Three-Dimensional Potential Distribution and Electromagnetic Wave Propagation Simulation

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Abstract

Electric field is the focus of this paper. Laplacian and Helmholtz, respectively, correspond to the static stable situation and dynamic situation.

First, the three-dimensional electric potential distribution bounded by different boundary conditions is studied. Laplacian Equation essentially describes the distribution. In this article, the finite difference method is used in solving Laplacian; therefore, a 3D matrix will be characterizing the potential distribution. For visualizing, matrices will be presented by contour charts of cross sections.

Based on the electric potential, the Electric field distribution is provided. The 3D vectorgraph corresponding to the field is plotted as well.

The following question is to find the dispersion for propagating electromagnetic wave in a waveguide. This article is aiming on solving Helmholtz equations and then simulating the 3D fluctuations. The two target functions, which represent electric field intensity and magnetic induction intensity, respectively, can be rewritten as a product of three partial functions by variable separation method. A given mode determines the partial functions with respect to X and Y (the pair (x,y) represents the position in waveguide), and the function of Z and T (Z denotes the propagation distance, and T denotes time) is solved by Forward Euler Method after variable separating again. Based on the results, a three-dimensional animated plot is shown. Additionally, the exact solutions and the corresponding plot are also provided for comparison.

Key words: Laplacian, Helmholtz, Electromagnetic Wave

1. Introduction

In this paper, two problems will be discussed.

The first one is to solve the electric potential distributing in a cube region. Specifically, we first treat the region as a three-dimensional matrix. Each of the element represents the value in its position. The relations among the points are characterized by Laplacian Equation. Deal with central points, Dirichlet boundary points, Neumann Boundary points differently. The solution will be given after enough iterations.

The second problem is to find the dispersion of the electromagnetic wave in a rectangular waveguide. Starting from Maxwell's Equations, the propagation of EM wave is described by Helmholtz Equation [1]. The solution, which is EM wave function, is solved by variable separation method; i.e. write the function as the product of three partial functions. In a waveguide region, the partial function about position (x,y) can be determined by mode type [2]. The rest of partial function about propagating distance z and time t is considered as the real part of a complex function [3]. Separating the complex function to four parts, namely, Z-real, Z-imaginary, T-real, and T-imaginary [3]. They satisfy second order ordinary differential equations, and the corresponding initial conditions can be determined by Maxwell's Equations and Waveguide boundary condition. Therefore, we solve them by Forward Euler.

2. Formulation

(1) Potential Distribution

The Laplacian equation characterizes the potential distribution.

$$\nabla^2\varphi = 0$$

In three dimensions, iterative scheme is given by [4]

$$\varphi_{i,j,k} = \frac{1}{6}(\varphi_{i+1,j,k} + \varphi_{i-1,j,k} + \varphi_{i,j+1,k} + \varphi_{i,j-1,k} + \varphi_{i,j,k+1} + \varphi_{i,j,k-1})$$

Thus, the central points can be calculated.

For the boundary points, the values depend on boundary condition.

a. Dirichlet Boundary

The values of points on Boundary are constant.

$$\varphi_D = constant$$

b. Neumann Boundary

Only consider a simple case,

$$\frac{\partial\varphi}{\partial n} = 0$$

According to the condition, Laplacian is reduced to 2D case for the surface points.

$$\varphi_{i,j} = \frac{\varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1}}{4}$$

Similarly, Laplacian is reduced to 1D case for the edge points.

$$\varphi_i = \frac{\varphi_{i+1} + \varphi_{i-1}}{2}$$

Finally, compute the electric field intensity by

$$\begin{cases} E^x_{i,j,k} = \dfrac{\varphi_{i+1,j,k} + \varphi_{i,j,k}}{\Delta x} \\ E^y_{i,j,k} = \dfrac{\varphi_{i,j+1,k} + \varphi_{i,j,k}}{\Delta x} \\ E^z_{i,j,k} = \dfrac{\varphi_{i,j,k+1} + \varphi_{i,j,k}}{\Delta x} \end{cases}$$

