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ANALYSES OF MASS TRANSFER IN HEMODIALYZERS FOR LAMINAR BLOOD FLOW AND HOMOGENEOUS DIALYSATE

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Abstract—Theoretical analyses of mass transfer in hemodialyzers which contain flowing blood and dialysate streams separated by a semi-permeable membrane are presented. Semi-infinite parallel-plate and cylindrical tube geometries are considered. Solutions are obtained in terms of well-known functions, a method which avoids difficulties associated with computing the higher eigenvalues encountered in previous analyses. Applications of the mathematical model to systems used in clinical practice are discussed.

Mass and heat transfer with laminar flow in cylindrical and parallel-plate geometries have been treated mathematically by a very large number of investigators. Assumptions about the nature of the 'wall' resistance, as reflected in the wall Sherwood number N_{Sh_w}, constitute a primary difference between the works of various authors. For $N_{Sh_w} \cong \infty$ (zero wall resistance) one has the classical Graetz[1, 2] type of problem (constant temperature or concentration at the wall). For $N_{Sh_w} \cong 0$, the limiting case of constant wall flux is obtained[3–5]. Several investigators have presented analyses for intermediate N_{Sh_w} values[3–5] as well as for continuously varying N_{Sh_w}[22].

The essential difficulty one faces in such analyses is the accurate computation of the higher eigenvalues and eigenfunctions which are needed if accurate solutions are desired for the entrance region. Requirements for large amounts of computer time or difficulties with round-off error have prevented investigators from attaining accurate results. The most exact computations performed to date are those of Brown[6] who, using a computer capable of manipulating 50 digit decimal numbers, computed the first ten eigenvalues and other constants for the case of $N_{Sh_w} = \infty$.

In this paper we wish to show how such problems can be treated conveniently in terms of the confluent hypergeometric function (CHF). Eigenvalues are readily obtained as the zeros of certain transcendental equations written in terms of the CHF, and higher eigenfunctions are obtained using available asymptotic forms of the CHF.

There are some obvious advantages in presenting solutions in terms of well-known tabulated functions. Not only are asymptotic solutions available but also properties of the functions (e.g. derivatives, recurrence relations, etc.) are known[7].

In illustrating our technique, we have chosen to analyze mass transfer in hemodialyzers. These systems have been given great attention recently[8, 9] owing to the increasing use of hemodialysis as a means of treating chronic renal failure. Since both parallel-plate and cylindrical-tube units are commonly utilized in clinical practice (e.g. the Kiil and hollow-fiber artificial kidneys, respectively), we have considered both types of geometries.

PROBLEM DESCRIPTION

We wish to consider mass transfer between two flowing fluids (blood, dialysate) separated by a semipermeable membrane. The fluid from which solute is to be extracted (blood) is assumed to enter the system in fully-developed laminar flow, and at a uniform concentration C_i. The other fluid (the dialysate) is assumed to have a constant *bulk* concentration C_0 at all axial positions in the dialyzer (the dialysate concentration adjacent to the membrane need not be constant, however). Furthermore, we assume:

(1) The blood is Newtonian, homogeneous, and has constant physical properties (because it is dilute with respect to the transferable solute).

(2) No sources or sinks exist in the system.

(3) Only purely diffusive transport occurs in the membrane (i.e. no convection).

(4) The membrane permeability and dialysate-side mass transfer coefficient are constant.

(5) The solute partition coefficients between the fluids and the membrane are not necessarily the same, nor equal to unity, and are defined as:

K_B = concentration in membrane/concentration in adjacent blood

K_D = concentration in membrane/concentration in adjacent dialysate

K_B and K_D are taken to be constant and independent of z.

(6) Axial diffusion in the blood is negligible compared to axial convection, in the range of interest.

(7) Steady state conditions exist.

(8) Diffusion in the blood is Fickian.

