3B Limits of Integrals and Integrals of limit

Def. $(X, \mathcal{S}, \mu)$ measure space.

$E \in \mathcal{S}$. $f: X \to [-\infty, +\infty]$. Define $\int_E f \, d\mu = \int f \mathbb{1}_E \, d\mu$

Lemma: $\left| \int_E f \, d\mu \right| \le \mu(E) \cdot \sup_E |f|$ bounding an integral

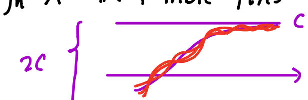
Pf: $|f| \mathbb{1}_E(x) \le c \cdot \mathbb{1}_E(x)$, where $c = \sup |f|$.

$$\left| \int_E f \, d\mu \right| = \left| \int f \mathbb{1}_E \, d\mu \right| \le \int |f \cdot \mathbb{1}_E| \, d\mu = \int |f| \cdot \mathbb{1}_E \, d\mu \le \int c \cdot \mathbb{1}_E \, d\mu = \mu(E) \cdot \sup_E |f| .$$

Prop (Bounded convergence thm)

$(X, \mathcal{S}, \mu)$, $f_n : X \to \mathbb{R}$, mble ftns

- $\mu(X) < \infty$
- $\lim_{n \to \infty} f_n(x) = f(x)$, $\forall x \in X$ 实际上可以 a.e.
- $\exists c \in (0, +\infty)$ s.t. $|f_n(x)| \le c$, $\forall x \in X$, $\forall n \in \mathbb{N}$ uniformly bounded



$\Rightarrow \lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$

Monotone Convergence Thm

$0 \le f_1(x) \le \cdots \quad \forall x \in X$

let $f(x) = \lim_{n \to \infty} f_n(x) \in [0, \infty]$

Then $\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$

(Pf). $|f(x)| \le c$, $\forall x \in X$.

Fix $\varepsilon > 0$.

By Egorov's Thm $\Rightarrow \exists E \in \mathcal{S}$, s.t. $\mu(E^c) < \frac{\varepsilon}{4c}$ and $f_n \to f$ uniformly.

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{2\mu(X)}$, which is true $\forall n \ge N$, $\forall x \in E$

$$\left| \int f_n - \int f \right| = \left| \left(\int_E f_n + \int_{E^c} f_n \right) - \left(\int_E f + \int_{E^c} f \right) \right| \le \left| \int_E f_n - \int_E f \right| + \left| \int_{E^c} f_n \right| + \left| \int_{E^c} f \right| .$$

$$< \left| \frac{\varepsilon}{2\mu(X)} \cdot \mu(E) \right| + \left| c \cdot \frac{\varepsilon}{4c} \right| + \left| c \cdot \frac{\varepsilon}{4c} \right| \le \varepsilon$$

☒

By this theorem, ex. $\lim_{n \to \infty} \int_0^1 \frac{n \cdot \sin(\frac{x}{n})}{x \cdot (1 + x^2)} = \frac{\pi}{4}$!

② Sets of Measure 0 and integration

Def. $(X, \mathcal{S}, \mu)$ measure space.

Let $P(x)$ be a statement for each $x \in X$

We say $P(x)$ holds μ-almost everywhere (μ-a.e.), if $\exists E \in \mathcal{S}$, $\mu(E^c) = 0$, $P(x)$ holds for $x \in E$.

example: ① $(\mathbb{R}, \mathcal{L}, \lambda)$, $f(x) = \frac{1}{x}$ $\quad P(x)$: 'f cts at x' $\Rightarrow$ P holds λ-a.e.

② $1_{\mathbb{Q}}(x) = 0$ a.e.

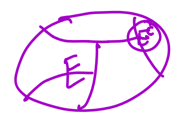
③ $f(x) = x^2$, $g(x) = \begin{cases} x^2 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$, $f(x) = g(x)$ a.e.

lemma: (X, S, μ)

$f, g : X \longrightarrow [-\infty, \infty]$ mble ftns, If $f = g$ a.e. $\Longrightarrow$ $\int f = \int g$

Pf: $\exists$ $E \in S$, $\mu(E^c) = 0$, $f(x) = g(x)$ $\forall x \in E$.

