① Integration of non-neg fcns

Def: Let (X, S) mble space

An S-mble partition of X is $P = \{A_1, \dots, A_m\}$, s.t. $A_i \in S$, $A_1, \dots, A_m$ disjoint $\bigcup_{i=1}^{\infty} A_i = X$.

back in Riemann Int, we use intervals. Here, our partitions will be something much more complicated.

Def. (X, S, μ) measure space $\quad \int f \, d\mu$.

$f: X \to [0, \infty]$, S-mble fcns. For a S-mble partition $P = \{A_1, \dots, A_m\}$. defines lower Lebesgue sum of f using P as $\mathcal{L}(f, P) = \sum_{i=1}^{m} (\inf_{A_i} f) \cdot \mu(A_i)$ Just the low Lebesgue sum to define Leb Int

Define: $\int f \, d\mu = \sup \{ \mathcal{L}(f, P) \mid P \text{ is a } S\text{-mble partition of } X \}$. much more flexible than Rie Int.

Lemma:

- If $E \in S$, $\int 1_E \, d\mu = \mu(E)$

Pf: For $P = \{E, E^c\} \Rightarrow \mathcal{L}(1_E, P) = \mu(E)$ $\Rightarrow \int 1_E \, d\mu \geq \mu(E)$

$\forall P = \{A_1, \dots, A_m\}$ be an arbitrary partition. $\mathcal{L}(1_E, P) = \sum_{i=1}^{m} \left(\inf_{A_i} 1_E\right) \mu(A_i)$

$\hookrightarrow = \begin{cases} 1 & A_i \subseteq E \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow \mathcal{L}(1_E, P) = \sum_{i: A_i \subseteq E} \mu(A_i) \leq \mu(E)$ $\Rightarrow \int 1_E \, d\mu \leq \mu(E)$.

$\quad \sum_{i: A_i \subseteq E} \mu(A_i) = \mu\left(\bigcup_{i: A_i \subseteq E} A_i\right)$

Done. #.

Notation: $\lambda(E) = |E|$ leb. measure on $\mathbb{R}$.

Ex. For Dirichlet Function

① $\int_{\mathbb{R}} 1_{\mathbb{Q}} \, d\lambda = \lambda(\mathbb{Q}) = 0$

② $\int_{\mathbb{R}} 1_{\underbrace{[0,1] \setminus \mathbb{Q}}_{\text{measure } = 1}} \, d\lambda = 1$

③ $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting measure})$ $f: \mathbb{N} \mapsto [0, \infty)$ $\quad f(i) = b_i \in [0, +\infty)$

$$\int f \, d\mu = \sum_{i=1}^{\infty} b_i$$

Lemma: (X, S, μ) measure space

$E_1, \dots, E_N \in S$ disjoint. $c_1, \dots, c_N \in [0, +\infty] \Rightarrow \int \underbrace{\left(\sum_{k=1}^{N} c_k 1_{E_k}\right)}_{\psi} d\mu = \sum_{k=1}^{N} c_k \, \mu(E_k)$

Pf: ① Let $E_{N+1} = X \setminus \left(\bigcup_{k=1}^{N} E_k\right) \Rightarrow \{E_1, \dots, E_{N+1}\}$ S-partition of X

Let $c_{N+1} = 0$.

$\Rightarrow \psi = \sum_{k=1}^{N+1} c_k 1_{E_k}$

For $P = \{E_1, \dots, E_{N+1}\}$, $\quad \mathcal{L}(\psi, P) = \sum_{k=1}^{N+1} c_k \, \mu(E_k) = \sum_{k=1}^{N} c_k \, \mu(E_k)$.

$\Rightarrow \int \psi \, d\mu \geq \sum_{k=1}^{N} c_k \, \mu(E_k)$

② Let $\mathcal{P} = \{A_1, \dots, A_m\}$ be a $\mathcal{S}$-partition of X

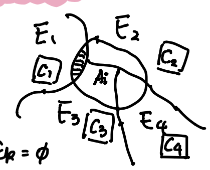
$$\mathcal{L}(\psi, \mathcal{P}) = \sum_{i=1}^{m} \inf_{A_i} \psi \, \mu(A_i) = \sum_{i=1}^{m} \inf_{A_i} \psi \sum_{k=1}^{N+1} \mu(A_i \cap E_k)$$

$$= \sum_{k=1}^{N+1} \sum_{i=1}^{m} \left(\inf_{A_i} \psi\right) \mu(A_i \cap E_k)$$

ψ: $\int \left(\sum_{i=1}^{N} c_i 1_{E_i} \right) d\mu$

$$\le \sum_{k=1}^{N+1} \sum_{i=1}^{m} \left(\inf_{A_i \cap E_k} \psi\right) \mu(A_i \cap E_k)$$

