ass4

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Q1.1

$$LHS = 1 + x + x^2/2 + x^3/6$$

$$RHS = ((x/3+1)x/2+1)x + 1 = (x^2/6 + x/2 + 1)x + 1 = 1 + x + x^2/2 + x^3/6$$

$LHS = RHS$

Q1.2

weirstrauss approximation therom states that: If f(x) is a continuous real-valued function on $[a, b]$ and if any $\epsilon > 0$ is given, then there exists a polynomial p on $[a, b]$ such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$. In words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy.

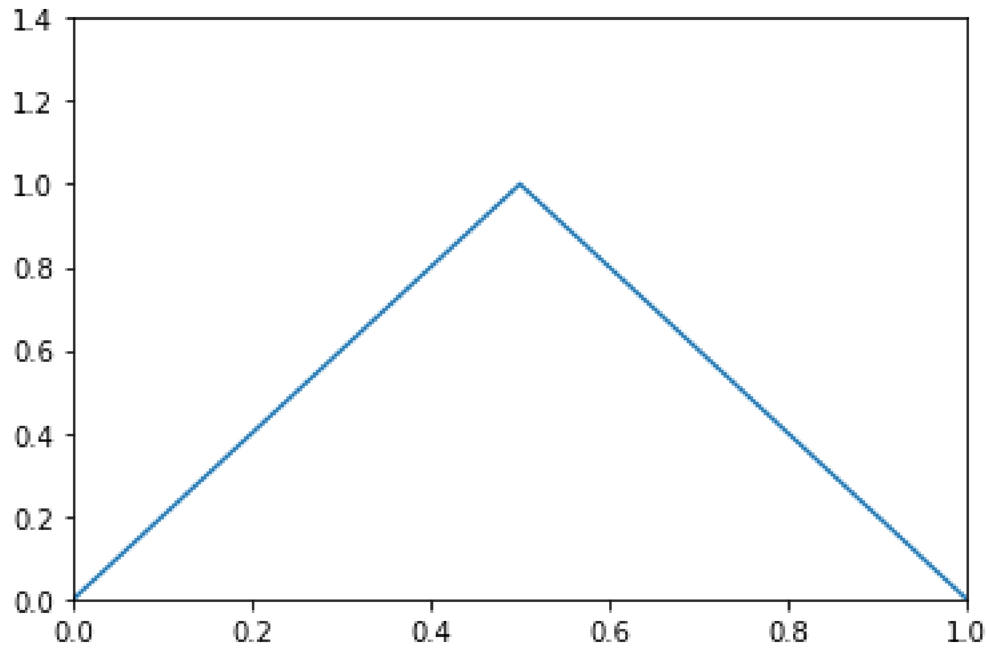
As

$$f(x) = max(1 - |2x - 1|, 0) = \begin{cases} 2x & 1 \leq x < \dfrac{1}{2} \\ 2 - 2x & \dfrac{1}{2} < x \leq 1 \end{cases} \tag{1}$$

f(x) is a continuous function (as $f(\frac{1}{2}-) = f(\frac{1}{2}+) = 1$) as shown in plot below. So if $\epsilon = 10^{-9}$ is given. There exists a polynomial p on $[0, 1]$ such that $|f(x) - p(x)| < \epsilon$.

So this function can be approximated to an error less than 10^{-9} using only polynomials.

In [3]:
```
import numpy as np
import matplotlib.pyplot as plt
xv=np.linspace(0,1,101)
yv=[]
for x in xv:
    if x>=0 and x<1/2:
        y=2*x
    if x>=1/2 and x<=1:
        y=2-2*x
    if x<=0 and x>=1:
        y=0
    yv.append(y)
plt.plot(xv,yv)
plt.axis([0,1,0,1.4])
plt.show()
```



Q1.3

For $p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$, then the interpolation conditions lead to a linear system of equations for the a_k:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

If $p_1(x) = a_0 + a_1 x$

collocation points:

i	0	1
x_i	0	1
y_i	1	1

- system of equations for a_k:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$a_0 = 1$, $a_1 = 0$ linear intepolation $p_1(x) = 1$ when x is close to 0.5, f(x) is close to positive infinity. Hence $|f(x) - 1| \to +\infty$ The largest error in $[0, 1]$ is $+\infty$.

