Paths in Graphs: Dijkstra's Algorithm

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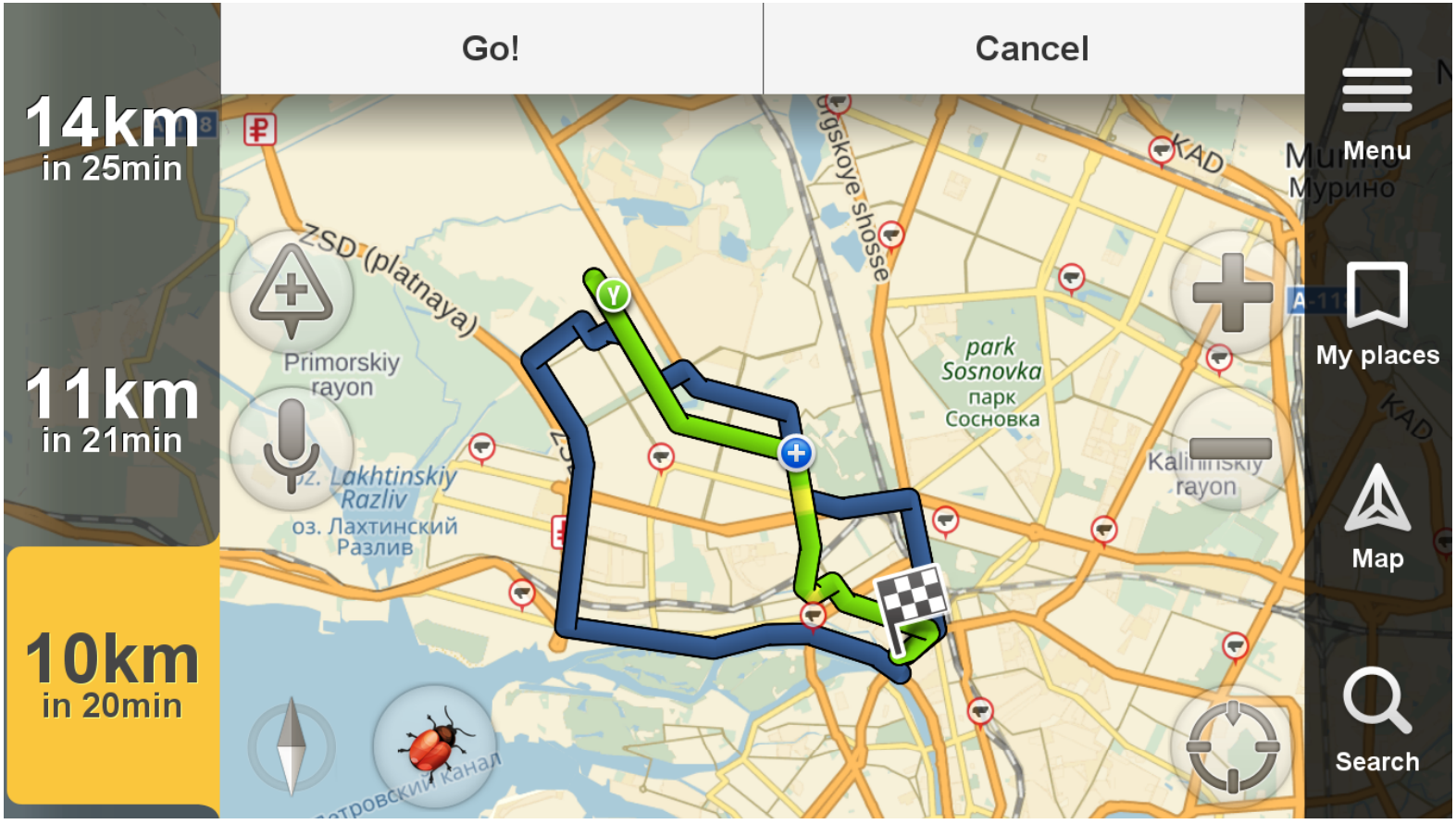
University of California, San Diego

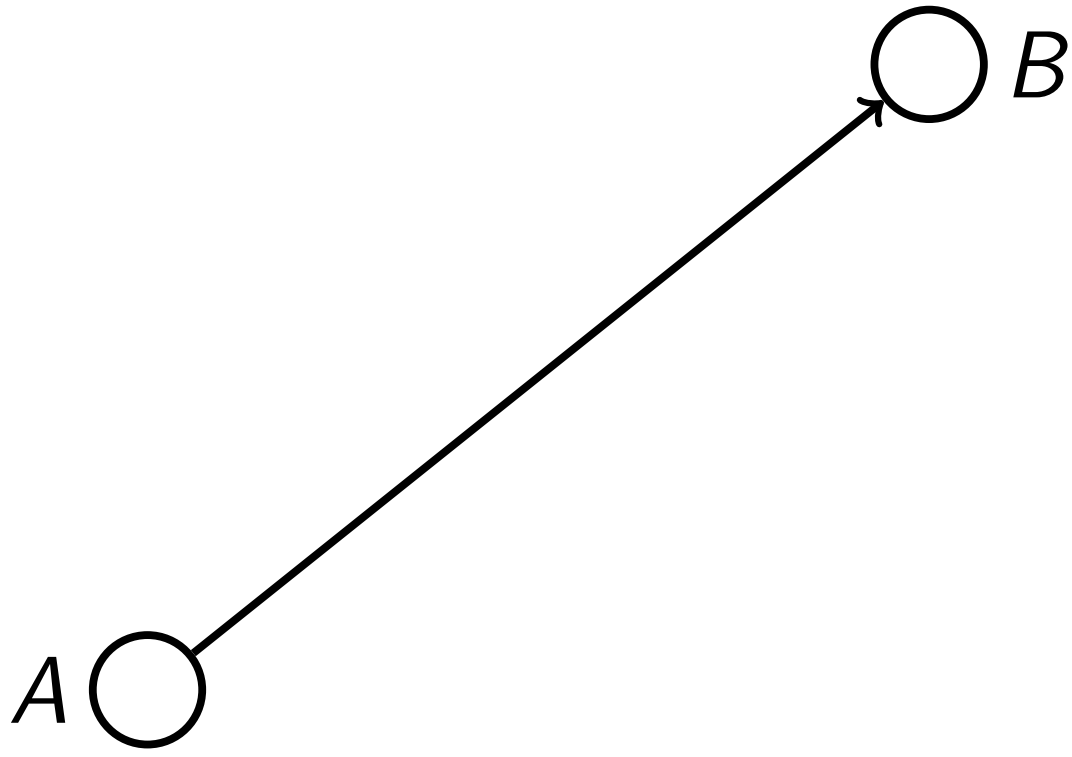
Graph Algorithms
Algorithms and Data Structures

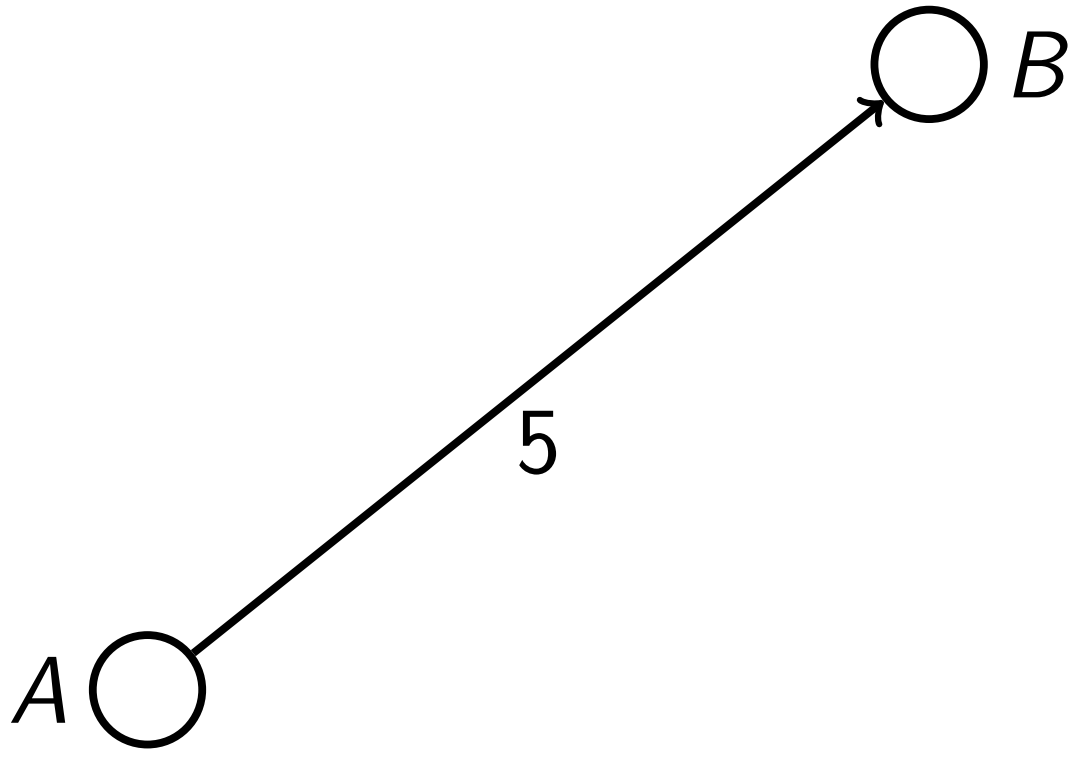
Outline

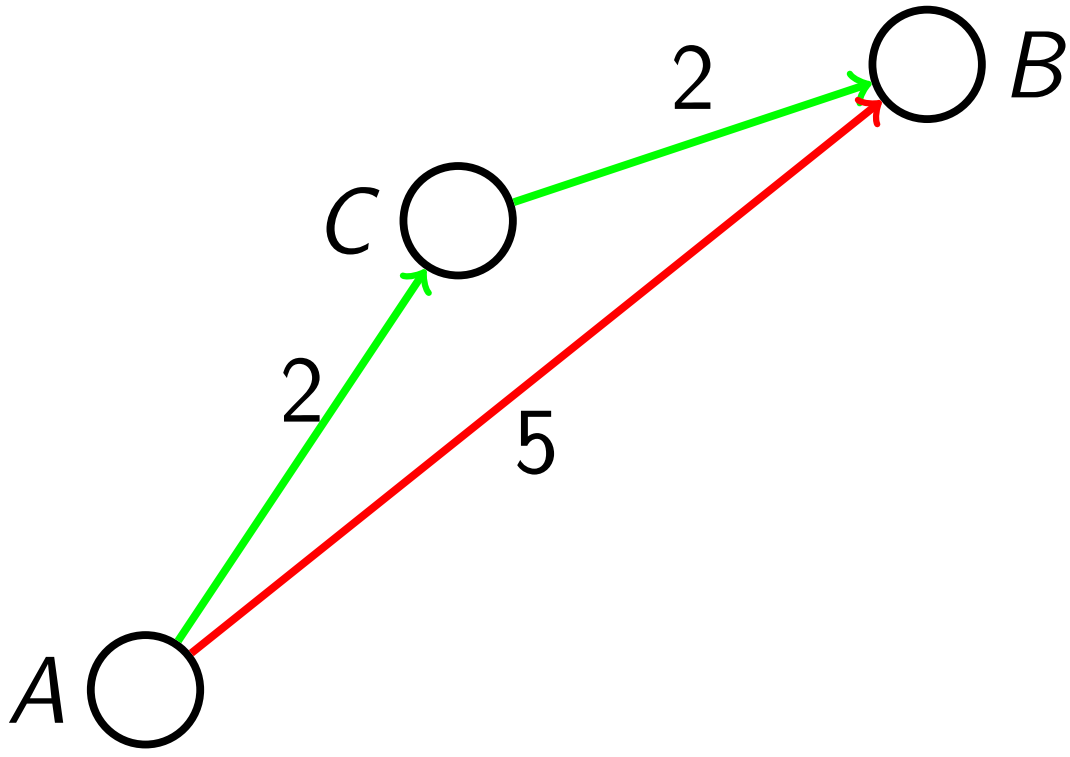
Fastest Route

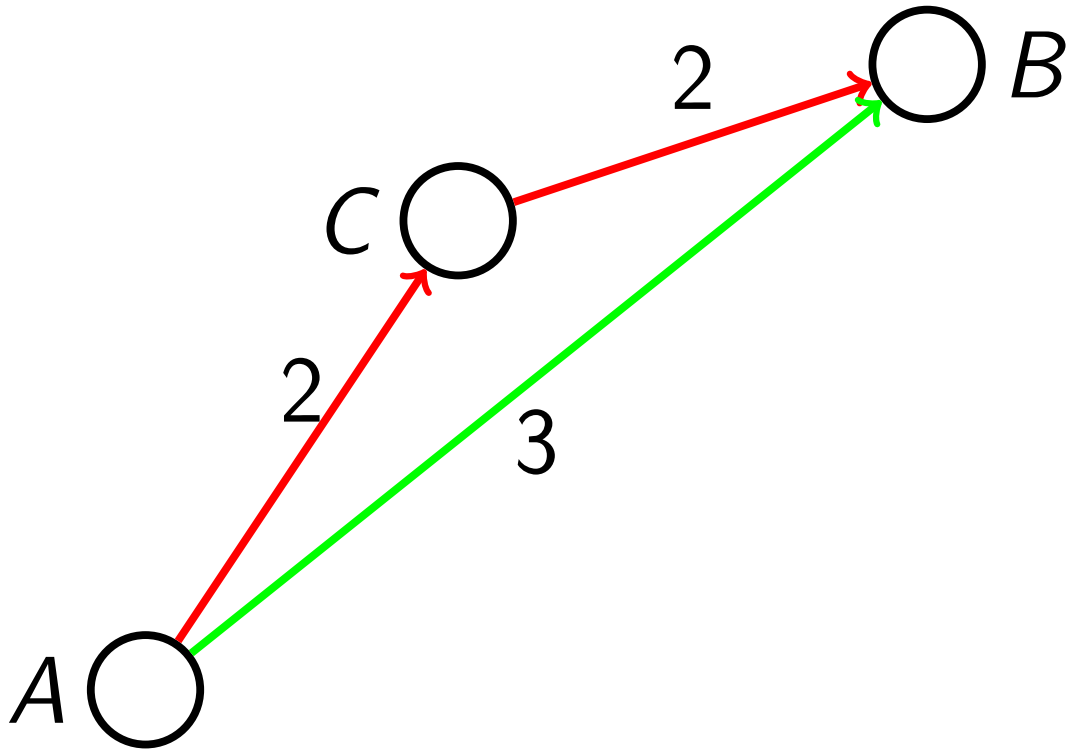
What is the fastest route to get home from work?




B
A

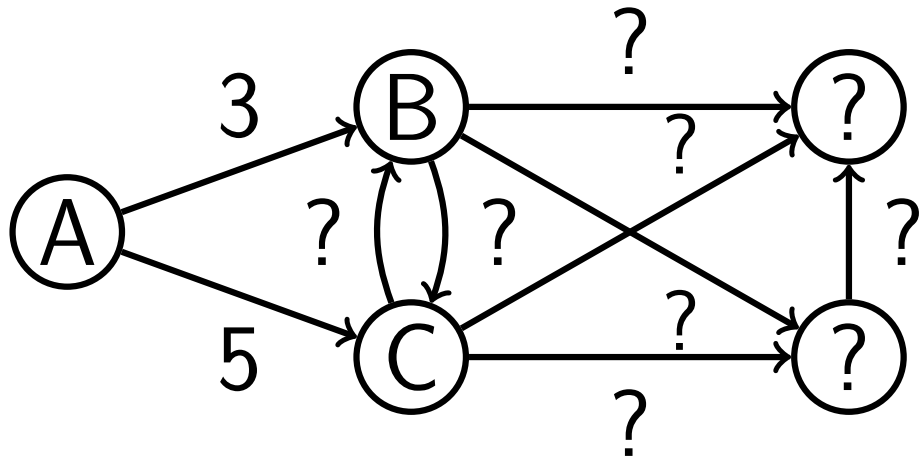

B
5
A


A
B
C
2
2
5


A
C
B
2
2
3

Intuition

- Assume that we stay at A and observe two outgoing edges:



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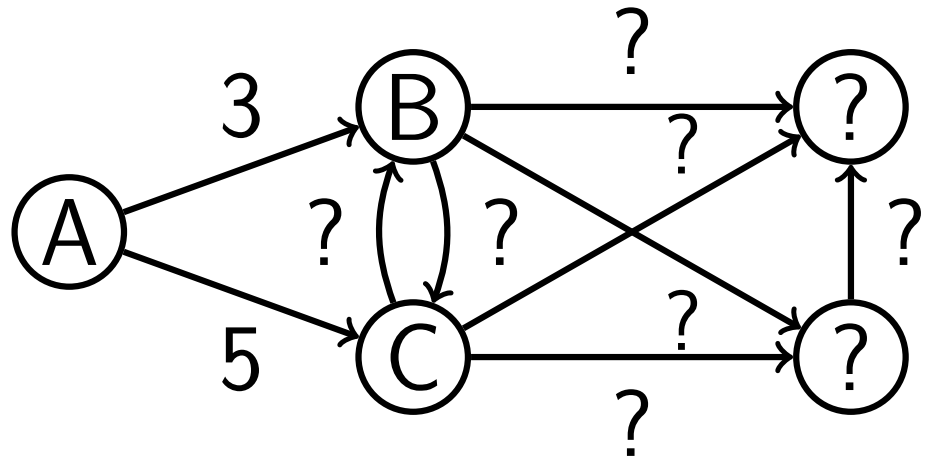

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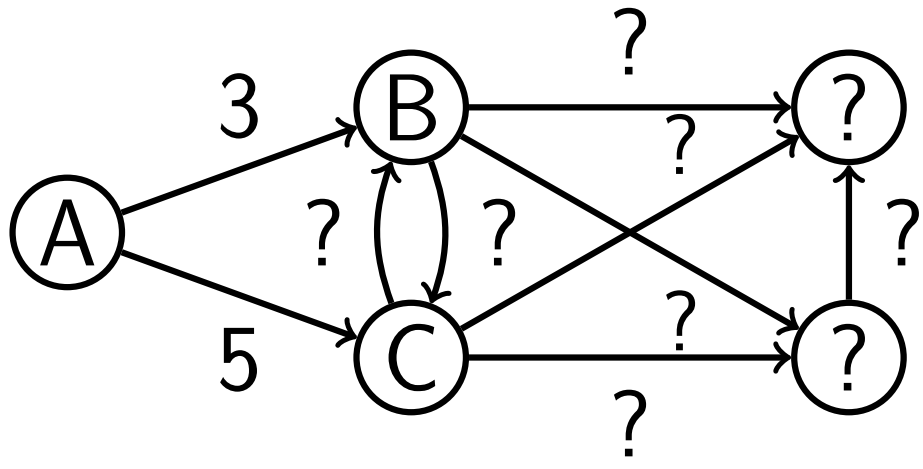

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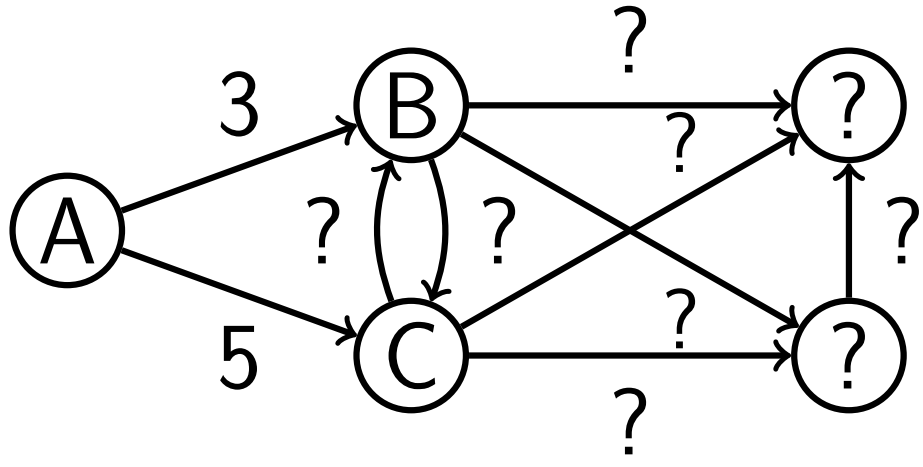

- No, because the weight of the edge (B, C) might be equal to, say, 1.

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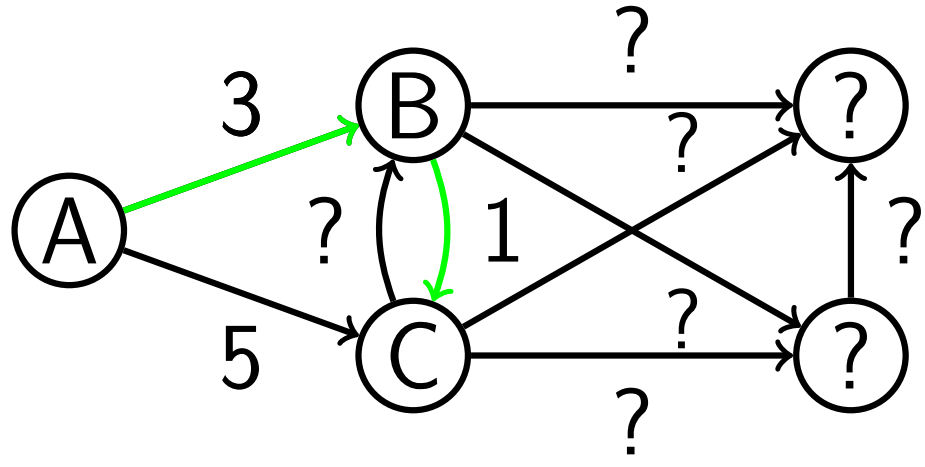

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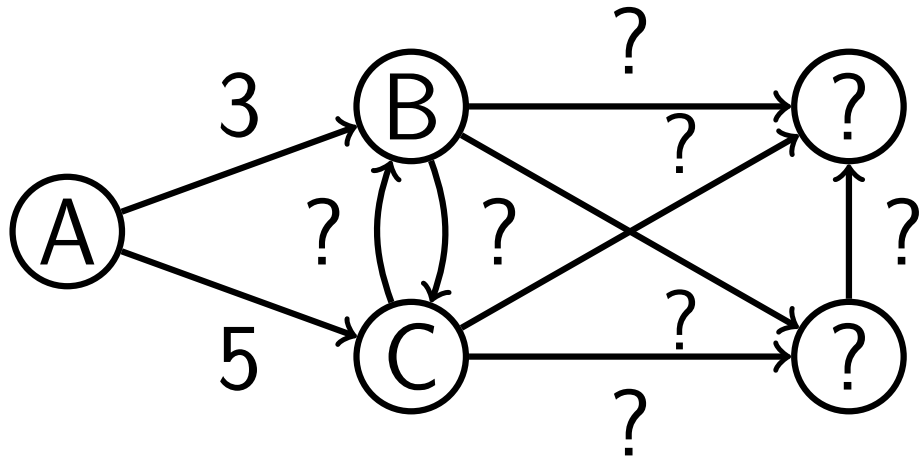
Intuition

- Can we be sure that the distance from A to B is 3?



