1.5 Decimal Rounding

The decimal floating point numbers with two digits $\mathbb{F}_{10}(2)$

- Any positive (normalised) $x \in \mathbb{F}_{10}(2)$ is of the form

$$x = \frac{n}{100} 10^e, \quad n = 10, \dots, 99, \; e \in \mathbb{Z}$$

- There are 91 such floating point numbers between 0.1 and 1:

$$S = \{0.10, 0.11, 0.12, \dots, 0.99, 1.0\}$$

- Every number $x \in S$ which is less than one has a *successor*

$$\operatorname{succ} x = x + 0.01$$

- We will also use the set of 90 *midpoints* between the floating point numbers

$$M = \{0.105, 0.115, 0.125, \dots, 0.985, 0.995\}$$

and $M \not\subset \mathbb{F}_{10}(2)$

Rounding

A *rounding function* $\phi : \mathbb{R} \to \mathbb{F}_{10}(2)$ has the following properties

- $\phi(x) = x$ for $x \in \mathbb{F}_{10}(2)$
- if $x \leq y$ then $\phi(x) \leq \phi(y)$ (monotonicity)
- $\phi(-x) = -\phi(x)$
- $\phi(10x) = 10\,\phi(x)$

It follows from the first two properties that any value $\phi(x)$ is either equal to the next lower or next higher floating point number, for example

$$\phi(0.12456) \in \{0.12, 0.13\}$$

The third and fourth property lets us extend the definition of ϕ from the interval $[0.1, 1]$ to the whole set of real numbers $\mathbb{R}$. For example, one has

$$\phi(124.56) = 100\,\phi(0.12456)$$

applying the fourth property twice

Examples of rounding functions include

- *truncation* where

$$\phi(0.x_1x_2x_3\ldots) = 0.x_1x_2$$

  and thus $\phi(0.1256) = 0.12$

- *rounding towards zero*
  - $\phi(0.x_1x_2x_3\ldots) = 0.x_1x_2$ if $x_3 \in \{0, 1, 2, 3, 4\}$
  - $\phi(0.x_1x_2x_3) = 0.x_1x_2$ if $x_3 = 5$ (midpoints)
  - $\phi(0.x_1x_2x_3x_4\ldots) = 0.x_1x_2 + 0.01$ if $x_3 = 5$ and $x_i > 0$ for some $i > 4$

and thus $\phi(0.1256) = 0.13$, $\phi(0.125) = 0.12$ and $\phi(0.124) = 0.12$

- *rounding used in most computers* is the same as rounding towards zero except that the second condition for the midpoints is replaced by two cases:
  - $\phi(0.x_1x_2x_3) = 0.x_1x_2$ if $x_3 = 5$ and x_2 is even
  - $\phi(0.x_1x_2x_3) = 0.x_1x_2 + 0.01$ if $x_3 = 5$ and x_2 is odd

thus $\phi(0.125) = 0.12$ but $\phi(0.135) = 0.14$
this condition corrects for the bias towards zero

Finally, a rounding function ϕ is *optimal* if it minimises the *rounding error* $|\phi(x) - x|$, i.e., if

$$|\phi(x) - x| \leq |y - x|, \quad \text{for all } y \in \mathbb{F}_{10}(2)$$

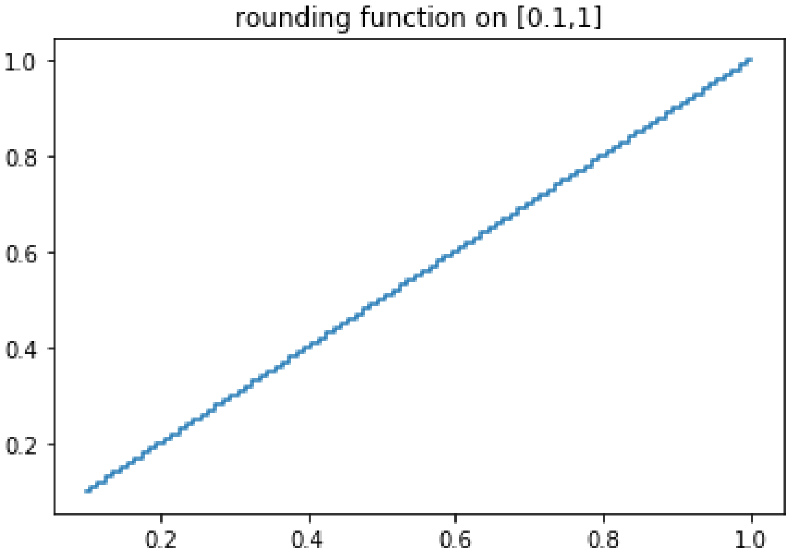
Truncation is not optimal, but both rounding towards zero and the rounding used in most computers are optimal.
Note that the rounding function used in computers rounds to a different set $\mathbb{F}_2(53)$, however, it uses the same tie-breaking strategy for the midpoints.

