NUMERICAL ANALYSIS

ERROR ANALYSIS

Error analysis means a deviation below or above from the true. It is also the difference between the truth value of a quantity and the Computed or approximate values. In general, difference δ may be +ve or -ve in sign therefore as the absolute value of the difference

Error δ = True Value − Approximate Value

$|\delta|$ = |True Value − Approximate Value|

SOURCES OF ERROR

There are different source from which error can occur in computation. We look briefly at these 3 sources

a) Gross Error
b) Rounding Error
c) Truncation Error

E TRUTH: When you know the truth, you act on that truth then you are set free from error

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GROSS ERROR : This are errors caused by human mistakes of the computer or through carelessness with ~~transposition~~ decimal point signs and parathesis (bracket). It happens in transposition of digits from a register of the machine into paper or wrongly transcribing a number that contains repeated digits or inaccurate use of mathematical table

ROUNDING ERROR : This error introduce by rounding off number of significant figure, decimal places or nearest whole number and so on. e.g If the distance between place in Kaduna state is 76.142 km, It can be rounded off as 76km that mean Error = true value − approximated Value

Error = 76.142km − 76km = 0.142

We can simply say error of 0.142km has been introduc divided by 20

$x^2 - 5x + 7 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\times7\times1}}{2}$$

$$x = \frac{+5 \pm \sqrt{25 - 28}}{2}$$

$$x = \frac{5 \pm i\sqrt{3}}{2}$$

EXERCISE

A function passes through the following points (0, −1), (3, 8) and (5, 24) using Lagrange's formula. If $f(x) = 3$ then find x. (ii) Find $f(3.5)$

Note

$$(E^{-1}\Delta - \nabla)f_K = E^{-1}\Delta f_K - \nabla f_K$$

$$= E^{-1}(f_{K+1} - f_K) - (f_K - f_{K-1})$$

$$= E^{-1}f_{K+1} - E^{-1}f_K - f_K + f_{K-1}$$

$$f_K - f_{K-1} - f_K + f_{K-1} = 0$$

SECANT METHOD

$f(x) = x^3 + 4x - 1$. Using the secant method with $x_0 = 0$ and x [reduce] find x_3

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

If $f(x_n) = x_n^3 + x_n - 1$ since

$x_0 = 0$, then $f(x_0) = x_0^3 + x_0 - 1$

$f(0) = 0^3 + 0 - 1 = -1$ also

$x_1 = 1.2$ then:

$f(x_1) = x_1^3 + x_1 - 1$

$f(1.2) = 1.2^3 + 1.2 - 1 = 1.928$

when $n = 1$

$$x_{1+1} = x_1 - \frac{f(x_1)(x_1 - x_{1-1})}{f(x_1) - f(x_{1-1})}$$

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = 1.2 - \frac{1.928(1.2 - 0)}{(1.928) - (-1)}$$

$$x_2 = 1.2 - \frac{2.3136}{2.928}$$

$x_2 = 1.2 - 0.790 = 0.4098$

when $n = 2$

$$x_{2+1} = x_2 - \frac{f(x_2)(x_2 - x_{2-1})}{f(x_2) - f(x_{2-1})}$$

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THE TRUTH: Bible says buy the truth but do not sell the truth. But Bible never says donot tell the truth.

Also, Pie (π) is 3.14215926535 is rounding up as 3.142 or 3.1426. It means some error has been introduced!

In general, If a number is correct to n decimal places It has a rounding error $|\delta| \leq \frac{1}{2} \times 10^{-n}$ that is the maximum error you can introduce

EXAMPLE

3.14735 approximated to 3.15

Absolute error = |True − approximate|

$$|\delta| = |3.14735 - 3.15|$$

$$|\delta| = |-0.00265|$$

Actual Error; $\delta = 0.00265$

Now,

We have said that the maximum error you can introduce

$$|\delta| \leq \frac{1}{2} \times 10^{-2}$$

where n is the number of decimal place of the approximate or computed value

from the previous example you can see that the true value is approximated to 2 decimal places i.e $n = 2$

Thus

$$|\delta| \leq \frac{1}{2} \times 10^{-2}$$

$$|\delta| \leq 0.5 \times 10^{-2}$$

$$|\delta| \leq 0.005$$

where $|\delta|$ has been gotten to be 0.00265. Then

$$0.00265 \leq 0.005$$

The physical interpretation of this is that ~~[illegible]~~ I approximate 3.1473 the error can not be great than 0.005 provided it is t 2 decimal places.

THE TRUTH . If Lie exists for 20years, one day truth will Catch up with it

If $n = 3$

$$|\delta| \leq \frac{1}{2} \times 10^{-3}$$

$$|\delta| \leq 0.5 \times 10^{-3}$$

$$|\delta| \leq 0.0005$$

EXERCISE

Caculate $|\delta|$ when 3.14735 is approximated to 3 decimal places

TRUNCATION ERROR: This are introduce when we try to approximate the sum of infinite and finite representation e.g the binomial Expansion

$\frac{1}{1-x} = (1-x)^{-1}$. using

$(1-x)^n = 1 + nx +$

(where n is negative or fraction)

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$\simeq 1 + x + x^2$ you have truncated some values.

PRECISION AND ACCURATE

Precision is the number of digit to which a number is expressed or presented irrespective of the correctness of these digit.

Accuracy on the hand is the number of digit to which an answer is correct e.g if

$\pi = 3.14127354$ (precision)

$\pi = 3.141$ (to 3 s.f which is accuracy)

Definition:

ABSOLUTE ERROR (AE): This is the actual absolute difference between the true value or actual ~~absolute~~ value and the approximate or computed value.

(AE) is denoted by

$AE = |\delta| = |TV - CV|$. If the true value is 3.42715 and the rounded value is 3.4271 then

$AE = |\delta| = |3.42715 - 3.4271|$

$= 0.00005$

RELATIVE ERROR [RE]: This is the ratio of the absolute error to the true value. That is,

$$RE = \frac{\text{Absolute error}}{\text{Actual Value}}$$

e.g from the Last example

$$RE = \frac{0.00005}{3.42715}$$

$$RE = 0.000014589$$

PERCENTAGE ERROR [PE]: When relative error is expressed in percentage, we called it percentage error i.e

$$PE = \frac{\text{Absolute error}}{\text{Actual value}} \times 100 = RE \times 100$$

$$PE = 0.000014589 \times 100$$

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$$PE = 0.0014589 38\%$$

EXERCISE

Compute AE, RE, PE if the True Value and Computed Value is given as follows

	True Value	Computed Value
(a)	0.6×10^{-4}	0.3×10^{-4}
(b)	600	5982
(c)	67.324	66.999

SOLUTION

True Value $= 0.6 \times 10^{-4}$

$= 0.00006$

Computed Value $= 0.3 \times 10^{-4}$

$= 0.00003$

Absolute Error = |True Value − Computed Val|

$= |0.00006 - 0.00003|$

$= 0.00003$

RE $= \dfrac{\text{Absolute Error}}{\text{True Value}} = \dfrac{0.00003}{0.00006}$

$= 0.5$

PE = RE × 100

$= 0.5 \times 100$

$= 50\%$

(b) True Value = 6000

Computed Value = 5982

$AE = |\delta| = |TV - CV|$

$= |6000 - 5982|$

$= 18$

$RE = \frac{AE}{\text{True Value}} = \frac{18}{6000} = 0.003$

$PE = RE \times 100 = 0.003 \times 100 = 0.3\%$

(c) True Value = 67.324

Computed Value = 66.999

$AE = |\delta| = |TV - CV|$

$= |67.324 - 66.999|$

$= 0.325$

$RE = \frac{AE}{\text{True Value}} = \frac{0.325}{67.324}$

$RE = 0.0048274\cancel{0}$

$PE = RE \times 100 = 0.0048274\cancel{0} \times 100$

$= 0.482740181\%$

THE TRUTH: There is nothing a man can do against the truth but for the truth.

EFFECT OF ROUNDING ERROR IN ARITHMETIC COMPUTATION

[ERROR ACCUMULATION]

1. ADDITION: Let X_1 and X_2 be two positive numbers with error δ_1 and δ_2 respectively.
If $Z = X_1 + X_2$ then Z has error δ_Z. If Z is going to be error free

$$Z - \delta_Z = (X_1 - \delta_1) + (Z_2 - \delta_2)$$

$$Z - \delta_Z = (X_1 + X_2) - (\delta_1 + \delta_2)$$

But $Z = X_1 + X_2$ then

$$Z - \delta_Z = Z - (\delta_1 + \delta_2)$$

$$\cancel{-}\delta_Z = +(\delta_1 + \delta_2)$$

$$\delta_Z = (\delta_1 + \delta_2)$$

Taking the absolute values

$$|\delta Z| = |\delta_1 + \delta_2|$$

MAXIMUM ABSOLUTE ERROR

By triangular Inequality laws of Vector Addition

$$|\delta z| \leq |\delta_1| + |\delta_2|$$

The maximum absolute error is $Z \leq |\delta_1| + |\delta_2|$

THE TRUTH: Some times when I see the wicked-ness in high places of this land, I would ~~be~~ sad and angry but Bible says man's anger does not produces God's righteousness. Apart from that we wrestle not against flesh and blood (human being) but against principalities (satan and his agents)

EXAMPLE

If $X_1 = 2.34$ and $X_2 = 5.215$

$Z = X_1 + X_2 = 2.34 + 5.215 = 7.565$

recall $|\delta| \leq \frac{1}{2} \times 10^{-n}$

$X_1 = 2.34$ is to 2 d.p i.e $n = 2$

$|\delta_1| \leq \frac{1}{2} \times 10^{-2}$

$X_2 = 5.215$ is to 3 d.p i.e $n = 3$

$|\delta_2| \leq \frac{1}{2} \times 10^{-3}$ Thus

$$|\delta z| \leq |\delta_1| + |\delta_2|$$

$$|\delta z| \leq \frac{1}{2} \times 10^{-2} + \frac{1}{2} \times 10^{-3}$$

$$|\delta z| \leq 0.5 \times 10^{-2} + 0.5 \times 10^{-3}$$

6 $$|\delta z| \leq 5 \times 10^{-3} + 0.5 \times 10^{-3}$$

$$|\delta z| \leq 10^{-3} [5 + 0.5]$$

$|\delta z| = 5.5 \times 10^{-3}$

Thus the maximum absolute error $= 0.0055$

Maximum And Minimum Range

Range $= Z \pm$ Absolute Error

from previous example

Range $= (Z - 0.0055, \quad Z + 0.0055)$

where $Z = 7.565$

Range $= (7.565 - 0.0055, \; 7.565 + 0.0055)$

Range $= (7.5495, \quad 7.5605)$

2. SUBTRACTION: If $Z = X_1 - X_2$ such that X_1 has error δ_1 and X_2 has error δ_2, then

$$Z - \delta_2 = (X_1 - \delta_1) - (X_2 - \delta_2)$$

$$Z - \delta_2 = (X_1 - X_2) - (\delta_1 - \delta_2)$$

But $Z = X_1 - X_2$

$$Z - \delta_2 = Z - (\delta_1 - \delta_2)$$

$$-\delta_2 = -(\delta_1 - \delta_2)$$

$$\delta_2 = (\delta_1 - \delta_2)$$

$$|\delta_2| = |\delta_1 - \delta_2)$$

Also by triangular Inequality laws of Vector subtraction

$|\delta z| \leq |\delta_1 - \delta_2|$

$|\delta z| \leq |\delta_1| - |\delta_2|$

$|\delta z| \leq |\delta_1| + |-\delta_2|$

$|\delta z| \leq |\delta_1| + |\delta_2|$

In general x_i has error δ_i

$i = 1, 2, 3, 4 \cdots n$

$$Z = \sum_{i=1}^{n} x_i$$

then

$$|\delta z| \leq \sum_{i=1}^{n} |\delta_i|$$

$$RE = \left|\frac{i\delta z}{Z}\right|$$

3. MULTIPLICATION: If X_1 and X_2 are rounded with errors of δ_1 and δ_2 and $Z = X_1 \cancel{*} X_2$ then

$$Z - \delta_2 = (X_1 - \delta_1)(X_2 - \delta_2)$$

$$Z - \delta_2 = X_1 X_2 - X_1\delta_2 - X_2\delta_1 + \delta_1\delta_2$$

If the error $\delta_1\delta_2$ tends to zero and $Z = X_1 X_2$ then

$$\cancel{Z} - \delta_2 = \cancel{Z} - X_1\delta_2 - X_2\delta_1 + 0$$

$$-\delta_2 = -X_1\delta_2 - X_2\delta_1$$

$$\delta_2 = X_1\delta_2 + X_2\delta_1$$

dividing both side by Z

$$\frac{\delta z}{z} = \frac{X_1\delta_2}{z} + \frac{X_2\delta_1}{z}$$ but

Substitute $z = X_1 X_2$

$$\frac{dz}{z} = \frac{\cancel{X_1}\delta_2}{\cancel{X_1}X_2} + \frac{\cancel{X_2}\delta_1}{X_1\cancel{X_2}}$$

$$\frac{dz}{z} = \frac{\delta_2}{X_2} + \frac{\delta_1}{X_1}$$

taking the absolute Value

$$\left|\frac{dz}{z}\right| = \left|\frac{\delta_2}{X_2}\right| + \left|\frac{\delta_1}{X_1}\right|$$

By triangular inequality laws of Vector ~~[illegible]~~

$$\left|\frac{\delta z}{z}\right| \leq \left|\frac{\delta z}{x_2}\right| + \left|\frac{\delta_1}{x_1}\right|,$$

$$\left|\frac{\delta z}{z}\right| \leq \left|\frac{\delta_1}{x_1}\right| + \left|\frac{\delta_2}{x_2}\right| \rightarrow (1)$$