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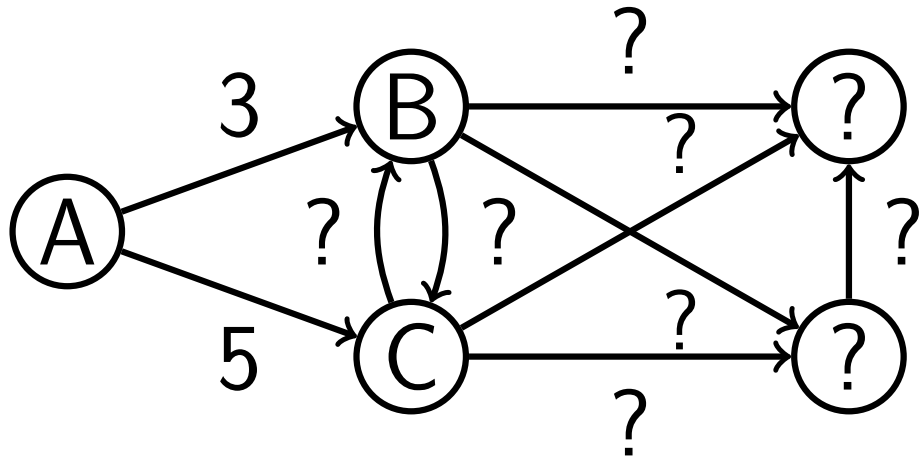


- Yes, because there are no negative weight edges.

Outline

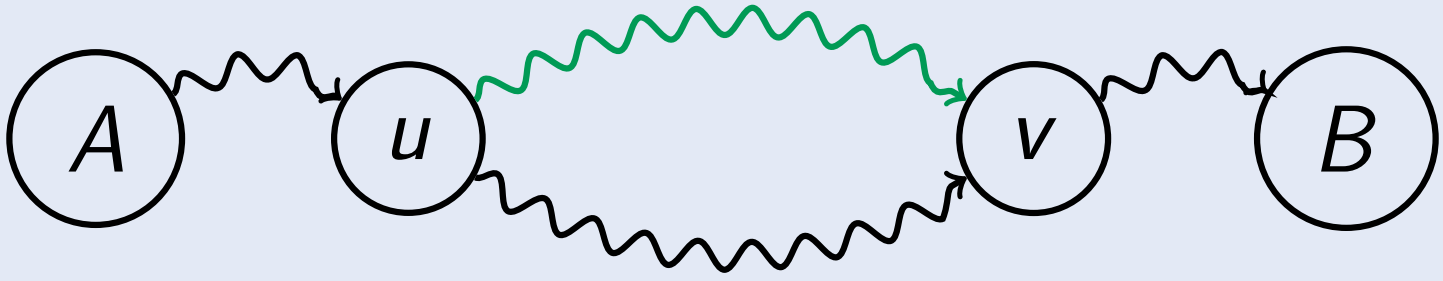
Optimal substructure

Observation

Any subpath of an optimal path is also optimal.

Proof

Consider an optimal path from A to B and two vertices u and v on this path. If there were a shorter path from u to v we would get a shorter path from A to B.



Corollary

If $A \to \ldots \to u \to B$ is a shortest path from A to B, then

$$d(A, B) = d(A, u) + w(u, B)$$

Edge relaxation

- `dist`[v] will be an upper bound on the actual distance from A to v.

Edge relaxation

- `dist`[v] will be an upper bound on the actual distance from A to v.
- The edge relaxation procedure for an edge (u, v) just checks whether going from A to v through u improves the current value of `dist`[v].

`Relax`$((u, v) \in E)$

`if` $dist[v] > dist[u] + w(u, v)$:

