Limits

Definitions

Precise Definition : We say $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

"Working" Definition : We say $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

Right hand limit : $\lim_{x\to a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

Left hand limit : $\lim_{x\to a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

Limit at Infinity : We say $\lim_{x\to\infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim_{x\to-\infty} f(x) = L$ except we require x large and negative.

Infinite Limit : We say $\lim_{x\to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$.

There is a similar definition for $\lim_{x\to a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one-sided limits

$$\lim_{x\to a} f(x) = L \;\Rightarrow\; \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \qquad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \;\Rightarrow\; \lim_{x\to a} f(x) = L$$

$$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) \;\Rightarrow\; \lim_{x\to a} f(x) \text{ Does Not Exist}$$

Properties

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and c is any number then,

1. $\lim_{x\to a}\left[cf(x)\right] = c\lim_{x\to a} f(x)$
2. $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x)\; \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$
6. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$

Basic Limit Evaluations at $\pm\infty$

Note : $\operatorname{sgn}(a) = 1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

1. $\lim_{x\to\infty} \mathbf{e}^x = \infty$ & $\lim_{x\to-\infty} \mathbf{e}^x = 0$
2. $\lim_{x\to\infty} \ln(x) = \infty$ & $\lim_{x\to 0^+} \ln(x) = -\infty$
3. If $r > 0$ then $\lim_{x\to\infty} \dfrac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim_{x\to-\infty} \dfrac{b}{x^r} = 0$
5. n even : $\lim_{x\to\pm\infty} x^n = \infty$
6. n odd : $\lim_{x\to\infty} x^n = \infty$ & $\lim_{x\to-\infty} x^n = -\infty$
7. n even : $\lim_{x\to\pm\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
8. n odd : $\lim_{x\to\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
9. n odd : $\lim_{x\to-\infty} a\,x^n + \cdots + c\,x + d = -\operatorname{sgn}(a)\infty$

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Evaluation Techniques

Continuous Functions

If $f(x)$ is continuous at a then $\lim_{x\to a} f(x) = f(a)$

Continuous Functions and Composition

$f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$$

Factor and Cancel

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} = \lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}$$

$$= \lim_{x\to 2}\frac{x+6}{x} = \frac{8}{2} = 4$$

Rationalize Numerator/Denominator

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81} = \lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}\,\frac{3+\sqrt{x}}{3+\sqrt{x}}$$

$$= \lim_{x\to 9}\frac{9-x}{(x^2-81)(3+\sqrt{x})} = \lim_{x\to 9}\frac{-1}{(x+9)(3+\sqrt{x})}$$

$$= \frac{-1}{(18)(6)} = -\frac{1}{108}$$

Combine Rational Expressions

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

L'Hospital's Rule

If $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then,

$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ a is a number, ∞ or $-\infty$

Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute $\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}$ factor largest power of x in $q(x)$ out of both $p(x)$ and $q(x)$ then compute limit.

$$\lim_{x\to-\infty}\frac{3x^2-4}{5x-2x^2} = \lim_{x\to-\infty}\frac{x^2\left(3-\frac{4}{x^2}\right)}{x^2\left(\frac{5}{x}-2\right)} = \lim_{x\to-\infty}\frac{3-\frac{4}{x^2}}{\frac{5}{x}-2} = -\frac{3}{2}$$

Piecewise Function

$\lim_{x\to -2} g(x)$ where $g(x) = \begin{cases} x^2+5 & \text{if } x < -2 \\ 1-3x & \text{if } x \ge -2 \end{cases}$

Compute two one sided limits,

$$\lim_{x\to -2^-} g(x) = \lim_{x\to -2^-} x^2+5 = 9$$

$$\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+} 1-3x = 7$$

One sided limits are different so $\lim_{x\to -2} g(x)$ doesn't exist. If the two one sided limits had been equal then $\lim_{x\to -2} g(x)$ would have existed and had the same value.

Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

1. Polynomials for all x.
2. Rational function, except for x's that give division by zero.
3. $\sqrt[n]{x}$ (n odd) for all x.
4. $\sqrt[n]{x}$ (n even) for all $x \ge 0$.
5. $\mathbf{e}^x$ for all x.
6. $\ln x$ for $x > 0$.
7. $\cos(x)$ and $\sin(x)$ for all x.
8. $\tan(x)$ and $\sec(x)$ provided $x \ne \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$
9. $\cot(x)$ and $\csc(x)$ provided $x \ne \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that $a < c < b$ and $f(c) = M$.

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Derivatives

Definition and Notation

If $y = f(x)$ then the derivative is defined to be $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

If $y = f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}\left(f(x)\right) = Df(x)$$

If $y = f(x)$ all of the following are equivalent notations for derivative evaluated at $x = a$.

$$f'(a) = y'\big|_{x=a} = \frac{df}{dx}\bigg|_{x=a} = \frac{dy}{dx}\bigg|_{x=a} = Df(a)$$

Interpretation of the Derivative

If $y = f(x)$ then,

1. $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x-a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$.
3. If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object at $x = a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(c\,f)' = c\,f'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(f\,g)' = f'\,g + f\,g'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c) = 0$
6. $\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)g'(x)$

   This is the **Chain Rule**

Common Derivatives

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

$\frac{d}{dx}\left(a^x\right) = a^x \ln(a)$

$\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x$

$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}, \quad x > 0$

$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}, \quad x \neq 0$

$\frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x \ln a}, \quad x > 0$

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Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}\left(\left[f(x)\right]^n\right) = n\left[f(x)\right]^{n-1} f'(x)$
2. $\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\mathbf{e}^{f(x)}$
3. $\frac{d}{dx}\left(\ln\left[f(x)\right]\right) = \frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}\left(\sin\left[f(x)\right]\right) = f'(x)\cos\left[f(x)\right]$
5. $\frac{d}{dx}\left(\cos\left[f(x)\right]\right) = -f'(x)\sin\left[f(x)\right]$
6. $\frac{d}{dx}\left(\tan\left[f(x)\right]\right) = f'(x)\sec^2\left[f(x)\right]$
7. $\frac{d}{dx}\left(\sec\left[f(x)\right]\right) = f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\frac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right) = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Higher Order Derivatives

The Second Derivative is denoted as

$f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2}$ and is defined as

$f''(x) = \left(f'(x)\right)'$, *i.e.* the derivative of the first derivative, $f'(x)$.

The n^th^ Derivative is denoted as

$f^{(n)}(x) = \frac{d^n f}{dx^n}$ and is defined as

$f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$, *i.e.* the derivative of the $(n\text{-}1)^{\text{st}}$ derivative, $f^{(n-1)}(x)$.

Implicit Differentiation

Find y' if $\mathbf{e}^{2x-9y} + x^3y^2 = \sin(y) + 11x$. Remember $y = y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). After differentiating solve for y'.

$$\mathbf{e}^{2x-9y}(2-9y') + 3x^2y^2 + 2x^3y\,y' = \cos(y)\,y' + 11$$
$$2\mathbf{e}^{2x-9y} - 9y'\mathbf{e}^{2x-9y} + 3x^2y^2 + 2x^3y\,y' = \cos(y)\,y' + 11 \quad\Rightarrow\quad y' = \frac{11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2}{2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)}$$
$$\left(2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)\right)y' = 11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2$$

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x = c$ is a critical point of $f(x)$ provided either

1. $f'(c) = 0$ or **2.** $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x) < 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x = c$ is a inflection point of $f(x)$ if the concavity changes at $x = c$.

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Extrema

Absolute Extrema

1. $x = c$ is an absolute maximum of $f(x)$ if $f(c) \geq f(x)$ for all x in the domain.
2. $x = c$ is an absolute minimum of $f(x)$ if $f(c) \leq f(x)$ for all x in the domain.

Fermat's Theorem

If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$.

Extreme Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ then there exist numbers c and d so that, **1.** $a \leq c, d \leq b$, **2.** $f(c)$ is the abs. max. in $[a,b]$, **3.** $f(d)$ is the abs. min. in $[a,b]$.

Finding Absolute Extrema

To find the absolute extrema of the continuous function $f(x)$ on the interval $[a,b]$ use the following process.

1. Find all critical points of $f(x)$ in $[a,b]$.
2. Evaluate $f(x)$ at all points found in Step 1.
3. Evaluate $f(a)$ and $f(b)$.
4. Identify the abs. max. (largest function value) and the abs. min.(smallest function value) from the evaluations in Steps 2 & 3.

Relative (local) Extrema

1. $x = c$ is a relative (or local) maximum of $f(x)$ if $f(c) \geq f(x)$ for all x near c.
2. $x = c$ is a relative (or local) minimum of $f(x)$ if $f(c) \leq f(x)$ for all x near c.

1st Derivative Test

If $x = c$ is a critical point of $f(x)$ then $x = c$ is

1. a rel. max. of $f(x)$ if $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$.
2. a rel. min. of $f(x)$ if $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$.
3. not a relative extrema of $f(x)$ if $f'(x)$ is the same sign on both sides of $x = c$.

2nd Derivative Test

If $x = c$ is a critical point of $f(x)$ such that $f'(c) = 0$ then $x = c$

1. is a relative maximum of $f(x)$ if $f''(c) < 0$.
2. is a relative minimum of $f(x)$ if $f''(c) > 0$.
3. may be a relative maximum, relative minimum, or neither if $f''(c) = 0$.

Finding Relative Extrema and/or Classify Critical Points

1. Find all critical points of $f(x)$.
2. Use the 1st derivative test or the 2nd derivative test on each critical point.

Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) then there is a number $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Newton's Method

If x_n is the n^{th} guess for the root/solution of $f(x) = 0$ then $(n+1)^{\text{st}}$ guess is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)$ exists.

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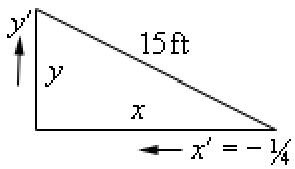
Related Rates

Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to *t* using implicit differentiation (*i.e.* add on a derivative every time you differentiate a function of *t*). Plug in known quantities and solve for the unknown quantity.

Ex. A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at $\frac{1}{4}$ ft/sec. How fast is the top moving after 12 sec?



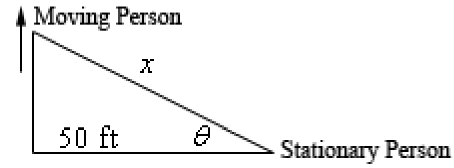
x' is negative because x is decreasing. Using Pythagorean Theorem and differentiating,

$$x^2 + y^2 = 15^2 \quad \Rightarrow \quad 2x\,x' + 2y\,y' = 0$$

After 12 sec we have $x = 10 - 12\left(\frac{1}{4}\right) = 7$ and so $y = \sqrt{15^2 - 7^2} = \sqrt{176}$. Plug in and solve for y'.

$$7\left(-\tfrac{1}{4}\right) + \sqrt{176}\; y' = 0 \;\Rightarrow\; y' = \frac{7}{4\sqrt{176}} \text{ ft/sec}$$

Ex. Two people are 50 ft apart when one starts walking north. The angle θ changes at 0.01 rad/min. At what rate is the distance between them changing when $\theta = 0.5$ rad?



We have $\theta' = 0.01$ rad/min. and want to find x'. We can use various trig fcns but easiest is,

$$\sec\theta = \frac{x}{50} \quad \Rightarrow \quad \sec\theta\tan\theta\;\theta' = \frac{x'}{50}$$

We know $\theta = 0.5$ so plug in θ' and solve.

$$\sec(0.5)\tan(0.5)(0.01) = \frac{x'}{50}$$

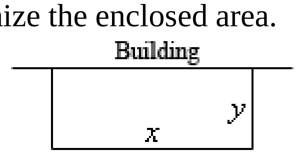
$$x' = 0.3112 \text{ ft/min}$$

Remember to have calculator in radians!

Optimization

Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

Ex. We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.



Maximize $A = xy$ subject to constraint of $x + 2y = 500$. Solve constraint for x and plug into area.

$$x = 500 - 2y \quad \Rightarrow \quad \begin{aligned} A &= y(500 - 2y) \\ &= 500y - 2y^2 \end{aligned}$$

Differentiate and find critical point(s).

