

Decomposition of Graphs: Exploring Graphs

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Graph Algorithms
Data Structures and Algorithms

Learning Objectives

- Implement the explore algorithm.
- Figure out whether or not one vertex of a graph is reachable from another.

Outline

- 1 Problem Discussion
- 2 Ideas
- 3 Explore
- 4 Correctness
- 5 DFS

Motivation

You're playing a video game and want to make sure that you've found everything in a level before moving on.
How do you ensure that you accomplish this?

Examples

This notion of exploring a graph has many applications:

- Finding routes
- Ensuring connectivity
- Solving puzzles and mazes

Paths

We want to know what is reachable from a given vertex.

Definition

A **path** in a graph G is a sequence of vertices v_0, v_1, \dots, v_n so that for all i , (v_i, v_{i+1}) is an edge of G .

Formal Description

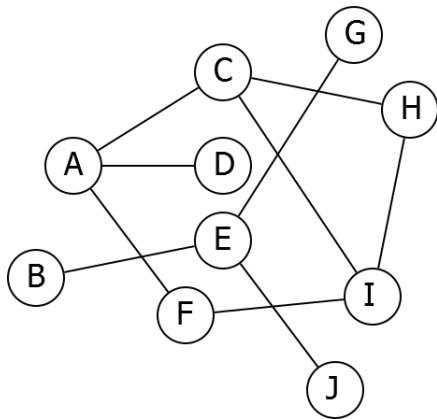
Reachability

Input: Graph G and vertex s

Output: The collection of vertices v of G so that there is a path from s to v .

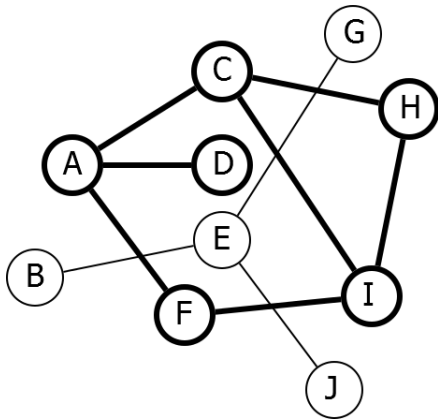
Problem

Which vertices are reachable from *A*?



Solution

A, C, D, F, H, I.

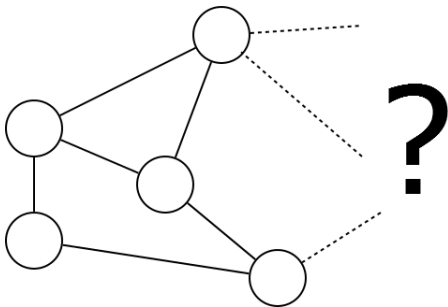


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Basic Idea

We want to make sure that we have explored every edge leaving every vertex we have found.



Pseudocode

Component(*s*)

```
DiscoveredNodes  $\leftarrow \{s\}$   
while there is an edge e leaving  
DiscoveredNodes that has not been  
explored:  
    add vertex at other end of e to  
    DiscoveredNodes  
return DiscoveredNodes
```

Formal Specification

We need to do some work to handle the bookkeeping for this algorithm.

- How do we keep track of which edges/vertices we have dealt with?
- What order do we explore new edges in?

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Visit Markers

To keep track of vertices found:

Give each vertex boolean `visited(v)`.

Unprocessed Vertices

Keep a list of vertices with edges left to check.

This will end up getting hidden in the program stack.

Depth First Ordering

We will explore new edges in **Depth First** order. We will follow a long path forward, only backtracking when we hit a dead end.

