MATH 526: Discrete State Stochastic Processes
Lecture 21

Chapter 5: Martingales

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5.3: Gambling Strategies, Stopping Times

Assume that $(M_n)_{n=0}^{\infty}$ is the price process of a financial asset (stock). At the beginning of each trading period (at each discrete moment in time), we can buy or sell the asset in any quantity. Denote by H_n, the number of units of the asset that we hold in the n-th period (between times $n-1$ and n).

Our total gain or loss from the trading strategy (H_n), at time n, is:

$$W_n = \sum_{i=1}^{n} H_i (M_i - M_{i-1}).$$

Alternatively, $M_n - M_{n-1}$ can be viewed as an outcome of the n-th gambling bet.

dependent on ...

Notice that the trading strategy (H_n) can be a stochastic process. Why?

More precisely, assume that the observed information is given by (X_n). Then, (M_n) should be adapted to (X_n): at time n, the value of the asset M_n is known, given all the available information. On the other hand, the value of the strategy H_n should be determined by $X_0, \ldots, X_{n-1}$: we should know how much asset we hold over the n-th period before that period begins (i.e. at time $n-1$).

Definition (Predictable)

A stochastic process $(H_n)_{n=1}^{\infty}$ is predictable w.r.t. $(X_n)_{n=0}^{\infty}$ if, for all $n \geq 1$, we have:

$$H_n = f_n(X_0, \ldots, X_{n-1}),$$

with some (measurable) functions f_n.

Assume (M_n) is a martingale w.r.t. some information flow (X_n). Notice that a martingale can be interpreted as a fair game. In particular, buying one unit of the asset at time $k \geq 0$, and holding it until time $n > k$ (buy and hold strategy), will give you zero average return, regardless of what happens before the trade begins. In other words:

$$\mathbb{E}\left(M_n - M_k \,|\, \mathcal{F}_k^X\right) = 0.$$

Question

Can we generate a positive average return by trading dynamically: using more complicated strategies than "buy and hold"?

Assume that the return has to be realized by a given deterministic time, then, under some additional constraints, we can show that by trading a martingale, one can only generate a martingale and hence not generate profit. That is, $\mathbb{E}(W_T - W_0) = 0$, for any deterministic T.

Theorem (Theorem 5.9)

Let (M_n) be a martingale, and let (H_n) be predictable. Assume, in addition, that, for each $n \geq 1$, we have: $|H_n| \leq c_n$, with probability one, where c_n's are finite constants. Then (W_n), given by

$$W_n = W_0 + \sum_{i=1}^{n} H_i(M_i - M_{i-1}),$$

is a martingale, for any initial condition $W_0 \in \mathbb{R}$.

Proof.

$$\mathbb{E}|W_n| \le |W_0| + \sum_{i=1}^{n} c_i(\mathbb{E}|M_i| - \mathbb{E}|M_{i-1}|) < \infty$$

$$\mathbb{E}(W_{n+1} | \mathcal{F}_n) = W_n + \mathbb{E}(H_{n+1}(M_{n+1} - M_n) | \mathcal{F}_n)$$

$$= W_n + H_{n+1}\mathbb{E}(M_{n+1} - M_n | \mathcal{F}_n) = W_n$$

$$H_n = f_n(X_0, \dots, X_{n-1})$$

5.3: Gambling Strategies, Stopping Times

The theorem stated in the previous slide has a very important corollary, which goes beyond financial applications. Consider the following trading strategy:

$$H_n = \begin{cases} 1, & n = 1, \ldots, \tau, \\ 0, & n > \tau, \end{cases}$$

with some random time τ. In words: we hold one unit of asset until some random time τ, and we sell it then.

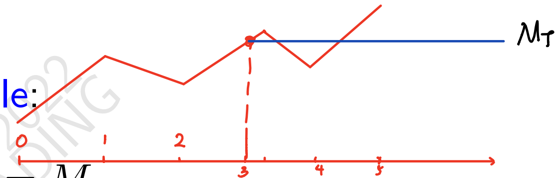
In order for this strategy to be predictable, τ needs to be a stopping time: at each time, we need to know whether to sell or not, based only on the available information at that time.

