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First Semester 2019
Mathematical Sciences Institute
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Assignment 5

This assignment must be submitted by **31st May 5pm**. Late Submissions will incur a 5% penalty per working day. Assignment submissions will close on the **7th June 5pm**. Submissions after this time will be invalid.

Question 1 (Quadrature) [50 pts]

1. Recall the formula for a (composite) trapezoidal rule $T_n(u)$ for $I = \int_a^b u(x)dx$ which requires n function evaluations at equidistant quadrature points and where the first and the last quadrature points coincide with the integration bounds a and b, respectively. [10pts]

2. For a given $v(x)$ with $x \in [0,1]$ do a variable transformation $x = g(\xi) = \alpha\xi + \beta$ such that $g(-1) = 0$ and $g(1) = 1$. Use this to transform the integral $I = \int_0^1 v(x)dx$ to an integral over $[-1,1]$. What are α and β? Show that for $u(\xi) = \alpha v(\alpha\xi + \beta)$ one has $I = \int_{-1}^1 u(\xi)d\xi = \int_0^1 v(x)dx$. [10pts]

3. Consider the function $v(x) = x(x-1/2)(x-1)$ for $x \in [0,1]$. Determine the transformed function $u(\xi)$ introduced in the previous question. Show that $\int_{-1}^1 u(\xi)d\xi = 0$. (Hint: you can do this without evaluating the function.) Determine the values of the midpoint rule,the simple trapezoidal rule (with two points) and of the Gaussian rule with 2 quadrature points. What do you observe about the accuracy of these rules? [10pts]

4. Consider the function $u(\xi) = (\xi^2 - 1)^2$ for $\xi \in [-1,1]$. Compute the values of the (composite) trapezoidal rule for equidistant points, and 2,3 and 5 points. Construct the corresponding Romberg tableau $R_{i,j}$ as in the lectures. Show that exactly one of the 3 computed values is exact. Give the reason why this is the case. (Hint: this follows from the Euler-Maclaurin formula.) [10pts]

5. How many Gauss points are required if the Gauss quadrature rule should provide the exact value of the integral $I = \int_{-1}^1 u(\xi)d\xi$ for the function $u(\xi) = (\xi^2 - 1)^2$? Prove that your answer is a consequence of a major result covered in the lectures about Gaussian quadrature.Provide the values of the weights and points (and show how to compute them) for this rule.[10pts]

Question 2 (Differences) [30 pts]

1. Use the method of undetermined coefficients to compute the coefficients of a finite difference approximation for $u'(\xi)$ using the values $u(0)$,$u(1)$ and $u(2)$. Choose the coefficients such that the formula is exact for polynomials with degree less or equal to 2. Can you use these ecoefficients to get an approximation for a first derivative based on function values $v(x)$,$v(x+h)$ and $v(x+2h)$? At which point z and for which functions $v(x)$ is this approximation equal to $v'(z)$? Determine the coefficients exactly for $\xi = 3/4$. [10pts]
2. Evaluate these formulas to determine $v'(\xi h)$ using $v(0)$, $v(h)$, $v(2h)$ for $v(x) = \exp(x)$ and $h = 1, 0.5, 0.25$. Determine the error for these three values of h. Based on these values, what would you guess the dependence of the error on h to be - linear, quadratic, cubic or of higher order? For this problem choose $\xi = 3/4$. [10pts]
3. Recall the error formula from the lecture and apply them to this problem. Derive upper and lower bounds, and compare them with the previously computed errors. [10pts]

Question 3 (Euler's method) [20pts]

Consider the initial value problem given by $u'(t) = u(t)$ and $u(0) = 1$.

1. What is the exact solution? [3pts]
2. Apply Euler's method to this problem. Show that you get

   $$u_k = (1+h)^k, \quad k = 0, 1, ...$$

   as the approximation for $u(t_k)$, with $t_k = hk$. [7pts]
3. Use the formula $\log(1+h) = h - 0.5h^2 + O(h^3)$ to show that

   $$u(t_k) - u_k = 0.5ht_ke^{t_k} + O(h^2).$$

   [10pts]