Binary Search Trees: Split and Merge

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Data Structures
Data Structures and Algorithms

Learning Objectives

- Implement merging and splitting of AVL trees.
- Analyze the runtime of these operations.

New Operations

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New Operations

Another useful feature of binary search trees is the ability to recombine them in interesting ways. We discuss two new operations:

- Merge Combines two binary search trees into a single one.
- Split Breaks one binary search tree into two.

Outline

1. Merge

2. Split

Merge

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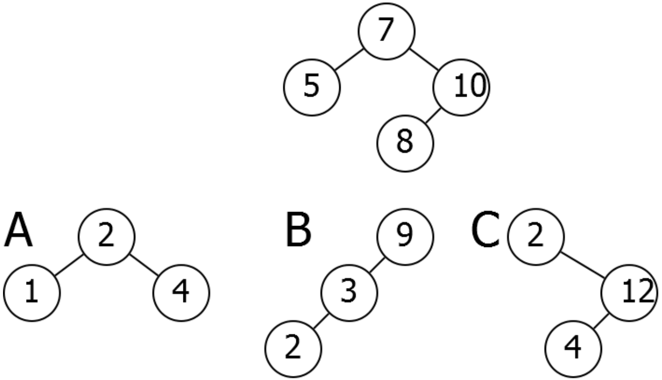
Merge

Input: Roots R_1 and R_2 of trees with all keys in R_1's tree smaller than those in R_2's

Output: The root of a new tree with all the elements of both trees

Problem

Which tree can be merged with the given one?



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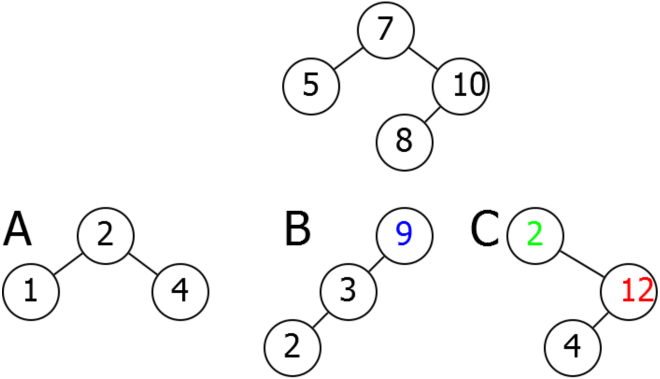
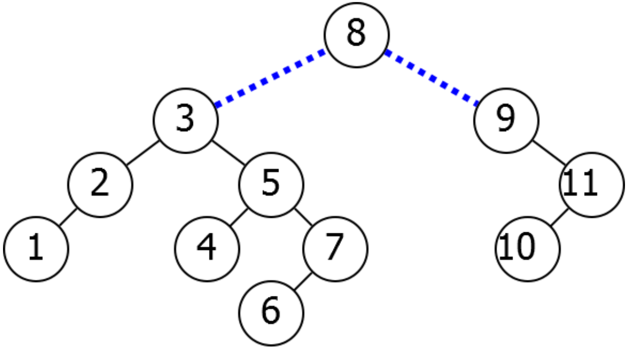


Extra Root

Easy if you have an extra node to add as root.



Implementation

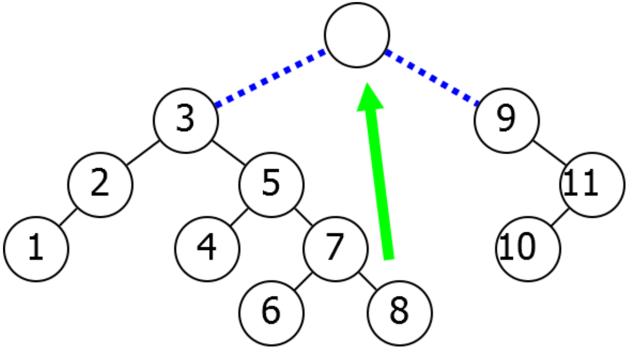
MergeWithRoot(R_1, R_2, T)

T.Left $\leftarrow R_1$
T.Right $\leftarrow R_2$
R_1.Parent $\leftarrow T$
R_2.Parent $\leftarrow T$
return T

Time $O(1)$.

Get Root

Get new root by removing largest element of left subtree.



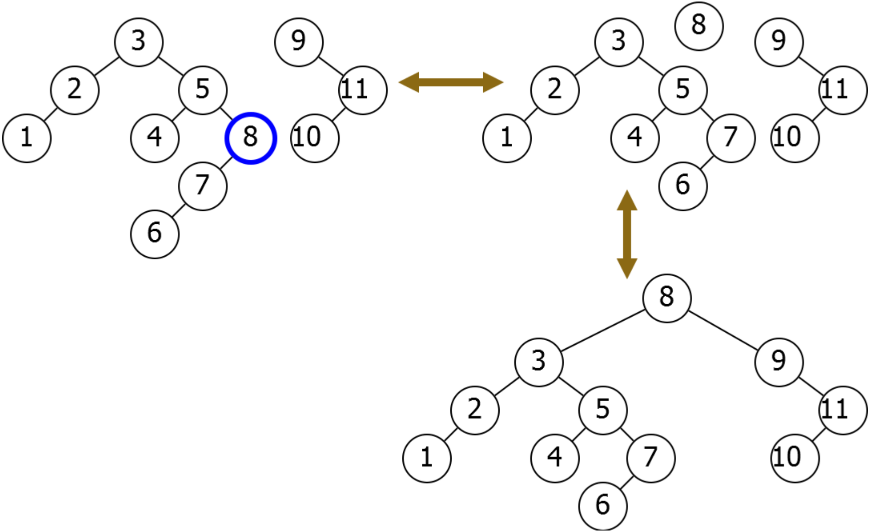
Merge

$\texttt{Merge}(R_1, R_2)$

$T \leftarrow \texttt{Find}(\infty, R_1)$
$\texttt{Delete}(T)$
$\texttt{MergeWithRoot}(R_1, R_2, T)$
$\texttt{return } T$