Definition (Stopping Time)

A random variable τ, with values in $\{0, 1, 2, \ldots, \infty\}$, is a stopping time w.r.t. $(X_n)_{n=0}^{\infty}$ if, for any $n \geq 0$, $\mathbf{1}_{\{\tau \leq n\}}$ is a function of $X_0, \ldots, X_n$.

R.V.

If τ is a stopping time, the above strategy is predictable: $H_n = 0$ if and only if $\tau \le n-1$, which is determined by the values of $X_0, \dots, X_{n-1}$. Hence, the value of H_n is determined by $(X_0, \dots, X_{n-1})$.



The resulting wealth process is a stopped martingale:

$$W_n = M_n \mathbf{1}_{\{n<\tau\}} + M_\tau \mathbf{1}_{\{n \ge \tau\}} = M_{n \wedge \tau}.$$

minimum

We know, from the above theorem, that $W_n = M_{n\wedge\tau}$ is, again, a martingale. In fact, this is one of the most important properties of martingales, known as the Optional Sampling theorem.

Theorem (Optional Sampling, Theorem 5.10)

Let $(M_n)_{n=0}^\infty$ be a martingale (submartingale, supermartingale) and let τ be a stopping time, w.r.t. the same information. Then, the stopped process $(M_{n\wedge\tau})_{n=0}^\infty$ is a martingale (submartingale, supermartingale) as well. In particular,

$$\mathbb{E}M_{\boxed{n\wedge\tau}} = (\ge, \le)\, \mathbb{E}M_{\boxed{0}},$$

($\mathbb{E}M_0 = M_0$ if M_0 is a constant)

$\mathbb{E}M_{n\wedge\tau} = \mathbb{E}M_0$

This theorem has many useful applications. In particular, it allows us to compute exit probabilities for a large class of stochastic processes, in a rather simple way.

Example (Symmetric simple random walk). Let $\{X_n\}$ be i.i.d. random variables, taking values -1 and 1, with probability $1/2$.

For any initial integer x, the process $(M_n = x + \sum_{i=1}^{n} X_i)$ is a martingale w.r.t. (X_n).

Fix integers a and b, such that $a \leq x \leq b$, and recall the exit times V_a and V_b.

Question

$\mathbb{P}(V_a < V_b) =?$

Let us define the time of the first exit from (a, b):

τ is a stopping time.

$$\tau = V_a \wedge V_b = \min\{n \geq 0 \, : \, M_n \notin (a, b)\}.$$

Notice that $\tau \leq \boxed{n}$ if and only if there exists a time index $k \in \{0, \dots, n\}$, s.t. $M_k \notin (a, b)$? Thus, the random variable $\mathbf{1}_{\{\tau \leq n\}}$ is a function of $(M_1, \dots, M_n)$, which, in turn, are determined by $(X_0, \dots, X_n)$. Hence, τ is a stopping time.

$M_{n \wedge T} = \tilde{M}_n \leftarrow$ mG

$\mathbb{E}(\tilde{M}_n) = \mathbb{E}(\tilde{M}_0)$

By the Optional Sampling theorem, $(M_{n\wedge\tau})$ is a martingale, and

$$\mathbb{E}M_{n\wedge\tau} = \mathbb{E}M_{\tau\wedge 0} = \mathbb{E}M_0 = \boxed{x}.$$

Recall that (M_n) eventually hits either a or b. *We had shown this using the irreducible finite state space MC, which arises from the present one when it is absorbed at a and b.* Thus, $\mathbb{P}(\tau < \infty) = 1$.

every state is recurrent.

Since τ is always finite, we have: $n \wedge \tau \to \tau$, as $n \to \infty$, with probability one. The finiteness of τ also implies that, with probability one,

$$M_{n\wedge\tau} \to M_\tau \quad ③ \qquad \text{as} \qquad n \to \infty.$$

In addition,

$$|M_{n\wedge\tau}| \leq \max(|a|, |b|), \quad ①$$

bounded

for all n, with probability one.