Suppose $P = \{A_1, \cdots, A_n\}$ S-partition



$P' = \{A_1 \cap E, \underbrace{A_1 \cap E^c}_{\mu = 0}, \cdots\}$

$\mathcal{L}(f, P) \le \mathcal{L}(f, P') = \sum \inf_{A_i \cap E} f \; \mu(A_i \cap E) = \sum \inf_{A_i \cap E} g \; \mu(A_i \cap E) = \mathcal{L}(g, P')$

$\Rightarrow \mathcal{L}(f, P) \le \mathcal{L}(g, P') \le \int g$. $\Rightarrow$ $\int f \le \int g$, similarly, $\int g \le \int f$

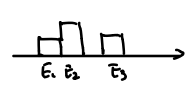
$\Rightarrow \int f = \int g$. ☒

Thm: a.e. version of BCT.

- $\mu(X) < \infty$
- $\lim_{n \to \infty} f_n(x) = f(x)$ a.e.
- $\exists$ $c \in (0, \infty)$, $|f_n(x)| \le c$ a.e. $\forall n \ge N$.

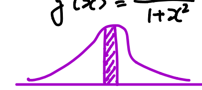
$\Rightarrow \lim_{n \to \infty} \int f_n = \int f$

③ Dominated Convergence Thm

ψ: 

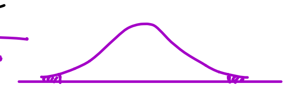
$\boxed{\text{Prop}}$ (X, S, μ), $g : X \longrightarrow [0, \infty]$ mble. Suppose $\int g \, d\mu < \infty$

$g(x) = \frac{1}{1+x}$ ✗
$g(x) = x$ ✗
$g(x) = \frac{1}{1+x^2}$ ✓



(a) $\forall \varepsilon > 0$, $\exists$ $\delta > 0$ s.t. $\int_E g < \varepsilon$. with $E \in S$, $\mu(E) < \delta$

(b) $\forall \varepsilon > 0$, $\exists$ $E \in S$ s.t. $\mu(E) < \infty$, $\int_{E^c} g < \varepsilon$



Pf: fix $\varepsilon > 0$. Then, $\overbrace{\exists}^{g \text{ is mble}}$ mble simple ftn $0 \le \boxed{\psi(x)} \le g(x)$ *

$\forall x \in X$. $\boxed{\int g - \int \psi < \frac{\varepsilon}{2}}$ *, $\psi = \sum_{k=1}^{N} c_k 1_{E_k}$, $c_k \in (0, \infty)$, E_i disjoint

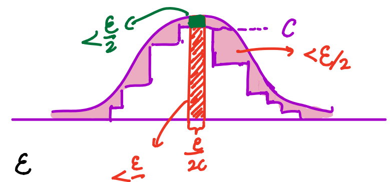
(a) Let $c = \max\{c_1, \cdots, c_N\}$. $\Rightarrow$ $c \in (0, \infty)$ and $\psi(x) \le c$

let $\delta = \frac{\varepsilon}{2c}$.

Suppose $B \in S$ with $\mu(B) < \delta$

$\Rightarrow \int_B g = \int_B g-f + \int_B f < \int_X g-f + \int_B f < \frac{\varepsilon}{2} + \frac{\varepsilon}{2c}\cdot c = \varepsilon$

$\delta \cdot \max\{c_1 \cdots, c_N\}$.



(b) let $E = \bigcup_{k=1}^{N} E_k$, $\forall x \notin E$. $\psi(x) = 0$.

