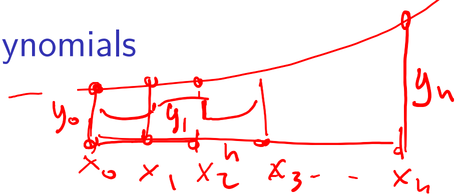
Piecewise polynomials

Piecewise linear (affine) polynomials



Definition:

A *piecewise linear function* $s : [a, b] \to \mathbb{R}$ for a grid $a = x_0 < x_1 < \cdots < x_n = b$ has function values

$$s(x) = a_k x + b_k, \quad x \in [x_{k-1}, x_k], \quad k = 1, \ldots, n$$

where a_k, b_k are constants

- for a general choice of a_k, b_k the function $s(x)$ is not continuous and has jumps at the grid points
- in the following we consider *continuous functions* $s(x)$
- a grid is *equidistant* if

$$x_k = a + kh, \quad k = 0, \ldots, n$$

and $h = (b - a)/n$.

pw linear interpolant

Definition: Given (x_k, y_k) for $k = 0, \dots, n$ with $x_0 < x_1 < \cdots < x_n$ the *linear interpolant* is the continuous piecewise linear function $s(x)$ satisfying the interpolation conditions

$$\boxed{s(x_k) = y_k,} \quad k = 0, \dots, n.$$

- the coefficients a_k, b_k are obtained by solving an interpolation problem for every subinterval $[x_{k-1}, x_k]$ and one gets

$$\begin{bmatrix} 1 & x_{k-1} \\ 1 & x_k \end{bmatrix} \begin{bmatrix} a_k \\ b_k \end{bmatrix} = \begin{bmatrix} y_{k-1} \\ y_k \end{bmatrix}$$

and the coefficients are

$$b_k = \frac{y_k - y_{k-1}}{x_k - x_{k-1}}, \quad a_k = y_k - b_k x_k = \frac{-y_k x_{k-1} + y_{k-1} x_k}{x_k - x_{k-1}}$$

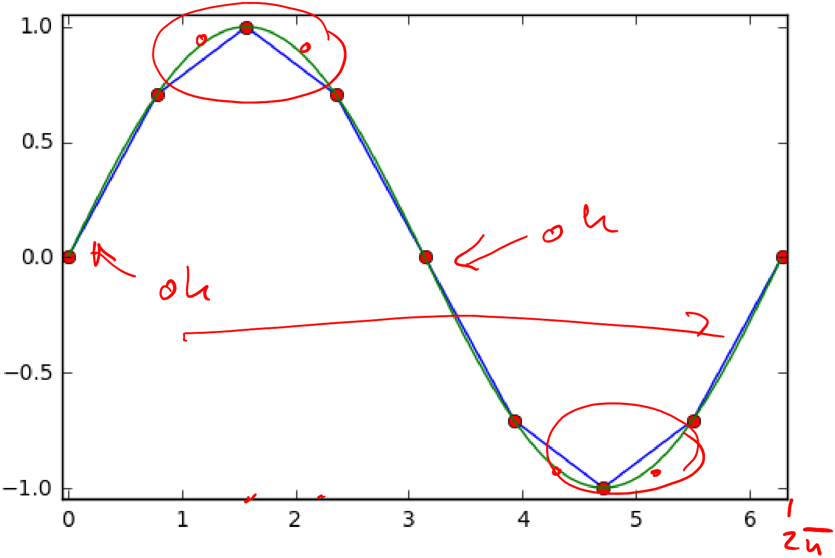
pw linear interpolant of sin(x)

```
x = np.linspace(0, 2*np.pi, 9)
y = np.sin(x)

s = interpolate.InterpolatedUnivariateSpline(x, y, k=1)
            # k=1: pw linear, s is a function

xnew = np.linspace(0,2*np.pi,100)
```

