Revealed Preference Tests for Price Competition in Multi-product Differentiated Markets[*]

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January 13, 2020

Assumptions on competitive structure are often crucial to estimate marginal costs and to obtain counterfactual predictions. In this paper, tests for price competition among multi-product firms are introduced. The tests are based on firm's revealed preference (in other words, revealed profit function). In contrast to other approaches based on estimated demand functions such as the conduct parameter estimation, those tests do not require any IVs even though the models can accommodate structural error terms. In this paper, I employ a class of demand structure introduced by Nocke and Schutz (2018), discrete-continuous choice model, which nests the multinomial logit demand and CES demand functions. Even though discrete-continuous choice model has IIA property, any data on price and quantity can rationalize the price competition under discrete-continuous choice model and increasing marginal cost. By adding more assumptions on demand function, such as logit, CES, or *co-evolving* and *log-concave* property, we can obtain some falsifiable restrictions.

1 Introduction

In the literature of Industrial Organization, we often assume specific competitive structures, such as price competition or quantity competition. In empirical research, such assumptions on competitive structure are crucial in many cases. For instance, we often back out marginal costs from the first order conditions based on estimated demand functions and competitive structures. Results of counterfactual analysis, which often provides the main policy implication in research with structural models, also depends on the imposed competitive structures. Even though we can obtain estimates

[*]An earlier version of this paper has been submitted as 2nd-year-paper.

of parameters in a structural model, which fits to data the best, it is still possible that the structural model itself does not fit to the data. In other words, data might not rationalize the model under any possible parameters. Furthermore, the model might not be rationalized by any realizations of structural error terms. This happens because some data points can be out of the support of the structural model. In this paper, I provide a systematic way to detect such inconsistency between data and price competition among single/multi-product firms under a class of demand structure.

For consumer's behavior, Afriat (1967) shows that finite data satisfies GARP if and only if it is rationalized by utility maximization given a finite set of price vectors. In other words, if the data violates GARP, then it cannot be explained by any (locally non-satiated) utility functions. Brown and Matzkin (1996) extend this idea to the general equilibrium framework. Carvajal et al. (2013) apply the idea to Cournot competition, and show that the Cournot rationalizability can be checked by the existence of parameters which satisfy some inequalities. Carvajal et al. (2014) introduced a few variants of Carvajal et al. (2013); a test for multi-market-contact Cournot competition, and a test for a price competition in a differentiated market. However, for the price competition, they focus on a competition where each firm produces a single product, while price competition with multi-product firms are often examined in the empirical IO literature (e.g., Berry et al. (1995), Goldberg (1995)). One of the main difficulties to extend Carvajal et al. (2014)'s test to competition among multi-product firms arises from substitution effects among products produced by the same firm (or, cannibalization effects). We can circumvent such a difficulty by employing an important class of demand structure, discrete-continuous choice model introduced by Nocke and Schutz (2018), which nests the multinomial logit demand function and CES demand function as special cases.

In order to test the competitive structure, we can also estimate the *conduct parameter*. Bresnahan (1982) shows that we can identify the conduct parameter in an industry if there are rotation of demand functions over time. (See Bresnahan (1989) for estimations and its applications.) Recently, some researchers estimate how much firms internalize other firms profits, which is closely related to the conduct parameter (e.g., Miller and Weinberg (2017) and Sullivan (2016)). Alternatively, if we have data on cost structure, we could compare marginal costs backed out from the model and the actual cost data since different competitive structures give different FOCs, which returns different estimates of marginal costs (e.g., Wolfram (1999)). The revealed preference test examined in this paper provides an alternative approach with advantages and disadvantages. First advantage is that the revealed preference test in this paper does not require any IVs while both the estimation of conduct parameter and the approach by Wolfram (1999) require appropriate IVs. The reason why we don't need IVs is that we don't to estimate parameters to test the model, but check restrictions valid for any parameter values. It is analogous to Afriat (1967)'s theorem which characterize data restrictions satisfied as long as consumers maximize their own utility, regardless of the specification of the underlying utility function. Second, we only need the market level price and quantity data, but not other characteristics, to implement the test. Due to this parsemoneous data requirement, this test could be used as a pretest/sanity-check before detailed estimation.

A disadvantage is that the test is a joint test of competitive structure and demand/cost functions. Therefore, rejection of the model might imply either other types of competition under a discrete-continuous demand structure, price competition under other demand functions, or other competitions under other demand functions. Second, even though discrete-continuous choice model is general enough to includes the logit demand function and CES demand function as special cases, it still has IIA property. Therefore, the main theorem in this paper does not hold for random coefficient logit model (e.g., Berry et al. (1995)). Another issue is that, in the tested model, cost functions are assumed to be invariant over time, even though it does not imply the constant marginal costs since the cost functions are assumed to be convex while time-invariant. Therefore, the test should be implemented for short panel data where cost structure is not supposed to change during the time range. In practice, if a researcher have a long panel data, then data can be cut into many short panels, the test can be implemented to each short data, and the rejection ratio can be reported (e.g., Carvajal et al. (2014)). This issue is alleviated in Section 3 by incorporating the observed cost shifter.

In terms of the power of tests, any data satisfy the condition to rationalize the price competition under the general discrete-continupus demand function. This is a contrasting result to Carvajal et al. (2013) and Carvajal et al. (2014). The key difference is that they consider demand changes due to shocks common over the different firms implicitly and explicitly.[1] Naturally, by imposing the

[1]Carvajal et al. (2013) consider the Cournot competition in the homogenous good market, where the demand shock is common among all the firms. Carvajal et al. (2014) consider the multi-market contact Cournot competition and and a differentiated price competition. In the differentiated price competition, they introduce additional restrictions which captures the idea of the common demand shock.

similar additional restriction, which is compatible with the discrete-continuous demand functions, we can obtain falsifiable restrictions. We can also obtain falsifiable restrictions by restricting the underlying demand function into a class of discrete-continuous demand function, which still nests both logit and CES as its special cases. This also implies the falfiability of the price competition under the logit or the CES demand function.

In general, the restrictions can be checked by evaluating a loss function similar to ones used for the moment inequalities, so, in principle, it shares some computational issues with the the moment inequalities. However, by focusing on the logit demand function and considering a slightly modified data requirement, which must be always satisfied when researchers estimate logit demand function, we can characterize the restriction as a set of linear constraints over parameters. Then, we can implement the test through standard algorithms for linear constraints.

The remaining of the paper is organized as follows. In section 2, I introduce the main model and its special cases. In this section, I first exemplify a revealed preference test under a logit demand function, and then, I formalize and generalize the result. In section 3, I discuss some extensions of the test with (i) additional demand restrictions discussed in earlier research, (ii) observed cost shifters, and (iii) possibility of collusive conduct among firms. In Section 4, I discuss algorithms for the tests, and I summarize in section 5.

2 The model

In this paper, I consider a standard framework for a competition in a differentiated market, where each firm produces different products. Each products can be similar but not completely same. Demand functions are assumed to change over time, potentially because of change in consumer's taste or product's characteristics (which might be observed or unobserved by the econometrician). I denote $\mathcal{J} = \{1, 2, ..., J\}$ as a set of products and $Q_{j,t} : R_+^J \to R_+$ as a demand function of product $j \in \mathcal{J}$ at time $t \in \{1, ..., T\}$. The demand is assumed to be in a class of *discrete/continuous* model explained later. Firm $f \in \{1, ..., F\}$ produces a set of products $\mathcal{J}_f \subset \mathcal{J}$ s.t. $\mathcal{J}_f \cap \mathcal{J}_g = \phi$ for $f \neq g$ and denote $J_f = |\mathcal{J}_f|$. A cost function of product $j \in \mathcal{J}$, $C_j : R_+ \to R$, is assumed to be convex and twice continuously differentiable. In this section, I focus on time-invariant cost functions, which plays the similar role as time-invariant preference in Afriat (1967).[2] Then, the profit function for firm f at time t is written as $\pi_{f,t}(p) = \sum_{j\in\mathcal{J}_f}\{Q_{j,t}(p)\,p_j - C_j(Q_{j,t}(p))\}$.

