ASTR 2013 Pro Set 2.

1. From $\frac{d\log P}{d\log \rho} = \gamma$

$d\log P = \gamma\, d\log \rho$

By integral we get $\log P = \gamma \log \rho + \log K$, where K is a constant

Hence: $\log P = \log(K\rho^{\gamma})$

1

$P = K\rho^{\gamma}$

2. For a fully convective star, $\frac{dT}{dr}$ is determined by adiabatic index γ and not radiation α: $\frac{dT}{dr} = \frac{\gamma-1}{\gamma}\frac{T}{P}\frac{dP}{dr}$ (2.1)

We also have equations for stellar structures:

$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$ (2.2)

$\frac{dM}{dr} = 4\pi r^2 \rho$ (2.3)

We define the dimensionless ~~unit~~ quantity $M' = M/M_\odot$ $r' = r/R_\odot$ (2.4)

Applying (2.4) to (2.3) we get

$\frac{dM'}{dr'} = 4\pi r'^2 \rho \frac{R_\odot^3}{M_\odot}$ (2.5) 1

We can rewrite ~~eq~~ (2.1) as:

$\frac{1}{T}\frac{dT}{dr} = \frac{\gamma-1}{\gamma}\frac{1}{P}\frac{dP}{dr} \Rightarrow \frac{d\ln T}{dr} = (1-\frac{1}{\gamma})\frac{d\ln P}{dr}$

$\frac{d\ln T}{dr'} = (1-\frac{1}{\gamma})\frac{d\ln P}{dr'} = (1-\frac{1}{\gamma})\frac{\gamma\, d\ln\rho}{dr'} = (\gamma-1)\frac{d\ln\rho}{dr'}$

So $\frac{d\ln T}{dr'} = (\gamma-1)\frac{d\ln\rho}{dr'}$ (2.6) 1

Finally $\frac{d\ln\rho}{dr'} = \frac{1}{\gamma}\frac{d\ln P}{dr'} = \frac{1}{\gamma P}\frac{dP}{dr'} = -\frac{1}{\gamma P}\frac{GM'\rho}{r'^2}\frac{M_\odot}{R_\odot}$

$\frac{d\ln\rho}{dr'} = -\frac{1}{\gamma}\frac{GM'\rho}{P\cdot r'^2}\frac{M_\odot}{R_\odot}$ (2.7) 1

(2.5) (2.6) (2.7) together forms alternative equations for stellar structure based on the three derivatives: $\frac{dM'}{dr'}$, $\frac{d\ln\rho}{dr'}$, $\frac{d\ln T}{dr'}$.

3. Plots are in the Appendix 5

The two models are significantly different in the near-surface layers. The greatest cause of the difference is that H and He are not fully ionised in the near-surface layers since its temperature is much lower than the center. Hence the simplified EOS is not valid in that region.

4. Convection stops when: 2

$$\left|\left(\frac{dT}{dr}\right)_{rad}\right| < \frac{\gamma-1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right|$$

Eq of stellar structure tells us:

$$\left(\frac{dT}{dr}\right)_{rad} = -\frac{3L\kappa\rho}{64\pi r^2 \sigma_{SB} T^3}$$

We define $\left|\left(\frac{dT}{dr}\right)_{convective}\right| = \frac{\gamma-1}{\gamma}\frac{T}{P}\cdot\left|\frac{dP}{dr}\right|$ and compare it with $\left|\left(\frac{dT}{dr}\right)_{rad}\right|$ in the same plot.

We find that convection stops at $\frac{r}{r_\odot} \approx 1.827$. At this radius, the temperature is $T \approx 4223K$. The detailed code of Q4 is in the Appendix also.

5. By calculation, the temperature and luminosity of ~~sta~~ star are $T=4223K$ and $L=0.96L_\odot$. In HR diagram, this is a dwarf star in the main-sequence. The mass of the star is roughly $0.75M_\odot$. Its subsequent evolution over the Gyr are as follows:

After the main sequence star has burnt out of its hydrogen in the core. The core ~~contrast~~ contracts and inner temperature rises.

causing hydrogen begins to burn surrounding the core. This process increase the luminosity and lower the temperature, which is called the red giant phase. As the red-giant phase progresses, Helium Ignition begins and the star moves quickly to the horizontal branch's left side and evolve slowly to the right. Once He burning is complete, the core contracts until supported by degeneracy pressure; the star ascends the asymptotic giant branch. In this phase ~~Hydro~~ hydrogen burning resumes with occasional ~~substanc~~ substantive He burning called thermal pulse. The resulting high luminosity drives the surface layer off, leaving behind an inert C, N, O core called a white dwarf.

