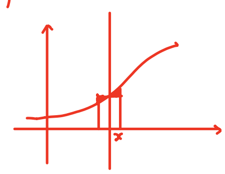
4B

Thm (Lebesgue differentiation thm — 1^{st} version) ← key



If $f \in \mathcal{L}'(\mathbb{R})$, then $\lim_{t\to 0} \frac{1}{2t}\int_{x-t}^{x+t} |f(y)-f(x)|\,dy = 0$ for a.e x

Thm (LDT - 2^{nd}) Let $f \in \mathcal{L}'(\mathbb{R})$

Define $F(x) = \int_{-\infty}^{x} f(y)\,dy$. Then $F'(x) = f(x)$ for a.e. x.

$$\left|\frac{F(x+t)-F(x)}{t} - f(x)\right| = \left|\frac{1}{t}\int_x^{x+t} f(y)\,dy - f(x)\right| = \left|\frac{1}{t}\int_x^{x+t} (f(y)-f(x))\,dy\right|$$

$$\le \frac{1}{t}\int_x^{x+t} |f(y)-f(x)|\,dy \le 2\cdot\frac{1}{2t}\int_{x-t}^{x+t} |f(y)-f(x)|\,dy \longrightarrow 0 \quad \text{as } t\to 0$$

Pf for 1^{st} version:

① If f is cts, the result holds for every x ← check. ✓

$\frac{1}{2t}\int_{b-t}^{b+t} |f-f(b)| \le \sup\{|f(x)-f(b)| : x\in B_t(b)\}$

when f is cts at b ⇒ RHS = 0

② Assume $f\in\mathcal{L}'(\mathbb{R})$

Let $Q(x) = \limsup_{t\to 0} \frac{1}{2t}\int_{x-t}^{x+t} |f(y)-f(x)|\,dy$ 的最大值在$t\to 0$的值.

WTS. $|\{x : Q(x) > 0\}| = 0$.

Note: $\{x : Q(x)>0\} = \bigcup_{n=1}^{\infty} \{x : Q(x) > \frac{1}{n}\}$.

$\limsup_{t\to 0} = \lim_{\varepsilon\to 0}(\sup\{f(x) : x\in E \cap B(\theta,\varepsilon) \setminus \{a\}\})$

⇒ Enough to show: $|\{x : Q(x) > \alpha\}| = 0$ $\forall \alpha > 0$

• Fix $\alpha>0$, $\varepsilon>0$.

$\exists\, g \in C_c(\mathbb{R})$ s.t. $\int_{\mathbb{R}} |\underbrace{f(x)-g(x)}_{|h|}|\,dx < \varepsilon$ 用连续函数逼近!

$$\frac{1}{2t}\int_{x-t}^{x+t} |f(y)-f(x)| \le \underbrace{\frac{1}{2t}\int_{x-t}^{x+t} |\overbrace{f(y)-g(y)}^{h}|\,dy}_{h^*(x)} + \frac{1}{2t}\int_{x-t}^{x+t} |\overbrace{g(y)-g(x)}^{\text{cts}}|\,dy + \frac{1}{2t}\int_{x-t}^{x+t} |g(x)-f(x)|\,dy$$

$$Q(x) \le h^*(x) + 0 + |g(x)-f(x)|$$

$$\{x : Q(x) > \alpha\} \subseteq \{x : h^*(x) > \tfrac{\alpha}{2}\} \cup \{x : |g(x)-f(x)| > \tfrac{\alpha}{2}\} \quad ✓$$

$$\Rightarrow |\{x : Q(x)>\alpha\}| \le |\{x : h^*(x) > \tfrac{\alpha}{2}\}| + |\{x : |g(x)-f(x)| > \tfrac{\alpha}{2}\}|.$$

✓ Markov Inequality

$$\le \frac{6}{\alpha}\cdot\int_{\mathbb{R}} |h(x)| + \frac{2}{\alpha}\cdot\int_{\mathbb{R}} |g(x)-f(x)|$$

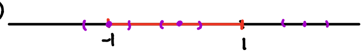
$$= \frac{8}{\alpha}\cdot\int_{\mathbb{R}} |h(x)| < \frac{8}{\alpha}\cdot\varepsilon \quad \text{which is arbitrarily small.}$$

Corollary: If $f\in\mathcal{L}'(\mathbb{R})$, $\lim_{t\to 0}\frac{1}{2t}\int_{x-t}^{x+t} f(y)\,dy = f(x)$ a.e. x

$$\left|\frac{1}{2t}\int_{x-t}^{x+t}(f(y)-f(x))\,dy\right| \le \frac{1}{2t}\int_{x-t}^{x+t}|f(y)-f(x)|\,dy \longrightarrow 0 \quad \text{a.e. } x$$

Def Let $E \subset \mathbb{R}$ Leb mble set. the density of E at x is

$$D_E(x) = \lim_{t \to 0} \frac{|E \cap (x-t, x+t)|}{2t} \text{ if exists}$$

e.g. ①  $E = [-1, 1]$

$$D_E(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \\ \frac{1}{2} & x = \pm 1 \end{cases}$$

② $E = \bigcup_{n=2}^{\infty} \left(\frac{1}{n}, \frac{1}{n} + \frac{1}{3n(n-1)} \right)$

$D_E(0) = \frac{1}{3} \times \frac{1}{2}$

Thm: (Lebesgue density Thm)

$E \subset \mathbb{R}$, Leb. mble set.

① $D_E(x) = 1$ for a.e. x in E

② $D_E(x) = 0$ for a.e. x in E^c

$\boxed{\text{Ex,}}$ → used in Midterm 6.

Proof First suppose $|E| < \infty$. Thus $\chi_E \in \mathcal{L}^1(\mathbf{R})$. Because

$$\frac{|E \cap (b-t, b+t)|}{2t} = \boxed{\frac{1}{2t} \int_{b-t}^{b+t} \chi_E} = 1_E(b)$$

for every $t > 0$ and every $b \in \mathbf{R}$, the desired result follows immediately from 4.21.

Now consider the case where $|E| = \infty$ [which means that $\chi_E \notin \mathcal{L}^1(\mathbf{R})$ and hence 4.21 as stated cannot be used]. For $k \in \mathbf{Z}^+$, let $E_k = E \cap (-k, k)$. If $|b| < k$, then the density of E at b equals the density of E_k at b. By the previous paragraph as applied to E_k, there are sets $F_k \subset E_k$ and $G_k \subset \mathbf{R} \setminus E_k$ such that $|F_k| = |G_k| = 0$ and the density of E_k equals 1 at every element of $E_k \setminus F_k$ and the density of E_k equals 0 at every element of $(\mathbf{R} \setminus E_k) \setminus G_k$.

Let $F = \bigcup_{k=1}^{\infty} F_k$ and $G = \bigcup_{k=1}^{\infty} G_k$. Then $|F| = |G| = 0$ and the density of E is 1 at every element of $E \setminus F$ and is 0 at every element of $(\mathbf{R} \setminus E) \setminus G$. ■

bound E to E_k and apply 4.21 (the Corollary)