Additional words to Q3 (Prove T is a tensor)

We now show that T is a tensor

In two coordinate system

$$\cancel{\nabla} \; T^{c}_{ab}\nabla_{c} f = \nabla_b \nabla_a f - \nabla_a \nabla_b f$$

$$T^{c'}_{a'b'}\nabla_{c'} f = \nabla_{b'}\nabla_{a'} f - \nabla_{a'}\nabla_{b'} f$$

Covariant derivative is ~~to~~ a tensor, ~~whose~~ which tells us the transformation law:

$$\cancel{\nabla_{c'} f =} \; \nabla_c f = \frac{\partial x^{c'}}{\partial x^{c}} \nabla_{c'} f \qquad \nabla_b \nabla_a f = \frac{\partial x^{b'}}{\partial x^{b}} \cdot \frac{\partial x^{a'}}{\partial x^{a}} \nabla_{b'}\nabla_{a'} f$$

$$\nabla_a \nabla_b f = \frac{\partial x^{a'}}{\partial x^{a}} \frac{\partial x^{b'}}{\partial x^{b}} \nabla_{a'}\nabla_{b'} f$$

So $$T^{c}_{ab}\nabla_c f = T^{c}_{ab}\frac{\partial x^{c'}}{\partial x^{c}}\nabla_{c'} f = \frac{\partial x^{a'}}{\partial x^{a}} \cdot \frac{\partial x^{b'}}{\partial x^{b}} \left(\nabla_{b'}\nabla_{a'} f - \nabla_{a'}\nabla_{b'} f\right).$$

$$= \frac{\partial x^{a'}}{\partial x^{a}} \cdot \frac{\partial x^{b'}}{\partial x^{b}} \cdot T^{c'}_{a'b'}\nabla_{c'} f$$

So $$T^{c}_{ab}\nabla_{c'} f = \frac{\partial x^{c}}{\partial x^{c'}} \frac{\partial x^{a'}}{\partial x^{a}} \cdot \frac{\partial x^{b'}}{\partial x^{b}} T^{c'}_{a'b'}\nabla_{c'} f$$

which gives us $$T^{c}_{ab} = \cancel{T^{c'}_{a'b'}} \frac{\partial x^{a'}}{\partial x^{a}} \cdot \frac{\partial x^{b'}}{\partial x^{b}} \cdot \frac{\partial x^{c}}{\partial x^{c}} T^{c'}_{a'b'}.$$

So T^{c}_{ab} is a tensor.