$X \times Y = \{(x,y): x \in X, y \in Y\}$

a rectangle in $X \times Y$ is a set of the form $A \times B$, $A \subseteq X$, $B \subseteq Y$

e.g. $\mathbb{Q} \times \mathbb{Q}$ is a rectangle

Def (X,S), (Y,T) mble spaces.

① $S \otimes T$ is the smallest σ-alg containing $\{A \times B : A \in S, B \in T\}$ 所有 mble rectangle.

② a mble rectangle is $A \times B$, $A \in S$, $B \in T$.

ex. E is an open unit ball in $\mathbb{R} \times \mathbb{R}$. E is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ mble.

$$E = \bigcup_{\substack{a,b,c,d \in \mathbb{Q} \\ (a,b) \times (c,d) \subseteq E}} (a,b) \times (c,d)$$

$\rightarrow$ Same for every open set in $\mathbb{R}^2$, they are $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ mble. open sets are mble

Thm: $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ ☆

Def: For $E \subset X \times Y$, the cross sections of E are

$[E]_a = \{y \in Y \mid (a,y) \subseteq E\}$ mble?

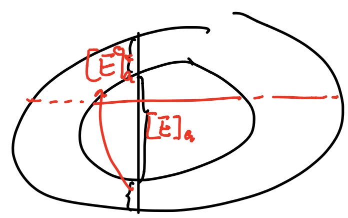
$[E]^b = \{x \in X \mid (x,b) \subseteq E\}$ set mble $\Rightarrow$ slice mble

Thm: If $E \in S \otimes T$, then $[E]_a \in T$, $[E]^b \in S$, $\forall a \in X, b \in Y$

(Pf): let $\mathcal{E} = \{E \subset X \times Y : [E]_a \in T, [E]^b \in S\}$.

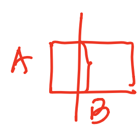
WTS: ① $\mathcal{E}$ is σ-alg

② $\{A \times B \mid A \in S, B \in T\} \subseteq \mathcal{E}$



① $\begin{cases} [E^c]_a = ([E]_a)^c \in T. \\ [\cup E]_a = \cup [E]_a \in T. \end{cases}$ σ-alg

② $[A \times B]_a = \begin{cases} B & a \in A \\ \phi & a \notin B \end{cases}$



Thm Note: The inverse is not true. Even if all cross sections are mble, the set might not be mble

Def: For $f: X \times Y \rightarrow \mathbb{R}$ the cross sections ftn of f are

$[f]_a = f(a,y)$

$[f]^b = f(x,b)$

e.g. $[1_E]_a = 1_{[E]_a}$ $[1_E]^b = 1_{[E]^b}$.

Thm: (Cross section ftns of mble ftns are mble)

If $f: X \times Y \rightarrow \mathbb{R}$ $S \otimes T$ mble, then

$[f]_a$ is T-mble, $[f]_b$ is S-mble, $a \in X$, $b \in Y$

Pf: For open set D

$$[f]_a^{-1}(D) = \{ y : f(a,y) \in D \}$$
$$= [f^{-1}(D)]_a \in T.$$

skip Monotone class lemma p. 120 – 123

Def: $(X, \mathcal{S}, \mu)$ measure space.

- μ $\boxed{\text{finite measure}}$ if $\mu(X) < \infty$
- μ $\boxed{\sigma\text{-finite}}$ if $\exists$ mble sets $X_1, X_2, \cdots$ s.t. $X = \bigcup_{i=1}^{\infty} X_i$ and $\mu(X_i) < \infty$, $\forall i$

(ex) $(\mathbb{R}, \mathcal{L}, \lambda)$ is σ-finite.

Thm. (X, S, μ), (Y, T, ν) σ-finite measure spaces

Suppose $E \in S \otimes T$. Then

- $x \longmapsto \nu([E]_x)$ is a S-mble ftn
- $y \longmapsto \mu([E]^y)$ is a T-mble ftn.

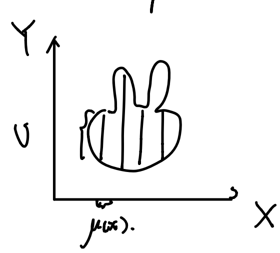
Pf. $\mathcal{E} = \{ E \subset X \times Y : \text{the result holds} \}$
(a) $\mathcal{E}$ is σ-alg $\rightarrow$ Monotone Class Thm
(b) $\forall E \in S \otimes T$, $E \in \mathcal{E}$.

Def. (product measure)

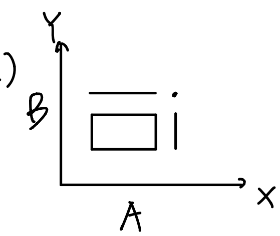
(X, S, μ), (Y, T, ν) σ-finite measure spaces.

Define $\mu \times \nu : S \otimes T \longrightarrow [0, \infty]$ by

$$(\mu \times \nu)(E) = \int_X \nu([E]_x) \, d\mu(x).$$
$$= \int_X \left[\int_Y 1_E(x,y) \, d\nu(y) \right] d\mu(x)$$



ex)



$E = A \times B$, $A \in S$, $B \in T$.

$\Rightarrow (\mu \times \nu)(E) = \mu(A) \cdot \nu(B)$

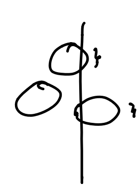
Lemma: $\mu \times \nu$ is a measure on $(X \times Y, S \otimes T)$.

$$\begin{cases} ① \ \mu \times \nu(\phi) = 0 \\ ② \ (\mu \times \nu)(\cup E) = \Sigma (\mu \times \nu) E \end{cases}$$

Pf for ②: Let $E_1, E_2, \cdots$ disjoint in $S \otimes T$.

$$(\mu \times \nu)\left(\bigcup_{i=1}^{\infty} E_i \right) = \int_X \nu\left(\left[\bigcup_{i=1}^{\infty} E_i \right]_x \right) d\mu(x)$$

$\sum^{\infty} \nu [E_i]_x$



$$= \int_X \sum_{i=1}^{\infty} \nu\,[E_i]_x \, d\mu(x) \quad \text{By Monotone Convergence Thm}$$

$$= \sum_{i=1}^{\infty} \int_X \nu\,[E_i]_x \, d\mu(x) = \sum_{i=1}^{\infty} (\mu \times \nu)(E_i) \qquad \blacksquare$$

Thm (X, S, μ), (Y, T, ν) σ-finite measure spaces. $E \in S \otimes T$.

$\Rightarrow \int_X \int_Y 1_E(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X 1_E(x,y)\, d\mu(x)\, d\nu(y)$

i.e. $\int_X \nu([E]_x)\, d\mu(x) = \int_Y \mu([E]^y)\, d\nu(y)$