$\boxed{2D}$ Lesbesgue Measure.

对于 Outer Measure

① $|A \cup G| = |A| + |G|$ if G is open

① Additivity of Outer Measure on Borel sets

Prop: Additivity if one of the sets is $\boxed{\text{open}}$. i.e.

$A, G \subset \mathbb{R}$, disjoint, G is open $\Rightarrow |A \cup G| = |A| + |G|$.

Pf. ① if $|G| = \infty$. done!

② Assume $|G| < \infty$

WTS: $|A \cup G| \geq |A| + |G|$.

Use open set can be countable open interval.

(i) Assume $G = (a, b)$

· we know $|A \cup \{a\}| = |A|$

$\Rightarrow$ we can discard $\{a\}, \{b\}$ if they are in A.

A. G disjoint $\Downarrow$

for any $\{I_n\}_{n \in \mathbb{N}}$ be an open cover for $A \cup G$

then $K_n = I_n \cap (a, b)$
$L_n = I_n \cap (b, +\infty)$
$J_n = I_n \cap (-\infty, a)$
$>$ Since we discard $\{a\}, \{b\}$.

$G \subset \bigcup K$

$A \subset \bigcup (L \cup J)$

· $\ell(I_n) = \ell(K_n) + \ell(L_n) + \ell(J_n)$

$$\Rightarrow \sum \ell(I_n) = \sum(\ell(L_n) + \ell(J_n)) + \sum \ell(K_n)$$
$$\geq |A| + |G|$$
$$|A \cup G| = \inf(\sum \ell(I_n)) \geq |A| + |G|$$

将 $A \cup G$ 的 I_n 分为 L_n, J_n.
$\geq |A|$, $\geq |G|$

ii) For finite union.

$$|A \cup G| = \left|A \cup \left(\bigcup^{m-1} G_i\right)\right| + |G_m| = \cdots = |A| + \cdots + |G_m|$$
$$|G| = |G_1| + \cdots + |G_m|$$
$$\Rightarrow |A \cup G| = |A| + |G|$$

iii) General G. $G = \bigcup_{i=1}^{\infty} G_i$ countable

$$|A\cup G| \ge \left|A \cup \bigcup_{i=1}^{N} G_i\right| = |A| + \sum_{i=1}^{N} |G_i| = |A| + \sum_{i=1}^{N} \ell(G_i)$$

take $N\to\infty$ both sides $\Rightarrow$ $|A\cup G| \ge |A| + \sum_{i=1}^{\infty} \ell(G_i) \ge |A| + |G|$.

Prop: Additivity if one of the sets is $\boxed{\text{closed}}$. i.e. ② $|A\cup F| = |A| + |F|$ if F is closed

$A, F \subset \mathbb{R}$, disjoint, F is closed $\Rightarrow$ $|A\cup F| = |A| + |F|$.

Pf. let $A\cup F \subset \boxed{\bigcup_{n=1}^{\infty} I_n = G}$.

Then, G is open $\Rightarrow$ $G\setminus F = G\cap F^c$ is open $\Rightarrow$ $|G| = |G\setminus F| + |F|$.

Since A, F disjoint $\Rightarrow$ $A \subseteq G\setminus F$ $\Rightarrow$ $|A| + |F| \le |G\setminus F| + |F| = |G| = \left|\bigcup_{n=1}^{\infty} I_n\right| \le \sum_{n=1}^{\infty} \ell(I_n)$

take inf on both sides $\Rightarrow$ $|A| + |F| \le |A\cup F|$

The other direction is simple.

$|A| = \inf\{\sum_{k=1}^{\infty} \ell(I_k) \mid \underbrace{\bigcup_{k=1}^{\infty} I_k}_{G} \supset A\}$ By def of outer measure. $|G| - |A| \le \varepsilon$.

$\Rightarrow$ $\forall \varepsilon > 0$. $|A| + \varepsilon \ge \sum_{k=1}^{\infty} \ell(I_k) \ge |G|$, $\forall \varepsilon > 0$, $\exists$ open $G \supset A$, s.t. $|A| + \varepsilon \ge |G|$

However, this $\not\Rightarrow$ $|G\setminus A| \le \varepsilon$. for Borel B, $\exists$ closed $F\subset B$. $|A\setminus F| < \varepsilon$

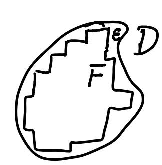
Prop: (approximation of Borel sets from below by closed sets)

$B \subset \mathbb{R}$, Borel subsets.

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists$ closed $F \subset B$ s.t. $|B\setminus F| < \varepsilon$.

