Advanced Shortest Paths: Bidirectional Dijkstra

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Graph Algorithms

Data Structures and Algorithms

Outline

Shortest Path

Input: A graph G with *non-negative* edge weights, a source vertex s and a target vertex t.

Output: The shortest path between s and t in G.

Why not just Dijkstra?

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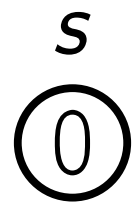
- Millions of users of Google Maps want the result in a blink of an eye, all at the same time
- Need something significantly faster

Dijkstra Progression



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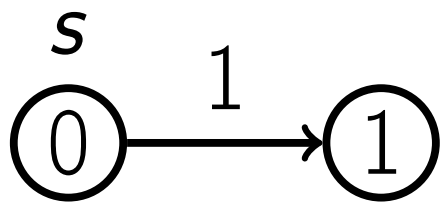

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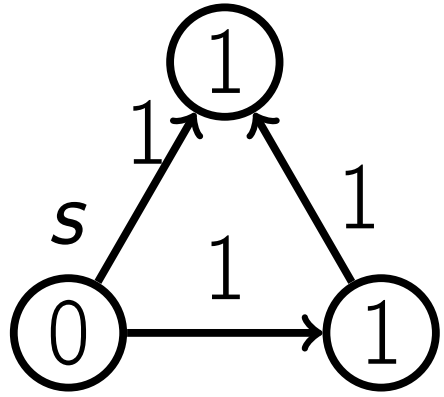

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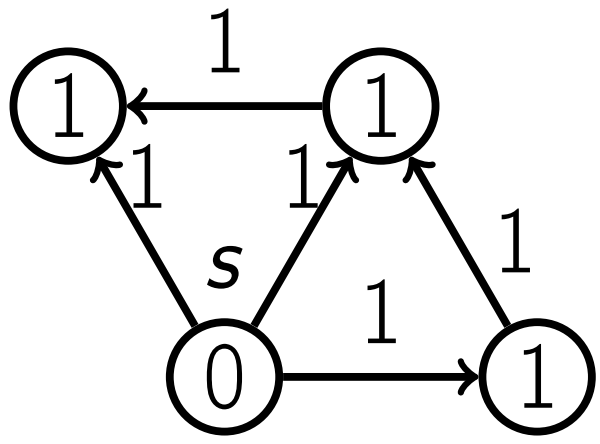

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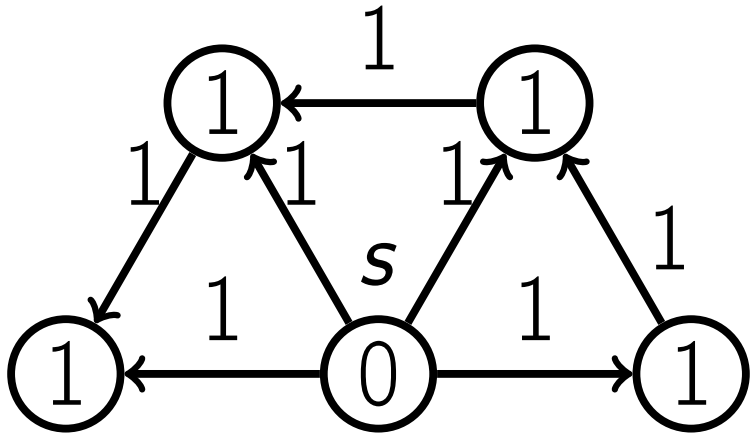

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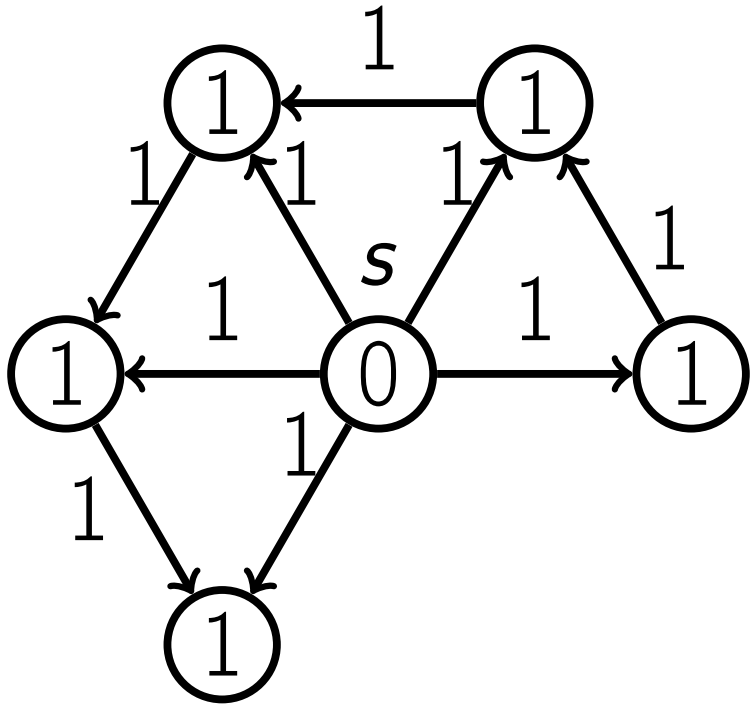

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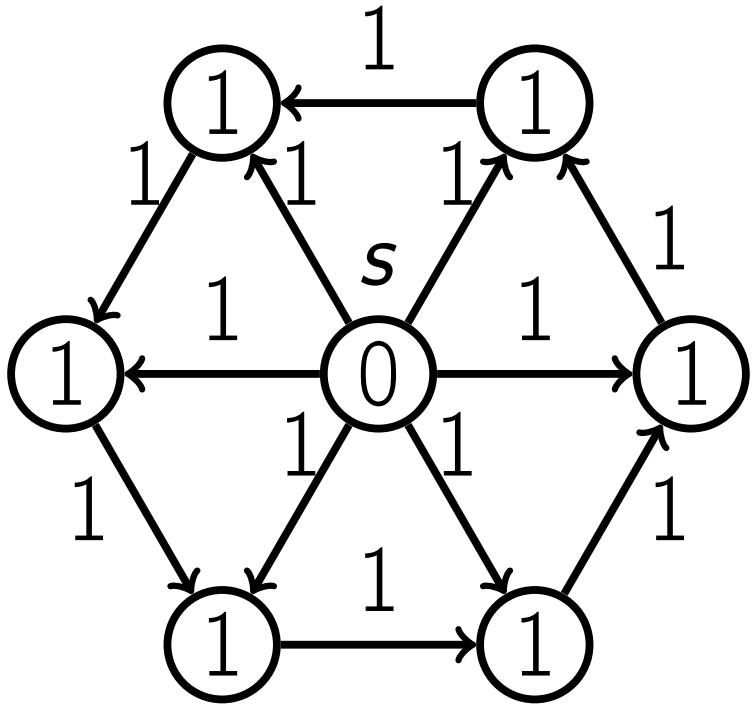

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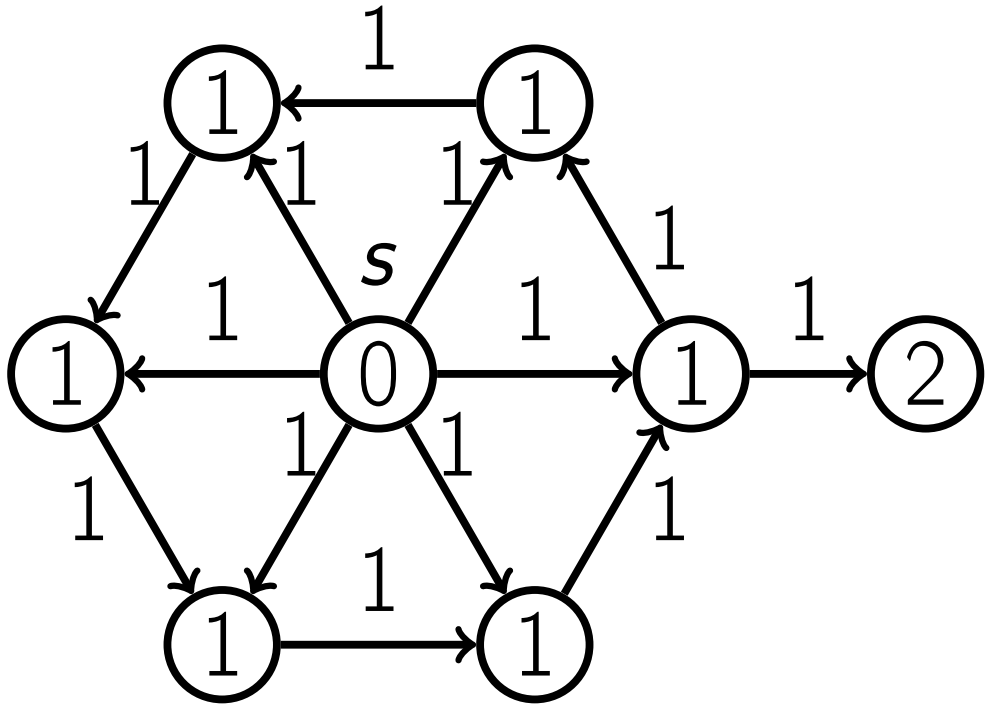

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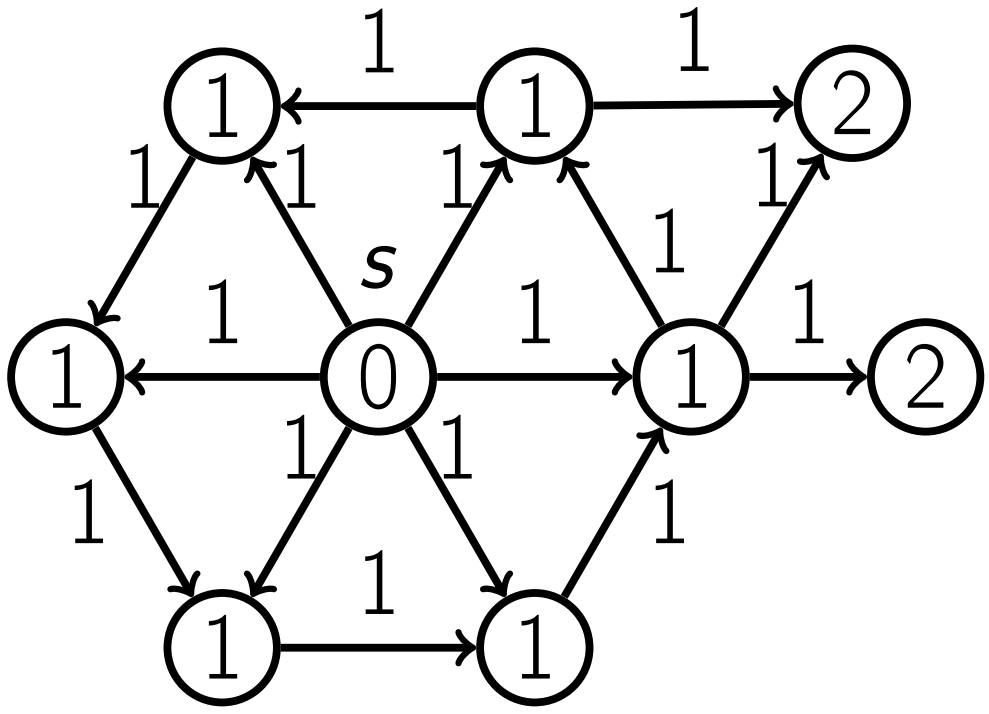

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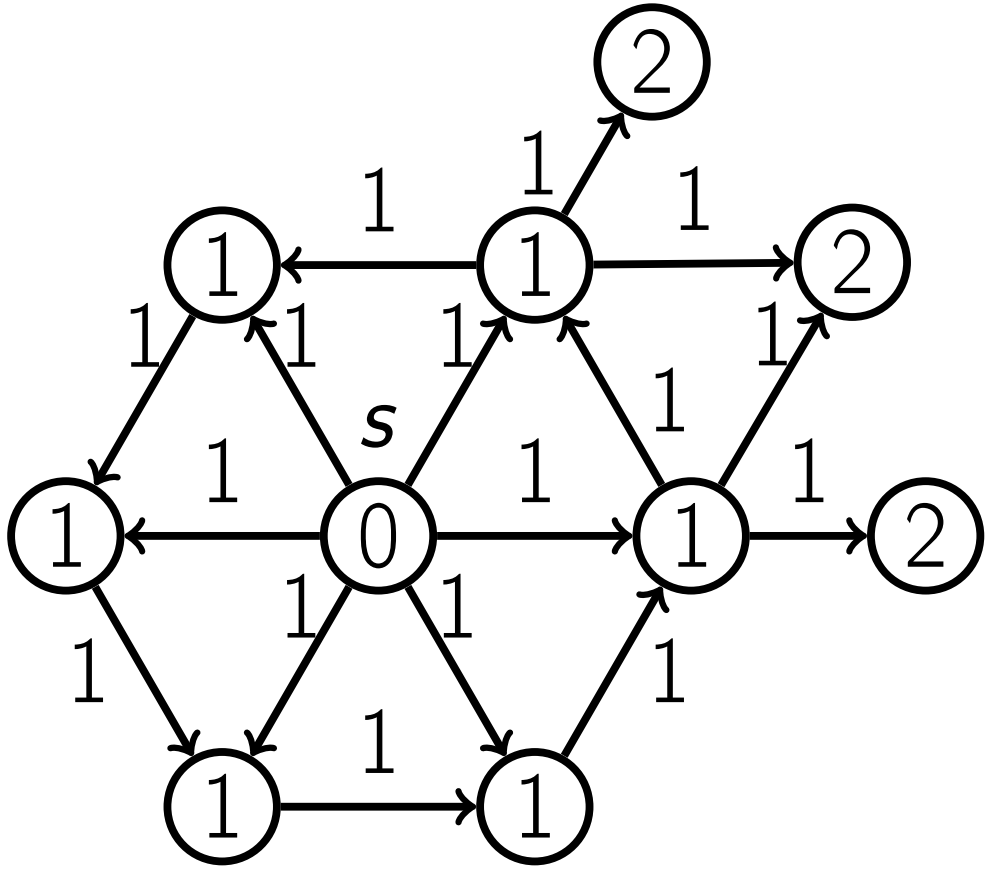

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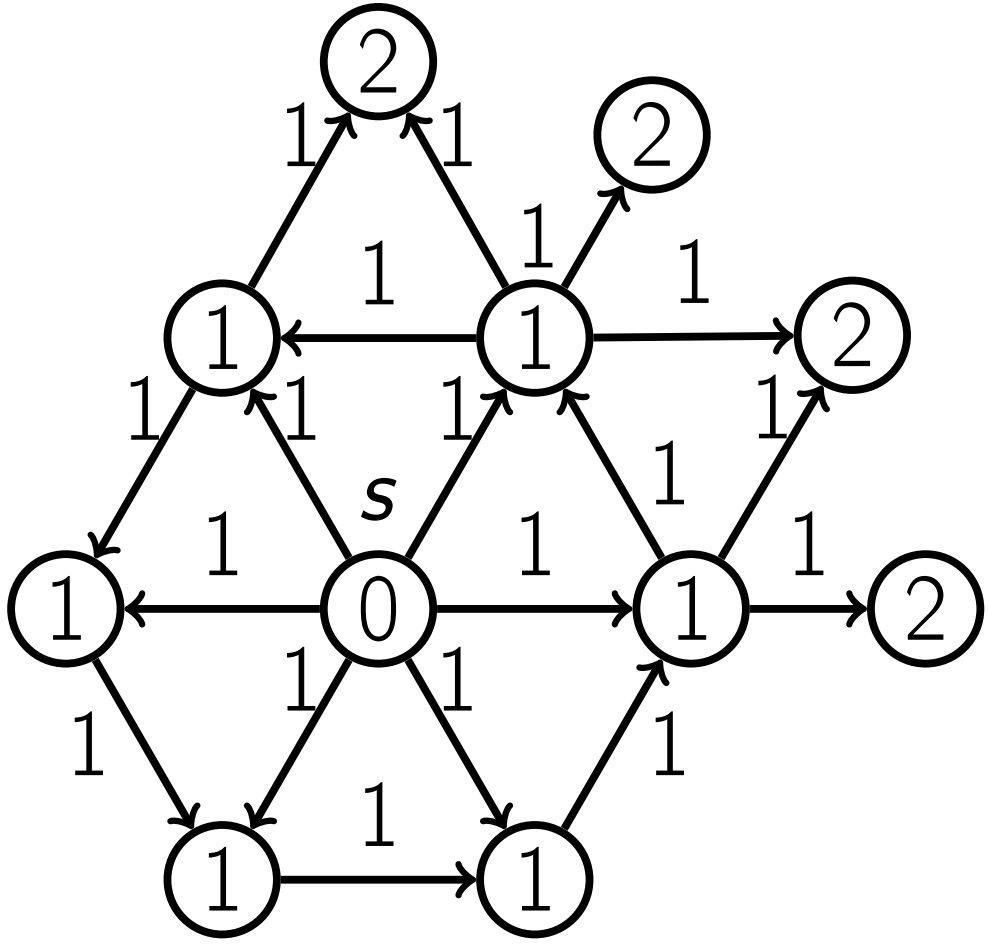

