e.x of vector space. over scalar field $\mathbb{R}$ ($\mathbb{C}$)

① $V = \mathbb{R}^n$ over $\mathbb{R}$

$V = \mathbb{C}^n$ over $\mathbb{R}$ or $\mathbb{C}$

$V = \{ a = (a_1, a_2, \cdots) : a_i \in \mathbb{C} \}$

$V = \ell^\infty = \{ a = (a_1, \cdots) : a_i \in \mathbb{C},\ \sup_{i \in \mathbb{N}} |a_i| < \infty \}$

$V = \ell^1 = \{ a = (a_1, \cdots) : \sum_i |a_i| < \infty \}$ any ℓ^p is a vector space

$V = \ell^2 = \{ a = (a_1, \cdots) : \sum_i |a_i|^2 < \infty \}$

$$\begin{aligned}
&\sum |a_i + b_i|^2 \\
&\le \sum (|a_i| + |b_i|)^2 \\
&= \sum |a_i|^2 + \sum |b_i|^2 + \sum 2 |a_i| \underline{|b_i|} \\
&\le \sum |a_i|^2 + \sum |b_i|^2 + \sum (|a_i|^2 + |b_i|^2) \\
&= 2 \sum |a_i|^2 + 2 \sum |b_i|^2 < \infty
\end{aligned}$$

$V = C([0,1])$ (cts on $[0,1]$)

$V = \mathcal{L}^1(X, S, \mu)$

$V = \mathcal{L}^2(X, S, \mu)$ $\int |f|^2 < \infty$

V over $\mathbb{R}$ $\mathbb{C}$

Inner Product condition: $\langle \,,\, \rangle : V \times V \to \mathbb{R}$ ($\mathbb{C}$). ! here, it means inner product must converge.

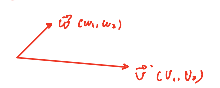
① $\langle f, f \rangle$ non-negative. $\forall f \in V$

② $\langle f, f \rangle = 0 \Leftrightarrow f = 0$ $\quad \langle f, \alpha g + \beta h \rangle = \bar{\alpha} \langle f, g \rangle + \bar{\beta} \langle f, h \rangle$ $\mathbb{C}$

③ $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$ $\forall f, g, h \in V$. $\forall \alpha, \beta \in \mathbb{R}$

④ $\langle f, g \rangle = \langle g, f \rangle$ $\quad \langle f, g \rangle = \overline{\langle g, f \rangle}$

$\mathbb{C}$:



$v \cdot \omega = v_1 \bar{\omega}_1 + v_2 \cdot \bar{\omega}_2$ over $\mathbb{C}$.

E.x. 1. $V = \ell^2$, $\langle a, b \rangle = \sum_{i=1}^{\infty} a_i \bar{b}_i$ in $\mathbb{C}$ 数列

① Need to check $\langle a, b \rangle$ converges.

$$\sum |a_i \bar{b}_i| = \sum |a_i| |b_i| \le \sum (|a_i|^2 + |b_i|^2) < \infty$$

$\Rightarrow$ conditionally converges.

② Props. $\langle f, f \rangle \ge 0$, "=" iff $f = 0$ - - -

2. $V = C([0,1])$. $\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)}\, dx$ 连续函数.

① compact $\Rightarrow$ bounded $\Rightarrow$ converges.

② cts ftn $\Rightarrow$ 0 every where

3. $\mathcal{L}^2(X, S, \mu) = \{ f : X \to \mathbb{R} : f \text{ mble } \int |f|^2 < \infty \}$.

L^2 函数 $\langle f, g \rangle = \int f \bar{g}\, d\mu$

$\int f \bar{f} = \int |f|^2 = 0 \Rightarrow f = 0$ a.e. $\to$ $L^2(X, S, \mu)$. here f and g are identified if $f = g$ a.e.

Inner product space.