$= 0$ if $A_i \cap E_k = \phi$



$$= \sum_{k=1}^{N+1} \sum_{i : A_i \cap E_k \neq \phi} C_k \, \mu(A_i \cap E_k)$$

$$= \sum_{k=1}^{N+1} C_k \cdot \mu(E_k) \quad \blacksquare$$

Lemma: $(X, \mathcal{A}, \mu)$

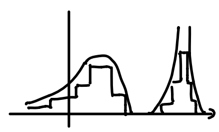
$f, g : X \longrightarrow [0, \infty)$ $\mathcal{S}$-mble fns

If $f(x) \le g(x)$, $\forall x \in X$, then $\int f \, d\mu \le \int g \, d\mu$.

Pf: For any partition $\mathcal{P} = \{A_1, \dots, A_m\}$. $\inf_{A_i} f \le \inf_{A_i} g \Rightarrow \sum_i \inf_{A_i} f \cdot \mu(A_i) \le \sum_i \inf_{A_i} g \cdot \mu(A_i)$

$\Rightarrow \int f \, d\mu \le \int g \, d\mu$

② Monotone convergence Thm



Important def for $\int f d\mu$.

Prop. $(X, \mathcal{S}, \mu)$. $f : X \longrightarrow [0, \infty]$ mble-ftn.

$\Rightarrow \int f \, d\mu = \underbrace{\sup}_{\text{RHS}} \left\{ \sum_{j=1}^{N} c_j \mu(E_j) \,\middle|\, E_1, \dots, E_N \text{ disjoint}, \ c_j \in [0, \infty), \ \sum_{j=1}^{N} c_j 1_{E_j}(x) \le f(x), \ \forall x \in X \right\}$

Pf: ① By Monotonicity. $\sum c_j \cdot \mu(E_j) = \int c_j 1_{E_j} < \int f \Rightarrow \underline{\text{RHS} \le \int f}$.

② (1) Assume $\inf_B f < \infty$, $\forall B \in \mathcal{S}$ with $\mu(B) > 0$.

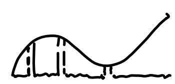
let $\mathcal{P} = \{A_1, \dots A_m\}$ be a $\mathcal{S}$-partition of X.

discard A_i s.t. $\mu(A_i) = 0$. and call the remaining ones $E_1, \dots E_N$

A particular choice $\mathcal{L}(f, \mathcal{P})$.

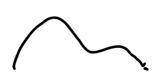
WT have this.

$\mathcal{L}(f, \mathcal{P}) = \sum_{i=1}^{N} c_i \, \mu(E_i)$, $c_i = \inf_{E_i}(f)$ $\Rightarrow$ c_i is finite.



$\Rightarrow \sup \sum_{j=1}^{N} c_j \mu(E_j) \ge \sum \inf_{E_i}(f) \, \mu(E_i)$?

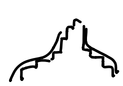
$\Rightarrow$ we have $\text{RHS} \ge \mathcal{L}(f, \mathcal{P})$ for arbitrary $\mathcal{P}$.



$\Rightarrow$ $\underline{\text{RHS} \ge \int f}$

(2) Assume $\exists B \in \mathcal{S}$ s.t. $\mu(B) > 0$, $\inf_B f = \infty$

check $\begin{cases} \int f = \infty \\ \text{RHS} = \infty \end{cases}$



$\Rightarrow \text{RHS} = \int f$.

Thm: (Monotone Convergence Thm) $(X, \mathcal{S}, \mu)$

$0 \le f_1(x) \le f_2(x) \le \cdots$, $\forall x \in X$, mble fns

Let $f(x) = \lim_{n \to \infty} f_n(x) \in [0, \infty]$

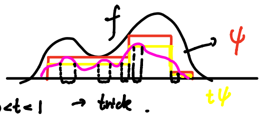
Then $\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$ We can switch it !

(pf) $\alpha_n = \int f_n \, d\mu$ an increasing seq

$\Rightarrow \lim_{n\to\infty} \alpha_n$ converges $\Rightarrow \lim_{n\to\infty} \int f_n \, d\mu$ converges in $[0, \infty]$

(a) $f(x) \geq f_n(x) \ \forall n, \forall x \Rightarrow \boxed{\int f \geq \lim_{n\to\infty} \int f_n}$

(b) consider $\psi = \sum_{j=1}^{N} c_j 1_{A_j}$ where $c_j \in [0, \infty)$, $A_j \in S$ disjoint. $\psi(x) \leq f(x) \ \forall x \in X$



- Fix $0 < t < 1$ → trick.

