Hashing: Hash Functions

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Data Structures Fundamentals
Algorithms and Data Structures

Outline

Phone Book

Design a data structure to store your contacts: names of people along with their phone numbers. The following operations should be fast:

- Add and delete contacts,
- Call person by name,
- Determine who is calling given their phone number.

- We need two Maps:
  (phone number $\rightarrow$ name) and
  (name $\rightarrow$ phone number)

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- Implement these Maps as hash tables

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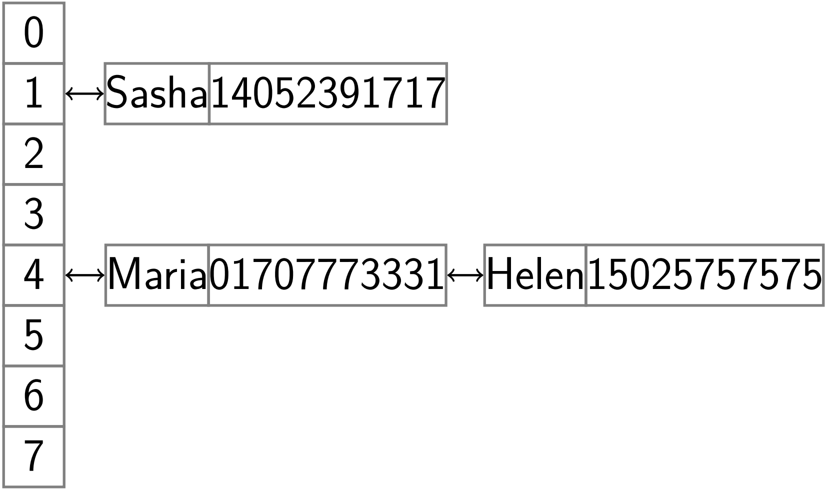
  (phone number $\rightarrow$ name) and
  (name $\rightarrow$ phone number)
- Implement these Maps as hash tables
- First, we will focus on the Map from phone numbers to names

Chaining for Phone Book

- Select hash function h of cardinality m
- Create array `Chains` of size m
- Each element of `Chains` is a list of pairs (`name`, `phoneNumber`), called *chain*
- Pair (`name`, `phoneNumber`) goes into chain at position h(`ConvertToInt`(`phoneNumber`)) in the array `Chains`

Chaining for Phone Book



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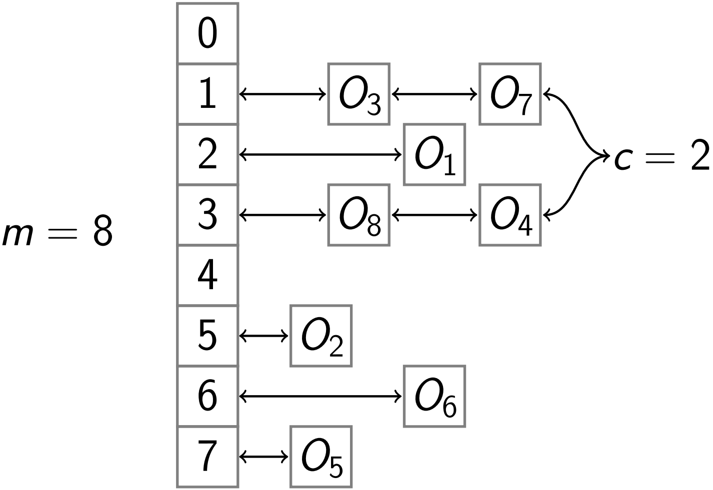
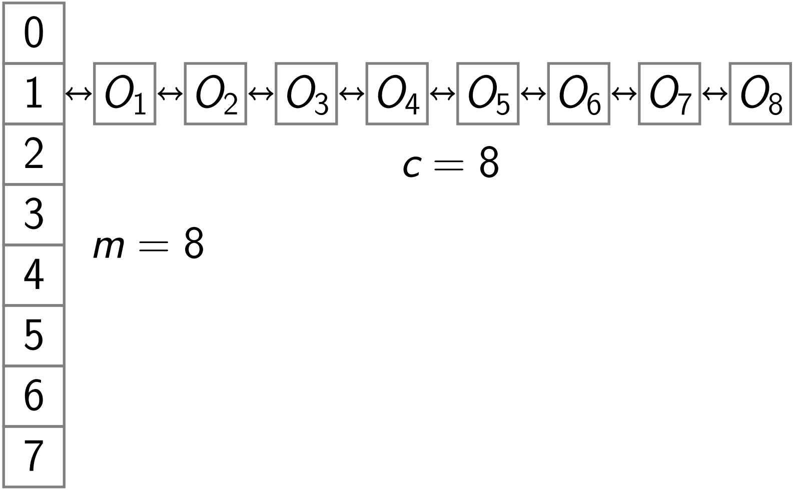
- You want small m and c! (but $c \geq \frac{n}{m}$)