The analysis will be performed for the semiinfinite parallel plate and cylindrical tube geometries (blood between two flat membranes and inside a tubular membrane, respectively) shown in Fig. 1. Because these systems are quite similar, we need not treat them separately and shall write only one set of starting equations and boundary conditions. These are

$$u_{\max}\left(1-\frac{x^2}{R^2}\right)\frac{\partial C}{\partial z}=D_B\frac{1}{x^p}\frac{\partial}{\partial x}\left(x^p\frac{\partial C}{\partial x}\right) \tag{1}$$

$p=0$ for flow between parallel plates

$p=1$ for flow in a cylindrical tube

$$\text{BC I}\quad C=C_i \quad \text{at} \quad z=0 \quad \text{for} \quad 0\le x\le R \tag{2}$$

$$\text{BC II}\quad \left.\frac{\partial C}{\partial x}\right|_{x=0}=0 \quad \text{for} \quad z>0 \tag{3}$$

$$\text{BC III}\quad -D_B\left.\frac{\partial C}{\partial x}\right|_{x=R}=\frac{k_w}{K}\left[C|_{x=R}-KC_0\right] \tag{4}$$

where k_w is a mass transfer coefficient which characterizes the transport resistance of the membrane *and* the dialysate stream, and $K=K_D/K_B$. This coefficient may be expressed in more detail (as is done in the case of heat exchangers) by

$$\frac{1}{k_w}=\frac{[R/R_{lm}]^p}{K_DP_m}+\frac{(R/R')^p}{k_D} \tag{5}$$

where R_{lm} is the logarithmic mean of R and R', P_m is the membrane permeability, K_D is the membrane/dialysate partition coefficient, and k_D is a mass transfer coefficient between the bulk of the dialysate and the adjacent membrane surface. In this work, we have taken P_m to be the ratio of the solute mass flux to the solute concentration difference existing *in the membrane itself.* Equations (4) and (5) are essentially the same as those used by Colton *et al.*[9].

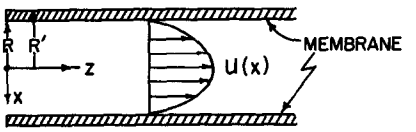

Fig. 1. Sectional view of parallel-plate and cylindrical-tube dialyzers.

Equation (4) requires some comment since it involves the assumption of constant *bulk* dialysate concentration. In clinical practice, dialysate flow rates in flat plate or tubular dialyzers are often large compared to blood flow rates. In such cases, C_0 is always very low and may reasonably be taken to be zero everywhere. When the parallel plate configuration is wound into a spiral and placed in a well-mixed bath of dialysate (e.g. in the case of widely used 'coil' units), Eq. (1) still governs the concentration field (providing that no secondary flows occur due to the curved channel configuration) and the bulk concentration C_0, although not close to zero, may be assumed to be the same at all axial positions because of the vigorous mixing of the dialysate fluid.

When the dialysate concentration is not constant (whether large or negligibly small) it is not possible to state the boundary condition at $x=R$ *a priori.* In this event the concentration fields in the dialysate and blood phases are coupled through mutual boundary conditions at $x=R$, and the problem must be treated as a conjugated boundary value problem.

DIMENSIONLESS VARIABLES AND GROUPS

Introducing the following definitions

$$C^*=\frac{C-KC_0}{C_i-KC_0} \tag{6}$$

$$x^*=\frac{x}{R} \tag{7}$$

$$z^*=\frac{z}{RPe} \quad \text{where} \quad Pe=\frac{u_{\max}R}{D_B} \tag{8}$$

$$N_{Sh_w}=\frac{k_wR}{KD_B} \tag{9}$$

the equations and BC's transform to

$$(1-x^{*2})\frac{\partial C^*}{\partial z^*}=\frac{1}{x^{*p}}\frac{\partial}{\partial x^*}\left(x^{*p}\frac{\partial C^*}{\partial x^*}\right) \tag{10}$$

BC I $C^* = 1$ at $z^* = 0$ for $0 \le x^* \le 1$ (11)

BC II $\left.\frac{\partial C^*}{\partial x^*}\right|_{x^*=0}$ for $z^* > 0$ (12)

BC III $-\left.\frac{\partial C^*}{\partial x^*}\right|_{x^*=1} = N_{Sh_w} C^*|_{x^*=1}.$ (13)

Note that N_{Sh_w} accounts for the transport conductance of the dialysate-side boundary layer and the conductance of the membrane.

