MATH 526: Discrete State Stochastic Processes
Lecture 18

Chapter 4: Continuous Time Markov Chains

Prakash Chakraborty

University of Michigan

In this section, we address the following questions:

- what is the **probability that a CTMC visits one state before the other one**?
- and **what is the expected time of reaching a certain set of states**?

Definition

The **time of the first visit** to state i (**hitting time** of state i) is

$$V_i = \min\{t \geq 0 \,:\, X_t = i\}.$$

Definition

The **time of the first visit** to a set of states $A \subset \mathbb{S}$ (**hitting time** of set A) is

$$V_A = \min\{t \geq 0 \,:\, X_t \in A\}.$$

Remark

Recall that, in discrete-time, we had two notions: the **exit time** and the **time of the first jump** to a state. One of them takes into account all the states that the process visits, the other one ignores the initial state. In the case of continuous-time, we will always need to take into account all the states, and we refer to the resulting random time as the **time of the first visit** to a state/set.

Example (Barbershop). Recall the Barbershop example, where the generator is given by:

$$Q = \begin{array}{c|cccc} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \hline \mathbf{0} & -2 & 2 & 0 & 0 \\ \mathbf{1} & 3 & -5 & 2 & 0 \\ \mathbf{2} & 0 & 3 & -5 & 2 \\ \mathbf{3} & 0 & 0 & 3 & -3 \end{array}$$

Question

If there is currently one customer in the shop, what is the **probability that the shop becomes full before it becomes empty**?

In view of the above definition, we can formulate the problem as

$$\mathbb{P}_1(V_3 < V_0) = ?$$

4.4: Exit distributions and hitting times

The solution to this problem is very similar to the discrete-time case. Notice that, to answer the above question, we **don't need to know when the process jumps**, we only **need to know where it is jumping to**. Thus, we can make a connection to the discrete-time case by looking at the **embedded chain** of the CTMC. Earlier, we wrote the transition rates of the embedded Markov chain, Now, we write its **dynamics**.

Theorem

For a CTMC (X_t), we denote by $\{T_k\}$ the times of its jumps. Then, $(Y_n)_{n=1}^{\infty}$, given by

$$Y_n = X_{T_n},$$

is the embedded Markov chain of (X_t). Namely, the transition probabilities of (Y_n) are given by

$$\frac{q(i,j)}{\lambda_i} = \frac{q(i,j)}{\sum_{j\neq i} q(i,j)} = -\frac{q(i,j)}{q(i,i)}$$

$$p(i,j) = -\frac{Q(i,j)}{Q(i,i)}, \qquad i \neq j, \qquad p(i,i) = 0.$$

Thus, we see that **a CTMC reaches state i before j, if and only if so does its embedded MC**.

In the present case, the embedded MC has the following transition matrix:

$$p = \begin{array}{c|cccc} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & 0 & 1 & 0 & 0 \\ \mathbf{1} & 3/5 & 0 & 2/5 & 0 \\ \mathbf{2} & 0 & 3/5 & 0 & 2/5 \\ \mathbf{3} & 0 & 0 & 1 & 0 \end{array}$$

We know how to find the probability that the embedded MC reaches state 3 before 0, starting from 1. Namely, we need to find function $h : \mathbb{S} \to [0, 1]$, s.t.: $h(0) = 0$, $h(3) = 1$ and

$$\sum_{k=0}^{3} p(i, k)h(k) = h(i), \quad i = 1, 2.$$

Let us formulate this in terms of Q:

$$-\sum_{k \neq i} \frac{Q(i, k)}{Q(i, i)} h(k) = h(i), \quad i = 1, 2,$$

$$\Rightarrow \sum_{k \neq i} Q(i, k)h(k) = -Q(i, i)h(i), \quad i = 1, 2,$$

$$\Rightarrow \sum_{k=0}^{3} Q(i, k)h(k) = 0, \quad i = 1, 2,$$

$$\Rightarrow h(2) = 10/19, \quad h(1) = 4/19.$$

Motivated by the above computations, we formulate the general result.