Pf: Let $\mathcal{L} = \{D \subset \mathbb{R} \mid \forall \varepsilon > 0,\ \exists \text{ closed } F \subset D \text{ s.t. } |D\setminus F| < \varepsilon\}$ *

$\forall$ D closed, $D \in \mathcal{L}$



WTS: $\mathcal{L} \supset \mathcal{B}(\mathbb{R})$

$\Rightarrow$ Enough to prove $\begin{cases} ① & \mathcal{L} \text{ is a } \sigma\text{-algebra.} \\ ② & \mathcal{L} \text{ contains all closed subsets } \checkmark \end{cases}$

② is clear

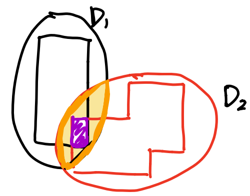
① (a) $\emptyset \in \mathcal{L}$ ✓

(b) Countable intersections

Let $D_1, \cdots \in \mathcal{L}$. WTS $\bigcap_{i=1}^{\infty} D_i \in \mathcal{L}$?

Fix $\varepsilon > 0$ for each $k \in \mathbb{N}$, $\exists$ closed $F_k \subset D_k$

s.t. $|D_k \setminus F_k| < \frac{\varepsilon}{2^k}$.



Let $F = \bigcap_{k=1}^{\infty} F_k$

By observation:

- F closed ✓ • $F \subset \bigcap_{k=1}^{\infty} D_k$ ✓
- $\left|\bigcap_{k=1}^{\infty} D_k \setminus F\right| \leqslant \left|\bigcup_{k=1}^{\infty} (D_k \setminus F_k)\right| \leqslant \sum_{k=1}^{\infty} |D_k \setminus F_k| < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$.

$\Rightarrow \bigcap_{k=1}^{\infty} D_k \in \mathcal{L}$ since $F \subseteq \bigcap D_k$ and $|\bigcap D_k \setminus F| < \varepsilon$.

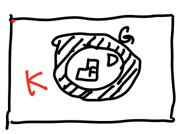
(c) complement

$\boxed{\text{The hardest part}}$

Fix $\varepsilon > 0$. By def $\Rightarrow \exists$ open $G \supset D$. s.t. $|D| + \frac{\varepsilon}{2} > |G|$. ①

Let $K = G^c = \mathbb{R} \setminus G$ closed. $K \subset D^c$

$D^c \setminus K = G \setminus D$



要证 $|D^c \setminus F| < \varepsilon$
我们只知道 $|G| - |D| < \varepsilon$.
要证 $|G \setminus D| < \varepsilon$
通过 $G \setminus F$ 来放缩
因为 $|G \setminus F| = |G| - |F|$

(i) Assume $|D| < \infty$

Since $D \in \mathcal{L}$, $\exists$ closed $F \subset D$, s.t. $|D \setminus F| < \frac{\varepsilon}{2}$.

$D^c \setminus K = G \setminus D \subset G \setminus F$

$|G \setminus F| + |F| = |G| \Rightarrow |G \setminus F| = |G| - |F|$

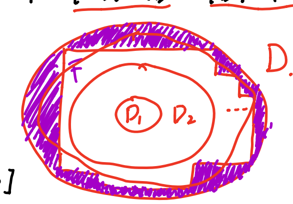
$\boxed{\begin{array}{l} F \text{ closed} \\ G \setminus F,\ F \text{ disjoint !} \end{array}}$

$|F| \leq |D| < \infty$.

Need to pay attention !

Hence $|D^c \setminus K| \leq |G \setminus F| = |G| - |F| = \underline{|G| - |D|} + \underline{|D| - |F|} < \varepsilon$

Thus $D^c \in \mathcal{L}$.



ii) Assume $|D| = \infty$

$D = \bigcup_{k=1}^{\infty} D_k$. $D_k = D \cap [-k, k]$

将 D^c 转化为 $\left(\bigcap_{k=1}^{\infty} D_k\right)^c$

Since $D \in \mathcal{L}$.
$\exists$ closed $F \subset D$. $|D \setminus F| < \varepsilon$.

Let $F_k = \underbrace{F \cap [-k, k]}_{\text{closed.}} \subset D_k$ $\quad |D_k \setminus F_k| \le |D \setminus F| < \epsilon$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |(D \setminus F) \cap [-k, k]|$

$\Rightarrow D_k \in \mathcal{L}$

since $|D_k| \le 2k < \infty$

$\Rightarrow D_k^c \in \mathcal{L}$

$\Rightarrow D^c = \bigcap_{k=1}^{\infty} D_k^c \in \mathcal{L}$

(d) countable union

If $D_1, D_2, \cdots \in \mathcal{L}$,

$\bigcup_{k=1}^{\infty} D_k = \left(\bigcap_{k=1}^{\infty} D_k^c\right)^c \in \mathcal{L}$

The above shows that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}$ $\Rightarrow$ all Borel sets (B) have closed subset (F) s.t. $|B \setminus F| < \epsilon$.

