## $\boxed{2C}$ Measures and their properties.

**Def:** Let $(X, S)$ be a mble space.
A measure on $(X, S)$ is a function $\mu: S \longmapsto [0, +\infty]$

s.t. ① $\mu(\phi) = 0$

② countable additivity: If $E_1, E_2, \cdots \in S$ and disjoint
$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$

(which also leads to finite additivity)

example:
- Dirac measure on $x_0$: $\mu(E) = 1_{\{x_0 \in E\}}$.
- Counting ms: $\mu(E) = \# E$
- $\{c_0, c_1, \cdots\}$.
  $\mu(E) = \sum_{i \in E} c_i$ $\qquad \mu(\{1, 6\}) = c_1 + c_6$.

Note: Outer Measure is not a measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$.
however, it is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

**Def:** $(X, S, \mu)$ is called a measure space. [ $(X, S)$ is 'measurable' space ]

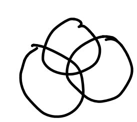
→ Simple Properties:

① Monotonicity: $D, E \in S$, $D \subset E$, $\mu(E) \geq \mu(D)$

② $D, E \in S$, $D \subset E$. if $\mu(D) < \infty$ $\Rightarrow$ $\mu(E \setminus D) = \mu(E) - \mu(D)$
Note: $\infty - \infty$ is not defined

③ $A, B \in S$, if $\mu(A \cap B) < \infty$ $\Rightarrow$ $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$

④ Countably subaddibivity: $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} (\mu(E_i))$



→ $F_1 = E_1$, $F_2 = E_2 \setminus E_1$
$F_3 = E_3 \setminus (E_2 \cup E_1)$, $\cdots$.

$\mu(\bigcup E_i) = \mu(\bigcup F_i) = \sum(\mu(F_i)) \leq \sum(\mu(E_i))$
$(\mu(F_i) \leq \mu(E_i))$

→ **Thm** $(X, S, \mu)$ measure space

① upward continuity of measure

$E_1, \cdots \in S$, $E_1 \subset E_2 \subset \cdots$

$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n)$

② downward continuity of measure

$E_1, \cdots \in S$, $E_1 \supset E_2 \supset \cdots$ and $\mu(E_1) < \infty$

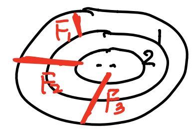
$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n)$


Pf: ① $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, $\cdots$

$\mu\left(\bigcup_{1}^{\infty} E_i\right) = \mu\left(\bigcup_{1}^{\infty} F_i\right) = \sum_{1}^{\infty} \mu(F_i) = \lim_{N\to\infty} \sum_{1}^{N} \mu(F_i) = \lim_{N\to\infty} \mu(E_N)$

② let $F_k = E_1 \setminus E_k$ ⇒ $F_k$ is mble

($E_1$ → mble, $E_k$ → mble)




Don't forget!

( if $\mu(E_k)$ finite )

• $F_1 \subseteq F_2 \subseteq \cdots$.

Apply ① $\mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} \mu(F_k)$

$\mu(E_1 \setminus E_k)$

$\text{LHS} = \mu\left(E_1 \setminus \bigcap_{k=1}^{\infty} E_k\right)$

$= \underbrace{\mu(E_1)}_{<\infty} - \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$

$\text{RHS} = \underbrace{\mu(E_1)}_{<\infty} - \lim_{k\to\infty} \mu(E_k)$

$\Rightarrow \lim_{k\to\infty} \mu(E_k) = \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$

ex. $(\mathbb{N}, \mathcal{P}(\mathbb{N})$, counting mea$)$

$E_k = \{k, k+1, \cdots\}$

$E_1 \supseteq E_2 \supseteq \cdots$

$\mu(E_k) = \infty$, for any $k$.

$\overset{?}{\Rightarrow} \bigcap_{k=1}^{\infty} E_k = \phi$ ,