Part1:

Use the return before 2023.12.31 as the training period data to calculate the market excess return, set the SPY data as the market benchmark at the same time. For each stocks in portfolio, using OLS regression to fit the model, print the beta below:

```
   Symbol      Beta
0     WFC  1.140628
1     ETN  1.116652
2    AMZN  1.532365
3    QCOM  1.479601
4     LMT  0.320696
..    ...       ...
94   MSFT  1.169683
95    PEP  0.376748
96     CB  0.459826
97   PANW  1.172476
98    BLK  1.243292
```

The ex post imputation function is then written to split daily portfolio excess returns into systematic (factor) and idiosyncratic (residual) components by updating the market capitalization weights on a daily basis, using a pre-fitted factor beta, and normalizing the arithmetic returns into additive geometric imputations using Cariño K coefficients, ultimately returning a complete time series of returns and residuals as well as cumulative imputation data.

Taking a table of positions as input, the weights are first calculated based on the opening market capitalization, then expost_factor is called to obtain the factor/trait return imputations and volatility imputations for the overall portfolio and each sub-portfolio, and the results are formatted into a summary table for easy comparison. After running the function, the results are listed below:

```
                 Value       SPY     Alpha  Portfolio
0          TotalReturn  0.261373 -0.035969   0.204731
1  Return Attribution  0.244039 -0.039309   0.204731
2     Vol Attribution  0.007207 -0.000131   0.007076

A portfolio attribution:
                 Value       SPY     Alpha  Portfolio
0          TotalReturn  0.261373 -0.095555   0.136642
1  Return Attribution  0.242621 -0.105980   0.136642
2     Vol Attribution  0.007056  0.000348   0.007404

B portfolio attribution:
                 Value       SPY     Alpha  Portfolio
0          TotalReturn  0.261373 -0.028626   0.203526
1  Return Attribution  0.234259 -0.030733   0.203526
2     Vol Attribution  0.006411  0.000442   0.006854

C portfolio attribution:
                 Value       SPY     Alpha  Portfolio
0          TotalReturn  0.261373  0.022337   0.281172
1  Return Attribution  0.255627  0.025546   0.281172
2     Vol Attribution  0.007230  0.000678   0.007908
```

The results show that the cumulative excess return of all the portfolios is around 26.1%, while the SPY (systematic part) contributes 23-24 percentage points to the “Return Attribution”, indicating that most of the return is driven by the market beta. The difference in returns across portfolios is mainly due to the difference in alpha(idiosyncratic part).

Part2:

Along with the CAPM parameters fitted in Part 1, the average market excess return and the risk-free rate are calculated before the holding period. The CAPM formula is subsequently fitted under the assumption of 0 alpha to obtain the expected daily return for each stock. For each sub-portfolio, the SLSQP optimization problem is solved using the expected return and historical covariance matrices to find the annualized maximum Sharpe ratio weights. The expected annualized return, expected annualized volatility, and expected

Sharpe ratio of each portfolio are output and compared with the original position ratio. The optimized portfolio weights and optimal Sharpe ratios are obtained as shown below:

```
portfolio A after optimized:
expected return: 0.250315
expected vol: 0.137065
Sharpe ratio(annual): 1.463484

weight comparison:
       Original Weight  Optimized Weight
WFC           0.023048      1.795470e-02
ETN           0.024155      3.816275e-02
AMZN          0.023657      9.117423e-02
QCOM          0.030724      1.089794e-02
LMT           0.031591      2.760596e-02
KO            0.031983      5.694559e-02
JNJ           0.036804      2.373200e-02
ISRG          0.021696      4.279053e-02
XOM           0.031014      0.000000e+00
MDT           0.033995      0.000000e+00
DHR           0.034305      1.851121e-02
PLD           0.041948      3.179576e-02
BA            0.052936      1.590679e-02
PG            0.030005      7.704113e-02
MRK           0.036638      4.618981e-02
AMD           0.040414      9.954577e-03
BX            0.025060      4.157826e-02
PM            0.025169      4.264925e-02
SCHW          0.031208      2.168404e-19
```

Optimized positions are converted to actual quantities passed into run_attribution to get return attribution and volatility imputation for each portfolio, the calculation method is like the part1 did, results are listed below:

	Value	SPY	Alpha	Portfolio
0	TotalReturn	0.261373	0.011515	0.283866
1	Return Attribution	0.269944	0.013922	0.283866
2	Vol Attribution	0.008035	-0.000501	0.007535

