Paths in Graphs: Breadth-First Search

Michael Levin

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University of California, San Diego

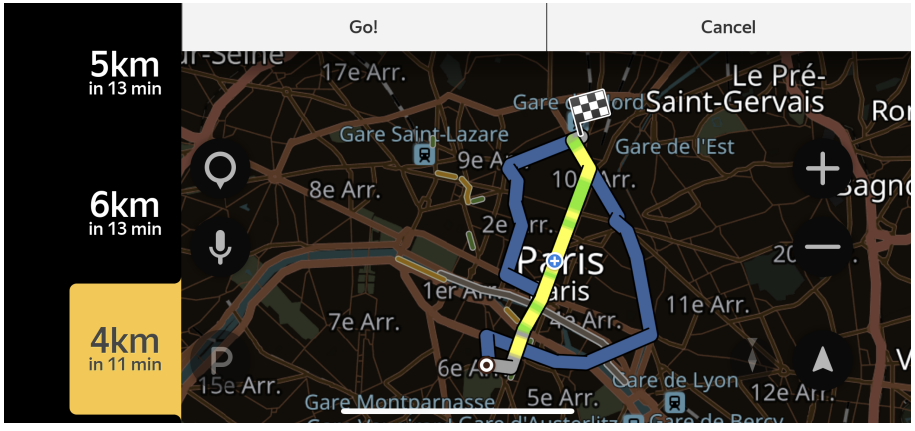
Graph Algorithms
Algorithms and Data Structures

Outline

Applications

Applications

Applications



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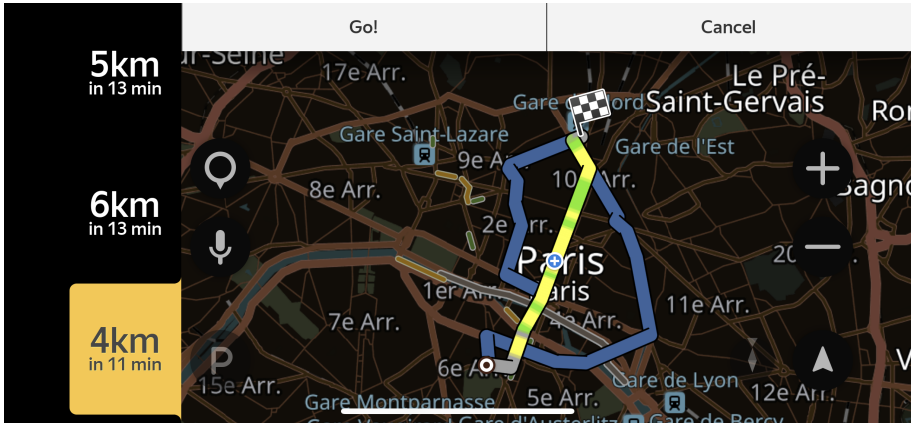

Minimize Transfers

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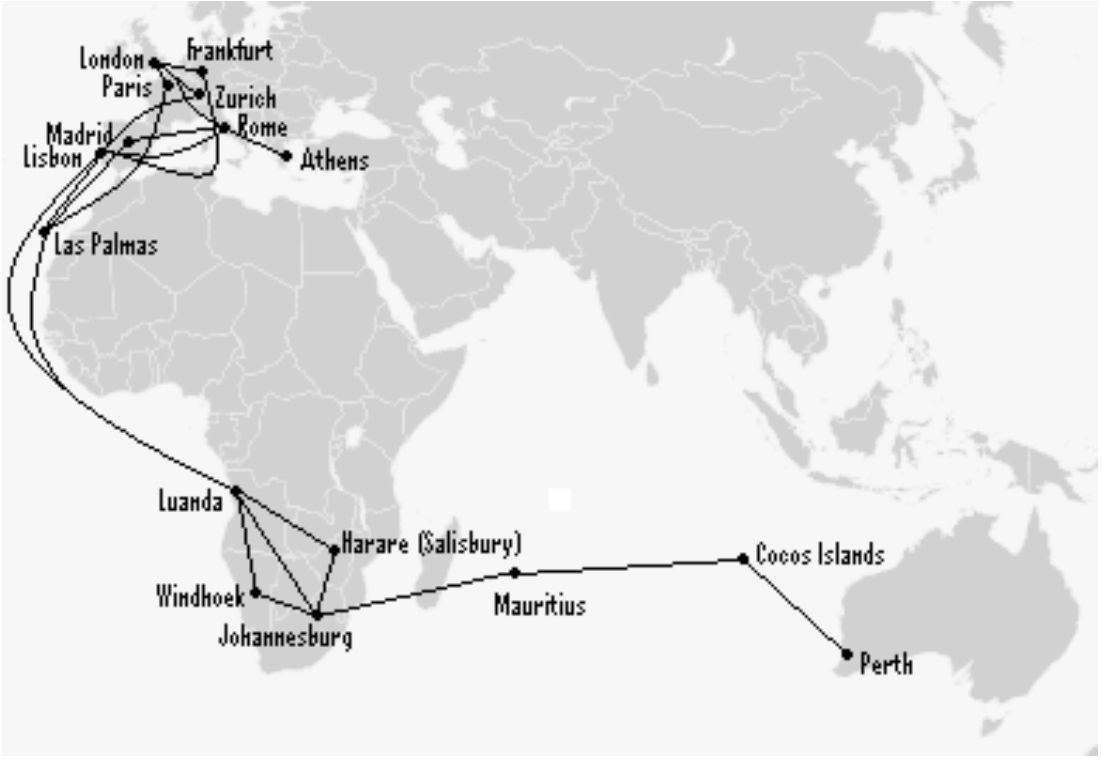

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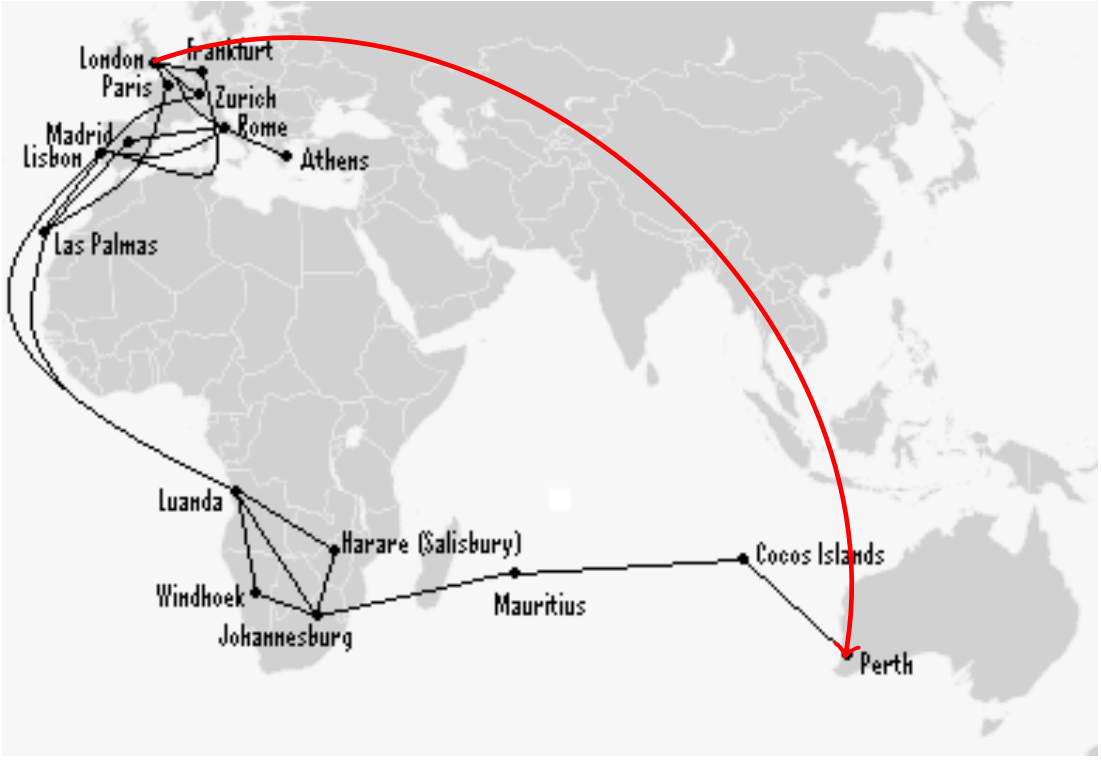

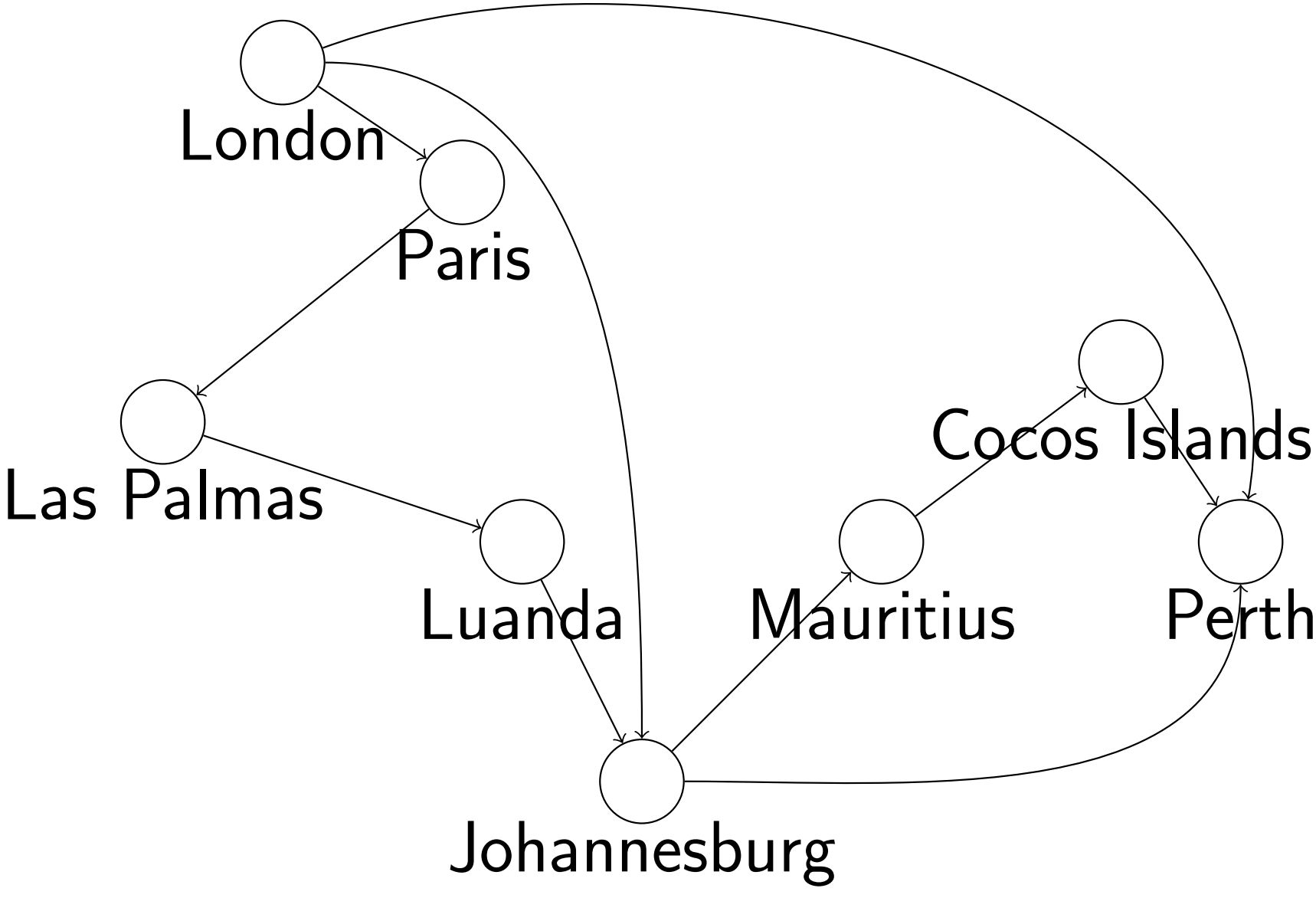
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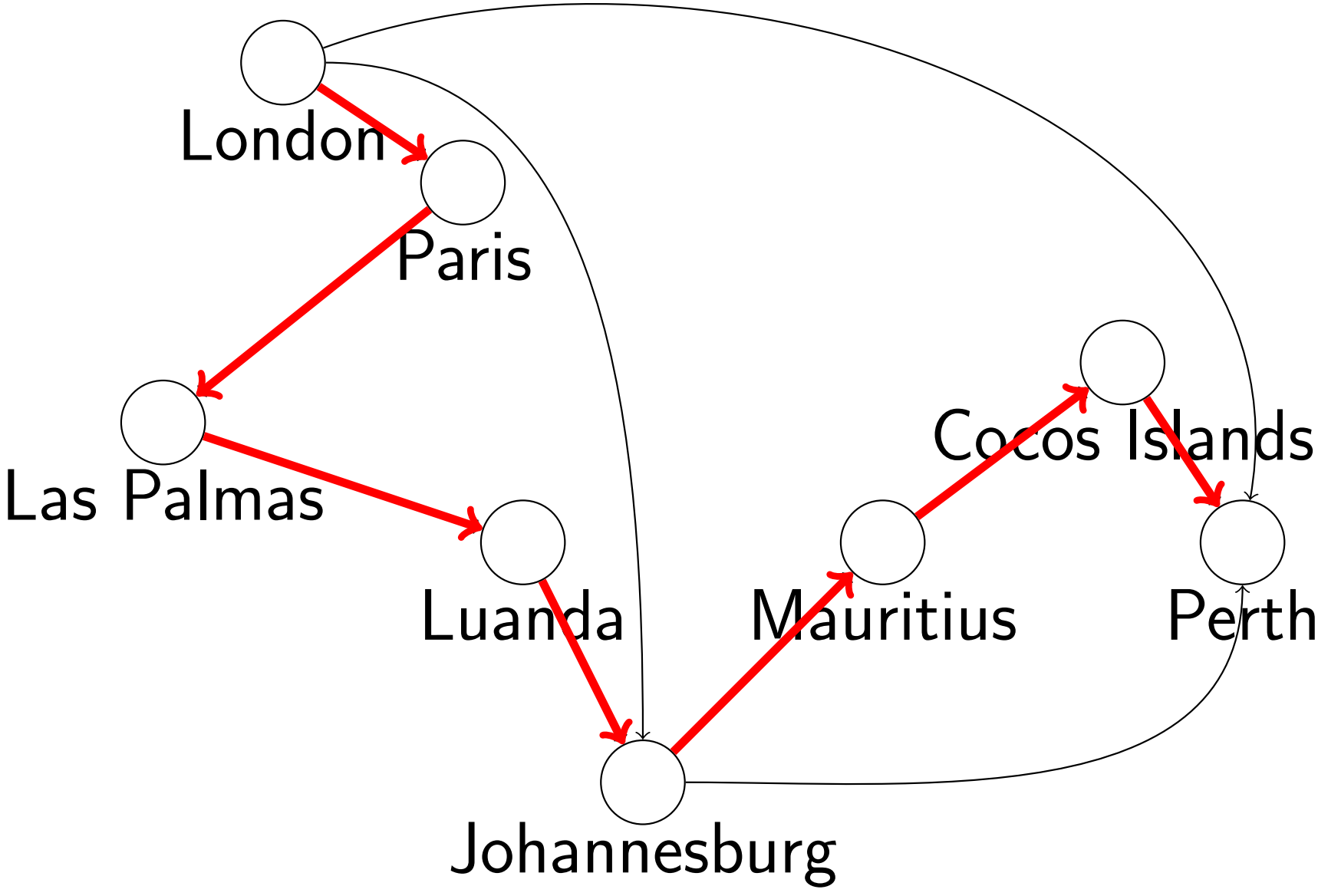
Luanda
Cocos Islands
Mauritius
Perth
Johannesburg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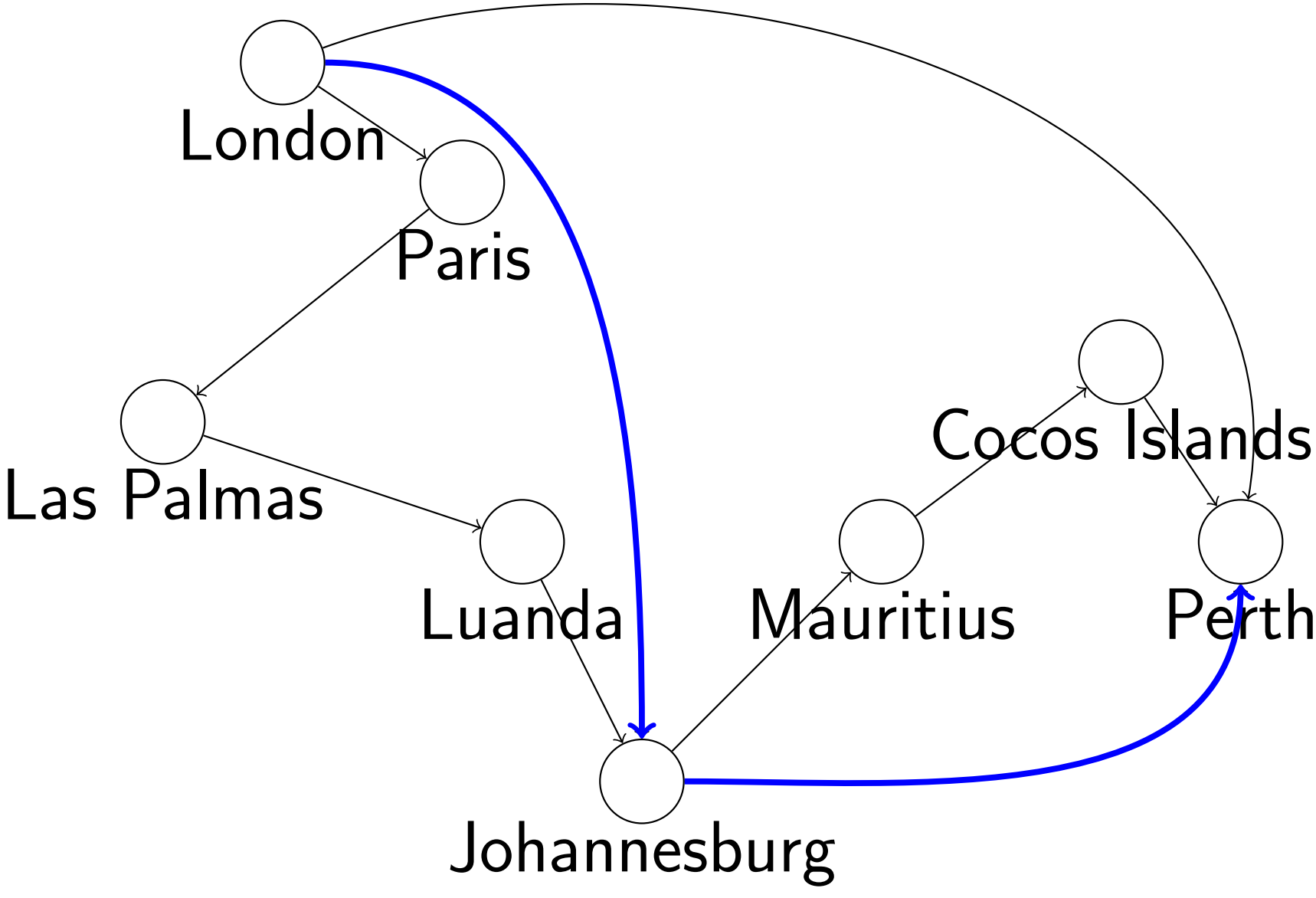
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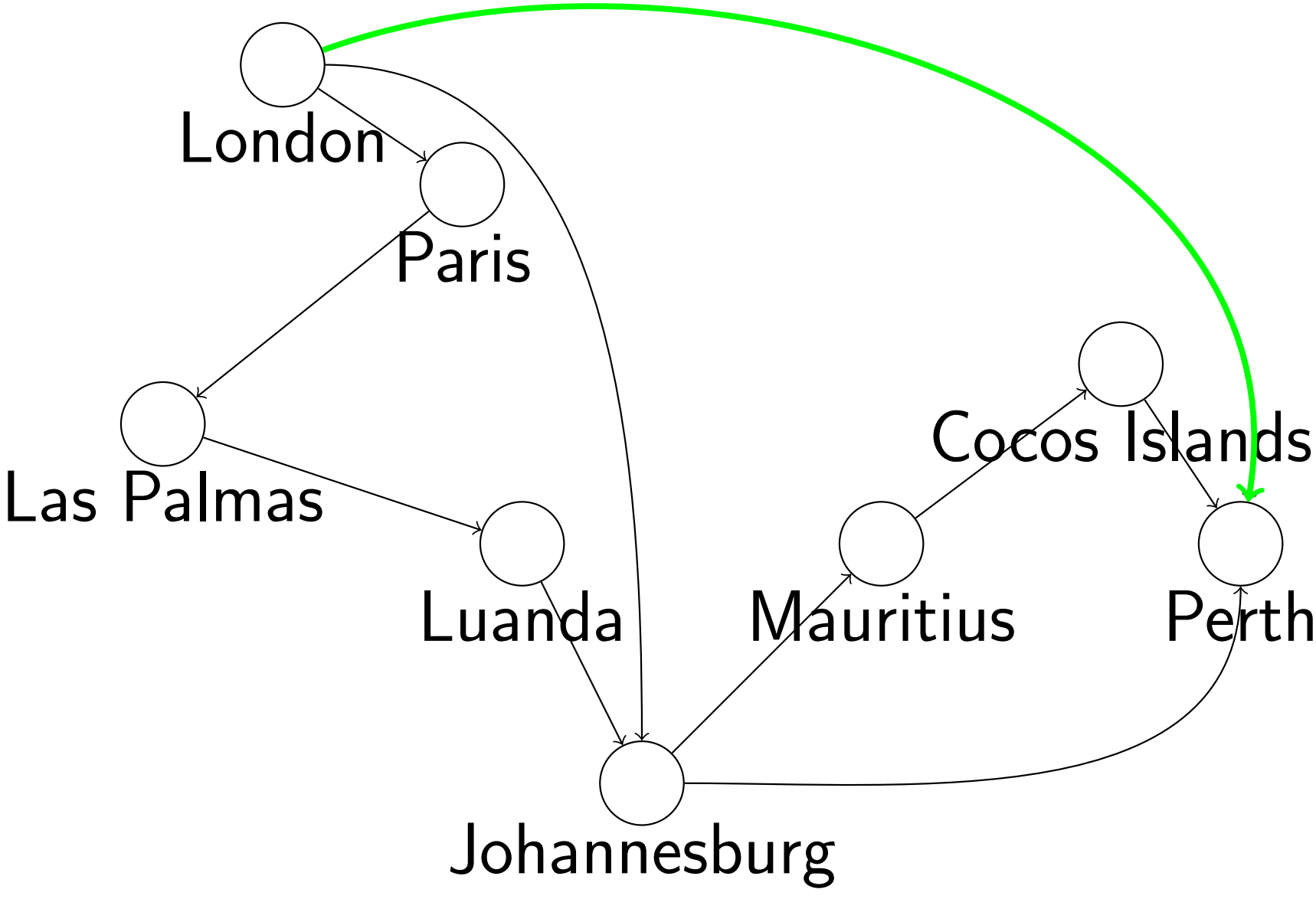
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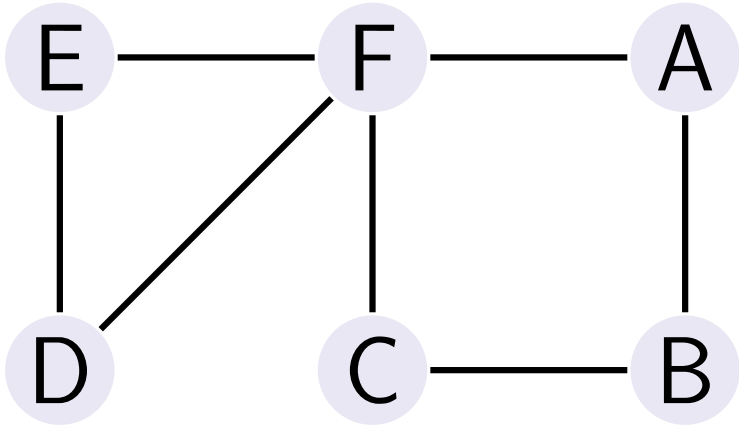
Outline

Path Length

Path length $L(P)$ is the number of edges in the path.



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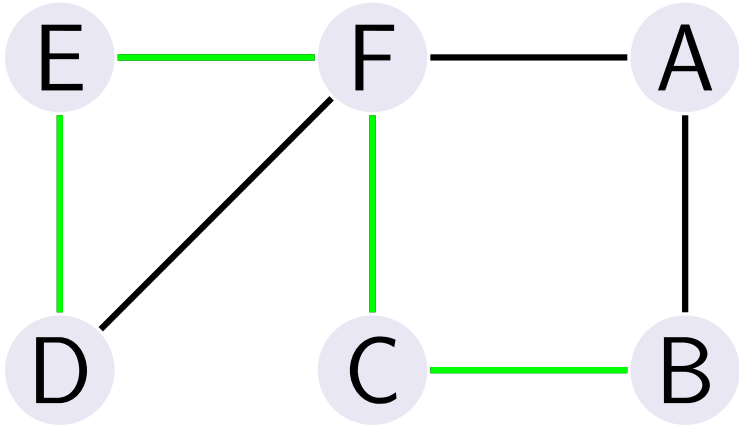

$L(B - C - F - E - D) = 4$

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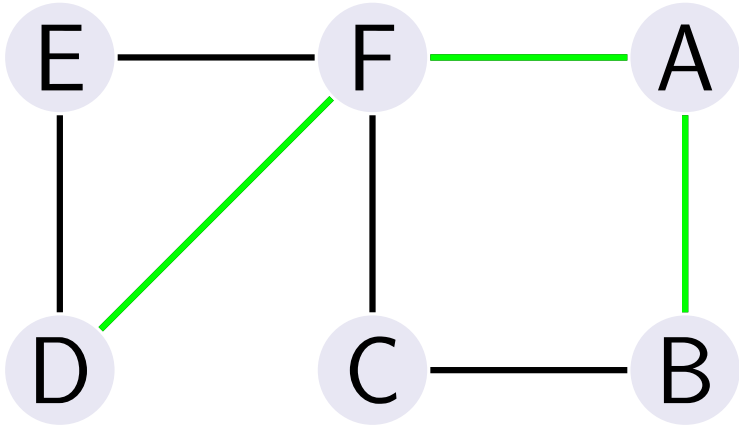
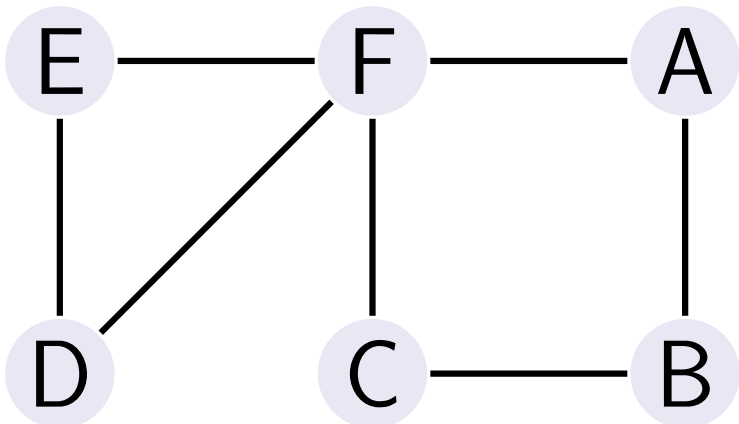


$L(B - C - F - E - D) = 4$

$L(B - A - F - D) = 3$

Distance

The distance between two nodes is the length of the shortest path between them.



Distance

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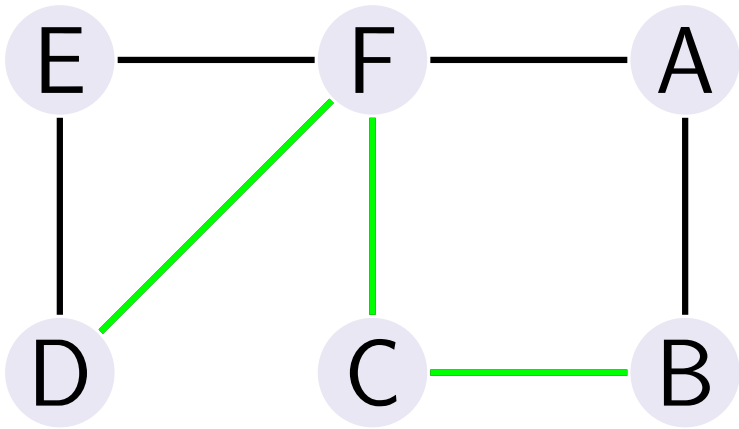
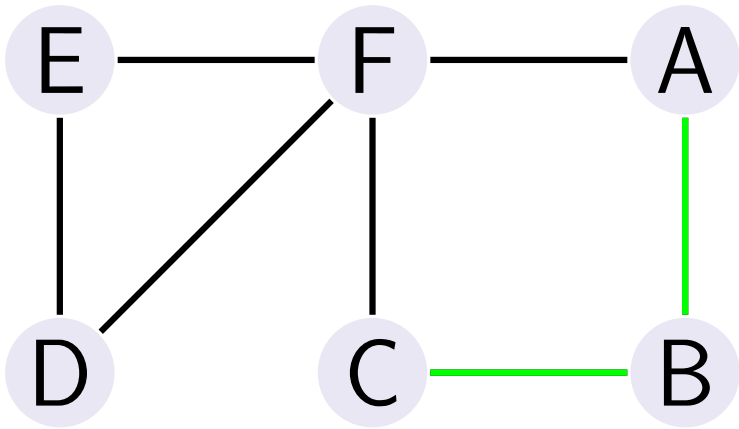


$d(B, D) = 3$

Distance

The distance between two nodes is the length of the shortest path between them.

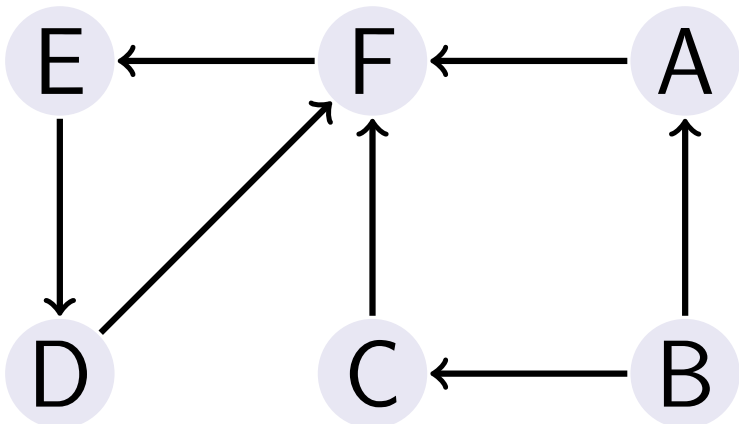


$d(B, D) = 3$

$d(A, C) = 2$

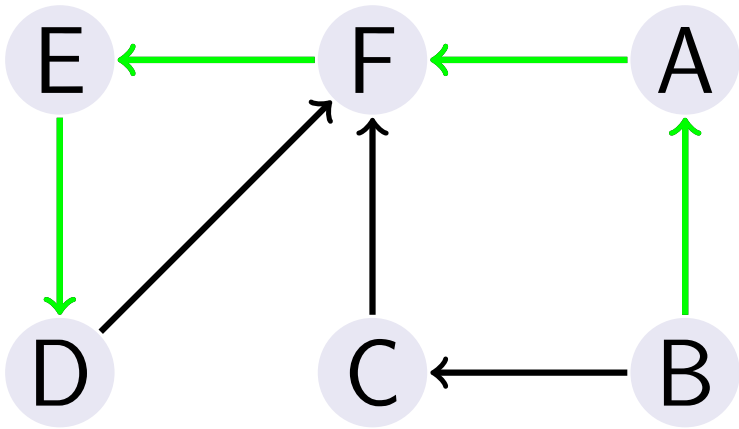
Distance

The distance between two vertices is the length of the shortest path between them.



Distance

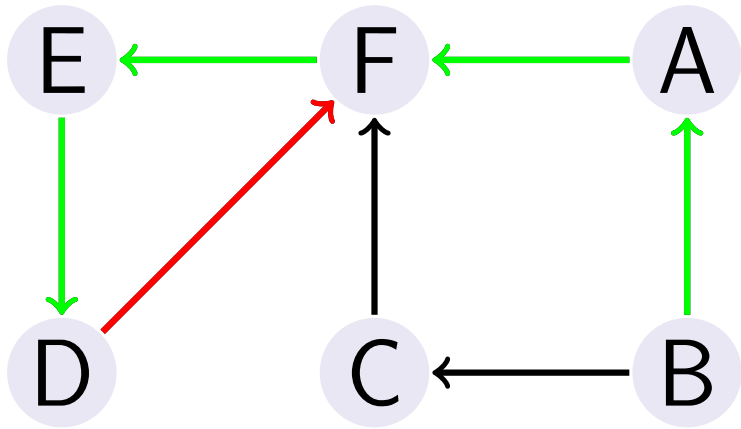
The distance between two vertices is the length of the shortest path between them.



$d(B, D) = 4$

Distance

The distance between two vertices is the length of the shortest path between them.



$d(B, D) = 4$

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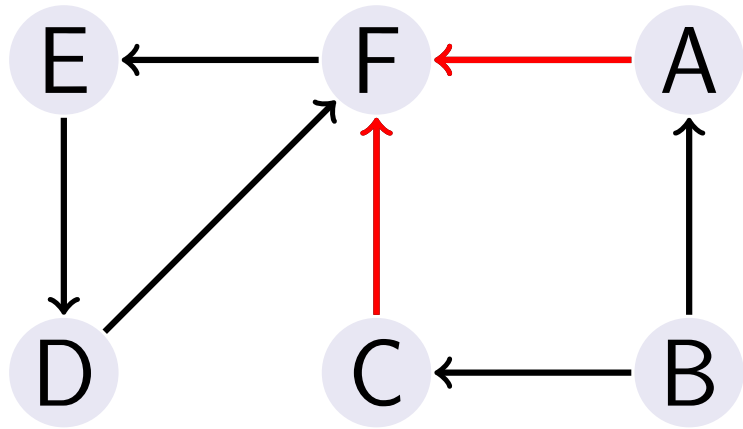

$d(B, D) = 4$

$d(A, C) = \infty$

Distance

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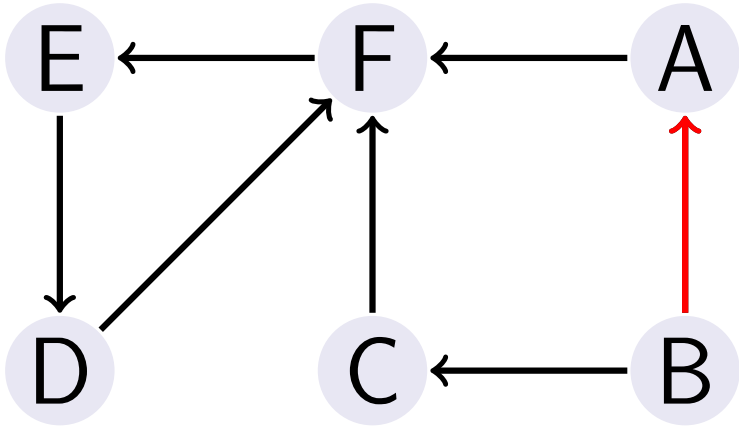

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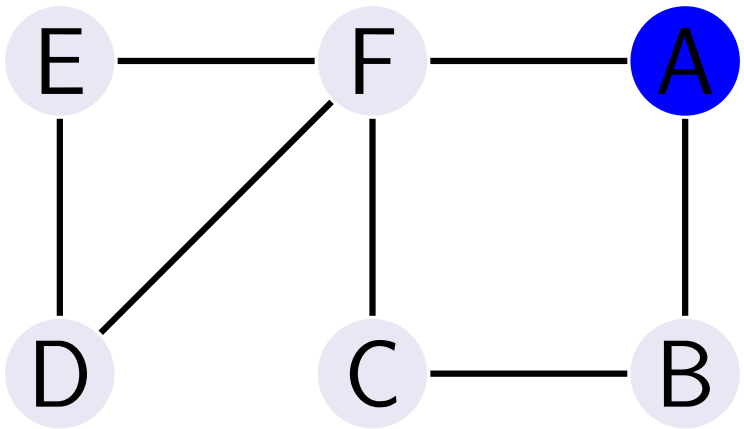
Distance



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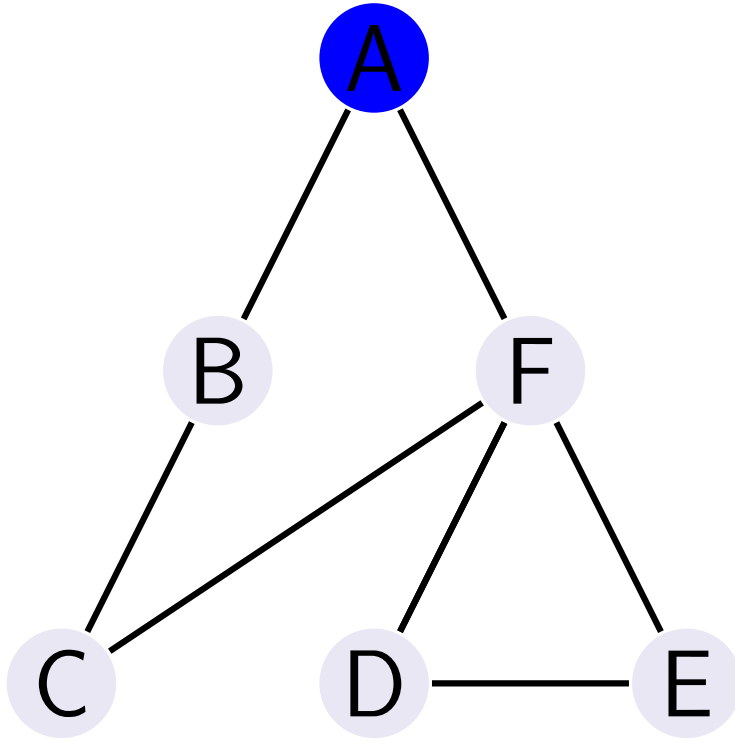

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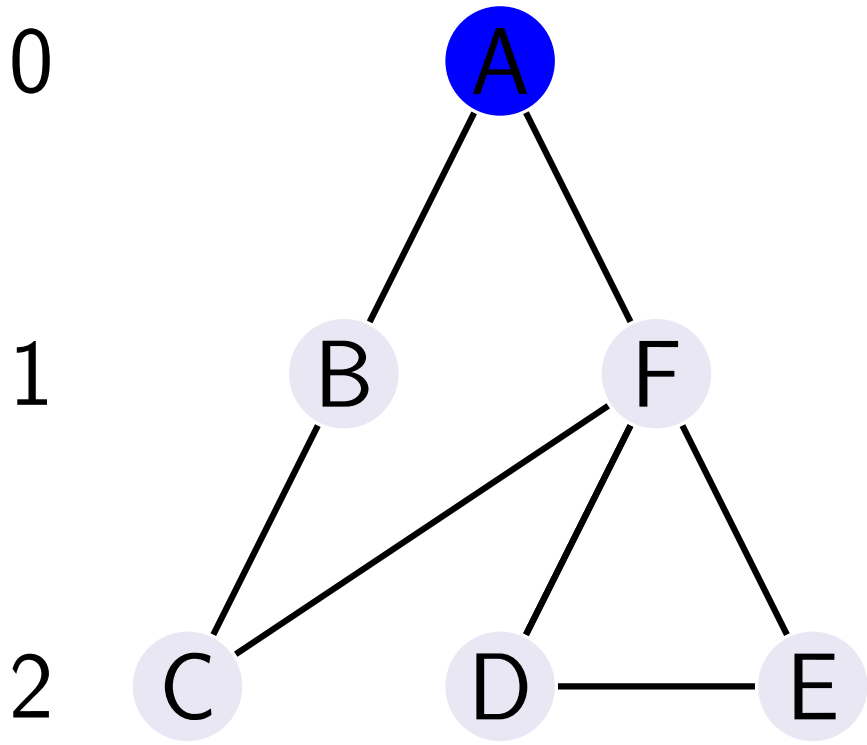

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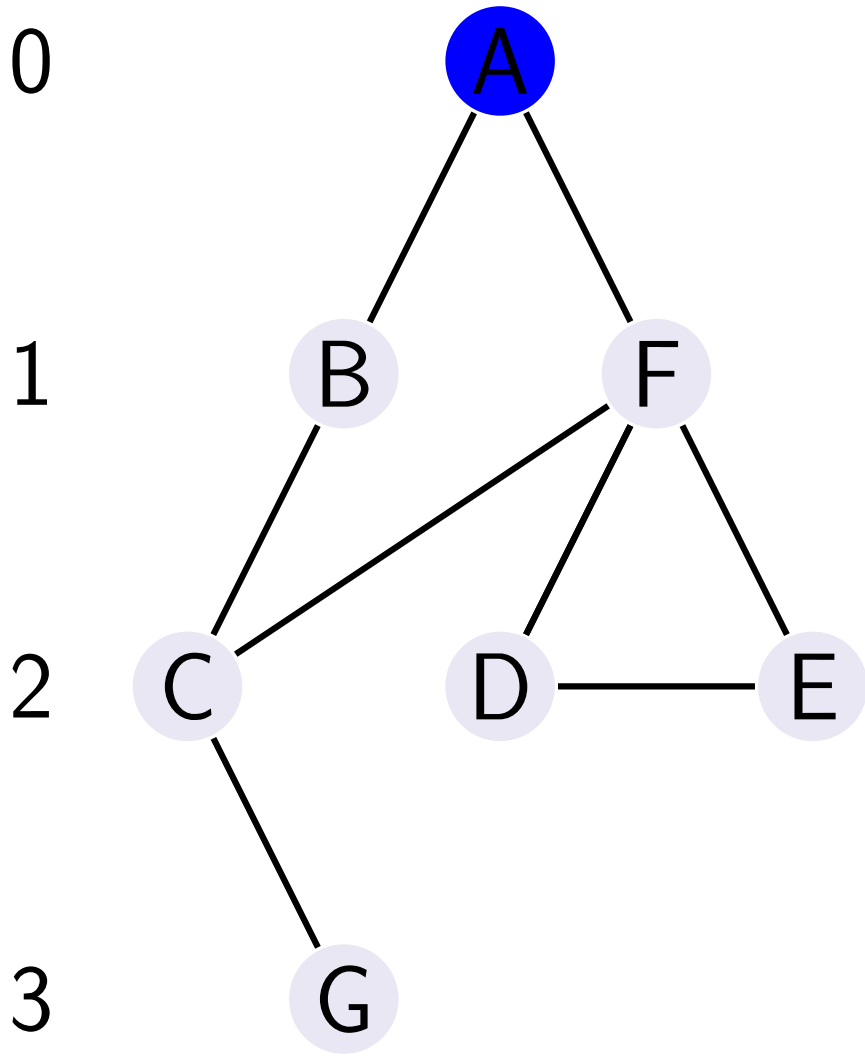

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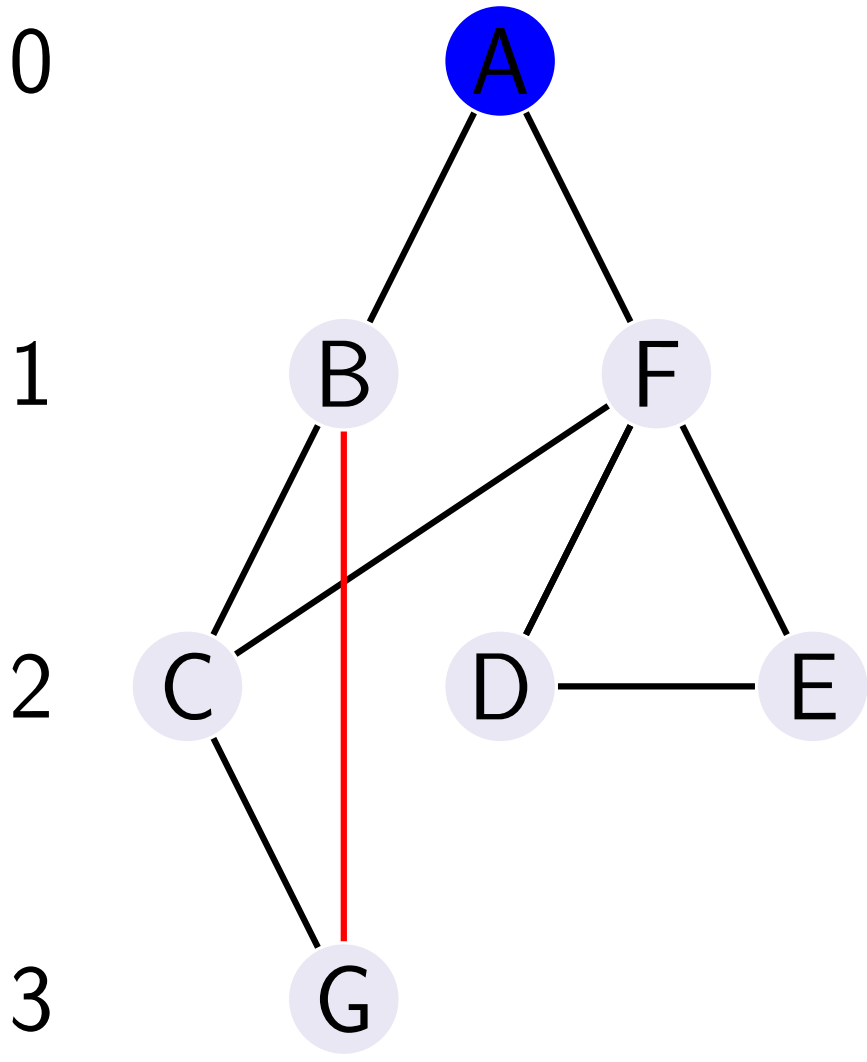