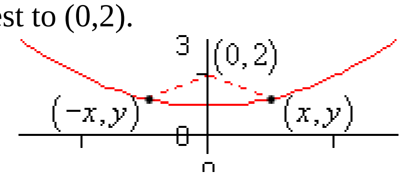
$$A' = 500 - 4y \quad \Rightarrow \quad y = 125$$

By 2nd deriv. test this is a rel. max. and so is the answer we're after. Finally, find *x*.

$$x = 500 - 2(125) = 250$$

The dimensions are then 250 x 125.

Ex. Determine point(s) on $y = x^2 + 1$ that are closest to (0,2).



Minimize $f = d^2 = (x-0)^2 + (y-2)^2$ and the constraint is $y = x^2 + 1$. Solve constraint for x^2 and plug into the function.

$$x^2 = y - 1 \;\Rightarrow\; f = x^2 + (y-2)^2$$

$$= y - 1 + (y-2)^2 = y^2 - 3y + 3$$

Differentiate and find critical point(s).

$$f' = 2y - 3 \quad \Rightarrow \quad y = \tfrac{3}{2}$$

By the 2nd derivative test this is a rel. min. and so all we need to do is find *x* value(s).

$$x^2 = \tfrac{3}{2} - 1 = \tfrac{1}{2} \quad \Rightarrow \quad x = \pm\tfrac{1}{\sqrt{2}}$$

The 2 points are then $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$.

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Integrals

Definitions

Definite Integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval.

Then $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$.

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x)=f(x)$.

Indefinite Integral : $\int f(x)\,dx = F(x)+c$

where $F(x)$ is an anti-derivative of $f(x)$.

Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then

$g(x)=\int_a^x f(t)\,dt$ is also continuous on $[a,b]$

and $g'(x)=\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$ (*i.e.* $F(x)=\int f(x)\,dx$)

then $\int_a^b f(x)\,dx = F(b)-F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)\,f[u(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -v'(x)\,f[v(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = u'(x)\,f[u(x)]-v'(x)\,f[v(x)]$$

Properties

$\int f(x)\pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int cf(x)\,dx = c\int f(x)\,dx$, c is a constant

$\int_a^b f(x)\pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant

$\int_a^a f(x)\,dx = 0$

$\int_a^b c\,dx = c(b-a)$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right|dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any value of c.

If $f(x)\ge g(x)$ on $a\le x\le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

If $f(x)\ge 0$ on $a\le x\le b$ then $\int_a^b f(x)\,dx \ge 0$

If $m\le f(x)\le M$ on $a\le x\le b$ then $m(b-a)\le \int_a^b f(x)\,dx \le M(b-a)$

Common Integrals

$\int k\,dx = k\,x+c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+c,\ n\ne -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x|+c$

$\int \frac{1}{a\,x+b}\,dx = \frac{1}{a}\ln|ax+b|+c$

$\int \ln u\,du = u\ln(u)-u+c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u+c$

$\int \cos u\,du = \sin u + c$

$\int \sin u\,du = -\cos u + c$

$\int \sec^2 u\,du = \tan u + c$

$\int \sec u\tan u\,du = \sec u + c$

$\int \csc u\cot u\,du = -\csc u + c$

$\int \csc^2 u\,du = -\cot u + c$

$\int \tan u\,du = \ln|\sec u|+c$

$\int \sec u\,du = \ln|\sec u+\tan u|+c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)+c$

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Standard Integration Techniques

Note that at many schools all but the Substitution Rule tend to be taught in a Calculus II class.

***u* Substitution :** The substitution $u = g(x)$ will convert $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ using $du = g'(x)\,dx$. For indefinite integrals drop the limits of integration.

Ex. $\int_1^2 5x^2\cos(x^3)\,dx$

$u = x^3 \Rightarrow du = 3x^2dx \Rightarrow x^2dx = \frac{1}{3}du$

$x = 1 \Rightarrow u = 1^3 = 1 \;::\; x = 2 \Rightarrow u = 2^3 = 8$

$$\int_1^2 5x^2\cos(x^3)\,dx = \int_1^8 \tfrac{5}{3}\cos(u)\,du$$
$$= \tfrac{5}{3}\sin(u)\Big|_1^8 = \tfrac{5}{3}\left(\sin(8) - \sin(1)\right)$$

Integration by Parts : $\int u\,dv = uv - \int v\,du$ and $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v = \int dv$.

Ex. $\int x\mathbf{e}^{-x}\,dx$

$u = x \quad dv = \mathbf{e}^{-x} \quad \Rightarrow \quad du = dx \quad v = -\mathbf{e}^{-x}$

$$\int x\mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} + \int \mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} - \mathbf{e}^{-x} + c$$

Ex. $\int_3^5 \ln x\,dx$

$u = \ln x \quad dv = dx \quad \Rightarrow \quad du = \frac{1}{x}dx \quad v = x$

$$\int_3^5 \ln x\,dx = x\ln x\Big|_3^5 - \int_3^5 dx = \left(x\ln(x) - x\right)\Big|_3^5$$
$$= 5\ln(5) - 3\ln(3) - 2$$

Products and (some) Quotients of Trig Functions

For $\int \sin^n x\cos^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 sine out and convert rest to cosines using $\sin^2 x = 1 - \cos^2 x$, then use the substitution $u = \cos x$.
2. ***m* odd.** Strip 1 cosine out and convert rest to sines using $\cos^2 x = 1 - \sin^2 x$, then use the substitution $u = \sin x$.
3. ***n* and *m* both odd.** Use either 1. or 2.
4. ***n* and *m* both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n x\sec^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2 x = \sec^2 x - 1$, then use the substitution $u = \sec x$.
2. ***m* even.** Strip 2 secants out and convert rest to tangents using $\sec^2 x = 1 + \tan^2 x$, then use the substitution $u = \tan x$.
3. ***n* odd and *m* even.** Use either 1. or 2.
4. ***n* even and *m* odd.** Each integral will be dealt with differently.

Trig Formulas **:** $\sin(2x) = 2\sin(x)\cos(x)$, $\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$, $\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$

Ex. $\int \tan^3 x\sec^5 x\,dx$

$$\int \tan^3 x\sec^5 x\,dx = \int \tan^2 x\sec^4 x\tan x\sec x\,dx$$
$$= \int \left(\sec^2 x - 1\right)\sec^4 x\tan x\sec x\,dx$$
$$= \int \left(u^2 - 1\right)u^4\,du \qquad (u = \sec x)$$
$$= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + c$$

Ex. $\int \frac{\sin^5 x}{\cos^3 x}\,dx$

$$\int \frac{\sin^5 x}{\cos^3 x}\,dx = \int \frac{\sin^4 x\sin x}{\cos^3 x}\,dx = \int \frac{(\sin^2 x)^2\sin x}{\cos^3 x}\,dx$$
$$= \int \frac{(1-\cos^2 x)^2\sin x}{\cos^3 x}\,dx \qquad (u = \cos x)$$
$$= -\int \frac{(1-u^2)^2}{u^3}\,du = -\int \frac{1-2u^2+u^4}{u^3}\,du$$
$$= \tfrac{1}{2}\sec^2 x + 2\ln\left|\cos x\right| - \tfrac{1}{2}\cos^2 x + c$$

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Trig Substitutions : If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$\sqrt{a^2-b^2x^2} \Rightarrow x=\frac{a}{b}\sin\theta$	$\sqrt{b^2x^2-a^2} \Rightarrow x=\frac{a}{b}\sec\theta$	$\sqrt{a^2+b^2x^2} \Rightarrow x=\frac{a}{b}\tan\theta$
$\cos^2\theta=1-\sin^2\theta$	$\tan^2\theta=\sec^2\theta-1$	$\sec^2\theta=1+\tan^2\theta$

Ex. $\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx$

$x=\frac{2}{3}\sin\theta \Rightarrow dx=\frac{2}{3}\cos\theta\,d\theta$

$\sqrt{4-9x^2}=\sqrt{4-4\sin^2\theta}=\sqrt{4\cos^2\theta}=2\left|\cos\theta\right|$

Recall $\sqrt{x^2}=\left|x\right|$. Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute θ's and remove absolute value bars based on that and,

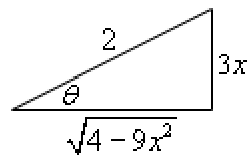
$$\left|x\right|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

In this case we have $\sqrt{4-9x^2}=2\cos\theta$.

$$\int \frac{16}{\frac{4}{9}\sin^2\theta(2\cos\theta)}\left(\tfrac{2}{3}\cos\theta\right)d\theta=\int\frac{12}{\sin^2\theta}\,d\theta$$

$$=\int 12\csc^2 d\theta=-12\cot\theta+c$$

Use Right Triangle Trig to go back to x's. From substitution we have $\sin\theta=\frac{3x}{2}$ so,



From this we see that $\cot\theta=\frac{\sqrt{4-9x^2}}{3x}$. So,

$$\int\frac{16}{x^2\sqrt{4-9x^2}}\,dx=-\frac{4\sqrt{4-9x^2}}{x}+c$$

Partial Fractions : If integrating $\int\frac{P(x)}{Q(x)}\,dx$ where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Ex. $\int\frac{7x^2+13x}{(x-1)(x^2+4)}\,dx$

$$\int\frac{7x^2+13x}{(x-1)(x^2+4)}\,dx=\int\frac{4}{x-1}+\frac{3x+16}{x^2+4}\,dx$$

$$=\int\frac{4}{x-1}+\frac{3x}{x^2+4}+\frac{16}{x^2+4}\,dx$$

$$=4\ln\left|x-1\right|+\tfrac{3}{2}\ln\left(x^2+4\right)+8\tan^{-1}\left(\tfrac{x}{2}\right)$$

Here is partial fraction form and recombined.

$$\frac{7x^2+13x}{(x-1)(x^2+4)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}=\frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

Set numerators equal and collect like terms.

$$7x^2+13x=(A+B)x^2+(C-B)x+4A-C$$

Set coefficients equal to get a system and solve to get constants.

$$A+B=7 \qquad C-B=13 \qquad 4A-C=0$$

$$A=4 \qquad B=3 \qquad C=16$$

An alternate method that *sometimes* works to find constants. Start with setting numerators equal in previous example : $7x^2+13x=A\left(x^2+4\right)+\left(Bx+C\right)\left(x-1\right)$. Chose *nice* values of x and plug in. For example if $x=1$ we get $20=5A$ which gives $A=4$. This won't always work easily.

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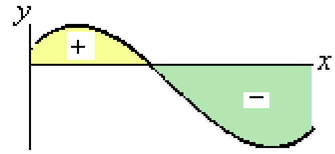
Applications of Integrals

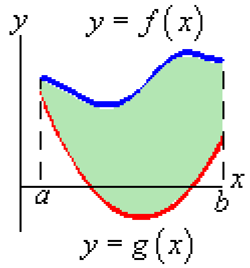
Net Area : $\int_a^b f(x)\,dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.



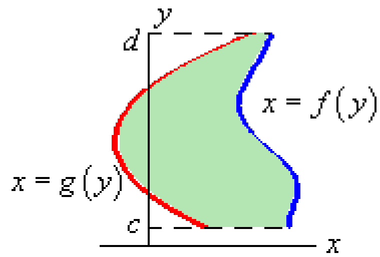
Area Between Curves : The general formulas for the two main cases for each are,

$$y = f(x) \Rightarrow A = \int_a^b [\text{upper function}] - [\text{lower function}]\,dx \quad \& \quad x = f(y) \Rightarrow A = \int_c^d [\text{right function}] - [\text{left function}]\,dy$$

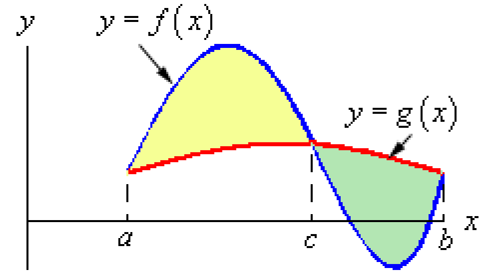
If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



$$A = \int_a^b f(x) - g(x)\,dx$$



$$A = \int_c^d f(y) - g(y)\,dy$$

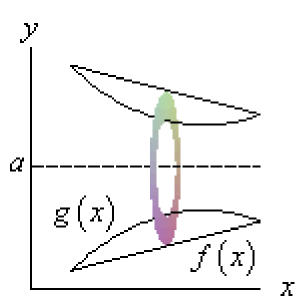


$$A = \int_a^c f(x) - g(x)\,dx + \int_c^b g(x) - f(x)\,dx$$

Volumes of Revolution : The two main formulas are $V = \int A(x)\,dx$ and $V = \int A(y)\,dy$. Here is some general information about each method of computing and some examples.

