

# Disjoint Sets: Efficient Implementations

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Data Structures  
Data Structures and Algorithms

# Outline

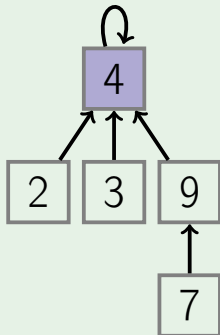
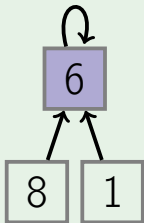
- 1 Trees
- 2 Union by Rank
- 3 Path Compression
- 4 Analysis

- Represent each set as a rooted tree

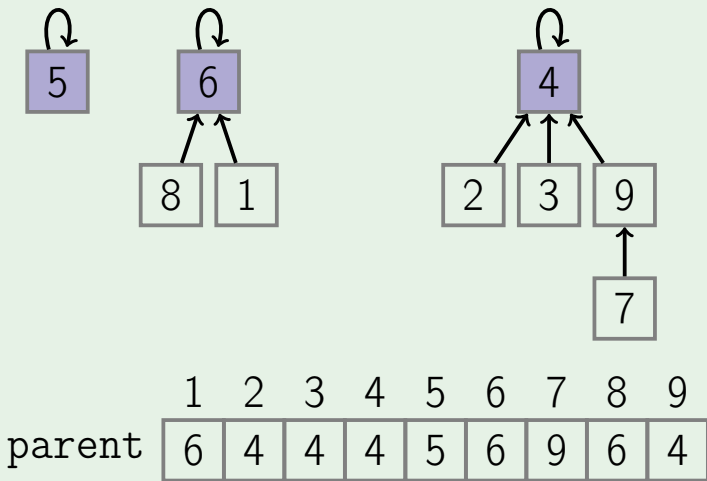
- Represent each set as a rooted tree
- ID of a set is the root of the tree

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- ID of a set is the root of the tree
- Use array  $\text{parent}[1 \dots n]$ :  $\text{parent}[i]$  is the parent of  $i$ , or  $i$  if it is the root

# Example



# Example



MakeSet( $i$ )

parent[ $i$ ]  $\leftarrow i$



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Running time:  $O(1)$

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Find( $i$ )

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return  $i$

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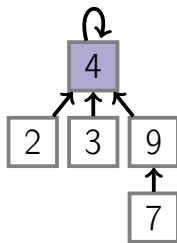
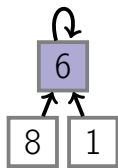
Running time:  $O(\text{tree height})$

- How to merge two trees?

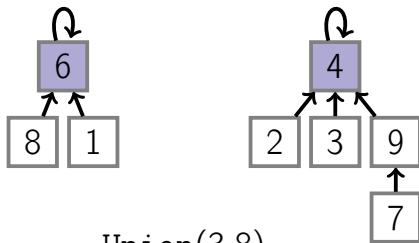
- How to merge two trees?
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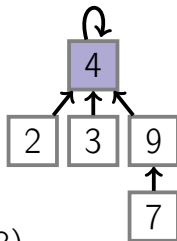
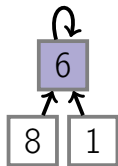
- How to merge two trees?
- Hang one of the trees under the root of the other one
- Which one to hang?
- A shorter one, since we would like to keep the trees shallow



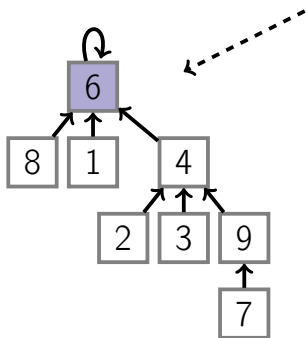


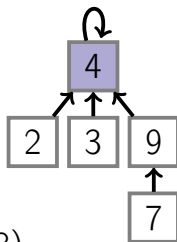
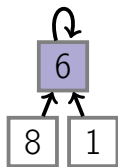


