$\boxed{2B}$ Measurable spaces and functions.

→ power set $\mathcal{P}(X)$ of X is collection of all subsets of X

- consider $\mathbb{N}$: if $\phi : \mathcal{P}(\mathbb{N}) \longrightarrow [0,1)$ exists

$\Rightarrow$ $\mathbb{N}$ is countable. $[0,1)$ is not $\Rightarrow$ $\mathcal{P}(\mathbb{N})$ is not countable.

where ϕ is easy to construct: $\phi(A) = (0.a_1a_2\cdots)_2$, $A \in \mathcal{P}(\mathbb{N})$

$$a_i = \begin{cases} 1 & i \text{ in } A \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow$ ϕ is surjective $\Rightarrow$ $\underbrace{\#\mathcal{P}(\mathbb{N})}_{\text{uncountable!}} \geq \#([0,1))$.

$\mathbb{N}$: countable.

Outer Measure is a function $\mu : \mathcal{P}(\mathbb{R}) \longrightarrow [0,\infty]$

B which is smaller

w/ property $\begin{cases} \text{monotone} \\ \text{countable subadditive.} \xrightarrow{?} \text{finite additive} \\ \mu([a,b]) = b-a \end{cases}$

$\boxed{1}$ σ – algebras.

Def: let X be a set. Let S be a collection of subsets of X.

We say S is a σ-algebra on X if

① $\phi \in S$

② If $E \in S$, then $E^c \in S$ complements

③ If $E_1, E_2, \cdots \in S$ (repetition allowed), then $\bigcup_{k=1}^{\infty} E_k \in S$

countable union

Example:

(a) $\mathcal{S} = \mathcal{P}(X)$ power σ-algebra.

(b) $\mathcal{S} = \{\phi, X\}$ trivial σ-algebra.

lemma: Let $\mathcal{S}$ be a σ-algebra on X, then

① $X \in \mathcal{S}$

② $D, E \in \mathcal{S}$. $D^c, E^c \in \mathcal{S} \Rightarrow D \cap E = (D^c \cup E^c)^c \in \mathcal{S}$

$D \setminus E = D \cap E^c \in \mathcal{S}$

$\bigcap_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} E_n^c\right)^c$

Terminology:

Def: A pair $(X, \mathcal{A})$ is called a measurable space ($\mathcal{A} \to$ σ-alg on X)

A set $E \in \mathcal{A}$ is called a $\mathcal{A}$-measurelle set

[2] Borel subsets of $\mathbb{R}$

lemma: (a) If $\mathcal{S}_1, \mathcal{S}_2$ are 2 σ-algebras on X, then $\mathcal{S}_1 \cap \mathcal{S}_2$ is an σ-algebra on X.

can be proved easily

can be extended to uncountably infinite.

(b) If $\mathcal{S}_\alpha$ is a σ-algebra on X, $\bigcap_\alpha \mathcal{S}_\alpha$ is also a σ-algebra on X

(ex.) let $A \subset X$, $A \neq \phi, X$. let $\mathcal{A} = \{A, X\}$.

then $\mathcal{A} \subseteq \begin{cases} \mathcal{P}(X) \\ \{\phi, A, A^c, X\} \ * \to \text{smallest} \\ \vdots \ \text{tons of} \end{cases}$

lemma. let X be a set. $\mathcal{A}$ be a collection of subsets of X.

Let $\langle\mathcal{A}\rangle$ be the intersections of all σ-algebras containing $\mathcal{A}$.

then $\langle\mathcal{A}\rangle$ is the smallest σ-algebra containing $\mathcal{A}$

The important part is that: we know there's such a thing even though we don't know how to construct it

Def: The Borel σ-algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$ is the smallest σ-algebra on $\mathbb{R}$ that contains all open subsets of $\mathbb{R}$. An element in $\mathcal{B}(\mathbb{R})$ is called a Borel subset of $\mathbb{R}$

(ex)
$$\left.\begin{array}{r} (-\infty, b) \\ (a, +\infty) \\ (a, b) \\ (-\infty, b] \\ [a, +\infty) \\ [a, b] \\ (a, b], [a, b) \\ \{a\} \\ \mathbb{Q} \\ \mathbb{R} \setminus \mathbb{Q} \end{array}\right| \in \mathcal{B}(\mathbb{R})$$

$\{r_1, r_2, \cdots\} = \mathbb{Q}$.

$$E = \bigcup_{k=1}^{\infty} \left(r_k - \frac{1}{10^k}, r_k + \frac{1}{10^k}\right) \in \mathcal{B}(\mathbb{R})$$

$E = \{x \in [0,1] \mid$ decimal expansion has inf 3's$\}$

Vitali not

lemma: $\mathcal{B}(\mathbb{R})$ is also the smallest σ-algebra containing all open intervals (a, b), $a < b$.

Pf: Let S be the smallest ... open intervals ..

