MATH 526: Discrete State Stochastic Processes
Lecture 23

Chapter 5: Martingales

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Martingales, submartingales, and supermartingales have a tendency to converge. Why?

From analysis: a decreasing sequence, bounded from below, always has a limit. Supermartingales can be viewed as the stochastic analogue of decreasing sequences (note that a supermartingale, itself, does NOT have to be decreasing in time, but its expectation is). Hence, it is natural to expect that, if a supermartingale is bounded form below, it has a limit.

Theorem (Martingale Convergence)

Let $X_n \geq C$ be a supermartingale. Then, there exists a random variable X_∞, satisfying:

convergence.

$$X_\infty = \lim_{n\to\infty} X_n, \quad \textit{with probability one,}$$

$\mathbb{E}(M_{n+1} \mid \mathcal{F}_n^X) \leq M_n$

and

$$\mathbb{E}X_\infty \leq \mathbb{E}X_0$$

Theorem (Martingale Convergence 2)

Let X_n be a supermartingale, such that $\mathbb{E}|X_n| \leq C$. Then, there exists a random variable X_∞, satisfying:

$$X_\infty = \lim_{n \to \infty} X_n, \quad \textit{with probability one,}$$

and

$$\mathbb{E}X_\infty \leq \mathbb{E}X_0$$

Remark

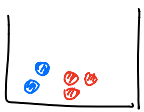
- any martingale is a supermartingale, hence, the above theorems hold for martingales as well;
- if X_n is a submartingale, then $-X_n$ is a supermartingale, hence, we can reformulate the above convergence theorems for submartingales:

The martingale convergence theorems, often, make it easy to prove that a given stochastic process has a limit. However, it tells us nothing about the distribution of the limiting random variable (compare to the case of Markov processes, where the limiting distribution is given by the stationary distribution of the process, and it can be computed via its generator).

For martingales, we, typically, first, apply the convergence theorem to conclude that the limit exists, and, then find the limit.

Example (Polya's urn)

Initially $K=2$



X_n = fraction of red balls at time n.

$\mathbb{P}(X_1 = \frac{2}{3}) = \frac{1}{2}$

- X_n adapts to itself. $|X_n| < \infty$

$\mathbb{E}(X_{n+1} \mid \mathcal{F}_n^X) = X_n$

$X_n \cdot \frac{(k+n)X_n+1}{k+n+1} + (1-X_n) \cdot \frac{(k+n) \cdot X_n}{k+n+1} = X_n$

Apply the Thm:

$X_n \to X_\infty$

$\mathbb{P}(X_n = \frac{j}{n+2})$

$= \binom{n}{j} \frac{1}{2} \times \cdots \times \frac{j-1}{j} \times \frac{1}{j+1} \times \cdots \times \frac{n-j+1}{n+1}$

$= \frac{1}{n+1}$

- Initially $K = r+b$

$\sim \text{Beta}(r, b)$

$\frac{x^{r-1}(1-x)^{b-1}}{\text{Beta}(r,b)}$

$\to \frac{\Gamma(r+b)}{\Gamma(r)\Gamma(b)}$

Example (Asymmetric simple random walk) Let $\{X_n\}$ be i.i.d. random variables, taking values 1 and -1, with probabilities p and $q = 1 - p$, respectively. Introduce

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i$$

p. q positive

$\Rightarrow M_n \geq 0$

$\to 0$

Assume that $p \neq q$. Then $(M_n = (q/p)^{S_n})$ is a martingale.

Since $M_n \geq 0$, we can apply the first Martingale Convergence theorem, to conclude that there is a random variable M_∞, such that:

$$(q/p)^{S_n} \to M_\infty,$$

$$\begin{cases} \frac{q}{p} < 1, & S_n \to +\infty \\ \frac{q}{p} > 1, & S_n \to -\infty \end{cases}$$

Intuition

as $n \to \infty$.

Let us show that $M_\infty = 0$, with probability one.

WTS: 如果 $M_\infty \neq 0$，那么

Pick any random outcome ω, such that $M_\infty(\omega) \neq 0$. Let us show that, for such ω,

$\omega \in A$ $\quad \lim_{n\to\infty} M_n(\omega) \neq M_\infty(\omega)$.

$$\lim_{n\to\infty} M_n(\omega) \neq M_\infty(\omega),$$

and, hence, the probability of all such ω is zero. We will prove it by contradiction.