(2) Electromagnetic Wave

In a region with no charge and no current, the Maxwell's Equation can be written as

$$\begin{cases} \nabla D = 0 \\ \nabla B = 0 \\ \nabla \times E = -\frac{\partial B}{\partial t} \\ \nabla \times H = \frac{\partial D}{\partial t} \end{cases} \qquad (1)$$

Taking the curl of the curl equations gives

$$\begin{cases} \nabla \times \nabla \times E = -\varepsilon\mu \dfrac{\partial^2 B}{\partial t^2} \\ \nabla \times \nabla \times B = -\varepsilon\mu \dfrac{\partial^2 E}{\partial t^2} \end{cases}$$

Simplify these two equations by $\nabla E = 0$ and $\nabla B = 0$, it reads

$$\begin{cases} \nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \\ \nabla^2 B - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = 0 \end{cases} \quad (2)$$

where $c = \frac{1}{\sqrt{\epsilon\mu}}$ is the speed of light.

Equations (2) are Helmholtz equations, and it can be rewritten as following format [2]

$$\nabla^2 u + k^2 u = 0 \quad (3)$$

where $k^2 = (\frac{\omega}{c})^2$ is defined as the wavenumber.

Expand it as [2]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2}{\partial z^2} + k^2\right) u = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k_c^2 u = 0 \quad (4)$$

where $k_c^2 = k^2 + \beta^2$ is defined as the cutoff wavenumber.

Both E and B are 3D vector, and the relations are given by Maxwell's Equation (1) and Homlhetz Equations (2). Noticed that the components H_z and E_z are independent, the others can be shown in terms of them [2],

$$\begin{cases} H_x = \frac{i}{k_c^2}(\omega\varepsilon \frac{\partial E_z}{\partial y} - \beta \frac{\partial H_z}{\partial x}) \\ H_y = -\frac{i}{k_c^2}(\omega c \frac{\partial E_z}{\partial x} + \beta \frac{\partial H_z}{\partial y}) \\ E_x = -\frac{i}{k_c^2}(\beta \frac{\partial E_z}{\partial x} + w\mu \frac{\partial H_z}{\partial y}) \\ E_y = \frac{i}{k_c^2}(-\beta \frac{\partial E_z}{\partial y} + w\mu \frac{\partial H_z}{\partial x}) \end{cases} \quad (5)$$

In a waveguide, there are only two categories of modes, namely, TE mode ($H_z \neq 0$ and $E_z = 0$) and TM mode ($H_z = 0$ and $E_z \neq 0$).

In this article, only TE mode will be discussed, i.e.

$$\begin{cases} E_z = 0 \\ H_z \neq 0 \end{cases}$$

Therefore, equations (5) are reduced to [2]

$$\begin{cases} H_x = -\frac{i\beta}{k_c^2}\frac{\partial H_z}{\partial x} \\ H_y = -\frac{i\beta}{k_c^2}\frac{\partial H_z}{\partial y} \\ E_x = -\frac{iw\mu}{k_c^2}\frac{\partial H_z}{\partial y} \\ E_y = \frac{iw\mu}{k_c^2}\frac{\partial H_z}{\partial x} \end{cases} \quad (6)$$

For solving H_z, apply variable separation method based on equation (4),

$$H_z = H_0 X(x) Y(y) f(z,t)$$

In rectangular waveguides, mode numbers are designated by two suffix numbers attached to the mode type. It is named TE_{mn}, where m is the number of half-wave patterns across the width of the waveguide and n is the number of half-wave patterns across the height of the waveguide.

In this case, we just consider the simplest mode TE_{10}, so we obtain

$$\begin{cases} X(x) = \cos(\frac{\pi}{a}\mathrm{x}) \\ Y(y) = 1 \end{cases}$$

where **a** is constant denoting the length of the waveguide.

