Australian National University

Student Number:

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Mathematical Sciences Institute

EXAMINATION: First Semester, 2018

MATH3511/6111 — Scientific Computing

Exam Duration: 180 minutes.
Reading Time: 15 minutes.

Materials Permitted In The Exam Venue:

- Any number of copied or hand written pages.
- Unmarked English-to-foreign-language dictionary (no approval from MSI required).
- No printed books are permitted other than the dictionary.
- **No electronic aids are permitted e.g. laptops, phones, calculators.**

Materials To Be Supplied To Students:

- Scribble Paper.

Instructions To Students:

- A good strategy is not to spend too much time on any question. Read them through first and attack them in the order that allows you to make the most progress.
- *You must justify your answers. Please be neat.*

Q1	Q2	Q3	Q4	Q5	Q6
20	20	20	20	20	20

Total / 120

Question 1 **20 pts**

(a) Which of the following real numbers can be represented exactly using the IEEE standard 64 bit floating point numbers:

- 0.75
- π
- $10^{30} - 1$
- $\frac{27}{192}$
- 0.0005

Explain why these numbers can / cannot be represented exactly. *5 pts*

(b) Discuss the errors you get when you compute $a + b$ for some real a and b on your computer. Give the best possible upper bound on the errors in terms of the rounding error

$$\epsilon = \sup_{x \neq 0} \frac{|\phi(x) - x|}{|x|}.$$

Assume that you do not have any over- or under-flow. *5 pts*

(c) Consider the evaluation of the formula $a^2 + b^2 - 2ab\cos(\phi)$. Rewrite this as a sequence of simple operations where each operation gives rise to one rounding error and evaluations of the sequence results in the value of the formula. Include rounding errors in the inputs. *5 pts*

(d) Provide and error bound for the (relative) error occurring when solving the linear system of equations $Ax = b$ where the solver used is exact but the data vector b has been replaced by $b + e$ where $\|e\| \leq \delta$ for some small $\delta > 0$. Compare this of the error of evaluating $y = Ab$ when the matrix vector product is exact but we only know the perturbed value $b + e$. *5 pts*

Question 2 **20 pts**

(a) Let

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

For which of the matrices A_1 and A_2 does the linear system of equations $A_i x = b$ always have a unique solution? Why or why not? Which of the matrices A_i has an LU decomposition and what is it? *5 pts*

(b) The direct solution of $Ax = b$ with cyclic matrices A can be done using only $O(n \log_2(n))$ real multiplications instead of the $O(n^3)$ generally required for Gaussian elimination. Explain the steps required to achieve this and in particular the $O(n \log_2(n))$ multiplication count. *5 pts*

(c) Does the Jacobi method converge for the matrix

$$A = \begin{bmatrix} 9 & 2 & 3 \\ 6 & 8 & -1 \\ -4 & 0 & 5 \end{bmatrix}?$$

Why or why not? *5 pts*

(d) Consider the solution of $Ax = b$ with

$$A = \begin{bmatrix} 1.4 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2.7 \end{bmatrix}.$$

Assume that you are using exact arithmetic. Would you expect the gradient descent method or the conjugate gradient method to be faster for this problem and why? *5 pts*

Question 3 **20 pts**

(a) Assume that you have been using a fixed point method to solve a nonlinear equations and observe an error with an absolute value less or equal than 0.1 after some steps. Furthermore, assume that after a further iteration step, you find this error bound has now dropped to 0.02. What error would you expect after a further step? What would you expect if you did the same experiment but using Newton's method instead of the fixed point method? *7 pts*

(b) Is it possible that a fixed point iteration converges at the same rate as the bisection method? Why or why not? *6 pts*

(c) Assume that you happen to know that some function f has the values $f(0) = -1$, $f(1) = 7$ and $f(2) = 13$. How many bisection iterations are required to obtain and approximation of a zero of f with an error less or equal than 0.5? Give necessary conditions for the existence of such a zero and the convergence of the bisection method, respectively. Assume that you happen to know that some function f has the values $f(0) = -1$, $f(1) = 7$ and $f(2) = 13$. How many bisection iterations are required to obtain and approximation of a zero of f with an error less or equal than 0.5? Give necessary conditions for the existence of such a zero and the convergence of the bisection method, respectively. *7 pts*

