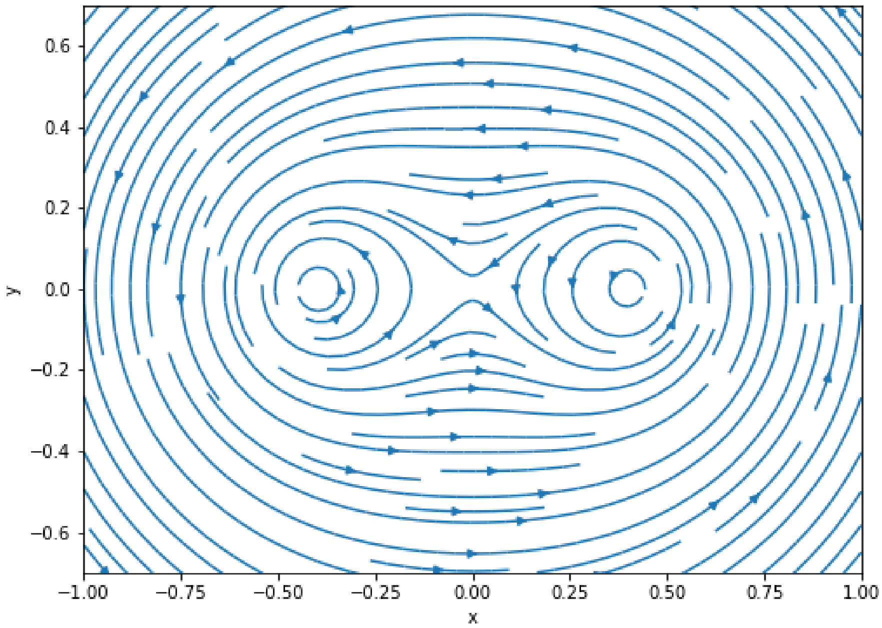
Chaotic Advection & Blinking Vortices

PHYS 2201 – Computational Assignment 2

Due: Friday 25 October, 5pm

Imagine you have a prescribed flow field, like the one depicted in the figure below. Here, you can see the flow field induced by two same-signed vortices.



The equation for this dipole system can be framed in terms of the velocity vector, $\mathbf{u} = (u, v)$, where u is the velocity in the x direction and v the velocity in the y direction, and is written:

$$u(x, y) = \frac{-\alpha y}{(x-\beta)^2 + y^2} - \frac{\alpha y}{(x+\beta)^2 + y^2} \tag{1a}$$

$$v(x, y) = \frac{\alpha(x-\beta)}{(x-\beta)^2 + y^2} + \frac{\alpha(x+\beta)}{(x+\beta)^2 + y^2} \tag{1b}$$

where we have used $\alpha = 1$ and $\beta = 0.4$. To solve for the position, $\mathbf{x} = (x, y)$ of a fluid parcel in this flow field, you simply need to integrate

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}. \tag{2}$$

If you follow the motion of a single parcel of fluid (that is, like the Lagrangian description of a fluid, as explained in week 8 video lectures) then you will complete a regular orbit around one, or both, of these vortices. However, the system may become chaotic when it is not integrable; this can be achieved via a *blinking vortex*, where we write

$$u(x, y, t) = \frac{-\alpha y}{(x-\beta)^2 + y^2} \tag{3a}$$

$$v(x, y, t) = \frac{\alpha(x-\beta)}{(x-\beta)^2 + y^2} \tag{3b}$$

$$\beta(t) = \beta_0 \,\mathrm{sign}(\sin(2\pi t)) \tag{3c}$$

which acts to alternately turn the left and right vortices on and off. (See Aref, 1984, for further details.)

In this computational assignment, you should:

1. Construct an algorithm to integrate particle trajectories forward in time, given the initial particle position and the stead dipole velocity field described by Eq. (1). Use this algorithm to show that particle paths follow regular orbits in this flow field, as expected. [20 Marks]

2. Demonstrate that the numerical accuracy of your system depends on the timestep you use. [20 Marks]

3. Now, apply your algorithm to the blinking vortex system (Eq. 3) and demonstrate that it can create chaotic advection, by calculating a Lyupanov exponent (see Week 9 lectures). [60 Marks]

Additional Information

To get you started ...

We are providing some assistance to help solve question 1. Here you will need to effectively integrate an ordinary differential equation (ODE) forward in time – and there are hundreds of options for this. You may like to write you own, simple, ODE solver, or you may want to base your algorithm on an existing packaged solver. We have provided templates in python, and you have already seen mathematica examples in lectures, which will help you to do this.

Your report

You should submit a report outlining the answers to the above 3 questions. Write enough words that your simulation makes sense to someone who has not seen these instructions.

You should also submit your code – and be sure to comment your code! To assess your understanding of the numerical algorithm, we will mark your comments as well as the correctness of the calculations you present.

Getting help vs Plagiarism

This project will take some people out of their comfort zone. Computational work can be frustrating all the way up to the point where things suddenly start working. Feel free to discuss this with lecturers/tutors/friends/each other. Feel free to look up books or webpages, or similar.

But you have to submit your own work, and write your code yourself. Reference anything you use. If you work with people, put a list of their names on the cover. For this project let's use the rule: do not copy and paste from anything or anyone. Look at someone's code by all means (and then tell us you did, as noted above), then go write your own version. Read a treatise on the system, but then go write your own analysis. Do not paste or transcribe.

References

H. Aref (1984), "Stirring by Chaotic Advection", *Journal of Fluid Mechanics*, **145**, 1–21.