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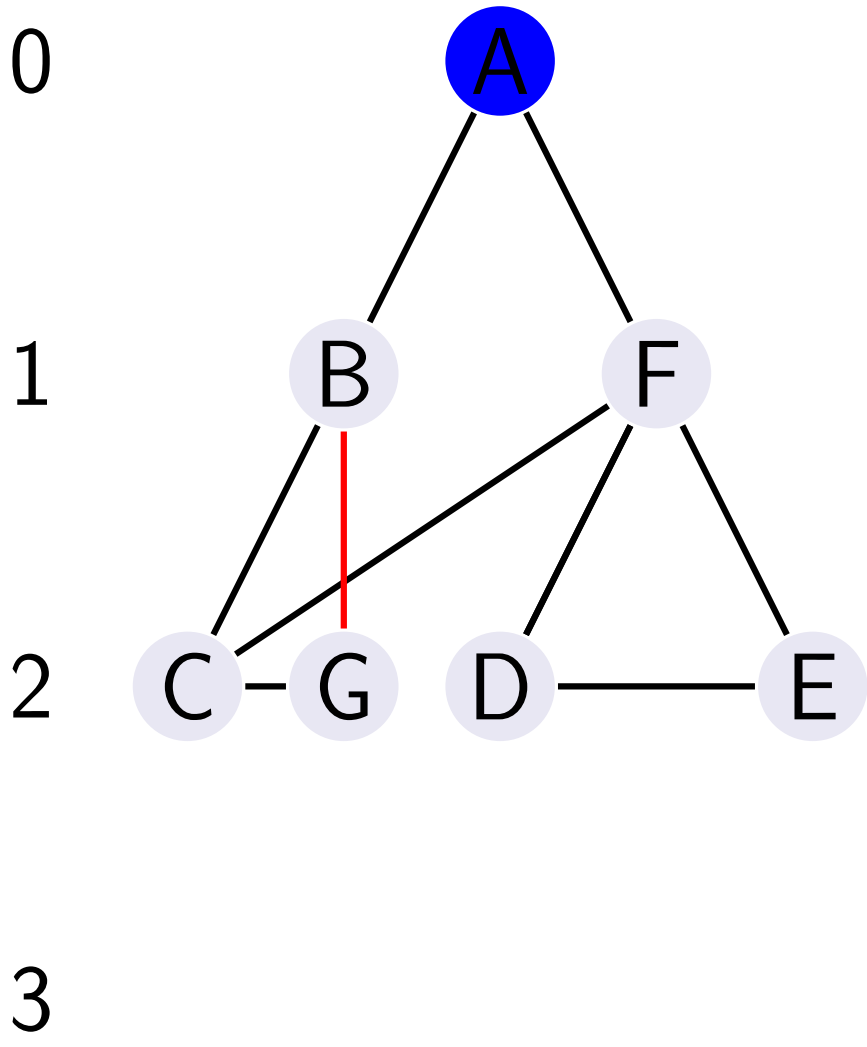

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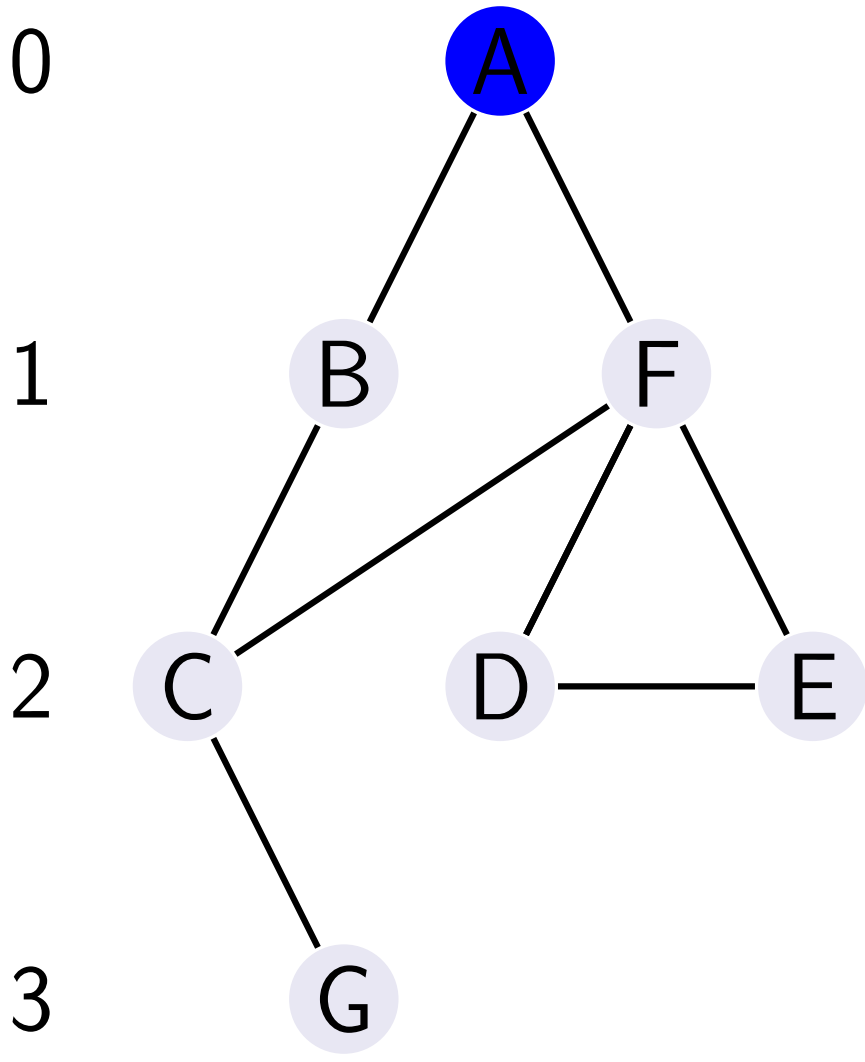

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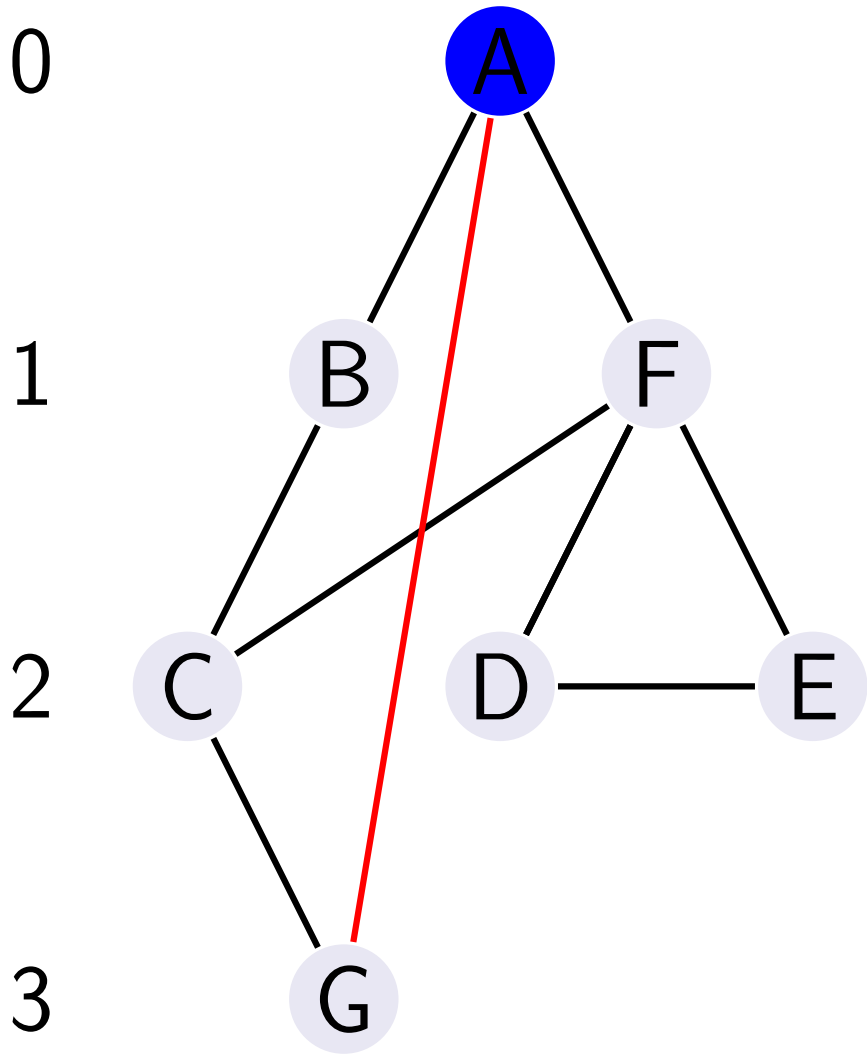

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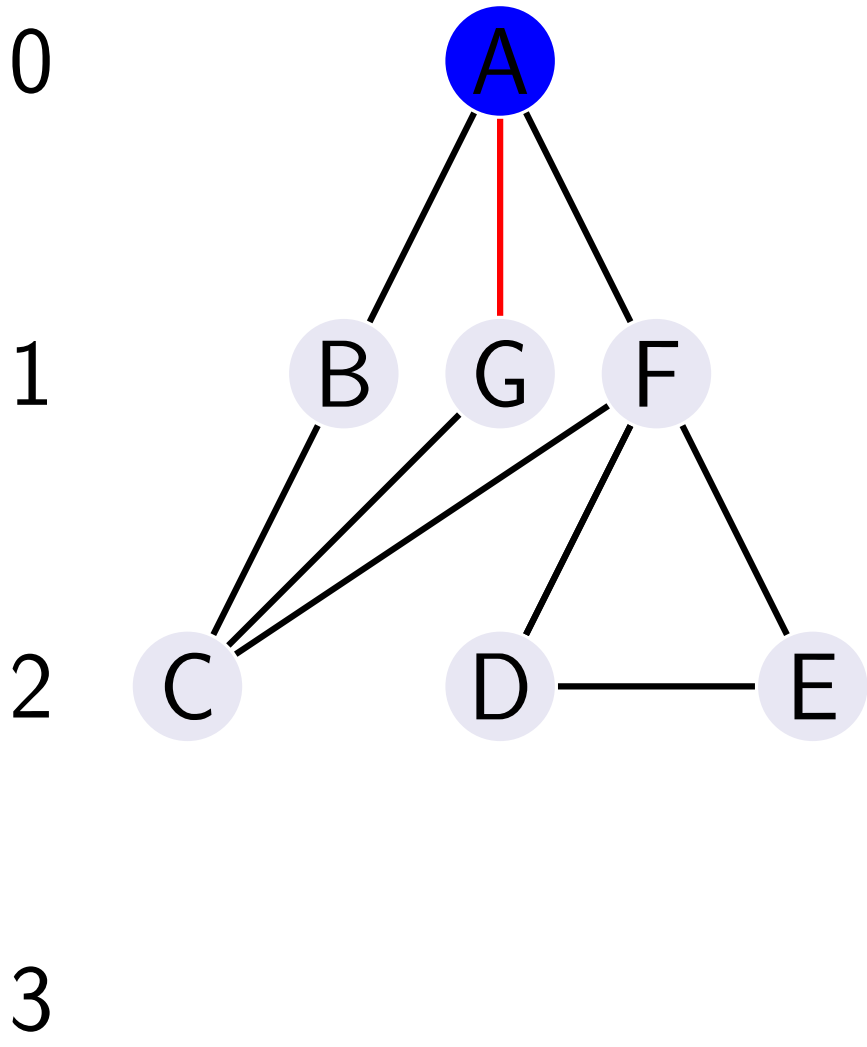

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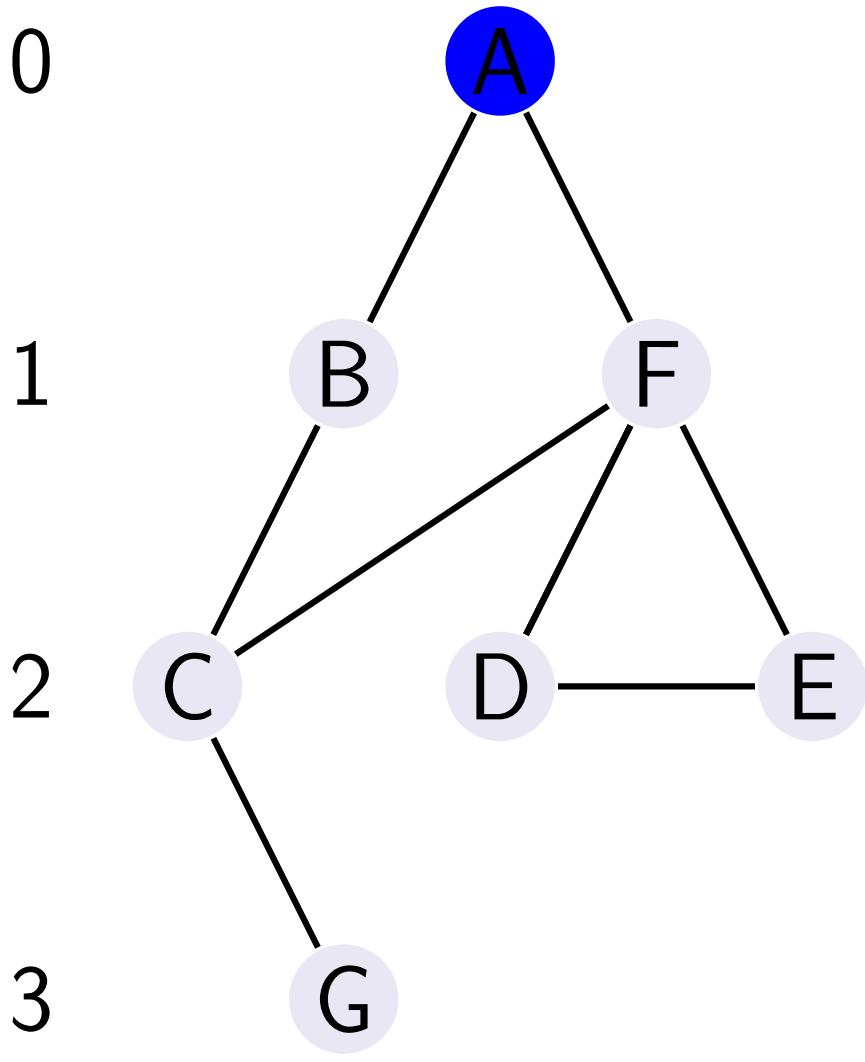

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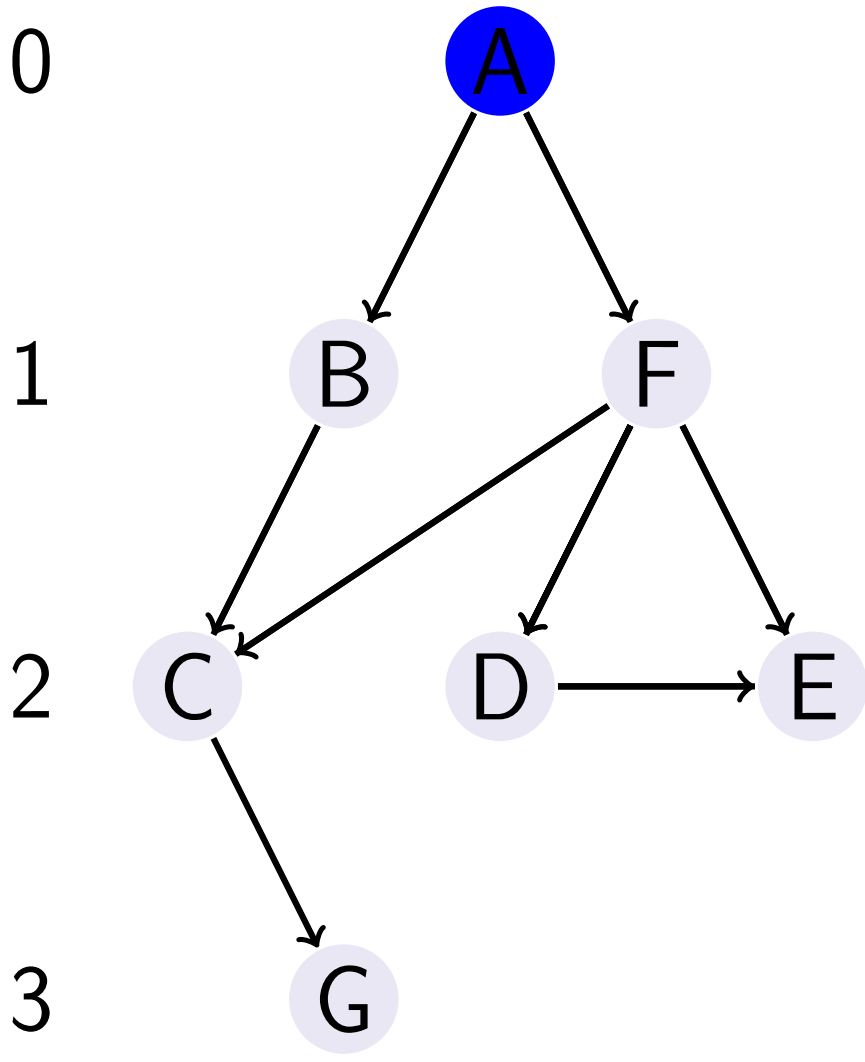

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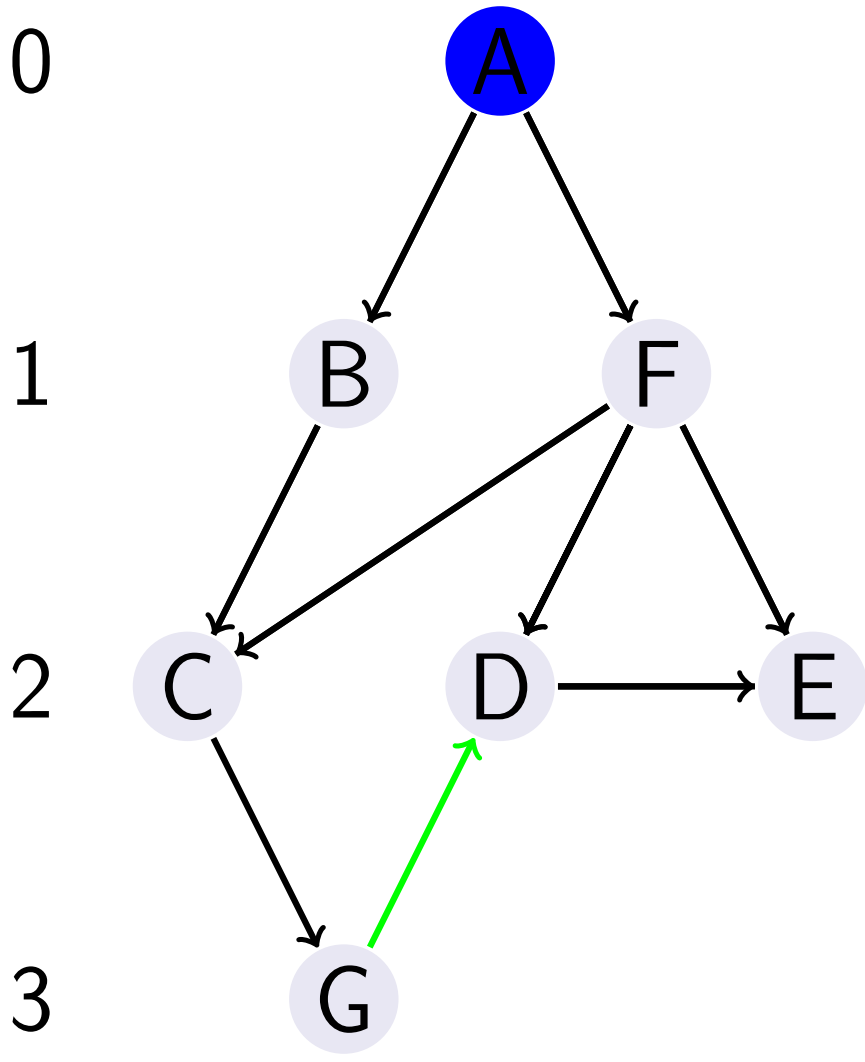

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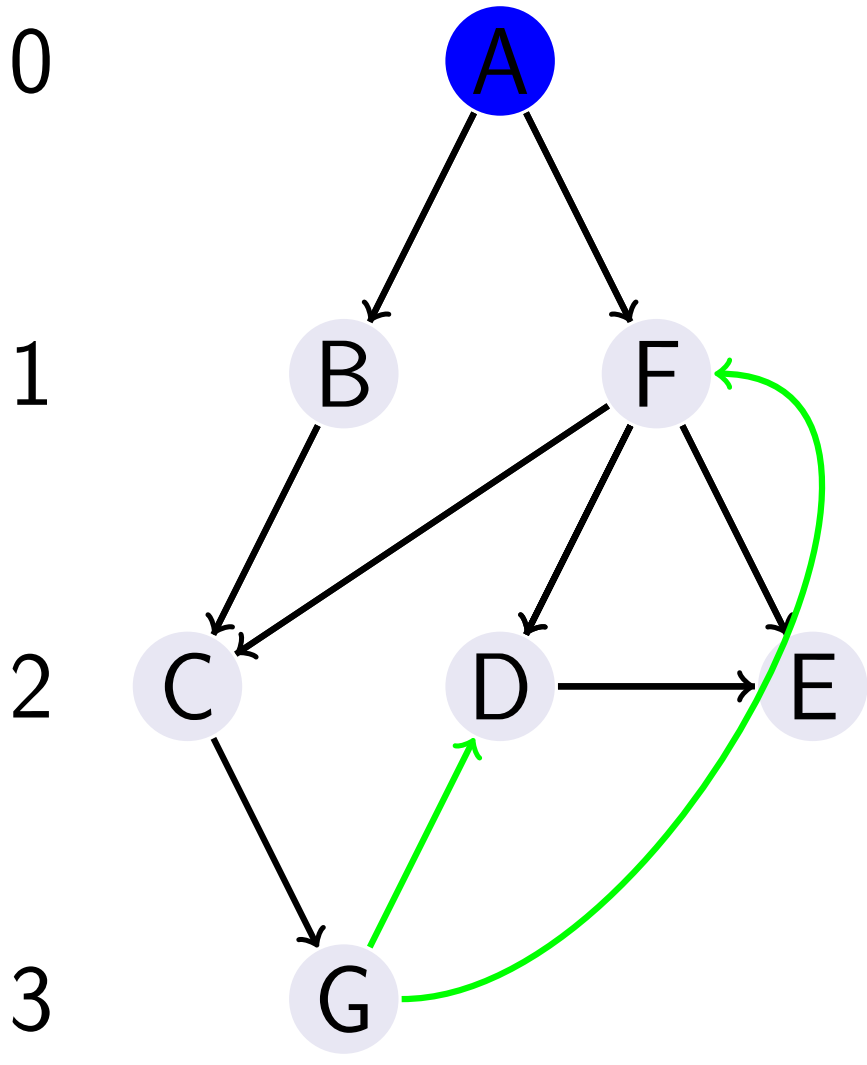

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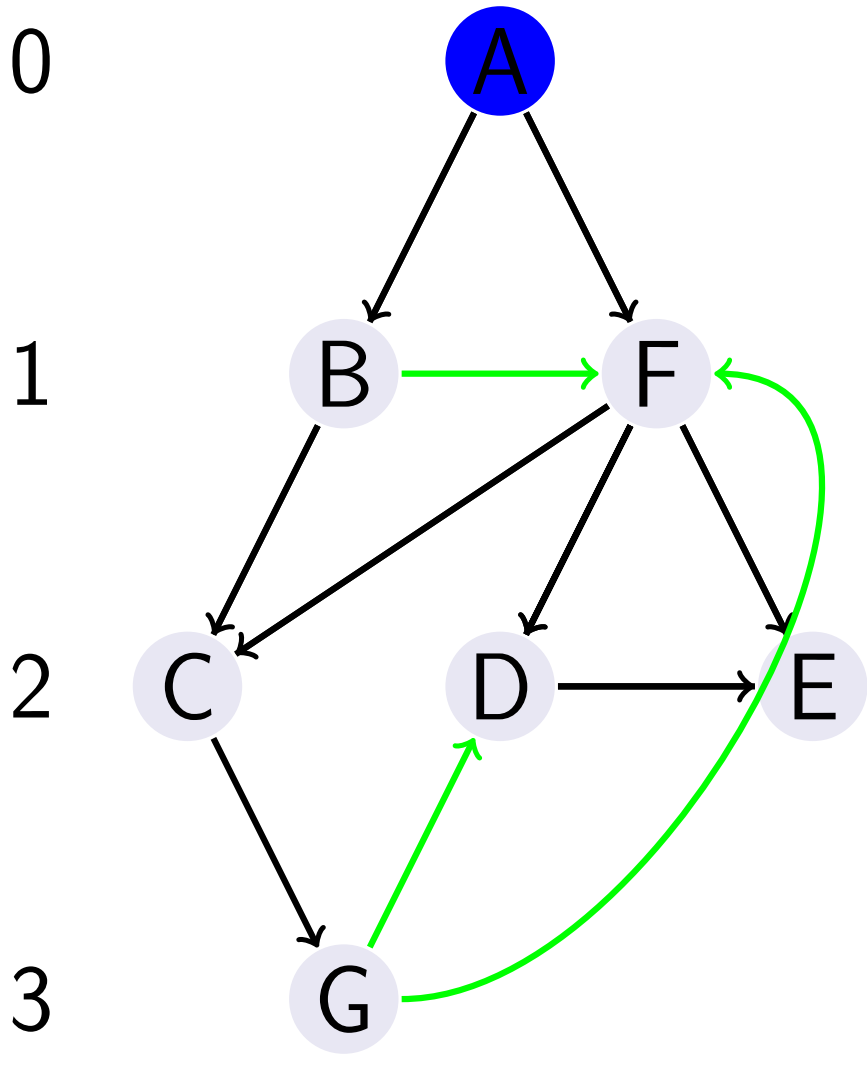

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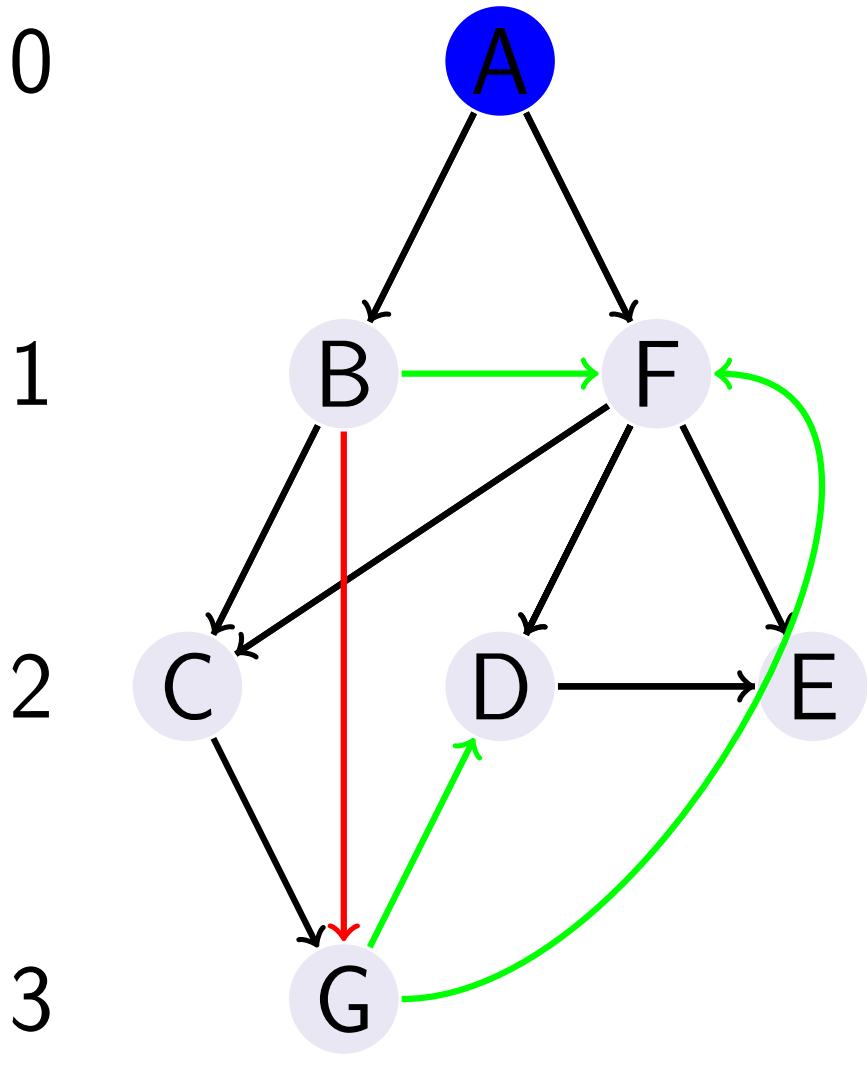