$dist[v] \leftarrow dist[u] + w(u, v)$

$prev[v] \leftarrow u$

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Naive approach

Naive(G, A)

```
for all u ∈ V:
    dist[u] ← ∞
    prev[u] ← nil
dist[A] ← 0
do:
    relax all the edges
while at least one dist changes
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Correct distances

Lemma

After the call to `Naive` algorithm all the distances are set correctly.

Proof

- Assume, for the sake of contradiction, that no edge can be relaxed and there is a vertex v such that $\text{dist}[v] > d(A, v)$.

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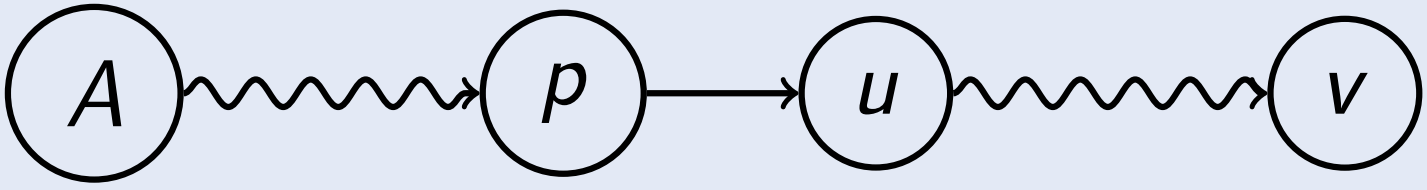
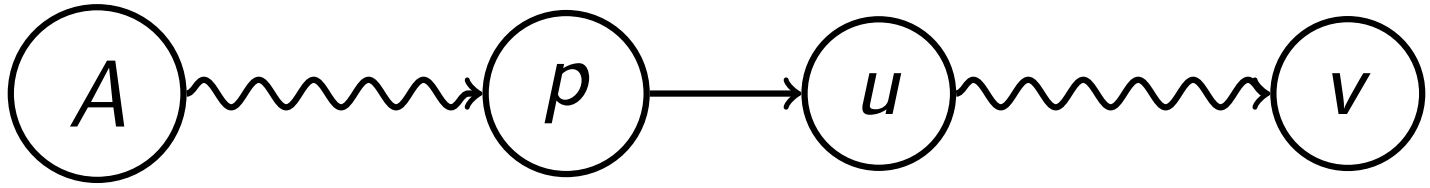
- Assume, for the sake of contradiction, that no edge can be relaxed and there is a vertex v such that $\text{dist}[v] > d(A, v)$.
- Consider a shortest path from A to v and let u be the first vertex on this path with the same property. Let p be the vertex right before u.

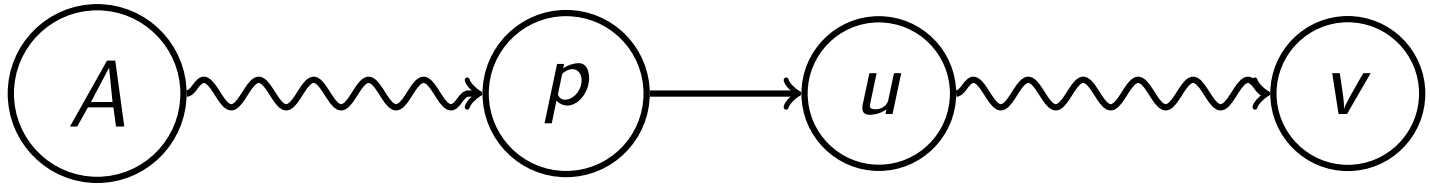


Proof (continued)



- Then $d(A, p) = \text{dist}[p]$ and hence $d(A, u) = d(A, p) + w(p, u) = \text{dist}[p] + w(p, u)$

Proof (continued)

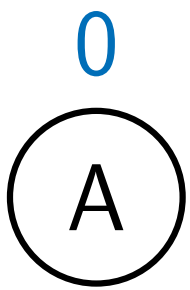


- Then $d(A, p) = \text{dist}[p]$ and hence $d(A, u) = d(A, p) + w(p, u) = \text{dist}[p] + w(p, u)$
