

Decomposition of Graphs: Computing Strongly Connected Components

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Graph Algorithms
Data Structures and Algorithms

Learning Objectives

- Efficiently compute the strongly connected components of a directed graph.

Last Time

- Connectivity in directed graphs.
- Strongly connected components.
- Metagraph.

Problem

Strongly Connected Components

Input: A directed graph G

Output: The strongly connected components of G .

Easy Algorithm

EasySCC(G)

```
for each vertex  $v$ :  
    run explore( $v$ ) to determine  
        vertices reachable from  $v$   
for each vertex  $v$ :  
    find the  $u$  reachable from  $v$  that  
        can also reach  $v$   
these are the SCCs
```

Runtime $O(|V|^2 + |V||E|)$. Want faster.

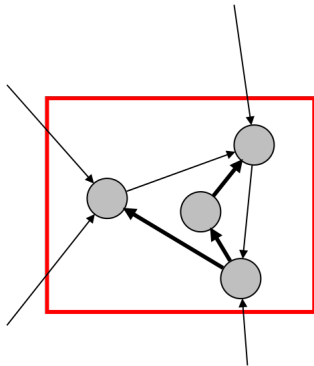
Outline

1 Sink Components

2 Algorithm

Sink Components

Idea: If v is in a sink SCC, $\text{explore}(v)$ finds vertices reachable from v . This is exactly the SCC of v .



Finding Sink Components

Need a way to find a sink SCC.

Theorem

Theorem

If \mathcal{C} and \mathcal{C}' are two strongly connected components with an edge from some vertex of \mathcal{C} to some vertex of \mathcal{C}' , then largest post in \mathcal{C} bigger than largest post in \mathcal{C}' .

Proof

Cases:

- Visit \mathcal{C} before visit \mathcal{C}'
- Visit \mathcal{C}' before visit \mathcal{C}

Case I

Visit \mathcal{C} first.

- Can reach everything in \mathcal{C}' from \mathcal{C} .
- Explore all of \mathcal{C}' while exploring \mathcal{C} .
- \mathcal{C} has largest post.

Case II

Visit \mathcal{C}' first.

- Cannot reach \mathcal{C} from \mathcal{C}'
- Must finish exploring \mathcal{C}' before exploring \mathcal{C}
- \mathcal{C} has largest post.

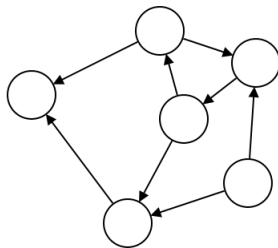
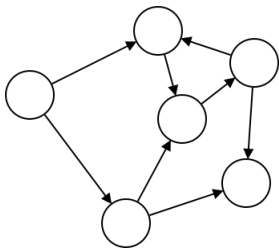
Conclusion

The vertex with the largest postorder number is in a source component!

Problem: We wanted a sink component.

Reverse Graph

Let G^R be the graph obtained from G by reversing all of the edges.



Reverse Graph Components

- G^R and G have same SCCs.
- Source components of G^R are sink components of G .

Find sink components of G by running DFS on G^R .

Problem

Which of the following is true?

- The vertex with largest postorder in G^R is in a sink SCC of G .
- The vertex with the largest preorder in G is in a sink SCC of G .
- The vertex with the smallest postorder in G is in a sink SCC of G .

Solution

Which of the following is true?

- The vertex with largest postorder in G^R is in a sink SCC of G .
- The vertex with the largest preorder in G is in a sink SCC of G .
- The vertex with the smallest postorder in G is in a sink SCC of G .