1.5

Appendix starts in next page!

~~Append~~

Convective Star

August 19, 2019

First, import all the modules we need

[5]:
```
import numpy as np
import matplotlib.pyplot as plt
import astropy.constants as c
import astropy.units as u
from scipy.integrate import solve_ivp
import scipy.optimize as op
```

Next, define the complex equation of state (EOS) functions

[6]:
```
def solarmet():
    """Return solar metalicity abundances by number and masses for low mass␣
 ↪elements.
    From Asplund et al (2009), up to Oxygen only"""
    abund = 10**np.array([0.00,-1.07,-10.95,-10.62,-9.3,-3.57,-4.17,-3.31])
    masses=  np.array([1.0, 4.0, 6.94, 9.01, 10.81, 12.01, 14.01, 16.00])
    return abund, masses

def simplified_eos_rho_T(rho, T):
    """Assume that H and He are ionised. Return the gas pressure for this␣
 ↪density and
    the adiabatic gamma

    Parameters
    ----------
    rho: Density, including units from astropy units.
    T: Temperature, including units from astropy units.

    Returns
    -------
    Pressure, including units from astropy.units
    Adiabatic index (dimensionless)
    """
    #Input the abundances of the elements
    abund, masses = solarmet()

    #Find the number density of H
    n_h = rho/(np.sum(abund*masses)*u.u)
    #Assume that H and He are totally ionized. Ignore heavier elements.
    n_e = n_h*(abund[0] + 2*abund[1])

    #Now find total pressure.
    P = ((n_h*np.sum(abund) + n_e)*c.k_B*T).cgs

    return P, 5/3

def saha(n_e, T):
    """Compute the solution to the Saha equation as a function of electron number
    density and temperature. This enables the problem to be a simple Ax=b linear␣
 ↪problem.
    Results from this function can be used to solve the Saha equation as e.g. a␣
 ↪function
    of rho and T via e.g. tabulating or solving.

    Parameters
    ----------
    n_e: the dimensioned electron number density
    T: Temperature in K.

    Returns
    -------
    rho: astropy.units quantity compatible with density
    mu: Mean molecular weight (dimensionless, i.e. to be multiplied by the AMU)
    ns: A vector of number densities of H, H+, He, He+, He++

    """
    #Input the abundances of the elements
    abund, masses = solarmet()

    #This will break for very low temperatures. In this case, fix a stupidly low
    #ionization fraction
    if (T<1500*u.K):
        ns = n_e*1e15*np.array([abund[0],0,abund[1],0,0])
    else:
        #The thermal de Broglie wavelength
        debroglie=np.sqrt(c.h**2/2/np.pi/c.m_e/c.k_B/T)

        #Hydrogen ionization. We neglect the excited states because
        #they are only important when the series diverges...
        h1  = 2./debroglie**3 *1/2*np.exp(-13.6*u.eV/c.k_B/T)

        #Helium ionization. NB excited states are still nearly ~20ev higher.
        he1 = 2./debroglie**3 *2/1*np.exp(-24.580*u.eV/c.k_B/T)
        #Helium double-ionization
        he2 = 2./debroglie**3 *1/2*np.exp(-54.403*u.eV/c.k_B/T)

        #Now construct our matrix of 5 equations defining these number densities.
        A = np.zeros( (5,5) );
        A[0,0:2]=[-h1/n_e,1]
        A[1,2:4]=[-he1/n_e,1]
        A[2,3:5]=[-he2/n_e, 1]
        A[3,:]  =[abund[1],abund[1],-abund[0],-abund[0],-abund[0]]
        A[4,:] =[0,1,0,1,2]
        #This has dimensions
        b=[0,0,0,0,n_e.to(u.cm**(-3)).value]
        ns =np.linalg.solve(A,b)*u.cm**(-3)

    #The next lines ensure ionization at high T, due to nuclei being separated␣
↪by less.
    #than the Debye length. Somewhat of a hack, but eventually the Saha equation␣
↪does
    #break down...
    ns_highT=[0,abund[0],0,0,abund[1]]
    ns_highT=ns_highT/(abund[0]+2*abund[1])*n_e
    if (T > 2e6*u.K):
        ns=ns_highT
    elif (T > 1e6*u.K):
        frac=(T.to(u.K).value-1e6)/1e6
        ns = frac*ns_highT + (1-frac)*ns

    #For normalization... we need the number density of Hydrogen
    #nuclei, which is the sum of the number densities of H and H+.
    n_h = np.sum(ns[:2])

    #Density. Masses should be scalars.
    rho = n_h*np.sum(abund*masses)*u.u

    #Fractional "abundance" of electrons.
    f_e = n_e/n_h

    #mu is mean "molecular" weight, and we make the approximation that
    #electrons have zero weight.
    mu = np.sum(abund*masses)/(np.sum(abund) + f_e)

    #Finally, we should compute the internal energy with respect to neutral gas.
    #This is the internal energy per H atom, divided by the mass in grams per H␣
↪atom.
