Def: V be a vector space. $\|\cdot\|$ be the norm,
Let $f_n \in V$, $n \in \mathbb{N}$. Let $f \in V$. $f_n \to f$ is defined as $\|f_n - f\| \to 0$

Def: Cauchy Seq $\forall \varepsilon > 0$, $\exists N$, $\forall m, n > N$, $\|f_m - f_n\| < \varepsilon$

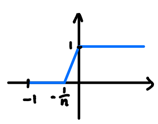
If every Cauchy seq convergent, we say the normed space is complete.

e.g. $\mathbb{R}$ is complete

- $\mathbb{R}^2$ $\sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} < \varepsilon$ $\Rightarrow \begin{cases} \|x_n - x_m\| < \varepsilon \\ - - - - \end{cases}$

$\Rightarrow x_n \to x, \ y_n \to y$

e.x. $C([-1,1])$ with $\|f\| = \sqrt{\int_{-1}^{1} f(x)\, dx}$ is not complete.



Limit not in the space!

(Riesz - Fischer Thm)
Thm: $L^2(X, S, \mu)$ is complete — Lebesgue Measurable fns

Def: Hilbert space = complete inner product space.

e.g. $\mathbb{R}^2$ is a HS
$L^2(X, S, \mu)$ is a HS. but $C([-1,1])$ w/ $\langle f, g\rangle = \int fg$ is not a HS
ℓ^2 is a HS

① Orthogonal Projection.

Dist (f, S)

Def: V normed vector space, S non emtpy subset of V, $f \in V$
defines $\text{dist}(f, S) = \inf\{\|f - g\| : g \in S\}$.

Thm 2.
$\boxed{\mathbb{R}^n \text{ 的特别形式}}$
因和 closed and bounded = compact

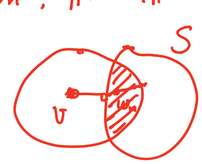
Thm

In $\mathbb{R}^2$, S closed set in $\mathbb{R}^2$. $v \in \mathbb{R}^2$
$\Rightarrow \exists w \in S$ s.t. $\text{dist}(v, S) = \|v - w\|$.

pf: Assume $v \notin S$.
$\exists$ seq, $w_n \in S$. $\|v - w_n\| \to d = \text{dist}(v, S)$

根据定义, $\exists\ w_n$, $\|v - w_n\| \to d$.

$\exists N$. s.t. $\forall n > N$, $\|v - w_n\| \le d + 1$
Let $K = S \cap \overline{B(v, d+1)}$ closed and bounded.
$\Rightarrow K$ is a compact set
$\Rightarrow$. $\exists$ subseq w_n' of w_n $\Rightarrow \lim\limits_{n\to\infty} w_n' = w \in S$
$d = \|v, w\| \le d$ #



$\boxed{\text{构造 } B(v, d+\varepsilon) \text{ 来获取一个 closed and bounded set.}}$

e.g. Not true for infinite dimension Useful construction

ℓ^2. $S = \{g_1, g_2, \cdots\}$. $g_n = (0, \cdots 0, 1+\frac{1}{n}, 0 \cdots)$

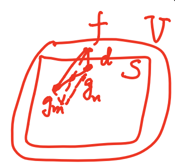
$\text{dist}(0, S) = 1$, S is closed. since every convergent seq converges in S
(there's no convergent seq $d(g_i, g_j) > \sqrt{2}$).

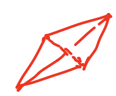
Def: A subset S of a vector space V is called convex iff
$f, g \in V$, $t \in [0,1] \Rightarrow t \cdot f + (1-t) \cdot g \in V$

ex. any linear subspace W of convex space V is convex.

Thm 2. General 形式.

Thm. V a Hilbert space
S closed convex subset of V. (S not empty).
$f \in V$. $\Rightarrow$ $\exists$ unique $\boxed{g} \in S$ s.t. $\text{dist}(f, S) = \|g - f\|$.





Pf. existence: ① if $f \in S$, take $g = f$.
② else. Assume $f \notin S$.

parallelogram law

$\exists g_n \in S$. s.t. $\lim_{n\to\infty} \|f - g_n\| = d = \text{dist}(f, S)$ key step

想证 Cauchy

$$\|g_n - g_m\|^2 = \|(g_n - f) - (g_m - f)\|^2 = 2\|g_n - f\|^2 + 2\|g_m - f\|^2 - \|g_n + g_m - 2f\|^2.$$

By convex: $\frac{1}{2} g_n + \frac{1}{2} g_m \in S$.

$$\Rightarrow \|g_n + g_m - 2f\|^2 = 4\left\|\frac{g_n + g_m}{2} - f\right\|^2 \geq 4d^2.$$

$$\Rightarrow \lim_{m,n\to\infty} \|g_n - g_m\|^2 = 2d^2 + 2d^2 - 4d^2 = 0$$

先是 $g \in V$, 再是 closed. $g_n \in S$

$\Rightarrow$ g_n is a Cauchy sequence. $\Rightarrow$ g_n is convergent $\Rightarrow \lim_{n\to\infty} g_n = g \in S$.

$$\Rightarrow \quad d \leq \|g - f\| \leq \|g - g_n\| + \|g_n - f\| \xrightarrow[n\to\infty]{\lim} d$$

Uniqueness.

