Def The subset $\{e_1, e_2, \cdots\}$ of an inner product space V is called an orthonormal set if

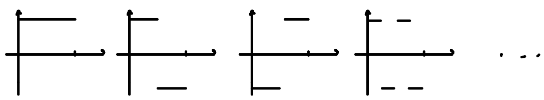
$$\langle e_m, e_n\rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n. \end{cases}$$

ex. ① $\ell^2([-\pi,\pi])$

$$e_n(x) = \begin{cases} \frac{1}{\sqrt{\pi}} \sin(nx) & n = 1, 2, \cdots \\ \frac{1}{\sqrt{2\pi}} & n = 0 \\ \frac{1}{\sqrt{\pi}} \cos(nx) & n = -1, -2, \cdots \end{cases}$$

$$\int_{-\pi}^{\pi} e_n(x)\, \overline{e_m(x)}\, dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

② $L^2([0,1])$



Lemma: Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal ON set of V.

For $f \in V$, let $c_n = \langle f, e_n\rangle$. Then,

① $\|f\|^2 = \sum_{n=1}^{N} |c_n|^2 + \| f - \sum_{n=1}^{N} c_n \cdot e_n\|^2$.

② $\sum_{n=1}^{N} |c_n|^2 \leq \|f\|^2$ (Bessel's inequality)

为什么 ≤ ?
因为 $\{e_n\}$ 不一定涵盖所有的.
This trick again!

Pf: ②: follows from ① obviously.

$\|f+g\|^2 = \|f\|^2 + \|g\|^2 + 2\mathrm{Re}\{\langle f, g\rangle\}$

①: Start from RHS. $\|f - \sum_{n=1}^{N} c_n \cdot e_n\|^2 = \|f\|^2 + \|\sum_{n=1}^{N} c_n \cdot e_n\|^2 - 2\,\mathrm{Re}\{\langle f, \sum_{n=1}^{N} c_n \cdot e_n\rangle\}$. Be careful

Pythagorean trick!

$= \|f\|^2 + \sum_{n=1}^{N} \|c_n \cdot e_n\|^2 - 2\,\mathrm{Re}\{\sum_{n=1}^{N} \overline{c_n} \underbrace{\langle f, e_n\rangle}_{c_n}\}$

遇事不决: 展开.

$= \|f\|^2 + \sum_{n=1}^{N} \|c_n\|^2 - 2\sum_{n=1}^{N} \|c_n\|^2$

$= \|f\|^2 - \sum_{n=1}^{N} |c_n|^2$

☆ **Def**: An orthonormal set $\{e_n\}$ ① is called an orthonormal basis of V if $W = \{$finite linear comb of $e_n\}$ satisfies $\overline{W} = V$ ②.

i.e. $\forall f \in V$, $\forall \varepsilon > 0$, $\exists N$, $\exists \alpha_1, \alpha_2, \cdots, \alpha_n$ s.t. $\| f - \sum_{i=1}^{n} \alpha_i e_i\| < \varepsilon$.

Approximation

$\begin{cases} \text{orthonormal} \\ \overline{W} = V \end{cases}$

$W^\perp = \{0\}$!

$\vec{v} \in W^\perp$.

$\langle \vec{v}, \Sigma \cdots\rangle = 0$

Thm. ① $\{e_n = (0, \cdots 0, \overset{n^{th}}{1}, 0, \cdots)\}_n$ is a ONB of ℓ^2

② Fourier basis of $L^2([-\pi,\pi])$

Pf: ① • clearly it's an orthonormal set.

• $W = \{$finite linear comb of $e_n\}$. WTS $\overline{W} = V$

△ Useful trick: $\Leftrightarrow W^{\perp} = \{0\}$.

$\forall \vec{a} \in W^{\perp}, \vec{b} \in W, \quad \langle \vec{a}, \vec{e_i} \rangle = a_i = 0 \Rightarrow W^{\perp} = \{0\}$!

Thm: Every Hilbert space has an ONB.

(skip proof) for countable --- Gram-Schmidt Process

$\Downarrow$

Thm: let $\{e_n\}_{n=1}^{\infty}$ be an ONB of Hilbert Space V. Then,

① $\sum_{n=1}^{\infty} |C_n|^2 = \|f\|^2$ where $\boxed{C_n = \langle f, e_n \rangle}$ 类似 Pythagorean?