In the following, I mainly utilize the FOC of the profit functions and cost convexity to derive testable data restrictions, which should be satisfied regardless of the level of parameters and structural error terms. Using the profit function defined above, FOC w.r.t. p_j is written as

$$0 = Q_{j,t}(p) + \sum_{k\in J_f}\{p_k - C'_k(Q_{k,t}(p))\}\frac{\partial Q_{k,t}(p)}{\partial p_j}.$$

2.1 Example: Logit Demand Function

Before going to the main result with a general specification, I exemplify that some data cannot be explained by price competition with the logit demand function, which is a special case of the discrete/continuous model. By using a logit demand function, $Q_{j,t}(p) = M_t\frac{\exp(v_{jt}-\alpha p_{jt})}{1+\sum_k\exp(v_{kt}-\alpha p_{kt})}$ for some M_t, $\alpha\in R_+$ and $(v_{j,t})_{j\in\mathcal{J}}\in R^J$, the first order condition is rewritten as follows:[3]

$$0 = Q_{j,t}(p) - \{p_j - C'_j(Q_{j,t}(p))\}\,\alpha Q_{j,t}(p) + \sum_{k\in J_f}\{p_k - C'_k(Q_{k,t}(p))\}\frac{\alpha}{M_t}Q_{k,t}(p)\,Q_{j,t}(p)$$

By rearranging it, we obtain the following equation:

$$p_j - C'_j(Q_{j,t}(p)) = \frac{1}{\alpha} + \frac{1}{M_t}\sum_{k\in J_f}\{p_k - C'_k(Q_{k,t}(p))\}\,Q_{k,t}(p).$$

In the above equation, RHS is common among goods produced by the same firm. Therefore, at the equilibrium,

$$p_j - C'_j(Q_{j,t}(p)) = p_k - C'_k(Q_{k,t}(p)) \text{ for any } j,\,k\in\mathcal{J}_f \tag{1}$$

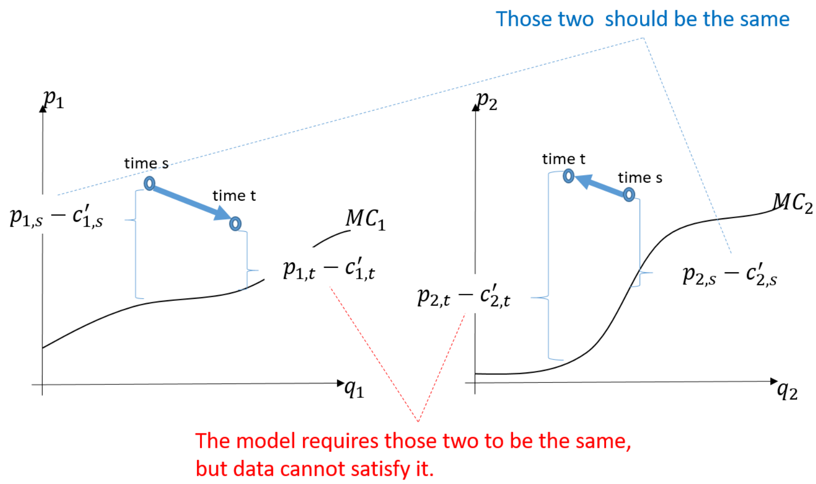
must hold. In this paper, I call this property as "common mark-up property." [4] By combining the common mark-up property with the increasing marginal cost assumption, the model is rejected under the following data ; $(\bar{p}_{j,\tau},\,\bar{q}_{j,\tau})_{j=1,2,\,\tau=s,t}$ s.t. $\{1,2\}\subset\mathcal{J}_f$, $\bar{p}_{1,s} > \bar{p}_{1,t}$, $\bar{p}_{2,s} < \bar{p}_{2,t}$, $\bar{q}_{1,s} < \bar{q}_{1,t}$, and $\bar{q}_{2,s} > \bar{q}_{2,t}$. That is, the price and quantity of good 1 and those of good 2 moves in the opposite

[2] In section 3, I discuss an extension with time variant and observed cost shifters.

[3] The following argument holds more generally with time variant α_t, instead of time-invariant α. I use time–invariant α simply because it is used more commonly in the literature.

[4] More generally, Nocke and Schutz (2018) call it as "common $\iota-$markup property" under the discrete/continuous choice model.

Figure 1: Example: Logit Demand Function



direction (see Fig. 1). Suppose that data satisfy eq.(1) at time s. Then, if marginal costs are (weakly) increasing in own quantity, then $\bar{p}_{1,t} - C_1'(\bar{q}_{1,t}) < \bar{p}_{1,s} - C_1'(\bar{q}_{1,s}) = \bar{p}_{2,s} - C_2'(\bar{q}_{2,s}) < \bar{p}_{2,t} - C_2'(\bar{q}_{2,t})$. Therefore, eq.(1) cannot be satisfied at time t. Thus, this data $(\bar{p}_{j,\tau}, \bar{q}_{j,\tau})_{j=1,2,\, \tau=s,t}$ cannot be explained by (a repetition of static) price competition under logit demand functions. It means that this data cannot be explained by any sets of parameters, α, m_t, and $v_{j,t}$ and non-parametric cost functions, $C_j(\cdot)$. This highlights a few important features of those restrictions on data.

No need for IVs In the logit specification, we can understand the revealed preference test in comparison with an alternative procedure to check the competitive structure. A parameter of demand function, α, is often estimated from aggregated data (using IVs to deal with unobserved heterogeneity potentially correlated with prices), and δ's are backed out from the first order condition. Potentially, we could check whether the obtained δ's are reasonable or not. In the revealed preference test, alternatively, the similar procedure is done for any possible $\alpha > 0$, instead of estimate of α pinned down by IVs. Therefore, we do not need IVs even if there are unobserved heterogeneity potentially correlated with price or some other variables.

Interpretation of rejection Note that the rejection or acceptance is not probabilistic even if the model has (only) the structural error term in the logit demand function. When we estimate logit demand functions from aggregate data, $v_{j,t}$ is decomposed to $v_{j,t} = x'_{j,t}\beta + \xi_{j,t}$ where $x_{j,t}$ is a vector of product j's observed characteristics, $\xi_{j,t}$ is unobserved characteristics, and β is a vector of parameters. In the logit demand estimation, $\xi_{j,t}$ is treated as a structural error term. However, eq.(1) should be satisfied regardless of the realization of $\xi_{j,t}$ as long as firms are competing in prices under logit demand functions. (Recall that I did not put any assumptions on $v_{j,t}$.) This is because the realization of $(\xi_{j,t})_{j\in\mathcal{J}}$ is assumed to be known to each firm (but not to the econometrician), which is often assumed in the literature of empirical IO. Therefore, any rejection of the model is not in probability, and cannot be attributed to a peculiar realization of structural error terms.

(No) data restrictions by each assumption Revealed preference tests in this article are joint tests of the demand and cost specifications and the competitive structure. However, it is worth noting that each of them alone is not rejected by any data $\{\bar{p}, \bar{q}\}$, but those can be rejected only if combined together. If we only assume logit demand function, for any data $(p_{j,t}, q_{j,t})_{j=1,2}$ at each t, we can back out the corresponding $(v_{j,t})_{j\in\mathcal{J}}$ by the inversion of market share function as in Berry (1994). Thus, logit demand can fit to the data since any changes in data over time can be captured by changes in $(v_{j,t})_{j\in\mathcal{J}}$ over time. As to the assumption of price competition and cost functions, I explain in Section 2.2 that any data can be rationalized by price competition under a more general demand function and convex time-invariant cost function. This emphasize that each assumption in this article is not trivially restrictive especially when we have only price and quantity data.

In the following part, I provide a set of inequalities as a systematic way to detect data inconsistent with price competitions, and show that such conditions are actually sufficient for rationalization by the price competition. Instead of the logit demand function, I employ a class of demand functions by Nocke and Schutz (2018), which nests the logit demand function and CES demand function.

2.2 Discrete-Continuous Demand Function

In the following, I employ the discrete-continuous demand function introduced by Nocke and Schutz (2018), where the demand function for product j is written as $Q_j(p) = m\frac{-h'_j(p_j)}{h_0+\sum_{k\in I} h_k(p_k)}$ where $h_j(\cdot)$ is decreasing and log-convex for every j, and m is a positive constant. An important example of this demand function is the logit model: $h_j(p_j) = \exp(v_j - \alpha p_j)$ and $m = M/\alpha$ where $v_j \in R$

is the value of goods j, $\alpha > 0$ is a coefficient for prices, $M > 0$ is the size of the market, and h_0 is the exponentiated value of the outside option.[5] Another important example is CES model: $h_j(p_j) = a_j p_j^{1-\sigma}$ and $m = \frac{I}{\sigma-1}$, where I is the income level of the consumer and σ is the elasticity of substitution.

In this paper, I utilize the fact that we can express the partial derivatives of discrete-continuous demand function in a simple form;

$$\begin{aligned}
\frac{\partial Q_{k,t}(p)}{\partial p_j} &= m\frac{-h'_{k,t}(p_k)}{\left(h_{0,t} + \sum_{k\in I} h_{k,t}(p_k)\right)^2}\left(-h'_{j,t}(p_j)\right). \\
&= m^{-1} Q_{k,t}(p) \cdot Q_{j,t}(p) \quad \forall k \neq j
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial Q_{j,t}(p)}{\partial p_j} &= m_t \frac{-h''_{j,t}(p_j)}{h_{0,t} + \sum_{k\in I} h_{k,t}(p_k)} + m_t\left(\frac{-h'_{j,t}(p_k)}{h_{0,t} + \sum_{k\in I} h_{k,t}(p_k)}\right)^2 \\
&= -Q_{j,t}(p)\frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)} + m_t^{-1}\left(Q_{j,t}(p)\right)^2 \\
&= -m_t^{-1} Q_{j,t}(p)\left\{m_t \frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)} - Q_{j,t}(p)\right\}.
\end{aligned}$$

It is worth noting that $m_t \frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)} - Q_{j,t}(p)$ is positive because of the log concavity of $h_j(\cdot)$. [6]

[5] As discussed by Nocke and Schutz (2018), discrete-continuous choice model with the outside option can be normalized to discrete-continuous choice model without the outside option, $\tilde{Q}_j(p) = m\frac{-\tilde{h}'_j(p_j)}{\sum_{k\in I}\tilde{h}_k(p_k)}$, by letting $\tilde{h}_j(p_j) = \frac{1}{J}h_0 + h_j(p_j)$. In this paper, h_0 is explicitly denoted just to explain intuition of the results in later parts.