- $\text{dist}[u] > d(A, u) = \text{dist}[p] + w(p, u) \Rightarrow$ edge (p, u) can be relaxed — a contradiction.

□

Outline

Intuition



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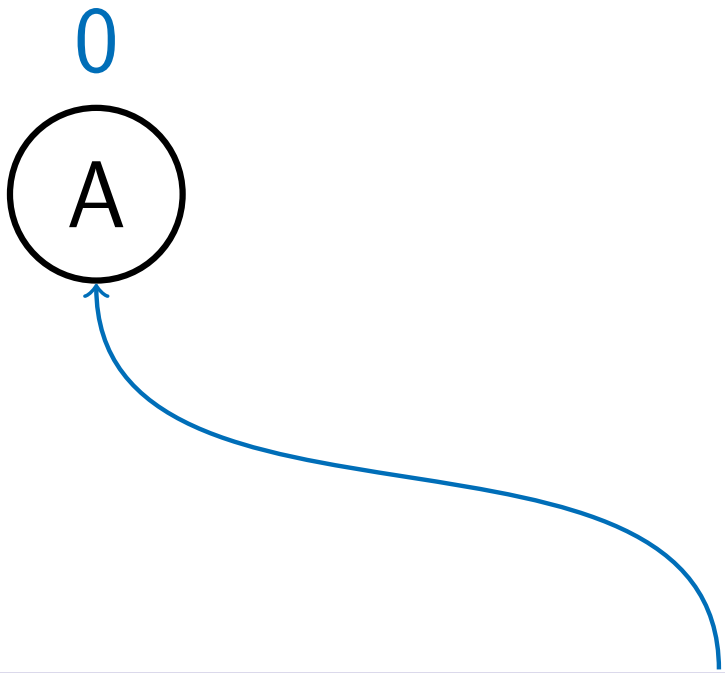
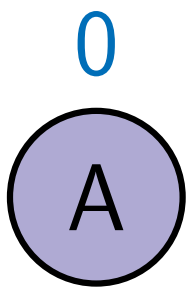
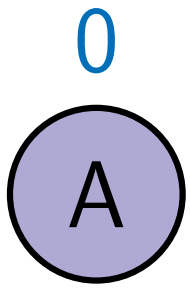


initially, we only know the distance to A

Intuition

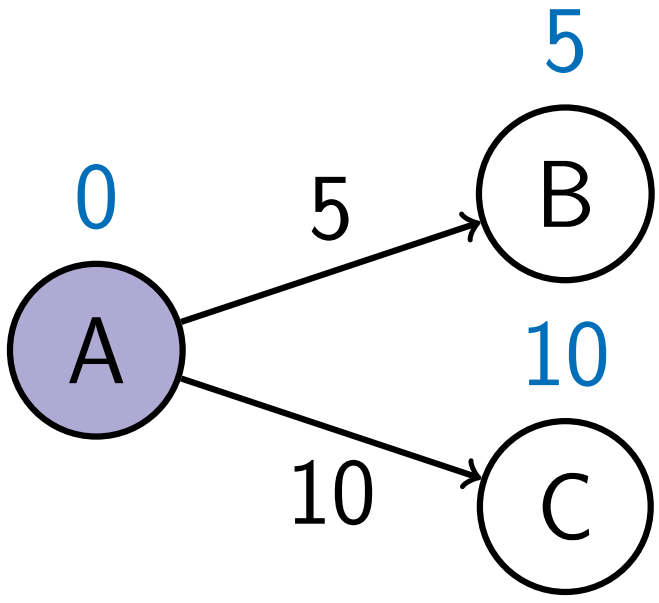


Intuition



let's relax all the edges from A

Intuition



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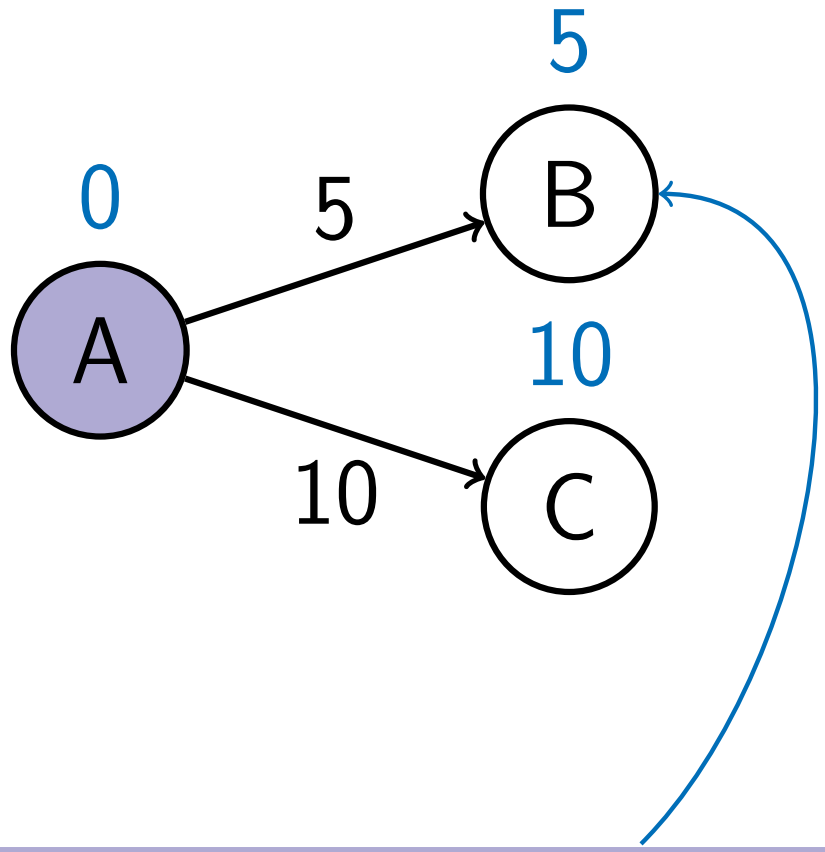
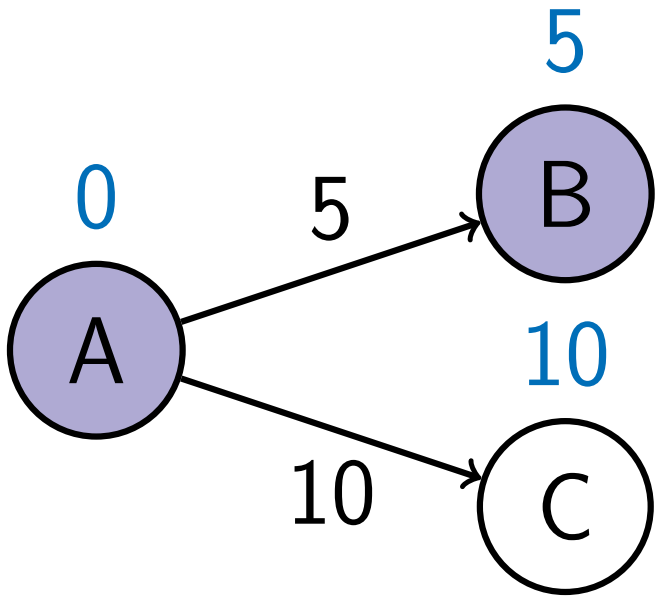


we now know the distance for B

Intuition



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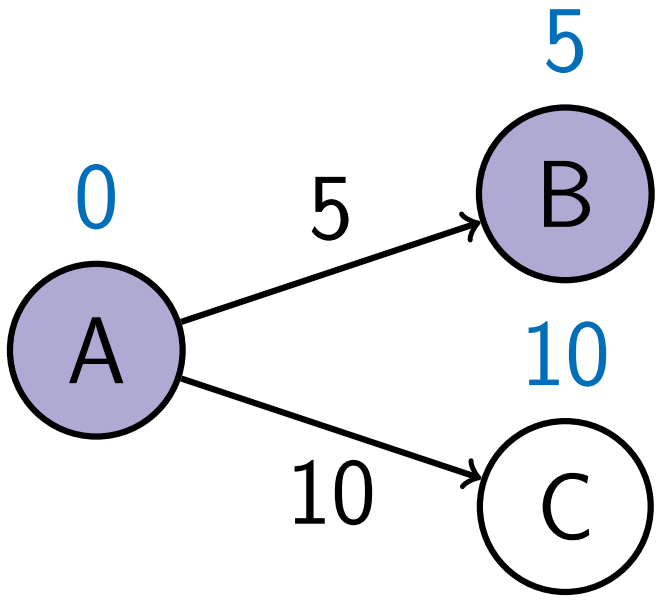

now, let's relax all the edges from B

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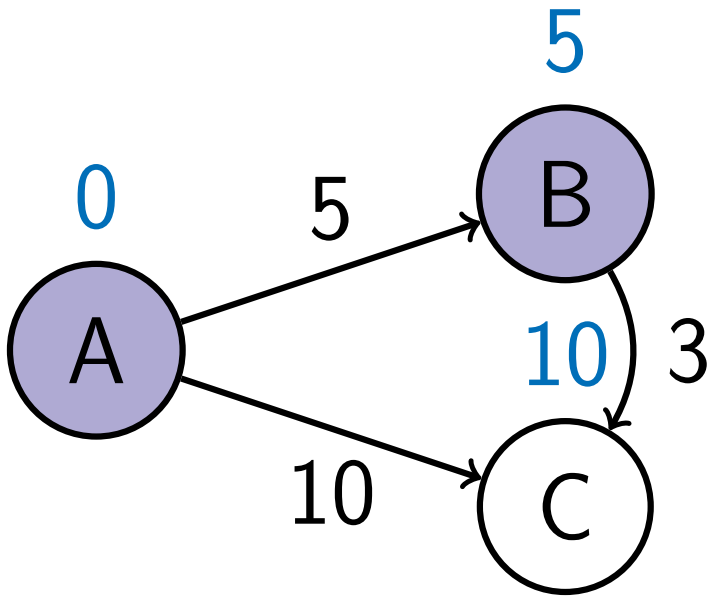

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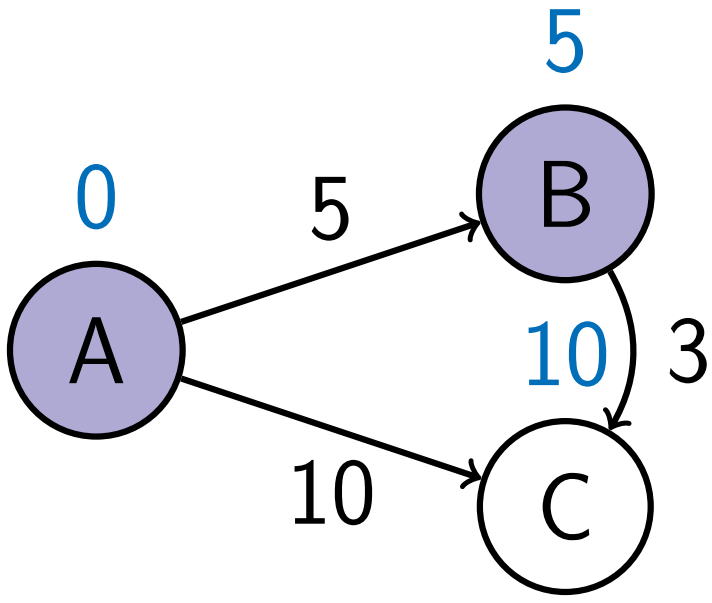

we discover an edge (B, C) of weight 3 that updates dist$[C]$

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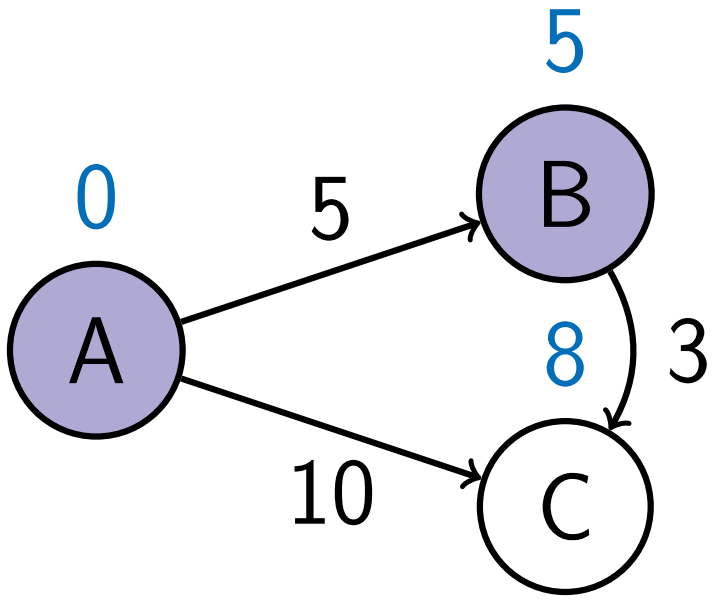

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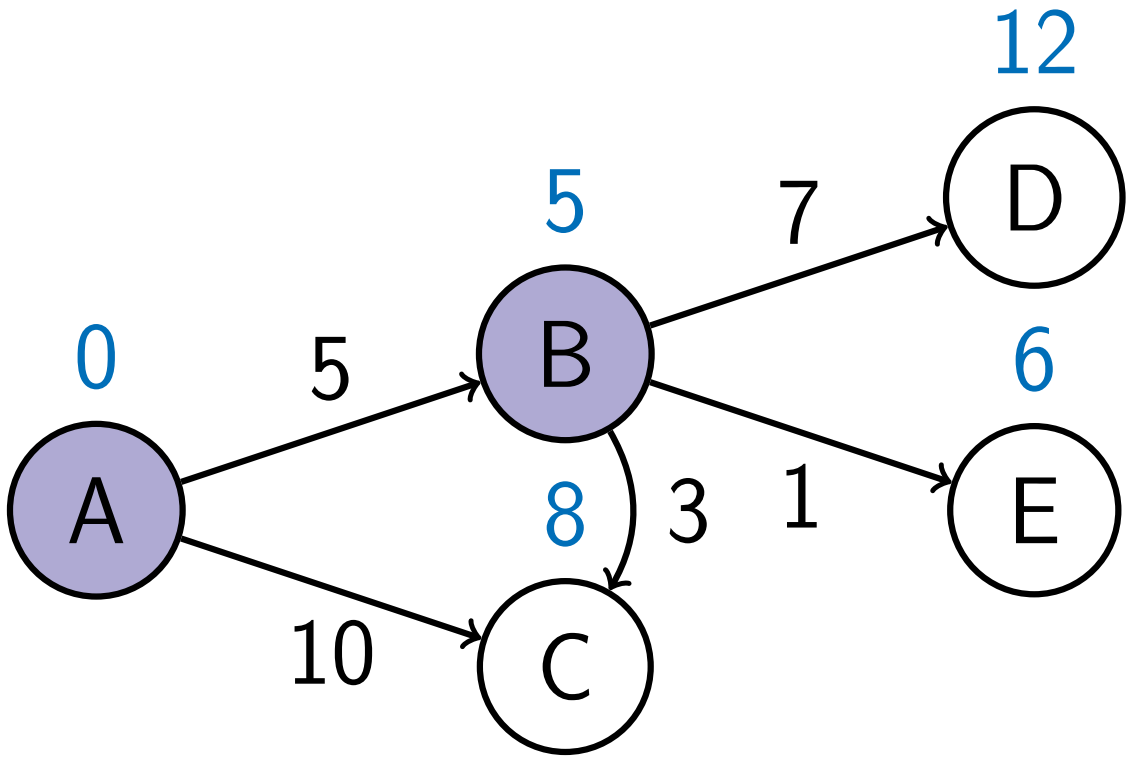

we also discover a few more outgoing edges