You already know that

$$RE = \frac{AE}{\text{True value}}$$

Introducing triangular inequalities

$AE \leq RE \times \text{True value}$

$AE \leq RE \times Z$

where $RE = \left|\frac{\delta z}{z}\right|$

$AE \leq \left|\frac{\delta z}{z}\right| \times Z$ Also

from equation (1)

$$AE \leq \left(\left|\frac{\delta_1}{x_1}\right| + \left|\frac{\delta_2}{x_2}\right|\right) Z$$

Hence the relation error of the modulus of the product of Z Number does not

8 exceed the sum of the

Relative error moduli of the given number

EXAMPLE

Find the Range in which the two answer lie if $Z = \pi r$ given $\pi = 3.141$ and $r = 5.34$

SOLUTION

STEPS

* Z is a function of r and π
* find the maximum absolute error of r and π
* find the absolute relative error of r and π
* Then find the absolute relative error of Z (RE)
* Proceed to Caculation of Absolute error of Z (AE)
* finally, Range $= (Z \pm AE)$

Step II (absolute error)

$$|\delta| \leq \frac{1}{2} \times 10^{-n}$$

$\pi = 3.141$ to 3-d.p $n=3$

$$|\delta\pi| \leq \frac{1}{2} \times 10^{-3}$$

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$r = 5.34$ to 2.d.p $n = 2$

$$|\delta r| \leq \frac{1}{2} \times 10^{-3}$$

Step III (absolute relative error)

$$\left|\frac{\delta \pi}{\pi}\right| = \frac{\frac{1}{2} \times 10^{-3}}{3.141}$$

$$\left|\frac{\delta r}{r}\right| = \frac{\frac{1}{2} \times 10^{-2}}{5.34}$$

Step III (RE of Z)

$$\left|\frac{\delta z}{z}\right| \leq \left|\frac{\delta \pi}{\pi}\right| + \left|\frac{\delta r}{r}\right|$$

$$\left|\frac{\delta z}{z}\right| \leq \frac{\frac{1}{2} \times 10^{-3}}{3.141} + \frac{\frac{1}{2} \times 10^{-2}}{5.34}$$

$$\left|\frac{dz}{z}\right| \leq \frac{0.5 \times 10^{-3}}{3.141} + \frac{0.5 \times 10^{-2}}{5.34}$$

$$\left|\frac{dz}{z}\right| \leq \frac{0.5 \times 10^{-3}}{3.141} + \frac{5 \times 10^{-3}}{5.34}$$

$$\left|\frac{dz}{z}\right| \leq 10^{-3}\left(\frac{0.5}{3.141} + \frac{5}{5.34}\right)$$

$$\left|\frac{dz}{z}\right| \le 10^{-3}(0.15918497 + 0.93632959)$$

$$\left|\frac{dz}{z}\right| \le 1.09551456 \times 10^{-3}$$

$$\left|\frac{dz}{z}\right| \le 0.00109551$$

Step IV (AE of Z)

$$AE = RE \times Z$$

$$Z = \pi r = 3.141 \times 5.34$$

$$Z = 16.77294$$ thus

$$AE = 0.00109551 \times 16.77294$$

$$= 0.01837492$$

Step V (Range

$$Range = Z \pm AE$$

$$Range = (16.77294 - 0.01837492,\ 16.77294 + 0.01837492)$$

$$Range = (16.75457,\ 16.79131)$$

THE TRUTH: For it has been appointed for every man to die once after which is judgement. Don't delay Come to Jesus today, He will save you

4. DIVISION: If $\frac{X_1}{X_2}$ is rounded with error $\frac{\delta_1}{\delta_2}$ then

$$Z - \delta_Z = \frac{X_1 - \delta_1}{X_2 - \delta_2}$$

$$Z - \delta_Z = \frac{X_1\left(1 - \frac{\delta_1}{X_1}\right)}{X_2\left(1 - \frac{\delta_2}{X_2}\right)}$$

Using binomial Expansion

$$\left(1 - \frac{\delta_2}{X_2}\right)^{-1} = 1 + \frac{\delta_2}{X_2} + \frac{\delta_2^2}{X_2^2} + \cdots$$

Since $Z - \delta_Z = \frac{X_1}{X_2}\left(1 - \frac{\delta_1}{X_1}\right)\left(1 - \frac{\delta_2}{X_2}\right)^{-1}$

and $Z = X_1/X_2$ Then

$$Z - \delta Z = Z\left(1 - \frac{\delta_1}{X_1}\right)\left(1 + \frac{\delta_2}{X_2} + \frac{\delta_2^2}{X_2^2} + \cdots\right)$$

$$Z - \delta Z = Z\left(1 + \frac{\delta_2}{X_2} - \frac{\delta_1\delta_2}{X_1X_2} - \frac{\delta_1}{X_1} + \frac{\delta^2}{X_2^2} - \frac{\delta_1\delta_2^2}{X_2^2}\right)$$

Where

$\frac{\delta_2^2}{X_2^2}$, $\frac{\delta_1\delta_2^2}{X_2^2}$ and $\frac{\delta_1\delta_2}{X_1X_2}$ tend to zero

Thus

$$\left(1-\frac{\delta x}{x}\right)^n \simeq \left(1-\frac{n\delta x}{x}\right) \longrightarrow \text{II}$$

Substitute II in I where

$$z = x^n$$

$$z-\delta z = x^n\left(1-\frac{\delta x}{x}\right)^n$$

$$z-\delta z = z\left(1-\frac{n\delta x}{x}\right)$$

$$z-\delta z = z\left(1-\frac{n\delta x}{x}\right)$$

$$\cancel{z}-\delta z = \cancel{z} - \frac{zn\delta x}{x}$$

$$\cancel{-}\delta z = \cancel{-}\frac{zn\delta x}{x}$$

divide through z

$$\frac{\delta z}{z} = \frac{n\delta x}{x}$$

taking the absolute value

Hence $\Rightarrow \left|\frac{\delta z}{z}\right| \leq |n|\left|\frac{\delta x}{x}\right|$

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EXAMPLE

Given that $Z = \sqrt{48.425}$, determine the maximum AE, RE and the Range in which the two answer lies.

SOLUTION

$$Z = \sqrt{48.425} = 48.425^{1/2}$$

$$Z = x^n = 48.425^{1/2} \text{ where}$$

$$x = 48.425 \quad n = \tfrac{1}{2}$$

since x is to 3 d.p $n = 3$ For

$$|\delta x| \le \frac{1}{2} \times 10^{-n}$$

$$|\delta x| \le \frac{1}{2} \times 10^{-3}$$

$$|\delta x| \le 0.5 \times 10^{-3}$$

Using

$$RE = \left|\frac{\delta z}{z}\right| \le |n| \left|\frac{\delta x}{x}\right|$$

$$RE = \left|\frac{\delta z}{z}\right| \le \left|\frac{1}{2}\right| \left|\frac{\frac{1}{2} \times 10^{-3}}{48.425}\right|$$

$$RE = \left|\frac{\delta z}{z}\right| \le 0.00000516 2$$

since $Z = \sqrt{X} = \sqrt{48.425} = 6.958807369$

Recall $AE \leq RE \times Z$

$AE \leq 0.000005162 \times 6.958807369$

$AE \leq 0.000035921$

The range in which the two lies

Range $= Z \pm AE$

$= (6.958807369 - 0.000035921,$

$6.958807369 + 0.000035921)$

$= (6.958771445, \ 6.958843290)$

EXAMPLE

Find the range in which the two answer lies

Given $Z = \sqrt[3]{\dfrac{6.2345 \times 0.82137}{2.7268}}$

Solution

let $X_1 = 6.2345 \rightarrow$ to 4 d.p

$X_2 = 0.82137 \rightarrow$ to 5 d.p

$X_3 = 2.7268 \rightarrow$ to 4 d.p

Thus

$$Z = \sqrt[3]{\frac{X_1 X_2}{X_3}} = \left(\frac{X_1 X_2}{X_3}\right)^{1/3} \quad n = 1/3$$

Thus

$$\left(1-\frac{\delta x}{x}\right)^n \simeq \left(1-\frac{n\delta x}{x}\right) \longrightarrow (11)$$

Substitute (11) in (1) where

$$Z = X^n$$

$$Z-\delta Z = X^n\left(1-\frac{\delta x}{x}\right)^n$$

$$Z-\delta Z = Z\left(1-\frac{n\delta x}{x}\right)$$

$$Z-\delta Z = Z\left(1-\frac{n\delta x}{x}\right)$$

$$\cancel{Z}-\delta Z = \cancel{Z} - \frac{Zn\delta x}{x}$$

$$\cancel{-}\delta Z = \cancel{-}\frac{Zn\delta x}{x}$$

divide through Z

$$\frac{\delta Z}{Z} = \frac{n\delta x}{x}$$

taking the absolute value

Hence $\Rightarrow \left|\frac{\delta Z}{Z}\right| \leq |n|\left|\frac{\delta x}{x}\right|$

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Since $|\delta| \leq \frac{1}{2} \times 10^{-n}$

for x_1 $|\delta_1| = \frac{1}{2} \times 10^{-4}$

for x_2 $|\delta_2| = \frac{1}{2} \times 10^{-5}$

for x_3 $|\delta_3| = \frac{1}{2} \times 10^{-4}$

for three variables

$$\left|\frac{\delta z}{z}\right| \leq \left|\frac{1}{n}\right| \left(\left|\frac{\delta_1}{x_1}\right| + \left|\frac{\delta_2}{x_2}\right| + \left|\frac{\delta_3}{x_3}\right| \right)$$

$$\left|\frac{\delta z}{z}\right| \leq \left|\frac{1}{3}\right| \left(\left|\frac{\frac{1}{2}\times 10^{-4}}{6.2345}\right| + \left|\frac{\frac{1}{2}\times 10^{-5}}{0.82137}\right| + \left|\frac{\frac{1}{2}\times 10^{-4}}{2.7268}\right| \right)$$

$$\leq \frac{1}{3}\left[\frac{0.5\times 10^{-4}}{6.2345} + \frac{0.5\times 10^{-5}}{0.82137} + \frac{0.5\times 10^{-4}}{2.7268}\right]$$

$$\leq \frac{1}{3}\left[\frac{5\times 10^{-5}}{6.2345} + \frac{0.5\times 10^{-5}}{0.82137} + \frac{5\times 10^{-5}}{2.7268}\right]$$

$$\leq \frac{1}{3}\times 10^{-5}\left[\frac{5}{6.2345} + \frac{0.5}{0.82137} + \frac{5}{2.7268}\right]$$

$$\leq \frac{1}{3}\times 10^{-5}\left(0.801988932 + 0.608739057 + 1.833651166\right)$$

$$\left|\frac{\delta z}{z}\right| \leq \frac{1}{3}\times 10^{-5} \times 3.24437915$$

$$RE = \left|\frac{\delta z}{z}\right| \leq 1.081459716 \times 10^{-5}$$

$$RE \leq \left|\frac{\delta z}{z}\right| \leq 0.00001081$$

$$z = \sqrt[3]{\frac{6.2345 \times 0.82137}{2.7268}} = 1.233755382$$

$$AE \leq RE \times z$$

$$AE \leq 0.00001081 \times 1.233755382$$

$$AE \leq 0.000013369$$

Range in which the two answer Lies $= (Z - AE, \quad Z + AE)$

$$\text{Range} = \begin{pmatrix} 1.233755382 - 0.000013369, \\ 1.233755382 + 0.000013369 \end{pmatrix}$$

$$\text{Range} = (1.233742013, \; 1.233768751)$$

EXAMINATION 2008/2009

Use the following expression to answer $Q_6 - Q_{10}$

$Z_1 = \sqrt[n]{\dfrac{2.0314^2}{0.12 \times 10.093}}$ $\quad Z_2 = e^x$

Give your answer to 4-d.p

Q_6: If $n = 3$ what is the relative error (RE) that ~~satisfy~~ satisfy Z_1?

SOLUTION

let $X_1 = (2.0314)^2 = 4.1266$ to 4 d.p

$X_2 = 0.12$ to 2 d.p

$X_3 = 10.093$ to 3 d.p

Thus

$$Z_1 = \sqrt[3]{\frac{X_1}{X_2 X_3}} = \left(\frac{X_1}{X_2 X_3}\right)^{1/3} \qquad n = 1/3$$

Since $|\delta| \le \frac{1}{2} \times 10^{-n}$

for X_1 $|\delta_1| \le \frac{1}{2} \times 10^{-4} \le 0.5 \times 10^{-4}$

for X_2 $|\delta_2| \le \frac{1}{2} \times 10^{-2} \le 0.5 \times 10^{-2}$

for X_3 $|\delta_3| \le \frac{1}{2} \times 10^{-3} \le 0.5 \times 10^{-3}$

For three Variable

$$\left|\frac{\delta z}{z}\right| \le |n| \left(\left|\frac{\delta_1}{X_1}\right| + \left|\frac{\delta_2}{X_2}\right| + \left|\frac{\delta_3}{X_3}\right| \right)$$

$$\left|\frac{\delta z}{z}\right| \leq \frac{1}{3}\left[\frac{0.5\times10^{-4}}{4.1266} + \frac{0.5\times10^{-2}}{0.12} + \frac{0.5\times10^{-3}}{10.093}\right]$$

$$\left|\frac{\delta z}{z}\right| \leq \frac{1}{3}\left[\frac{0.5\times10^{-4}}{4.1266} + \frac{50\times10^{-4}}{0.12} + \frac{5\times10^{-4}}{10.093}\right]$$

$$\left|\frac{\delta z}{z}\right| \leq \frac{1}{3}\times10^{-4}\left[\frac{0.5}{4.1266} + \frac{50}{0.12} + \frac{5}{10.093}\right]$$

$$\left|\frac{\delta z}{z}\right| \leq \frac{1}{3}\times10^{-4}\left[0.121165123 + 416.666667 + 0.495392846\right]$$

$$\left|\frac{\delta z}{z}\right| \leq \frac{1}{3}\times10^{-4} \times 417.283225$$

$$\left|\frac{dz}{z}\right| \leq 0.01390944$$

$$\left|\frac{dz}{z}\right| \leq 0.0139 \quad 4\text{ d.p}$$

Q7. If $n = 2$, the absolute error [AE] that satisfies z, would be what?