A portfolio attribution:

	Value	SPY	Alpha	Portfolio
0	TotalReturn	0.261373	0.009659	0.288602
1	Return Attribution	0.276657	0.011945	0.288602
2	Vol Attribution	0.007980	0.000035	0.008014

B portfolio attribution:

	Value	SPY	Alpha	Portfolio
0	TotalReturn	0.261373	-0.004927	0.257900
1	Return Attribution	0.262293	-0.004393	0.257900
2	Vol Attribution	0.007451	-0.000099	0.007352

C portfolio attribution:

	Value	SPY	Alpha	Portfolio
0	TotalReturn	0.261373	0.031075	0.305896
1	Return Attribution	0.270254	0.035641	0.305896
2	Vol Attribution	0.007652	0.000553	0.008205

	Original Portfolio	Optimized Portfolio	Difference
Total Return	0.204731	0.283866	0.079135
Systematic Return(SPY)	0.244039	0.269944	0.025905
Specific Return(Alpha)	-0.039309	0.013922	0.053231
Portfolio Volatility	0.007076	0.007535	0.000459
Systematic Volatility	0.007207	0.008035	0.000829
Specific Volatility	-0.000131	-0.000501	-0.000370

```
portfolio A comparison
                         Original Portfolio  Optimized Portfolio  Difference
Total Return                       0.136642             0.288602    0.151960
Systematic Return(SPY)             0.242621             0.276657    0.034036
Specific Return(Alpha)            -0.105980             0.011945    0.117924
Portfolio Volatility               0.007404             0.008014    0.000611
Systematic Volatility              0.007056             0.007980    0.000924
Specific Volatility                0.000348             0.000035   -0.000313

portfolio B comparison
                         Original Portfolio  Optimized Portfolio  Difference
Total Return                       0.203526             0.257900    0.054374
Systematic Return(SPY)             0.234259             0.262293    0.028034
Specific Return(Alpha)            -0.030733            -0.004393    0.026340
Portfolio Volatility               0.006854             0.007352    0.000498
Systematic Volatility              0.006411             0.007451    0.001040
Specific Volatility                0.000442            -0.000099   -0.000541

portfolio C comparison
                         Original Portfolio  Optimized Portfolio  Difference
Total Return                       0.281172             0.305896    0.024724
Systematic Return(SPY)             0.255627             0.270254    0.014628
Specific Return(Alpha)             0.025546             0.035641    0.010096
Portfolio Volatility               0.007908             0.008205    0.000297
Systematic Volatility              0.007230             0.007652    0.000422
Specific Volatility                0.000678             0.000553   -0.000125
```

The result shows that optimized portfolios have higher systematic return contributions, indicating increased exposure to market factors. Moreover, the alpha of each portfolio increases to varying degrees after optimization, indicating higher returns on specific stock choices.

In the comparison between expected value and realized value of idiosyncratic risk in each stock, we first calculate the market capitalization weight w_i of each stock in the portfolio based on the starting price and the number of positions, and then apply the formula to obtain the "predicted" contribution of each stock to the portfolio idiosyncratic risk using the residual variance from the CAPM regression in Part 1. Meanwhile, from the volatility attribution results in Part 2, the values of each stock in the Vol Attribution's Alpha row are extracted as their "realized" idiosyncratic contribution to the portfolio volatility during the actual holding period. Finally, the original and optimized weights, as well as the difference between them, are summarized in a table, and the comparison can be used to visualize the overestimation or underestimation of the idiosyncratic risk of a single stock by the CAPM model.

```
      Original Weight  Optimized Weight  Expected Idiosyncratic Risk  \
WFC          0.008068          0.006286                     0.014745   
ETN          0.008456          0.013360                     0.013937   
AMZN         0.008282          0.031918                     0.016538   
QCOM         0.010756          0.003815                     0.015578   
LMT          0.011059          0.009664                     0.011105   
...               ...               ...                          ...   
MSFT         0.010636          0.034650                     0.012555   
PEP          0.013260          0.003147                     0.008976   
CB           0.010042          0.017261                     0.012310   
PANW         0.009783          0.002006                     0.022179   
BLK          0.009395          0.009646                     0.009458   

      Realized Idiosyncratic Risk  Difference  
WFC                      0.016993    0.002249  
ETN                      0.012814   -0.001123  
AMZN                     0.012703   -0.003835  
QCOM                     0.018785    0.003207  
LMT                      0.010996   -0.000109  
...                           ...         ...  
MSFT                     0.008313   -0.004243  
PEP                      0.010668    0.001692  
CB                       0.010762   -0.001548  
PANW                     0.025223    0.003044  
BLK                      0.009646    0.000188  

[99 rows x 5 columns]
```

