Romberg

introduction

$$I = \int_a^b f(x)\,dx$$

- (composite) trapezoidal rule for quadrature

$$T(f,h) = \frac{h}{2}\left(f(a) + 2\sum_{k=1}^{n-1} f(a+kh) + f(b) \right)$$

  where $h = (b-a)/n$ for some integer n
- the Romberg method is Richardson extrapolation applied to the trapezoidal rule
- first we need to establish an error formula for the trapezoidal rule
- while in the case of central differences one considers a sequence of increasingly coarser grids, for Romberg one consider a sequence of increasingly finer grids

example

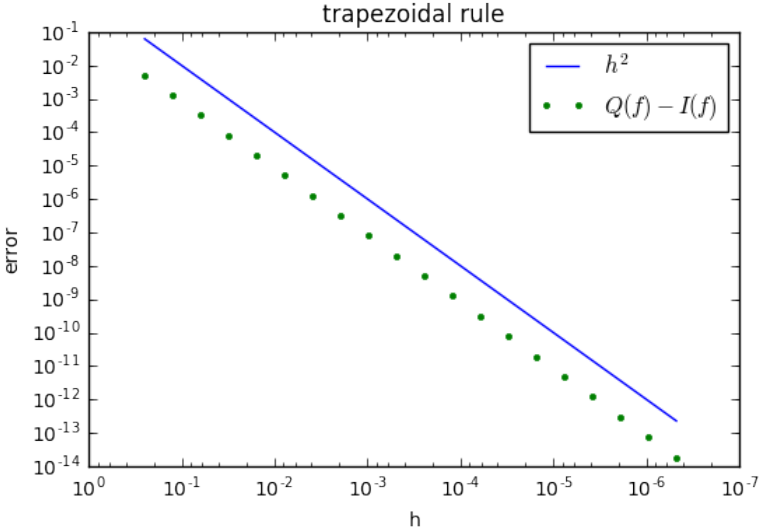
$f(x) = e^{-x}$

```
f   = lambda x : np.exp(-x)
intf = 1.0 - 1/np.e

# trapezoidal rule
T = lambda f,n,a=0.0,b=1.0 : (b-a)/n*((f(a)+f(b))/2.0 \
    + np.sum(f(np.linspace(a+(b-a)/n,b-(b-a)/n,n-1))))
d = 20   # might have to set lower ...
q = np.zeros(d)
nv = 2**np.arange(d)*4
for i,n in enumerate(nv):
    q[i] = T(f,n)
h = 1.0/nv

def pl1():  # plotting commands
    plt.title('trapezoidal rule')
    plt.gca().invert_xaxis();plt.loglog(h,h*h,label="$h^2$'
    plt.xlabel('h');plt.ylabel('error')
    plt.loglog(h,abs((q - intf)/intf),'.',label="$Q(f)-I(f
```

```
pl1()
```



trapezoidal rule
h^2
$Q(f) - I(f)$
error
h
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-14}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}

Euler-Maclaurin

Theorem

Let $f \in C^{2k+1}[a,b]$ and $h = (b-a)/n$. Then

$$T(f,h) - \int_a^b f(x)\,dx = \sum_{k=1}^{m-1} h^{2k} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a) \right) + h^{2m} \frac{B_{2m}}{(2m)!} (b-a) f^{(2m)}(\xi)$$

where B_{2k} are the Bernoulli numbers.

- proof by induction and integration by parts
- first few Bernoulli numbers

$$B_0 = 1,\ B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42,\ B_8 = -1/30, \ldots$$

- improve quadrature using derivatives
- case of $m = 1$ see earlier discussion of trapezoidal rule
- f periodic and analytic: error $o(h^m)$ for all m
  - in this case: trapezoidal rule better than Gauss quadrature!

(see Wikipedia)

Recursive computation of $T(2^{-k}h)$

- compute sequence $T(f, 2^{-k}(b-a)$, for $k = 0, \ldots, m$ by

$$
\begin{aligned}
T(f, b-a) &= \frac{(b-a)(f(a)+f(b))}{2} \\
T\left(f, (b-a)/2\right) &= \frac{T(f, b-a)}{2} + \frac{(b-a) f\left(a + (b-a)/2\right)}{2}, \\
T(f, (b-a)/4) &= \frac{T(f, (b-a)/2)}{2} \\
&\quad + \frac{(b-a)\left(f(a+(b-a)/4) + f(a+3(b-a)/4)\right)}{4} \\
&\vdots \\
T(f, (b-a)/2^m) &= \frac{T(f, (b-a)/2^{m-1})}{2} \\
&\quad + \frac{(b-a)\sum_{i=1}^{2^{m-1}} f\left(a + (2i-1)(b-a)/2^m\right)}{2^m}
\end{aligned}
$$