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For each $n \in \mathbb{N}$

let $E_n = \{x \in X : f_n(x) \geq t\,\psi(x)\}$.

$\Rightarrow E_1 \subseteq E_2 \subseteq \cdots$, $\bigcup_{n=1}^{\infty} E_n = X$ ✓

$f_n(x) \geq f_n(x) 1_{E_n}(x) \geq t\,\psi(x) \cdot 1_{E_n}(x) = t \cdot \sum_{j=1}^{N} c_j 1_{A_j \cap E_n}$

$\Rightarrow \int f_n \, d\mu \geq t \cdot \sum_{j=1}^{N} [c_j \, \mu(A_j \cap E_n)]$

$\Rightarrow \lim_{n\to\infty} \int f_n \, d\mu \geq t \cdot \sum_{j=1}^{N} c_j \underbrace{\lim_{n\to\infty} \mu(A_j \cap E_n)}$

$$\left(\begin{aligned} &= \mu\Big(\bigcup_{n=1}^{\infty} (A_j \cap E_n) \Big) \\ &= \mu(A_j) \end{aligned} \right)$$

$\Rightarrow \lim_{n\to\infty} \int f_n \, d\mu \geq \sum_{j=1}^{\infty} c_j \cdot \mu(A_j)$, take sup of c_j, A_j

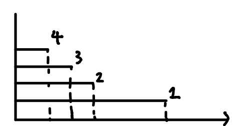
$\Rightarrow \boxed{\lim_{n\to\infty} \int f_n \, d\mu \geq \int f}$.

examples : ① $f_n = 1_{[n, n+1)}$, then $\lim_{n\to\infty} \int f_n \, d\lambda \neq \int f \, d\lambda$.

without $f_1 \leq f_2 \leq \cdots$

② $f_n = \frac{1}{n} 1_{[0, n)}$, then $\lim_{n\to\infty} \int f_n \, d\lambda \neq \int f \, d\lambda$.

③ $f_n = n \, 1_{[0, \frac{1}{n})}$, then $\lim_{n\to\infty} \int f_n \, d\lambda \neq \int f \, d\lambda$.

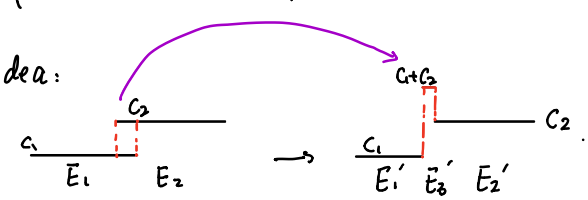


lemma : (X, S, μ)

$$\int \Big(\sum_{k=1}^{N} c_k 1_{E_k} \Big) d\mu = \sum_{k=1}^{N} c_k \, \mu(E_k), \quad \forall c_k \in [0, \infty], \ \forall E_k \in S$$

↑ not disjoint

Pf Idea:



Cor: $f, g : X \to [0, \infty]$ mble ftns on (X, S, μ)

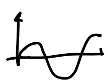
$\int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu$

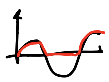
Pf: f, g mble $\Rightarrow \exists$ simple ftns $\phi_1, \phi_2, \cdots \to f$ (pw).
$\psi_1, \psi_2, \cdots \to g$. (pw)

$$\int f+g \, d\mu \overset{\text{MCT}}{=} \lim_{n\to\infty} \int \phi_n + \psi_n \, d\mu \underset{\text{Lemma}}{=} \lim_{n\to\infty} \left(\int \phi_n \, d\mu + \int \psi_n \, d\mu \right) \overset{\text{MCT}}{=} \int f \, d\mu + \int g \, d\mu.$$

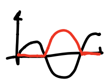
② Integration of real-valued ftns

Def: For $f : X \longrightarrow [-\infty, \infty]$ $\quad f = f^+ - f^-$

f 

$f^+ =$ 

$|f| = f^+ + f^-$

$f^- =$ 

$f^+, f^- : X \longrightarrow [0, +\infty)$

if $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$, Define $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu$

Note: $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty \Leftrightarrow \int |f| \, d\mu < \infty$

Prop:
- $\int cf = c \int f$
- $\int f+g = \int f + \int g$
- $f \le g$, $\int f \le \int g$
- $\left| \int f \right| \le \int |f|$

Pf: let $h = f+g$, $h^+ - h^- = f^+ + g^+ - f^- - g^- \Rightarrow h^+ + f^- + g^- = f^+ + g^+ + h^-$

WTS: $\int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^-$

$\Downarrow$

$\int h^+ + \int f^- + \int g^-$
$\quad \parallel$
$\int f^+ + \int g^+ + \int h^-$

$\uparrow$ Mid term
