

# Decomposition of Graphs: Connectivity

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Graph Algorithms  
Data Structures and Algorithms

# Learning Objectives

- Understand the importance of connected components of a graph.
- Compute the connected components of a graph.

# Outline

① Connected Components

② Algorithm

# Reachability

Want to understand which vertices in  $G$  are reachable from which others.

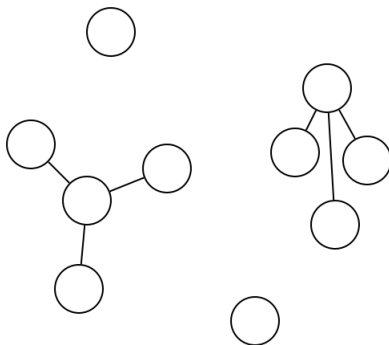
# Connected Components

## Theorem

The vertices of a graph  $G$  can be partitioned into **Connected Components** so that  $v$  is reachable from  $w$  if and only if they are in the same connected component.

# Problem

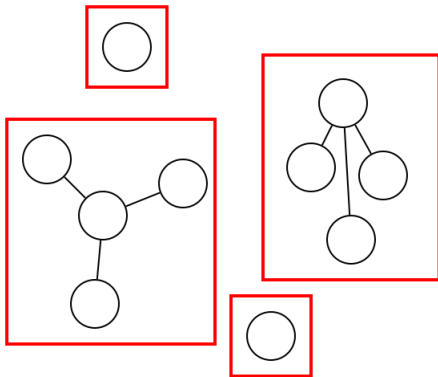
How many connected components does the graph below have?



# Solution

How many connected components does the graph below have?

4.



# Proof

## Proof.

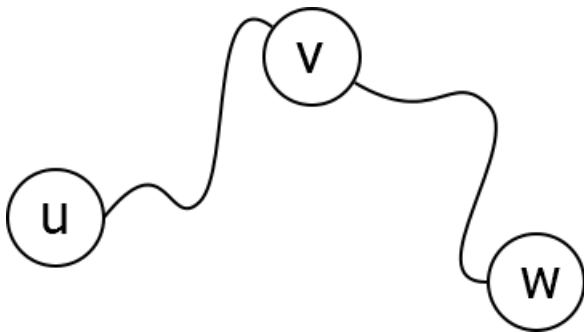
Need to show reachability is an **equivalence relation**. Namely:

- $v$  is reachable from  $v$ .
- If  $v$  reachable from  $w$ ,  $w$  reachable from  $v$ .
- If  $v$  reachable from  $u$ , and  $w$  reachable from  $v$ ,  $w$  reachable from  $u$ .





## Proof (continued)



# Problem

## Connected Components

Input: Graph  $G$

Output: The connected components of  $G$

# Outline

① Connected Components

② Algorithm

# Idea

`Explore( $v$ )` finds the connected component of  $v$ . Just need to repeat to find other components.

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Modify DFS to do this.

Modify goal to label connected components.

# Modification of Explore

Explore( $v$ )

visited( $v$ )  $\leftarrow$  true

CCnum( $v$ )  $\leftarrow$   $cc$

for ( $v, w$ )  $\in E$ :

    if not visited( $w$ ):

        Explore( $w$ )

# Modifications of DFS

DFS( $G$ )

for all  $v \in V$  mark  $v$  unvisited

$cc \leftarrow 1$

for  $v \in V$ :

    if not visited( $v$ ):

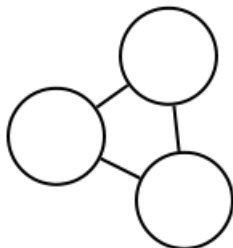
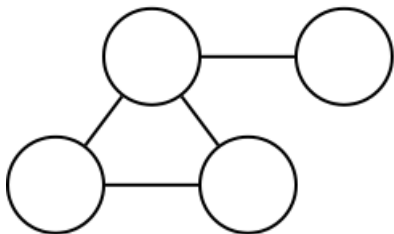
        Explore( $v$ )

$cc \leftarrow cc + 1$



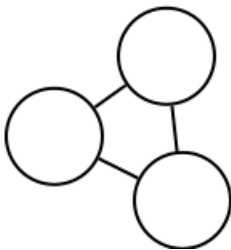
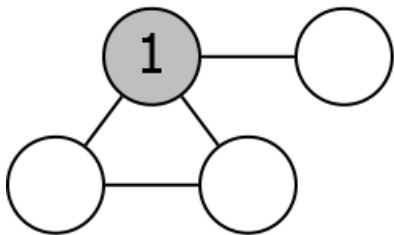
# Example