    Ui=(ns[1]*13.6 + ns[3]*24.58 + ns[4]*(54.403+24.58))*u.eV/n_h/np.
↪sum(abund*masses*u.u);
```

```
    return rho, mu, Ui, ns

def saha_solve(log_n_e_mol_cm3, T, rho_0_in_g_cm3):
    """Dimensionless version of the Saha equation routine, to use in np.solve to
    solve for n_e at a fixed density."""
    n_e = np.exp(log_n_e_mol_cm3[0])*c.N_A.value/u.cm**3
    rho, mu, Ui, ns = saha(n_e, T)

    return rho.to(u.g/u.cm**3).value - rho_0_in_g_cm3

def eos_rho_T(rho, T, full_output=False):
    """Compute the key equation of state parameters via the Saha equation

    Parameters
    ----------
    rho: rho: astropy.units quantity compatible with density
    T: gas Temperature

    Returns
    -------
    P: Gas Pressure
    n_e: Electron number density
    ns: Number densities of H, H+, He, He+, He++
    mu: Mean molecular weight in atomic mass units
    Ui: Internal energy per unit mass due to ionization
    """
    rho_in_g_cm3 = rho.to(u.g/u.cm**3).value
    #Start with the electron number density equal in mol/cm^3 equal to the␣
 ↪density
    #in g/cm^3
    x0 = np.log(rho_in_g_cm3)
    res = op.fsolve(saha_solve, x0, args=(T, rho_in_g_cm3), xtol=1e-6)

    #Now evaluate the saha equation one more time
    n_e = np.exp(res[0])*c.N_A.value/u.cm**3
    rho_check, mu, Ui, ns = saha(n_e, T)

    #The total gas pressure is just the sum of the number densities multiplied␣
 ↪by kT
    #!!! We should take heavier elements into account here as well, and wrap␣
 ↪this in
    #a function. Maybe simpler just to add in more elements to Saha?
    P = ((np.sum(ns) + n_e)*c.k_B*T).cgs

    #Next, find the adaibatic exponent. As we are neglecting radiation,␣
 ↪pressure, there
```

```
    #is only a single gamma (e.g. https://ui.adsabs.harvard.edu/abs/2002ApJ...
↪581.1407S/abstract)
    #We will do this two different ways, to double-check. I'm double checking␣
↪because
    #the results didn't seem to exactly match Unsold's 1968 book:
    #Physik der Sternatmosphaeren MIT besonderer Beruecksichtigung der Sonne,␣
↪which is
    #referenced in e.g.
    #http://www.ifa.hawaii.edu/users/kud/teaching/4.Convection.pdf

    #In both cases, we need to numerically compute derivatives. We do this by␣
↪slightly
    #increasing temperature and density, and re-calculating.
    dlog = 1e-4
    #Increase temperature
    res_Tplus = op.fsolve(saha_solve, res[0], args=(T*np.exp(dlog),␣
↪rho_in_g_cm3), xtol=1e-6)
    n_e_Tplus = np.exp(res_Tplus[0])*c.N_A.value/u.cm**3
    rho_check, mu_Tplus, Ui_Tplus, ns_Tplus = saha(n_e_Tplus, T*np.exp(dlog))
    P_Tplus = ((np.sum(ns_Tplus) + n_e_Tplus)*c.k_B*T*np.exp(dlog)).cgs
    #Increase Density
    res_rhoplus = op.fsolve(saha_solve, res[0], args=(T, rho_in_g_cm3*np.
↪exp(dlog)), xtol=1e-6)
    n_e_rhoplus = np.exp(res_rhoplus[0])*c.N_A.value/u.cm**3
    rho_check, mu_rhoplus, Ui_rhoplus, ns_rhoplus = saha(n_e_rhoplus, T)
    P_rhoplus = ((np.sum(ns_rhoplus) + n_e_rhoplus)*c.k_B*T).cgs
    #Compute the 4 logarithmic derivatives. We scale internal energy by rho/P to␣
↪make
    #it dimensionless.
    dUidlrho_scaled = float( (Ui_rhoplus - Ui)/dlog*rho/P )
    dUidlT_scaled = float( (Ui_Tplus - Ui)/dlog*rho/P )
    dlPdlrho = float( (P_rhoplus - P)/dlog/P )
    dlPdlT = float( (P_Tplus - P)/dlog/P )

    #Now the tricky bit. We have to use partial derivative relations to move␣
↪from
    #Ui(rho, T) to Ui(rho, P), which we call UU.
    dUUdlrho_scaled = dUidlrho_scaled - dUidlT_scaled * (dlPdlrho/dlPdlT)
    dUUdlP_scaled = dUidlT_scaled / dlPdlT

    #The following comes directly from the definition of adiabatic, from e.g.␣
↪the derivation
    #on page 5 of https://websites.pmc.ucsc.edu/~glatz/astr_112/lectures/notes6.
↪pdf
    gamma = (5/2 - dUUdlrho_scaled) / (3/2 + dUUdlP_scaled)
    # For method 2, see equation 18.8 on page 571 of Cox and Guili.
    #We need an additional two normalised logarithmic derivatives.
    #They are all zero in the absence of a phase change
    dlmudlT = float( (mu_Tplus - mu)/mu/dlog )
    dlmudlrho = float( (mu_rhoplus - mu)/mu/dlog )
    dUidlT_scaled = float( (Ui_Tplus - Ui)/dlog*mu*u.u/c.k_B/T)

    #Composition quantities to match Cox and Guili's equations. As far as I can␣
↪tell,
    #this gamma is identical to the gamma above, so a great check.
    chi_T     = 1 - dlmudlT
    chi_rho   = 1 - dlmudlrho
    gamma_old = chi_rho + chi_T**2/(3/2*chi_T + dUidlT_scaled)

    if full_output:
        return P, n_e, ns, mu, Ui, gamma
    else:
        return P, gamma
```