① 要证 $\psi(x)$ 的 $E<\infty$

测度

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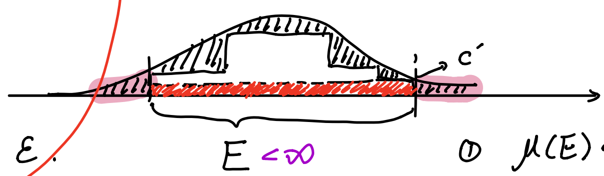
let $c' = \boxed{\min}\{c_1, \cdots, c_N\} > 0 \Rightarrow c' \cdot 1_E(x) \leq \psi(x) \leq g(x) \quad \forall x \in X$.

$\Rightarrow c \cdot \mu(E) \leq \int \psi \leq \int g < \infty$

$\Rightarrow \mu(E) < \infty$

$\int_{E^c} g = \int_{E^c} g-\psi + \int_{E^c} \psi \leq \frac{\varepsilon}{2} + 0 < \varepsilon$.

$= 0$



① $\mu(E) < \infty$

Existence.

② bound.

Thm (Dominated Convergence Thm).

(X, S, μ) $\forall n \in N$, $f_n : X \longrightarrow [-\infty, +\infty]$ mble.

$f : X \longrightarrow [-\infty, +\infty]$ mble.

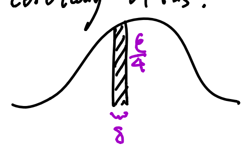
- $\lim_{n\to\infty} f_n(x) = f(x)$ a.e.
- $\exists g : X \longmapsto [-\infty, +\infty]$ s.t. $\int g < \infty$, and $|f_n(x)| \leq g(x)$ a.e. $\forall n \in N$

finding g will be the main problem.

$\Rightarrow \lim_{n\to\infty} \int f_n = \int f$

$\Rightarrow$ Bounded Convergence Thm is a corollary of this.

Pf: Note $|f_n(x)| \leq g(x)$ $\boxed{\text{a.e.}}$



① Assume $\mu(X) < \infty$

Fix $\varepsilon > 0$, By Prop (a), $\exists \delta > 0$ s.t. $\int_B g < \frac{\varepsilon}{4}$, $\forall B \in S$, s.t. $\mu(B) < \delta$

For this $\delta > 0$, by Egorov's Thm $\exists E \in S$, s.t. $\mu(E^c) < \delta$.

and $f_n \to f$ uniformly on E

$\Rightarrow \exists N \in N$, s.t. $\forall n \geq N$, $\forall x \in E$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2\mu(X)}$

$\left|\int f_n - \int f\right| \leq \left|\int_E (f_n - f)\right| + \left|\int_{E^c} f_n\right| + \left|\int_{E^c} f\right| \leq \int_E |f - f_n| + \int_{E^c} |f_n| + \int_{E^c} |f| \leq \int_E \frac{\varepsilon}{2\mu(X)} + \int_{E^c} g + \int_{E^c} g$

$\leq \mu(E) \cdot \frac{\varepsilon}{2\mu(X)} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \varepsilon$.

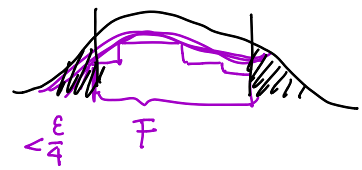
② General case. Assume $\underline{\mu(X)=\infty}$

Since $\int g$ is finite, by Prop (b), $\int g < \infty$

$\exists F \in S$ $\mu(F)<\infty$. and $\int_{F^c} g < \frac{\varepsilon}{4}$.

$\Rightarrow |\int f_n - \int f| \leq \left|\int_F f_n - f\right| + \int_{F^c} |f_n| + \int_{F^c} |f| \leq \frac{\varepsilon}{2} < \varepsilon$.

by ① $\lim_{n\to\infty} \int_F f_n - f = 0$



ex. $\lim_{n\to\infty} \int_0^1 \underbrace{\frac{(1-x)^n \cos(\frac{n}{x})}{\sqrt{x}}}_{f_n} dx$

$f_n(x) \leq \frac{1}{\sqrt{x}} = g(x)$

$\Rightarrow \int_0^1 \frac{1}{\sqrt{x}} = 2$ finite.

fix $x \in (0,1)$ $(1-x)^n = 0$. $n\to\infty$ $\Rightarrow \lim_{n\to\infty} f_n(x) = 0$ a.e.