Thm: (Additivity of outer measure if one of the sets is a Borel subsets).

$A, B \subset \mathbb{R}$, disjoint, B borel. $\qquad B \in \mathcal{L}$

$\Rightarrow |A \cup B| = |A| + |B| \qquad$ WTS $|A \cup B| \ge |A| + |B|$

Pf: Fix $\epsilon > 0$.

$\exists$ closed $F \subset B$, $|B \setminus F| < \epsilon$.

$|A \cup B| \ge |A \cup F| \overset{F \text{ closed}}{=} |A| + |F| = |A| + |B| - |B \setminus F| > |A| + |B| - \epsilon$

$\left(|B| = |B \setminus F| + |F| \right.$ (F closed) $\Rightarrow |F| = |B| - |B \setminus F|$. $\{|B \setminus F| < \infty\}$

$\Rightarrow |A \cup B| \ge |A| + |B|$

$|A \cup B| \le |A| + |B|$

$\Rightarrow |A \cup B| = |A| + |B|$

which leads to

- finite additivity on $\mathcal{B}(\mathbb{R})$ ✓
- countable subaddibivity on $\mathcal{P}(\mathbb{R}) \supset \mathcal{B}(\mathbb{R})$ ✓
- monotonicity on ... ✓

These 3 properties lead to theorem.

Thm (Outer measure is a measure on Borel sets)

The outer measure is a measure on $\mathcal{B}(\mathbb{R})$

Recall: a measure satisfies 2 prop

$\mu(\phi) = 0$

$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$

(pf) $\cdot\ |\phi| = 0$

$\cdot\ B_1, B_2, \cdots \in \mathcal{B}(\mathbb{R})$, disjoint.

$\boxed{1}$ $\left|\bigcup_{k=1}^{\infty} B_k\right| \ge \left|\bigcup_{k=1}^{N} B_k\right| = \sum_{k=1}^{N} B_k \qquad \forall N \in \mathbb{N}$

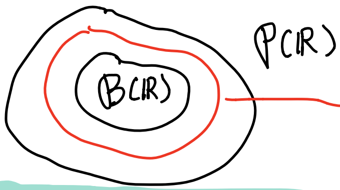
take $N \to \infty$ from both sides: $\left|\bigcup_{k=1}^{\infty} B_k\right| \ge \sum_{k=1}^{\infty} B_k$.

$\boxed{2}$ On the other hand, by countable subadditivity, $|\bigcup_{k=1}^{\infty} B_k| \le \sum_{k=1}^{\infty} B_k$.

$\Rightarrow |\bigcup_{k=1}^{\infty} B_k| = \sum_{k=1}^{\infty} B_k$

Def: The **lebesgue measure** is the measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by the outer measure

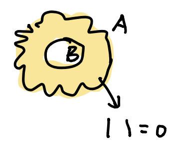
$\Rightarrow$



In fact there exists a slightly bigger σ-alg for the Lebesgue measure: "The Lebesgue σ-alg"

② Lebesgue measurable sets

Def: $A \subset \mathbb{R}$ is called Leb. mble. if $\exists$ Borel set $B \subset A$ s.t. $|A \setminus B| = 0$
($\Rightarrow$ Every Borel set B is a Lebesgue mble set)



$\longrightarrow$ **Thm** (Equ Definition of Leb. mble sets — Inner operation)
Suppose $A \subset \mathbb{R}$. Then, the followings are equ.
(a) A is Leb. mble
(b) $\forall \varepsilon > 0$, $\exists$ closed $F \subset A$ s.t. $|A \setminus F| < \varepsilon$.
(c) $\exists$ closed $F_1, F_2, \cdots \subset A$ s.t. $|A \setminus \bigcup_{k=1}^{\infty} F_k| = 0$
(d) $\exists$ Borel $B \subset A$, s.t. $|A \setminus B| = 0$ $\longrightarrow$ the main def

$\} \rightarrow \mathcal{L}$ is the collection of Leb mble sets.

$a \Leftrightarrow d \Leftarrow c$, $d \Downarrow b$, $b \Rightarrow c$, $g \Rightarrow b$, $b \Downarrow e$, $e \Rightarrow f$, $f \Rightarrow g$

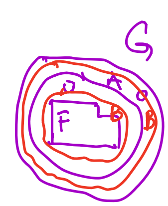
Pf: (a) $\Leftrightarrow$ (d) by def
(c) $\Rightarrow$ (d) take $B = \bigcup_{k=1}^{\infty} F_k$

(b) $\Rightarrow$ (c) $\forall n$, $\exists F_n \subset A$, s.t. $|A \setminus F_n| < \frac{1}{n}$.

