$\boxed{2A}$ Outer Measures on $\mathbb{R}$

$[0,1] \cap \mathbb{Q}$, $[0,1] \setminus \mathbb{Q}$. $A = \bigcup_{i=1}^{\infty} (r_i - \frac{1}{2^i}, r_i + \frac{1}{2^i})$

$A = \{x \in [0,1] : \text{a decimal expansion of } x \text{ contains a string of '333'}\}$

↳ Probability.

Def. For an open interval I, the length is defined as

$$\ell(I) = \begin{cases} b-a & I = (a,b) \\ \infty & I = (-\infty, b),\ (a, +\infty)\ (-\infty, +\infty) \\ 0 & I = \phi \end{cases}$$

Def. The outer measure of $A \subseteq \mathbb{R}$ is

$$|A| = \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; A \subset \bigcup_{i=1}^{\infty} I_k,\ I_k \text{ open interval} \right\}$$

1. finite sets have outer measure 0

(ex.) 1. $A = \{a_1, \cdots, a_N\} \subseteq \mathbb{R}$. $|A| = 0$. — $\boxed{|A| < \varepsilon}$ ↑ for finite.

$I_k = (a_k - \frac{\varepsilon}{2N}, a_k + \frac{\varepsilon}{2N})$.

$\Rightarrow A \subseteq \bigcup_{k=1}^{N} I_k \quad \Rightarrow \quad |A| \le \sum_{k=1}^{N} \ell(I_k) = N \cdot \frac{\varepsilon}{N} = \varepsilon$.

2. countably infinite sets also have outer measure 0

2. $A = \{a_1, a_2, \cdots\} \subseteq \mathbb{R}$ countable set — $\boxed{\frac{\varepsilon}{2^k} \text{ trick}}$

let $I_k = (a_k - \frac{\varepsilon}{2^k}, a_k + \frac{\varepsilon}{2^k})$

$\Rightarrow |A| \le \sum_{k=1}^{\infty} \ell(J_k) = 2\varepsilon \Rightarrow |A| = 0$

↑ Probably useful for countable sets!

$\Rightarrow |\mathbb{Q}| = 0$ $\qquad$ $|\mathbb{Q}| = 0$ 因为它可数!

- Good Properties of Outer Measure.

Lemma: ① $A \subseteq B \Rightarrow |A| \le |B|$

② $|t+A| = |A| \quad \forall t \in \mathbb{R},\ \forall A \subseteq \mathbb{R}.$

(translate of A by t)

① $B \subseteq \bigcup_{k=1}^{\infty} I_k \Rightarrow A \subseteq \bigcup_{k=1}^{\infty} I_k.$

$$\Rightarrow |A| \le \inf \left| \bigcup_{k=1}^{\infty} I_k \right| = |B|$$

② $\ell(I) = \ell(I+A)$

(Tao, Thm 0.0.2)

Recall: Tonelli's Thm for sequences. a_{ij} must be positive.

If $a_{ij} \in [0,\infty)$. $\forall i,j \in \mathbb{N}$, then.

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

Thm. countable subadditivity of outer measure.

$$\left| \bigcup_{k=1}^{\infty} A_k \right| \le \sum_{k=1}^{\infty} |A_k|$$

even if A_k are mutually disjoint, we can still only have '$\le$' rather than '=' !!

'room of ε'

WTS. $\left| \bigcup_{k=1}^{\infty} A_k \right| \le \sum_{k=1}^{\infty} |A_k| + \varepsilon$

Pf: [1] if $|A_i| = \infty$. rhs $= \infty$. nothing to prove

[2] assume $|A_i| < \infty$. $\forall i \in \mathbb{N}$

fix ε, $\forall k \in \mathbb{N}$, there're open intervals — room of ε, + $\frac{\varepsilon}{2^k}$ trick.

$I_{1,k}, I_{2,k}, \dots$ s.t. $A_k \subseteq \bigcup_{j=1}^{\infty} I_{j,k}$.

s.t. $|A_k| + \frac{\varepsilon}{2^k} \ge \sum_{j=1}^{\infty} \ell(I_{j,k})$ (since $|A_k|$ is the inf, $\inf + \varepsilon \ge \dots$)

For LHS, $\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} I_{j,k} = \bigcup_{(j,k)\in\mathbb{N}} I_{j,k}$.

$$\Rightarrow \left|\bigcup_{k=1}^{\infty} A_k\right| \le \sum_{(j,k)\in\mathbb{N}} \ell(I_{j,k}) \overset{\text{Tornelli}}{=\!=} \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \ell(I_{j,k}) \le \sum_{k=1}^{\infty} |A_k| + \varepsilon .$$

$\uparrow$ use $\frac{\varepsilon}{2^k}$.

③ Outer measure of intervals. if we have

Thm $|[a,b]| = b-a = \ell([a,b])$

Pf: $|[a,b]| \le b-a+\varepsilon$ here. we just need a particular

$\ge b-a-\varepsilon$ here we need a general one.