Q1.4

Basis functions $n_0(x) = 1$ and

$$n_{j+1}(x) = \prod_{k=0}^{j} (x - x_k)$$

interpolant for points $(x_0, y_0), \ldots, (x_k, y_k)$:
$p_0(x) = y_0$
$p_k(x) = \sum_{j=0}^{k} c_j n_j(x)$
From recursion formula: $p_{k+1}(x) = p_k(x) + c_{k+1} n_{k+1}(x)$
$c_{k+1} = \frac{y_{k+1} - p_k(x_{k+1})}{n_{k+1}(x_{k+1})}$
$x_k = \frac{k}{4}$ $k = 0, ..., 4$ *Horner's rule*

$$p_n(x) = a_0 + x(a_1 + x(a_2 + x(a_3 + \cdots)))$$

$$p_4(x) = y_0 + (x - x_0)(c_1 + (x - x_1)(c_2 + (x - x_2)(c_3 + (x - x_3)c_4))) = y_0 + x(c_1 + (x - \frac{1}{4})(c_2 + (x - \frac{1}{2})(c_3 + (x - \frac{3}{4}$$

When $x = \pi$

$$p_4(x) = y_0 + \pi(c_1 + (\pi - \frac{1}{4})(c_2 + (\pi - \frac{1}{2})(c_3 + (\pi - \frac{3}{4})c_4)))$$

which needs 4 multiplications.
When $x = -1$

$$p_4(x) = y_0 - (c_1 + (-1 - \frac{1}{4})(c_2 + (-1 - \frac{1}{2})(c_3 + (-1 - \frac{3}{4})c_4))) = y_0 - c_1 + \frac{5}{4}(c_2 - \frac{3}{2}(c_3 - \frac{7}{4}c_4))$$

which needs 3 multiplications.
Q2.1
Interpolation points:

i	0	1	2
x_i	0	1	1/2
y_i	1	8/3	79/48

Newton's functions are

$$n_0(x) = 1,$$
$$n_1(x) = x,$$
$$n_2(x) = x(x-1)$$

and so

$$p_0(x) = y_0 = 1,$$
$$c_1 = \frac{\frac{8}{3} - 1}{1} = \frac{5}{3},$$
$$p_1(x) = 1 + \frac{5}{3}x,$$
$$c_2 = \frac{79/48 - (1 + \frac{5}{3}\frac{1}{2})}{\frac{1}{2}(\frac{1}{2} - 1)} = \frac{3}{4},$$
$$p_2(x) = 1 + \frac{5}{3}x + \frac{3}{4}x(x-1)$$

Q2.2

Interpolation points:

i	0	1	2	3	4
x_i	0	1	1/2	1/4	3/4
y_i	y_0	y_1	y_2	y_3	y_4

Newton's functions are

$$\begin{aligned} n_0(x) &= 1, \\ n_1(x) &= x, \\ n_2(x) &= x(x-1), \\ n_3(x) &= x(x-1)(x-1/2), \\ n_4(x) &= x(x-1)(x-1/2)(x-1/4) \end{aligned}$$

and so

$$\begin{aligned} p_0(x) &= y_0, \\ c_1 &= \frac{y_1 - y_0}{1} = y_1 - y_0 \\ p_1(x) &= y_0 + (y_1 - y_0)x, \\ c_2 &= \frac{y_2 - [y_0 + (y_1 - y_0)\frac{1}{2}]}{\frac{1}{2}(\frac{1}{2} - 1)} = 2(y_1 + y_0) - 4y_2 \\ p_2(x) &= y_0 + (y_1 - y_0)x + [2(y_1 + y_0) - 4y_2]x(x-1) \\ c_3 &= \frac{y_3 - y_0 + (y_1 - y_0)\frac{1}{4} + [2(y_1 + y_0) - 4y_2]\frac{1}{4}(\frac{1}{4} - 1)}{\frac{1}{4}(\frac{1}{4} - 1)(\frac{1}{4} - 1/2)} = \frac{64y_3 - 48y_2 + 8y_1 - 24y_0}{3} \\ p_3(x) &= y_0 + (y_1 - y_0)x + [2(y_1 + y_0) - 4y_2]x(x-1) + \frac{64y_3 - 48y_2 + 8y_1 - 24y_0}{3}x(x-1)(x-1/2) \\ c_4 &= \frac{y_4 - p_3(\frac{3}{4})}{n_4(\frac{3}{4})} = \frac{32y_0 + 32y_1 + 192y_2 - 128y_3 - 128y_4}{3} \\ p_4(x) &= y_0 + (y_1 - y_0)x + [2(y_1 + y_0) - 4y_2]x(x-1) + \frac{64y_3 - 48y_2 + 8y_1 - 24y_0}{3}x(x-1)(x-1/2) + \frac{32y_0 + 32y}{} \end{aligned}$$

Q2.3

From equations in Q2.2 we can calculate that:

$$\frac{p_2(x) - p_4(x)}{p_1(x) - p_2(x)} = \frac{c_3 x(x-1)(x-1/2) + c_4 x(x-1)(x-1/2)(x-1/4)}{c_2 x(x-1)} = \frac{c_3 + c_4(x-1/4)}{c_2}(x-1/2)$$

, which is dependent on x.

Using error formula and recursion formula, we get:

$$\frac{p_2(x) - f(x)}{p_1(x) - f(x)} = \frac{p_1(x) - f(x) + p_2(x) - p_1(x)}{p_1(x) - f(x)} = 1 + \frac{p_2(x) - p_1(x)}{\frac{1}{2!}f''(\xi)x(x-1)} = 1 + \frac{c_2 x(x-1)}{\frac{1}{2!}f''(\xi)x(x-1)} = 1 + \frac{2c_2}{f''(\xi)}$$

which is independent of x.