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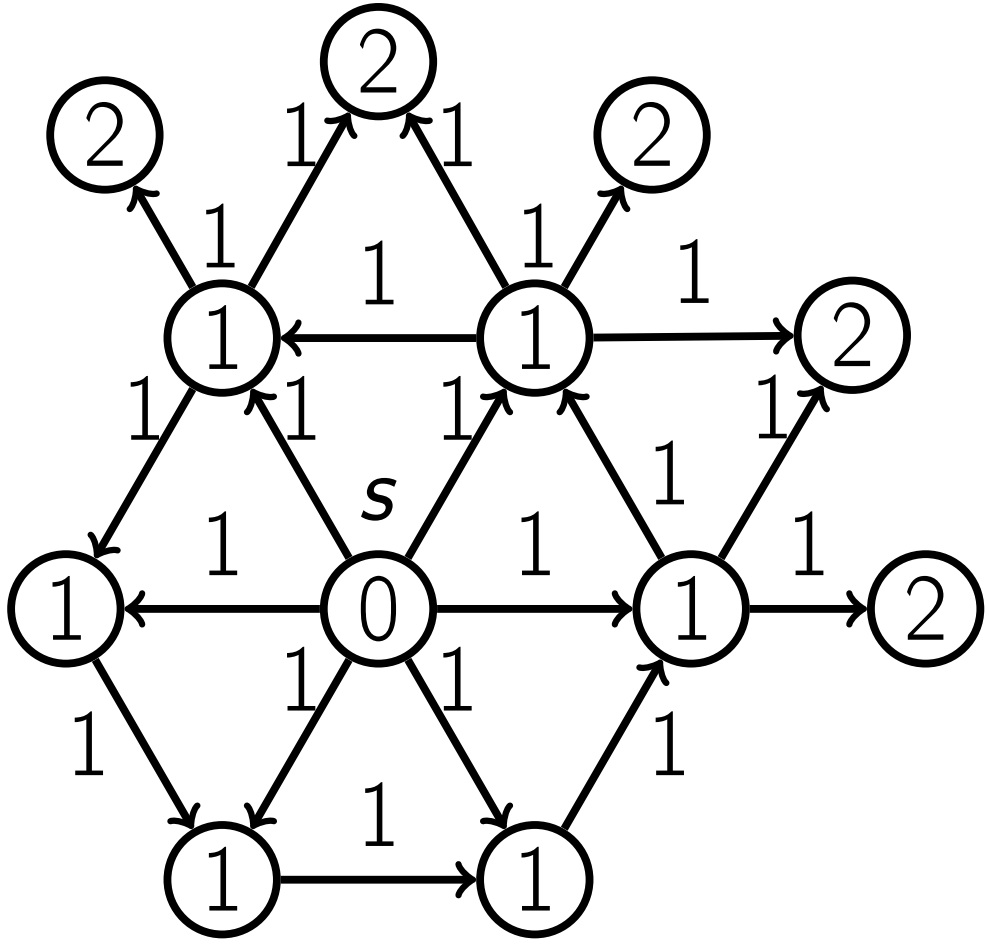

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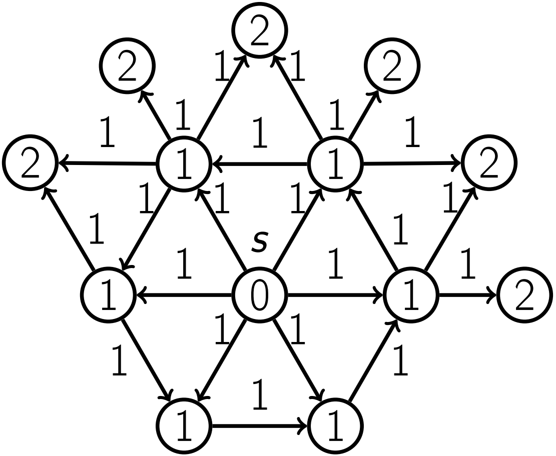

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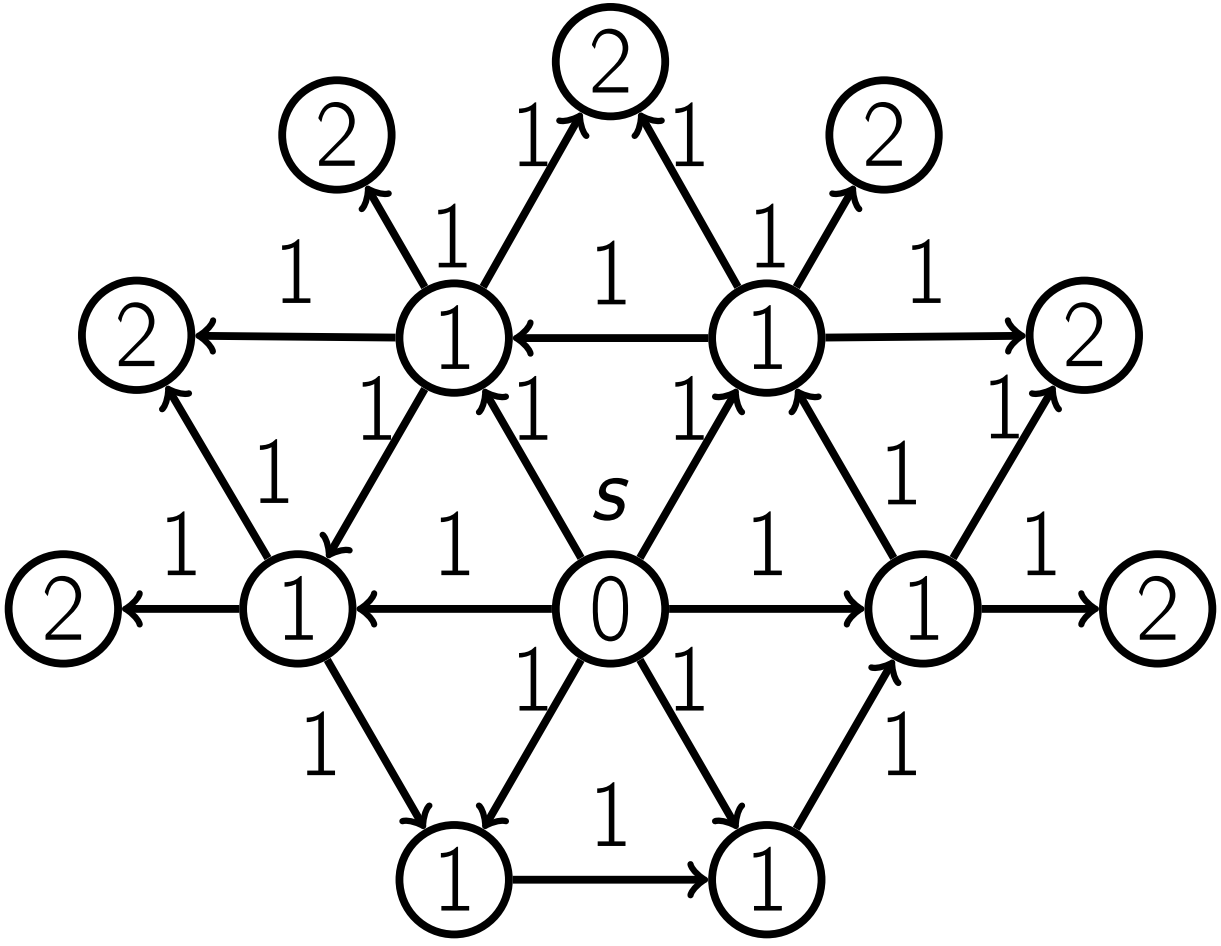

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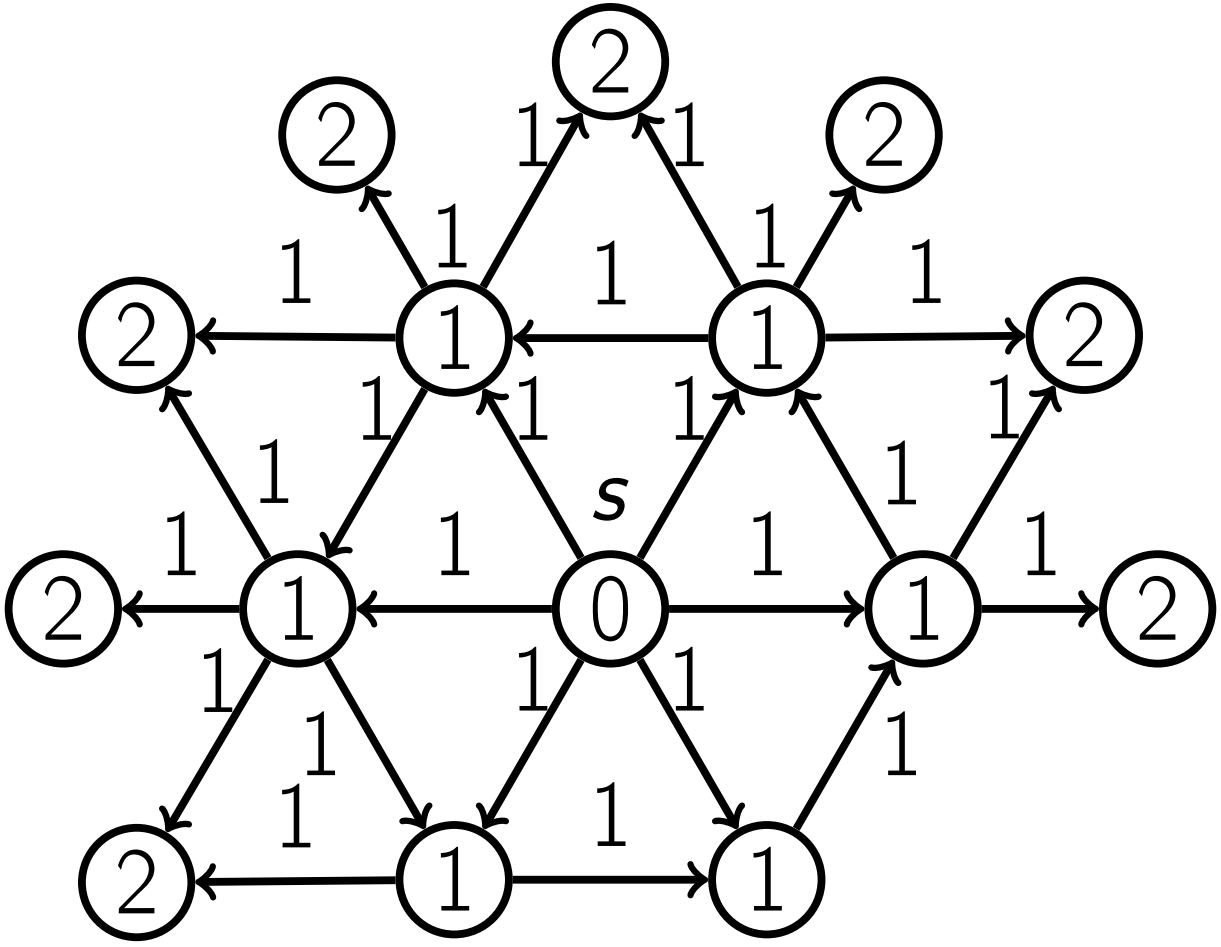

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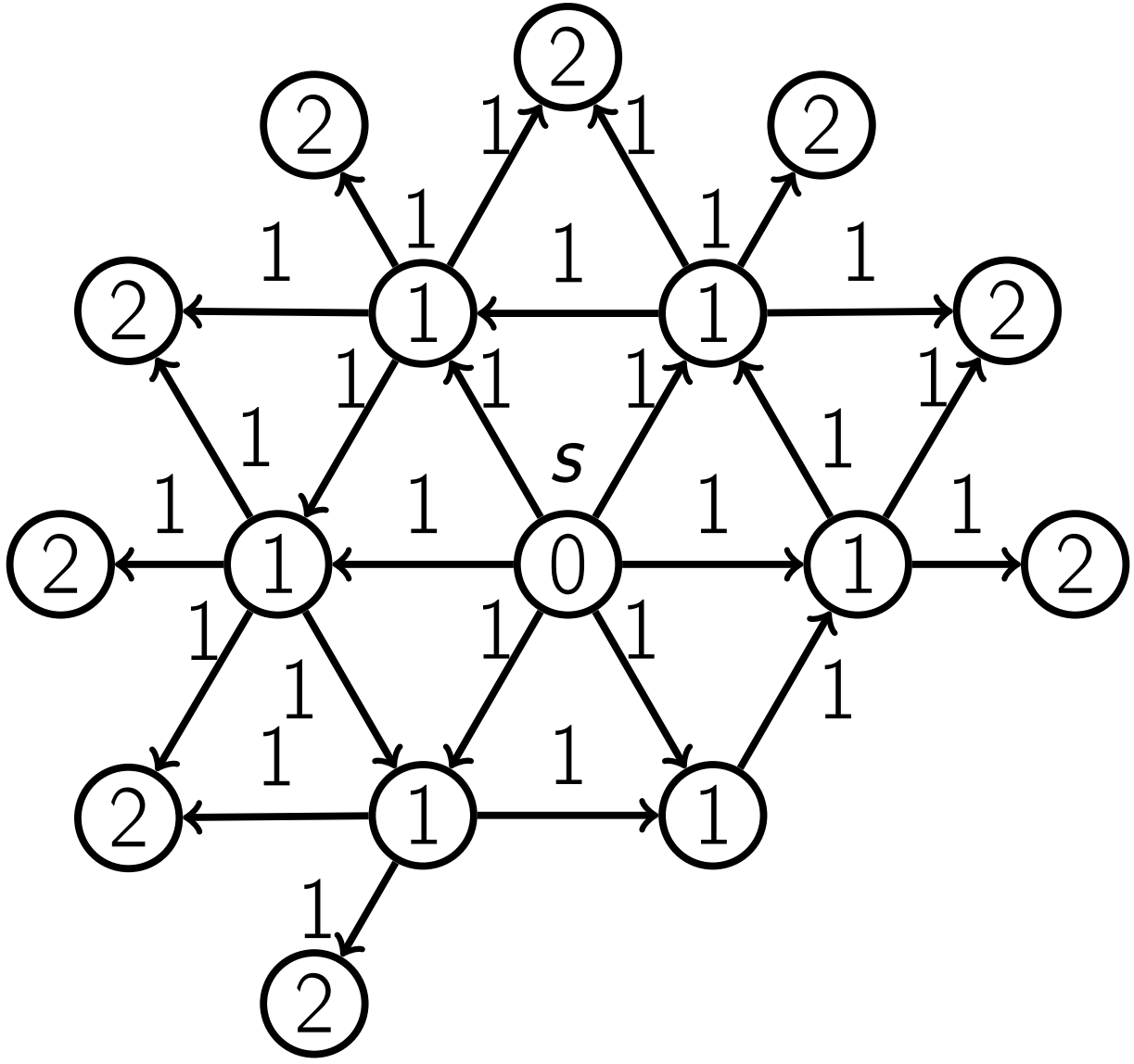

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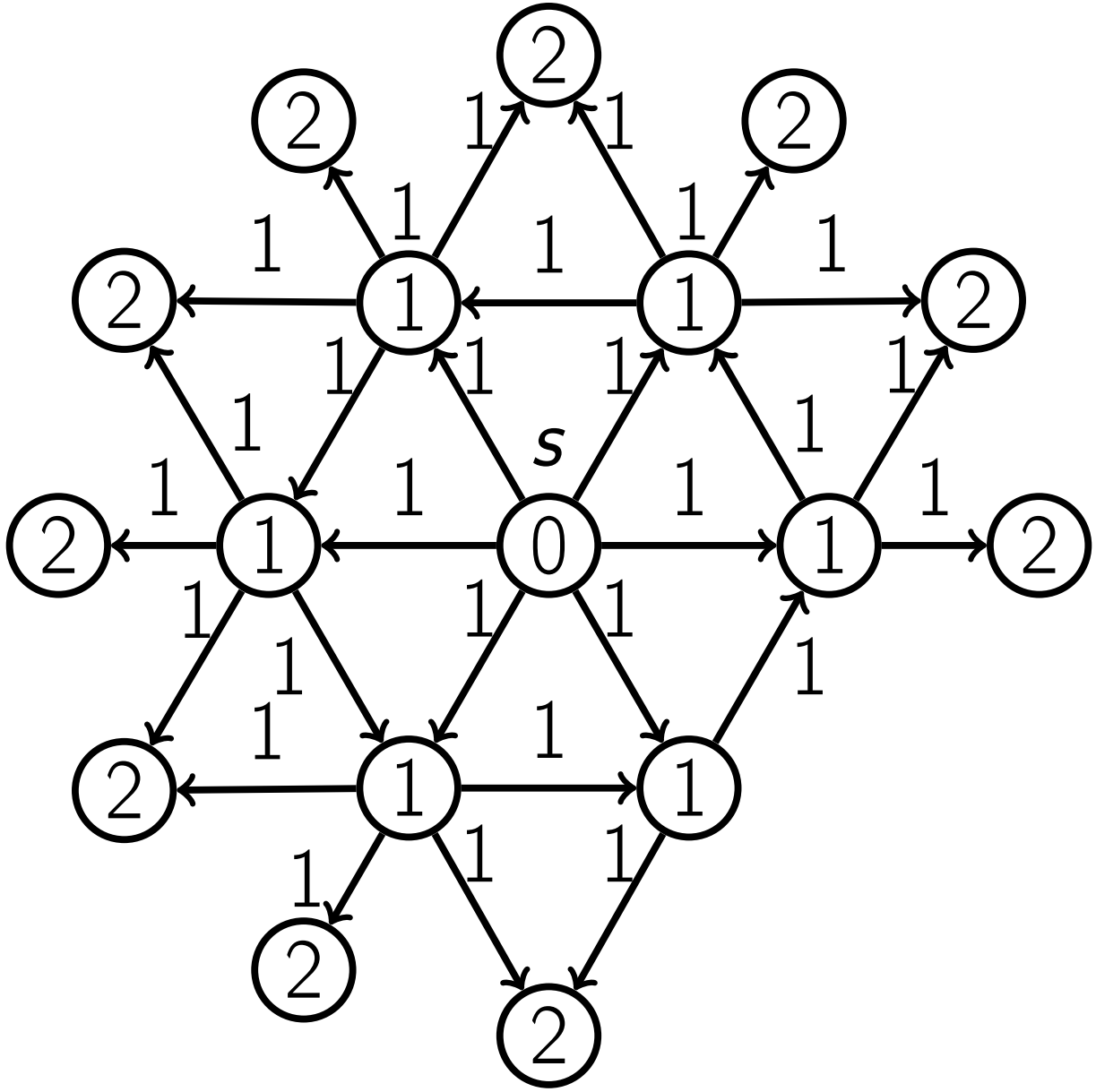