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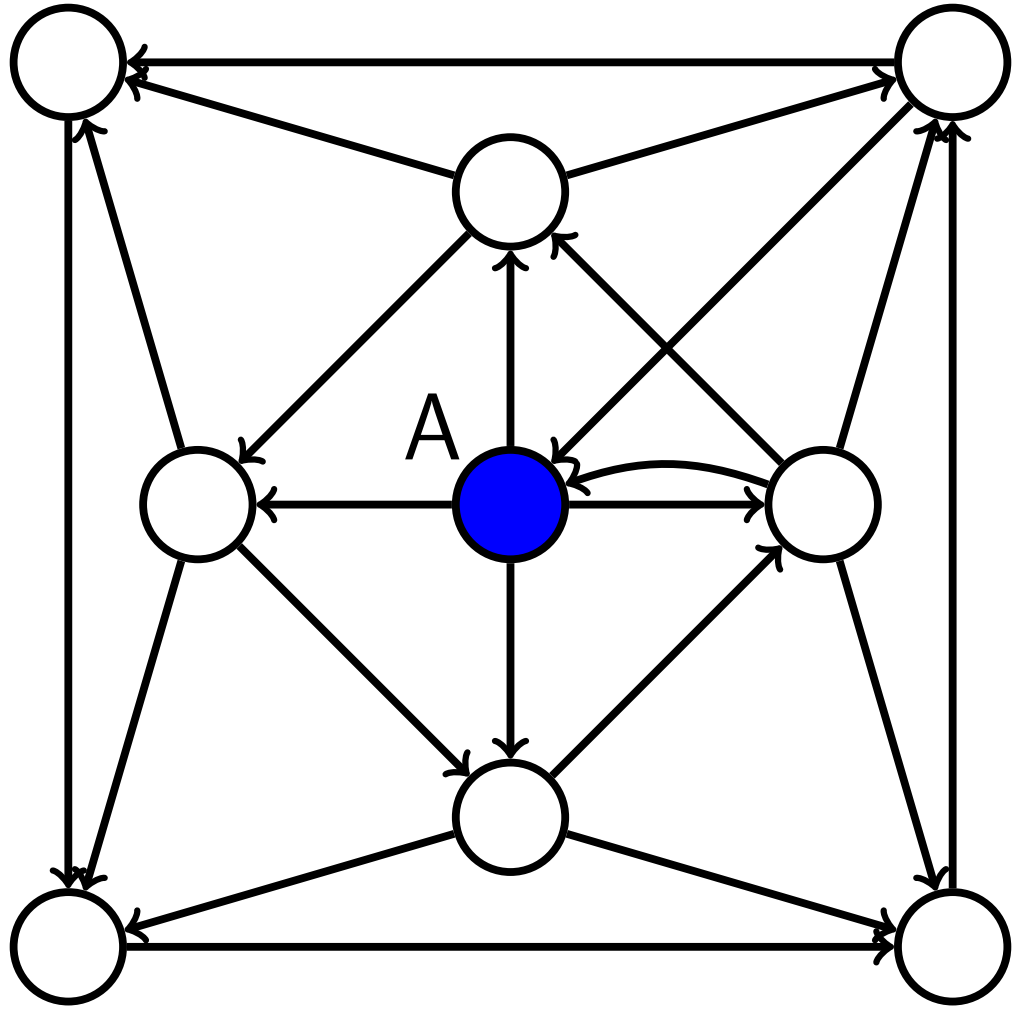
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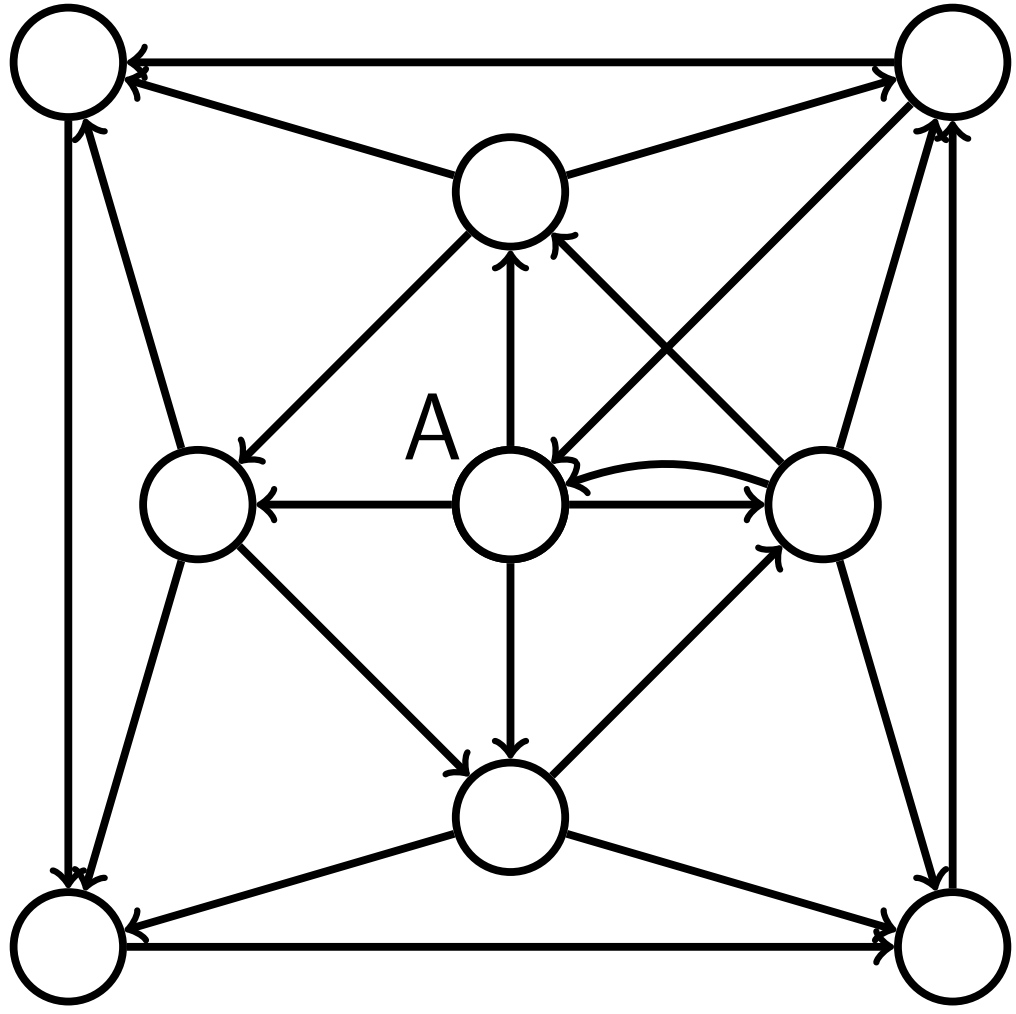
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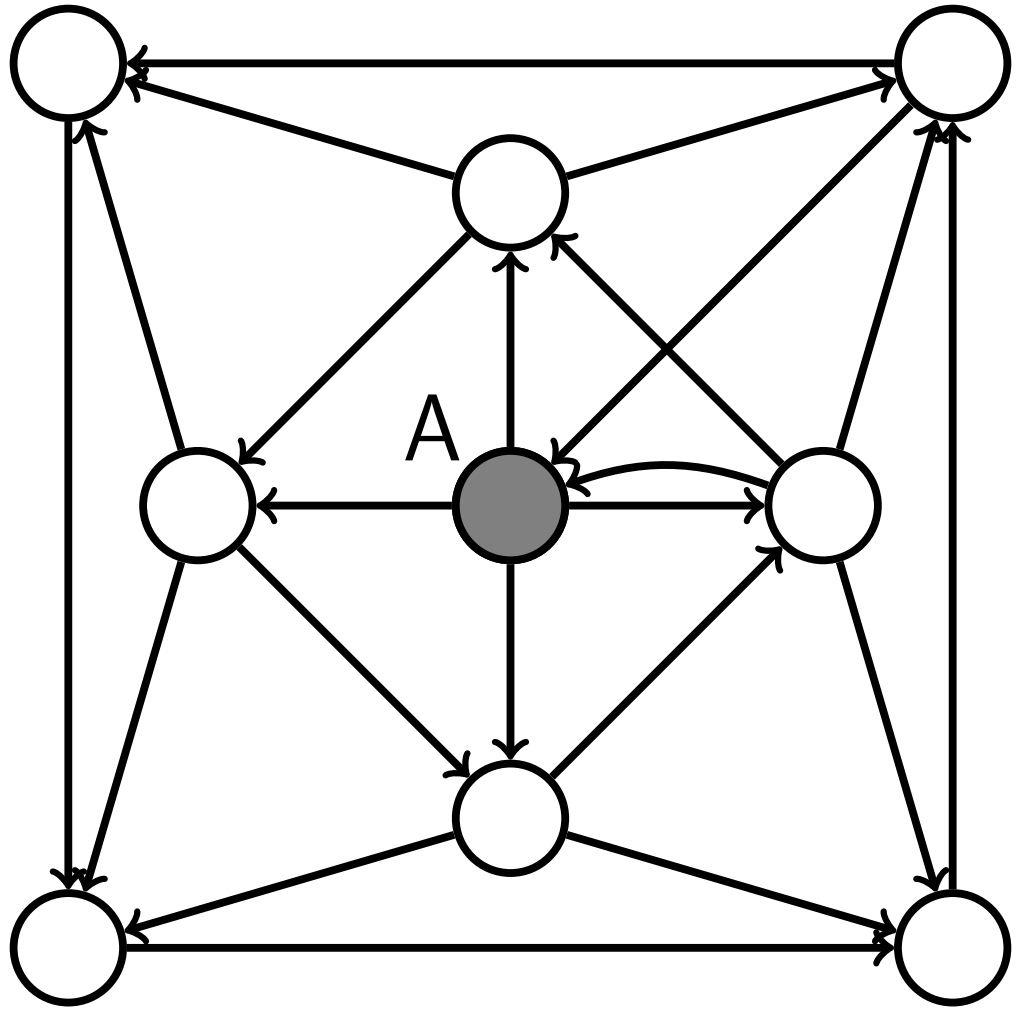
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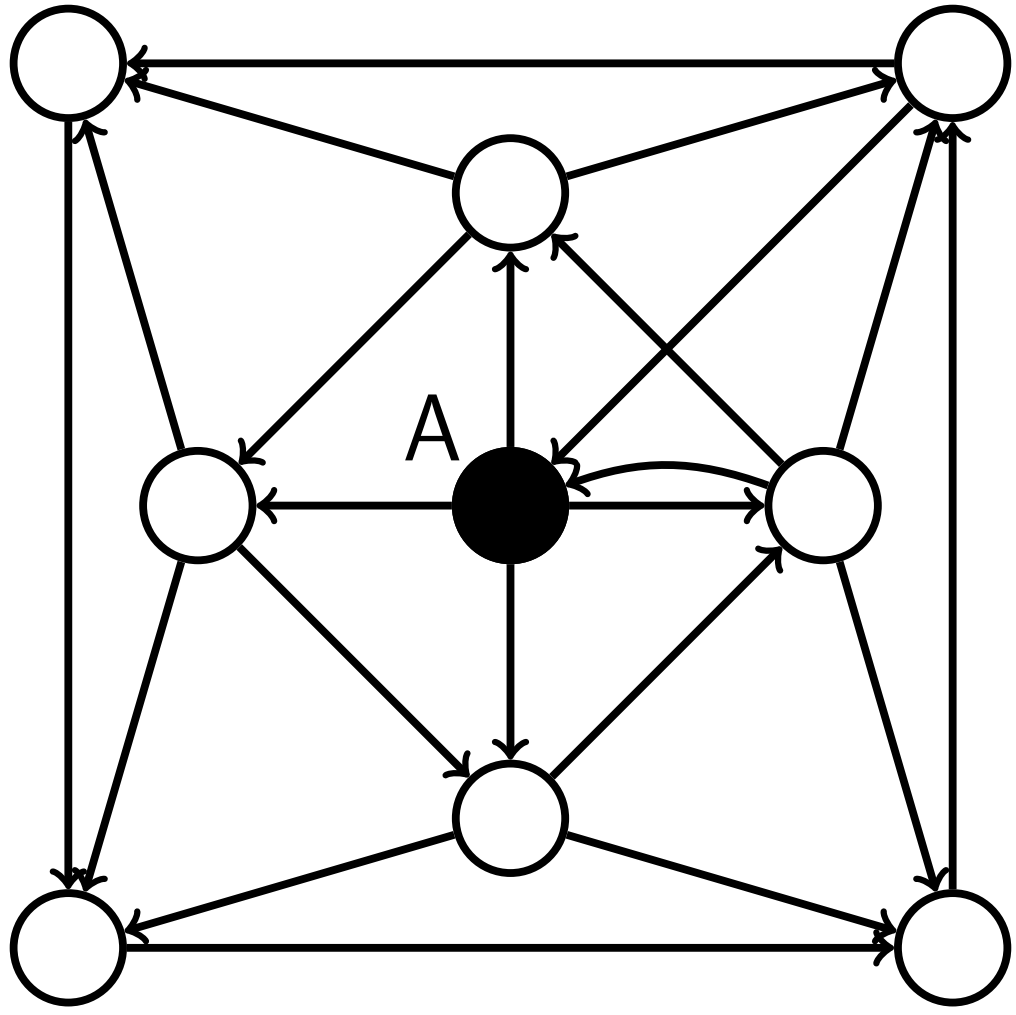
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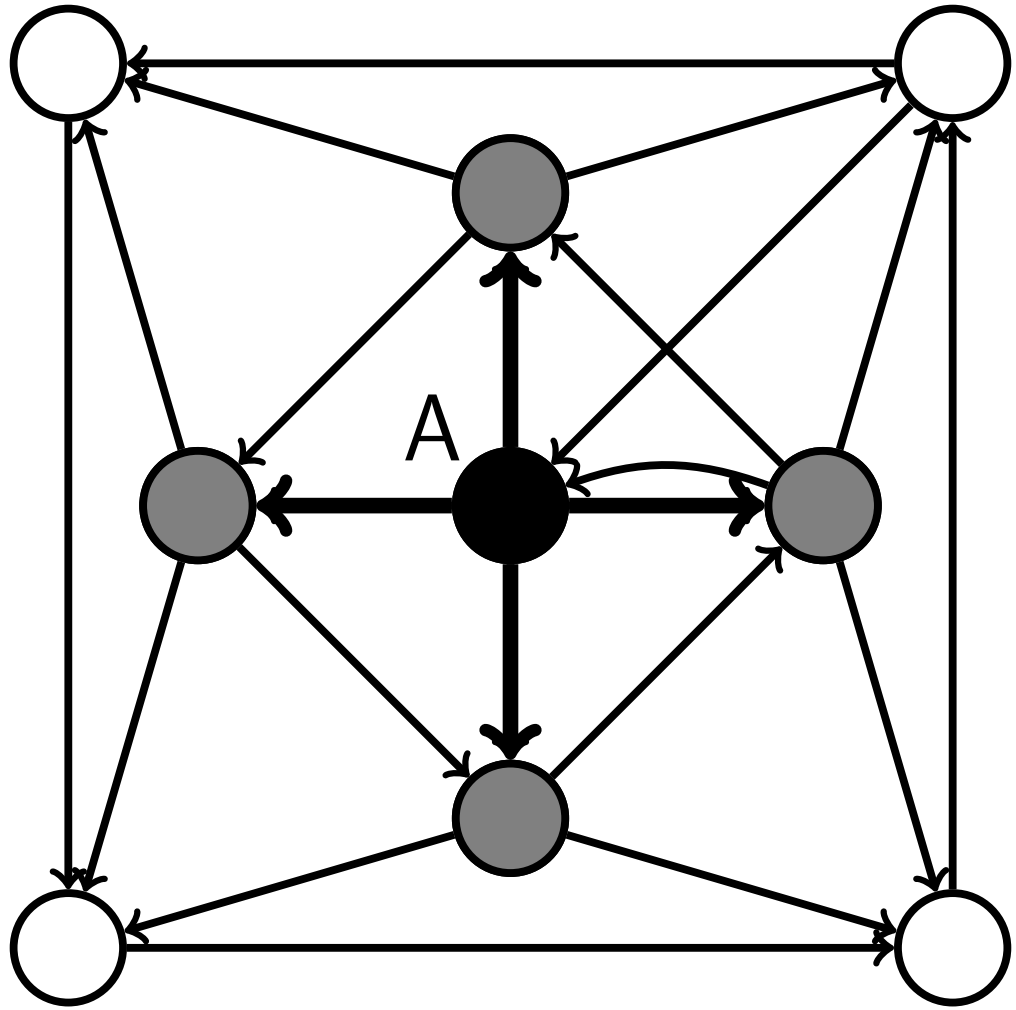
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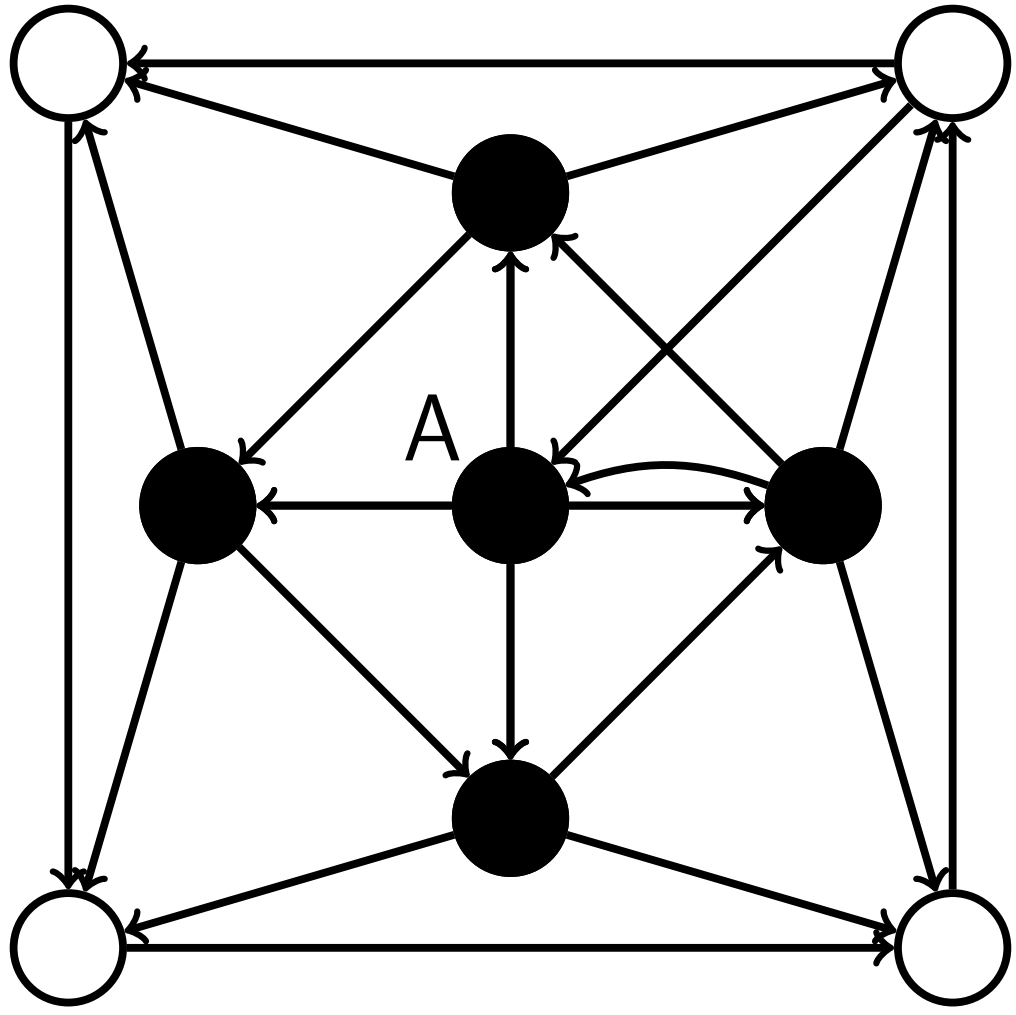
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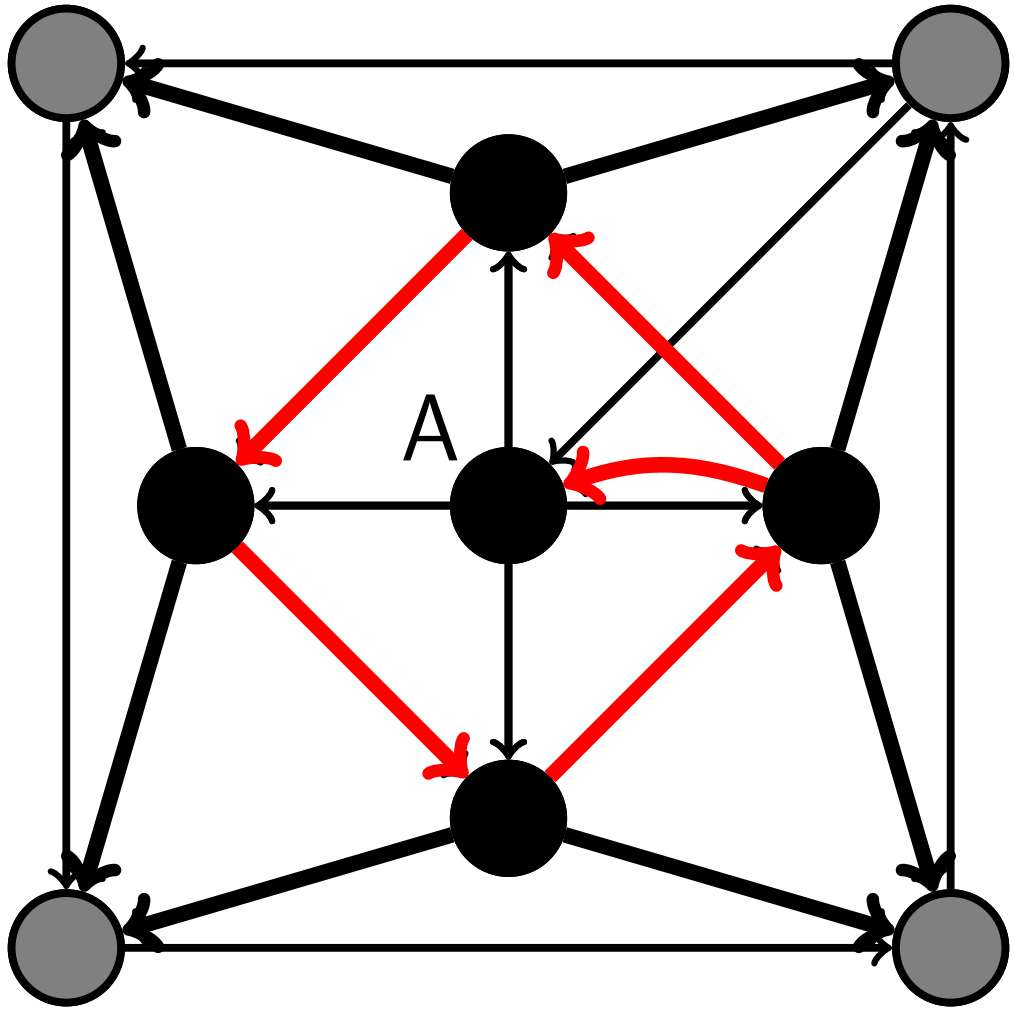
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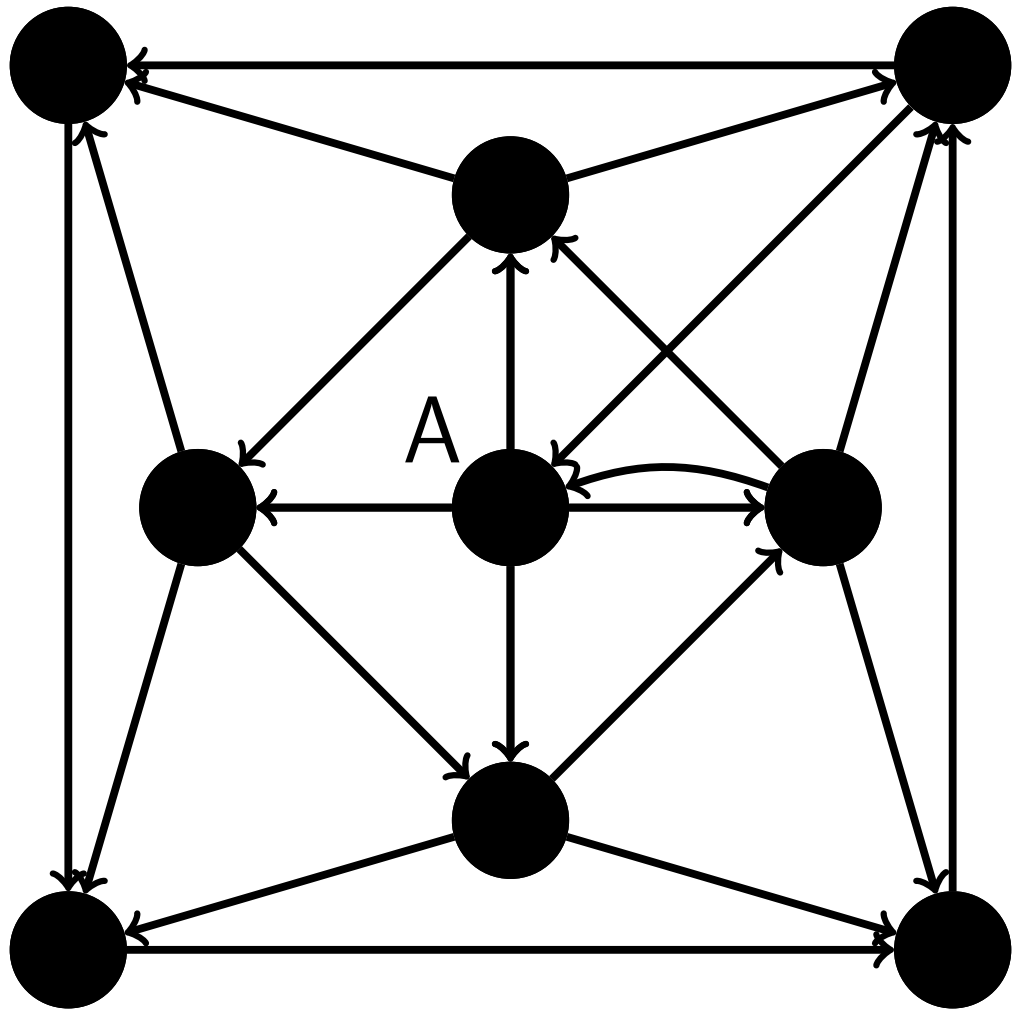
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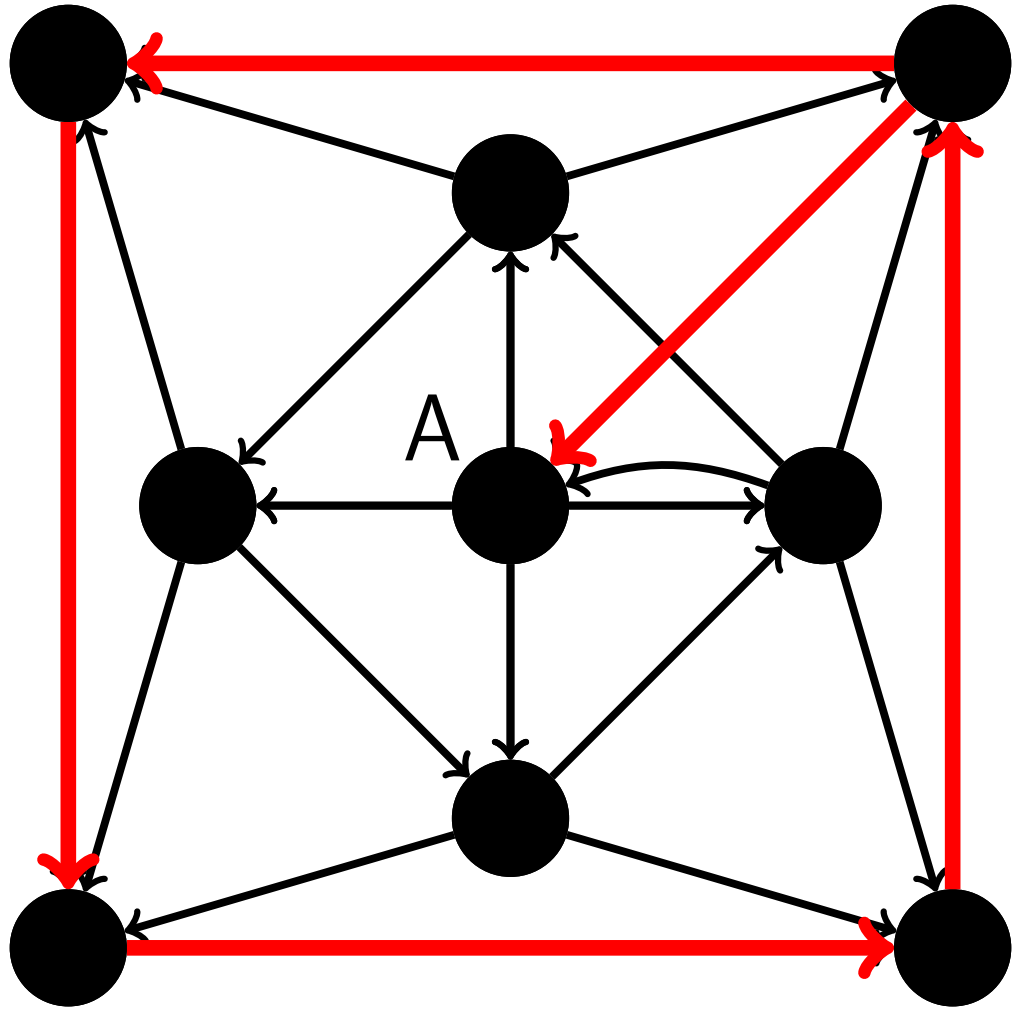
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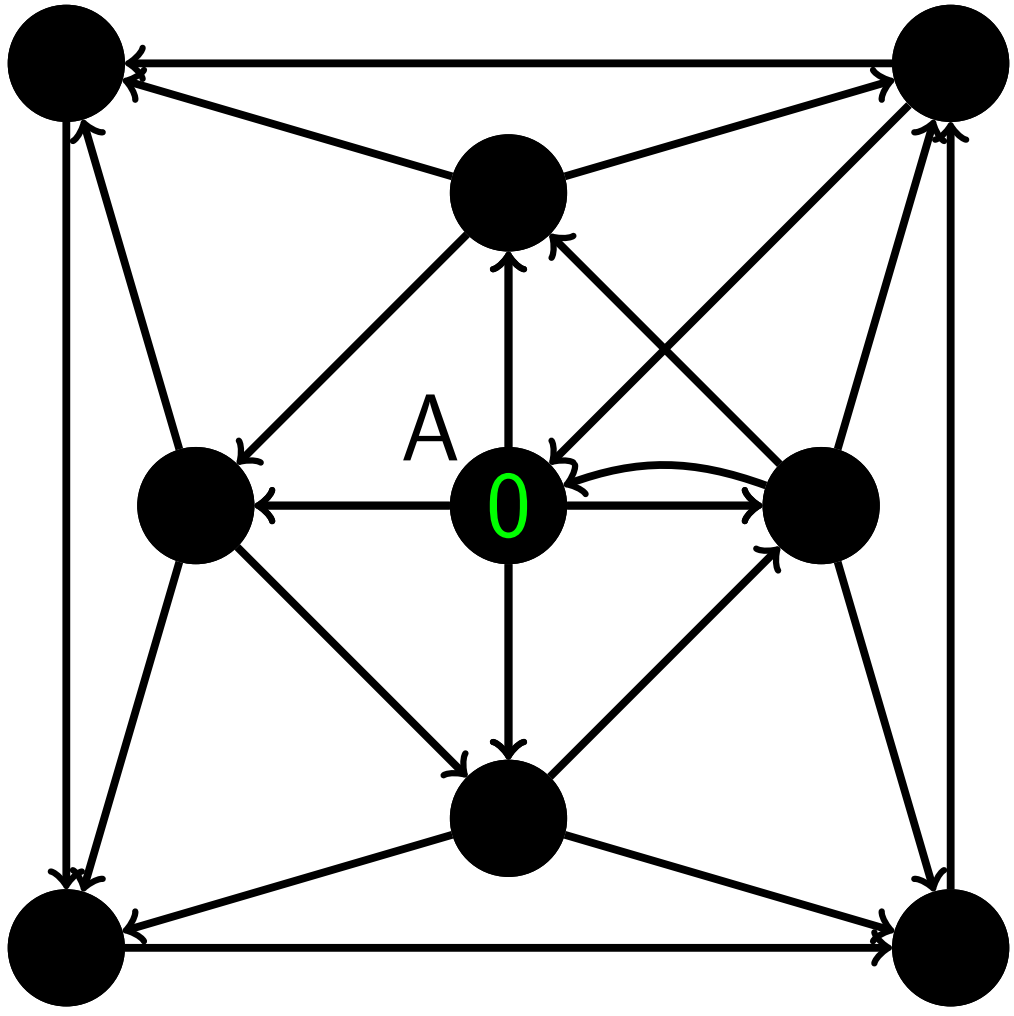
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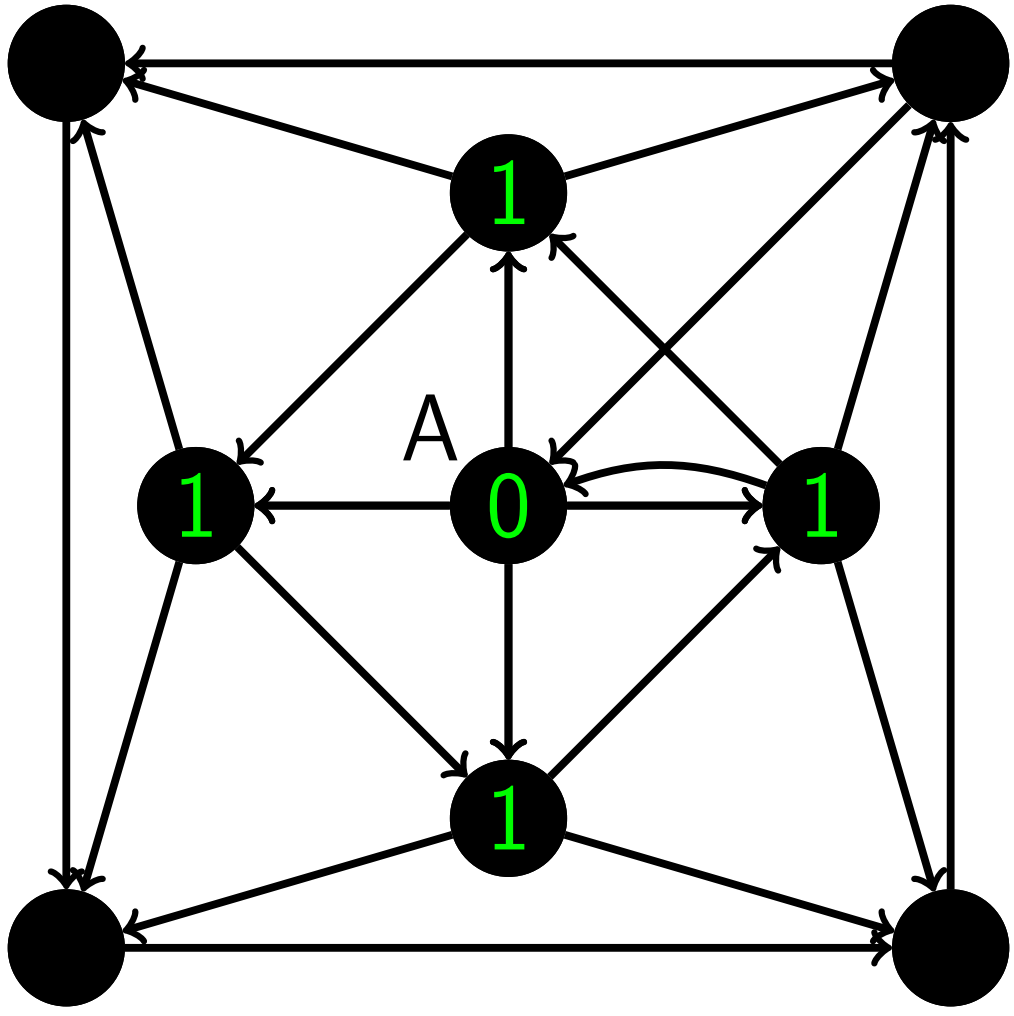
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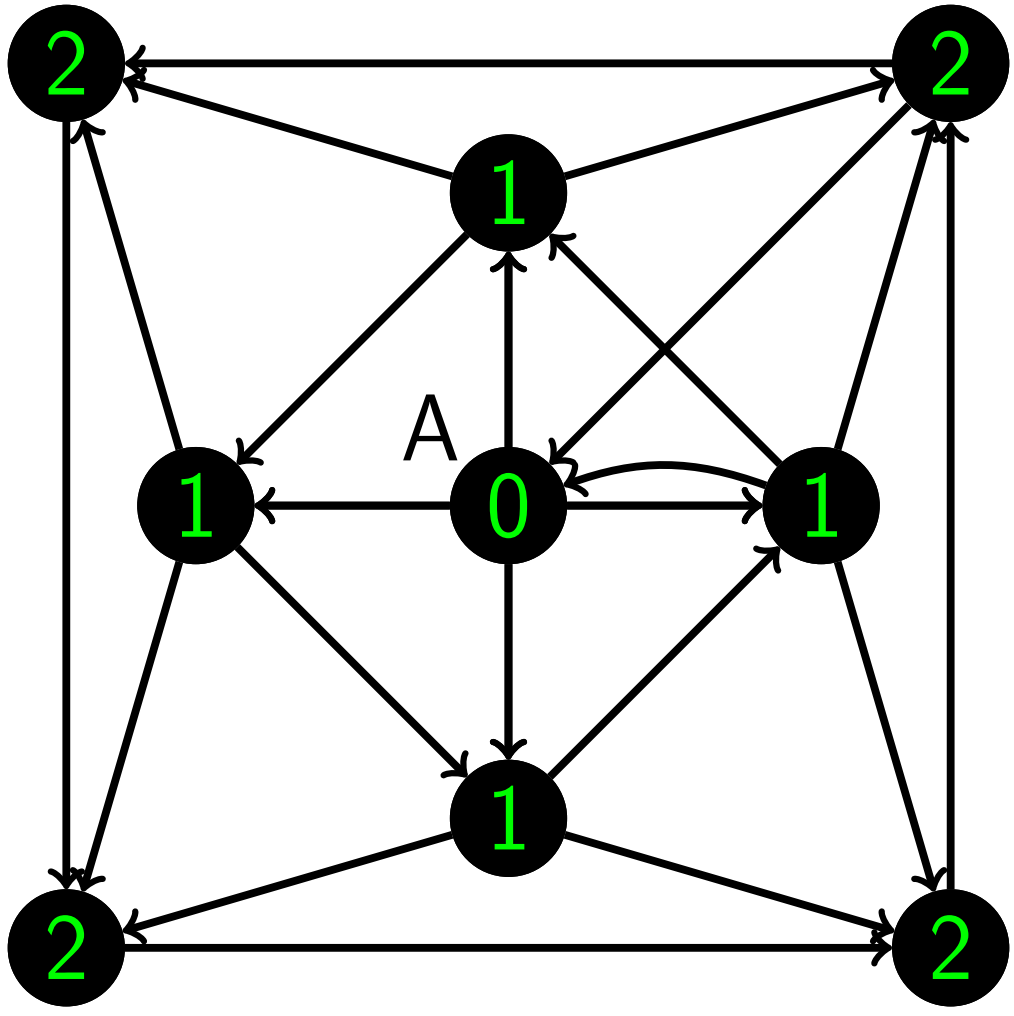
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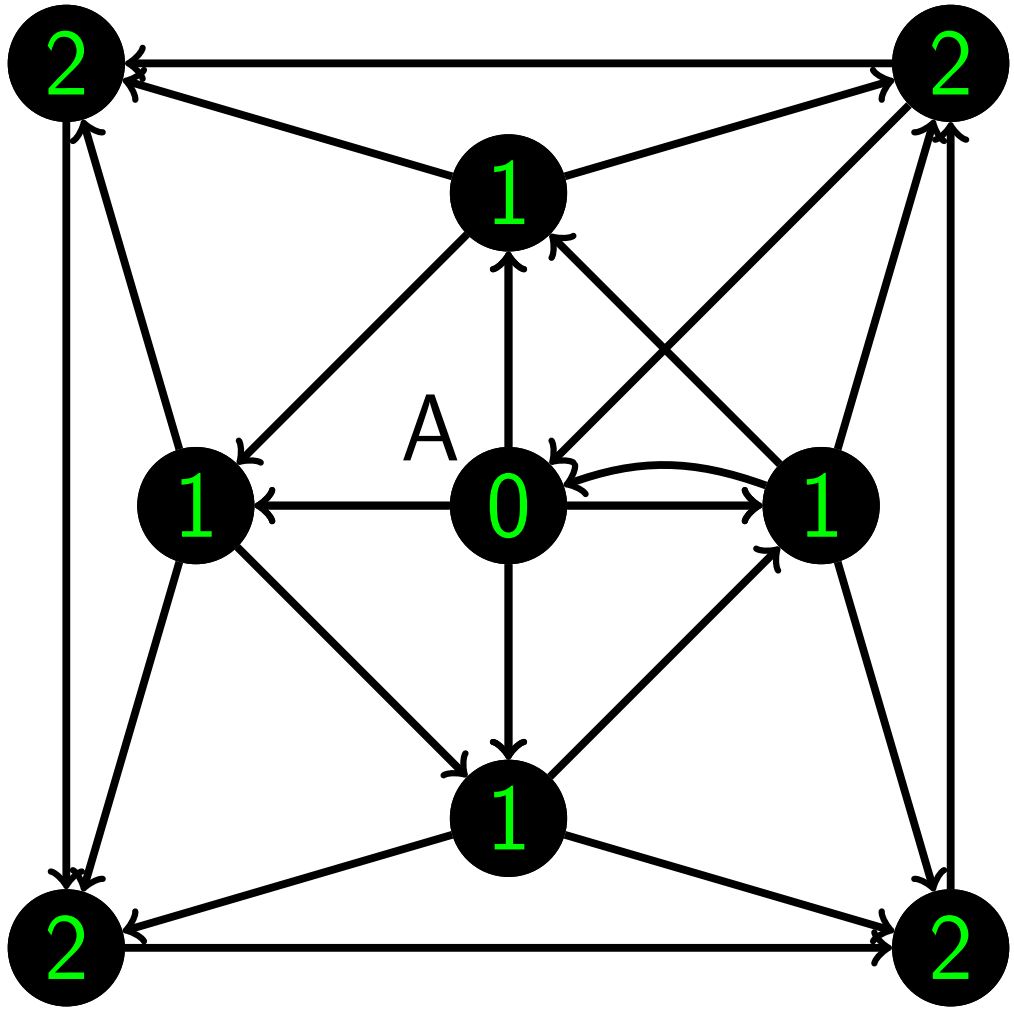
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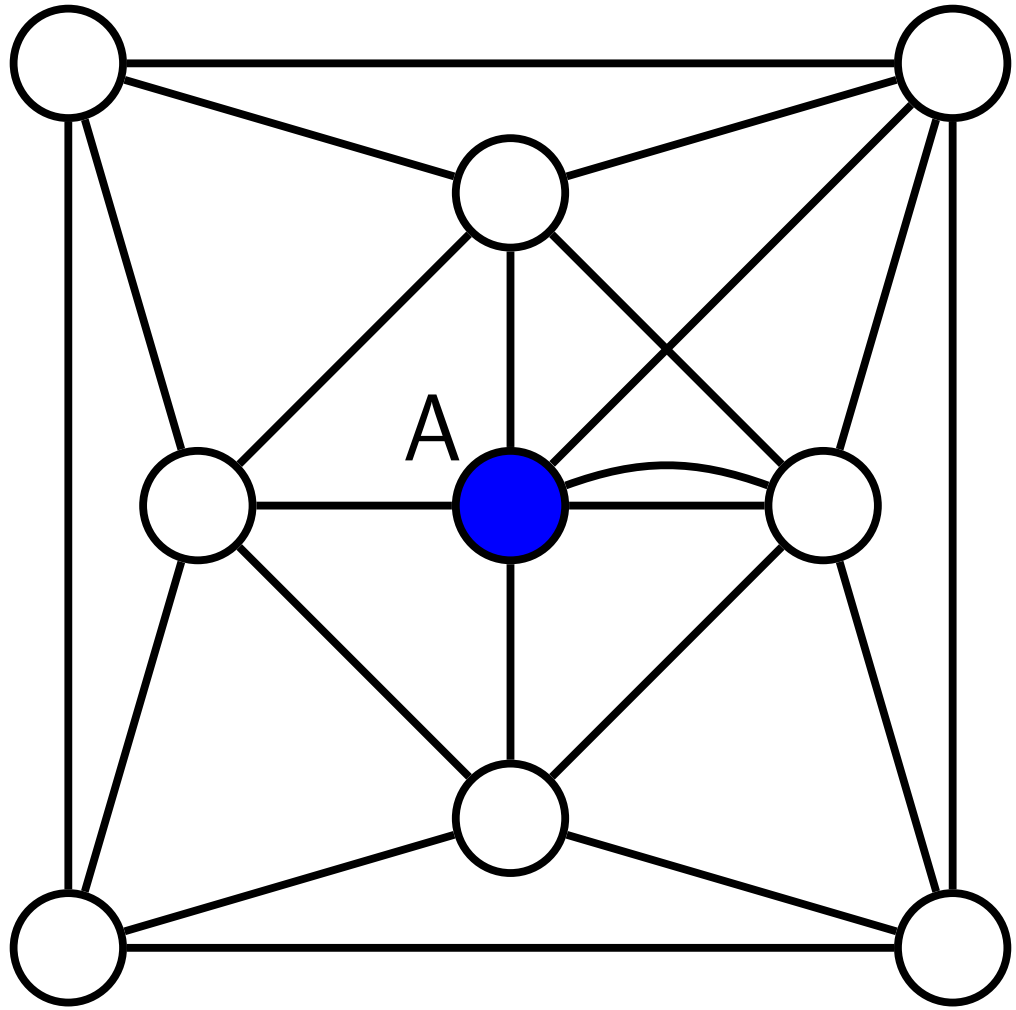
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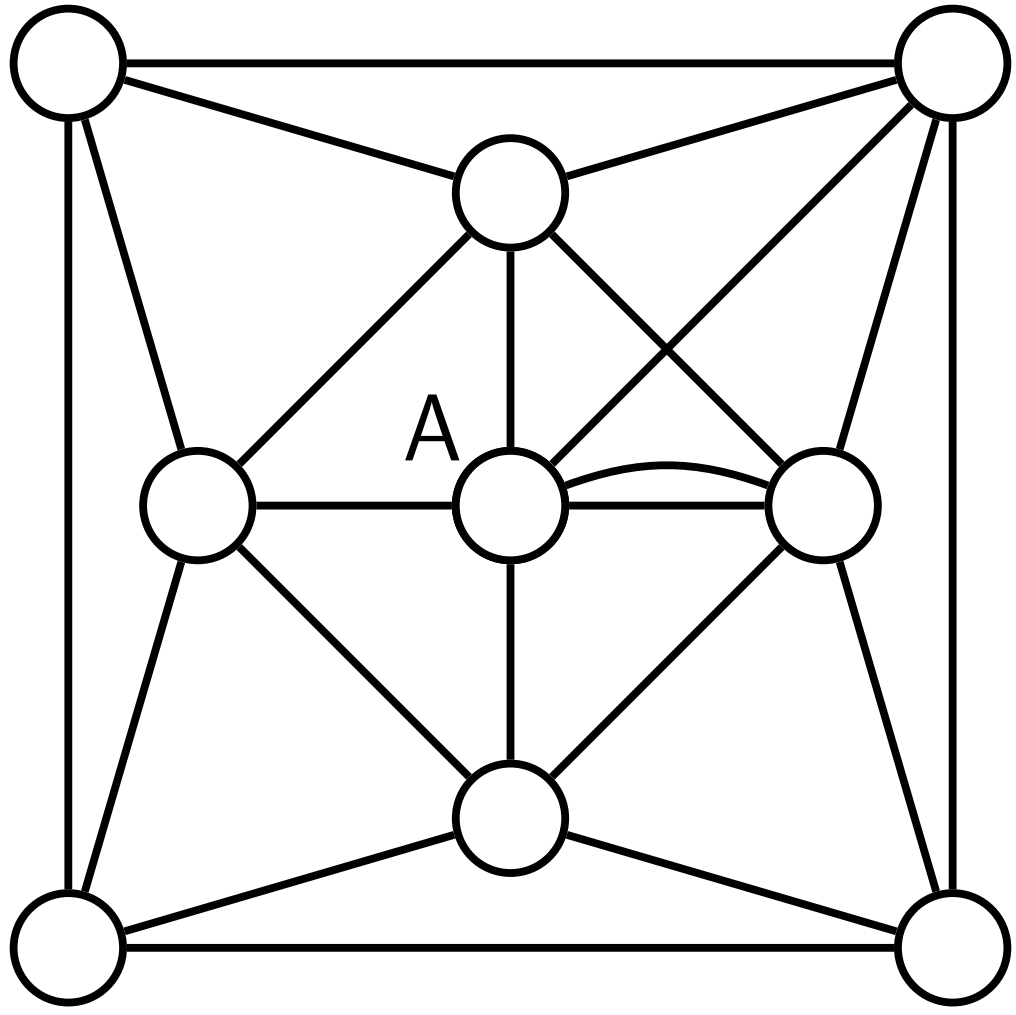