WTS: $\mathcal{B}(\mathbb{R}) = S$

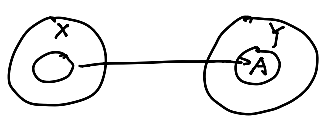
- $\mathcal{B}(\mathbb{R})$ is a σ-algebra containing all open intervals. But, S is the smallest such σ-algebra $\Rightarrow \mathcal{B}(\mathbb{R}) \supset S$
  - Fact: $\forall$ open sets in $\mathbb{R}$ is at most countable union of open intervals ?
- Since S is a σ-algebra and contains all open intervals, every open set (which is at most countable union of open intervals)

3 Inverse Images

Def: $f: X \to Y$, $A \subset Y$. then $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

(ex) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

$f^{-1}([4, +\infty)) = (-\infty, -2] \cup [2, +\infty)$



→ **lemma:** (a) $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.

i.e. $f^{-1}(A^c) = (f^{-1}(A))^c$.

(b) $f^{-1}(\bigcap_\alpha A_\alpha) = \bigcap_\alpha f^{-1}(A_\alpha)$

(c) $f^{-1}(\bigcup_\alpha A_\alpha) = \bigcup_\alpha f^{-1}(A_\alpha)$

Pf: • $\forall\, x \in f^{-1}(\bigcap_\alpha A_\alpha)$. $f(x) \in \bigcap_\alpha A_\alpha$

$\Rightarrow f(x) \in A_\alpha \Rightarrow \alpha \in f^{-1}(A_\alpha) \Rightarrow \alpha \in \bigcap_\alpha f^{-1}(A_\alpha) \Rightarrow$ '$\supseteq$'

• $\forall\, y \in f^{-1}(\bigcap_\alpha A_\alpha)$ $\quad f(y) \in \bigcap_\alpha A_\alpha \Rightarrow f(y) \in A_\alpha \Rightarrow y \in f^{-1}(A_\alpha)$

$\Rightarrow y \in \bigcap_\alpha f^{-1}(A_\alpha) \Rightarrow$ '$\subseteq$' $\quad\Rightarrow$ '$=$'

$\boxed{\chi_E}$

$\boxed{4}$ Measurable Functions.

Def: $E \subset X$

≠ the one in Probability Theory

The characteristic function on E (indicator function) is $\chi_E : X \to \mathbb{R}$

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Def: Let $(X, \mathcal{S})$ be a measurable space.

A function $f: X \to \mathbb{R}$ is called a $\mathcal{S}$-measurable function, if $\forall B \in \mathcal{B}(\mathbb{R})$. $f^{-1}(B) \in \mathcal{S}$

(Ex.) $f(x) = x$

分段常函数是 $\mathcal{S}_3$-measurable.

$\mathcal{S}_1 = \mathcal{P}(\mathbb{R})$. ✓ $\mathcal{S}_2 = \mathcal{B}(\mathbb{R})$. ✓ $\mathcal{S}_3 = \{\phi, (1, +\infty), (-\infty, -1], \mathbb{R}\}$. × $\mathcal{S}_4 = \{\phi, \mathbb{R}\}$. ×

$\Rightarrow$ Every function is $\mathcal{S}_1$ measurable. only constant functions are $\mathcal{S}_4$ measurable.

Prop: Let $(X, \mathcal{S})$ be a measurable space.

A function $f: X \to \mathbb{R}$ is a $\mathcal{S}$-measurable function, if $\forall a \in \mathbb{R}$ $f^{-1}((a, \infty)) \in \mathcal{S}$, then f is $\mathcal{S}$-mble.

$\Rightarrow f^{-1}(B) \in \mathcal{S}$

证明思路: $f^{-1}((a, \infty)) \in \mathcal{S}$

要证 $\forall B \in \mathcal{B}(\mathbb{R})$, $f^{-1}(B) \subseteq \mathcal{S}$

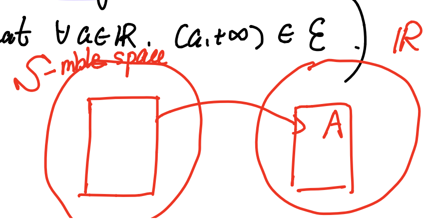
Coro: Every cts func from $\mathbb{R} \longrightarrow \mathbb{R}$ is $\mathcal{B}(\mathbb{R})$-mble $\Rightarrow$ 即所有使 $f^{-1}(A) \in \mathcal{S}$ 的A集合是 $\mathcal{B}\mathcal{S}$.

$\Rightarrow f^{-1}((a, +\infty))$ is open $\Rightarrow f^{-1}((a, +\infty)) \in \mathcal{B}(\mathbb{R})$ $\Leftrightarrow \{A\} \supseteq \mathcal{B}(\mathbb{R})$

Pf of Prop:

This is the key

Let $\mathcal{E} = \{A \subset \mathbb{R} : f^{-1}(A) \in \mathcal{S}\}$. (We know that $\forall a \in \mathbb{R}$, $(a, +\infty) \in \mathcal{E}$)



WTS: $\mathcal{E} \supset \mathcal{B}(\mathbb{R})$

① $\mathcal{E}$ is a σ-alg $\begin{cases} \phi \in \mathcal{E}. \\ A \in \mathcal{E} \quad A^c \in \mathcal{E} \end{cases}$

$A_i \in \mathcal{E}, \ \bigcup A_i \in \mathcal{E}$

② $\mathcal{E}$ contains all open intervals. ...