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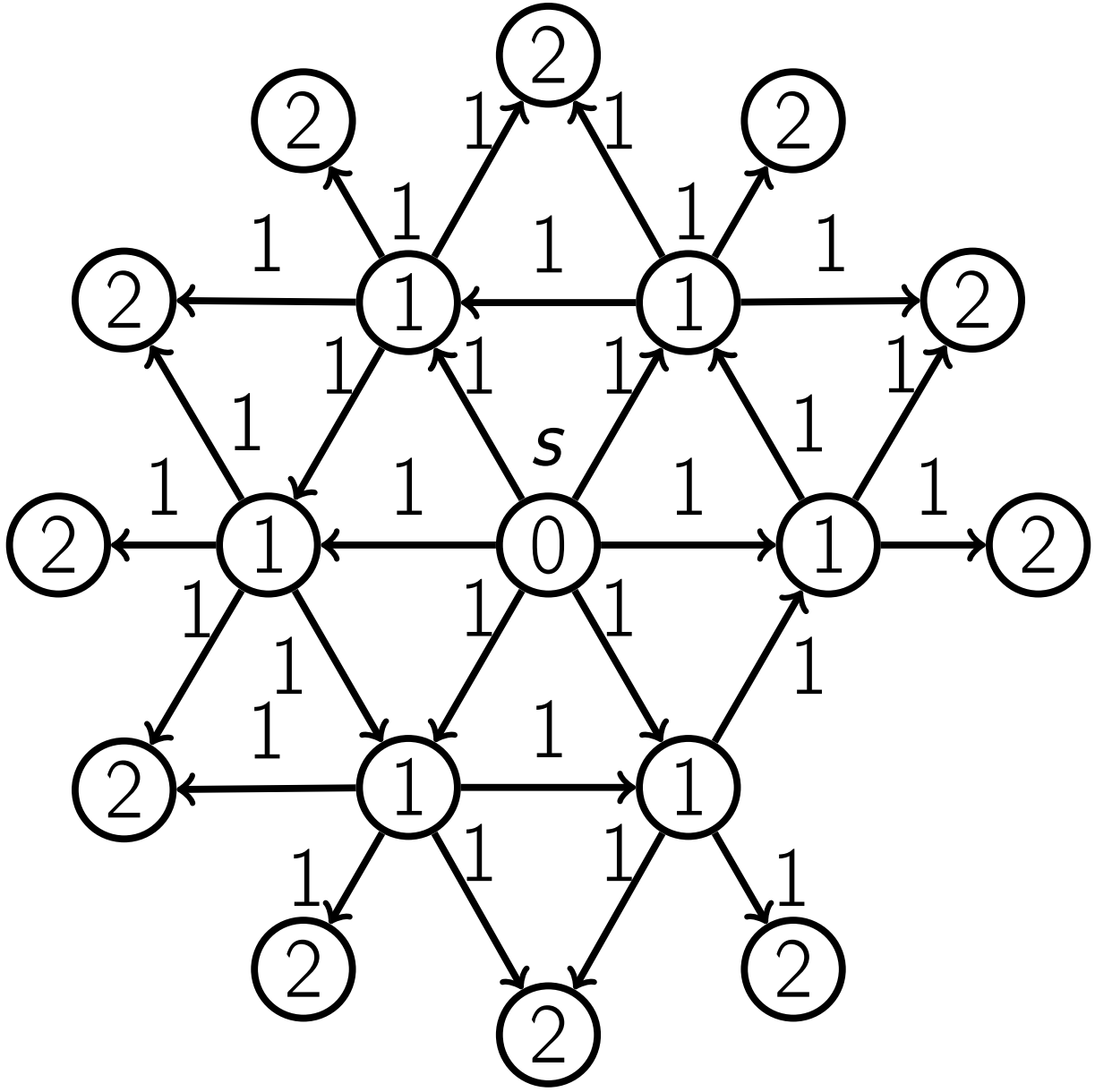

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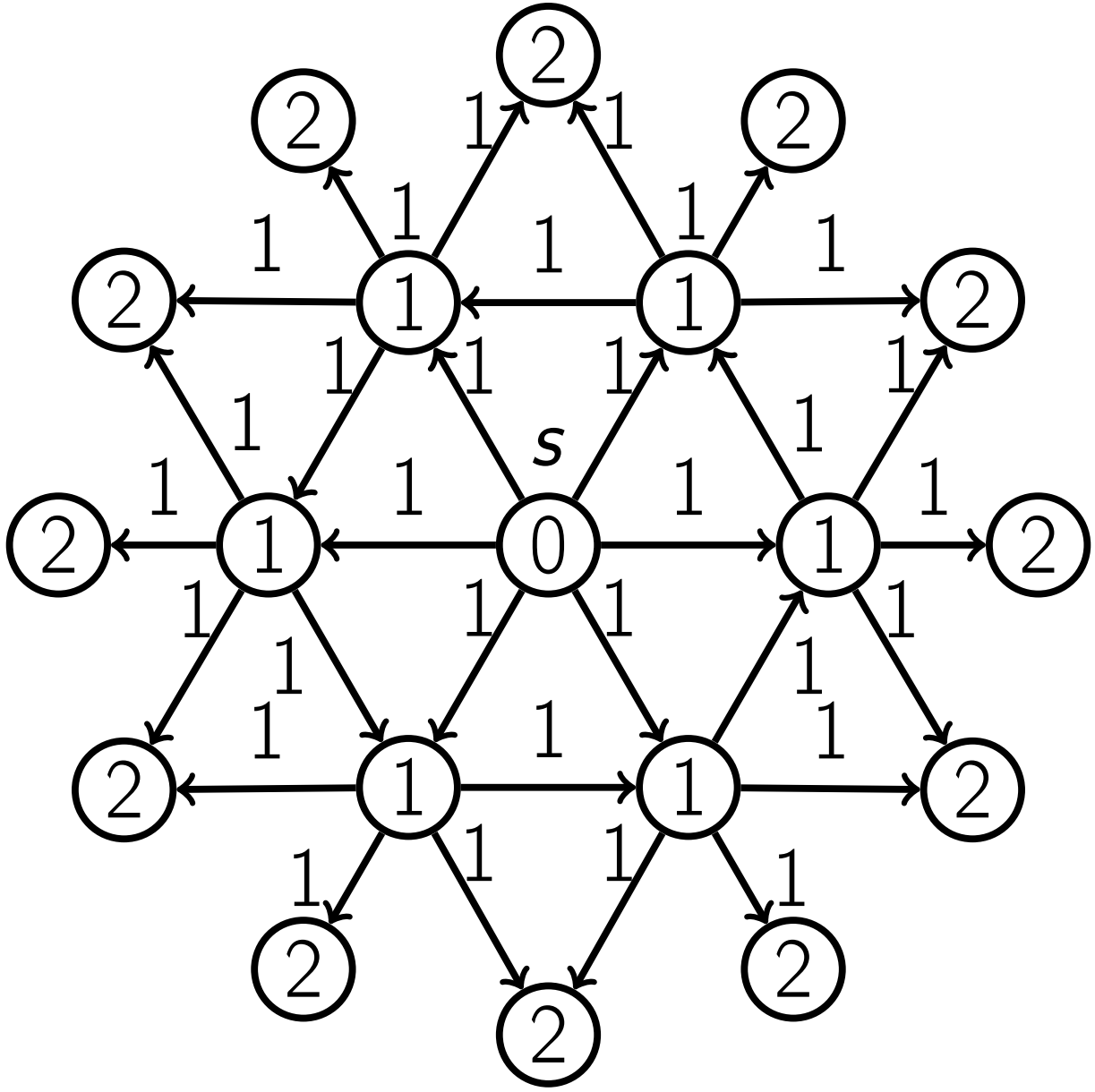

Idea: Growing Circle

Lemma

When a vertex u is selected via `ExtractMin`, $\text{dist}[u] = d(s, u)$.

- When a vertex is extracted from the priority queue for processing, all the vertices at smaller distances have already been processed

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When a vertex u is selected via `ExtractMin`, $\text{dist}[u] = d(s, u)$.

- When a vertex is extracted from the priority queue for processing, all the vertices at smaller distances have already been processed
- A "circle" of processed vertices grows

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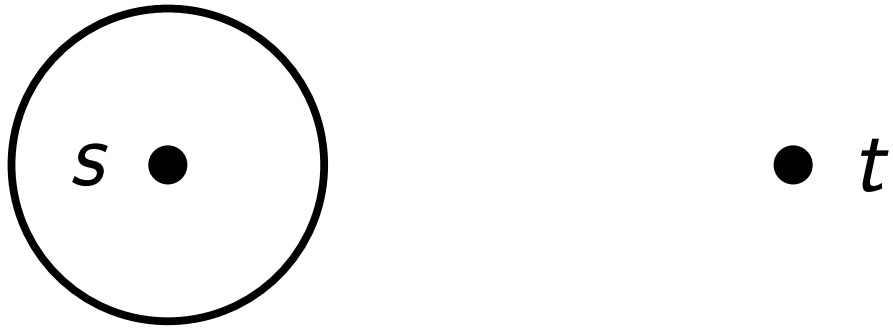

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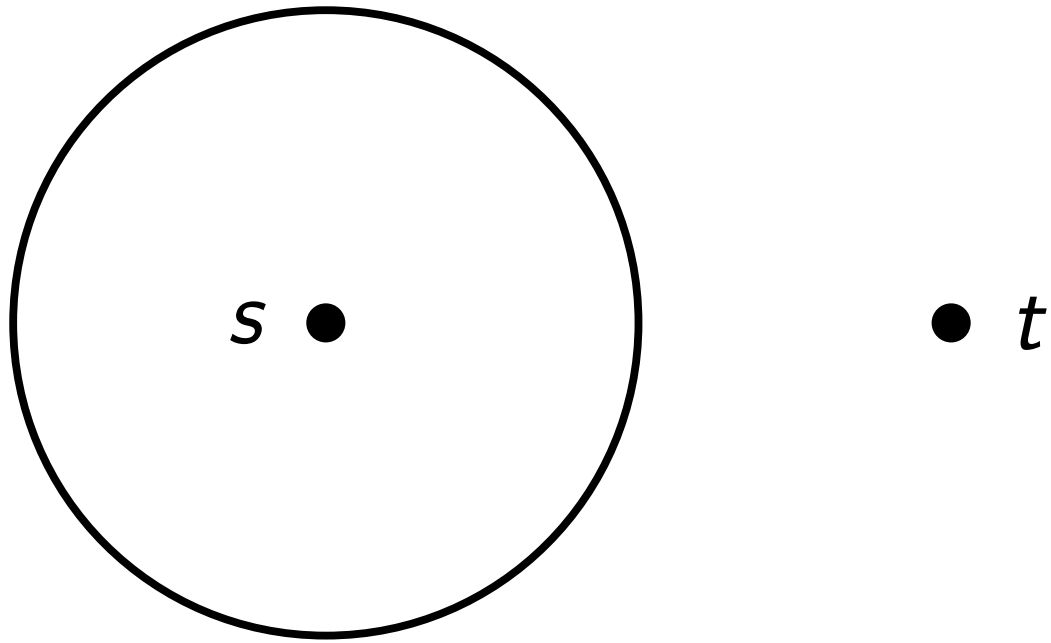

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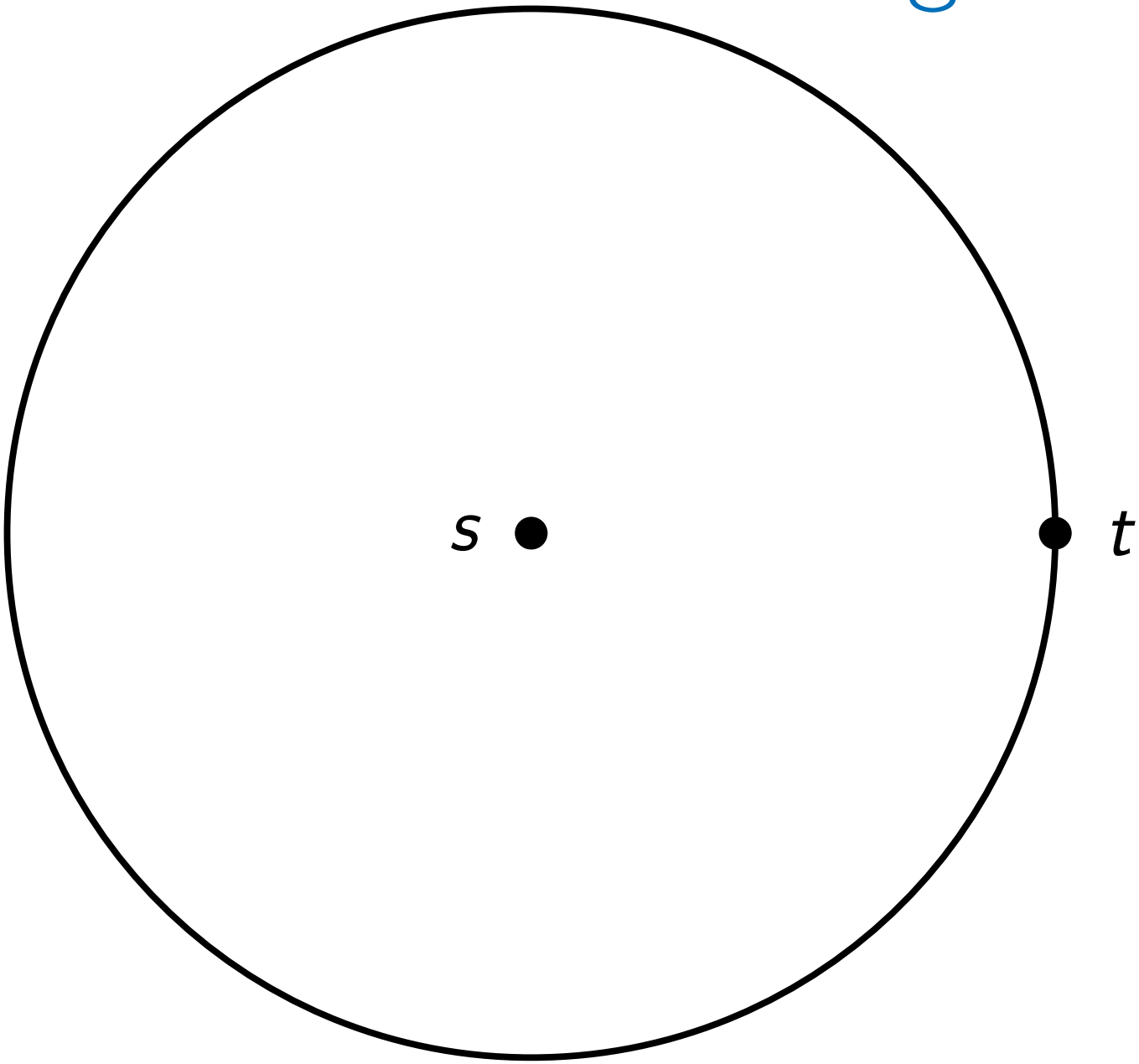

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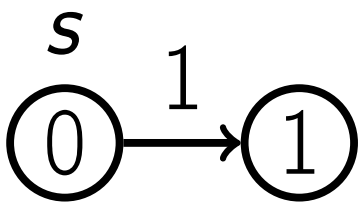

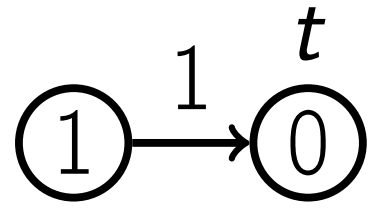

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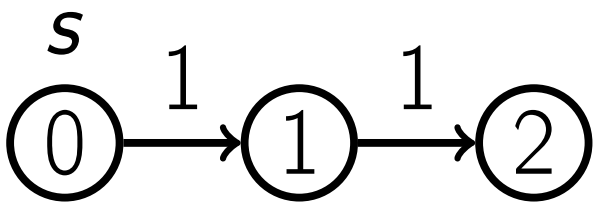

