Lecture - 1

Thursday, 28 July 2016 (15:20 - 16:10)

Puzzles : Monty Hall Show, Coin Tossing Game

1 The Monty Hall Show

The Monty Hall problem is based on the American reality show - "Let's make a deal" and is named after its host, Monty Hall.

Consider a scenario where there are 3 doors. One door contains BMW car and other 2 contain goats. Your goal is to choose one door and if the door has BMW hidden inside, you win.

In this case, probability of winning a BMW = 1/3, probability of losing=2/3.

But, the Monty Hall Game introduces a twist here.

Suppose Monty Hall enters the scene. He asks you to choose one door. You choose. After asking the choice of your door, out of the remaining two doors, he opens the one which has goat[1]. After showing you, out of the remaining two doors, which has the goat, he asks you if you want to change your choice and choose the third gate which is unopened and previously not chosen by you. This has been shown in Figure **??**

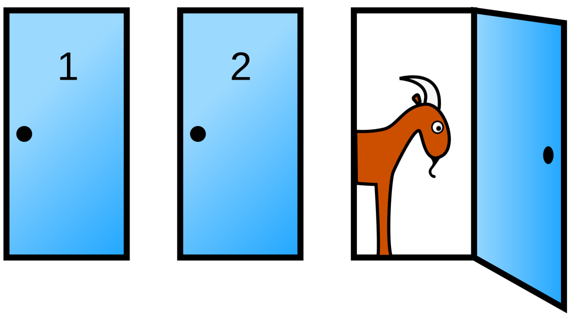

Fig. 1: The Monty Hall Game

SHOULD YOU SWAP? Does swapping make any difference to your probability of winning?

Case 1: You do not swap: So, you have chosen one door and after knowing which of the other two doors has a goat, you do not swap your choice. So, this extra knowledge does not matter to your answer. Hence, Probability (Win) = 1/3 and Probability (Losing) = 2/3

[1] Monty knows which door has what. That's why he is capable of opening a door which has a goat hidden.

Case 2: You swap: Case 2.a : You have chosen the door hiding a goat previously: Probability (You have chosen the door hiding goat) = 2/3. Now Monty shows you the other door having a goat. So, when you swap, you definitely pick the door having BMW. Hence, your probability of winning = 2/3.

Case 2.b : You have chosen the door hiding BMW previously: Probability (You have chosen the door hiding BMW) = 1/3. Now Monty shows you the other door having a goat. So, when you swap, you definitely pick the door having another goat. Hence, your probability of losing= 1/3.

Thus, you should always swap.

Quick Brainteaser:What is the probability of winning if participant tosses an unbiased coin. If head comes, she swaps, else does not swap?
Answer:1/2

2 The Coin Tossing Game

The Game: In a contest, a participant tosses a coin until he gets first head. He wins cash amount of $100 \times number\ of\ coins\ tossed$. What is the expected value of cash price won by the participant?

Answer: Let X be a random variable which counts the number of tosses to get the first head.

We know that X can take the values 1, 2, 3, 4.......
$E[X]\ =\ 1 \times pr(X = 1) + 2 \times pr(X = 2) + 3 \times pr(X = 3) + ...$

$= \sum_{i=1}^{\infty} i \times pr(X = i)$

Since, $pr(X = i) = (1/2)^i$

Hence, $E[X]\ = \sum_{i=1}^{\infty} i/2^n$

$= 1/2 + (2/2^2) + (3/2^3) + (4/2^4) + ...\ =\ \alpha$, say

Now, $\alpha = 1/2 + (2/2^2) + (3/2^3) + (4/2^4) + ...(1)$

$\alpha/2 = (2/2^2) + (3/2^3) + (4/2^4) + ...(2)$

Subtract 2 from 1

$\alpha/2 = (1/2) + (1/2^2) + (1/2^3) + ...$

It can be seen that it is a geometric progression a, ar, ar^2, ar^3......, with $a = 1$ and $r = 1/2$.

We know, that the sum of an infinite Geometric Progression = $\frac{a}{1-r}$ = $\frac{1/2}{1-1/2}$ (in this case) = 1.

Hence, $\alpha/2\ = 1$

$\alpha = 2$

$E[X] = 2$
(3)

So, the expected amount of money won by the participant = 2×100 \$ = 200 \$.

Now, we know that the expected amount of money won by the participant is 200 \$. But, when the game is actually played, the participant can win 100 \$ in one case, yet 500 \$ in other case, yet 10000 \$ in other case. Hence, it is important to look at the standard deviation of the random variable X, in addition to its expected value.

The formula for Standard Deviation, $\sigma(X)$, is given as :

$\sigma(X)$= $\sqrt{(E[X-\mu])^2}$, where $\mu = E[X]$

$= \sqrt{E[X^2 - 2X\mu + \mu^2]}$

$= \sqrt{E[X^2] - 2E[X]\mu + E[\mu^2]}$

$= \sqrt{E[X^2] - 2\mu^2 + \mu^2}$

$= \sqrt{E[X^2] - \mu^2}$

$= \sqrt{E[X^2] - (E[X])^2}$

Hence, $\sigma(X)$= $\sqrt{E[X^2] - (E[X])^2}$ (4)

X^2 is also a random variable, which takes the values $1^2, 2^2, 3^2, 4^2$..........

$Pr(X^2 = 1^2) = Pr(X = 1) = 1/2$

$Pr(X^2 = 2^2) = Pr(X = 2) = 1/2^2$

... and so on

According the the expectation formula, $E[X^2] = 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{2^2} + 3^2 \times \frac{1}{2^3} + ... = \alpha$ say

$\alpha \ = \ \frac{1^2}{2^1} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + ...$ (5)

$\alpha/2 \ = \ \frac{1^2}{2^2} + \frac{2^2}{2^3} + \frac{3^2}{2^4} + \frac{4^2}{2^5} + ...$ (6)

Subtract (6) from (5)

$\alpha/2 \ = \ \frac{1^2}{2^1} + \frac{2^2-1^2}{2^2} + \frac{3^2-2^2}{2^3} + \frac{4^2-3^2}{2^4} + ...$

or, $\alpha/2 \ = \ \frac{1^2}{2^1} + \frac{(2+1)(2-1)}{2^2} + \frac{(3+2)(3-2)}{2^3} + \frac{(4+3)(4-3)}{2^4} + ...$

or, $\alpha/2 \ = \ \frac{1}{2^1} + \frac{3}{2^2} + \frac{5}{2^3} + \frac{7}{2^4} + ...$ (7)

Divide by 2

or, $\alpha/4 \ = \ \frac{1}{2^2} + \frac{3}{2^3} + \frac{5}{2^4} + \frac{7}{2^5} + ...$ (8)

Subtract (8) from (7)

$\alpha/4 \ = \ \frac{1}{2} + \frac{2}{2^2} + \frac{2}{2^3} + \frac{2}{2^4} + ...$

or, $\alpha/4 \ = \ 1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + ..$

or, $\alpha/4 \ = \frac{3}{4}$, (applying the formula for sum of Geometric Progression)

$\alpha = 6$,

$E[X^2] = 6$ (9)

Putting 3 and 9 in 4, $\sigma(X)= \sqrt{6-2^2}$

$= \sqrt{2}$
Brain Teaser: 1. State whether series is convergent $\sum_{n=1}^{\infty} n/2^n$
2. State whether the series is convergent $\sum_{n=1}^{\infty} 1/n$