Finding the Stable State by Upwind Scheme

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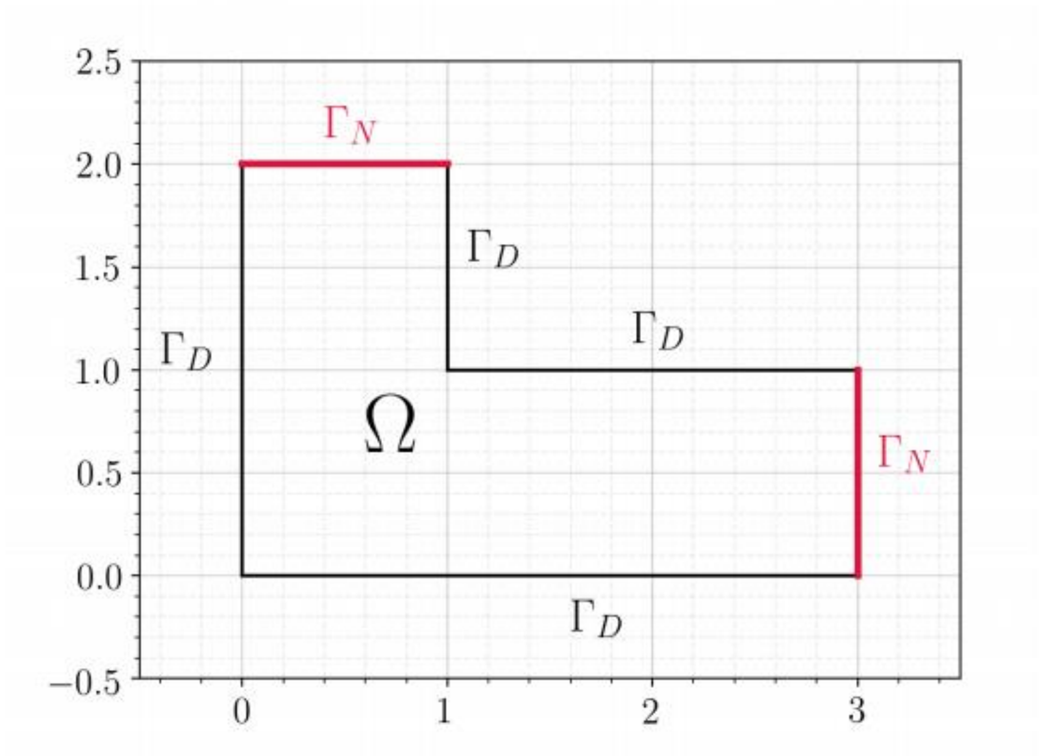
1. Description

The differential equation that needs to be solved is given

$$\frac{\partial \varphi}{\partial t} = D\nabla^2\varphi$$

Where $D = 1$.



The initial condition is given by

$$(1)\ \ \varphi(0) = 0 \qquad in\ \Omega$$

The boundary conditions are written as

$$(2)\ \ \varphi = f \qquad on\ \Gamma_D$$

$$(3)\ \ \frac{\partial \varphi}{\partial n} = \frac{\partial f}{\partial n} \qquad on\ \Gamma_N \quad \Rightarrow \begin{cases} \dfrac{\partial \varphi}{\partial x} = \dfrac{\partial f}{\partial x} & x = 3, 0 < y < 1 \\ \dfrac{\partial \varphi}{\partial y} = \dfrac{\partial f}{\partial y} & 0 < x < 1,\ y = 2 \end{cases}$$

Where

$$f(x,y) = x^3y - xy^3 + \frac{1}{2}x^2$$

2. Discretization

(1) Position relations

Convert RHS to discretized form by $\Delta x = \Delta y$.

The distribution φ(x,y) will be described by the matrix $[\varphi_{i,j}]$; i.e. the position (x,y) will be represented by column i and row j, respectively.

a. For the points in domain Ω, the relation reads

$$
\begin{aligned}
RHS &= D\nabla^2\varphi \\
&= D\frac{\partial^2\varphi}{\partial x^2} + D\frac{\partial^2\varphi}{\partial y^2} \\
&= D\frac{\varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j-1} + \varphi_{i,j+1} - 4\varphi_{i,j}}{\Delta x^2}
\end{aligned}
$$

b. For the points on Neumann boundary, the relations depend on the specific boundary conditions

$x = 3, 0 < y < 1$ RHS= $D\frac{\partial^2\varphi}{\partial x^2} + D\frac{\partial^2\varphi}{\partial y^2}$

$$
\begin{aligned}
&= D\frac{\partial \frac{\partial\varphi}{\partial x}}{\partial x} + D\frac{\partial^2\varphi}{\partial y^2} \\
&= D\frac{\partial^2 f}{\partial x^2} + D\frac{\partial^2\varphi}{\partial y^2} \\
&= D\frac{\partial^2 f}{\partial x^2} + D\frac{\varphi_{i,j-1} + \varphi_{i,j+1} - 2\varphi_{i,j}}{\Delta x^2}
\end{aligned}
$$