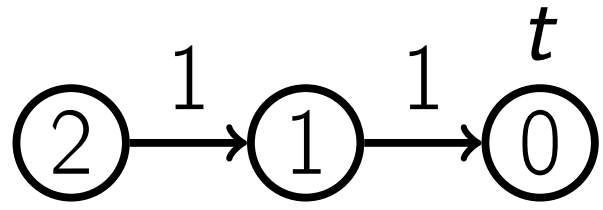

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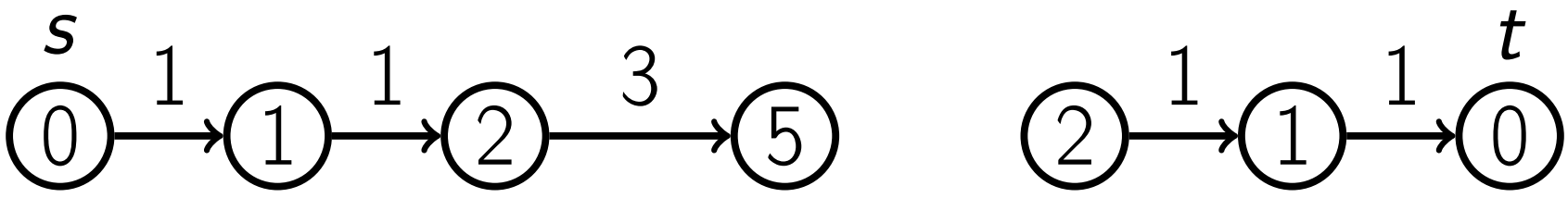

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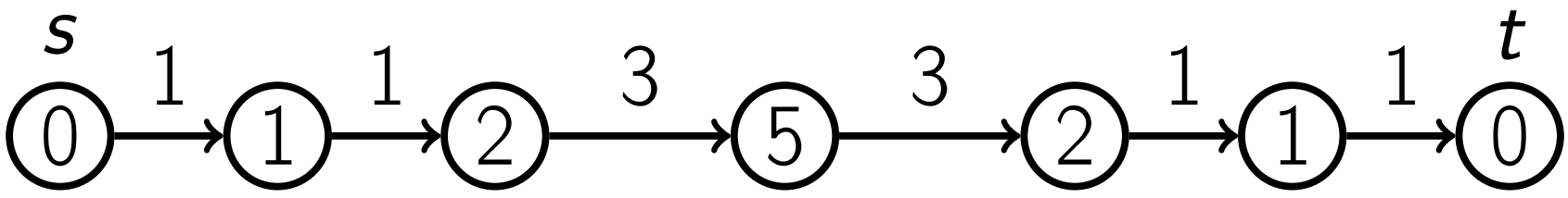

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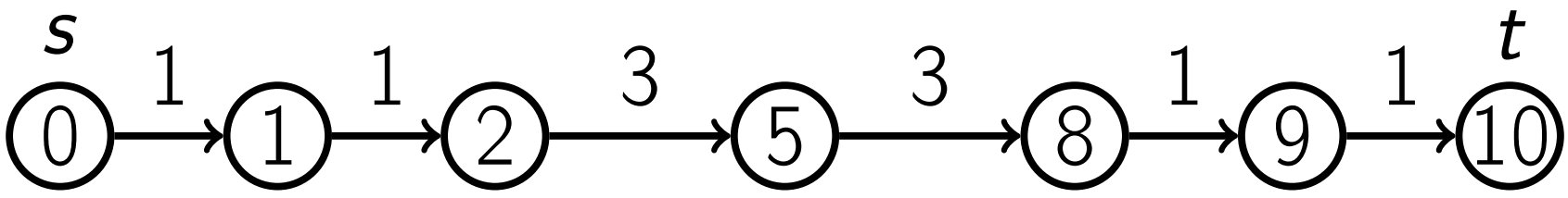

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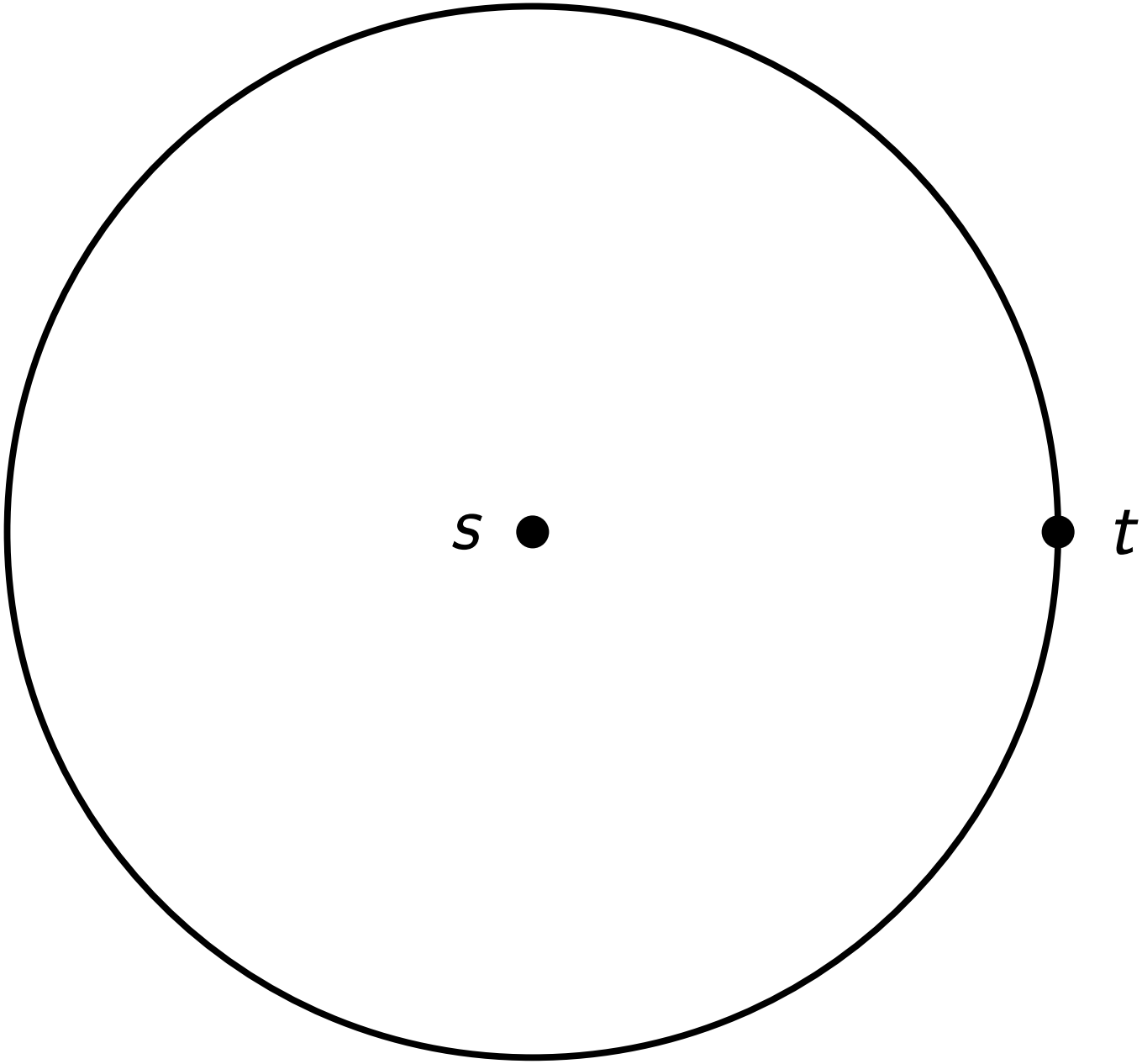

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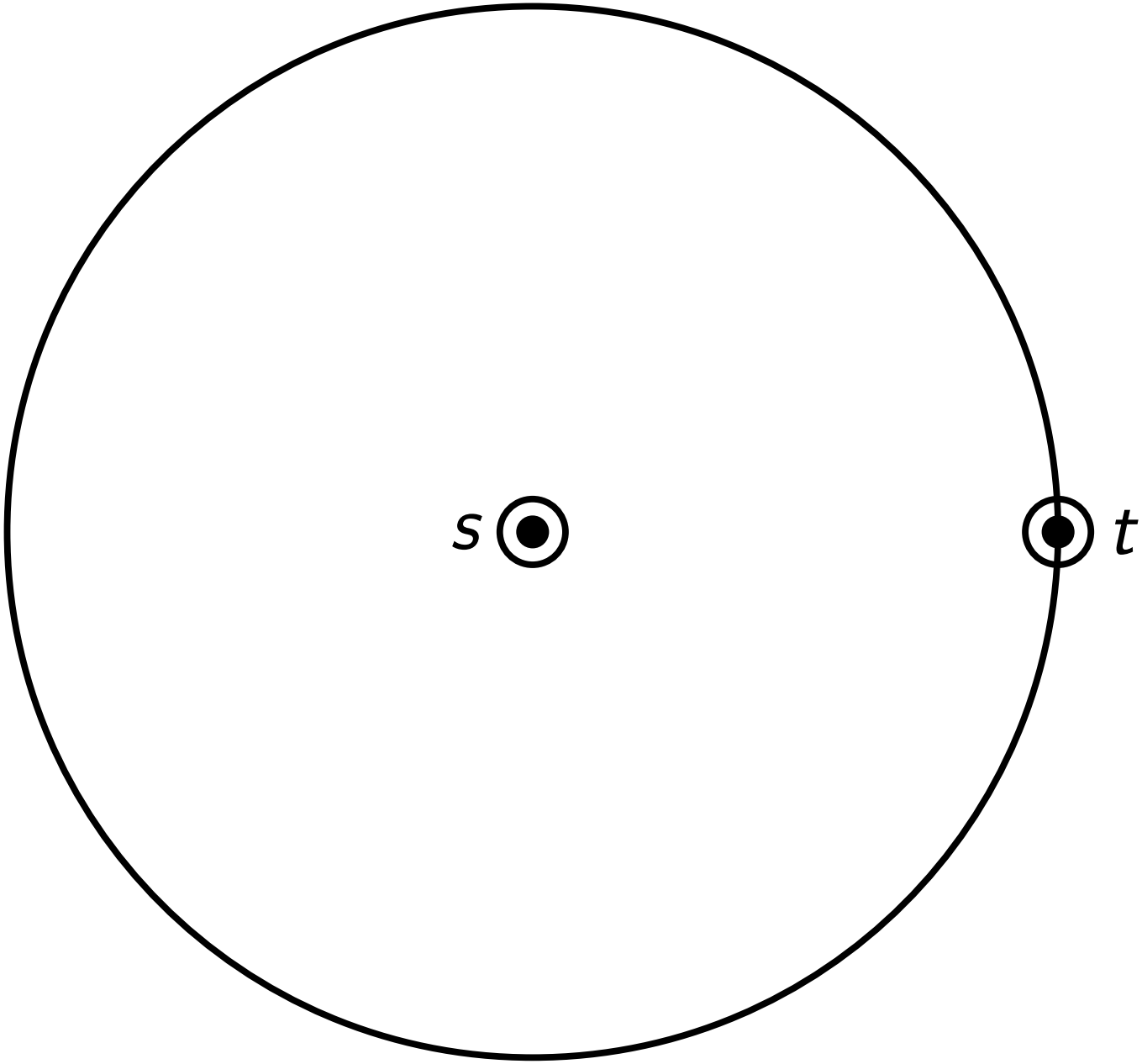

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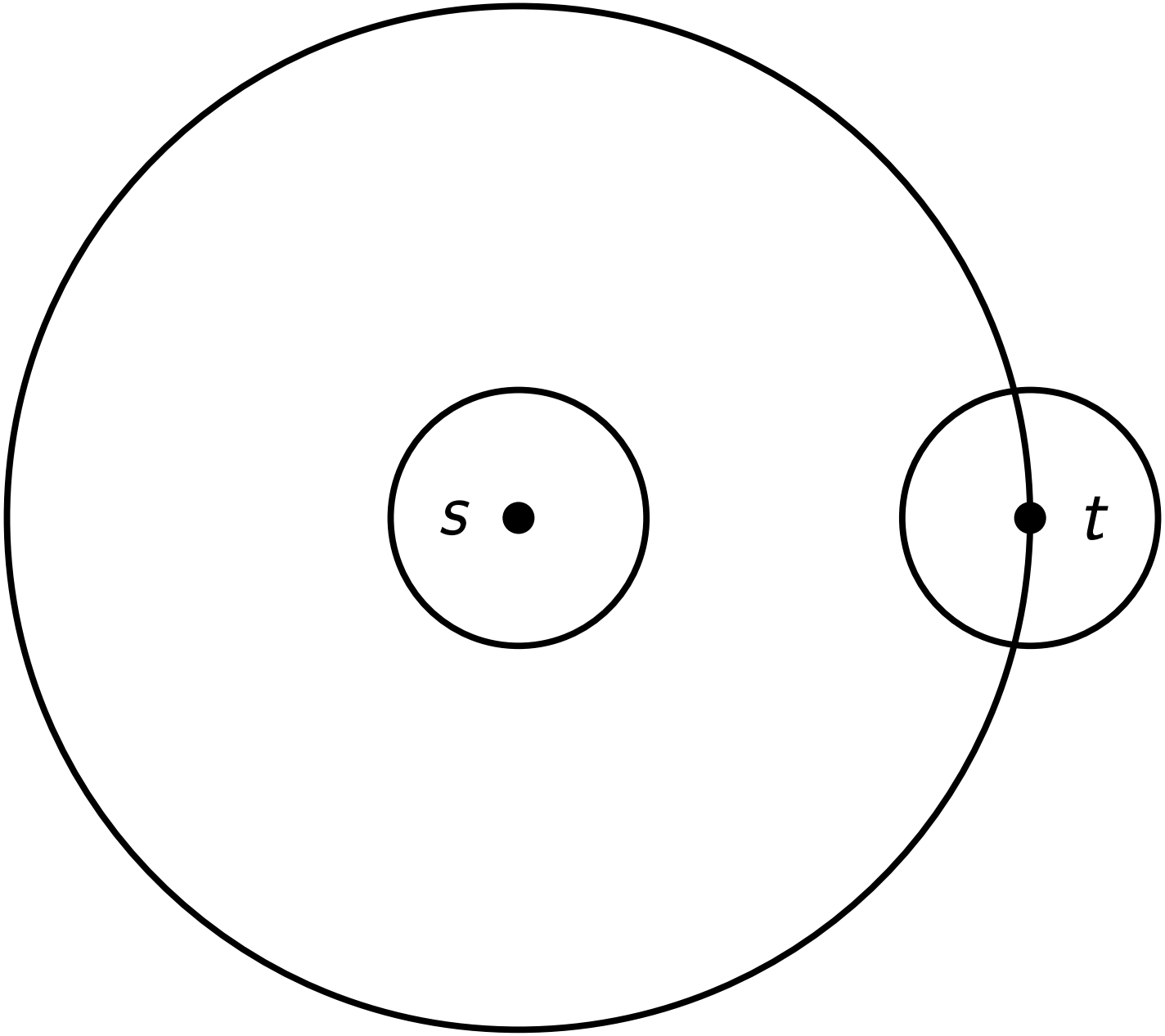

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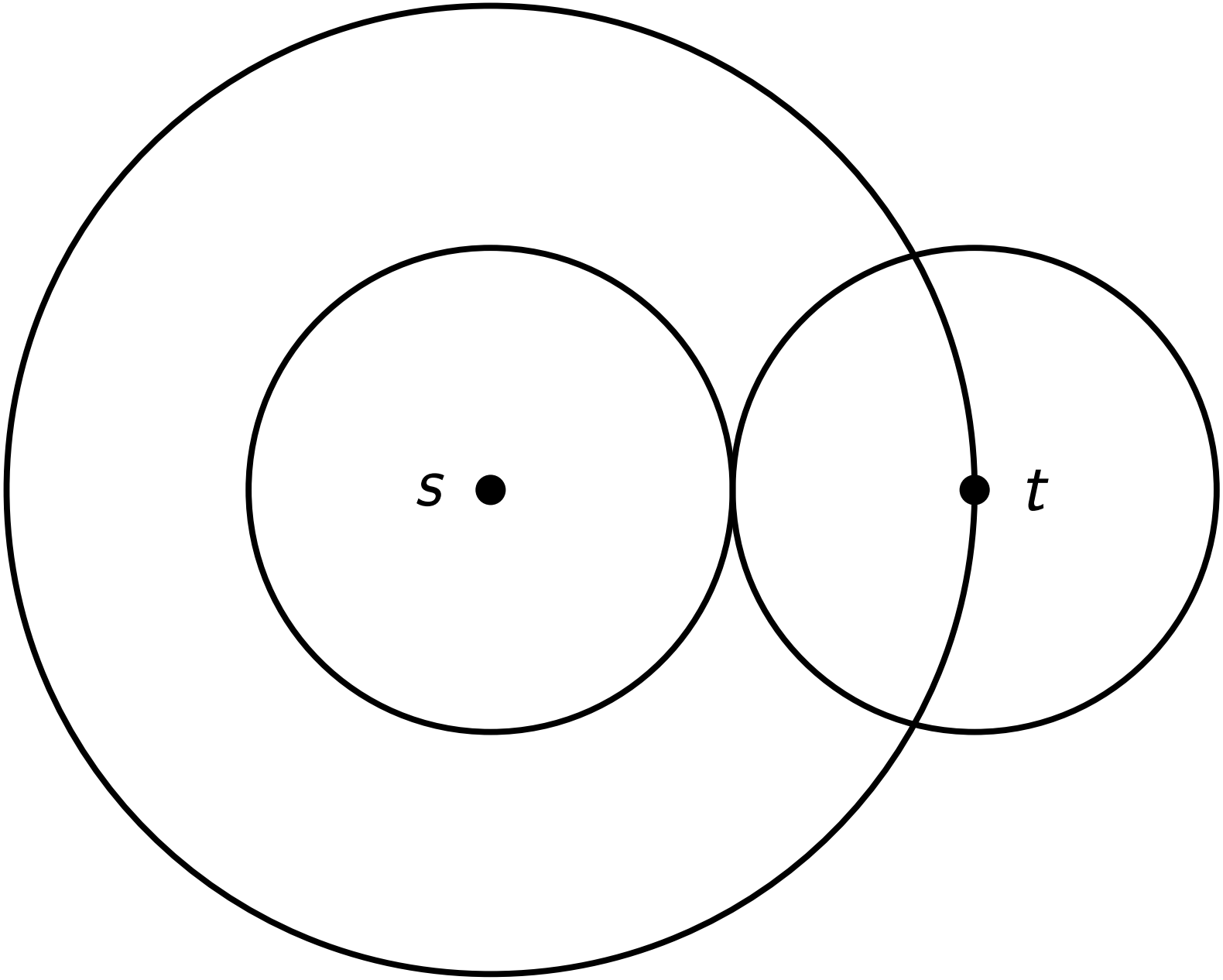

