Def: Ω set.
$\mathcal{F}$ σ-alg on Ω

- A probability measure on $(\Omega, \mathcal{F})$ is a measure $\mathbb{P}$, s.t. $\mathbb{P}(\Omega)=1$
- Ω sample space
- An $\mathcal{F}$-mble set A is called an event.
- For an event A, $\mathbb{P}(A)$ is called the probability of A
- $(\Omega, \mathcal{F}, \mathbb{P})$ is called a prob space

example: [1] $\Omega = \{0,1,2,\cdots,n\}$.
$\mathcal{F}$ = power σ-alg
① $\mathbb{P} = \frac{1}{n+1}$ counting measure. ② $\mathbb{P}(A) = \dfrac{\sum_{i\in A} 2^i}{2^{n+1}-1}$ ③ $\mathbb{P}(A) = \frac{1}{2^n} \sum_{i\in A} \frac{n!}{i!\,(n-i)!}$

[2] $\Omega = [0,1]$. $\mathcal{F}$ = Borel σ-alg ① $\mathbb{P} = \lambda$ Leb measure. $\Rightarrow$ $\mathbb{P}([0,\tfrac{1}{3}]) = \tfrac{1}{3}$
$\mathbb{P}([0,1]\setminus \mathbb{Q}) = 1$

[3] $\Omega = \mathbb{R}$. $\mathcal{F}$ = Borel σ-alg ① $\mathbb{P}$ = Dirac $\mu(E) = \frac{1}{3}(\delta_0 + 2\delta_1)$ for example.
② $\mathbb{P}(E) = \int_E \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$
③ $\mathbb{P}(E) = \int_E e^{-x} \cdot 1_{\mathbb{R}_+}(x)\, dx$

$f \in L^1(\mathbb{R})$, $\mu(E) = \int_E f\, dm$.

for any $f \in L^1(\mathbb{R}, m)$, with $\int_{\mathbb{R}} f\, dx = 1$ $\mathbb{P}(E) = \int_E f\, dx$.

→ Def: $(\Omega, \mathcal{F}, \mathbb{P})$ An event A happens almost surely if $\mathbb{P}(A^c) = 0$
i.e. $\mathbb{P}(A) = 1$.
↓
$\mathbb{P}(\Omega) = \mathbb{P}(A) + \mathbb{P}(A^c)$

→ Def: $(\Omega, \mathcal{F}, \mathbb{P})$ prob space

- A random variable is a mble ftn $X: \Omega \to \mathbb{R}$

$\Leftrightarrow$ $\forall$ borel subset U, $X^{-1}(U) \in \mathcal{F}$
$\Leftrightarrow$ $X^{-1}((-\infty, a]) \in \mathcal{F}$.

- If $X \in \mathcal{L}^1(P)$
$\mathbb{E}X = \int_\Omega X\, dP$ is the expected value.

e.g. $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}, \lambda)$, $X(\omega) = \omega^2$.
$\Rightarrow \mathbb{E}X = \int_0^1 \omega^2\, d\omega = \frac{1}{3}$

○ Note: Discrete ?

$(\{1,\cdots,n\},\ \text{discrete } \sigma\text{-alg},\ \overset{\mu}{\text{counting measure}})$

$1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$

We can write: $f(x) = \sum f(i) \cdot 1_{\{i\}}(x)$

$$\int f \, d\mu = \sum_{i=1}^{n} f(i) \int 1_{\{i\}} \, d\mu = \sum_{i=1}^{n} f(i) \cdot \mu(\{i\}) = \sum_{i=1}^{n} f(i)$$ ← Be careful.

e.x.) $(\{1,\cdots,n\},\ \mathcal{F} = \mathcal{P}(\Omega),\ P = \frac{1}{n+1} \cdot \text{counting measure})$

$X(\omega) = 2^{\omega}, \quad \mathbb{E}X = \dfrac{2^{n+1}-1}{n+1}$

e.x.) $(\Omega = \{HH, HT, TH, TT\},\ \mathcal{F} = \mathcal{P}(\Omega),\ P = \text{Unif prob measure})$

$X(\omega) = \#$ of H in Ω. $\Rightarrow \mathbb{E}X = 1$.

