MATH 526: Discrete State Stochastic Processes
Lecture 22-23

Chapter 5: Martingales

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Choosing the proper martingale, we can estimate certain probabilities, even if they cannot be computed in closed form.

Example (Probability of ruin). Assume that the company's value on day n is given by

$$S_n = S_{n-1} + X_n, \quad S_0 > 0,$$

where $\{X_n\}$ are i.i.d. normal random variables, with mean $\mu > 0$ and variance $\sigma^2 > 0$: $X_n \sim N(\mu, \sigma^2)$. Clearly, we have

$$S_n = S_0 + \sum_{i=1}^{n} X_i, \quad S_0 > 0.$$

The company goes bankrupt when its value drops below zero. Denote by τ the first time when the process (S_n) drops below zero: $\tau = \min\{n \geq 0 \,:\, S_n \leq 0\}$.

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$e^{-\frac{2\mu}{\sigma^2} S_n}$

Question

How can we estimate the probability that the company will ever go bankrupt: $\mathbb{P}(\tau < \infty) \leq$?

First, we set up the proper martingale: $M_n = \exp(-2\mu S_n/\sigma^2)$. To see that (M_n) is, indeed, a martingale, we check the "main" property. (At home, verify the finite expectation of the absolute value).

- adaptness $\{X_i\}$
- absolute finite Expectation

$e^{-\frac{2\mu}{\sigma^2}}$

- $$\begin{aligned}\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n^X) &= M_n \mathbb{E}(\exp(-2\mu X_{n+1}/\sigma^2)) \\ &= M_n \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} e^{-2\mu y/\sigma^2 - (y-\mu)^2/(2\sigma^2)}\, dy \\ &= M_n \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(\frac{y+\mu}{\sigma})^2}\, dy \\ &= M_n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}y^2}\, dy \\ &= M_n.\end{aligned}$$

X_{n+1}

$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

↓ 换元

$\mathbb{E} e^x = \int f e^x$

$\mathbb{E}[\exp(-cX_i)]^n$

$\mathbb{E}\, e^{(-c)X}$.

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The Optional Sampling theorem implies:

$$\exp(-2\mu S_0/\sigma^2) = \mathbb{E}M_0 = \mathbb{E}M_{n\wedge\tau} = \mathbb{E}\exp(-2\mu S_{n\wedge\tau}/\sigma^2).$$

When $\tau \le n$, we have $0 \le -2\mu S_\tau = -2\mu S_{n\wedge\tau}$. So,

$$\mathbf{1}_{\{\tau\le n\}} \le \exp(-2\mu S_{n\wedge\tau}/\sigma^2)\mathbf{1}_{\{\tau\le n\}} \le \exp(-2\mu S_{n\wedge\tau}/\sigma^2),$$

Taking expectations in the inequality, we obtain:

$$\mathbb{P}(\tau \le n) \le \mathbb{E}\exp(-2\mu S_{n\wedge\tau}/\sigma^2) = \exp(-2\mu S_0/\sigma^2).$$

Note: 这个缩放有点太大了吧?

Due to the continuity of probability, the l.h.s. of the above converges to $\mathbb{P}(\tau < \infty)$, as $n \to \infty$. Thus, we obtain

$$\mathbb{P}(\tau < \infty) \le \exp(-2\mu S_0/\sigma^2).$$

Since $S_0 > 0$

$\mu > 0$

$\Rightarrow$ RHS < 1

$\Rightarrow$ good bound

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Martingales can also be used to compute the expectations of exit times, as well as the expectations of some other stopping times.

Example (Symmetric simple random walk) $\{X_n\}$ are i.i.d. random variables, taking values 1 and -1 with probability $1/2$.

$$S_n = x + \sum_{i=1}^{n} X_i, \quad S_0 = x.$$

Choose integers $a < x < b$ and introduce:

$$\tau = V_a \wedge V_b = \min\{n \geq 0 \,:\, S_n \notin (a,b)\}.$$

上次是 $P(V_a < V_b)$
这次计算 $E\tau$

Question

What is the expected time until the process (S_n) leaves the interval (a, b): $\mathbb{E}\tau = ?$

As usual, we, first, need to find the proper martingale:

$$(M_n := S_n^2 - n)_{n=0}^{\infty}.$$

这里的 mG 是 Compensated Square of mG, S_n 本身是一个 mG.

Check: M_n is a martingale w.r.t. (X_n):

Due to the Optional Sampling theorem, $(M_{n\wedge\tau})$ is a martingale as well. In particular

$$x^2 = \mathbb{E}M_0 = \mathbb{E}M_{n\wedge\tau}.$$

这里的 mG 非 bounded.

Notice that $|M_{n\wedge\tau}|$ is not bounded uniformly, hence, we cannot apply the DCT to pass to the limit, as $n\to\infty$, inside the expectation!

Indeed, we need to make few more transformations before we can pass to the limit:

DCT MCT

$$x^2 = \mathbb{E}M_{n\wedge\tau} = \mathbb{E}\left(S^2_{n\wedge\tau}\right) - \mathbb{E}\left(n\wedge\tau\right).$$

WT Find another method ~~to~~ express it!

We have shown before (using Markov chains) that

simple symmetric RW is recurrent.

$$\mathbb{P}(\tau<\infty) = 1.$$

Therefore, with probability one, we have,

$$\lim_{n\to\infty}\tau\wedge n = \tau, \qquad \lim_{n\to\infty} S^2_{n\wedge\tau} = S^2_\tau.$$

Since $|S^2_{n\wedge\tau}| \le \max(a^2, b^2)$, we can apply DCT to conclude that

$$\lim_{n\to\infty}\mathbb{E}S^2_{n\wedge\tau} = \mathbb{E}S^2_\tau.$$

S_n 是一个 bounded mG.