Theorem (Exit probabilities)

Consider a CTMC with generator Q. Assume that states n and m are such that $\mathbb{P}_i(V_n \wedge V_m < \infty) > 0$, for all $i \in \mathbb{S}$. If there exists a function $h : \mathbb{S} \to [0,1]$, such that:

- $h(n) = 1$, $h(m) = 0$,
- *and*

cts

$$\sum_{k \in \mathbb{S}} Q(i,k) h(k) = 0, \qquad \text{for all } i = \mathbb{S} \setminus \{n, m\},$$

then $\mathbb{P}_i(V_n < V_m) = h(i)$, *for all* $i \in \mathbb{S}$.

Example (Barbershop continued)

Question

If there is currently one customer in the store, what is the **expected time until the shop becomes full**?

In this case, the **times of jumps do matter**, so we cannot simply refer to the results we had in the discrete-time case. Nevertheless, the derivations are very similar, except that now we need the **strong Markov property** of a CTMC.

Definition (Stopping Time)

Assume we are given a stochastic process $(X_t)_{t\geq 0}$ and a r.v. T, with values in the interval $[0, \infty]$. We call T a **stopping time** (with respect to (X_t)) if, for any time $t \geq 0$, the occurrence or non-occurrence of the event "we stop at or before time t", $\{T \leq t\}$, is determined only by looking at the past and present values of the process $(X_u)_{u\in[0,t]}$.

Theorem (Strong Markov property of a CTMC)

Let (X_t) be a **regular** *CTMC and let T be a stopping time (with respect to (X_t)). Conditional on $T < \infty$ and $X_T = y$, any other information about $X_0, \ldots, X_T$ is irrelevant for the future distribution of the CTMC. Namely, the new process $(\tilde{X}_t = X_{T+t})_{t \geq 0}$ is also a CTMC, with the same transition probabilities and with initial state y, and it is independent of T and of the past values $(X_u)_{u \in [0,T]}$.*

Now, we can proceed with the conditioning argument. Let τ_1 be the time of the first jump of X and let V_3 be the first time X hits 3. As we did in the discrete-time case, set $\tilde{X}_t = X_{\tau_1+t}$ and $\tilde{V}_3 = V_3 - \tau_1$. Then,

T_1 is a stopping time

$$g(i) = E_i(V_3) = E(V_3 \mid X_0 = i)$$

$$= E(T_1 \mid X_0 = i) + \sum_k E[V_3 - T_1 \mid X_{T_1} = k, X_0 = i) \, P(X_{T_1} = k \mid X_0 = i).$$

$$= E_i(T_1) + \sum_k E[\tilde{V}_3 \mid \tilde{X}_0 = k] \cdot P(X_{T_1} = k \mid X_0 = i)$$

$$= -\frac{1}{Q(i,i)} - \sum_{k\neq i} g(k)\frac{Q(i,k)}{Q(i,i)},$$

$$= E_i(T_1) + \sum_{k\neq i} E(V_3 \mid X_0 = k) \, p(i,k)$$

$$\frac{1}{\lambda} = \frac{1}{\sum Q(i,j)} = \frac{1}{-Q(i,i)}$$

$$-\frac{Q(i,k)}{Q(k,k)}$$

where we used the **strong Markov property of** (X_t).

Rearranging the terms, we obtain

$$\sum_{k\in\mathbb{S}} Q(i,k)g(k) = -1, \quad \text{for all } i \neq 3,$$

and together with $g(3) = 0$, we get the following system:

$$\begin{cases} -2g(0) + 2g(1) = -1, \\ 3g(0) - 5g(1) + 2g(2) = -1, \\ 3g(1) - 5g(2) = -1. \end{cases}$$

Solving the above, we find $g(1) = 29/8$.

The above computations motivate the following theorem.

Theorem (Expected Exit Times)

Consider a CTMC, with generator Q. Assume that a set of states $A \subset \mathbb{S}$ is such that $\mathbb{P}_i(V_A < \infty) > 0$, for all $i \in \mathbb{S}$. If there exists a function $g : \mathbb{S} \to [0, \infty)$, such that:

- $g(i) = 0$, *for all* $i \in A$,
- *and*

$$\sum_{k \in \mathbb{S}} Q(i,k) g(k) = -1, \quad \text{for all } i = \mathbb{S} \setminus A,$$

then $\mathbb{E}_i V_A = g(i)$, *for all* $i \in \mathbb{S} \setminus A$.

Suggested Exercises (Durrett, 3rd ed.), not for submission

Exercises 4.15, 4.16, 4.17, 4.18.