[6] The log-concavity implies $\frac{h''_j(p)}{-h'_j(p)} > \frac{-h'_j(p)}{h_j(p)} \left(> \frac{-h'_j(p)}{h_0 + \sum h_k(p)}\right)$.

By using the above expression, the FOC w.r.t. p_j is written as follows;

$$\begin{aligned}
0 &= 1+\sum_{k\in J_f}\left\{p_k - C_k'\left(Q_{k,t}\left(p\right)\right)\right\}\frac{\partial Q_{k,t}\left(p\right)}{\partial p_j}\frac{1}{Q_{j,t}\left(p\right)} \\
&= 1-m_t^{-1}\left\{p_j - C_j'\left(Q_{j,t}\left(p\right)\right)\right\}\left\{m_t\frac{h_{j,t}''\left(p_k\right)}{-h_{j,t}'\left(p_k\right)}-Q_{j,t}\left(p\right)\right\} \\
&\quad +m_t^{-1}\sum_{k\in J_f,\,k\neq j}\left\{p_k - C_k'\left(Q_{k,t}\left(p\right)\right)\right\}Q_{k,t}\left(p\right) \\
&= m_t-\left\{p_j - C_j'\left(Q_{j,t}\left(p\right)\right)\right\}m_t\frac{h_{j,t}''\left(p_k\right)}{-h_{j,t}'\left(p_k\right)}+\sum_{k\in J_f}\left\{p_k - C_k'\left(Q_{k,t}\left(p\right)\right)\right\}Q_{k,t}\left(p\right)
\end{aligned}$$

Therefore, if the data $\{\bar{p}, \bar{q}\}$ is generated by the price competition with (unknown) discrete-continuous demand function, there exists α_{jt}, δ_{jt}, which corresponds to $\frac{h_{j,t}''(\bar{p}_j)}{-h_{j,t}'(\bar{p}_j)}$ and $C_j'\left(\bar{q}_{j,t}\right)$, respectively, such that

$$0 = m_t - \left\{\bar{p}_j - \delta_{j,t}\right\}m_t\alpha_{j,t} + \sum_{k\in J_f}\left\{\bar{p}_k - \delta_{j,t}\right\}\bar{q}_{k,t}. \tag{2}$$

On the other hand, since $\delta_{j,t}$ corresponds to $C_j'\left(\bar{q}_{j,t}\right)$ and $C_j'\left(\cdot\right)$ is assumed to be increasing, $\delta_{j,t}$ must be greater than $\delta_{j,s}$ $(s\neq t)$ if $\bar{q}_{j,t}$ is greater than $\bar{q}_{j,s}$. This is summarized as an inequality;

$$0 \leq \left(\delta_{j,s}-\delta_{j,t}\right)\left(\bar{q}_{j,s}-\bar{q}_{j,t}\right). \tag{3}$$

By combining eq.(2) and eq.(3), we can obtain a set of necessary conditions for the data to rationalize the model. Furthermore, it turns out to be also sufficient conditions of the rationalization. It is summarized in the following theorem.

Theorem 1. *(Discrete-Continuous): The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under convex cost function and discrete/continuous demand function if and only if there are real numbers $\alpha_{j,t}$, $\delta_{j,t}$, m_t for any $t\in\mathcal{T}$ and $j\in\mathcal{J}$, such that the following holds;*

1. $\alpha_{j,t}>0$, $\delta_{j,t}>0$, $m_t>0$;
2. $0 = m_t - \left\{\bar{p}_{j,t}-\delta_{j,t}\right\}m_t\alpha_{j,t} + \sum_{k\in J_f}\left\{\bar{p}_{k,t}-\delta_{k,t}\right\}\bar{q}_{k,t}$; *and*
3. $0\leq\left(\delta_{j,t'}-\delta_{j,t}\right)\left(\bar{q}_{j,t'}-\bar{q}_{j,t}\right)$.

The first set of conditions comes from underlying specifications of demand and cost functions; $\alpha_{j,t} > 0$ comes from an assumption that h_j is decreasing and log-convex, $\delta_{j,t} > 0$ comes from increasing cost functions, $m_t > 0$ comes from the assumption that quantity of each goods are non-negative. The proof of sufficiency consists of two steps. First, given $\{\alpha_{j,t}\}$ and $\{\delta_{j.t}\}$ which satisfy the conditions, I re-construct demand functions $\left\{\bar{Q}_{j,\,t}\left(\cdot\right)\right\}$ and cost functions $\left\{\bar{C}_{j}\left(\cdot\right)\right\}$ to satisfy $\frac{\bar{h}''_{j,t}(\bar{p}_j)}{-\bar{h}'_{j,t}(\bar{p}_j)} = \alpha_{j,t}$ and $\bar{C}'_j\left(\bar{q}_{j,t}\right) = \delta_{j.t}$. [7]Then, data $\{\bar{p},\, \bar{q}\}$ satisfies FOC under the re-constructed demand and cost functions. In the second step, I show that the FOC is a sufficient condition for profit maximization given other firms' prices, given the re-constructed demand cost functions. It is not trivial since the profit function does not satisfy quasi-concavity. In this paper, the sufficiency is proved by the unique solution of FOC, which comes from the unique "common ι-markup" and a mapping from ι-markup to price vectors as in Nocke and Schutz (2018). See the Appendix for the full proof.[8]

2.3 Tests for more restrictive models

For more restrictive specifications, such as logit or CES demand functions, we can easily derive a necessary condition for data to rationalize the models, by simply adding restrictions on the second condition in the above tests. Sufficiency of the restriction, however, is less trivial. In the proof of the sufficiency in Theorem 1, I re-construct the demand function as $\left\{\bar{Q}_{j,\,t}\left(\cdot\right)\right\}$, which still nests the logit demand function, but not CES. Such a reconstruction is sufficient for Theorem 1 since the re-constructed demand function $\left\{\bar{Q}_{j,\,t}\left(\cdot\right)\right\}$ is in the class of the demand function we are interested in. For the test of a model with the logit demand, the similar reconstruction can be applied $\left\{\bar{Q}_{j,\,t}\left(\cdot\right)\right\}$ and the remaining is proved analogously. On the other hand, for a test of a model with CES demand, such a re-construction is not longer valid. As of today, I have not found a re-constructed CES demand function which gives the suffiiciency of FOC under arbitrary convex cost functions[9].

[7]In the proof of the sufficiency, re-constructed demand functions, $\left\{\bar{Q}_{j,\,t}\left(\cdot\right)\right\}$ can be different from the actual demand functions $\{Q_{j,\,t}\left(\cdot\right)\}$. For instance, in the proof of Theorem 1, even if data $\{\bar{p},\, \bar{q}\}$ is actually generated from CES demand, which is a special case of discrete-continuous choice model, re-constructed $\left\{\bar{Q}_{j,\,t}\left(\cdot\right)\right\}$ can be non-CES demand as long as it is another special case of discrete-continuous choice model. Further discussion is found in the next subsection.

[8]It is worth noting that the second condition is not linear in general because of an interaction of $\delta_{j,t}$ and $\alpha_{j,t}$ in contrast to Carvajal et al. (2013, 2014). It prevents us from using algorithms for linear programming. One way to implement the above test is to consider an algorithm similar to moment inequalities.

[9]A matrix corresponding to a negative semi-definite matrix in the proof of Theorem 1 would no longer be symmetric in a case of CES. Therefore, discussion about negative semi definiteness does not apply.

If we assume a constant marginal cost in addition to CES demand, we can regain the sufficiency by applying the proof of Nocke and Schutz (2018) for unique solution of FOC.

Those conditions mentioned above are articulated in the following propositions. (Summary of the specifications and results can be found in Table 1 in the Appendix.)

Proposition 1. *(Logit) The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under convex cost function and logit demand function* ***if and only if*** *there are real numbers α_t, $\delta_{j,t}$, m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;*

1. $\alpha > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_t + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$; *and*
3. $0 \leq (\delta_{j,t'} - \delta_{j,t}) (\bar{q}_{j,t'} - \bar{q}_{j,t})$.

In the above statement, α_t is allowed to vary over time for the sake of generality. We can easily replace α_t with time-invariant α. Such a simplified version is proved analogously to Proposition 1.