Part3:

NIG distribution contains four parameters:

1.α: tail heaviness control parameter

2.β: skewness control parameter

3.μ: location parameter

4.δ: scale parameter

And it also has the following characteristics:

1.Can exhibit significant skewness and kurtosis

2.Features semi-heavy tails, between exponential and power-law tails

3.May have a sharp peak at the origin

4.Possesses stable additive properties (the sum of NIG random variables remains an NIG distribution)

Based on the above characteristics, the practical application of NIG in financial industry mainly focus on:

Asset Return Modeling: Financial asset returns often exhibit skewness and heavy-tail characteristics that don't align with traditional normal distribution assumptions. The NIG distribution can capture both skewness and heavy tails simultaneously, making it excellent for modeling returns of stocks, options, and other financial assets.

Risk Measure Calculation: When calculating risk measures such as VaR and ES (Expected Shortfall), the NIG distribution can more accurately capture the risk of extreme events, avoiding the problem of traditional normal distributions underestimating extreme risks.

Derivatives Pricing: In option pricing models, the NIG distribution can replace the normal distribution assumption in the standard Black-Scholes model, generating prices more consistent with market realities.

Stochastic Volatility Models: In stochastic volatility models, the NIG distribution can be used to describe the conditional distribution of returns, better capturing the volatility clustering phenomenon in financial markets.

Skew Normal Distribution contains three parameters:

1.ξ: location parameter

2.ω: scale parameter

3.α: shape parameter (controls skewness)

Key characteristics of the Skew Normal Distribution:

1.Can exhibit varying degrees of left or right skewness

2.Reduces to the standard normal distribution when $\alpha=0$

3.Has lighter tails compared to the NIG distribution

4.Mathematically more tractable, with close relationships to the normal distribution

Financial Applications of Skew Normal Distribution:

Asset Return Modeling: When markets exhibit mild to moderate skewness, while tail risks are not particularly extreme, the Skew Normal distribution provides a simpler but still effective model choice compared to NIG.

Portfolio Construction: In extensions of Markowitz portfolio theory, using the Skew Normal distribution can better reflect the asymmetry of asset returns, thus constructing portfolios more aligned with investor risk preferences.

Risk Analysis: The Skew Normal distribution can be used to model risk factors, especially in studies of macroeconomic factors' impact on financial markets, where many risk factors exhibit significant skewness but without extremely heavy tails.

Part4:

Four marginal distributions - normal, generalized t, NIG (normal inverse generalized), and skewed normal - are first fitted sequentially to each stock's historical return series and the optimal model is automatically selected using the AIC indicator, store the fitted model result in a data frame to display its best model and parameters:

	Symbol	Best_Model	Parameters
0	SPY	Normal	[0.0009849959688576436, 0.008230265429090224]
1	AAPL	GenT	[0.001760002237799804, 0.010701217623969377, 7...
2	NVDA	GenT	[0.003606099947648441, 0.021823711275835114, 5...
3	MSFT	GenT	[0.0016598623360141202, 0.012845638087603707, ...
4	AMZN	GenT	[0.0021398343483106886, 0.016486799520298173, ...
...	...	...	...
95	KKR	GenT	[0.002555003948757106, 0.01616928163457702, 5....
96	MU	NIG	[1.45304422688949, 0.5480757220672373, -0.0077...
97	PLD	GenT	[0.0009519211835215854, 0.01339075971165784, 5...
98	LRCX	NIG	[1.625477094139498, 0.555045323080846, -0.0064...
99	EQIX	GenT	[0.001171868414197325, 0.01209382954651547, 5....

100 rows × 3 columns

The joint returns are then simulated based on the fitted parameters of the optimal distribution for each stock by Gaussian Copula (first generating multivariate normal correlation samples and then doing the inverse CDF transformation of the marginal distribution), and also separately and directly by multivariate normal. Finally, the 95% VaR and ES are calculated for each subportfolio and the simulated returns of the entire portfolio to cross-sectionally compare the difference in risk measures under the Copula approach and the pure MVN approach. The results comparison are listed below:

	Portfolio	VaR_Copula	ES_Copula	VaR_MVNorm	ES_MVNorm
0	A	31.663157	40.699725	41.248642	52.684674
1	B	20.947818	26.334076	28.290255	35.628622
2	C	26.182631	33.464783	28.496079	36.674058
3	Total	71.856060	92.555855	91.859442	118.059750

From the generated VaR and ES values, we can see that the multivariate normal method outperforms Gaussian Copula by about 20%-30% for both VaR and ES.