Equation (10) is easily solved by separation of variables to yield

$$C^* = \sum_{n=1}^{\infty} A_n \exp(-\Lambda_n^2 z^*) X_n(x^*) \tag{14}$$

where Λ_n and $X_n(x^*)$ are the eigenvalues and eigenfunctions of

$$\frac{1}{x^{*p}} \frac{d}{dx^*}\left(x^{*p} \frac{dX}{dx^*}\right) + \Lambda^2(1 - x^{*2})X = 0 \tag{15}$$

with the associated BC's

$$\left.\frac{dX}{dx^*}\right|_{x^*=0} = 0 \tag{16}$$

$$-\left.\frac{dX}{dx^*}\right|_{x^*=1} = N_{Sh_w} X|_{x^*=1}. \tag{17}$$

Previous investigators [1–6, 8, 9] have solved Eq. (15) either by assuming a power series solution or by numerical solution of the ordinary differential equation. An equivalent, but more convenient approach, is to write the solution of Eq. (15) in terms of known functions as indicated by Davis [10]. As Davis has reviewed the applications of the CHF to Graetz problems it is not necessary to review that literature here.

Using the transformations

$$y = \Lambda x^{*2} \tag{18}$$

$$W = \exp(\Lambda x^{*2}/2) X. \tag{19}$$

Eq. (15) transforms to Kummer's equation

$$y\frac{d^2W}{dy^2} + \left(\frac{1+p}{2} - y\right)\frac{dW}{dy} - \left(\frac{1+p-\Lambda}{4}\right) W = 0 \tag{20}$$

which has the solution, under the BC of symmetry [Eq. (16)]

$$W = M\left(\frac{1+p-\Lambda}{4}, \frac{1+p}{2}, \Lambda x^{*2}\right) \tag{21}$$

M is Kummer's function, defined by

$$M(a, b, y) = 1 + \frac{ay}{b} + \frac{(a)_2 y^2}{(b)_2 2!} + \cdots + \frac{(a)_n y^n}{(b)_n n!} + \cdots \tag{22}$$

with

$$(a)_n = a \times (a+1) \cdots (a+n-1) \tag{23a}$$

$$(b)_n = b \times (b+1) \cdots (b+n-1) \tag{23b}$$

A computationally more useful form of Eq. (22) is

$$M(a, b, y) = 1 + \frac{a}{b} y \left\{1 + \frac{a+1}{b+1}\frac{y}{2}\left[1 + \frac{a+2}{b+2}\frac{y}{3} \times \left(1 + \frac{a+3}{b+3}\frac{y}{4}(1 + \cdots)\right)\right]\right\}. \tag{24}$$

It should be pointed out that the notation ${}_1F_1(a, b, y)$ is also used in the literature. We shall use the notation $M(a, b, y)$, following the convention of Ref. [7].

The advantages of writing the solution to this problem in terms of the Kummer function are:

(1) When using the power-series form of solution $X = \sum_{i=0}^{\infty} a_i x^i$, one finds that in evaluating the coefficients a_i from the previous coefficients a_{i-2} and a_{i-4} using a recurrence relation, errors tend to build up rapidly [11]. This is especially true for the higher eigenvalues, since Λ_n^2 multiplies the coefficients a_{i-2} and a_{i-4} in the recurrence formula.