Corollary 1. *(CES): If the set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under convex cost function and CES demand function, then there are real numbers σ_t, $\delta_{j,t}$, m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;*

1. $\sigma_t > 1$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \frac{\bar{p}_{j,t} - \delta_{j,t}}{\bar{p}_{j,t}} m_t \sigma_t + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$; *and*
3. $0 \leq (\delta_{j,t'} - \delta_{j,t}) (\bar{q}_{j,t'} - \bar{q}_{j,t})$.

As explained earlier, this necessary condition is derived as a special case of (the necessity part of) Theorem 1. It is worth noting that the necessary condition alone works well to reject the model, but it would be harder to interpret when the model is not rejected, i.e., data may or may not be rationalized by the model. If we assume constant (and time-invariant) marginal cost, we can easily prove the sufficiency since we can apply the proof in Nocke and Schutz (2018) for unique solution of FOC.

Proposition 2. *(CES and constant MC): The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under linear cost function and CES demand function if and only if there are real numbers σ_t, δ_j, m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;*

1. $\sigma_t > 1$, $\delta_j > 0$, $m_t > 0$;
2. $0 = m_t - \frac{\bar{p}_{j,t} - \delta_j}{\bar{p}_{j,t}} m_t \sigma_t + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_j\} \bar{q}_{k,t}$

Note that δ_j does not have time index since constant and time-invariant marginal costs are imposed, instead of increasing marginal costs.

2.4 Falsifiability

Theorem 1 characterizes the necessary and sufficient condition for data to rationalize the model of price competition under discrete/continuous choice model and time-invariant convex cost function. Meanwhile, readers might wonder how restrictive the restriction on data is. It turns out that the restriction in Theorem 1 is so loose that any data can rationalize the model with the general discrete-continuous demand function. It can be surprising considering that even the general discrete-continuous choice model satisfies IIA property. Any changes of prices and quantities along a fixed discrete-continuous demand function must satisfy IIA property, while demand functions themselves are allowed to change over time in the model of Theorem 1. To be more clear about how any data satisfy the restrictions, consider the following. For any given $\delta_{j,t}$ and m_t, the remaining parameter $\alpha_{j,t}$, which characterizes the demand functions, can be determined only through the FOC w.r.t. $p_{j,t}$, independently from the FOC w.r.t. $p_{k,s}$ where $k \neq j$ or $s \neq t$. (In contrast, under the logit demand, α_t is common for all goods so that there are restrictions on data across different products.) Thus, for any data, we can find corresponding $\alpha_{j,t}$, $\delta_{j,t}$, m_t, and then, the sufficiency implies that any data is rationalized by the model.

Corollary 2. *Any data $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under convex cost function and discrete/continuous demand function.*

Even though the price competition under the general discrete-continuous choice model is not falsifiable, a model can be falsifiable under a more restrictive demand model such as logit demand function as shown in the example. One natural question is how general this falsifiability is. I can show that a subclass of discrete-continuous choice model which still nests both logit and CES is falsifiable by the similar logic as in the example. Consider discrete-continuous demand function generated by $h_{j,t}(\cdot)$ such that $\frac{h''_{j,t}(p_j)}{-h'_{j,t}(p_j)} = \frac{1}{a_t p_{j,t} + b_t}$ for some $a_t \geq 0$ and $b_t \geq 0$. Now, we can express the logit demand function by setting $a_t = 0$ and the CES by setting $b_t = 0$. I call this class of demand function as discrete-continuous model with HARA h since h is characterized analogously to hyperbolic absolute risk averse vNM utility function, which nests CARA and CRRA as special cases. First, I introduce a modified version of necessary condition for data rationalized by price competition under the modified specification. The statement is only for the necessity of the data restriction by the same reason as in Corollary 1.

Corollary 3. *If the set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under convex cost function and discrete/continuous demand function with HARA h, then there are real numbers a_t, b_t, $\delta_{j,t}$, m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;*

1. $a_t > 0$, $b_t > 0$ $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \frac{1}{a_t p_{j,t} + b_t} + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$; *and*
3. $0 \leq \left(\delta_{j,t'} - \delta_{j,t}\right)\left(\bar{q}_{j,t'} - \bar{q}_{j,t}\right)$.

Note that the modified model is falsifiable, i.e., the model can be rejected under some data. By the second condition in the set of restriction, data must satisfy $\frac{\bar{p}_{j,t} - \delta_{j,t}}{a_t \bar{p}_{j,t} + b_t} = \frac{\bar{p}_{k,t} - \delta_{k,t}}{a_t \bar{p}_{k,t} + b_t}$ for all j, $k \in \mathcal{J}_f$ and all $t \in \mathcal{T}$. Consider a case where one firm produces products 1, 2, and 3 , and data $\{\bar{p}, \bar{q}\}$ such that $\bar{p}_{j,t} = \bar{p}_{j,s} \equiv \bar{p}_j$ and $\bar{q}_{j,t} = \bar{q}_{j,s} \equiv \bar{q}_j$ for $j = 1, 2$ and for some t, $s \in \mathcal{T}$, and $\bar{p}_{3,t} < \bar{p}_{3,s}$ and $\bar{q}_{3,t} > \bar{q}_{3,s}$.

Then, the above equality is rewritten as follows:

$$\frac{\bar{p}_1 - \delta_1}{\bar{p}_1 + b_t/a_t} = \frac{\bar{p}_2 - \delta_2}{\bar{p}_2 + b_t/a_t} = \frac{\bar{p}_{3,t} - \delta_{3,t}}{\bar{p}_{3,t} + b_t/a_t} \tag{4}$$

and

$$\frac{\bar{p}_1 - \delta_1}{\bar{p}_1 + b_s/a_s} = \frac{\bar{p}_2 - \delta_2}{\bar{p}_2 + b_s/a_s} = \frac{\bar{p}_{3,s} - \delta_{3,s}}{\bar{p}_{3,s} + b_s/a_s}. \tag{5}$$

Note that equalities for goods 1 and 2 in eqs. (4) and (5) implies $b_t/a_t = b_s/a_s \equiv b/a$. Therefore, all the terms in eqs. (4) and (5) must be the same. Thus, for good 3, $\frac{\bar{p}_{3,t} - \delta_{3,t}}{\bar{p}_{3,t} + b/a} = \frac{\bar{p}_{3,s} - \delta_{3,s}}{\bar{p}_{3,s} + b/a}$ must hold.

It is a contradiction to $\bar{p}_{3,t} < \bar{p}_{3,s}$ and $\bar{q}_{3,t} > \bar{q}_{3,s}$.

3 Extensions

In this section, some extensions of the revealed preference tests are introduced: (i) additional assumptions on demand function introduced by Carvajal et al. (2014), (ii) observable cost shifters as discussed in Carvajal et al. (2014), (iii) collusive price competition, which can also work as an alternative hypothesis of the above tests.

3.1 Additional restrictions on demand

Even though the above results provides some testable restrictions, the general model is not falsifiable. We can obtain a more strict constraint by combining the demand assumption introduced by Carvajal et al. (2014). Even though it is straightforward that the additional assumption in the model provides additional constraints in data, it is less clear that discrete-continuous demand function and the additional assumptions have a non-empty intersection.

In order to define the additional restrictions, I introduce some notations first. Denote $\epsilon_{jt}(p)$: $R_{+}^{J} \to R$ as the relative decrease in the demand of good j at time t in response to an infinitesimal increase in its price. That is, given the demand function Q_{jt} for good j at time t ,

$$\epsilon_{jt}(p) = -\frac{\partial Q_{j,t}(p_j, p_{-j})}{\partial p_j}\frac{1}{Q_{jt}(p)}$$

Therefore, the own price elasticity is expressed as $p_j \epsilon_{jt}(p)$.

Then we can define the following properties of demand functions.

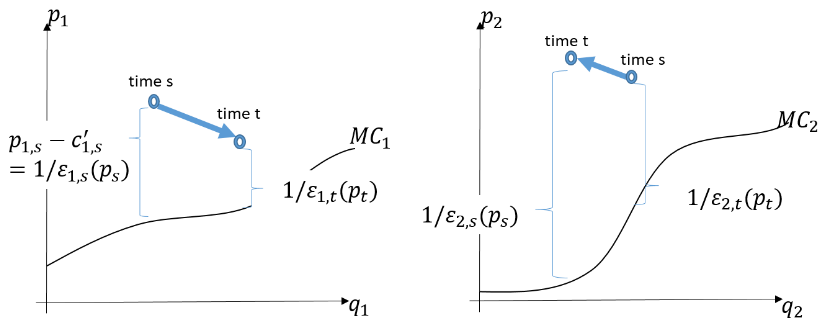
Definition: A system of demand functions satisfies *co-evolving* property if, for any s and $t \in \mathcal{T}$, either

$$\epsilon_{js}(p) \geq \epsilon_{jt}(p) \text{ for all } p \in R_{+}^{J} \text{ and all } j \in \mathcal{J}\text{, or} \tag{6}$$

$$\epsilon_{js}(p) \leq \epsilon_{jt}(p) \text{ for all } p \in R_{+}^{J} \text{ and all } j \in \mathcal{J} \tag{7}$$

The co-evolving demand property captures an idea of *common demand shock* in Carvajal et al.

