# Solving A 2D Laplacian by Symmetric Matrix

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## 1. Description

This problem is to solve the Laplacian Equation in 2D.

$$\nabla^2\varphi = 0$$

With following boundary condition

$$\varphi|_{x=3,0<y<1} = \varphi|_{y=0} = 0$$

$$\frac{\partial\varphi}{\partial x}|_{x=3,1<y<2} = \frac{\partial\varphi}{\partial x}|_{x=0} = 0$$

$$\varphi|_{x=3,2<y<3} = \varphi|_{y=3} = Q$$

## 2. Discretization

$$\nabla^2\varphi = 0$$

$$\Rightarrow \frac{\varphi_{i+1,j} - 2\varphi_{i,j}+\varphi_{i-1,j}}{\Delta x^2} + \frac{\varphi_{i,j+1} - 2\varphi_{i,j}+\varphi_{i,j+1}}{\Delta y^2} = 0$$

Therefore, the equations are written as

$$\varphi_{i+1,j}+\varphi_{i,j+1}+\varphi_{i-1,j} + \varphi_{i,j-1}-4\varphi_{i,j} = 0$$

For the boundary point,

a. Dirichlet boundary conditions

$$\varphi = 0 \; or \; \varphi = Q$$

b. Neumann boundary conditions

$$\frac{\partial\varphi}{\partial x} = 0$$

Thus, use forward for x=0,

$$\frac{\varphi_{i,j} - \varphi_{i-1,j}}{\Delta x} = 0$$

$$\Rightarrow \varphi_{i,j} = \varphi_{i+1,j}$$

The equation is given by

$$\varphi_{i+1,j}+\varphi_{i,j+1} + \varphi_{i,j-1}-3\varphi_{i,j} = 0$$

Use backward for x=3,

$$\varphi_{i,j} = \varphi_{i-1,j}$$

The equation is given by

$$\varphi_{i+1,j}+\varphi_{i,j+1} + \varphi_{i,j-1}-3\varphi_{i,j} = 0$$

## 3. Matrix

Take dx=1 as an example.

the linear equations are given by

$$A\varphi = b$$

$$\begin{bmatrix}
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & & \\
1 & & & & -3 & 1 & & & 1 & & & & & & & \\
 & 1 & & & 1 & -4 & 1 & & & 1 & & & & & & \\
 & & 1 & & & 1 & -4 & 1 & & & 1 & & & & & \\
 & & & & & & & 1 & & & & & & & & \\
 & & & & 1 & & & & -3 & 1 & & & 1 & & & \\
 & & & & & 1 & & & 1 & -4 & 1 & & & 1 & & \\
 & & & & & & 1 & & & 1 & -4 & 1 & & & 1 & \\
 & & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & & & 1 & & & \\
 & & & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & & & 1
\end{bmatrix}
\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \\ \varphi_8 \\ \varphi_9 \\ \varphi_{10} \\ \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \\ \varphi_{16} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Apparently, matrix A is not symmetric.

To have a symmetric matrix, eliminate the points with Dirichlet boundary conditions (A(i,i)=1), and then construct a new matrix X.

In this case, $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_8, \varphi_{12}, \varphi_{13}, \varphi_{14}, \varphi_{15}, and\ \varphi_{16}$ will not show up in X, but $\varphi_5, \varphi_6, \varphi_7, \varphi_9, \varphi_{10}, and\ \varphi_{11}$ will remain. The new equations are

$$X\varphi = b'$$

$$\begin{bmatrix}
-3 & 1 & & 1 & & \\
1 & -4 & & & 1 & \\
 & & -4 & & & 1 \\
1 & & & -3 & 1 & \\
 & 1 & & 1 & -4 & 1 \\
 & & 1 & & 1 & -4
\end{bmatrix}
\begin{bmatrix} \varphi_5 \\ \varphi_6 \\ \varphi_7 \\ \varphi_9 \\ \varphi_{10} \\ \varphi_{11} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -2 \end{bmatrix}$$

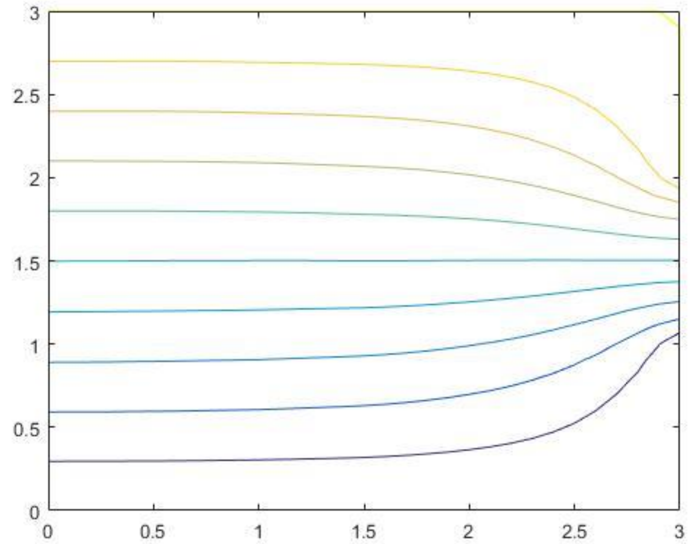
This symmetric matrix can be solved conjugate gradient method.