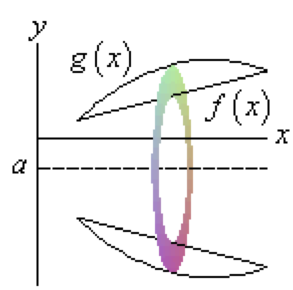
Rings		**Cylinders**	
$A = \pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right)$		$A = 2\pi(\text{radius})(\text{width / height})$	
Limits: *x*/*y* of right/bot ring to *x*/*y* of left/top ring		Limits : *x*/*y* of inner cyl. to *x*/*y* of outer cyl.	
Horz. Axis use $f(x)$, $g(x)$, $A(x)$ and *dx*.	Vert. Axis use $f(y)$, $g(y)$, $A(y)$ and *dy*.	Horz. Axis use $f(y)$, $g(y)$, $A(y)$ and *dy*.	Vert. Axis use $f(x)$, $g(x)$, $A(x)$ and *dx*.

Ex. Axis : $y = a > 0$



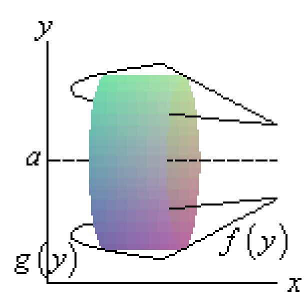
outer radius : $a - f(x)$
inner radius : $a - g(x)$

Ex. Axis : $y = a \le 0$



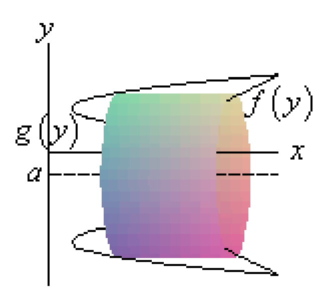
outer radius: $|a| + g(x)$
inner radius: $|a| + f(x)$

Ex. Axis : $y = a > 0$



radius : $a - y$
width : $f(y) - g(y)$

Ex. Axis : $y = a \le 0$



radius : $|a| + y$
width : $f(y) - g(y)$

These are only a few cases for horizontal axis of rotation. If axis of rotation is the x-axis use the $y = a \le 0$ case with $a = 0$. For vertical axis of rotation ($x = a > 0$ and $x = a \le 0$) interchange x and y to get appropriate formulas.

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Work : If a force of $F(x)$ moves an object in $a \le x \le b$, the work done is $W = \int_a^b F(x)\,dx$

Average Function Value : The average value of $f(x)$ on $a \le x \le b$ is $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Arc Length Surface Area : Note that this is often a Calc II topic. The three basic formulas are,

$L = \int_a^b ds$ $\quad SA = \int_a^b 2\pi y\,ds$ (rotate about *x*-axis) $\quad SA = \int_a^b 2\pi x\,ds$ (rotate about *y*-axis)

where *ds* is dependent upon the form of the function being worked with as follows.

$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ if $y = f(x),\ a \le x \le b$ $\qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ if $x = f(t), y = g(t),\ a \le t \le b$

$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ if $x = f(y),\ a \le y \le b$ $\qquad ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ if $r = f(\theta),\ a \le \theta \le b$

With surface area you *may* have to substitute in for the *x* or *y* depending on your choice of *ds* to match the differential in the *ds*. With parametric and polar you will always need to substitute.

Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called convergent if the limit exists and has a finite value and divergent if the limit doesn't exist or has infinite value. This is typically a Calc II topic.

Infinite Limit

1. $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
2. $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$
3. $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ provided BOTH integrals are convergent.

Discontinuous Integrand

1. Discont. at *a*: $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$
2. Discont. at *b* : $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$
3. Discontinuity at $a < c < b$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided both are convergent.

Comparison Test for Improper Integrals : If $f(x) \ge g(x) \ge 0$ on $[a,\infty)$ then,

1. If $\int_a^\infty f(x)\,dx$ conv. then $\int_a^\infty g(x)\,dx$ conv.
2. If $\int_a^\infty g(x)\,dx$ divg. then $\int_a^\infty f(x)\,dx$ divg.

Useful fact : If $a > 0$ then $\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges for $p \le 1$.

Approximating Definite Integrals

For given integral $\int_a^b f(x)\,dx$ and a *n* (must be even for Simpson's Rule) define $\Delta x = \frac{b-a}{n}$ and divide $[a,b]$ into *n* subintervals $[x_0,x_1]$, $[x_1,x_2]$, … , $[x_{n-1},x_n]$ with $x_0 = a$ and $x_n = b$ then,

Midpoint Rule : $\int_a^b f(x)\,dx \approx \Delta x\left[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)\right]$, x_i^* is midpoint $[x_{i-1},x_i]$

Trapezoid Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)++2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Simpson's Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$

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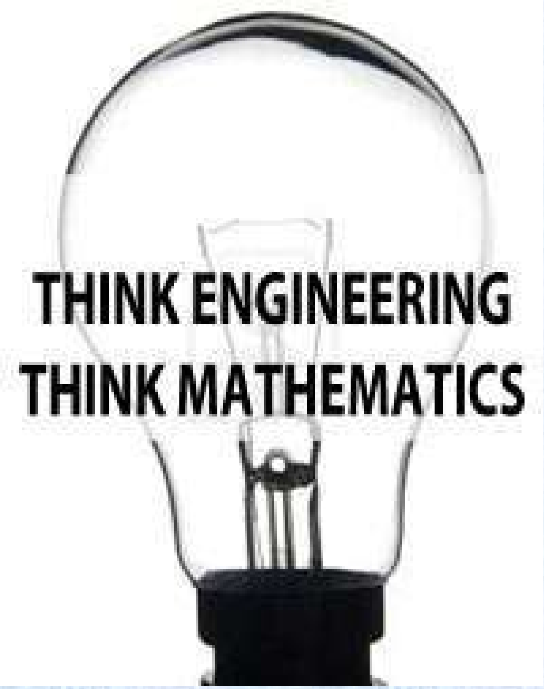
ENGINEERING MATHEMATICS-I
SECTION-B



DIFFERENTIAL CALCULUS

- Successive Differentiation
- Leibnitz Theorem and Applications
- Taylor's and Maclaurin's Series
- Curvature
- Asymptotes
- Curve tracing
- Functions of Two or More Variables
- Partial Derivatives of First and Higher Order
- Euler’s Theorem on Homogeneous Functions

- Differentiation of Composite and Implicit unctions
- Jacobians
- Taylor’s Theorem For A Function of Two Variables
- Maxima and Minima of Functions of Two Variables
- Lagrange's Method of Undetermined Multipliers
- Differentiation Under Integral Sign.

E-LEARNING

Topic :Taylor's series.
E-learning: http://nptel.ac.in/courses/122104017/11

Topic :Maclaurin's series.
E-learning: http://nptel.ac.in/courses/122104017/11

Topic : Partial derivatives of first order & its higher order.
E-learning: http://nptel.ac.in/courses/122101003/31

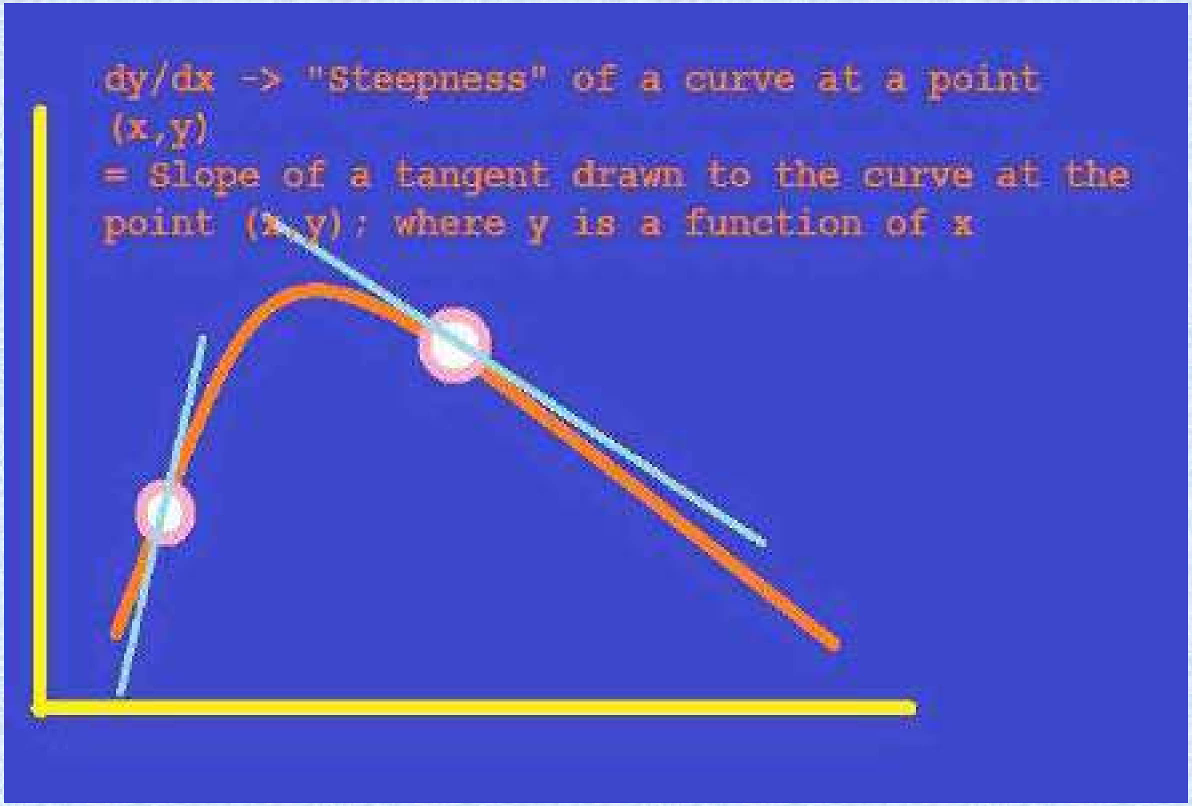
Topic :Euler's theorem on homogeneous functions .
E-learning: www.nptel.ac.in/courses/122101003/downloads/Lecture-31.pdf.

Topic :Total differential,Derivatives of composite and implicit function.
E-learning: http://nptel.ac.in/courses/122101003/32
http://nptel.ac.in/courses/122101003/33

Topic :Maxima and minima of function of two variables.
E-learning: http://nptel.ac.in/courses/122104017/10
http://nptel.ac.in/courses/122101003/37
http://nptel.ac.in/courses/122104017/26

Topic :Lagrange's method of undermined multipliers.
E-learning: http://nptel.ac.in/courses/122104017/27

SUCCESSIVE DIFFERENTIATION



The Process of Differentiating a function again and again is called successive Differentiation.