Romberg integration = Richardson extrapolation for trapozoidal rule

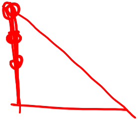
$k=1 \quad k=2$

- we use

$$\underline{T(f,h) = \cancel{a_1 h^2} + \cancel{a_2 h^4} + a_3 h^6 + \cdots}$$

  and eliminate a_1, then a_2, etc

- establish lower triangular tableau $R[j,k]$ with



  - leftmost column

$$\underline{R[j,0] = T(2^{-j}h)}$$

  - elements

$$R[j,k] = \frac{4^k R[j,k-1] - R[j-1,k-1]}{4^k - 1}$$

  - use the efficient method to update $T(2^{-j}h)$

- error approximation $e_h \approx R[j,j] - R[j+1,j+1]$

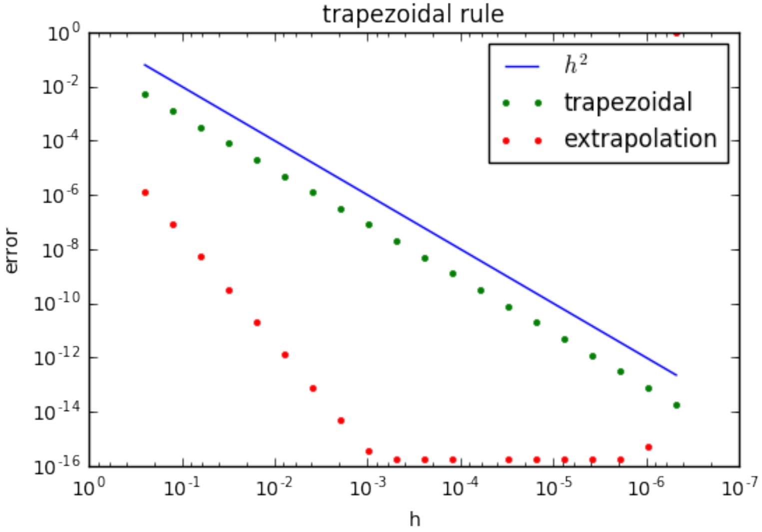
example $f(x) = \exp(-x)$

h 2h

```
R  = lambda f,n: (4*T(f,n) - T(f,n/2))/3
qr = np.zeros(d)
for i,n in enumerate(nv[1:]):
    qr[i] = R(f,n)

def pl2():  # plotting commands
    plt.title('trapezoidal rule')
    plt.gca().invert_xaxis();plt.loglog(h,h*h,label="$h^2$'
    plt.xlabel('h');plt.ylabel('error')
    plt.loglog(h,abs((q - intf)/intf),'.',label="trapezoida
    plt.loglog(h,abs((qr - intf)/intf),'.',label="extrapola
    plt.legend();
```

`pl2()`



trapezoidal rule
h^2
trapezoidal
extrapolation
error
h
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}

Romberg tableau

- all the $R(i,j)$ required for $R(0,k)$:

h^2 $\quad$ h^4 $\quad$ h^6

$$
\begin{array}{cccccc}
R(0,0) & & & & & \\
R(1,0) & R(1,1) & & & & \\
R(2,0) & R(2,1) & R(2,2) & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
R(n-1,0) & R(n-1,1) & R(n-1,2) & \cdots & R(n-1,n-1) & \\
R(n,0) & R(n,1) & R(n,2) & \cdots & \cdots & R(n,n)
\end{array}
$$

- follows from

$$R[j,k] = \frac{4^k R[j,k-1] - R[j-1,k-1]}{4^k - 1}$$

- use $R(j,k)$ for error estimate on diagonal

$$\underline{e_j = R(j,j) - \phi(0) \approx R(j,j) - R(j+1,j+1)}$$

example

```
mr = 5; h = 0.1
R = np.zeros((mr,mr))
for n in range(mr):
    R[n,0] = T(f,2**n)
for k in range(1,mr):
    for j in range(k,mr):
        R[j,k] = (4**k*R[j,k-1] - R[j-1,k-1])/(4**k-1)
print(R)
print("exact value: {}".format(intf))
```

```
[[ 0.68393972  0.          0.          0.          0.
 [ 0.64523519  0.63233368  0.          0.          0.
 [ 0.63540943  0.63213418  0.63212088  0.          0.
 [ 0.63294342  0.63212141  0.63212056  0.63212056  0.
 [ 0.63232631  0.63212061  0.63212056  0.63212056  0.63212
exact value: 0.6321205588285577
```

errors

```
for j in range(mr):
    for k in range(j+1):
        print("{:1.2e}".format(R[j,k]-intf),end='    ')
    print("\n")
```

```
5.18e-02

1.31e-02   2.13e-04

3.29e-03   1.36e-05   3.16e-07

8.23e-04   8.56e-07   5.06e-09   1.23e-10

2.06e-04   5.36e-08   7.96e-11   4.94e-13   1.21e-14
```

error estimate on diagonal

```
# error approximation for diagonal
print("approx",end='    ')
for k in range(mr-1):
    print("{:1.4e}".format(R[k,k]-R[k+1,k+1]),end='  ')
print("\nexact",end='    ')
for k in range(mr-1):
    print("{:1.4e}".format(R[k,k]-intf),end='  ')
```

```
approx   5.1606e-02  2.1280e-04  3.1606e-07  1.2341e-10
exact   5.1819e-02  2.1312e-04  3.1618e-07  1.2342e-10
```

comments

- practical implementation will require stopping criteria based on error estimate
- the columns converge according to the Euler-Maclaurin formula
- the diagonal may not converge!
- in contrast to differentiation, rounding errors for integration are usually not a problem