Assignment 3

May 3, 2019

Q1.a

By matrix calcualtion

$$AS = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$SA = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

Hence AS=SA, A is a circulat matrix.

Circulant matrix A can be of this form

$$A = \sum_{k=0}^{n-1} a_k S^k$$

From lecture note

$$Ae_1 = a$$

where $e_1 = (1, 0, \ldots, 0)^T$.

by calculation $a = (a_0, \ldots, a_{n-1})^T = (1, -1, 0, 0)^T$

We can prove that:

$$[Ax]_k = \sum_{m=0}^{n-1} a_m [S^m x]_k = \sum_{m=0}^{n-1} a_m x_{k-m} = \sum_{m=0}^{n-1} a_{k-m} x_m = [a * x]_k$$

Hence

$$Ax = a * x$$

$$a = (1, -1, 0, 0)^T$$

Q1.b

The eigenvalue of S are of the form:

$$\lambda_k = \exp(-2\pi i k/4) = \omega_4^{-k}$$

$$\omega_4 = i$$

The eigenvalue of S are: $\lambda_0 = 1$ $\lambda_1 = -i$ $\lambda_2 = -1$ $\lambda_3 = i$

- the eigenvectors are of the form
$$v_k = (1, i^k, i^{2k}, i^{3k})^T$$
Hence the eigenvactors of S are:
$$v_0 = (1,1,1,1)^T$$
$$v_1 = (1,i,-1,-i)^T$$
$$v_2 = (1,-1,1,-1)^T$$
$$v_3 = (1,-i,-1,i)^T$$

- the *discrete Fourier transform matrix* is then the matrix
$$F_4 = [v_0, v_1, v_2, v_3] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

Hence we have
$$SF_4 = F_4 W_4$$
$$W_4 = diag(1,-i,-1,i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

Q1.c

From Q1.a we solve the following equation:
$$a * x = Ax = b$$

Fourier transform the both sides gives us:
$$F_4 a \cdot F_4 x = F_4 b$$

$$F_4 b = (0, 1+i, 2, 1-i)^T$$

$$F_4 a = (0, 1-i, 2, 1+i)^T$$

Hence
$$F_4 x = (f_0, i, 1, -i)^T$$
where
$$f_0 \in R$$

Q1.d

$$M = B4(I2 \otimes F2) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{bmatrix}$$

In the following formula

$$F_4 = MP_4$$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{bmatrix}$$

We interchange colomn 2 and 3 of M by manipulate the permutation matirx

$$P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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In [ ]:
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