Union(3,8)

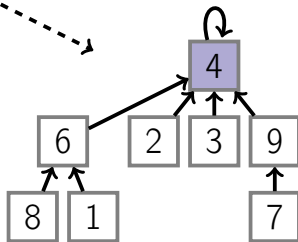
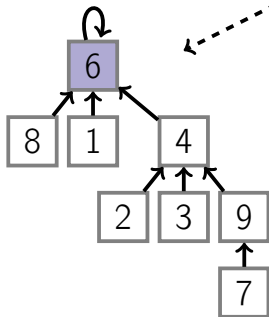


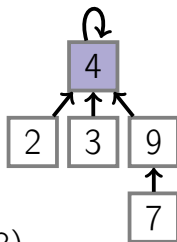
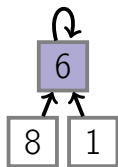
Union(3,8)



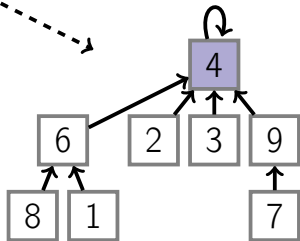
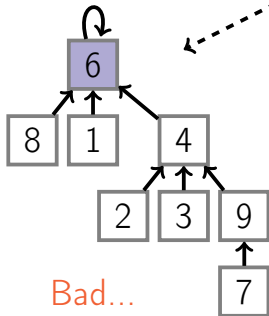


Union(3,8)





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- When merging two trees we hang a shorter one under the root of a taller one

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- (The reason we call it `rank`, but not `height` will become clear later)
- Hanging a shorter tree under a taller one is called a **union by rank heuristic**

## MakeSet( $i$ )

```
parent[ $i$ ]  $\leftarrow i$   
rank[ $i$ ]  $\leftarrow 0$ 
```

## Find( $i$ )

```
while  $i \neq \text{parent}[i]$ :  
     $i \leftarrow \text{parent}[i]$   
return  $i$ 
```

## Union( $i, j$ )

```
 $i\_id \leftarrow \text{Find}(i)$   
 $j\_id \leftarrow \text{Find}(j)$   
if  $i\_id = j\_id$ :  
    return  
if  $\text{rank}[i\_id] > \text{rank}[j\_id]$ :  
     $\text{parent}[j\_id] \leftarrow i\_id$   
else:  
     $\text{parent}[i\_id] \leftarrow j\_id$   
    if  $\text{rank}[i\_id] = \text{rank}[j\_id]$ :  
         $\text{rank}[j\_id] \leftarrow \text{rank}[j\_id] + 1$ 
```

# Example

Query:

|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent |   |   |   |   |   |   |
| rank   |   |   |   |   |   |   |

# Example

Query:

MakeSet(1)

MakeSet(2)

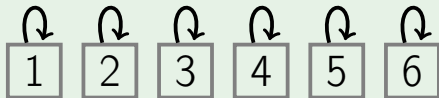
...

MakeSet(6)

|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent |   |   |   |   |   |   |
| rank   |   |   |   |   |   |   |

# Example

Query:

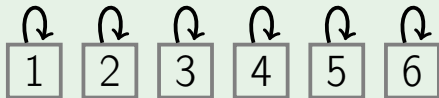


|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent | 1 | 2 | 3 | 4 | 5 | 6 |
| rank   | 0 | 0 | 0 | 0 | 0 | 0 |

# Example

Query:

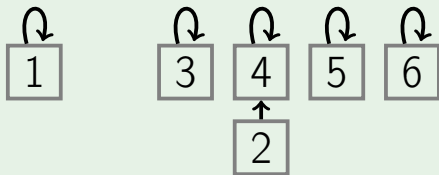
Union(2, 4)



|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent | 1 | 2 | 3 | 4 | 5 | 6 |
| rank   | 0 | 0 | 0 | 0 | 0 | 0 |