SOLUTION

$$z_1 = \sqrt[2]{\frac{x_1}{x_2 x_3}} = \left(\frac{x_1}{x_2 x_3}\right)^{1/2} \qquad n = 1/2$$

other steps remain the same

$$\left|\frac{\delta z}{z}\right| \leq |n| \left(\left|\frac{\delta_1}{x_1}\right| + \left|\frac{\delta_2}{x_2}\right| + \left|\frac{\delta_3}{x_3}\right| \right)$$

$$\left|\frac{\delta z}{z}\right| \leq \frac{1}{2}\left[\frac{0.5\times10^{-4}}{4.1266} + \frac{0.5\times10^{-2}}{0.12} + \frac{0.5\times10^{-3}}{10.073}\right]$$

$$\left|\frac{\delta z}{z}\right| \leq \frac{1}{2}\times10^{-4}\times 417.283225$$

$$\left|\frac{\delta z}{z}\right| \leq 0.02086416$$

$$RE = \left|\frac{\delta z}{z}\right| \leq 0.02086416$$

Also $AE \leq RE \times Z$

$$Z = \sqrt[2]{\frac{2.0314^2}{0.12\times10.073}} = 1.84542701$$

$$AE \leq 0.02086416 \times 1.84542701$$

$AE \leq 0.0385$ to 4-dp

Q8 If $n=1$ the true value of Z_1 would lie at what interval?

SOLUTION

$$Z_1 = \sqrt[n]{\frac{2.0314^2}{0.12 \times 10.093}} = \left(\frac{2.0314^2}{0.12\times10.093}\right)^{1/n}$$

since $n = 1$ then

$$Z_1 = \frac{2.0314^2}{0.12\times 10.093} = \frac{4.1266}{0.12\times10.093}$$

By DIVISION with three variables

$$\left|\frac{\delta z}{z}\right| \le \left|\frac{\delta_1}{x_1}\right| + \left|\frac{\delta_2}{x_2}\right| + \left|\frac{\delta_3}{x_3}\right|$$

$$\left|\frac{\delta z}{z}\right| \le \left[\frac{0.5\times10^{-4}}{4.1266} + \frac{0.5\times10^{-2}}{0.12} + \frac{0.5\times10^{-3}}{10.093}\right]$$

$$\left|\frac{dz}{z}\right| \le 10^{-4} \times 417.283225$$

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$$\left|\frac{dz}{z}\right| \le 0.041728322$$

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$AE \leq RE \times Z$

$Z = \frac{4.1266}{0.12 \times 10.093} = 3.407146867$

$AE \leq 0.0417728322 \times 3.407146867$

$AE \leq 0.14217$

Range $= (Z - AE, \quad Z + AE)$

Range $= (3.4071 - 0.1422, \quad 3.4071 + 0.1422)$

Thus the true value of Z_1 lies in the interval $(3.2649, \quad 3.5493)$

Q4 If $x = 0.1234$ what is the relative error (RE) that would satisfy Z_2?

SOLUTION

$Z_2 = e^{x} = e^{0.1234}$

$Z_2 = 1.131$ to 3 d.p

$|\delta| \leq \frac{1}{2} \times 10^{-n}$ (Absolute Error)

$|\delta| \leq \frac{1}{2} \times 10^{-3} = 0.5 \times 10^{-3}$

$|\delta| \leq 0.0005$ to 4 d.p

$RE = \left| \frac{\delta z}{z} \right| = \frac{0.0005}{1.131} = 0.0004$

Q_{10}. If $x = 0.1234$ the true value of Z_2 would lie at what interval?

Solution

Range $= Z \pm$ Absolute Error

Range $= (1.131 - 0.0005,\ 1.131 + 0.0005)$

Range $= (1.1305,\ 1.1315)$

the true value of Z_2 will lies between 1.1305 to 1.1315

ERROR ACCUMULATION IN FUNCTIONAL RELATION

Let $f(x)$ be an approximate function with error δx and $\delta f(x)$ for x and $f(x)$ respectively then the Relative Error (RE) is given by

$$RE \leq |\delta x| \left| \frac{f'(x)}{f(x)} \right|$$

This function or expression can be extended to function of several variables i.e $f(x) = f(x_1, x_2, x_3 \cdots x_n)$

$|\delta_x| \left| \frac{f'(x)}{f(x)} \right|$ as $AE = |\delta f| \leq \sum_{i=1}^{n} |\delta x_i| \left| \frac{\delta f}{\delta x_i} \right|$

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where δx_i is absolute error of x_i and
$\frac{\delta f}{\delta x_i}$ are the partial derivative of
f(x) with respect to x

Proof

let $Z = f(x)$

$Z + \delta x = f(x + \delta x)$

Expanding by Taylor's series expansion

$$Z + \delta z = f(x) + \delta x f'(x) + \frac{\delta x^2}{2!} f''(x) + \ldots$$

Taking δx so small that $\delta^2 x$ and ~~highe~~ higher power tend to zero then we have $Z + \delta z = f(x) + \delta x\, f'(x)$.

where $Z = f(x)$

$$\cancel{Z} + \delta z = \cancel{Z} + \delta x \cdot f'(x)$$

$$\delta z = \delta x\, f'(x)$$

divide through by Z

$$\frac{\delta z}{z} = \frac{\delta x}{z} \cdot f'(x)$$

since $Z = f(x)$

$$\frac{\delta z}{z} = \delta x \frac{f'(x)}{f(x)}$$

taking Absolute of both side

$$\left|\frac{\delta z}{z}\right| \leq |\delta x|\left|\frac{f'(x)}{f(x)}\right|$$

EXAMPLE

If $x = 0.2345$ find the AE, RE and Error bound for (a) $f(x) = e^x$ (b) $f(x) = \sin x$ (c) $f(x) = \cos x$. (d) $f(x) = \tan x$ (e) $f(x) = 1/x$. (f) $f(x) = \log(1+x)$ (g) $f(x) = x + 1/x$

SOLUTION

(a)

$f(x) = e^x \Rightarrow f(0.2345) = e^{0.2345}$

$f'(x) = e^x \Rightarrow f(0.2345) = e^{0.2345}$

since x 0.2345 to 4 d.p

$|\delta x| \leq \frac{1}{2} \times 10^{-n} \leq \frac{1}{2} \times 10^{-4}$

$$RE = \left|\frac{\delta z}{z}\right| \leq |\delta x|\left|\frac{f'(x)}{f(x)}\right|$$

$$RE \leq 0.5 \times 10^{-4}\left|\frac{1.26427642}{1.26427642}\right|$$

Since $e^{0.2345} = 1.26427642$

18 thus $RE \leq 5 \times 10^{-5}$

$RE \leq 0.00005$

$AE \leq RE \times f(x)$

$AE \leq 0.00005 \times 1.26427642$

$AE \leq 0.000063213$

Error bound or Error range or Simple range (intervals of Error)

$= f(x) \pm AE$

$= \begin{cases} 1.2642764 - 0.000063213, \\ 1.2642764 + 0.000063213 \end{cases}$

$= (1.2642132, 1.264339636)$

(b)

$f(x) = \sin x = \sin 0.2345$

$f(0.2345) = 0.004092786$

$f'(x) = \cos x = \cos 0.2345$

$f'(0.2345) = 0.999991624$

Since $x = 0.2345$ is to 4 d.p

$|\delta x| \leq \frac{1}{2} \times 10^{-4}$

$RE \leq |\delta x| \left| \frac{f'(x)}{f(x)} \right|$

$RE \leq 0.5 \times 10^{-4} \left| \frac{0.999991624}{0.004092786} \right|$

$RE \leq 0.5 \times 10^{-4} \times 244.3302984$

$RE \leq 0.0122165174$

$AE = RE \times f(x)$

$AE = 0.012216514 \times 0.004092786$

0.0000499996

$\text{Range} = f(x) \pm AE$

$R = (0.004092786 - 0.0000499996$
$0.004092786 + 0.0000499996$

$R = (0.004042786, \; 0.004142786)$

(c)

$f(x) = \cos x \Rightarrow f(0.2345) = \cos 0.2345$

$f(0.2345) = 0.999991624$

$f'(x) = -\sin x \Rightarrow f'(x) = -\sin 0.2345$

$f'(0.2345) = -0.004092786$

$|\delta x| = 1/2 \times 10^{-4}$

$$RE = |\delta x| \left| \frac{f'(x)}{f(x)} \right|$$

$$RE \leq 0.5 \times 10^{-4} \left| \frac{0.004092786}{0.999991624} \right|$$

$RE \leq 0.000000205$

$AE = RE \times f(x)$

19 $AE = 0.000000205 \times 0.999991624$

$AE = 0.000000205$

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Range $= f(x) \pm AE$

$R = (0.999991624 - 0.000000205,$
$0.999991624 + 0.000000205)$

$R = (0.999991142, 0.999991829)$

(d)

$f(x) = \tan x$

$f(0.2345) = \tan 0.2345 = 0.004092819\,15$

$f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$

$f'(0.2345) = \frac{1}{(\cos 0.2345)^2} = \frac{1}{0.99998325}$

$f'(0.2345) = 1.00001675$

$|\delta x| \leq \frac{1}{2} \times 10^{-4}$

$RE \leq |\delta x| \left| \frac{f'(x)}{f(x)} \right|$

$RE \leq 0.5 \times 10^{-4} \left| \frac{1.00001675}{0.004092819\,95} \right|$

$RE \leq 0.5 \times 10^{-4} (244.3344)$

$RE \leq 0.01221672$

$AE \leq RE \times f(x)$

$AE \leq 0.01221672 \times 0.004092819\,95$

$AE \leq 0.00005$

$$\text{Range} = f(x) \pm AE$$

$$R = \left(\begin{array}{l} 0.004092281995 - 0.00005, \\ 0.004092281995 + 0.00005 \end{array}\right)$$

$$R = (0.004042819,\ 0.\cancel{0}04142819)$$

(e)

$$f(x) = \log(1+x)$$

$$f(0.2345) = \log(1+0.2345) = \log 1.2345$$

$$f(0.2345) = 0.091491094$$

$$f'(x) = \frac{d}{dx}(1+x) \times \frac{1}{1+x} = \frac{1}{1+x}$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0.2345) = \frac{1}{1+0.2345}$$

$$f'(x) = \frac{1}{1.2345} = \frac{1}{1.2345} = 0.8100446$$

$$|\delta x| = \tfrac{1}{2} \times 10^{-4}$$

$$RE \leq |\partial x| \left| \frac{f'(x)}{f(x)} \right|$$

$$RE \leq 0.5 \times 10^{-4} \left| \frac{0.8100446}{0.091491094} \right|$$

$$RE \leq 0.5 \times 10^{-4} \times 8.853808219$$

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$$RE \leq \cancel{0}\,.00044269$$

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Range $= f(x) \pm AE$

$R = (0.999991624 - 0.000000205,$
$0.999991624 + 0.000000205)$

$R = (0.999991142, \quad 0.999991829)$

(d)

$f(x) = \tan x$ ~~$\Rightarrow f(0.2345)$~~

$f(0.2345) = \tan 0.2345 = 0.004092819\,95$

$f'(x) = \sec^2 x = \dfrac{1}{\cos^2 x}$

$f'(0.2345) = \dfrac{1}{(\cos 0.2345)^2} = \dfrac{1}{0.99998325}$

$f'(0.2345) = 1.00001675$

$|\delta x| \leq 1/2 \times 10^{-4}$

$RE \leq |\delta x| \left| \dfrac{f'(x)}{f(x)} \right|$

$RE \leq 0.5 \times 10^{-4} \left| \dfrac{1.00001675}{0.004092819\,95} \right|$

$RE \leq 0.5 \times 10^{-4} (244.3344)$

$RE \leq 0.01221672$

$AE \leq RE \times f(x)$

$AE \leq 0.01221672 \times 0.004092819\,95$

$AE \leq 0.00005$

$$\text{Range} = f(x) \pm AE$$

$$R = \begin{pmatrix} 0.00409281995 - 0.00005, \\ 0.00409281995 + 0.00005 \end{pmatrix}$$

$$R = (0.004042819, 0.004142819)$$

(e)

$$f(x) = \log(1+x)$$

$$f(0.2345) = \log(1+0.2345) = \log 1.2345$$

$$f(0.2345) = 0.091491094$$

$$f'(x) = \frac{d}{dx}(1+x) \times \frac{1}{1+x} = \frac{1}{1+x}$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0.2345) = \frac{1}{1+0.2345}$$

$$f'(x) = \frac{1}{1.2345} = \frac{1}{1.2345} = 0.8100446$$

$$|\delta x| = \tfrac{1}{2} \times 10^{-4}$$

$$RE \leq |\partial x| \left| \frac{f'(x)}{f(x)} \right|$$

$$RE \leq 0.5 \times 10^{-4} \left| \frac{0.8100446}{0.091491094} \right|$$

$$RE \leq 0.5 \times 10^{-4} \times 8.853808219$$

20 $RE \leq 0.00044269$

$AE \leq RE \times f(x)$

$AE \leq 0.00044269 \times 0.091491094$

$AE \leq 0.0000405$

$\text{Range} = f(x) \pm AE$

$R = (0.091491094 + 0.0000405, \; 0.09141094 - 0.0000405)$

$R = (0.09153159, \; 0.0913 7044)$

(f)