Outline

1 Sink Components

2 Algorithm

Basic Algorithm

$\text{SCCs}(G)$

run $\text{DFS}(G^R)$

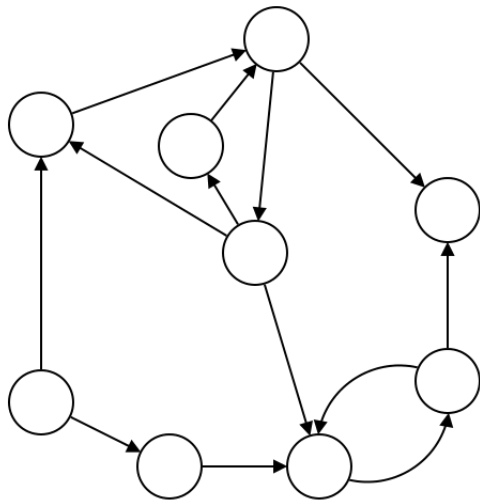
let v have largest post number

run $\text{Explore}(v)$

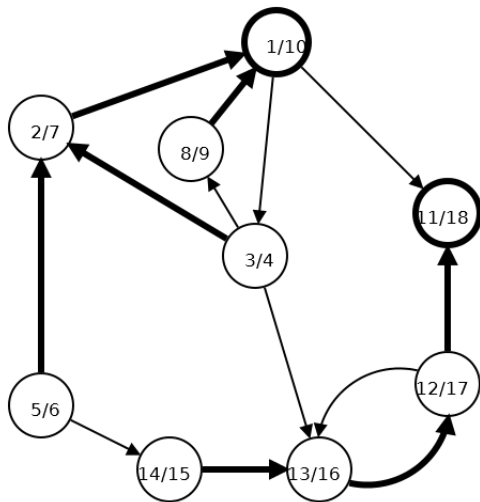
vertices found are first SCC

Remove from G and repeat

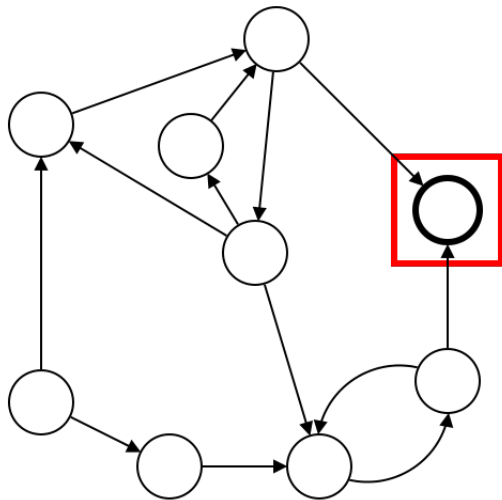
Example



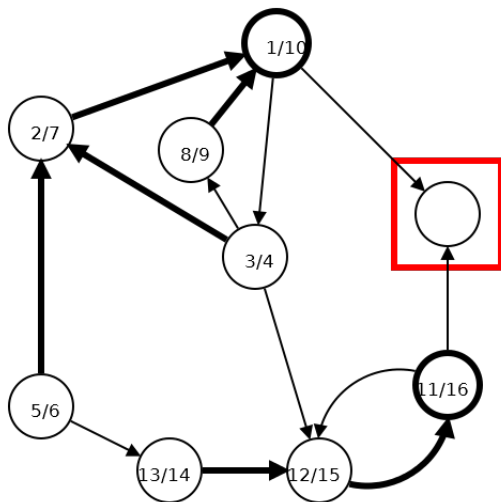