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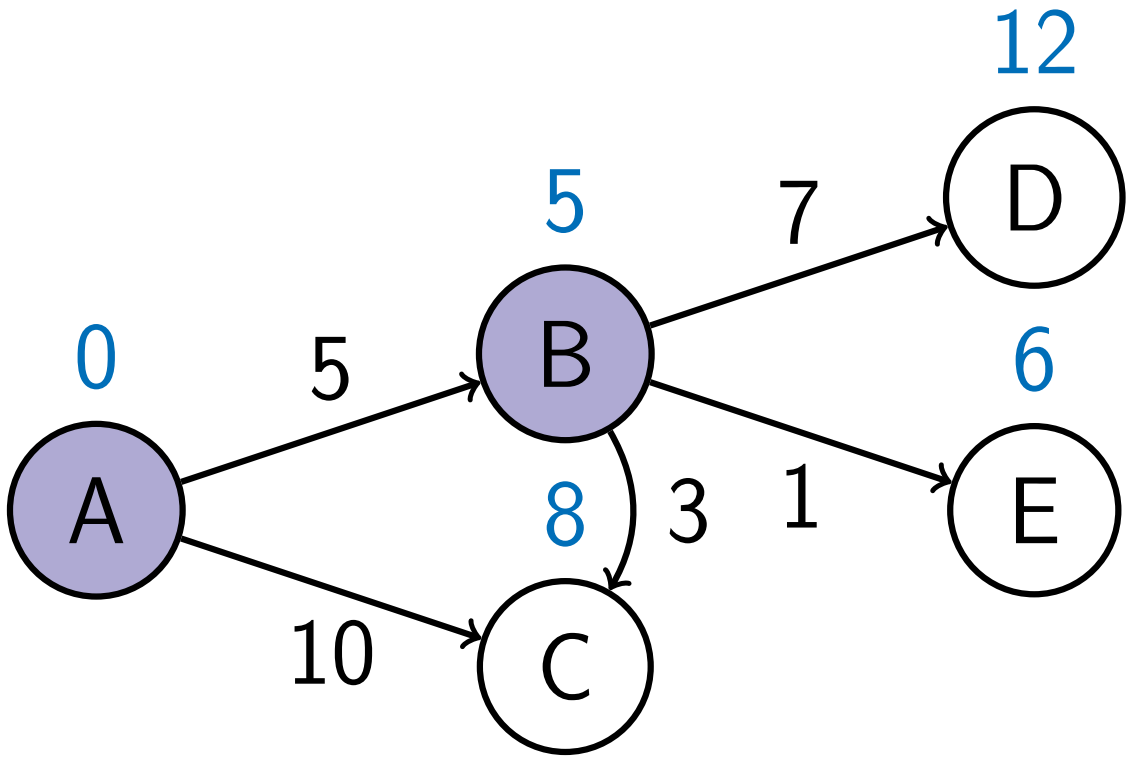

what is the next vertex for which we already know the correct distance?

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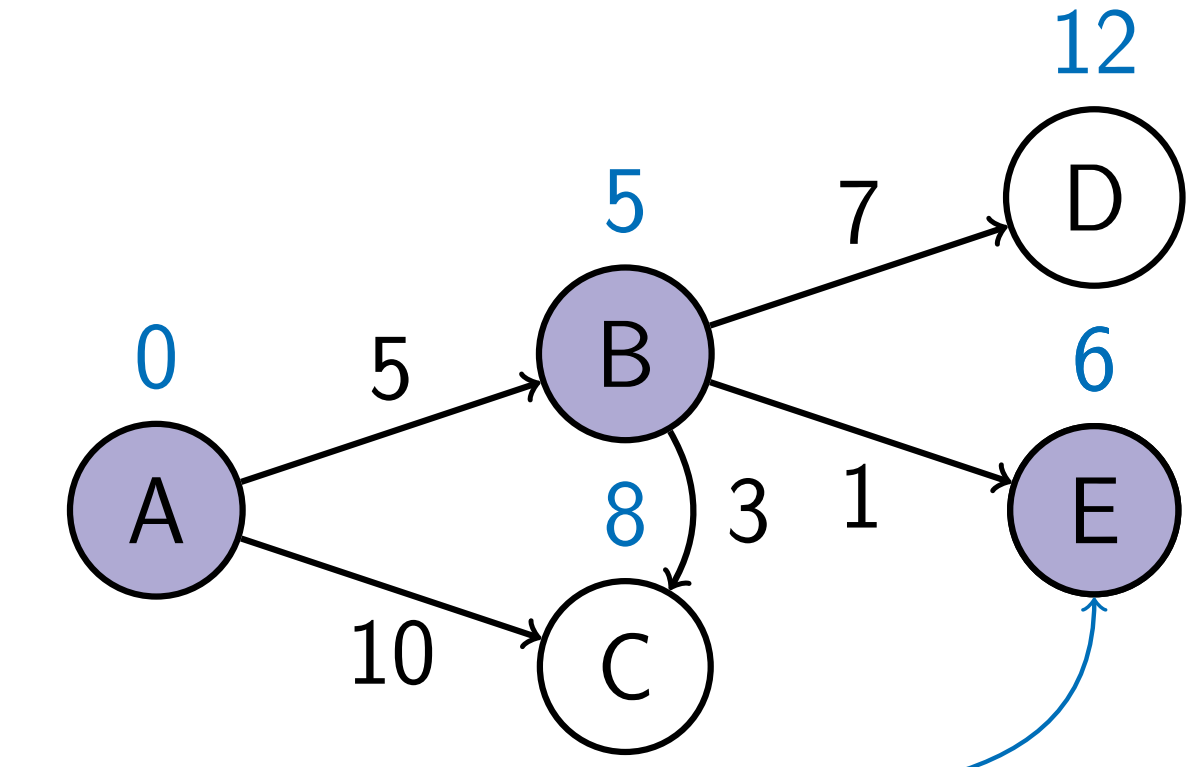

it is E

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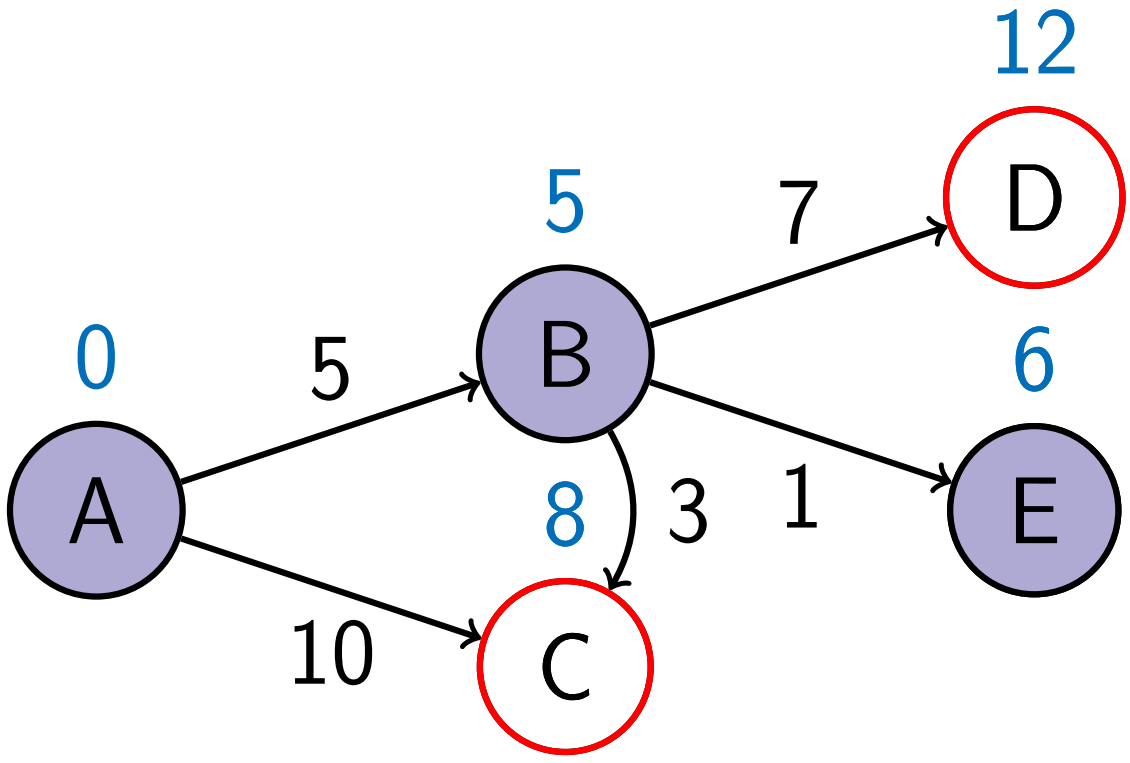

while for C and D it is possible that their `dist` values are larger than actual distances

Main ideas of Dijkstra's Algorithm

- We maintain a set R of vertices for which `dist` is already set correctly ("known region").

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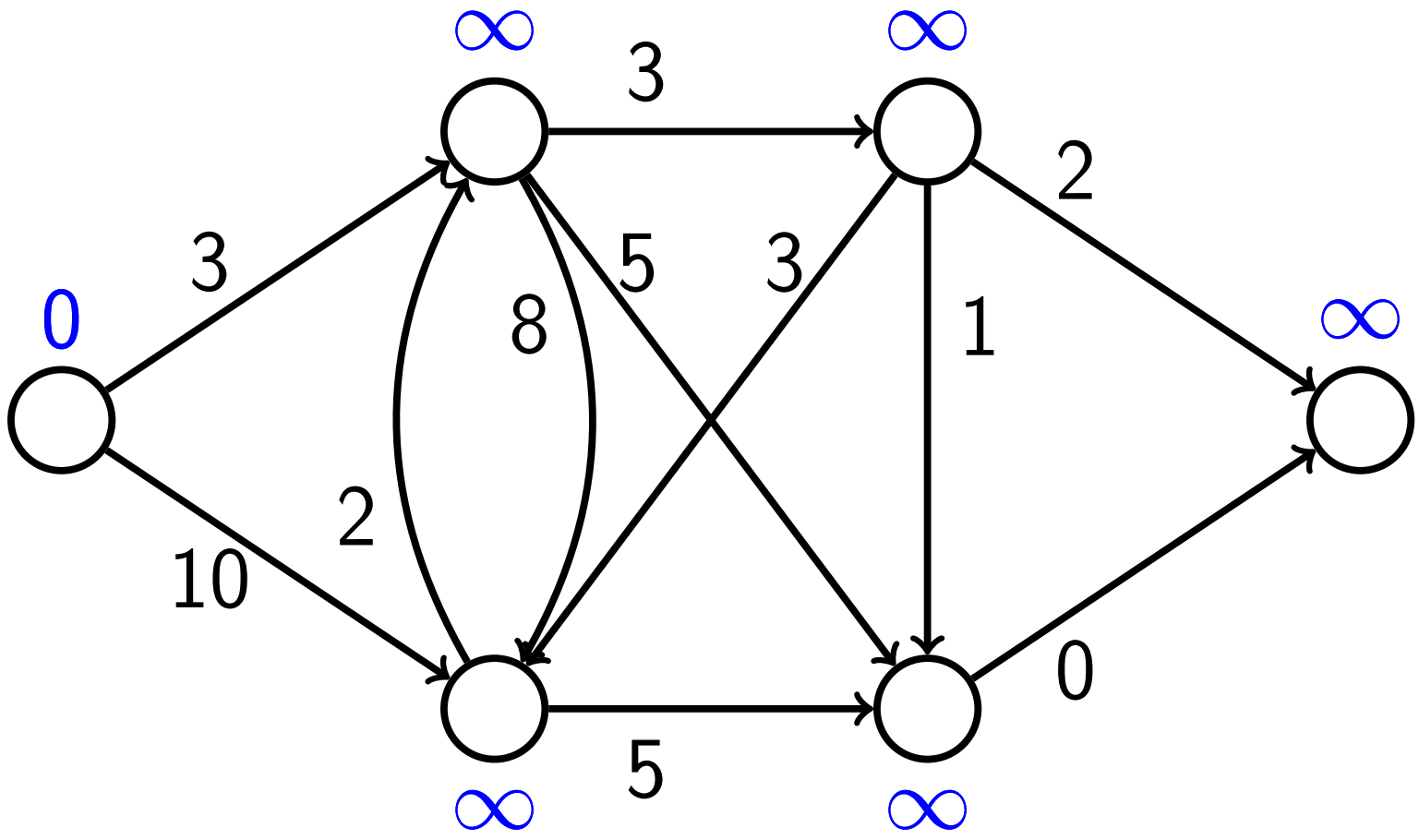
- We maintain a set R of vertices for which `dist` is already set correctly ("known region").
- The first vertex added to R is A.
- On each iteration we take a vertex outside of R with the minimal `dist`-value, add it to R, and relax all its outgoing edges.

Outline

Example



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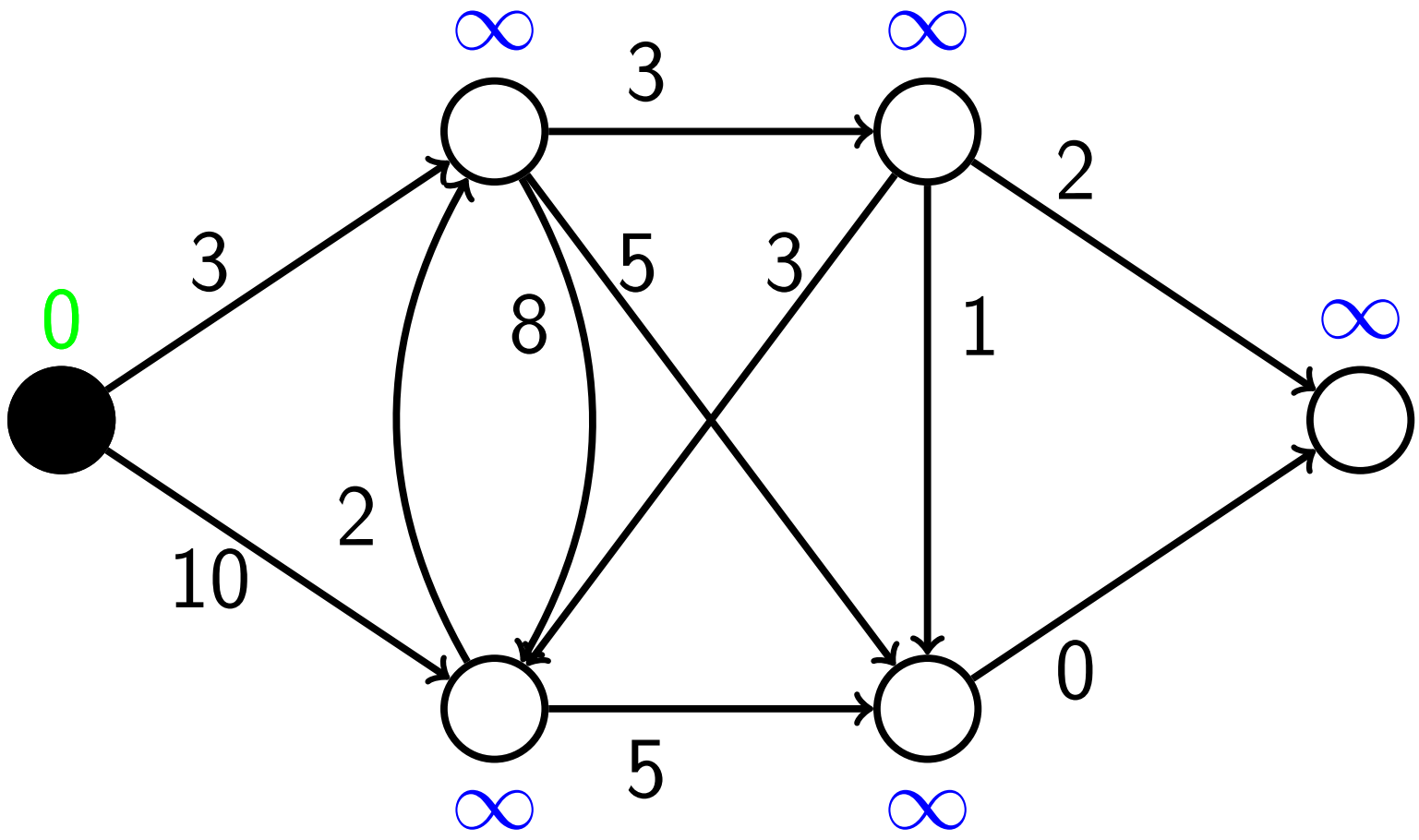

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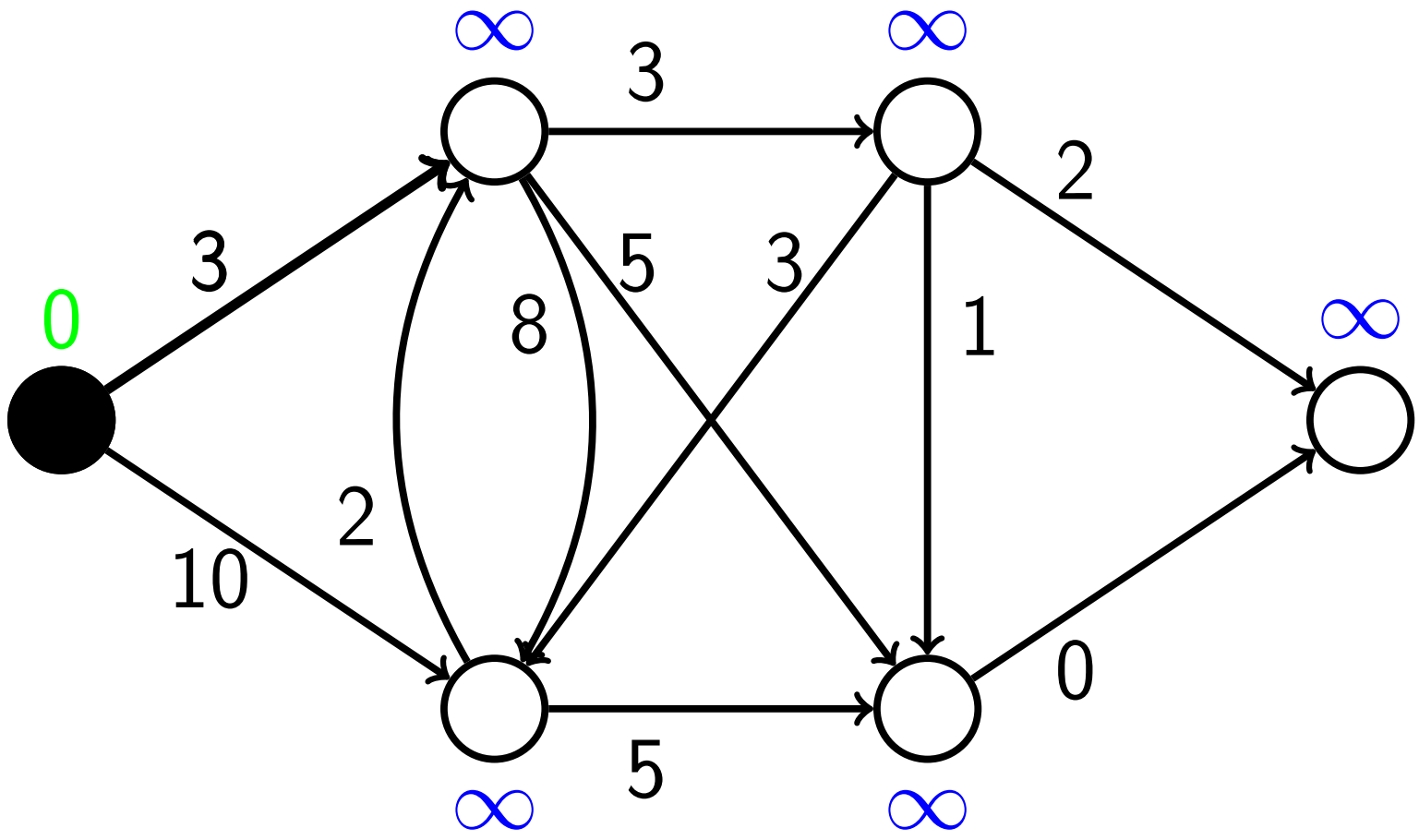

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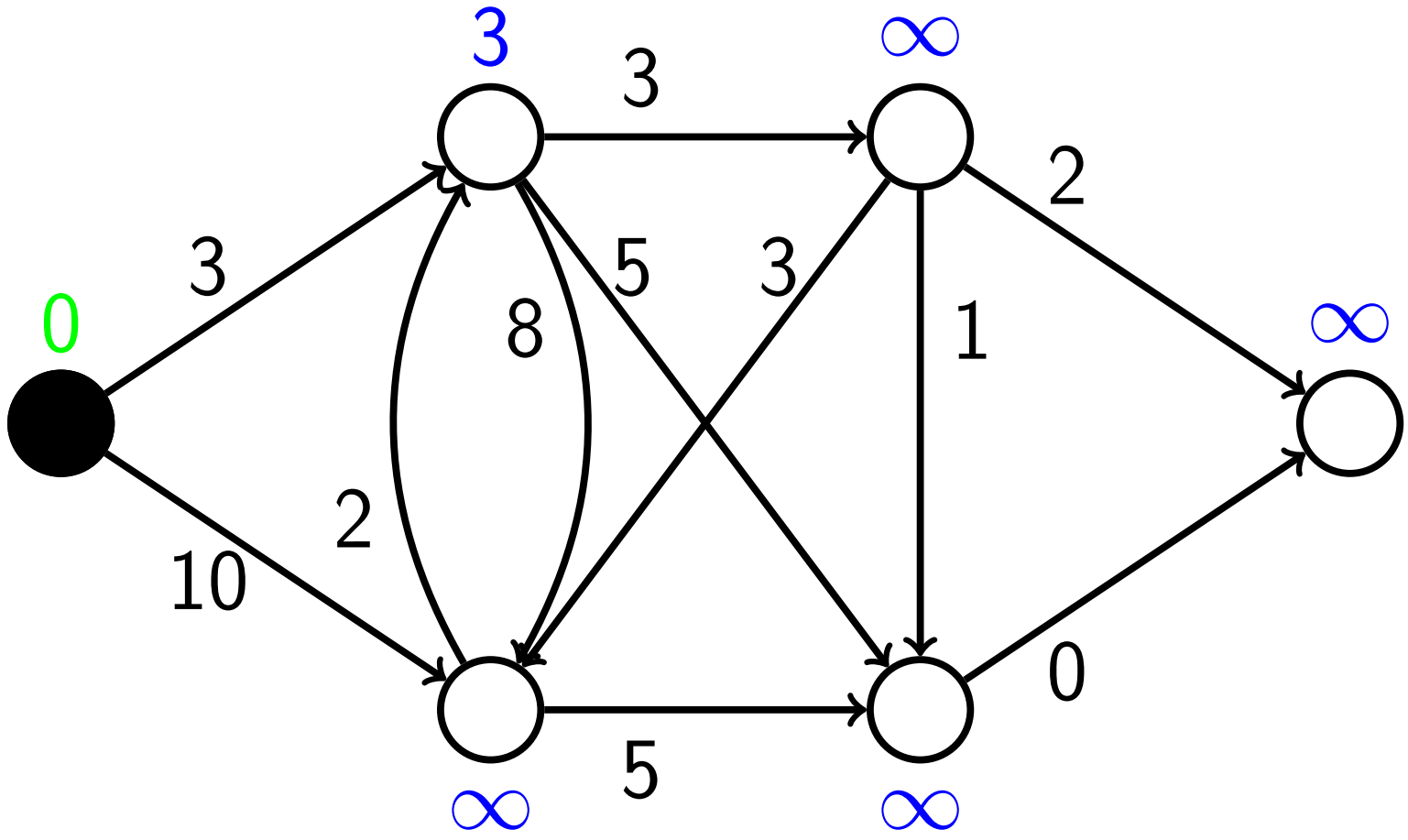