In the following we plot the graph of the rounding function ϕ both for $[0.1, 1]$ and for $[0.1, 10]$. Note that on each intervall $[10^{e-1}, 1^e]$ the rounding function is a step function with constant steps at the midpoints between the floating point numbers. The height of the step is proportional to 10^e.

```
# optimal rounding functions

%matplotlib inline
from decimal import Decimal, getcontext
getcontext().prec = 3
from pylab import plot, title, loglog
t = 2
h = Decimal('0.1')**t
x = Decimal('0.1')
xg = [x,]
yg = [x,]
```

```python
for i in range(9*10**(t-1)):
    xg.append(x+h/2) # midpoint
    yg.append(x)
    xg.append(x+h/2) # midpoint
    yg.append(x+h)
    xg.append(x+h)
    yg.append(x+h)
    x += h
```

```
title('rounding function on [0.1,1]')
plot(xg,yg);
```



```
xg += [10*x for x in xg]
yg += [10*y for y in yg]
```

```
title('rounding function on [0.1,10]')
loglog(xg,yg);
```

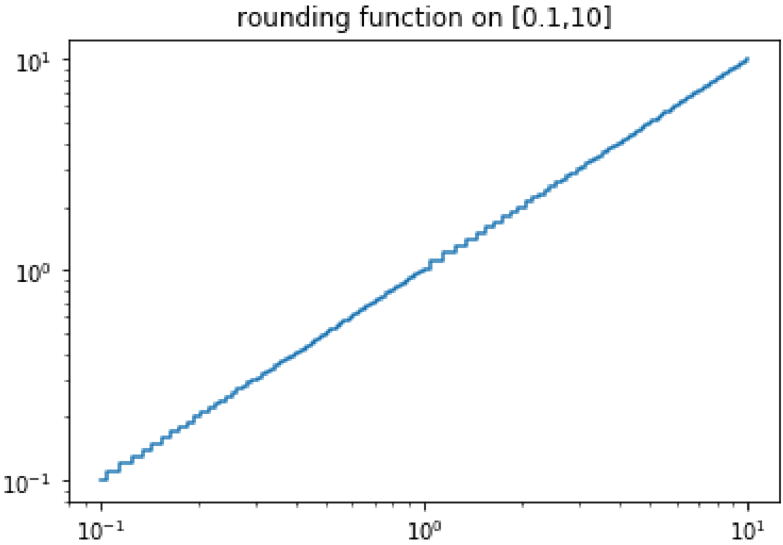


Figure 2: png

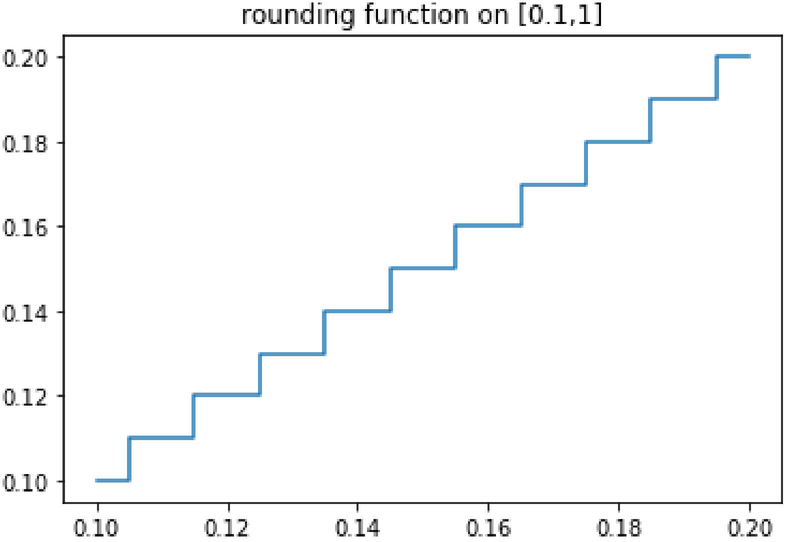
Rounding errors

In the following plots we have a closer look at the rounding function and the absolute and relative rounding errors.