Explore

Explore(v)

visited(v) \leftarrow true

for (v, w) $\in E$:

 if not visited(w):

 Explore(w)

Explore

Explore(v)

visited(v) \leftarrow true

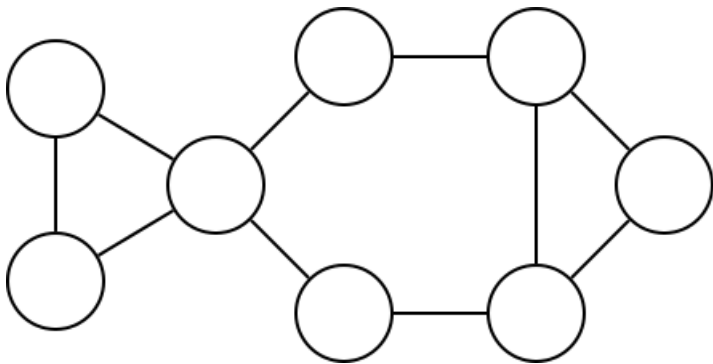
for (v, w) $\in E$:

 if not visited(w):

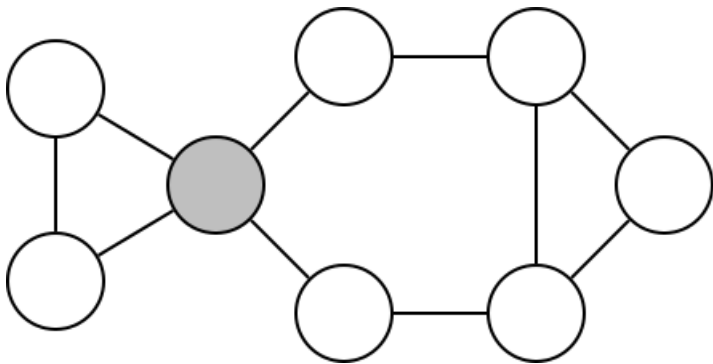
 Explore(w)

Need adjacency list representation!

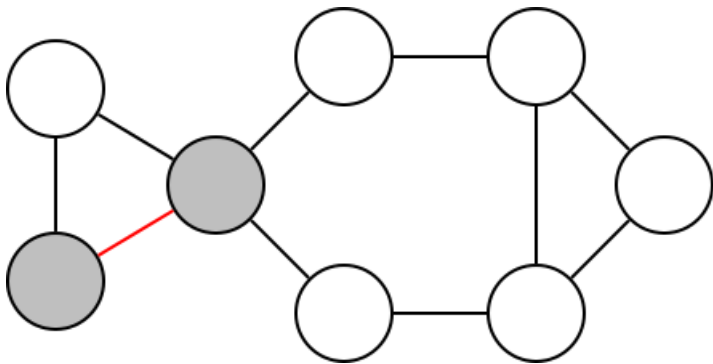
Example



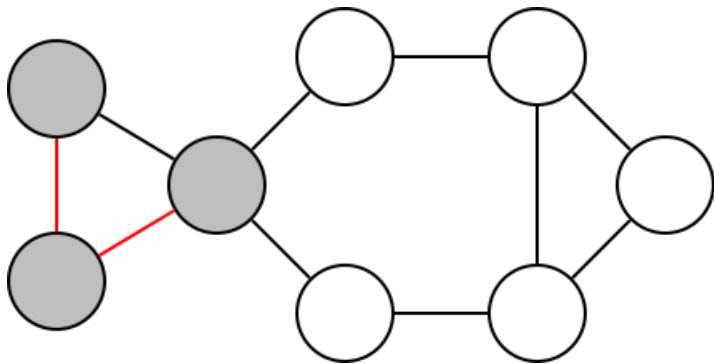
Example



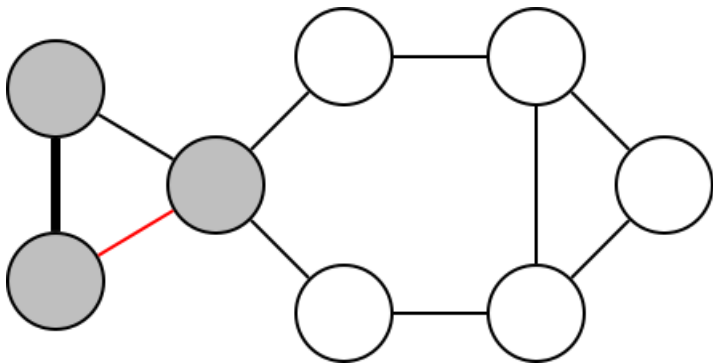
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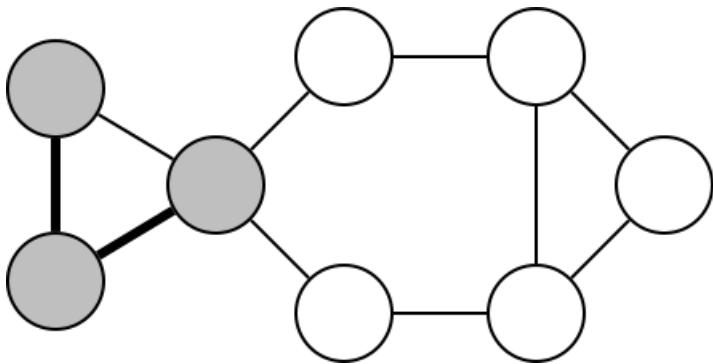
Example



