Australian National University

Student Number:

u							

Mathematical Sciences Institute

EXAMINATION: Semester 1 — First Semester, 2017

MATH3511/6111 — Scientific Computing

Exam Duration: 180 minutes.
Reading Time: 15 minutes.

Materials Permitted In The Exam Venue:

- Any number of copied or hand written pages.
- Unmarked English-to-foreign-language dictionary (no approval from MSI required).
- No printed books are permitted other than the dictionary.
- **No electronic aids are permitted e.g. laptops, phones, calculators.**

Materials To Be Supplied To Students:

- Scribble Paper.

Instructions To Students:

- A good strategy is not to spend too much time on any question. Read them through first and attack them in the order that allows you to make the most progress.
- *You must justify your answers. Please be neat.*

Q1	Q2	Q3	Q4	Q5	Q6
20	20	20	20	20	20

Total / 120

Question 1 **20 pts**

(a) Consider evaluting the function $f(x) = \cos(x) - 1$ for some real number x. Assume that you are using a computer for which the (relative) rounding error is bounded by $\epsilon = 0.0001$. Derive an error bound for the rounding error which arrises. *5 pts*

(b) Discuss the relative error for very small $x \approx 0$. What is the effect called which occurs for very small x? *5 pts*

(c) Give an alternative formula for $f(x)$ which gives better results for small x. *5 pts*

(d) Compare the (relative) rounding errors occurring for the evaluation of $x * y * z$ and $x + y + z$. Assume that x, y and z are floating point numbers. Show that for multiplication the associative law holds up to a relative error which is proportional to the rounding error bound. Show that this is not the case for addition. *5 pts*

Question 2 **20 pts**

(a) Show that every 2 by 2 matrix A which only has nonzero elements has an LU factorisation. What are the L and the U? Is it always safe to use the this LU factorisation to solve $Ax = b$ in this case? Why or why not? *5 pts*

(b) Consider $Ax = b$ for a large n by n circulant matrix A. Describe a direct solver for the system of equations $Ax = b$ with this matrix A which is faster than Gaussian elimination for sufficiently large n. What are the main computational steps required for this solver? How does the amount of arithmetic operations grow with the dimension n for this method and compare with the number of operations of order n^3 required for Gaussian elimination?

5 pts

(c) Consider the equation $Ax = b$ with circulant matrix A for which the first row is $(1, q, q^2, \ldots, q^{n-1})$ with $0 < q < 0.5$. Show that the Jacobi method applied to this problem converges and give a bound for the spectral radius of A. *5 pts*

(d) Let $A = I + uu^T$ for some vector u with (Euclidean) norm $\|u\| < 1$. Show that the fixpoint iteration with matrix $T = I - A$ converges for this matrix. By how much does the error decrease in k iteration steps? *5 pts*

Question 3 **20 pts**

(a) Assume that you have a calculator which computes $7^{1/4}$ to two digit accuracy. Give a method based on bisection to compute the value to five digit accuracy with Python. Use the calculator approximation as a starting point. How many function evaluations do you require? (Hint: Identify the equation solved here. You do *not* need to provide running Python code.) *7 pts*

(b) Consider the fixed point iteration to solve $x = F(x)$ for some continuous function $F : [a, b] \to \mathbb{R}$ for some fixed real a, b. Assume that the function F is (strictly) monotonically decreasing and the boundary values satisfy $a < F(b) < F(a) < b$. Give a condition on F which guarantees that the fixed point iteration converges. Show that in this case one has $(a - F(a))(b - F(b)) \leq 0$. *6 pts*

(c) Let $f(x)$ be twice continuously differentiable. Assume that Newton's method converges for f and some starting value x_0. Assume that you know the error for the first three iterates x_0, x_1 and x_2 is bounded by 0.1, 0.01 and 0.0001. How many iterations would you expect Newton's method to require to get an error of at most 10^{-12}? Justify your answer. How would you estimate the error of the x_k for $k = 0, 1, 2, \ldots$? *7 pts*

Question 4 **20 pts**

(a) Consider the polynomial defined by $p(x) = \sum_{k=0}^{n} x_k l_k(x)$ where $l_k(x)$ are the cardinal or Lagrangian functions for the points $x_0, \ldots, x_n$. Show that $p(x) = x$. (Use the uniqueness of the interpolation polynomial). *5 pts*

(b) Let $p_3(x)$ be the interpolation polynomial of degree 3 which interpolates a function $f(x)$ at the points x_0, x_1, x_2, x_3. Specify a polynomial $q(x)$ of degree 4 such that the function $p_3(x) + q(x)$ interpolates $f(x)$ in the points $x_0, x_1, \ldots, x_4$. *5 pts*

(c) Consider the Chebyshev interpolant of degree k of $f(x) = \exp(x)$ in the interval $[-1, 1]$. Use the error formula to estimate how much the error decreases if the degree of the interpolant is increased to $k + 1$. *5 pts*

(d) Show that the Chebyshev polynomials $q_k(x)$ of degree k are even if k is even. (Hint: use the trigonometric identity $\cos(\pi - x) = -\cos(x)$.) *5 pts*

Question 5 **20 pts**

(a) What is the composite Gauss quadrature rule for approximating $\int_0^1 f(x)\,dx$ using four 3-point Gauss rules for the intervals $[k/4, (k+1)/4]$ and $k = 0, 1, 2, 3$? *6 pts*

(b) Assume that you would like to compute the integral $\int_0^1 p(x)\,dx$ for a polynomial of degree 6 exactly, using the trapezoidal rule and Romberg extrapolation. How many evaluations of $p(x)$ would you require and how many columns do you need in the Romberg scheme? *7 pts*

(c) Provide a bound for the (approximation) error of the central difference scheme to compute an approximate of the derivative $f'(0)$ using function values $f(-h), f(0)$ and $f(h)$ for some (small) $h > 0$. What is the actual approximation formula for this method? Use rounding error analysis to show how the rounding error grows with decreasing h where you assume that $h = 2^{-j}$ for some positive integers j. *7 pts*

Question 6 **20 pts**

Consider the solution of the initial value problem (IVP)

$$\frac{d^2x}{dt^2} + \sin(x) = 0$$
$$x(0) = 1$$
$$\frac{dx}{dt}(0) = 0$$

using Euler's method.

(a) rewrite this as a system of first order differential equations. *5 pts*

(b) show that the IVP has a solution $x(t)$ for $t \in [0, T]$ and some $T > 0$ *5 pts*

(c) show that Euler's method converges for the stepsize $h \to 0$ *5 pts*

(d) assume you have done some experiments and found that the error of the solution $x(1)$ is approximately 0.1 when a stepsize $h = 0.001$ is used. Which error would you expect for the stepsize $h = 10^{-6}$. Justify your expectation. *5 pts*