Recall:

sup: least upper bound $\quad S \subset \mathbb{R}$ $\qquad$ sup and inf

$a = \sup\{x \mid x \in S\}$ and $a \in \mathbb{R}$

$\Leftrightarrow$ $a \ge x$ $\forall x \in S$; $\forall \varepsilon > 0 . \exists x \in S$, $x > a - \varepsilon$.

inf is similar . .

Ch 1. Riemann Integral.

→ $\boxed{\text{1A.}}$ Review of -- (Darboux version) $\qquad$ Lower/Upper Riemann Integral

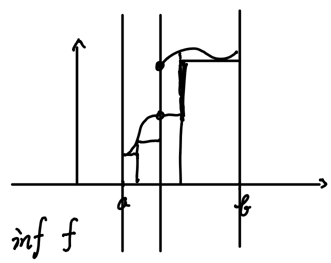
A partition of $[a,b]$ is a finite seq

$$a = x_0 < x_1 < \cdots < x_n = b .$$

→ Notation: $\sup\limits_A f = \sup\{f(x), x \in A\}$.

$$| \quad | \le C$$

let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function



for partition $P = \{x_0 = a, x_1 = \cdots, x_n = b\}$. of $[a,b]$

Lower Riemann Sum : $L(f, P, [a,b]) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf\limits_{[x_{i-1}, x_i]} f$

Upper .. $\qquad$: $U(f, P, [a,b]) =$ ——— $\sup f$ $\qquad < c \cdot [b-a]$.

(Since bounded, these sums are finite !)

→ lemma $\quad$ for any partition P, P'.

$$L(f, P, [a,b]) \le U(f, P', [a,b])$$

Pf: P'' be the refinement of P and P'

$$\boxed{L(P) \le L(P'') \le U(P'') \le U(P')}$$

Def <u>Riemann Integrable</u>:

$f: [a,b] \to \mathbb{R}$ be a bounded fcn, then f is called Riemann Integrable if

$$\sup_P L(f, \underset{\sim}{P}, [a,b]) = \inf_P U(f, P, [a,b]) \quad \int_a^b f = U = L$$

Denote Lower Riemann Integrable: $L(f, [a,b])$

Upper — — : $U(f, [a,b])$

It will be too crazy to do the sup by hand.

To find which function is Riemann Integrable, we need some Theorems!

① Recall: A cts fcn on a compact (= closed & bounded) interval is uniformly cts* [1] and bounded [2].

$\boxed{1}$ Pf. fix ε, $\forall x$, $\exists \delta_x$,

cts $\Rightarrow f(B_{\delta_x}(x)) \subseteq B_\varepsilon(f(x))$, $O \to O$

$\Rightarrow \{B_{\delta_x/2}(x)\}_{x \in S}$ is an open cover, $\Rightarrow \exists$ finite cover $\{B_{\delta_{i}/2}(x_i)\}_{i \in I}$

$\Rightarrow \exists \min \delta_i/2 = \delta$ (*)

$\forall x \in B_{\frac{\delta_i}{2}}(x_i)$, if $d(x,y) \le \delta$,

$\Rightarrow d(x_i, y) \le d(x_i, x) + d(x,y) \le \frac{1}{2}\delta_i + \frac{1}{2}\delta_i = \delta_i$.

$\Rightarrow y \in B_{\delta_i}(x_i)$

Which means $f(B_\delta(x)) \subseteq f(B_{\delta_i}(x_i)) \subseteq B_\varepsilon(f(x))$.

$\Rightarrow$ Uniformly Continuous.

$\boxed{2}$ Suppose $f: X \to \mathbb{R}$ not bounded, pick $\{B_n(0)\}_{n \in \mathbb{N}}$,

$f^{-1}(B_n) \subseteq X$ is open since B_n is open.

$X = \bigcup f^{-1}(B_n) \Rightarrow \exists$ finite open subcover

$\boxed{CTS} + C\;\;) \Rightarrow$ Riemann Integrable

$\Rightarrow$ closed and bounded interval. Idea:

$\boxed{\text{Thm:}}$ A cts ftn on a compact interval is Riemann integrable.

To show $c = 0$.
$\Updownarrow$
$\forall \varepsilon,\ -\varepsilon < c < \varepsilon$

Pf: We know $L \leq U$, $\boxed{\text{WTS: } \forall \varepsilon,\ \varepsilon > U - L \geq 0}$

$\boxed{\text{Fix } \varepsilon > 0}$. By cts $\longrightarrow$ uniformly cts

Then $\exists\, \delta > 0$, s.t. $|f(s) - f(t)| < \varepsilon$ $\forall s, t \in [a, b]$ w/ $|s-t| < \delta$

Pick $n \in \mathbb{N}$, s.t. $\frac{b-a}{n} < \delta$.

