ASTR2013 Tutorial 4

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Everyone should do questions (1) and (2). Think about and attempt (3) and (4) if you get time - we can go through solutions next week. Textbook questions 4.2 and 4.4 are also recommended.

1. For a general degenerate electron gas, we can define the Fermi momentum based on an electron density n_e:

$$p_F = \frac{h}{2}\left(\frac{3n_e}{\pi}\right)^{1/3} \tag{1}$$
$$= \frac{h}{2}\left(\frac{3\rho\mathcal{Z}}{\pi m_H A}\right)^{1/3} \tag{2}$$

   We can write the equation of state in differential form (not in the textbook, but easy to derive from with some relativity):

$$\frac{dP_e}{d\rho} = \frac{p_F^2 c}{3m_h\sqrt{p_F^2 + m_e^2c^2}}\left(\frac{\mathcal{Z}}{A}\right) \tag{3}$$

   Assuming that pressure is dominated by this electron pressure, show that the equation of hydrostatic equilibrium can be written:

$$\frac{d\log(\rho)}{dr} = \frac{-GM(r)}{r^2} \Bigg/ \frac{dP_e}{d\rho} \tag{4}$$

2. This equation and the equation above have been coded for you in a Jupyter notebook on the course website.
   (a) Compare density versus radius plots for a central density where the Fermi momentum is much larger than m_ec, and where the Fermi momentum is much smaller than m_ec. What do you notice? Repeat the same for mass versus radius.
   (b) For a temperature of 10,000K and central density of 10^6 $\mathrm{g\,cm^{-3}}$, at what density does the assumption of a degenerate electron gas break down? As I haven't given you the complex analytic solution to electron pressure, you probably want to use 4.27 in the text and the ideal gas law.
   (c) Create an evenly spaced `numpy` array consisting of 30 values of $\log(\rho_c)$ between $\log(10^4)$ and $\log(10^{11})$ with ρ_c in $\mathrm{g\,cm^{-3}}$ as required for the function input, using `np.linspace`. Plot surface radius versus total mass, both with lines and dots. What is the maximum mass for a white dwarf?

3. The textbook approximates nuclear reactions as being determined by point-like particles and slightly complex quantum mechanics. An alternative, approximate, simpler approach might be to base our assumptions on physics experiments (e.g. the Geiger-Marsden experiments). These have shown that the nuclear density is approximately constant, giving:

$$r_{\text{nuc}} \approx A^{1/3} \times 1.2 \times 10^{-15} \text{ m}, \tag{5}$$

where A is the atomic mass number (1 for ^{1}H, 4 for ^{4}He, 12 for ^{12}C). Using SI units, Coulomb's law can be used to find a potential energy between two charges:

$$E = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r} \tag{6}$$

What is the characteristic energy of two Hydrogen nucleii approaching eacy other to within one nuclear radius? What about one H and one C or one He and one C? Express your answer in eV and a characteristic temperature by dividing by Boltzmann's constant.

4. The p-p chain fusion is complicated by the small cross-section of reactions, but we'd naievely expect the CNO cycle to have a high cross section if a H can (classically) reach the C nucleus. Lets compute the number of interactions per second. At a characteristic temperature of a star where the CNO cycle dominates of 100 MK, what is the typical velocity of a proton? At a characteristic density of 100 g cm^{-3} what is the number density? Combine the velocity with the cross section of the carbon nucleus to get a rate of close approaches (within a nuclear radius) per Hydrogen nuclei, and the number of opportunities to react in the lifetime of a star that survives for 100 Myr. Given that the classical chance of a collision on each close approach is $\exp(-E/k_B T)$, would the chance of a nuclear collision be high enough for a star of central temperature 10^8 K.

5. Find the limiting high density at the core of a white dwarf, when ^{1}2C nucleii are touching. What does this limit the white dwarf mass to? [NB in practice, a thermonuclear explosion due to high temperatures caused by compression occurs prior to this point]