$\Rightarrow \lim_{n\to\infty} \int_0^1 \frac{(1-x)^n \cos(\frac{n}{x})}{\sqrt{x}} dx = 0$

Thm (X, S, μ). $f: X \to [0,\infty]$ mble (HW)

If $\int f \, d\mu = 0 \Leftrightarrow f = 0$ a.e.

4 Riemann Integrals and Lebesgue Integrals.

Thm: $f: [a,b] \to \mathbb{R}$, bounded fn

$\lambda(E)=0$.

(a) f is Riemann Integrable $\Leftrightarrow$ f is cts (Lebesgue a.e.)

(b) f is Riemann Integrable $\Rightarrow$ $\begin{cases} f \text{ is Lebesgue mble.} \\ \int_{[a,b]} f \, d\lambda = \int_a^b f(x)\,dx \end{cases}$

P_n: $I_1, \cdots, I_{2^n}$.

each $\frac{b-a}{2^n}$.

(a) skip. $\longrightarrow$ $|x : f \text{ not cts at } x| = |\{x \in [a,b] : f^U(x) \neq f^L(x)\}| = 0$

(b) sketch: $\exists M>0$ s.t. $|f(x)| \leq M$, $\forall x \in [a,b]$.

R.I.: $\exists P_1 \subset P_2 \subset \cdots$ s.t.

$L(P_n) \nearrow$ $U(P_n) \searrow$

L.I.: If $P_n = \{a = x_0 < x_1 < \cdots < x_m = b\}$, define

$\phi_n = \sum_{i=1}^m \left(\inf_{[x_{i-1}, x_i]} f\right) 1_{[x_{i-1}, x_i]}$

$\psi_n = \sum_{i=1}^m \left(\sup_{[x_{i-1}, x_i]} f\right) 1_{[x_{i-1}, x_i]}$

$\Rightarrow \begin{cases} \int_{[a,b]} \phi_n \, d\lambda = L(P_n), \quad \int_{[a,b]} \psi_n \, d\lambda = U(P_n) \\ -M \leq \phi_1(x) \leq \cdots \leq f(x) \leq \cdots \leq \psi_1(x) \leq M \end{cases}$

Let $\phi(x) = \sup\{\phi_n(x) \mid n \in \mathbb{N}\} = \lim_{n\to\infty} \phi_n(x)$, $\psi(x) = \lim_{n\to\infty} \psi_n(x)$

$\Rightarrow \begin{cases} \phi, \psi \text{ are Leb mble ftns} \\ \phi(x) \le f(x) \le \psi(x) \quad \forall x \in [a,b] \end{cases}$

By BCT $\Rightarrow$

$\int_{[a,b]} \phi \, d\lambda = \lim_{n\to\infty} \int_{[a,b]} \phi_n \, d\lambda = \lim_{n\to\infty} L(P_n) = \int_a^b f(x)\, dx$, similar for $\int \psi \, d\lambda$.

$\Rightarrow \int_{[a,b]} \phi \, d\lambda = \int_{[a,b]} \psi \, d\lambda \Rightarrow \int_{[a,b]} \phi - \psi \, d\lambda = 0 \Rightarrow \phi = \psi$ a.e.

$\Rightarrow f$ is Lebesgue mble ftn. ∎

skip

Now, $\int_0^1 1_{\mathbb{Q}}(x)\, dx = 0$

- Improper Riemann Integral

e.g. $f(x) = (-1)^n \frac{1}{n}$, $x \in [n, n+1)$. $\int_0^\infty f(x)\, dx$ exists,

however, $\int_{[0,\infty)} f \, d\mu = \int f^+ - \int f^-$ do not exist .