Figure 2: Example: Rejection by Co-evolving Property



(2013), which is a key component to obtain non-trivial restriction on data in their work. As we can see in the above equations, if a relative slope of demand is higher for firm j in a market t, then so are the relative slopes of other firms $k \neq j$. In other words, we can construct a well defined order of demands over $\mathcal{T}$, which is common for all firms according to the relative slopes.

The power of the co-evolving property is emphasized by two products produced by different firms. Consider the same prices and quantities as the previous example, but two goods are produced by different firms; $(p_{j,\tau}, q_{j,\tau})_{j=1,2,\,\tau=s,t}$ s.t. $\mathcal{J}_1 = \{1\}$, $\mathcal{J}_2 = \{2\}$, $p_{1,s} > p_{1,t}$, $p_{2,s} < p_{2,t}$, $q_{1,s} < q_{1,t}$, and $q_{2,s} > q_{2,t}$ (see Fig. 2).[10] Since two goods are produced by different firms, eq.(1) is no longer satisfied. However, the co-evolving property gives us an alternative restriction even under the general discrete/continuous model (instead of the logit demand function). If they are single product firms, the FOC is re-written in the following form; $p_j - C'_j\left(Q_{j,t}\left(p\right)\right) = 1/\epsilon_{j,t}\left(\bar{p}_{j,t}\right)$. Since the marginal costs are increasing, we can obtain the inequality about the profit margins; $1/\epsilon_{1,s}\left(\bar{p}_s\right) = \bar{p}_{1,s} - C'_1\left(Q_{1,s}\left(\bar{p}_s\right)\right) > \bar{p}_{1,t} - C'_1\left(Q_{1,t}\left(\bar{p}_t\right)\right) = 1/\epsilon_{1,t}\left(\bar{p}_t\right)$ for firm 1. Similarly, we also have $1/\epsilon_{2,s}\left(\bar{p}_s\right) < 1/\epsilon_{2,t}\left(\bar{p}_t\right)$. Therefore, the data implies $\epsilon_{1,s}\left(\bar{p}_s\right) < \epsilon_{1,t}\left(\bar{p}_t\right)$ and $\epsilon_{2,s}\left(\bar{p}_s\right) > \epsilon_{2,t}\left(\bar{p}_t\right)$. By combining a property that $\epsilon_{j,s}\left(\cdot\right)$ is non-decreasing in own price and decreasing in other's price, we have $\epsilon_{1,s}\left(p\right) < \epsilon_{1,t}\left(p\right)$ but $\epsilon_{1,s}\left(p\right) > \epsilon_{1,t}\left(p\right)$, which is a contradiction to the co-evolving property. In the following, I refer $\epsilon_{j,t}\left(\cdot\right)$ non-decreasing in own price as *log-concave* following to Carvajal et

[10] The same logic is applied to multi-product firms just by ignoring other products.

al. (2014).[11]

Before going to a proposition, I exemplify that the discrete-continuous model and co-evolving property have a non-empty intersection. In the multinomial logit demand, the co-evolving property is satisfied when v_{jt} and v_{kt} move similarly over time. Since the logit demand function (with common M_t over t) requires $\epsilon_{jt}(p) = \alpha - \frac{\alpha}{M} Q_{j,t}(p)$, $\epsilon_{jt}(p) \geq \epsilon_{js}(p)$ holds if and only if $Q_{j,t}(p) \geq Q_{j,s}(p)$ holds. The co-evolving property under the logit demand function requires that $Q_{j,t}(p) \geq Q_{j,s}(p)$ if and only if $Q_{k,t}(p) \geq Q_{k,s}(p)$. It can be satisfied when the change of relative value of outside option dominates the change in demand functions. Thus, there is a non-empty intersection of the two properties. Log-concavity (of $Q_{j,t}(p)$) is also satisfied if $\frac{h_j''(p_j)}{-h_j'(p_j)}$ is non-decreasing in p_j. In the following proposition, I combine the discrete-continuous choice model and the co-evolving property to derive a set of necessary conditions for data to rationalize price competition.

Proposition 3. *The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under convex cost function and discrete-continuous demand function with log-concavity and co-evolving property only if there is a permutation of $\mathcal{T}$, denoted by the function $\sigma : \mathcal{T} \to \mathcal{T}$, and real numbers $\alpha_{j,t}$, $\delta_{j,t}$, m_t for all s, $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;*

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_j - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{k \in J_f} \{\bar{p}_k - \delta_{j,t}\} \bar{q}_{k,t}$;
3. $0 \leq (\delta_{j,t'} - \delta_{j,t}) (\bar{q}_{j,t'} - \bar{q}_{j,t})$; *and*
4. *if* $\bar{p}_{j,t} \geq \bar{p}_{j,s}$, $\bar{p}_{-j,t} \leq \bar{p}_{-j,s}$ *and* $\sigma(t) < \sigma(s)$, *then* $\alpha_{j,t} - m_t^{-1} \bar{q}_{j,t} \leq \alpha_{j,s} - m_s^{-1} \bar{q}_{j,s}$

See Appendix for the proof.

The last condition comes from the co-evolving property and log-concavity, which characterize the common order of $\epsilon_{jt}(p)$ over time. Under the discrete-continuous demand model; $\epsilon_{jt}(\bar{p}_t) = \frac{h_{j,t}''(\bar{p}_{j},t)}{-h_{j,t}'(\bar{p}_{j},t)} - m_t^{-1} Q_{j,t}(\bar{p}_t) = \alpha_{j,t} - m_t^{-1} \bar{q}_{j,t}$. The permutation σ is constructed to provide the common order of $\epsilon_{jt}(p)$ (if there exists). It is worth noting that co-evolving is defined by comparing $\epsilon_{jt}(p)$ and $\epsilon_{js}(p)$ for all p, but we only observe values corresponding to $\epsilon_{jt}(\bar{p}_t)$ and $\epsilon_{js}(\bar{p}_s)$, where $\bar{p}_t$ and

[11] Carvajal et al. (2014) impose another condition, *substitut*es *condition*, that $\epsilon_{j,t}(\cdot)$ is decreasing in others' prices. Under the discrete-continuous demand, however, this condition is always satisfied and plays the similar role as in the previous research.

$\bar{p}_s$ can take different values. To deal with this subtlety, inequalities "$\bar{p}_{j,t} \geq \bar{p}_{j,s}, \bar{p}_{-j,t} \leq \bar{p}_{-j,s}$" are added in the last condition. In this proposition, I proved only the necessity of the conditions. For the proof of sufficiency, I need to re-construct demand functions satisfying both discrete-continuous structure and co-evolving property from any parameters satisfying the conditions 1-4.

3.2 Observed cost shock

One of the important assumptions in the above tests is the time-invariant cost function. In reality, however, the cost functions shifts over time because of change in input prices, for instance. We can accommodate such a shift to the revealed preference tests if cost shifters are observed.

Now, assume the following cost functions: $C_j\left(q_j,\, w_j\right)$ where $\frac{\partial C_j(q_j,\, w_j)}{\partial q_j}$ is increasing both in q_j and w_j, and assume that we observe the cost shifter w in addition to price and quantity. Denote the observed price, quantity, and cost shifter as follows: $\{\bar{p},\, \bar{q},\, \bar{w}\}$ where $\bar{x} = \left(\bar{x}_1',\, ...,\, \bar{x}_T'\right)'$ and $\bar{x}_t = \left(\bar{x}_{1,t},\, ...,\, \bar{x}_{J,t}\right)'$ for $x = p,\, q,\, w$. Then, the restriction is modified as follows.

Remark 1. The set of observations $\{\bar{p},\, \bar{q},\, \bar{w}\}$ is Bertrand rationalizable under marginal cost function increasing in own quantity and a cost shifter and discrete/continuous demand function only if there are real numbers $\alpha_{j,t}$, $\delta_{j,t}$, m_t for any $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \left\{\bar{p}_{j,t} - \delta_{j,t}\right\} m_t \alpha_{j,t} + \sum_{k \in J_f} \left\{\bar{p}_{k,t} - \delta_{k,t}\right\} \bar{q}_{k,t}$; and
3. $\delta_{j,t'} \geq \delta_{j,t}$ whenever $\left(\bar{q}_{j,t'},\, \bar{w}_{j,t'}\right) > \left(\bar{q}_{j,t},\, \bar{w}_{j,t}\right)$.

In the above claim, I use a partial order for the third condition since, if $\bar{q}_{j,t'} > \bar{q}_{j,t}$ and $\bar{w}_{j,t'} < \bar{w}_{j,t}$, then we cannot tell when the marginal cost is higher. Tests for the the price competition under logit, CES, and HARA h, can be also derived analogously.