(2) The Kummer function may be evaluated quickly by using simple programming strategies. If the Kummer series is to be evaluated in a forward direction, the n^{th} term is computed making use of the value of the $(n-1)^{\text{st}}$ term. If evaluation of the right hand side of Eq. (24) is performed in a backward direction, the nesting property

$$S_n = 1 + \left(\frac{a+n-1}{b+n-1}\right)\frac{y}{n} S_{n+1} \tag{25}$$

is useful. Here S_n is the sum from the n^{th} term through the last term. For good convergence n should be selected such that

$$\left(\frac{a+n-1}{b+n-1}\right)\frac{y}{n} \ll 1.$$

When Λ_n is sufficiently large it is unnecessary to use the series form of $M(a, b, y)$, for the following asymptotic form [7] can be applied

$$M(a, b, y) = \frac{\Gamma(b) \sin(a\pi) \exp[(b-2a)(\frac{1}{2}\sinh 2\theta - \theta + \cosh^2\theta)]}{[(b-2a)\cosh\theta]^{b-1}[\pi(\frac{1}{2}b - a)\sinh 2\theta]^{1/2}}$$

where

$$\cosh^2\theta = y/(2b - 4a).$$

Such an asymptotic formula was not used in the present study, however, since computation times

needed for evaluating the more general expressions were reasonably small.

Now application of BC III yields the following transcendental equation, which is solved to obtain the eigenvalues for the problem

$$(\Lambda_n - N_{Sh_w})\, M\left(\frac{1+p-\Lambda_n}{4}, \frac{1+p}{2}, \Lambda_n\right) - \Lambda_n \left(\frac{1+p-\Lambda_n}{1+p}\right) M\left(\frac{5+p-\Lambda_n}{4}, \frac{3+p}{2}, \Lambda_n\right) = 0 \qquad n = 1, 2, \cdots \tag{26}$$

The zeros of Eq. (26) were found by numerical methods, and a number of eigenvalues are given in Table 1, as discussed below.

The solution to our problem is, therefore,

$$C^* = \sum_{n=1}^{\infty} A_n \exp(-\Lambda_n^2 z^{*2}) \exp(-\Lambda_n x^{*2}/2) \times M\left(\frac{1+p-\Lambda_n}{4}, \frac{1+p}{2}, \Lambda_n x^{*2}\right). \tag{27}$$

Application of BC I, the inlet condition, leads to the requirement that

$$\sum_{n=1}^{\infty} A_n X_n(x^*) = 1. \tag{28}$$

Since Eq. (15) and its BC's constitute a Sturm–Liouville system, A_n can be shown to be given by

$$A_n = \frac{\int_0^1 x^{*p}(1-x^{*2}) X_n(x^*)\, dx^*}{\int_0^1 x^{*p}(1-x^{*2}) X_n^2(x^*)\, dx^*}. \tag{29}$$

The definite integrals may be transformed so as to produce the computationally more useful form:

$$A_n = -\frac{2}{\Lambda_n} \frac{\dfrac{dX_n(1)}{dx^*}}{\dfrac{dX_n(1)}{dx^*}\dfrac{dX_n(1)}{d\Lambda_n} - X_n(1)\dfrac{d^2X_n(1)}{dx^*\, d\Lambda_n}}. \tag{30}$$

In determining the derivatives $dX/d\Lambda_n$ Newton's first-order backward difference formula was employed, with the size of the Λ_n increment ($\Delta\Lambda_n \simeq 10^{-6}$) selected to optimize the trade-off between truncation error and significant digits. The derivatives were determinable to at least seven significant decimal digits via this approach.

The dimensionless mixing-cup concentration for the blood stream is defined by

$$C_m^* = \frac{\int_0^1 x^{*p}(1-x^{*2}) C^*(x^*)\, dx^*}{\int_0^1 x^{*p}(1-x^{*2})\, dx^*} = \sum_{n=1}^{\infty} B_n \exp(-\Lambda_n^2 z^*) \tag{31}$$

where

$$B_n = \frac{-(p^2+4p+3) A_n}{2\Lambda_n^2} \frac{dX_n(1)}{dx^*}. \tag{32}$$

Since the constants B_n decrease faster with increase in n than do the constants A_n, the mixing-cup concentration formula converges more rapidly than the local concentration result.