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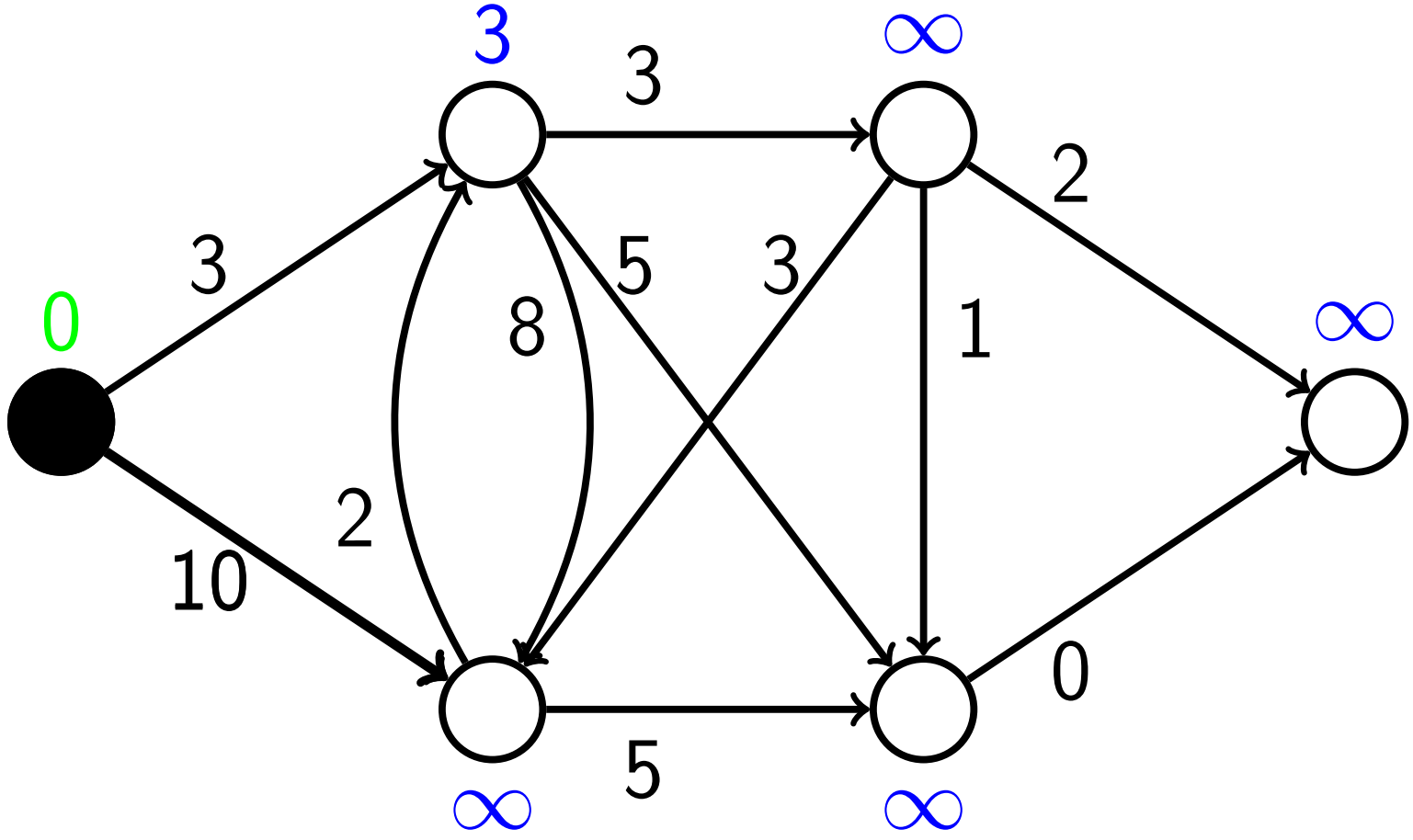

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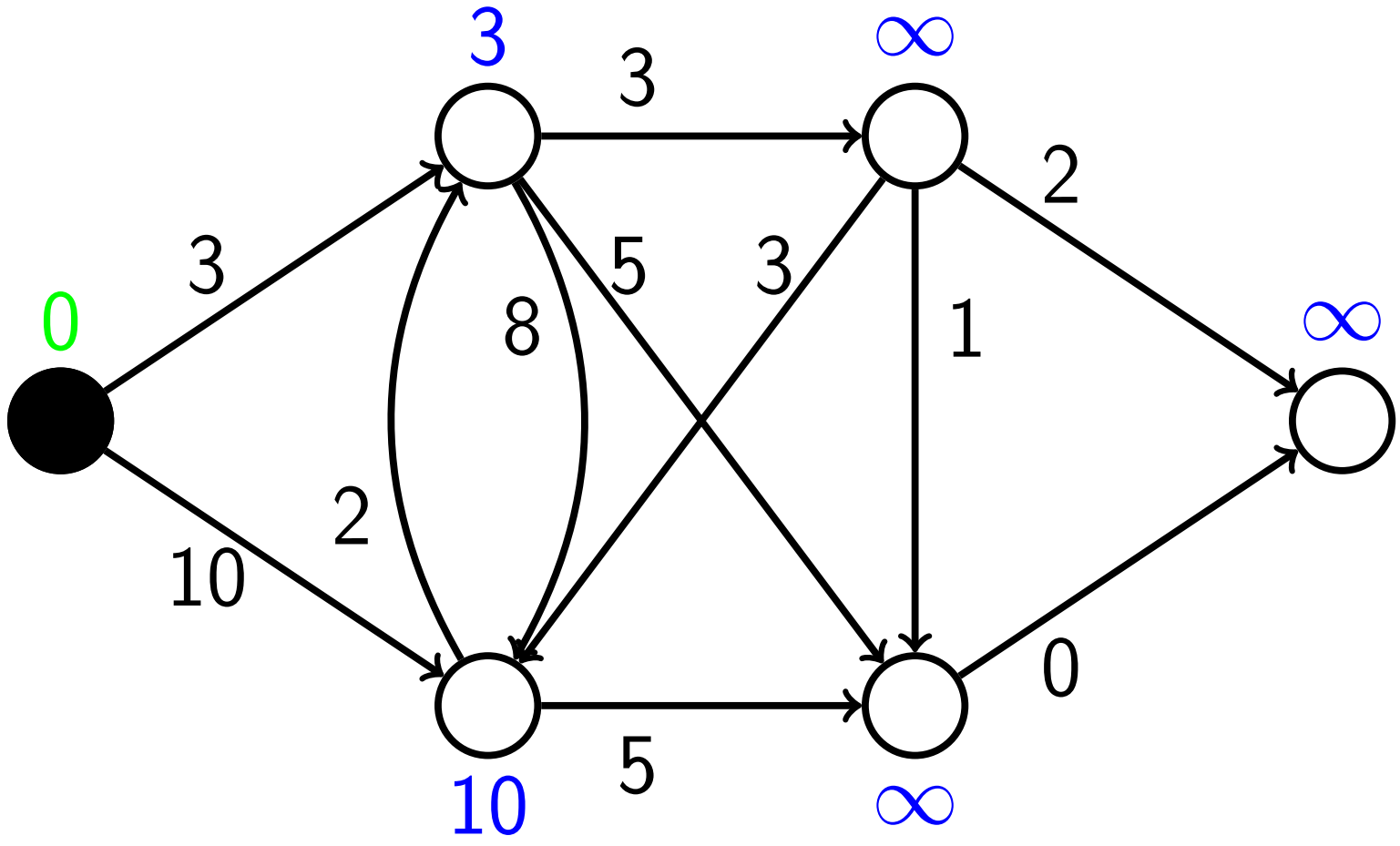

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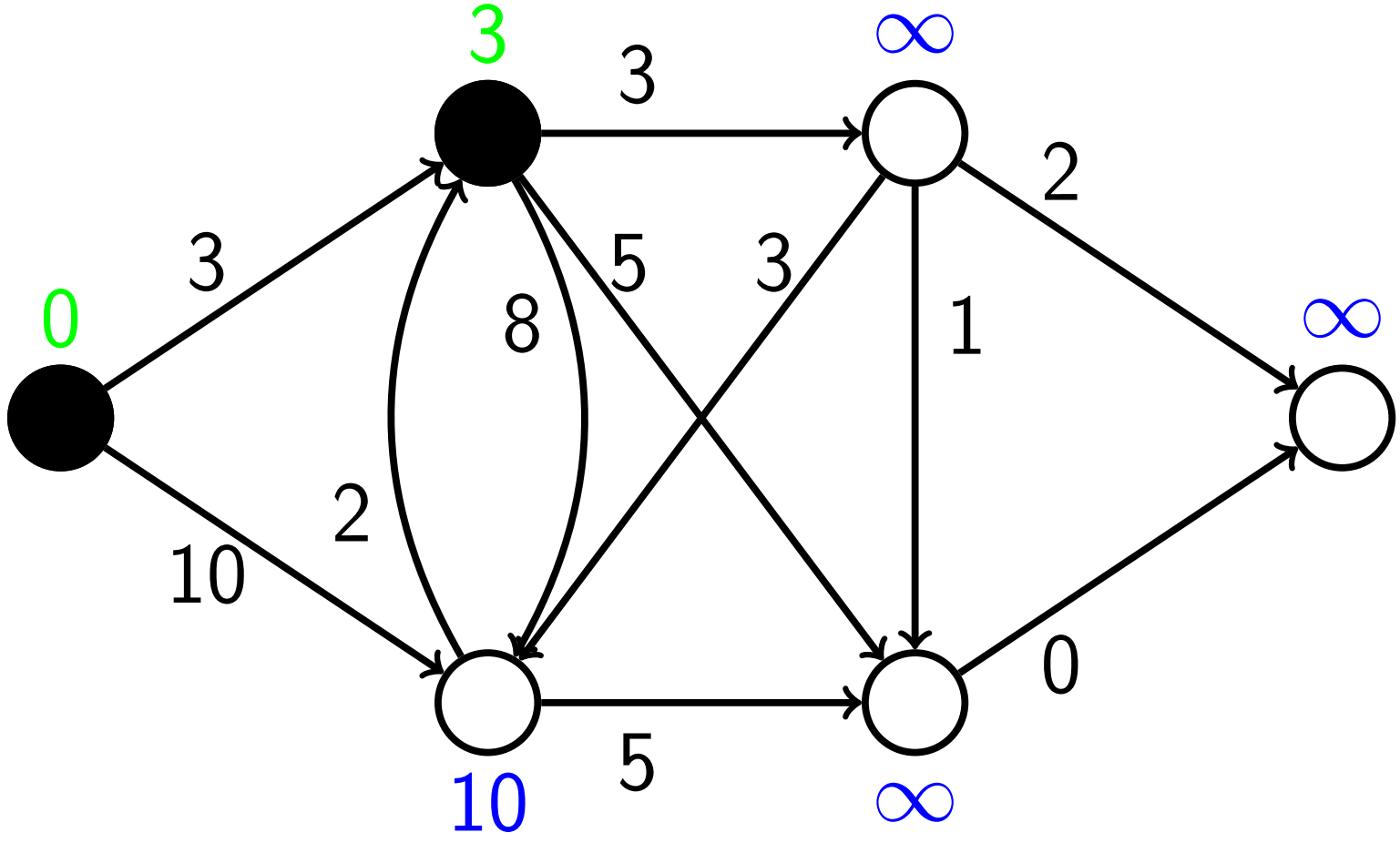

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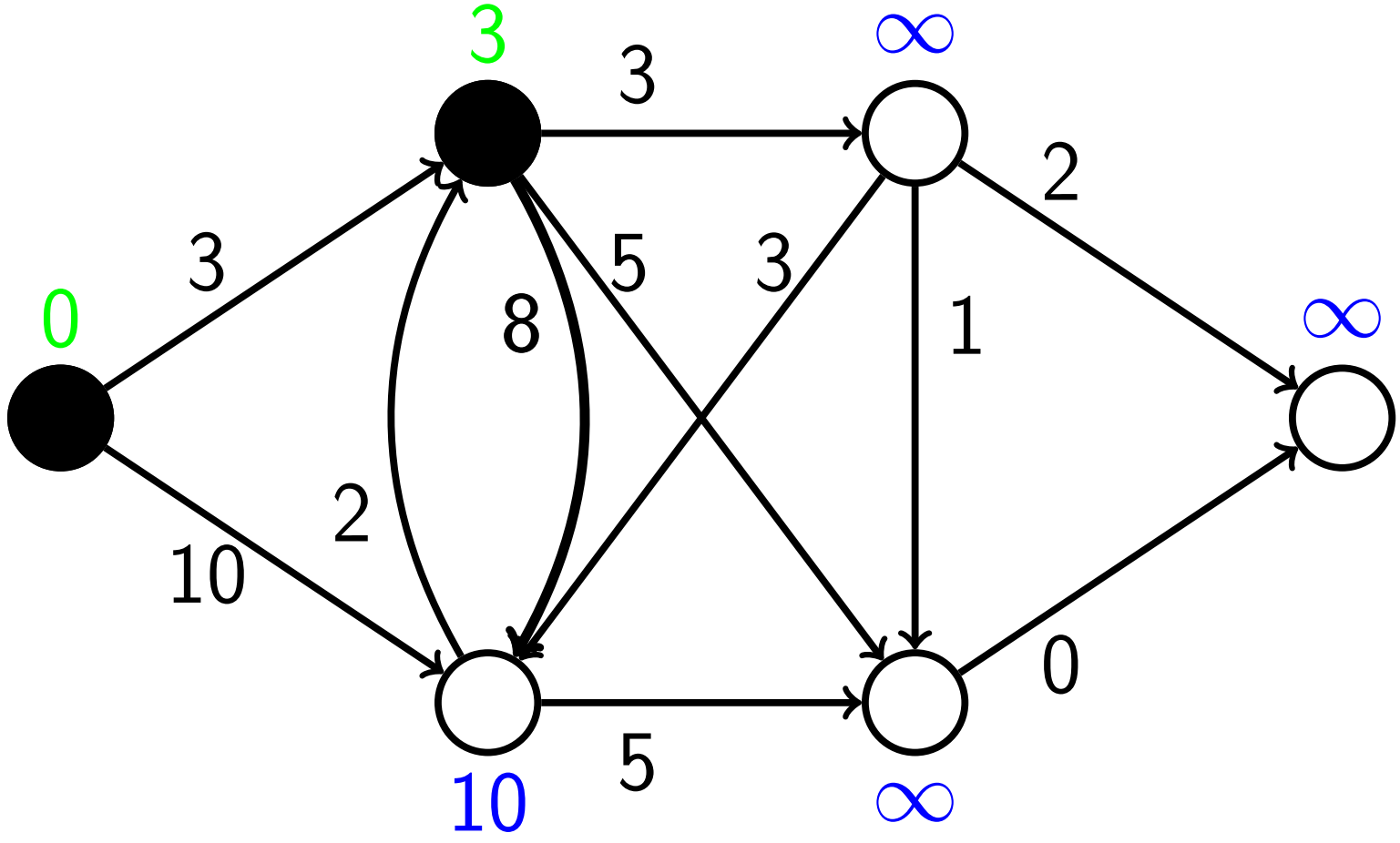

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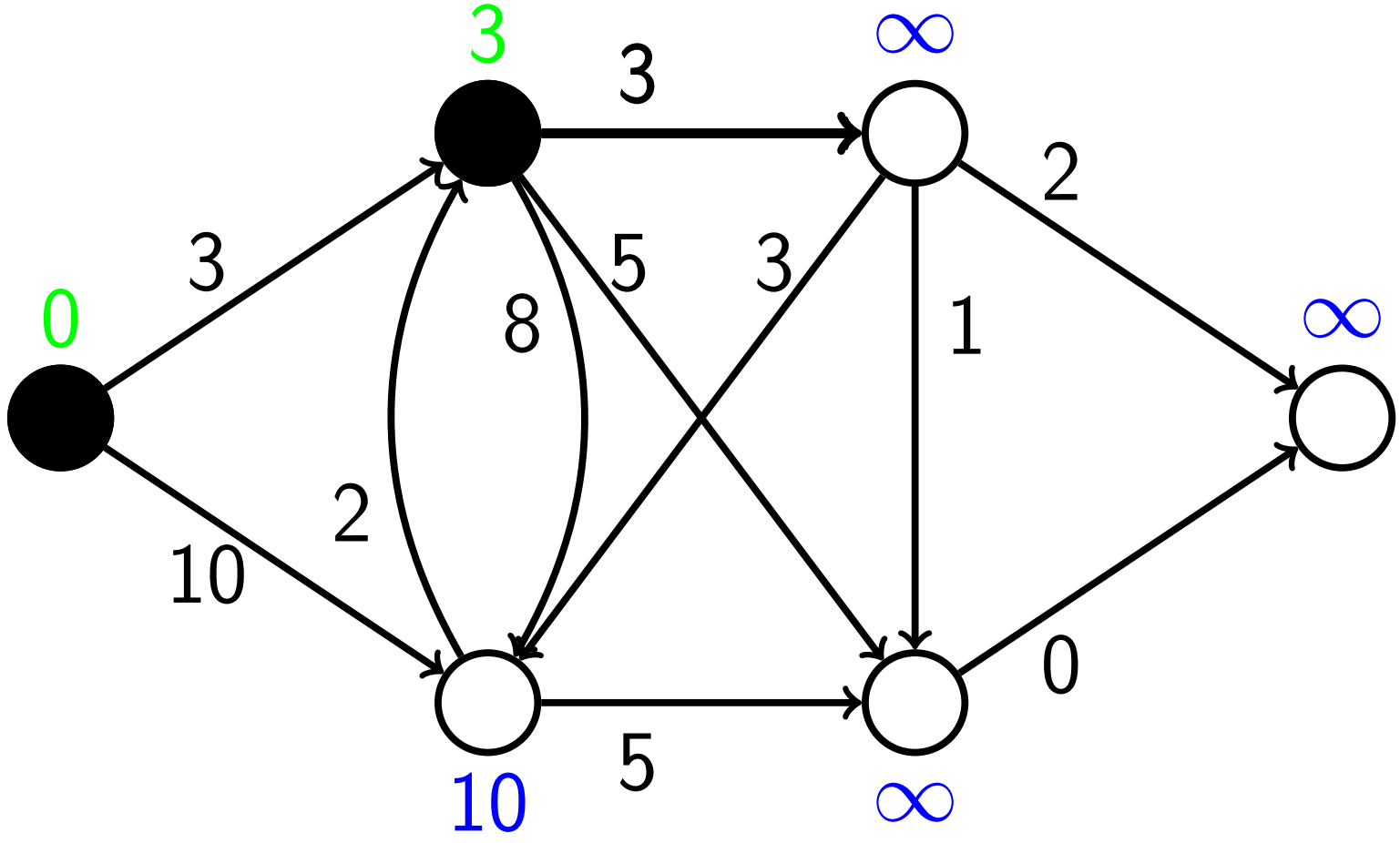

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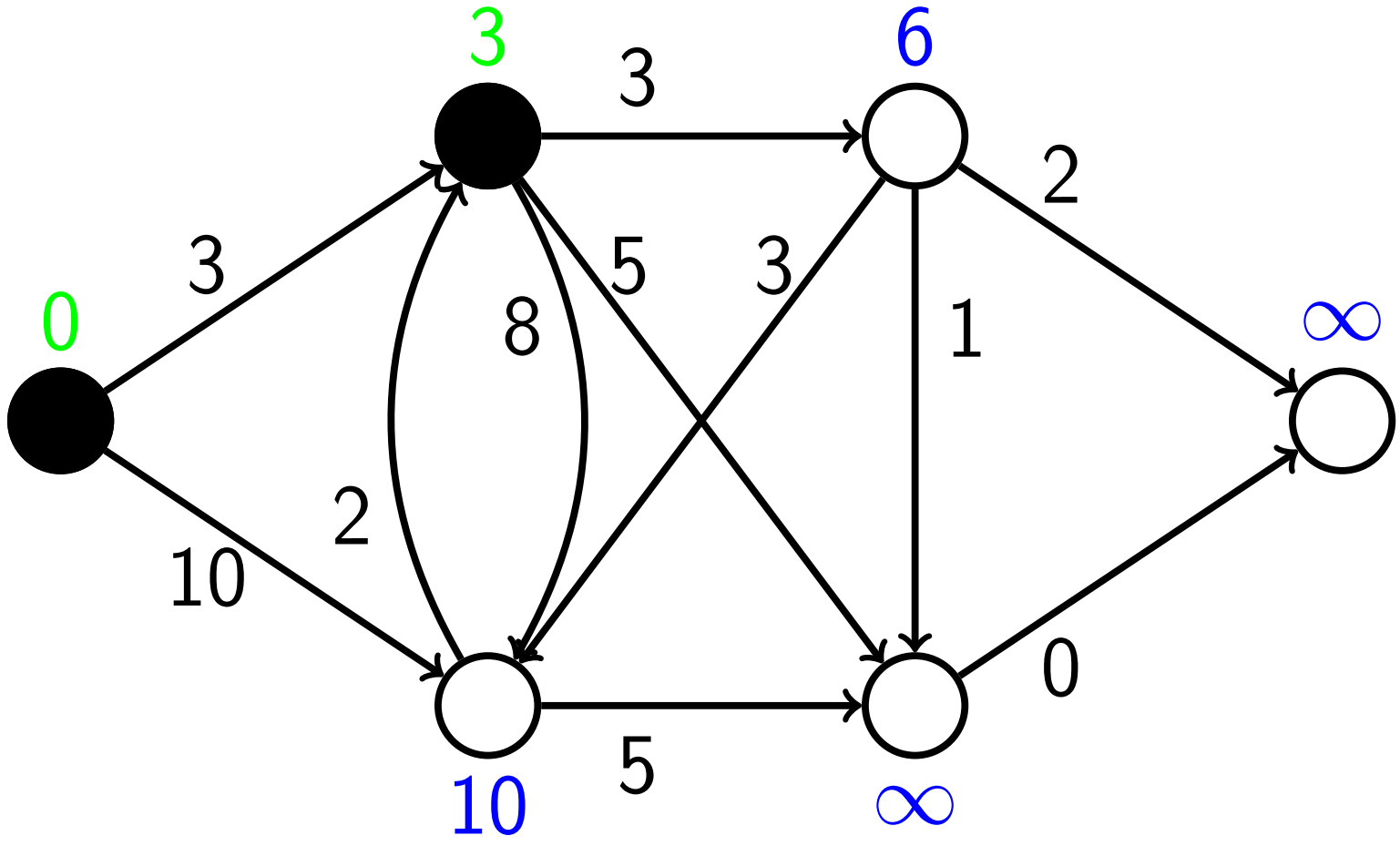

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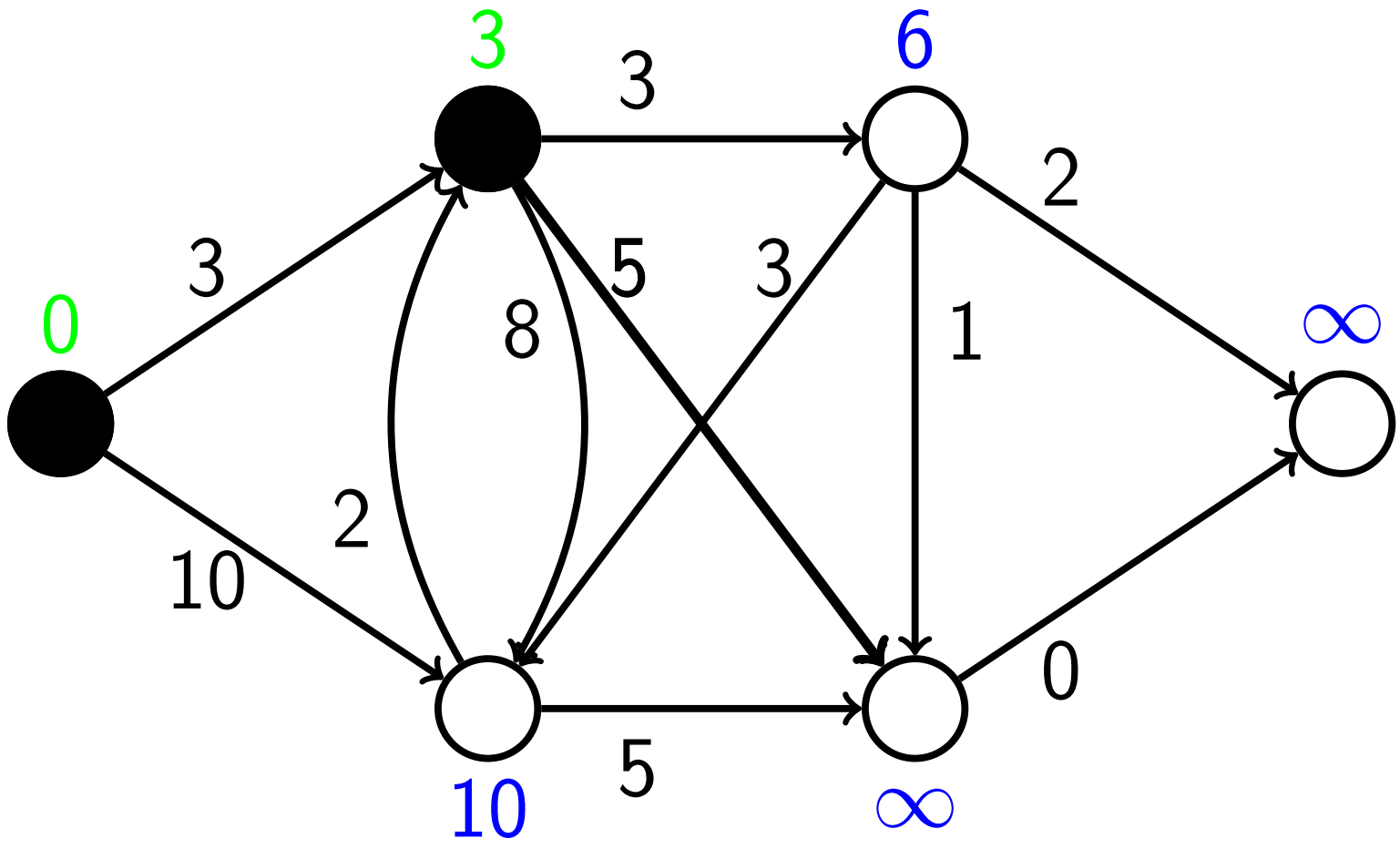

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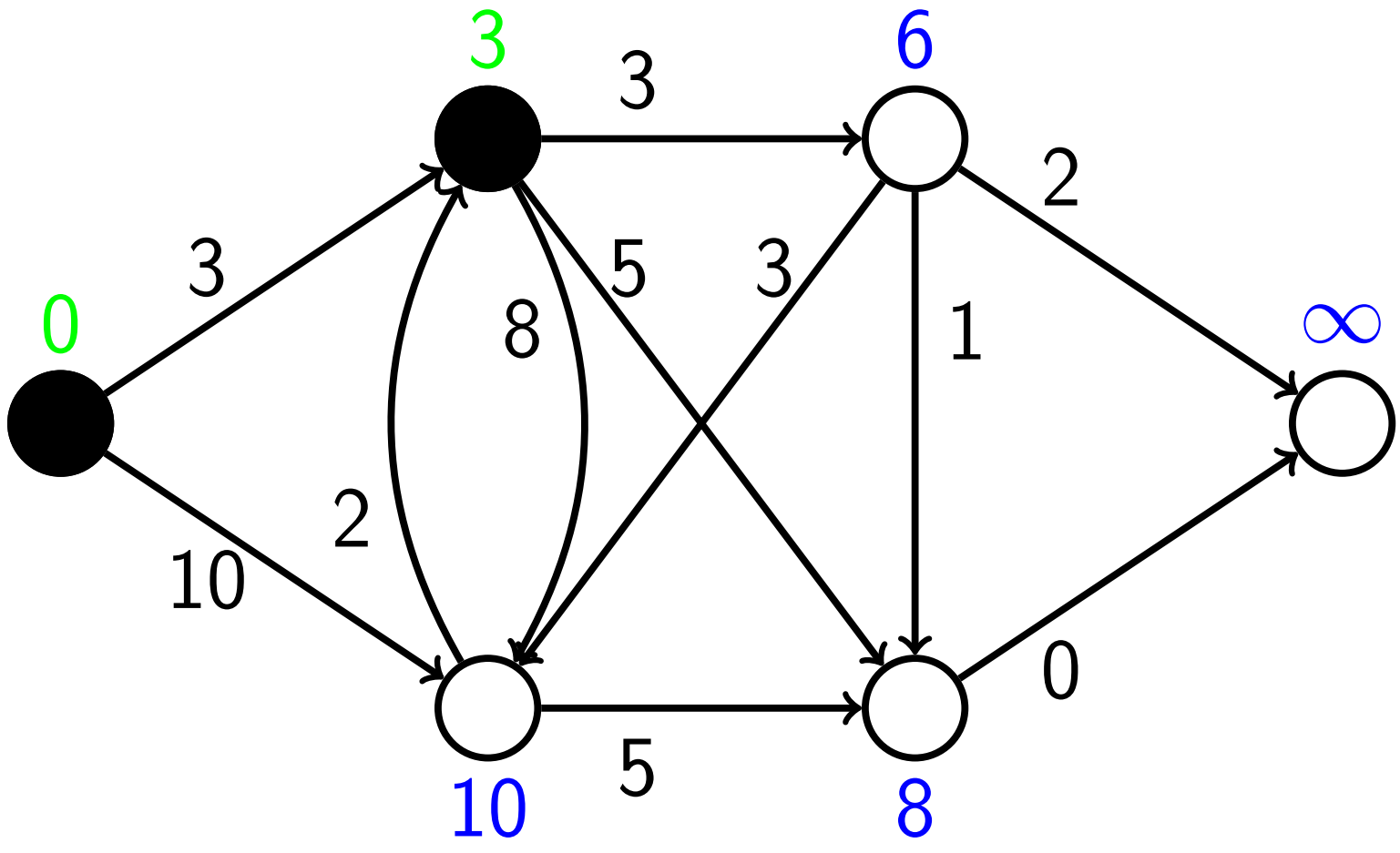

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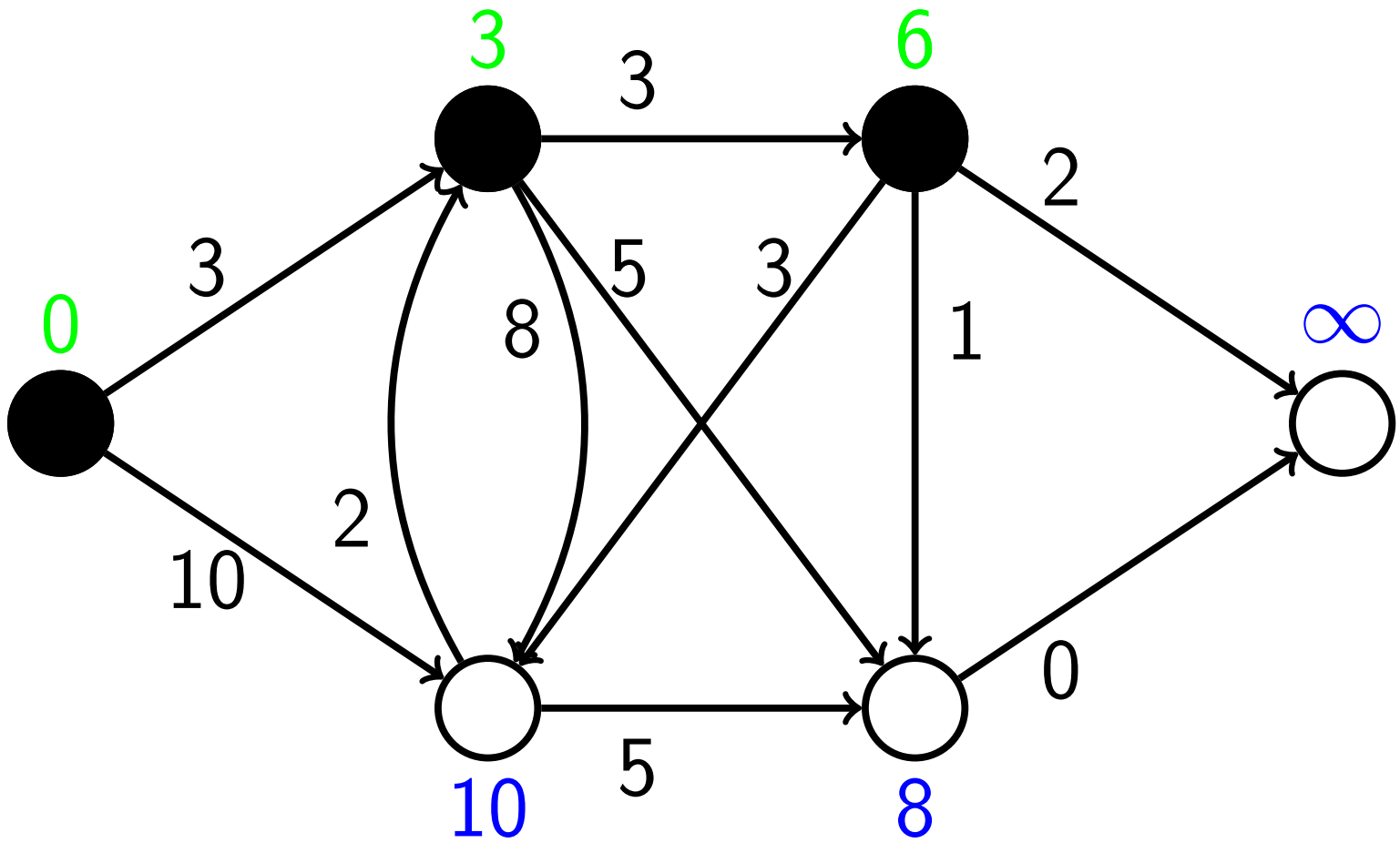

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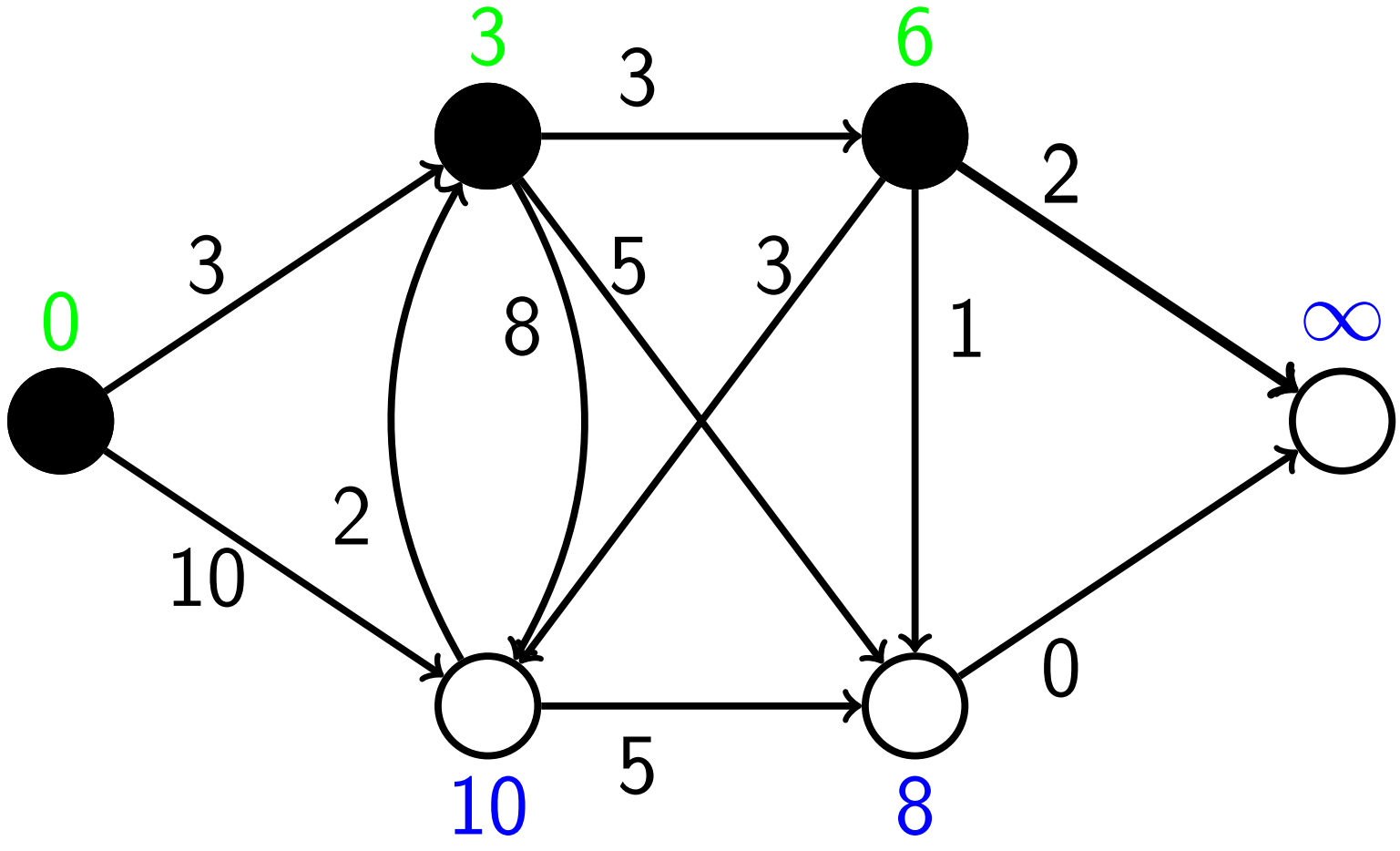

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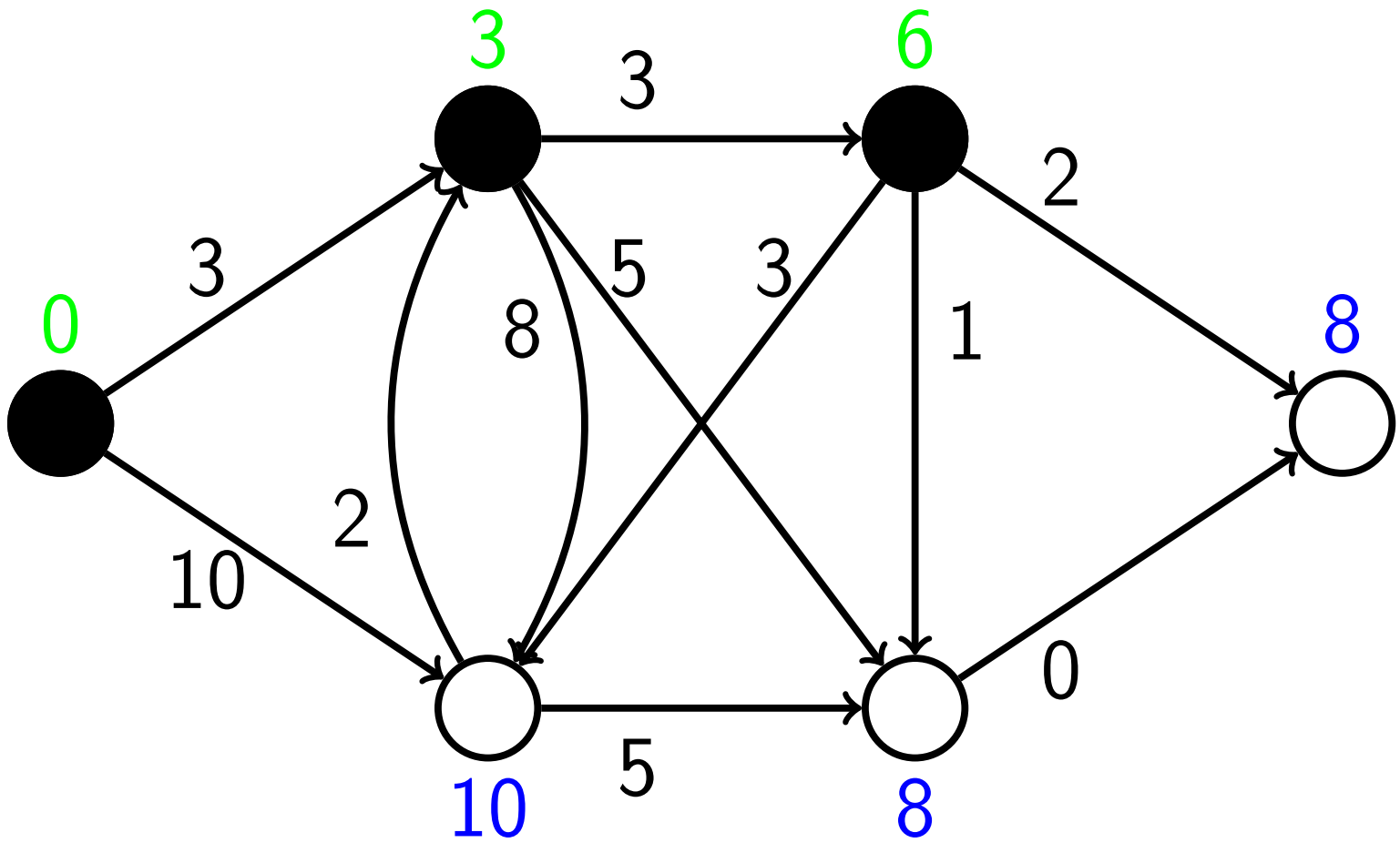

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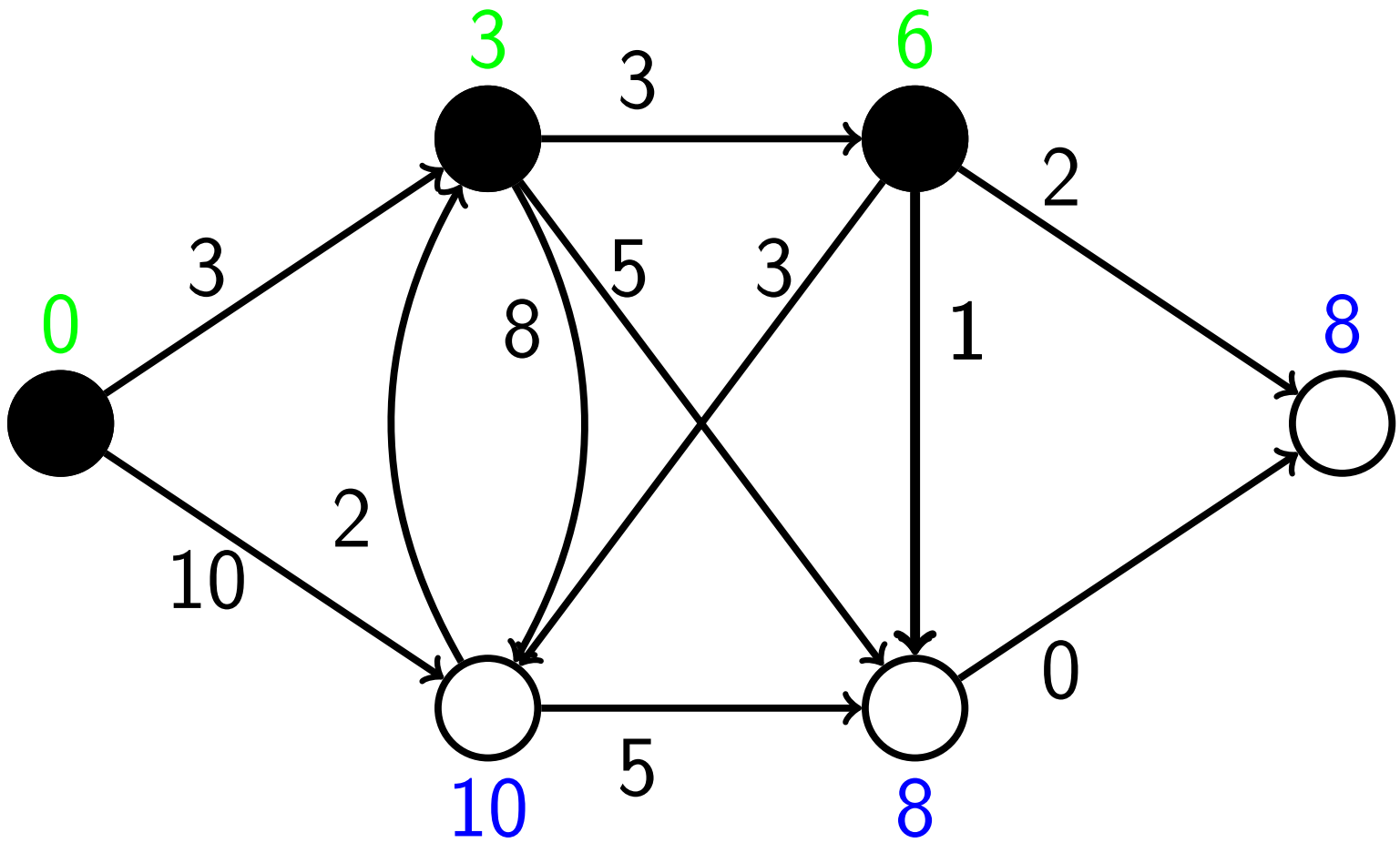

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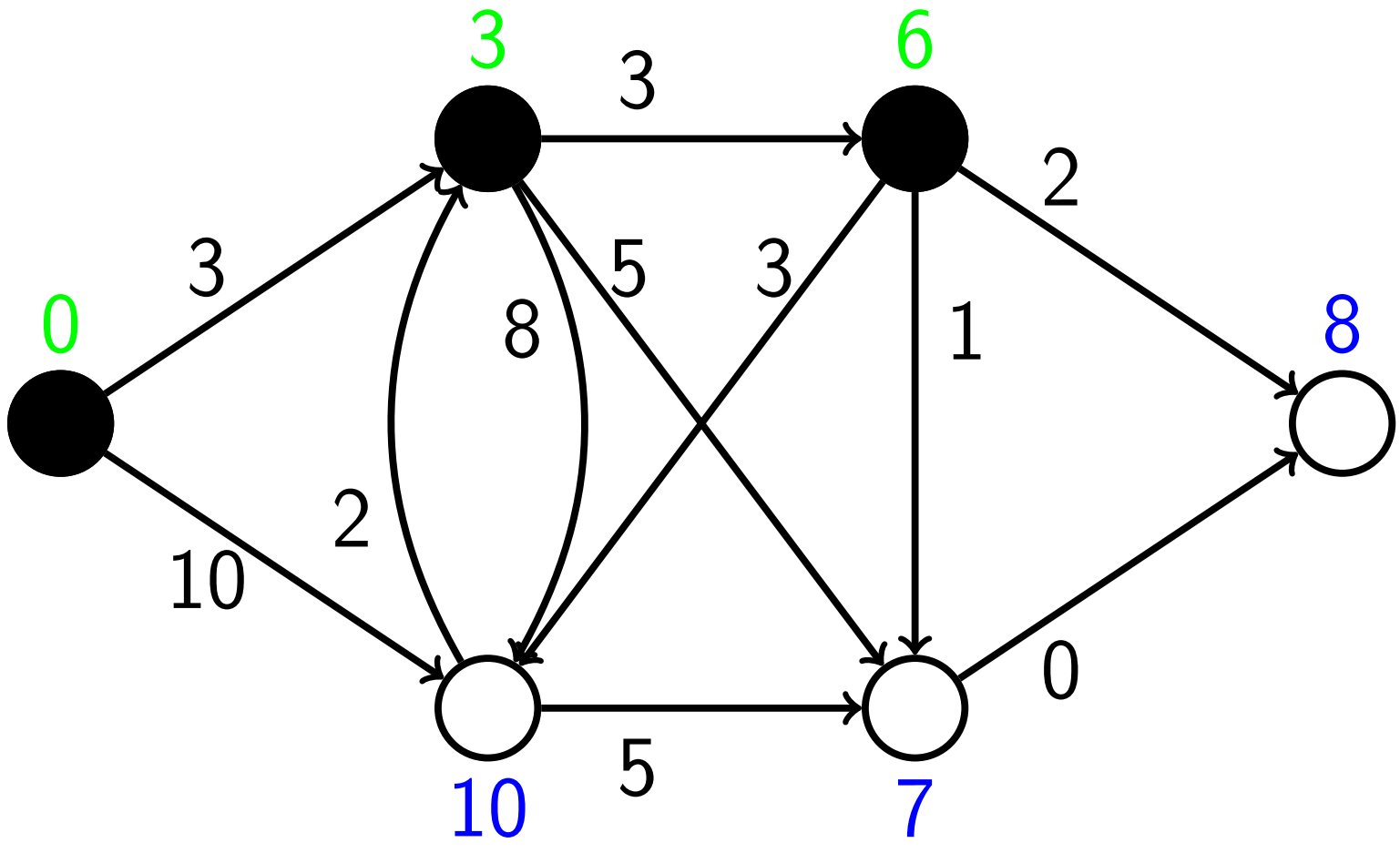

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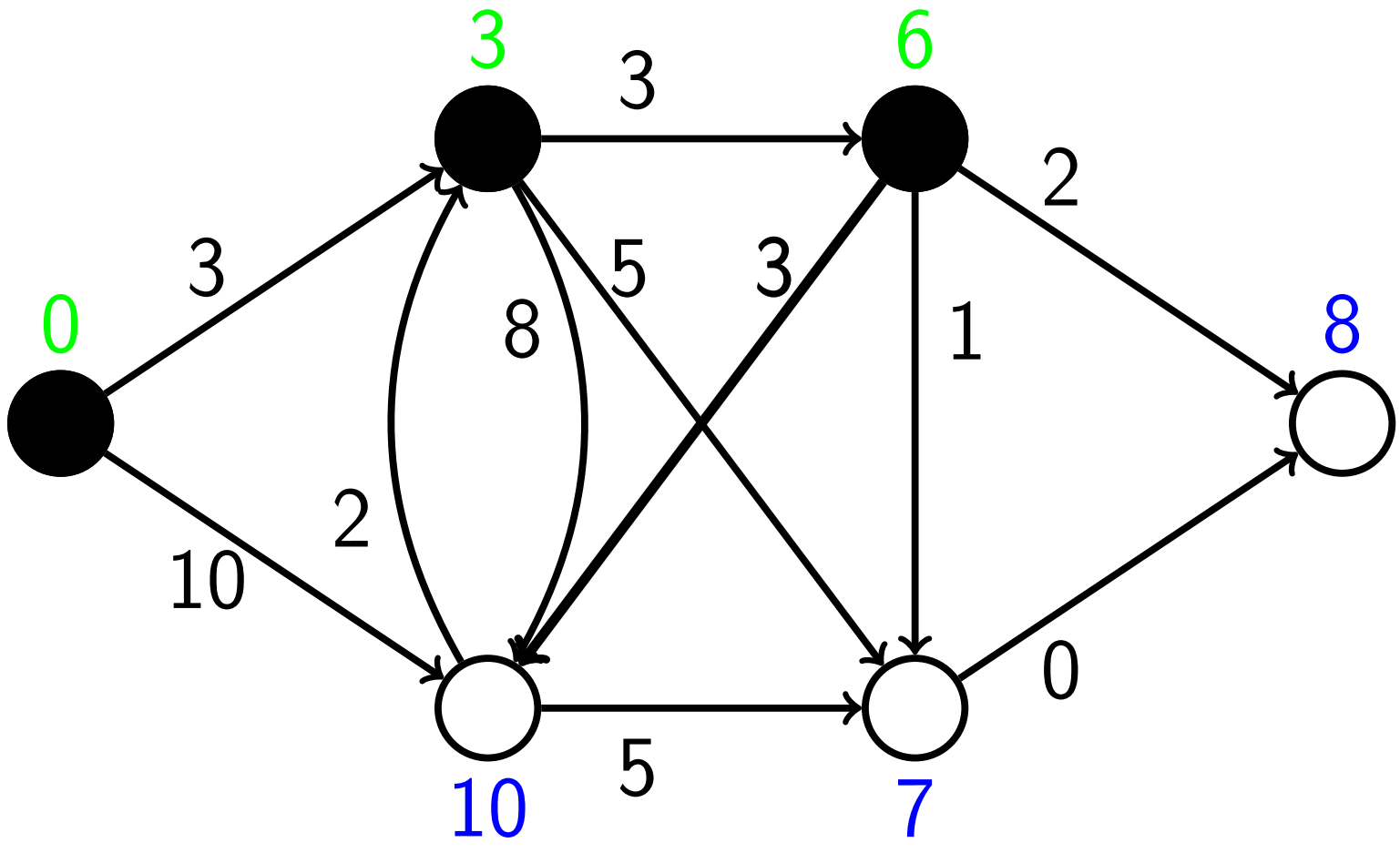