3.3 Collusive price competition

In this section, a revealed preference test of collusive price competition is discussed. Each firm is assumed to choose their own price while (partially) internalizing the effect on the other firms as in Miller and Weinberg (2017) and Sullivan (2016). More specifically, firm f maximizes the following objective function

$$\pi_f\left(p\right)=\sum_{f'\in\mathcal{F}}\left[\phi_{f,f'}\sum_{k\in J_{f'}}\left\{p_k-C_k'\left(Q_{k,t}\left(p\right)\right)\right\}Q_{k,t}\left(p\right)\right]$$

given the others' prices at any time t, where $\phi_{f,\,f'}\in[0,\,1]$ is firm f's weight on firm f''s profit. FOC w.r.t. p_j is written as follows:

$$0 \;=\; Q_{j,t}\left(p\right)+\sum_{f'\in\mathcal{F}}\left[\phi_{f,f'}\sum_{k\in J_{f'}}\left\{p_k-C_k'\left(Q_{k,t}\left(p\right)\right)\right\}\frac{\partial Q_{k,t}\left(p\right)}{\partial p_j}\right].$$

Then, by employing discrete/continuous choice model and dividing both sides by the market share of product j, the FOC requires the following constraint

$$0=m_t-\left\{\bar{p}_{j,t}-\delta_{j,t}\right\}m_t\alpha_{j,t}+\sum_{f'\in\mathcal{F}}\phi_{f,f'}\sum_{k\in J_f}\left\{\bar{p}_{k,t}-\delta_{k,t}\right\}\bar{q}_{k,t}.$$

Then, a modified version of revealed preference test is stated as follows.

Remark 2. The set of observations $\{\bar{p},\,\bar{q}\}$ rationalizes a collusive price competition under convex cost functions and logit demand functions *only if* there are real numbers α_t, $\delta_{j,t}$, m_t for all $t\in\mathcal{T}$ and $j\in\mathcal{J}$, such that the following holds;

1. $\alpha_t>0$, $\delta_{j,t}>0$, $m_t>0$;
2. $0=m_t-\left\{\bar{p}_{j,t}-\delta_{j,t}\right\}m_t\alpha_t+\sum_{f'\in\mathcal{F}}\phi_{f,f'}\sum_{k\in J_f}\left\{\bar{p}_{k,t}-\delta_{k,t}\right\}\bar{q}_{k,t}$; and
3. $0\leq\left(\delta_{j,t'}-\delta_{j,t}\right)\left(\bar{q}_{j,t'}-\bar{q}_{j,t}\right)$.

In the above test, demand function is assumed to be the logit since any data rationalize the model with the general discrete/continuous model, and the additional parameter $\phi_{f,f'}$ would further loosen the restriction. Under the logit demand, on the other hand, the common markup property still holds, so the data shown in Subsection 2.1 do not rationalize the collusive price competition.

We can also write the model of perfect collustion, as a special case of the aboe specification, where $\phi_{f,\,f'}=1$ for all f and f'. Furthermore, it turns out to be the mathematically same model as the baseline model with a different definition of a firm, therefore, we obtain the same result as in Section 2 with the different definition of the firm. To be more specific, I write an immediate corollary of Theorem 1 (characterization of the test) and Corollary 1 (unfalsifiability) for the perfect collustion under a general discrete-continuous demand model.

Corollary 4. *The set of observations $\{\bar{p}, \bar{q}\}$ is Collusion rationalizable under convex cost function and discrete/continuous demand function if and only if there are real numbers $\alpha_{j,t}$, $\delta_{j,t}$, m_t for any $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;*

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{f' \in \mathcal{F}} \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{k,t}\} \bar{q}_{k,t}$; *and*
3. $0 \leq \left(\delta_{j,t'} - \delta_{j,t}\right)\left(\bar{q}_{j,t'} - \bar{q}_{j,t}\right)$.

Furthermore, any data $\{\bar{p}, \bar{q}\}$ is Collusion rationalizable under convex cost function and discrete/continuous demand function.

By restricting a class of the demand to the logit, we can also obtain falsifiable model, which is summarized as an immediate corollary of Proposition 1.

Corollary 5. *(Logit) The set of observations $\{\bar{p}, \bar{q}\}$ is Collusion rationalizable under convex cost function and logit demand function if and only if there are real numbers α_t, $\delta_{j,t}$, m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;*

1. $\alpha > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_t + \sum_{k \in J_f} \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$; *and*
3. $0 \leq \left(\delta_{j,t'} - \delta_{j,t}\right)\left(\bar{q}_{j,t'} - \bar{q}_{j,t}\right)$.

4 Implementation

The existence of parameters satisfying the inequality constraints can be checked by minimizing a loss function over a set of parameters, given the data observed, and checking whether the minimized value is close to zero. For instance, for an inequality $g\left(\theta; \bar{p}, \bar{q}\right) \geq 0$, we can construct a loss function

$(\min\{0,\, g(\theta;\bar{p},\,\bar{q})\})^2$. Similarly, for a vector of equality constraints $\boldsymbol{h}(\theta;\bar{p},\,\bar{q}) = \mathbf{0}$ and a vector of inequality constraints $\boldsymbol{g}(\theta;\bar{p},\,\bar{q}) \geq \mathbf{0}$, we can construct a loss function

$$\boldsymbol{h}(\theta;\bar{p},\,\bar{q})^T \boldsymbol{h}(\theta;\bar{p},\,\bar{q}) + \tilde{\boldsymbol{g}}(\theta;\bar{p},\,\bar{q})^T \tilde{\boldsymbol{g}}(\theta;\bar{p},\,\bar{q})$$

where $\tilde{\boldsymbol{g}}(\theta;\bar{p},\,\bar{q}) = [\min\{0,\, g_i(\theta;\bar{p},\,\bar{q})\}]_i$. In general, this minimization faces a computational issues similar to the estimiation with moment inequalities.[12] Under the logit demand and a slightly modified data requirement, the inequalities are written as linear constraints on parameters so that we can check the existence of parameters using off-the-shelf tools for linear constraints. In the following, I assume that the market size $\{\bar{M}_t\}_t$ in addition to prices $\{\bar{p}_{j,\,t}\}$ and quantities $\{\bar{q}_{j,\,t}\}$ are observable, which is always the case when researchers can estimate the logit demand function. Then, the market shares of products at each time $\{\bar{s}_{j,\,t}\}$ are also observable since $\bar{s}_{j,\,t} = \frac{\bar{q}_{j,\,t}}{\bar{M}_t}$. Then, by remembering that $m = \frac{M}{\alpha}$ under the logit demand and by replacing $\frac{1}{\alpha} = \tilde{\alpha}$, the data restriction for the price competition under the logit demand function is chracterized by a set of parameter constraints linear in parameters, $\tilde{\alpha}_{j,\,t}$, and $\delta_{j,\,t}$.

Corollary 6. *(Logit) The set of observations $\{\bar{p},\,\bar{q},\,\bar{M}\}$ is Bertrand rationalizable under convex cost function and logit demand function* ***if and only if*** *there are real numbers $\tilde{\alpha}_t$, $\delta_{j,t}$, m_t for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following holds;*

1. $\tilde{\alpha}_t > 0$, $\delta_{j,t} > 0$;
2. $0 = \bar{M}_t\tilde{\alpha}_t - \{\bar{p}_{j,t} - \delta_{j,t}\} + \sum_{k\in J_f}\{\bar{p}_{k,t} - \delta_{j,t}\}\,\bar{q}_{k,t}$; *and*
3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

Proof. The proof is an immediate corollary of Proposition 1. □

Thus, we can use standard algorithms for linear constraints to check the constraint.

[12] The loss function tends to have a basin at the bottom with kinks around it. Therefore, the standard optimization algorithms do not work well.

5 Summary

In this paper, I modify a test for Bertrand assumption introduced by Carvajal et al. (2014) so that it is implementable for Bertrand competition among multi-product firms. To deal with difficulties caused by cannibalization effects, I employ the discrete/continuous demand function introduced by Nocke and Schutz (2018), which includes the multinomial logit demand function and CES demand function as special cases. In the main theorem, I provide the necessary and sufficient condition for data to be rationalized by Bertrand competition among multi-product firms under the discrete-continuous model. The test is implementable without any IVs, and the rejection by the suggested test deterministically implies misspecification of the model rather than peculiar realizations of structural error terms. Under the general discrete/continuous model, any data satisfies the necessary and sufficient condition to rationalize the price competition, while some data does not rationalize the price competition under more restrictive demand specifications such as logit demand function, CES, or a discrete/continuous model with HARA h. I also discuss additional restrictions on demand function, which played the main role in Carvajal et al. (2014), a test with observed cost shifters, and a test of collusive price competition, and a simple implementation for the case of the logit demand function.