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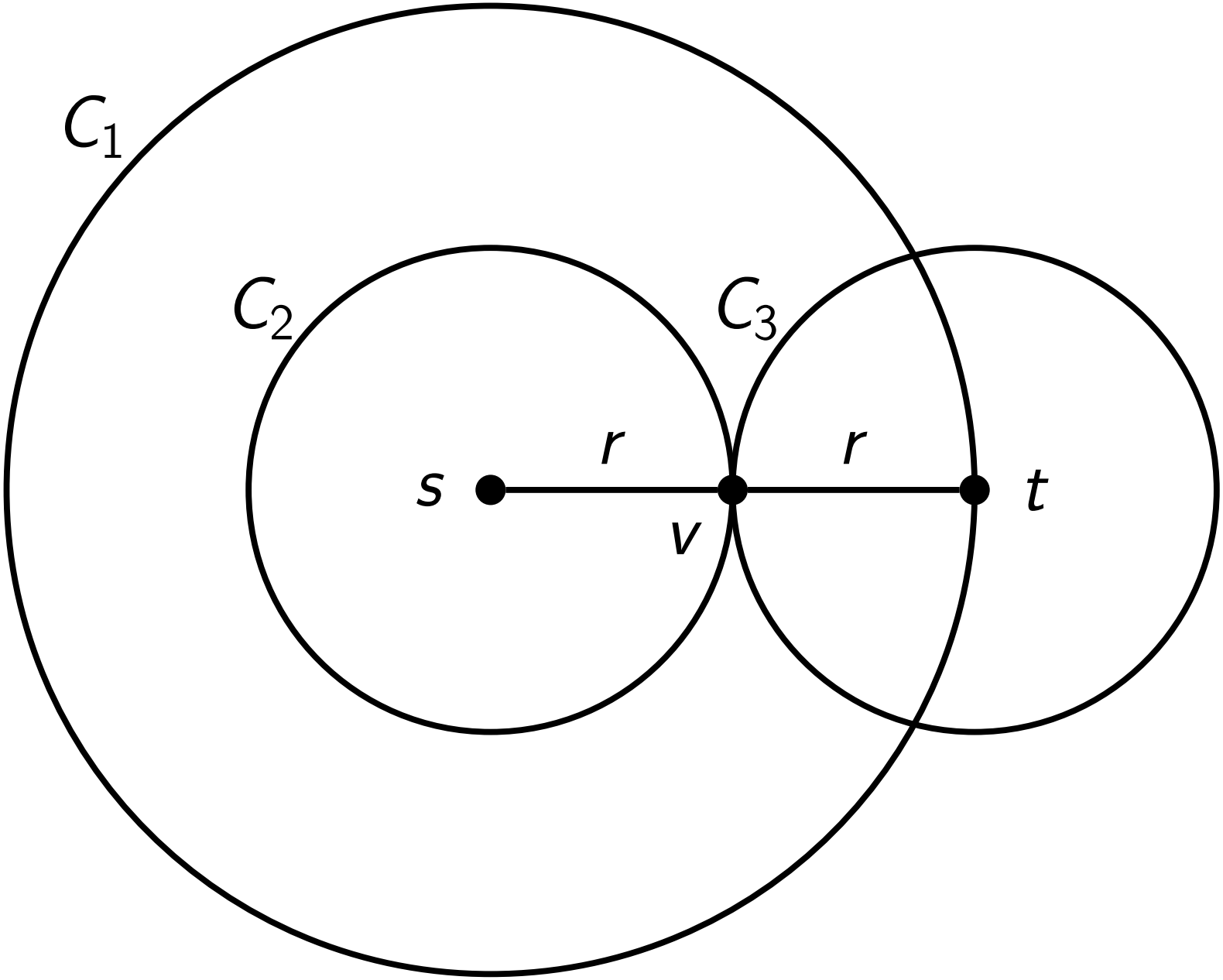

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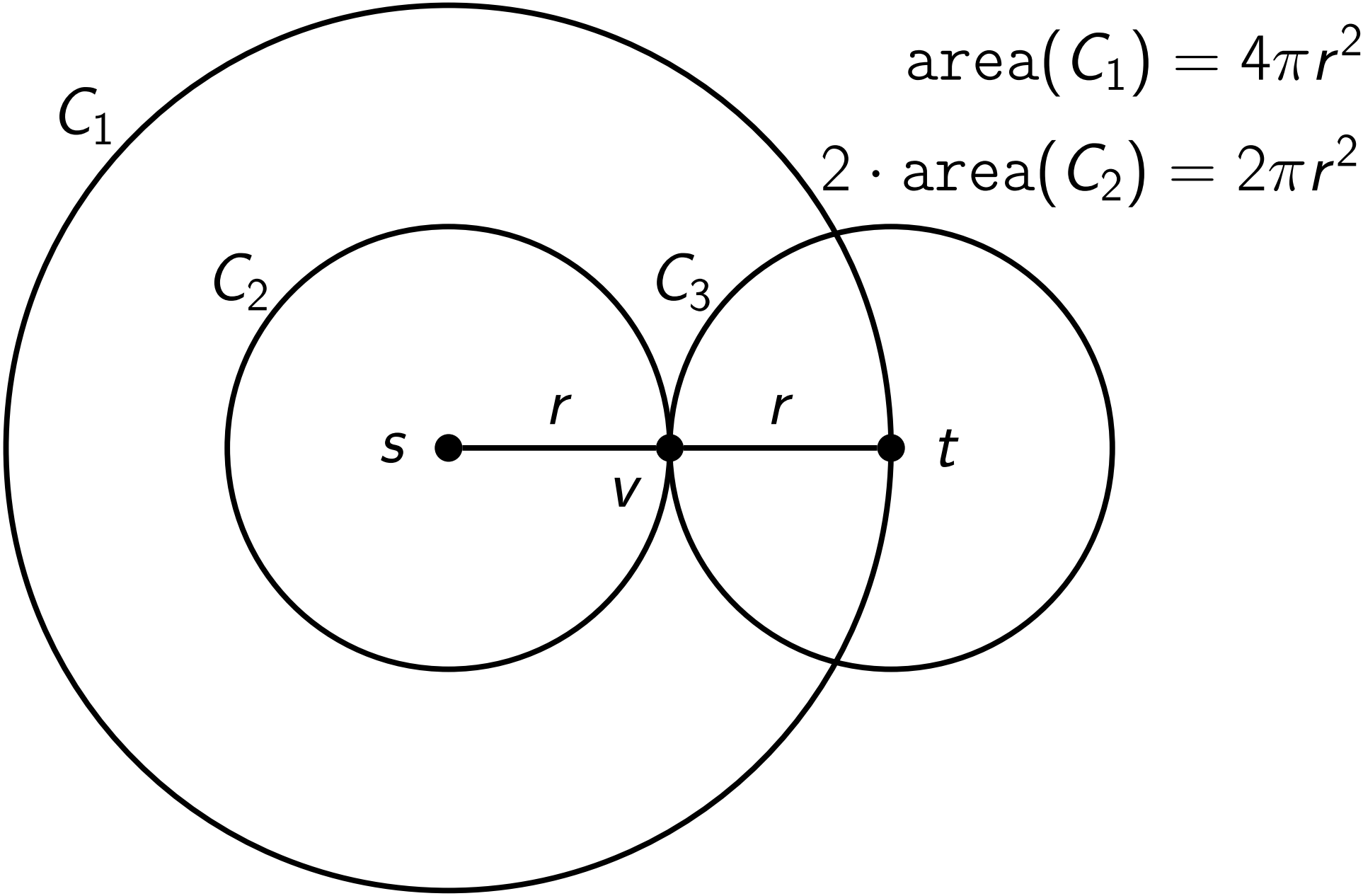
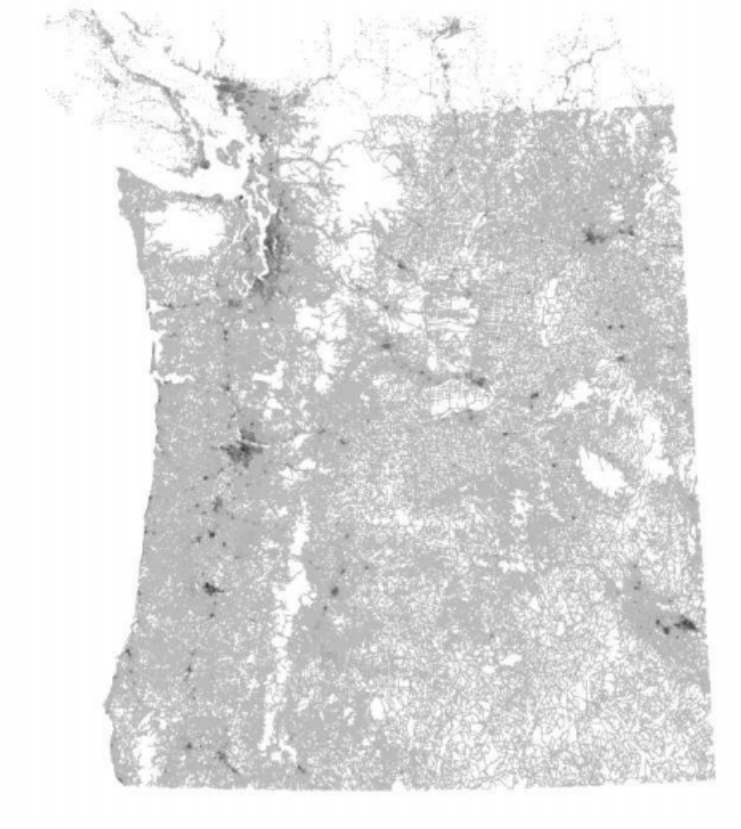


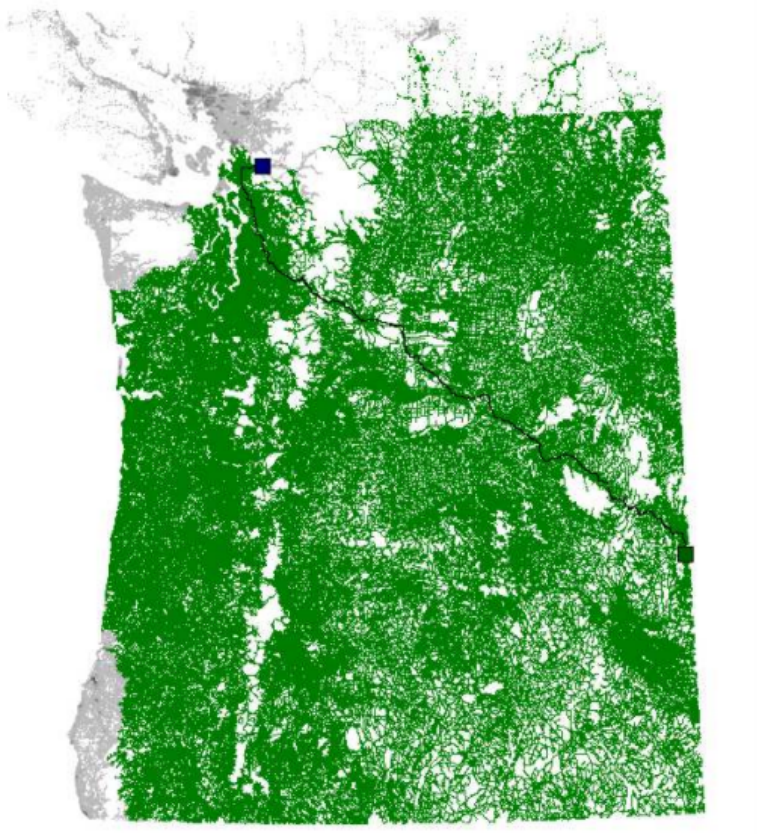
Road networks



1.6M vertices, 3.8M arcs, travel time metric.

Picture by Andrew Goldberg.

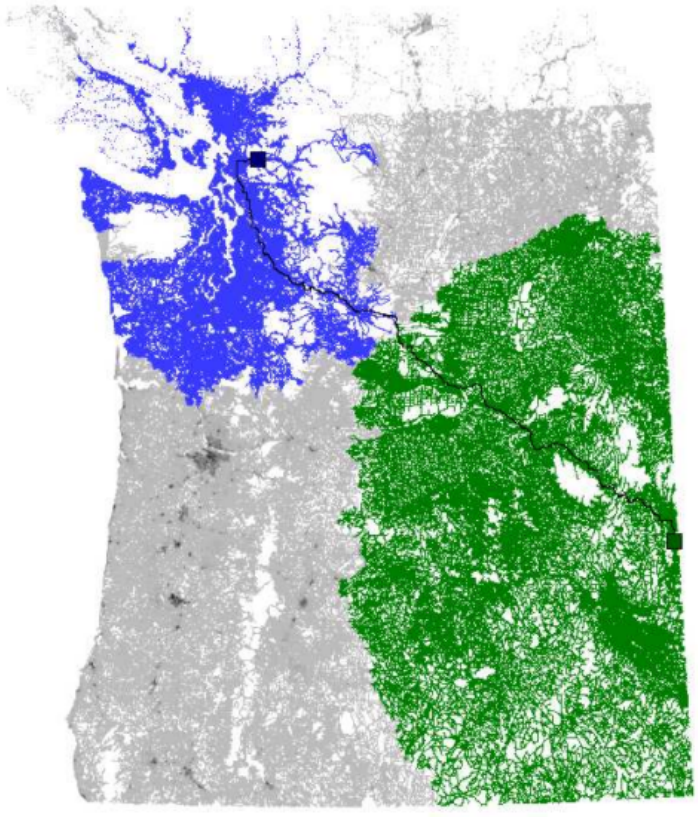
Road networks



Searched area

Picture by Andrew Goldberg.

Road networks



forward search/ reverse search

Picture by Andrew Goldberg.

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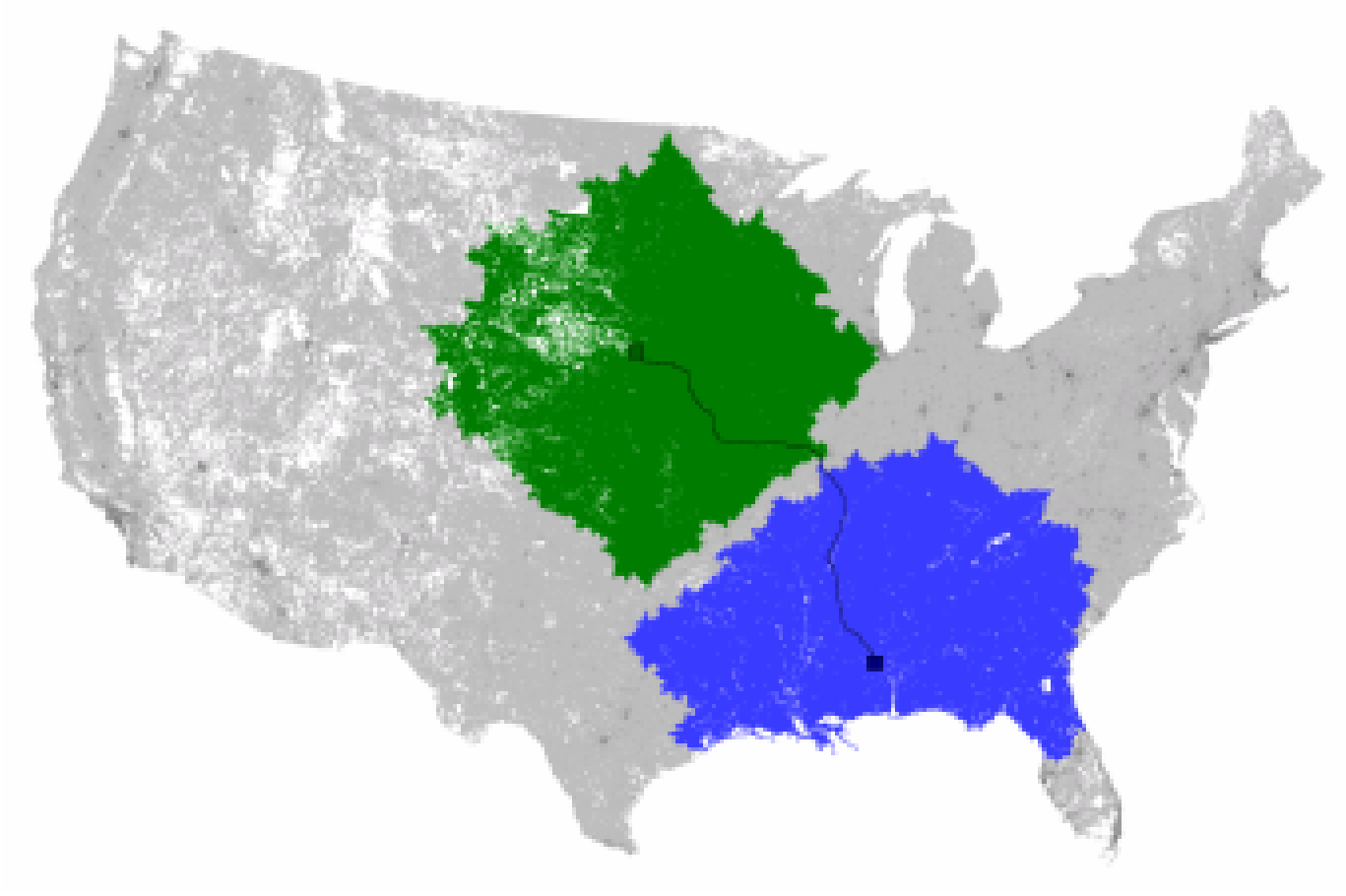