$f(x) = 1/x \Rightarrow f(0.2345) = \dfrac{1}{0.2345}$

$f(0.2345) = 4.\cancel{2}64392324$

$f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}(x^{-1}) = -1 x x^{-2}$

$f'(0.2345) = \dfrac{-1}{(0.2345)^2} = -18.1850418 9$

$|\delta x| = 1/2 \times 10^{-4}$

$RE \leq |\delta x| \left| \dfrac{f'(x)}{f(x)} \right|$

$$RE \leq 0.5\times10^{-4}\left|\frac{-18.18504189}{4.264392324}\right|$$

$$RE \leq 0.5\times10^{-4}\times 4.264392314$$

$$RE \leq 0.0002132196$$

$$AE = RE \times f(x)$$

$$AE = 0.0002132196 \times 4.264392324$$

$$AE = 0.000909252$$

$$Range = f(x) \pm AE$$

$$R = \begin{pmatrix} 4.264392314 + 0.000909252, \\ 4.264392314 - 0.000909252 \end{pmatrix}$$

$$R = (4.265301576, 4.263483072)$$

(9)

$$f(x) = x + 1/x \Rightarrow f(0.2345) = 0.2345 + \frac{1}{0.2345}$$

$$f(0.2345) = 4.498892324$$

$$f'(x) = \frac{d}{dx}\left(x + \frac{1}{x}\right) = 1 - \frac{1}{x^2}$$

$$f'(0.2345) = 1 - \frac{1}{(0.2345)^2}$$

$$f'(0.2345) = 1 - 18.18504189$$

21 $f'(0.2345) = -17.18504189$

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$$|\delta x| = 1/2 \times 10^{-4}$$

$$RE \leq |\delta x| \left|\frac{f'(x)}{f(x)}\right|$$

$$RE \leq 0.5\times10^{-4} \left|\frac{-17.18504189}{4.498892324}\right|$$

$$RE \leq 0.5\times10^{-4} \times 3.819838452$$

$$RE \leq 0.0001909919$$

$$AE = RE \times f(x)$$

$$= 0.0001909919 \times 4.498892324$$

$$= 0.0008592 39$$

$$Range = f(x) \pm AE$$

$$R = \begin{pmatrix} 4.498892324 - 0.000859239 \\ 4.498892324 + 0.000859239 \end{pmatrix}$$

$$R = (4.498033085,\ 4.499751563)$$

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EVALUATION OF FUNCTION BY SERIES EXPANSION

We have two types of series

(1) TAYLOR'S SERIES

It is considered to the foundation of Numerical analysis. It is also used as a basis for many Numerical formula. The theorem defining the series as follows:

Taylor's theorem

If a continuous function $f(x)$ has a continuous $(n+1)^{th}$ on the interval $[x_0, x]$ then it can be represented by a finite taylor series of the form

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + f'''$$

$$+ - - - - \frac{(x-x_0)^n}{n!}f^n(x_0) + R_{n+1}(x)$$

where

$$R_{n+1}(x) = \frac{x^{n+1}}{(n+1)!}f^{n+1}\xi$$

EXAMPLE

Obtain a 2nd degree polynomial approximation of $f(x) = \sqrt{1+x}$. Using the taylor's series about the point $x_0 = 0$. Calculate the truncation for $x = 0.4$ and Range in which the answer lies

Solution

$$f(x) = f(x_0) + \cdots R_{n+1}(x)$$

$n = 2$ (2nd degree)

Hence to obtain the last term By taylor series

$$R_{n+1}(x) = \frac{x^{n+1}}{(n+1)!} f^{n+1}(\xi) \quad x_0 \le \xi \le x$$

$$R_{2+1}(x) = \frac{x^{2+1}}{(2+1)!} f^{2+1}(\xi)$$

— 3rd derivative

$$R_3(x) = \frac{x^3}{3!} f'''(\xi)$$

From taylor Expansion at $x_0 = 0$.

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \cdots$$

substitute $x_0 = 0$

$$f(x) = f(0) + (x-0)f'(0) + \frac{(x-0)^2}{2!}f''(0) + \frac{(x-0)^3}{3!}f'''(0) + \cdots$$

from the question

$$f(x) = \sqrt{1+x}$$

$$f(0) = \sqrt{1+0} = 1$$

$$f'(x) = \frac{d}{dx}(\sqrt{1+x}) = \frac{d}{dx}\left((1+x)^{1/2}\right)$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad f'(0) = \frac{1}{2}(1+0)^{-1/2} = 1/2$$

$$f''(x) = \frac{d^2}{dx^2}(1+x)^{1/2} = \frac{d}{dx}\left(\frac{1}{2}(1+x)^{-1/2}\right)$$

$$f''(x) = \frac{-1}{4}(1+x)^{-3/2}$$

$$f''(0) = \frac{-1}{4}(1+0)^{-3/2} = -1/4$$

$$f'''(x) = \frac{d^3}{dx^3}(1+x)^{1/2} = \frac{d}{dx}\left(\frac{-1}{4}(1+x)^{-3/2}\right)$$

$$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$$

$$f'''(0) = \frac{3}{8}(1+0)^{-5/2} = \frac{3}{8}$$

Substitute for $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ in previous expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0)$$

$$f(x) = 1 + x \times \frac{1}{2} - \frac{x^2}{2\times1} \times 1/4 + \frac{x^3 \times 3/8}{3\times2\times1} + $$

$$f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$$

to obtain second degree polynomial, truncate last term

$$f(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$$

$$f(x) = 1 + 0.5x - 0.125x^2$$

note $AE \leq R_{n+1}(\xi)$ $\quad 0 \leq \xi \leq 0.4$

$$AE \leq \frac{x^3}{3!} f'''(\xi)$$

$$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$$

when $x_0 = 0$

$$f'''(0) = \frac{3}{8} = 0.375$$

when $x = 0.4$

$$f'''(0.4) = \frac{3}{8}(1+0.4)^{-5/2}$$

0.064

$$f'''(0.4) = \frac{3}{8}(1.4)^{-5/2}$$

$$f'''(0.4) = 3/8\left(\frac{1}{(\sqrt{1.4})^5}\right) = 0.161700431$$

$$AE_1 \le \frac{x^3}{3!} f'''(0) = \frac{(0.4)^3}{3\times2\times1} \times 0.375$$

$$AE_1 \le 0.004$$

$$AE_2 \le \frac{x^3}{3!} f'''(x) = \frac{(0.4)^3}{3\times2\times1} \times 0.161700431$$

$$AE_2 \le 0.0017248$$

If

$$Z = f(x) = \sqrt{1+x} \quad at \quad x = 0.4$$

$$z = \sqrt{1+0.4} = \sqrt{1.4} = 1.183215957$$

Hence the range in which the two answer lie in between

$$= f'''(x) \pm AE \quad at \quad x = 0.4$$

$$\Rightarrow f'''(0.4) \pm AE$$

$$= \begin{pmatrix} 0.161700431 + 0.004 \\ 0.161700431 - 0.004 \end{pmatrix}$$

$$= (0.165700431, \quad 0.157700431)$$

THE TRUTH: Bible says godliness with Contentment is great gain, for we came with nothing. The love of money is the root of all kind of evils

24

1.024

EXAMPLE

Find a 2^{nd} degree polynomial approximation of $f(x) = e^x$, $f(x) = \sin x$, $f(x) = \cos x$ using Taylor series about a point $x_0 = 0.4$ and $x = 0.17$

SOLUTION

2nd degree polynomial of taylor series is

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0)$$

(a)

when $f(x) = e^x$

$$f'(x) = \frac{d}{dx}(e^x) = e^x$$

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}(e^x) = e^x$$

since $x_0 = 0.4$

$f'(x_0) = e^x = e^{0.4} = 1.4918$

$f''(x_0) = e^x = e^{0.4} = 1.4918$

$f(x_0) = e^x = e^{0.4} = 1.4918$

Substitute for x, x_0, $f(x_0)$, $f'(x_0)$, $f''(x_0)$ in the taylor series

259.13716058

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + $$

$$f(x) = 1.4918 + (x-0.4)(1.4918) + \frac{(x-0.4)^2}{2\times 1}(1.4918) + - - -$$

$$f(x) = 1.4918\left[1 + (x-0.4) + \frac{(x-0.4)^2}{2} + - - -\right]$$

$$f(x) = 1.4918\left[1 + x - 0.4 + \frac{x^2 + 0.4^2 - 2(x)(0.4)}{2}\right]$$

$$f(x) = 1.4918\left[1 + x - 0.4 + \frac{x^2 - 0.8x + 0.16}{2}\right]$$

at $x = 0.17$

$$f(x) = 1.4918\left[1 + (0.17-0.4) + \frac{0.17^2 - 0.8(0.17) + 0.16}{2}\right]$$

$$f(x) = 1.4918\left[1 - 0.23 + 0.02645\right]$$

$$f(x) = 1.4918 \times 0.79645$$

$$f(x) \approx 1.188144$$

(b)

$F(x) = \sin x^{\circ}$

$f'(x) = \frac{d}{dx} \sin x^{\circ} = \cos x^{\circ}$

$f''(x) = \frac{d}{dx}(F'(x)) = \frac{d}{dx} \cos x^{\circ} = -\sin x^{\circ}$

$180^{\circ} = 1\pi$ radian

$180^{\circ} = 3.142$

Covert 0.4 to degree

$180^{\circ} = 3.142$

$x^{\circ} = 0.4$

$x^{\circ} = \frac{180^{\circ} \times 0.4}{3.142}$

Apart from that, some example of taylor series

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} \quad -\infty < x < \infty$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-n)^{n} x^{2n+1}}{(2n+1)!} \quad -\infty < x < \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2n}}{(2n)!} \quad -\infty < x < \infty$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \quad -1 < x < x$$

$$\tan^{-1} x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1} \quad |x| < x$$

Determine the number of terms of the series

$f(x) = 1.4918[1 + x - 0.4 + x^2 - 0.8x + 0.16] +$

that can give the value of e^{x^2}

Correct to 8 d.p when $x = 1$

solution

since $R_{n+1}(x) = \frac{x^{n+1}}{(n+1)!} e^{\xi}$

For the exponential function

since $x = 1$

$R_{n+1}(x) = \frac{1^{n+1}}{(n+1)!} e^{\xi} \longrightarrow (1)$

$|R_{n+1}(x)| \leq \frac{1}{2} \times 10^{-n}$

since is corrected to 8 d.p $n = 8$

$|R_{n+1}(x)| \leq \frac{1}{2} \times 10^{-8} \longrightarrow (11)$

Substitute (11) in (1)

$\frac{1^{n+1}}{(n+1)!} e^{\xi} \leq \frac{1}{2} \times 10^{-8} \quad 0 \leq \xi \leq 1$

Cross multiply

$\frac{2 \times 10^{8} (1^{n+1}) e^{\xi}}{(n+1)!} \leq 1$

note $1^{n+1} = 1$

$2 e^{\xi} \times 10^8 \leq (n+1)!$

$x_0 \leq \xi \leq x \quad 0 \leq \xi \leq 1 \quad$ i.e $\xi = 1$

$2 e^1 \times 10^8 \leq (n+1)!$

$2 \times 2.7182818 \times 10^8 \leq (n+1)!$

$543656366 \leq (n+1)!$

Now which number's factorial will give greater value than 543656366 ?

Guess

$10! = 3628800$

$11! = 39916800$

$12! = 479001600$

$13! = 6227020800$

It is $\underline{\underline{13}}$

therefore

$n+1 = 13$

$n = 13 - 1 = \underline{\underline{12}}$

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2) MACLAURIN'S SERIES

Let $f(x)$ be a continuous function and its derivative upon nth and higher order exist then

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$\frac{x^n}{n!}f^n(0) + \cdots$$

STANDARD EXPANSION

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots$$

$$\tanh^{-1} = \frac{1}{2}\log\frac{1+x}{1-x}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

FINITE DIFFERENCES

Let $f(x)$ be a function define over an interval $[x_0, x_n]$ if the interval is partitioned into small sub-interval of equal width i.e $x_0, x_1, x_2, x_3 \cdots x_n$ $[x_0, x_1], [x_1, x_2], [x_2, x_3] \cdots [x_{n-1}, x_n]$.