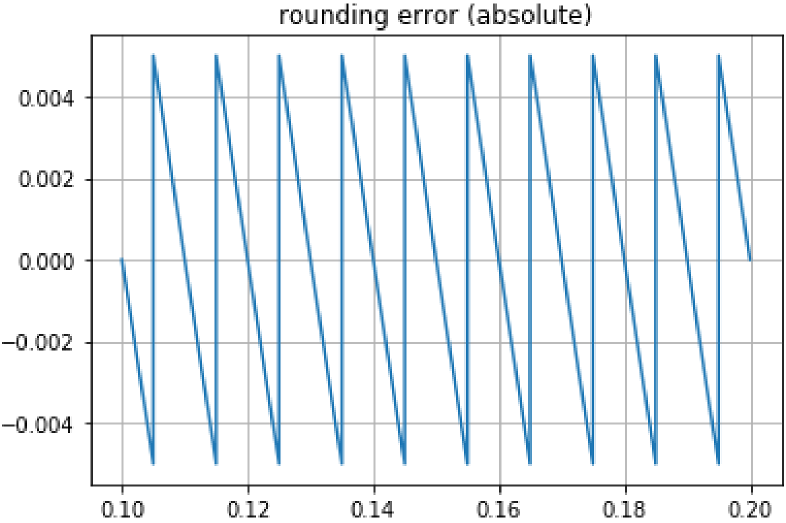
```
# rounding  function and error

%matplotlib inline
from decimal import Decimal, getcontext
getcontext().prec = 6
from pylab import plot,  title, loglog, grid
h = Decimal('0.01')
hg = h/20
x = Decimal('0.1')
y = Decimal('0.1')
xg = [x,]
yg = [y,]
nx = 10
```

```python
for k in range(nx):
    for i in range(10):
        x += hg
        xg.append(x)
        yg.append(y)
    y += h
    xg.append(x)   # double up midpoint
    yg.append(y)
    for i in range(10):
        x += hg
        xg.append(x)
        yg.append(y)
```

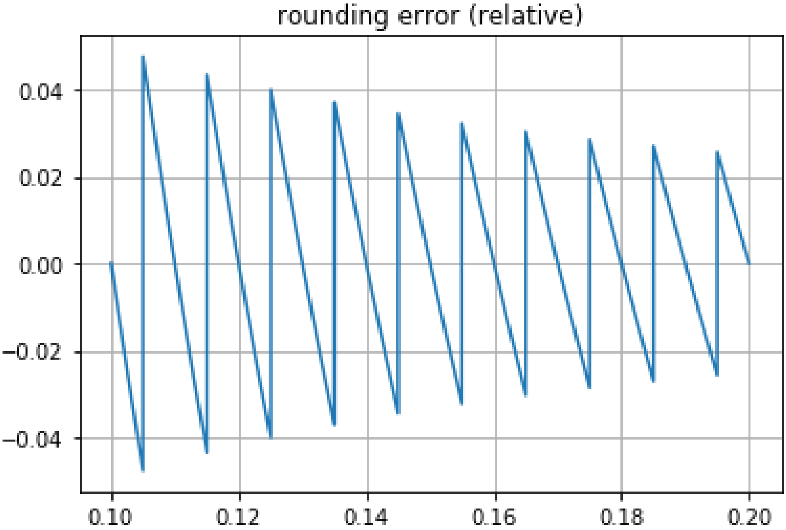
```
title('rounding function on [0.1,1]')
plot(xg,yg);
```



```
title('rounding error (absolute)'); grid('on');
eg = [yg[i]-xg[i] for i in range(len(xg))]
plot(xg, eg);
```



```
title('rounding error (relative)'); grid('on');
erg = [(yg[i]-xg[i])/xg[i] for i in range(len(xg))]
plot(xg,erg);
```



One can see that the maximal relative rounding error occurs at the first midpoint $x = 0.105$. This is rounded to $\tilde{x} = 0.1$ and the error is thus $\tilde{x} - x = -0.005$. The absolute value of the maximal relative error is then

$$\delta = \left| \frac{\tilde{x} - x}{x} \right| = \frac{0.005}{0.105} = 0.0476.$$

This is close to the upper bound of $0.5B^{-t+1}$ (with $B = 10$ and $t = 2$) given in the lectures and the bound gets better for larger mantissa sizes t.

Suggestion: Study the plots and derive a formal proof that the maximum of the rounding error is indeed obtained in the first mid point.