→ Def: if $X \in \mathcal{L}^2(P)$, $\mathrm{Var}(X) = \sigma^2(X) = \mathbb{E}\left((X - \mathbb{E}X)^2\right)$

e.x.) let A be an event of $\mathcal{F}$

$X = 1_A, \quad \mathbb{E}X = \int_\Omega 1_A \, dP = P(A)$.

$$\sigma^2(X) = \int_\Omega (1_A - P(A))^2 \, dP = \int_\Omega \underbrace{1_A(\omega)^2}_{1_A(\omega)} - 2\, 1_A(\omega) \cdot P(A) + P(A)^2 \, dP(\omega)$$

$$= P(A) - 2P^2(A) + P^2(A) = P(A) - P^2(A)$$

Lemma:
- $\mathbb{E}(aX+b) = a\,\mathbb{E}X + b$. if $X \in \mathcal{L}^1(P)$
- $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2$

Def: $(\Omega, \mathcal{F}, P)$ prob space, X r.v. on Ω $(X: \Omega \to \mathbb{R})$

The probability distribution of X is the probability measure P_X on $(\mathbb{R}, \mathcal{B})$

defined by $P_X(B) = P(X^{-1}(B)) = P(\{\omega \in \Omega : X(\omega) \in B\})$

$(\mathbb{R}, \mathcal{B}, P_X)$

$$\begin{array}{ccc} P & & P_X \\ \Omega & \longrightarrow & \mathbb{R} \\ & & \downarrow \\ X^{-1}(B) & \longleftarrow & B \end{array}$$

① $\begin{cases} \mathcal{B} \times \mathcal{B} \text{ is Borel.} \\ \mathcal{L} \times \mathcal{L} \text{ not } \mathcal{L}. \end{cases}$

② $X^{-1}(B) \in \mathcal{F}$
$X^{-1}(\mathcal{L})$ not ...

• The distribution ftn of X is : $F_X(s) = P_X([-\infty, s])$

Textbook : $\tilde{X}$ for $F_X(s)$

Lemma : Suppose $g : \mathbb{R} \to \mathbb{R}$ Borel mble ftn, s.t. either $g(x) \geq 0 \ \forall x$ or $g \in \mathcal{L}^1(\mathbb{R}, P_X)$.

$\Rightarrow \mathbb{E}(g(X)) = \int_\Omega g(X(\omega))\, dP(\omega) = \int_{\mathbb{R}} g(s)\, d\underset{\text{distribution}}{P_X(s)}$

① g being simple ftn
② MCT , DCT ⋯

Pf: ① Assume $g = 1_B$, $B \in \mathcal{B}$.

$\int_\Omega 1_B(X(\omega))\, dP(\omega)$

$= \int_\Omega 1_{X^{-1}(B)}(\omega)\, dP(\omega) = P(X^{-1}(B)) = \int_{\mathbb{R}} 1_B(s)\, dP_X(s) = P_X(B)$

② simple ftns ✓

③ for $g \geq 0$ case

$\exists$ Increasing $\Phi_1(x) \leq \cdots \leq g(x)$ & $\lim_{n\to\infty} \Phi_n(x) = g(x)$

$$\lim_{n\to\infty} \int_\Omega \Phi_n(X(\omega))\, P(\omega) \underset{\text{MCT}}{=} \int_\Omega g(X(\omega))\, P(\omega)$$

$\|$

$$\lim_{n\to\infty} \int_{\mathbb{R}} \Phi(s)\, dP_X(s) \underset{\text{MCT}}{=} \int_{\mathbb{R}} g(s)\, dP_X(s)$$

④ for $g \in \mathcal{L}^1(\mathbb{R}, P_X)$ case

$g = g^+ - g^-$ . . ⋯

Def: distribution ftn for X is $F_X : \mathbb{R} \to [0,1]$, given by $F_X(s) = P_X((-\infty, s]) = P(\{\omega \in \Omega : X(\omega) \leq s\})$