Question 4 **20 pts**

(a) Let $p(x)$ be the interpolation polynomial for the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ thus

$$p(x_k) = y_k, \quad k = 0, \ldots, n.$$

What is the degree of the interpolating polynomial (think carefully)? Consider the same interpolation points but in different order $(x_1, y_1), (x_0, y_0), (x_2, y_2), \ldots, (x_n, y_n)$. What is the interpolation polynomial for this reordered set? How do the coefficients and basis functions for our three interpolation methods relate to the coefficients and basis functions of the same method for the orginal order of the points? *5 pts*

(b) Explain why you expect the Chebyshev points to give a better approximation in general than equidistant points both in terms of the location of the points and the error formula. *5 pts*

(c) Use Newton's interpolation method to prove that the interpolant of a polynomial function $f(x)$ of degree two is itself of degree two no matter how many interpolation points you use. (NB Do not use the uniqueness result of polynomial interpolation but determine the coefficients of the Newton interpolant and use them to show the result.) *5 pts*

(d) Assume that you have computed an interpolant for the interpolation points $x_0 = 0, x_1 = 2, x_2 = 4$ and $x_3 = 6$ of some function $f(x)$. Now assume that you observe that the interpolant is not good for $x = 3$ so that you would like to add another interpolation point. How can you update your previous interpolation polynomial using this one point without having the compute the full interpolant from scratch? *5 pts*

Question 5 **20 pts**

(a) Consider the problem of computing the integral $I(f) = \int_0^{2\pi} f(x)\,dx$ using 3 quadrature points. What is the highest degree of polynomials you can integrate exactly with three points? How do you have two choose the points such that you achieve this optimal accuracy? Now if you would know that your function is periodic such that all the derivatives are 2π periodic, how would you choose your points and weights then? Explain. *6 pts*

(b) Get upper bounds for the absolute value of the approximation error of a central difference approximation of the derivative $f'(x)$ if f is sufficiently smooth. Assume that the function values $f(x)$ have a rounding error with an absolute value which is bounded by δ. How large is the error in the central difference approximation originating from the rounding error? How would you choose h such that the corresponding approximation error is of the same order as the rounding error? Why would this be a good choice? *7 pts*

(c) Assume you have computed a composite equi-distant trapezoidal rule approximation of an integral $\int_0^1 f(x)\,dx$. Assume also that you observe that the accuracy of your method is not sufficient so you determine an approximation with about twice the number of points. If you have not the stored the function values from the first approximation, only the value of the quadrature, can you still determine the improved value of the finer rule without having to recompute the function values? If the accuracy is still not good enough, can you then get an even better approximant without having to compute any new function values? What is your method? *7 pts*

Question 6 **20 pts**

(a) Rewrite the 2nd order differential equation

$$\frac{d^2u(t)}{dt} + \alpha\frac{du(t)}{dt} + u(t) = 0$$

as an equivalent system of two 1st order differential equations of the form

$$\frac{dv(t)}{dt} = Av(t).$$

What is the 2 by 2 matrix A? How are the 2 components of $v(t)$ connected to $u(t)$? What are the initial conditions for $v(t)$ if the initial conditions for $u(t)$ are $u(0) = 1$ and $u'(0) = 0$ where u' is the derivative of u. *5 pts*

(b) Give a formula for the exact solution $v(t)$ of the system of differential equations from (a). Explain what the terms you used in this formula mean. *5 pts*

(c) Show that Euler's method applied to the system in (a) is consistent and stable. Why are these properties relevant? *5 pts*

(d) Show that for the system in (a) Runge-Kutta's 4th order method is of the form

$$v^{n+1} = Mv^n$$

where v^n denotes the approximation of the exact solution $v(t)$ for $t = nh$. What is the matrix M? Compare this to the Taylor series for the exact solution $v(t)$. (Hint: $M = p(A)$ for some polynomial p.) *5 pts*