Example



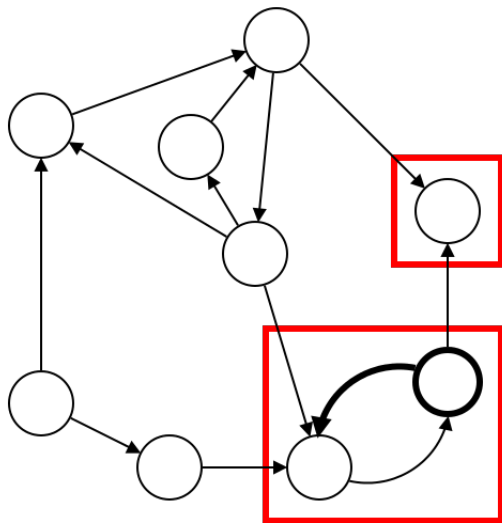
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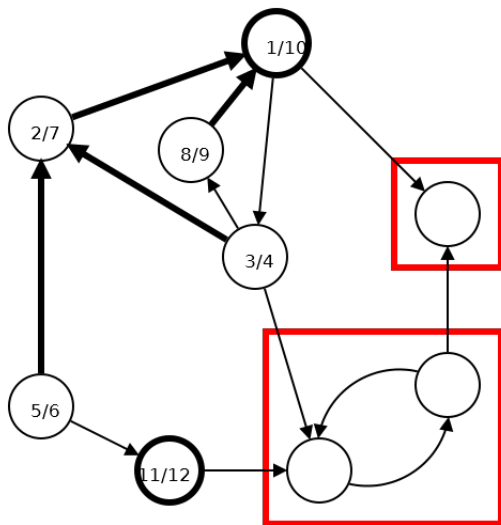
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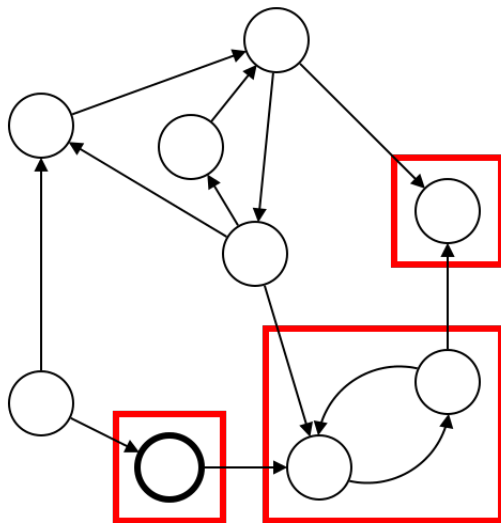
Example



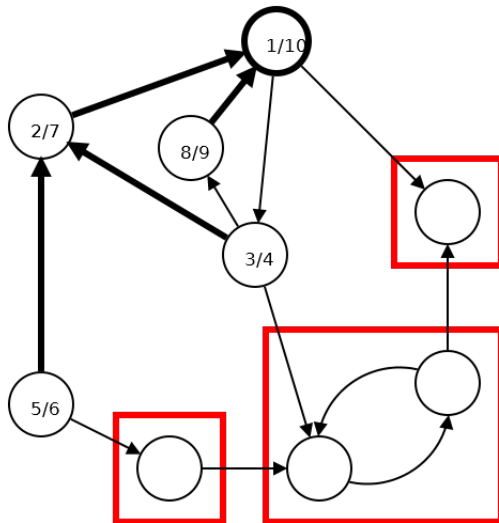
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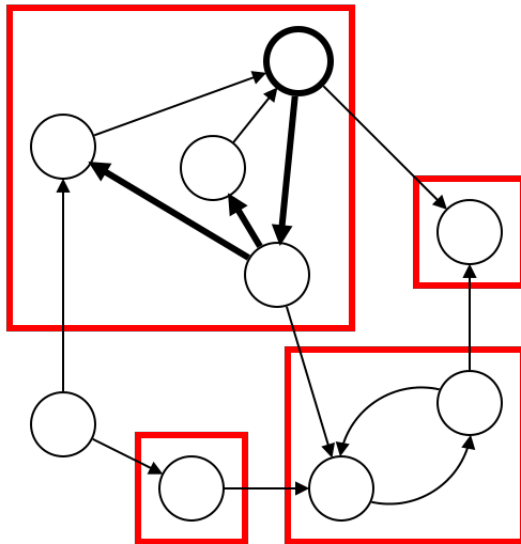
Example