Remark

Recall the Dominated Convergence theorem (DCT): Let the random variables X, Z and $\{X_n\}_{n=1}^{\infty}$, satisfy: ① $|X_n| \le Z$, for all $n \ge 1$, ② $\mathbb{E}Z < \infty$, and ③ $\lim_{n\to\infty} X_n = X$, with probability one. Then,

bounded

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X.$$

Applying DCT, we obtain:

$$x = \lim_{n\to\infty} \mathbb{E}M_{n\wedge\tau} = \mathbb{E}M_\tau. = \mathbb{E}(M_0)$$

Next, notice that M_τ can only take two values, a or b, depending on which state is reached first. Thus, we have:

因为 Martingale

$$\begin{aligned} x = \mathbb{E}M_\tau &= \mathbb{E}\left(a\mathbf{1}_{\{V_a<V_b\}} + b\mathbf{1}_{\{V_b<V_a\}}\right) \\ &= a\mathbb{P}(V_a < V_b) + b\mathbb{P}(V_a \ge V_b) \\ &= a\mathbb{P}(V_a < V_b) + b(1 - \mathbb{P}(V_a < V_b)), \end{aligned}$$

$\mathbb{E}(M_T) = \lim_{n\to\infty} \mathbb{E}(M_{n\wedge T})$.

$= E_0 := x$.

and, as a result:

$$\Rightarrow \quad \mathbb{P}(V_a < V_b) = \frac{b-x}{b-a}.$$

5.3: Gambling Strategies, Stopping Times

As shown above, we will typically be interested in $\mathbb{E}M_\tau$, rather than $\mathbb{E}M_{n\wedge\tau}$. To compute the desired expectation from the Optional Sampling theorem, we need to pass to the limit. This requires additional properties, such as the boundedness of the stopped process $|M_{n\wedge\tau}|$, uniformly over all n.

Question

Is the above assumption of boundedness really necessary? or is it just a technical detail that can be ignored?

Indeed, without these additional assumptions, things may not work out as we want. Namely, it is possible that the expected return of the strategy is non-zero, $\mathbb{E}(M_\tau - M_0) \neq 0$, if the return is realized at some unbounded stopping time τ.

Example (Doubling strategy). Consider a symmetric simple random walk: $\{X_n\}$ are i.i.d. random variables, taking values -1 and 1, with probability $1/2$.

$$S_n = \sum_{i=1}^{n} X_i, \qquad S_0 = 0.$$

初始为0

Think of S_n as a price of an asset that starts from zero, or, think of X_n as an outcome of the n-th gambling bet.

Introduce the stopping time

$$\tau = \min\{n \geq 0 \,:\, X_n = 1\},$$

首次成功.

which is the time of the first successful bet. We define the doubling strategy as:

$$H_n = \begin{cases} 2^{n-1}, & n = 1, \ldots, \tau, \\ 0, & n > \tau. \end{cases}$$

指数投注

In words, we keep doubling our bets, until the first time we win, then we stop betting.

The wealth process of this strategy, (W_n), is given by: $W_0 = 0$ and

$$W_n = \sum_{i=1}^{n} H_n X_n.$$

The wealth at time τ is

$$W_\tau = -1 - 2 - \cdots - 2^{\tau-2} + 2^{\tau-1} = -(2^{\tau-1} - 1) + 2^{\tau-1} = 1.$$

Starting from zero, we will always win 1!

Remark

The doubling strategy works even for the asymmetric random walk.

Remark

If $p \geq 1/2$, there is an even simpler strategy: buy one unit of the asset at time zero (for zero price), and sell it whenever the price hits one. It will hit level one with probability one, and, of course, the wealth at that (random) hitting time is 1.

There are two main problems with the above strategies:

- we do not control the time when the strategy pays out (in fact, the expected duration of the game in the second simplified strategy is infinite),
- the drawdowns of the wealth process, generated by these strategies, are potentially unlimited (in other words, we need to have infinite monetary reserves, to implement these strategies).

Mathematically speaking, the second problem outlined above means that $|W_{n\wedge\tau}|$ is not bounded over $n \geq 1$. As a result, the DCT cannot be applied, and $\mathbb{E}W_{n\wedge\tau}$ may not converge to $\mathbb{E}W_\tau$, as $n \to \infty$. Indeed, for the strategies presented above,

$$1 = \mathbb{E}W_\tau \neq \mathbb{E}W_0 = 0.$$

Question

How can we fix the above problem?