Assume $X \in \mathcal{B}(\mathbb{R})$

Def. $f: X \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a Borel-measurable ftn if $\forall B \in \mathcal{B}(\mathbb{R}),\ f^{-1}(B) \in \mathcal{B}(\mathbb{R})$

cts : B-mble

Lemma: If $f: X \subset \mathbb{R} \to \mathbb{R}$ is a $\boxed{\text{cts ftn}}$, f is a Borel-Mble ftn.

- Pf is easy and mentioned above.

Example: $\frac{1}{x} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is a Borel-mble ftn.

Composition : $f^2, -f, |f| \cdots$ are $\mathcal{S}$-mble

Lemma. $(X, \mathcal{S})$ mble space

cts are Bm

$X \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$. f is $\mathcal{S}$-mble ; g Borel-mble

$\Rightarrow$ $g \circ f$ is an $\mathcal{S}$-mble ftn.

$\boxed{\text{Borel} \circ \mathcal{S}}$

$\boxed{\mathcal{S}}$

Pf: $(g \circ f)^{-1}(B) = f^{-1} \circ g^{-1}(B) = f^{-1}(B') \in \mathcal{S}$.

Example: if f is $\mathcal{S}$-mble ftn.

① f^2: $g = x^2$. cts $\Rightarrow$ Borel-mble $\Rightarrow$ $g \circ f = f^2$ is $\mathcal{S}$-mble.

② $-f$ --

③ $|f|$ $\quad |x|$

$+, -, \times, \div$ 保持 $\mathcal{S}$-mble.

$\boxed{\textbf{Prop}}$: $(X, \mathcal{S})$ mble space. $f, g: X \to \mathcal{S}$-mble ftns.

$\Rightarrow \boxed{f+g,\ f-g,\ fg,\ \frac{f}{g}}$ are $\mathcal{S}$-mble ftns. $(g \neq 0)$

$\boxed{f+g}$ is $\mathcal{S}$-mble.

① $(f+g)^{-1}((a, +\infty)) \in \mathcal{S}$?

countable is important

Here's a claim: $(f+g)^{-1}((a, +\infty)) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}(r, +\infty) \cap g^{-1}(a-r, +\infty) \right)$

'$\supseteq$' $\quad x \in RHS \Rightarrow \exists\, r \in \mathbb{Q}$, s.t. $f(x) > r$, $g(x) > a - r$.
$\Rightarrow (f+g)(x) > a$

Switch it will be better.

'$\subseteq$' $\quad x \in LHS \Rightarrow (f+g)(x) > a \Rightarrow g(x) > a - f(x) \Rightarrow \exists\, r \in \mathbb{Q}$, s.t. $r \in (a - f(x), g(x))$

$\Rightarrow \quad g(x) > r, \quad r > a - f(x) \quad \Rightarrow \begin{cases} f(x) > a - r. \\ g(x) > r \end{cases}$ $\boxtimes$

② $f \cdot g\,(x) = \dfrac{(f+g)^2 - f^2 - g^2}{2}$ is $\mathcal{S}$-mble ftn.

③ $f/g\,(x) = f(x)/g(x)$

Prop: $(X, \mathcal{S})$ mble-space. $f_1, f_2, \cdots : X \mapsto \mathbb{R}$ $\mathcal{S}$-mble ftns

Suppose $\lim\limits_{n\to\infty} f_n(x)$ converges, $\forall x \in X$

$\Rightarrow f(x) = \lim\limits_{n\to\infty} f_n(x)$ is $\mathcal{S}$-mble.

$\mathcal{S}$-mble 的极限也是 $\mathcal{S}$-mble.

(pf). $f^{-1}((a, +\infty)) = \bigcup\limits_{j=1}^{\infty} \bigcup\limits_{m=1}^{\infty} \bigcap\limits_{k=m}^{\infty} f_k^{-1}((a+\frac{1}{j}, +\infty))$

$\forall x \in RHS$:

$\exists j$. $\exists m$. $\forall n > m$. $f_n(x) > a + \frac{1}{j}$

$\Rightarrow \exists j$. $f(x) \geq a + \frac{1}{j}$. $\Rightarrow f(x) > a \Rightarrow x \in LHS$

$\forall x \in LHS$:

$f(x) > a \Rightarrow \exists j > 0$ s.t. $f(x) > a + \frac{1}{j} \Rightarrow \exists m$. s.t. $\forall n > m$. $f_n(x) > a + \frac{1}{j}$

$\Rightarrow x \in RHS$

Read P37 ?