① Ptwise and uniform convergence.

Def $f_1, \cdots : X \to \mathbb{R}$, $f : X \to \mathbb{R}$. $S \subset X$

1. $f_n \to f$ ptwise on S means

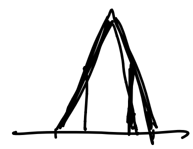
$\forall x \in S$, $\lim_{n\to\infty} f_n(x) = f(x)$ → depends on x and ε.

i.e. $\forall x \in S$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$ $|f_n(x) - f(x)| < \varepsilon$.

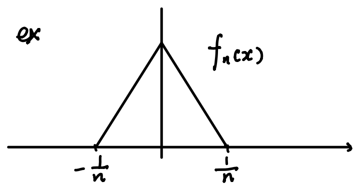
2. $f_n \to f$ uniformly on S means

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall x \in S, \forall n > N$ $|f_n(x) - f(x)| < \varepsilon$

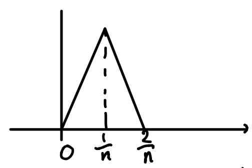
→ depends on ε.



ex



$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$ $\quad f_n \not\to f$ uniformly

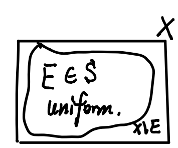


$g(x) = 0$ $\quad g_n \not\to g$ uniformly.

- $$S = \overset{\forall}{\bigcap_{k=1}^{\infty}} \overset{\exists}{\bigcup_{N=1}^{\infty}} \overset{\forall}{\bigcap_{n=N}^{\infty}} \left\{ x \in X : |f_n(x) - f(x)| < \frac{1}{k} \right\} \iff f_n \to f \text{ on } S \text{ pw.}$$

- $\exists N_1, N_2, N_3 \cdots \in \mathbb{N}$ s.t.
$$S = \underset{\forall}{\bigcap_{k=1}^{\infty}} \underset{\forall}{\bigcap_{n=N_k}^{\infty}} \left\{ x \in X : |f_n(x) - f(x)| < \frac{1}{k} \right\} \iff f_n \to f \text{ on } S \text{ uniformly.}$$

Thm (Ergorov Thm)



- Suppose (X, S, μ) a finite measure space (i.e. $\mu(X) < \infty$)

$f_1, f_2, \cdots : X \to \mathbb{R}$ mble, $f : X \to \mathbb{R}$ mble, Then

$f_n \to f$ pw on $X \Rightarrow \forall \varepsilon > 0$, $\exists E \in S$ w/ $\mu(E^c) < \varepsilon$. s.t. $f_n \to f$ uniformly on E

Pf.

$$\Rightarrow X = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \underbrace{\left\{ x \in X : |f_n(x) - f(x)| < \frac{1}{k} \right\}}$$ → denote as $B_{n,k}$.

$$\Rightarrow \phi = \bigcup_k \bigcap_N \bigcup_n B_{n,k}^{\,c} \Rightarrow \forall k. \ \bigcap_N \boxed{\bigcup_{n=N} B_{n,k}^{\,c}} = \phi$$

$\to E_N^k = \bigcup_{n=N}^{\infty} \left\{ x \in X : |f_n(x) - f(x)| \geq \frac{1}{k} \right\}$

$$\Rightarrow \forall k \quad E_1^k \supset E_2^k \supset \cdots \Rightarrow \mu(E_1^k) \leq \mu(X) < \infty$$?

$\Rightarrow \forall k \in \mathbb{N} \quad \lim_{N\to\infty} \mu(E_N^k) = \mu(\phi) = 0 \quad \Rightarrow \forall k \in \mathbb{N}, \ \exists N_k \in \mathbb{N}. \ \mu(E_{N_k}^k) < \frac{\varepsilon}{2^k}$

let $E = \bigcup_{k=1}^{\infty} E_{N_k}^k \quad \Rightarrow \mu(E) < \varepsilon$

$E^c = \bigcap_{k=1}^{\infty} (E_{N_k}^k)^c = \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k} \quad \Rightarrow f_n \to f$ unif on E^c

$\left(E_{N_k}^k = \bigcup_{n=N_k}^{\infty} B_{n,k}^c\right)$

② Approx by simple func

Def: $\varphi : X \to \mathbb{R}$ is called simple ftn if it takes only finitely many points

- If image of f is $\{c_1, c_2, \cdots c_N\}$. let $E_i = f^{-1}(\{c_i\})$

then $X = \bigcup_{i=1}^{N} f^{-1}(\{c_i\})$, and $\varphi = \sum_{i=1}^{N} c_i \cdot 1_{E_i}$

Important!

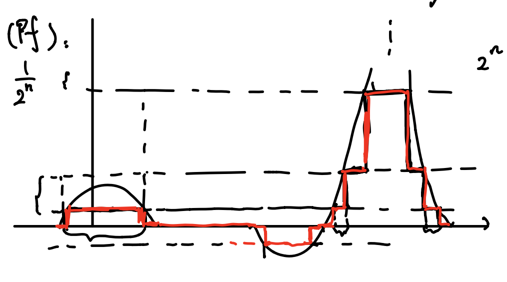
Thm. (X, S) mble space $f : X \to [-\infty, +\infty]$ S-mble ftn

$\Rightarrow \exists$ S-mble simple ftns $\varphi_1, \varphi_2, \cdots \ X \to \mathbb{R}$ • S-mble simple ftns : $\sum c_i \cdot 1_{E_i}$, $E_i \in S$.

s.t. $0 \le |\varphi_1(x)| \le \cdots \le |f(x)| \quad \forall x \in X$, $\lim_{n\to\infty} \varphi_n(x) = f(x) \quad \forall x \in X$

pw.

(Pf).



let $F_n = f^{-1}([2^n, \infty])$, $G_n = f^{-1}([-\infty, -2^n])$ $\quad$ S-mble.

S-mble

For $-2^{2n} \le k \le 2^{2n} - 1$

$E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)$

Define $\varphi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{E_{n,k}} + 2^n 1_{F_n} - 2^n 1_{G_n} + \sum_{k=-2^{2n}}^{-1} \frac{k+1}{2^n} 1_{E_{n,k}}$

$\Rightarrow$ • For $x \notin f^{-1}(\{-\infty, +\infty\})$,
$x \in X \setminus (F_n \cup G_n)$ for all large enough n $\Rightarrow \lim_{n\to\infty} |\varphi_n(x) - f(x)| = 0$

$\left(\text{since } |\varphi_n(x) - f(x)| < \frac{1}{2^n}\right)$

• For $x \in f^{-1}(\{\infty\})$

$x \in F_n$, $\forall n$, $\varphi_n(x) = 2^n \Rightarrow \lim_{n\to\infty} \varphi_n(x) = \infty = f(x)$

Cor.: Suppose $A \subset X$, s.t. $\exists\ L > 0$ w/ $\underbrace{|f(x)| \leq L}_{\text{bounded}}$, $\forall x \in A$,

$\Rightarrow \varphi_n \to f$ uniformly on A.