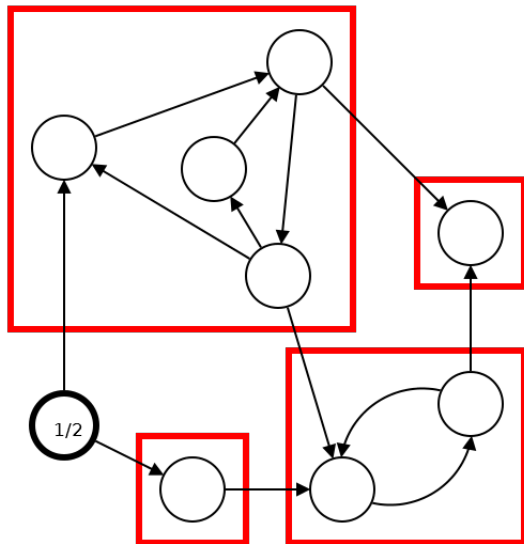
Example



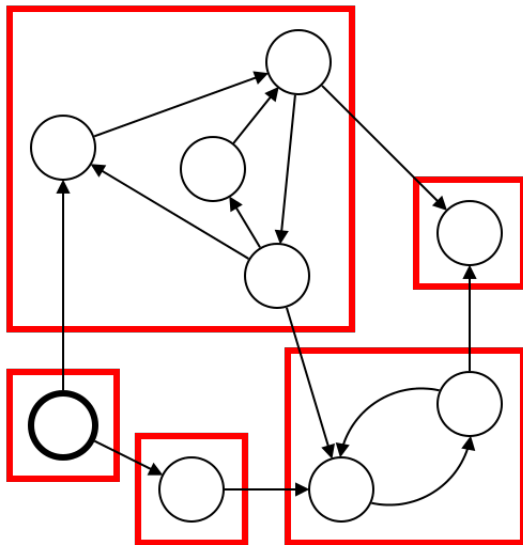
Example



Example



Example



Improvement

- Don't need to rerun DFS on G^R .
- Largest remaining post number comes from sink component.

New Algorithm

$\text{SCCs}(G)$

Run $\text{DFS}(G^R)$

for $v \in V$ in reverse postorder:

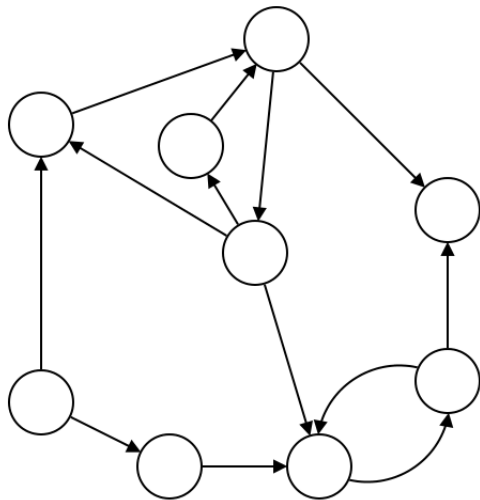
 if not visited(v):

 Explore(v)