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- Let's look at social networks

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- Can pass a message from any person to any person in at most 6 handshakes
- This is close to truth according to experiments and is called a "six handshakes" or "six degrees of separation" idea

Facebook

- Suppose an average person has around 100 Facebook friends

Facebook

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- Then 10000 friends of friends

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- 1 trillion people at six handshakes

Facebook

- Suppose an average person has around 100 Facebook friends
- Then 10000 friends of friends
- 1000000 friends of friends of friends
- ...
- 1 trillion people at six handshakes
- Not possible, as there are only about 7 billion people on earth

Facebook

- Find the shortest path from Michael to Bob via friends connections

Facebook

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- For the two "farthest" people, Dijkstra has to look through 2 billion people
- If we only consider friends of friends of friends for both Michael and Bob, we will find a connection
- Roughly 1M friends of friends of friends
- $1M + 1M = 2M$ people — 1000 times less

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Meet-in-the-middle

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- Instead of searching for all possible objects, search for first halves and for second halves separately
- Then find "compatible" halves
- Typically roughly $O(\sqrt{N})$ instead of $O(N)$, including the previous Facebook example

Conclusion

- Dijkstra goes in "circles"
- Bidirectional search idea can reduce the search space
- Roughly 2x speedup for road networks
- Meet-in-the-middle — $\sqrt{N}$ instead of N
- 1000 times faster for social networks
- Next video — Bidirectional Dijkstra algorithm

Outline

Dijkstra Reminder

- To find the shortest path from s to t

Dijkstra Reminder

- To find the shortest path from s to t
- Initialize `dist`$[s]$ to 0, all other distances to ∞

Dijkstra Reminder

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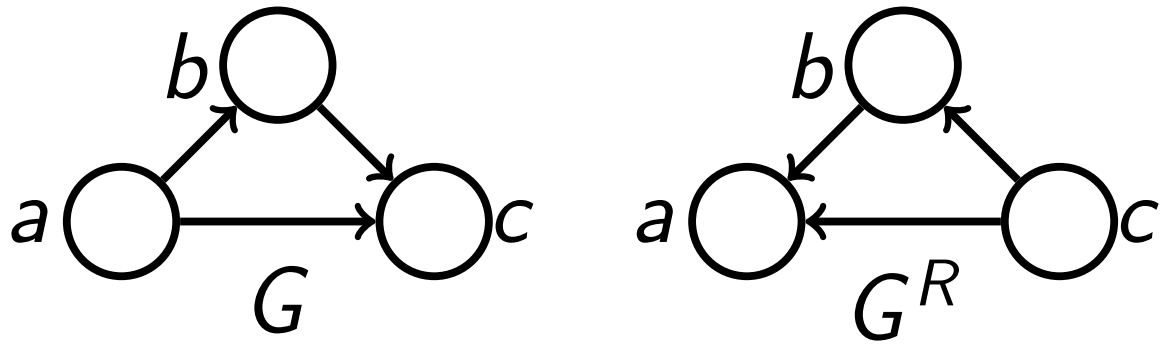
- Repeat until t is processed

Reversed Graph

Definition

Reversed graph G^R for a graph G is the graph with the same set of vertices V and the set of reversed edges E^R, such that for any edge $(u, v) \in E$ there is an edge $(v, u) \in E^R$ and vice versa.



Bidirectional Dijkstra

- Build G^R

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- Alternate between Dijkstra steps in G and in G^R
- Stop when some vertex v is processed both in G and in G^R
- Compute the shortest path between s and t

Computing Distance

Let v be the first vertex which is processed both in G and in G^R. Does it follow that there is a shortest path from s to t going through v?

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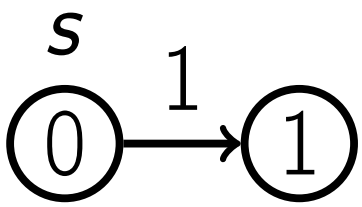

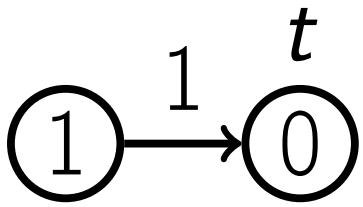

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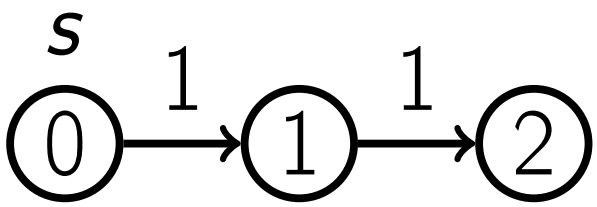

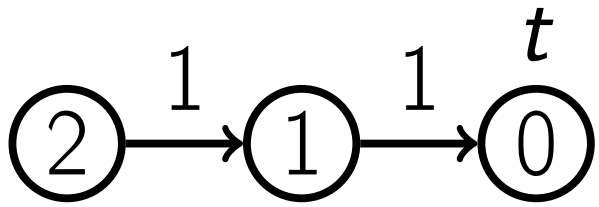

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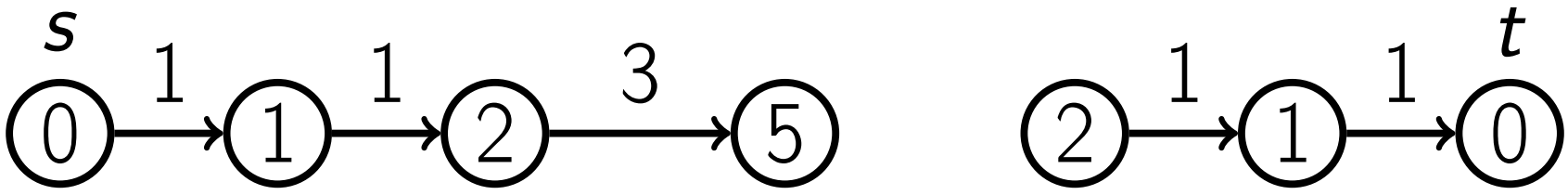

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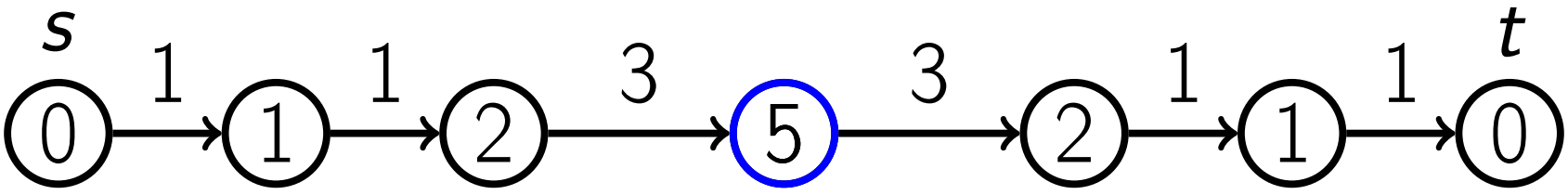

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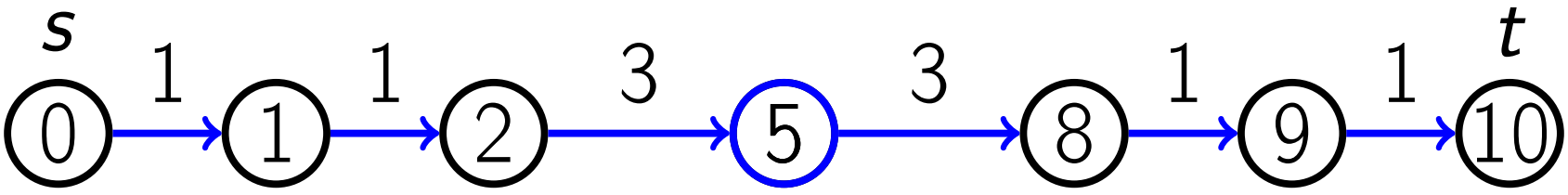

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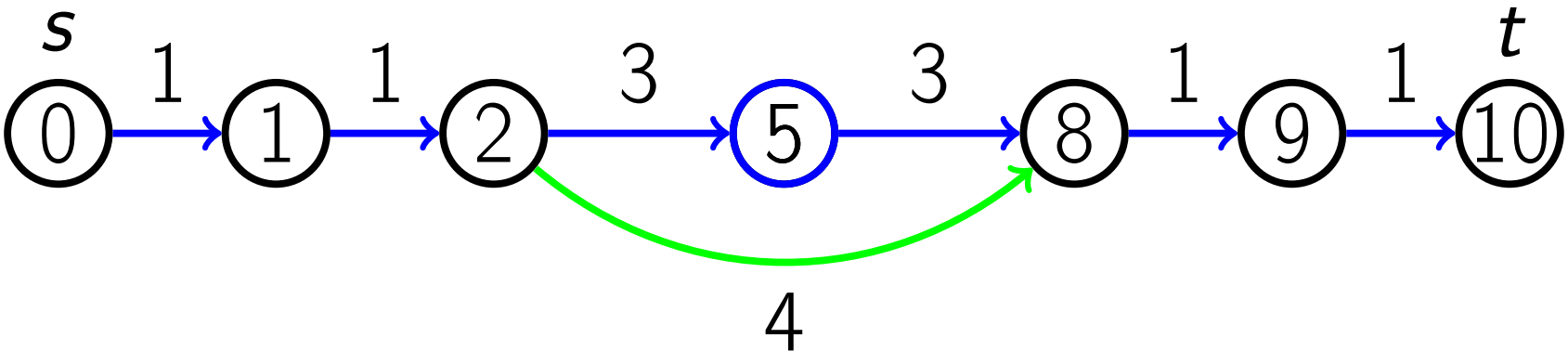

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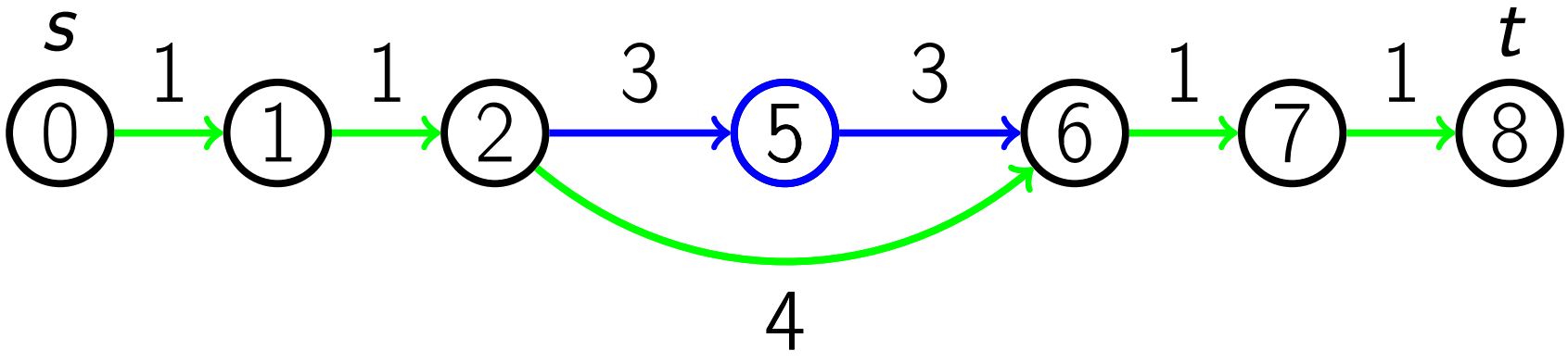
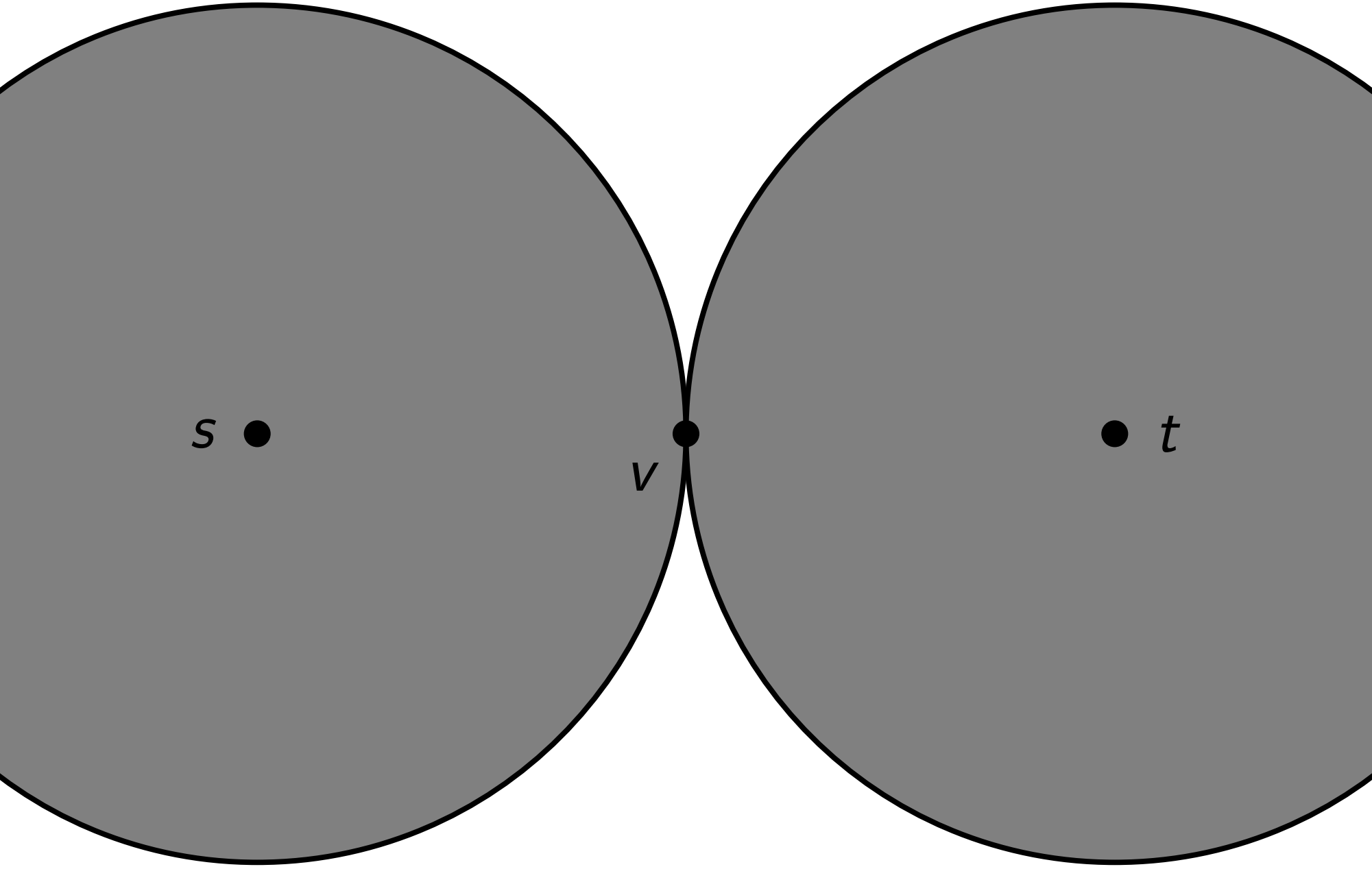


Computing Distance

Lemma

Let $\texttt{dist}[u]$ be the distance estimate in the forward Dijkstra from s in G and $\texttt{dist}^R[u]$ — the same in the backward Dijkstra from t in G^R. After some node v is processed both in G and G^R, some shortest path from s to t passes through some node u which is processed either in G, in G^R, or both, and $d(s,t) = \texttt{dist}[u] + \texttt{dist}^R[u]$.