Good Example



Bad Example



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- $h(425\text{–}234\text{–}55\text{–}67) =$
  $h(425\text{–}123\text{–}45\text{–}67) =$
  $h(425\text{–}223\text{–}23\text{–}23) = \cdots = 425$

Last Digits

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- Problem if many phone numbers end with three zeros

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- Hash function must be deterministic

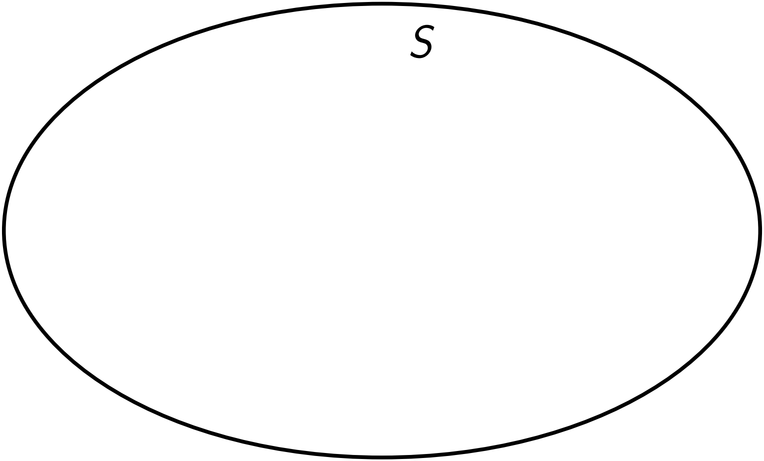
Good Hash Functions

- Deterministic
- Fast to compute
- Distributes keys well into different cells
- Few collisions

No Universal Hash Function

Lemma

If the number of possible keys is big ($|S| \gg m$), for any hash function h there is a bad input resulting in many collisions.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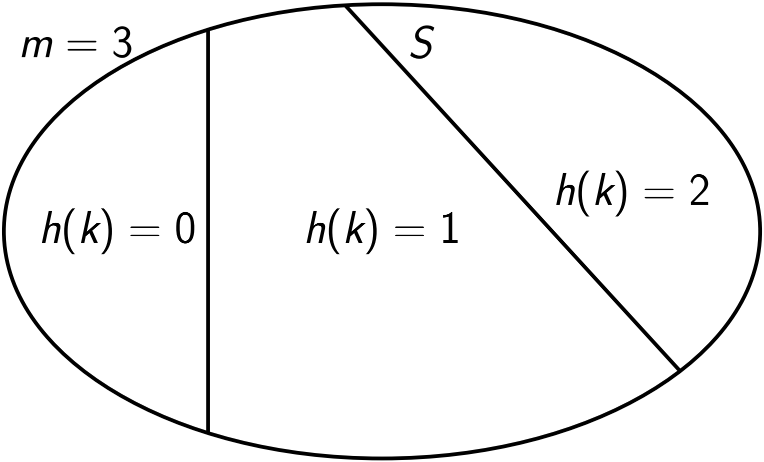

