Chebyshev Interpolation

Interpolation error revisited

- Error of interpolation of function $f \in \mathbb{C}^{n+1}[a, b]$ by n-th degree polynomial polynomial $p \in P_n$ ~~at~~ with interpolation points $x_0, \ldots, x_n$ is

$$e(x) = p(x) - f(x) = -\frac{f^{(n+1)}(\xi)}{(n+1)!} w(x)$$

  where $w(x) = \prod_{i=0}^{n}(x - x_i)$
  - $1/(n+1)!$ suggests high degree polynomials p
  - $f^{(n+1)}(\xi)$ suggests smooth f are approximated well
  - $w(x)$ suggest how to choose x_i
- this section is about the interpolation points and a choice which gives small $w(x)$

- Python modules and control plotting

```
import numpy as np
import scipy as sc
import pylab as plt

%matplotlib inline
plt.rcParams['savefig.dpi'] = 75;
plt.rcParams['figure.dpi']  = 200;
plt.tight_layout();
plt.rcParams['figure.figsize'] = 12, 4;
```

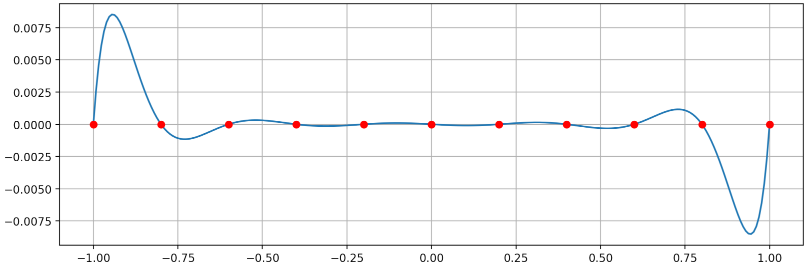
```
<Figure size 1200x800 with 0 Axes>
```

$w(x)$ for equidistant points

```
xk = np.linspace(-1,1,11) # equidistant interp. points

def w(x,xk=xk):
    wx = 1.0
    nk = xk.shape[0]
    for k in range(nk):
        wx = wx*(x-xk[k])
    return wx

xg = np.linspace(-1,1,257) # for display
yg = w(xg)

# Suggestion: think about how to make w(x) faster ...
```

problem: value of $w(x)$ is large close to the boundaries

```
plt.plot(xg,yg,xk,np.zeros(xk.shape[0]),'ro')
plt.grid(True)
```

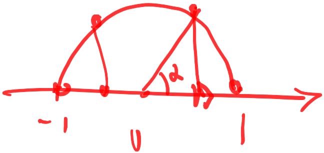


Chebyshev points

- idea: choose more points close to the boundary
- motivation: on the circle, equidistant points are optimal
- Chebyshev points = x-coordinates of equidistant circular points

α

$$x_k = \cos\left(\frac{2k+1}{2n+2}\,\pi\right), \quad k = 0, \ldots, n$$



Example $n = 0$

$$x_0 = \cos(\pi/2) = 0$$

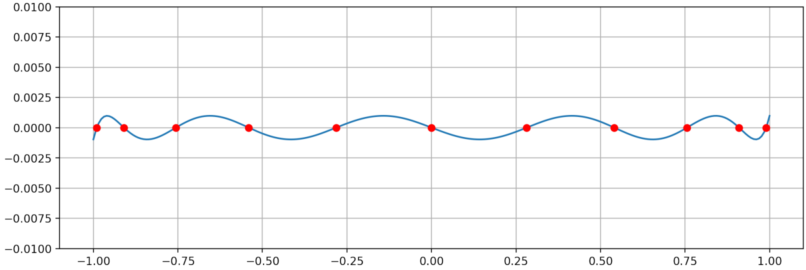
Example $n = 1$

$$x_0 = \cos(\pi/4) = 1/\sqrt{2}$$
$$x_1 = \cos(3\pi/4) = -1/\sqrt{2}$$

$w(x)$ for Chebyshev points

```
nk = 10
xkc = np.cos(np.linspace(np.pi/(2*nk+2.0),
                         np.pi*(2*nk+1.0)/(2*nk+2.0),nk+1))

xg = np.linspace(-1,1,257)
yg = w(xg,xkc)
```

```python
plt.plot(xg,yg,xkc,np.zeros(xkc.shape[0]),'ro')
plt.axis(ymin=-0.01,ymax=0.01)
plt.grid(True)
```



Chebyshev polynomials

$$\boxed{T_n(x) = \cos(n \arccos(x))}$$

- Examples:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \end{aligned}$$

- obviously, T_n are polynomials of degree n for $n = 0, 1, 2, 3$
- naming T_n (instead of C_n) due to earlier transliteration from Russian as Tshebyshev (or Tschebyscheff in German)

Induction: all $T_n(x)$ are polynomials

- addition theorem of cos for T_{n+1} and T_{n-1}

$$\begin{aligned}
T_{n+1}(x) &= \cos((n+1)\arccos(x)) \\
&= \cos(n\arccos(x))\cos(\arccos(x)) \\
&\quad - \sin(n\arccos(x))\sin(\arccos(x)) \\
&= xT_n(x) - \sin(n\arccos(x))\sin(\arccos(x) \\
T_{n-1}(x) &= \cos((n-1)\arccos(x)) \\
&= xT_n(x) + \sin(n\arccos(x))\sin(\arccos(x)
\end{aligned}$$