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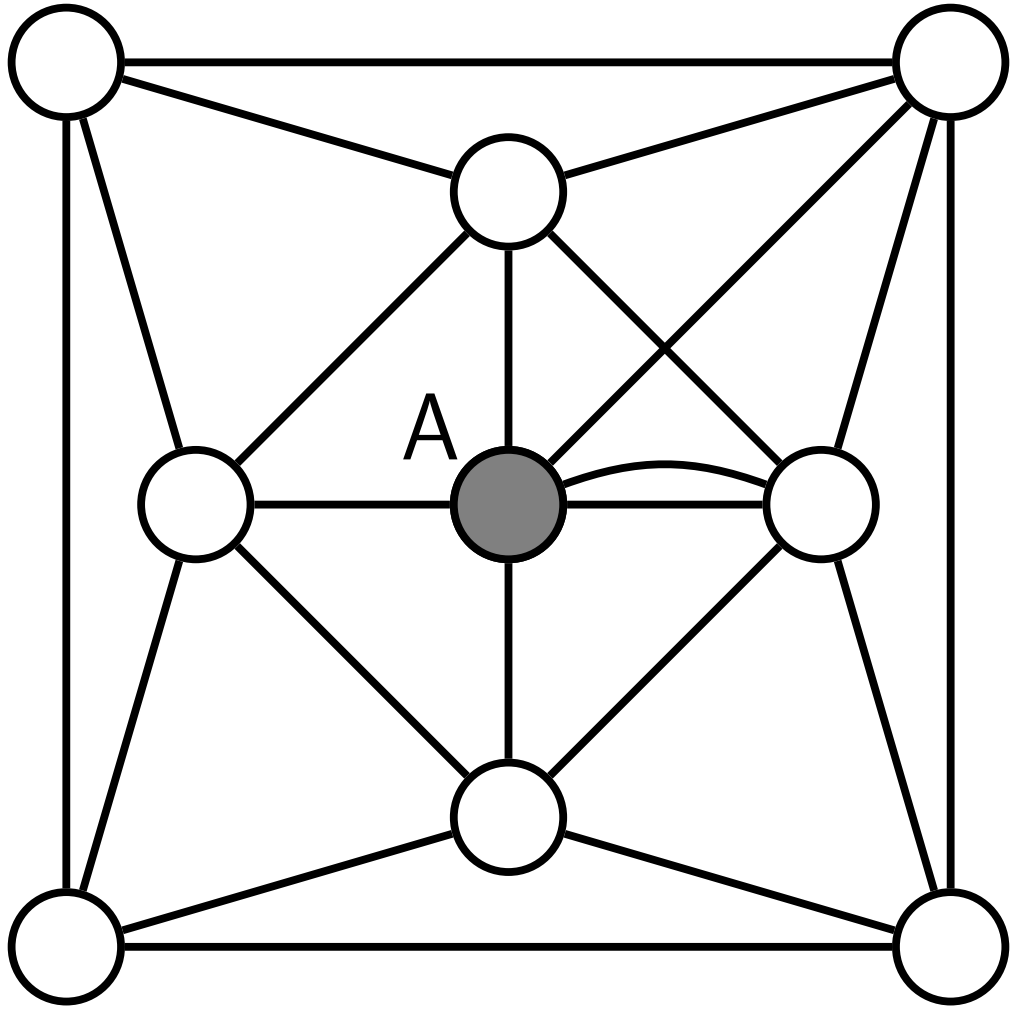
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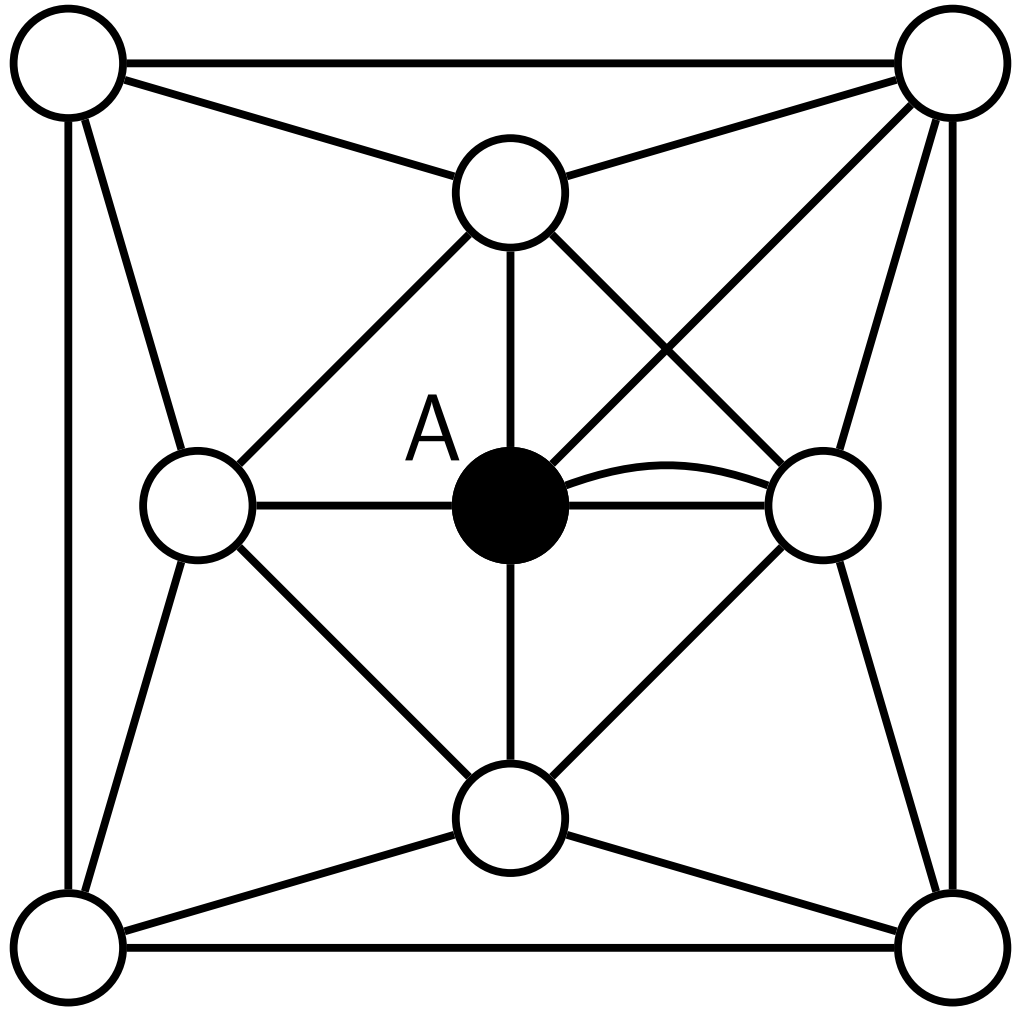
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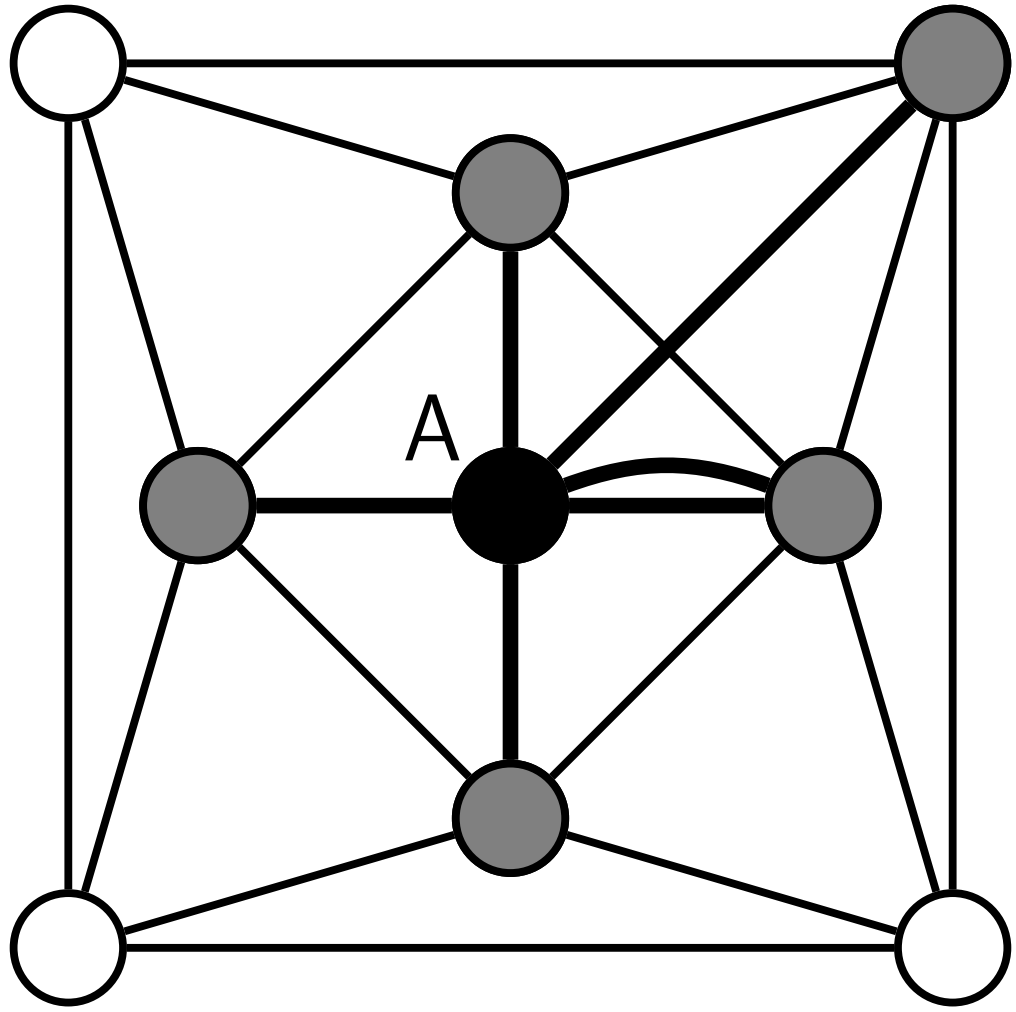
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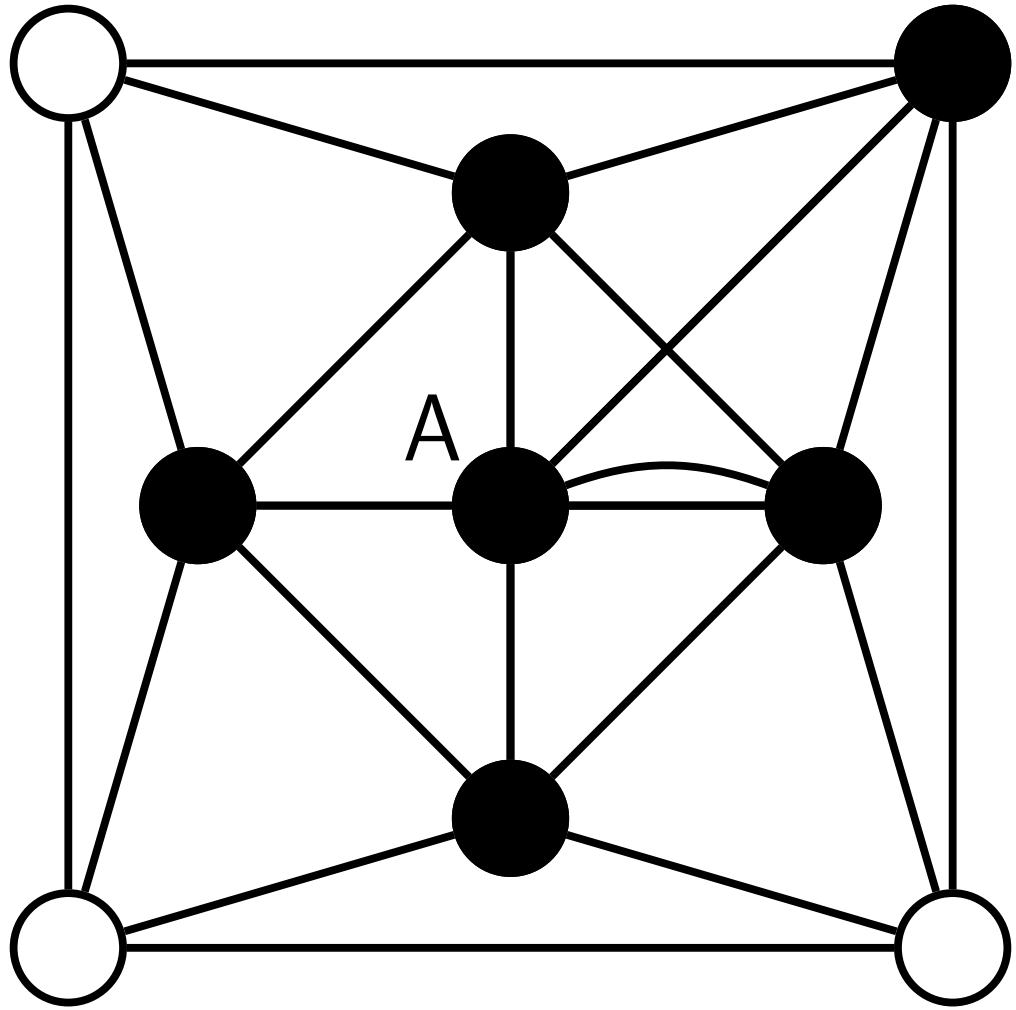
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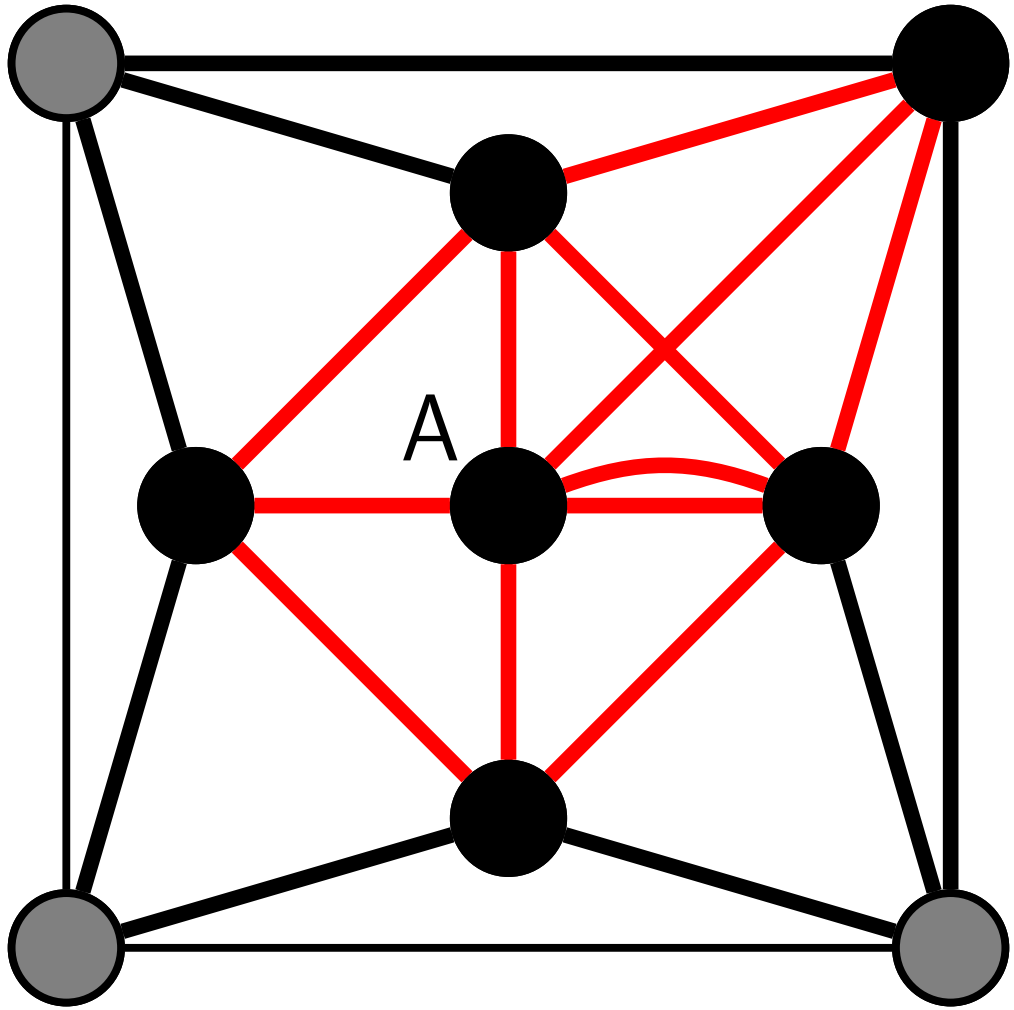
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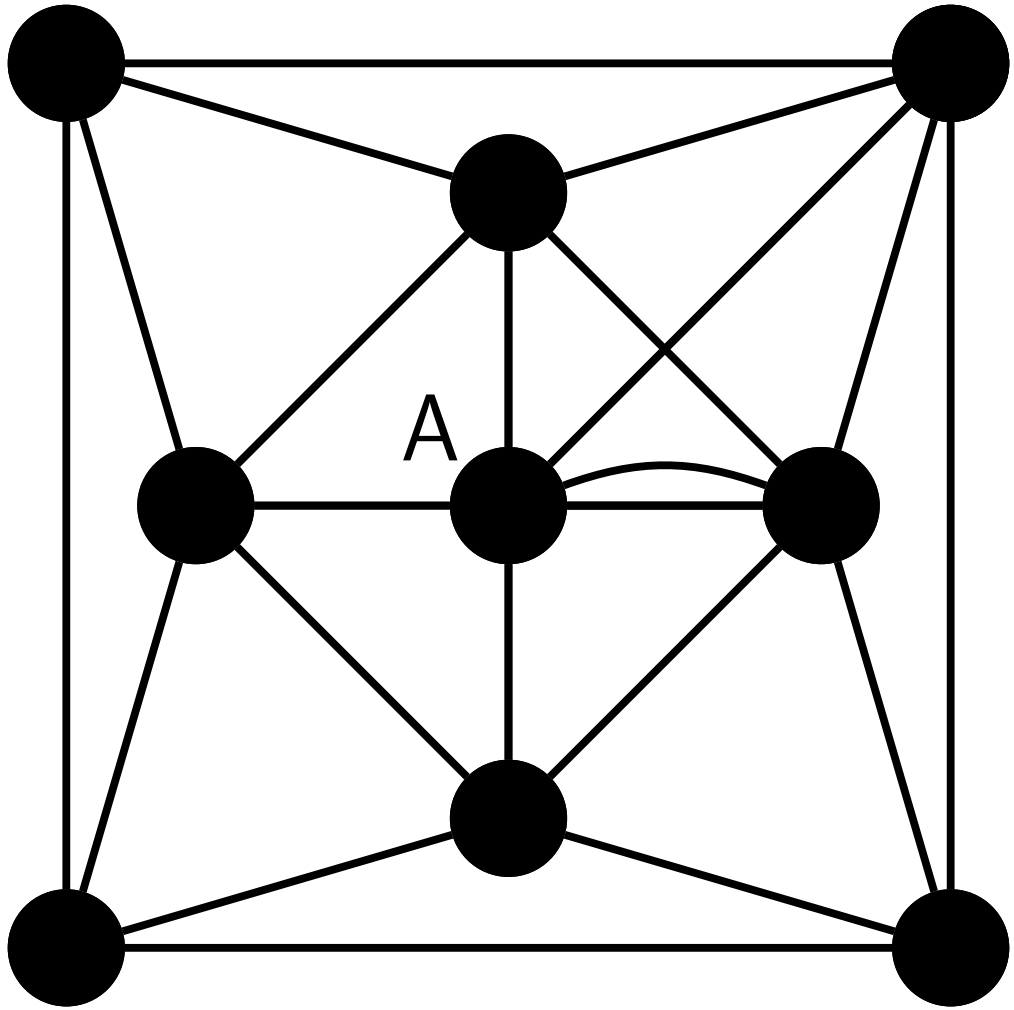
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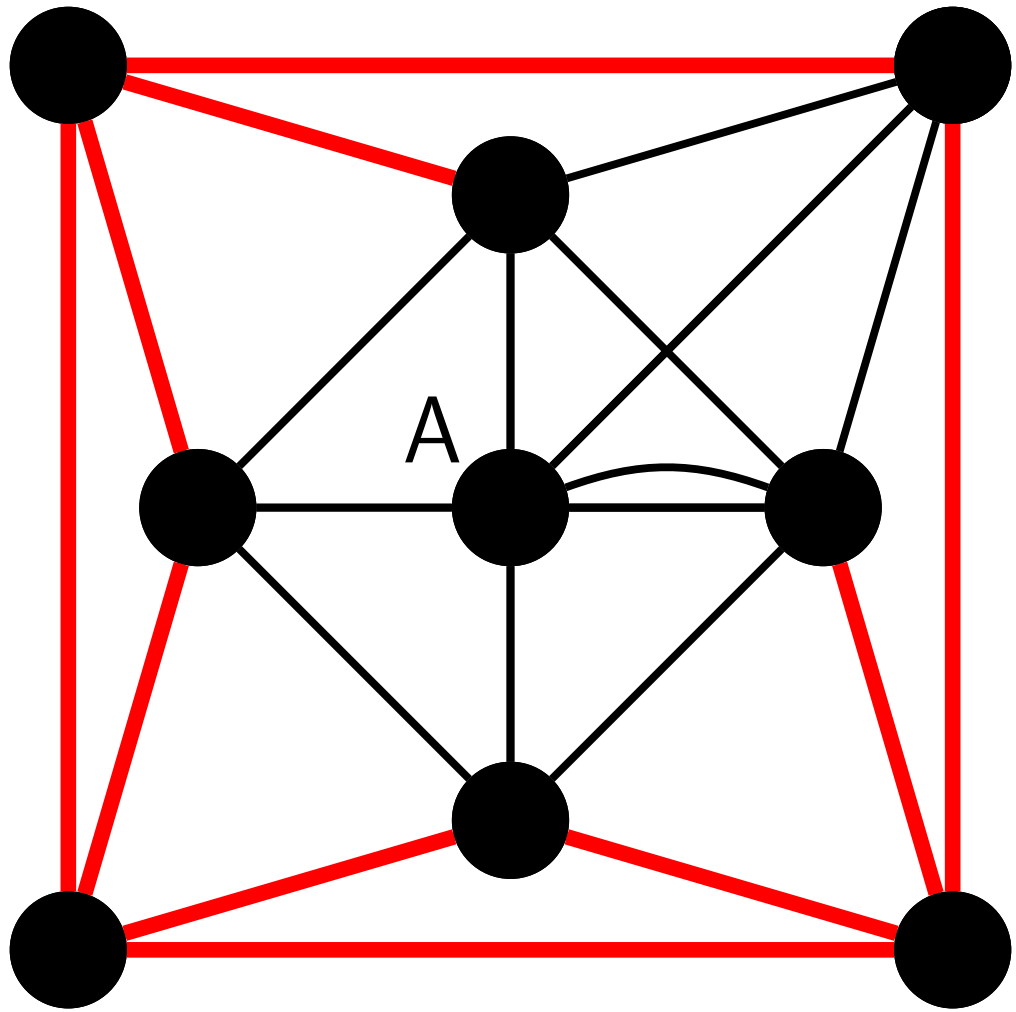
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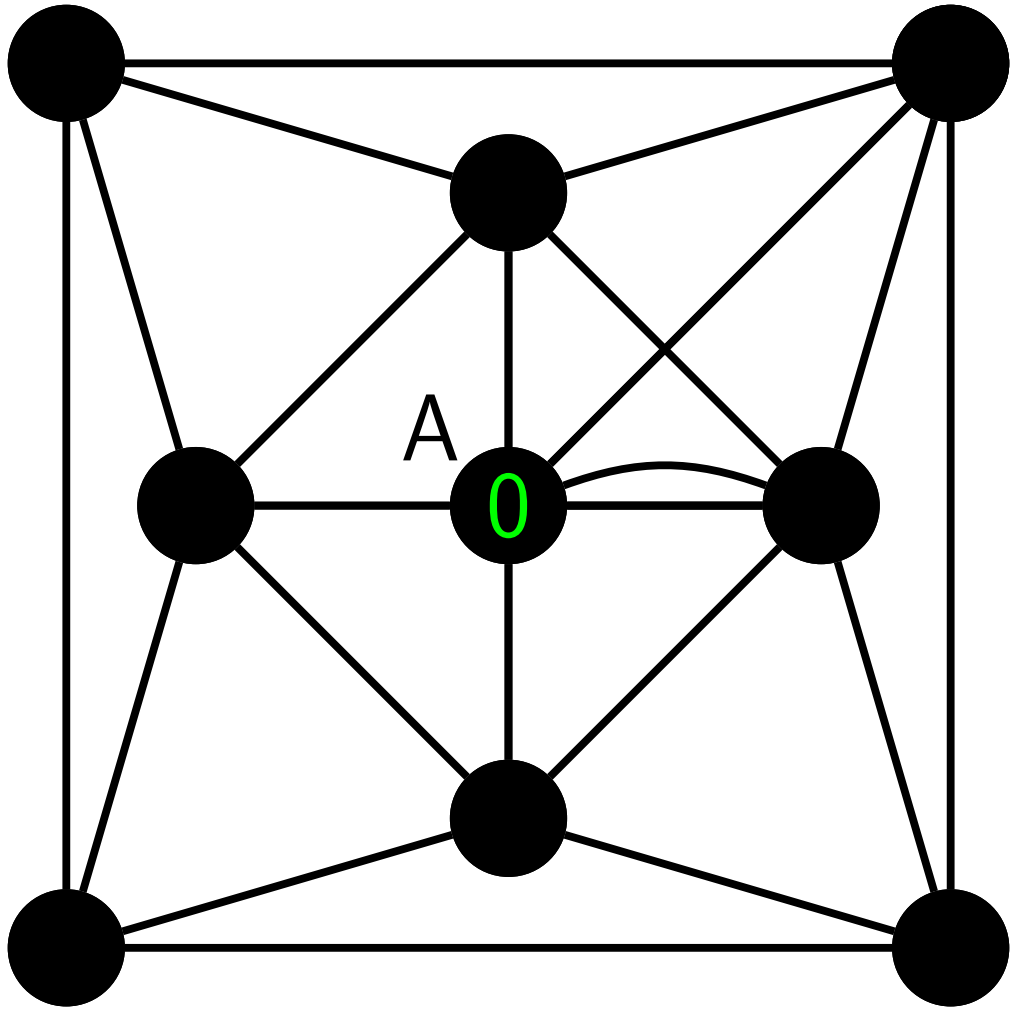
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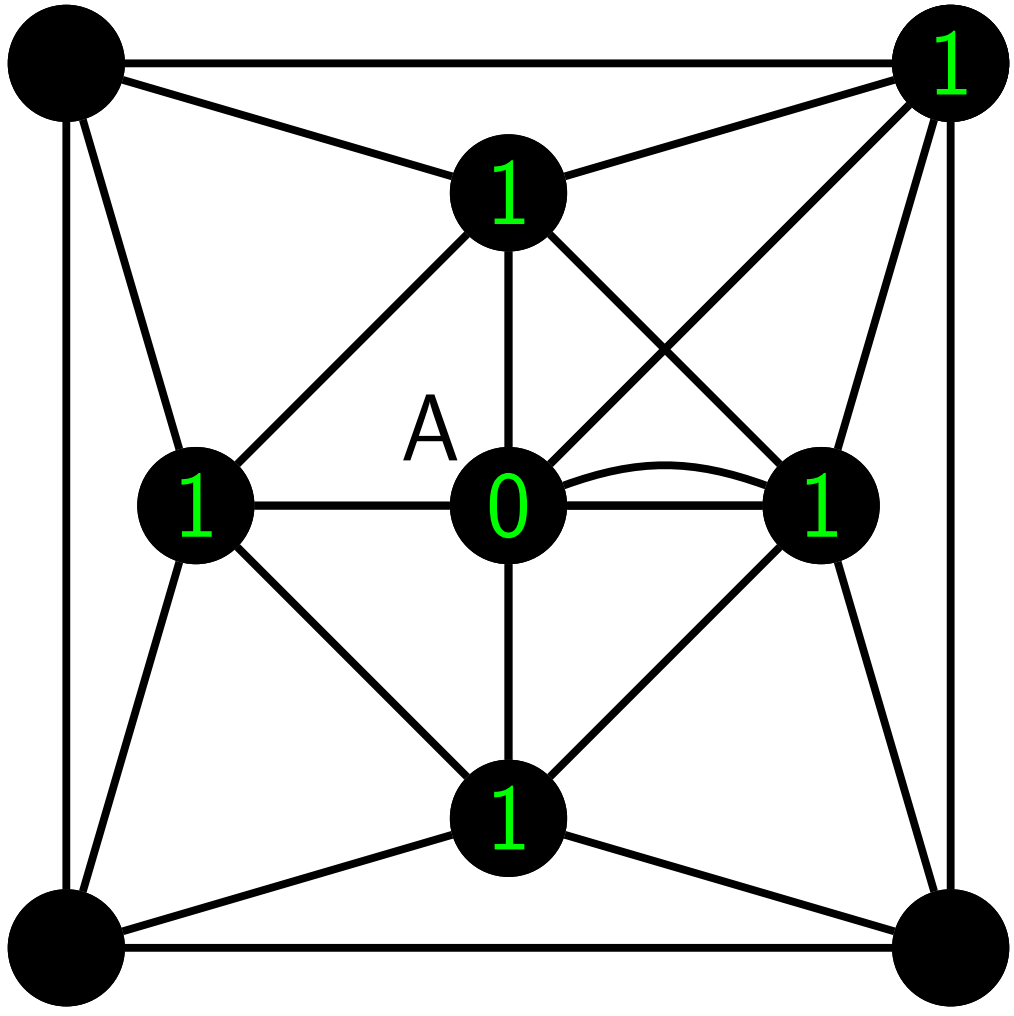
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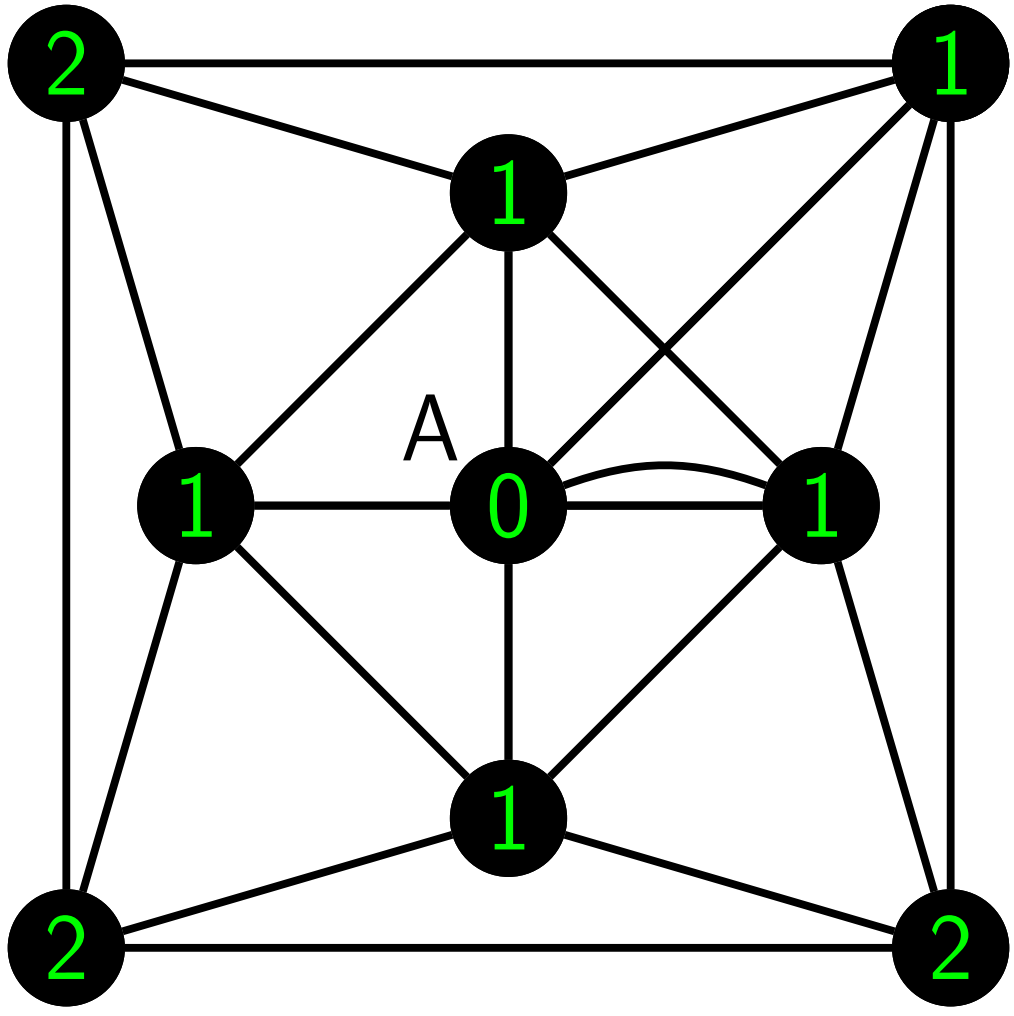
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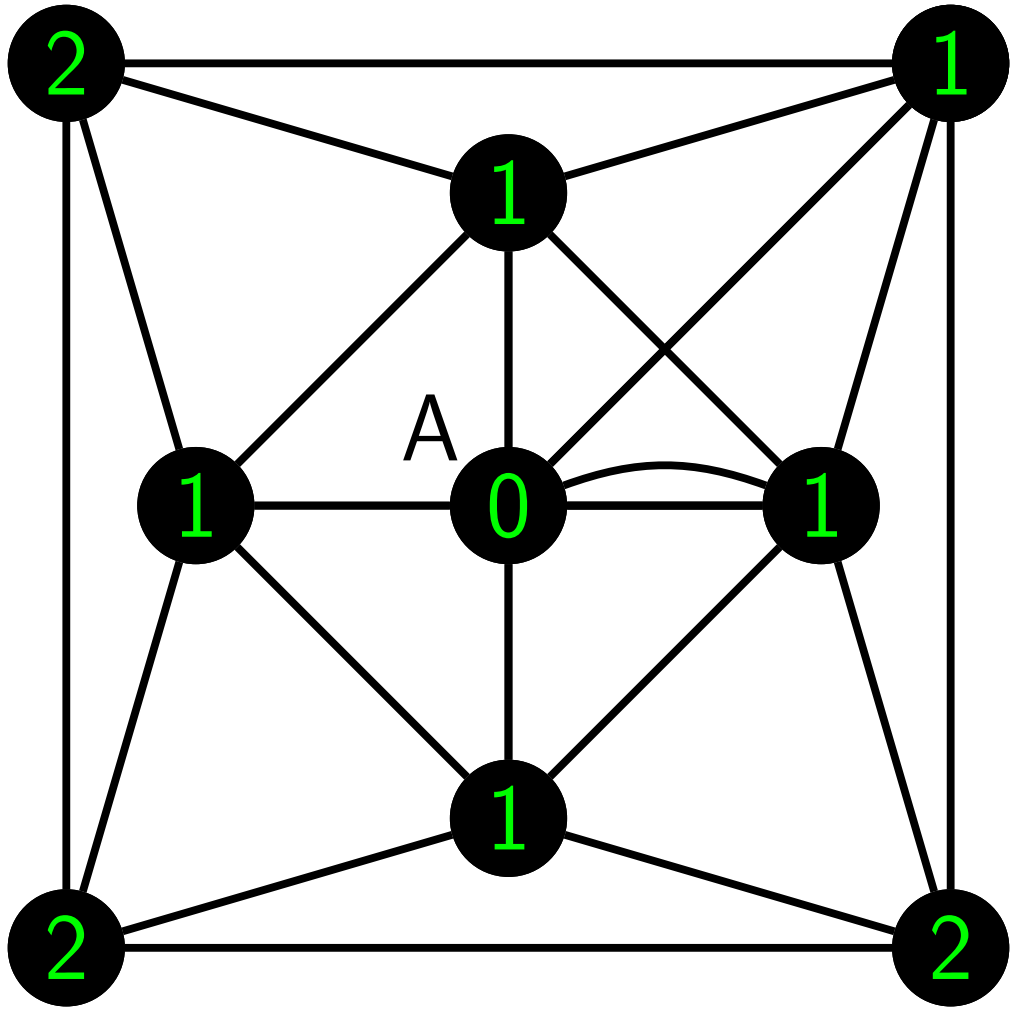
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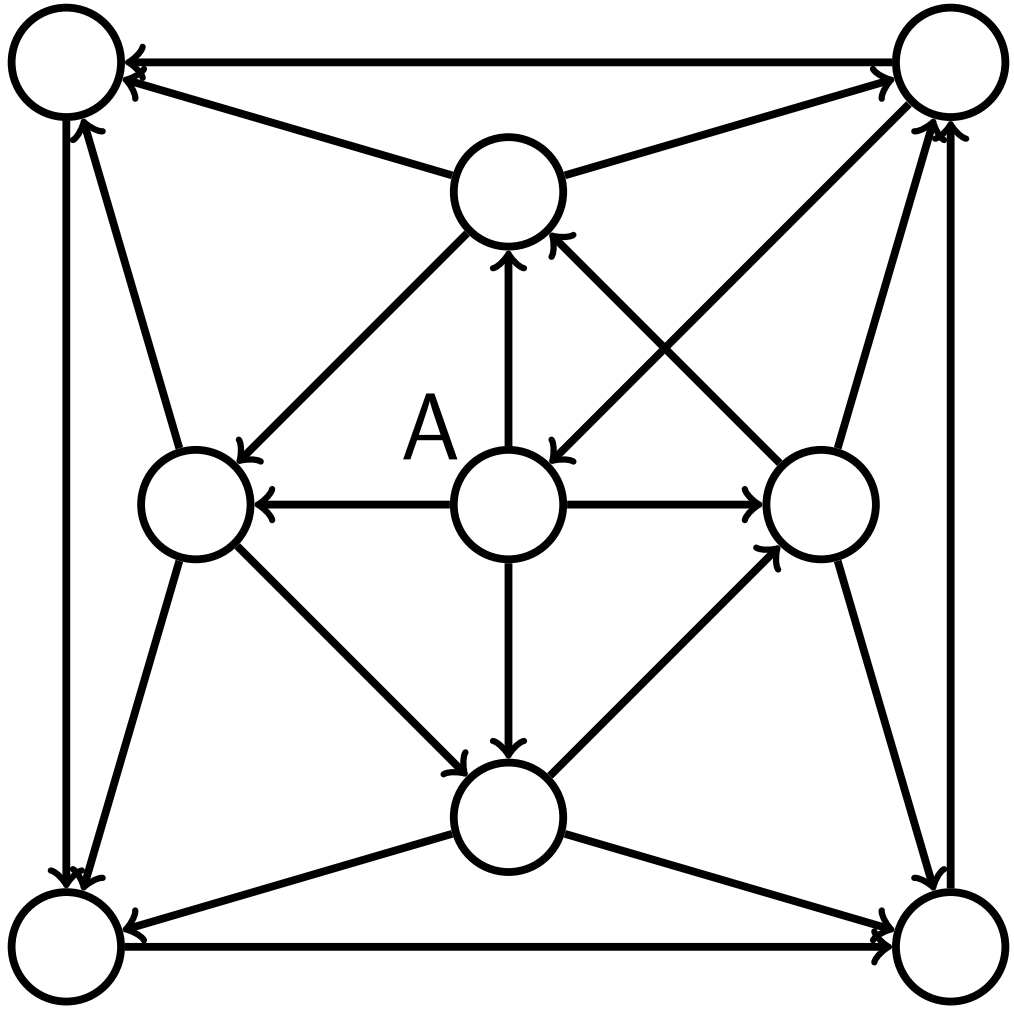
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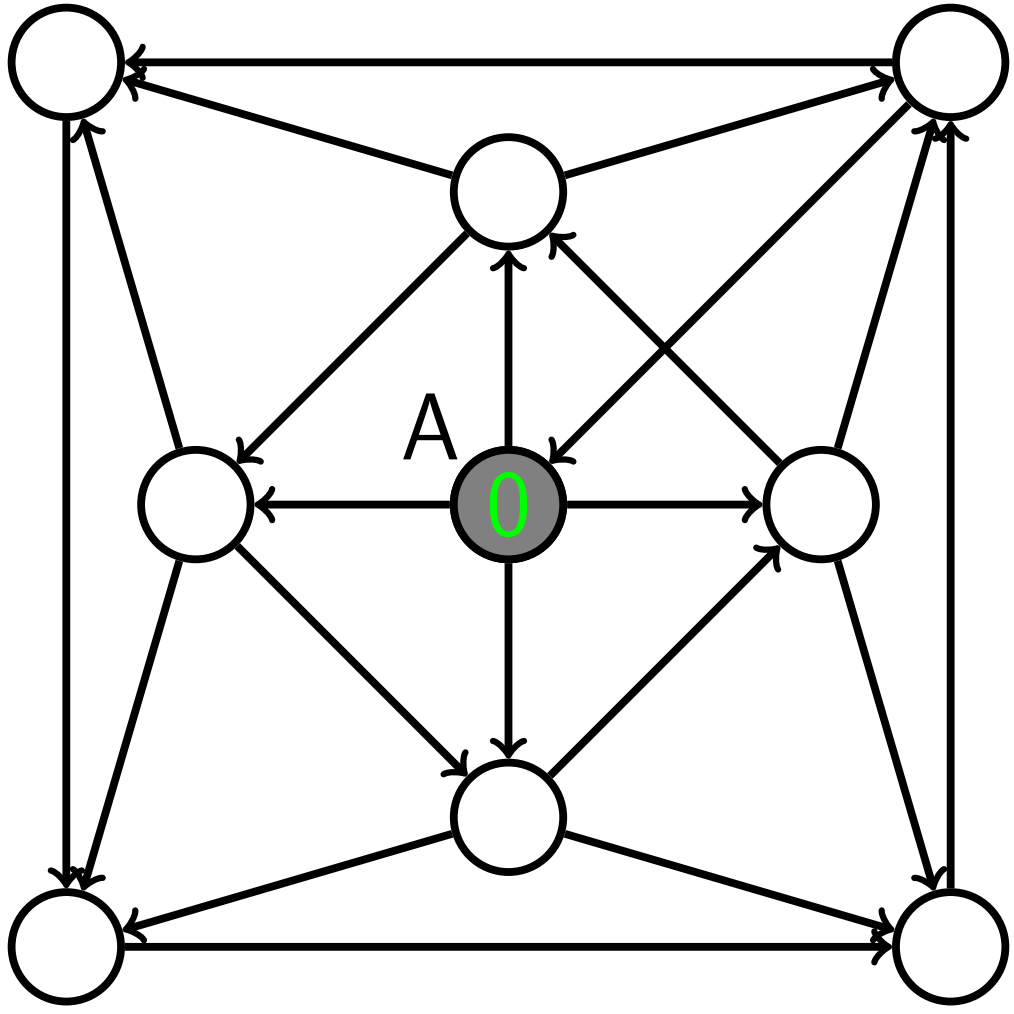
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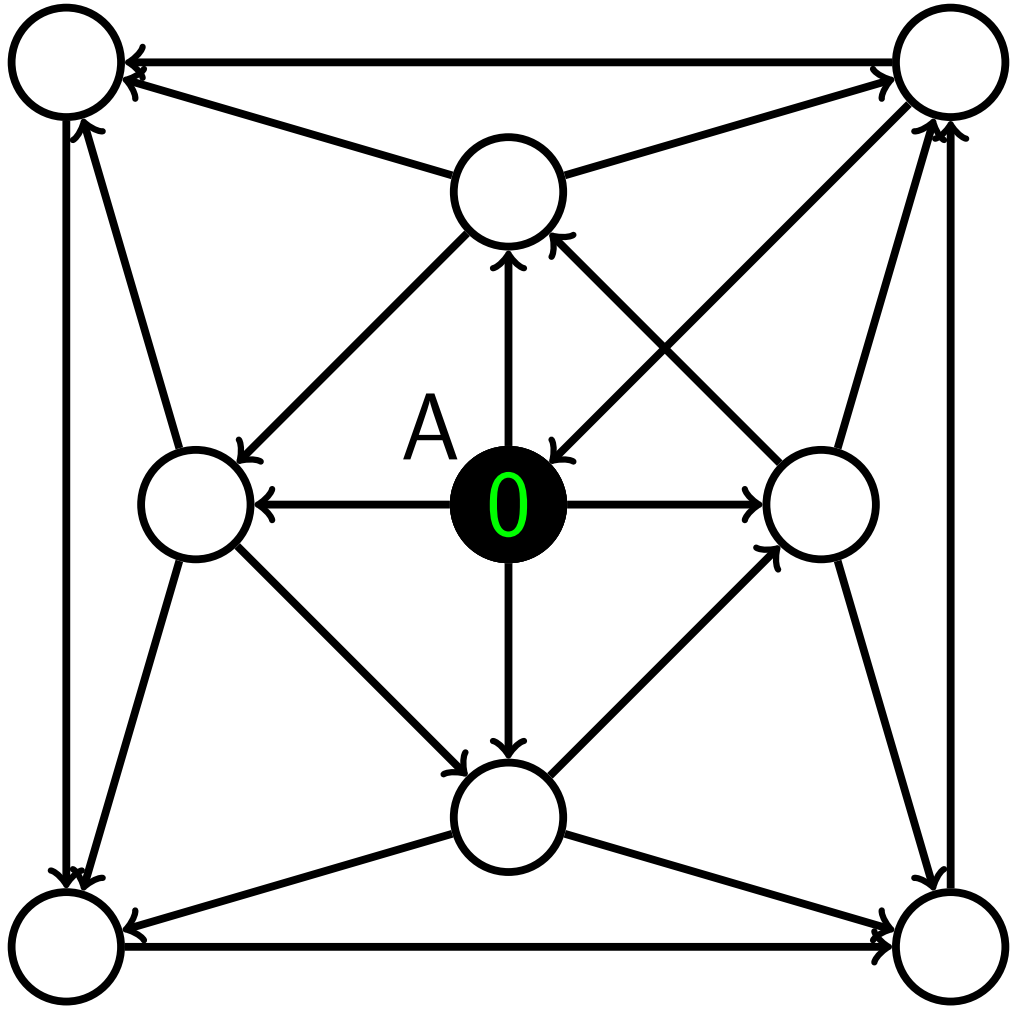
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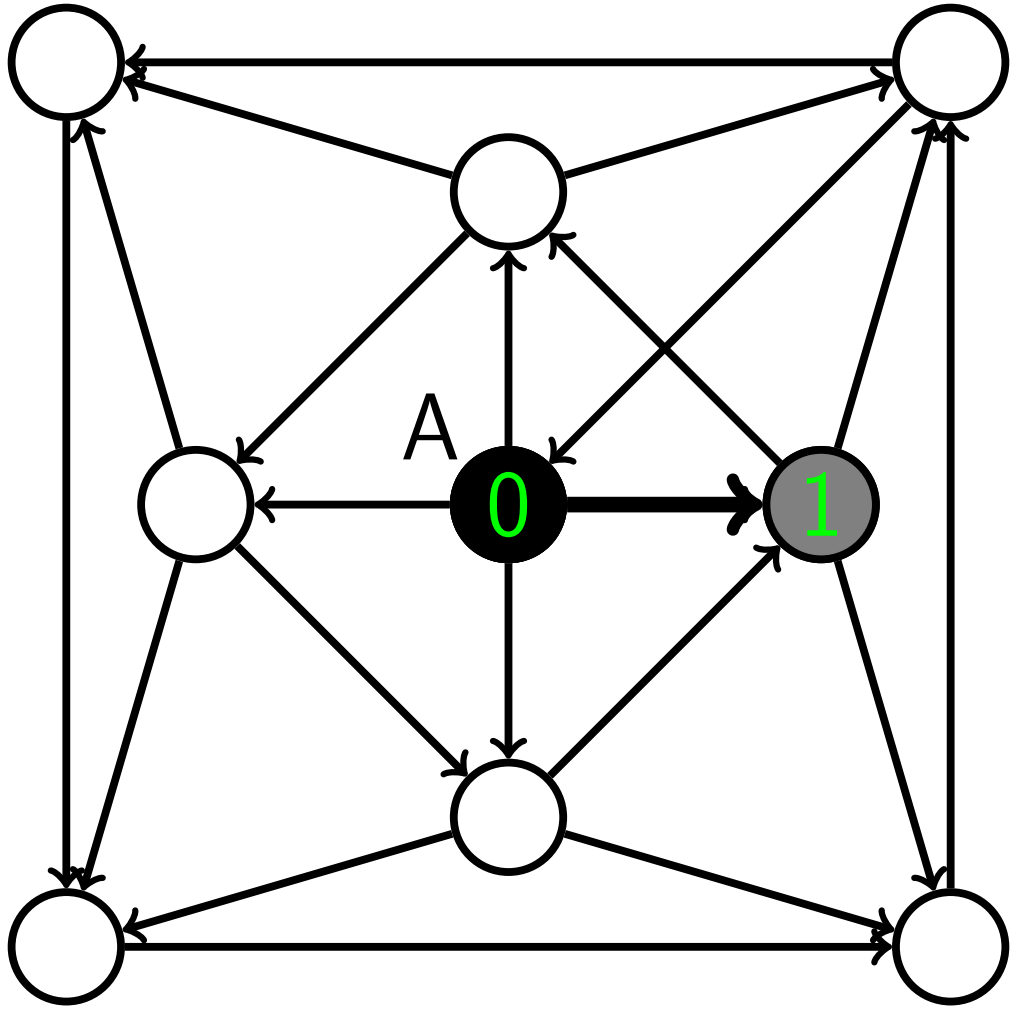