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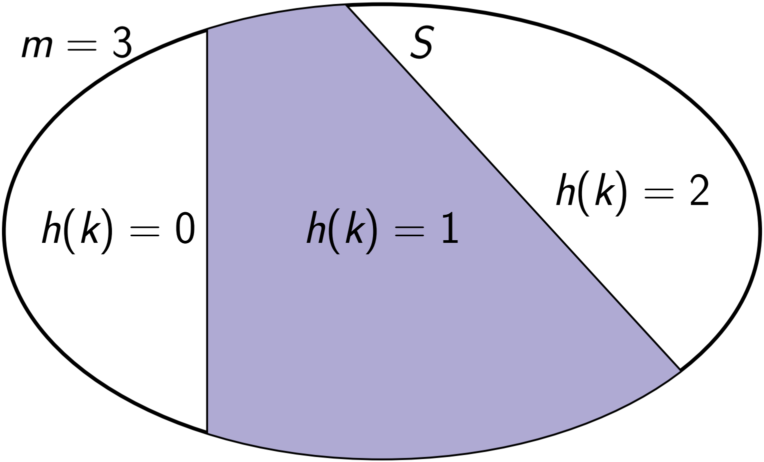
$h(k) = 2$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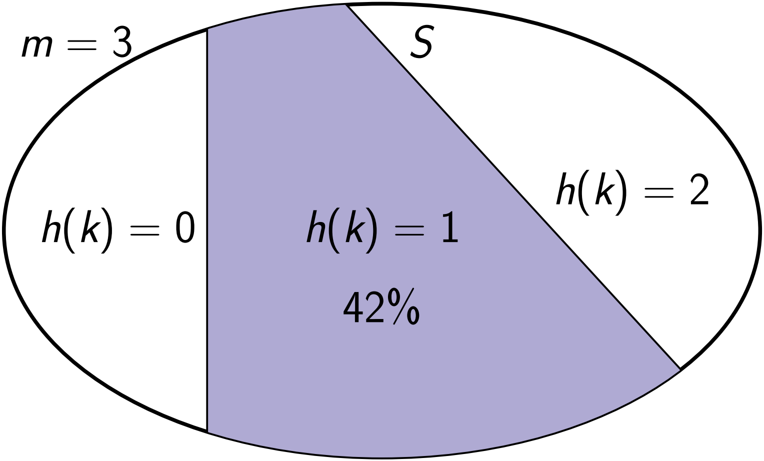
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- Choose random function from the family

Universal Family

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$$\Pr[h(x) = h(y)] \leq \frac{1}{m}$$

Universal Family

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means that a collision $h(x) = h(y)$ for any fixed pair of different keys x and y happens for no more than $\frac{1}{m}$ of all hash functions $h \in \mathcal{H}$.

How Randomization Works

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- All hash functions in $\mathcal{H}$ are deterministic
- Select a random function h from $\mathcal{H}$
- Fixed h is used throughout the algorithm

Load Factor

Definition

The ratio $\alpha = \frac{n}{m}$ between number of objects n stored in the hash table and the size of the hash table m is called load factor.

Running Time

Lemma

If h is chosen randomly from a universal family, the average length of the longest chain c is $O(1 + \alpha)$, where $\alpha = \frac{n}{m}$ is the load factor of the hash table.

Corollary

If h is from universal family, operations with hash table run on average in time $O(1 + \alpha)$.

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- Choose new hash function and rehash all the objects

Rehashing

Keep load factor below 0.9:

Rehash(T)

$loadFactor \leftarrow \frac{T.\texttt{numberOfKeys}}{T.\texttt{size}}$
`if` $loadFactor > 0.9$:
  `Create` T_{new} `of size` $2 \times T.\texttt{size}$
  `Choose` h_{new} `with cardinality` $T_{new}.\texttt{size}$
  `For each object in` T:
    `Insert object in` T_{new} `using` h_{new}
  $T \leftarrow T_{new}, h \leftarrow h_{new}$

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Rehash Running Time

You should call `Rehash` after each operation with the hash table

Similarly to dynamic arrays, single rehashing takes $O(n)$ time, but amortized running time of each operation with hash table is still $O(1)$ on average, because rehashing will be rare

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- As a result, any phone number will be converted to an integer less than 10^{15}
- If we come up with a universal family for integers up to 10^{15}, we will be able to map phone numbers to names efficiently using chaining

Hashing Integers

Lemma

$\mathcal{H}_p = \left\{ h_p^{a,b}(x) = ((ax + b) \bmod p) \bmod m \right\}$ for all $a, b : 1 \leq a \leq p - 1, 0 \leq b \leq p - 1$ is a universal family for the set of integers between 0 and $p - 1$, for any prime p.