Proof



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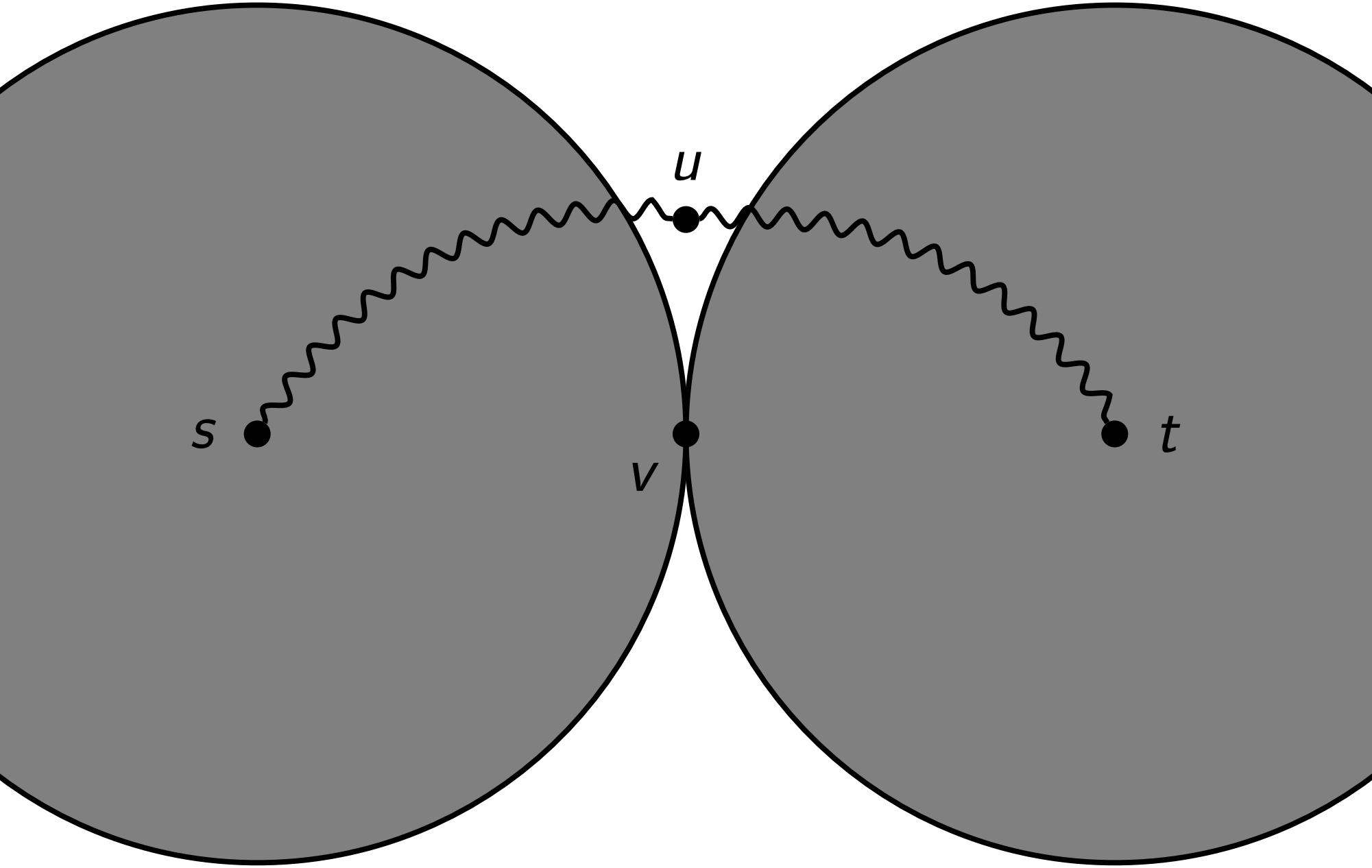

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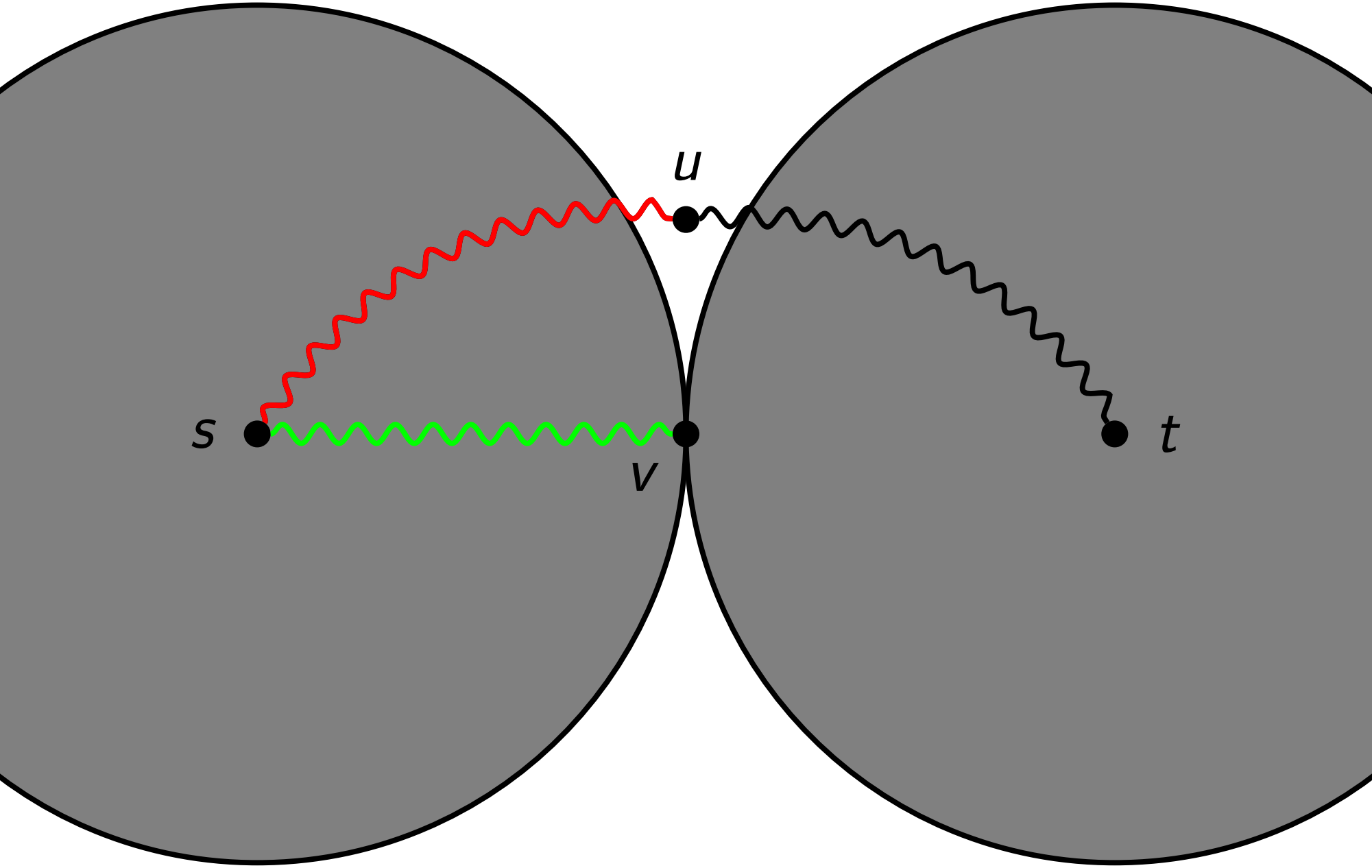

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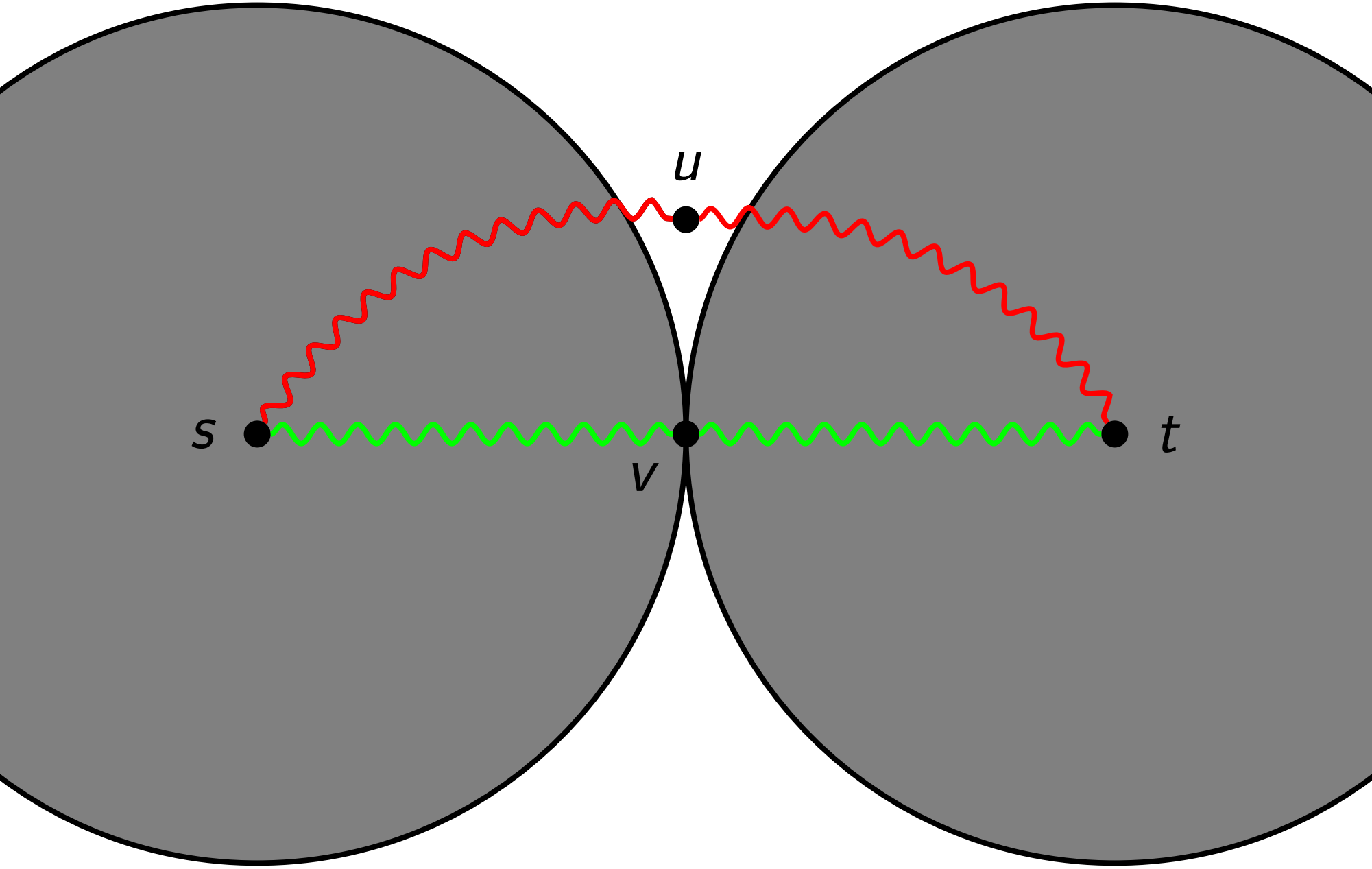

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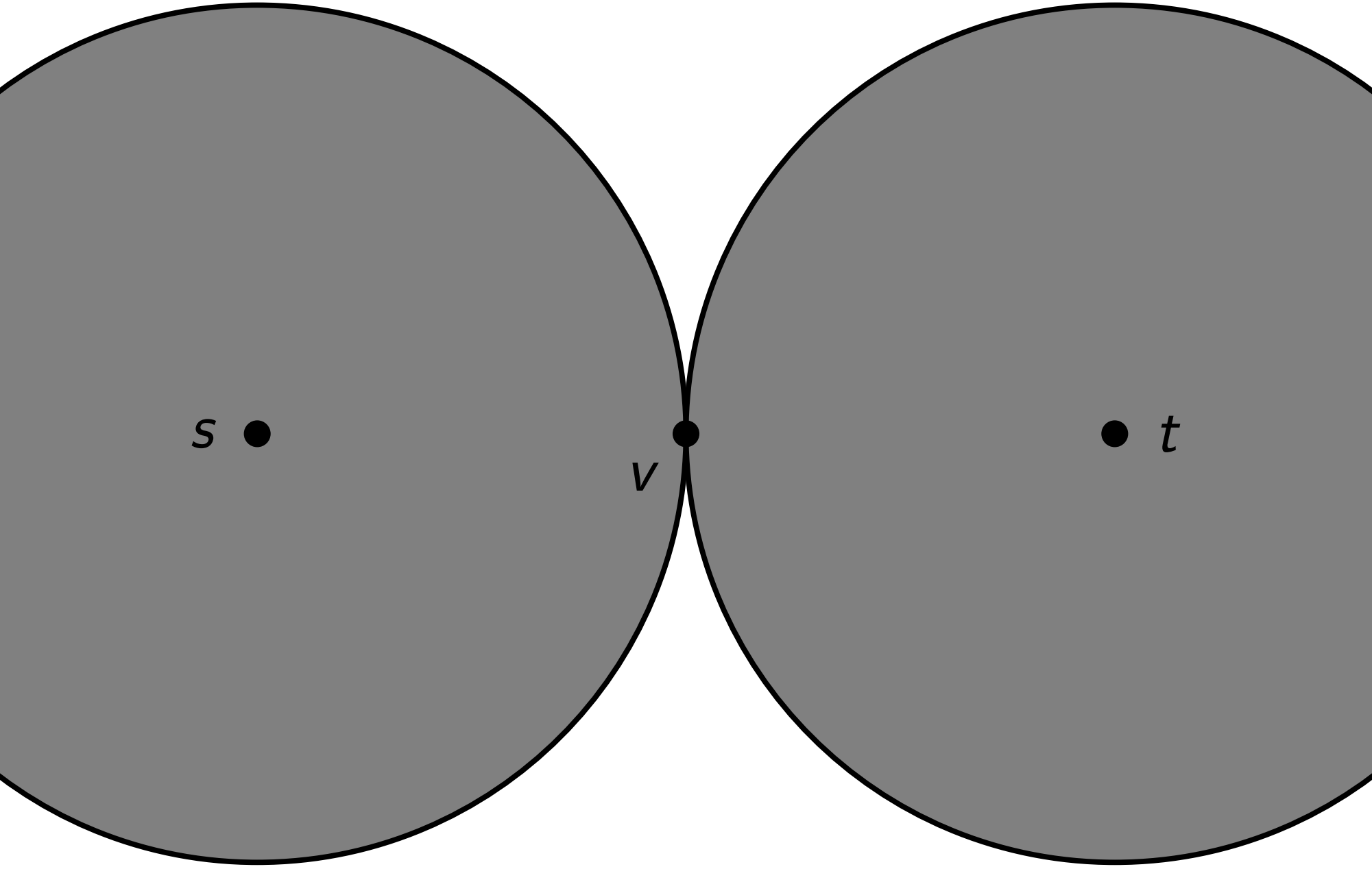

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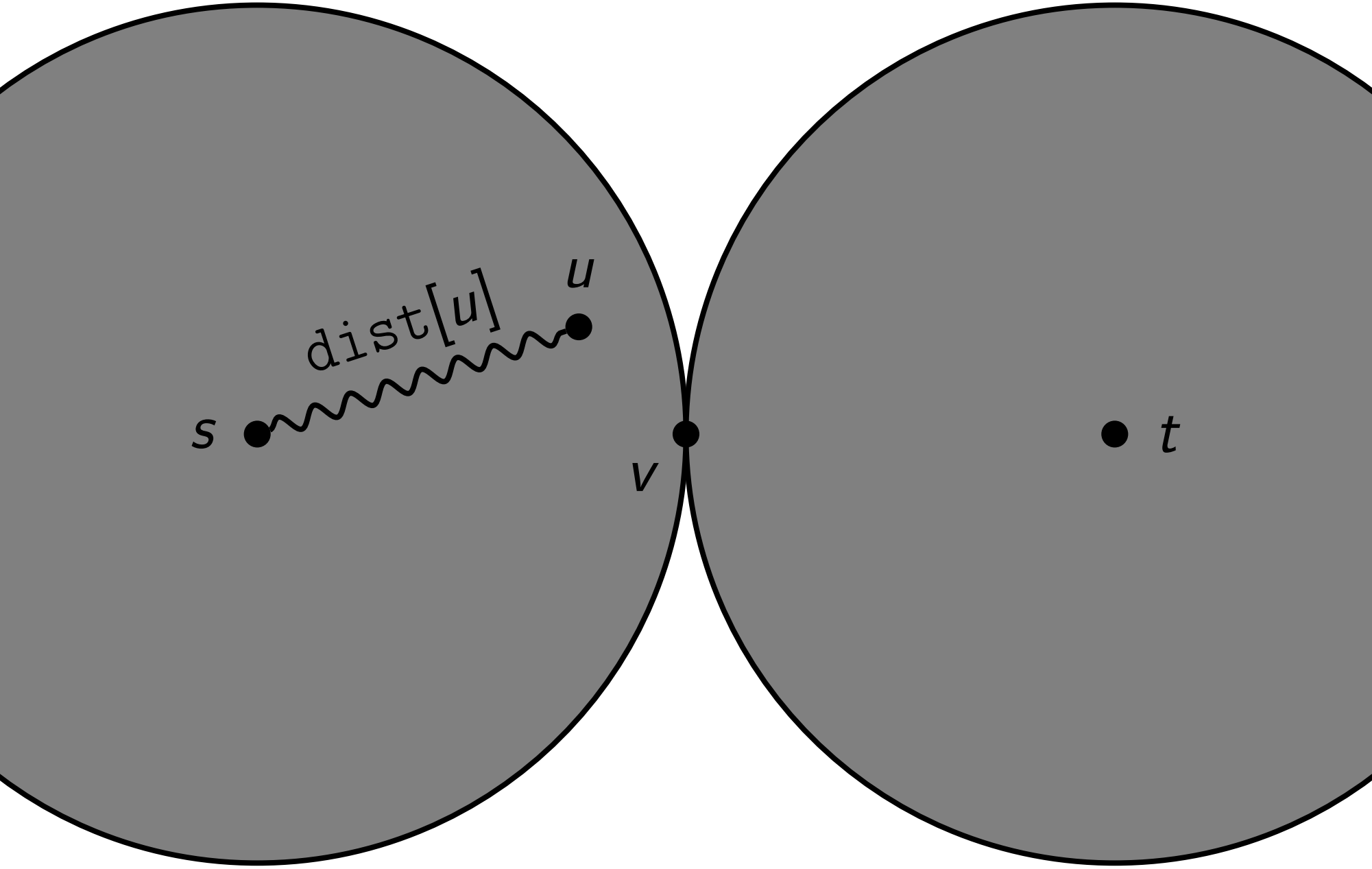