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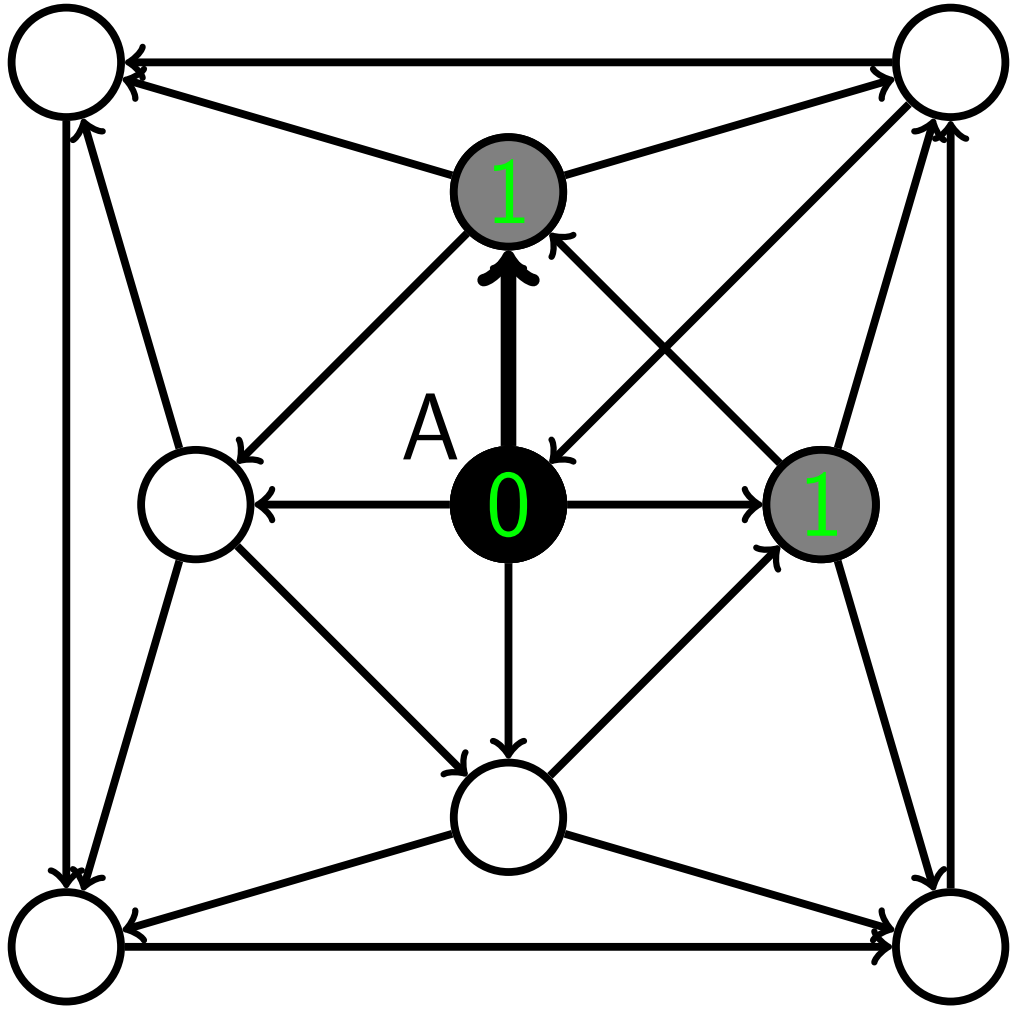
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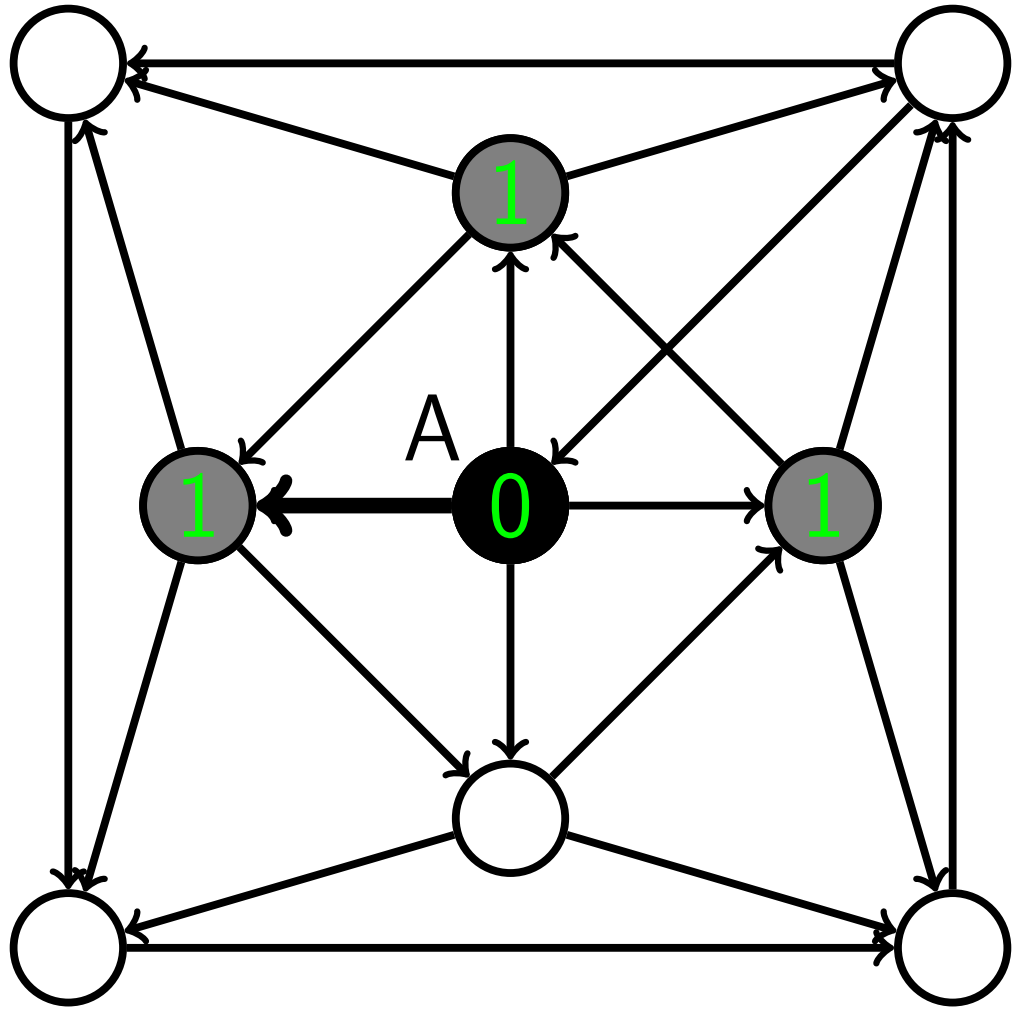
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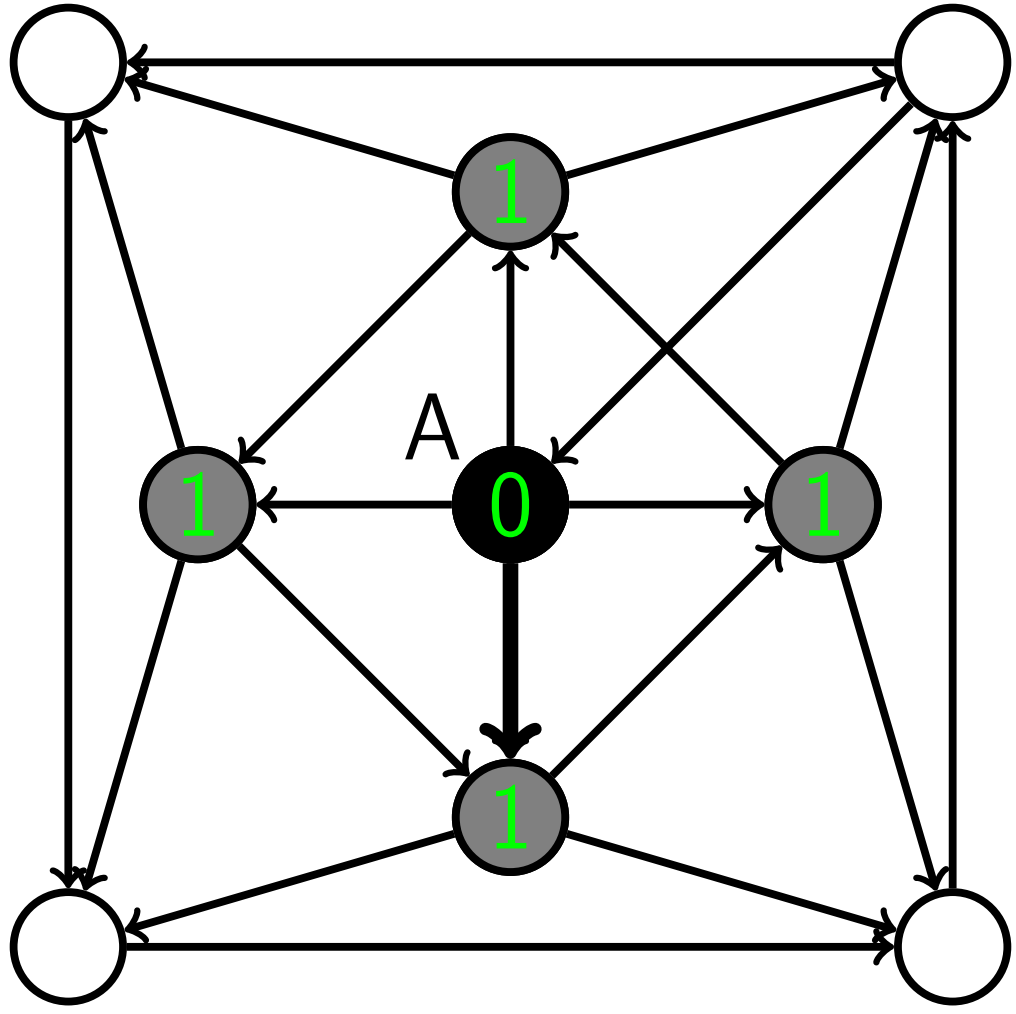
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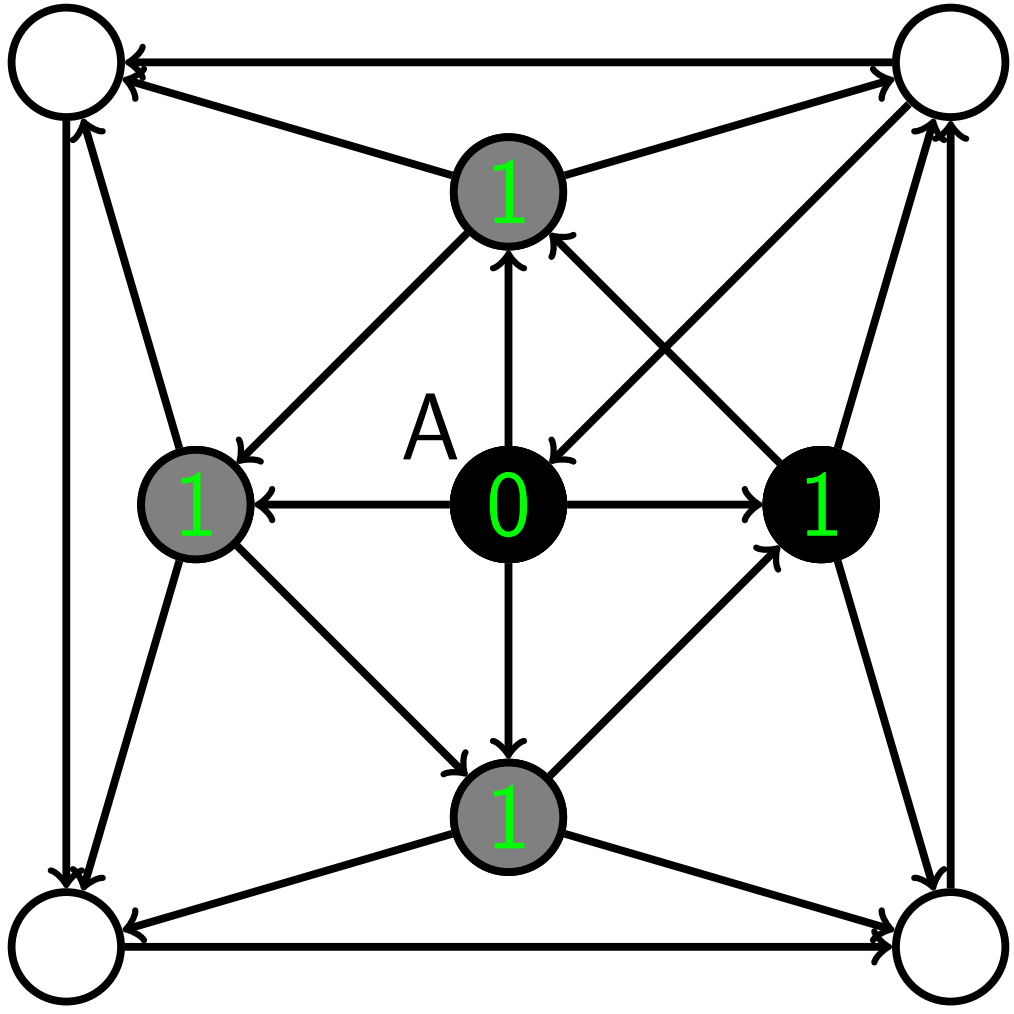
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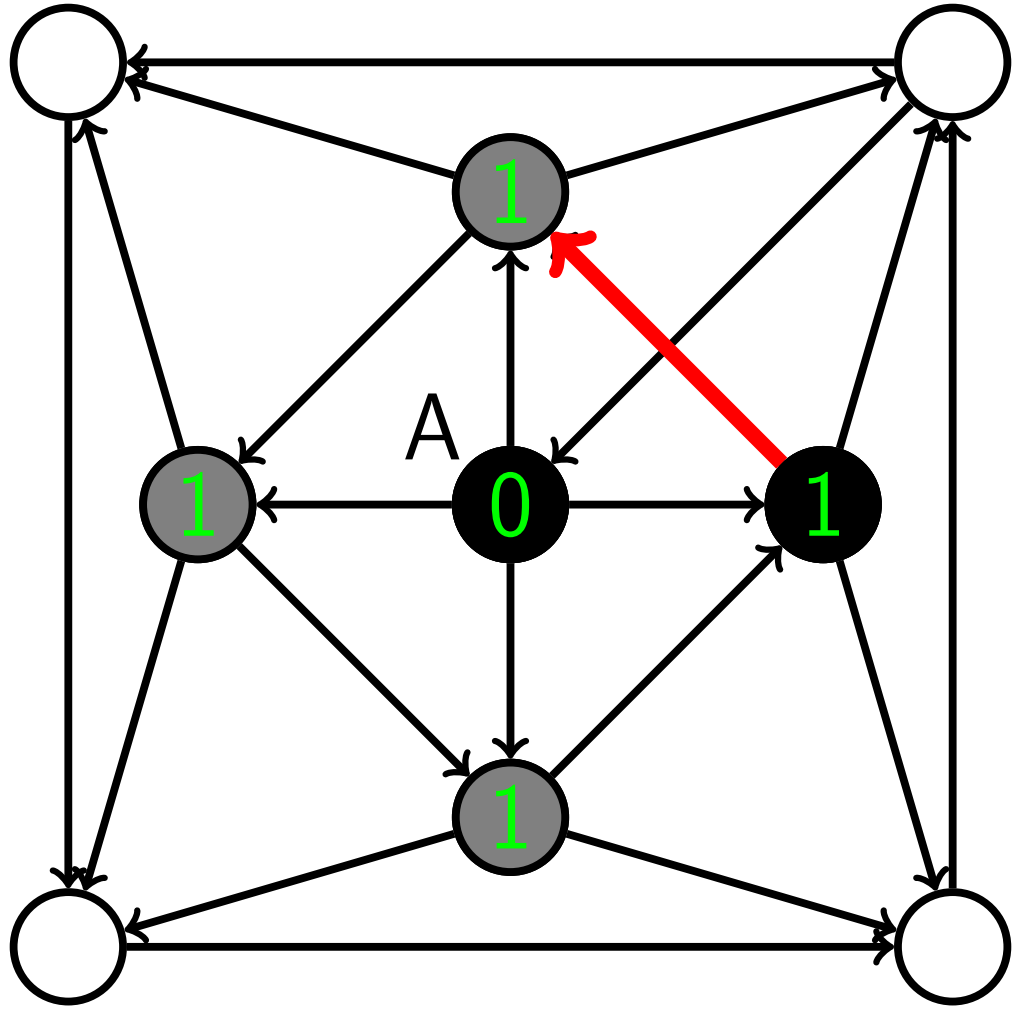
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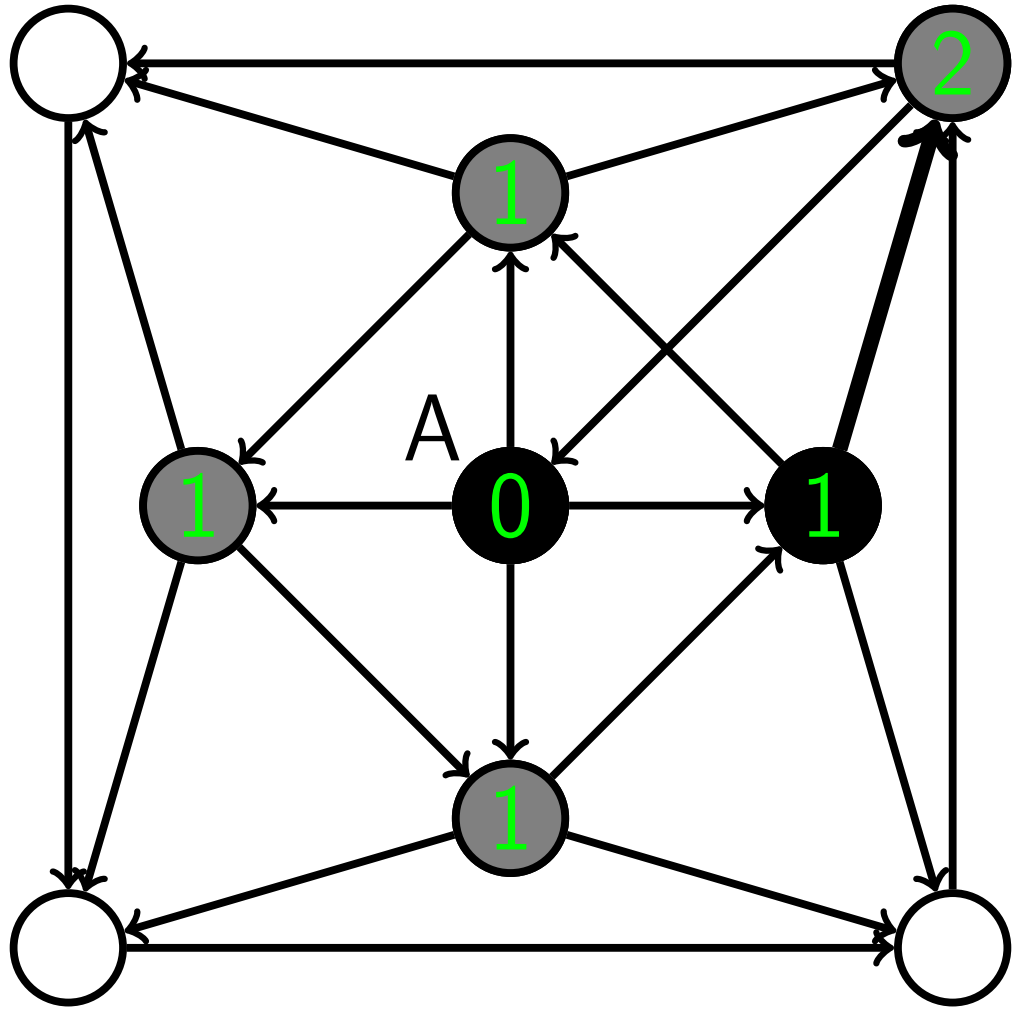
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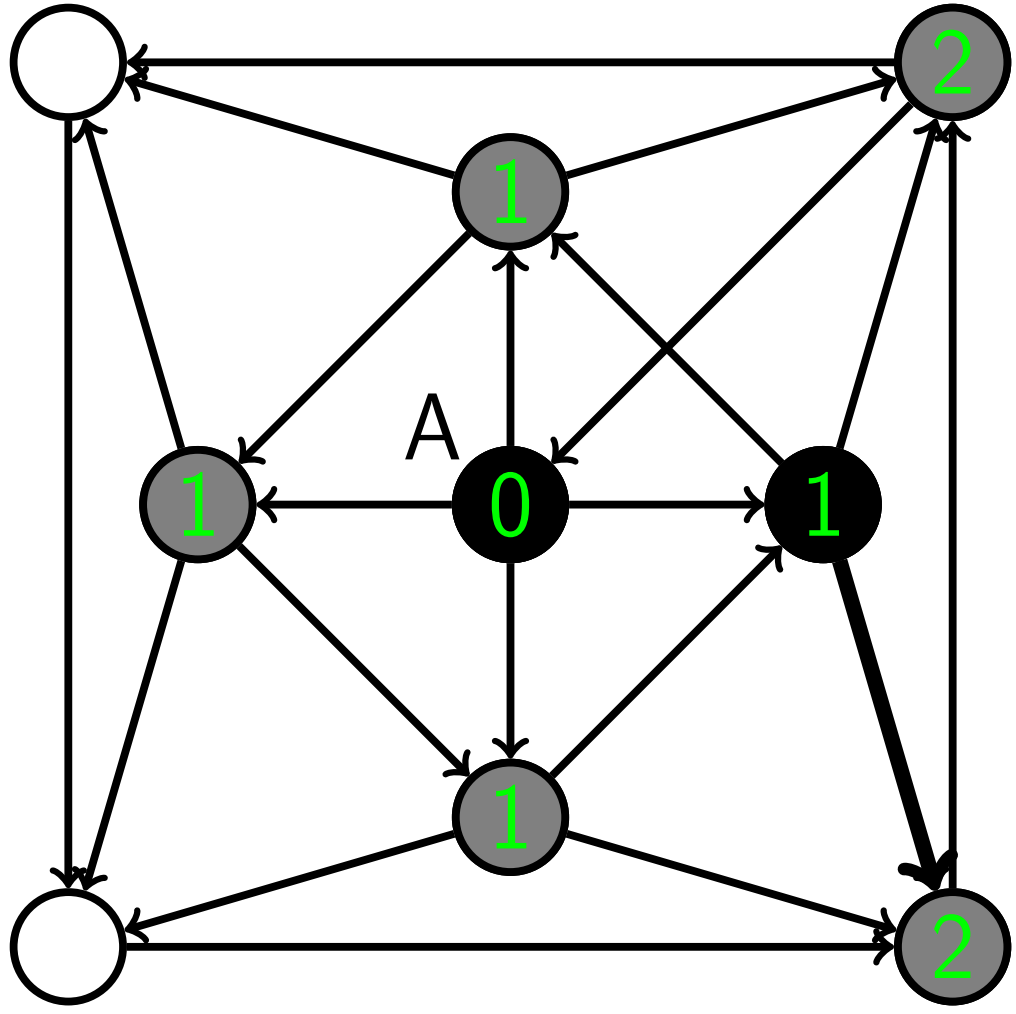
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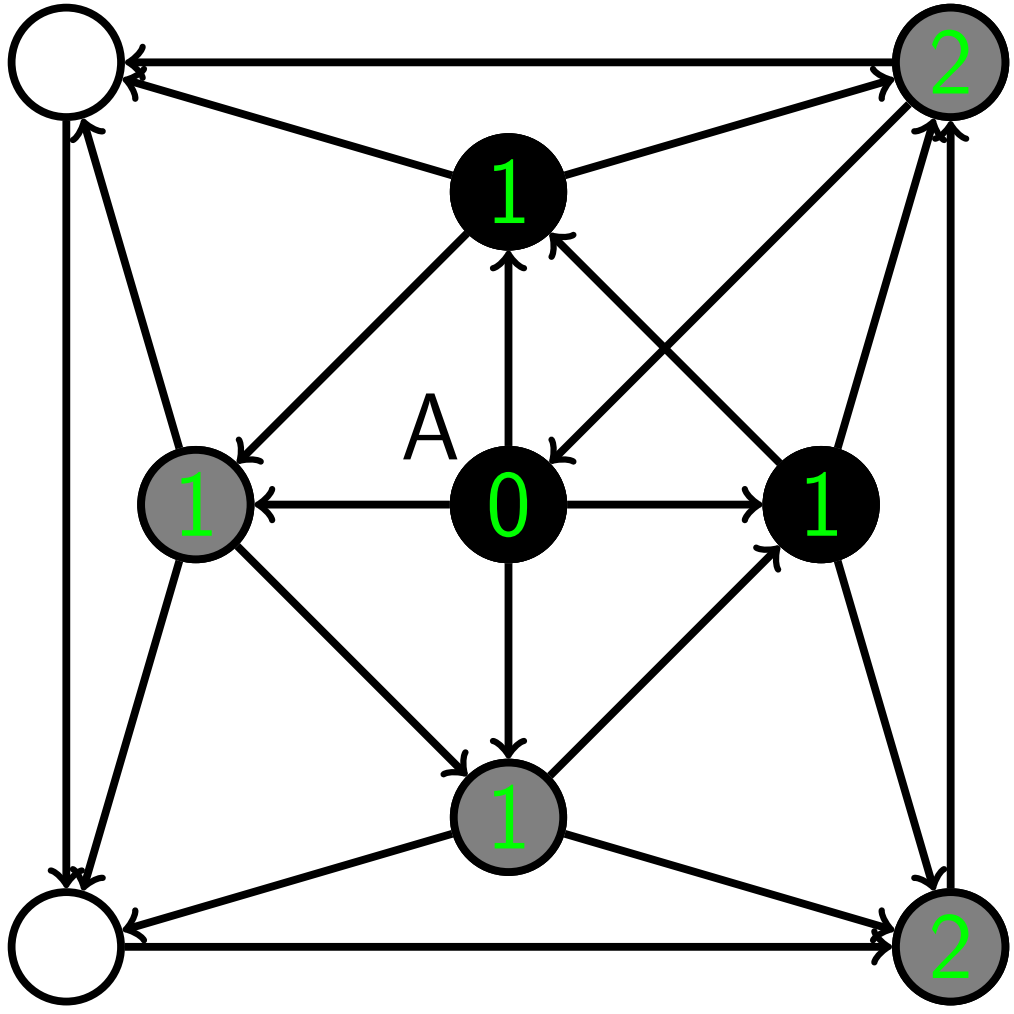
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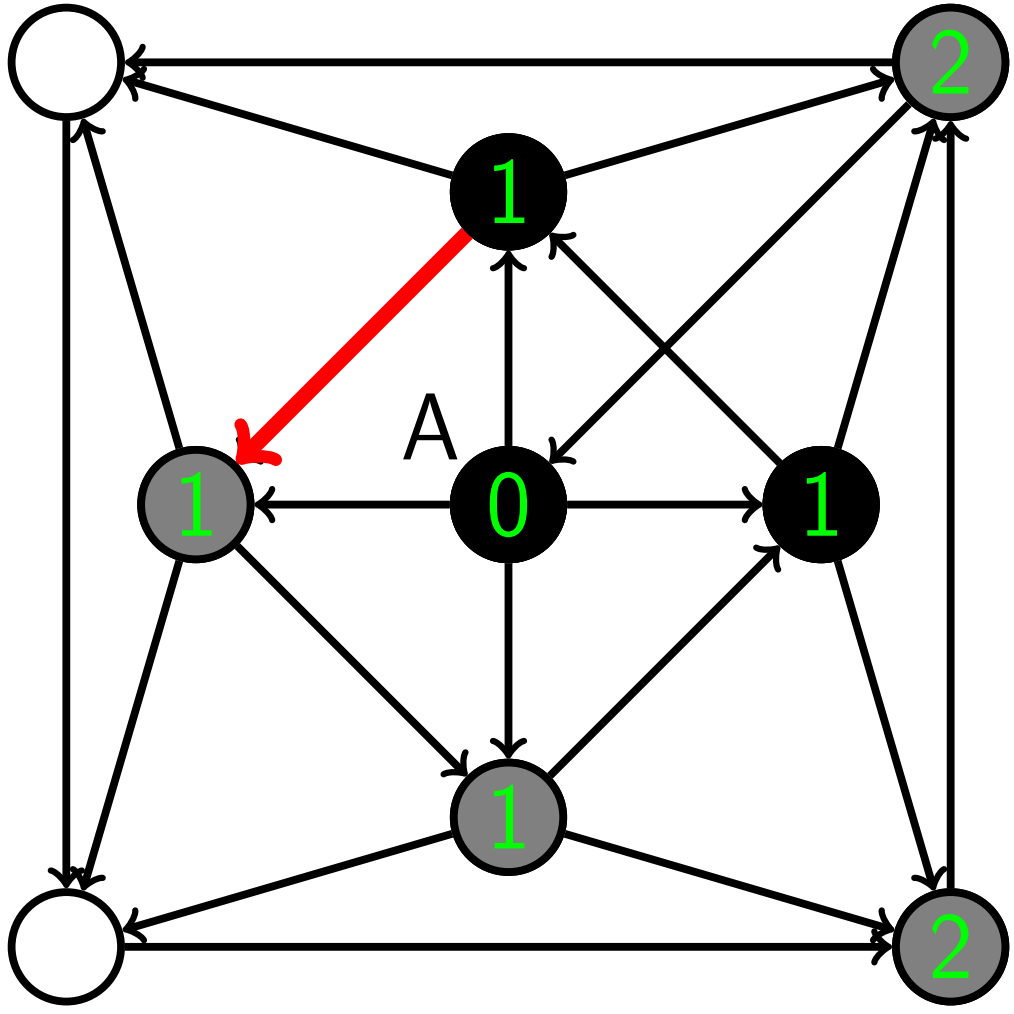
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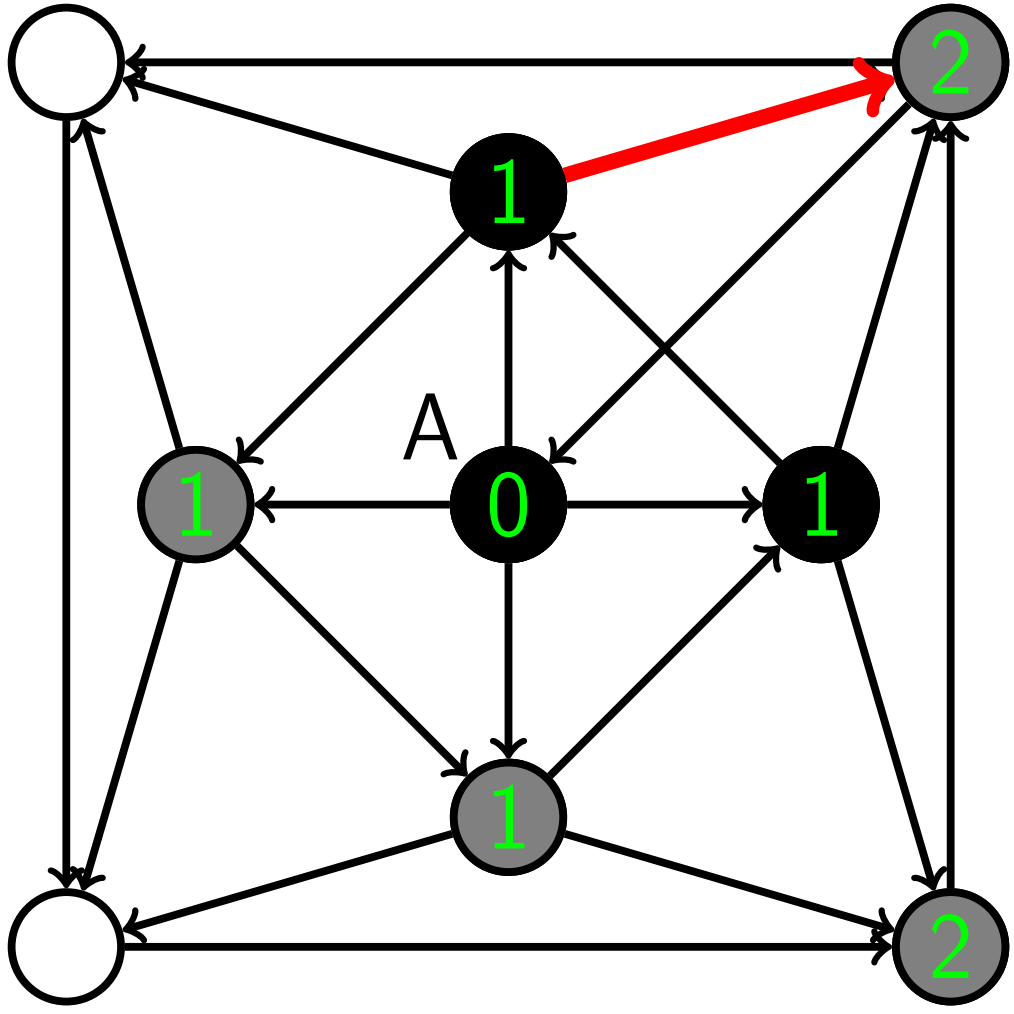
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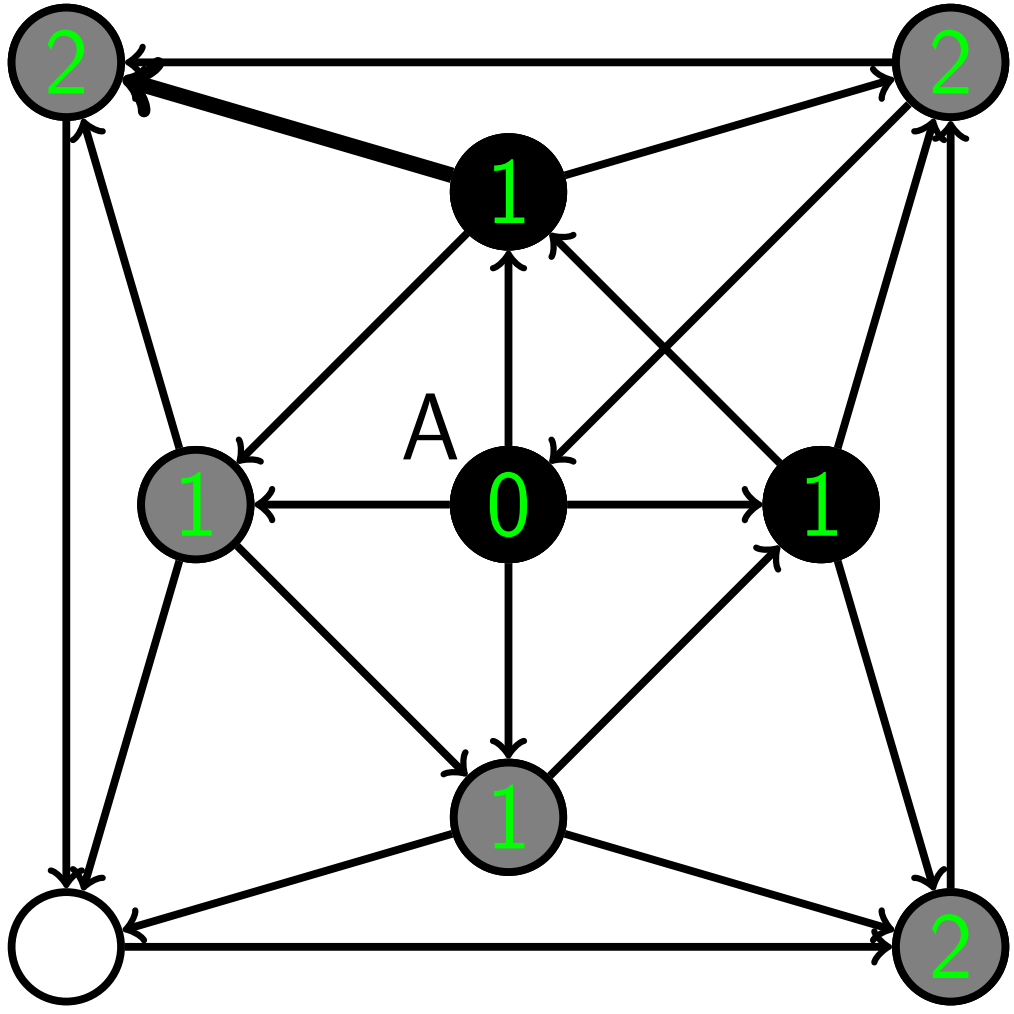
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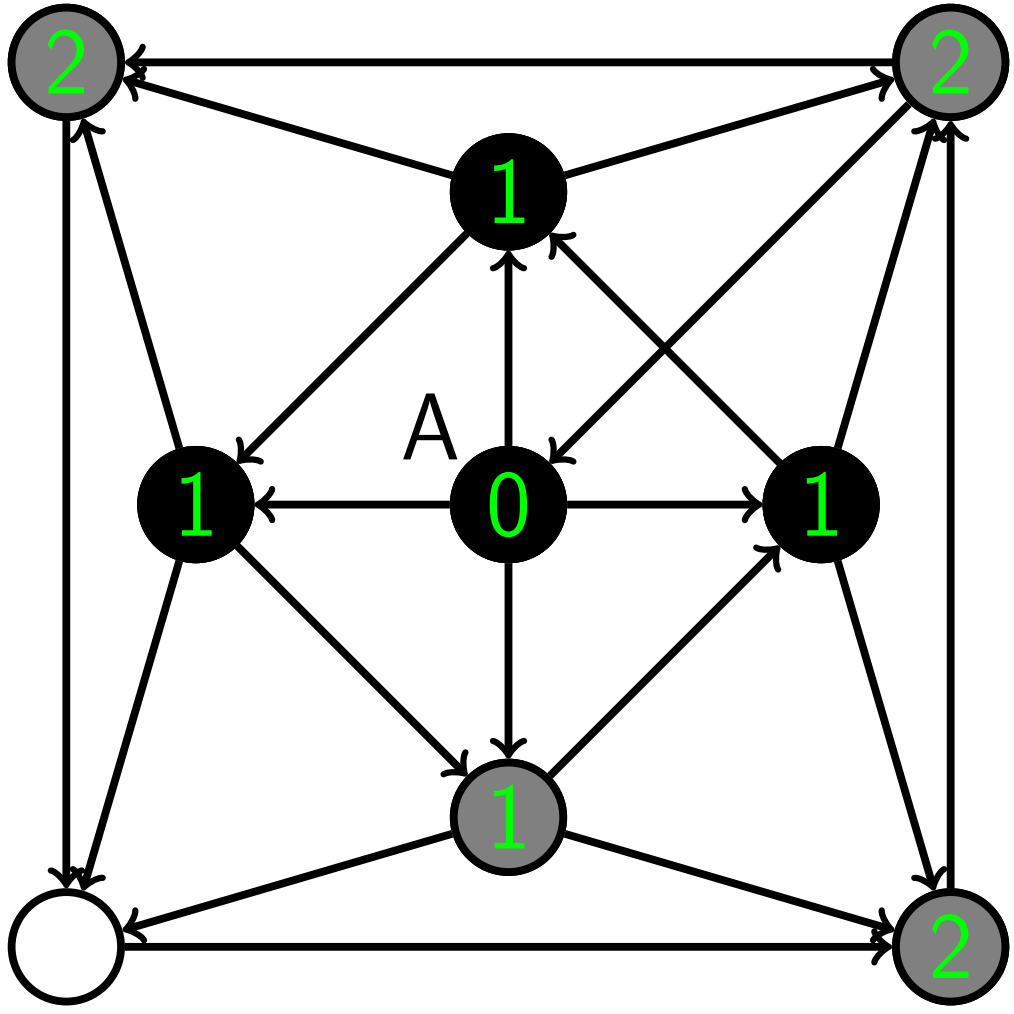
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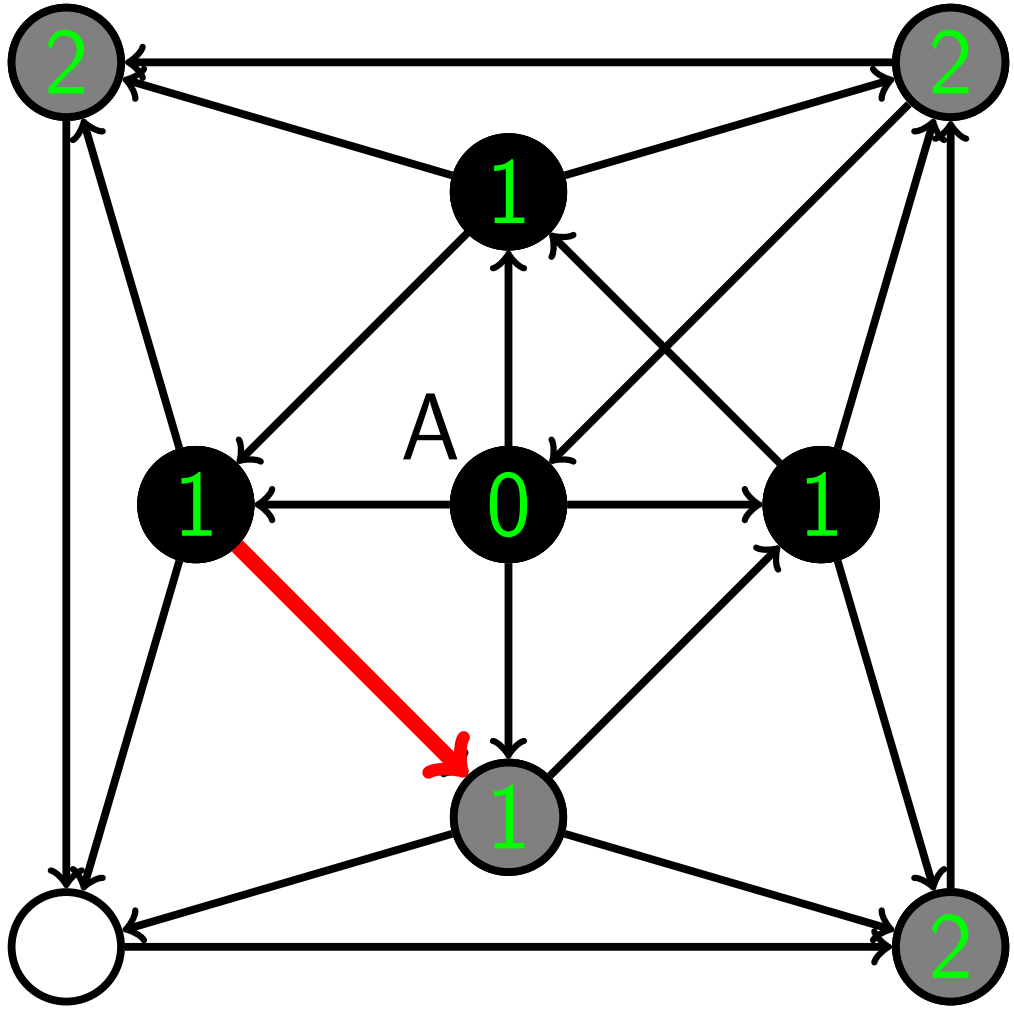
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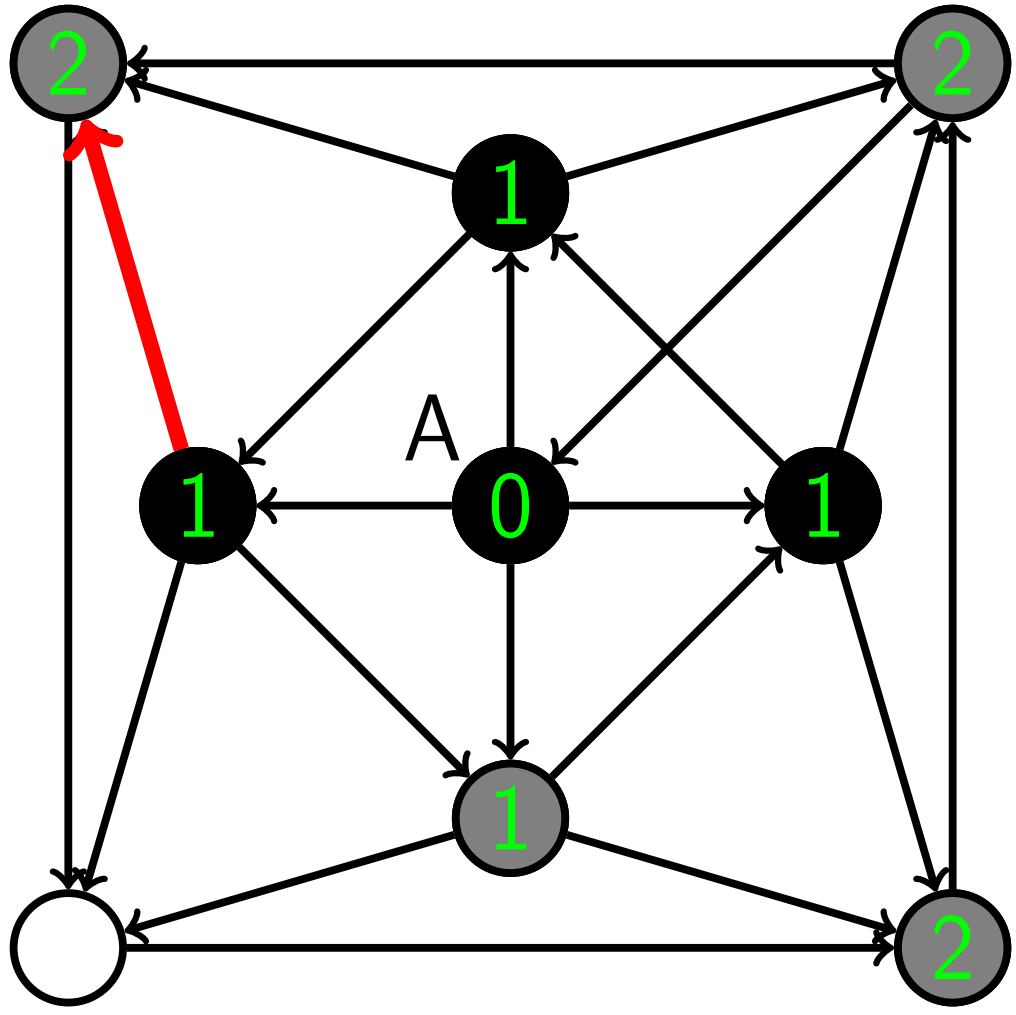
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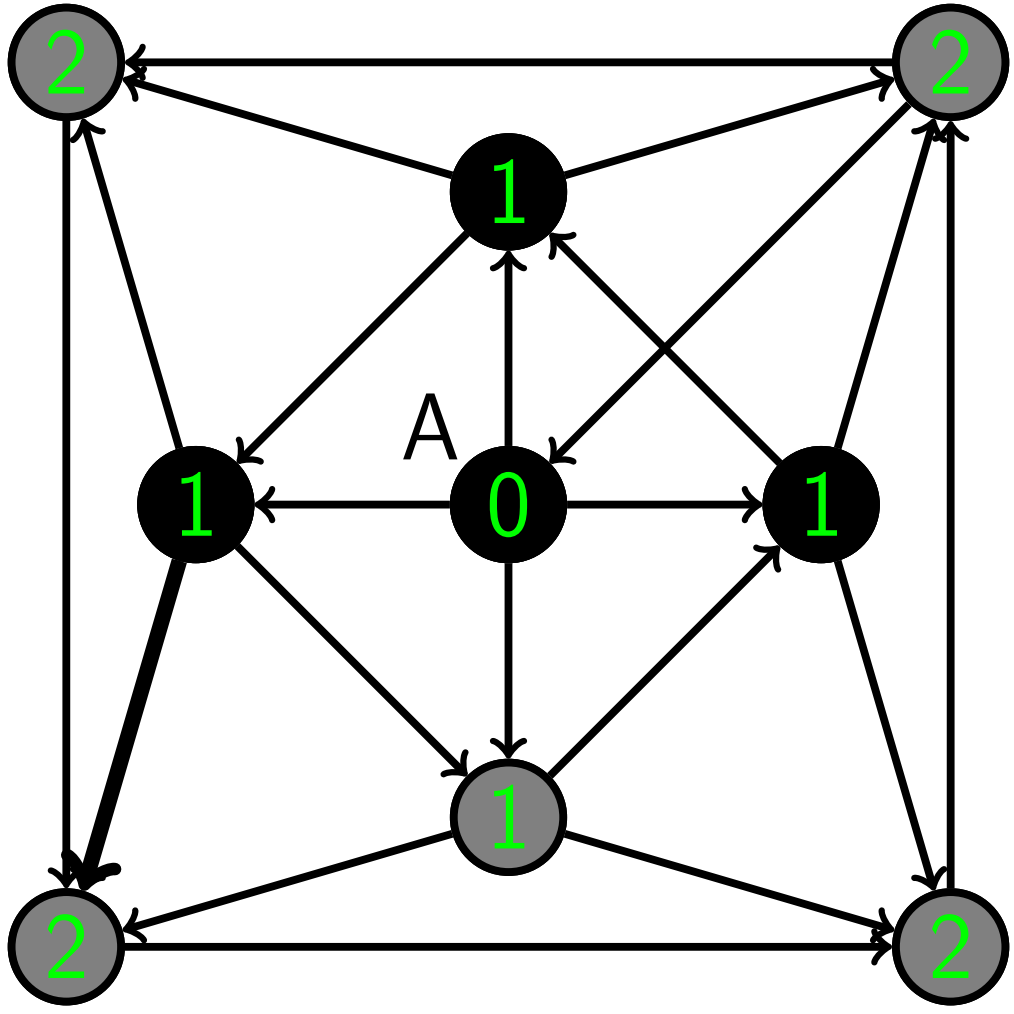