**CC: 1**



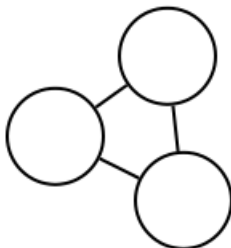
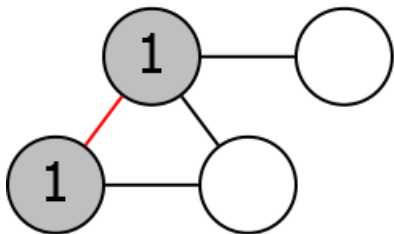
# Example

CC: 1



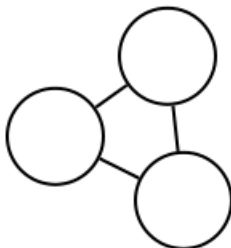
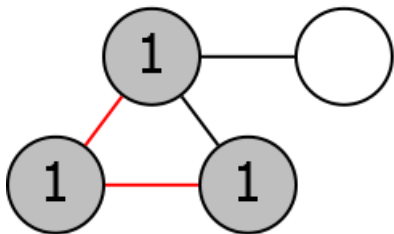
# Example

CC: 1



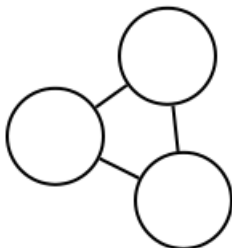
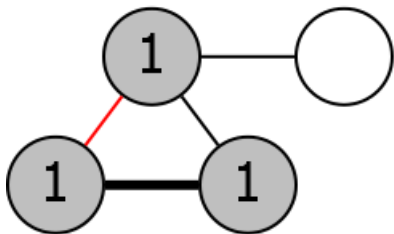
# Example

CC: 1



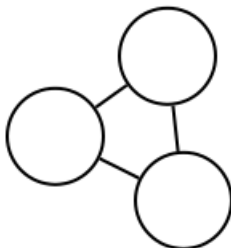
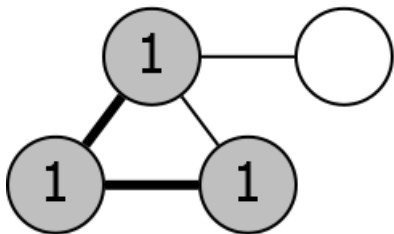
# Example

CC: 1



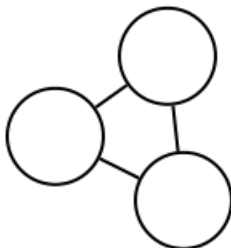
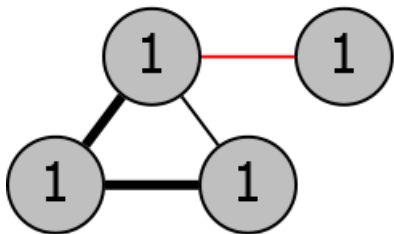
# Example

CC: 1



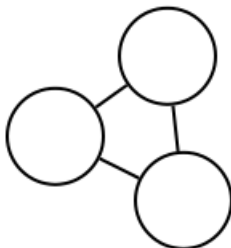
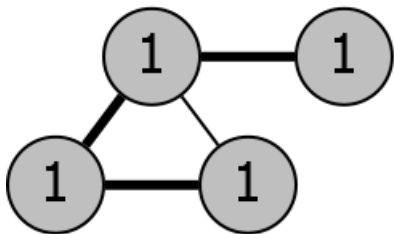
# Example

CC: 1



# Example

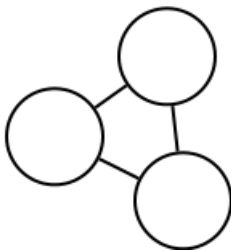
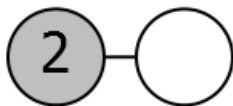
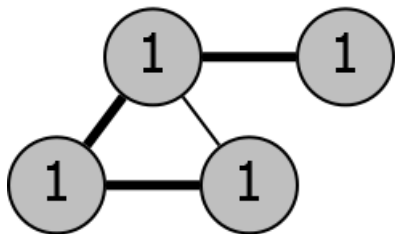
CC: 1





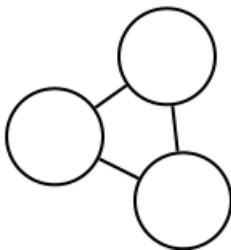
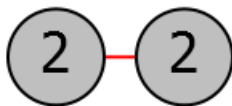
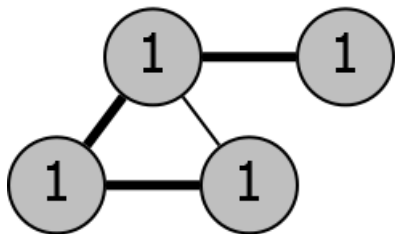
# Example