Importantly, the partial function $f(z,t)$ is the real part of a complex function. [3]

$$f(z,t) = real(f_{complex}(z,t))$$

Thus, we separate $f_{complex}$ as not only z component and t component but also real part and imaginary part. [3]

$$\begin{aligned} f_{complex}(z,t) &= Z(z)T(t) \\ &= \big(Z_{real}(z) \\ &\quad + iZ_{ima}(z)\big)\big(T_{real}(t) \\ &\quad + iT_{ima}(t)\big)\big] \\ &= (Z_{real}T_{real} - Z_{ima}T_{ima}) \\ &\quad + i(Z_{real}T_{ima} + Z_{ima}T_{real}) \end{aligned}$$

Therefore, after taking those partial functions in function (6) and letting $H_0 = 1$, we have

$$\begin{cases} E_x = 0 \\ E_y = \frac{\pi w \mu}{k_c^2 a} \sin\left(\frac{\pi}{a}x\right)(Z_{real}T_{ima} + Z_{ima}T_{real}) \\ E_z = 0 \end{cases}$$

(7)

$$\begin{cases} H_x = \frac{\pi \beta}{k_c^2 a} \sin\left(\frac{\pi}{a}x\right)(Z_{real}T_{ima} + Z_{ima}T_{real}) \\ H_y = 0 \\ H_z = \cos\left(\frac{\pi}{a}x\right)(Z_{real}T_{real} - Z_{ima}T_{ima}) \end{cases}$$

(8)

The microwave is bounded by waveguide boundary conditions

$$\begin{aligned} x = 0, a \qquad E_x = E_y = \frac{\partial E_x}{\partial x} = 0 \\ y = 0, a \qquad E_x = E_z = \frac{\partial E_y}{\partial y} = 0 \end{aligned} \qquad (9)$$

Based on (7), (8), and (9), the initial conditions are obtained as

$$T(0) = 1 = 1 + 0i$$

$$Z(0) = 1 = 1 + 0i$$
$$T'(0) = 0 + 1i$$
$$Z'(0) = 0 + 1i$$

The equations about t and z to be solved are written as

$$\begin{aligned} w^2 T_{real} - \frac{\partial^2 T_{real}}{\partial t^2} = 0 \\ w^2 T_{ima} - \frac{\partial^2 T_{ima}}{\partial t^2} = 0 \\ \beta^2 Z_{real} - \frac{\partial^2 Z_{real}}{\partial z^2} = 0 \\ \beta^2 Z_{ima} - \frac{\partial^2 Z_{ima}}{\partial z^2} = 0 \end{aligned} \qquad (10)$$

Those four equations share the same form.

$$\alpha^2 u + \frac{\partial^2 u}{\partial h^2} = 0$$

Discretizing it by Forward Euler Method, the scheme is given by

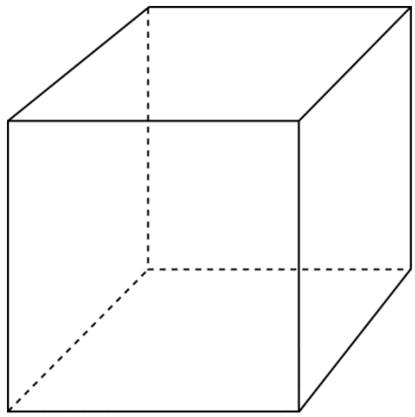
$$\begin{aligned} u_{i+1} &= u_i + (\Delta h)u_i \\ u'_{i+1} &= u'_i + (\Delta h)u'_i \\ u''_{i+1} &= -\alpha^2 u_{i+1} \end{aligned}$$

Finally, plug the results in (7) and (8)

3. Results

(1) Potential distribution

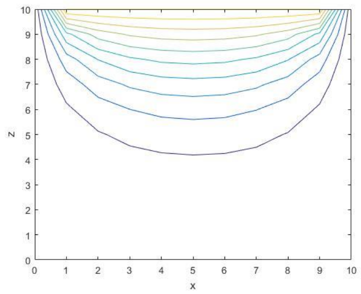
Create a cube with length a=10.