$\Rightarrow$ $\boxed{\int_0^\infty |f(x)|\, dx < \infty \Rightarrow \int_{[0,\infty)} f \, d\mu = \int_0^\infty f(x)\, dx}$

5 Approximation by nice ftns

Def: (X, S, μ)

For $f: X \to [-\infty, \infty]$ mble

define $\|f\|_1 = \int |f| \, d\mu$, called $\mathcal{L}^1$ - norm of f

$\|f\|_p = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}$

$\mathcal{L}^1(X, S, \mu) = \mathcal{L}^1(\mu) = \{ f: X \to [-\infty, \infty] \text{ mble} : \|f\|_1 < \infty \}$ 所有 L_1-norm $<\infty$ 的 ftn.

(example) • $f(x) = x^2$ on $\mathbb{R}$. $f \notin \mathcal{L}^1$

• $f(x) = \frac{\sin x}{1+x^2}$, $f \in \mathcal{L}^1$

• $f = \sum c_i 1_{E_i}$ $E_1, \dots, E_N$.

then $f \in \mathcal{L}^1 \Leftrightarrow \mu(E_i) < \infty$, $\forall i$

• $(X, S, \mu) = (\mathbb{N}, P(\mathbb{N}), \text{counting measure})$

$f: \mathbb{N} \to [-\infty, +\infty]$

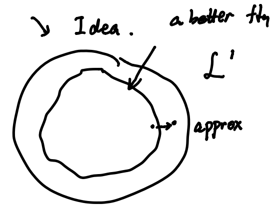
$f(i) = a_i$, $\|f\|_1 = \sum_{i=1}^{\infty} |a_i|$ $\quad \ell_1 = \{ (a_i)_{i \in \mathbb{N}} \mid \sum |a_i| < \infty \}$ → abs convergent.

↳ infinite dim vector space

lemma: (X, S, μ), $f, g \in \mathcal{L}^1(X, \mu)$

- $\|f\|_1 \geq 0$
- $\|f\|_1 = 0 \iff f = 0$ a.e.
- $\|cf\|_1 = |c| \cdot \|f\|_1$
- $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$ $\quad (\Rightarrow f+g \in \mathcal{L}_1(X, \mu))$

Cor $\mathcal{L}^1(X, S, \mu)$ is a vector space. over the real field.



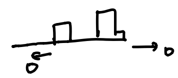
prop (X, S, μ). $f \in \mathcal{L}^1(X, \mu)$ $(\Rightarrow f$ is S-mble$)$

$\Rightarrow \forall \varepsilon > 0$, $\exists$ simple fn $\psi \in \mathcal{L}^1(X, \mu)$ s.t. $\|f - \psi\|_1 < \varepsilon$

→ Notation $\mathcal{L}(\mathbb{R}) = \mathcal{L}(\lambda)$ on either Borel or Leb

Def: Step fn: $g = \sum_{i=1}^{N} c_i \mathbb{1}_{I_i}$ $\quad$ (N ← finite) $\quad I_i$ interval

If $I_1, \cdots I_N$ disjoint, $c_i \in (0, \infty)$. then $g \in \mathcal{L}^1(\mathbb{R}) \iff I_i$ are bounded $\forall i$

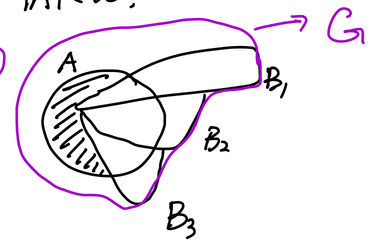


Lemma: Suppose A is a Leb mble set in $\mathbb{R}$ with $|A| < \infty$.

Then $\forall \varepsilon > 0$, $\exists$ finitely many disjoint open intervals

$B_1 = (a_1, b_1), \cdots, B_m = (a_m, b_m)$ s.t.