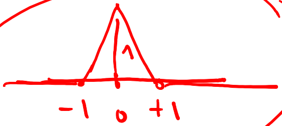
equidistant

s(4)

```
plt.plot(xnew, s(xnew), x,y,'ro',xnew,np.sin(xnew));
plt.axis([-0.05, 6.33, -1.05, 1.05]);
```



hat function



we will use the function

$$h(x) = (1 - |x|)_+, \quad x \in \mathbb{R}$$

where

- $|x|$ is the *absolute value* of x
- $(x)_+$ is the *positive value* of x such that
  - $(x)_+ = x$ if $x \geq 0$
  - $(x)_+ = 0$ if $x < 0$

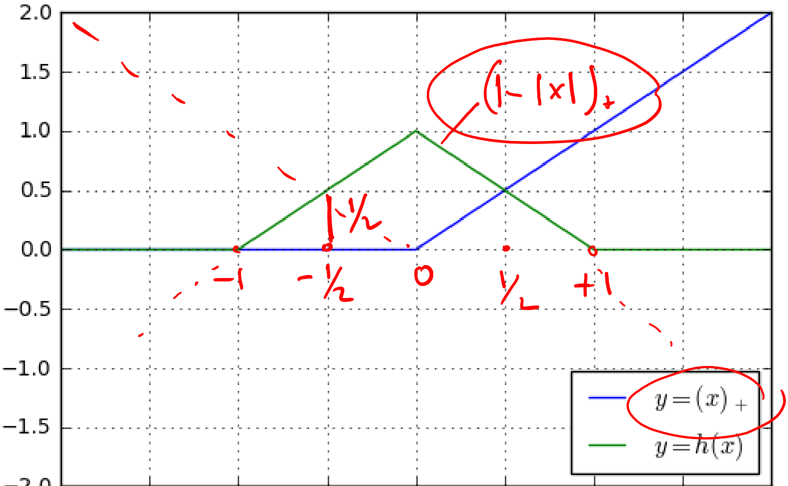
$$(4)_+ = 4$$
$$(-\pi)_+ = 0$$

- one then has

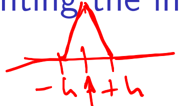
$$x = (x)_+ - (-x)_+$$
$$|x| = (x)_+ + (-x)_+$$

graph of hat function

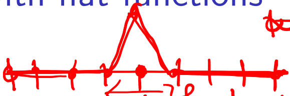
```
x  = np.linspace(-2,2,257)
plt.plot(x,np.maximum(x,0),label='$y=(x)_+$')
plt.plot(x,np.maximum(1-abs(x),0),label='$y=h(x)$')
plt.legend(loc=4);plt.grid('on');plt.axis(ymin=-2);
```

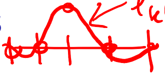


representing the interpolant with hat functions



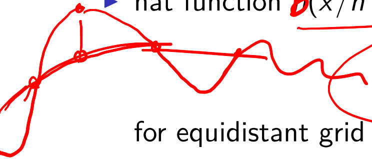
$x_k = kh$





$\delta_{jk} = \begin{cases} 1 & j=k \\ 0 & \text{else} \end{cases}$

$x_j = jh$

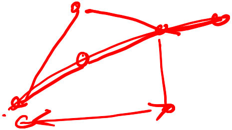


- hat function $h(x/h - k)$ satisfies

$$h(x_j/h - k) = \delta_{jk}$$

for equidistant grid $x_k = kh$ where $h = 1/n$ and $k = 0, \ldots, n$

- interpolating $s(x)$ takes Lagrangian form:



$$\boxed{s(x) = \sum_{k=0}^{n} y_k h\left(\frac{x}{h} - k\right)}$$

φ

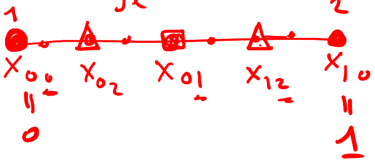
φ_k

- hat functions are *nodal basis* $b_k(x) = h(x/h - k)$
- support $\operatorname{supp} b_k = [(k-1)h, (k+1)h]$ thus y_k affects s only locally

hierarchical basis

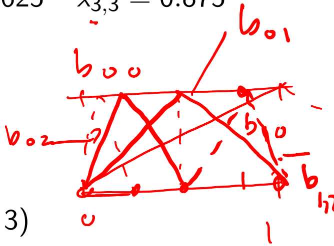
- using hat functions, one defines a Newton-style interpolation formula
- introduce two indices for grid points $x_{j,l}$ where l is called level

  x_{jl}
- interval $[0, 1]$:



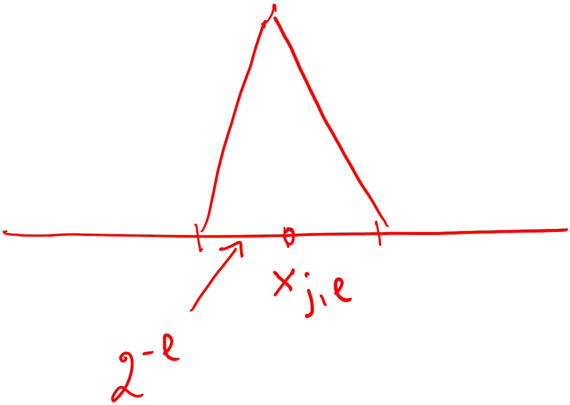
$$x_{0,0} = 0 \qquad x_{1,0} = 1$$
$$x_{0,1} = 0.5$$
$$x_{0,2} = 0.25 \qquad x_{1,2} = 0.75$$
$$x_{0,3} = 0.125 \qquad x_{1,3} = 0.375 \qquad x_{2,3} = 0.625 \qquad x_{3,3} = 0.875$$

- basis functions:



$$b_{0,0}(x) = 1 \qquad b_{1,0}(x) = x$$
$$b_{0,1}(x) = h(2x - 1)$$
$$b_{0,2}(x) = h(4x - 1) \qquad b_{1,2}(x) = h(4x - 3)$$
$$b_{0,3}(x) = h(8x - 1) \qquad b_{1,3}(x) = h(8x - 3) \qquad b_{2,3}(x) = \ldots$$

graphs of $b_{j,l}(x)$



coefficients of interpolants

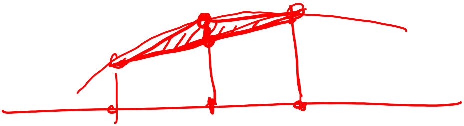
$$\boxed{s(x) = c_{0,0} b_{0,0}(x) + c_{1,0} b_{1,0}(x) + \sum_{l=1}^{m} \sum_{j=0}^{2^{l-1}-1} c_{j,l} b_{j,l}(x)}$$

where

$$c_{0,0} = y_{0,0} \qquad c_{1,0} = y_{1,0} - y_{0,0}$$

$$c_{0,1} = y_{0,1} - 0.5(y_{0,0} + y_{1,0})$$

$$c_{0,2} = y_{0,2} - 0.5(y_{0,0} + y_{0,1}) \qquad c_{1,2} = y_{1,2} - 0.5(y_{0,1} + y_{1,0})$$



minimisation property of piecewise linear function

Proposition
Let $s(x)$ be the piecewise linear interpolant of the points $(x_k, y_k), \quad k = 0, \ldots, n$ and $g(x)$ be any continuous function with $g(x_k) = y_k$ which is continuously differentiable in each subintervall (x_{k-1}, x_k) then

$$\int_{x_0}^{x_n} s'(x)^2 \, dx \leq \int_{x_0}^{x_n} g'(x)^2 \, dx.$$

Euler eqns

$s'' = 0$

Proof. Calculus of variation, show this is true for C^2 functions, then take the limit.

- thus s minimises the average squared change of the function.