Again, parallelogram law

Suppose $\exists g, h \in S$. s.t. $\|g - f\| = \|h - f\| = d$.

$$\|g - h\|^2 = 2\|g - f\|^2 + 2\|h - f\|^2 - \|g + h - 2f\|^2 \leq 0$$

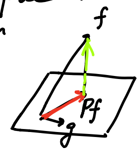
$\Rightarrow g = h$.

Def: V Hilbert space.
S closed convex subset of V (not empty)
The orthogonal projection of V onto S is the map $\boxed{P_S : V \to S,\ f \mapsto g}$, s.t.

$\to$ $P_S(f) = g$ where g is the unique vector in S . s.t. $\operatorname{dist}(f, S) = \|f-g\|$

Thm (Orthogonal Projection onto closed subspace)

W a closed subspace of Hilbert space V.
write $P_W = P$. let $f \in V$. Then
$\|f - P_S f\| \le \|f-h\| \quad \forall h \in S$

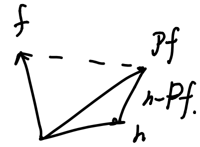


① $f - Pf \perp g \quad \forall g \in W$
② If $h \in W$, s.t. $f-h \perp g$. $\forall g \in W$.
then $h = Pf$
③ $P: V \to W$ is a linear map. & $\|Pf\| = \|f\| \Leftrightarrow f \in W$
④ $\|Pf\| \le \|f\|$

Pf: ① let $g \in W$. let α be any scalar   WTS $\langle f-Pf, g\rangle = 0$

$\|f-Pf\|^2 \le \|f-(Pf+\alpha g)\|^2 = \|(f-Pf)-\alpha g\|^2 = \|f-Pf\|^2 - 2\operatorname{Re}\{\bar\alpha \langle f-Pf, g\rangle\} + |\alpha|^2\|g\|^2$

$\Rightarrow$ $2\operatorname{Re}\,\bar\alpha\langle f-Pf, g\rangle \le |\alpha|^2\|g\|^2$. $\forall$ real scalar α
let $\alpha = t\langle f-Pf, g\rangle$. $t>0$



$\Rightarrow$ $2\,|\langle f-Pf, g\rangle|^2 \le t\,|\langle f-Pf, g\rangle|^2\|g\|^2 \quad \forall t>0$

$\Rightarrow$ $\langle f-Pf, g\rangle = 0$

② $\|h-Pf\|^2 = \langle h-Pf, h-Pf\rangle \overset{\text{by def}}{=} \langle h-f + f-Pf, h-Pf\rangle = 0$

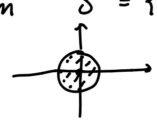
③ $\langle \alpha f - P(\alpha f), g\rangle = 0$ by ①
$\langle(\alpha f - \alpha Pf), g\rangle = \alpha\langle f-Pf, g\rangle \overset{\forall g}{=} 0$

By ② $\alpha Pf = P\alpha f$ Another, similar.

④ $\|f\|^2 = \|f-Pf+Pf\|^2 = \|f-Pf\|^2 + \|Pf\|^2 \ge \|Pf\|^2$
$\langle f-Pf, Pf\rangle = 0$

Def. S subset of inner product space V
$S^\perp = \{h \in V : \langle h, g\rangle = 0 \quad \forall g \in S\}$.

e.g.1 $V = \mathbb{R}^2$. $S = \{v : \|v\| < 1\}$. then $S^\perp = \{0\}$. $(S^\perp)^\perp \supseteq S$.



e.g.2. $V = L^2([-1,1])$. $S = \{g : g \text{ is constant fn}\}$.

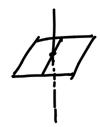
$\Rightarrow S^{\perp} = \{f \in L^2([-1,1]) : \int_{-1}^{1} f = 0\}$

Thm: S subset of inner product space V. then,

① $S^{\perp}$ is a closed subspace of V

② $S^{\perp} \cap S = \{0\}$.

③ $W \subset S \Rightarrow W^{\perp} \supset S^{\perp}$



④ closure $(\overline{S})^{\perp} = S^{\perp}$

⑤ $(S^{\perp})^{\perp} \supset S$

* closed set → convergence ∈ closed set.