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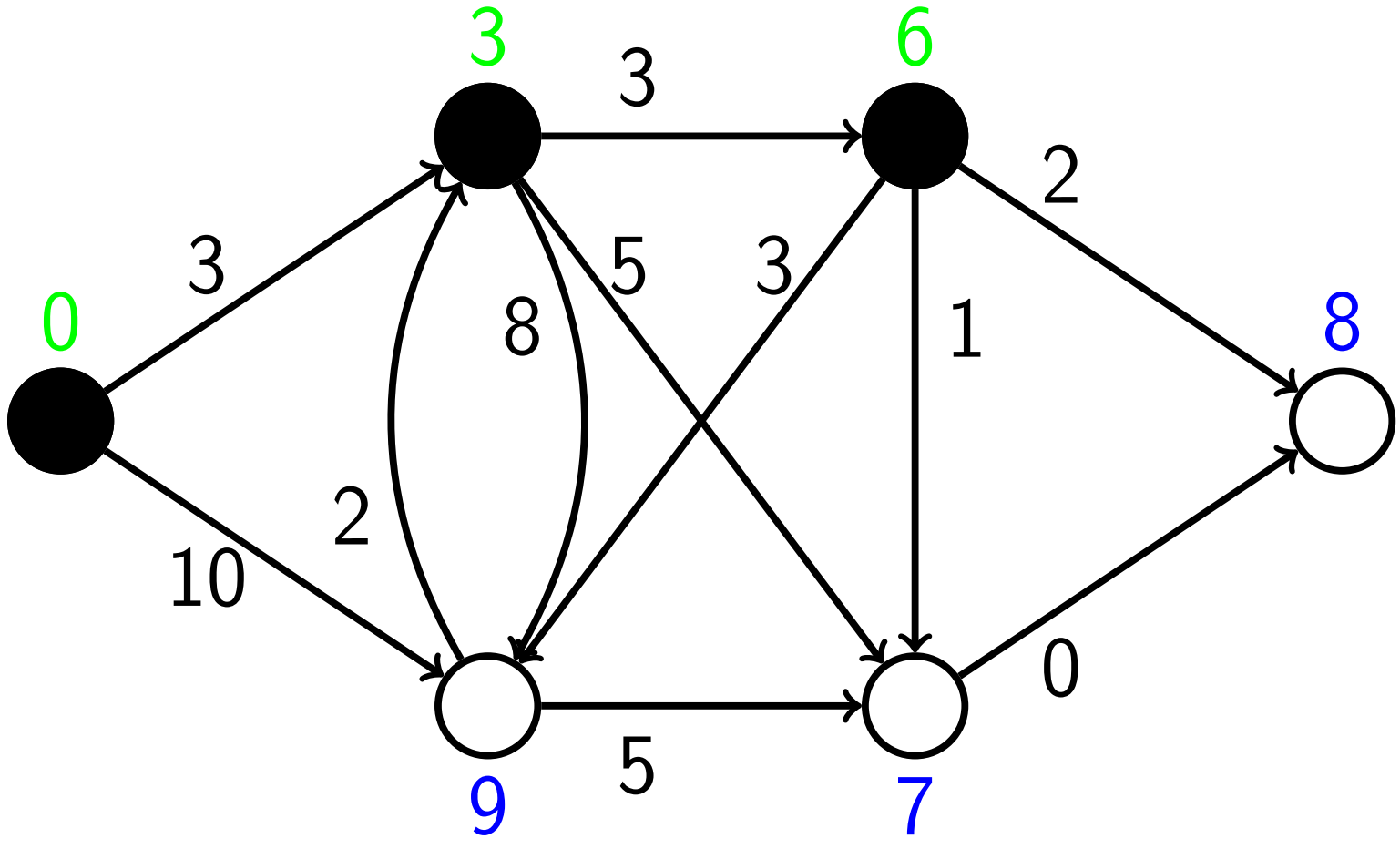

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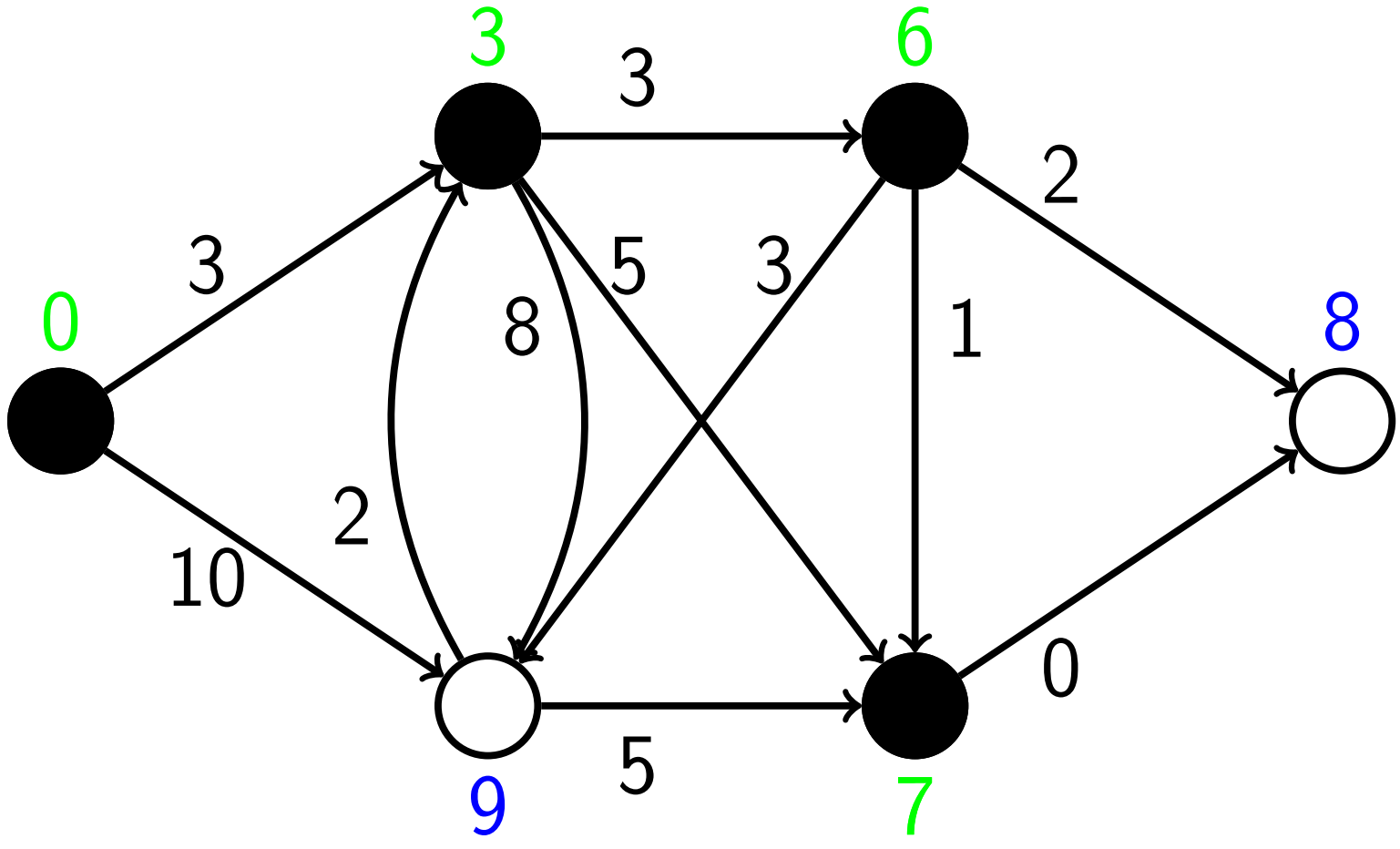

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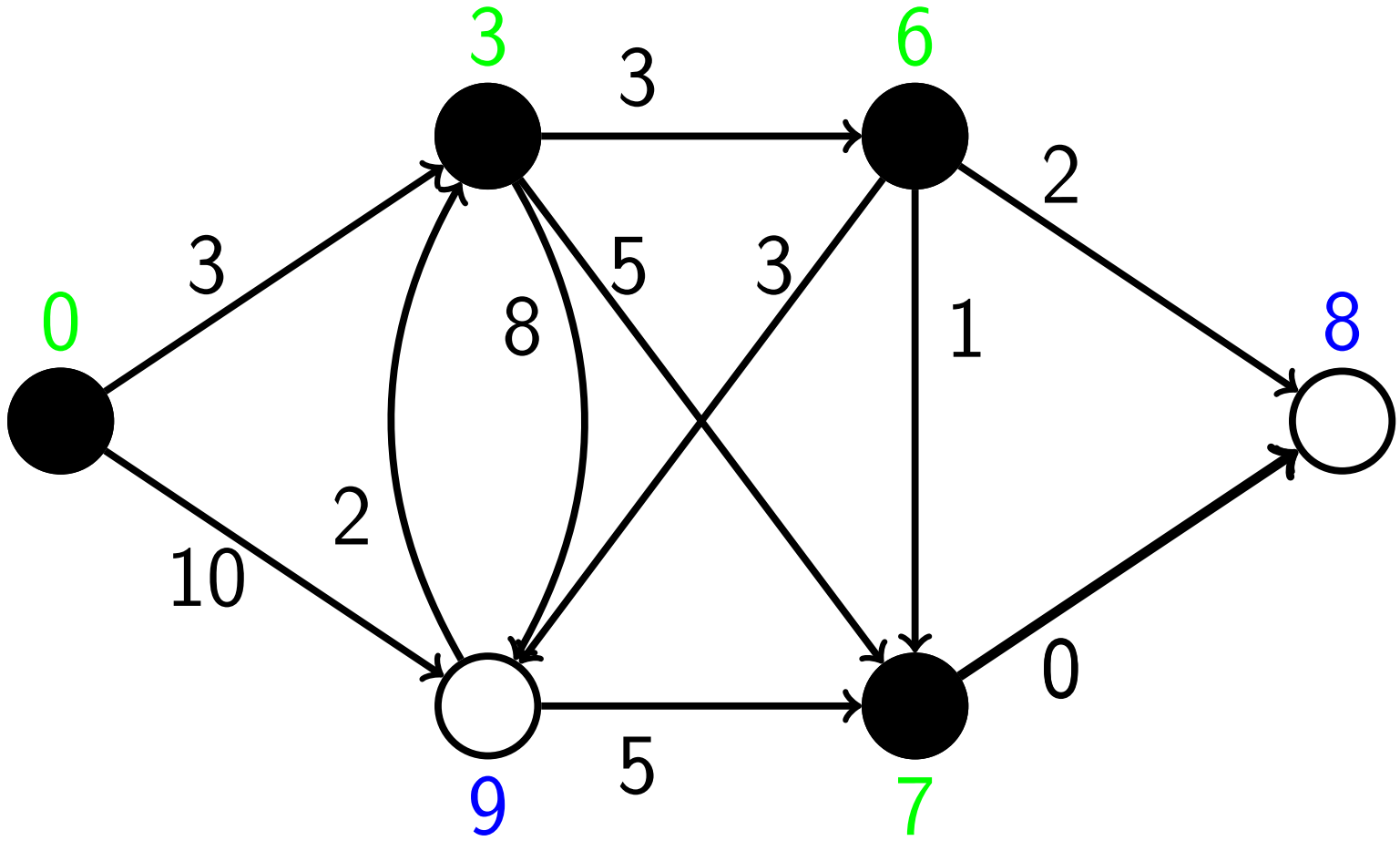

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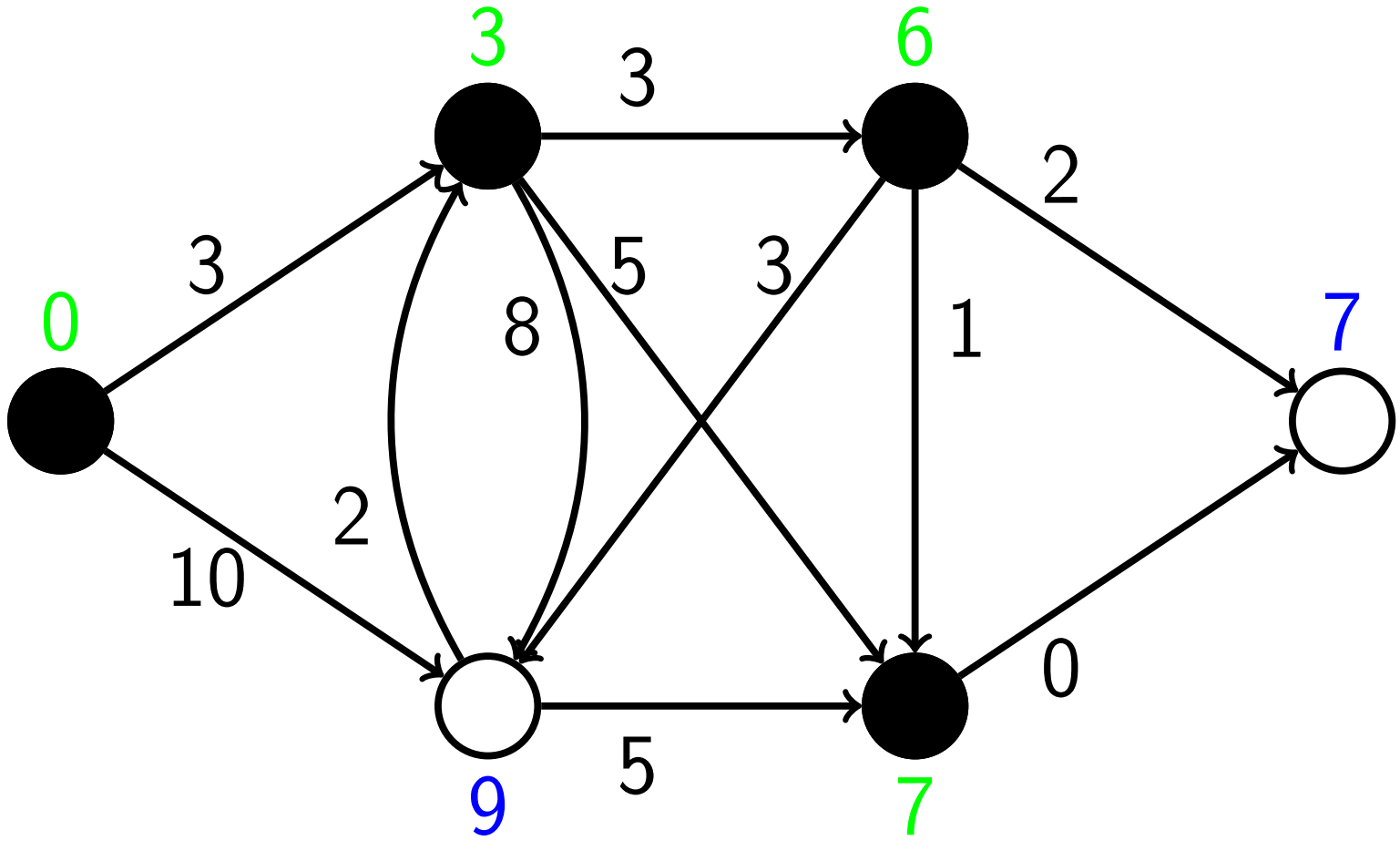

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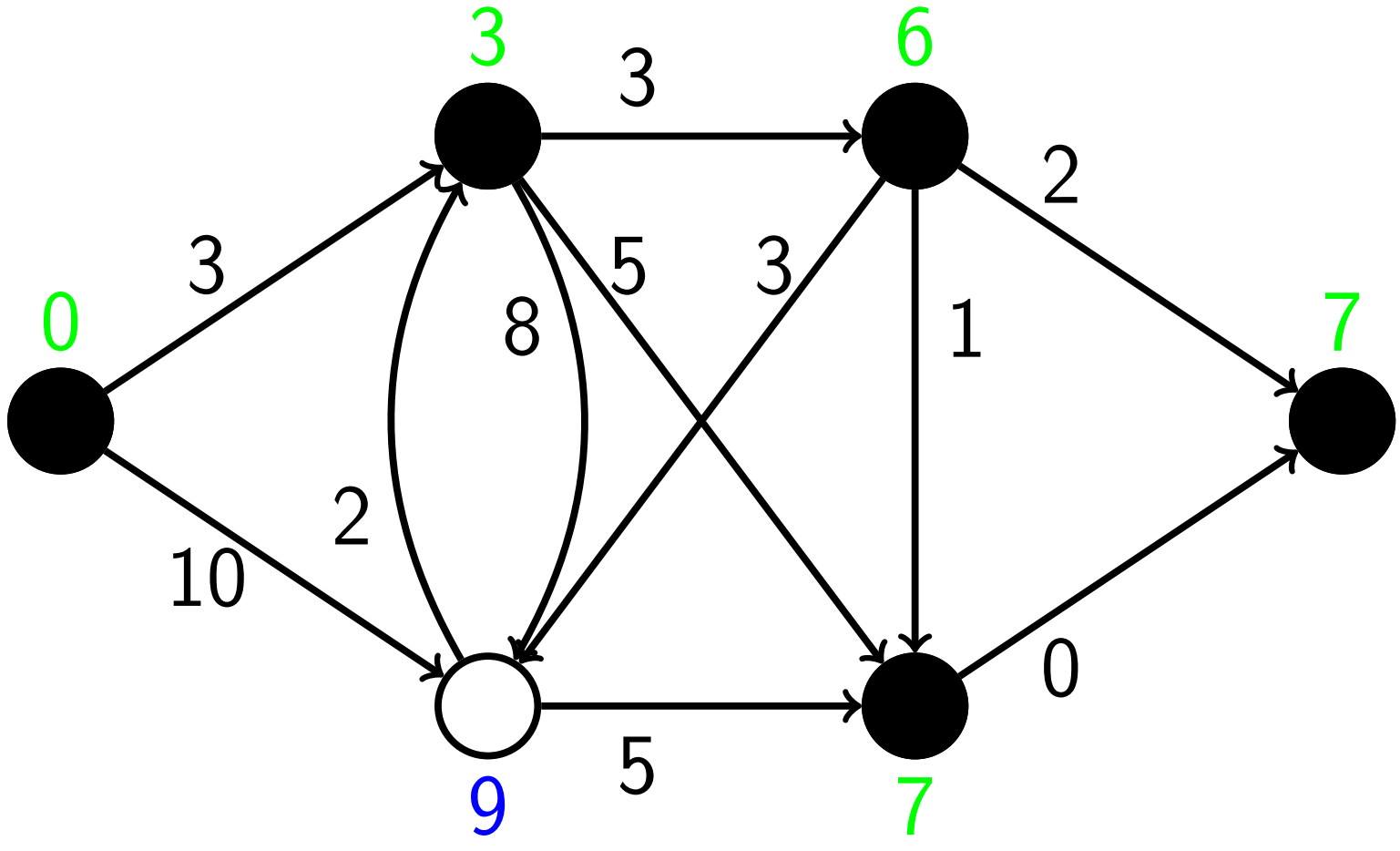

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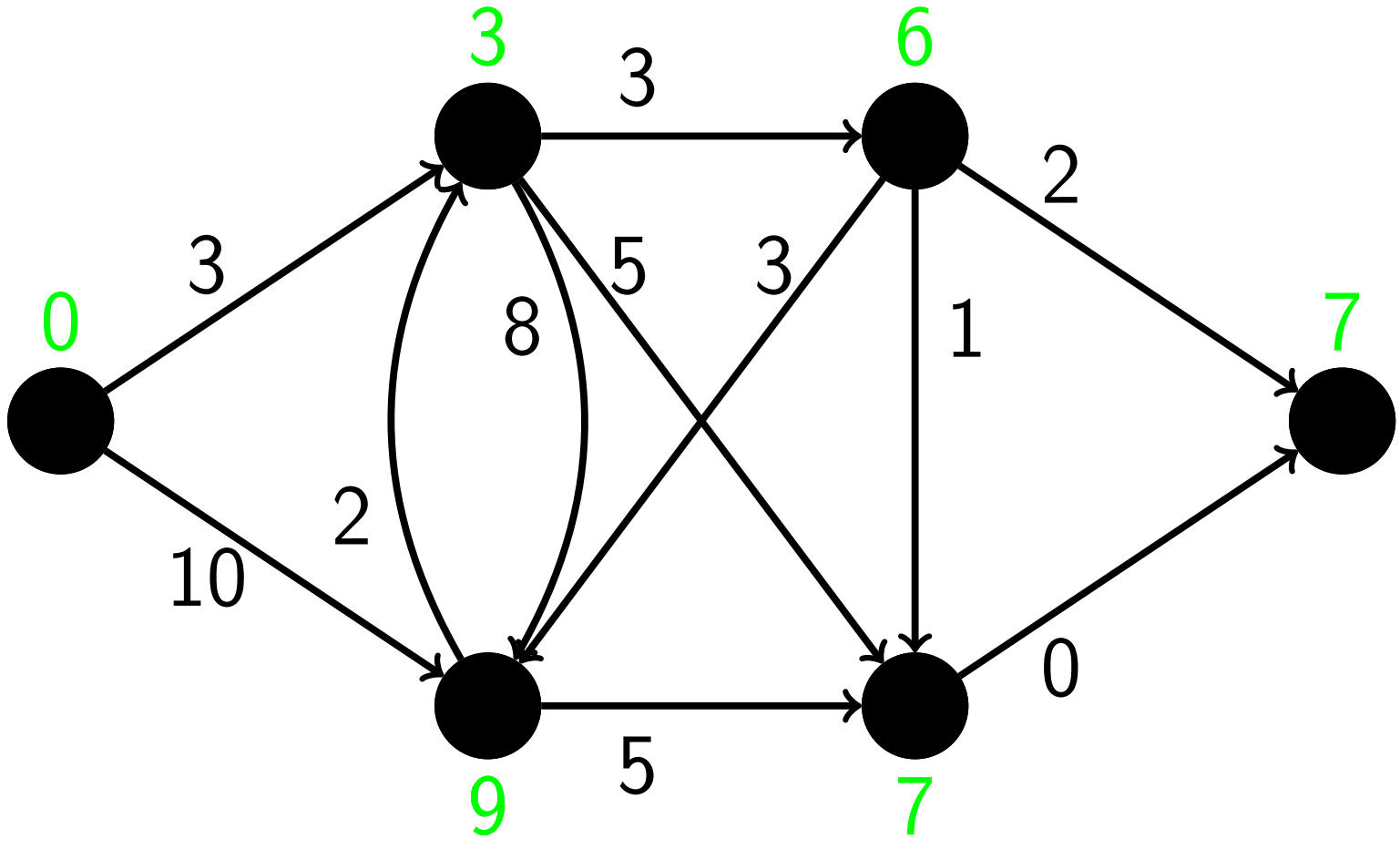
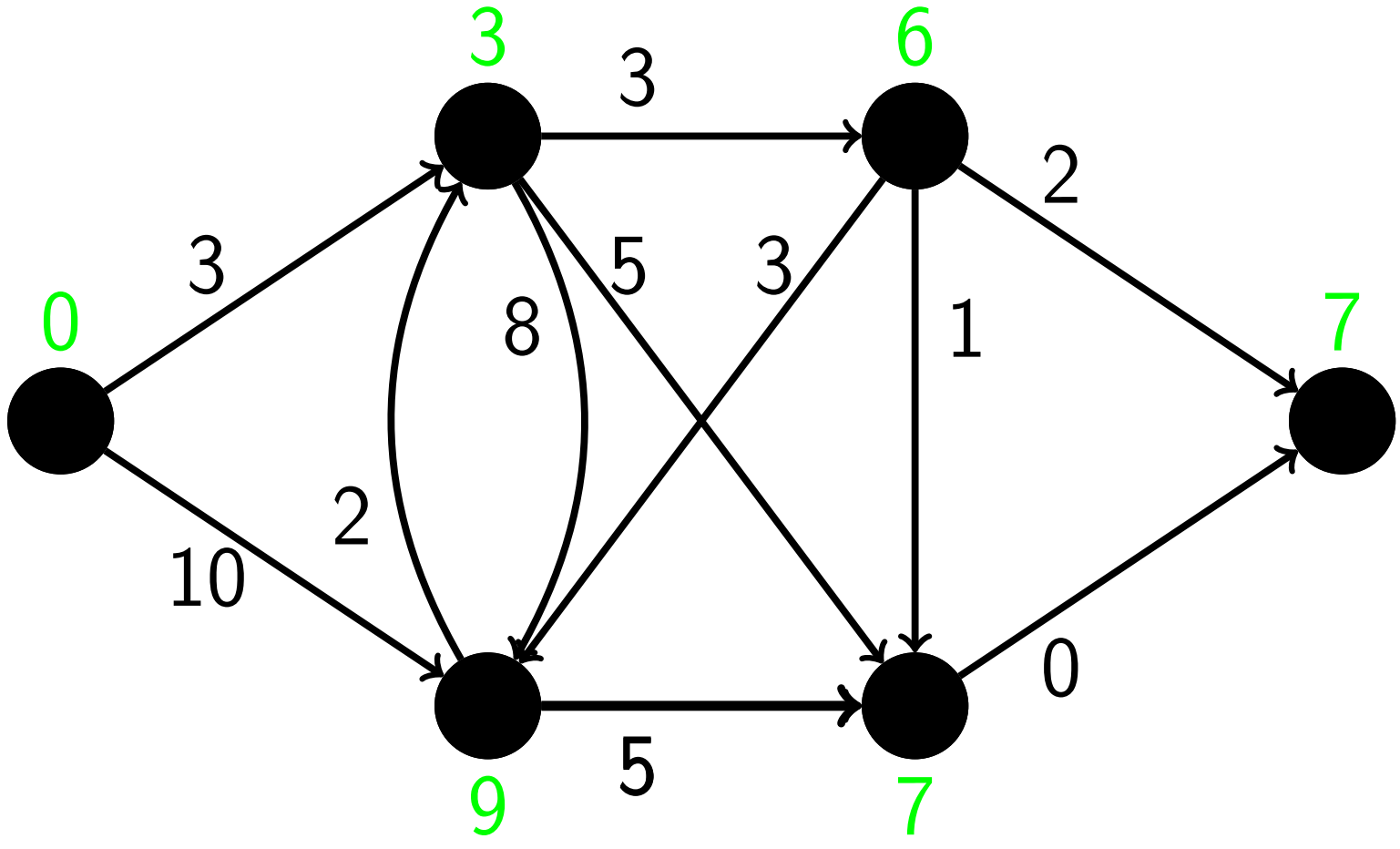


Example



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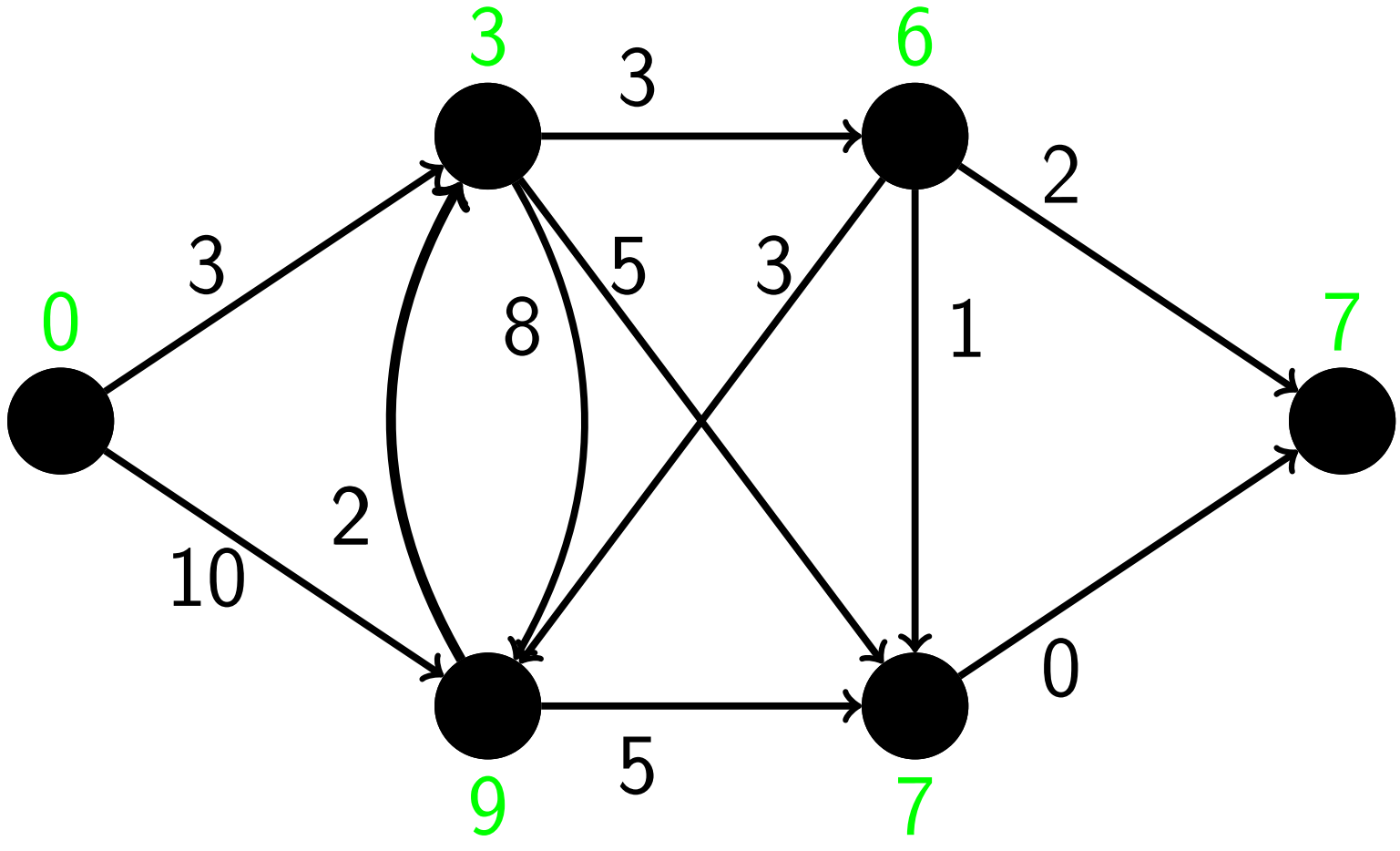

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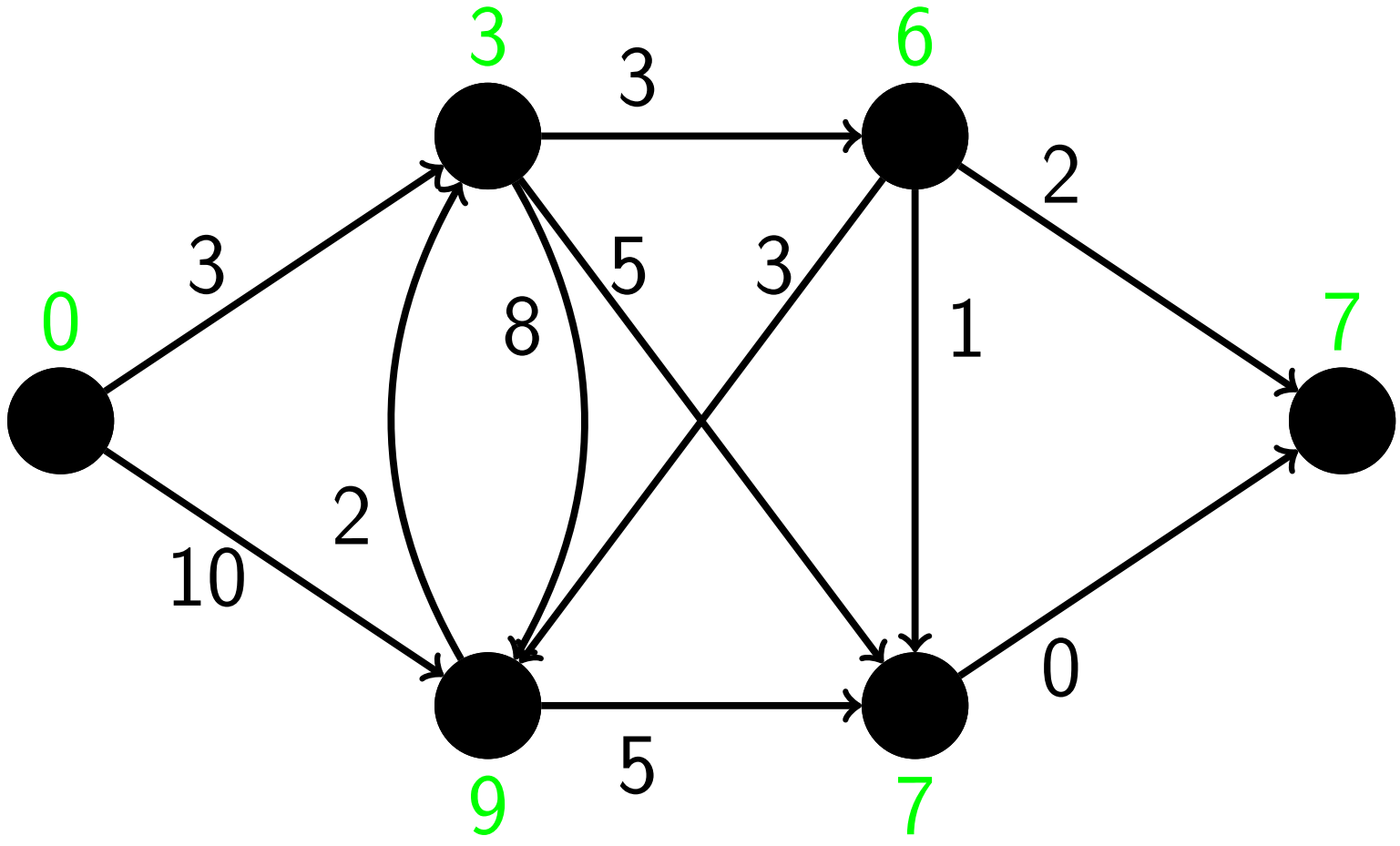

Outline

Pseudocode

Dijkstra(G, A)

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for all u ∈ V:
  dist[u] ← ∞, prev[u] ← nil
dist[A] ← 0
H ← MakeQueue(V) {dist-values as keys}
while H is not empty:
  u ← ExtractMin(H)
  for all (u, v) ∈ E:
    if dist[v] > dist[u] + w(u, v):
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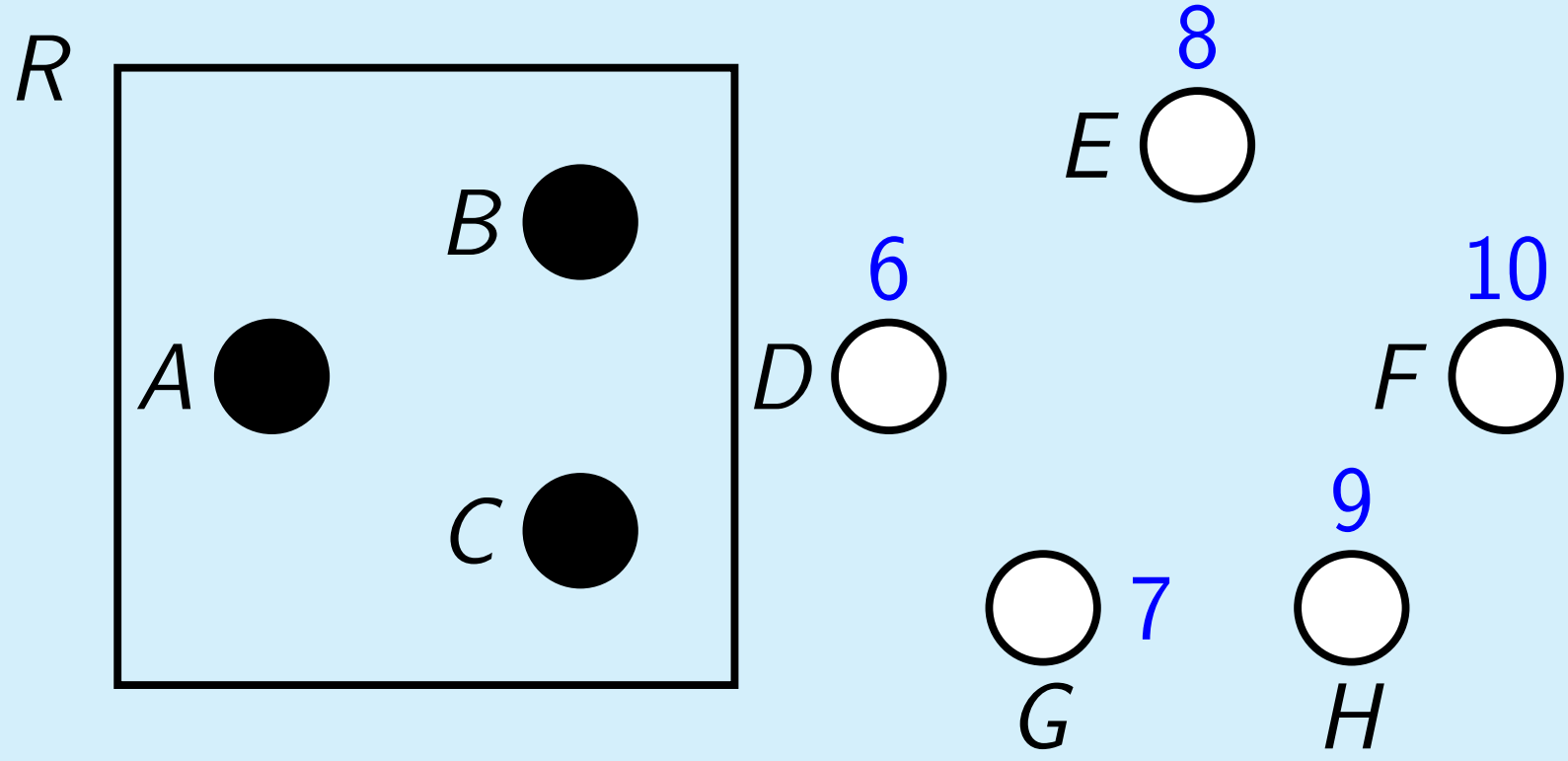
Correct distances

Lemma

When a node u is selected via `ExtractMin`, $\text{dist}[u] = d(A, u)$.