Hashing Phone Numbers

Example

Select $a = 34$, $b = 2$, so $h = h_p^{34,2}$ and consider $x = 1\,482\,567$ corresponding to phone number 148-25-67. $p = 10\,000\,019$.

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$$407185 \bmod 1000 = 185$$

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$407185 \bmod 1000 = 185$

$h(x) = 185$

Proof Ideas

- For any pair of different keys (x, y), any two of the $p(p-1)$ hash functions in $\mathcal{H}_p$ hash them into different pairs (r, s) of different remainders modulo p

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- Thus any pair (r, s) of different remainders modulo p has equal probability $\frac{1}{p(p-1)}$
- The ratio of pairs (r, s) of different remainders modulo p such that $r \equiv s \pmod{m}$ is less than $\frac{1}{m}$ $\square$

General Case

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- Choose random hash function from universal family $\mathcal{H}_p$ (choose random $a \in [1, p-1]$ and $b \in [0, p-1]$)

Outline

Lookup Phone Numbers by Name

- Time to implement Map from names to phone numbers

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- Hash arbitrary strings of bounded length
- You will learn how string hashing is implemented in Java!

String Length Notation

Definition

Denote by $|S|$ the length of string S.

Examples

$|\text{"edx"}| = 3$

$|\text{"ucsd"}| = 4$

$|\text{"chaining"}| = 8$

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- Otherwise there will be many collisions
- For example, if $S[0]$ is not used, then $h(\text{"aa"}) = h(\text{"ba"}) = \cdots = h(\text{"za"})$

Preparation

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Polynomial Hashing

Definition

Family of hash functions

$$\mathcal{P}_p = \left\{ h_p^x(S) = \sum_{i=0}^{|S|-1} S[i]x^i \bmod p \right\}$$

with a fixed prime p and all $1 \leq x \leq p-1$ is called polynomial.

PolyHash(S, p, x)

```
hash ← 0
for i from |S| − 1 down to 0:
  hash ← (hash · x + S[i]) mod p
return hash
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2. hash $\leftarrow S[2] \bmod p$
3. hash $\leftarrow S[1] + S[2]x \bmod p$
4. hash $\leftarrow S[0] + S[1]x + S[2]x^2 \bmod p$

Java Implementation

The method `hashCode` of the built-in Java class `String` is very similar to our `PolyHash`, it just uses $x = 31$ and for technical reasons avoids the $\pmod p$ operator.

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You now know the implementation of the function that is used trillions of times a day in many thousands of programs!

Outline

1. Phone Book Data Structure
2. Universal Family
3. Hashing Phone Numbers
4. Hashing Names
5. Analysis of Polynomial Hashing

Lemma

For any two different strings s_1 and s_2 of length at most $L+1$, if you choose h from $\mathcal{P}_p$ at random (by selecting a random $x \in [1, p-1]$), the probability of collision $\Pr[h(s_1) = h(s_2)]$ is at most $\frac{L}{p}$.

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Proof idea

This follows from the fact that the equation $a_0 + a_1 x + a_2 x^2 + \cdots + a_L x^L = 0 \pmod{p}$ for prime p has at most L different solutions x.

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$$h_m(S) = h_{a,b}(h_x(S)) \bmod m$$

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Lemma

For any two different strings s_1 and s_2 of length at most $L + 1$ and cardinality m, the probability of collision $\Pr[h_m(s_1) = h_m(s_2)]$ is at most $\frac{1}{m} + \frac{L}{p}$.

Polynomial Hashing

Corollary

If $p > mL$, for any two different strings s_1 and s_2 of length at most $L + 1$ the probability of collision $\Pr[h_m(s_1) = h_m(s_2)]$ *is* $O(\frac{1}{m})$.

Proof

$$\frac{1}{m} + \frac{L}{p} < \frac{1}{m} + \frac{L}{mL} = \frac{1}{m} + \frac{1}{m} = \frac{2}{m} = O\left(\frac{1}{m}\right) \qquad \square$$

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- Computing $\texttt{PolyHash}(S)$ runs in $O(|S|)$
- If lengths of the names in the phone book are bounded by constant L, computing $h(S)$ takes $O(L) = O(1)$ time

Conclusion

- You learned how to hash integers and strings
- Phone book can be implemented as two hash tables
- Mapping phone numbers to names and back
- Search and modification run in $O(1)$ on average!