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Table 1: Summary of results

	Demand	Cost	Extention	Neccesity	Sufficiency	Falsifiability
Theorem 1	DC	concave	-	Y	Y	N
Proposition 1	Logit	concave	-	Y	Y	Y
Corollary 1	CES	concave	-	Y	N	Y
Proposition 2	CES	Linear	-	Y	Y	Y
Corollary 3	DC with HARA h	concave	-	Y	N	Y
Proposition 3	*co-evolving* DC	concave	-	Y	N	Y
Remark 1	—	concave	cost shifter	Y	N	—
Remark 2	Logit	concave	potential collusion	Y	N	Y
Corollary 4	DC	concave	full collusion	Y	Y	N
Proposition 4	Logit	concave	full collusion	Y	Y	Y

Appendix

Proof of Theorem 1. For sufficiency, it is enough to construct cost functions and demand functions for each firm which construct a profit function maximized at $\bar{p}_{j,t}$, $\bar{q}_{j,t}$.

First, consider the re-construction of demand function. If data satisfies the restriction defined in Theorem 1, we should be able to find $\alpha_{j,t}$ which corresponds to $\frac{h''_{j,t}(\bar{p}_{j,t})}{-h'_{j,t}(\bar{p}_{j,t})}$ for each observation, where $h_{j,t}: R_+ \to R$ composes the true data generating process. In the reconstruction of demand functions, I consider $\bar{h}_{j,t}: R_+ \to R$ s.t. $\frac{\bar{h}''_{j,t}(p_j)}{-\bar{h}'_{j,t}(p_j)} = \alpha_{j,t}$ for all $p_j \in R_+$. This is an analogous to the construction of utility function in Afriat (1967), where the gradient of utility function is assumed to be locally constant. Since the constant $\frac{\bar{h}''_{j,t}(p_j)}{\bar{h}'_{j,t}(p_j)}$ implies that $\bar{h}_{j,t}(p_j)$ can be represented as CARA function with risk averseness $\alpha_{j,t}$, $\bar{h}_{j,t}(p_j) = \exp\{v_{jt} - \alpha_{jt}p_j\}$ for some v_{jt}. Then, we can construct a demand function, $\bar{Q}_{j,t}(p) = m_t \frac{-\bar{h}'_{j,t}(p_j)}{H_0 + \sum_k \bar{h}_{k,t}(p_j)} = m_t \frac{\alpha_{jt}\exp\{v_{jt} - \alpha_{jt}p_j\}}{H_0 + \sum_k \exp\{v_{kt} - \alpha_{kt}p_j\}}$. Here, I denote the reconstructed demand function as $\bar{Q}_{j,t}(p)$ in order to distinguish from the demand function in true data generating process, $Q_{j,t}(p)$. Now, v_{jt} can be chosen to satisfy a system of K equations; $m_t \frac{\alpha_{jt}\exp\{v_{jt} - \alpha_{jt}\bar{p}_{jt}\}}{H_0 + \sum_k \exp\{v_{kt} - \alpha_{kt}\bar{p}_{jt}\}} = \bar{q}_{jt}$ for all j, in the same way as the inversion of share function in logit specifications discussed in Berry (1994).

Since (δ_{jt}, q_{jt}) satisfies co-monotone property, we can use monotone cubic interpolation to re-construct increasing and continuously differentiable $\bar{C}'(\cdot)$. Then, we can re-construct $\bar{C}(q) = \int_0^q \bar{C}'(x)\,dx$, which is convex and twice continuously differentiable. [13]

The remaining step is to prove that $(\bar{p}, \bar{q})$ is an equilibrium under reconstructed demand and

[13] Carvajal et al. (2013, 2014) re-construct a cost function as an upper envelop of linear cost functions, whose slope is determined by $\delta_{j,t}$'s. Instead, in this paper, I use cubic interpolation to have the differentiability, which is necessary to invert ι-markup.

cost functions. Since the re-constructed profit function is continuously differentiable, FOC must be satisfied at the optimal price. Therefore, it is enough to show that there is the unique solution of FOC for each firm given other firms' strategies. To show that, I use the common ι-markup property examined in Nocke and Schutz (2018). The following part is closely related to the proofs in Nocke and Schutz (2018) (especially, Lemma F), but there are a few differences. First, we don't need to prove the existence of the equilibrium since we already have data as a candidate of equilibrium. Therefore, we just need to show that those data can be an equilibrium. Second, we consider more general cost specification than Nocke and Schutz (2018). It complicates the inversion from μ^f to price vectors since marginal cost is not a constant, but a function of quantity of the product. Third, the re-constructed demand function is a special case of the demand function in Nocke and Schutz (2018). Therefore, we can circumvent a difficulty from a general cost function by specifying shape of the demand function.

In the following, I omit the subscript for time t since I consider repetition of static NE and the following logic is applied for each t. Then, I denote the reconstructed demand function as $\bar{Q}_j(p) = m\frac{-\bar{h}'_j(p_j)}{H_0+\sum_k \bar{h}_k(p_j)} = m\frac{\alpha_j \exp\{v_j-\alpha_j p_j\}}{H_0+\sum_k \exp\{v_k-\alpha_k p_j\}}$ and $\bar{h}'_j(p_k) = -\alpha_j \exp\{v_j - \alpha_j p_j\}$, $\bar{h}''_j(p_k) = \alpha_j^2 \exp\{v_j - \alpha_j p_j\}$, and $\frac{\bar{h}''_j(p_k)}{-\bar{h}'_j(p_k)} = \alpha_j$. Since we now consider a maximization problem of a specific firm given other firm's strategy, let denote $\bar{h}_0 + \sum_{k\notin J_f} \bar{h}_k(p_k) = H_0$ and $J_f = \{1, ..., n\}$ without loss of generality. By the FOC, we have the following for any j

$$\left\{p_j - \bar{C}'_j\left(\bar{Q}_j(\mathbf{p})\right)\right\}\frac{\bar{h}''_j(p_j)}{-\bar{h}'_j(p_j)} = 1 + m^{-1}\sum_{k\in J_f}\left\{p_k - \bar{C}'_k\left(\bar{Q}_k(\mathbf{p})\right)\right\}\bar{Q}_k(\mathbf{p}) \tag{8}$$

Since RHS is same for any $j \in J_f$, the solution of system of equations defined by (8) for any $j \in J_f$ satisfies

$$\nu_j(\mathbf{p}) \equiv \left\{p_j - \bar{C}'_j\left(\bar{Q}_j(\boldsymbol{p})\right)\right\}\alpha_j = \mu^f$$

for any $j \in J_f$. Let $\nu(\mathbf{p}) = [\nu_1(\mathbf{p}), ..., \nu_n(\mathbf{p})]'$. Then, $\mathbf{p} = \nu^{-1}\left(\mathbf{1}\mu^f\right) \equiv r\left(\mu^f\right) \equiv \left[r_1\left(\mu^f\right), ..., r_n\left(\mu^f\right)\right]'$

at the solution of (8). Then, we can rewrite the condition (8) as

$$
\begin{aligned}
\mu^f &= 1 + m^{-1} \sum_{k \in J_f} \left\{ r_k\left(\mu^f\right) - \bar{C}_k'\left(r\left(\mu^f\right)\right) \right\} \bar{Q}_k\left(r\left(\mu^f\right)\right) \\
&= 1 + m^{-1} \sum_{k \in J_f} \underbrace{\left\{ r_k\left(\mu^f\right) - \bar{C}_k'\left(r\left(\mu^f\right)\right) \right\} \alpha_k}_{\mu^f} \frac{1}{\alpha_k} \bar{Q}_k\left(r\left(\mu^f\right)\right) \\
&= 1 + m^{-1} \mu^f \sum_{k \in J_f} \frac{1}{\alpha_k} \bar{Q}_k\left(r\left(\mu^f\right)\right) \\
\Leftrightarrow 0 &= 1 + \mu^f \left\{ m^{-1} \sum_{k \in J_f} \frac{1}{\alpha_k} \bar{Q}_k\left(r\left(\mu^f\right)\right) - 1 \right\} \equiv \psi\left(\mu^f\right)
\end{aligned}
$$

Then, the uniqueness of the solution of FOC is proved by strict monotonicity of $\psi\left(\mu^f\right)$. Again, the existence of the solution can be omitted since the data satisfies FOC by the construction of $\left(\bar{Q}_j(\cdot), \bar{C}_j(\cdot)\right)_{j \in J_f}$. By taking a derivative w.r.t. μ^f

$$
\begin{aligned}
\psi'\left(\mu^f\right) &= \underbrace{\sum_{k \in J_f} \frac{\exp\left\{v_k - \alpha_k r_k\left(\mu^f\right)\right\}}{H_0 + \sum_l \exp\left\{v_l - \alpha_l r_l\left(\mu^f\right)\right\}} - 1}_{<0} \\
&\quad \underbrace{+ \mu^f m^{-1} \sum_{k \in J_f} \frac{1}{\alpha_k} \underbrace{\frac{\partial \bar{Q}_k(\mathbf{p})}{\partial \mathbf{p}}|_{\mathbf{p}=r\left(\mu^f\right)}}_{1 \times n} \underbrace{r'\left(\mu^f\right)}_{n \times 1}}_{\equiv A}
\end{aligned}
$$

It is enough to show that $A \leq 0$.

$$
\begin{aligned}
A &= \mu^f m^{-1} \underbrace{\left[\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}\right]}_{1 \times n} \underbrace{\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=r\left(\mu^f\right)}}_{n \times n} \underbrace{\frac{\partial \nu^{-1}(\mathbf{m})}{\partial \mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f}}_{n \times n} \underbrace{\mathbf{1}}_{n \times 1} \\
&= \mu^f m^{-1} \underbrace{\mathbf{1}'}_{1 \times n} \underbrace{\Lambda^{-1} \underbrace{\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=r\left(\mu^f\right)}}_{n \times n} \underbrace{\frac{\partial \nu^{-1}(\mathbf{m})}{\partial \mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f}}_{n \times n}}_{\equiv B} \underbrace{\mathbf{1}}_{n \times 1}
\end{aligned}
$$

where $\Lambda = \begin{bmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{bmatrix}$. Since $\mu^f > 0$ and $m > 0$, it is enough to show that B is negative semi-definite.