$A \setminus \bigcup_{k=1}^{\infty} F_k \subset A \setminus F_n \Rightarrow |A \setminus \bigcup_{k=1}^{\infty} F_k| \le |A \setminus F_n| < \frac{1}{n} \Rightarrow_{n\to\infty} |A \setminus \bigcup_{k=1}^{\infty} F_k| = 0$.

(d) $\Rightarrow$ (b)
Recall $\mathcal{L}$ is a σ-alg ①, and $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}$ ②, also all sets of outer measure $0 \in \mathcal{L}$ ③

$A = \underbrace{B}_{\text{borel} \in \mathcal{L}} \cup \underbrace{(A \setminus B)}_{\text{measure } 0 \in \mathcal{L}} \Rightarrow A \in \mathcal{L}$ Done.



Thm: (approx from outside!)

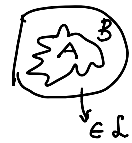
$a \Leftrightarrow d \Leftarrow c$, $d \Downarrow b$, $b \Rightarrow c$, $g \Rightarrow b$, $b \Downarrow e$, $e \Rightarrow f$, $f \Rightarrow g$

(e) $\forall \varepsilon > 0$, $\exists$ open $G \supset A$ s.t. $|G \setminus A| < \varepsilon$.
(f) $\exists$ open $G_1, G_2, \cdots \supset A$ s.t. $|\bigcap_{k=1}^{\infty} G_k \setminus A| = 0$
(g) $\exists$ Borel $B \supset A$, s.t. $|B \setminus A| = 0$

$\boxed{\begin{array}{l} e \Rightarrow f \ \checkmark \\ f \Rightarrow g \ \checkmark \end{array}}$

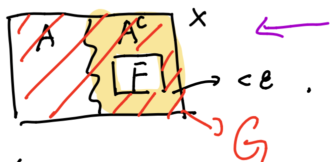
$\boxed{g \Rightarrow b}$: g holds $\Rightarrow \exists$ Borel $B \supset A$, $|B \setminus A| = 0$

$$A = \underbrace{B}_{\in \mathcal{L}} \setminus \underbrace{(B \setminus A)}_{\in \mathcal{L}} \in \mathcal{L}.$$



$\boxed{b \Rightarrow e}$: $\boxed{b \Leftrightarrow A \in \mathcal{L}}$ $\Rightarrow A^c \in \mathcal{L} \Rightarrow \forall \varepsilon > 0$, $\exists$ closed $F \subset A^c$, $|A^c \setminus F| < \varepsilon$

Let $G = F^c \Rightarrow$ open, $A \subset G \Rightarrow G \setminus A = A^c \setminus F \Rightarrow |G \setminus A| < \varepsilon$.



显而易见.

$$A = \underbrace{B}_{\bigcup_{k=1}^{\infty} F_k} \cup N = \underbrace{B'}_{\bigcap_{k=1}^{\infty} G_k} \setminus N'.$$

Thm:

by (b) $\Rightarrow$ $\mathcal{L}$ we defined is such collection

(a) Let $\mathcal{L}$ be the collection of Lebesgue mble sets. Then, $\mathcal{L}$ is a σ-alg on $\mathbb{R}$. Furthermore, $\mathcal{L}$ contains all Borel sets and sets of outer measure zero

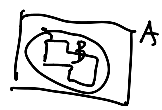
(b) Outer measure is a measure on $(\mathbb{R}, \mathcal{L})$

Pf: (a) done

(b) Suppose $A_1, A_2, \cdots \in \mathcal{L}$, disjoint

$\Rightarrow \forall k \in \mathbb{N}$, $\exists$ Borel $B_k \subset A_k$, $|A_k \setminus B_k| = 0$

$$\left|\bigcup_{k=1}^{\infty} A_k\right| \geq \left|\bigcup_{k=1}^{\infty} B_k\right| = \sum_{k=1}^{\infty} |B_k| = \sum_{k=1}^{\infty} |A_k|.$$



$$\left(A_k = \underbrace{B_k}_{\text{borel}} \cup (A_k \setminus B_k) \quad \Rightarrow |A_k| = |B_k| + |A_k \setminus B_k| = |B_k| \right)$$

③ The Cantor Set (The middle thirds Cantor Sets)

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

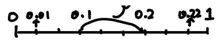
$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

$\vdots$

Let $C = \bigcap_{i=1}^{\infty} C_i$ is called Cantor set

What is left?

Fact :

- $x \in C \iff$ 

$x = (0.a_1 a_2 \cdots)_3$
none of a_i is 1. $a_i \in \{0, 2\}$.
- This implies that C is uncountable.
similar to $\mathbb{R}$.
- C is a compact set
$|C| \le |C^k| = \frac{2^k}{3^k} \Rightarrow |C| = 0$. uncountable $\Rightarrow 0$!