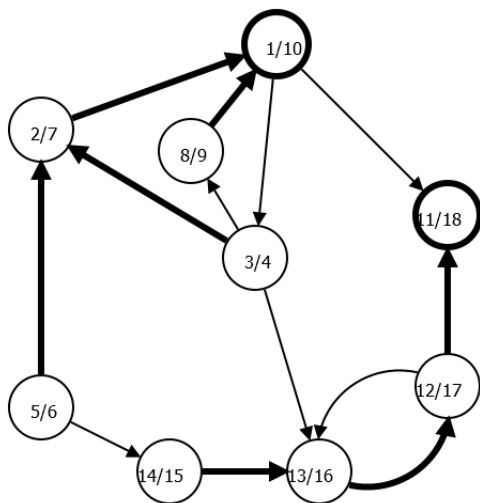
 mark visited vertices

 as new SCC

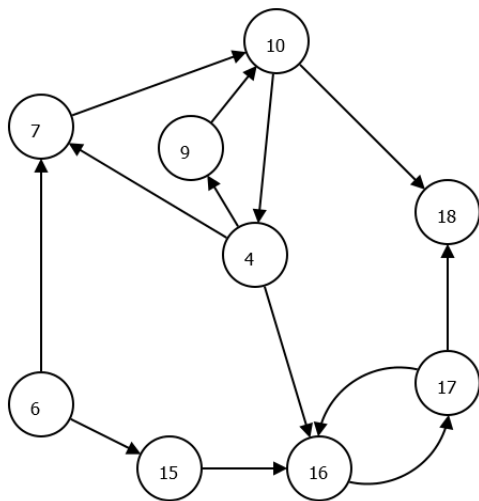
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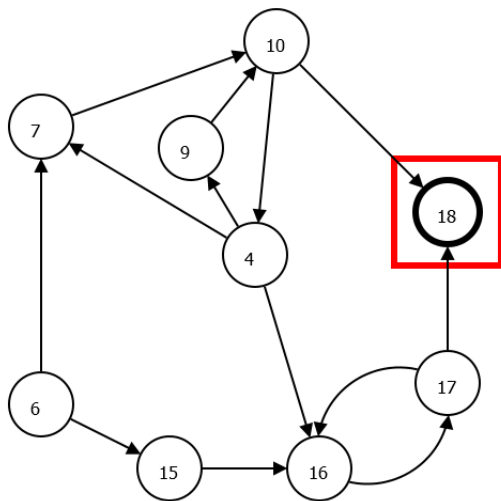
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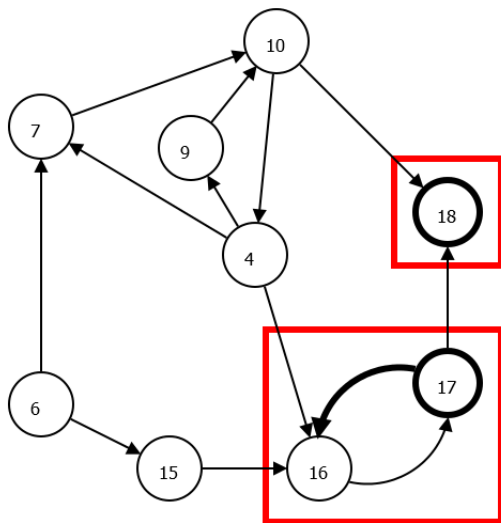
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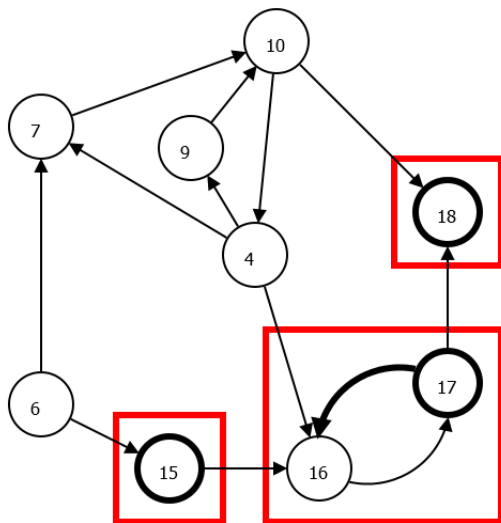
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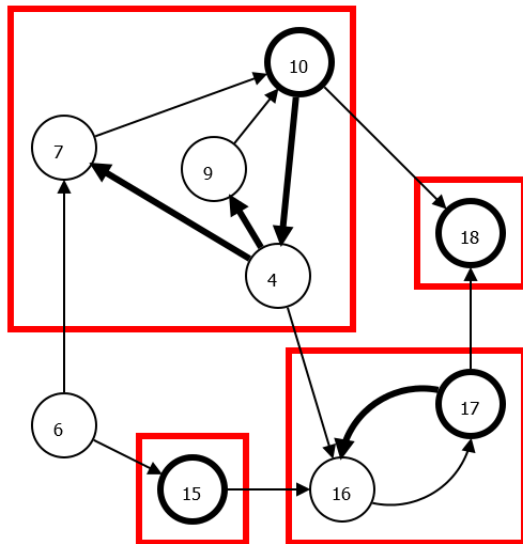
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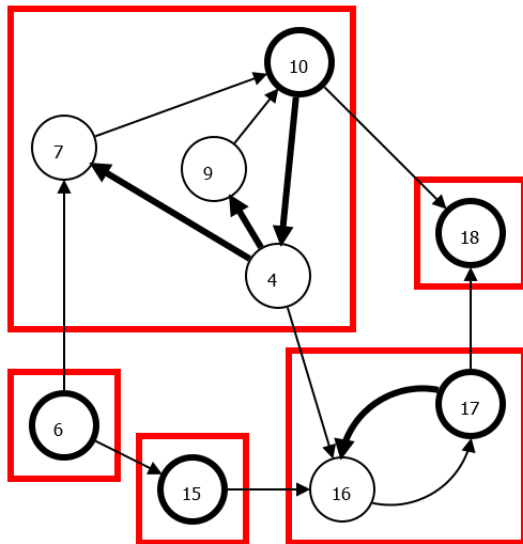
Example



Example



Example



Runtime

- Essentially DFS on G^R and then on G .
- Runtime $O(|V| + |E|)$.