Thm. (Tonelli Thm)

(X, S, μ), (Y, T, ν) σ-finite measure space.

Suppose $f: X\times Y \to [0, +\infty]$ is $S\otimes T$ mble ftn.

then. ① $x \longmapsto \int_Y f(x,y)\, d\nu(y)$ is a S-mble ftn

② $y \longmapsto \int_X f(x,y)\, d\mu(x)$ --- T ---

③ $\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y) = \int_{X\times Y} f \, d(\mu\times\nu)$

$\int_X \int_Y 1_E(x,y)\, d\nu(y)\, d\mu(x)$

$= \int_Y \int_X 1_E(x,y)\, d\mu(x)\, d\nu(y)$

$\Rightarrow \left(\int_Y 1_E(x,y)\, d\nu(y)\right) = \nu([E]_x)$ is S-mble

Pf: (a) Assume $f = 1_E$, $E \in S\otimes T$, true by previous Thms

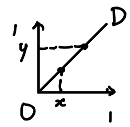
(b) Assume $f = \sum_{i=1}^{N} c_i 1_{E_i}$, also true.

(c) Assume $f \geq 0$ mble, $\exists$ simple ftns. $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$, $\lim_{n\to\infty} \phi_n(x,y) = f(x,y)$ $\forall x, y$.

MCT $\Rightarrow \cdots$

(ex) $X = Y = [0,1]$. $S = T = \mathcal{B}([0,1])$, μ = counting measure, λ = Leb. measure

$D = \{(x,x) : x \in [0,1]\}$



counting measure μ is not σ-finite!

$\Rightarrow$ Tonelli does not apply.

$\mu(X_n)$ is finite iff X_n has finitely many pts.

$\Rightarrow [0,1] = \bigcup_{i=1}^{\infty} X_i$ is only countably many pts. contradiction.

① $\int_{[0,1]} \underbrace{\int_{[0,1]} 1_D(x,y)\, d\lambda(y)}\, d\mu(x)$

$\lambda([D]_x) = \lambda(\{x\}) = 0$

$\Rightarrow \int_{[0,1]} 0 \, d\mu(x) = 0$

不等！

② $\int_{[0,1]} \underbrace{\int_{[0,1]} 1_D(x,y)\, d\mu(x)}\, d\lambda(y)$

$\mu([D]^y) = \mu(\{y\}) = 1$

$\int_{[0,1]} \int_{[0,1]} 1_D(x,y)\, d\mu(x)\, d\lambda(y) = \int_{[0,1]} 1 \, d\lambda(y) = \underline{1}$.

Thm (Fubini) (X, S, μ), (Y, T, ν) σ-finite.

$f: X\times Y \to \underline{[-\infty, +\infty]}$ $S\otimes T$-mble

if $\int_{X\times Y} |f| \, d(\mu\times\nu) < \infty$ i.e. $f \in \mathcal{L}^1(X\times Y, \mu\times\nu)$, then

① $g(x) \overset{def}{=} \int_Y f(x,y)\, d\nu(y)$ is defined (convergent) for a.e. x and is an S-mble ftn.

② $h(y) \overset{def}{=} \int_X f(x,y)\, d\mu(x)$ is defined (convergent) for a.e. y and is an T-mble ftn.

③ $\int_{X\times Y} f \, d\mu\times\nu = \int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y)$

Pf: write $f = f^+ - f^-$ $\Rightarrow$ $|f| = f^+ + f^-$ $\int f^+ < \infty$, $\int f^- < \infty$. $\Leftarrow \int |f| < \infty$

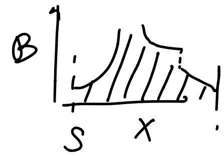
By Tonelli, $\int_{X\times Y} f^+ = \int_X \Big[\underbrace{\int_Y f^+ \, d\nu(y)}_{g(x)}\Big]\, d\mu(x) < \infty$

$\int_X g(x)\,\mu(x) < \infty \Rightarrow g(x) < \infty$ for a.e x

By Tonelli, $g(x)$ $\mathcal{S}$-mble.

Def: region under graph of f, $f: X \to [0,\infty]$

$$U_f = \{(x,t) \in X\times(0,\infty) \mid 0<t<f(x)\}.$$



Thm: $(X,\mathcal{S},\mu)$ σ-finite, $f: X\to[0,\infty]$ $\mathcal{S}$-mble ftn. then

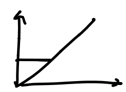
positive ftn

① U_f is $(\mathcal{S}\times\mathcal{B})$-mble set.

② $(\mu\times\lambda)(U_f) = \int_X f\,d\mu$

$$= \int_{(0,\infty)} \mu(\{x\in X: f(x)>t\})\,d\lambda(t)$$

$$\int_0^\infty P(X>t)\,dt = \int_0^\infty\int_x^\infty f_X(t)\,dt\,dx = \int_0^\infty \int_0^t f_X(t)\,dx\,dt = \int_0^\infty t\cdot f_X(t)\,dt = E(X)$$

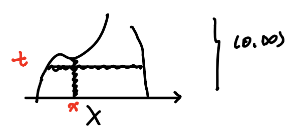


Pf: ② Suppose ① correct.

$$(\mu\times\lambda)(U_f) = \int_{X\times(0,\infty)} 1_{U_f}\, d(\mu\times\lambda)$$

$$\overset{\text{Tonelli}}{=} \int_X \underbrace{\int_{(0,\infty)} 1_{U_f}(x,t)\,d\lambda(t)}\, d\mu(x) \qquad \mathcal{B} \quad \mathcal{S}$$

$$= \begin{cases} f(x) & f(x)>0 \\ 0 & f(x)=0\end{cases}$$



$$= \int_X f(x)\,d\mu(x)$$

$$\overset{\text{Tonelli}}{=} \int_{(0,\infty)}\underbrace{\int_X 1_{U_f}(x,t)\,d\mu(x)}\,d\lambda(t) = \int_{(0,\infty)}\mu(\{x\in X: f(x)>t\})\,d\lambda(t)$$

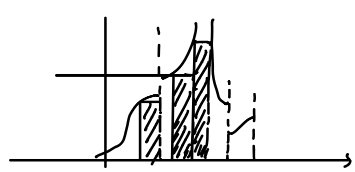
$$= \mu[U_f]^t$$

$$= \mu(\{x: f(x)>t\})$$

① For each $k\in\mathbb{N}$

let $F_k = f^{-1}([k,\infty))\times(0,k) \subseteq X\times(0,\infty)$

$$E_k = \bigcup_{m=0}^{k^2-1} f^{-1}\left(\left[\tfrac{m}{k}, \tfrac{m+1}{k}\right)\right)\times\left(0,\tfrac{m}{k}\right).$$



$\Rightarrow F_k, E_k \in \mathcal{S}\otimes\mathcal{B}$

$\Rightarrow U_f = \bigcup_{k=1}^\infty (E_k\cup F_k)\in\mathcal{S}\otimes\mathcal{B}$. ∎