In order to consider derivatives of $\nu^{-1}(\mathbf{m})$, we first consider the derivative of ν. Recall that $\nu_j(\mathbf{p}) \equiv \left\{p_j - C_j'\left(\bar{Q}_j(p)\right)\right\}\alpha_j$. then, partial derivatives are;

$$\begin{aligned}
\frac{\partial \nu_k(\mathbf{p})}{\partial p_k} &= \alpha_k \left(1 - c''\left(\bar{Q}_k(\mathbf{p})\right) \frac{\partial \bar{Q}_k(\mathbf{p})}{\partial p_k}\right) \\
\frac{\partial \nu_k(\mathbf{p})}{\partial p_j} &= -\alpha_k c''\left(\bar{Q}_k(\mathbf{p})\right) \frac{\partial \bar{Q}_k(\mathbf{p})}{\partial p_j}
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial \nu(\mathbf{p})}{\partial \mathbf{p}'} &= \Lambda \left\{ I - \begin{bmatrix} c''\left(\bar{Q}_1(\mathbf{p})\right) \frac{\partial \bar{Q}_1(\mathbf{p})}{\partial p_1} & c''\left(Q_1(\mathbf{p})\right) \frac{\partial Q_1(\mathbf{p})}{\partial p_n} \\ \\ c''\left(Q_n(\mathbf{p})\right) \frac{\partial Q_n(\mathbf{p})}{\partial p_1} & c''\left(Q_n(\mathbf{p})\right) \frac{\partial Q_n(\mathbf{p})}{\partial p_n} \end{bmatrix} \right\} \\
&= \Lambda \left\{ I - \Gamma(\mathbf{p}) \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} \right\}
\end{aligned}$$

where $\Gamma(\mathbf{p}) = \begin{bmatrix} c''\left(Q_1(\mathbf{p})\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c''\left(Q_n(\mathbf{p})\right) \end{bmatrix}$. Then,

$$\begin{aligned}
B &= \Lambda^{-1} \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \frac{\partial r(\mathbf{m})}{\partial \mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f} \\
&= \Lambda^{-1} \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \left[\frac{\partial \nu(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \right]^{-1} \\
&= \Lambda^{-1} \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \left[\Lambda \left\{ I - \Gamma(\mathbf{p}) \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} \right\} \right]^{-1} \\
&= \Lambda^{-1} \left(\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \right)^{-1} \right)^{-1} \left\{ I - \Gamma(\mathbf{p}) \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} \right\}^{-1} \Lambda^{-1} \\
&= \Lambda^{-1} \left(\left\{ I - \Gamma(\mathbf{p}) \frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} \right\} \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \right)^{-1} \right)^{-1} \Lambda^{-1} \\
&= \Lambda^{-1} \left(\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \right)^{-1} - \Gamma(\mathbf{p}) \right)^{-1} \Lambda^{-1} \\
&= -\Lambda^{-1} \left(\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}} \right)^{-1} \right)^{-1} \Lambda^{-1}
\end{aligned}$$

Now, B is negative definite as long as $\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}$ is negative semi definite;

$$\begin{aligned}
\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'} &= \begin{bmatrix} -m^{-1}Q_1(p)\{\alpha_1 m - Q_1(p)\} & m^{-1}Q_1(p)Q_2(p) & \cdots & m^{-1}Q_1(p)Q_n(p) \\ m^{-1}Q_1(p)Q_2(p) & -m^{-1}Q_2(p)\{\alpha_2 m - Q_2(p)\} & & m^{-1}Q_2(p)Q_n(p) \\ \vdots & & \ddots & \vdots \\ m^{-1}Q_1(p)Q_n(p) & m^{-1}Q_2(p)Q_n(p) & \cdots & -m^{-1}Q_n(p)\{\alpha_n m - Q_n(p)\} \end{bmatrix} \\
&= m^{-1}\begin{bmatrix} -Q_1(p)\{\alpha_1 m - Q_1(p)\} & Q_1(p)Q_2(p) & \cdots & Q_1(p)Q_n(p) \\ Q_1(p)Q_2(p) & -Q_2(p)\{\alpha_2 m - Q_2(p)\} & & Q_2(p)Q_n(p) \\ \vdots & & \ddots & \vdots \\ Q_1(p)Q_n(p) & Q_2(p)Q_n(p) & \cdots & -Q_n(p)\{\alpha_n m - Q_n(p)\} \end{bmatrix} \\
&= m^{-1}\left\{\begin{bmatrix} Q_1(p)Q_1(p) & \cdots & Q_1(p)Q_n(p) \\ \vdots & \ddots & \vdots \\ Q_1(p)Q_n(p) & \cdots & Q_n(p)Q_n(p) \end{bmatrix} - m\begin{bmatrix} \alpha_1 Q_1(p) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n Q_n(p) \end{bmatrix}\right\}
\end{aligned}$$

Then,

$$\begin{aligned}
x'\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}x &= m^{-1}\left\{x'Q(p)Q(p)'x - mx'\begin{bmatrix} \alpha_1 Q_1(p) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n Q_n(p) \end{bmatrix}x\right\} \\
&= m^{-1}\left\{\left(\sum_i x_i Q_i\right)^2 - m\sum_i x_i^2 \alpha_i Q_i\right\} \\
&= m^{-1}\left(\sum_i x_i Q_i\right)^2 - m^{-1}\sum_i x_i^2 Q_i^2 + m^{-1}\sum_i x_i^2 Q_i^2 - \sum_i x_i^2 \alpha_i Q_i \\
&= -m^{-1}\underbrace{\left\{\sum_i x_i^2 Q_i^2 - \left(\sum_i x_i Q_i\right)^2\right\}}_{>0} - m^{-1}\left\{\sum_i x_i^2 Q_i \underbrace{(m\alpha_i - Q_i)}_{>0}\right\} < 0
\end{aligned}$$

Then, $-\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}$ is positive definite, so as $\left(-\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}$. Therefore, $\Gamma(\mathbf{p}) - \left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}$ is positive definite since $\Gamma(\mathbf{p})$ is a diagonal matrix with positive components. Therefore,

$$\begin{aligned} x'Bx &= -x'\Lambda^{-1}\left(\Gamma(\mathbf{p})-\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}\right)^{-1}\Lambda^{-1}x \\ &= -\left(\left(\Lambda^{-1}\right)'x\right)'\left(\Gamma(\mathbf{p})-\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}\right)^{-1}\Lambda^{-1}x \\ &= -\left(\Lambda^{-1}x\right)'\left(\Gamma(\mathbf{p})-\left(\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right)^{-1}\right)^{-1}\Lambda^{-1}x \\ &< 0 \end{aligned}$$

Therefore, B is negative definite, which gives $\psi'\left(\mu^f\right)<0$.

Proof of Proposition 1:

For Proposition 1, we need to derive the last condition as a necessary condition.

By co-evolving property, we can find a permutation such that $\sigma(t)<\sigma(s)$ implies $\epsilon_{j,t}(p)\leq\epsilon_{j,s}(p)$ for all $j\in J$ and for all p. Then, if $\bar{p}_{i,t}\geq\bar{p}_{i,s}$, $\bar{p}_{-i,t}\leq\bar{p}_{-i,s}$ and $\sigma(t)<\sigma(s)$, then $\alpha_{j,t}-m_t^{-1}\bar{q}_{j,t}=\epsilon_{j,t}(\bar{p}_{jt},\bar{p}_{-jt})\leq\epsilon_{j,t}(\bar{p}_{js},\bar{p}_{-jt})\leq\epsilon_{j,s}(\bar{p}_{js},\bar{p}_{-js})\leq\epsilon_{j,s}(\bar{p}_{js},\bar{p}_{-js})=\alpha_{j,s}-m_s^{-1}\bar{q}_{j,s}$. Thus, $\alpha_{j,t}-m_t^{-1}\bar{q}_{j,t}\leq\alpha_{j,s}-m_s^{-1}\bar{q}_{j,s}$ $\square$