那么 $P(A) = 0$

$M_n = (\frac{q}{p})^{S_n} \geqslant 0$

Contradiction: $M_\infty(\omega) > 0$

Assume that $\lim_{n\to\infty} M_n(\omega) = M_\infty(\omega)$,
Since $M_\infty(\omega)$ is a limit of nonnegative numbers, it has to be nonnegative itself. Since, in addition, it is not equal to zero, we have: $M_\infty(\omega) > 0$. This means that $M_n(\omega)$ has a positive limit, and, therefore,

由 MCT $\Rightarrow \mathbb{P} = 0$

$$\log(M_n(\omega))$$

has a finite limit, as $n \to \infty$.
Then, as a difference of two sequences with the same limit,

如果 $M_\infty(\omega) > 0$
$\Rightarrow \lim_{n\to\infty} \log(M_n(\omega)) < \infty$

$$|S_{n+1}(\omega)\log(q/p) - S_n(\omega)\log(q/p)| \to 0,$$

as $n \to \infty$.

$|S_{n+1}(\omega) - S_n(\omega)| = 1$

On the other hand, we notice that

$$|S_{n+1}(\omega)\log(q/p) - S_n(\omega)\log(q/p)| = \log(q/p) = \text{const} \neq 0,$$

which is a contradiction.
Hence, when $M_\infty(\omega) \neq 0$, $M_\infty(\omega)$ cannot coincide with the limit of $M_n(\omega)$, and, due to the Martingale Convergence theorem, the set of all such random outcomes ω has probability zero. In other words, $M_\infty = 0$, with probability one.

The previous example provides an alternative proof of the fact that a simple random walk with $p < q$, hits any lower level, and, similarly, a simple random walk with $p > q$ hits any upper level, with probability one. Indeed, if $p < q$, then, $q/p > 1$ and

$$(q/p)^{S_n} \to 0$$

is equivalent to

$$S_n \to -\infty$$

Similar argument applies in the case $p > q$.

Remark

The convergence of S_n to $+\infty$ or $-\infty$, in turn, implies that

$$\mathbb{P}_x(\tau < \infty) = 1,$$

where $\tau = V_a \wedge V_b$, with some integers $a \le x \le b$. Recall that the fact that τ is finite is needed for the computation of exit probabilities and expected exit times, when we apply the Optional Sampling Theorem. We used to argue for the finiteness of τ using the results of Markov chain theory. As the above examples shows, the same conclusion can be obtained by using martingale convergence instead.

recurrence.
transient

Example Consider a sequence $\{X_i\}_{i=1}^{\infty}$ of non-constant independent identically distributed random variables with zero mean, such that, with probability one, $|X_i| \leq C$ for any $i \geq 1$, with a fixed constant C. Consider

$$S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0,$$

$\longrightarrow$ mG.

$\mu = 0,\ |X_i| \leq C$

$\Rightarrow T_a < \infty$ w/ $p=1$

and the stopping time

$$T_a = \min\{n \geq 0 \,:\, S_n \geq a\}, \quad \text{for } a > 0.$$

Q: Let us show that $\mathbb{P}(T_a < \infty) = 1$.

Due to Optional Sampling Theorem, $(S_{n \wedge T_a})$ is a martingale. In addition, $S_{n \wedge T_a} \leq a + C$. Therefore, due to the Martingale Convergence Theorem, with probability one,

$$\lim_{n \to \infty} S_{n \wedge T_a} = S_\infty,$$

for some random variable S_∞.

$T_a = \infty \Rightarrow S_n(\omega)$ 不 converge

Let us show that the convergence is only possible if $T_a < \infty$. $\Leftrightarrow$ converge

Pick ω such that $T_a(\omega) = \infty$. Then, $S_{n \wedge T_a}(\omega) = S_n(\omega)$, and, hence, the convergence is possible only if $S_n(\omega)$ converges to a finite limit, as $n \to \infty$. Assume that there is a positive probability that $T_a = \infty$. Then, there is a positive probability that (S_n) converges to a finite limit as $n \to \infty$. Therefore, for $\varepsilon > 0$, there is a positive probability that at least for one n, the event

$$A_n = \{|X_n| \leq \varepsilon, |X_{n+1}| \leq \varepsilon, \ldots\}$$

$\lim_{n\to\infty} |S_{n+1} - S_n| = 0$

$\Rightarrow X_{n+1} = 0 \cdot \forall n > N.$

occurs. In other words

$$\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) > 0$$

Let us choose $\varepsilon > 0$ so that $\mathbb{P}(|X_n| \leq \varepsilon) = \delta < 1$. Then, $\mathbb{P}(A_n) = \delta^{\infty} = 0$, and, since there are countably many A_n's, the probability of their union is also zero. Thus, we obtain a contradiction. Therefore, $\mathbb{P}(T_a = \infty) = 0$.