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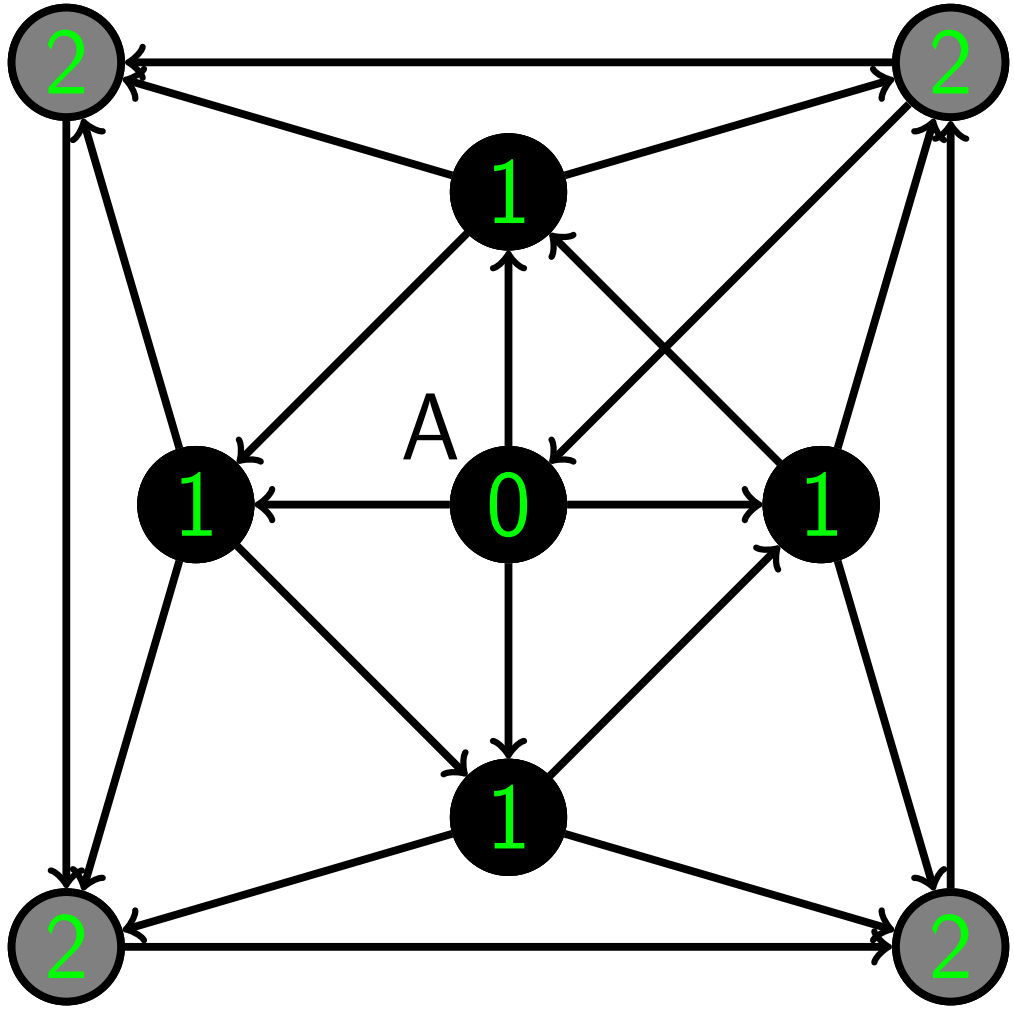
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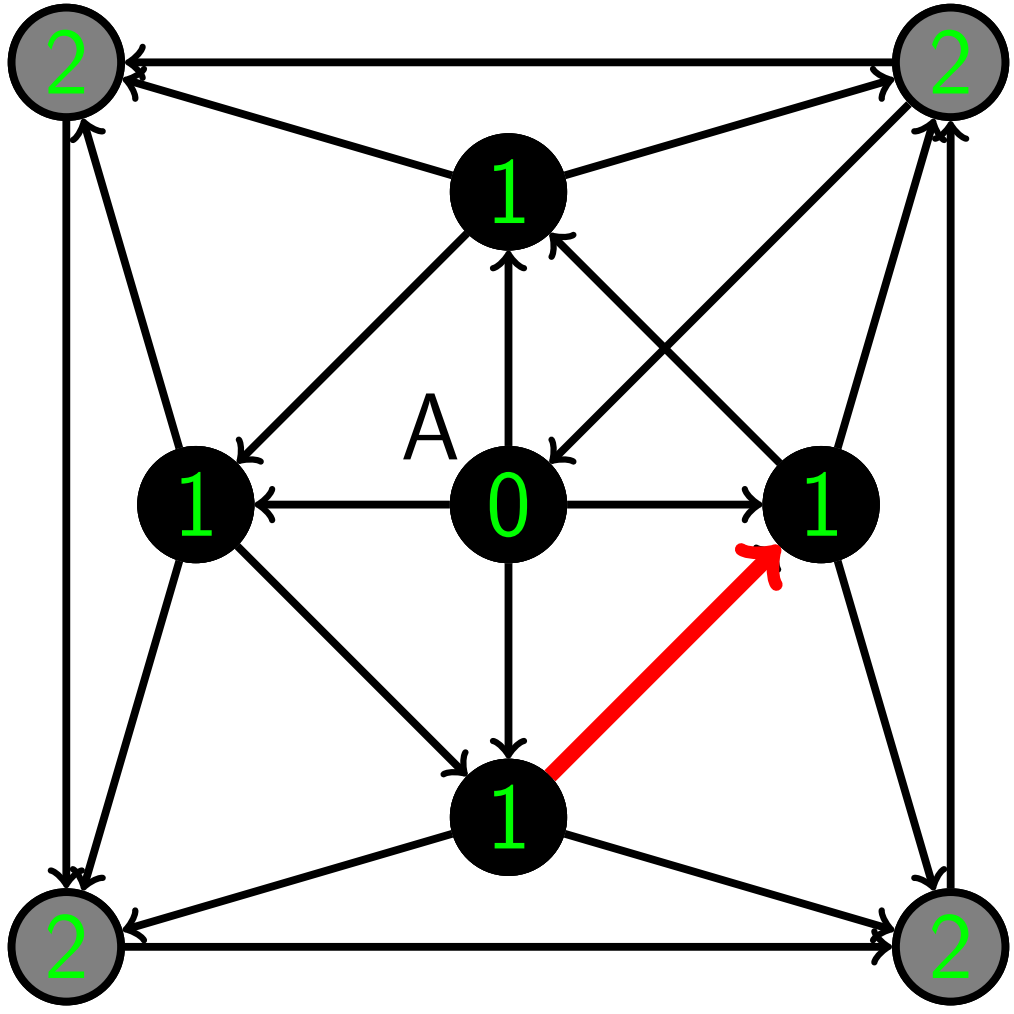
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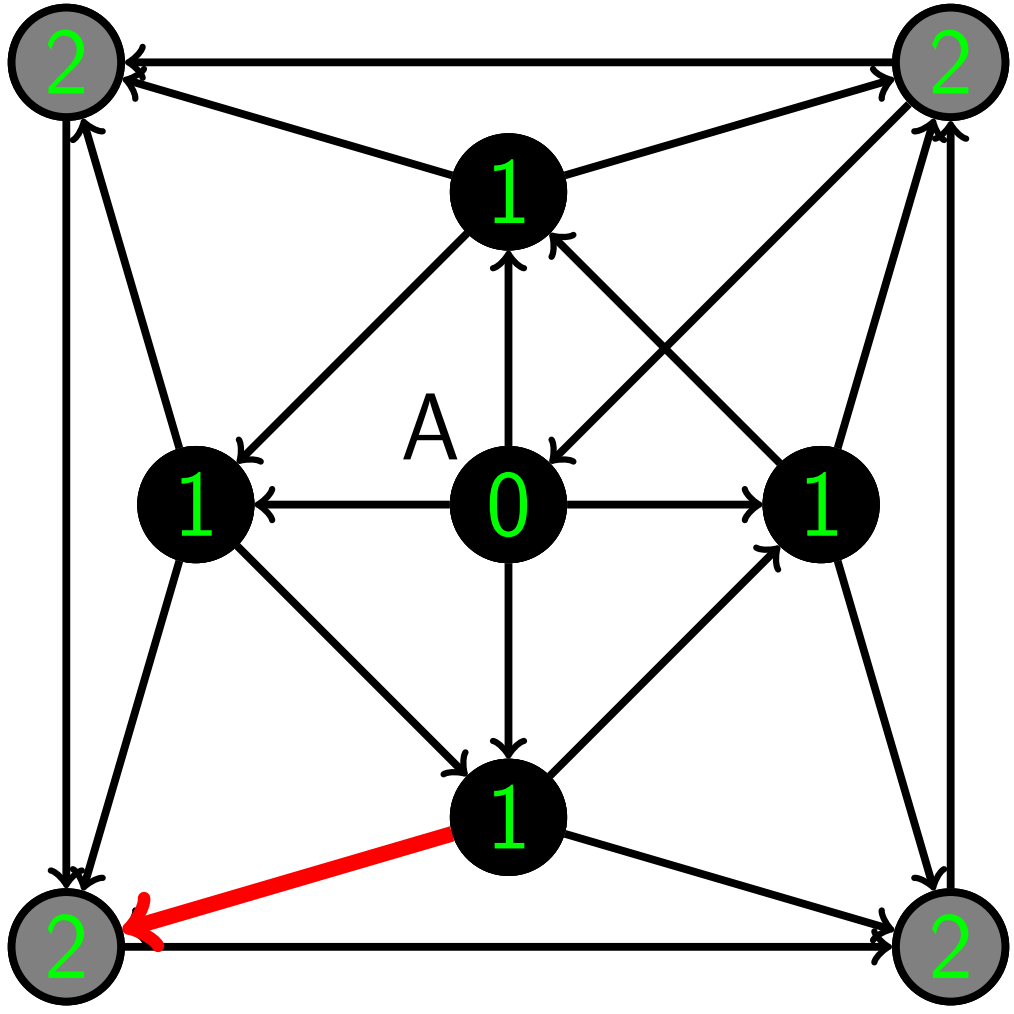
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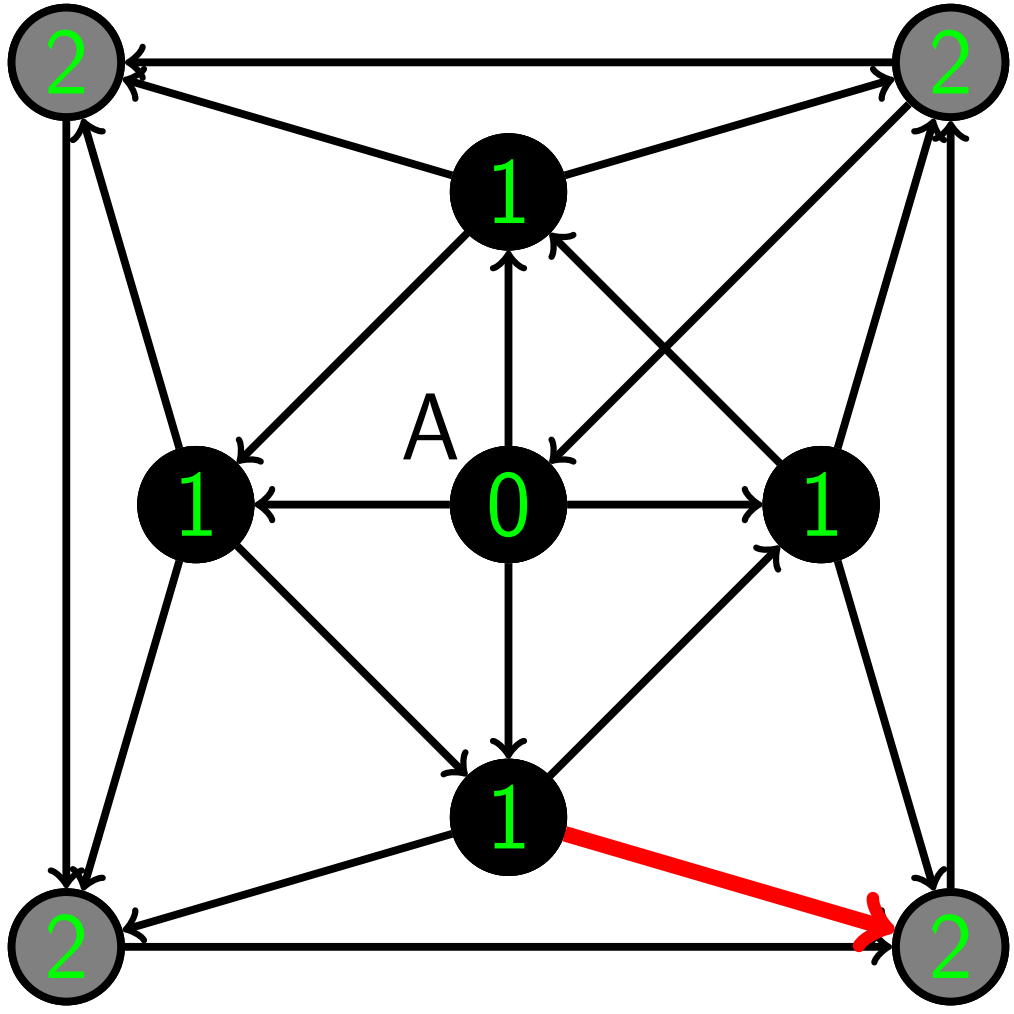
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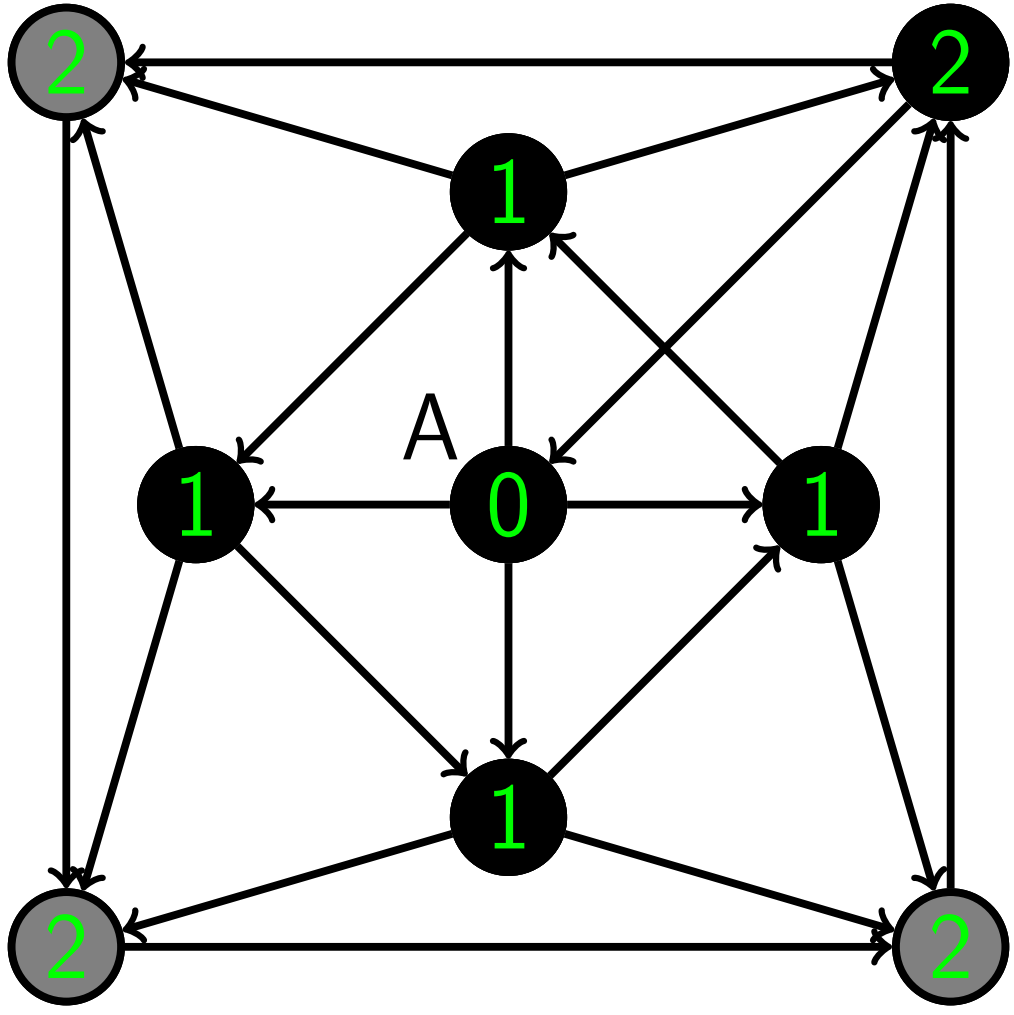
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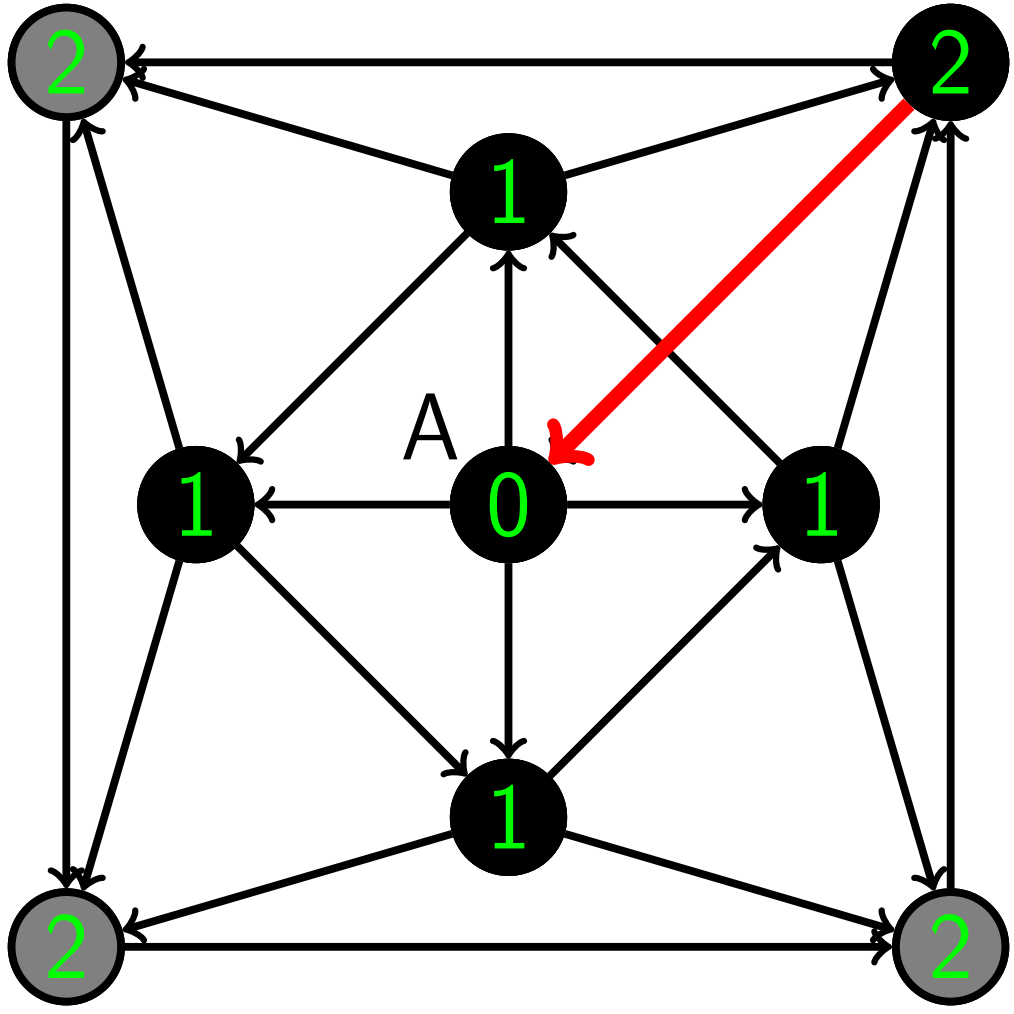
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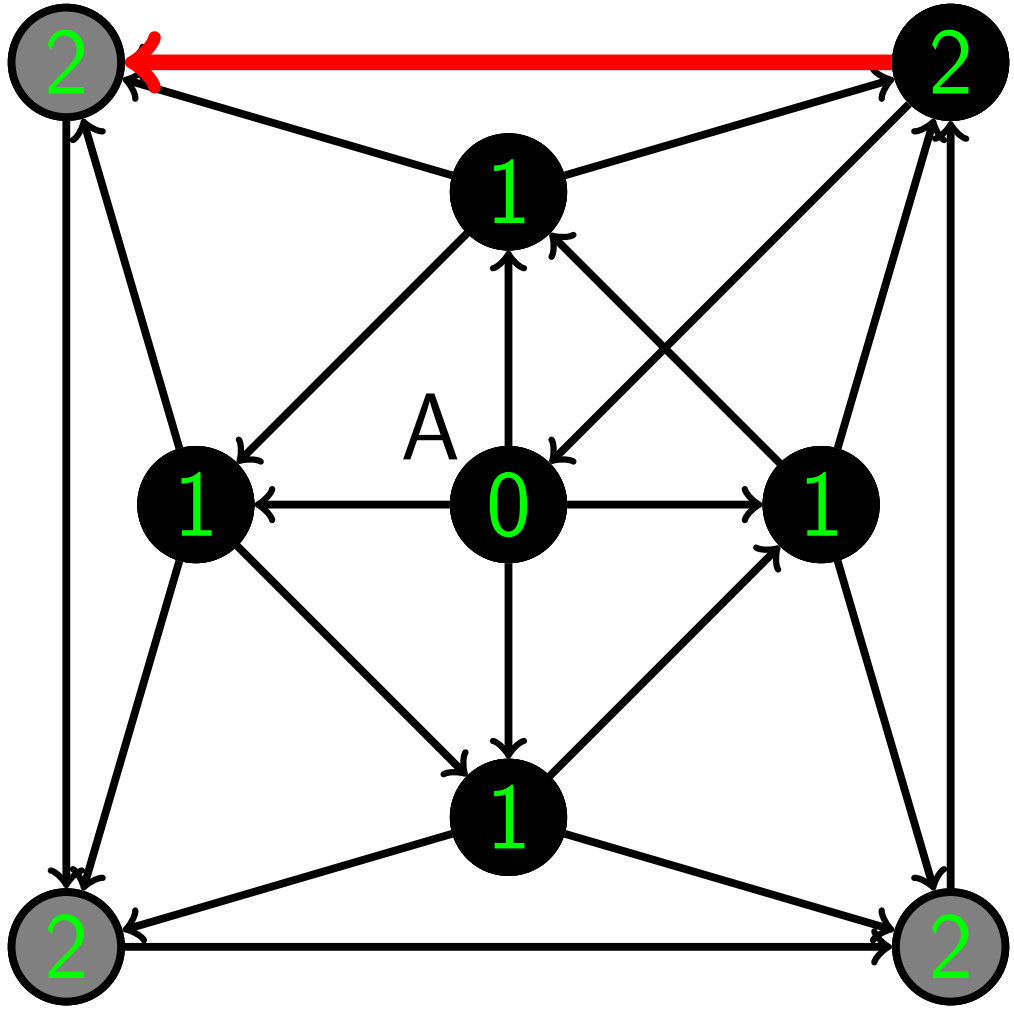
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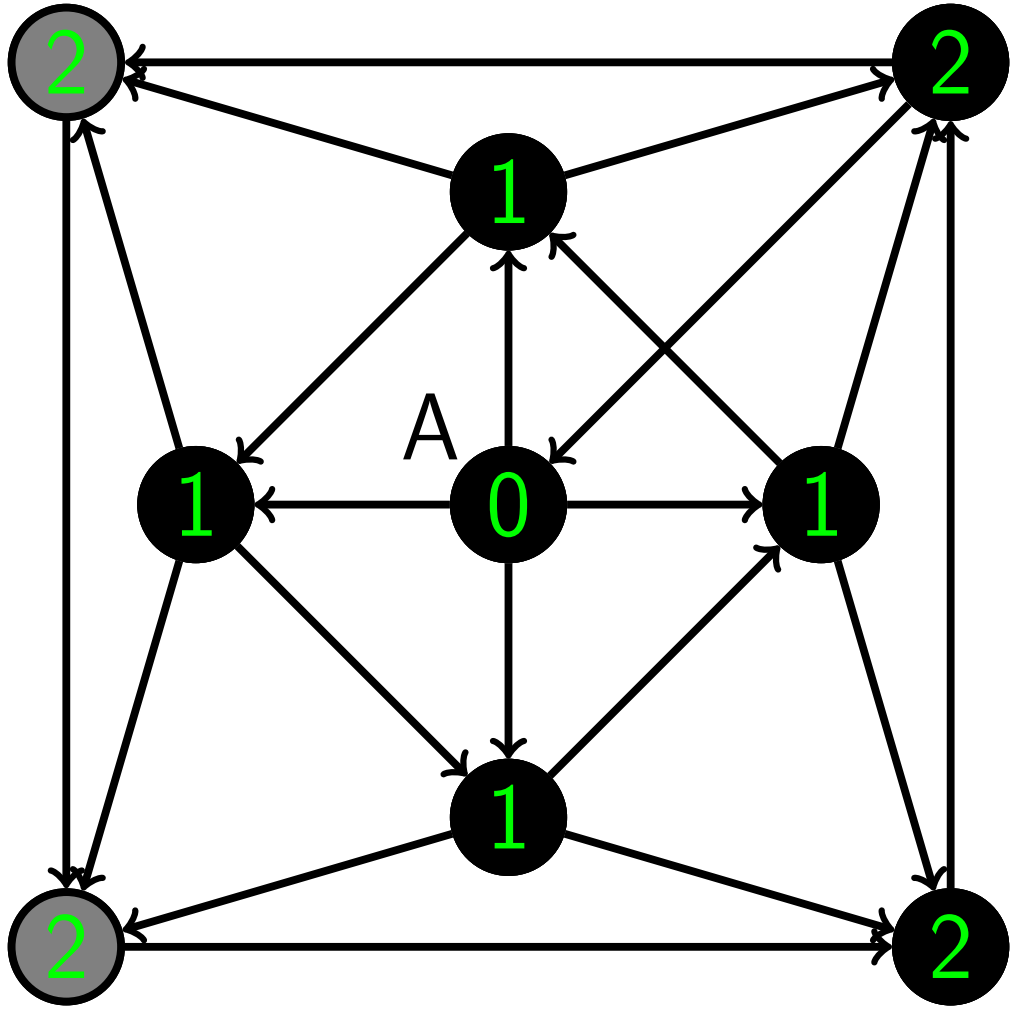
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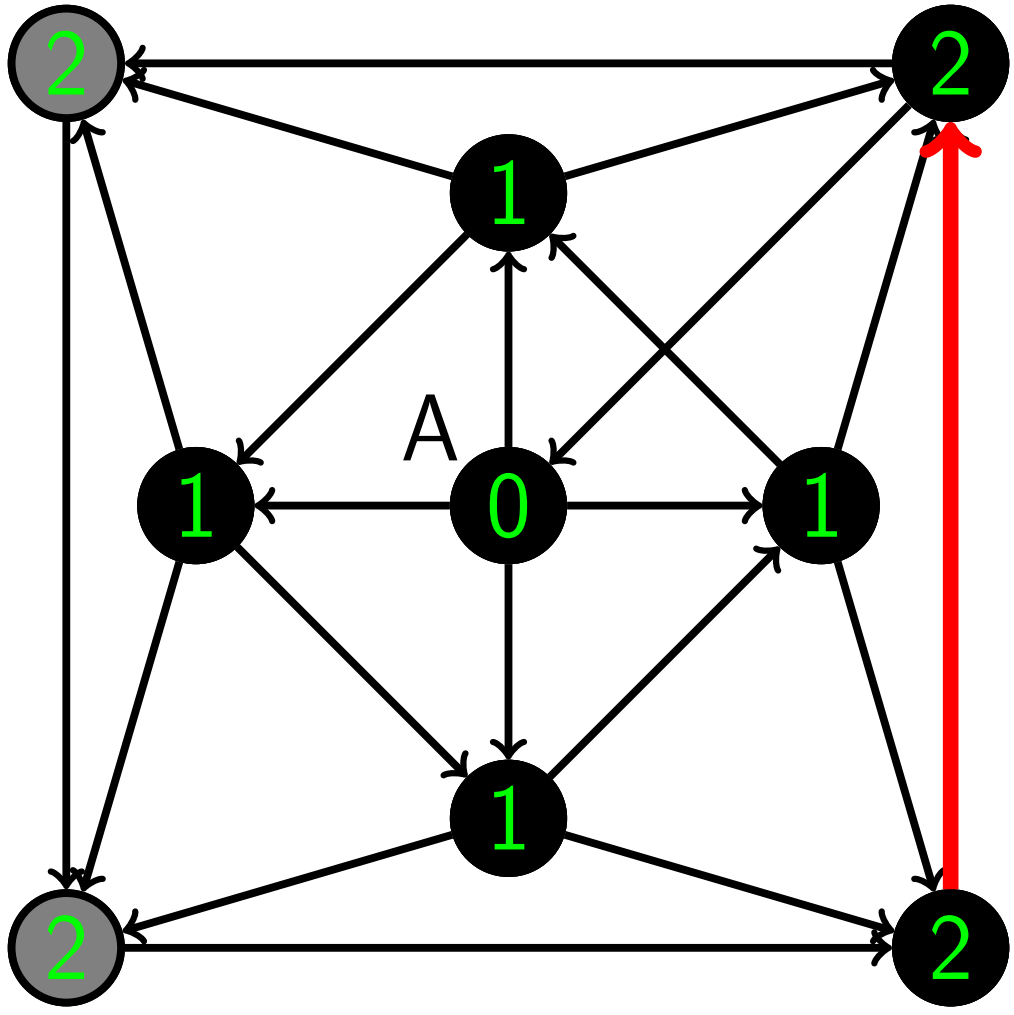
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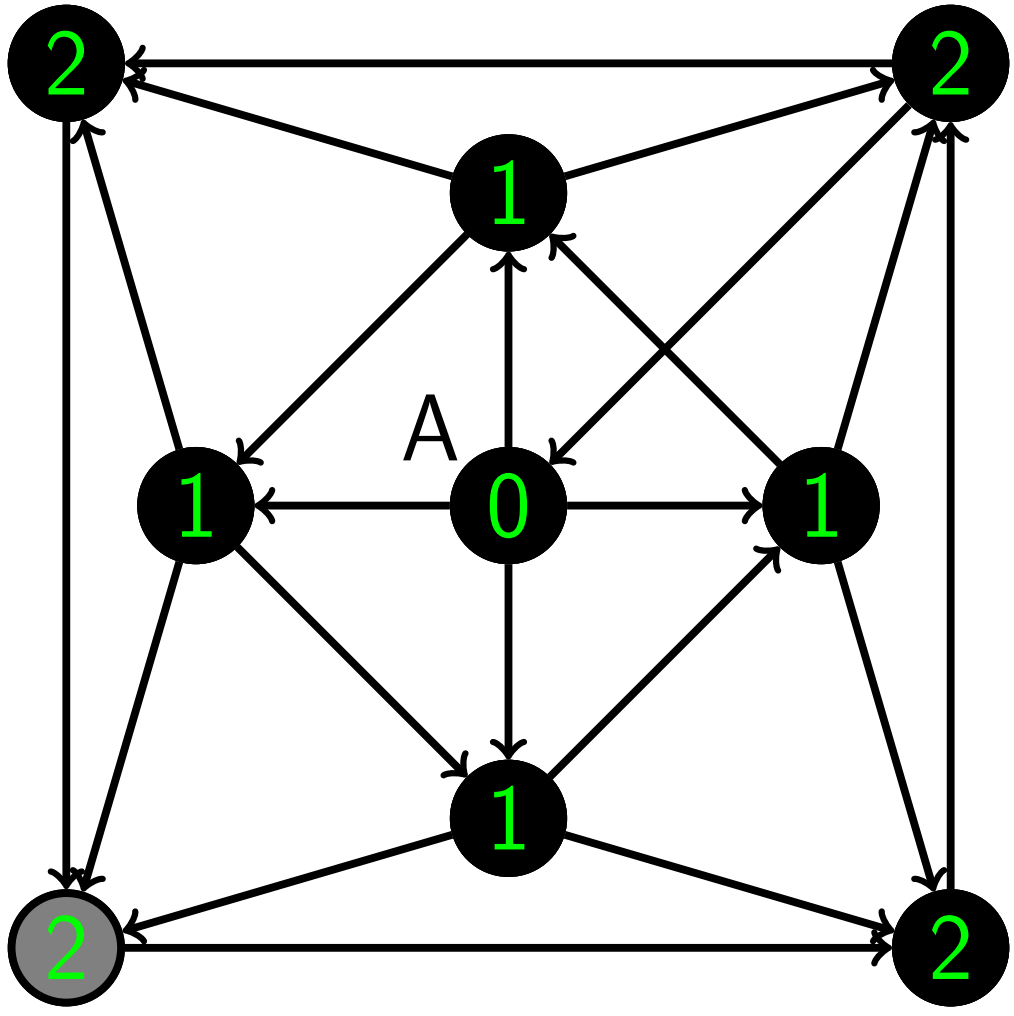
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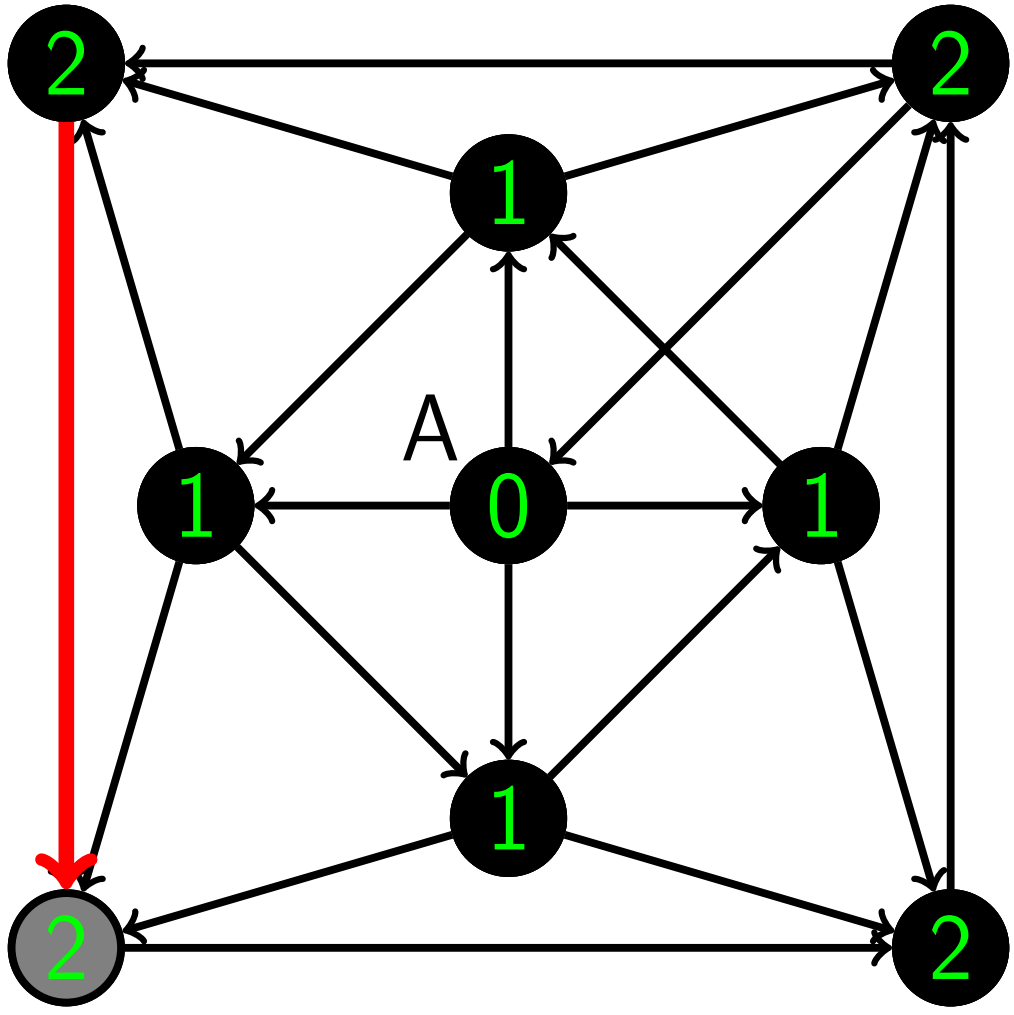
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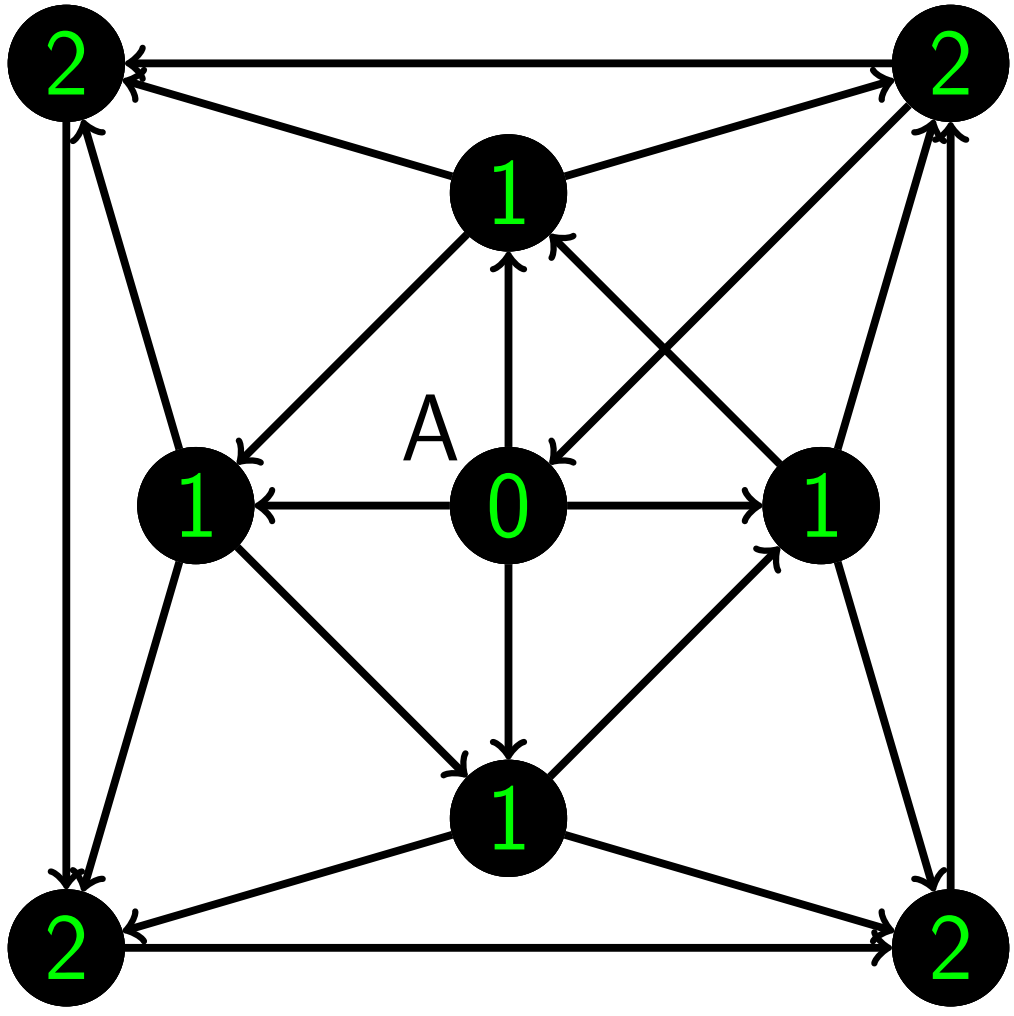
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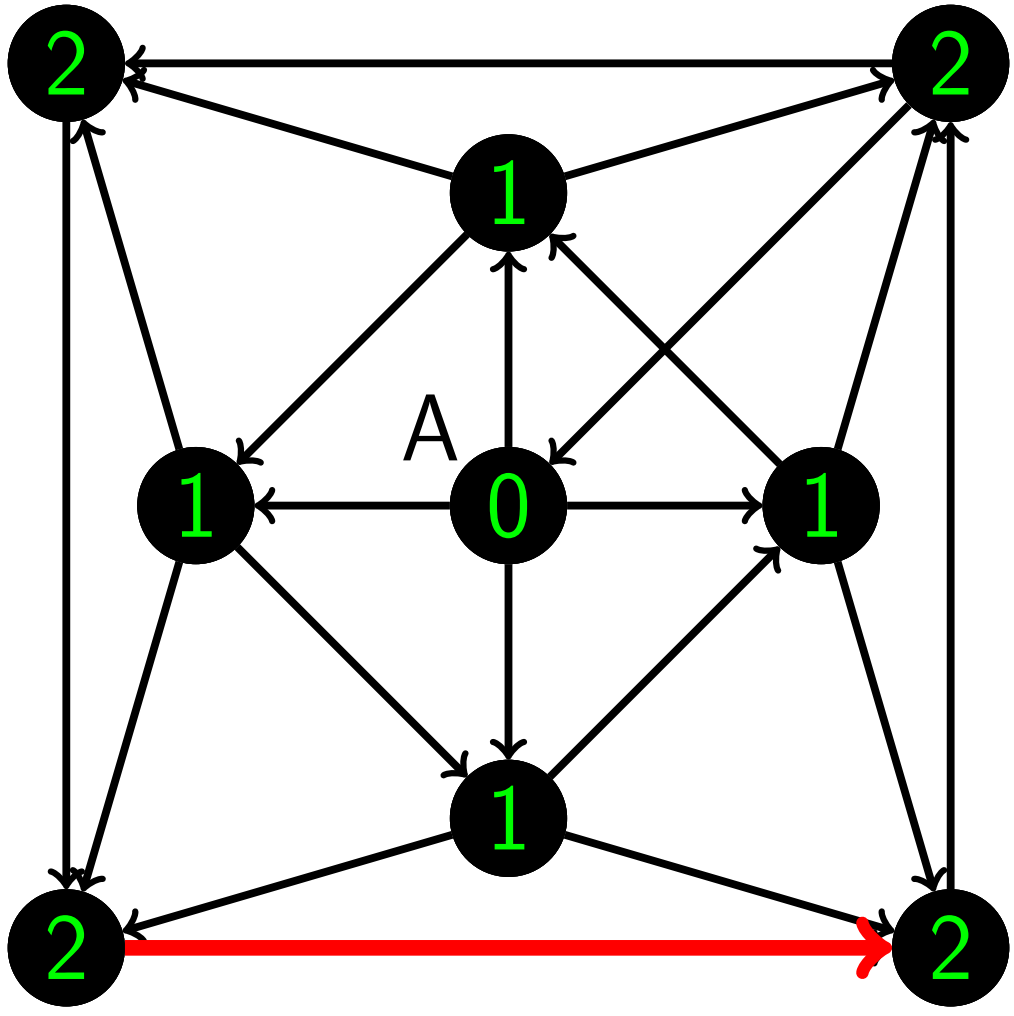
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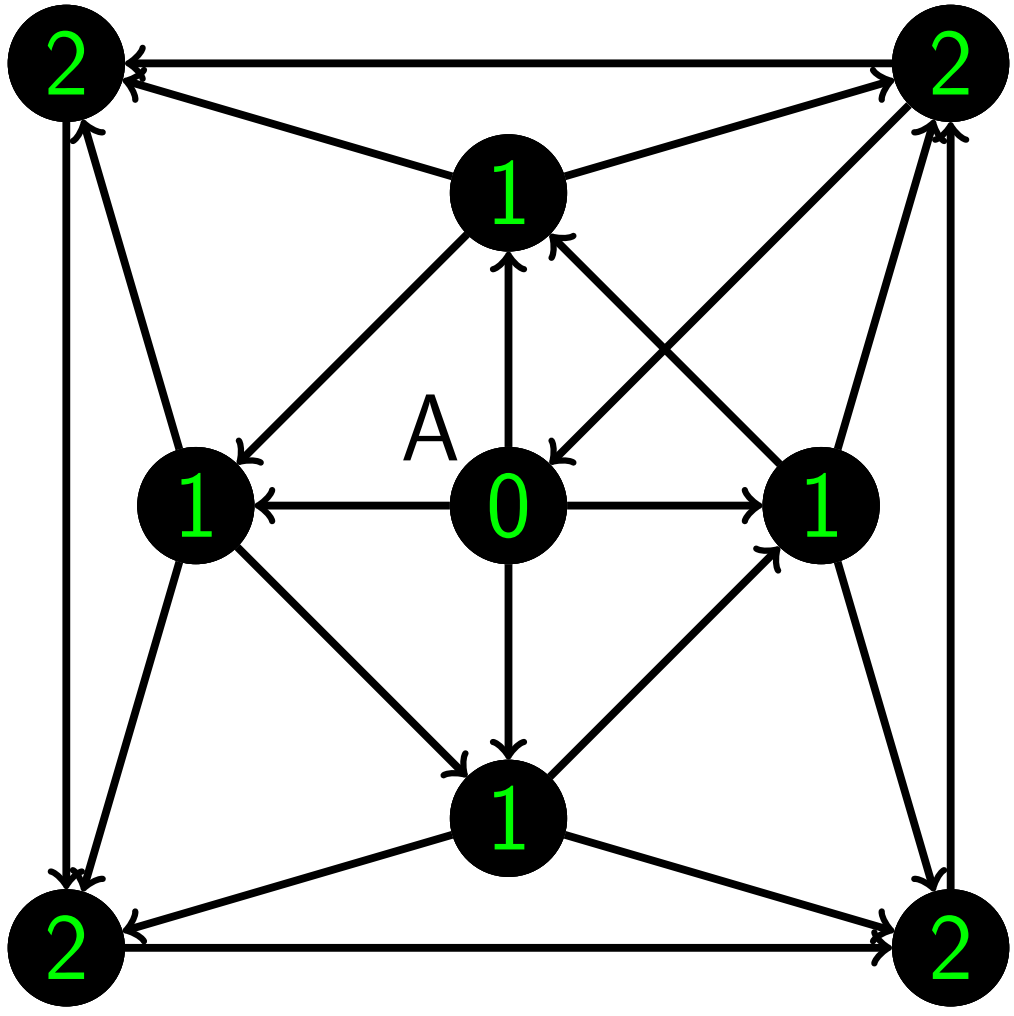
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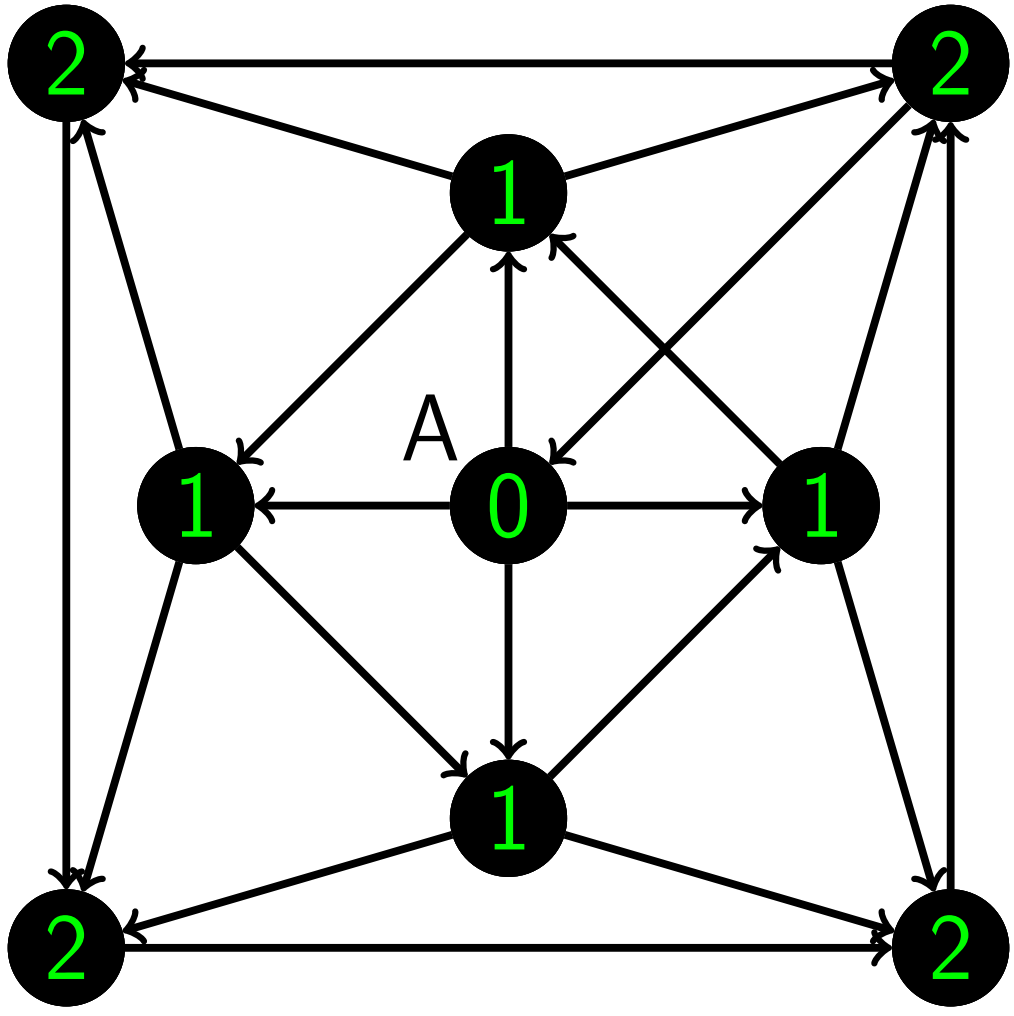
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Outline

