① Borel σ-alg on $\mathbb{R}^n$.

Notation: $\mathcal{B}_n = \mathcal{B}[\mathbb{R}^n]$.

= the smallest σ-alg containing all open subsets of $\mathbb{R}^n$.

Thm: $\mathcal{B}_m \otimes \mathcal{B}_n = \mathcal{B}_{m+n}$ (Lebesgue σ-alg doesn't have such property)

Pf. We have prove $\mathcal{B} \otimes \mathcal{B} = \mathcal{B}_2 \rightarrow$ rational end pt rectangle.

$\mathcal{B}_m \otimes \mathcal{B}_n \subseteq \mathcal{B}_{m+n}$.

On the other hand. skip ...

② Def: The Lebesgue measure λ_n on $(\mathbb{R}^n, \mathcal{B}_n)$ is $\lambda_n = \lambda_{n-1} \times \lambda$,

$= \lambda \times \cdots \times \lambda$.

i.e. $\lambda_n(E) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} 1_E(x_1, \dots, x_n)\, dx_1 \cdots dx_n$

Need Tonelli, Fubini to make it valid.

Thm: ① $\lambda_n(E+a) = \lambda_n(E)$ $\quad E \in \mathcal{B}_n,\ a \in \mathbb{R}^n$.

② $\lambda_n(tE) = t^n \lambda_n(E)$ $\quad E \in \mathcal{B}_n.\ t>0$

③ linear Transformation.

$\lambda_n(T(E)) = |\det(T)| \cdot \lambda_n(E)$

e.g.: $B_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : |(\cdots)| \le 1\}$.

$$\lambda_n(B_n) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^{n-2}} 1_{B_n}(x, y)\, dy\, dx.$$

$x \qquad y$

$$= \int_{\mathbb{R}^2} \lambda_{n-2}\left(\sqrt{1-x_1^2-x_2^2} \cdot B_{n-2}\right) \cdot 1_{B_2}(x)\, dx$$

$$= \int_{\mathbb{R}^2} (1-x_1^2-x_2^2)^{\frac{n-2}{2}} \cdot \lambda_{n-2}(B_{n-2}) \cdot 1_{B_2}(x)\, dx$$

$$= \lambda_{n-2}(B_{n-2}) \int_{B_2} (1-x_1^2-x_2^2)^{\frac{n-2}{2}} \cdot dx$$

$$= \lambda_{n-2}(B_{n-2}) \int_0^{2\pi} \int_0^1 (1-r^2)^{\frac{n-2}{2}}\, r \cdot dr\, d\theta$$

$$= \lambda_{n-2}(B_{n-2}) \cdot 2\pi \left(-\frac{1}{n}(1-r^2)^{\frac{n}{2}}\right)\Big|_0^1 = \lambda_{n-2}(B_{n-2}) \cdot \frac{2\pi}{n}$$

$\lambda_1(B_1) = 2$

$\lambda_2(B_2) = \pi$

$\Rightarrow \lambda_{2k+1}(B_{2k+1}) = \frac{1}{2k+1} \cdot \frac{1}{2k-1} \cdots \cdots (2\pi)^k \cdot 2$

$\lambda_{2k}(B_{2k}) = \cdots$