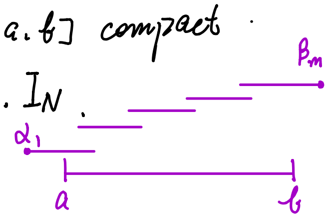
① Fix $\varepsilon>0$. let $I_1 = [a-\varepsilon, b+\varepsilon]$, $I_k = \phi$ for $k\ge 2$.

then $[a,b] \subseteq \bigcup I_k \Rightarrow |[a,b]| \le \sum \ell(I_k) = \underbrace{b-a+2\varepsilon}$

$\Rightarrow |[a,b]| \le b-a$. ✓ 构造一组 open interval s.t $\sum \ell = b-a+\varepsilon$

② $\forall I_k$ s.t. $[a,b] \subseteq \bigcup_{k=1}^{\infty} I_k$, since $[a,b]$ compact.

$\Rightarrow \exists$ finite open subcover, calling them $I_1, I_2, \cdots, I_N$.



Discard I_k s.t. ① I_k does not intersect $[a,b]$.

② $\exists I_j$ s.t. $I_k \subseteq I_j$

Relabel I_k, $I_k = [\alpha_k, \beta_k]$. $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ $\swarrow < N$

also. $\begin{cases} \alpha_k < \beta_{k-1} \\ \alpha_1 < a \\ \beta_m > b \end{cases}$

$$\Rightarrow \sum_{k=1}^{m} I_k = \beta_1 - \alpha_1 + \beta_2 - \alpha_2 + \cdots + \beta_m - \alpha_m$$

$$= \beta_m - \alpha_1 + (\beta_1 - \alpha_2) + (\beta_2 - \alpha_3) + \cdots > \beta_m - \alpha_1 > b-a.$$

$\Rightarrow |[a,b]| \ge b-a$ with $\boxed{1}$

$\Rightarrow |[a,b]| = b-a$

Cor. $|(a,b)| = |(a,b]| = |[a,b)| = b-a$

① $|(a,b)| \le |[a,b]| = b-a$

② $[a+\frac{\varepsilon}{2}, b-\frac{\varepsilon}{2}] \subseteq (a,b)$

By Thm & monotonicity

$$\Rightarrow |[a+\tfrac{\varepsilon}{2}, b-\tfrac{\varepsilon}{2}]| = b-a-\varepsilon \le |(a,b)| \le b-a.$$

Vitali Set

③ **Outer Measure is not necessarily additive.**

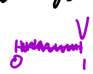
Lemma. $\forall\, x \in [0,1]$.

$$B_x = \{ t \in [0,1] \mid t-x \in \mathbb{Q} \} \qquad (x+\mathbb{Q} \cap [0,1])$$

then B_x, B_y are either disjoint or equal. —— $\boxed{\text{equ class}}$

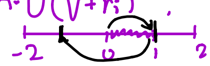
→ pick one element from each class, and form a set V[1] 'Vitali Set'

different.



$\subseteq [0,1]$

$A = \bigcup (V + r_i)$.



$r_i : \mathbb{Q} \cap [-2,2]$

Lemma: let $[-2,2] \cap \mathbb{Q} = \{r_1, r_2, \dots\}$.

then (a) $(-1,1) \subset \bigcup_i (r_i + V) \subset (-3,3)$

(b) $r_i + V$ are all disjoint

$$\Rightarrow 2 = |[-1,1]| \underset{\text{monotone}}{\le} \Big|\bigcup_i (r_i+V)\Big| \underset{\substack{\text{subadditive} \\ \text{countable}}}{\le} \sum_i |r_i+V| \underset{\text{translation of vertical}}{=} \Sigma|V|$$

Then, we have Thm: (a) $|V| > 0$

(b) There's $n \in \mathbb{N}$ s.t. $\left|\bigcup_{i=1}^{n} (r_i+V)\right| < \sum_{i=1}^{n} |r_i+V|$

Pf: (a) $2 = |[-1,1]| \le \left|\bigcup_i (r_i+V)\right| \le \sum_i |r_i+V| = \sum_{i=1}^{\infty} |V| \Rightarrow |V| > 0$

monotonicity — countable sub-additivity — translation

(b) pick $n > \frac{6}{|V|}$.

$\left|\bigcup_{i=1}^{n} (r_i+V)\right| \leq \sum_{i=1}^{n} |r_i+V|$

$$\left|\bigcup_{i=1}^{n} (r_i+V)\right| \leq \left|\bigcup_{i=1}^{\infty} (r_i+V)\right| \leq |(-3,3)| = 6$$

$$\sum_{i=1}^{n} |r_i+V| = \sum_{i=1}^{n} |V| > 6$$

Corollary #: There are disjoint subsets A and B of $\mathbb{R}$ s.t. $|A \cup B| < |A| + |B|$

By contradiction, Suppose $|E \cup F| = |E| + |F|$ for $\forall$ E, F disjoint

which means $\left|\bigcup_{i=1}^{n} E_i\right| = \sum_{i=1}^{n} |E_i|$.

let $E_i = r_i + V$, will give a contradiction.

$\Rightarrow \exists\ |A \cup B| < |A| + |B|$