$0 < x < 1, y = 2$ RHS= $D\frac{\partial^2\varphi}{\partial x^2} + D\frac{\partial^2\varphi}{\partial y^2}$

$$
\begin{aligned}
&= D\frac{\partial^2\varphi}{\partial x^2} + D\frac{\partial \frac{\partial\varphi}{\partial y}}{\partial y} \\
&= D\frac{\partial^2\varphi}{\partial x^2} + D\frac{\partial^2 f}{\partial y^2} \\
&= D\frac{\varphi_{i-1,j} + \varphi_{i+1,j} - 2\varphi_{i,j}}{\Delta x^2} + D\frac{\partial^2 f}{\partial y^2}
\end{aligned}
$$

(2) Time evolution

Discrete the LHS by Forward Euler method or 4th Runge-Kutta Method.

a. Forward Euler Method

The scheme is given by

$$
\frac{\partial\varphi}{\partial t} = \frac{\varphi_{k+1} - \varphi_k}{\Delta t}
$$

Thus, for the points in domain Ω, the time-evolution scheme is given by

$$\frac{\partial \varphi}{\partial t}=\frac{\varphi_{i,j,k+1}-\varphi_{i,j,k}}{\Delta t}=\mathrm{D}\frac{\varphi_{i+1,j,k}+\varphi_{i-1,j,k}+\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-4\varphi_{i,j,k}}{\Delta x^2}$$

$$\Rightarrow \varphi_{i,j,k+1}=\varphi_{i,j,k}+\frac{D\Delta t}{\Delta x^2}\left(\varphi_{i+1,j,k}+\varphi_{i-1,j,k}+\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-4\varphi_{i,j,k}\right)$$

And for the points on Neumann boundary, we have

$x=3, 0<y<1$ $$\frac{\partial \varphi}{\partial t}=D\frac{\varphi_{i,j,k+1}-\varphi_{i,j,k}}{\Delta t}=D\frac{\partial^2 f}{\partial x^2}+D\frac{\varphi_{i,j-1}+\varphi_{i,j+1}-2\varphi_{i,j}}{\Delta x^2}$$

$$\Rightarrow \varphi_{i,j,k+1}=\varphi_{i,j,k}+D\Delta t\frac{\partial^2 f}{\partial x^2}|_{x=3,y=(\Delta x)(j-1)}+\frac{D\Delta t}{\Delta x^2}\left(\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-2\varphi_{i,j,k}\right)$$

$0<x<1, y=2$ $$\frac{\partial \varphi}{\partial t}=D\frac{\varphi_{i,j,k+1}-\varphi_{i,j,k}}{\Delta t}=D\frac{\varphi_{i-1,j,k}+\varphi_{i+1,j,k}-2\varphi_{i,j,k}}{\Delta x^2}+D\frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow \varphi_{i,j,k+1}=\varphi_{i,j,k}+\frac{D\Delta t}{\Delta x^2}\left(\varphi_{i-1,j,k}+\varphi_{i+1,j,k}-2\varphi_{i,j,k}\right)+D\Delta t\frac{\partial^2 f}{\partial y^2}|_{x=(\Delta x)(i-1),y=2}$$

b. Runge-Kutta 4th order

The scheme is given by

$$\varphi_{i,j,k+1}=\varphi_{i,j,k}+\frac{\Delta t}{6}(X_1+2X_2+2X_3+X_4)$$

For the points in domain Ω, we have

$$\begin{cases} X_1=\frac{D}{\Delta x^2}(\varphi_{i+1,j,k}+\varphi_{i-1,j,k}+\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-4\varphi_{i,j,k}) \\ X_2=\frac{D}{\Delta x^2}(\varphi_{i+1,j,k}+\varphi_{i-1,j,k}+\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-4X_1) \\ X_3=\frac{D}{\Delta x^2}(\varphi_{i+1,j,k}+\varphi_{i-1,j,k}+\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-4X_2) \\ X_4=\frac{D}{\Delta x^2}(\varphi_{i+1,j,k}+\varphi_{i-1,j,k}+\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-4X_3) \end{cases}$$