Table 1. Eigenvalues and constants A_n

		Cylindrical tube		Parallel-plates	
N_{Shw}	n	Λ_n	A_n	Λ_n	A_n
	1	2·70436 441988	1·4764354	1·68159 532224	1·2008304
	2	6·67903 144935	−0·8061239	5·66985 734590	−0·2991607
∞	3	10·67337 953805	0·5887622	9·66824 246251	0·1608265
	4	14·76107 846274	−0·4758504	13·66766 144261	−0·1074366
	5	18·66987 186445	0·4050218	17·66737 356535	0·0796461
	1	1·64124 968025	1·2013443	1·0	1·0882367
	2	5·47830 895912	−0·2929399	4·65613 740363	−0·1164139
1	3	9·43596 343412	0·1466828	8·56201 999716	0·0417816
	4	13·41524 289536	−0·0930493	12·51583 353451	−0·0216177
	5	17·40259 122647	0·0662907	16·48764 826504	0·0133105
	1	0·0	1·0	0·0	1·0
	2	5·06750 550093	0	4·28722 494563	0
0	3	9·15760 642631	0	8·30372 447753	0
	4	13·19722 473505	0	12·31060 606272	0
	5	17·22022 936397	0	16·31452 169609	0

DETERMINATION OF SHERWOOD NUMBERS

The overall Sherwood number is defined as $N_{Sho} = k_0 R/D_B$, where k_o is an overall mass transfer coefficient. This dimensionless group can be shown to be equal to

$$N_{Sho} = \frac{-(\partial C^*/\partial x^*)|_{x^*=1}}{C_m^*}. \quad (33)$$

The blood-side Sherwood number may be expressed in similar fashion as

$$N_{Sh_B} = -\frac{(\partial C^*/\partial x^*)|_{x^*=1}}{(C_m^* - C^*|_{x^*=1})}. \quad (34)$$

These Sherwood numbers are related to the 'wall' (membrane plus dialysate) Sherwood number via the usual 'addition-of-resistance' type of equation

$$\frac{1}{N_{Sh0}} = \frac{1}{N_{Sh_B}} + \frac{1}{N_{Sh_w}}. \quad (35)$$

Therefore, any one N_{Sh} may be calculated knowing the other two.

TYPICAL RESULTS

Table 1 presents the first five eigenvalues and constants A_n for both the cylindrical and parallel-plate geometries. It is easy to calculate any reasonable number of additional such values, and in fact we have done so. However, we shall omit presenting them, and also shall omit listing values for other important quantities like B_n, $X_n(1)$, $dX_n(1)/dx^*$, and $dX_n(1)/d\Lambda_n$ in the present paper. The eigenvalues in Table 1 are presented to one more significant figure than those given by Brown[6], although we have computed them to even higher accuracy. As expected, they agree exactly with Brown's results.

Figures 2 and 3 present, for the parallel-plate case, computed information on axial concentration profiles for various N_{Sh_w} values and transverse concentration profiles at several axial positions for the specific and realistic case of $N_{Sh_w} = 1$. The type of information given in this last figure has generally not been presented by earlier investigators (for larger N_{Sh_w} values, particularly, the transverse profile computations require being able to determine a fair number of Λ_n's to quite good accuracy). We also determined the dependence of N_{Sh_B} on axial distance for various N_{Sh_w} values, but omit graphical presentation here, since previous investigators[22, 23] have already shown such information.

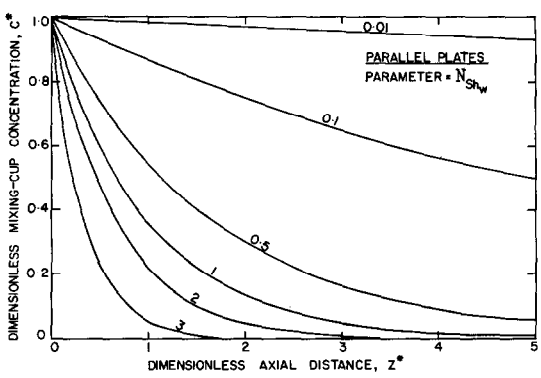

Fig. 2. Axial mixing-cup concentration profiles in parallel-plate dialyzers.