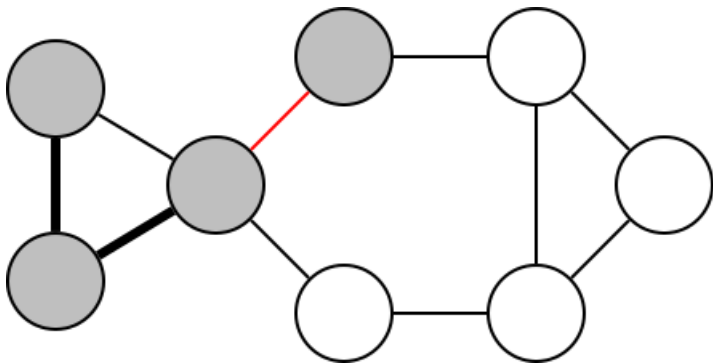
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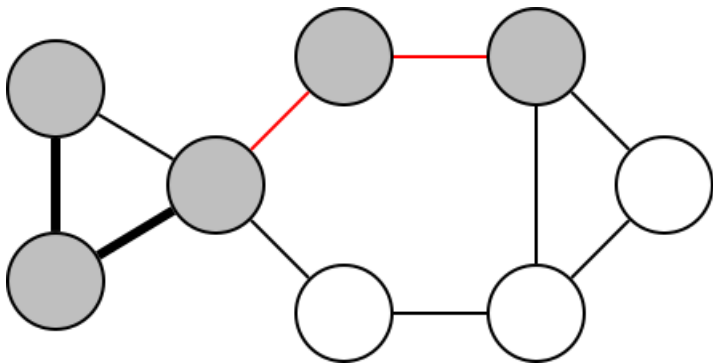
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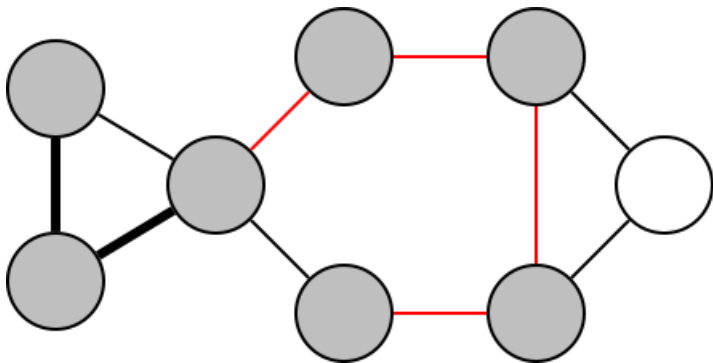
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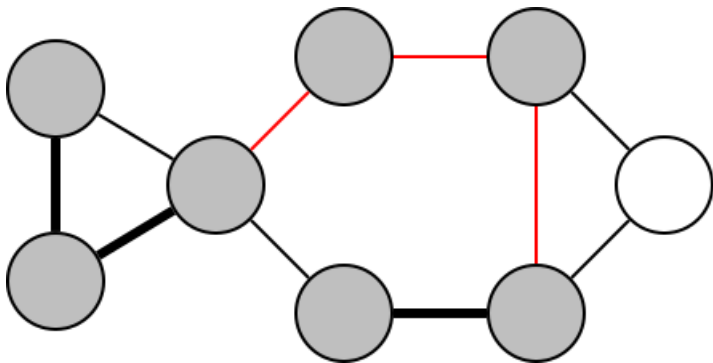
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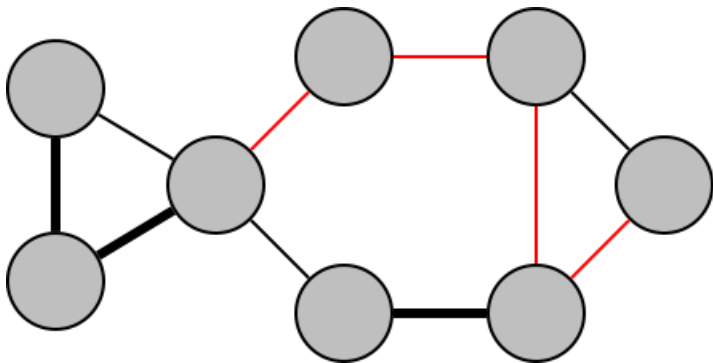