And for the points on Neumann boundary, we have

$x=3, 0<y<1$

$$\begin{cases} X_1=D\frac{\partial^2 f}{\partial x^2}|_{x=3,y=(\Delta x)(j-1)}+\frac{D}{\Delta x^2}\left(\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-2\varphi_{i,j,k}\right) \\ X_2=D\frac{\partial^2 f}{\partial x^2}|_{x=3,y=(\Delta x)(j-1)}+\frac{D}{\Delta x^2}\left(\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-2X_1\right) \\ X_3=D\frac{\partial^2 f}{\partial x^2}|_{x=3,y=(\Delta x)(j-1)}+\frac{D}{\Delta x^2}\left(\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-2X_2\right) \\ X_4=D\frac{\partial^2 f}{\partial x^2}|_{x=3,y=(\Delta x)(j-1)}+\frac{D}{\Delta x^2}\left(\varphi_{i,j-1,k}+\varphi_{i,j+1,k}-2X_3\right) \end{cases}$$

$0<x<1, y=2$ $$\frac{D\Delta t}{\Delta x^2}\left(\varphi_{i-1,j,k}+\varphi_{i+1,j,k}-2\varphi_{i,j,k}\right)+D\Delta t\frac{\partial^2 f}{\partial y^2}|_{x=(\Delta x)(i-1),y=2}$$

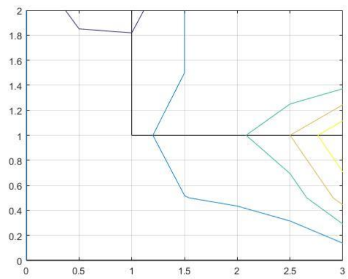
$$
\begin{cases}
X_1 = \dfrac{D}{\Delta x^2}\left(\varphi_{i-1,j,k} + \varphi_{i+1,j,k} - 2\varphi_{i,j,k}\right) + D\dfrac{\partial^2 f}{\partial y^2}\Big|_{x=(\Delta x)(i-1),y=2} \\
X_2 = \dfrac{D}{\Delta x^2}\left(\varphi_{i-1,j,k} + \varphi_{i+1,j,k} - 2X_1\right) + D\dfrac{\partial^2 f}{\partial y^2}\Big|_{x=(\Delta x)(i-1),y=2} \\
X_3 = \dfrac{D}{\Delta x^2}\left(\varphi_{i-1,j,k} + \varphi_{i+1,j,k} - 2X_2\right) + D\dfrac{\partial^2 f}{\partial y^2}\Big|_{x=(\Delta x)(i-1),y=2} \\
X_4 = \dfrac{D}{\Delta x^2}\left(\varphi_{i-1,j,k} + \varphi_{i+1,j,k} - 2X_3\right) + D\dfrac{\partial^2 f}{\partial y^2}\Big|_{x=(\Delta x)(i-1),y=2}
\end{cases}
$$

3. Results

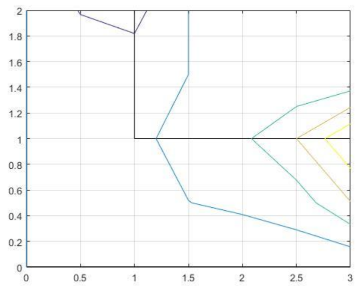
The scheme is conditional stable for $\frac{D\Delta t}{\Delta x^2} < \frac{1}{4}$

When choosing dt and dx, checking stability is necessary.

(1) dx=0.5 dt=0.001

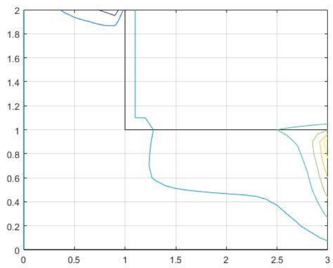


Euler

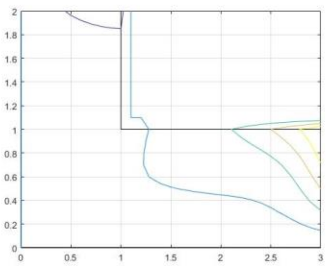


RK4

(2) dx=0.1 dt=0.0001

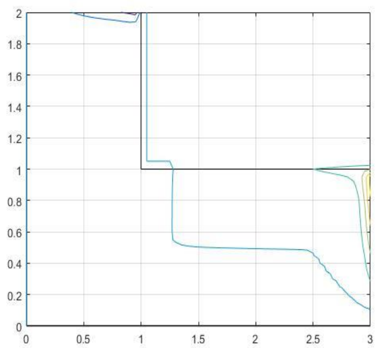


Euler



RK4

(3) dx=0.05 dt=0.0001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1.8
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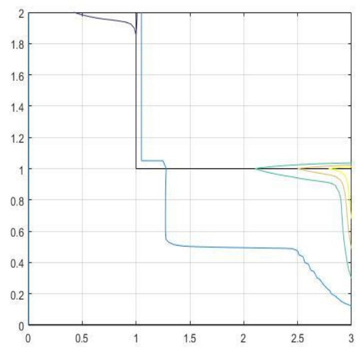
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3

Euler


2
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RK4