Let h be the width of each sub-interval

$$x_1 - x_0 = h$$
$$x_2 - x_1 = h$$
$$x_1 - x_0 = h$$
$$\Rightarrow \quad x_1 = x_0 + h \quad \cdots\cdots * _1$$
$$x_2 - x_1 = h$$
$$x_2 - x_0 - h = h$$
$$\Rightarrow \quad x_2 = x_0 + 2h \quad \cdots\cdots * _2$$
$$x_3 - x_0 - 2h = h \qquad \text{Since}$$
$$x_3 - x_2 = h \qquad \text{then}$$
$$x_3 - x_0 = h + 2h$$
$$x_3 = x_0 + 3h \quad \cdots\cdots * _3$$

Suppose $f(x)$ define on the interval

$\therefore X_p = X_0 + ph$

$[a, b] \to R$ we write

$f(x_k) = f(x_0 + kh) = f_k$

based on this we define the following operations

DIFFERENCE OPERATOR

Forward Difference Operator (Δ) Delta

by Definition

$\Delta f_k = f_{k+1} - f_k \Rightarrow \Delta^2 f_k = \Delta(\Delta f_k)$

$\Delta f_1 = f_2 - f_1$ e.g. when $k = 1$

Thus $\Delta^2 f_k = \Delta f_{k+1} - \Delta f_k$

$\Delta^2 f_k = (f_{k+2} - f_{k+1}) - (f_{k+1} - f_k)$

$\Delta^2 f_k = f_{k+2} - 2f_{k+1} + f_k$

Also

$\Delta^3 f_k = \Delta(\Delta^2 f_k)$

$\Delta^3 f_k = \Delta(f_{k+2} - 2f_{k+1} + f_k)$

$\Delta^3 f_k = \Delta f_{k+2} - 2\Delta f_{k+1} + \Delta f_k$

$\Delta^3 f_k = f_{k+3} - f_{k+2} - 2(f_{k+2} - f_{k+1}) + f_{k+1} - f_k$

$\Delta^3 f_k = f_{k+3} - f_{k+2} - 2f_{k+2} + 2f_{k+1} + f_{k+1} - f_k$

$\Delta^3 f_k = f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k$

Also

$\Delta^4 f_k = \Delta(\Delta^3 f_k)$

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$$\Delta^4 f_K = \Delta(f_{K+3} - 3f_{K+2} + 3f_{K+1} + f_K)$$

$$\Delta^4 f_K = \Delta f_{K+3} - 3\Delta f_{K+2} + 3\Delta f_{K+1} + \Delta f_K$$

$$\Delta^4 f_K = (f_{K+4} - f_{K+3}) - 3(f_{K+3} - f_{K+2})$$
$$+ 3(f_{K+2} - f_{K+1}) + (f_{K+1} - f_K)$$

$$\Delta^4 f_K = f_{K+4} - f_{K+3} - 3f_{K+3} + 3f_{K+2} +$$
$$3f_{K+2} - 3f_{K+1} + f_{K+1} - f_K$$

$$\Delta^4 f_K = f_{K+4} - 4f_{K+3} + 6f_{K+2} + 4f_{K+1} - f_K$$

OR (Binomial theorem)

$$\Delta^4 f_K = \Delta(\Delta^3 f_K) = \Delta^3 f_{K+3} - \Delta^3 f_{K+2}$$

$$\Delta(\Delta^3 f_K) = (f_{K+4} - 3f_{K+3} + 3f_{K+2} - f_{K+1}) -$$
$$(f_{K+3} - 3f_{K+2} + 3f_{K+1} - f_K)$$

$$\Delta^4 f_K = f_{K+4} - 4f_{K+3} + 6f_{K+2} - 4f_{K+1} + f_K$$

Where 1, 3 3 1 are Binomial Coefficient in Expansion

Also

$$\Delta^3 f_K = \Delta(\Delta^2 f_K) = \Delta^2 f_{K+2} - \Delta^2 f_{K+1}$$

$$\Delta^3 f_K = (f_{K+3} - 2f_{K+2} + f_{K+1}) - (f_{K+2} - 2f_{K+1} + f_K)$$

$$\Delta^3 f_K = f_{K+3} - 3f_{K+2} + 3f_{K+1} - f_K$$

In general

$$\Delta^n f_K = f_{K+n} - n f_{K+n-1} + \frac{n(n-1)}{2!} f_{K+n-2} - -$$

$$+ (-1)^{n-1} n f_{K+1} + (-1)^n f_K$$

Consider the function $y = f(x)$

Suppose $y_0 = f(a)$ & $y_1 = f(a+k)$

The expression $y_1 - y_0$ is called the first difference of y_0

Thus $y_1 - y_0 = \Delta y_0$

Multiplying by Δ (delta operator)

$$\Delta(y_1 - y_0) = \Delta(\Delta y_0)$$

$$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$$

$\Delta^2 y_0$ is called the second difference of y_0. The delta operator Δ, denotes the forward difference operator, the values of x in $f(x)$ e.g. a, $a+k$ are called argument and that of y is called the entry. In $y_1 = f(a+k)$ the argument a is increased by k in the entry, the operation of increasing this argument a by the k in the entry is denoted by operator E e.g.

$$Ef(a) = f(a+k)$$

which implies $Ey_0 = y_1 \longrightarrow (1)$

30 Since $y_0 = f(a)$ and

$$y_1 = f(a+k)$$

Thus, $\Delta y_0 = y_1 - y_0$ or

$$y_1 = \Delta y_0 + y_0 \longrightarrow (11)$$

Substitute for y_1 in eq (1)

$$Ey_0 = y_1$$

$$Ey_0 = \Delta y_0 + y_0$$

$$\Delta y_0 = Ey_0 - y_0$$

$$\Delta \cancel{y_0} = \cancel{y_0}[E-1]$$

$$\Delta = E - 1$$

$$\boxed{E = 1 + \Delta}$$

$\nabla \Rightarrow$ Nabla operator

If ∇ denotes backward difference operators, the Operator E^{-1} denotes the operation of decreasing the argument by k in the entry

e.g $y_0 = f(a)$, $y_1 = f(a-k)$

$$\nabla y_0 = y_0 - y_1 \longrightarrow (1)$$

$$\nabla f(a) = f(a) - f(a-k)$$

$$E^{-1}f(a) = f(a-k) \text{ or}$$

$$E^{-1}y_0 = y_1 \longrightarrow (11)$$

Substitute eq (11) in (1)

$$\nabla y_0 = y_0 - E^{-1}y_0$$

$$\nabla \cancel{y_0} = \cancel{y_0}[1 - E^{-1}]$$

$$\boxed{\nabla = 1 - E^{-1}}$$

Also

Since

$\nabla f(a) = f(a) - f(a-k)$

$E\nabla f(a) = E(f(a) - f(a-k)) \rightarrow (1)$

Also if

$E^{-1} f(a) = f(a-k)$

$\frac{1}{E} f(a) = f(a-k)$

$f(a) = E f(a-k) \rightarrow (11)$

from eq (1)

$E\nabla f(a) = E f(a) - E f(a-k)$

$E\nabla f(a) = E f(a) - f(a)$

$E\nabla \cancel{f(a)} = \cancel{f(a)}[E-1]$

$E\nabla = E-1$

since $\Delta = E-1$ then

$\boxed{E\nabla = \Delta}$

Likewise

$E f(a) = f(a+k)$

find ~~Nabla~~ operator of both side

$\nabla E f(a) = \nabla f(a+k)$

31 $\nabla E f(a) = f(a+k) - f(a)$

$\nabla E f(a) = \Delta f(a)$

$$\boxed{\nabla E = \Delta}$$

Similarly,

$$\nabla f(a) = f(a) - f(a-k)$$

find the delta operator of both side

$$\Delta\nabla f(a) = \Delta[f(a) - f(a-k)]$$

$$\Delta\nabla f(a) = \Delta f(a) - \Delta f(a-k)$$

where $E^{-1}f(a) = f(a-k)$

$$\Delta\nabla f(a) = \Delta f(a) - \Delta E^{-1} f(a)$$

$$\Delta\nabla = \Delta - \Delta E^{-1}$$

recall $E\nabla = \Delta$ or

$$\nabla = \frac{\Delta}{E} = \Delta E^{-1}$$

Thus $\boxed{\Delta\nabla = \Delta - \nabla}$

Also

$$E = 1 + \Delta \rightarrow (\text{I})$$

$$E^{-1} = 1 - \nabla \rightarrow (\text{II})$$

multiply (I) by (II)

$$E \times E^{-1} = (1+\Delta)(1-\nabla)$$

CHAPTER - V = 1

EXAMPLES

(1) Evaluate $\left(\frac{\Delta^2}{E}\right)x^2$ where the difference of Interval is K

SOLUTION

$$\left(\frac{\Delta^2}{E}\right)x^2 = \frac{(E-1)^2}{E}x^2$$

$$\left(\frac{E^2+1-2E}{E}\right)x^2 = \left(E+\frac{1}{E}-2\right)x^2$$

$$= [E+E^{-1}-2]x^2$$

$$= Ex^2 + E^{-1}x^2 - 2x^2$$

$$= (x+K)^2 + (x-K)^2 - 2x^2$$

$$(x^2+K^2+2Kx) + (x^2+K^2-2Kx) - 2x^2$$

$$2x^2 + 2K^2 - \cancel{2x^2}$$

$$= \underline{2K^2}$$

prove that

$$\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

Solution

$$\Delta \log f(x) = \log f(x+k) - \log f(x)$$

by law of logarithms

$$\Delta \log f(x) = \log\left[\frac{f(x+k)}{f(x)}\right]$$

$$\Delta \log f(x) = \log\left[\frac{f(x)+f(x+k)-f(x)}{f(x)}\right]$$

$$\Delta \log f(x) = \log\left[\frac{f(x)}{f(x)} + \frac{f(x+k)-f(x)}{f(x)}\right]$$

$$\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

(3) Evaluate $(\nabla+\Delta)^2(x^2+x)$ given that $k = 1$

Solution

$$\Delta = E - 1 \rightarrow (i)$$

$$\nabla = 1 - E^{-1} \rightarrow (ii)$$

where

$(\nabla+\Delta)^2(x^2+x)$ then

$$=(\cancel{1}-E^{-1}+E-\cancel{1})^2(x^2+x)$$

$$=(-E^{-1}+E)^2(x^2+x)$$

$$=(E^{-2}+E^2-2\times E\times E^{-1})(x^2+x)$$

$= (E^{-2} + E^{2} - 2)(x^2 + x) =$

$E^{-2}x^2 + E^2x^2 - 2x^2 + E^{-2}x$

$+ E^2x - 2x$

$\Rightarrow$ note $E^{n}x^{m} = (x+nk)^{m}$

and $E^{-n}x^{m} = (x-nk)^{m}$

Thus $E^{-2}x^2 + E^2x^2 - 2x^2 +$

$E^{-2}x + E^2x - 2x$

$= (x-2k)^2 + (x+2k)^2 - 2x^2 +$

$(x-2k) + (x+2k) - 2x$

Since $k = 1$ Then

$= (x-2)^2 + (x+2)^2 - 2x^2 + (x-2) + (x+2)$

$- 2x$

$= \cancel{x^2} + 4 \cancel{- 4x} + \cancel{x^2} + 4 \cancel{+ 4x} \cancel{- 2x^2}$

$\cancel{+ 2x} \cancel{- 2x}$

$= 4 + 4 = \underline{8}$

Central difference and Average operator

$\delta \rightarrow$ Central difference Operator

Suppose

$$\delta f(x) = f\left(x + \frac{k}{2}\right) - f\left(x - \frac{k}{2}\right)$$

33 $$\delta f(x) = E^{1/2} f(x) - E^{-1/2} f(x)$$

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

Square both side

$$\delta^2 = \left[E^{1/2} - E^{-1/2}\right]^2$$

$$\delta^2 = E^1 + E^{-1} - 2 \times E^{1/2} \times E^{-1/2}$$

$$\delta^2 = E + E^{-1} - 2$$

$$\Rightarrow E\delta^2 = E\left[E + E^{-1} - 2\right]$$

$$E\delta^2 = E^2 - 2E + 1$$

$$\Rightarrow E\delta^2 = (E-1)^2$$

note: $E - 1 = \Delta$

and $E = \Delta + 1$

$$(1+\Delta)\delta^2 = \Delta^2$$

$$\delta^2 + \delta^2\Delta = \Delta^2$$

$$\Rightarrow \Delta^2 - \delta^2\Delta - \delta^2 = 0$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-(-\delta^2) \pm \sqrt{(-\delta^2)^2 - 4 \times 1 \times -\delta^2}}{2}$$

$$\Delta = \frac{\delta^2 \pm \sqrt{\delta^4 + 4\delta^2}}{2}$$

$$\Delta = \frac{\delta^2}{2} \pm \frac{\sqrt{\delta^4 + 4\delta^2}}{2}$$

$$\Delta = \frac{\delta^2}{2} \pm \sqrt{\frac{\delta^4 + 4\delta^2}{4}}$$

$$\Delta = \frac{\delta^2}{2} \pm \delta\sqrt{\frac{\delta^2 + 4}{4}}$$

$$\boxed{\Delta = \frac{\delta^2}{2} \pm \delta\sqrt{1 + \frac{\delta^2}{4}}}$$

Suppose

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$$

$$\mu f(x) = \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$$

$$\delta = E^{1/2} - E^{-1/2}$$

multiply μ by δ

$$\delta\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]\left[E^{1/2} - E^{-1/2}\right]$$

difference of two square

$$\delta\mu = \frac{1}{2}\left(E^{1/2}\right)^2 - \left(E^{-1/2}\right)^2$$

$$\mu\delta = \frac{1}{2}[E - E^{-1}]$$

$$\mu\delta = \frac{1}{2}(\cancel{1}+\Delta + \nabla \cancel{-1})$$

$$\boxed{\mu\delta = \frac{1}{2}(\Delta+\nabla)}$$

Also

Prove that

$$(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta$$

Solution

$E = = 1+\Delta$ thus

$$(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} =$$

$$[E^{1/2}+E^{-1/2}][E]^{1/2}$$

$$= [E^{1/2} + E^{-1/2}][E^{1/2}]$$

$$= E^{1/2}\times E^{1/2} + E^{1/2}\times E^{-1/2}$$

$$= E+1 \quad \text{therefore}$$

$$(E^{1/2}+E^{-1/2})(1+\Delta)^{1/2} = E+1$$

$$[E^{1/2}+E^{-1/2}][1+\Delta]^{1/2} = 1+\Delta+1$$

$$[E^{1/2}+E^{-1/2}][1+\Delta]^{1/2} = 2+\Delta \text{ proved}$$

Prove that $\nabla = \delta E^{-1/2} = 1 - E^{-1}$

Solution

$$\delta = E^{1/2} - E^{-1/2}$$

$$\delta E^{-1/2} = E^{-1/2}[E^{1/2} - E^{-1/2}]$$

$$\delta E^{-1/2} = E^{-1/2} \times E^{1/2} - E^{-1/2} \times E^{-1/2}$$

$$\delta E^{-1/2} = 1 - E^{-1}$$

where $\nabla = 1 - E^{-1}$

$\therefore \delta E^{-1/2} = \nabla$ proved

prove $E^{1/2} = \mu + \frac{1}{2}\delta \rightarrow (1)$

$$\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$$

$$\delta = E^{1/2} - E^{-1/2}$$

Substitute for μ and δ in eq (1)

$$\frac{1}{2}[E^{1/2} + E^{-1/2}] + \frac{1}{2}[E^{1/2} - E^{-1/2}]$$

$$\tfrac{1}{2}E^{1/2} + \cancel{\tfrac{1}{2}E^{-1/2}} + \tfrac{1}{2}E^{1/2} - \cancel{\tfrac{1}{2}E^{-1/2}}$$

$$\tfrac{1}{2}E^{1/2} + \tfrac{1}{2}E^{1/2} = E^{1/2}$$ proved

EXERCISE

prove $E^{-1/2} = \mu - \frac{1}{2}\delta$

prove $\sqrt{1 + \delta^2\mu^2} = 1 + \frac{1}{2}\delta^2$

prove $\Delta^3 = E^3 - 3E^2 + 3E - 1$

prove $E\Delta = \Delta E$

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prove $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta$