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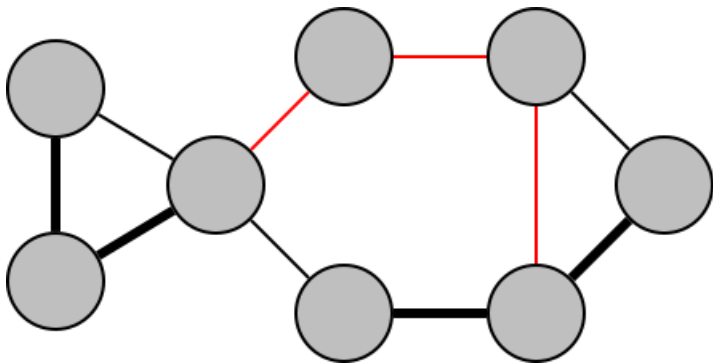
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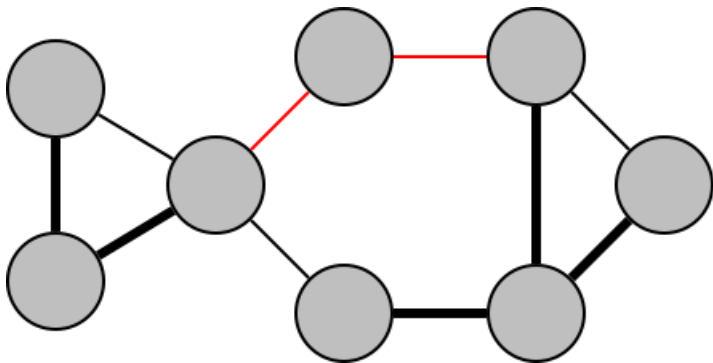
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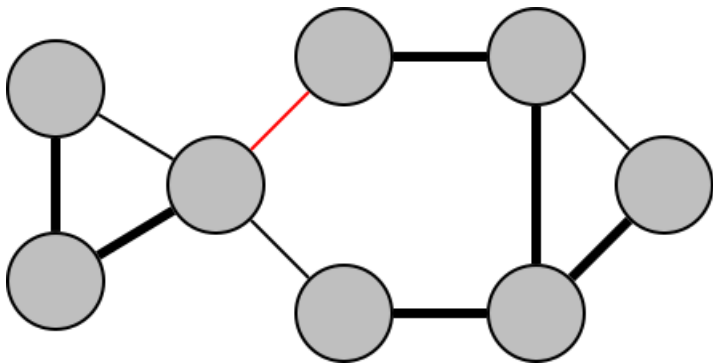
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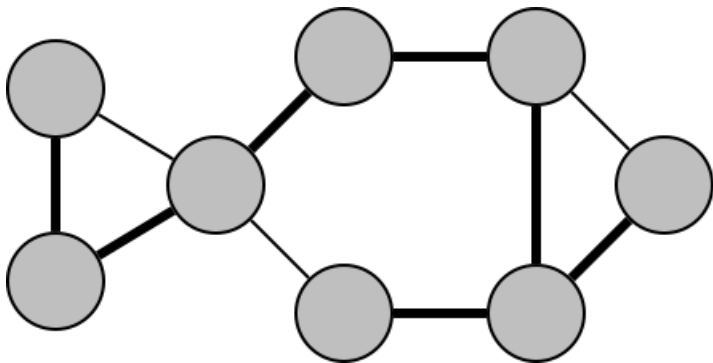
Example