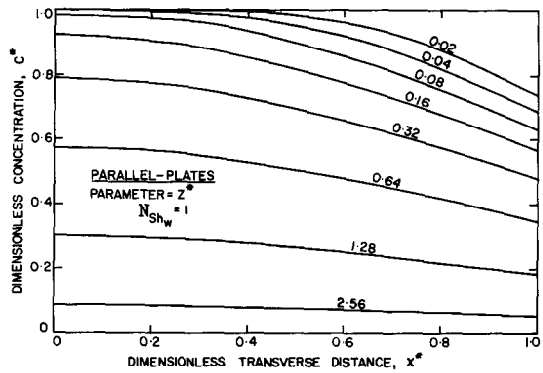

Fig. 3. Transverse concentration profiles in parallel-plate dialyzers for $N_{Sh_w} = 1$.

For any single value of N_{Sh_w}, such information on the Λ_n's and A_n's, the axial and transverse concentration profiles, and the axial variation of N_{Sh_B} were computed to acceptable accuracy in less than 30 sec execution time on an IBM 360 Model 44 computer.

APPLICATION TO REAL FLAT-PLATE HEMODIALYZERS

Since our equations constitute a reasonable model for parallel-plate hemodialyzers, it is of value to consider their application to real systems. Typical dialysis units of this kind (e.g. the Kiil device) have two side-by-side 15 cm × 90 cm blood channels per 'layer' (i.e. per pair of membrane sheets). This gives a membrane area of about 0·54 m^2 per layer. Usually 2 or 4 layers are employed. Static blood channel heights of about 0·025 cm are usually used (Colton[12] cites values of 0·023, 0·026, and 0·03 cm for Kiil units), since studies[13] have shown this to be about optimum, but values for dynamic (flow) conditions are higher[12]. We shall assume a dynamic blood channel height of 0·03 cm. Blood flow rates are normally on the order of 100 cm^3/min per layer.

To compute C_m^* vs z^* in such a device, a value of N_{Sh_w} must be determined. For a solute like urea (the solute which one removes in largest quantity in hemodialysis), Colton *et al.*[9] give values of D_B for various blood hematocrit values, at 27°C. Since uremic patients typically have hematocrits of 25 or even less a D_B value of $\sim 0{\cdot}8 \times 10^{-5}$ cm^2/sec, deduced from Colton's data for 20–25 per cent hematocrit, will be assumed. For 37°C, the usual temperature of dialyzer operation, the Stokes–Einstein equation suggests that D_B would be somewhat larger, e.g. $1{\cdot}0 \times 10^{-5}$ cm^2/sec. For the types of membranes that are most commonly used (e.g. Cuprophane PT-150), Colton *et al.*[9] cite a urea membrane diffusivity (37°C) of $0{\cdot}286 \times 10^{-5}$ cm^2/sec, while Farrell and Babb[14] give $0{\cdot}265 \times 10^{-5}$ cm^2/sec. Using $D_m \cong 0{\cdot}276 \times 10^{-5}$ cm^2/sec and a wet membrane thickness of 1 mil[12], a membrane P_m of about 0·0011 cm/sec is indicated. The membrane 'resistance', R_m, is therefore about 15 min/cm. Various investigators[9, 15, 16] reporting on experiments in 'test' dialyzers (small units about $13 \times 15 \times 2{\cdot}5$ cm in size and 90 cm^2 in membrane area) found R_m values of roughly 15 to 16 min/cm for Cuprophane PT-150.

For the dialysate side, one must normally determine the resistance (R_D) by experiment, since various types of mixing devices (e.g. multiple cone-shaped supports) are often built into the dialysate side and no simple flow field exists. Colton *et al.*[9] have presented a specific correlation for foamed nickel support material. They, Popovich *et al.*[16] and Babb[15] estimated $R_D \cong 5$–7 min/cm for reasonable flow rates in their 'test' dialyzers. In regular Kiil dialyzers, however, R_D is probably much larger (e.g. Babb *et al.*[15] cite an estimate of 16 min/cm for urea at 37°C) since foamed metal is not used and dialysate velocities are much lower than in typical test units.