Comparison, the first expression in dependent on x, while second one is independent of x.

Q3.1

degree $n = 3$

Interpolation points are: $x_k = 2k/3 - 1 \quad k = 0,1,2,3$
$x_0 = -1, x_1 = -1/3, x_2 = 1/3, x_3 = 1$
For function

$$f(x) = x^4 - 1.2356x^2 \quad x \in [-1,1]$$

$$f^{(4)}(x) = 24$$

the interpolating polynomial p of degree 3 satisfies

$$f(x) - p(x) = \frac{1}{4!} f^{(4)}(\xi)\, w(x) = w(x), \quad \text{for some } \xi \in [-1,1]$$

where $w(x) = (x - x_0) \cdots (x - x_3) = (x-1)(x-1/3)(x+1/3)(x+1)$
so

$$e(x) = (x-1)(x-1/3)(x+1/3)(x+1) = x^4 - \frac{10}{9}x^2 + \frac{1}{9}$$

We calculate the derivative:

$$e'(x) = 4x^3 - \frac{20}{9}x$$

$$\text{the maximum or minimum points may occur at } x = 0 \quad x = \pm\frac{\sqrt{5}}{3}$$

$$e''(x) = 12x^2 - \frac{20}{9} \quad e''(0) < 0 \quad e''(\pm\sqrt{5}/3) > 0$$

Hence $e_{max} = e(0) = \frac{1}{9} \quad e_{min} = min\{e(\pm\frac{\sqrt{5}}{3})\} = -\frac{16}{81}$
So the interpolation error

$$e(x) \in [-\frac{16}{81}, \frac{1}{9}]$$

Q3.2
polynomials p(x) of degree 5 for which $p(k/4) = 0 \quad k = 0,1,2,3,4$ has roots: $x_0 = 0x_1 = 1/4, x_2 = 1/2, x_3 = 3/4, x_4 = 1$.
So a general formula for all polynomials p(x) of this kind is:

$$p(x) = ax(x-1/4)(x-1/2)(x-3/4)(x-1)$$

Where a is a arbitrary coefficient which is not equal to zero.
Q3.3
For a function $f \in C^{n+1}[-1,1]$, if $|f^{(n+1)}(x)| \leq C$ for all $x \in [-1,1]$ where $C = \sup_{x\in[-1,1]} |f^{(n+1)}(x)|$
the error bound for interpolating polynomial p of degree n is

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) w(x) \leq \frac{1}{(n+1)!} C w(x), \quad \text{for some } \xi \in [-1,1]$$

where $w(x) = (x - x_0) \cdots (x - x_n)$
From lecture notes the upper error bound for Chebyshev intepolation

$$\frac{1}{2^n(n+1)!} \sup_{x\in[-1,1]} |f^{(n+1)}(x)| = |e(x)| \leq \frac{1}{2^n(n+1)!} C$$

The centre a=0 $b = \pm 1$ then the upper error bound of Taylor polynomial is

$$\frac{u^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \le \frac{(b-a)^{n+1}}{(n+1)!}C = \frac{1}{(n+1)!}C$$

For n=2 and a function $f \in C^3[-1,1]$, for the following 3 cases, we calculate the upper error bound:

- the interpolant with points $x_k = -1, 0, 1$,

$$|e(x)| \le \frac{1}{3!}C|(x+1)x(x-1)| \le \frac{\sqrt{3}}{27}C$$

In the last step we do the following calculation: $\frac{d}{dx}(x+1)x(x-1) = 3x^2 - 1 = 0$
$x = \pm\frac{1}{\sqrt{3}}$
$|(x+1)x(x-1)| < max(|(-\frac{1}{\sqrt{3}}+1)(-\frac{1}{\sqrt{3}})(-\frac{1}{\sqrt{3}}-1)|, |(\frac{1}{\sqrt{3}}+1)(\frac{1}{\sqrt{3}})(\frac{1}{\sqrt{3}}-1)|) = \frac{2\sqrt{3}}{9}$

- the interpolant of degree 2 with Chebyshev interpolation points,

$$|e(x)| \le \frac{1}{2^2 3!}C = \frac{1}{24}C$$

- the Taylor polynomial of degree 2 centred at x = 0.

$$|e(x)| \le \frac{1}{3!}C = \frac{1}{6}C$$

where $C = \sup_{x\in[-1,1]} |f^{(3)}(x)|$
In general case:
the upper error bound for interpolating polynomial p of degree n is

$$f(x) - p(x) \le \frac{1}{(n+1)!}Cw(x) \le \frac{2^{n+1}}{(n+1)!}C$$

The upper bound for interpolating polynomial p of degree n, Chebyshev intepolation, Taylor polynomial are

$$\frac{2^{n+1}}{(n+1)!}C, \frac{1}{2^n(n+1)!}C, \frac{1}{(n+1)!}C$$

respectively.

The upper bound for Chebyshev interpolation is much smaller than the other two. Hence the Chebyshev interpolation points give the best approximation.