Proof



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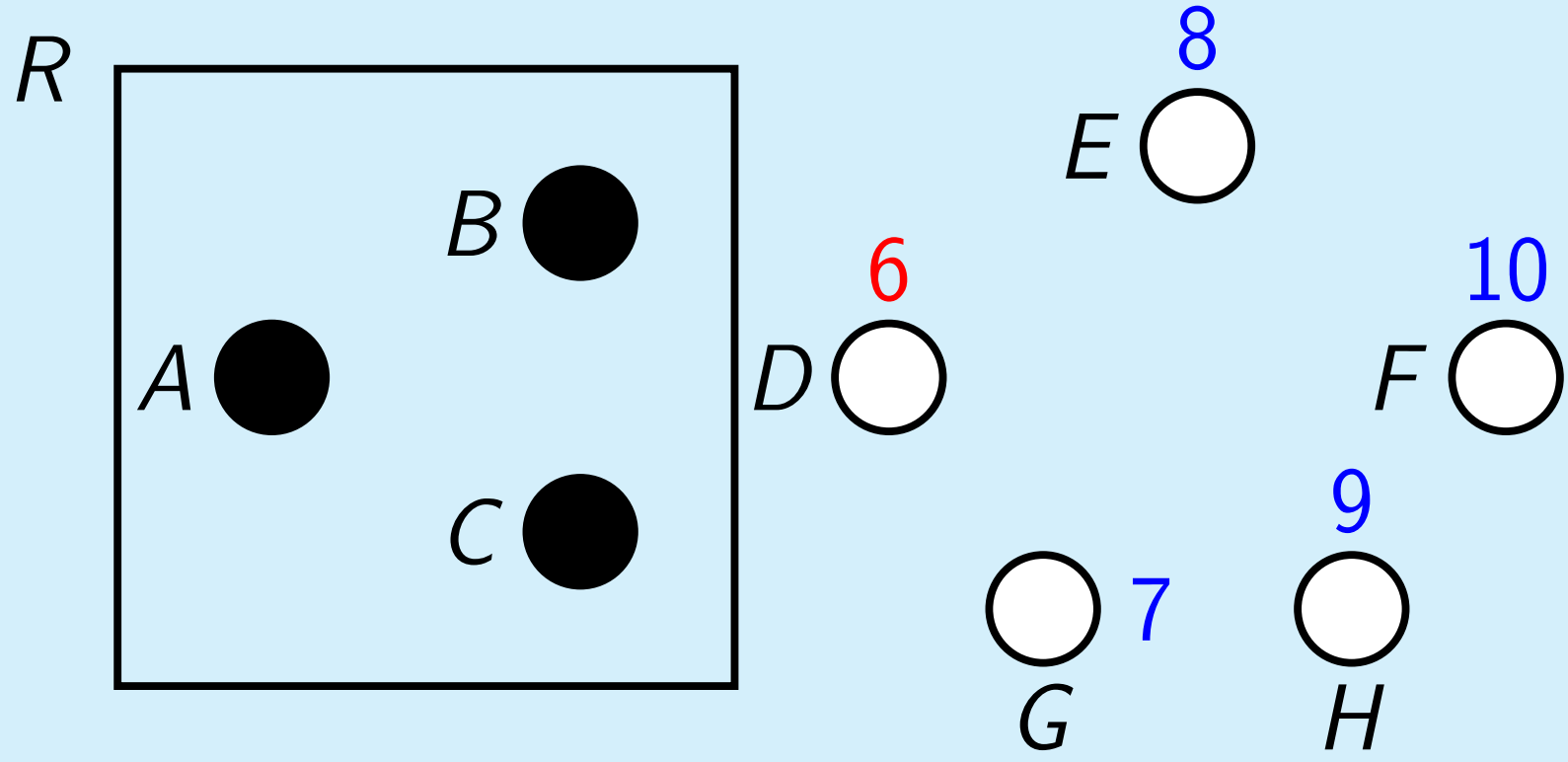

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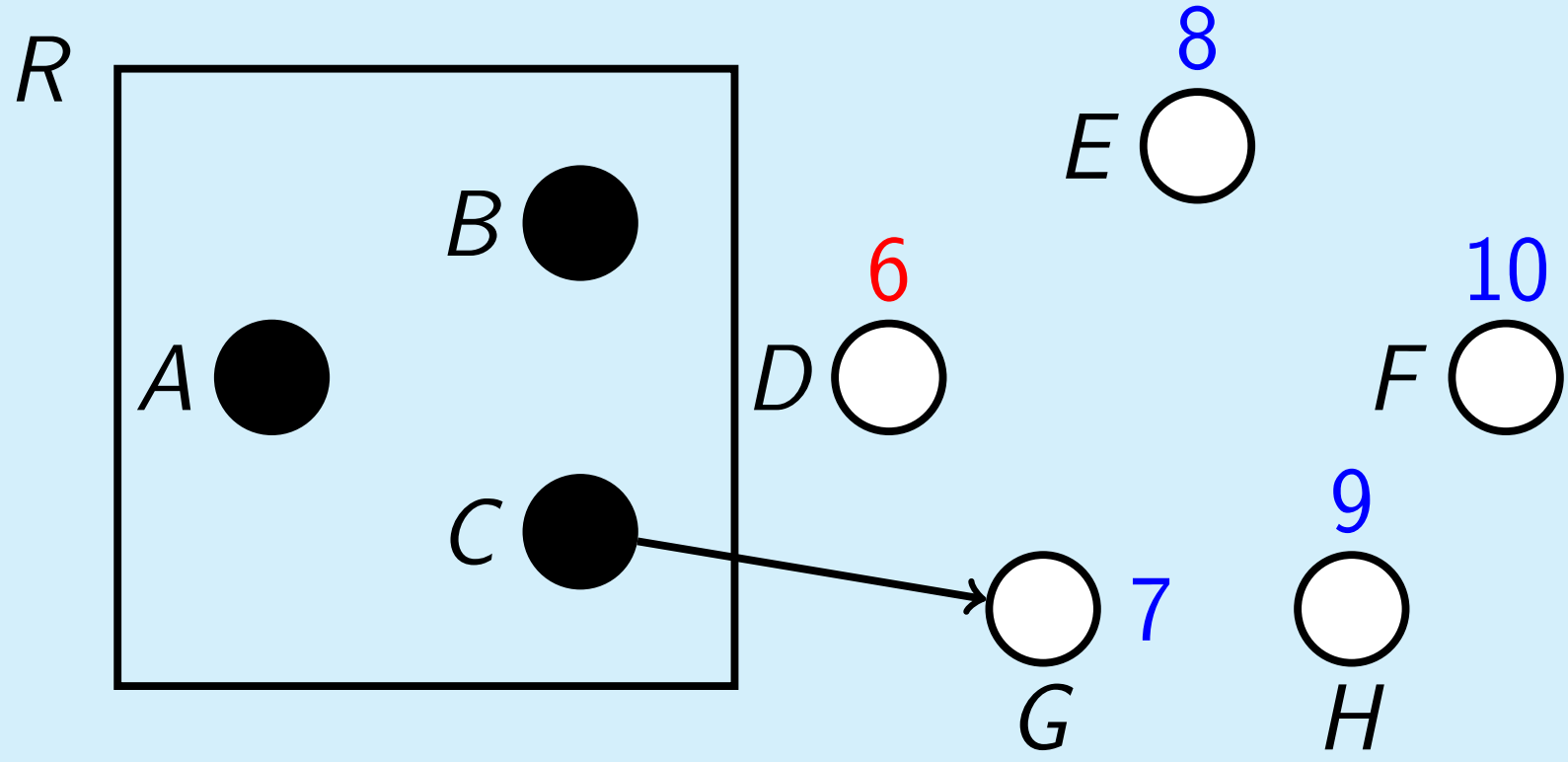

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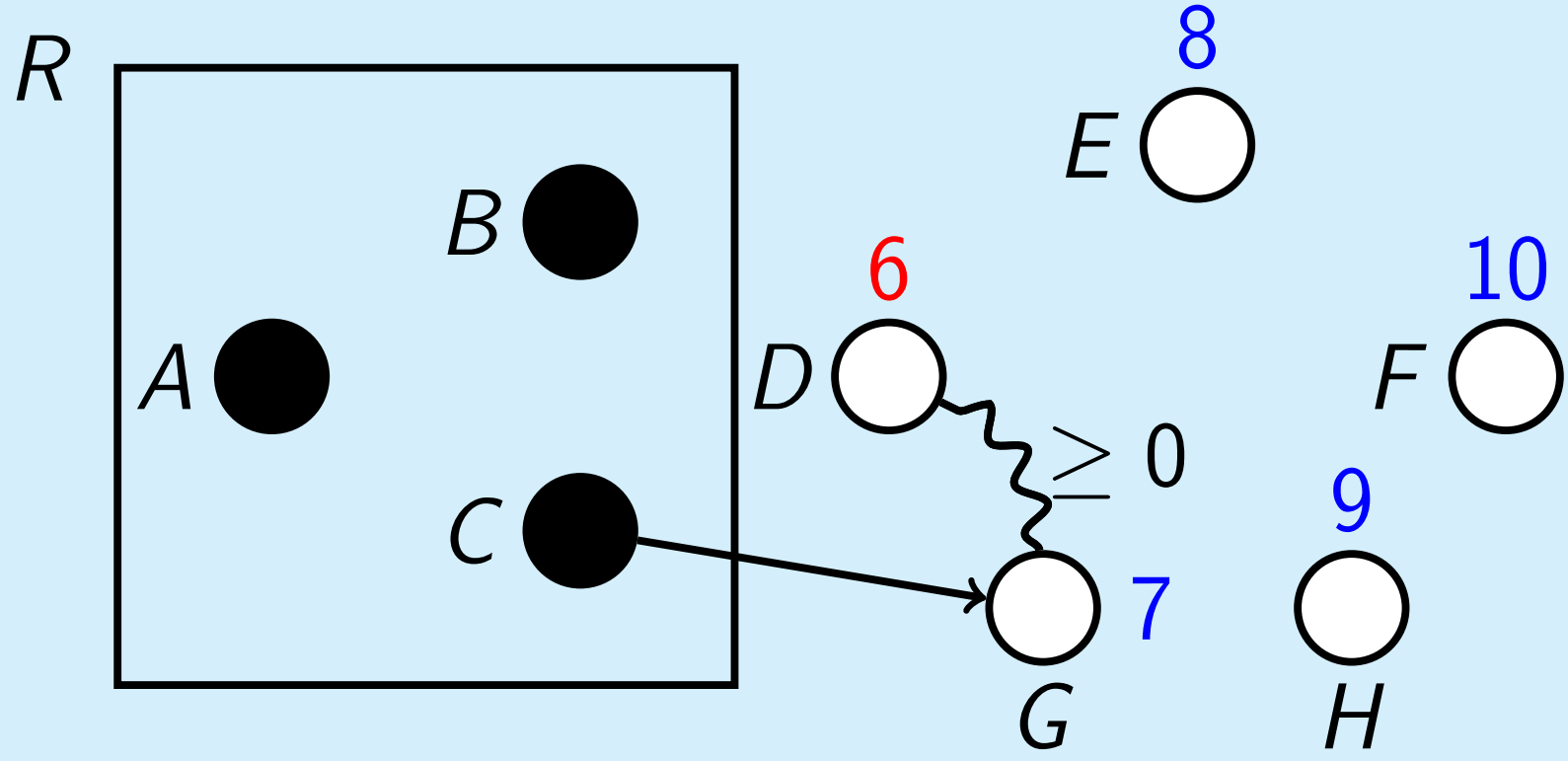

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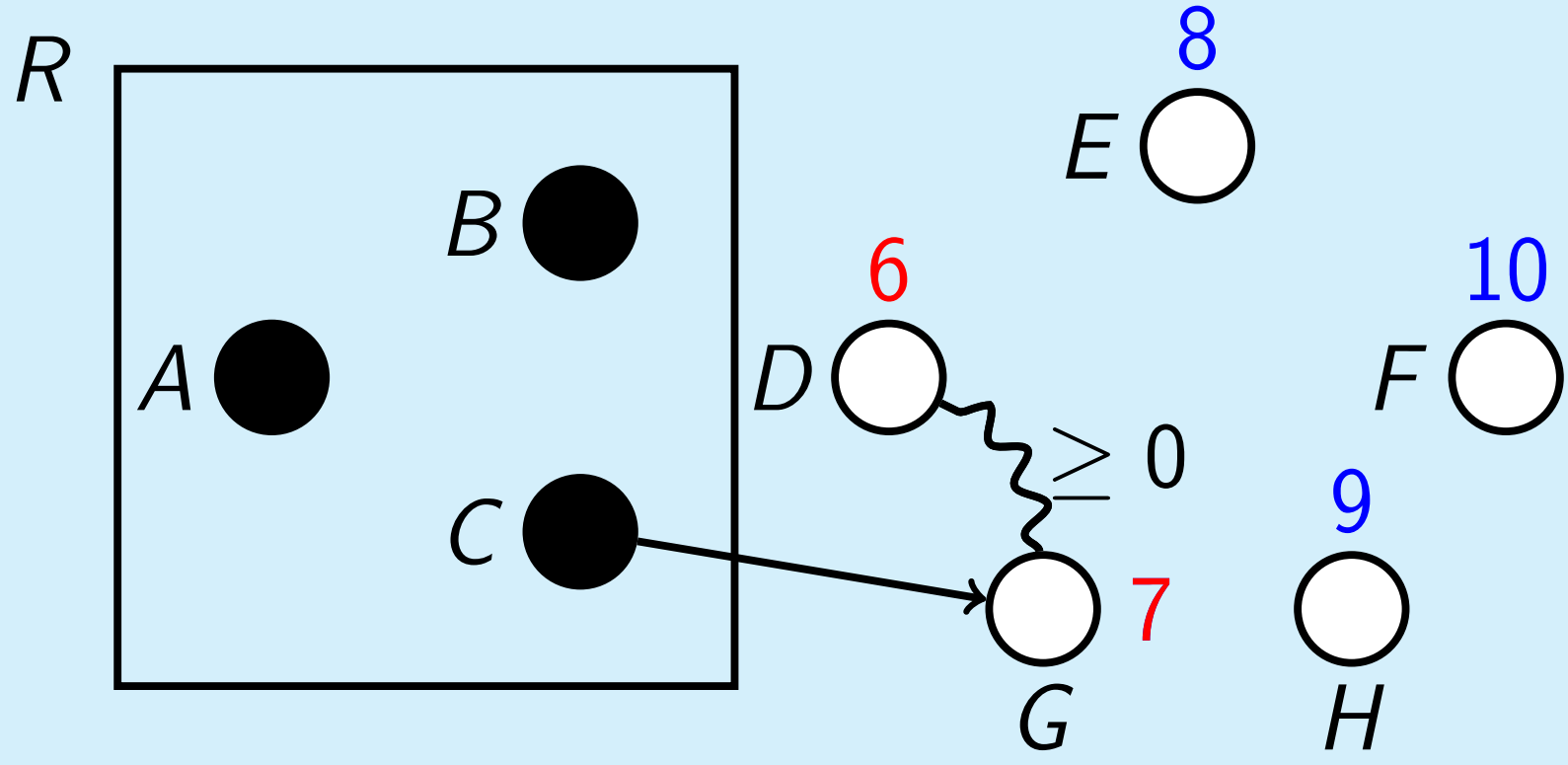

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Running time

Total running time:

$$\begin{aligned} & O(V) + T(\texttt{MakeQueue}) \\ & \quad + |V| \cdot T(\texttt{ExtractMin}) \\ & \quad + |E| \cdot T(\texttt{ChangePriority}) \end{aligned}$$

Priority queue implemented as array:
$O(|V| + |V| + |V|^2 + |E|) = O(|V|^2)$

Running time

Total running time:

$$
\begin{aligned}
& O(V) + T(\texttt{MakeQueue}) \\
& \quad + |V| \cdot T(\texttt{ExtractMin}) \\
& \quad + |E| \cdot T(\texttt{ChangePriority})
\end{aligned}
$$

Priority queue implemented as binary heap:
$O(|V| + |V| + |V| \log |V| + |E| \log |V|) =$
$O((|V| + |E|) \log |V|)$

Conclusion

- Can find the minimum time to get from work to home
- Can find the fastest route from work to home
- Works for any graph with non-negative edge weights
- Works in $O(|V|^2)$ or $O((|V| + |E|)\log(|V|))$ depending on the implementation