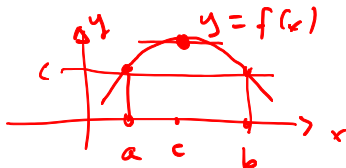


Error Formulae for Polynomial Collocation

Rolle's Theorem

Theorem



- ▶ If $f : [a, b] \rightarrow \mathbb{R}$ continuous
- ▶ and $f(x)$ differentiable for $a < x < b$
- ▶ and $f(a) = f(b)$

then there exists $c \in (a, b)$ such that

$$f'(c) = 0$$

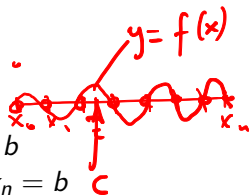
Proof considers maximum of function f
applications in calculus

- ▶ proof of mean value theorem
- ▶ proof of Taylor's theorem

[https://en.wikipedia.org/wiki/Rolle%27s_theorem]

Theorem (Generalisation of Rolle's Theorem)

- ▶ if $f \in C^{n-1}[a, b]$
- ▶ if $f(x)$ is n times differentiable for $a < x < b$
- ▶ if $f(x_k) = 0$ for $x_0 = a < x_1 < x_2 < \dots < x_n = b$



then there exists $c \in (a, b)$ such that

$$f^{(n)}(c) = 0$$

Proof

- ▶ induction over n
- ▶ apply Rolle's theorem to subintervals $[x_k, x_{k+1}]$

Error Formula

Theorem (Error of Lagrangian interpolation)

- ▶ if p_n polynomial (Lagrangian) interpolant of $f(x) \in C^{(n+1)}[a, b]$
- ▶ and $x_i, x \in [a, b]$ distinct real numbers, $i = 0, \dots, n$

then the interpolating polynomial p of degree n satisfies

$$p(x) - f(x) = \frac{-1}{(n+1)!} f^{(n+1)}(\xi) w(x), \text{ for some } \xi \in [a, b]$$

where $w(x) = (x - x_0) \cdots (x - x_n)$ $= x^{n+1} + \dots$

$$w(x) = (x - x_0) \cdots (x - x_n) \\ = x^{n+1} + \underbrace{x^n \cdots}_{\text{...}} + \dots$$

$$G(t) = e(x)w(t) - e(t)w(x)$$

$$G(x_i) = e(x)w(x_i) - e(x_i)w(x)$$

$$= e(x) \overset{0}{w(x)} - \overset{0}{e(x_i)} \overset{0}{w(x)} = 0$$

Proof



$$\bullet G(t) := (f(x) - p(x))w(t) - (f(t) - p(t))w(x)$$

$$\bullet \text{ then } G(x_i) = G(x) = 0, \text{ Rolle: } G^{(n+1)}(\xi) = 0 \text{ for some } \xi$$

$$\bullet G^{(n+1)}(t) = (f(x) - p(x))(n+1)! - f^{(n+1)}(t)w(x) \cdot x, t$$

D.A. Arnold, 'A Concise Introduction to Numerical Analysis', 2001,
[<http://www.ima.umn.edu/~arnold>]

$$G^{(n+1)}(t) = e(x) \cdot \frac{w^{(n+1)}(t)}{(n+1)!} - \frac{e^{(n+1)}(t)}{f^{(n+1)}(t) - p^{(n+1)}(t)} w(x)$$

$$(f(x) - p(x))(n+1)! - f^{(n+1)}(\xi)w(x) = 0$$



$$\xi \in \text{span}(x_0, \dots, x_n, x)$$

Example: error of interpolating $f(x) = \sin(x)$ on $[0, 1]$

- ▶ all derivatives f are either $\pm \sin(x)$ or $\pm \cos(x)$ and it follows that