Cubic splines

Proposition

Consider the class V of functions which are continuous and C^2 on each interval $[x_{k-1}, x_k]$. Then there exists a function $s(x)$ in that class which interpolates $s(x_k) = y_k$, for $k = 0, \ldots, n$ which satisfies

$$\boxed{\int_{x_0}^{x_n} s''(x)^2 \, dx \leq \int_{x_0}^{x_n} g''(x)^2 \, dx.} \longrightarrow s^{(4)}(x) = 0$$

for all $g \in V$ which interpolate $g(x_k) = y_k$ for $k = 0, \ldots, n$.

This function s is called the cubic spline interpolant, it is continuously differentiable and is a piecewise cubic polynomial.

Proof. similar as for piecewise linear interpolant.

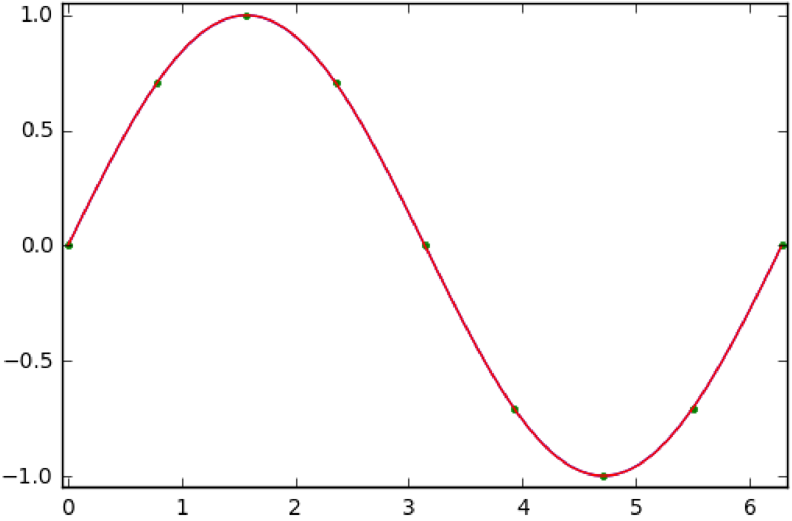
- the cubic spline minimises the average squared second derivative which is a substitute for the curvature
- this cubic spline has zero second derivative at the boundary, an alternative is to impose values of the first or second derivative on the boundary

Scipy cubic spline interpolant

- cannot see the difference between the interpolant and the exact function in the plot!

```
# Cubic spline (Lagrange) interpolation from Scipy

from scipy import interpolate
from scipy.interpolate import CubicSpline

x = np.linspace(0, 2*np.pi, 9)
y = np.sin(x)
s = CubicSpline(x, y, bc_type="natural")  # natural bounda
xnew = np.linspace(0,2*np.pi,100)
```

```
plt.plot(xnew, s(xnew), x,y,'.',xnew,np.sin(xnew));
plt.axis([-0.05, 6.33, -1.05, 1.05]);
```



Hermite Interpolation with Piecewise Cubic Functions

- Hermite interpolant $H(x)$ of a function $f(x)$ satisfies:
  - interpolation condition for function value at x_k
  - interpolation condition for derivative at x_k
- This gives for conditions per interval $[x_{i-1}, x_i]$:

$$
\begin{aligned}
H(x_i) &= f(x_i) = y_i, & H'(x_i) &= f'(x_i) = y'_i \\
H(x_{i+1}) &= f(x_{i+1}) = y_{i+1}, & H'(x_{i+1}) &= f'(x_{i+1}) = y'_{i+1}
\end{aligned}
$$

- These conditions uniquely determine the polynomials of degree three in the intervals

Parametrisation

$$H_i(x) = a_i + b_i(x - x_i) + (x - x_i)^2[c_i + d_i(x - x_{i+1})]$$

- With $h_i = x_{i+1} - x_i$ the four interpolation conditions give

$$\begin{aligned} a_i &= y_i, & b_i &= y_i', \\ c_i &= \frac{y_{i+1} - y_i}{h_i^2} - \frac{y_i'}{h_i}, & d_i &= \frac{y_{i+1}' + y_i'}{h_i^2} - \frac{2(y_{i+1} - y_i)}{h_i^3} \end{aligned}$$

- Approximation often similar to the B-spline interpolant