Example



Example



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Result

Theorem

If all vertices start unvisited, $\text{Explore}(v)$ marks as visited exactly the vertices reachable from v .

Proof

Proof.

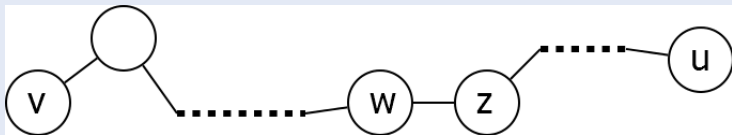
- Only explores things reachable from v .
- w not marked as visited unless explored.
- If w explored, all neighbors explored.



Proof (continued)

Proof.

- u reachable from v by path.
- Assume w furthest along path explored.



- Must explore next item.



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Reach all Vertices

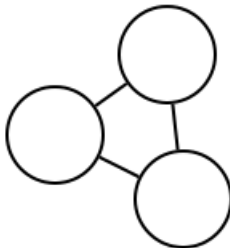
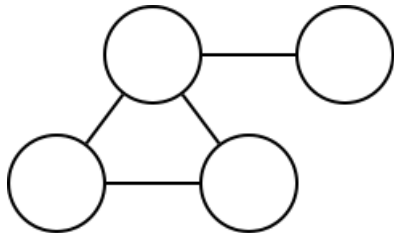
Sometimes you want to find all vertices of G , not just those reachable from v .

DFS

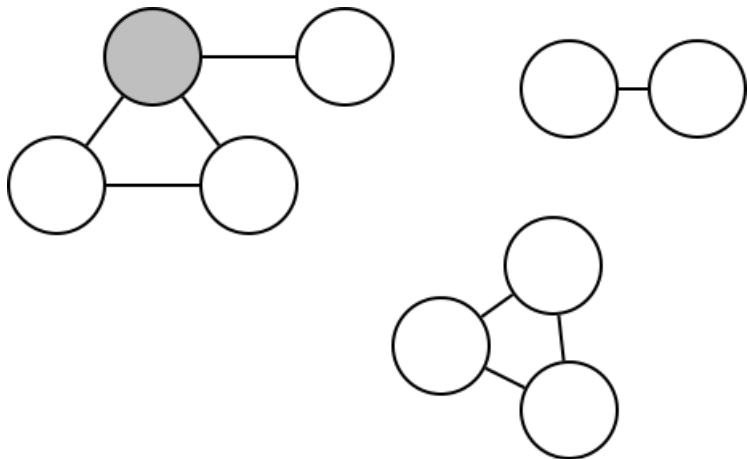
DFS(G)

```
for all  $v \in V$ :      mark  $v$  unvisited
for  $v \in V$ :
    if not visited( $v$ ):
        Explore( $v$ )
```