② $\langle f, g \rangle = \sum_{n=1}^{\infty} C_n \overline{d_n}$ $\quad d_n = \langle g, e_n \rangle$

$\rightarrow$ Need to check Im and Re and ①

e.g. use ① $f(x) = x$, e_n = Fourier basis

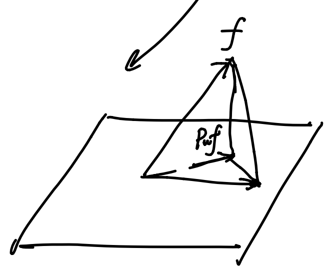
$$\frac{2}{3}\pi^3 = \sum_{n=1}^{\infty} \frac{4\pi}{n^2} = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$\rightarrow$ **Lemma**: $W = \{\sum_{i=1}^{N} a_i e_i \mid a_1, \dots, a_N \in \mathbb{R} \text{ or } \mathbb{C}\}$, $\underline{P_W f = \sum_{i=1}^{N} C_i e_i}$

Pf: $\forall f \in V$. $P_W f = \sum_{i=1}^{N} C_i e_i$, $C_i = \langle f, e_i \rangle$.

let $h = P_W f \Leftrightarrow f - h \perp e_i$

$\langle f - \sum_{i=1}^{N} C_i e_i, e_n \rangle = \langle f, e_n \rangle - C_n = 0 \Rightarrow \sum_{i=1}^{N} C_i e_i = h$



Pf: ① (a) By Bessel Inequality $\sum_{n=1}^{\infty} |C_n|^2 \leq \|f\|^2$

(b) Since ONB, $\|f - \sum_{i=1}^{N} d_i e_i\| < \varepsilon$. even smaller

By Lemma, we can find $\|f - \sum_{i=1}^{N} C_i e_i\|^2 < \varepsilon^2$.

$\Rightarrow \|f\|^2 - 2\,\mathrm{Re}\langle f, \sum_{i=1}^{N} C_i e_i \rangle + \|\sum_{i=1}^{N} C_i e_i\|^2 < \varepsilon^2$.

$\quad\quad \downarrow 2\sum_{i=1}^{N} \overline{C_i} C_i \quad\quad \downarrow$

$\Rightarrow \|f\|^2 - 2\sum_{i=1}^{N} |C_i|^2 + \sum_{i=1}^{N} |C_i|^2 < \varepsilon^2$

$\Rightarrow \|f\|^2 \leq \sum_{i=1}^{\infty} |C_i|^2 + \varepsilon^2$. since ε arbitrarily small

$\Rightarrow \|f\|^2 \leq \sum_{i=1}^{\infty} |C_i|^2$

$\Rightarrow \|f\|^2 = \sum_{i=1}^{\infty} |C_i|^2$.

Re:

② $\|f+g\|^2 = \sum_{n=1}^{\infty} |c_n+d_n|^2$ by ①

$\Rightarrow \|f\|^2 + 2Re\langle f,g\rangle + \|g\|^2 = \sum_{n=1}^{\infty}\left(|c_n|^2 + 2Re\,(c_n\overline{d_n}) + |d_n|^2\right)$

$\Rightarrow Re\langle f,g\rangle = \sum_{n=1}^{\infty} Re\,(c_n\overline{d_n})$

Im: $\|f+ig\|^2 = \sum_{n=1}^{\infty} |c_n+i\,d_n|^2$ by ①

$\Rightarrow \|f\|^2 + 2Re\langle f,ig\rangle + \|g\|^2 = \sum_{n=1}^{\infty}\left(|c_n|^2 + 2Re\,(c_n\overline{i\cdot d_n}) + |i\cdot d_n|^2\right)$

$\Rightarrow Im\langle f,g\rangle = \sum_{n=1}^{\infty} Im\,(c_n\overline{d_n})$

$Re\langle f,ig\rangle = Im\langle f,g\rangle$

Pf: $Re\langle f,ig\rangle = Re\,(-i\langle f,g\rangle)$

$= Im\langle f,g\rangle$.