Assignment 1

a1739440 Yao Yao

Question1

For photon's polarization state $|\rightarrow\rangle$, $|\uparrow\rangle$ satisfies the following normal orthogonal relation: $\langle\rightarrow|\rightarrow\rangle=\langle\uparrow|\uparrow\rangle=1$, $\langle\rightarrow|\uparrow\rangle=0$

① The polaroid A remains horizontally polarized to measure.

This state of photons is $|\varphi\rangle$. $|\varphi\rangle=\cos\varphi\,|\rightarrow\rangle+\sin\varphi\,|\uparrow\rangle$.

Through A's light is $|\rightarrow\rangle$, so the amplitude of through A's photon's polarization is

$\langle\rightarrow|\varphi\rangle = \langle\uparrow|\,(\cos\varphi|\rightarrow\rangle+\sin\varphi|\uparrow\rangle|) = \sin\varphi$

Therefore, the probability is in $|\sin\varphi|^2= \sin^2\varphi$.

For each photon generated by the laser pointer has random polarization, so I need to do a statistical average. For this problem, the initial state of $|\varphi\rangle$ of photon to screen to calculate, but the state is all photons through A is $|\rightarrow\rangle$, so the right result is the average of probability of through A's is

$$\frac{1}{2\pi}\int_0^{2\pi} |\langle\rightarrow|\varphi\rangle|^2 d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \sin^2\varphi\, d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\cos 2\varphi}{2}\, d\varphi$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}\varphi - \frac{1}{4}\sin 2\varphi\right]_0^{2\pi}$$

$$= \frac{1}{2}$$

$$\sin^2\varphi = 1-\cos^2\varphi = \frac{(1-\cos 2\varphi)}{2}$$

The probability of random polarized light passing through A is 1/2.

② The incident ray of B is $|\rightarrow\rangle$, through B's light is $|v\rangle=\cos\theta\,|\rightarrow\rangle+\sin\theta\,|\uparrow\rangle$, so the amplitude of through B's photon's polarization is

$\langle v|\rightarrow\rangle= (\cos\theta\,\langle\rightarrow|+\sin\theta\langle\uparrow|)\,|\rightarrow\rangle=\cos\theta$.

Therefore, the probability is in $|\cos\theta|^2= \cos^2\theta$.

③ The incident ray of C is $|v\rangle$, through C's light is $|\uparrow\rangle$, so the amplitude of through C's photon's polarization is $\langle\uparrow|v\rangle=\langle\uparrow|\,(\cos\theta\,|\rightarrow\rangle+ \sin\theta\,|\uparrow\rangle|) =\sin\theta$.

Therefore, the probability is in $|\sin\theta|^2 = \sin^2\theta$.

In summary, the probability P of photons from the laser pointer reaching the screen is $P=1/2\ \sin^2\cos^2\theta=1/8\ \sin^2 2\theta$ (When $\theta=\pi/4=45^\circ$, the maximum is 1/8)

Question2

Firstly, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle),\ |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$

$$|\mathbf{i}\rangle = \frac{1}{\sqrt{2}}(|0\rangle+\mathbf{i}|1\rangle), \qquad |-\mathbf{i}\rangle = \frac{1}{\sqrt{2}}(|0\rangle-\mathbf{i}|1\rangle)$$

$\{|+\rangle. |-\rangle\}$, $\{|i\rangle, |-i\rangle\}$ respectively to form orthogonal basis.

- It's obviously the same state.

  $|x\rangle=|0\rangle$, $|y\rangle=-|0\rangle$

  $|\langle x|y\rangle|=1$

  The global phase is π ($e^{i\pi}=-1$)

- It's obviously the same state.

  $|x\rangle=|1\rangle$, $|y\rangle=i|1\rangle$

  $|\langle x|y\rangle|=1$

  The global phase is $\pi/2$ ($e^{i\pi/2}=\cos\pi/2+i\sin\pi/2=i$)

- It's not the same state it's two states.

  $|x\rangle=|+\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, $|y\rangle=-|-i\rangle=\frac{1}{\sqrt{2}}(-|0\rangle+i|1\rangle)$

  $|x^\perp\rangle=|-\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$

  $|y^\perp\rangle=|i\rangle=\frac{1}{\sqrt{2}}(|0\rangle+\mathbf{i}|1\rangle)$

  $\langle x|y\rangle=1/2\ (\langle 0|+\langle 1|)\ (-|0\rangle+i|1\rangle)=(-1+i)/2$

  When the absolute value of $\langle x|y\rangle$ is 1, they have the same state. So, it's not the same state, it's two states.