If y be a function of x, then its successive derivatives are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots \dots \dots \dots \dots \dots \dots, \frac{d^ny}{dx^n}$$

$$y_1, y_2, y_3, \dots \dots \dots \dots \dots \dots \dots \dots \dots, y_n$$

$$y', y'' y''', \dots \dots \dots \dots \dots \dots \dots \dots \dots y^n$$

Example 1. Find the fourth derivative of tan x at $x = \frac{\pi}{4}$

Example 2. $if\ y = Ae^{mx} + Be^{nx}, Prove\ that \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

SOME STANDARD RESULTS

1. n^{th} $derivative\ of\ x^m = \frac{m!}{(m-n)!} x^{m-n}\ \ if\ m\epsilon N, m > n.$
2. n^{th} $derivative\ of\ (ax+b)^m = m(m-1)(m-2)\ldots\ldots\ldots\ldots\ldots\ldots(m-n+1)(ax+b)^{m-n}\ \ a^n\ \ if\ m\epsilon N, m > n.$
3. $Find\ the\ n^{th}\ derivative\ of\ \frac{1}{ax+b} \quad = \frac{(-1)^n\ n!\ a^n}{(ax-b)^{n+1}}$
4. $Find\ the\ n^{th}\ derivative\ of\ log(ax+b) = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$
5. n^{th} $derivative\ of\ a^{mx} = m^n a^{mx}\ (\log a)^n$
6. n^{th} $derivative\ of\ e^{mx} = m^n e^{mx}$
7. n^{th} $derivative\ of\ \sin(ax+b) = a^n \sin\left(ax+b+n\frac{\pi}{2}\right)$
8. n^{th} $derivative\ of\ \cos(ax+b) = a^n \cos\left(ax+b+n\frac{\pi}{2}\right)$
9. n^{th} $derivative\ of\ \mathrm{e}^{\mathrm{ax}} \sin(bx+c) = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx+c+ntan^{-1}\frac{b}{a}\right)$
10. n^{th} $derivative\ of\ \mathrm{e}^{\mathrm{ax}} \cos(bx+c) = (a^2+b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx+c+ntan^{-1}\frac{b}{a}\right)$

Leibnitz's Theorem

Statement:- if y=uv where u and v are function of x, having derivative of nth order, then

$$y_n = n_{C_0} u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots \dots \dots \dots .. + n_{C_r} u_{n-r} v_r + \cdots \dots \dots .. + n_{C_n} u v_n$$

$$\textit{where suffixes denote the number of derivatives.}$$

Example 1. $If\ y = x^n \log x\ , prove\ that\ y_{n+1} = \frac{n!}{x}$

$Example$ 2.

$If\ y = \cos\ (m \log x), prove\ that\ x^2 y_{n+2} + (2n+1) x y_{n+1} + (m^2 + n^2) y_n = 0$

LINK FOR REFERENCE

- Leibnitz's Theorem for successive differentiation.
- https://www.youtube.com/watch?v=67uJGwsZz-Q

TAYLOR AND MACLAURIN'S SERIES

- The Taylor's series is named after the English mathematician Brook Taylor (1685–1731).

- The Maclaurin's series is named for the Scottish mathematician Colin Maclaurin (1698–1746).

- This is despite the fact that the Maclaurin's series is really just a special case of the Taylor's series.

APPLICATIONS OF TAYLOR'S AND MACLAURIN'S SERIES

- Expressing the complicated functions in terms of simple polynomials.
- Complicated functions can be made smooth.
- Differentiation of the such functions can be done as often as we please.
- In the field of Ordinary Differential Equations when finding series solution to Differential Equations.
- In the study of Partial Differential Equations.

GENERAL TAYLOR'S SERIES

(I) Expressing f(x + h) in ascending integral powers of h.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots\cdots$$

provided that all derivatives of f(x) are continuous and exist in the interval [x   x+h]

(II) Expressing f(x) in ascending integral powers of (x – a)

$$f(x) = f\big(a + (x-a)\big)$$

$$= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots\cdots$$

GUIDELINES FOR FINDING TAYLOR SERIES

Expanding f(x) about x = a

Differentiate f(x) several times

Evaluate each derivative at *x* = *a*

Evaluate $f(a), f'(a), f''(a)$

Substitute the above values in

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots\cdots$$

Example:

Find the Taylor series for f(x) = sin x at $c = \pi/4$

$f(x) = \sin x$ $$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$f'(x) = \cos x$ $$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$f''(x) = -\sin x$ $$f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$f'''(x) = -\cos x$ $$f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$f^{(4)}(x) = \sin x$ $$f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

Cont….

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x-c)^n}{n!}=f(c)+f'(c)(x-c)+\ldots..\frac{f^{(n)}(c)}{n!}(x-c)^n+\ldots.$$

$$=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})-\frac{\sqrt{2}}{2\cdot 2!}(x-\frac{\pi}{4})^2-\frac{\sqrt{2}}{2\cdot 3!}(x-\frac{\pi}{4})^3+\frac{\sqrt{2}}{2\cdot 4!}(x-\frac{\pi}{4})^4\ldots.$$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x-c)^n}{n!}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})-\frac{\sqrt{2}}{2\cdot 2!}(x-\frac{\pi}{4})^2\ldots..\frac{\sqrt{2}}{2n!}(x-\frac{\pi}{4})^n+\ldots.$$

$$=\frac{\sqrt{2}}{2}\left[1+(x-\frac{\pi}{4})-\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}-\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}+\frac{\left(x-\frac{\pi}{4}\right)^4}{n!}+\ldots.\right]$$

MACLAURIN'S SERIES

The Maclaurin's series is simply the Taylor's series about the point x = 0

It is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots\cdots$$

Find the Maclaurin's series for f(x) = ln(x² + 1)

$$f(x) = \ln(x^2+1) \qquad f(0) = 0$$

$$f'(x) = \frac{2x}{1+x^2} \qquad f'(0) = 0$$

$$f''(x) = \frac{2-2x^2}{(x^2+1)^2} \qquad f''(0) = 2$$

$$f'''(x) = \frac{4x(x^2-3)}{(x^2+1)^3} \qquad f'''(0) = 0$$

$$f^{(4)}(x) = \frac{12(-x^4+6x^2-1)}{(x^2+1)^4} \qquad f^{(4)}(0) = -12$$

$$f^{(5)}(x) = \frac{48x(x^4-10x^2+5)}{(x^2+1)^5} \qquad f^{(5)}(0) = 0$$

Cont….

$$f^{(5)}(x)=\frac{48x(x^4-10x^2+5)}{(x^2+1)^5}$$

$$f(0)=0 \qquad f'''(0)=0$$

$$f'(0)=0 \qquad f^{(4)}(0)=-12$$

$$f''(0)=2 \qquad f^{(5)}(0)=0$$

$$f^{(6)}(x)=\frac{-240(5x^6-15x^4+15x^2-1)}{(x^2+1)^6}$$

$$f^{(6)}(0)=240$$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}x^n}{n!}=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2.....\frac{f^{(n)}(0)}{n!}x^n+....$$

$$=0+0+\frac{2}{2!}x^2+\frac{0}{3!}x^3+\frac{-12}{4!}x^4+\frac{0}{5!}x^5+\frac{240}{6!}x^6.........$$

$$=x^2-\frac{x^4}{2}x^4+\frac{x^6}{3}.........$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}$$

Find the Taylor series for f(x) = e^{-2x} at c = 0

f(x) = e^{-2x} **f(0) = 1**

f ’(x) = $-2e^{-2x}$ **f ’(0) = -2**

f ’’(x) = $4e^{-2x}$ **f ’’(0) = 4**

f ’’’(x) = $-8e^{-2x}$ **f ’’’(0) = -8**

$f^{(4)}(x)$ = $16e^{-2x}$ **$f^{(4)}(0)$ = 16**

$$\sum_{n=0}^{\infty} \frac{f^{(n)} x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \ldots\ldots \frac{f^{(n)}(0)}{n!}x^n + \ldots\ldots$$

$$= 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \ldots\ldots \frac{2^n x^n}{n!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$$

MACLAURIN'S SERIES

We defined:

- *the nth Maclaurin polynomial for a function as*

$$\sum_{k=0}^{n} \frac{f^{k}(0)}{k!} x^{k} = f(0) + f^{/}(0)x + \frac{f^{//}(0)}{2!} x^{2} + \ldots + \frac{f^{n}(0)}{n!} x^{n}$$

- *the nth Taylor polynomial for f about x = x_0 as*

$$\sum_{k=0}^{n} \frac{f^{k}(x_0)}{k!} (x - x_0)^{k} = f(x_0) + f^{/}(x_0)(x - x_0) + \frac{f^{//}(x_0)}{2!} (x - x_0)^{2} + \ldots + \frac{f^{n}(x_0)}{n!} (x - x_0)^{n}$$

Example

Derive the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The Maclaurin series is simply the Taylor series about the point x=0

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + \underset{e^x}{f}'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4} + f'''''(x)\frac{h^5}{5} + \cdots$$

$$f(0+h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + f''''(0)\frac{h^4}{4} + f'''''(0)\frac{h^5}{5} + \cdots$$

DERIVATION FOR MACLAURIN SERIES FOR

e^x

Since $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, ..., $f^n(x) = e^x$ and
$f^n(0) = e^0 = 1$

the Maclaurin series is then

$$f(h) = (e^0) + (e^0)h + \frac{(e^0)}{2!}h^2 + \frac{(e^0)}{3!}h^3 ...$$

$$= 1 + h + \frac{1}{2!}h^2 + \frac{1}{3!}h^3 ...$$

So,

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

DERIVATION (CONT.)

Find the Maclaurin polynomial for f(x) = x cos x

We find the Maclaurin polynomial cos x and multiply by x

f(x) = cos x | **f(0) = 1**

f '(x) = -sin x | **f '(0) = 0**

f ''(x) = - cos x | **f ''(0) = - 1**

f '''(x) = sin x | **f '''(0) = 0**

$f^{(4)}(x)$ = cos x | **$f^{(4)}(0)$ = 1**

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \ldots\ldots \frac{f^{(n)}(0)}{n!}x^n + \ldots\ldots$$

$$= 1 + 0 - \frac{4x^2}{2!} - 0 + .\frac{x^4}{4!}\ldots\ldots \quad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\ldots\ldots$$

$$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!}\ldots\ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!}$$

Find the Maclaurin polynomial for f(x) = sin 3x

We find the Maclaurin polynomial sin x and replace x by 3x

f(x) = sin x	**f(0) = 0**
f '(x) = cos x	**f '(0) = 1**
f "(x) = - sin x	**f "(0) = 0**
f "'(x) = - cos x	**f "'(0) = -1**
$f^{(4)}(x)$ = sin x	**$f^{(4)}(0)$ = 0**

$$\sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \ldots\ldots \frac{f^{(n)}(0)}{n!}x^n + \ldots.$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0\ldots\ldots \quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\ldots\ldots$$

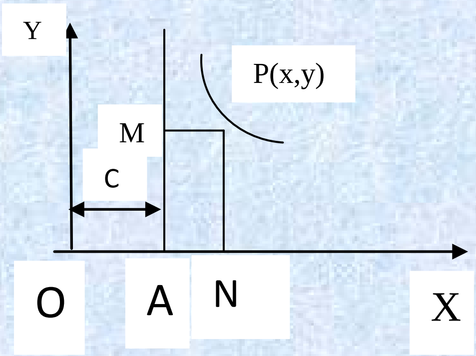
$$\sin 3x = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!}\ldots\ldots = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

- Taylor’s & Maclaurin’s Theorem for one variable.
- http://nptel.ac.in/courses/122104017/11
- http://www.creativeworld9.com/2011/02/iit-guest-lecture-mathematics-iii-video.html

ASYMPTOTES

Definition: An **asymptote** of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity. In other words..

A Straight line at a finite distance from the origin, is said to be an asymptote of an **infinite branch of a curve**, if the perpendicular distance of a point P on that branch from the straight line tends to zero as P tends to infinity along the branch of the curve.



A Curve With Finite Branches	A Curve With Infinite Branches
Ellipse :	**Hyperbola :**

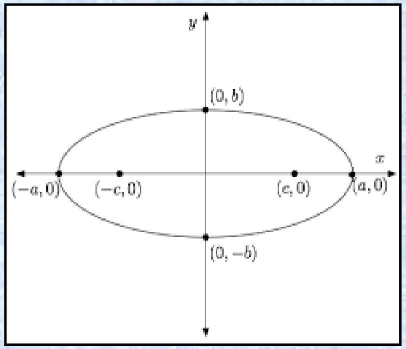
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Its two branches are

y= $b\sqrt{1-\frac{x^2}{a^2}}$ and y= -b$\sqrt{1-\frac{x^2}{a^2}}$

(upper half) (lower half)

(Both branches lie within x=a,x= -a ,y=b , y=-b.)