Based on their characteristics, Gaussian Copula uses the distribution that is measured for each stock - some are really light-tailed (approximately normal) and some are thick-tailed (Gen-t/NIG). Thus, the extreme losses modeled by Copula are pulled down when the portfolio has about half light-tailed components.

The multivariate normal method, however, treats all stocks as normal and uses symmetrically exponentially decaying normal tails regardless of how thick or thin the original data tails are, which “artificially thickens” the tails of stocks that are otherwise light-tailed and thus pushes the VaR/ES passages upward.

In the A, B, and C sub-portfolios, Copula ES falls around 26-40, while MVN ES jumps to 36-52. suggesting that even though some stocks are inherently thicker-tailed than others (the C portfolio ES has the smallest gap), there are more lighter-tailed components in general, and thus Copula pulls down the tail of the portfolio.

Part5:

First, the es function is written, and then, using the finite difference method, the portfolio ES is recalculated by increasing the weight of each asset by ε one by one, and the CES is derived to obtain the marginal contribution of each asset to the portfolio ES.

The CES vectors for all assets are centered, squared and summed. Set the optimization objective to make the marginal ES contributions of all assets as equal as possible, i.e., minimize the above sum of squares.

Construct the function optimize_risk_parity to solve the above nonlinear minimization problem with SLSQP under the constraint that "the sum of the weights is 1, and each weight $\in [0,1]$" to obtain the risk parity weights for each sub-portfolio. The weights and ES contribution results of each stock and portfolio are listed below: we can see that after optimization, regardless of whether it's combination A, B, or C, the values for ES contribution in each row are firmly clustered between 3.3×10^{-4} and 3.8×10^{-4} , indicates the marginal ES contribution of each stock is almost the same.

```
Optimization terminated successfully    (Exit mode 0)
            Current function value: 1.3485177973494507e-05
            Iterations: 582
            Function evaluations: 20152
            Gradient evaluations: 582
A portfolio risk parity weights:
   Symbol    Weight  ES Contribution
0     WFC  0.021611         0.000353
1     ETN  0.018818         0.000355
2    AMZN  0.027542         0.000354
3    QCOM  0.018642         0.000355
4     LMT  0.057359         0.000351
5      KO  0.055921         0.000357
6     JNJ  0.054840         0.000352
7    ISRG  0.024716         0.000356
8     XOM  0.027938         0.000358
9     MDT  0.035315         0.000357
10    DHR  0.027488         0.000354
11    PLD  0.021657         0.000352
12     BA  0.025079         0.000353
13     PG  0.054966         0.000359
14    MRK  0.041413         0.000357
15    AMD  0.018702         0.000357
16     BX  0.016946         0.000352
17     PM  0.036958         0.000353
18   SCHW  0.022304         0.000355
19     VZ  0.053901         0.000352
20    COP  0.029198         0.000356
21    ADI  0.023013         0.000356
22    BAC  0.015768         0.000353
```

```
Optimization terminated successfully    (Exit mode 0)
            Current function value: 0.00033623151268662377
            Iterations: 58
            Function evaluations: 2136
            Gradient evaluations: 58
B portfolio risk parity weights:
   Symbol    Weight  ES Contribution
0     AXP  0.021731         0.000325
1     HON  0.024865         0.000325
2    META  0.026082         0.000327
3    NFLX  0.025583         0.000345
4     PGR  0.059743         0.000318
5     LLY  0.057521         0.000321
6     JPM  0.019329         0.000325
7    VRTX  0.040556         0.000334
8     TJX  0.026609         0.000328
9    EQIX  0.023928         0.000321
10   AAPL  0.024885         0.000317
11     FI  0.033419         0.000339
12     DE  0.025451         0.000338
13   SBUX  0.027313         0.000327
14  GOOGL  0.025795         0.000326
15      T  0.047221         0.000306
16    ABT  0.037101         0.000330
17    BMY  0.035676         0.000308
18     MS  0.023100         0.000326
19    CRM  0.025791         0.000337
20    PFE  0.029354         0.000294
21   SPGI  0.022899         0.000330
22  BRK-B  0.031120         0.000319
```

```
Optimization terminated successfully    (Exit mode 0)
            Current function value: 0.0006516889821774453
            Iterations: 459
            Function evaluations: 16008
            Gradient evaluations: 459
C portfolio risk parity weights:
   Symbol    Weight  ES Contribution
0     IBM  0.048292         0.000347
1     TXN  0.018477         0.000351
2     ADP  0.030919         0.000340
3    GOOG  0.024549         0.000333
4    ORCL  0.037942         0.000325
5     BSX  0.050088         0.000335
6     UNH  0.061299         0.000368
7    TMUS  0.037021         0.000347
8     SYK  0.014651         0.000344
9      GS  0.025168         0.000336
10   UBER  0.018364         0.000357
11   AVGO  0.018808         0.000353
12    MMC  0.024583         0.000351
13   CSCO  0.037831         0.000316
14   PLTR  0.006897         0.000350
15     MA  0.031091         0.000350
16      C  0.025961         0.000348
17   BKNG  0.026063         0.000341
18    MCD  0.035068         0.000356
19    LOW  0.020141         0.000368
20     HD  0.027974         0.000367
21   INTU  0.018706         0.000357
22   LRCX  0.015757         0.000360
```