# Example

Query:



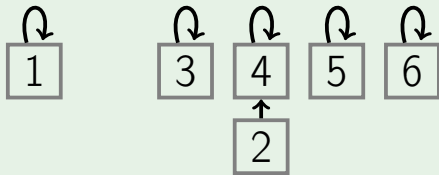
|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent | 1 | 4 | 3 | 4 | 5 | 6 |
| rank   | 0 | 0 | 0 | 1 | 0 | 0 |



# Example

Query:

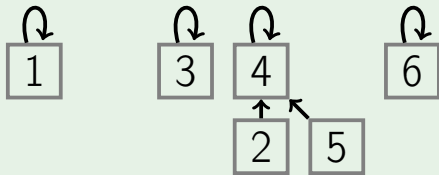
Union(5, 2)



|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent | 1 | 4 | 3 | 4 | 5 | 6 |
| rank   | 0 | 0 | 0 | 1 | 0 | 0 |

# Example

Query:

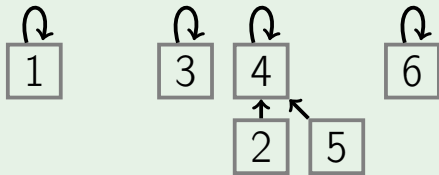


|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent | 1 | 4 | 3 | 4 | 4 | 6 |
| rank   | 0 | 0 | 0 | 1 | 0 | 0 |

# Example

Query:

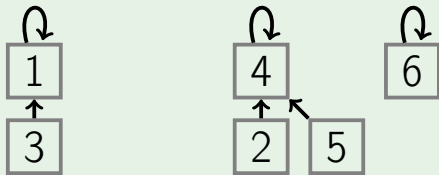
Union(3, 1)



|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent | 1 | 4 | 3 | 4 | 4 | 6 |
| rank   | 0 | 0 | 0 | 1 | 0 | 0 |

# Example

Query:

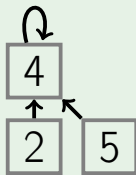
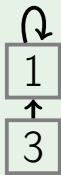


|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent | 1 | 4 | 1 | 4 | 4 | 6 |
| rank   | 1 | 0 | 0 | 1 | 0 | 0 |

# Example

Query:

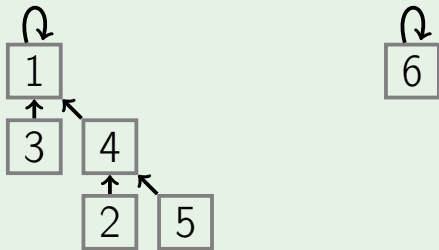
Union(2, 3)



|        |   |   |   |   |   |   |
|--------|---|---|---|---|---|---|
|        | 1 | 2 | 3 | 4 | 5 | 6 |
| parent | 1 | 4 | 1 | 4 | 4 | 6 |
| rank   | 1 | 0 | 0 | 1 | 0 | 0 |

# Example

Query:

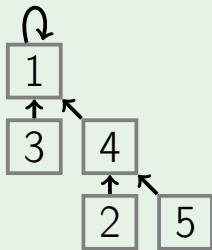


|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent | 1 | 4 | 1 | 1 | 4 | 6 |
| rank   | 2 | 0 | 0 | 1 | 0 | 0 |

# Example

Query:

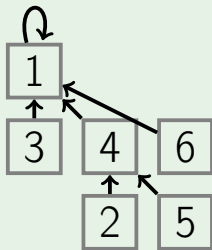
Union(2, 6)



|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent | 1 | 4 | 1 | 1 | 4 | 6 |
| rank   | 2 | 0 | 0 | 1 | 0 | 0 |

# Example

Query:



|        | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| parent | 1 | 4 | 1 | 1 | 4 | 1 |
| rank   | 2 | 0 | 0 | 1 | 0 | 0 |



Important property: for any node  $i$ ,  $\text{rank}[i]$  is equal to the height of the tree rooted at  $i$

## Lemma

The height of any tree in the forest is at most  $\log_2 n$ .