As far as the financial application is concerned, we can restrict the set of admissible trading strategies, allowing only for those strategies that generate bounded wealth processes (in fact, bounded from below is enough).

Mathematically, we can add the condition that $|M_{n\wedge\tau}|$ is bounded.

Theorem (Theorem 5.11)

Let (M_n) be a martingale, and let τ be a stopping time w.r.t. the same information. Assume that $\mathbb{P}(\tau<\infty)=1$ and $\mathbb{E}M_0 = \lim\limits_{n\to\infty} \mathbb{E}M_{n\wedge\tau} = \mathbb{E}M_\tau$

$$|M_{n\wedge\tau}|\le C,$$

for all $n\ge 1$ and some constant C. Then, $\mathbb{E}M_\tau=\mathbb{E}M_0$.

Let us illustrate the use of the above theorem for the computation of exit probabilities.

Remark

Recall the Dominated Convergence theorem (DCT): Let the random variables X, Z and $\{X_n\}_{n=1}^{\infty}$, satisfy: ① $|X_n|\le Z$, for all $n\ge1$, ② $\mathbb{E}Z<\infty$, and ③ $\lim_{n\to\infty}X_n=X$, with probability one. Then,

bounded

$$\lim_{n\to\infty}\mathbb{E}X_n=\mathbb{E}X.$$

Theorem (Optional Sampling, Theorem 5.10)

Let $(M_n)_{n=0}^{\infty}$ be a martingale (submartingale, supermartingale) and let τ be a stopping time, w.r.t. the same information. Then, the stopped process $(M_{n\wedge\tau})_{n=0}^{\infty}$ is a martingale (submartingale, supermartingale) as well. In particular,

$$\mathbb{E}M_{\boxed{n\wedge\tau}}=(\ge,\le)\,\mathbb{E}M_{\boxed{0}},$$

($\mathbb{E}M_0=M_0$ if M_0 is a constant)

$+1 \quad -1$

Example (Asymmetric simple random walk). Let $\{X_n\}$ be i.i.d. random variables, each taking values 1 and -1, with probabilities p and $q = 1 - p$, respectively. We assume that $p \neq q$ and $p, q > 0$.

$M_n = S_n - n(p-q)$ ← work on this

Choose integer x and define

$$S_n := x + \sum_{i=1}^{n} X_i, \quad S_0 = x.$$

$p \quad +1 \quad -1 \quad q \qquad b \quad a$

Choose two integers a and b, such that $a \leq x \leq b$, and recall the exit times: V_a and V_b.

Question

$\mathbb{P}(V_a < V_b) = ?$

if $M_n = S_n - n(p-q)$

M_n is not bounded!

First, we set up the proper martingale:

$$M_n := (q/p)^{S_n}.$$

$T = V_a \wedge V_b$.

$P(T < \infty) = 1$

$a \leq S_{n\wedge T} \leq b$

hence

$|M_{n\wedge T}| \leq \max\left(\left(\frac{q}{p}\right)^a, \left(\frac{q}{p}\right)^b \right)$ for all $n \geq 0$

Apply Thm, $\mathbb{E} M_T = \mathbb{E} M_0 = \left(\frac{q}{p}\right)^x$. $(S_0 = x)$

and

$\mathbb{E} M_T = \left(\frac{q}{p}\right)^a \mathbb{P}(V_a < V_b) + \left(\frac{q}{p}\right)^b \mathbb{P}(V_a \geq V_b)$.

$= \left(\frac{q}{p}\right)^x$

$\Rightarrow \mathbb{P}(V_a < V_b) = \dfrac{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^x}{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^a}$ $\qquad \dfrac{b-x}{b-a}$

Note that there may be many martingales associated with a given problem. In particular, in the above example, we could have constructed

$$M_n = S_n - (p - q)n$$

$\left(\frac{q}{p}\right)^{S_n}$

However, such a choice would not allow us to solve the problem.

The choice of the proper martingale is one of the most difficult tasks, for which there is no ready recipe, and one has to rely on his or her intuition!

Suggested Exercises (Durrett, 3rd ed.), not for submission

Exercises 5.6, 5.7.