② Cauchy - Schwarz Inequality

Def: V Inner product space

Define $\|f\| = \sqrt{\langle f, f\rangle}$

e.g. In $\mathcal{L}^2$, $\|f\| = \sqrt{\int |f|^2}$

Norm Property : ① $\|f\| \geq 0$

② $\|f\| = 0$ iff $f = 0$

③ $\|c\cdot f\| = |c|\cdot\|f\|$

④ $\|f\| + \|g\| \geq \|f+g\|$.

Def: $f, g \in V$ are orthogonal to each other means $\langle f, g\rangle = 0$

e.g. $L^2([-\pi,\pi])$, $f(x) = \sin(3x)$, $g(x) = \sin(4x)$

$\langle f, g\rangle = \int_{[-\pi,\pi]} \sin(3x)\cdot\sin(4x)\, dx = \frac{1}{2}\int_{[-\pi,\pi]} \cos(x) - \cos(7x)\, dx = 0$

Pf of $\|f\|$ is a norm:

③ $\|c\cdot f\|^2 = \left(\sqrt{\langle cf, cf\rangle}\right)^2 = \langle cf, cf\rangle = c\langle f, cf\rangle = c\cdot\bar{c}\langle f, f\rangle$

$= |c|^2\langle f, f\rangle = |c|^2\cdot\|f\|^2$

④ $\langle f, f\rangle + \langle g, g\rangle + 2\sqrt{\langle f, f\rangle}\cdot\sqrt{\langle g, g\rangle} \geq \langle f+g, f+g\rangle$

$= \langle f, f\rangle + \langle g, g\rangle + \langle f, g\rangle + \overline{\langle f, g\rangle}$

Lemma (Pythagorean Thm)

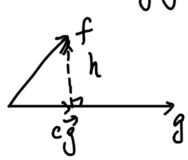
Useful Trick.

If $\langle g, f\rangle = 0$, then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$

Pf: $\|f+g\|^2 = \langle f, f\rangle + \langle g, g\rangle + \langle f, g\rangle + \langle g, f\rangle$

$= \langle f, f\rangle + \langle g, g\rangle + \overline{\langle g, f\rangle} + \langle g, f\rangle = \|f\|^2 + \|g\|^2$

Projection:



$\langle f - c\vec{g}, \vec{g}\rangle = 0$

$\Rightarrow c = \frac{\langle f, g\rangle}{\|g\|^2}$

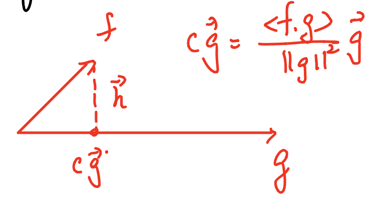
$h = f - cg$

Lemma (Orthogonal decomposition)

$\forall f, g,\ g \neq 0 \quad \exists\, h = f - \frac{\langle f, g\rangle}{\|g\|^2}\cdot g, \quad \langle h, g\rangle = 0$

Thm (Cauchy-Schwarz Ineq) Here $\|\cdot\| = |\langle \cdot, \cdot \rangle|$

$$|\langle f, g\rangle| \leq \|f\| \cdot \|g\|$$, "=" holds iff $g=0$ or $f = \alpha \cdot g$.



① $g=0$ ✓

② $$\|f\|^2 = \|h\|^2 + \left(\frac{\langle f, g\rangle}{\|g\|^2}\right)^2 \cdot \|g\|^2 \geq \frac{|\langle f, g\rangle^2|}{\|g\|^2}$$

taking sqrt: $\|f\| \cdot \|g\| \geq |\langle f, g\rangle|$

e.x.

$\mathbb{R}^2$ (v_1, v_2) (w_1, w_2). $\Rightarrow$ $v_1 w_1 + v_2 w_2 \leq \sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2}$

Then $\|f+g\| \leq \|f\| + \|g\|$. using CS-Inequality.

Geom:

$\|c\|^2 \leq \|a\|^2 + \|b\|^2$.

algebraically: use C-S.

$\Rightarrow$ $\|f\| = |\langle f, f\rangle|$ is a norm on V

Thm $\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$