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In addition, the following monotonicity property holds:

$$\tau \wedge n \le \tau \wedge (n+1).$$

Remark

Recall the Monotone Convergence theorem (MCT): Let X and $\{X_n\}$ be random variables, such that, with probability one: $X_n \le X_{n+1}$, for each n, and $\lim_{n\to\infty} X_n = X$. Then, $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$

$f_1 \le \cdots \le f_n \le \cdots$

Thus, due to MCT:

$$\mathbb{E}(n \wedge \tau) \to \mathbb{E}\tau,$$

as $n \to \infty$.

$\Rightarrow \lim_{n\to\infty} \mathbb{E} M_{n\wedge\tau} = \mathbb{E} M_\tau$

$\lim_{n\to\infty} \int f_n \, dx = \int \lim_{n\to\infty} f_n \, dx = \int f \, dx$

Collecting the above, we conclude:

$= \mathbb{E} M_{n\wedge\tau}$

$x^2 = \lim_{n\to\infty} \mathbb{E} M_{n\wedge\tau}$

$$x^2 = \mathbb{E}M_\tau = \mathbb{E}S_\tau^2 - \mathbb{E}\tau,$$

$$x^2 = a^2\mathbb{P}(V_a \le V_b) + b^2\left(1 - \mathbb{P}(V_a \le V_b)\right) - \mathbb{E}\tau$$

Since we also know the formula for $\mathbb{P}(V_a < V_b)$, we obtain:

$$\mathbb{E}\tau = a^2\frac{b-x}{b-a} + b^2\frac{x-a}{b-a} - x^2 = (b-x)(x-a).$$

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if $\mu \neq 0$, we can use MG

When the mean of X_n, denoted μ, is not zero, there is an even easier way to compute the expected exit time – by considering $M_n = S_n - \mu n$. Below is a general statement that summarizes this result.

Theorem

(Wald's first identity) Let $\{X_n\}$ be i.i.d. random variables, with mean μ. Let τ be a stopping time with respect to (X_n), such that $\mathbb{E}\tau < \infty$. Set for any $n \geq 1$,

$$S_n := \sum_{i=1}^{n} X_i.$$

Then,

$$\mathbb{E}S_\tau = \mu\mathbb{E}\tau.$$

Stronger

Recall Compound Poisson Process : $\{X_n\}$, $N_t \sim PP(\lambda)$

$X_t = \sum_{i=1}^{N_t} X_i$

Independent

$\mathbb{E}(X_t) = \mathbb{E}(\mathbb{E}(X_t \mid N_t)).$

$= \mu \mathbb{E}(X_i)$

Theorem

(Wald's first identity) Let $\{X_n\}$ be i.i.d. random variables, with mean μ. Let τ be a stopping time with respect to (X_n), such that $\mathbb{E}\tau < \infty$. Set for any $n \geq 1$,

$$S_n := \sum_{i=1}^{n} X_i.$$

$S_\tau := \sum_{i=1}^{\infty} X_i \, \mathbb{1}\{\tau \geq i\}.$

Then,

$$\mathbb{E}S_\tau = \mu \mathbb{E}\tau.$$

$|S_{\tau \wedge n}| = \left| \sum_{i=1}^{\infty} X_i \, \mathbb{1}\{\tau \wedge n \geq i\} \right|.$

Proof. The proof follows similar lines. Start with the monotone convergence theorem that yields $\mathbb{E}[\tau \wedge n] \to \mathbb{E}[\tau]$. Next,

$$|S_{\tau \wedge n}| \leq Y \quad \text{where} \quad Y = \sum_{m=1}^{\infty} |X_m| \mathbf{1}_{\{\tau \geq m\}}.$$

WTS $\mathbb{E}\,Y < \infty$ to use DCT

Since $\{\tau \geq m\} = \{\tau \leq m-1\}^c$ is determined by $X_1, \ldots, X_{m-1}$, it is independent of X_m, and

?

$\because \tau > 0$

$$\mathbb{E}[Y] = \mathbb{E}|X_1| \sum_{m=1}^{\infty} \mathbb{P}(\tau \geq m) = \mathbb{E}|X_1| \mathbb{E}[\tau] < \infty.$$

Finally, the dominated convergence theorem implies that $\mathbb{E}[S_{\tau \wedge n}] \to \mathbb{E}[S_\tau]$. As before, we use the martingale $\{S_n - \mu n\}_n$ to conclude that

$$0 = \mathbb{E}[S_0 - 0] = \mathbb{E}[S_{\tau \wedge n}] - \mu \mathbb{E}[\tau \wedge n] \to \mathbb{E}[S_\tau] - \mu \mathbb{E}[\tau]. \quad \blacksquare$$

Sec 5.5

Theorem (Maximal inequality for supermartingales)

Let (X_n) *be a* *supermartingale*, *such that* $X_n \geq 0$, *for all* n, *and consider an arbitrary constant* $\lambda > 0$. *Then,*

$$\mathbb{P}\left(\max_{n\geq 0} X_n > \lambda\right) \leq \frac{\mathbb{E}X_0}{\lambda}.$$

$\sum_{\lambda}$ $\int \mathbb{P}(X_n > \lambda)$

$= \mathbb{E}$

Proof. Think at home.

Suggested Exercises (Durrett, 3rd ed.), not for submission

Exercises 5.8.