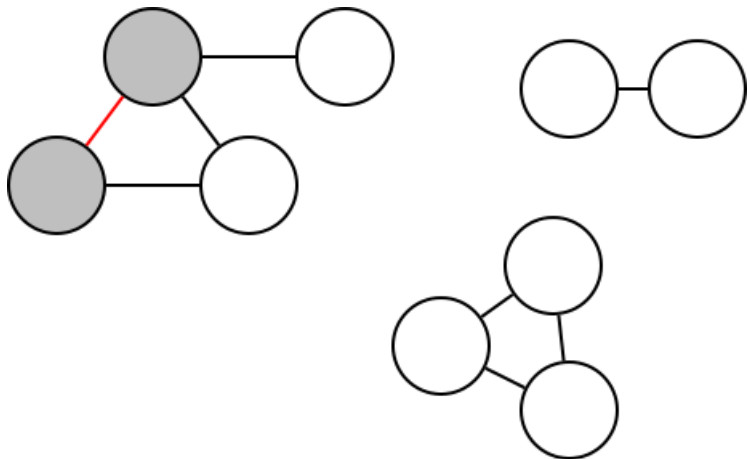
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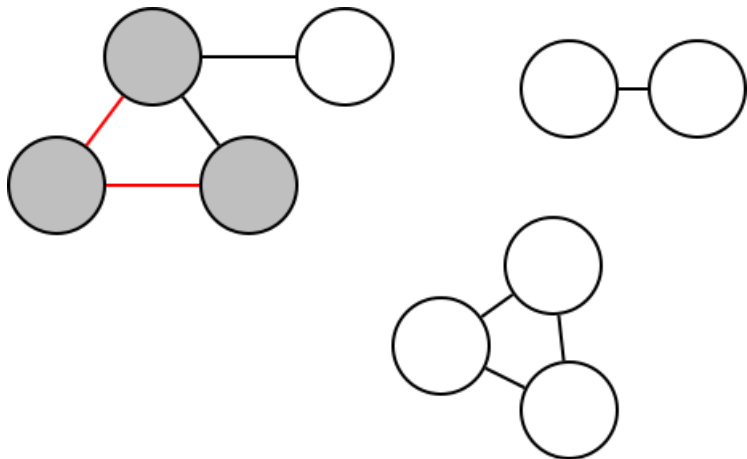
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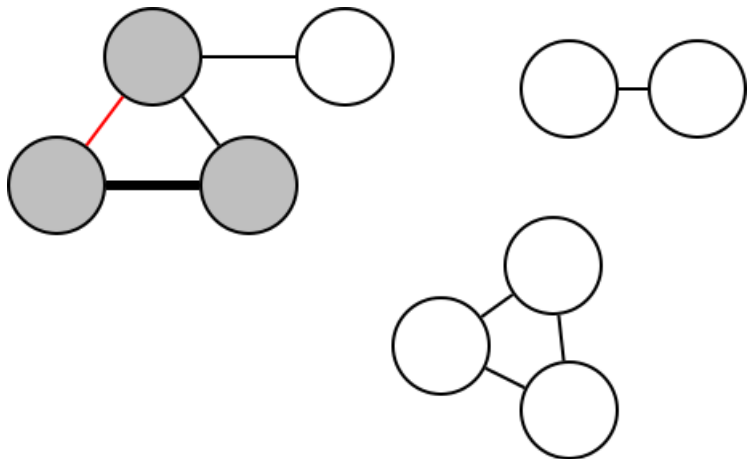
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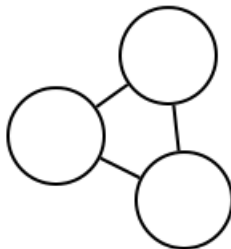
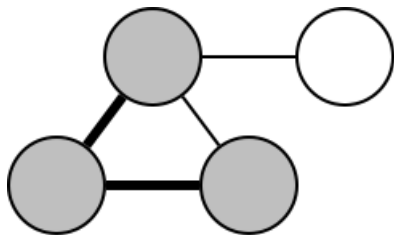
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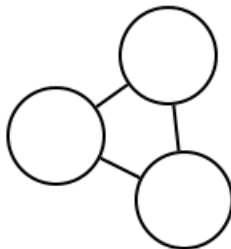
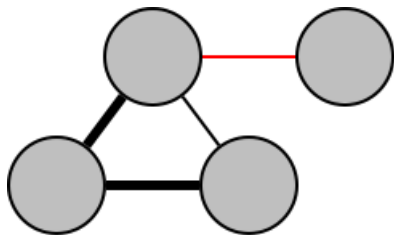
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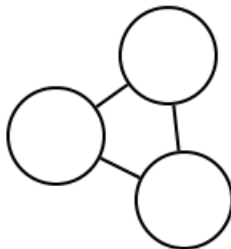
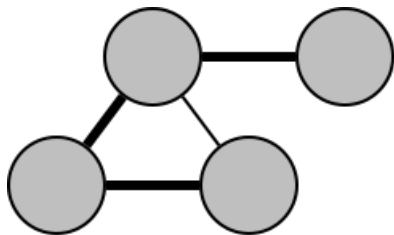
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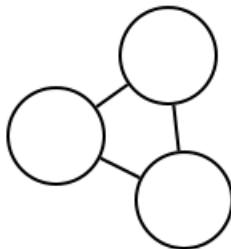