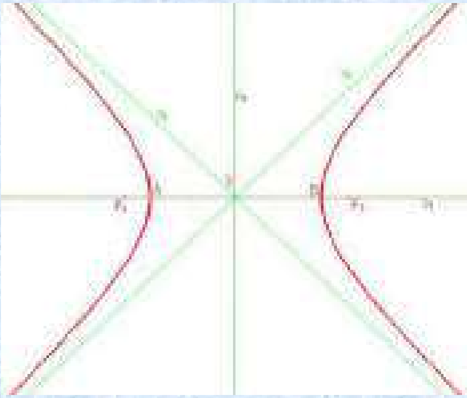


$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

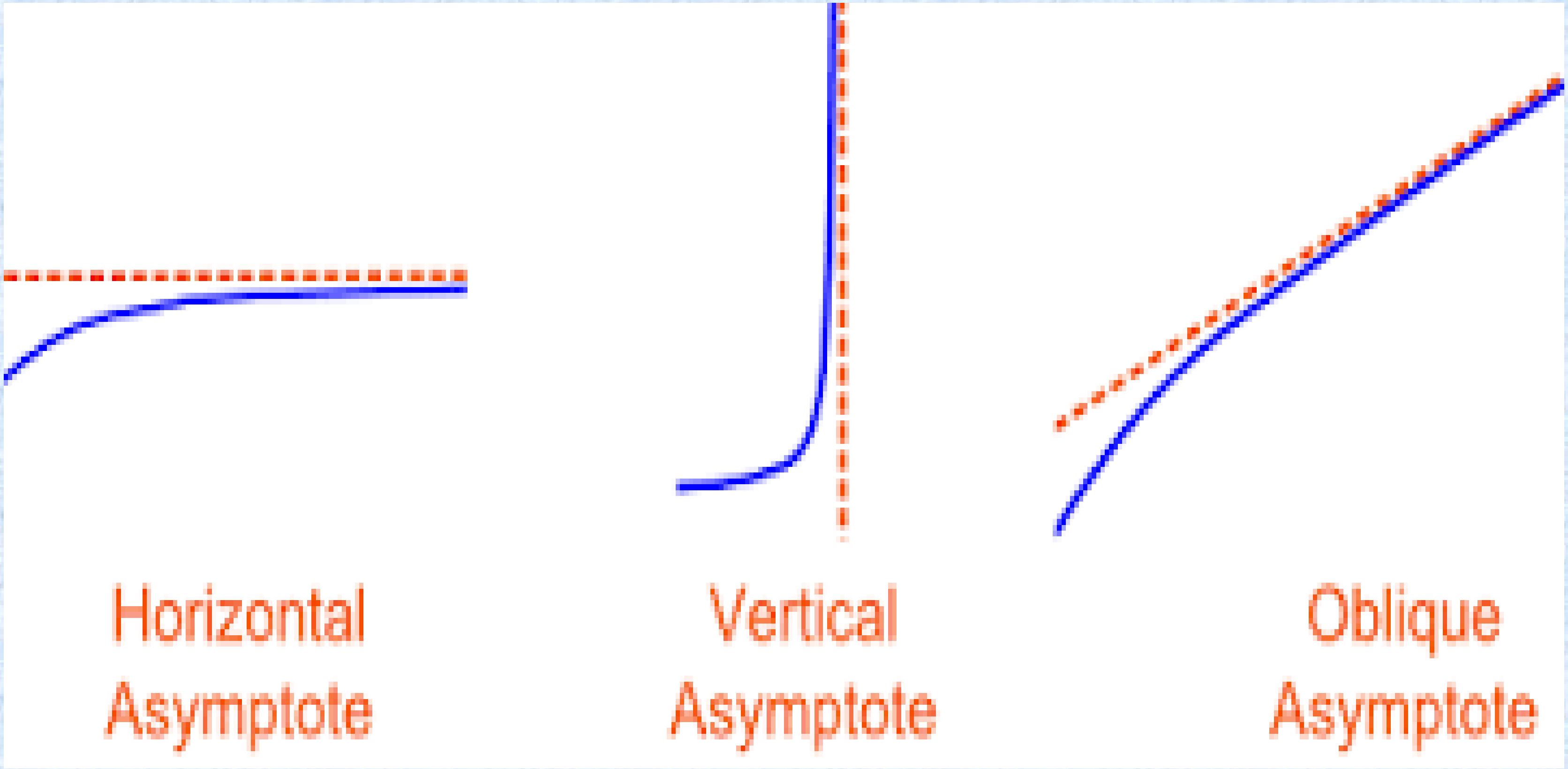
Its infinite branches are

$$y=\frac{b}{a}\sqrt{x^2-a^2},\quad y=-\frac{b}{a}\sqrt{x^2-a^2}$$

(Here $y\to\pm\infty$ as $x\to\pm\infty$)



KINDS OF ASYMPTOTES



ASYMPTOTE PARALLEL TO AXES

Asymptote Parallel to x-axis

Rule to find the asymptote || to X-axis, is to equate to zero the real linear factors in the co-efficient of the highest power of x in the equation of the curve.

Asymptote Parallel to y-axis

Rule to find the asymptote || to Y-axis, is to equate to zero the real linear factors in the co-efficient of the highest power of y in the equation of the curve.

$$Example\ 1. Find\ the\ x^2y^2 - a^2(x^2+y^2) - a^3(x+y) + a^4 Asmptote\ Parallel\ to\ axes$$

$$Example\ 6. Find\ the\ x^2y^2 - x^2y - xy^2 + x + y + 1 = 0\ Asmptote\ Parallel\ to\ axes$$

Oblique Asymptote

The equation of straight line y=mx+c is the oblique asymptote to the given curve

WORKING RULE FOR FINDING OBLIQUE ASYMPTOTES OF AN ALGEBRAIC CURVE OF THE NTH DEGREE

1. Find the $\emptyset_n(m)$. This can be obtained by putting x=1, y=m in the highest degree terms of the given equation of the curve.
2. Equate $\emptyset_n(m)$ to zero and solve for m.
   Let its roots be $m_1, m_2, m_3, \ldots\ldots\ldots..$
3. Find $\emptyset_{n-1}(m)$ by putting x=1 and y=m in the next lower terms of the equation. Similarly $\emptyset_{n-2}(m)$ may be found out by putting x=1 and y=m in the next lower degree terms in the curve and so on.
4. Find the values of c1,c2,c3,……………….. corresponding to the values $m_1, m_2, m_3, \ldots\ldots\ldots..$by using equation $c = \frac{\emptyset_{n-1}(m)}{\emptyset_n'(m)}$
5. Then the required asymptotes are $y = m_1x + c_1$,y= $m_2 + c_2$,…………………..

6. If $\emptyset_n^{'}(m) = 0\ for\ some\ value\ of\ m\ and\ \emptyset_{n-1}(m) \neq 0$corressponding to that value, then there will be no asymptote corresponding to that value of m.
7. If $\emptyset_n^{'}(m) = 0$ and $\emptyset_{n-1}(m) \neq 0$ for some value of m, the value of c are determined by

$$\frac{c^2}{2!}\emptyset_n^{''}(m) + c\emptyset_{n-1}^{'}(m) + \emptyset_{n-2}(m) = 0,$$

And this will determine two value of c and thus we shall have two parallel asymptotes corresponding to this value of m.

$Example\ 1. Find\ the\ asymtote\ of\ the\ curve\ (x-y)^2(x+2y-1) = 3x+y-7$

$Example\ 2. Find\ all\ the\ asymtote\ of\ the\ following\ curve$

$(i) y^2(x-2a) = x^3 - a^3$

$(ii) y^3 - 2xy^2 - x^2y + 2x^3 + 3y^2 - 7xy + 2x^2 + 2x + 2y + 1 = 0$

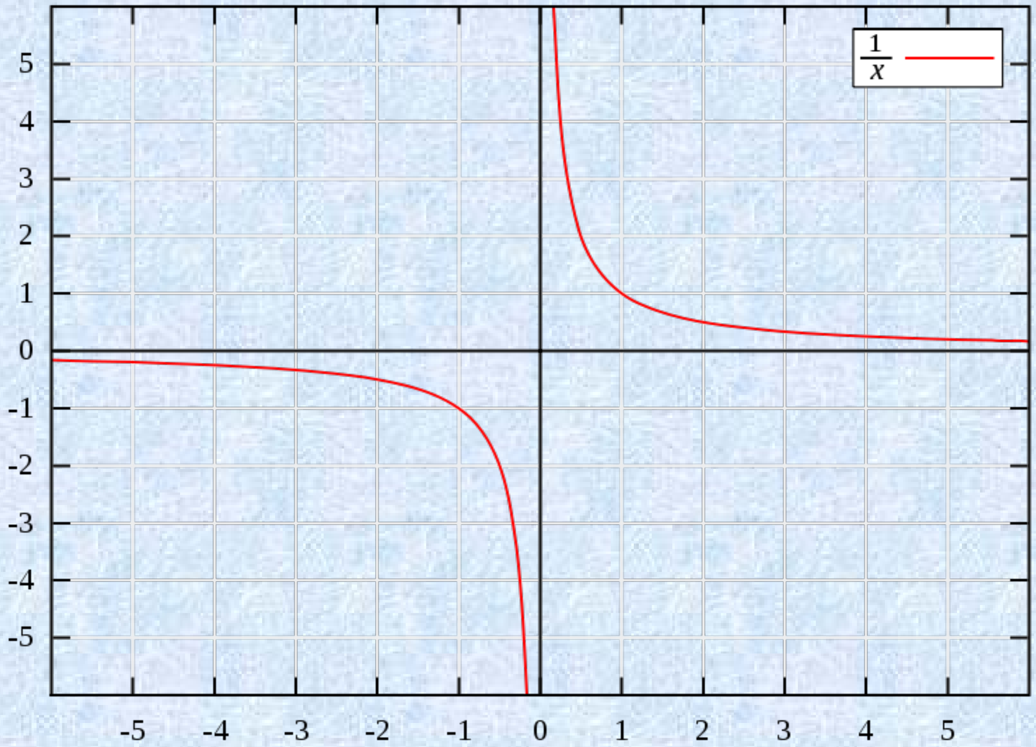
$(iii) y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 0$

$Example\ 3. show\ that\ the\ asymptotes\ of\ the\ curve\ x^2y^2 - a^2(x^2+y^2) - a^3(x+y) + a^4 = 0$

Form the square through two of whose vertices the curve passes.

PICTORIAL EXAMPLE 1

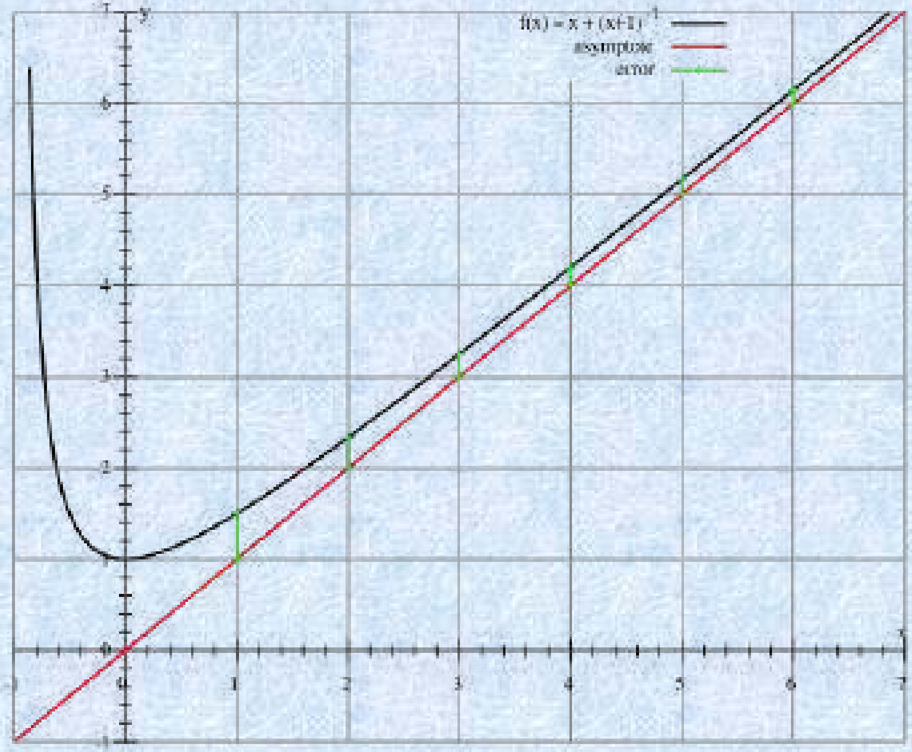
$f(x) = \frac{1}{x}$ **graphed on Cartesian coordinates.**



The *x* and *y*-axes are the asymptotes of the curve.