Now we can define our main function - something that returns the results of the three modified equations of stellar structure for a given r, M, rho and T

[7]:
```
def find_derivatives(r_in_rsun, M_lrho_lT, simplified_EOS=False):
    """Given an interior mass M, a density logarithm rho and a pressure␣
↪logarithm P,
    find the derivatives of M and P. Derivatives are in units of solar radii.
    Logarithms are natural logarithms in cgs units.

    Parameters
    ----------
    r_in_rsun: radius to compute derivatives in solar units
    M_lrho_lT: numpy array-like, including M in solar units, log(rho in g/cm^3),␣
↪and
        log(T in K).

    Returns
    -------
    derivatives: Derivatives of M in solar units, log(rho in g/cm^3), and
        log(T in K) with respect to r_in_rsun, as a numpy array-like variable.
    """
    M_in_Msun, lrho, lT = M_lrho_lT
    #Mass continuity
    dM_in_Msundr = 4*np.pi*r_in_rsun**2*np.exp(lrho) * float(c.R_sun**3*u.g/u.
↪cm**3/c.M_sun)

    #Equation of state.
    if simplified_EOS:
        P, gamma = simplified_eos_rho_T(np.exp(lrho)*u.g/u.cm**3, np.exp(lT)*u.K)
    else:
        P, gamma = eos_rho_T(np.exp(lrho)*u.g/u.cm**3, np.exp(lT)*u.K)

    #(logarithmic) density derivative. Firstly, avoid a divide by zero by taking␣
 ↪a limit
    #for r=0 and constant rho at the star's center.
    if M_in_Msun == 0:
        dlrhodr = 0
    else:
        #Put all parameters with units on the second line for neatness.
        dlrhodr = -1/gamma * (M_in_Msun)*np.exp(lrho)/P.cgs.value/r_in_rsun**2 *\
         float(c.G*c.M_sun/c.R_sun * (u.g/u.erg) )

    #(logarithmic) temperature derivative.
    dlTdr = (gamma - 1) * dlrhodr
    # dlTdr =

    return np.array([dM_in_Msundr, dlrhodr, dlTdr])
```

This function is wrapped for convenience, and surface stopping condition functions are defined

