fundamental theorem of calculus

Motivation:

known: $\lim_{t\to 0} \frac{1}{t}\int_x^{x+t} f(y)-f(x)\, dy = 0$

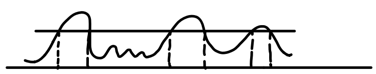
WTK: $\lim_{t\to 0} \frac{1}{2t}\cdot\int_{x-t}^{x+t} f(y)-f(x)\, dy = 0$?

$\boxed{4A}$ Hardy-Littlewood maximal ftn.

$P(X\geq a) \leq \frac{E(X)}{a}$.

① Thm. Markov Inequality.

(X, S, μ). $h\in \mathcal{L}^1(X,\mu)$, $\forall c>0$, $\mu(\underbrace{\{x\in X : |h(x)|\geq c\}}_{E}) \leq \frac{1}{c}\int |h|$



$$\int |h| \geq \int_E |h| \geq \int_E c = c\cdot\mu(E) \Rightarrow \mu(E) \leq \frac{1}{c}\cdot\int|h|.$$

②

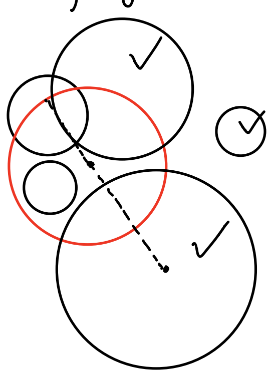
Notation: $B = B(a,r)$. $cB = B(a, c\cdot r)$ for $c>0$

Thm. (Vitali covering lemma)

Let $B_1,\cdots,B_n$ be open balls in $\mathbb{R}^d$.

Then, $\exists$ a subcollection $B_1',\cdots,B_k'$ s.t. they are disjoint and $\bigcup_{i=1}^n B_i \subseteq \bigcup_{j=1}^k (3B_j')$.

- Greedy Algorithm : "biggest one + disjoint".



Pf: for the red ball, it must intersect some selected balls $\Rightarrow \exists$ a ball with larger radius. $\Rightarrow$ it will be covered.

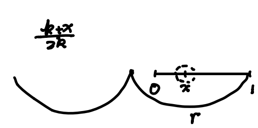
③ HL maximal inequality.

Def: (Hardy-Littlewood maximal ftn)

Let $h:\mathbb{R}\to\mathbb{R}$ be a Leb. mble ftn, then the HL maximal ftn for h is

$h^*:\mathbb{R}\to[0,\infty]$ given by $h^*(x) = \sup_{t>0}\frac{1}{2t}\int_{x-t}^{x+t}|h(y)|\,dy$.

Example: $h(x) = 1_{[0,1]}(x)$, $h^*(x) = \begin{cases} \frac{1}{2(1-x)} & (-\infty, 0] \\ 1 & (0,1) \\ \frac{1}{2x} & [1,\infty) \end{cases}$



Lemma: Assume $h \in L^1(\mathbb{R})$

$A_t(x) = \frac{1}{2t}\int_{x-t}^{x+t} |h(y)|\, dy$. is jointly cts in $(x,t) \in \mathbb{R} \times \mathbb{R}^+$

$\Rightarrow h^*$ is a Borel mble ftn

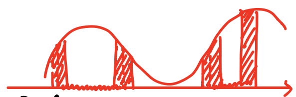
DCT ?

Pf: ...

~~Thm:~~ (HL maximal Inequality)

If $h \in L^1(\mathbb{R})$, then

$\forall c > 0$, $\mu(\{x \in \mathbb{R} : h^*(x) > c\}) \leq \frac{3}{c}\int |h|$



$h^*(x) = \sup_{t>0} \{\frac{1}{2t}\int_{x-t}^{x+t} |h(y)|\, dy\}$.

Thm: Linear Regularity of Leb measure

$|A| = \sup\{|K| : K$ closed and bounded, $K \subset A\}$.

Pf: ① $|A| > RHS$

② (a) A bounded, $\exists$ closed F s.t. $|A \setminus F| < \varepsilon \Rightarrow |A| - |F| < \varepsilon$ #

$\Rightarrow F$ is compact

(b) A unbounded, but $|A|$ finite

$A_n = A \cap [-n, n] \Rightarrow A = \bigcup_{n=1}^{\infty} A_n$

$|A| = \lim_{n\to\infty} |A_n| \Rightarrow \forall n > N$. $|A| - |A_n| < \frac{\varepsilon}{2}$.

A_n is bounded $\Rightarrow \exists \underset{\text{compact}}{F} \subseteq A_n \cdots$ $|A_n| - |F| < \frac{\varepsilon}{2}$

$\Rightarrow |A| - |F| < \varepsilon$ #

(c) A unbounded, and $|A|$ infinite

similarly, $|A| = \lim_{n\to\infty} |A_n| = \infty$. $\exists N \in \mathbb{N}$, fix L,

$\forall n > N$, $|A_n| > L+1$, $\exists F$, $|A_n| - |F| < \varepsilon \Rightarrow |F| > L \Rightarrow RHS = \infty$

Pf: Fix $\varepsilon > 0$, let $E = \{x \mid h^*(x) > c\} \to$ Borel mble

transform to Compact

WTS: $\forall$ compact set $K \subset E$, $|K| \leq \frac{3}{c}\int |h|$

fix compact $K \subset E$, for each $x \in K$, st $h^*(x) > c$

$\Rightarrow \exists t_x > 0$ s.t. $\frac{1}{2t_x}\int_{x-t_x}^{x+t_x} |h(y)|\, dy > c$

Subcover

do it for all $x \in K \Rightarrow K \subset \bigcup_{x\in K}(x - t_x, x + t_x)$ since K compact.

$\Rightarrow K \subset \bigcup_{i=1}^{n} \underbrace{(x_i - t_{x_i}, x + t_{x_i})}_{I_i}$

and $\frac{1}{|I_i|}\int_{I_i} |h(y)|\,dy > c \Rightarrow |I_i| < \frac{1}{c}\int_{I_i} |h(y)|\, dy$. Markov

By Vitalli Cover. $\Rightarrow \exists$ subcollection $J_1, \cdots, J_m$ disjoint and $K \subset \bigcup_{i=1}^{n} I_i \subset \bigcup_{i=1}^{m}(3J_i)$

$$|K| \le \sum_{i=1}^{m} |3J_i| = 3\sum_{i=1}^{m} |J_i| < \frac{3}{c}\sum_{j=1}^{m} \int_{J_i} |h(y)|\, dy \underset{\text{disjoint}}{\le} \frac{3}{c}\int_{\mathbb{R}} |h(y)|\, dy. \qquad \boxtimes$$