$

The experiment of selecting an element from a sample space with a uniform distribution is called **selecting an element of *S at random.***

Assigning Probabilities Contd

- EXAMPLE 2: Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

  Sol: *p(E) = 4/7.*

suppose that there are *n* equally likely outcomes; each possible outcome has probability 1/*n, because the sum of their* probabilities is 1. Suppose the event *E contains m outcomes.*

$$p(E) = \sum_{i=1}^{m} \frac{1}{n} = \frac{m}{n}$$

Because |*E*| = *m and* |*S*| = *n, it follows that*

$$p(E) = \frac{m}{n} = \frac{|E|}{|S|}$$

Probabilities of Complements and Unions of Events

- Recall that:
  - $p(E) = 1 - p(\overline{E})$,
  - where E *is the complementary event of the event* $\overline{E}$.
- And each outcome is either in E *or in* $\overline{E}$, *but not in both, so we have*
- $$\sum_{s \in S} p(s) = 1 = p(E) + p(\overline{E}).$$

Probabilities of Complements and Unions of Events Contd

- Recall that:
- $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$
  - whenever E_1 *and* E_2 *are events in a sample space S.*
- Note that, if the events E_1 *and* E_2 *are disjoint, then* $p(E_1 \cap E_2) = 0$*, which implies* that
  - $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$
    $= p(E1) + p(E2)$.

Probabilities of Complements and Unions of Events Contd

- THEOREM 1: If $E_1, E_2, \ldots$ is a sequence of pairwise disjoint events in a sample space S, then

$$p\left(\bigcup_i E_i\right) = \sum_i p(E_i)$$

  - (Note that this theorem applies when the sequence $E_1, E_2, \ldots$ consists of a finite number or a countably infinite number of pairwise disjoint events.)

Conditional Probability

- Suppose that we flip a coin three times, and all eight possibilities are equally likely.
- Moreover, suppose we know that the event *F, that the first flip comes up tails, occurs.*
- *Given this information,* what is the probability of the event *E, that an odd number of tails appears? Because the first* flip comes up tails, there are only four possible outcomes: *TTT, TTH, THT, and THH, where H* and *T represent heads and tails, respectively.*
- *An odd number of tails appears only for the* outcomes *TTT and THH. Because the eight outcomes have equal probability, each of the four* possible outcomes, given that *F occurs, should also have an equal probability of 1/4.*
- *This* suggests that we should assign the probability of 2*/4 = 1/2 to E, given that F occurs.*
- *This* probability is called the **conditional probability of *E given F.***

Conditional Probability Contd

- In general, to find the conditional probability of *E given F, we use F as the sample space.*
- For an outcome from *E to occur, this outcome must also belong to E ∩ F.*
- DEFINITION 3: Let *E and F be events with p(F) > 0. The conditional probability of E given F, denoted* by *p(E | F), is defined as*

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

Conditional Probability Contd

- EXAMPLE 3: A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely.)

Sol: $p(E|F) = 5/8$.

EXAMPLE 4: What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities *BB, BG, GB, and GG is equally* likely, where *B represents a boy and G represents a girl. (Note that BG represents a family with* an older boy and a younger girl while *GB represents a family with an older girl and a younger* boy.)

Sol: $p(E|F) = 1/3$.

Independence

- Suppose a coin is flipped three times, as described in the introduction to our discussion of conditional probability
  - Does knowing that the first flip comes up tails (event *F*) *alter the* probability that tails comes up an odd number of times (event *E*)?
  - *In other words, is it the case* that $p(E \mid F) = p(E)$?
  - *This equality is valid for the events E and F, because* $p(E \mid F) = ½$ and $p(E) = ½$.
- *Because this equality holds, we say that E and F are* ***independent events.***
- When two events are independent, the occurrence of one of the events gives no information about the probability that the other event occurs.
- Because $p(E \mid F) = p(E \cap F)/p(F)$, *asking whether* $p(E \mid F) = p(E)$ *is the same as* asking whether $p(E \cap F) = p(E)p(F)$.

Independence Contd

- DEFINITION 4 The events *E and F are independent if and only if* $p(E \cap F) = p(E)p(F)$.
- EXAMPLE 5: Suppose *E is the event that a randomly generated bit string of length four begins with a 1* and *F is the event that this bit string contains an even number of 1s. Are E and F independent,* if the 16 bit strings of length four are equally likely?

Sol: $p(E \cap F) = 1/4$

Are the events *E, that a family with three children has children of both sexes, and F, that this* family has at most one boy, independent? Assume that the eight ways a family can have three children are equally likely.

Sol: It follows that $p(E \cap F) = p(E)p(F)$, *soE and F are independent.*

Exercises

- What probability should be assigned to the outcome of heads when a biased coin is tossed, if heads is three times as likely to come up as tails? What probability should be assigned to the outcome of tails?
- Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die.
- Find the probability of each outcome when a biased die is rolled, if rolling a 2 or rolling a 4 is three times as likely as rolling each of the other four numbers on the die and it is equally likely to roll a 2 or a 4.
- What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

Exercises Contd

- What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?
- What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1? (Assume the probabilities of a 0 and a 1 are the same.)
- Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?
- Let E and F be the events that a family of n children has children of both sexes and has at most one boy, respectively. Are E and F independent if
- **a)** $n = 2$? **b)** $n = 4$? **c)** $n = 5$?

Exercises Contd

- Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has
  - **a)** exactly three boys?
  - **b)** at least one boy?
  - **c)** at least one girl?
  - **d)** all children of the same sex?
- Find the probability that a randomly generated bit string of length 10 does not contain a 0 if bits are independent and if
  - **a)** a 0 bit and a 1 bit are equally likely.
  - **b)** the probability that a bit is a 1 is 0.6.
  - **c)** the probability that the ith bit is a 1 is $1/2i$ for $i = 1, 2, 3, \ldots, 10$.
- Find the probability that a family with five children does not have a boy, if the sexes of children are independent and if
  - **a)** a boy and a girl are equally likely.
  - **b)** the probability of a boy is 0.51.
  - **c)** the probability that the ith child is a boy is $0.51 - (i/100)$.