Assignment 3

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Question1

The unary gate 0 operator acts on 1 bit, and returns 0 independently of the input. For 0 goes to 0 and 1 goes to 0.

$$N=2^n=2 \qquad \left\{ \begin{array}{lcl} n=1 & & \\ 0 & --- & 0 \\ 1 & --- & 0 \end{array} \right.$$

I will use a reversible operator like Not operator to switch the output. To make sure the output is 0.

So, 2 bits are necessary to implement the unary gate 0 operator.

Question2

Question 2.

OR operator from 2 bits to 1 bit.

$N = 2^n = 4$ $\begin{cases} 00 \to 0 \\ 01 \\ 10 \to 1 \\ 11 \end{cases}$

matrix form: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For OR operator:

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ not unitary not permutation.

$\because UU^\dagger = I \Leftarrow$ a unitary matrix

It's not a permutation and it's not unitary.

To get a reversible OR operator need to build the permutation like: x is n bits, y is m bits.

$\pi_f : (x, y) \to (x, y \oplus f(x))$

To make x is 2 bits and y is 1 bit: the corresponding matrix form:

$N = 2^{\overset{n=3}{n}} = 8$ $\begin{cases} 000 \to 000 \\ 001 \to 001 \\ 010 \to 011 \\ 011 \to 010 \\ 100 \to 101 \\ 101 \to 100 \\ 110 \to 111 \\ 111 \to 110 \end{cases}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

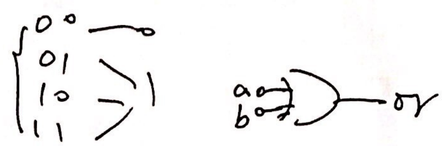
It's a permutation and it's unitary.

So, it's impossible to implement a reversible OR operator with less than 3 bits.

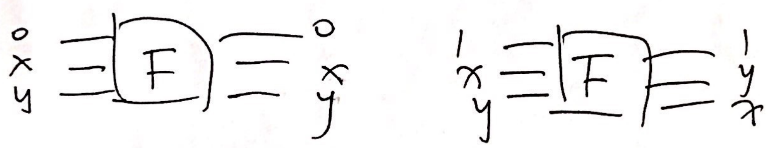
Question3

Question 3.

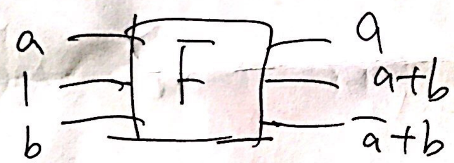
OR operator: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$



Fredkin gate: 3×3 gate (3 inputs, 3 outputs)



I use the Fredkin gate to implement a reversible OR operator.



(a)

Like this diagram (a), it can realize a reversible OR operator with Fredkin gate.

It's a permutation and unitary.

Question4

There is no reversible quantum operation f that transforms any input state |x> to a state orthogonal to it.

Let's suppose that there exists an operation f that maps the state of its input qubits to an orthogonal state. Reversible quantum operations can be simulated using unitary operators, so we can simulate f by a matrix U, such that $UU^\dagger =$ I, Since U is unitary, U is also normal (that is, U commutes with its transpose, Normal operators in the complex space are defined by $UU^\dagger = U^\dagger U$), so that we can use the spectral theorem to decompose U into eigenvalues and eigenvectors, $U = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|.$

So, because a quantum is reversible, i.e. Unitary, this means it is also Normal and hence can be diagonalized. So, it can be written in the form above Where the $|\psi i\rangle$ are the eigenvectors of U. So, assume the input state is in one of these eigenstates.

$U|\psi_n\rangle = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i||\psi_n\rangle. \langle\psi_i||\psi_n\rangle = 0\ for\ i \neq\ n\ U|\psi_n\rangle = \lambda_n|\psi_n\rangle$

So, $\lambda_n\langle\psi_n||\psi_n\rangle = \lambda_n \neq 0$. the |ψi⟩ is guaranteed to be orthogonal by spectral theorem or directly - diagonal matrix has orthogonal eigenvectors and unitary transformations preserve scalar products. And for every unitary matrix can be diagonalized as

$U = S^\dagger \Lambda S S \Lambda \lambda_i U \qquad \langle x|U|x\rangle = \langle x|S^\dagger \Lambda S|x\rangle = 0. \qquad |\tilde{x}\rangle \equiv S|x\rangle$

that there always exists a state $|\tilde{x}\rangle$

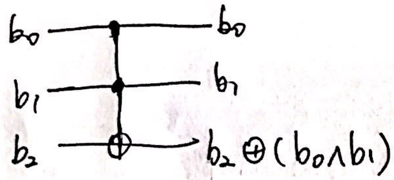
$$\langle\tilde{x}|\Lambda|\tilde{x}\rangle = \sum_i \lambda_i \tilde{x}_i^* \tilde{x}_i \neq 0.$$

So, there is no reversible quantum operation f that transforms any input state |x> to a state orthogonal to it. <f(x)|x>=0 for any |x>.

Question5

Question 5.

Toffoli gate matrix form:



000	000
001	001
010	010
011	011
100	100
101	101
110	111
111	110

(matrix form)

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{bmatrix}$$

Fredkin gate (Control - swap) matrix form:

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

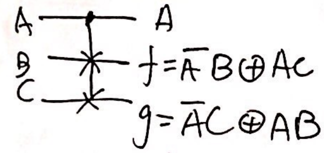
(matrix form)

000	000
001	001
010	010
011	011
100	100
101	110
110	101
111	111

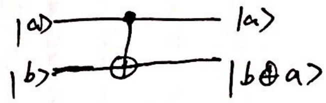
Question6

Question 6.

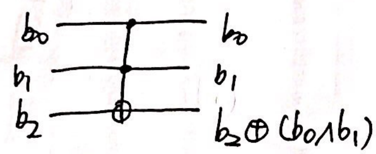
Fredkin gate ⇒



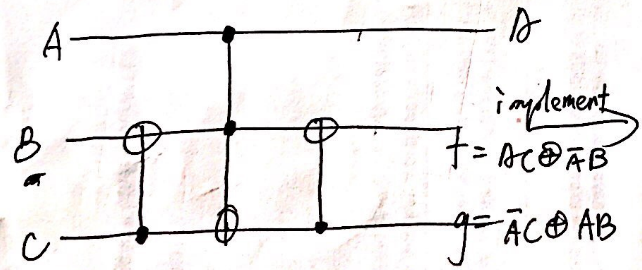
CNOT gate
- controlled NOT gate
- Acts on 2 ~~b~~ qubits.
- ~~CNOT~~ CNOT matrix form: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- circuit ~~repr~~ representation:

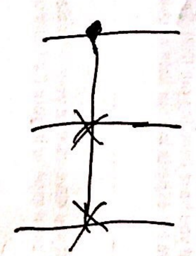


Toffoli gate ⇒



I use the Toffoli gate with CNOT gates to realize the Fredkin gate.





ABC	Afg
000	000
001	001
010	010
011	011
100	100
101	110
110	101
111	111

Realization of Fredkin gate using Toffoli gate and CNOT gates.

I use this quantum simulator to check the implementation results.
On the left is Fredkin gate. On the right are the Toffoli gate and Cnot gates.

These results can be verified by the comparison diagram.
Using the Toffoli gate and Cnot gates can realize the Fredkin gate.