Breadth-first search

BFS(G, A)

```
for all u ∈ V:
  dist[u] ← ∞
dist[A] ← 0
Q ← {A} {queue containing just A}
while Q is not empty:
  u ← Dequeue(Q)
  for all (u, v) ∈ E:
    if dist[v] = ∞:
      Enqueue(Q, v)
      dist[v] ← dist[u] + 1
```

Breadth-first search

BFS(G, A)

```
for all u ∈ V:
  dist[u] ← ∞
dist[A] ← 0
Q ← {A} {queue containing just A}
while Q is not empty:
  u ← Dequeue(Q)
  for all (u, v) ∈ E:
    if dist[v] = ∞:
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Breadth-first search

BFS(G, A)

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for all u ∈ V:
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Running time

Lemma

The running time of breadth-first search is $O(|E| + |V|)$.

Proof

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Proof

- Each vertex is enqueued at most once

Running time

Lemma

The running time of breadth-first search is $O(|E| + |V|)$.

Proof

- Each vertex is enqueued at most once
- Each edge is examined either once (for directed graphs) or twice (for undirected graphs)

Outline

Reachability

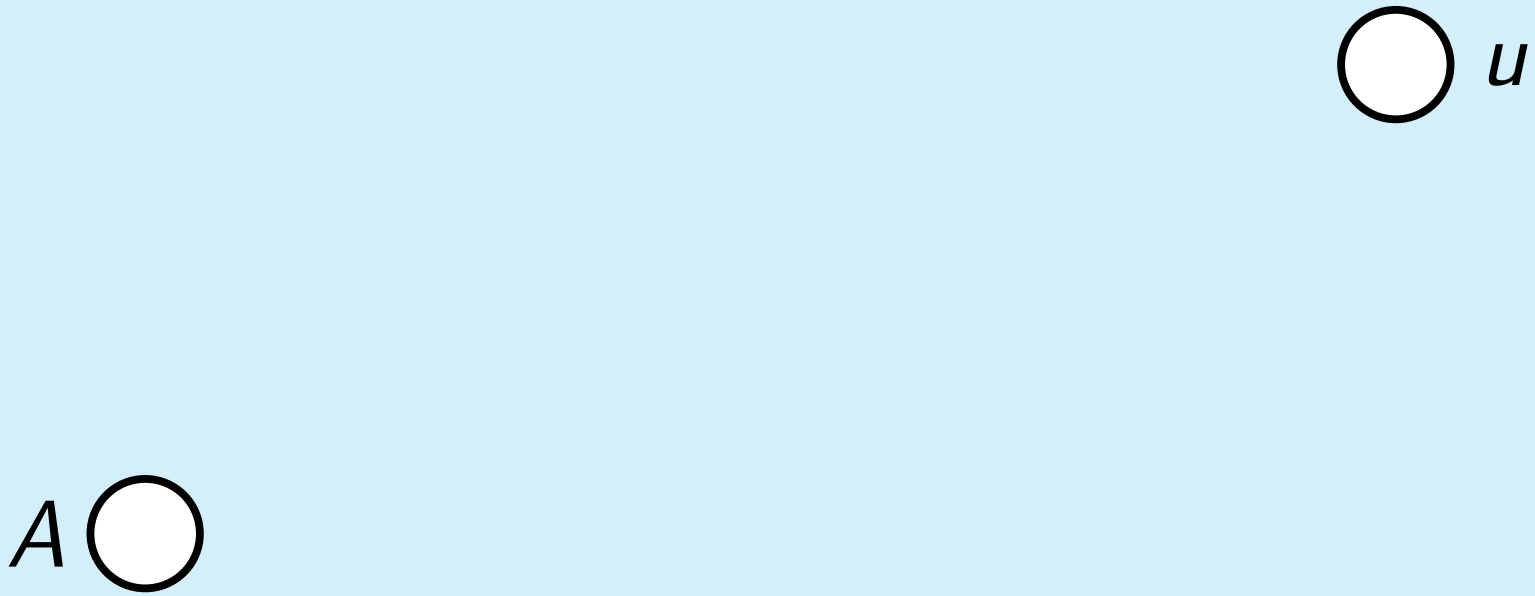
Definition

Node u is reachable from node A if there is a path from A to u

Lemma

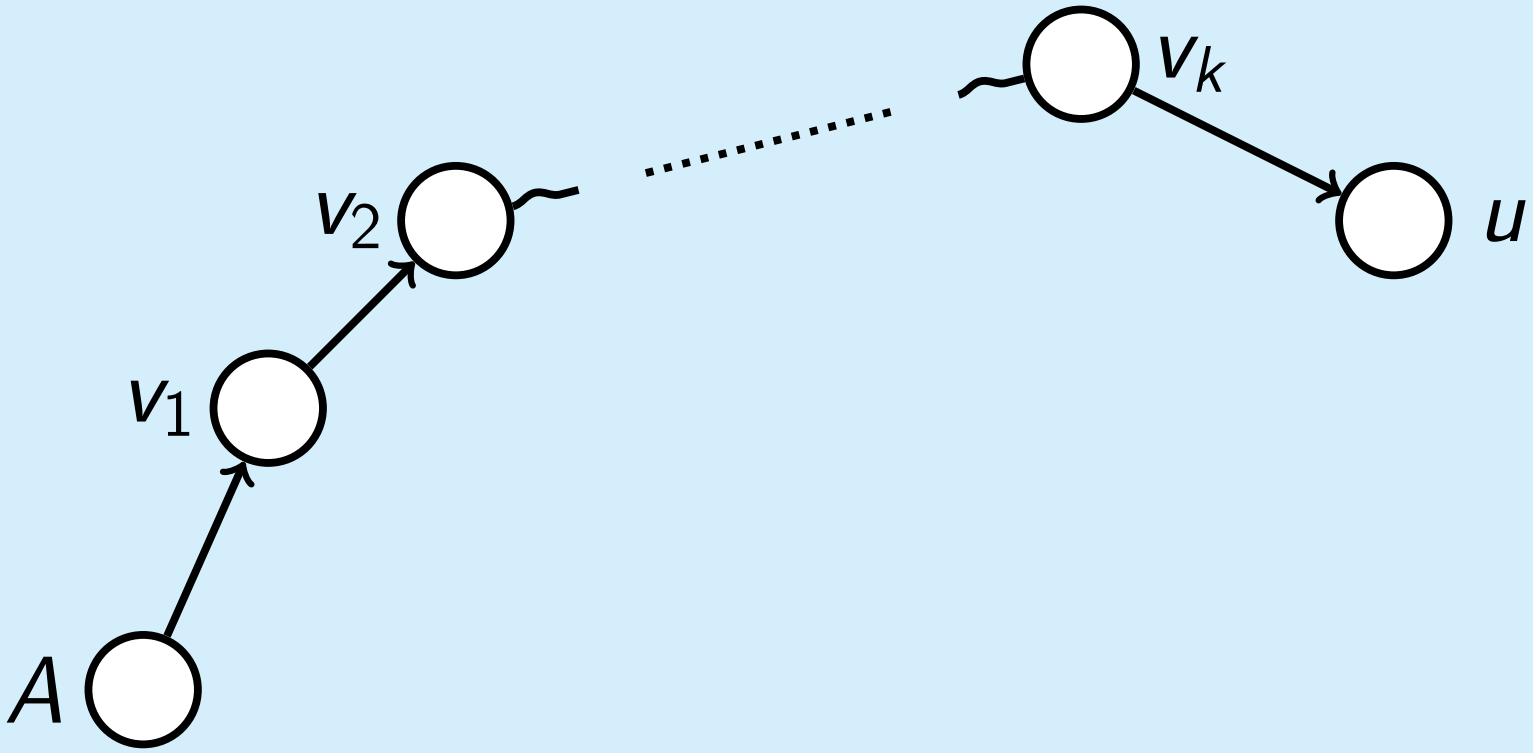
Reachable nodes are discovered at some point, so they get a finite distance estimate from the source. Unreachable nodes are not discovered at any point, and the distance to them stays infinite.

Proof



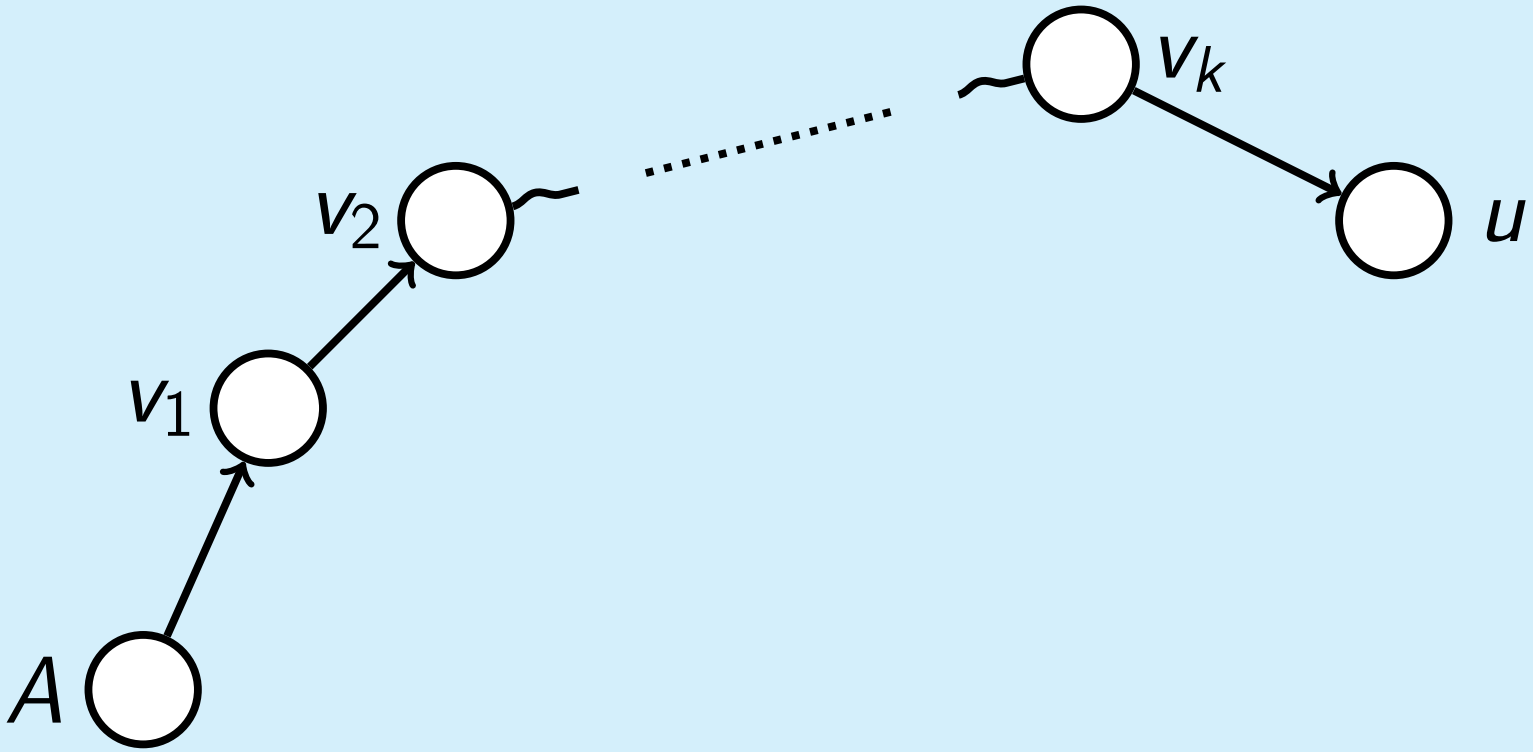
- u — reachable undiscovered closest to A

Proof



- u — reachable undiscovered closest to A
- $A - v_1 - \cdots - v_k - u$ — shortest path

Proof



- u — reachable undiscovered closest to A
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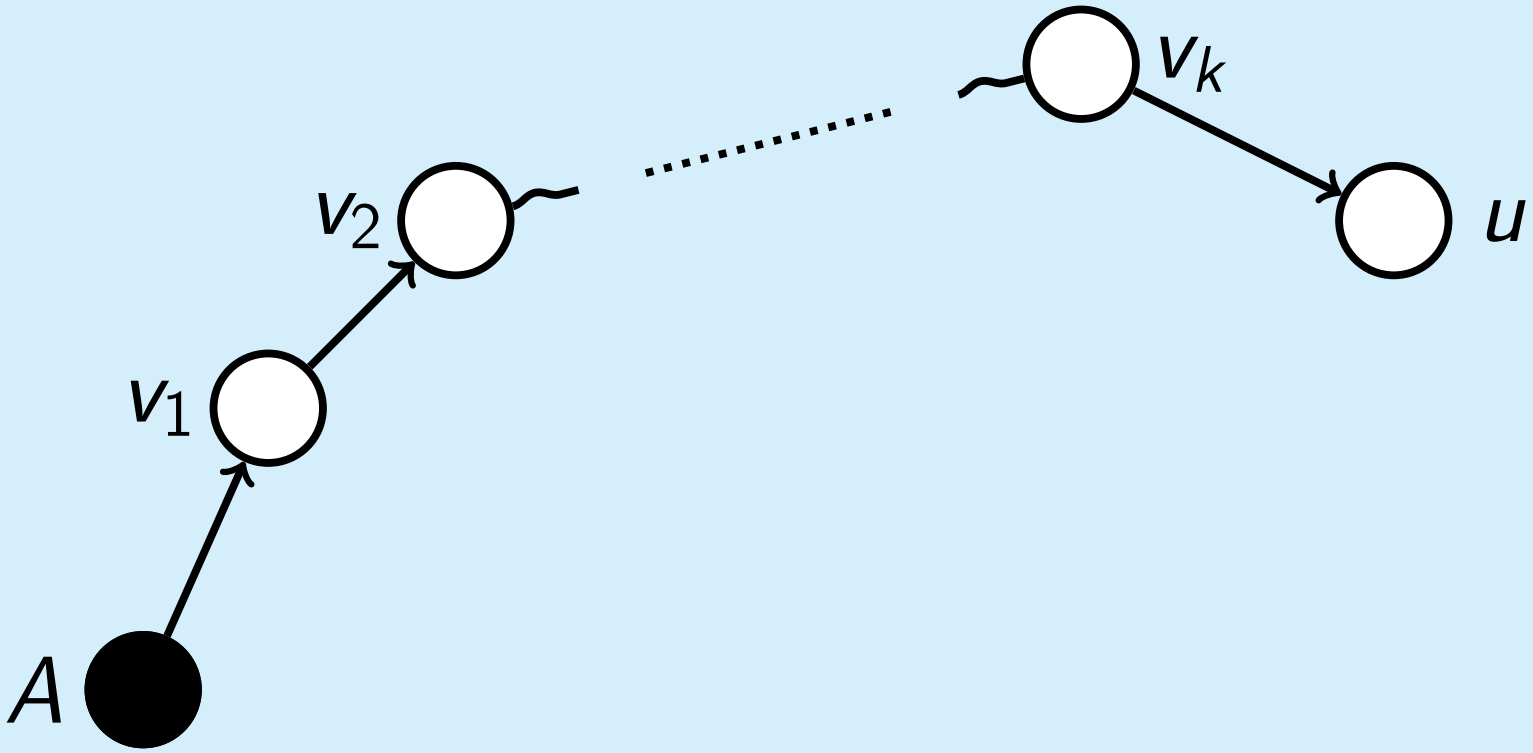
- u is discovered while processing v_k

Proof



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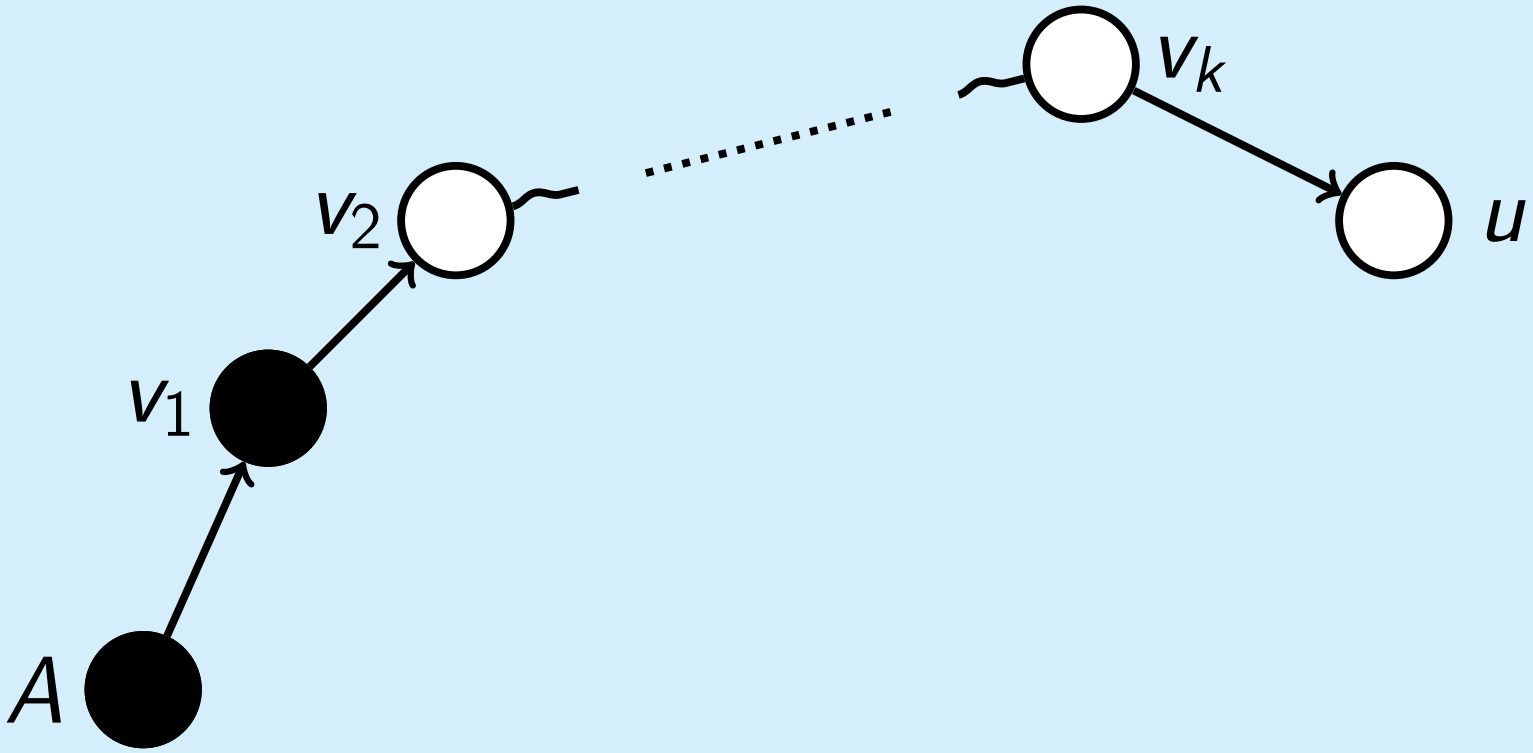

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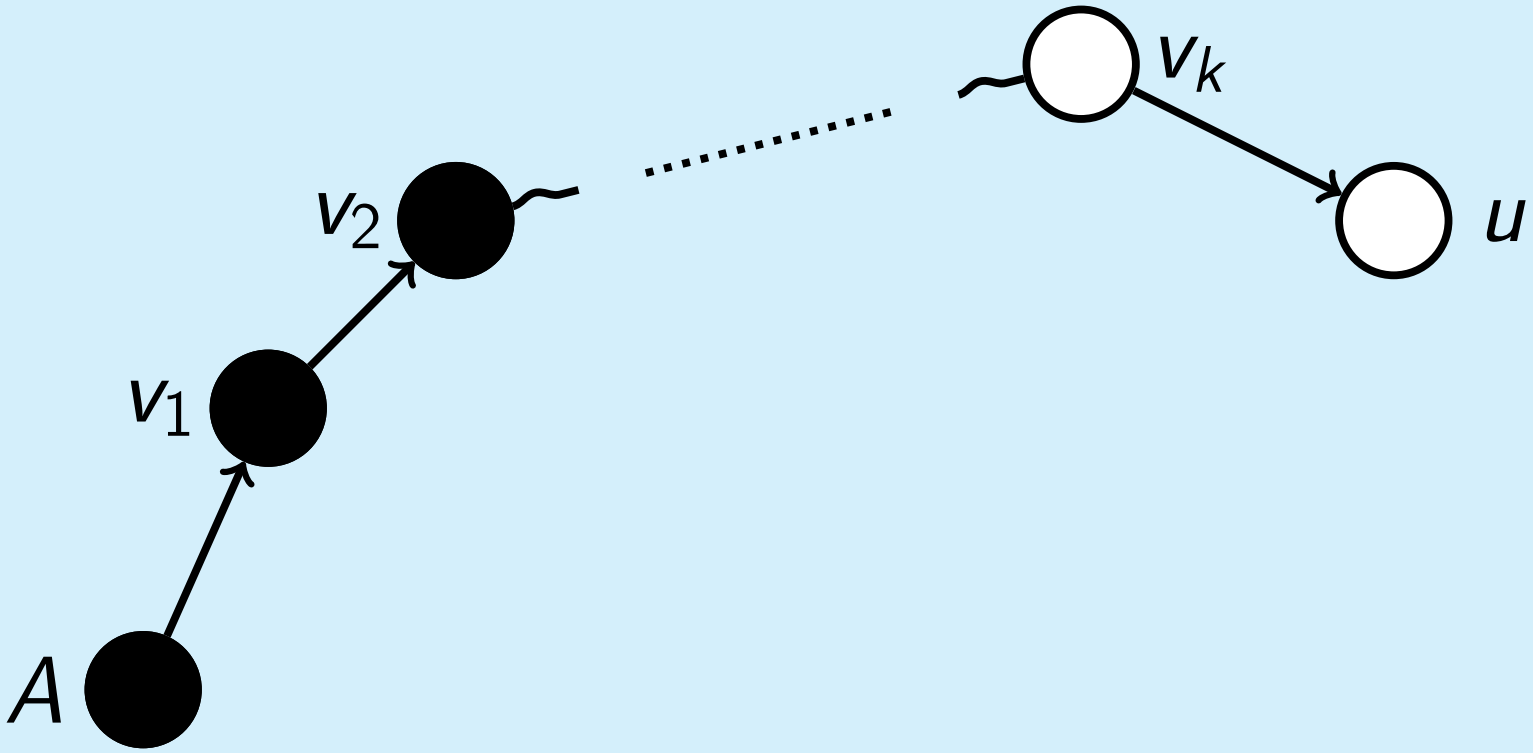

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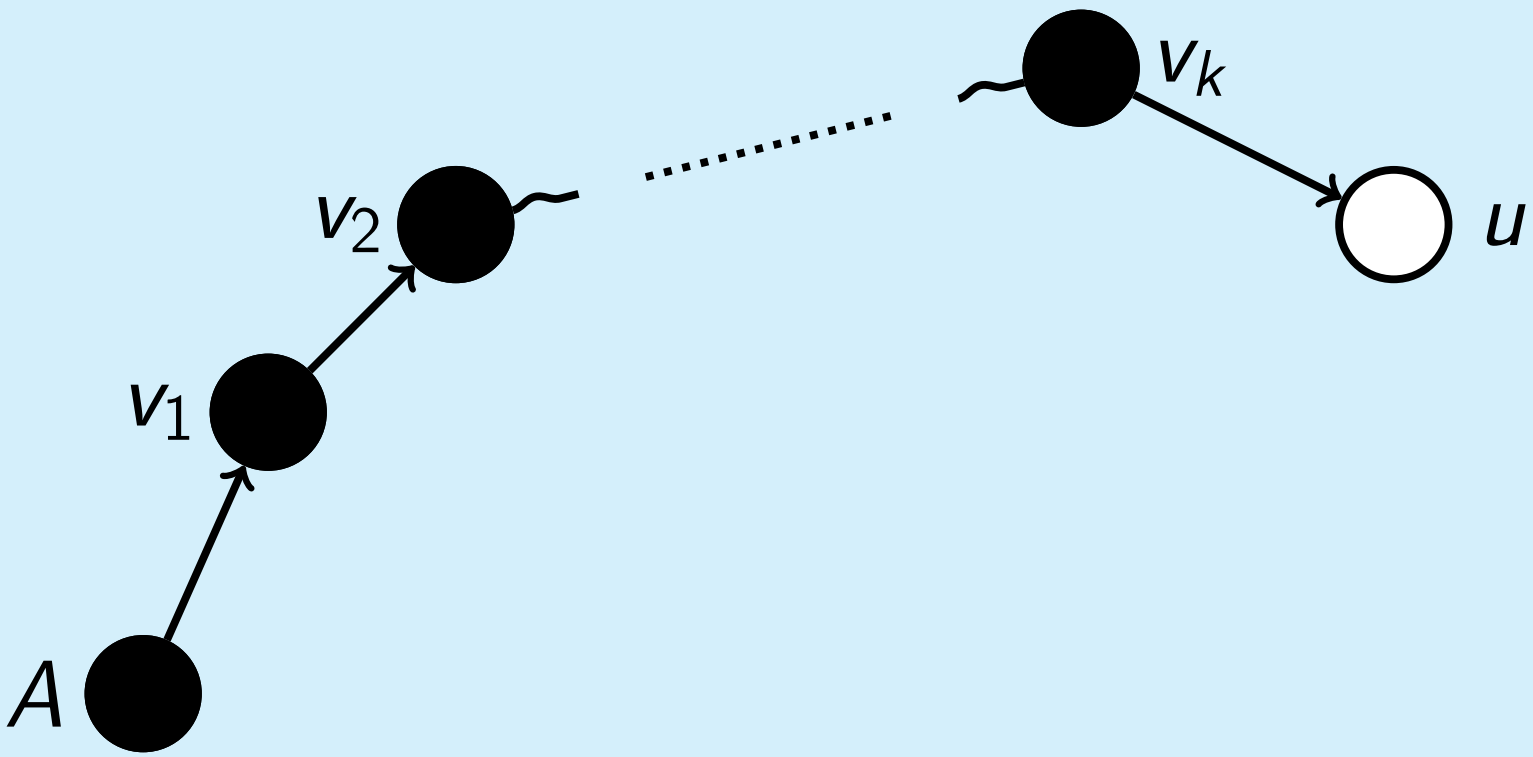

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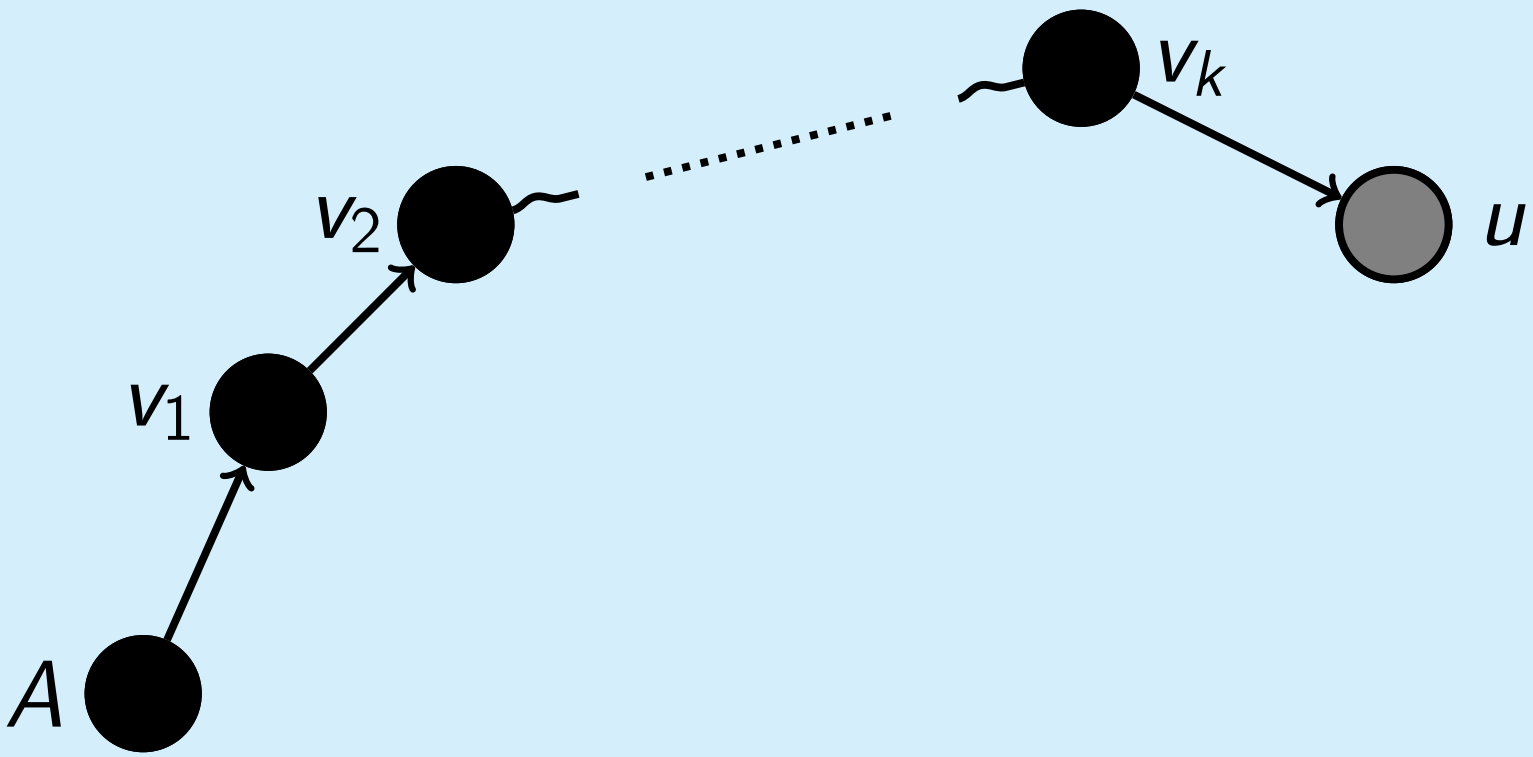

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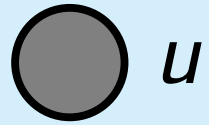
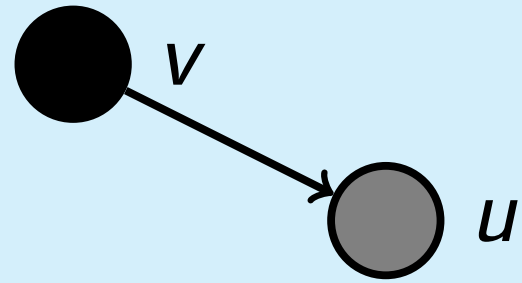


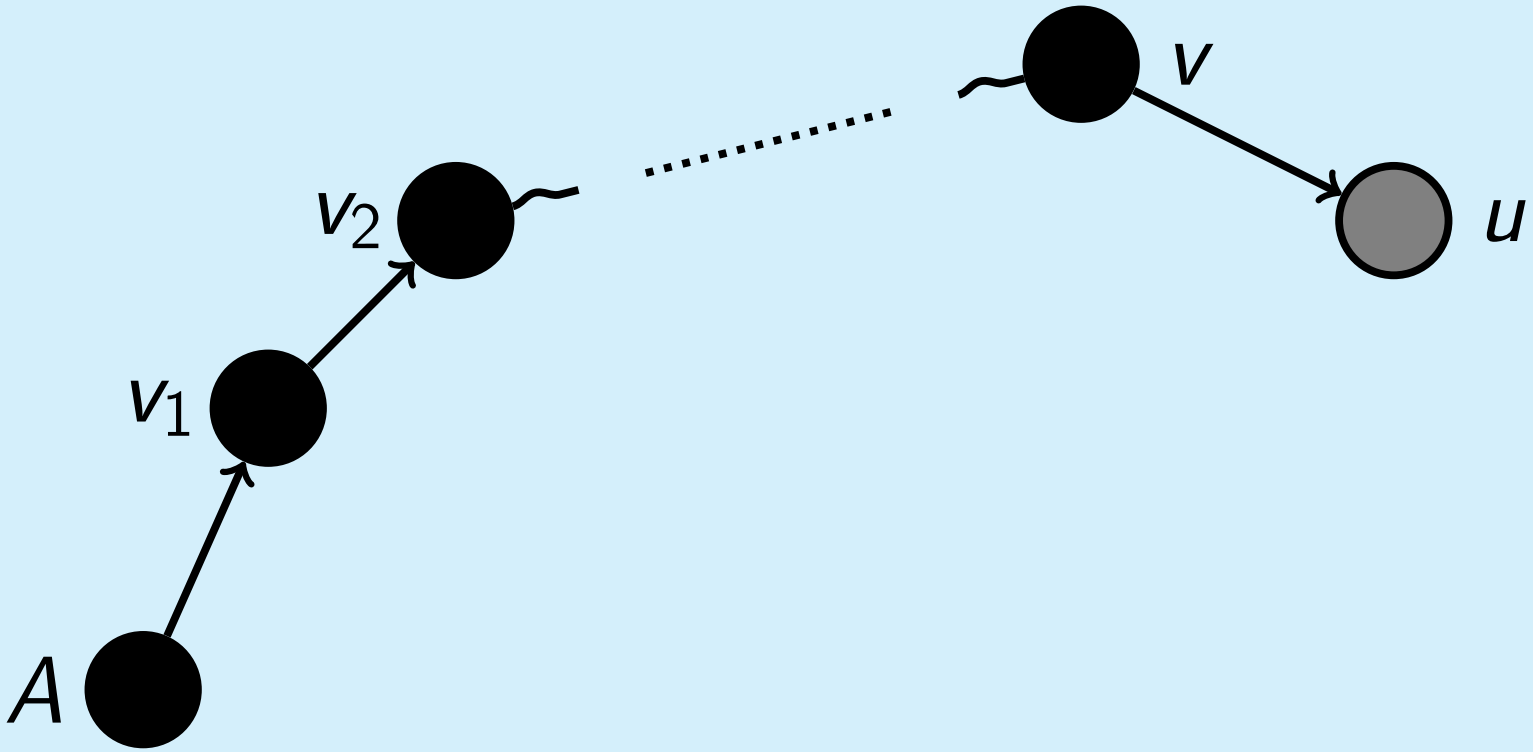
- u — first unreachable discovered

Proof



- u — first unreachable discovered
- u was discovered while processing v

Proof



- u — first unreachable discovered
- u was discovered while processing v
- u is reachable through v □

Order Lemma

Lemma

By the time node u at distance d from A is dequeued, all the nodes at distance at most d have already been discovered (enqueued).

Order Lemma Proof

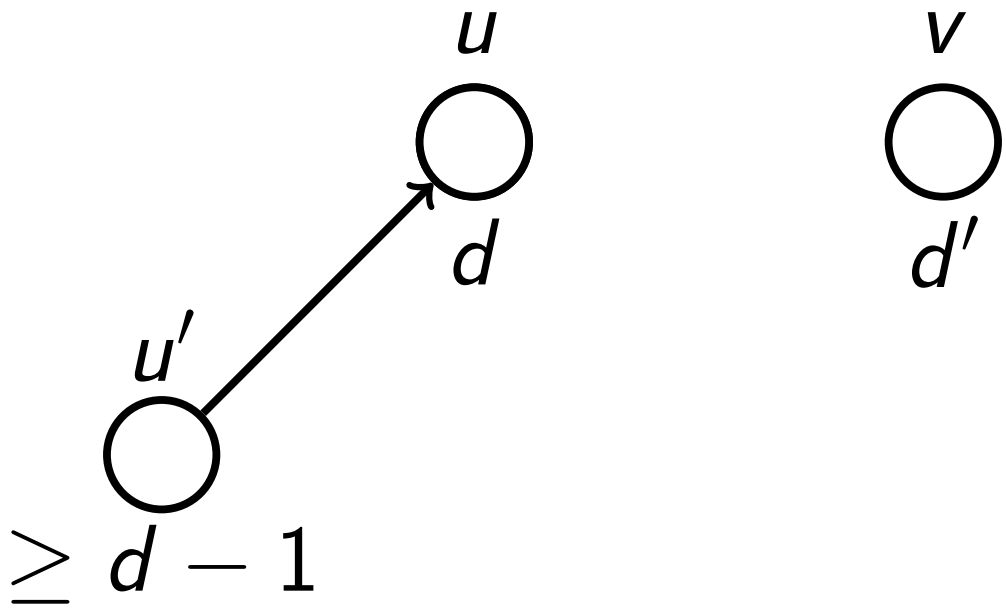
Consider the first time the order was broken

Order Lemma Proof

Consider the first time the order was broken

$d' \leq d$

Order Lemma Proof



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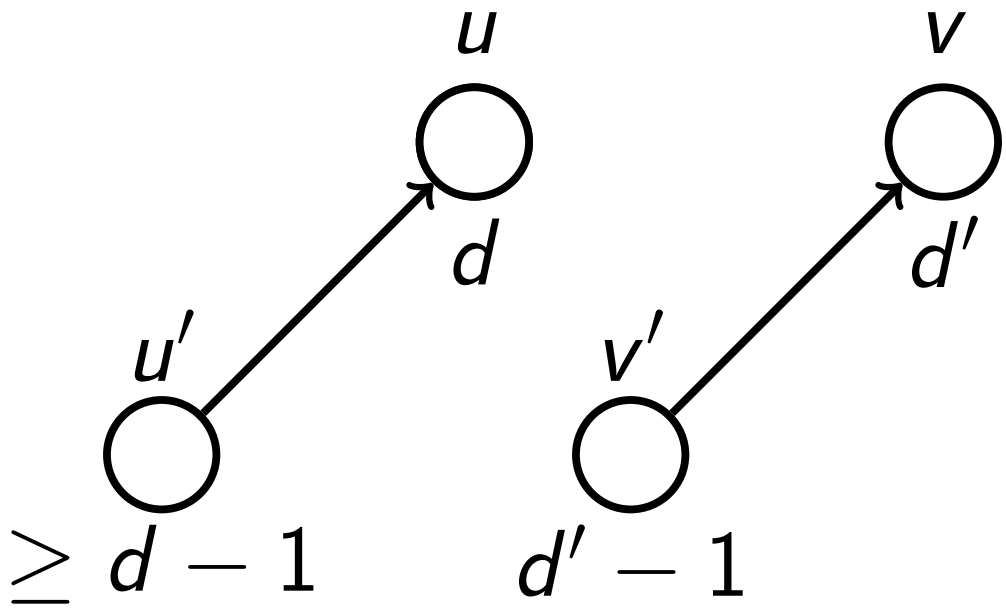

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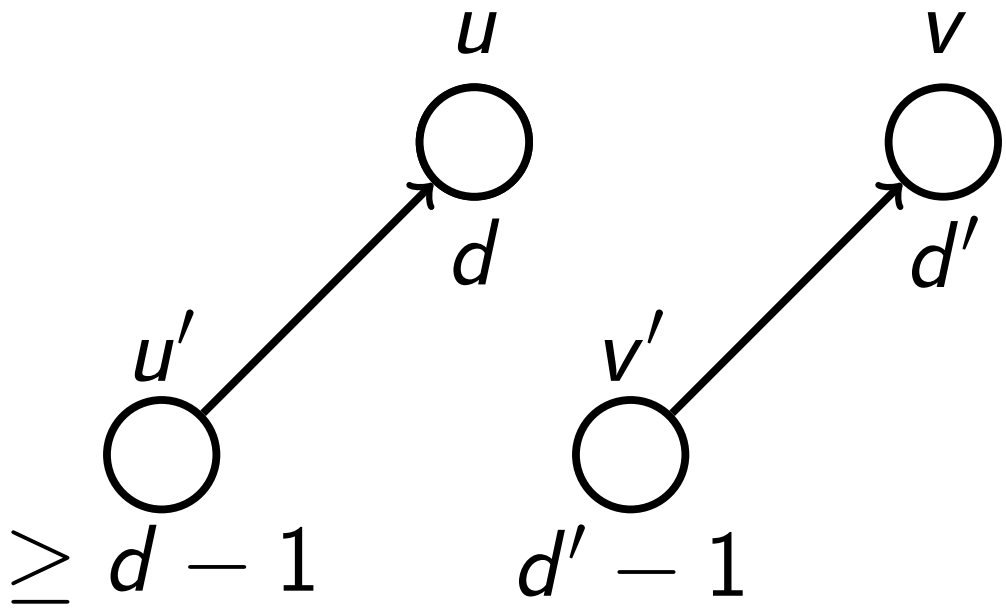
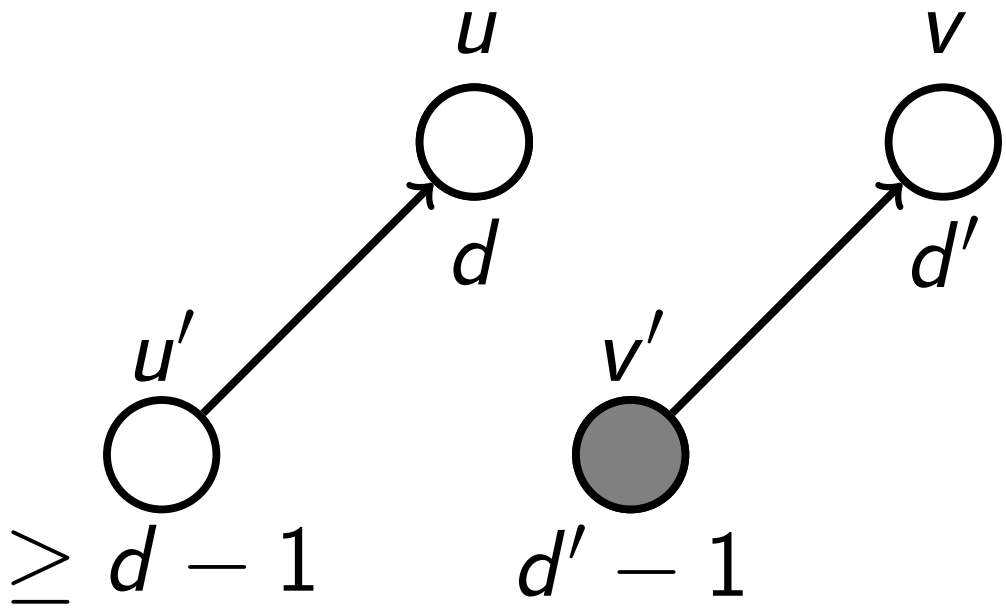


Consider the first time the order was broken

$d' \leq d \quad \Rightarrow d'-1 \leq d-1$, so v' was discovered before u' was dequeued

Order Lemma Proof



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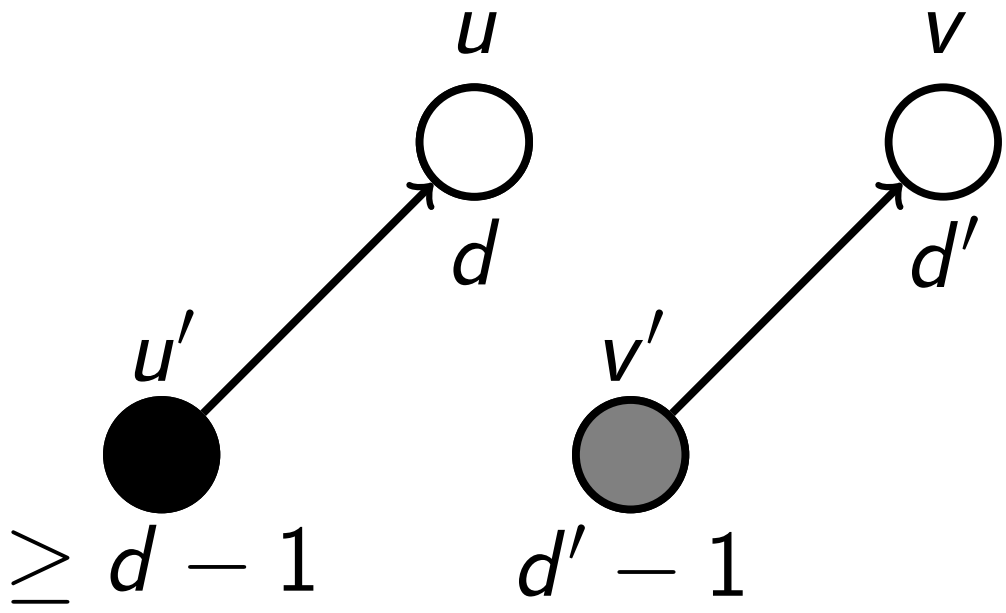
$d' \leq d \quad \Rightarrow d' - 1 \leq d - 1$, so v' was discovered before u' was dequeued

Order Lemma Proof



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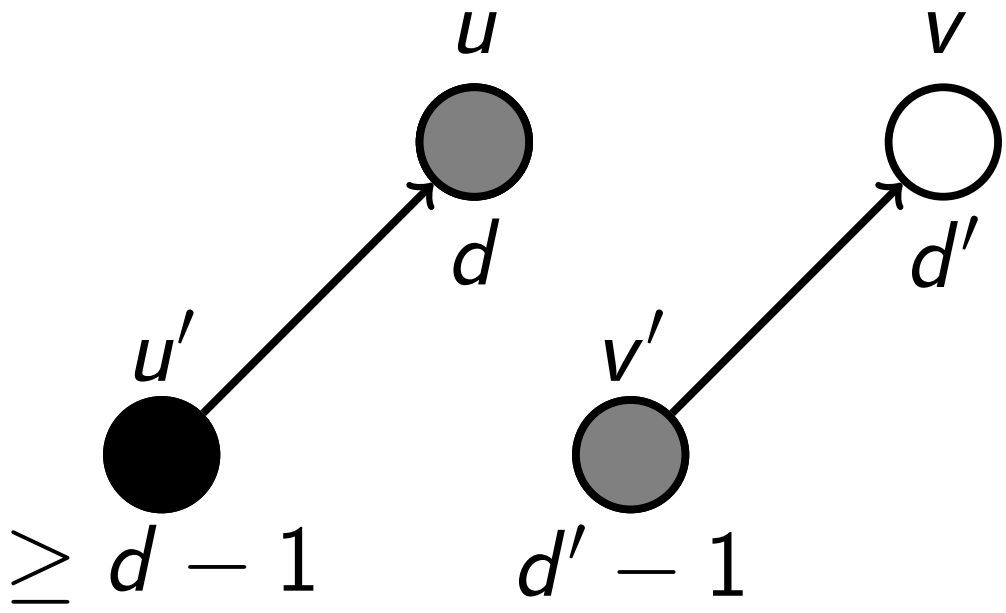

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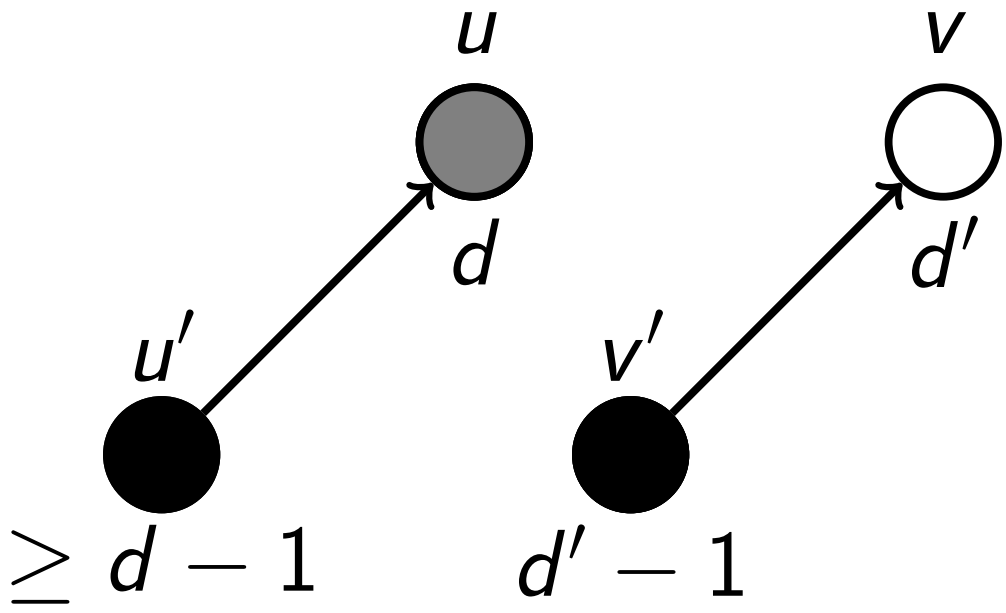

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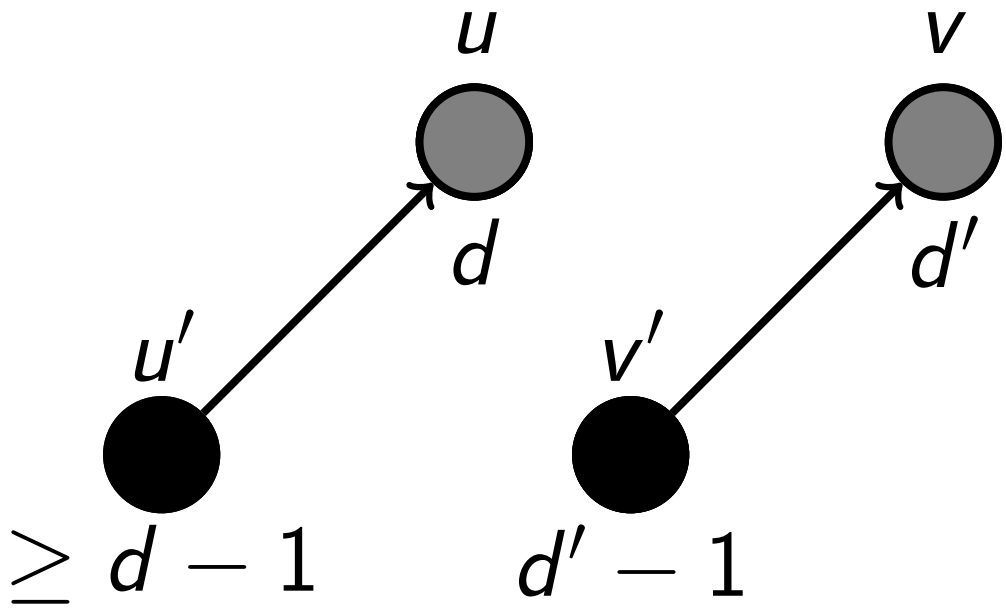

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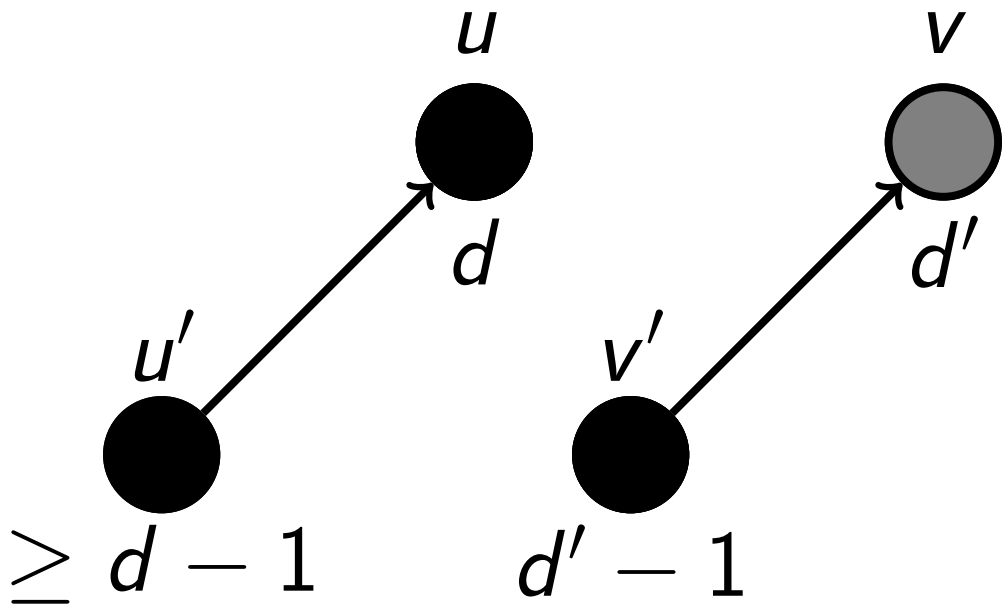

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$d' \leq d \quad \Rightarrow d'-1 \leq d-1$, so v' was discovered before u' was dequeued

Queue property

Queue:

d	d	d	$\ldots$	d	d	$d+1$	$d+1$	$\ldots$	$d+1$

Lemma

At any moment, if the first node in the queue is at distance d from A, then all the nodes in the queue are either at distance d from A or at distance $d+1$ from A. All the nodes in the queue at distance d go before (if any) all the nodes at distance $d+1$.