Pf: ①. 1° $S^{\perp}$ is a subspace.

2° suppose $f_1, f_2, \cdots \in S^{\perp}$ and $f_n \to f \in V$ (means $\lim_{n\to\infty} \|f_n - f\| = 0$)

WTS $f \in S^{\perp}$

想让它为0

$|\langle f, g\rangle| = |\langle f - f_n, g\rangle + \underbrace{\langle f_n, g\rangle}_{0}| = |\langle f - f_n, g\rangle| \overset{☆ \leftarrow CS.}{\leq} \|f_n - f\| \cdot \|g\|$

$\Rightarrow \lim_{n\to\infty} |\langle f, g\rangle| = |\langle f, g\rangle| \leq 0 \cdot \|g\| = 0$

用$\|f_n - f\|$来逼近$\langle f, g\rangle$.

Useful Trick.

$\Rightarrow |\langle f, g\rangle| = 0 \Rightarrow f \in S^{\perp}$.

②. ③. ⑤ DIY.

④ we know $S \subset \overline{S}$. By ③ $\overline{S}^{\perp} \subset S^{\perp}$.

WTS: $\overline{S}^{\perp} \supset S^{\perp}$

$\Leftrightarrow \forall\ u \in S^{\perp}$, $g \in \overline{S}$. $\langle u, g\rangle = 0$

$\hookrightarrow \exists\ g_1, g_2, \cdots \in S$ s.t. $\lim_{n\to\infty} g_n = g$.

Similar Trick $\Rightarrow |\langle u, g\rangle| = |\langle u, g - g_n\rangle| \leq \|u\| \cdot \|g - g_n\|$

$\Rightarrow \langle u, g\rangle = 0$

用$\|g - g_n\|$逼近. $\langle u, g\rangle$到0

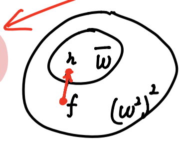
Thm: V Hilbert Space, W subspace of $V \Rightarrow (W^{\perp})^{\perp} = \overline{W}$

(若 W 已经是 closed 的, 那么 $(W^{\perp})^{\perp} = W$)

Pf: ① $(W^\perp)^\perp \supset W \Rightarrow \overline{(W^\perp)^\perp} \supset \overline{W}$ $\forall S,\ S^\perp$ closed $\Big\} \Rightarrow \overline{W} \subset (W^\perp)^\perp$.   '$\supset$'

② Suppose $f \in (W^\perp)^\perp$.

let $h = P_{\overline{W}}(f)$, WTS $h = f$ Trick!   '$\subset$'



• $\left.\begin{array}{l} f \in (W^\perp)^\perp \\ h \in \overline{W} \subset (W^\perp)^\perp \end{array}\right\} \Rightarrow f-h \in (W^\perp)^\perp$.

$f-h \perp \overline{W}$

$f-h \in (W^\perp)^\perp$.

$f-h \in (\overline{W})^\perp$
$\qquad\ \ \ \| $
$\qquad (W^\perp)$

• By def $f-h \perp g \quad \forall g \in \overline{W}$

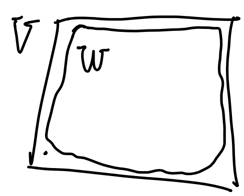
$\Rightarrow f-h \in (\overline{W})^\perp = W^\perp$.

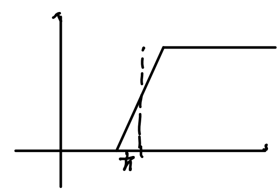
$\Rightarrow f-h \in W^\perp \cap (W^\perp)^\perp \Rightarrow f-h = 0 \Rightarrow f = h$.

| Cor: W subspace of Hilbert space of V. Then $\overline{W} = V \iff W^\perp = \{0\}$

Pf: (a) $\overline{W} = V \Rightarrow W^\perp = V^\perp = \{0\}$.

(b) $W^\perp = \{0\} \Rightarrow \overline{W} = (W^\perp)^\perp = V$ (By Thm). $(W^\perp)^\perp = \overline{W}$

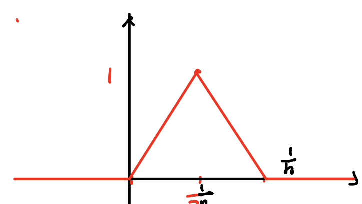




Not closed subspace.

(ex) $L^2([0,1]) \supset C([0,1])$.

$\left.\begin{array}{l} f_1, f_2, \cdots \in C([0,1]). \\ \lim_{n\to\infty} \int_0^1 |f - f_n|^2\, dx = 0 \end{array}\right\} \not\Rightarrow f \in C([0,1])$



※ $\overline{C([0,1])} = L^2([0,1])$, since we can approx $\forall f$ by cts ftn.