SOLUTION

$\Delta = E - 1$

$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}(E-1)E^{-1} + \frac{1}{2}(E-1)$

$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}(E \times E^{-1} - E^{-1}) + \frac{1}{2}(E-1)$

$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}(1 - E^{-1}) + \frac{1}{2}(E-1)$

$= \cancel{\frac{1}{2}} - \frac{1}{2}E^{-1} + \frac{1}{2}E - \cancel{\frac{1}{2}}$

$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}[E - E^{-1}]$

Recall $\mu\delta = \frac{1}{2}[E - E^{-1}]$

$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \mu\delta$ proved

It is to be noted that

$\Delta(f_k \pm g_k) = \Delta f_k \pm \Delta g_k$

$\Delta^m(\Delta^n f_k) = \Delta^{m+n} f_k$

$\Delta(c f_k) = c \Delta f_k$

where c is constant

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BACKWARD DIFFERENCE OPERATOR

By definition

$\nabla f_K = f_K - f_{K-1} \Rightarrow \nabla^2 f_K = \nabla(\nabla f_K)$

$\nabla^2 f_K = \nabla(f_K - f_{K-1}) = (f_K - f_{K-1}) - (f_{K-1} - f_{K-2})$

$\nabla^2 f_K = f_K - 2f_{K-1} + f_{K-2}$

Also

$\nabla^3 f_K = \nabla(\nabla^2 f_K) = \nabla(f_K - 2f_{K-1} + f_{K-2})$

$\nabla^3 f_K = \nabla f_K - 2\nabla f_{K-1} + \nabla f_{K-2}$

$\nabla^3 f_K = f_K - f_{K-1} - 2(f_{K-1} - f_{K-2}) + f_{K-2} - f_{K-3}$

$\nabla^3 f_K = f_K - 3f_{K-1} + 3f_{K-2} - f_{K-3}$

Likewise

$\nabla^4 f_K = \nabla(\nabla^3 f_K) = \nabla(f_K - 3f_{K-1} + 3f_{K-2} - f_{K-3})$

$\nabla^4 f_K = \nabla f_K - 3\nabla f_{K-1} + 3\nabla f_{K-2} - \nabla f_{K-3}$

$= f_K - f_{K-1} - 3(f_{K-1} - f_{K-2}) + 3(f_{K-2} - f_{K-3}) - (f_{K-3} - f_{K-4})$

$= f_K - f_{K-1} - 3f_{K-1} + 3f_{K-2} + 3f_{K-2} - 3f_{K-3} - f_{K-3} + f_{K-4}$

$= f_K - 4f_{K-1} + 6f_{K-2} - 4f_{K-3} + f_{K-4}$

A difference can be constructed for a forward difference operators as follow

FORWARD DIFFERENCE OPERATOR TABLE

x	f	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	f_0	f_1-f_0	$f_2-2f_1+f_0$	$f_3-3f_2+3f_1-f_0$	$f_4-4f_3+6f_2-4f_1+f_0$	$f_5-5f_4+10f_3-10f_2+5f_1-f_0$
x_1	f_1	f_2-f_1	$f_3-2f_2+f_1$	$f_4-3f_3+3f_2-f_1$	$f_5-4f_4+6f_3-4f_2+f_1$	$f_6-5f_5+10f_4-10f_3+5f_2-f_1$
x_2	f_2	f_3-f_2	$f_4-2f_3+f_2$	$f_5-3f_4+3f_3-f_2$	$f_6-4f_5+6f_4-4f_3+f_2$	$f_7-5f_6+10f_5-10f_4+5f_3-f_2$
x_3	f_3	f_4-f_3	$f_5-2f_4+f_3$	$f_6-3f_5+3f_4-f_3$	$f_7-4f_6+6f_5-4f_4+f_3$	
x_4	f_4	f_5-f_4	$f_6-2f_5+f_4$	$f_7-3f_6+3f_5-f_4$		
x_5	f_5	f_6-f_5	$f_7-2f_6+f_4$			
x_6	f_6	f_7-f_6				
x_7	f_7					

BACKWARD DIFFERENCE OPERATOR TABLE (∇) NABLA

x	f	∇	∇^2	∇^3	∇^4	∇^5
x_0	f_0	f_0-f_1	$f_0-2f_1+f_2$	$f_0-3f_1+3f_2-f_3$	$f_0-4f_1+6f_2-4f_3+f_4$	$f_0-5f_1+10f_2-10f_3+5f_4-f_5$
x_1	f_1	f_1-f_2	$f_1-2f_2+f_3$	$f_1-3f_2+3f_3-f_4$	$f_1-4f_2+6f_3-4f_4+f_5$	$f_1-5f_2+10f_3-10f_4+5f_5-f_6$
x_2	f_2	f_2-f_3	$f_2-2f_3+f_4$	$f_2-3f_3+3f_4-f_5$	$f_2-4f_3+6f_4-4f_5+f_6$	$f_2-5f_3+10f_4-10f_5+5f_6-f_7$
x_3	f_3	f_3-f_4	$f_3-2f_4+f_5$	$f_3-3f_4+3f_5-f_6$	$f_3-4f_4+6f_5-4f_6+f_7$	
x_4	f_4	f_4-f_5	$f_4-2f_5+f_6$	$f_4-3f_5+3f_6-f_7$		
x_5	f_5	f_5-f_6	$f_5-2f_6+f_7$			
x_6	f_6	f_6-f_7				
x_7	f_7					

We read the tables in forward direction

CENTRAL DIFFERENCE OPERATOR (δ)

Operator (δ) ~~and~~ is defined as follows

(1) $\delta f_K = f_{K+1/2} - f_{K-1/2}$

$$\delta^2 f_K = \delta(\delta f_K) = \delta(f_{K+1/2} - f_{K-1/2})$$

$$= \delta f_{K+1/2} - \delta f_{K-1/2}$$

$$= \left(f_{K+\frac{1}{2}+\frac{1}{2}} - f_{K-\frac{1}{2}+1/2}\right) - \left(f_{K-\frac{1}{2}+\frac{1}{2}} - f_{K-\frac{1}{2}-\frac{1}{2}}\right)$$

$$= f_{K+1} - f_K - f_K + f_{K-1}$$

$$= f_{K+1} - 2f_K + f_{K-1}$$

$$\delta^3 f_K = \delta(\delta^2 f_K) = \delta(f_{K+1} - 2f_K + f_{K-1})$$

$$\delta^3 f_K = \delta f_{K+1} - 2\delta f_K + \delta f_{K-1}$$

$$= f_{K+1+1/2} - f_{K+\frac{1}{2}} - 2(f_{K+\frac{1}{2}} - f_{K-\frac{1}{2}}) + f_{K-1+\frac{1}{2}} - f_{K-1-\frac{1}{2}}$$

$$= f_{K+\frac{3}{2}} - 3f_{K+\frac{1}{2}} + 3f_{K-1/2} - f_{K-\frac{3}{2}}$$

note: The table of the Central difference is the same with that of the forward and Backward table

Example

Given that $F(x) = x^2$, the following table represent the difference table for $f(x)$.

x	f	δ	δ^2	δ^3
0	0			
		1		
1	1		2	
		3		0
2	4		2	
		5		
3	9			

① This operator relate with each other

$$\Delta - \nabla = \Delta\nabla$$

Taking L.H.S $= (\Delta - \nabla) f_K = \Delta f_K - \nabla f_K$

$= (f_{K+1} - f_K) - (f_K - f_{K-1})$

$= f_{K+1} - f_K - f_K + f_{K-1}$

$= f_{K+1} - 2f_K + f_{K-1}$

Taking R.H.S $\Rightarrow \Delta\nabla f_K = \Delta(\nabla f_K)$

$= \Delta(f_K - f_{K-1}) = \Delta f_K - \Delta f_{K-1}$

$= (f_{K+1} - f_K) - (f_K - f_{K-1})$

$f_{K+1} - 2f_K + f_{K-1}$

L.H.S = R.H.S proved

② This operator relate with each other

$$\delta^2 = \Delta - \nabla$$

38 Taking L.H.S $= \delta^2 f_K = \delta(\delta f_K)$

where $\delta^2 f_k = f_{k+1} - 2f_k + f_{k-1}$

$\cancel{\delta^2 f_k = \delta(\delta f_k) = \delta(f_{k+1} - 2f_k + f_{k-1})}$

$\cancel{\delta^2 f_k = \delta f_{k+1} - 2\delta f_k + \delta f_{k-1}}$

$\cancel{\delta^2 f_k = (f_{k+1+\frac{1}{2}} - f_{k+1-\frac{1}{2}}) - 2(f_{k+\frac{1}{2}} - f_{k-\frac{1}{2}}) + (f_{k-1+\frac{1}{2}} - f_{k-1-\frac{1}{2}})}$

Taking R.H.S $= (\Delta - \nabla) f_k$

$(\Delta - \nabla) f_k = f_{k+1} - 2f_k + f_{k-1}$

L.H.S = R.H.S proved

3) This operators relate with each other $(1+\Delta)(1-\nabla) = 1$

Taking L.H.S

$(1+\Delta)(1-\nabla) f_k = (1+\Delta)(f_k - \nabla f_k)$

$= (1+\Delta)[f_k - (f_k - f_{k-1})]$

$= (1+\Delta)(f_k - f_k + f_{k-1})$

$= (1+\Delta)(f_{k-1})$

$= 1 f_{k-1} + \Delta f_{k-1}$

$= \cancel{f_{k-1}} + f_k - \cancel{f_{k-1}}$

$= f_k$

Taking R.H.S $1 f_k$

L.H.S = R.H.S Proved

4) The operations relate with each other

$\nabla = 1-(1+\Delta) = \nabla(1+\Delta)$

R.H.S $\nabla(1+\Delta) = \nabla + \nabla\Delta$

where $\Delta\nabla = \Delta - \nabla$

R.H.S $= \cancel{\nabla} + \Delta - \cancel{\nabla} = \Delta$

SHIFT OPERATION [E]

as increament or decrease in argument

$Ef_k = f_{k+1}$

$E^2 f_k = E(Ef_k) = E(f_{k+1}) = f_{k+2}$

$E^p f_k = f_{k+p}$

Relationship between the operators

$\Delta f_k = f_{k+1} - f_k$

$= Ef_k - f_k$

$\Delta \cancel{f_k} = (E-1)\cancel{f_k}$

$$\boxed{\Delta = E-1}$$

add one to both side

$\Delta + 1 = E \cancel{-1} \cancel{+1}$

$\Delta + 1 = E$

$$\boxed{E = 1+\Delta}$$

Similarly

$\nabla f_k = f_k - f_{k-1}$

But

$E^{-1} f_K = f_{K-1}$, Also $E^{1/2} f_K = f_{K+1/2}$

$E^{-1/2} f_K = f_{K-1/2}$ Therefore

$$\nabla f_K = f_K - f_{K-1} = f_K - E^{-1} f_K$$

$$\nabla \cancel{f_K} = (1 - E^{-1}) \cancel{f_K}$$

$$\boxed{\nabla = 1 - E^{-1}}$$

subtract one from both side

$$\nabla - 1 = \cancel{1} - E^{-1} - \cancel{1}$$

$$\nabla = 1 - E^{-1} \Rightarrow$$

$$\nabla - 1 = E^{-1}$$

$$\nabla - 1 = \frac{1}{E}$$

$$E = \frac{1}{\nabla - 1}$$

$$E = (\nabla - 1)^{-1}$$

Also

$$\delta f_K = f_{K+1/2} - f_{K-1/2}$$

$$E^{1/2} f_K - E^{-1/2} f_K$$

$$\delta \cancel{f_K} = \cancel{f_K} [E^{1/2} - E^{-1/2}]$$

$$\delta = E^{1/2} - E^{-1/2}$$

square both side

$$\delta^2 = (E^{1/2} - E^{-1/2})^2$$

$$\delta^2 = E + E^{-1} - 2$$

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AVERAGE / MEAN OPERATOR (μ)