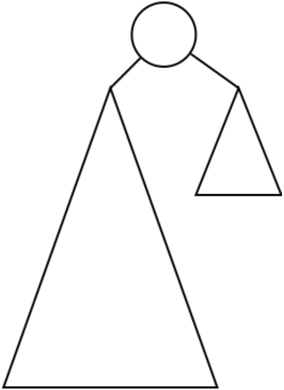
Time $O(h)$.

Merge



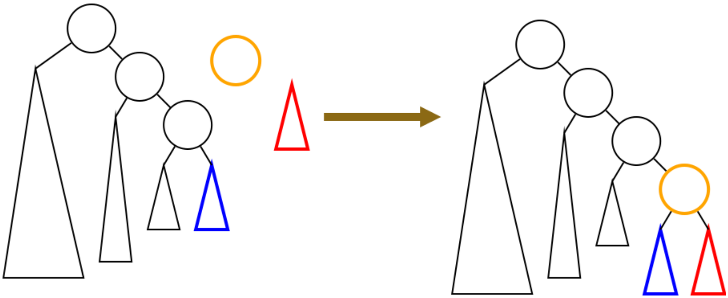
Balance

Unfortunately, this merge does not preserve balance properties.



Idea

Go down side of tree until merge with subtree of same height.



Implementation

$\texttt{AVLTreeMergeWithRoot}(R_1, R_2, T)$

$\texttt{if}\ |R_1.\texttt{Height} - R_2.\texttt{Height}| \leq 1:$

$\quad \texttt{MergeWithRoot}(R_1, R_2, T)$

$\quad T.\texttt{Ht} \leftarrow \max(R_1.\texttt{Height}, R_2.\texttt{Height}) + 1$

$\quad \texttt{return}\ T$

Implementation (continued)

```
AVLTreeMergeWithRoot(R_1, R_2, T)

else if R_1.Height > R_2.Height:
   R' ← AVLTreeMWR(R_1.Right, R_2, T)
   R_1.Right ← R'
   R'.Parent ← R_1
   Rebalance(R_1)
   return root
else if R_1.Height < R_2.Height:
   ...
```

Analysis

- Each step changes height difference by 1 or 2.
- Eventually within 1.
- Time $O(|R_1.\texttt{Height} - R_2.\texttt{Height}| + 1)$.

Outline

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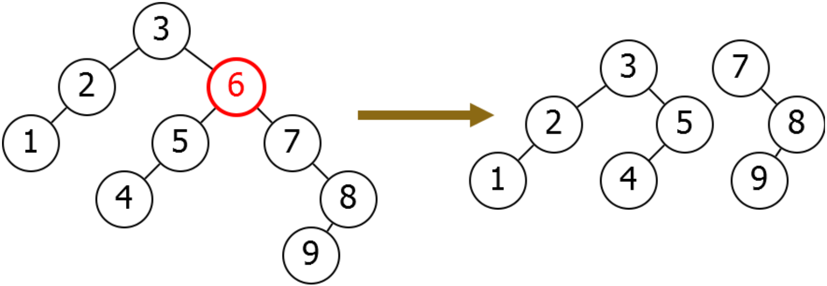
Split

Break tree into two trees:



Formal Definition

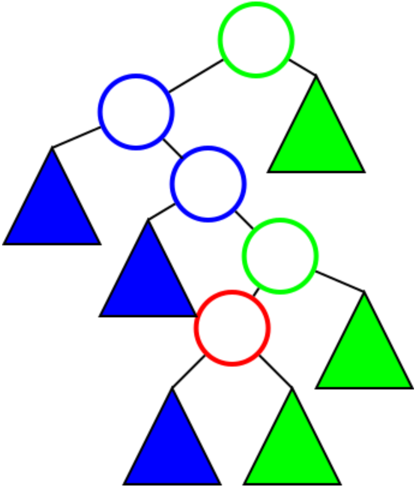
Split

Input: Root R of a tree, key x

Output: Two trees, one with elements $\leq x$, one with elements $> x$.

Idea

Search for x, merge subtrees.



Implementation

$\texttt{Split}(R, x)$

$\texttt{if}\ \ R = \mathit{null}:$

$\quad \texttt{return}\ \ (\mathit{null}, \mathit{null})$

$\texttt{if}\ \ x \leq R.\texttt{Key}:$

$\quad (R_1, R_2) \leftarrow \texttt{Split}(R.\texttt{Left}, x)$

$\quad R_3 \leftarrow \texttt{MergeWithRoot}(R_2, R.\texttt{Right}, R)$

$\quad \texttt{return}\ \ (R_1, R_3)$

$\texttt{if}\ \ x > R.\texttt{Key}:$

$\quad \ldots$

AVL Trees

- Using `AVLMergeWithRoot` maintains balance.
- Time $= \sum O(|h_i - h_{i+1}| + 1) = O(h_{max}) = O(\log(n))$.

Conclusion

Summary

- `Merge` combines trees.
- `Split` turns one tree into two.
- Both can be implemented in $O(\log(n))$ time for AVL trees.