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Follows from the following lemma.

## Lemma

Any tree of height  $k$  in the forest has at least  $2^k$  nodes.

# Proof

Induction on  $k$ .

- Base: initially, a tree has height 0 and one node:  $2^0 = 1$ .
- Step: a tree of height  $k$  results from merging two trees of height  $k - 1$ . By induction hypothesis, each of two trees has at least  $2^{k-1}$  nodes, hence the resulting tree contains at least  $2^k$  nodes.



## Summary

The union by rank heuristic guarantees that Union and Find work in time  $O(\log n)$ .

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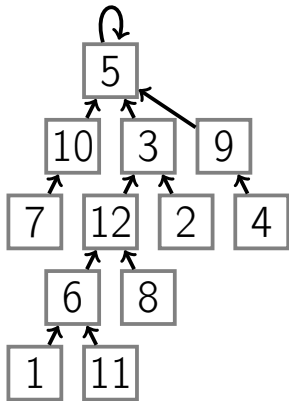
## Next part

We'll discover another heuristic that improves the running time to nearly constant!

# Outline

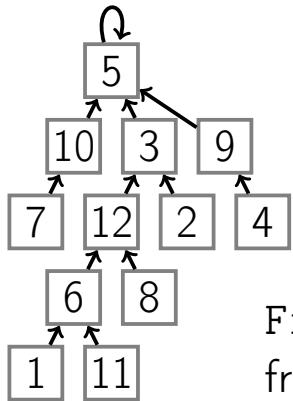
- 1 Trees
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# Path Compression: Intuition



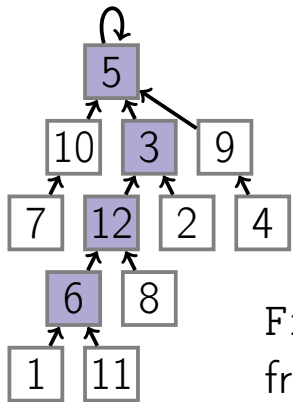


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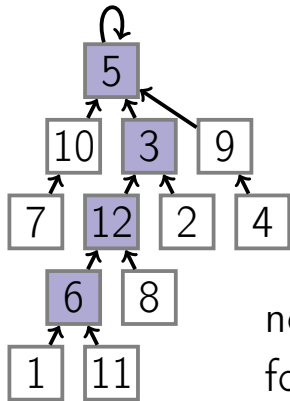
Find(6) traverses the path  
from 6 to the root

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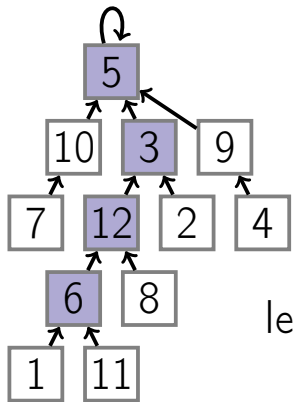
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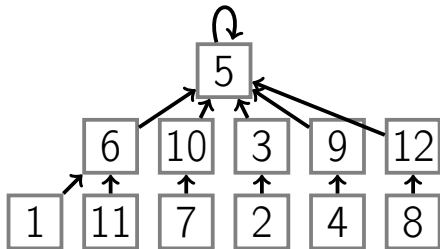
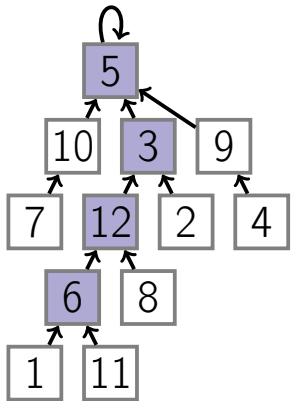
not only it finds the root  
for 6, it does so for all the  
nodes on this path