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

$$\mu f_k = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) f_k$$

$$\mu f_k = \frac{1}{2}\left(E^{1/2} f_k + E^{-1/2} f_k\right)$$

$$\mu f_k = \frac{1}{2}\left(f_{k+1/2} + f_{k-1/2}\right)$$

EXAMPLES

$$(1+\Delta)(1-\nabla) = 1$$

Taking L.H.S $(1+\Delta)(1-\nabla)$

$$= 1 - \nabla + \Delta - \Delta\nabla$$

$$= 1 - (f_k - f_{k-1}) + f_{k+1} - f_k - \Delta(\nabla f_k)$$

$$= 1 - f_k + f_{k-1} + f_{k+1} - f_k - \Delta(f_k - f_{k-1})$$

$$= 1 - f_k + f_{k-1} + f_{k+1} - f_k - \Delta f_k + \Delta f_{k-1}$$

$$= 1 - f_k + f_{k-1} + f_{k+1} - f_k - (f_{k+1} - f_k) + f_{k-1+1} - f_{k-1}$$

$$= 1 - \cancel{f_k} + \cancel{f_{k-1}} + \cancel{f_{k+1}} - \cancel{f_k} - \cancel{f_{k+1}} + \cancel{f_k} + \cancel{f_k} - \cancel{f_{k-1}}$$

$$= 1$$

L.H.S = R.H.S proved

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$$\Delta = (1 - \nabla)^{-1} - 1$$

$$\Delta = \frac{1}{1-\nabla} - 1$$

multiply through

$$\Delta(1-\nabla) = 1-(1-\nabla)$$

taking L.H.S

$$\Delta(1-\nabla) = \Delta - \Delta\nabla$$

$$(\Delta - \Delta\nabla)f_K = \Delta f_K - \Delta\nabla f_K$$

$$(\Delta - \Delta\nabla)f_K = f_{K+1} - f_K - \Delta(\nabla f_K)$$

$$(\Delta - \Delta\nabla)f_K = f_{K+1} - f_K - \Delta(f_K - f_{K-1})$$

$$= f_{K+1} - f_K - \Delta f_K + \Delta f_{K-1}$$

$$= f_{K+1} - f_K - (f_{K+1} - f_K) + (f_K - f_{K-1})$$

$$= \cancel{f_{K+1}} - \cancel{f_K} - \cancel{f_{K+1}} + \cancel{f_K} + f_K - f_{K-1}$$

$$= f_K - f_{K-1}$$

but $\nabla f_K = f_K - f_{K-1}$

$$R.H.S = 1-(1-\nabla) = 1-1+\nabla$$

$$R.H.S = \nabla \Rightarrow \nabla f_K$$

$$R.H.S = L.H.S \text{ proved}$$

III.

$$E = (1-\nabla)^{-1} = 1 + \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$$

Note the following

$$\delta f_K = f_{K+1/2} - f_{K-1/2} = E^{1/2} - E^{-1/2}$$

$$E f_K = f_{K+1}$$

$$\mu f_K = \frac{1}{2}(f_{K+1/2} - f_{K-1/2}) = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

$1 -$

$\Delta f_K = f_{K+1} - f_K = E - 1$

$\nabla f_K = f_K - f_{K-1} = 1 - E^{-1}$

THE TRUTH: Book of Isaiah says for God created us for Himself to show forth His praise and glory. Our Lifestyle should bring glory to God. Let Others see Christ in you is one of my favourite hymns.

taking the R-H-S

$1 + \frac{1}{2}\delta^2 + \delta\sqrt{1+\delta^2}$

$= 1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{\frac{4 + (E^{1/2} - E^{-1/2})^2}{4}}$

$= 1 + \frac{1}{2}[E + E^{-1} - 2] + [E^{1/2} - E^{-1/2}]\sqrt{\frac{4 + E - 2 + E^{-1}}{4}}$

$= 1 + \frac{1}{2}[E + E^{-1} - 2] + [E^{1/2} - E^{-1/2}]\sqrt{\frac{E + E^{-1} + 2}{4}}$

where $[E^{1/2} + E^{-1/2}]^2 = E + E^{-1} + 2$

$= 1 + \frac{1}{2}[E + E^{-1} - 2] + [E^{1/2} - E^{-1/2}]\sqrt{\frac{(E^{1/2} + E^{-1/2})^2}{4}}$

$= 1 + \frac{(E + E^{-1} - 2)}{2} + (E^{1/2} - E^{-1/2})\frac{(E^{1/2} + E^{-1/2})}{2}$

difference of two square

$(a-b)(a+b) = a^2 - b^2$

4/ $= 1 + \frac{E + E^{-1} - 2}{2} + \frac{E - E^{-1}}{2}$

multiply through by 2

$$= \frac{\cancel{2} + E + \cancel{E^{-1}} - \cancel{2} + E - \cancel{E^{-1}}}{2}$$

$$= \frac{E+E}{2} = \frac{2E}{2} = E$$

R.H.S = L.H.S proved

(IV)

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] = \left(1 + \frac{\Delta}{2}\right)\left(1+\Delta\right)^{-1/2}$$

$$\mu = \left(1 - \frac{\nabla}{2}\right)\left(1-\nabla\right)^{-1/2} = \sqrt{1 + \frac{\delta^2}{4}}$$

taking R.H.S $\sqrt{1+\frac{\delta^2}{4}}$

$$= \sqrt{\frac{4+\delta^2}{4}} = \sqrt{\frac{4+(E^{1/2} - E^{-1/2})^2}{4}}$$

$$= \sqrt{\frac{4 + E + E^{-1} - 2}{4}} = \sqrt{\frac{E + E^{-1} + 2}{4}}$$

$$= \sqrt{\frac{(E^{1/2} + E^{-1/2})^2}{4}} = \frac{E^{1/2} + E^{-1/2}}{2}$$

$$= \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

$$= \frac{1}{2}\left(f_{k+1/2} + f_{k-\frac{1}{2}}\right)$$

L.H.S = R.H.S

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V

$$\Delta = E^{1/2}\delta$$

taking the R.H.S

$$E^{1/2}\delta = E^{1/2}(\delta f_k) = E^{1/2}(f_{k+1/2} - f_{k-1/2})$$

$$= E^{1/2}[E^{1/2} - E^{-1/2}]$$

$$= E - 1 = \Delta$$

$\therefore$ R.H.S = L.H.S proved

VI

$$E\nabla = \nabla E = \Delta$$

taking L.H.S $E\nabla = E(\nabla f_k)$

$$= E(f_k - f_{k-1})$$

$$= Ef_k - Ef_{k-1}$$

$$= f_{k+1} - f_k$$

taking R.H.S $\nabla E = \nabla(Ef_k) = \nabla(f_{k+1})$

$$= f_{k+1} - f_k$$

R.H.S = L.H.S proved

VII

$$E = 1 + \mu\delta + \frac{1}{2}\delta^2$$

taking the R.H.S

$$1 + \mu\delta + \frac{1}{2}\delta^2 = 1 + \mu(E^{1/2} - E^{-1/2}) + \frac{1}{2}(E^{1/2} - E^{-1/2})^2$$

$$= 1 + \mu E^{1/2} - \mu E^{-1/2} + \frac{E + E^{-1} - 2}{2}$$

where $\mu E^{1/2} = \frac{1}{2}(E+1)$

$$\mu E^{-1/2} = \frac{1}{2}(1 + E^{-1})$$

$$= 1 + \frac{1}{2}(E+1) - \frac{1}{2}(1+E^{-1}) + \frac{E - 2 + E^{-1}}{2}$$

multiply by 2

$$= \frac{\cancel{2} + E + \cancel{1} - \cancel{1} - \cancel{E^{-1}} + E - \cancel{2} + \cancel{E^{-1}}}{2}$$

$$= \frac{E+E}{2} = \frac{2E}{2} = E$$

VIII

$$E = 1 + \delta \sqrt{E}$$

R.H.S

$$1 + \delta\sqrt{E} = 1 + \delta E^{1/2} = 1 + (E^{1/2} - E^{-1/2})E^{1/2}$$

$$= \cancel{1} + E - \cancel{1}$$

$$= E$$

R.H.S = R.H.S $\Rightarrow$ proved

FACTS ABOUT DIFFERENCE TABLE

1. For an exact polynomial $f(x) = x^n$ the n^{th} difference are constant or the same which make the n+1 difference equal to zero (0)
2. If on the other hand, the polynomial is not exact, there is rounding error, the difference will not be constant but will oscillate between $\pm 2^{n-1}$
3. In practice also, decimal point and leading zero's (0") are omitted from the value given in difference table

3

DETECTION AND CORRECTION OF ERROR IN DIFFERENCE TABLE.

Suppose an error occur in a difference table say δ then its effect on the difference spread out on the table below

X	F	Δ	Δ^2	Δ^3
X_1	0		0	
		0		δ
X_2	0	0	0	
				-3δ
X_3	0	$+\delta$	-2δ	
X_4	δ			3δ
		$-\delta$		
X_5	0		$+\delta$	$-\delta$
		0		
X_6	0		0	
		0		

The coefficient of factorization nth difference $(1-x)^n$ is taken out (factorized) in each case

δ $\qquad \delta(1)$

$$\delta(1, -1)$$

$$\delta(1, -2, 1)$$

$$\delta(1, -3, 3, -1)$$

Since the sign is alternating i.e $+ - + -$ we are approaching to binomial series

Example

The next table contains one (s) incorrect entry entry. Locate the error and profer a suitable correction and fixed the error

SOLUTION

x	F	Δ	Δ^2	Δ^3	Δ^4
0	18				
		19			
1	37		18		
		37		6	
2	74		24		0
		61		6	
3	135		30		-3
		91		3	
4	226		33		12
		124		15	
5	350		48		-18
		172		-3	
6	522		45		12
		217		9	
7	739		54		-3
		271		6	
8	1010		60		0

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note $\Delta_n = f_{n+1} - f_n$ eg $\Delta_1 = f_2 - f_1 = 74 - 37 =$

$\Delta_n^m = \Delta_{n+1}^m - f_n^m$ eg $\Delta_2^2 = f_3^2 - f_2^2 =$

$\Delta_2^2 = \Delta_3 - \Delta_2 = 61 - 37 = 24$

Now, where there is alternating of sign has been ruled out to the value of x, which is

$-3 \quad 12 \quad -18 \quad 12 \quad -3$

factorized Common factor -3

$-3 [1, -4, 6, -4, 1]$

-3 is factorized to have the binomial Coefficients of $(1-x)^n$

Thus $f(x)$ has and error δ

i.e $f(5)$ has error $\delta = -3$

then $f(5)$ should be be $350 - \delta$

i.e $f(5) \neq 350$ since there is error, to fix the error $f(5) = 350 - \delta = 350 - (-3)$

$f(5) = 353$

This difference table show that the polynomial function is of order 3 but the difference error is seen clearly

EXERCISE

Determine and fix the error in the following table

(a)

X	1	2	3	4	5	6	7	8
F	5	12	25	39	63	78	117	150

(b)

X	1	2	3	4	5	6	7	8
F	1	5	19	49	107	181	295	441

NEWTON - GREGORY FORMULA OF INTERPOLATION

This is also known as Newton's forward Interpolation for equal intervals, It is applied to interpolation near the beginning of the tabulated Values

$$f(a+nh) = E^n f(a)$$

$$f(a+nh) = (1+\Delta)^n f(a)$$

on Expanding $(1+\Delta)^n$ by Binomial theorem, we get

$$f(a+nh) = \left[1 + n\Delta + \frac{n(n-1)}{2}\Delta^2 + \cdots\right] f(a)$$

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or $f(a+nh) = f(a) + n\Delta f(a) + \frac{n(n-1)}{2}\Delta^2 f(a) + \cdots$

Example 2009/2010 Exam

x	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$f(x_i)$	0.000	0.182	0.336	0.470	0.587	0.693	0.788

take $x_0 = 0.0$ and $h = 0.2$

Using Gregory-Newton Formula, $f(0.75) =$

A) 0.559 B) 0.558 C) 0.555 D) 0.557

SOLUTION

x	F	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	0.000					
		0.182				
1	0.182		-0.028			
		0.154		0.008		
2	0.336		-0.02		-0.005	
		0.134		0.003		0.0[illegible]
3	0.470		-0.017		0.003	
		0.117		0.006		-0.0[illegible]
4	0.587		-0.011		-0.006	
		0.106		0.000		
5	0.693		-0.011			
		0.095				
6	0.788					

$F(0) + \Delta F(a)_{(a+nh)=x}$

$a + nh = x \quad a = 0 \quad h = 0.2$ then

$0 + 0.2n = x \Rightarrow n = \frac{x}{0.2} = 5x$

By Newton-Gregory Formular

$F(x) = F(a) + \Delta f(a) + \frac{n(n-1)}{2!}\Delta^2 f(a) +$

$\frac{n(n-1)(n-2)}{3!}\Delta^3 f(a) + \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^4 f(a)$

$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\Delta^5 f(a) + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!}\Delta^6 f(a)$

$\Delta^5 \quad \Delta^6$

But we are after

$f(0.75)$ i.e $f(0.75) = F(a+nh)$.