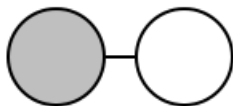
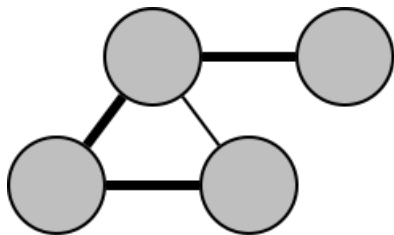
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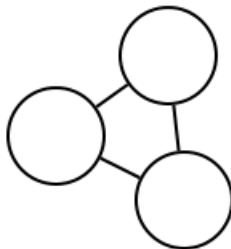
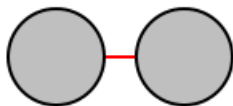
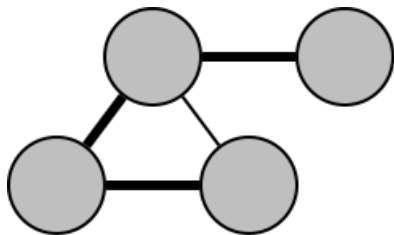
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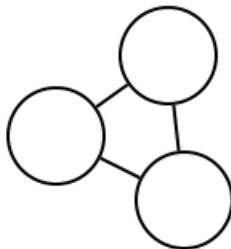
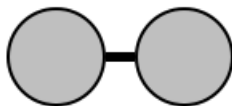
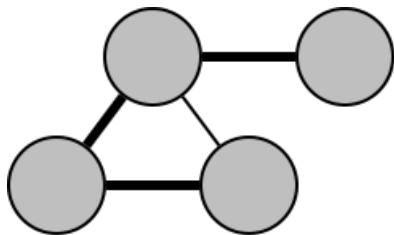
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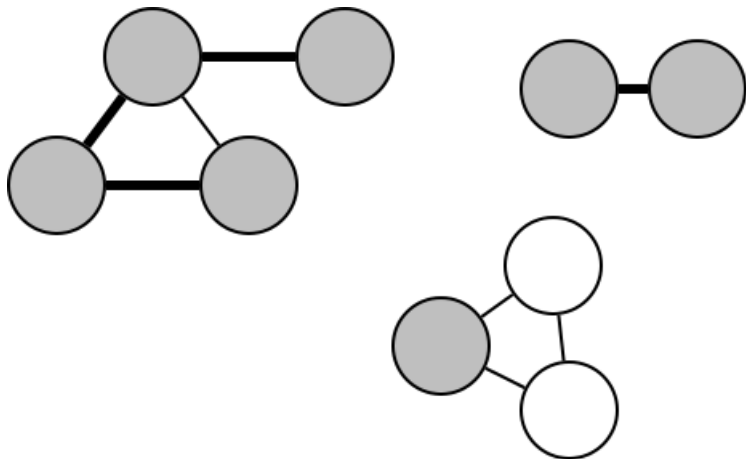
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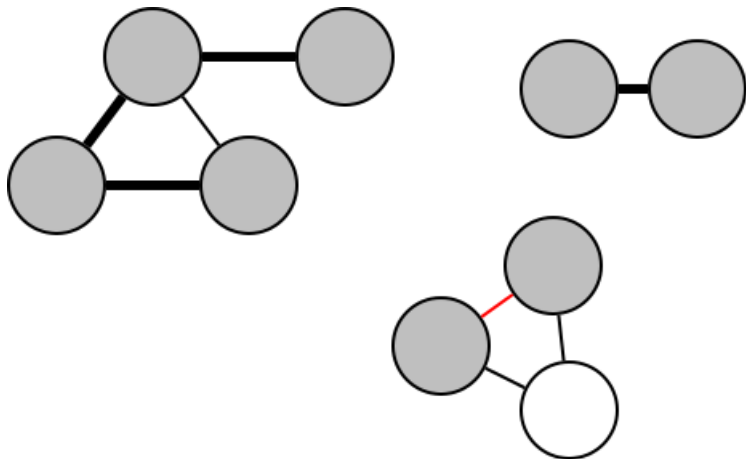
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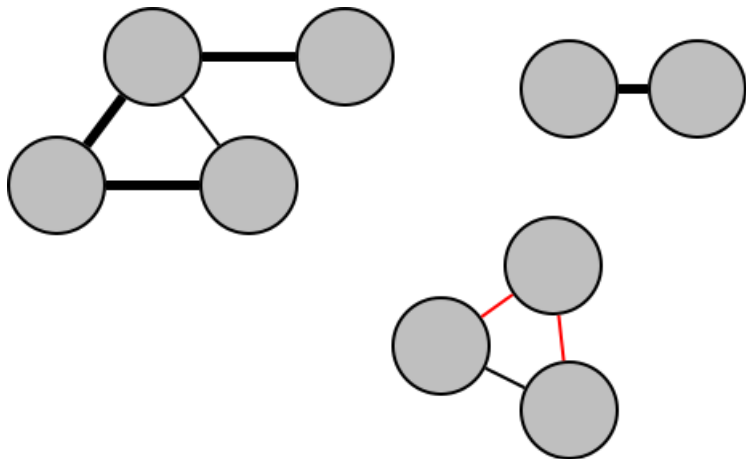