$$|f^{(k)}(x)| \leq 1$$

- ▶ for $x, x_k \in [0, 1]$ one has

$$w(x) = (x - x_0)(x - x_1) \dots$$

$$|w(x)| \leq 1$$

consequently, in this case the error is bounded by

$$|f(x) - p(x)| = \frac{1}{(n+1)!}$$

- ▶ if $n = 9$, i.e. 10 nodes then

$$|\sin x - p(x)| \leq \frac{1}{10!} < 2.8 \times 10^{-7}$$

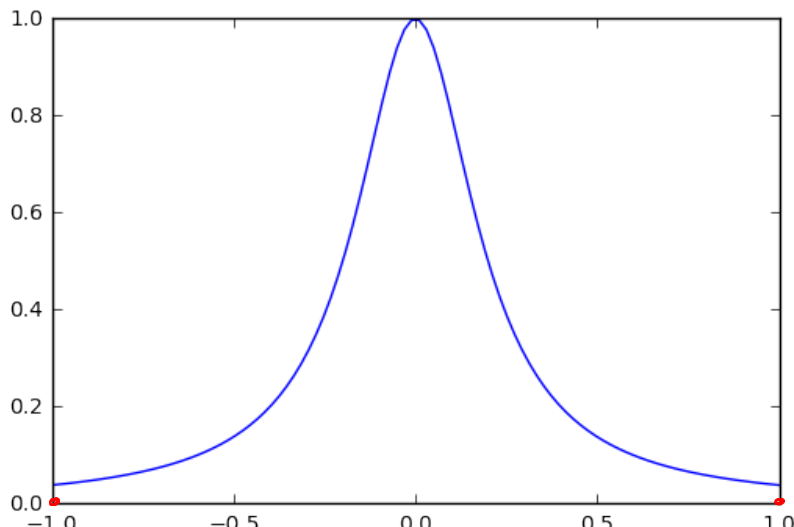
p(x) - sin(x)

Comment

- ▶ it is seen that often the error close to a multiple of $w(x)$ and thus choosing x_k to achieve small $|w(x)|$ is a good strategy

Example: Runge's function $f(t) = \frac{1}{1+25t^2}$

```
fr = lambda t : 1.0/(1.0+25*t**2)  
ng = 100; xg = np.linspace(-1.0,1.0,ng)  
plt.plot(xg, fr(xg), '-');
```



*# stable polynomial interpolation for Runge example
barycentric method and equidistant points*

`fr = lambda t : 1.0/(1.0+25*t**2)` *# Runge function*

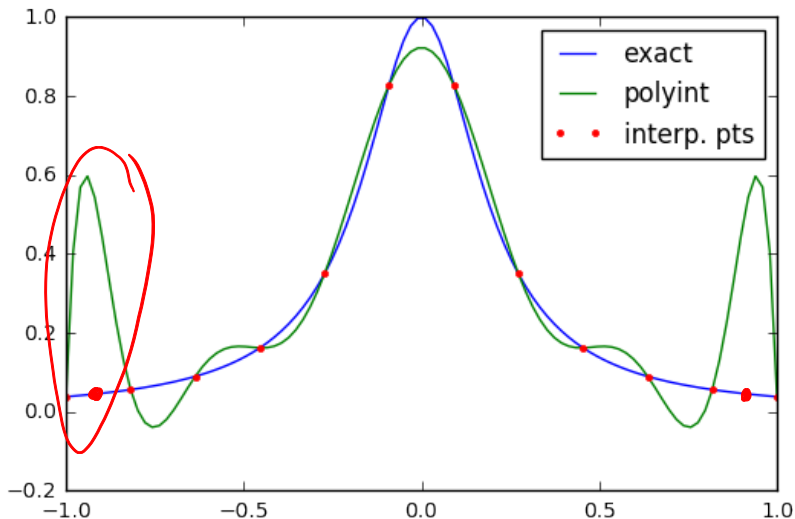
`ng = 12`

`xg = np.linspace(-1.0, 1.0, ng)`

`pr = spyint.BarycentricInterpolator(xg, fr(xg))`

`xplt = np.linspace(-1.0, 1.0, 100)`

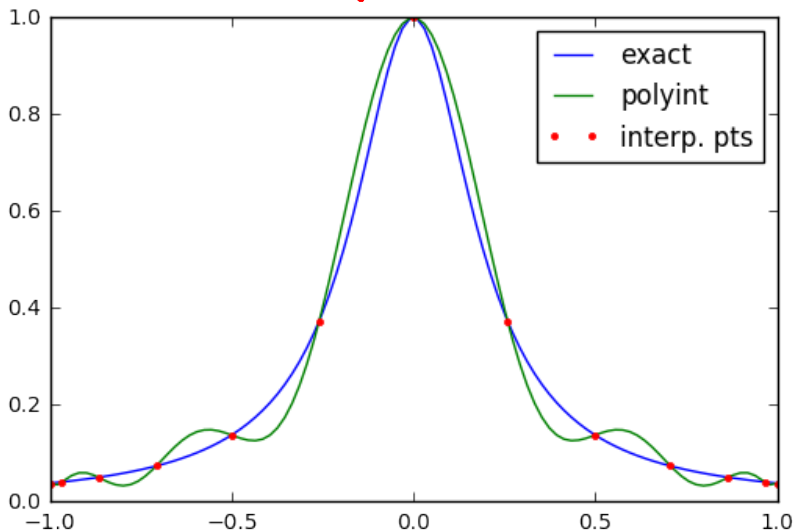

```
plt.plot(xplt, fr(xplt), '-', label='exact')  
plt.plot(xplt, pr(xplt), '-', label='polyint')  
plt.plot(xg, pr(xg), '.', label='interp. pts')  
plt.legend();
```



```
# stable polynomial interpolation for Runge example using
# barycentric method and Chebyshev points (see section below)
fr = lambda t : 1.0/(1.0+25*t**2)      # Runge function
ng = 13
xg = np.cos(np.linspace(0.0, np.pi, ng))    # Chebyshev points
pr = BarycentricInterpolator(xg, fr(xg))
xplt = np.linspace(-1.0, 1.0, 100)
```

```
plt.plot(xplt, fr(xplt), '--', label='exact')  
plt.plot(xplt, pr(xplt), '--', label='polyint')  
plt.plot(xg, pr(xg), '.', label='interp. pts')  
plt.legend();
```

