Consider the IVP

$$\left.\begin{aligned} \frac{du}{dt} &= \lambda u, \quad \lambda \in \mathbb{C}, \quad t > 0 \\ u(0) &= u_0 \end{aligned}\right\} \text{—} (*)$$

1. What is the solution $u(t)$ $\circledast$

2. For what values of λ is the solution

   (i) Unstable
   (ii) Stable
   (iii) Asymptotically stable.

3. Let $Re(\lambda) < 0$, and ~~Consider~~ Introduce a uniform discretization of the time variable with a constant step-size $h > 0$

   $t_K = Kh, \quad K = 0, 1, \ldots$

   Consider

   (a) $$\frac{u_{k+1} - u_k}{h} = \lambda u_k$$

   (b) $$\frac{u_{k+1} - u_k}{\Delta t} = \lambda u_{k+1}$$

   (c) $$\frac{u_{k+1} - u_{k-1}}{2h} = \lambda u_k$$

   (d) $$\frac{u_{k+1} - u_{k-1}}{2h} = \frac{\lambda}{2}(u_{k+1} + u_{k-1})$$

For each of the following methods (a) – (d)

(i) Show that the method is a consistent approximation of the IVP ⊛

(ii) What is the truncation error

(iii) Derive the region of absolute stability.

(iv) Is the method implicit or explicit
Give reasons!

(v) Is the method on-step or Multi-step

(vi) What is the maximum ~~allowable~~ stable time-step.

(vii) Is the method Conditionally stable, Unconditionally stable / Unstable!

2) Consider the scheme approximating the IVP

$$U^* = U_k + \lambda h U_k$$

$$U_{k+1} = U_k + \frac{\lambda h}{2}(U_k + U^*) \qquad (**)$$

i) What is the order of accuracy.

ii) Is this a one-step or two-step method
Why?

iii) Derive the growth factor and the region of absolute stability.

iv What is the maximum stable time-step.

d) For the method (**), ~~if~~ assume that you have computed the solution with the step size $h = 0.1$ and the error is $e = 0.1$. If you wanted to have 1% error, what is the ~~the~~ largest possible time-step required to achieve this.

Consider the IVP

$$\left.\begin{aligned} \frac{du}{dt} &= Au, \quad A \in \mathbb{R}^{n\times n}, \quad u \in \mathbb{R}^n \\ u(0) &= u_0 \end{aligned}\right\} \text{—} (+)$$

(4) What is the solution $u(t)$

~~(3) Consider the numerical method~~

(5) Let λ_i denote the i-th eigenvalue of A. For what values of λ_i is (+)

(i) Unstable

(ii) Stable

(iii) Asymptotically stable

6) Consider the numerical method for (+)

$$\cancel{U_{k+1} = U_k + hAU_k}$$

$$U^* = U_k + hAU_k$$

$$U_{k+1} = U_k + \frac{h}{2}\left(AU_k + AU^*_*\right)$$

(i) what is the order of accuracy of the approximat.

(ii) Derive the growth factor

(iii) What is the region of absolute stability.

(7) Write down the RK4 approximation of (*) and (+), respectively.

(8) Show that the numerical approximation (RK4) of (*) and (+) correspond to a truncated taylor series expansion of the solution.

(9) What is the truncation error

(10)

Consider the second order ODE

$$\left.\begin{aligned} &\frac{d^2u}{dt^2} + \beta\frac{du}{dt} + K^2 u = 0, \\ &u(0) = f, \quad \frac{du}{dt}(0) = g, \end{aligned}\right\} \quad \text{—— } (++)$$

where β and K are real parameters.

(10) Rewrite $(++)$ as a first order system

$$\frac{d\bar{u}}{dt} = A\bar{u} \qquad \text{————} \quad (+++)$$

(11) Determine the matrix A and the initial condition $\bar{u} = 0$?

(12) Detemine the conditions on β for which $(+++)$ is

a) unstable
b) stable
c) asymptotically stable.

(13) Let $\beta < 0$ and approximate $(+++)$ with a one-step method

$$u_{k+1} - u_k - (t_{k+1} - t_k)\,\Phi(t_k, u_k) = 0$$

i) $\Phi(t_k, u_k) = Au_k$

ii) $\Phi(t_k, u_k) = \frac{1}{2}\left(Au_k + A(u_k + Au_k)\right)$

For (i) and (ii)

~~Determ~~

(a) Determine the truncation error

b) What is the order of accuracy

c) Determine the growth factor and the region of absolute stability.

d) What is the maximum time-step required for stability.

(14) Consider the discretization of (++)

$$\frac{U_{k+1} - 2U_k + U_{k-1}}{h^2} + \beta \frac{U_{k+1} - U_{k-1}}{2h} + K^2 U_k = 0$$

(a) What is the truncation error and the order of accuracy.

(b) Is this a one-step or two-step method Why?

(c) Is this method explicit or Implicit.

(d) In particular if $\beta = 0$? explicit or Implicit

(e) What is the maximum time-step required for stability.