- It's not the same state it's two states

$|x\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|y\rangle = -|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\langle x \mid y\rangle = 1/2\ (\langle 0| + \langle 1|)\ (|0\rangle - |1\rangle) = 0$

$|x^{\perp}\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$|y^{\perp}\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- It's obviously the same state.

  $|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, $|y\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$

  $|\langle x \mid y\rangle| = |1/2\ (\langle 0| - \langle 1|)\ (|1\rangle - |0\rangle)\ | = 1$

  The global phase is π.

Question3

- This is a superposition with respect to the standard basis.

  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- This is not a superposition with respect to the standard basis.

  $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = |0\rangle$

- This is not a superposition with respect to the standard basis.

  $\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = |1\rangle$

- This is a superposition with respect to the standard basis.

  $\frac{\sqrt{3}}{2}|+\rangle - \frac{1}{2}|-\rangle = (\sqrt{6} - \sqrt{2})/4\ |0\rangle + (\sqrt{6} + \sqrt{2})/4\ |1\rangle$

- This is not a superposition with respect to the standard basis.

  $\frac{1}{\sqrt{2}}(|i\rangle - |-i\rangle) = i\ |1\rangle$

Question4

$(e^{i\theta} = \cos\theta + i\sin\theta)$

- θ is only available for global phases, so the two states are the same for any real number θ state. Any plural θ state can be converted to the same state if allowing for non-standardized states.

- $\frac{1}{\sqrt{2}}(|-i\rangle + e^{-i\theta}|i\rangle) = \frac{1}{\sqrt{2}} e^{-i\theta}(|i\rangle + e^{i\theta}|-i\rangle)$

So, for any θ in real numbers, the two states are the same. Any plural θ can go to the same state if allowing for the non-standardized state.

Question5

The state is $|\psi\rangle$, The measurement basis is $|x\rangle$, $|y\rangle$.

The measurement probability of $|x\rangle$ is $|\langle x|\psi\rangle|^2$.

The measurement probability of $|y\rangle$ is $|\langle y|\psi\rangle|^2$.

These probabilities are P_x and P_y.

- $$P_x = \left|\langle 0|\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)\right|^2 = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$$
$$P_y = \left|\langle 1|\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)\right|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$
$P_x=3/4$, $P_y=1/4$

- $$P_x = \left|\langle 0|\left(\frac{\sqrt{3}}{2}|1\rangle - \frac{1}{2}|0\rangle\right)\right|^2 = \left|\frac{\sqrt{3}}{2}\langle 0|1\rangle - \frac{1}{2}\langle 0|0\rangle\right|^2 = \left|-\frac{1}{2}\right|^2 = \frac{1}{4}$$
$$P_y = \left|\langle 1|\left(\frac{\sqrt{3}}{2}|1\rangle - \frac{1}{2}|0\rangle\right)\right|^2 = \left|\frac{\sqrt{3}}{2}\langle 1|1\rangle - \frac{1}{2}\langle 1|0\rangle\right|^2 = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$$
$P_x=1/4$, $P_y=3/4$

- $$P_x = |\langle 0|-i\rangle|^2$$
  $$= |\langle 0|(\tfrac{1}{\sqrt{2}}(|0\rangle - i|1\rangle))\rangle|^2$$
  $$= |\tfrac{1}{\sqrt{2}}\langle 0|0\rangle - \tfrac{1}{\sqrt{2}}i\langle 0|1\rangle|^2$$
  $$= |\tfrac{1}{\sqrt{2}}|^2 = \tfrac{1}{2}$$
  $$P_y = |\langle 1|-i\rangle|^2 = |\langle 1|(\tfrac{1}{\sqrt{2}}(|0\rangle - i|1\rangle))\rangle|^2$$
  $$= |\tfrac{1}{\sqrt{2}}\langle 1|0\rangle - \tfrac{1}{\sqrt{2}}i\langle 1|1\rangle|^2 = |\tfrac{1}{\sqrt{2}}i|^2 = \tfrac{1}{2}$$
  $P_x = 1/2$, $P_y = 1/2$
- $$P_x = |\langle +|0\rangle|^2 = |\langle \tfrac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|0\rangle|^2$$
  $$= |\tfrac{1}{\sqrt{2}}\langle 0|0\rangle + \tfrac{1}{\sqrt{2}}\langle 1|0\rangle|^2 = \tfrac{1}{2}$$
  $$P_y = |\langle -|0\rangle|^2 = |\langle \tfrac{1}{\sqrt{2}}(\langle 0| - \langle 1|)|0\rangle|^2$$
  $$= |\tfrac{1}{\sqrt{2}}\langle 0|0\rangle - \tfrac{1}{\sqrt{2}}\langle 1|0\rangle|^2 = \tfrac{1}{2}$$
  $P_x = 1/2$, $P_y = 1/2$
- $$P_x = |\langle i|1\rangle|^2 = |\langle \tfrac{1}{\sqrt{2}}(\langle 0| + i\langle 1|)|1\rangle|^2$$
  $$= |\tfrac{1}{\sqrt{2}}\langle 0|1\rangle + \tfrac{1}{\sqrt{2}}i\langle 1|1\rangle|^2$$
  $$= \tfrac{1}{2}$$
  $$P_y = |\langle -i|1\rangle|^2 = |\langle \tfrac{1}{\sqrt{2}}(\langle 0| - i\langle 1|)|1\rangle|^2$$
  $$= |\tfrac{1}{\sqrt{2}}\langle 0|1\rangle - \tfrac{1}{\sqrt{2}}i\langle 1|1\rangle|^2$$
  $$= \tfrac{1}{2}$$
  $P_x = 1/2$, $P_y = 1/2$