Let P be a partition. $x_k = k \cdot \frac{b-a}{n}$.

$$0 \leq U(f) - L(f) \leq U(f, P) - L(f, P)$$
$$= \sum_{i=1}^{n} \frac{b-a}{n} \Big(\sup_{[x_0 \cdots x_n]} (f) - \inf_{[x_0, \cdots, x_n]} (f) \Big) < (b-a)\, \varepsilon$$

$\Rightarrow U - L = 0 \Rightarrow U = L$.

Ex (1A.6)

Suppose $f\colon [a, b] \to \mathbf{R}$ is Riemann integrable. Suppose $g\colon [a, b] \to \mathbf{R}$ is a function such that $g(x) = f(x)$ for all except finitely many $x \in [a, b]$. Prove that g is Riemann integrable on $[a, b]$ and

$$\int_a^b g = \int_a^b f.$$

$$U(g) \leq \inf_n U(g, P) = U(f) = L(f) \leq L(g)$$

$\rightarrow \boxed{1B}$

$\boxed{1}$ ex 1. $f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & \text{otherwise} \end{cases}$ $\Rightarrow$ not Riemann Integrable

$L = 0 \neq 1 = U$.

ex 2. $f_n(x) = \begin{cases} 1 & x = r_1, \dots, r_n \\ 0 & \text{otw} \end{cases}$ ← finite. ($r_1, \dots$ is a list of rational num)

By Thm, Every $f_n(x)$ is [1] R-I- & [2] $\int_0^1 f_n(x)\,dx = 0$.

Property of Seq.

Note: $\lim_{n\to\infty} f_n(x) = f(x)$ in ex 1

⇒ pointwise limit of a sequence of Rie Intg ftns is not necessarily Rie Intg.

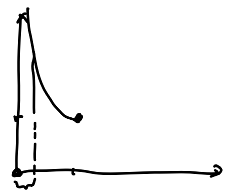
$f(x)$ $f_n(x)$

However, uniformly convergent.
$\Rightarrow \int_a^b f = \lim_{n\to\infty} \int_a^b f_n$

Unbounded.

[2] ex 1. $f : [0,1] \to \mathbb{R}$.

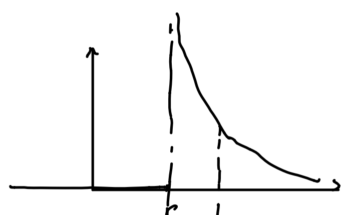
$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{\sqrt{x}} & 1 \geq x > 0 \end{cases}$



⇒ Every unbounded function is not Riemann Integrable.

However, $\lim_{a \downarrow 0} \int_a^1 f = \lim_{a\downarrow 0} \int_a^1 x^{-\frac{1}{2}}\,dx = \lim_{a\downarrow 0} (2 - 2a^{1/2}) = 2$

ex2. $f_r(x) = \begin{cases} 0 & , X \leq r \\ \frac{1}{\sqrt{x-r}} & , X > r \end{cases}$



$\forall\ r \in [0,1]$, $\lim_{a\downarrow r} \int_a^1 f_r(x)\,dx$ converges with value ≤ 2.

$g(x) = f_0(x) + \frac{1}{2} f_{\frac{1}{3}}(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{\sqrt{x}} & 0 < x \leq \frac{1}{3} \\ \frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x-\frac{1}{3}}} & x > \frac{1}{3} \end{cases}$

$S \leq 2 + 2 \times \frac{1}{2} = 3.$

What if do that inf times?

Define: $h(x)' = \sum_{i=1}^{\infty} \frac{f_{r_i}(x)}{2^i}$ where $r_i \cdots$ is list of rational #s.

h is unbounded on every open interval in $[0, 1]$

Riemann Integral of f is undefined on every subinterval

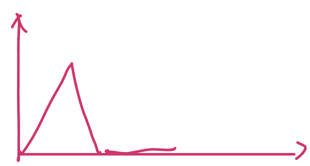
Nonetheless, the area below should be less than $\sum_{i=1}^{\infty} \frac{2}{2^i} = 2$.

$\boxed{3}$ $$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}\,dx \overset{?}{=} \int_0^1 \lim_{n\to\infty}\left(\frac{1+nx^3}{(1+x^2)^n}\right)dx$$

Are there easier criterias allowing us this switch?

Note for Chapter 1:

① Dirichlet Function is useful for counter examples.

②  so is this one!