The solution of nonlinear equations

Nonlinear equations

- general form of equation, x real

$$f(x) = 0$$

- exact solution $x = x^*$

examples:

1. geometric: compute length requires square root: $f(x) = x^2 - d$
2. decision problems: when cost $g(x)$ hits g_0: $f(x) = g(x) - g_0$
3. stationary states of dynamical systems
   - discrete time $x(t+1) = g(x(t))$: $f(x) = x - g(x)$
   - continuous time $x'(t) = g(x(t))$: $f(x) = g(x)$
4. optimisation of $g(x)$: maximum: $f(x) = g'(x)$

Question: What is your favorite nonlinear equation? Can you recall an application?

simple examples, explicit solutions

- solution of linear equation $f(x) = ax + b$, $a \neq 0$:

$$x^* = -b/a$$

- quadratic equation $x^2 - d$ (square root of d)
  - has two solutions $x^*_{1,2} = \pm\sqrt{d}$ if $d > 0$
  - has one solution $x^* = 0$ if $d = 0$
  - has no (real) solutions if $d < 0$
- the determination of the square root requires a numerical algorithm which is typically part of the system library (math in Python)
- for most equations it is not possible to find a formula for the solution
- range reduction: one only needs algorithm for $1 < d < 4$ as

$$\sqrt{4^k d} = 2^k \sqrt{d}$$

continuous functions f – getting help from calculus

Bolzano's theorem *If $f(x)$ is a continuous real valued function on the interval $[a, b]$ and $f(a) \cdot f(b) \leq 0$ then there exists a solution $x = x^* \in [a, b]$ of the equation $f(x) = 0$.*

- consequence of intermediate value theorem for continuous functions
- existence of solution: If we know real numbers a and b such that $f(a)$ and $f(b)$ have different signs then we know that there is a solution of $f(x) = 0$ between a and b

iterations – our class of solution methods

- determine a sequence $x_0, x_1, \dots$ by

$$x_{n+1} = F_n(x_0, x_1, \dots, x_n)$$

  such that

$$x_n \to x^*, \quad \text{for } n \to \infty$$

- we will study some very successful choices of F
- F has to depend on $f(x)$

The Bisection Method

Bisection

- construction of intervals I_k such that $I_{k+1} \subset I_k$ contain the solution x^* of
$$f(x) = 0$$
- solution is unique if $f(x)$ (strictly) increasing or decreasing
- size of the intervals
$$|I_{k+1}| = 0.5|I_k|$$
- as size of intervals goes to zero there is exactly one real number contained in
$$\bigcap_{k=1}^{\infty} I_k$$
- intervals chosen to maintain different signs of $f(x)$ on the two endpoints of I_k

```python
# First version of code

def bisect(f,a,b,n):
    # first simple bisection code
    fa, fb = f(a), f(b)
    for i in range(n):
        m = (a+b)/2.0
        fm = f(m)
        if fa*fm <= 0:
            b, fb = m, fm
        else:
            a, fa = m, fm
    return (a+b)/2.0, (b-a)/2.0, f((a+b)/2)  # output
```

- try this out and challenge the code with difficult examples, then fix the probems and retry