PICTORIAL EXAMPLE 2

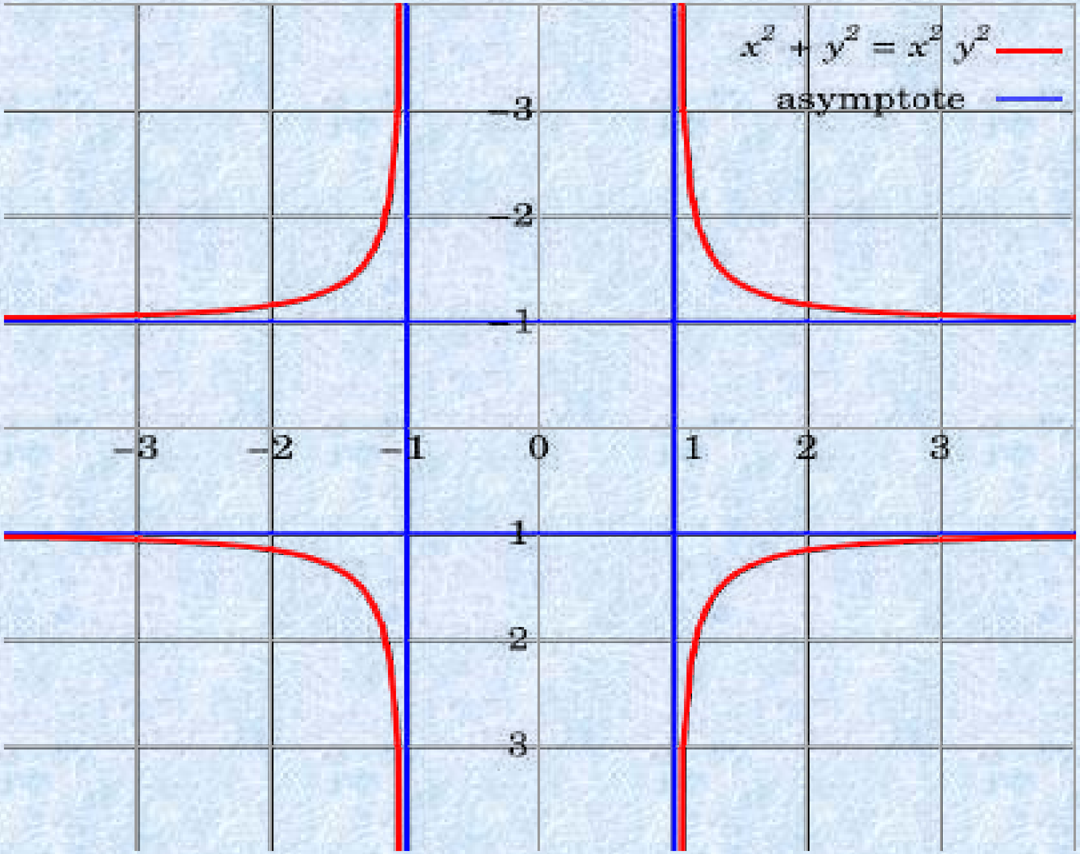
The graph of $f(x) = (x^2 + x + 1)/(x + 1)$



y = x is the Asymptote

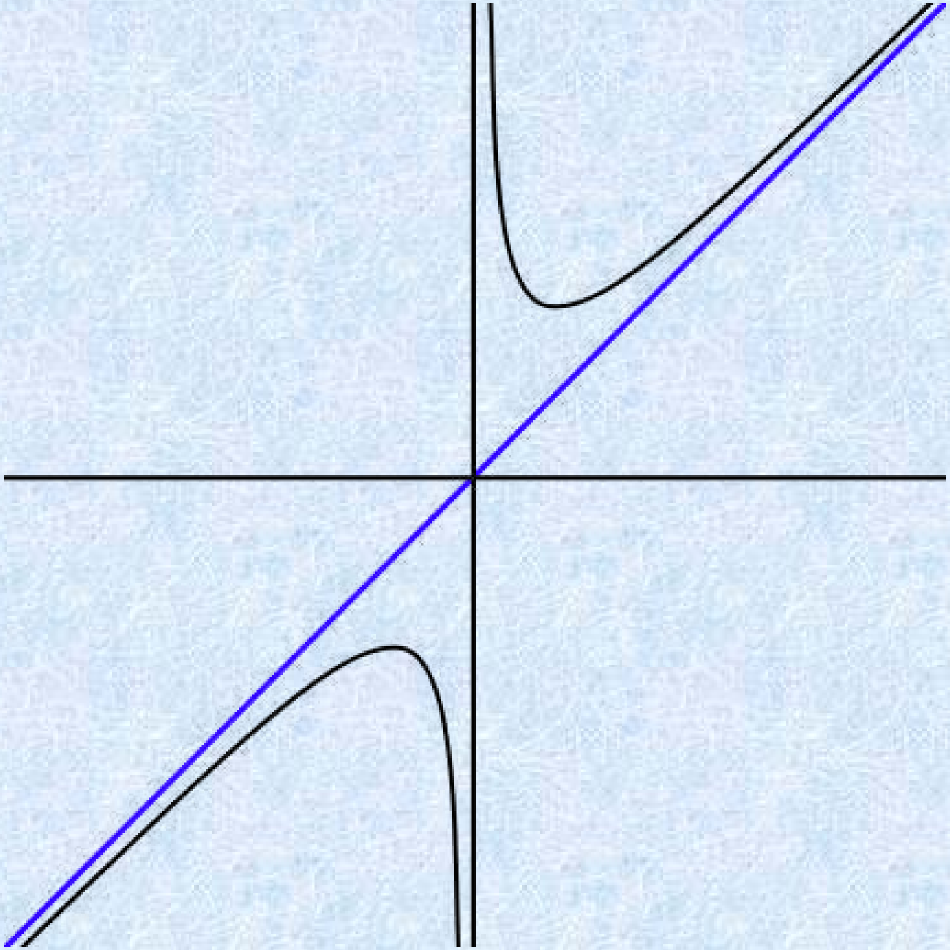
PICTORIAL EXAMPLE 3

The graph of $x^2 + y^2 = (xy)^2$, with 2 horizontal and 2 vertical asymptotes



PICTORIAL EXAMPLE 4

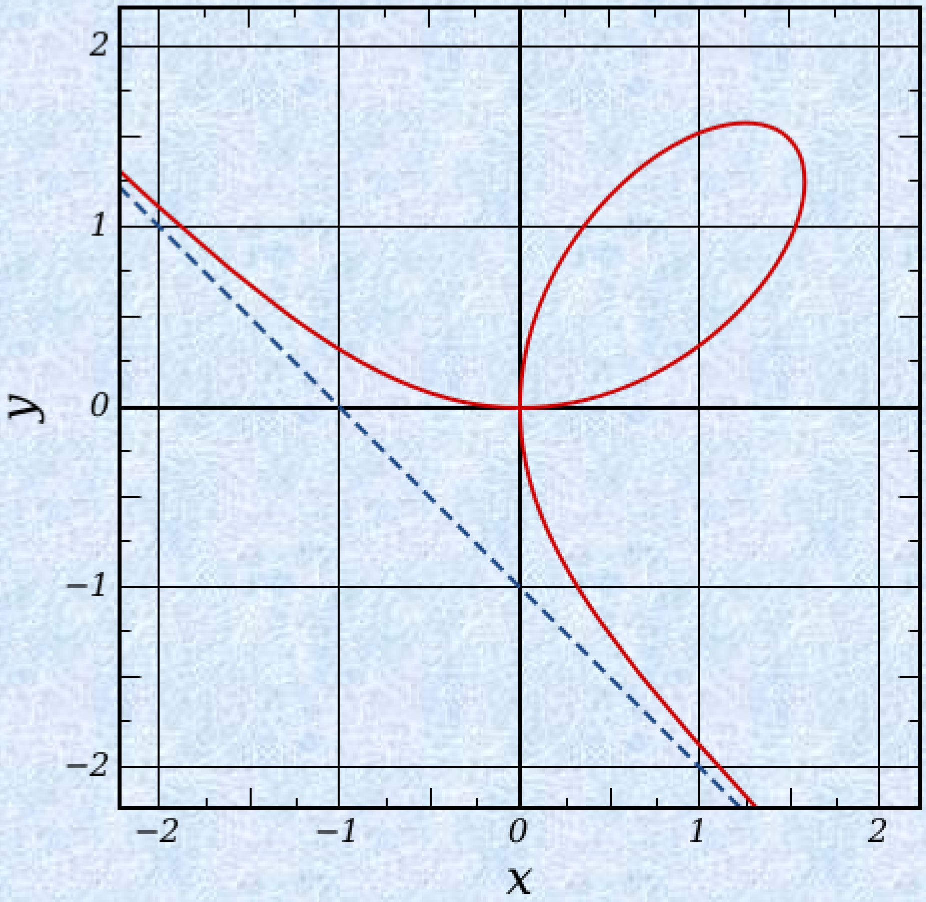
The graph of $f(x) = x + \frac{1}{x}$



The y-axis (x = 0) and the line y = x are both asymptotes

PICTORIAL EXAMPLE 5

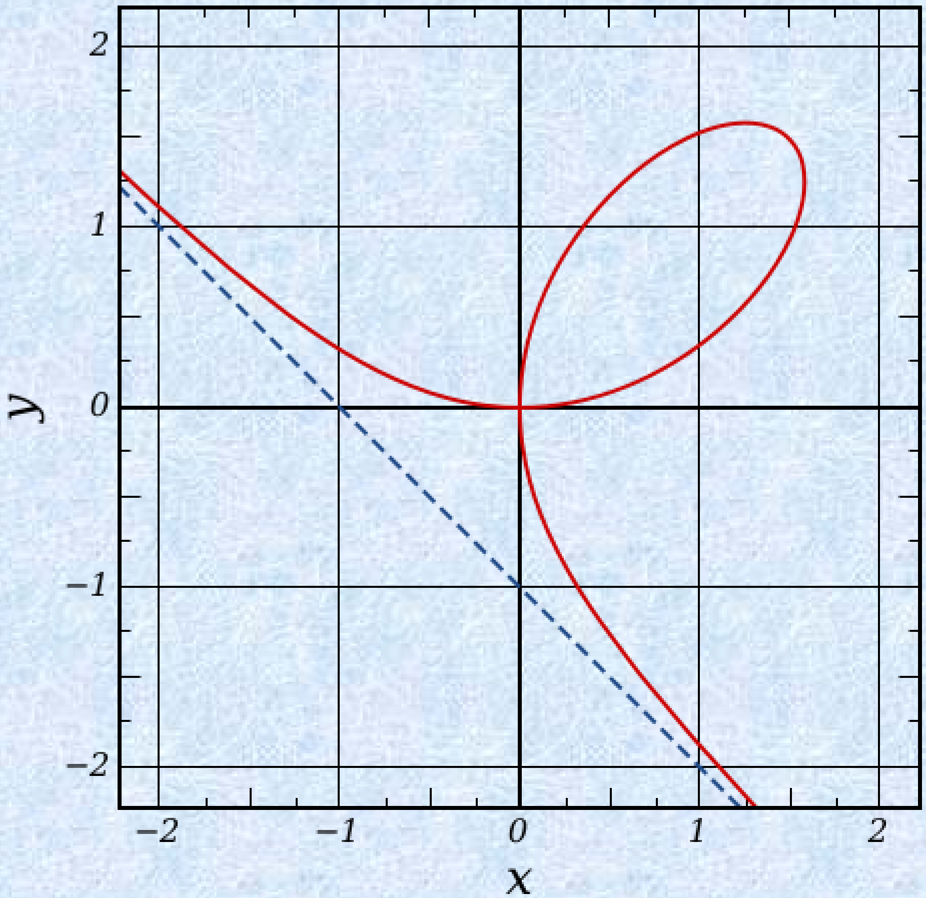
The graph of $x^3 + y^3 = 3axy$



A cubic curve, the folium of Descartes (solid) with a single real asymptote (dashed) given by $x + y + a = 0$.