## 4. Results

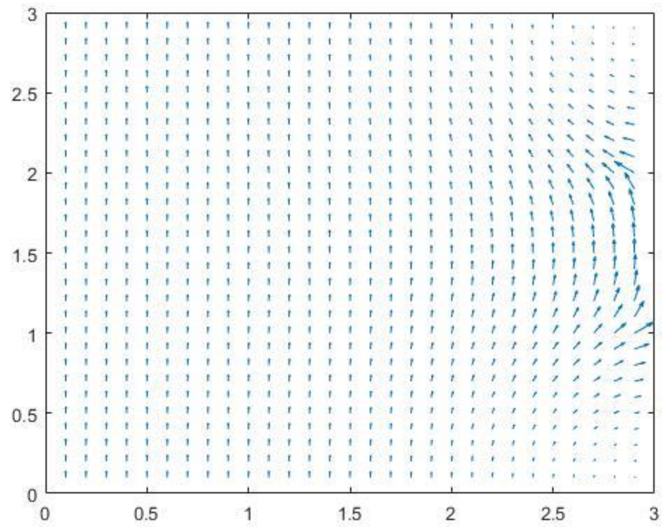
For dx=1, we have $\varphi$=

$$\begin{bmatrix}
1 & 1 & 1 & 1 \\
0.6703 & 0.6813 & 0.7363 & 1 \\
0.3297 & 0.3187 & 0.7363 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

For dx=0.1, plot the data.




Plot of $\varphi$



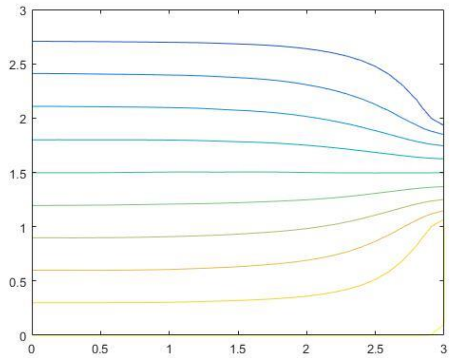

Plot of (U,V)

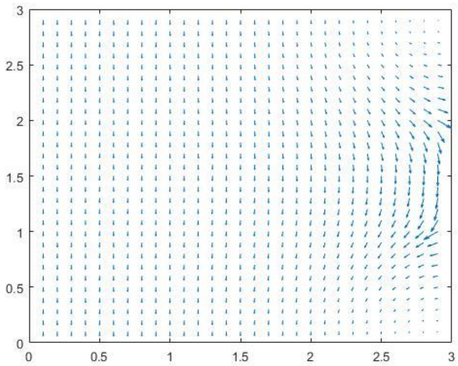
## 5. Discussion

a. Try different value of Q

Select Q=-1,Q=0, and Q=10.

For Q=-1, we have



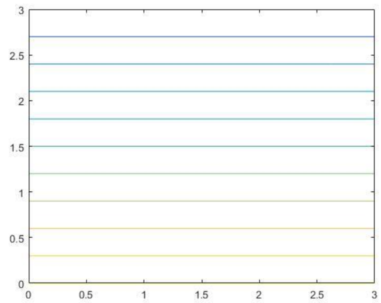

The plots of Q=1 and plots of Q=-1 are symmetric about x=1.5. Specifically, in the plot of $\varphi$, the monotonicity of y direction is inversed, and in the plot of (U,V), the arrows are inversed.

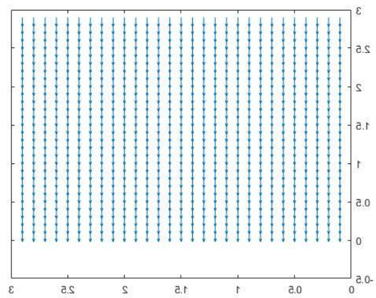
For Q=0, the value is zero everywhere.

For Q=10, the plots have no difference with Q=1.

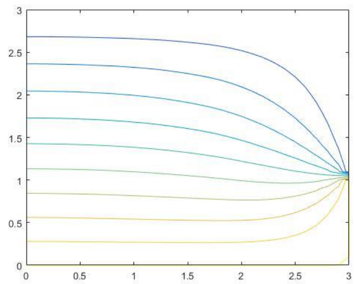
b. Change the positions with Neumann boundary conditions

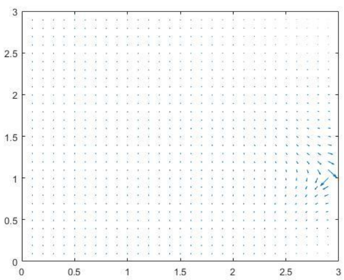
(1) $\frac{\partial\varphi}{\partial x}|_{x=0} = 0$, $\frac{\partial\varphi}{\partial x}|_{x=3} = 0$



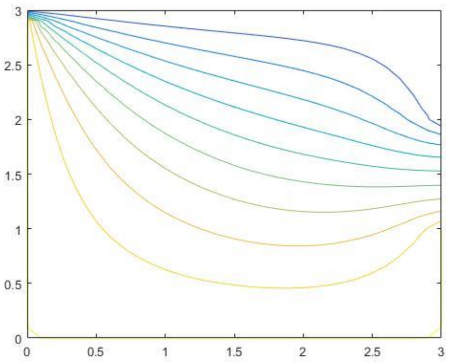



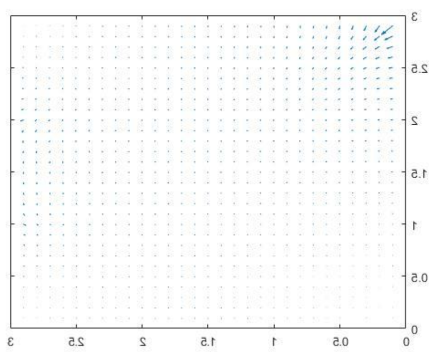
(2) $\frac{\partial\varphi}{\partial x}|_{x=0} = 0, \varphi|_{x=3,1<y<3} = \varphi|_{y=3} = Q$




(3) $\varphi|_{x=0} = 0$



c. For dx=0.01, since the 90601x90601 matrix cannot be created, the optimal solution is needed. (It has not been solved)