Queue property

Proof

- All nodes at distance d were enqueued before first such node is dequeued, so they go before nodes at distance $d + 1$

Queue property

Proof

- All nodes at distance d were enqueued before first such node is dequeued, so they go before nodes at distance $d + 1$
- Nodes at distance $d - 1$ were enqueued before nodes at d, so they are not in the queue anymore

Queue property

Proof

- All nodes at distance d were enqueued before first such node is dequeued, so they go before nodes at distance $d + 1$
- Nodes at distance $d - 1$ were enqueued before nodes at d, so they are not in the queue anymore
- Nodes at distance $> d + 1$ will be discovered when all d are gone

Outline

Correct distances

Lemma

When node u is discovered (enqueued), `dist[u]` is assigned exactly $d(A, u)$.

Correct distances

Proof

- Use mathematical induction

Correct distances

Proof

- Use mathematical induction
- Base: when A is discovered, `dist[A]` is assigned $0 = d(A, A)$

Correct distances

Proof

- Use mathematical induction
- Base: when A is discovered, `dist[A]` is assigned $0 = d(A, A)$
- Inductive step: suppose proved for all nodes at distance $\leq k$ from $A \rightarrow$ prove for nodes at distance $k + 1$

Correct distances

Proof

- Take a node v at distance $k+1$ from A

Correct distances

Proof

- Take a node v at distance $k+1$ from A
- v was discovered while processing u

Correct distances

Proof

- Take a node v at distance $k+1$ from A
- v was discovered while processing u
- $d(A,v) \leq d(A,u)+1 \Rightarrow d(A,u) \geq k$

Correct distances

Proof

- Take a node v at distance $k+1$ from A
- v was discovered while processing u
- $d(A,v) \leq d(A,u)+1 \Rightarrow d(A,u) \geq k$
- v is discovered after u is dequeued, so $d(A,u) < d(A,v) = k+1$

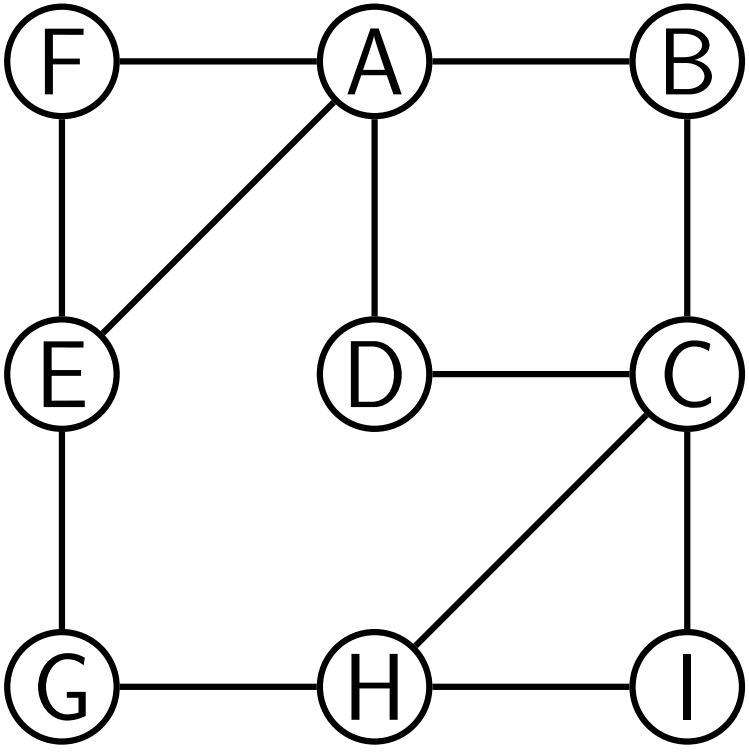
Correct distances

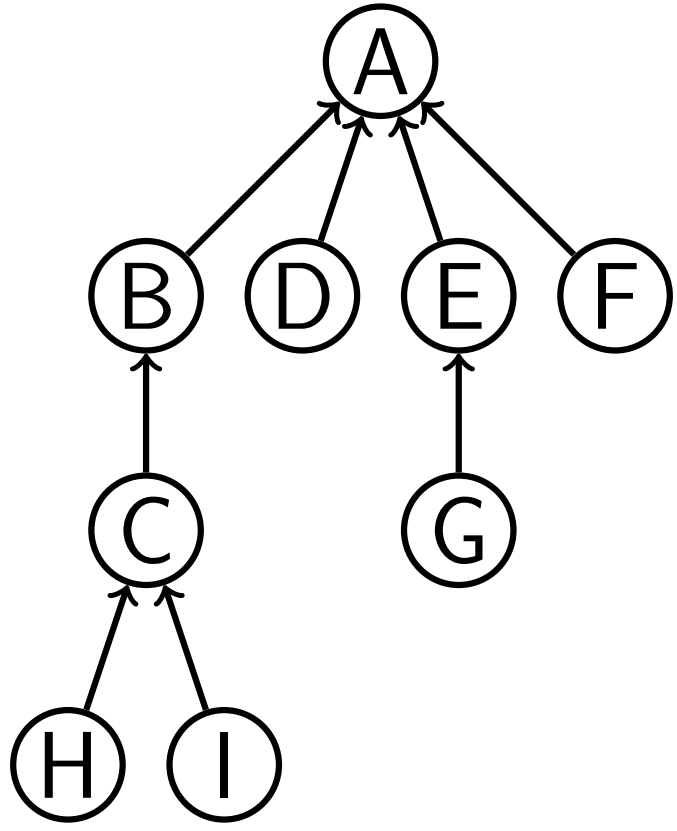
Proof

- Take a node v at distance $k+1$ from A
- v was discovered while processing u
- $d(A,v) \leq d(A,u)+1 \Rightarrow d(A,u) \geq k$
- v is discovered after u is dequeued, so $d(A,u) < d(A,v) = k+1$
- So $d(A,u) = k$, and $\text{dist}[v] \leftarrow \text{dist}[u]+1 = k+1$ □

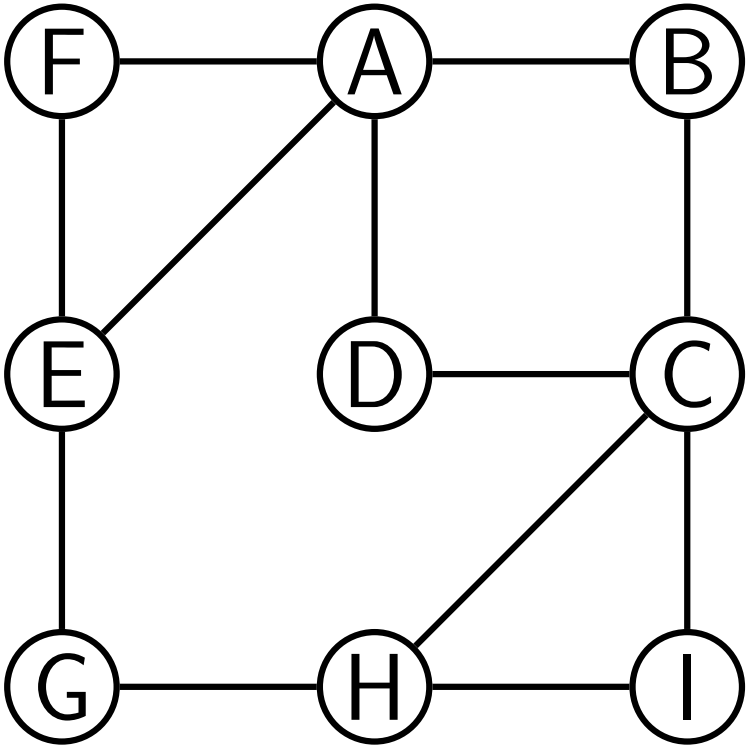
Outline

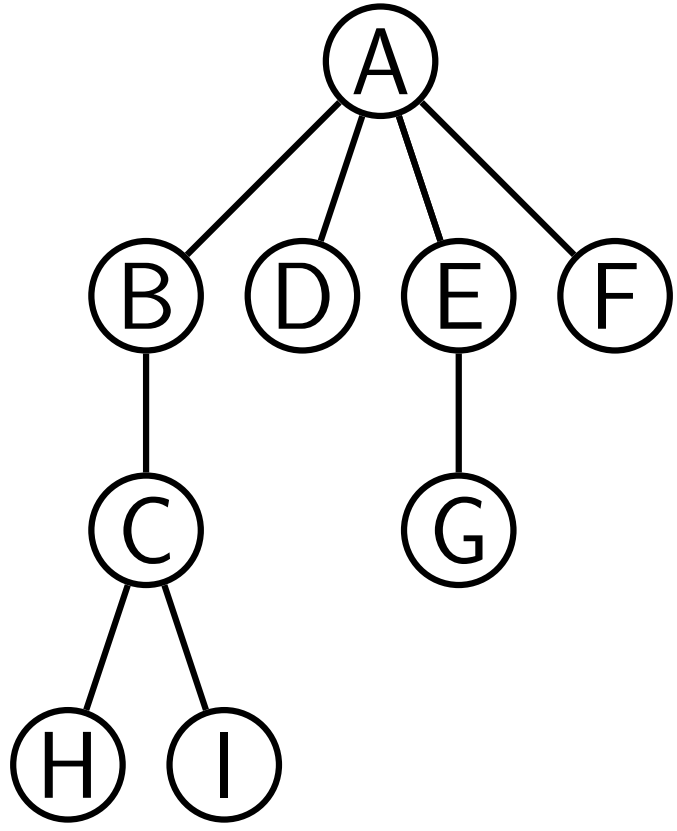
Shortest-path tree





Shortest-path tree

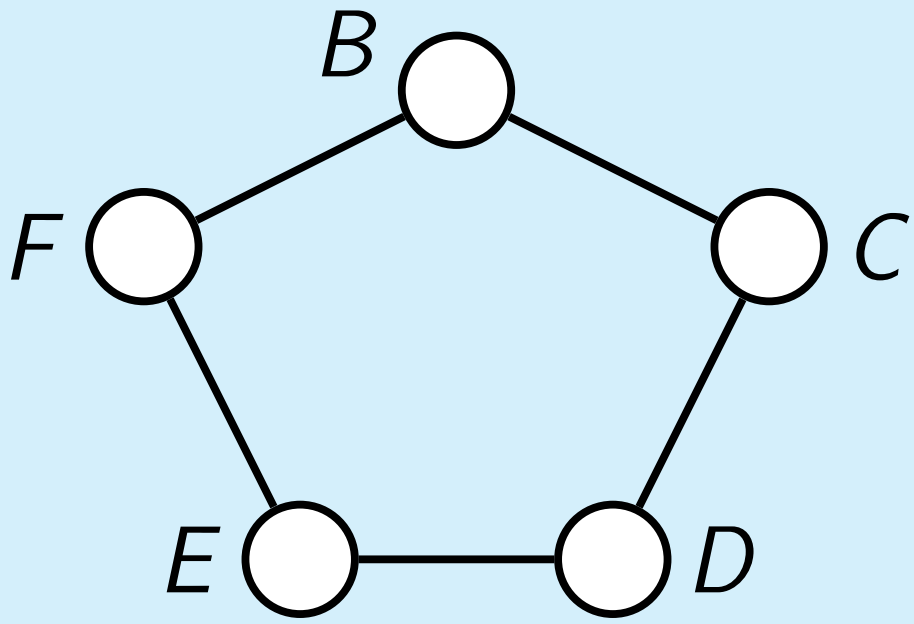




Lemma

Shortest-path tree is indeed a tree, i.e. it doesn't contain cycles (it is a connected component by construction).

Proof



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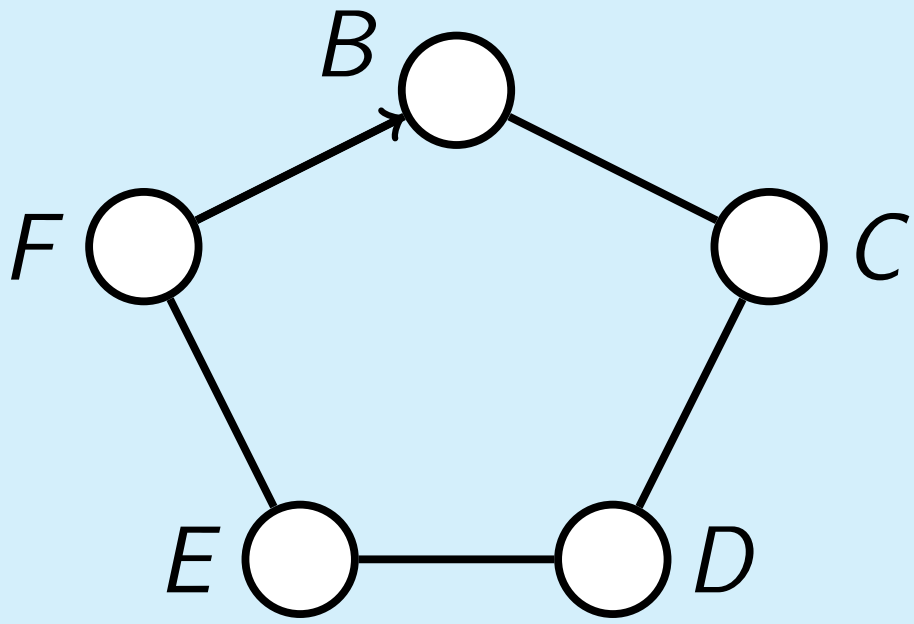

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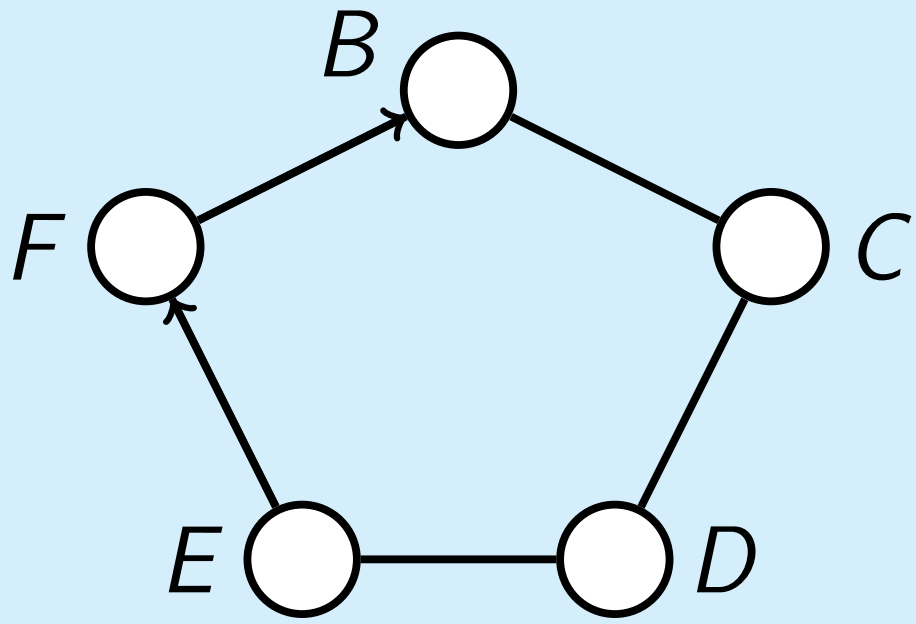
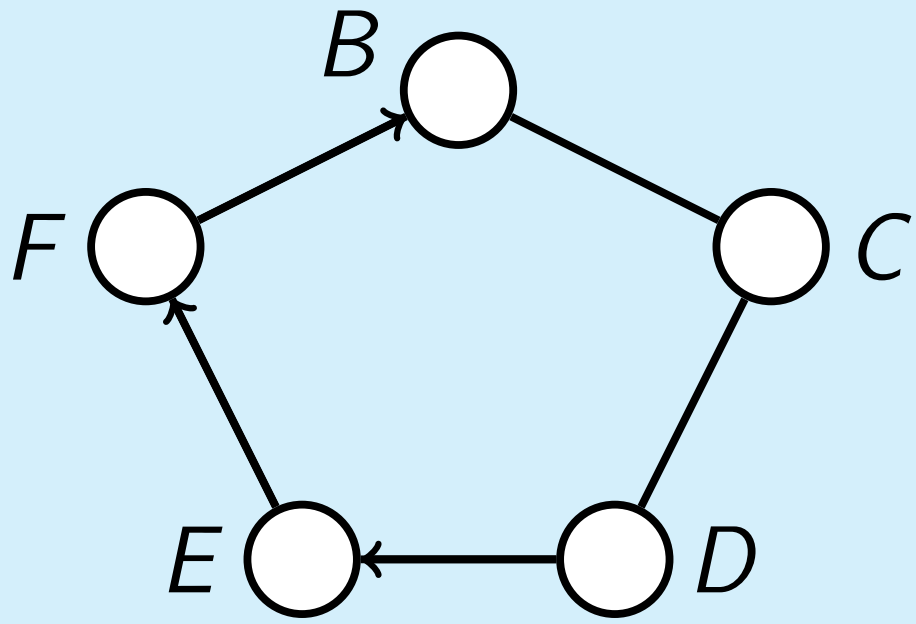


- Only one outgoing edge from each node

Proof



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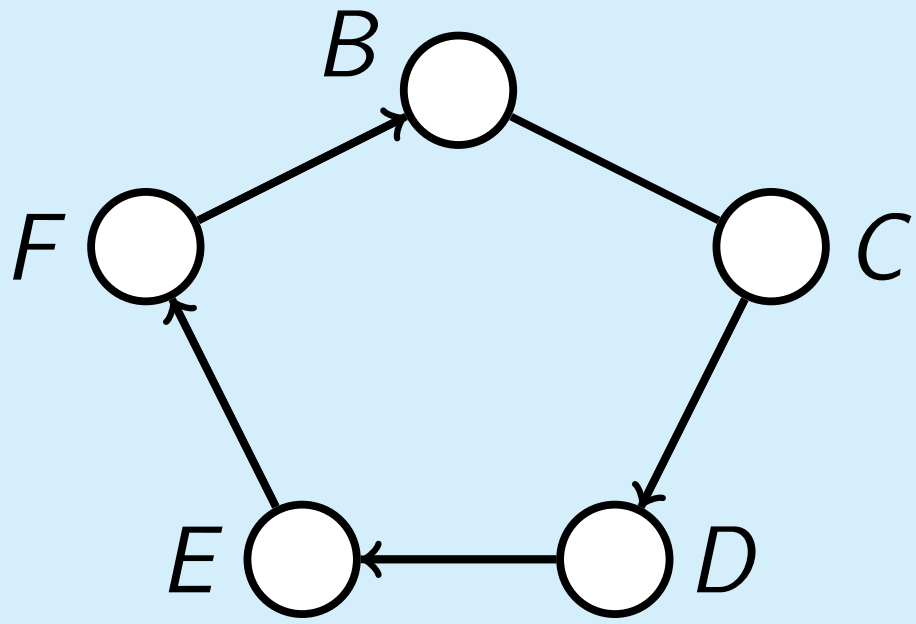

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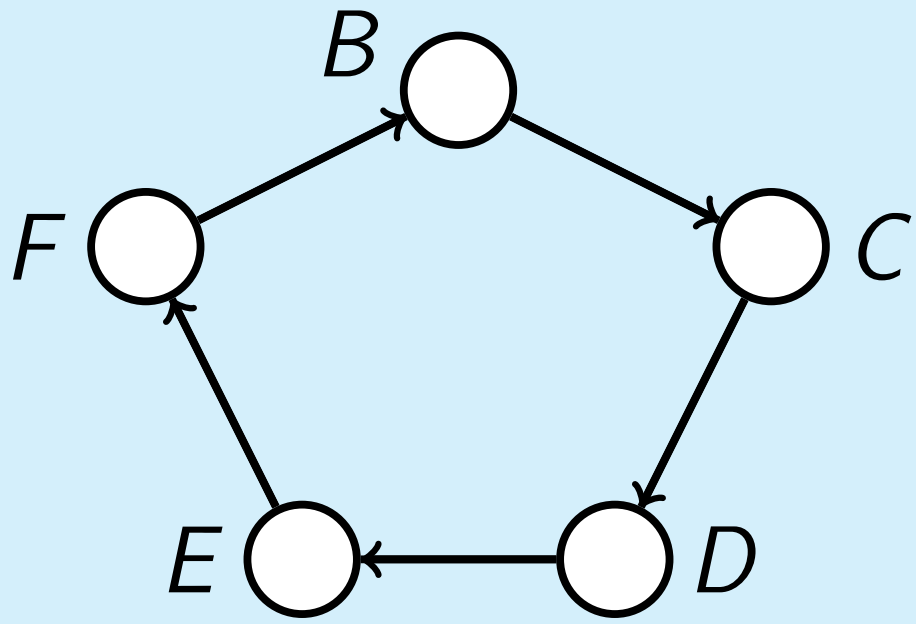

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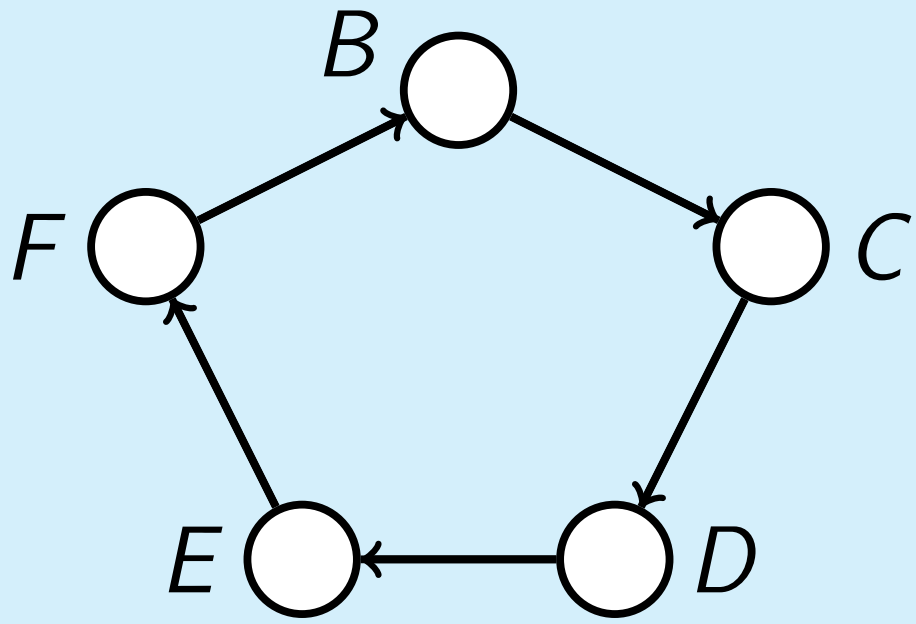
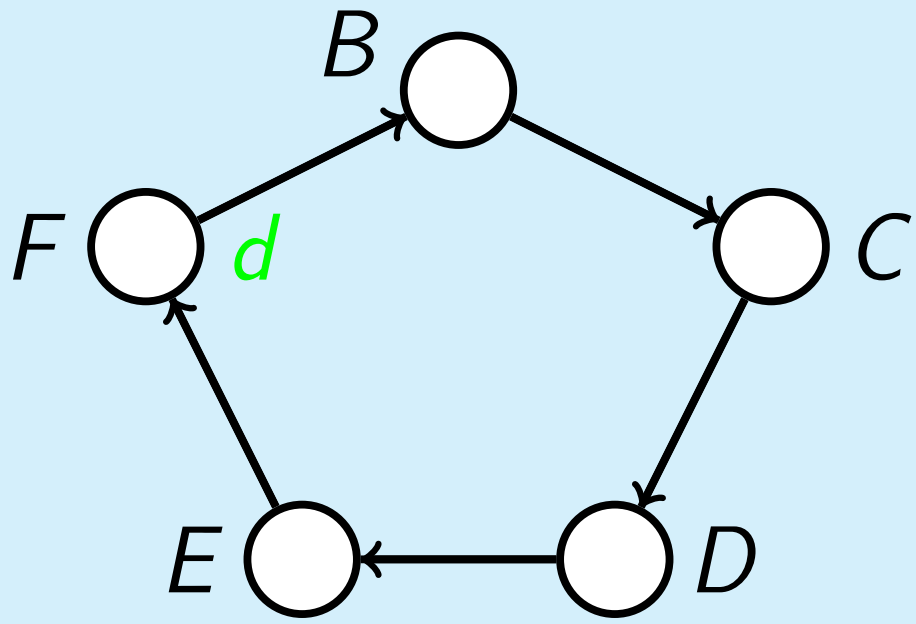


- Only one outgoing edge from each node
- Distance decreases after going by edge

Proof



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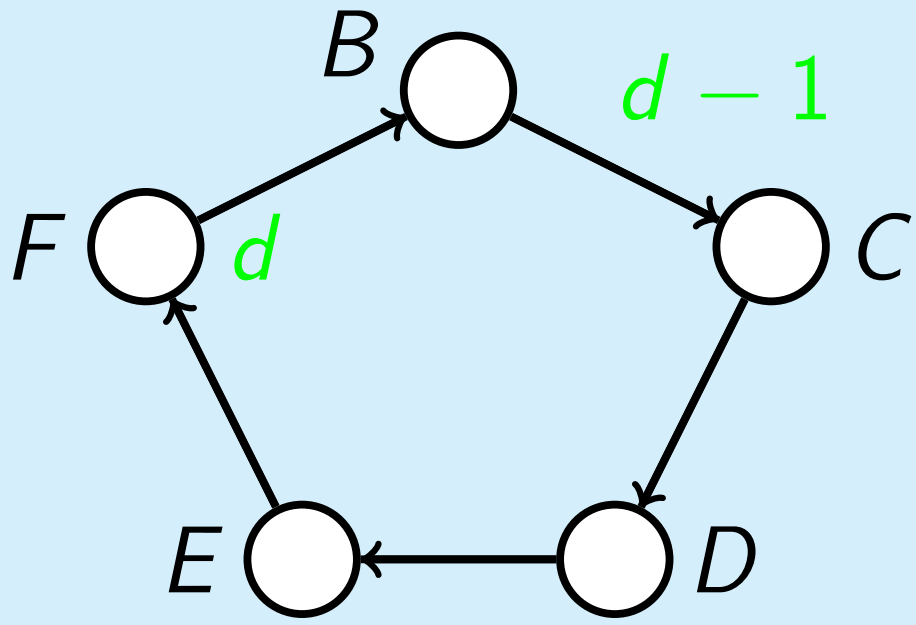
Proof



- Only one outgoing edge from each node
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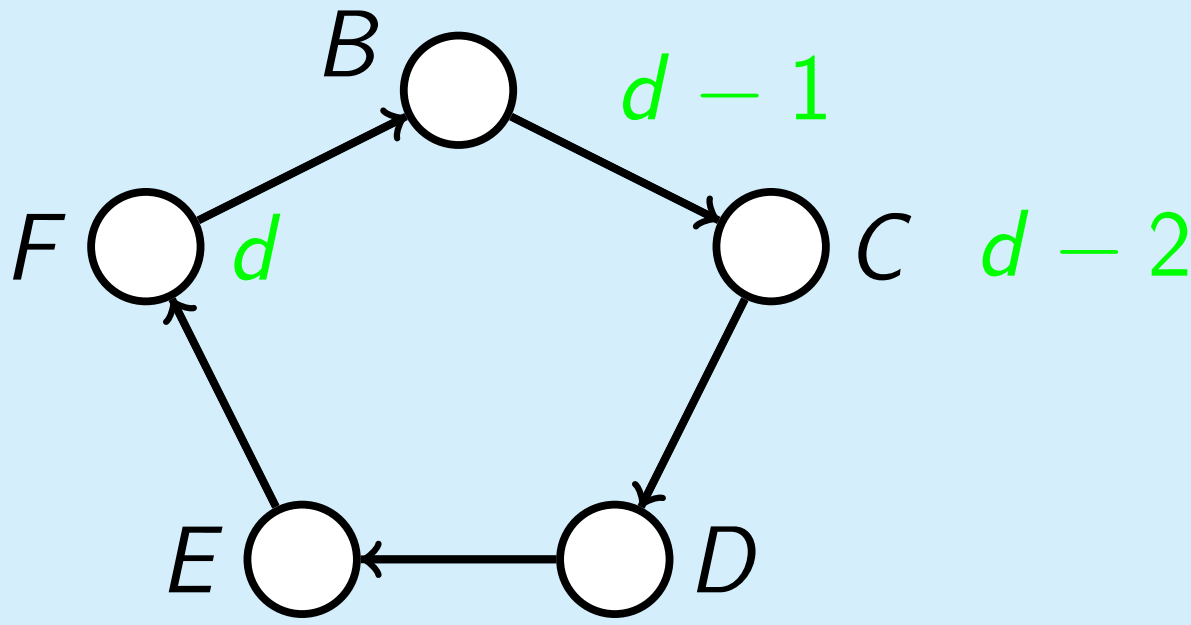

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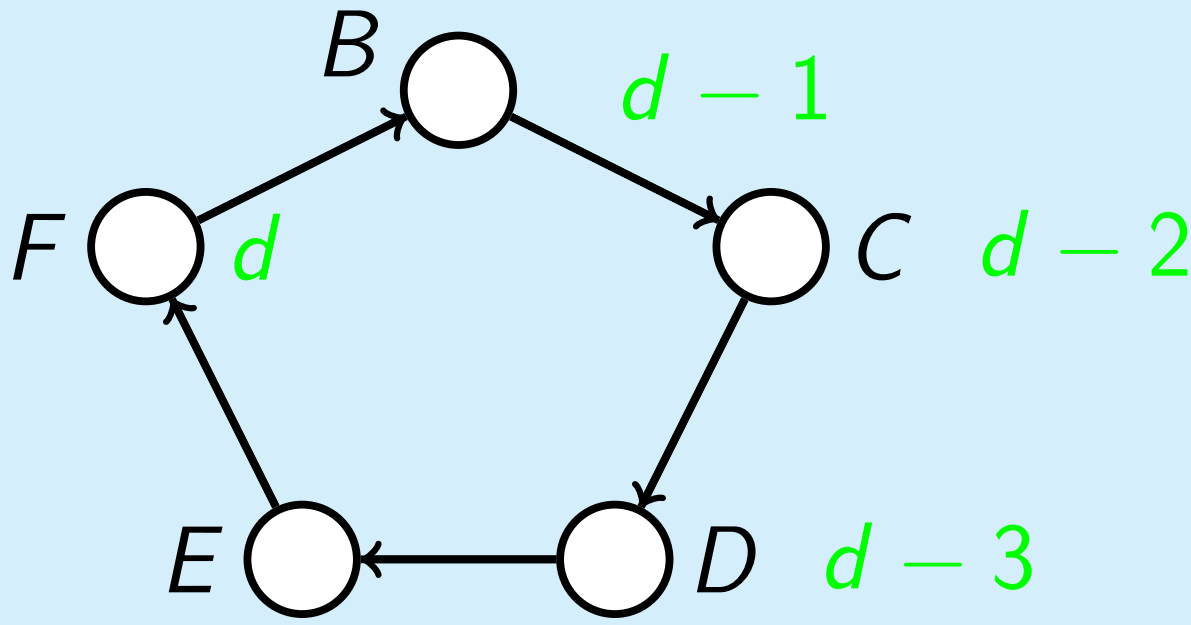

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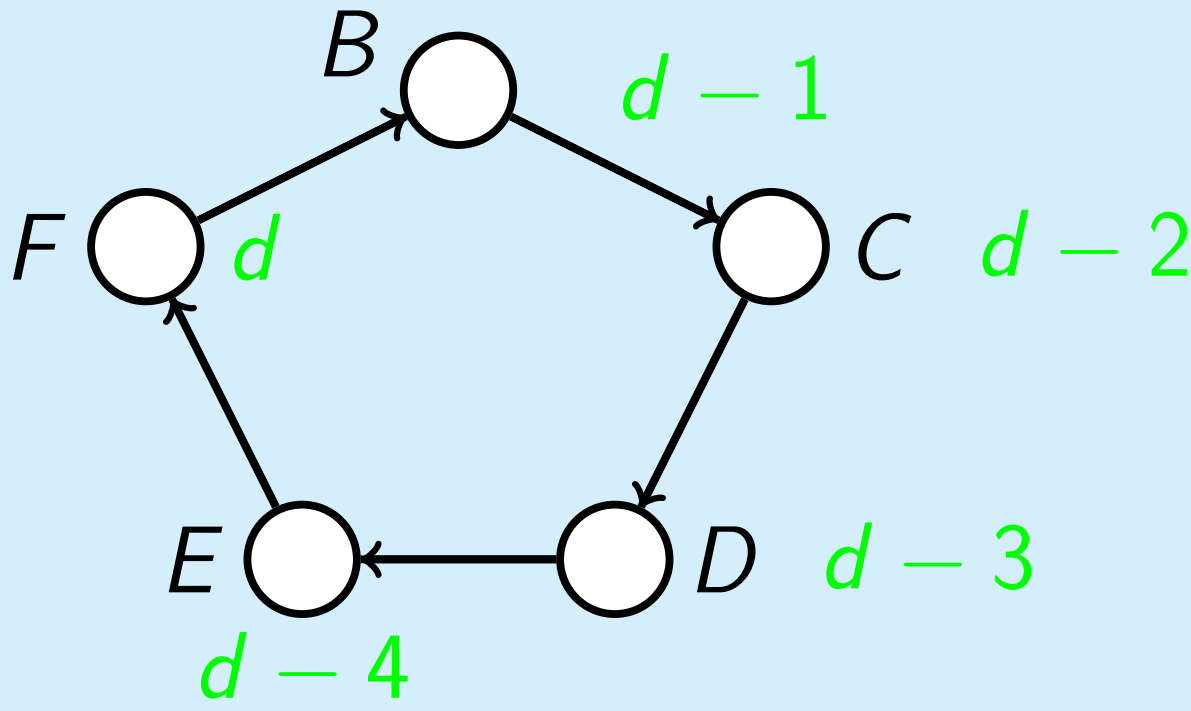

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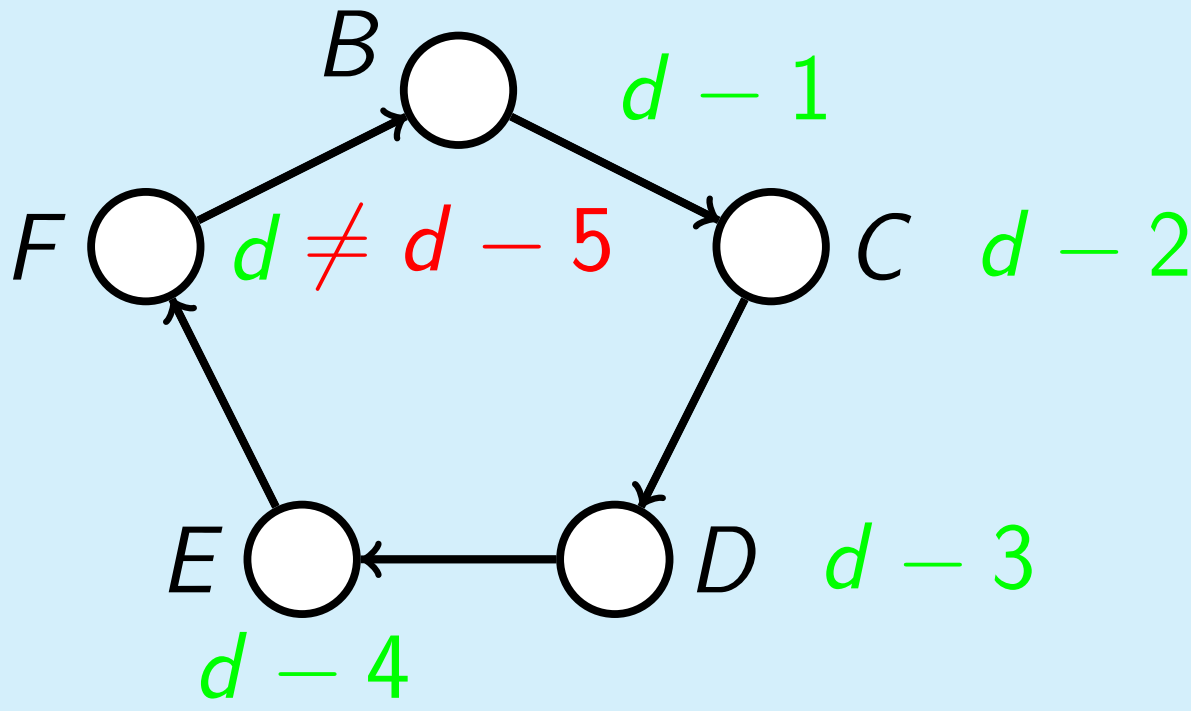

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Constructing shortest-path tree

BFS(G, A)

```
for all u ∈ V:
  dist[u] ← ∞, prev[u] ← nil
dist[A] ← 0
Q ← {A} {queue containing just A}
while Q is not empty:
  u ← Dequeue(Q)
  for all (u, v) ∈ E:
    if dist[v] = ∞:
      Enqueue(Q, v)
      dist[v] ← dist[u] + 1, prev[v] ← u
```

Reconstructing Shortest Path

ReconstructPath($A, u,$ prev)

```
result ← empty
while u ≠ A:
    result.append(u)
    u ← prev[u]
return Reverse(result)
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- Works in $O(|E| + |V|)$