• if $F(s) = \underbrace{F_X(s)}_{\parallel \ P_X((-\infty, s])}$, $\overbrace{\wedge}^{\text{for some } X}$, then F satisfies
① increasing
② right cts   e.g. not rcts   is rcts
③ $\lim_{s\to-\infty} F(s) = 0$, $\lim_{s\to+\infty} F(s) = 1$

• if $F : \mathbb{R} \to [0,1]$ satisfies ①⋯④,

$\exists$ unique Borel measure μ on $(\mathbb{R}, \mathcal{B})$, s.t. $F(s) = \mu((-\infty, s])$

(Lebesgue-Stieltjes measures) $\mu([a,b]) = F(b) - F(a)$

Cantor ftn : increasing, cts, derivative is 0 a.e from 0 to 1

derivative $\neq 0$ on cantor sets $\downarrow$ measure 0

Monotone differentiable Thm

If $F: \mathbb{R} \to \mathbb{R}$ is an $\nearrow$ ftn, then F is differentiable leb a.e.

Moreover $\int_a^b F'(x)\,dx \leq F(b) - F(a)$

Pf: Let $G = \begin{cases} F(a) & x < a \\ F(x) & a \leq x \leq b \\ F(b) & x > b \end{cases}$

increasing

$$\int_a^b F'(x)\,dx = \int_a^b G'(x)\,dx = \int_a^b \lim_{n\to\infty} \frac{G(x+\frac{1}{n}) - G(x)}{\frac{1}{n}}\,dx$$

Fatou's Lemma

$$\leq \liminf_{n\to\infty} \int_a^b \frac{G(x+\frac{1}{n}) - G(x)}{\frac{1}{n}}\,dx$$

$$= \liminf_{n\to\infty} \left[n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} G(x)\,dx - n\int_a^b G(x)\,dx \right]$$

$$= \liminf_{n\to\infty} \left[n\underbrace{\int_b^{b+\frac{1}{n}} G(x)\,dx}_{=\frac{1}{n}\cdot F(b)} - n\underbrace{\int_a^{a+\frac{1}{n}} G(x)\,dx}_{\geq \frac{1}{n} F(a)} \right]$$

$$\leq \liminf_{n\to\infty} [F(b) - F(a)] = F(b) - F(a)$$

Lebesgue-Radon-Nykodym Thm

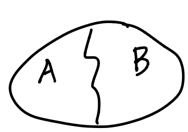
Borel measure $\mu = \overset{①}{\mu_{pp}} + \overset{②}{\mu_{sc}} + \overset{③}{\mu_{ac}}$

$\mu((-\infty, s]) = F(s)$

① pure point:
$\mu_{pp} = \sum_i c_i \delta_{x_i}$ at most countably many with $\sum |c_i| < \infty$ — dis R.V.

② abs cts:
$\mu_{ac}(B) = \int_B F'(x)\,dx$, $\forall B \in \mathcal{B}$. right cts, $\nearrow$ — cts R.V

③ singular cts:
$\mu_{sc}(\{a\}) = 0$, $\forall$ singleton set $\{a\}$. $\mathbb{R} = A \cup B$. $|A| = 0$. $\mu_{sc}(B) = 0$



$$F(b) - F(a) = \int_a^b F'(x)\,dx \quad \text{iff} \quad \underbrace{\mu_{pp}([a,b])}_{=0} + \underbrace{\mu_{sc}([a,b])}_{=0}$$

If F is abs cts means: (< uniformly cts)

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. for all finitely many disjoint open intervals $(a_1, b_1), \dots, (a_N, b_N)$ s.t. $\sum_{i=1}^{N} (b_i - a_i) < \delta$, we have $\sum_{i=1}^{N} |F(b_i) - F(a_i)| < \delta$

☆ If F is $\nearrow$ & AC, then μ_F is ac measure