[8]:
```
def find_derivatives_simplified(r_in_rsun, M_lrho_lT):
    """A wrapper function to force a simplified equation of state"""
    return find_derivatives(r_in_rsun, M_lrho_lT, simplified_EOS=True)


#The following two functions may be slightly confusing for people who aren't␣
 ↪python or
#object oriented programming experts, as variables (properties) are added to a␣
 ↪function.
#We can actually always add additional properties to functions, ad all variables␣
 ↪and
#functions in python are objects.
def cool_surface(r_in_rsun, M_lrho_lT):
    """Determine a surface condition by the surface becoming too cool. In␣
 ↪practice,
    our adiabatic approximation is likely to break before this!"""
    return M_lrho_lT[2] - np.log(2000)
cool_surface.terminal = True
cool_surface.direction = -1

def near_vacuum(r_in_rsun, M_lrho_lT):
    """Determine a surface condition by the surface becoming too cool. In␣
 ↪practice,
    our adiabatic approximation is likely to break before this!"""
    return M_lrho_lT[1] - np.log(1e-7)
near_vacuum.terminal = True
near_vacuum.direction = -1
```

Finally, we're up to defining our core function which creates a numerical approximation to a fully convective star

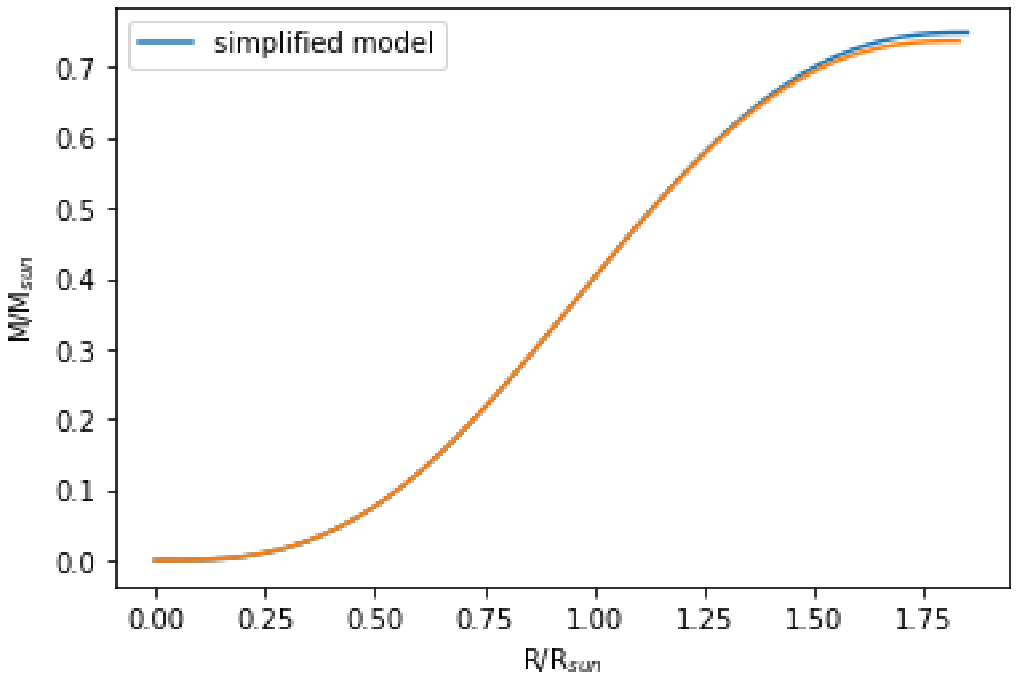
[277]:
```
def convective_star(rho_c, T_c, simplified_EOS=False):
    """Assuming a fully convective star, compute the structure using an equation␣
↪of
    state and the first two equations of stellar structure.

    Parameters
    ----------
    rho_c: Central density, including units from astropy.units
    T_c: Central temperature, including units from astropy.units
    simplified_EOS: True or False - do we use a simplified equation of state
        with a fixed gamma (from a fixed ionisation fraction)?
    """
    #Start the problem at the star center.
    y0 = [0, np.log(rho_c.to(u.g/u.cm**3).value), np.log(T_c.to(u.K).value)]

    #Don't go past 100 R_sun!
    rspan = [0,2]

    #Solve the initial value problem!
    if simplified_EOS:
        result = solve_ivp(find_derivatives_simplified, rspan, y0,␣
↪events=[cool_surface, near_vacuum], method='RK23',max_step = 0.01)
    else:
        result = solve_ivp(find_derivatives, rspan, y0, events=[cool_surface,␣
↪near_vacuum], method='RK23',max_step = 0.01)

    #Extract the results
    r_in_rsun = result.t
    M_in_Msun = result.y[0]
    rho = np.exp(result.y[1])*u.g/u.cm**3
    T = np.exp(result.y[2])*u.K
    return r_in_rsun, M_in_Msun, rho, T
```