- add the two results to get recursion

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

- thus: $T_n(x)$ is a polynomial of degree n and for $n \geq 1$:

$$T_n(x) = 2^{n-1}x^n + \cdots$$

The zeros of $T_{n+1}(x)$

$$T_{n+1}(x_k) = \cos((n+1)\arccos(x_k)) = 0$$

and so

$$(n+1)\arccos(x_k) = \frac{\pi}{2} + k\pi$$

thus

$$x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$$

- we get the Chebyshev points for $k = 0, \ldots n$
- then the function $w(x)$ for interpolation with the Chebyshev points $x_0, \ldots, x_n$ is

$$w(x) = 2^{-n} T_{n+1}(x)$$

The error bound for Chebyshev points

- insert the formula for $w(x)$ into the error formula for polynomial interpolation

$$e(x) = p(x) - f(x) = -\frac{f^{(n+1)}(\xi)}{2^n\,(n+1)!}\;T_{n+1}(x)$$

- as $T_{n+1}(x) = \cos((n+1)\arccos(x))$ its values are in $[-1,1]$ and so one gets the error bound

$$|e(x)| \le \frac{1}{2^n(n+1)!}\sup_{x\in[-1,1]}|f^{(n+1)}(x)|$$

Example $n = 1$

$$|e(x)| \le \frac{1}{4}\sup_{x\in[-1,1]}|f^{(2)}(x)|$$

bound for equidistant points $x_{0,1} = \pm 1$: $|e(x)| \le 0.5\sup_x |f^{(2)}(x)|$

Chebyshev points and for interval $[a, b]$

- transform interval $[-1, 1]$ to $[a, b]$

$$x \to z = \frac{a+b}{2} + \frac{b-a}{2}\,x$$

- gives Chebyshev interpolation points

$$z_k = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{2k+1}{2k+2}\,\pi\right)$$

Example $[0, 1]$

$$z_k = 0.5 + 0.5\cos\left(\frac{2k+1}{2k+2}\,\pi\right)$$

Error bound for interval $[a, b]$

- note the transformation of the derivative (and corresponding formula for higher derivatives)

$$f'(x) = \frac{b-a}{2} f'(z)$$

- insert this into error bound for interval $[-1, 1]$ to get

$$|p(x) - f(x)| \leq \frac{1}{2^n(n+1)!} \left(\frac{b-a}{2}\right)^{n+1} \max_{a \leq x \leq b} |f^{(n+1)}(x)|$$

Example $[0, 1]$, $n = 2$

$$|p(x) - f(x)| \leq \frac{1}{192} \max_{a \leq x \leq b} |f^{(3)}(x)|$$

Maxima and Minima of Chebyshev polynomials

- recall
$$T_n(x) = \cos(n \arccos(x))$$
- maxima/minima of $\cos(y)$ occur for $y = k\pi$
- thus maxima/minima of $T_n(x)$ occur for $n \arccos(\overline{x}_k) = k\pi$ and so
$$\overline{x}_k = \cos(k\pi/n)$$

```python
# Degree n interpolation with Chebyshev points and Chebyshe

np.set_printoptions(precision = 2)
n = 10

# Chebyshev points
xkc = np.cos(np.linspace(np.pi/(2*n+2.0),\
                         np.pi*(2*n+1.0)/(2*n+2.0),n+1))

def T(x,n=n+1): # Chebyshev polynomials
    if n==0:
        return 1.0
    elif n==1:
        return x
    else:
        return 2*x*T(x,n-1) - T(x,n-2)

# check that T is zero at the Chebyshev points
print(T(xkc))
```

Discrete Orthogonality

If x_k for $k = 1, 2, ...m$ are m zeros of $T_m(x)$, and assuming that $i, j < m$ then

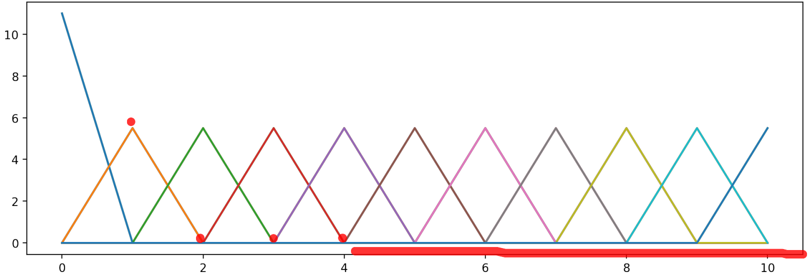
$$\sum_{k=1}^{m} T_i(x_k) T_j(x_k) = \begin{cases} 0 & i \neq j \\ \frac{m}{2} & i = j \neq 0 \\ m & i = j = 0 \end{cases}$$

which is a discrete orthogonality relation.

- Question: Why does this hold?
- It follows that the interpolation matrix A with elements $a_{k,j} = T_j(x_k)$ is orthogonal, and $D = A^T A$ is then a diagonal matrix
- Thus the interpolation problem $Ac = y$ is solved by solving

$$Dc = A^T y$$

```
# Collocation matrix for Chebyshev polynomials
A = np.zeros((n+1,n+1))
for k in range(n+1): A[:,k] = T(xkc,k)
# check the orthogonality of A
for j in range(n+1):    plt.plot(np.dot(A.T,A))
```



Solving interpolation problem with Chebyshev polynomials

```
f = lambda x : 1.0/(25*x*x+1)
ykc = f(xkc)   # function values

aty = np.dot(A.T,ykc)  # A.T times rhs
ata = np.dot(A.T, A)   # normal matrix (is diagonal)

c = aty/np.diag(ata)   # coeffs of Chebyshev polynomials

print(c)   #every second coefficient is zero, why?

[ 2.01e-01  6.77e-19 -2.74e-01  4.10e-18  1.91e-01  1.36e-1
  6.80e-17  1.06e-01 -1.24e-16 -9.11e-02]
```

```
xg = np.linspace(-1,1,257)
yg = np.zeros(257)
for k in range(n+1):
    yg += c[k]*T(xg,k)

plt.plot(xg, f(xg),xg,yg,xkc,ykc,'ro')
plt.grid(True)
```