Assignment_1_standard_2019

March 11, 2019

1 Assignment 1

This assignment must be submitted by **22nd March 5pm**. Late Submissions will incur a 5% penalty per working day. Assignment submissions will close on the **29th March 5pm**. Submissions after this time will be invalid.

1.1 Question 1 (Decimal and binary fractions)

1a. Which of the following five fractions are decimal fractions (in $\mathbb{Q}_{10}$)? Which are binary fractions (in $\mathbb{Q}_2$)?

$$3/16,\ 91/200,\ 8/60,\ 24/60,\ 99/250$$

Include your reasoning.

1b. Explain how you get a binary fraction which has the same first (significant) three (nonzero) binary digits as 0.07. What is (in decimal format) the approximation of 0.07 which you get in this way? The rounding function used here is trunction (to 3 binary digits).

1.2 Question 2 (Floating point numbers)

2a. Which of the following numbers are in $\mathbb{F}_{10}(2)$, which are in $\mathbb{F}_2(6)$.

$$0.4,\ 0.25,\ 0.125,\ 0.0125, 0.0625, 0.0025, 0.625, 6.25$$

Justify your answers.

2b. Which numbers in $[1/2, 1]$ are in the intersection of $\mathbb{F}_{10}(2)$ and $\mathbb{F}_2(6)$.

2c. Show that $\mathbb{F}_{10}(2)$ and $\mathbb{F}_7(2)$

- are not dense in $\mathbb{R}$
- do not contain all the integers (characterise the integers contained)

1.3 Question 3 (Rounding)

3a. What is the rounding error for the product $3.4 * 0.43$? Use decimal arithmetic or manual multiplication to compute the exact product and use Python to compute the numerical approximation. Is the Python implementation of this product optimal for this data.

3b. Prove that for an optimal implementation of multiplication, the rounding error is the same for the (numerical) products $x_1 * x_2$ and for $x_2 * x_1$. (Do not assume that x_i are floating point numbers.)

3c. Compare the rounding errors of $\mathbb{F}_{10}(2)$ and in $\mathbb{F}_2(7)$.

1.4 Question 4 (Expression errors)

4a. Anaylse the error for the mean $m = \frac{1}{N}\sum_{i=1}^{N} x_i$ for $x_i \in [0, 1]$. Consider all possible rounding errors including the ones from the input data and the summation. For values of $N = 10, 100, 500$.

4b. Get an error bound for the rounding error of

$$2 * \cos(x)^2 - 1 - \cos(2x)$$

for $x \in (0, \pi/2]$ when it is computed using optimal implementation or arithmetic operations and $\cos$ and the standard 64 bit IEEE floating point arithmetic. (use linearised model).

1.5 Question 5 (Condition numbers)

5a. Compute the condition number for the function $\sin(x)$ for $x \in [0, \pi/4]$.

5b. What is the condition number for $f(x) = \sum_{i=1}^{n} x_i$ for $x_i \in [0, 1]$.

1.6 Question 6 (Stability and error bounds)

6a. Show that the optimal implementation of $\sin(x)$ is backward stable on a suitable interval $[a, b] \in \mathbb{R}$. What is the interval $[a, b]$?

6b. Get an error bound for the optimal implementation of $\sin(x)$ using the condition number and the backward stability result. How does this bound compare to the bound obtained from the direct analysis of the rounding error?