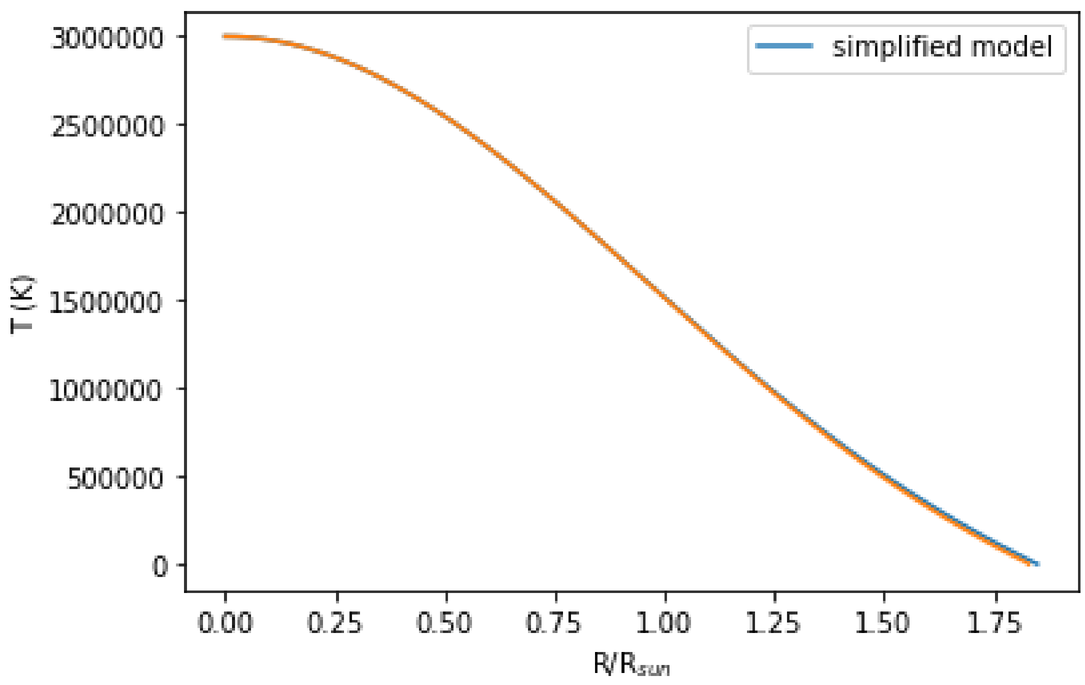
From this point, you'll have to run convective_star(), e.g. as `r_in_rsun, M_in_Msun, rho, T = convective_star(INSERT_DENSITY_HERE, INSERT_TEMPERATURE_HERE)` **... then you can try to make pretty plots etc. Enjoy! Note that without the simplified_EOS option, this takes several seconds to run.**

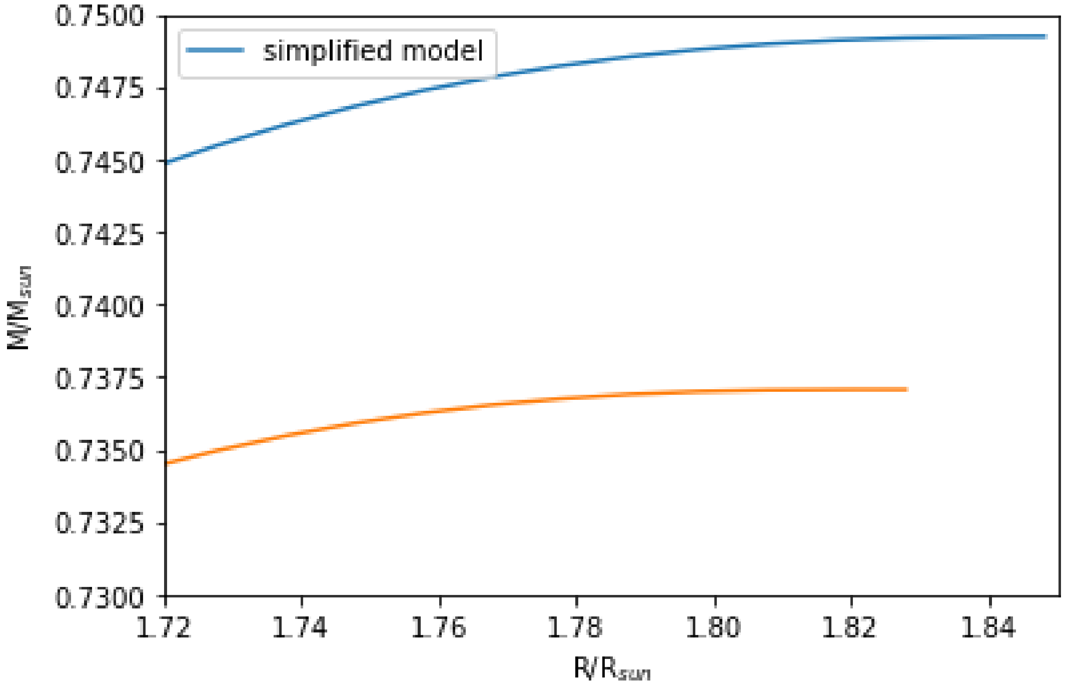
[278]:
```
r_in_rsun_sim, M_in_Msun_sim, rho_sim, T_sim = convective_star(1*u.g/u.cm**3,␣
↪3e6*u.K, simplified_EOS = True)
r_in_rsun, M_in_Msun, rho, T= convective_star(1*u.g/u.cm**3, 3e6*u.K)
```