CC: 2



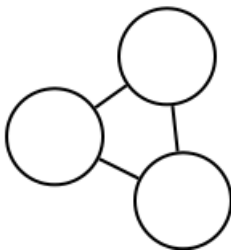
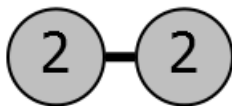
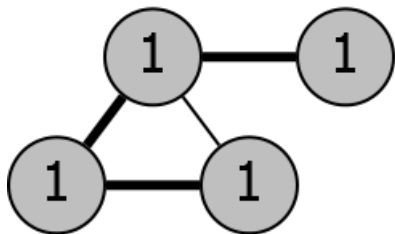
# Example

CC: 2



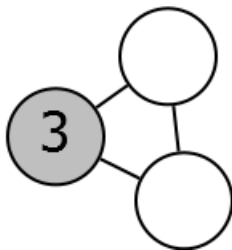
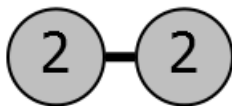
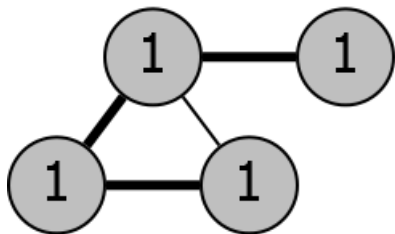
# Example

CC: 2



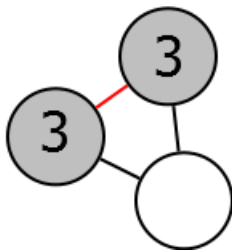
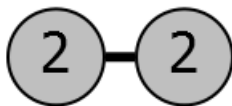
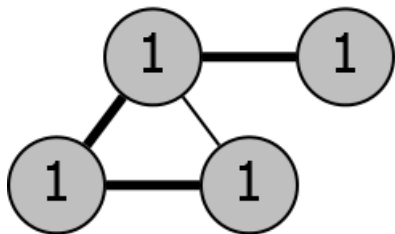
# Example

CC: 3



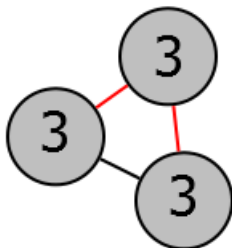
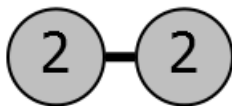
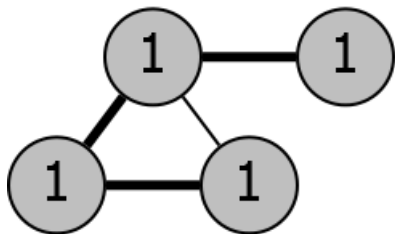
# Example

CC: 3



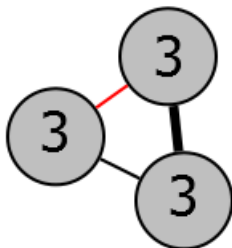
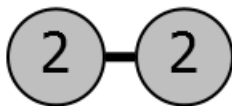
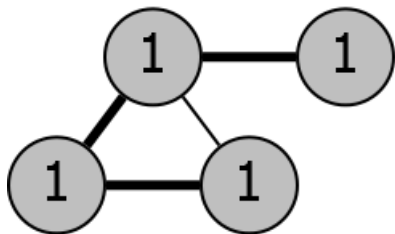
# Example

CC: 3



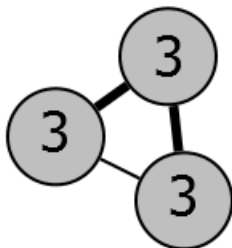
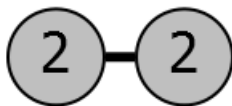
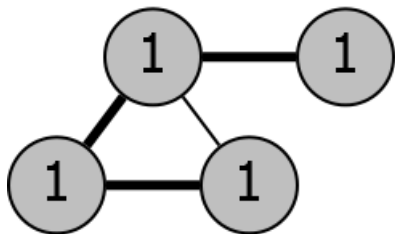
# Example

CC: 3



# Example

CC: 3





# Correctness

- Each new explore finds new connected component.
- Eventually find every vertex.
- Runtime still  $O(|V| + |E|)$ .

# Next Time

Other applications of DFS.