PICTORIAL EXAMPLE 5

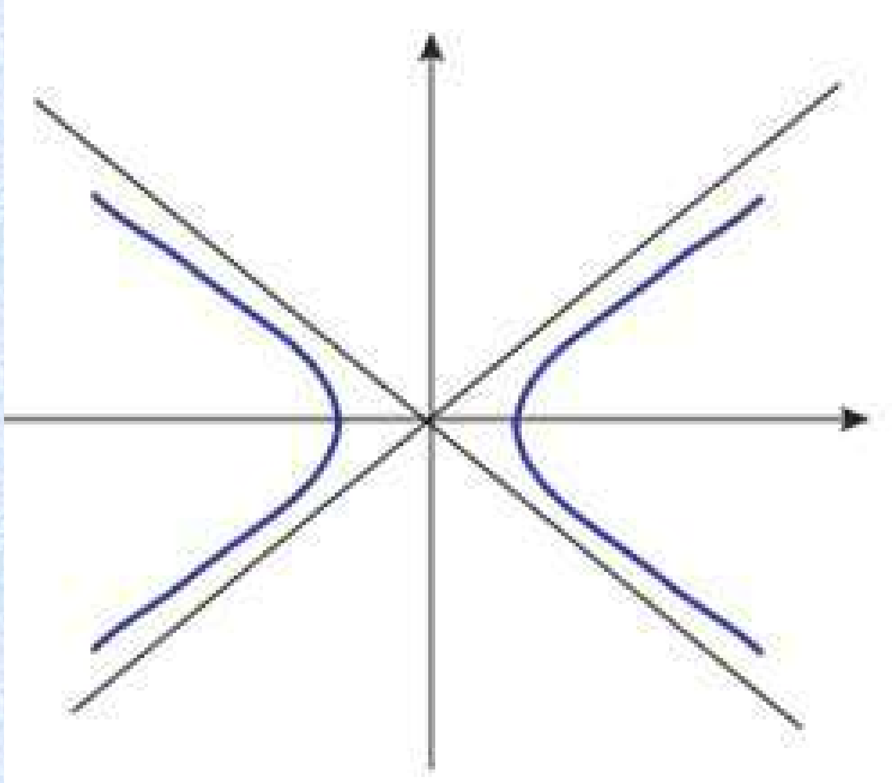
The graph of $x^3 + y^3 = 3axy$



A cubic curve, the folium of Descartes (solid) with a single real asymptote (dashed) given by $x + y + a = 0$.

PICTORIAL EXAMPLE 6

The graph of Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Its asymptotes are y = $\pm\frac{b}{a}x$

ASYMPTOTE OF THE POLAR CURVES

$$If\ \alpha\ is\ a\ root\ of\ the\ equation\ f(\theta) = 0, then\ \ r\sin(\theta-\alpha) = \frac{1}{f^{'}(\alpha)}\ is\ an\ asympote\ of\ the\ polar\ curve\ \frac{1}{r} = f(\theta)$$

Working rule for finding the asymptotes of polar curves.

1. Write down the given equation as $\frac{1}{r} = f(\theta)$
2. Equate $f(\theta)$ to zero and solve for $\theta = \theta_1, \theta_2, \theta_3, \dots \dots \dots \dots$
3. Find $f^{'}(\theta)$ and calculate $f^{'}(\theta)$ at $\theta = \theta_1, \theta_2, \theta_3, \dots \dots \dots \dots$
4. Then write asymptote as $r\sin(\theta-\theta_1) = \frac{1}{f^{'}(\theta_1)}, r\sin(\theta-\theta_2) = \frac{1}{f^{'}(\theta_2)}$,……………………..

IMPORTANT FORMULAS

$1. If\ \sin(\theta) = 0, then\ \theta = \frac{\pi}{4}\Big\}$

$2. If cos\theta = 0, then\ \theta = (2n+1)\frac{\pi}{2}$

$3. If sin\theta = sin\alpha\ , then\ \theta = n\pi + (-1)^n\alpha$

$4. If cos\theta = cos\alpha\ , then\ \theta = 2n\pi \pm \alpha$

$5. If tan\theta = tan\alpha\ , then\ \theta = n\pi + \alpha$

$6. \sin(n\pi + \theta) = (-1)^n \sin\theta$

$7. \cos(n\pi + \theta) = (-1)^n \cos\theta$

$8. \tan(n\pi + \theta) = tan\ \theta$

$n \in \mathrm{I}$

$Example\ \ find\ the\ asymptotes\ of\ the\ following\ polar\ curves$

$(i) r = a\tan\theta$

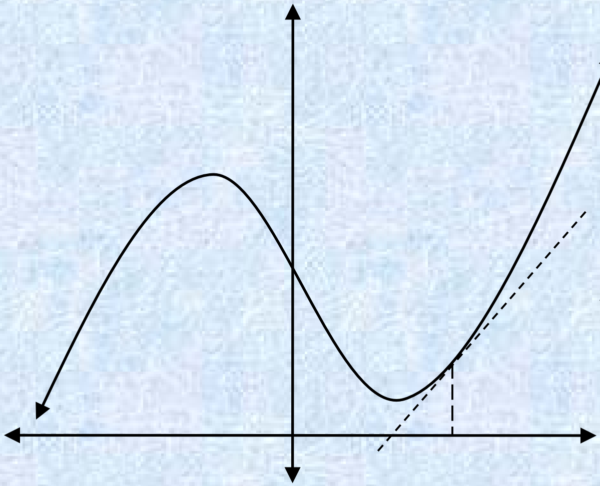
$(ii) r\sin\theta = 2\cos 2\theta$

Derivative of a function

Single-Variable Function

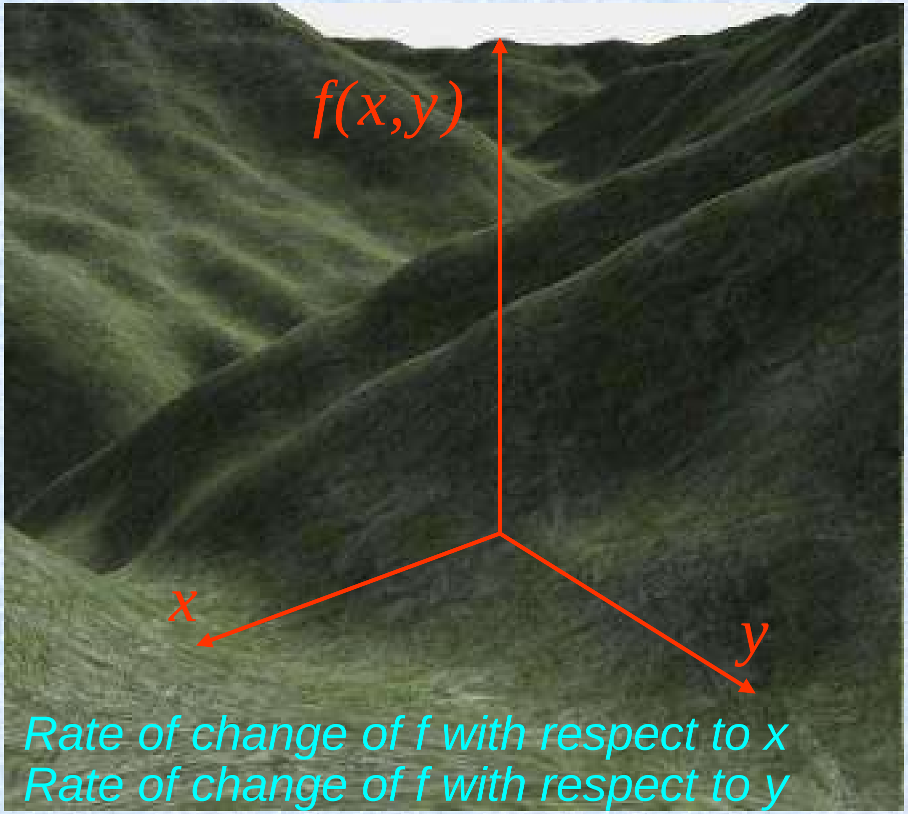
Recall how we find the derivative for a Single Variable function f(x)

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

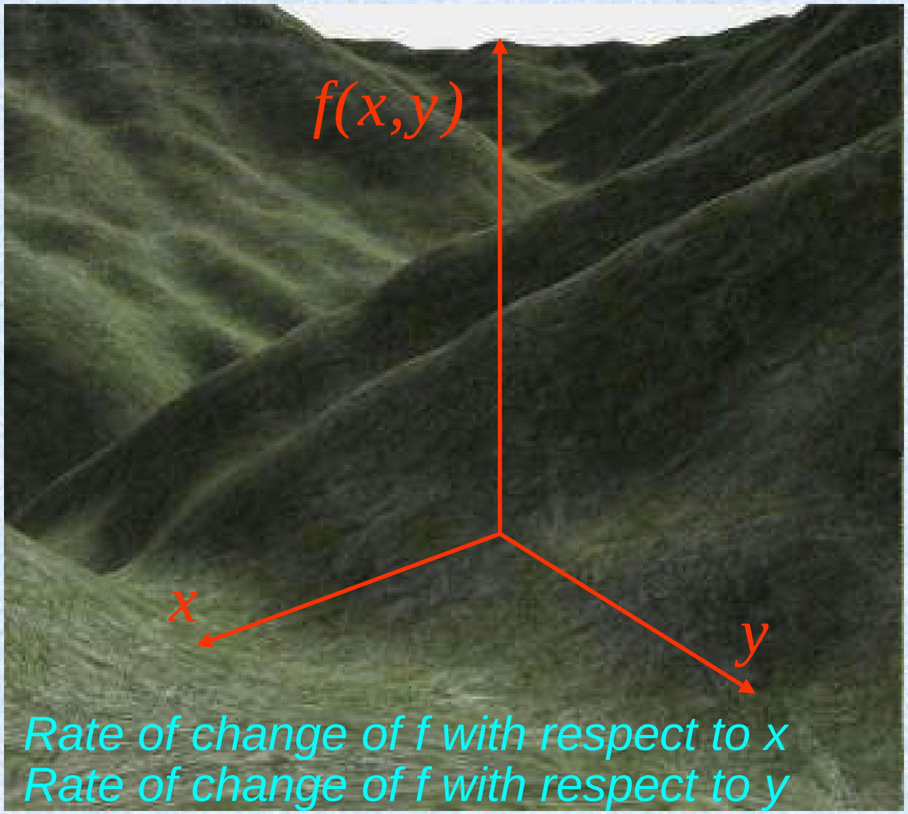


Rate of change of f with respect to x (slope)

Two-Variable Function



Partial derivatives of a function



Partial Derivative of f with respect to x
Partial Derivative of f with respect to y

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Remarks:

- It is called the Partial Derivative because it describes the derivative in one direction.
- Scripted “d”, not the regular “d” or “2”
- When differentiate f with respect to x, we treat y as if y were a constant, and vice versa.

Ex: Given $f(x,y) = x^3 - x^2y + xy + 3y^2$

$$\boxed{\text{Find} \quad \frac{\partial f}{\partial x}}$$

HERE: we treat “y” as a constant!!!!

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 - x^2y + xy + 3y^2)$$

$$= \frac{\partial}{\partial x}(x^3) - y\frac{\partial}{\partial x}(x^2) + y\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(3y^2)$$

$$= 3x^2 - y(2x) + y(1) + 0$$

$$= 3x^2 - 2xy + y$$

Assignment

If $w = x^2 - xy + y^2 + 2yz + 2z^2 + z,$

find $\dfrac{\partial w}{\partial x}, \quad \dfrac{\partial w}{\partial y}, \quad \text{and} \quad \dfrac{\partial w}{\partial z}.$

Example: A cellular phone company has the following production function for a certain product:

$$p(x,y) = 50x^{2/3}y^{1/3},$$

where p is the number of units produced with x units of labor and y units of capital.

a) Find the number of units produced with 125 units of labor and 64 units of capital.

b) Find the marginal productivities of labor and of capital.

c) Evaluate the marginal productivities at $x = 125$ and $y = 64$.