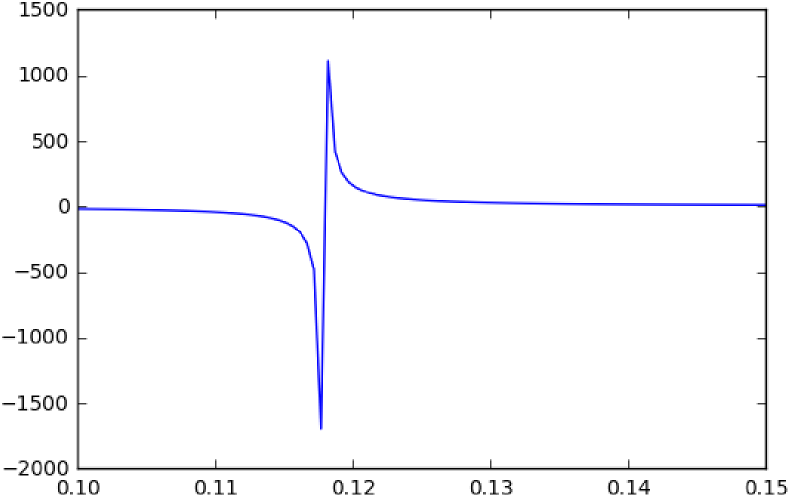
Example $f(x) = x^2 - 3$

- verify that $I_0 = [1, 2]$ contains the zero
- after first subdivision one gets $I_1 = [1.5, 2]$

```
f = lambda x : x*x - 3; xex=math.sqrt(3)
a = 1.0; b = 2.0; fa, fb = f(a), f(b)
print("exact solution:  sqrt(3.0)= %g"%(xex))
print("interval [%g,%g] contains the zero? %s"%(a,b,fa*fb <
xex = math.sqrt(3.0)
for k in range(6):
    m, e, fm = bisect(f,a,b,1) # one step
    a = m-e
    b = m+e
    print("[a,b]=[%6.4g,%6.4g],  e/ebound=%8.1e, f(m)=%8.1e
```

```
exact solution:  sqrt(3.0)= 1.73205
interval [1,2] contains the zero? True
[a,b]=[   1.5,     2],  e/ebound= 7.2e-02, f(m)= 6.2e-02
[a,b]=[   1.5,  1.75],  e/ebound=-8.6e-01, f(m)=-3.6e-01
[a,b]=[ 1.625,  1.75],  e/ebound=-7.1e-01, f(m)=-1.5e-01
[a,b]=[ 1.688,  1.75],  e/ebound=-4.3e-01, f(m)=-4.6e-02
[a,b]=[ 1.719,  1.75],  e/ebound= 1.5e-01, f(m)= 8.1e-03
[a,b]=[ 1.719, 1.734],  e/ebound=-7.0e-01, f(m)=-1.9e-02
```

```
# challenging example for testing
f = lambda x : (x**3+4.0*x**2+3.0*x+5.0)/(2.0*x**3-9.0*x**2
xg = np.linspace(0.1, 0.15, 100)
pl.plot(xg, f(xg));
```



```
a = 0.1; b = 0.15; fa, fb = f(a), f(b)
print("interval [%g,%g] contains the zero?"%(a,b),fa*fb<=0)
for k in range(10):
    m, e, fm = bisect(f,a,b,1) # one step
    a = m-e
    b = m+e
    print("[a,b]=[%8.5g,%8.5g], f(m)=%8.1e, eb=%8.1e"%(a,b,
```

```
interval [0.1,0.15] contains the zero? True
[a,b]=[      0.1,   0.125], f(m)=-6.3e+01, eb= 1.2e-02
[a,b]=[  0.1125,   0.125], f(m)= 3.9e+02, eb= 6.2e-03
[a,b]=[  0.1125, 0.11875], f(m)=-1.5e+02, eb= 3.1e-03
[a,b]=[ 0.11563, 0.11875], f(m)=-4.9e+02, eb= 1.6e-03
[a,b]=[ 0.11719, 0.11875], f(m)= 3.7e+03, eb= 7.8e-04
[a,b]=[ 0.11719, 0.11797], f(m)=-1.1e+03, eb= 3.9e-04
[a,b]=[ 0.11758, 0.11797], f(m)=-3.3e+03, eb= 2.0e-04
[a,b]=[ 0.11777, 0.11797], f(m)=-6.2e+04, eb= 9.8e-05
[a,b]=[ 0.11787, 0.11797], f(m)= 7.8e+03, eb= 4.9e-05
[a,b]=[ 0.11787, 0.11792], f(m)= 1.8e+04, eb= 2.4e-05
```

Bisection Algorithm: First Version – weaknesses

- no test for accuracy
- examples: small interval, error bounds
- rounding errors can cause problems – might even get sign wrong for small $f(x)$!
- infinite loops due to small intervals or negative δ
- discontinuities may mask as zeros
- what if $a > b$?

The test for accuracy is not specified. The obvious thing is to require the interval to be small enough such that the root is known to within a certain maximum error δ.

Bisection Algorithm: Second Version

```
Initialisation
Find a, b that surround a root [f(a),f(b)] of opposite sig
rename them if necessary so that a< b,
    and specify an iteration limit M, desired error bound 
    and a "zero threshold" \epsilon.
Evaluate u=f(a), v=f(b), e=b-a
stop [failure due to bad initial interval]
Initialise the count of steps done, n=0

loop-start-2
    Set e=e/2, m=a+e, w=f(m), n=n+1
    Stop: output m, e and w, and stop success
    Stop: output m, e and w, and stop "probable success"
    Stop: output m, e and w, and stop failure to converge
    set a=m, u=w set b=m, v=w
Repeat from step 7
```

- sign only used in choosing subinterval
- but still problem with sign for small intervals and uncertainty of root

```
# TODO: code an improved version of bisection in python
```

Convergence, Error Analysis and Convergence Rate

- automatic simple and reliable upper bound of error if $f(x)$ continous
- stops for good approximation
- error bound (by induction)

$$|x^* - m_n| < d_n/2 = (b_0 - a_0)/2^{n+1}, \to 0 \text{ as } n \to \infty$$

- ignoring rounding errors, one can make the error as small as desired by taking enough steps
- error tolerance achieved when $(b_0 - a_0)/2^{n+1} \leq \delta$
- requires $\log_2 10 \approx 3.3$ steps to reduce error by factor 10
    - we can predict accuracy!
- but:
    - only computes one of all possible zeros
    - may be tricked by a singularity