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First Semester 2019
Mathematical Sciences Institute
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Assignment 4

This assignment must be submitted by **17th May 5pm**. Late submissions will incur a 5% penalty per working day. Assignment submissions will close on the **24th May 5pm**. Submissions after this time will be invalid.

Question 1 (General, polynomial) [20pts]

- [5pts] Using only properties of basic real arithmetic show that

$$1 + x + x^2/2 + x^3/6 = ((x/3 + 1)x/2 + 1)x + 1.$$

- [5pts] Let $f(x) = \max(1 - |2x - 1|, 0)$, $x \in [0, 1]$. Can this function be approximated to an error less than 10^{-9} using only polynomials? Justify your answer.
- [5pts] Compute the linear interpolant of $f(x)$ with

$$f(x) = \begin{cases} \frac{1}{|2x-1|}, & x \neq 0.5, \\ 0, & x = 0.5, \end{cases}$$

  and the interpolation points $x_0 = 0$ and $x_1 = 1$. What is the largest interpolation error in the interval $[0, 1]$?
- [5pts] How many floating point multiplications do you require to evaluate a Newton inter-polant $p(x)$ of degree 4 and the interpolation points $x_k = k/4$ for $k = 0, 1, 2, 3, 4$ both for $x = \pi$ and $x = -1$. Use the Horner-like approach discussed in the lectures and take into account the special values of x_k and x.

Question 2 (Newton form) [20pts]

- [7pts] Use Newton the interpolation method with the interpolation points $x_0 = 0$, $x_1 = 1$, $x_2 = 1/2$ to compute the polynomial interpolant $p(x)$ of the function $f(x) = 1 + x + x^2/2 + x^3/6$. What is the error $e(x) = p(x) - f(x)$. Prove your error result without using Taylor'sremainder theorem.
- [7pts] What are the Newton interpolants $p_0(x)$, $p_1(x)$, $p_2(x)$ and $p_4(x)$ for the interpolationpoints $x_0 = 0$, $x_1 = 1$, $x_2 = 0.5$, $x_3 = 0.25$ and

$x_4 = 0.75$ for any function $f(x)$ with $f(x_k) = y_k$. What is the ratio $(p_2(x) - p_4(x))/(p_1(x) - p_2(x))$? We assume that $p_4(x) = c_0 + c_1 n_1(x) + c_2 n_2(x) + c_3 n_3(x) + c_4 n_4(x)$ where $n_k(x)$ are the basis functions of the Newton interpolant and all $c_k \neq 0$.

- [6pts] Compare the ratio $(p_2(x) - p_4(x))/(p_1(x) - p_2(x))$ from the previous question to the ratio $(p_2(x) - f(x))/(p_1(x) - f(x))$ and we assume that $f^{(k)}(x) \neq 0$ for all k

Question 3 (Error) [20pts]

- [7pts] What is the interpolation error $e(x) = p(x) - f(x)$ of the polynomial interpolant $p(x)$ of degree 3 for $x_k = 2k/3 - 1$ and $f(x) = x^4 - 1.2356x^2$, where $k = 0, 1, 2, 3$ and -1< x <1.
- [7pts] Give a general formula for all polynomials $p(x)$ of degree 5 for which $p(k/4) = 0$ for $k = 0, 1, 2, 3, 4$.
- [7pts] Use the error formulas to get an upper bound (not containing x) for the error of the approximation of a function $f \in C^3[-1, 1]$ using
  - the interpolant with points $x_k = -1, 0, 1$,
  - the interpolant of degree 2 with Chebyshev interpolation points,
  - the Taylor polynomial of degree 2 centred at $x = 0$.

Explain why the Chebyshev interpolation points give the best approximation in general, if possible, for any n.

Tutorial (By tutor at the beginning of the tutorial.)

- Revise Horner's formula, example $p(x) = 3x^2 + 2x - 1$ and for the general Newton interpolation formula of degree 2.
- Revise Newton's interpolation formula, how it computes degree k interpolants $p_k(x)$ by looking at the recursion $p_{k+1}(x) = p_k(x) + c_k n_k(x)$.
- Apply the error interpolation formula to the polynomial interpolation of $f(x) = exp(-x)$ for the interpolation points $x_0 = 0$, $x_1 = 1.0$ and $x_2 = 0.5$. (Note they are not ordered.)
- Definition of Chebyshev polynomials, Chebyshev interpolation formula and the error formulawhen using 2 Chebyshev interpolation points (say what they are) for the interval $[-1, 1]$.
- Theory of polynomials: uniqueness (up to constant factor) of polynomial of degree k with given zeros. Application of this result to show uniquenes of interpolant. Example for degree 2 and $x_0 = 0$, $x_1 = 1$ and $x_2 = 0.5$.