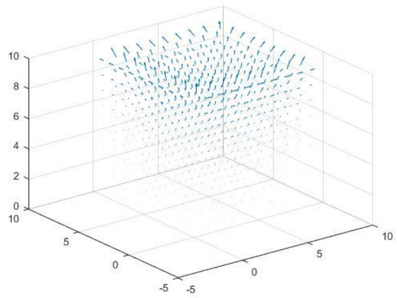
a. 6 sides with Dirichlet Boundary

The boundary conditions are given by

$$\varphi|_{x=0,y=0,z=0,x=a,y=a} = 0$$
$$\varphi|_{z=a} = 5$$



The contour of the cross section in Y=a/2

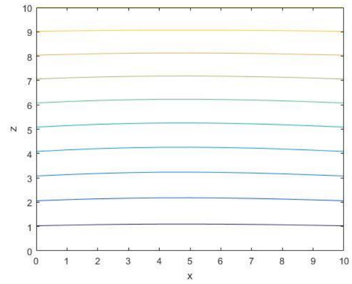


Electric field

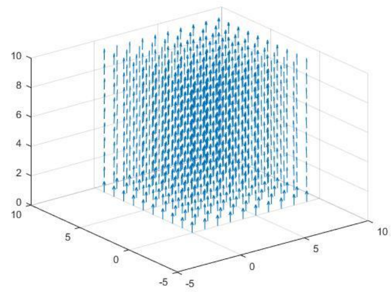
b. 4 sides with Neumann Boundary and 2 sides with Dirichlet Boundary

The boundary conditions are given by

$$\varphi|_{z=0} = 0$$
$$\varphi|_{z=a} = 5$$
$$\frac{\partial\varphi}{\partial n}|_{x=0,y=0,x=a,y=a} = 0$$



The contour of the cross section in Y=a/2

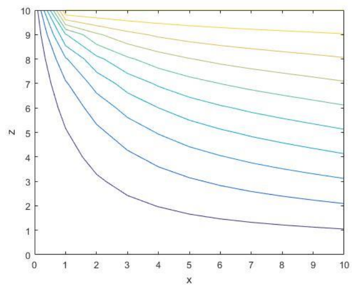


Electric field

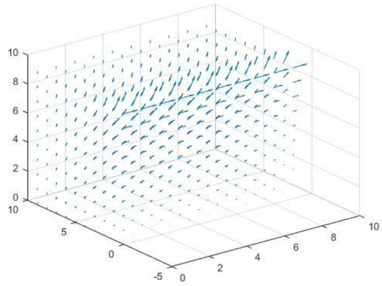
c. 3 sides with Neumann Boundary and 3 sides with Dirichlet Boundary

The boundary conditions are given by

$$\varphi|_{x=0,z=0} = 0$$
$$\varphi|_{z=a} = 5$$
$$\frac{\partial\varphi}{\partial n}|_{y=0,x=a,y=a} = 0$$



The contour of the cross section in Y=a/2



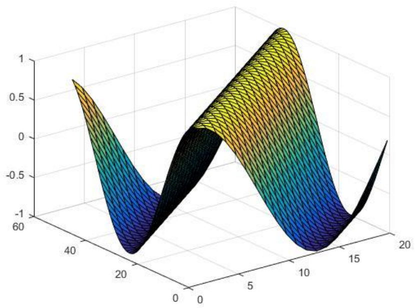
Electric field

(2) Electromagnetic Wave in rectangular waveguide

The exact solution for $f(z,t)$ is given by

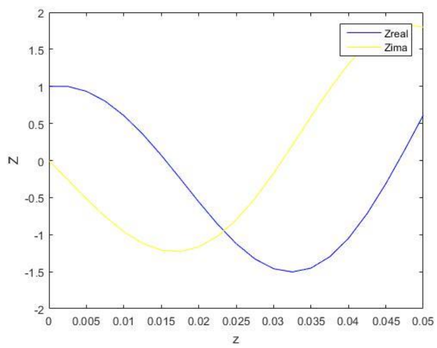
$$f(z,t) = \cos(wt - \beta z)$$

Which function is actually the 1D case of wavefunction.