$a + nh = 0.75$ which means as

$a = 0$ we can conclude that

$0 + 0.2n = 0.75$ or $n = 0.75/0.2$

$n = 0.75 \times 5 = 3.75$

0.008

-0.015

-0.009

$F(0.75) = f(0) + 0.182 + \frac{3.75(3.75-1)(-0.028)}{2}$

$+ \frac{3.75(3.75-1)(3.75-2)(0.008)}{3\times2\times1} + \frac{3.75(3.75-1)(3.75-2)(3.75-3)(-0.005)}{4\times3\times2\times1} + \frac{3.75(3.75-1)(3.75-2)(3.75-3)(3.75-4)(0.008)}{5\times4\times3\times2\times1}$

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or $f(a+nh) = f(a) + n\Delta f(a) + \frac{n(n-1)}{2}\Delta^2 f(a) + \cdots$

Example 2009/2010 exam

x	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$f(x)$	0.000	0.182	0.336	0.470	0.587	0.693	0.788

take $x_0 = 0.0$ and $h = 0.2$

Using Gregory-Newton formula $f(0.75) =$

A) 0.559 B) 0.558 C) 0.555 D) 0.557

SOLUTION

x	F	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	0.000						
		0.182					
1	0.182		-0.028				
		0.154		0.008			
2	0.336		-0.02		-0.005		
		0.134		0.003		0.008	
3	0.470		-0.017		0.003		-0.015
		0.117		0.006		-0.007	
4	0.587		-0.011		-0.006		
		0.106		0.000			
5	0.693		-0.011				
		0.095					
6	0.788						

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$f(0) + \Delta f(a)_{(a+nh) \to x}$

$a + nh = x \quad a = 0 \quad h = 0.2$ then

$0 + 0.2n = x \Rightarrow n = \frac{x}{0.2} = 5x$

By Newton-Gregory formula

$$f(x) = f(a) + \Delta f(a) + \frac{n(n-1)}{2!}\Delta^2 f(a) + \frac{n(n-1)(n-2)}{3!}\Delta^3 f(a) + \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^4 f(a)$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\Delta^5 f(a) + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!}\Delta^6 f(a)$$

But we are after

$f(0.75)$ i.e $f(0.75) = f(a+nh)$

$a + nh = 0.75$ which means as

$a = 0$ we can conclude that

$0 + 0.2n = 0.75$ or $n = 0.75/0.2$

$n = 0.75 \times 5 = 3.75$

$$f(0.75) = f(0) + 0.182 + \frac{3.75(3.75-1)(-0.028)}{2} + \frac{3.75(3.75-1)(3.75-2)(0.008)}{3\times2\times1} + \frac{3.75(3.75-1)(3.75-2)(3.75-3)(-0.005)}{4\times3\times2\times1} + \frac{3.75(3.75-1)(3.75-2)(3.75-3)(3.75-4)(0.008)}{5\times4\times3\times2\times1}$$

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$+3.75(3.75-1)(3.75-2)(3.75-3)(3.75-4)$
$\frac{(3.75-5)(-0.015)}{6\times5\times4\times3\times2\times1}$

Thus

$f(0.75) = 0 + 0.182 - 0.1444 + 0.024 - 0.00282$
$- 0.000226 - 0.00008812$

$f(0.75) = 0.0585$

Example 2008/2009 EXAM

x	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$f(x_i)$	0.0000	0.0392	0.1484	0.3075	0.4947	0.6931	0.8920

Take $x_0 = 0.0$ and $h = 0.2$

x	f
0	0.0000
1	0.0392
2	0.1484
3	0.
4	
5	
6	

SIMPSON's 1/3rd Rule

$$\text{Area} = \int_{x_1}^{x_2} f(x)\,dx = \frac{h}{3}[X + 2O + 4E]$$

where X = Sum of the first and the last ordenates i.e

$$X = y_1 + y_n$$

O = Sum of odd ordenates = $y_3 + y_5 + y_7 + \ldots$

E = Sum of even ordenates = $y_2 + y_4 + y_8 + \ldots$

h = interval or width of independent Variable

SIMPSON's 3/8th Rule

$$\text{Area} = \int_{x_1}^{x_2} f(x)\,dx = \frac{3h}{8}\left[y_1 + 3y_2 + 3y_3 + y_4\right]$$

TRAPEZOIDAL RULE

$$\text{Area} = \int_{x_1}^{x_2} f(x)\,dx = \frac{h}{2}\left[(y_1 + y_n) + 2(y_2 + y_3 \ldots y_{n-1})\right]$$

h = Range of interval

EXAMPLE 2010/2011 EXAM

x	0.20	0.50	0.80	1.10	1.40	1.70	2.00
f(x)	1.8775	2.2830	2.7183	3.1912	3.7075	4.2725	4.8910

Using Trapezoidal rule $\int_{0.50}^{1.70} f(x)\,dx =$

SOLUTION

Since the interval is between 1.70 to 0.50 then we exclude 0.50 and 2.00 column. We focus on 5 columns

0.5	0.80	1.10	1.4	1.7
2.2830	2.7183	3.1912	3.7075	4.2725

Interval $h = 0.8 - 0.5 = 0.3$

$$\int_{0.50}^{1.70} f(x)\,dx = \frac{0.3}{2}\left[(2.2830 + 4.2725) + 2(2.7183 + 3.1912 + 3.7075)\right]$$

$$= \frac{0.3}{2}\left((6.5555) + 2(9.617)\right)$$

$$= \frac{0.3}{2} \times 25.7895 = 3.868425$$

Using Simpson's $\frac{1}{3}$rd rule $= \int_{0.20}^{0.80} f(x)\,dx$

SOLUTION

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we exclude others

x	0.20	0.5	0.8
$f(x)$	1.8775	2.2830	2.7183

$$h = 0.3$$

$$\int_{0.20}^{0.80} f(x)dx = \frac{h}{3}\left[(y_1 + y_3) + 2(0) + 4(y_2)\right]$$

$$= \frac{0.3}{3}\left[(2.7183 + 1.8775) + 0 + 4(2.2830)\right]$$

$$= 1.372778$$

Using Simpson's $\frac{3}{8}$th rule $\int_{1.10}^{2.00} f(x)dx$

SOLUTION

x	1.10	1.4	1.70	2.00
$f(x)$	3.1912	3.7075	4.2725	4.8910

$$h = 0.3$$

$$\int_{x_1}^{x_4} f(x)dx = \frac{3}{8}h\left(y_1 + 3(y_2 + y_3) + y_4\right)$$

$$= \frac{3}{8} \times 0.3 \left(3.1912 + 3(3.7075 + 4.2725) + 4.8910\right)$$

$$= 3.602498$$

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x	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$f(x)$	0.000	0.182	0.336	0.470	0.587	0.693	0.788

$h = 0.2$ Using Simpson's ~~rule~~ $\frac{3}{8}$ th rule $\int_{x_1}^{x_4} f(x)\,dx = 0$

a) 0.239 b) 0.249 c) 0.259 d) 0.269

SOLUTION

x	x_1	x_2	x_3	x_4
$f(x)$	0.182	0.336	0.470	0.587

$$\int_{x_1}^{x_4} f(x)\,dx = \frac{3h}{8}\left(y_1 + 3(y_2 + y_3) + y_4\right)$$

$$= \frac{3 \times 0.2}{8}\left(0.182 + 3(0.336 + 0.470) + 0.587\right)$$

$$= \underline{0.239}$$

EXERCISE

x	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$f(x)$	0.000	1.234	2.345	3.456	4.567	5.678	6.789

Using Trapezoidal rule $\int_{x_0}^{x_5} f(x)\,dx$

Newton - Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example 2010/2011 Exam

$f(x) = x^3 + x - 1$

If $x_0 = 2$, using Newton - Raphson method find x_3

Solution

let $f(x_n) = x_n^3 + x_n - 1$

$f'(x_n) = 3x_n^2 + 1$

Substitute in the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^3 + x_n - 1)}{3x_n^2 + 1}$$

$$x_{n+1} = \frac{x_n(3x_n^2 + 1) - (x_n^3 + x_n - 1)}{3x_n^2 + 1}$$

$$x_{n+1} = \frac{3x_n^3 + x_n - x_n^3 - x_n + 1}{3x_n^2 + 1}$$

$$\boxed{x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 + 1}}$$

When $n = 0$

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$$x_{0+1} = \frac{2x_0^3 + 1}{3x_0^2 + 1} \text{ which means}$$

$$x_1 = \frac{2x_0^3+1}{3x_0^2+1} = \frac{2(2)^3+1}{3(2)^2+1} = \frac{16+1}{12+1}$$

$$x_1 = \frac{17}{13} = 1.3077$$

when $n = 1$

$$x_{1+1} = \frac{2x_1^3+1}{3x_1^2+1} = \frac{2(1.3077)^3+1}{3(1.3077)^2+1}$$

$$x_2 = \frac{5.4725}{6.1302} = 0.8927$$

when $n = 2$

$$x_{2+1} = \frac{2x_2^3+1}{3x_2^2+1} = \frac{2(0.8927)^3+1}{3(0.8927)^2+1}$$

$$x_3 = \frac{2.4228}{3.3907} = 0.7145$$

Thus $x_3 = \underline{\underline{0.7145}}$

In an attempt to evaluate $\sqrt[3]{20}$ by Newton-Raphson method if $x_0 = 4$ then find x_3

Solution

let $x_n = \sqrt[3]{20}$

Cube both side

$x_n^3 = 20$ or

$x_n^3 - 20 = 0$

$f(x_n) = x_n^3 - 20$

$f'(x_n) = 3x_n^2$

Using Newton-Rhapson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^3 - 20)}{3x_n^2}$$

$$x_{n+1} = \frac{x_n(3x_n^2) - (x_n^3 - 20)}{3x_n^2}$$

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + 20}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{2x_n^3 + 20}{3x_n^2}}$$

when $n = 0$

$$x_{0+1} = \frac{2x_0^3 + 20}{3x_0^2} = \frac{2(4)^3 + 20}{3(4)^2}$$

$$x_1 = \frac{128+20}{48} = \frac{148}{48} = 3.0833$$

when $n = 1$

$$x_{1+1} = \frac{2x_1^3+120}{3x_1^2} = \frac{2(3.0833)^3+120}{3(3.0833)^2}$$

$$x_2 = \frac{78.6243}{28.5202} = 2.7568$$

when $n = 2$

$$x_{2+1} = \frac{2x_2^3+120}{3x_2^2} = \frac{2(2.7568)^3+120}{3(2.7568)^2}$$

$$x_3 = \frac{61.9031}{22.7998} = 2.7151$$

Thus, $x_3 \doteq 2.7151$

Using Newton-Raphson method to find cube root of b, the recurring formula will be given by

$x_{n+1} =$ ____________

SOLUTION

Let $x_n \doteq \sqrt[3]{b}$

cube both side

$$x_n^3 = b$$

$$x_n^3 - b = 0$$

$$f(x_n) = x_n^3 - b$$

$$f'(x_n) = 3x_n^2$$

let $x_n = \sqrt[3]{20}$

Cube both side

$x_n^3 = 20$ or

$x_n^3 - 20 = 0$

$f(x_n) = x_n^3 - 20$

$f'(x_n) = 3x_n^2$

Using Newton-Rhapson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^3 - 20)}{3x_n^2}$$

$$x_{n+1} = \frac{x_n.(3x_n^2) - (x_n^3 - 20)}{3x_n^2}$$

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + 20}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{2x_n^3 + 20}{3x_n^2}}$$

when $n = 0$

$$x_{0+1} = \frac{2x_0^3 + 20}{3x_0^2} = \frac{2(4)^3 + 20}{3(4)^2}$$

$$x_1 = \frac{128 + 20}{48} = \frac{148}{48} = 3.0833$$

when $n = 1$

$$x_{1+1} = \frac{2x_1^3+20}{3x_1^2} = \frac{2(3.0833)^3+20}{3(3.0833)^2}$$

$$x_2 = \frac{78.6243}{28.5202} = 2.7568$$

when $n = 2$

$$x_{2+1} = \frac{2x_2^3+20}{3x_2^2} = \frac{2(2.7568)^3+20}{3(2.7568)^2}$$

$$x_3 = \frac{61.9031}{22.7998} = 2.7151$$

Thus, $x_3 = \underline{2.7151}$

Using Newton-Raphson method to find cube root of b, the recurring formula will be given by

$x_{n+1} =$ ____ ____ ____ ____

SOLUTION

Let $x_n = \sqrt[3]{b}$

cube both side

$$x_n^3 = b$$

$$x_n^3 - b = 0$$

$$F(x_n) = x_n^3 - b$$

$$F'(x_n) = 3x_n^2$$

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Using Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^3 - b)}{3x_n^2}$$

$$x_{n+1} = \frac{x_n(3x_n^2) - x_n^3 + b}{3x_n^2}$$

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + b}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + b}{3x_n^2}$$

EXERSISE

1) Find the recurring formular of
a) Inverse square of b $[x_n = 1/b^2]$
b) Inverse square root of b $[x_n = 1/\sqrt{b}]$
2. In an attempt to evaluate $\sqrt[3]{13}$ by Newton-Raphson method If $x_0 = 2$ Find x_3

LAGRANGE'S FORMULA

$$f(x) = \frac{(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} f(x_1) + \frac{(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} f(x_2) + \cdots \frac{(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} \times f(x_n)$$

EXAMPLE 2007/2010 Exam

A function passes through the following point (0, 1), (1, -3) and (5, 1)

a) Using Lagrange's formular $f(4.5) =$

b) If $f(x) = -16$ then $x = ?$

SOLUTION

note $f(x_n) = y_n$ e.g $f(x_1) = y_1$

when $x_1 = 0$, $f(x_1) = 1$, when $x_2 = 1$ $f(x_2) = -3$ when $x_3 = 5$ $f(x_3) = 1$

$$f(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$f(x) = \frac{(x-1)(x-5)}{(0-1)(0-5)} \times 1 + \frac{(x-0)(x-5)}{(1-0)(1-5)} \times -3 + \frac{(x-0)(x-1)}{(5-0)(5-1)} \times 1$$

$$F(x) = \frac{(x-1)(x-5)}{5} + \frac{3}{4}(x-0)(x-5) + \frac{1}{20}(\ldots$$

$$f(x) = \frac{x^2-6x+5}{5} + \frac{3}{4}(x^2-5x) + \frac{1}{20}(x^2-x)$$

$$f'(x) = \frac{1}{5}(2x-6) + \frac{3}{4}(2x-5) + \frac{1}{20}(2x-1)$$

$f'(4.5)$ is when $x = 4.5$.

$$f'(4.5) = \frac{1}{5}(2(4.5)-6) + \frac{3}{4}(2(4.5)-5) + \frac{1}{20}(2(4.5)-1)$$

$$f'(4.5) = 0.6 + 4 + 0.4 = \underline{5}$$

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If $f(x) = -6$ to find x

$$f(x) = \frac{x^2-6x+5}{5} + \frac{3}{4}(x^2-5x) + \frac{1}{20}(x^2-x)$$

$$-6 = \frac{x^2-6x+5}{5} + \frac{3}{4}(x^2-5x) + \frac{1}{20}(x^2-x)$$

multiply through by 20

$$20 \times -6 = 4(x^2-6x+5) + 15(x^2-5x) + (x^2-x)$$

$$-120 = 4x^2 - 24x + 20 + 15x^2 - 75x + x^2 - x$$

$$-120 - 20 = 20x^2 - 100x$$

$$-140 = 20x^2 - 100x$$