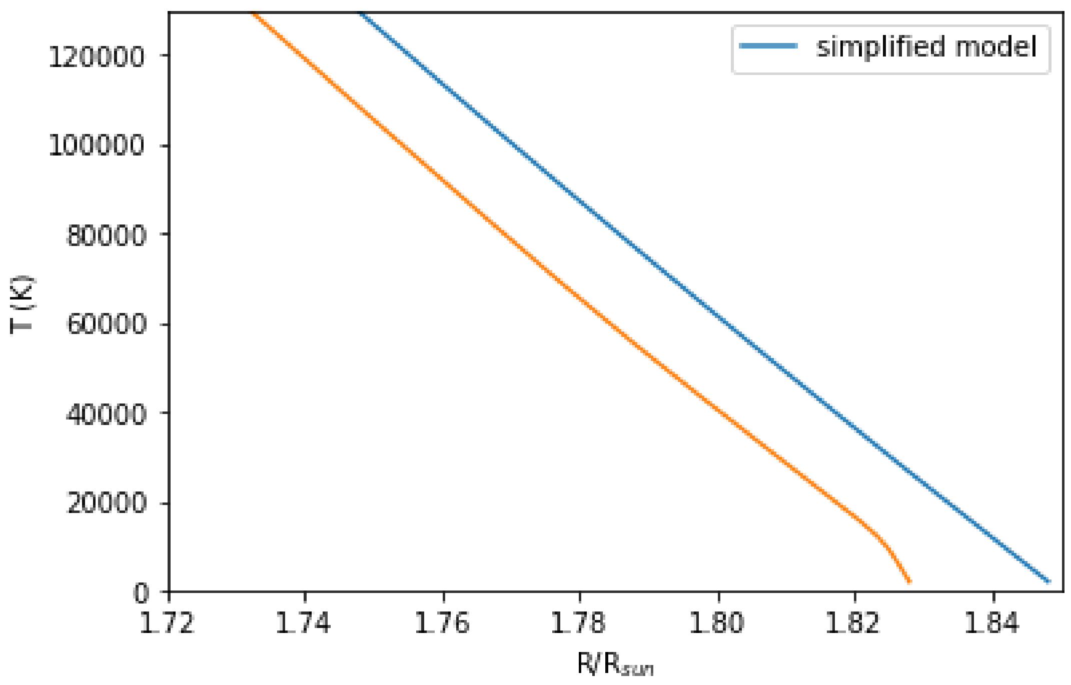
[279]:
```
plt.figure(1)
plt.plot(r_in_rsun_sim, M_in_Msun_sim,label = 'simplified model')
plt.plot(r_in_rsun, M_in_Msun)
plt.xlabel('R/R$_{sun}$');plt.ylabel('M/M$_{sun}$');plt.legend()

plt.figure(2)
plt.plot(r_in_rsun_sim, T_sim,label = 'simplified model')
plt.plot(r_in_rsun, T)
plt.xlabel('R/R$_{sun}$');plt.ylabel('T (K)');plt.legend()

plt.figure(3)
plt.plot(r_in_rsun_sim, M_in_Msun_sim,label = 'simplified model')
plt.plot(r_in_rsun, M_in_Msun)
plt.xlabel('R/R$_{sun}$');plt.ylabel('M/M$_{sun}$');plt.legend()
plt.axis([1.72, 1.85, 0.73, 0.75])

plt.figure(4)
plt.plot(r_in_rsun_sim, T_sim,label = 'simplified model')
plt.plot(r_in_rsun, T)
plt.xlabel('R/R$_{sun}$');plt.ylabel('T (K)');plt.legend()
plt.axis([1.72, 1.85, 0,130000])

plt.legend()
```

