

# Decomposition of Graphs: Directed Acyclic Graphs

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Graph Algorithms  
Data Structures and Algorithms

# Learning Objectives

- Understand the difference between directed and undirected graphs.
- Prove that graphs with cycles cannot be linearly ordered.

# Outline

- 1 Motivation
- 2 Event Ordering
- 3 DAGs

# Directed Graphs

Sometimes we want the edges of a graph to have a direction.

## Definition

A **directed graph** is a graph where each edge has a start vertex and an end vertex.

# Examples

Directed graphs might be used to represent:

- Streets with one-way roads.
- Links between webpages.
- Followers on social network.
- Dependencies between tasks.

# Directed DFS

Can still run DFS in directed graphs.

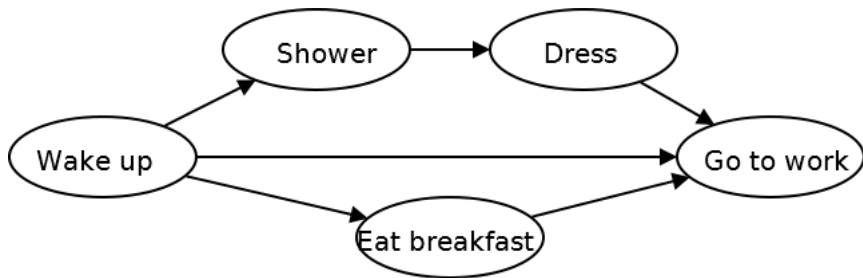
- Only follow **directed** edges.
- `explore( $v$ )` finds all vertices **reachable** from  $v$ .
- Can still compute pre- and post-orderings.

# Outline

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# Morning Routine

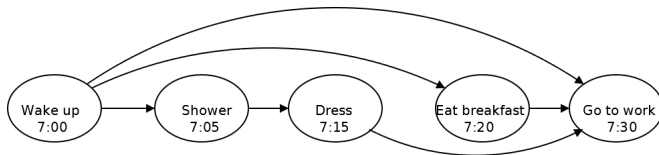
The following morning tasks must be performed some before others.





# Linear Ordering

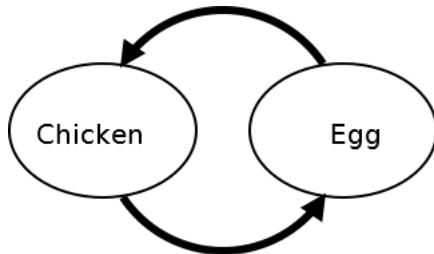
We would like to order tasks to respect dependencies as below.



Is it always possible to do this?

# Example

No!



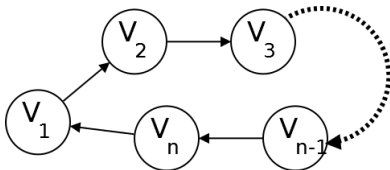
# Outline

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# Cycles

## Definition

A **cycle** in a graph  $G$  is a sequence of vertices  $v_1, v_2, \dots, v_n$  so that  $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)$  are all edges.



# Cycles

## Theorem

If  $G$  contains a cycle, it cannot be linearly ordered.

# Proof

## Proof.

- Has cycle  $v_1, \dots, v_n$ .
- Suppose linearly ordered.
- Suppose  $v_k$  comes first.
- Then  $v_k$  comes **before**  $v_{k-1}$ , contradiction.



# DAGs

## Definition

A directed graph  $G$  is a **Directed Acyclic Graph** (or DAG) if it has no cycles.

# DAGs

## Definition

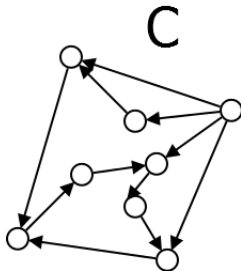
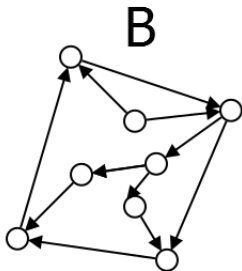
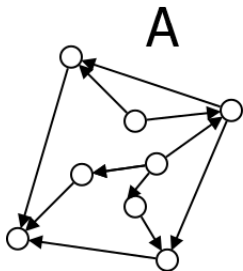
A directed graph  $G$  is a **Directed Acyclic Graph** (or DAG) if it has no cycles.

By the above being a DAG is necessary to linearly order. Is it sufficient?



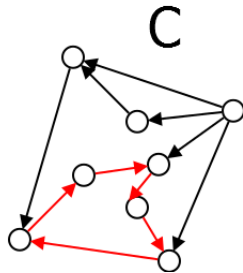
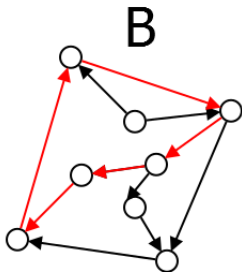
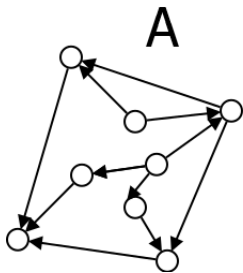
# Problem

Which of the following graphs is a DAG?



# Solution

A.



# Theorem

## Theorem

Any DAG can be linearly ordered.

# Next Time

- Prove Theorem
- Develop algorithm