# Path Compression: Intuition

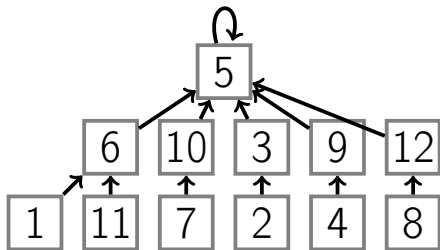
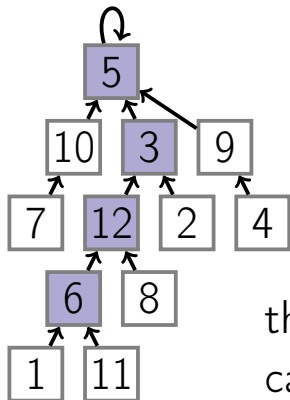


let's not lose this useful info

# Path Compression: Intuition



# Path Compression: Intuition



the resulting heuristic is called **path compression**

Find( $i$ )

```
if  $i \neq \text{parent}[i]$ :  
     $\text{parent}[i] \leftarrow \text{Find}(\text{parent}[i])$   
return  $\text{parent}[i]$ 
```

## Definition

The **iterated logarithm** of  $n$ ,  $\log^* n$ , is the number of times the logarithm function needs to be applied to  $n$  before the result is less or equal than 1:

$$\log^* n = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + \log^*(\log n) & \text{if } n > 1 \end{cases}$$



# Example

| $n$                                 | $\log^* n$ |
|-------------------------------------|------------|
| $n = 1$                             | 0          |
| $n = 2$                             | 1          |
| $n \in \{3, 4\}$                    | 2          |
| $n \in \{5, 6, \dots, 16\}$         | 3          |
| $n \in \{17, \dots, 65536\}$        | 4          |
| $n \in \{65537, \dots, 2^{65536}\}$ | 5          |

## Lemma

Assume that initially the data structure is empty. We make a sequence of  $m$  operations including  $n$  calls to MakeSet. Then the total running time is  $O(m \log^* n)$ .

In other words

The amortized time of a single operation is  $O(\log^* n)$ .

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Nearly constant!

For practical values of  $n$ ,  $\log^* n \leq 5$ .

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## Goal

Prove that when both union by rank heuristic and path compression heuristic are used, the average running time of each operation is nearly constant.

# Height $\leq$ Rank

- When using path compression,  $\text{rank}[i]$  is no longer equal to the height of the subtree rooted at  $i$

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- Still, the height of the subtree rooted at  $i$  is at most  $\text{rank}[i]$



# Height $\leq$ Rank

- When using path compression,  $\text{rank}[i]$  is no longer equal to the height of the subtree rooted at  $i$
- Still, the height of the subtree rooted at  $i$  is at most  $\text{rank}[i]$
- And it is still true that a **root node** of rank  $k$  has at least  $2^k$  nodes in its subtree: a root node is not affected by path compression

# Important Properties

- 1 There are at most  $\frac{n}{2^k}$  nodes of rank  $k$

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- 2 For any node  $i$ ,  
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# Important Properties

- 1 There are at most  $\frac{n}{2^k}$  nodes of rank  $k$
- 2 For any node  $i$ ,  
 $\text{rank}[i] < \text{rank}[\text{parent}[i]]$
- 3 Once an internal node, always an internal node