Higher-Order Derivatives

Single-Variable Function

$$f'(x) = \frac{df}{dx}$$ (derivative)

$$f''(x) = \frac{d^2 f}{dx^2}$$ (2nd derivative)

$$f'''(x) = \frac{d^3 f}{dx^3}$$ (3rd derivative)

Multi-Variable Function

$$f_x = \frac{\partial f}{\partial x}$$

(partial derivative of f wrt x)

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

(2nd partial derivative of f wrt x)

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3}$$

(3rd partial derivative of f wrt x)

Ex: Given $f(x,y) = x^3 - x^2y + xy + 3y^2$

We found $f_x = \frac{\partial f}{\partial x} = 3x^2 - 2xy + y$

Find f_{xxxx}

Find $\frac{\partial^2 f}{\partial y^2}$

Mixed Derivatives

$$f_{xy} = \left(f_x \right)_y = \left(\frac{\partial f}{\partial x} \right)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = \left(f_y \right)_x = \left(\frac{\partial f}{\partial y} \right)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

so $\qquad f_{xy} = f_{yx}$

Assignment

If $z = f(x,y) = x^2y^3 + x^4y + xe^y,$

find the following partial derivatives:

$$f_x =$$

$$f_{xx} =$$

$$f_{xy} =$$

$$f_y =$$

$$f_{yy} =$$

$$f_{yx} =$$

A Function f(x,y) is said to be homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form $x^n \emptyset\left(\frac{y}{x}\right) or\ y^n \emptyset\left(\frac{x}{y}\right)$

An alternative test for a function f(x,y) to be homogeneous of degree (or order) n is that

$$f(tx, ty) = t^n f(x, y)$$

For example, if $f(x, y) = \frac{x+y}{\sqrt{x}+\sqrt{y}}$, then

$$(i)\ f(x, y) = \frac{x(1 + \frac{y}{x})}{\sqrt{x}(1 + \sqrt{\frac{y}{x}}} = x^{1/2} \emptyset\left(\frac{y}{x}\right)$$

➡ f(x,y) is a homogeneous function of degree ½ in x and y.

Similarly, a function f(x,y,z) is said to be homogeneous of degree n in the variables x,y,z if

$$f(x, y, z) = x^n \emptyset\left(\frac{y}{z}, \frac{z}{x}\right) \quad or \quad y^n (\emptyset) \left(\frac{x}{y}, \frac{z}{y}\right) \quad or \quad z^n \emptyset\left(\frac{x}{z}, \frac{y}{z}\right)$$

Alternative test is f(tx,ty,tz)= t^n f(x,y,z)

Euler's Theorem on Homogeneous Functions

If u is a homogeneous function of degree n in x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Since u is a homogeneous function of degree n in x and y , it can be expressed as $u = x^n\, f\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = nx^{n-1}\, f\left(\frac{y}{x}\right) = \, x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = nx^n\, f\left(\frac{y}{x}\right) - \, x^{n-1}\, y\, f'\left(\frac{y}{x}\right) \qquad (i)$$

$$Also \qquad \frac{\partial u}{\partial y} = \, x^n\, f'\left(\frac{y}{x}\right)\frac{1}{x} = \, x^{n-1} f'\left(\frac{y}{x}\right)$$

$$y\frac{\partial u}{\partial y} = \, x^{n-1}\, y\, f'\left(\frac{y}{x}\right) \qquad (ii)$$

Adding (i) and (ii), we get $x\frac{\partial u}{\partial x}+\ y\frac{\partial u}{\partial y}=nx^n f\left(\frac{y}{x}\right)=nu$

If u is a Homogeneous function of degree n in x and y , then $x^2\frac{\partial^2 u}{\partial x^2}+$ $2xy\ \frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

$$Example\ 1.if\ u\ \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}\ ,prove\ that\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\ \frac{\partial^2 u}{\partial x\partial y}+\ y^2\frac{\partial^2 u}{\partial y^2}=\ -\ \frac{\sin u\cos 2u}{4\ cos^3 u}$$

Composit functions

$$(i)if\ u=f(x,y)\ where\ x=\emptyset(t),y=\varphi(t)$$

Then u is called a composit function of t and we can find du/dt

$$(ii)\ if\ z=f(x,y)where\ x=\emptyset(u,v),y=\varphi(u,v)$$

Then z is called a composite function of u and v so that we can find $\partial z/_{\partial u}\quad and\ \partial z/_{\partial v}$

Cor. 1. If u=f(x,y,z) and x,y,z are function of t, then y is a composite function of t and $\frac{du}{dt} = \frac{\partial u}{\partial x} . \frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{dy}{dt} + \frac{\partial u}{\partial z} . \frac{dz}{dt}$

Cor. 2. If z = f(x,y) and x and y are the functions of u and v, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial u} \quad ; \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial v}$$

Cor. 3. If u=f(x,y) where y= $\emptyset(x)$ then since x = $\varphi(x)$, u is a composite function of x

$$\frac{du}{dx} = \frac{\partial u}{\partial x} . \frac{dx}{dx} + \frac{\partial u}{\partial y} . \frac{dy}{dx} \quad => \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} . \frac{dy}{dx}$$

Cor. 4. If we are given a implicit function f(x,y) = c , then u=f(x,y) where u=c using cor. 3 , we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} . \frac{dy}{dx}$$

But du/dx=0

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} . \frac{dy}{dx} = 0 \qquad or \qquad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} . \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = - \frac{f_x}{f_y}$$

Hence the differential coefficient of f(x,y) w.r.t x is $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} . \frac{dy}{dx}$

Cor 5. If f(x,y) = c, then by cor 4, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$Differentiating\ again\ w.r.t.x, we\ get$$

$$\frac{d^2u}{d^2x} = -\frac{f_y\frac{d}{dx}(f_x) - f_x\frac{d}{dx}(f_y)}{f_y^2} = -\frac{f_y\left[\frac{\partial f_x}{\partial x} + \frac{\partial f_x}{\partial y}\cdot\frac{dy}{dx}\right] - f_x\left[\frac{\partial f_y}{\partial x} + \frac{\partial f_y}{\partial y}\cdot\frac{dy}{dx}\right]}{f_y^2}$$

$$= -\frac{f_y\left[f_{xx} - f_{yx}\cdot\frac{f_x}{f_y}\right] - f_x\left[f_{xy} - f_{yy}\cdot\frac{f_x}{f_y}\right]}{f_y^2}$$

$$= -\frac{f_{xx}f_y^2 - f_xf_yf_{xy} - f_xf_yf_{xy} - f_{yy}f_x^2}{f_y^3}$$

$$Hence\ \frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} - f_{yy}f_x^2}{f_y^3}$$

$Example\ 1. If\ z = 2xy^2 - 3x^2y\ and\ f\ increases\ at\ the\ rate\ of\ 2\ cm\ per\ second\ when\ it$

$passes\ through\ the\ value\ x = 3cm, show\ that\ if\ y\ is\ passing\ through\ the\ value\ y = 1\ cm,$

$y\ must\ be\ decreasing\ at\ the\ rate\ of\ 2\frac{2}{15}\ cm\ per\ second, in\ order\ that\ z\ shall\ remain\ constant.$

$Example\ 2. if\ u\ is\ a\ homogeneous\ function\ of\ nth\ degree\ in\ x, y, z,\quad prove\ that$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

$Example\ 3. Find\frac{dy}{dx}, when$

$(i)\ x^y + y^x = c \qquad (ii)\quad (\cos x)^y = (\sin y)^x$

NPTEL LINKS FOR REFERENCE

Partial derivatives	http://nptel.ac.in/courses/122101003/31
Partial derivatves and euler th.	www.**nptel**.ac.in/courses/122101003/downloads/Lecture-31.pdf.

JOCOBIANS

If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u,v with respect to x,y and is denoted by symbol $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$

Simolarly, if u,v,w be the function of x,y,z, then the Jacobian of u,v,w with respect to x,y,z is

$$J\left(\frac{u,v,w}{x,y,z}\right) \quad or \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of JACOBIANS

1. If u,v are functions of r,s where r, s are functions of x,y, then
$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)} \qquad [\, chain\ Rule\ for\ Jacobians]$$
2. If J_1 is the Jacobian of u,v, with respect to x,y and J_2 is the Jacobian of x,y with respect to u,v, then J_1J_2 =1 i.e., $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(u,v)} = 1$

$$Example\ 1.\ If\ x = r\ sin\theta\ cos\emptyset\ , z = r\cos\theta\ , show\ that\ \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\ \sin\theta$$

MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

- A function $f(x,y)$ is said to have a maximum value at $x = a$, $y = b$ if $f(a,b) > f(a+h,b+k)$, for small and independent values of h and k, positive or negative.
- A function $f(x,y)$ is said to have a minimum value at $x = a$, $y = b$ if $f(a,b) < f(a+h,b+k)$, for small and independent values of h and k, positive or negative.

RULE TO FIND THE EXTREME VALUES OF A FUNCTION

Let z = f(x,y) be a function of two variables

(i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(ii) Solve $\frac{\partial z}{\partial x}$ =0 and $\frac{\partial z}{\partial y}$ =0 simultaneously.

Let (a,b); (c,d)….. Be the solutions of these equations.

(iii) For each solution in step (ii), find r $\frac{\partial^2 z}{\partial x^2}$

s = $\frac{\partial^2 z}{\partial x \partial y}$, t = $\frac{\partial^2 z}{\partial y^2}$

(iv) (a) If $rt\ s^2 > 0$ and $r < 0$ for a particular solution (a,b) of step (ii),then z has a maximum value at (a,b).

(b)) If $rt\ s^2 > 0$ and $r > 0$ for a particular solution (a,b) of step (ii),then z has a minimum value at (a,b).

(c) If $rt\ s^2 < 0$ for a particular solution (a,b) of step (ii),then z has no extreme value at (a,b)

(d) If $rt\ s^2 = 0$, the case is doubtful and requires further investigation.

ASSIGNMENT

1. Examine the extreme values of $x^2 + y^2 + 6x + 12$
2. Find the points on the surface $z^2 = xy + 1$ nearest to the origin.
3. A rectangular box open at the top, is to have a volume of 32 c.c. Find the dimensions of the box requiring least material for its construction.
4. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum.

Differentiation Under Integral SIGN

If a function f(x,α) of the two variables x and α, α being called parameter, be integrated w.r.t. x between limits a and b, $\int_a^b f(x,\alpha)dx\ is\ a\ function\ of\ \alpha$. for example,

$$\int_0^{\frac{\pi}{2}} \sin\alpha\, dx = -\left[\frac{\cos\alpha}{\alpha}\right]_0^{\pi/2} = -\frac{1}{\alpha}\left(\cos\frac{\pi}{2}\,\alpha - 1\right)$$

$$= \frac{1}{\alpha}\left(1 - cos\frac{\pi}{2}\,\alpha\right)$$

$$thus\ in\ general\ \int_a^b f(x,\alpha)dx = F(\alpha)$$

$\boldsymbol{Leibnitz's\ \ Rule}$

$If\ f(x,\alpha) and\ \frac{\partial}{\partial x}\ [f(x,\alpha)] be\ continous\ functions\ of\ x\ and\ \alpha,$ then $\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)dx\right]$

$= \int_a^b \frac{\partial}{\partial x}\ [f(x,\alpha)]\ dx\ where\ a\ and\ b\ are\ constants\ independent\ of\ \alpha.$

$Example\ 1. Evaluate\ \int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)}\ dx\ (a \geq 0) by\ applying\ differentiation\ under\ the\ Integral\ sign.$

$Example\ 2. evaluate\ \int_0^{a} \frac{\log(1+ax)}{1+x^2}\ dx\ and\ hence\ show\ that\ \int_0^{1} \frac{\log(1+x)}{1+x^2}\ dx$

$= \frac{\pi}{8}\log 2$