Question6

The probability of transitioning between these states is 1.

For $|1\rangle$ and $-|1\rangle$'s probability of transitioning is $|\langle 1| (- \langle 1|) |^2 = |-1|^2 = 1$, So $|1\rangle$ and $-|1\rangle$ represent the same state.

For $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$'s probability of transitioning is

$|1/2\ (\langle 0|+\langle 1|)\ (|0\rangle-|1\rangle)\ |^2=|1/2(1-0+0-1)\ |^2=0 \neq 1$. So, these two states are not the same.

Question7

The standard basis $\{|0\rangle, |1\rangle\}$. Hadamard basis $\{|+\rangle, |-\rangle\}$.

Consider the case of Alice sending 1 qubit to Bob. To be specific, let Alice sends qubit $|0\rangle$. It doesn't matter if Eve gets the right bits, just pay attention to Bob. If Eve when measured with Hadamard basis, Bob's interception rate is 50%, the measurement results and the $(|+\rangle. |-\rangle)$ has nothing to do.

Bob uses the standard basis for his measurements. Eve measured it with Hadamard basis. Bob gets the determination of value $|1\rangle$.

The probability of monitoring a Bob per qubit is 12.5%(=1/8), since these two terms are 50% respectively. However, there is a conditional probability without including the bits to be discarded (because the probability of the bits not being discarded is 50%)

12.5% / 50% = 1/8 / 1/2 = 1/4

If you consider that Alice and Bob have the same basis and there are n qubits left, in one qubit, if Bob is aware of eavesdropping, he will find out that the whole process, which means that Bob is aware of the probability of eavesdropping is Pn:

$Pn=1-(3/4)^n$

More than 90% is:

$P8=1-(3/4)^8=0.89988\ldots$

$P9=1-(3/4)^9=0.92491\ldots$

Thus, I find that when n=9 (n=8, almost 90%).

Question8

The vector space based on these three vectors

$\{(1/\sqrt{2}, 1/\sqrt{2},0),(1/\sqrt{2},- 1/\sqrt{2},0),(0,0,1)\}$ is defined as V'.

This is essentially the same vector space as V.

First of all, as the two vector spaces, use the basis given on the problem of $V \otimes V$:

{(1,0,0,0,0,0,0,0,0),

(0,1,0,0,0,0,0,0,0),

(0,0,1,0,0,0,0,0,0),

(0,0,0,1,0,0,0,0,0),

(0,0,0,0,1,0,0,0,0),

(0,0,0,0,0,1,0,0,0),

(0,0,0,0,0,0,1,0,0),

(0,0,0,0,0,0,0,1,0),

(0,0,0,0,0,0,0,0,1)}

Then, one of the two vector spaces, $V \otimes V'$ is when using V':

{(1/√2,1/√2,0,0,0,0,0,0,0),

(1/√2, 1/√2,0,0,0,0,0,0,0),

(0,0,1,0,0,0,0,0,0),

(0,0,0, 1/√2, 1/√2,0,0,0,0),

(0,0,0, 1/√2,− 1/√2,0,0,0,0),

(0,0,0,0,0,1,0,0,0),

(0,0,0,0,0,0, 1/√2, 1/√2,0),

(0,0,0,0,0,0, 1/√2,− 1/√2,0),

(0,0,0,0,0,0,0,0,1)}

By changing the order of the second and different basis to get $(V' \otimes V)$:

{(1/√2,0,0, 1/√2,0,0,0,0,0),

(0, 1/√2,0,0, 1/√2,0,0,0,0),

(0,0, 1/√2,0,0, 1/√2,0,0,0),

(1/√2,0,0,− 1/√2,0,0,0,0,0),

(0, 1/√2,0,0,− 1/√2,0,0,0,0),

(0,0, 1/√2,0,0,− 1/√2,0,0,0),

(0,0,1,0,0,0,0,0,0),

(0,0,0,0,0,1,0,0,0),

(0,0,0,0,0,0,0,0,1)}

For $V' \otimes V'$ is:

{(1/2, 1/2,0, 1/2, 1/2,0,0,0,0),

(1/2, − 1/2,0, 1/2, − 1/2,0,0,0,0),

(0,0, 1/√2,0,0, 1/√2,0,0,0),

(1/2, 1/2,0, −1/2, −1/2,0,0,0,0),

(1/2, −1/2,0, − 1/2, 1/2,0,0,0,0),

(0,0, 1/√2,0,0,− 1/√2,0,0,0),

(0,0,1,0,0,0,0,0,0),

(0,0,0,0,0,1,0,0,0),

(0,0,0,0,0,0,0,0,1)}

Question9

Example 3.2.3 *Multiple meanings of entanglement.* We say that the four-qubit state

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle) = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$$

is entangled, since it cannot be expressed as the tensor product of four single-qubit states. That the entanglement is with respect to the decomposition into single qubits is implicit in this statement. There are other decompositions with respect to which this state is unentangled. For example, $|\psi\rangle$ can be expressed as the product of two two-qubit states:

$$|\psi\rangle = \frac{1}{2}(|0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4 + |0\rangle_1|1\rangle_2|0\rangle_3|1\rangle_4 + |1\rangle_1|0\rangle_2|1\rangle_3|0\rangle_4 + |1\rangle_1|1\rangle_2|1\rangle_3|1\rangle_4$$

$$= \frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_3 + |1\rangle_1|1\rangle_3) \otimes \frac{1}{\sqrt{2}}(|0\rangle_2|0\rangle_4 + |1\rangle_2|1\rangle_4),$$

where the subscripts indicate which qubit we are talking about. So $|\psi\rangle$ is not entangled with respect to the system decomposition consisting of a subsystem of the first and third qubit and a subsystem consisting of the second and fourth qubit. On the other hand, the reader can check that $|\psi\rangle$ is entangled with respect to the decomposition into the two two-qubit systems consisting of the first and second qubits and the third and fourth qubits.

To solve this problem, I read p40 of the textbook.

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

$$= \frac{1}{\sqrt{3}}(|0\rangle_0|0\rangle_1|1\rangle_2 + |0\rangle_0|1\rangle_1|0\rangle_2 + |1\rangle_0|0\rangle_1|0\rangle_2)$$

W_3 can't be factored into n qubits (tensor product). So, this state $|W_3\rangle$ is entangled.

Question10

The n-qubit system state is decomposed into n Qubit tensor products, which is basically just factorization. To decompose it, the problem of the first-order factorization of n elements (n variables) is as follows:

$$\begin{cases} |0\rangle_i \longmapsto 1 \\ |1\rangle_i \longmapsto X_i \end{cases}$$

X_i is the appropriate variable, X_i (i=0,1,2, 3, …, n-1).

So, if I do a transition to state W_n:

$|W_n\rangle \longmapsto 1/\sqrt{n}\ (X0+X1+X2+\ldots+Xn-1)$

Because this can't be factored any more, W_n can't be factored into n qubits. In other words, this state is entangled.

Question11

the following basis, the Bell basis for a two-qubit system, $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$,

$|\Phi^+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$

$|\Phi^-\rangle = 1/\sqrt{2}(|00\rangle - |11\rangle)$

$|\Psi^+\rangle = 1/\sqrt{2}(|01\rangle + |10\rangle)$

$|\Psi^-\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle)$,

- $$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle)$$

- $$|+\rangle|-\rangle = \frac{1}{2}(|0\rangle+|1\rangle)(|0\rangle-|1\rangle) = \frac{1}{2}(|00\rangle-|01\rangle+|10\rangle-|11\rangle) = \frac{1}{2}(|\Phi^-\rangle-|\Psi^-\rangle)$$

Question12

- $$\frac{1}{\sqrt{2}}(|+\rangle|+\rangle+|-\rangle|-\rangle) = \frac{1}{2\sqrt{2}}\{(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)+(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)\} = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle+|1\rangle|1\rangle)$$

- $$\frac{1}{\sqrt{2}}(|i\rangle|i\rangle+|-i\rangle|-i\rangle) = \frac{1}{2\sqrt{2}}[(|0\rangle+i|1\rangle)(|0\rangle+i|1\rangle)+(|0\rangle-i|1\rangle)(|0\rangle-i|1\rangle)] = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle-|1\rangle|1\rangle)$$