The new weights are converted to adjusted number of holdings based on the percentage of market capitalization of each stock in the sub-portfolio, generating risk_parity_df. Subsequently, using the same run_attribution function, do return attribution and volatility attribution for risk parity portfolio, verifying that they are more balanced in terms of the system vs. trait, and return vs. volatility splits. The new portfolio holding and attribution result are listed below:

```
   Portfolio Symbol     Holding
0          A    WFC  133.148909
1          A    ETN   23.372122
2          A   AMZN   53.555058
3          A   QCOM   38.833124
4          A    LMT   38.129321
..       ...    ...         ...
94         C   MSFT   21.255131
95         C    PEP   86.384231
96         C     CB   51.628163
97         C   PANW   60.266780
98         C    BLK    7.605671
```

```
                 Value       SPY     Alpha  Portfolio
0          TotalReturn  0.261373  0.032901   0.257971
1   Return Attribution  0.220770  0.037201   0.257971
2      Vol Attribution  0.006190  0.000158   0.006348

A portfolio attribution:
                 Value       SPY     Alpha  Portfolio
0          TotalReturn  0.261373 -0.024973   0.197382
1   Return Attribution  0.223995 -0.026613   0.197382
2      Vol Attribution  0.006307  0.000245   0.006552

B portfolio attribution:
                 Value       SPY     Alpha  Portfolio
0          TotalReturn  0.261373  0.062971   0.285237
1   Return Attribution  0.214865  0.070371   0.285237
2      Vol Attribution  0.005347  0.001029   0.006377

C portfolio attribution:
                 Value       SPY     Alpha  Portfolio
0          TotalReturn  0.261373  0.065839   0.296245
1   Return Attribution  0.223091  0.073154   0.296245
2      Vol Attribution  0.006013  0.001091   0.007104
```

Comparing to the other attribution results in part1 and part2, we can conclude that:

In Return Attribution part:

Part 1: The systematic (SPY) and idiosyncratic (Alpha) contributions of the original positions to returns are each low or negative, resulting in the lowest overall return at 20%.

Part 2: The Maximum Sharpe Ratio portfolio deliberately amplifies the systematic return, pulling the SPY contribution to 27% and increasing the portfolio return to 28%.

Part 5: The ES Risk Parity portfolio balances idiosyncratic returns (Alpha) more, with the Alpha contribution rising to 3-7%, and total returns between 25-30.

In Vol Attribution part:

Part 1: With the original position, the systematic volatility contribution is about 0.72% and the trait is almost negative or zero, indicating that the portfolio risk is highly dependent on market volatility.

Part 2: The Maximum Sharpe Ratio Portfolio slightly boosts systematic volatility and depresses idiosyncratic volatility in some sub-portfolios (negative idiosyncratic contribution occurs), with higher risk concentration.

Part 5: The risk parity portfolio increases the contribution of idiosyncratic volatility to a positive value and moderately reduces the contribution of market volatility.

If the goal is to maximize Sharpe, Part 2 gives the highest expected return, but is also prone to more concentrated market risk at the extremes. If 'tail risk balance' is desired, Part 5's ES Risk Parity significantly improves the diversification of idiosyncratic risk while sacrificing some of the systematic returns, making the portfolio more robust and less susceptible to being swayed by extreme losses in a few individual stocks.