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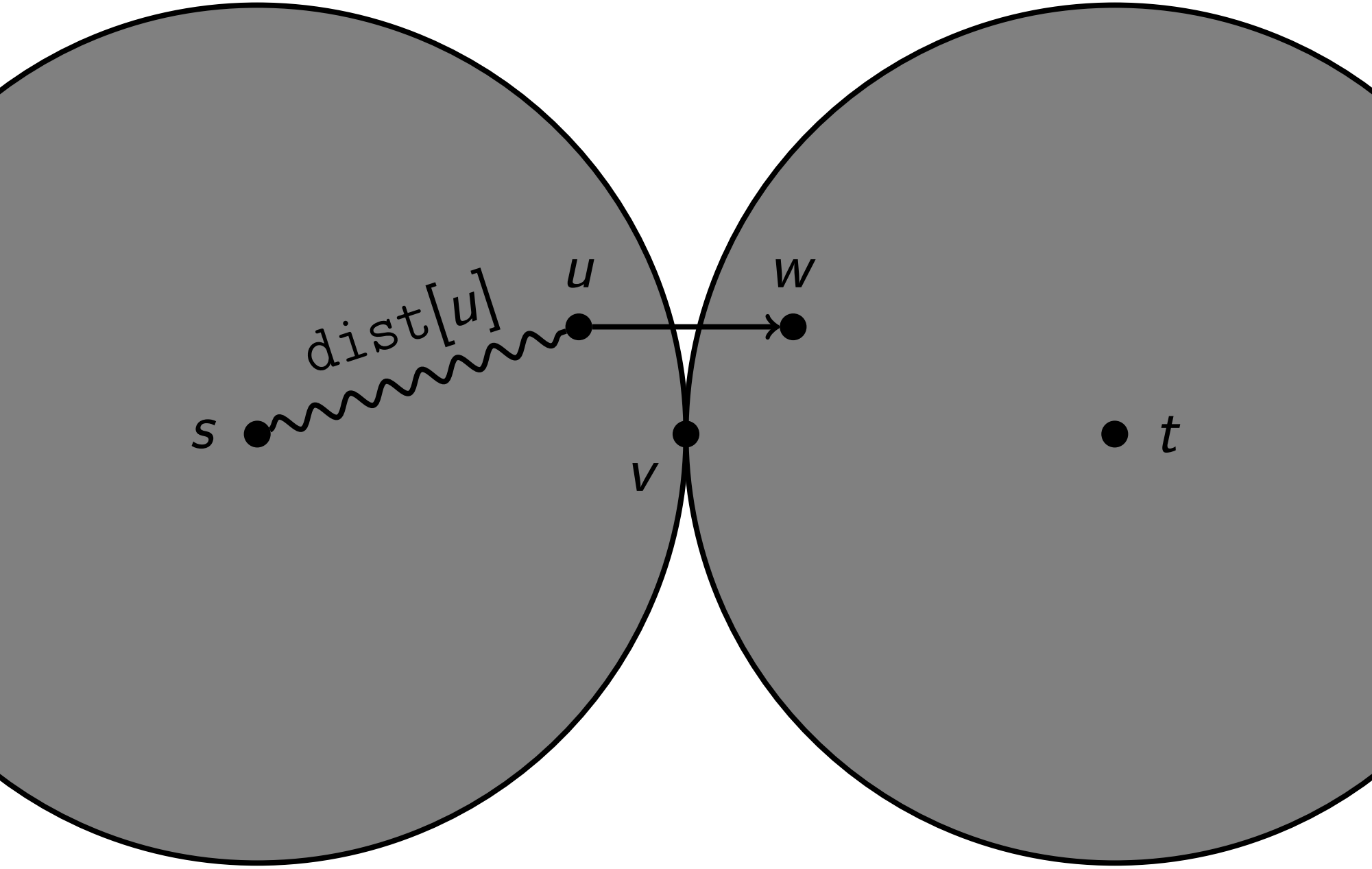

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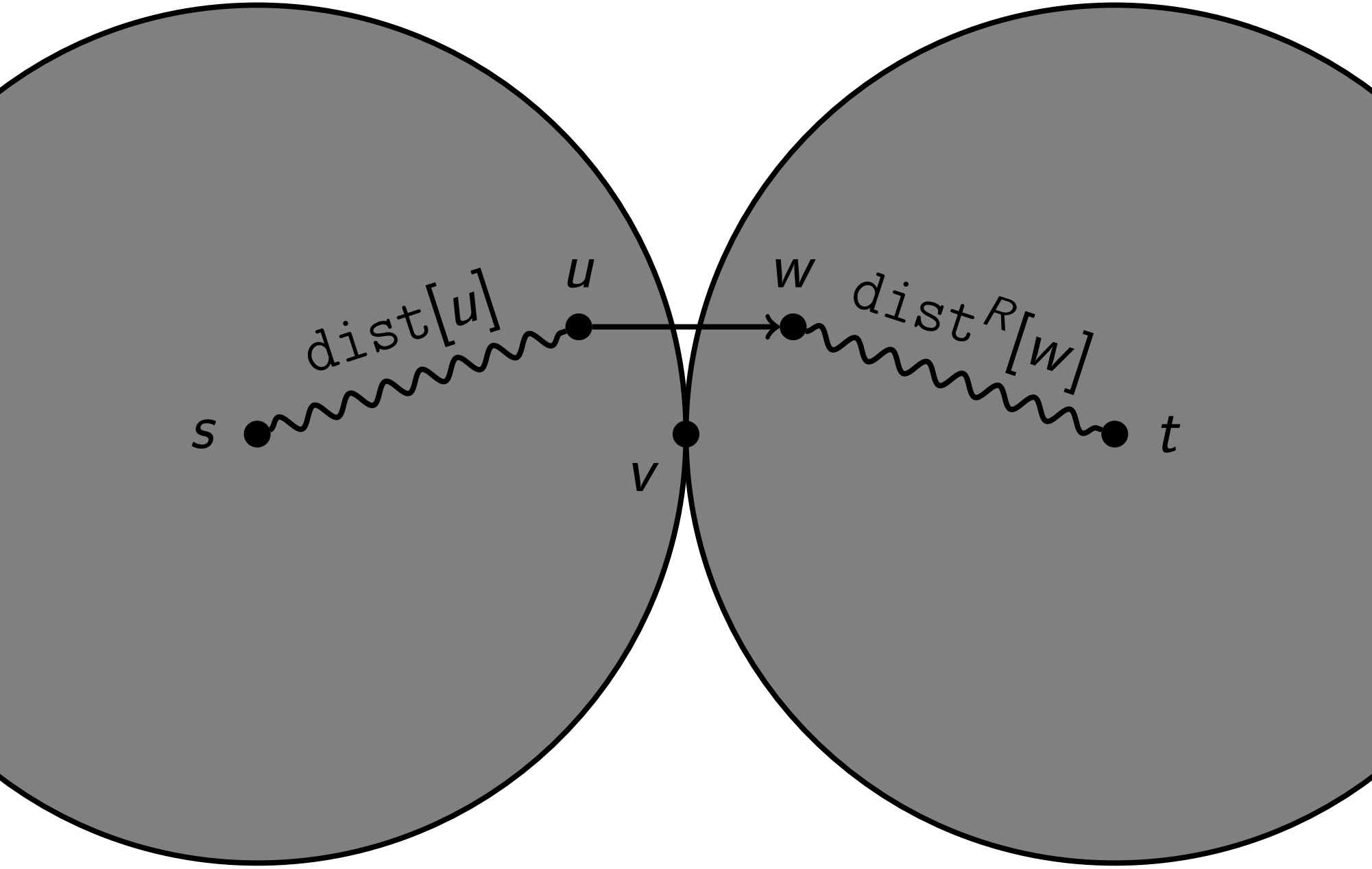

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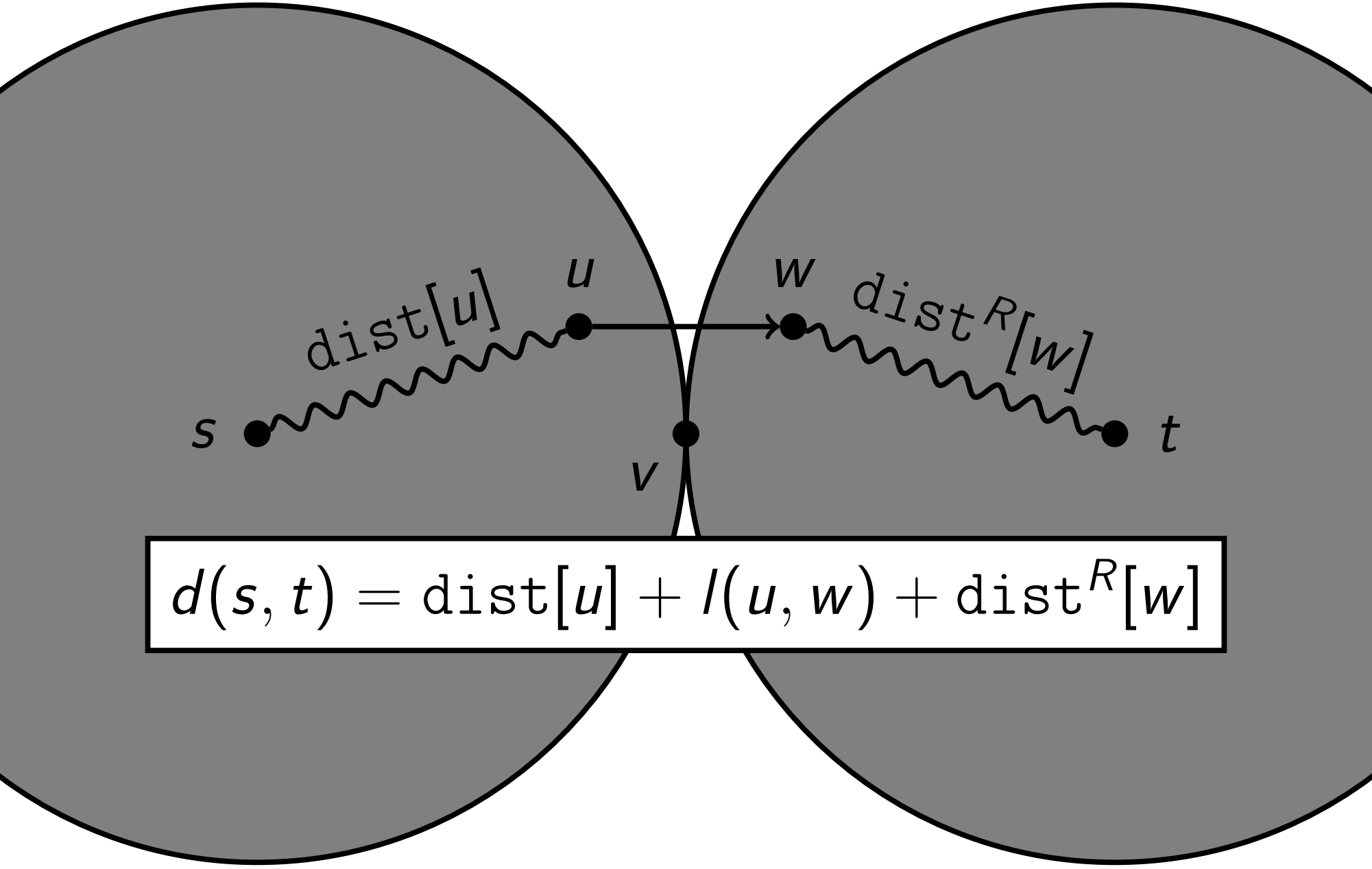

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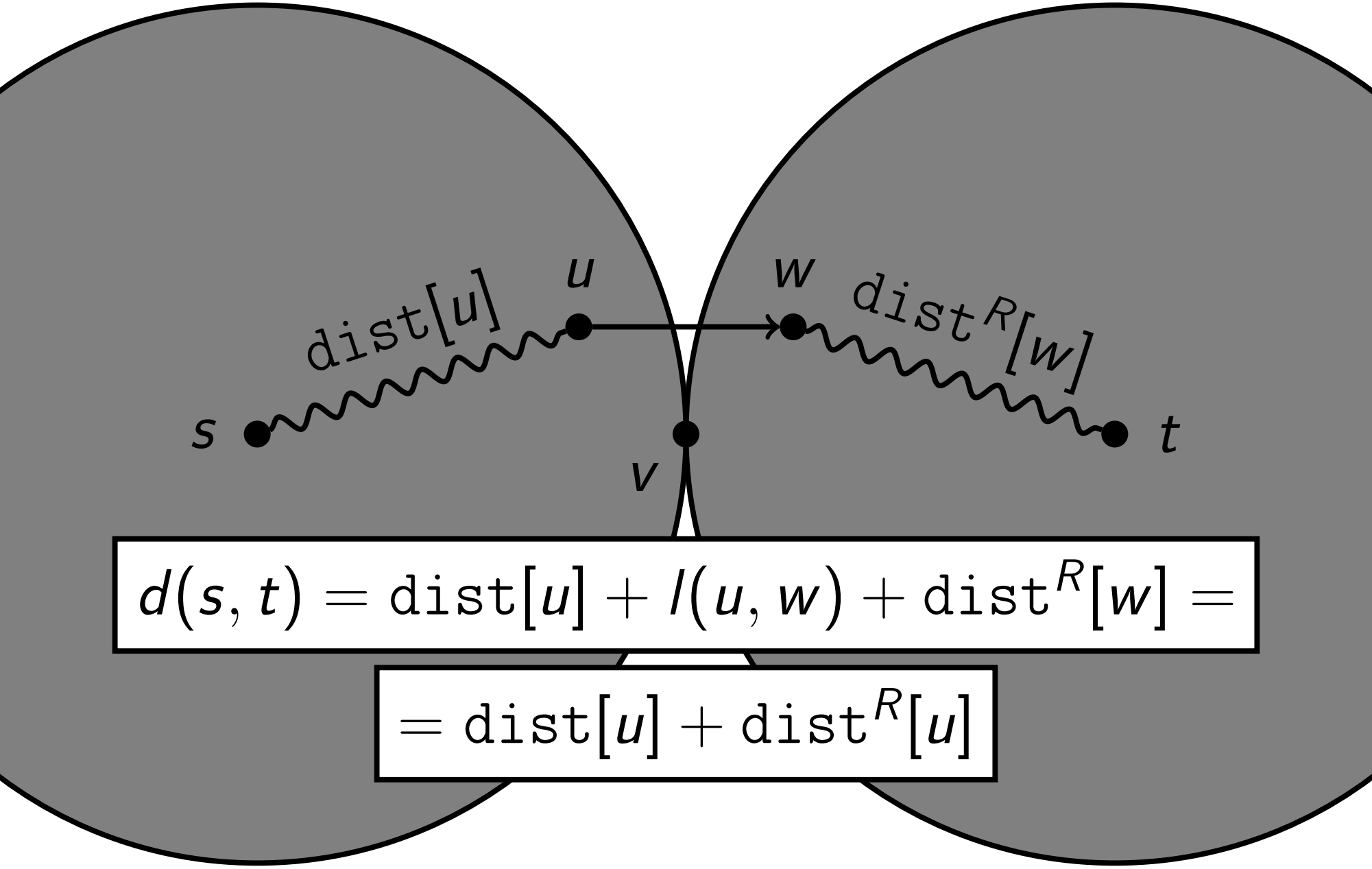

BidirectionalDijkstra(G, s, t)

```
G^R ← ReverseGraph(G)
Fill dist, dist^R with +∞ for each node
dist[s] ← 0, dist^R[t] ← 0
Fill prev, prev^R with None for each node
proc ← empty, proc^R ← empty
do:
  v ← ExtractMin(dist)
  Process(v, G, dist, prev, proc)
  if v in proc^R:
    return ShortestPath(s, dist, prev, proc, t, ...)
  v^R ← ExtractMin(dist^R)
  repeat symmetrically for v^R as for v
while True
```

Relax($u, v, \texttt{dist}, \texttt{prev}$)

if $\texttt{dist}[v] > \texttt{dist}[u] + w(u, v)$:
  $\texttt{dist}[v] \leftarrow \texttt{dist}[u] + w(u, v)$
  $\texttt{prev}[v] \leftarrow u$

Process($u, G, \texttt{dist}, \texttt{prev}, \texttt{proc}$)

```
for (u, v) ∈ E(G):
    Relax(u, v, dist, prev)
proc.Append(u)
```

ShortestPath(s, dist, prev, proc, t, dist^R, prev^R, proc^R)

```
distance ← +∞,  u_best ← None
for u in proc + proc^R:
    if dist[u] + dist^R[u] < distance:
        u_best ← u
        distance ← dist[u] + dist^R[u]
path ← empty
last ← u_best
while last ≠ s:
    path.Append(last)
    last ← prev[last]
path ← Reverse(path)
last ← u_best
while last ≠ t:
    last ← prev^R[last]
    path.Append(last)
return (distance, path)
```

Conclusion

- Worst-case running time of Bidirectional Dijkstra is the same as for Dijkstra
- Speedup in practice depends on the graph
- Memory consumption is 2x to store G and G^R
- You'll see the speedup on social network graph in the Programming Assignment