Assuming $R_D = 16$ min/cm and $R_m = 15$ min/cm, as shown above, R_w then would equal 31 min/cm (1860 sec/cm). Colton *et al.*[9] have shown that, for urea, $K \cong 1$. Therefore,

$$N_{Sh_w} = \frac{k_w R}{D_B K} = \frac{R}{R_w D_B K} = \frac{(0{\cdot}015)}{1860(1{\cdot}0 \times 10^{-5})} = 0{\cdot}8.$$

This question now arises as to what z^* value corresponds to the end of the real dialyzer. Since $Pe = u_{max}R/D_B \cong 4200$ for the parameter values we have selected, then

$$z^* = \frac{z}{RPe} \cong \frac{90}{0{\cdot}015\,(4200)} \cong 1{\cdot}4$$

at the end of the dialyzer.

Our computations (see also Refs. 22 and 23) indicate that N_{Sh_B} is within 10 per cent of its asymptotic value over a large part (about 92 per cent) of the length of the dialyzer. N_{Sh_B} approaches roughly 2·0 as an asymptotic value (Babb *et al.*[17] observed this also), and this corresponds to a blood-side resistance, R_B, of 20·2 min/cm. Babb *et al.*[15] cite a similar figure, $R_B \cong 24$ min/cm, for urea removal in a 2-layer Kiil unit at $Q_B = 200$ cm^3/min and $Q_D = 500$ cm^3/min. A typical distribution of resistances in a Kiil dialyzer is therefore: $R_{tot} = R_B + R_m + R_D \cong 22 + 15 + 16 = 53$ min/cm. The blood, membrane, and dialysate resistances constitute roughly 42, 28, and 30 per cent, respectively, of the overall resistance.

Table 2 summarizes available experimental data concerning the percentage of urea extracted from inlet blood streams at 37°C in Kiil type dialyzers. Using an R_m value of ~15 min/cm for Cuprophane PT-150 membranes, as established above, and Colton's[12] R_m value of ~24 min/cm for DuPont PD-215 cellophane membranes (these are about 2 mils thick when wet, hence they have higher resistance), one can establish the N_{Sh_w} numbers cited in

Table 2. Comparison of experimental and predicted urea removal values for parallel-flow Kiil dialyzers at 37°C

Membrane type	R_m (min/cm)	Area (m^2)	No. layers	Q_B (cm^3/min)	Q_D (cm^3/min)	N_{Sh_w}	z^* at end of dialyzer	Exptl. % Rem.	Pred. % Rem.	Ref.
Cuprophane PT-150	15	1·15	2	200	1000	1·0	1·2	56	69	[18]*
DuPont PD 215	24	1·15	2	130	1800	0·6	2·4	73	78	[19]†
DuPont PD 215	24	1·15	2	200	2000	0·8	1·2	54	63	[19]*
DuPont PD 215	24	2·1	4	400	4500	0·8	1·1	56	59	[19]*

**In vitro* with normal blood (D_B computed for 40 per cent hematocrit as $\sim 0{\cdot}8 \times 10^{-5}$ cm^2/sec).
†*In vivo* with uremic blood ($D_B \cong 1{\cdot}0 \times 10^{-5}$ cm^2/sec).

Table 2. The values of z^* at the end of each dialyzer were determined from the membrane area figures (assuming 15 cm channel widths) and the blood flow rate figures. Figure 2 then was used to find the predicted removal percentages. On average the predicted values are 13 per cent higher than the experimental ones (range $\cong$5–23 per cent). The agreement is worst at the lowest dialysate rate, which is expected, since this comparison is based on the assumption of zero dialysate solute concentration. At higher Q_D rates, for which the zero concentration assumption is more valid, the agreement is fairly good.