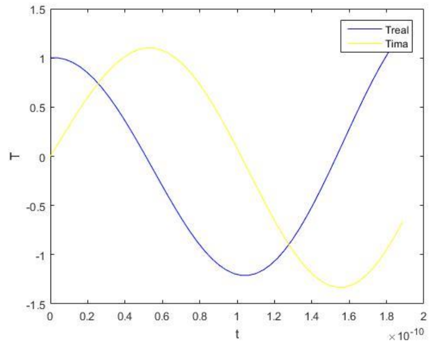


Plot of $f(z,t) = cos(wt - \beta z)$

Solved by Forward Euler, T(t) and Z(z) are shown as following



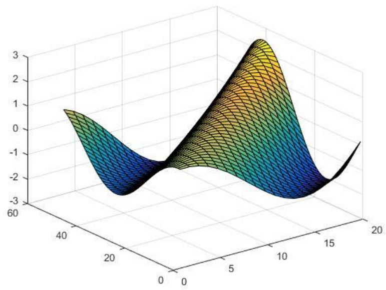
Plot of Z (real part-blue; imaginary prat-yellow)



Plot of T (real part-blue; imaginary prat-yellow)

Combine them together, and take the real part, we have the $f(z,t)$

$$f(z,t) = real\left(f_{complex}(z,t)\right)$$
$$= Z_{real}T_{real} - Z_{ima}T_{ima}$$



Plot of $f(z,t)$

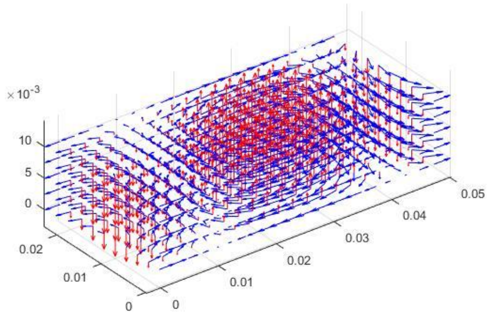
In order to approximate the practical problem, WR90 is chosen as the model.

The parameters are given as following:

$$a = 22.86mm$$
$$b = 10.15mm$$
$$f_{cutoff} = 6.56\ GHz$$
$$Operation\ frequency: 8.20 - 12.40\ GHz$$

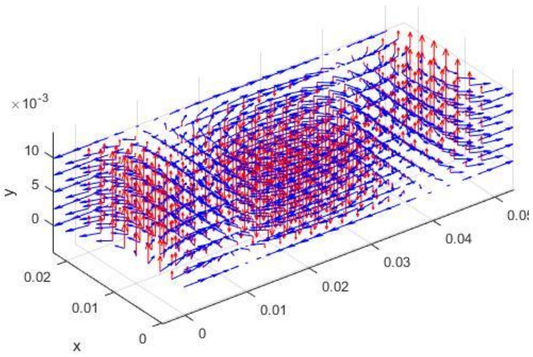
Select $f = 8.20\ GHz$ as input frequency.

Finally, present the result as 3D vectorgraph.



Simulation (E field-red; H field-blue)

compare with the exact solution



Exact Solution (E field-red; H field-blue)

4. Discussion

(1) Can we use Laplacian to solve Waveguide problem?

The answer is ‘no’.

Look at Laplacian and Helmholtz Equation,

$$\nabla^2\varphi = 0 \qquad \nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

They are pretty much similar. However, it does not mean that Helmholtz is able to convert to Laplacian. The only exception is TEM mode, which makes $H_z = 0$ and $E_z = 0$, but such mode does not exist in waveguide.

(2) Why do we focus on $f(z,t)$?

There are two reasons.

First, the other components can be expressed by the component in propagating direction.

Second, $f(z,t)$ is actually the 1D wavefunction.

(3) Comparison between numerical solution and analytical solution?

It is no doubts that analytical solution is more accurate, but using finite difference Method saves calculation time. It is 6.49s vs 5.42s.

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