$|A \setminus B| + |B \setminus A| < \varepsilon$, $\quad B = \bigcup_{j=1}^{m} B_j$

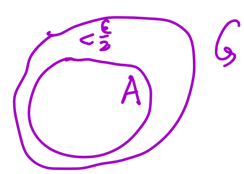


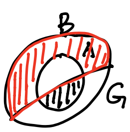
Pf. fix $\varepsilon > 0$.

Then $\exists$ open $G \subseteq A$ s.t. $|G \setminus A| < \frac{\varepsilon}{2}$ → Useful practice.

$G = \bigcup^{\infty} B_i$, $\Rightarrow \sum_{1}^{\infty} |B_i| = |G| < |A| + \frac{\varepsilon}{2} < \infty$

$\Rightarrow \exists m \in \mathbb{N}$, s.t. $\sum_{m+1}^{\infty} |B_i| < \frac{\varepsilon}{2}$ → key point.

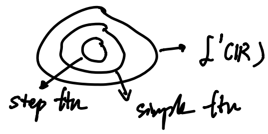




let $B = \bigcup_{i=1}^{m} B_i$.

$\Rightarrow |B \setminus A| + |A \setminus B| \leq |G \setminus A| + \underbrace{|G \setminus B|}_{\sum_{m+1}^{\infty} |B_i|} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thm. (Approx by step ftns) let $f \in \mathcal{L}^1(\mathbb{R})$. then $\forall \varepsilon > 0$. $\exists$ step $g \in \mathcal{L}^1(\mathbb{R})$ s.t. $\|f-g\|_1 < \varepsilon$



Pf:

① $\exists$ simple ftn $\in \mathcal{L}^1(\mu)$ s.t. $\left\|f - \sum_{k=1}^{n} a_k 1_{A_k}\right\| < \frac{\varepsilon}{2}$.

② $\forall A_k$. $\exists B_{k1}, \dots, B_{km}$. , $B_k = \bigcup_i B_{ki}$,

$\forall \varepsilon$, $|A_k \setminus B_k| + |B_k \setminus A_k| \le \varepsilon \Rightarrow |\mu(A_k) - \mu(B_k)| \leq \varepsilon$

$$\Rightarrow \left\|f - \sum_{k=1}^{n} a_k 1_{B_k}\right\| \leq \left\|f - \sum_{k=1}^{n} a_k 1_{A_k}\right\| + \left\|\sum_{k=1}^{n} a_k 1_{A_k} - \sum a_k 1_{B_k}\right\|$$

$$< \frac{\varepsilon}{2} + \sum |a_k| \cdot \|1_{A_k} - 1_{B_k}\|_1 < \varepsilon$$

Def Let $C_c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$. $\subseteq \mathcal{L}^1$

cts

compact support : $\{x \mid f(x) \neq 0\}$ is a bounded set.

for $f \in \mathcal{L}^1(\mathbb{R})$, $\forall \varepsilon > 0$

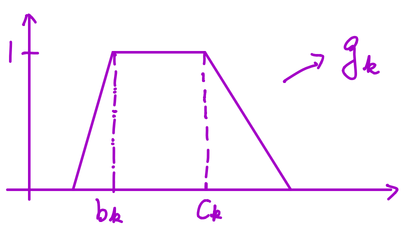
In other word . $\exists$ cts ftn $g : \mathbb{R} \to \mathbb{R}$.

$\|f-g\|_1 < \varepsilon$ and $\{x \in \mathbb{R} : g(x) \neq 0\}$ bounded

(Approximation by cts ftn)

Thm: let $f \in \mathcal{L}^1(\mathbb{R}) \Rightarrow \forall \varepsilon > 0$ $\exists h \in C_c(\mathbb{R})$ s.t. $\|f-h\|_1 < \varepsilon$.

Pf:



$g = \sum_{k=1}^{n} g_k$.

$$\|f-g\|_1 \leq \left\|f - \sum a_k 1_{[b_k, c_k]}\right\|_1 + \sum_{k=1}^{n} |a_k| \left\|1_{[b_k, c_k]} - g_k\right\|_1$$