adding bias: $|p(x) - f(x)| \leq \epsilon$



Bernstein approximation – an alternative to interpolation

- ▶ Bernstein polynomials

$$b_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, \dots, n, x \in [0, 1]$$

Handwritten notes: $\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = (x + 1 - x)^n = 1$

- ▶ Bernstein approximation of $f(x)$

$$p(x) = \sum_{k=0}^n f(k/n) b_{k,n}(x)$$

Handwritten note: A red circle with a cross through it, indicating that the approximation is not interpolation.

- ▶ can show that interpolant converges uniformly for continuous f
 - ▶ proof of Weierstrass theorem
- ▶ probabilistic interpretation:
 - ▶ $b_{k,n}(x)$ binomial probability of choosing k given n and $x \in [0, 1]$
 - ▶ Bernstein approximation is then the expectation of $f(k/n)$
 - ▶ approximate by sampling

Monte Carlo

[https://en.wikipedia.org/wiki/Bernstein_polynomial]

Bernstein approximation from scipy

`fr = lambda t : 1.0/(1.0+25*t**2)` *# Runge function*

`ng = 51`

`xg = np.linspace(-1.0, 1.0, ng)`

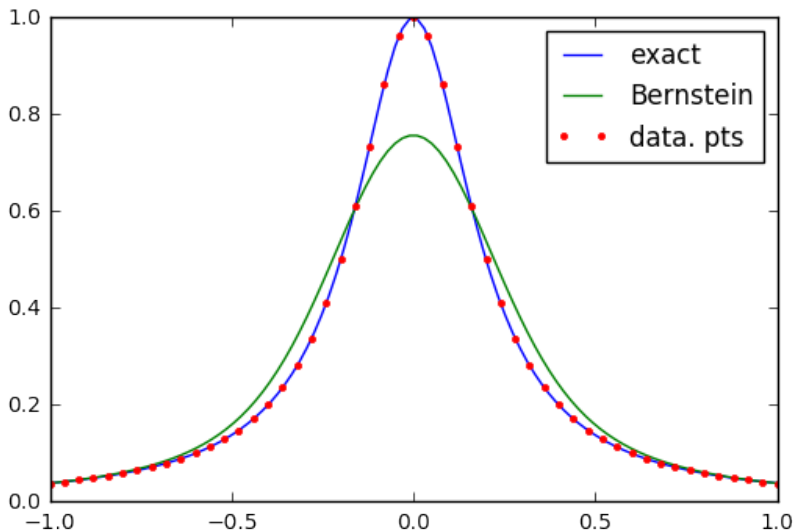
`yg = fr(xg)`

`yg.shape = yg.shape+(1,)`

`pr = BPoly(yg, [0,1])`

`xplt = np.linspace(-1.0, 1.0, 100)`

```
plt.plot(xplt, fr(xplt), '-', label='exact')  
plt.plot(xplt, pr((xplt+1)/2), '-', label='Bernstein')  
plt.plot(xg, fr(xg), '.', label='data. pts')  
plt.legend();
```



```
# simulation using samples from binomial distribution
from numpy.random import binomial

fr = lambda t : 1.0/(1.0+25*t**2)           # Runge function

# data points (random variable)
ng = 51      # data size
xg = np.linspace(-1.0, 1.0, ng)
yg = fr(xg)

nplt = 200    # evaluation grid
xplt = np.linspace(-1.0,1.0,nplt)
yplt = np.zeros(nplt)

ns = 100      # sample size
for k in range(nplt):
    i = binomial(ng-1, (xplt[k]+1)/2.0, ns)
    yplt[k] = np.mean(yg[i])
```

```
plt.plot(xplt, fr(xplt), '-', label='exact')  
plt.plot(xplt, yplt, '+', label='sample approx.')  
plt.plot(xg, fr(xg), '.', label='data. pts')  
plt.legend();
```