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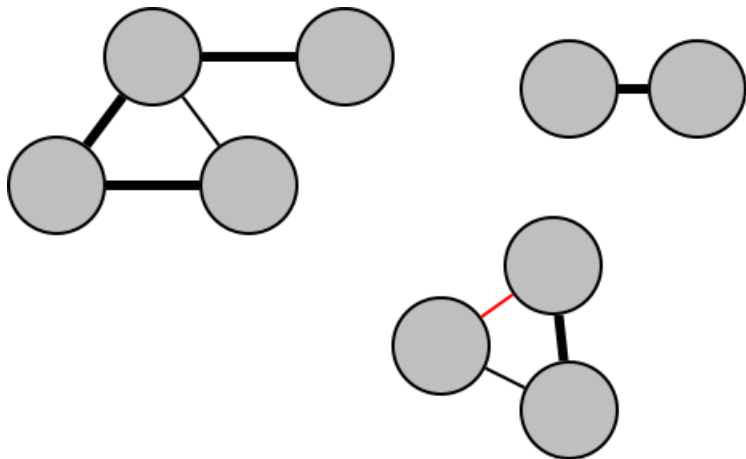
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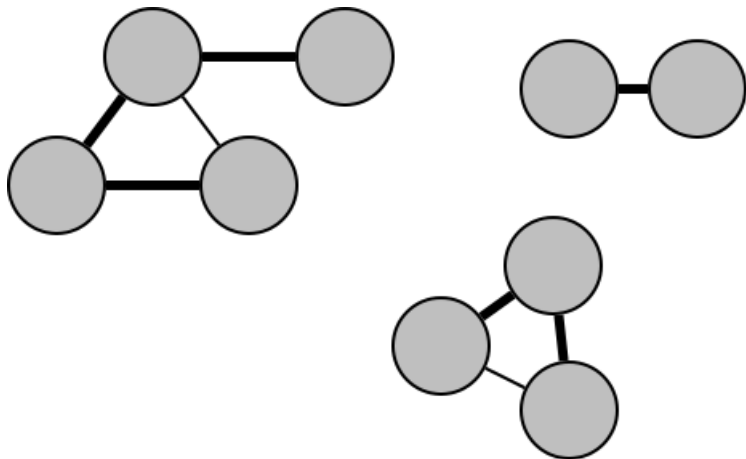
Example



Example



Example



Runtime

Number of calls to explore:

- Each explored vertex is marked visited.
- No vertex is explored after visited once.
- Each vertex is explored exactly once.

Runtime

Checking for neighbors:

- Each vertex checks each neighbor.
- Total number of neighbors over all vertices is $O(|E|)$.

Runtime

Total runtime:

- $O(1)$ work per vertex.
- $O(1)$ work per edge.
- Total $O(|V| + |E|)$.

Next Time

- More on reachability in graphs.
- Application of DFS.