The discrepancy between predicted solute removals and the experimentally determined levels probably results from the effects of non-uniform blood flow distribution among the channels of Kiil units. Freeman *et al.*[19] have actually determined the extent of maldistribution in such dialyzers, and they concluded that it does indeed cause appreciable loss of efficiency. Other factors such as the fact that membrane supports always tend to block a certain amount of membrane area (perhaps as much as 20 per cent[20]), the existence of finite urea concentrations in the dialysate, and distension of the membrane due to significant transmembrane pressure differences help to explain the disagreement also. Since most investigators fail to report one or more critical values for geometrical dimensions and operating variables, it is impossible to completely reconcile such discrepancies.

It is also very important to point out that the *theoretical* results are quite sensitive to the choices of certain parameter values. In particular, the value assigned to the dynamic blood channel height is critical. This quantity is difficult to measure accurately and is often not highly reproducible (the degree to which membranes are stretched during assembly of a Kiil unit and the resulting amount of distension which occurs during actual operation varies widely). Since R_B is directly proportional to the value of R, and z^* varies as R^{-2}, use of the theoretical model requires a reasonably accurate knowledge of R. The value of D_B is also of great importance. One must also be careful that accurate membrane thickness and D_m values[12] are employed. It might be remarked here that Smith *et al.*[21] have found that different samples of membranes of the same kind vary by as much as 20 per cent in thickness.

CONCLUSIONS

(1) A method of solving certain eigenvalue problems has been outlined which has significant computational advantages over previously used methods.

(2) A mathematical model for parallel-plate (or coiled) and cylindrical-tube hemodialyzers has been formulated and solved. Basic system characteristics have been presented for parallel-plate units. The effect of N_{Sh_w} on the shape of transverse concentration profiles in such systems has been illustrated. The approach of system behavior to asymptotic conditions has also been determined.

(3) Comparisons of theoretical and experimental results for parallel-plate hemodialysers has shown that:

(a) R_D values are a very significant part of the total resistance in full-sized units, and therefore careful experiments are needed to establish the magnitude of R_D.

(b) Actual R_B and per cent-solute-removal values for Kiil hemodialyzers disagree by about 20 per cent with predicted values. These discrepancies are explainable, however, in terms of blood flow maldistribution effects and in terms of the sensitivity of the theoretical model to the choices of parameter values.

(4) The theoretical model presented provides one with valuable insight into system characteristics, and serves as a rational basis for interpreting and correlating experimental data.

NOTATION

a, b, A_n, B_n constants in various series (defined in text)
C concentration in blood phase
C_i concentration in blood phase at inlet to dialyzer
C_m mixing-cup concentration in blood phase
C_o concentration in dialyzate phase
D_B diffusivity in blood phase
k mass transfer coefficient
K_B membrane/blood partition coefficient
K_D membrane/dialysate partition coefficient
K ratio K_D/K_B
M confluent hypergeometric function
N_{Sh} Sherwood number ($= kR/KD_B$)
p index equal to zero for parallel-plates, unity for cylindrical tube
Pe Peclet number ($= u_{max}R/D_B$)
P_m membrane permeability
Q_B, Q_D volumetric blood and dialysate flow rates, respectively

R half height of blood channel for parallel-plates; radius for cylindrical tube

R_B, R_D, R_m blood phase, dialysate phase, and membrane mass transfer resistances, respectively

u_{max} blood phase velocity at centerline of flow

W transformed eigenfunction defined by Eq. (19)

x transverse coordinate

X eigenfunction of Eq. (15)

y transformed dimensionless transverse coordinate defined by Eq. (18)

z axial coordinate

Greek symbol

Λ eigenvalue of Eq. (15)

Superscripts

* indicates dimensionless variables

′ indicates half-channel height or radius is measured from centerline to dialysate-side of membrane (see Fig. 1)

Subscripts on k *and* N_{Sh}

o, B, D, w indicate overall, blood phase, dialysate phase, and wall quantities, respectively

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