Countable Intersection of $P = 1$ is $P = 1$

Applications

Example. Suppose a sequence of independent events $\{A_n\}_{n=1}^{\infty}$ satisfies

$$\sum_{i=1}^{n} \mathbb{P}(A_i) \to \infty, \qquad \text{as } n \to \infty.$$

Notice that this fact implies

$$\prod_{i=1}^{n} (1 - \mathbb{P}(A_i)/2) \to 0, \qquad \text{as } n \to \infty.$$

$|A_i| > 0$

Consider the random variables $X_i = \mathbf{1}_{A_i}$ (each X_i takes value 1, if the event A_i occurs, and zero otherwise).

Q: Let us show that, with probability one, $Y_n = \prod_{i=1}^{n}(1 + X_i) \to \infty$, as $n \to \infty$.

Define $X_i = \mathbf{1}_{A_i}$, for $i = 1, 2, \ldots$, and

$M_{n+1} = M_n \cdot \frac{1}{(1 - \frac{\mathbb{P}(A_{n+1})}{2})(1 + X_{n+1})}$

$$M_n = \prod_{i=1}^{n} \frac{1}{(1 - \mathbb{P}(A_i)/2)(1 + X_i)}, \qquad M_0 = 1.$$

$1 - \mathbb{P}(A_i)$

Notice that (M_n) is a martingale:

M_n positive

$$\mathbb{E}|M_n| = \prod_{i=1}^{n} \frac{\mathbb{E}\,(1/(1 + X_i))}{1 - \mathbb{P}(A_i)/2} = \prod_{i=1}^{n} \frac{\mathbb{P}(A_i)/2 + 1 - \mathbb{P}(A_i)}{1 - \mathbb{P}(A_i)/2} = 1. \quad < \infty$$

same $= 1$

$$\mathbb{E}\left(M_{n+1} \,|\, \mathcal{F}_n^X\right) = \left(\prod_{i=1}^{n} \frac{1}{(1 - \mathbb{P}(A_i)/2)(1 + X_i)}\right) \frac{\mathbb{E}\,(1/(1 + X_{n+1}))}{(1 - \mathbb{P}(A_{n+1})/2)} = M_n.$$

Applications

Since $M_n \geq 0$, we can apply the Martingale Convergence Theorem to conclude that, with probability one, (M_n) converges to a finite limit M_∞, as $n \to \infty$. Recall that

$$\prod_{i=1}^{n}(1 - \mathbb{P}(A_i)/2) \to 0, \quad \text{as } n \to \infty.$$

Therefore,

$$\prod_{i=1}^{n}(1 + X_i) = \frac{1}{M_n \prod_{i=1}^{n}(1 - \mathbb{P}(A_i)/2)} \to \frac{1}{M_\infty \cdot 0} = \infty,$$

as $n \to \infty$.

Remark

This example proves the Borel-Cantelli lemma, which states that, under the conditions of this exercise, with probability one, an infinite number of events A_i occur.

说明有无限个 $X_i = 1$

$x \to x+1$ $\quad p_x$

$x \to x$ $\quad 1-p_x-q_x$

$x \to x-1$ $\quad q_x$

$$\frac{p_1}{q_1} + \frac{p_1}{q_1}\cdot\frac{p_2}{q_2} + \cdots + \frac{p_1\cdots p_x}{q_1\cdots q_x}.$$

$$\phi(x) = \sum_{y=1}^{x} \prod_{z=1}^{y-1} \frac{p_z}{q_z} \quad \frac{q_z}{p_z}.$$?

① $M_n = \phi(X_{n\wedge V_0})$ is mG.

$$M_n^x = \frac{p_1}{q_1} + \cdots + \frac{p_1\cdots p_{x-1}}{q_1\cdots q_{x-1}}.$$

p_x.

q_x

$1-p_x-q_x$.

① • $\phi(X_{n\wedge V_0})$ adapted to itself

• $\mathbb{E}|M_n| < \infty$

• $\mathbb{E}(M_{n+1} | \mathcal{F}_n^x) =$

② $P_x(V_a < V_b)$

$$= \frac{\phi(b)-\phi(x)}{\phi(b)-\phi(a)}$$

$$\frac{M_n - M_{n-1}}{M_{n-1} - M_2}.$$

M_{n-1}

$$\frac{p_x^2}{q_x}$$

$+$

$(1-q_x)$

$$\frac{q_x}{p_x}\cdot p_x$$

$+ \quad (1-q_x)$