[279]:
```
<matplotlib.legend.Legend at 0x1c126aee9e8>
```




simplified model
T (K)
3000000
2500000
2000000
1500000
1000000
500000
0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
R/R$_{sun}$


simplified model
M/M$_{sun}$
0.7500
0.7475
0.7450
0.7425
0.7400
0.7375
0.7350
0.7325
0.7300
1.72
1.74
1.76
1.78
1.80
1.82
1.84
R/R$_{sun}$

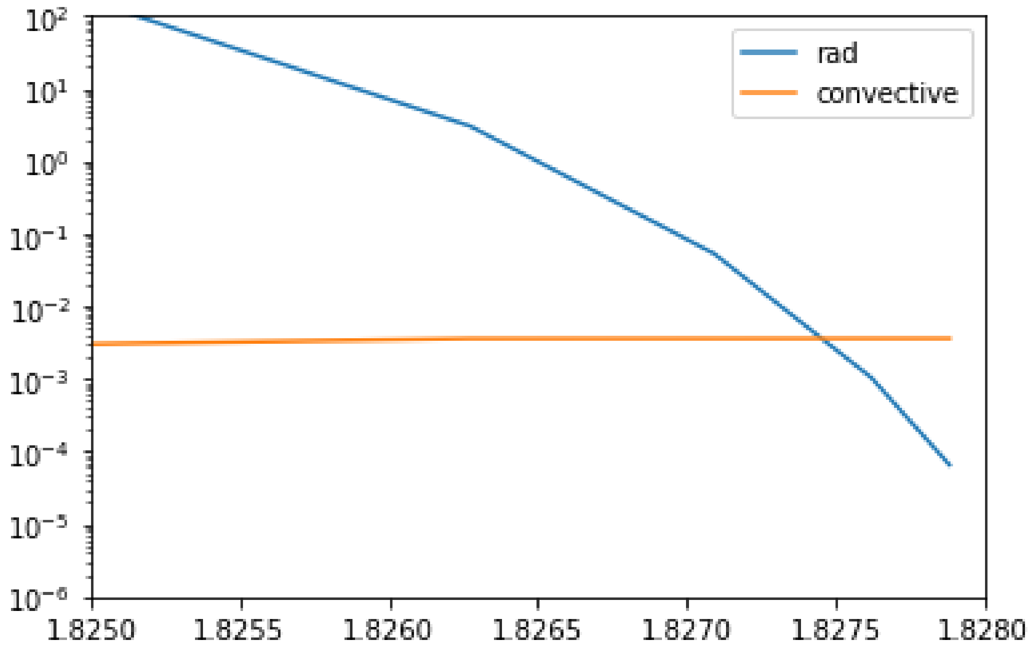


[280]:
```
dTdr_rad = 3.75e-17*T**7*rho**1.8/u.K**6/u.g**1.8*u.cm**5.4/u.m
```

[284]:
```
# compute the convective temperature gradient
dTdr = np.zeros(np.size(r_in_rsun))*u.K/u.m
for i in range(np.size(r_in_rsun)):
    P, gamma = eos_rho_T(rho[i], T[i])
    #dlrhodr = -1/gamma * (M_in_Msun_sim)*rho_sim/P.cgs.value/r_in_rsun_sim**2 *\
    #          c.G*c.M_sun/c.R_sun * (u.cm**3/u.erg)

    #(logarithmic) temperature derivative.
    dTdr[i] = (((gamma - 1)/gamma * T[i]/P *c.G*M_in_Msun[i]*c.M_sun*rho[i]/
 ↪r_in_rsun[i]**2/c.R_sun**2)).to(u.K/u.m)
```

[304]:
```
plt.semilogy(r_in_rsun, dTdr_rad,label = 'rad')
plt.semilogy(r_in_rsun, dTdr, label = 'convective')
plt.axis([1.825,1.828,1e-6,1e2])
plt.legend()
```

[304]:
```
<matplotlib.legend.Legend at 0x1c128024a20>
```



[300]:
```
# From above plot we know that the transition radius is about 1.827 R_sun
# We calculate the temperature at this radius by interpolation
from scipy import interpolate
from scipy.misc import derivative
f = interpolate.interp1d(r_in_rsun, T/u.K)

f(1.827)
```

[300]:
```
array(4223.74952076)
```

[298]:
```
# We calculate the luminosity of the star.
L = 4*np.pi*c.sigma_sb*1.827**2*c.R_sun**2*(4223*u.K)**4
L.to(u.L_sun)
```

[298]: $0.95643295\ L_\odot$