$T(\text{all calls to Find}) =$

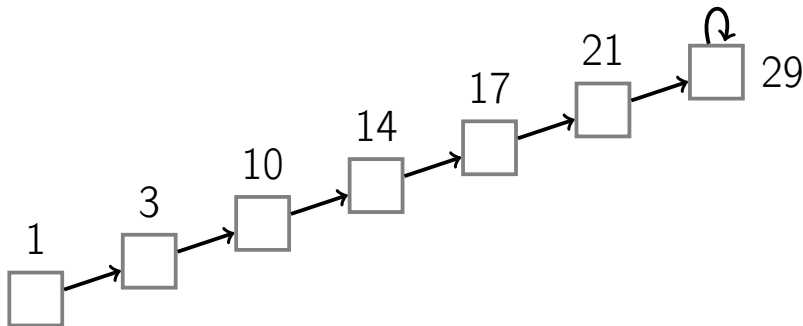
$\#(i \rightarrow j) =$

$\#(i \rightarrow j: j \text{ is a root}) +$

$\#(i \rightarrow j: \log^*(\text{rank}[i]) < \log^*(\text{rank}[j])) +$

$\#(i \rightarrow j: \log^*(\text{rank}[i]) = \log^*(\text{rank}[j]))$

$$\begin{aligned}
 T(\text{all calls to Find}) = \\
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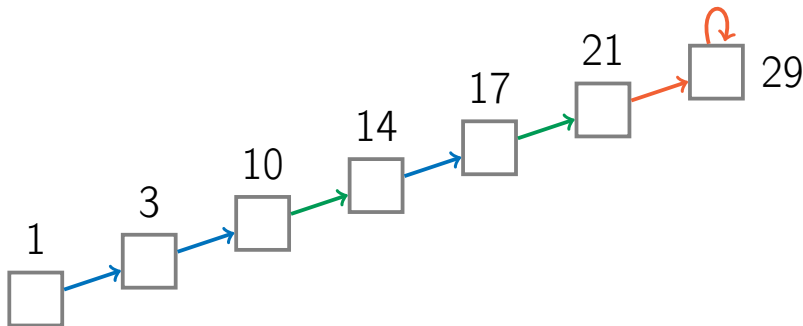
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Claim

$$\#(i \rightarrow j: j \text{ is a root}) \leq O(m)$$



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$$\#(i \rightarrow j: j \text{ is a root}) \leq O(m)$$

## Proof

There are at most  $m$  calls to Find.



# Claim

$$\begin{aligned} \#(i \rightarrow j: \log^*(\text{rank}[i]) < \log^*(\text{rank}[j])) \\ \leq O(m \log^* n) \end{aligned}$$

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## Proof

There are at most  $\log^* n$  different values for  $\log^*(\text{rank})$ . □

## Claim

$$\#(i \rightarrow j: \log^*(\text{rank}[i]) = \log^*(\text{rank}[j])) \leq O(n \log^* n)$$

# Proof

- assume  $\text{rank}[i] \in \{k+1, \dots, 2^k\}$

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- after a call to  $\text{Find}(i)$ , the node  $i$  is adopted by a new parent of strictly larger rank
- after at most  $2^k$  calls to  $\text{Find}(i)$ , the parent of  $i$  will have rank from a different interval



## Proof (Continued)

- there are at most  $\frac{n}{2^k}$  nodes with rank in  $\{k + 1, \dots, 2^k\}$

## Proof (Continued)

- there are at most  $\frac{n}{2^k}$  nodes with rank in  $\{k + 1, \dots, 2^k\}$
- each of them contributes at most  $2^k$


## Proof (Continued)

- there are at most  $\frac{n}{2^k}$  nodes with rank in  $\{k + 1, \dots, 2^k\}$
- each of them contributes at most  $2^k$
- the contribution of all the nodes with rank from this interval is at most  $O(n)$

## Proof (Continued)

- there are at most  $\frac{n}{2^k}$  nodes with rank in  $\{k + 1, \dots, 2^k\}$
- each of them contributes at most  $2^k$
- the contribution of all the nodes with rank from this interval is at most  $O(n)$
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## Proof (Continued)

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  - each of them contributes at most  $2^k$
  - the contribution of all the nodes with rank from this interval is at most  $O(n)$
  - the number of different intervals is  $\log^* n$
  - thus, the contribution of all nodes is  $O(n \log^* n)$
- 

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(constant for practical values of  $n$ )