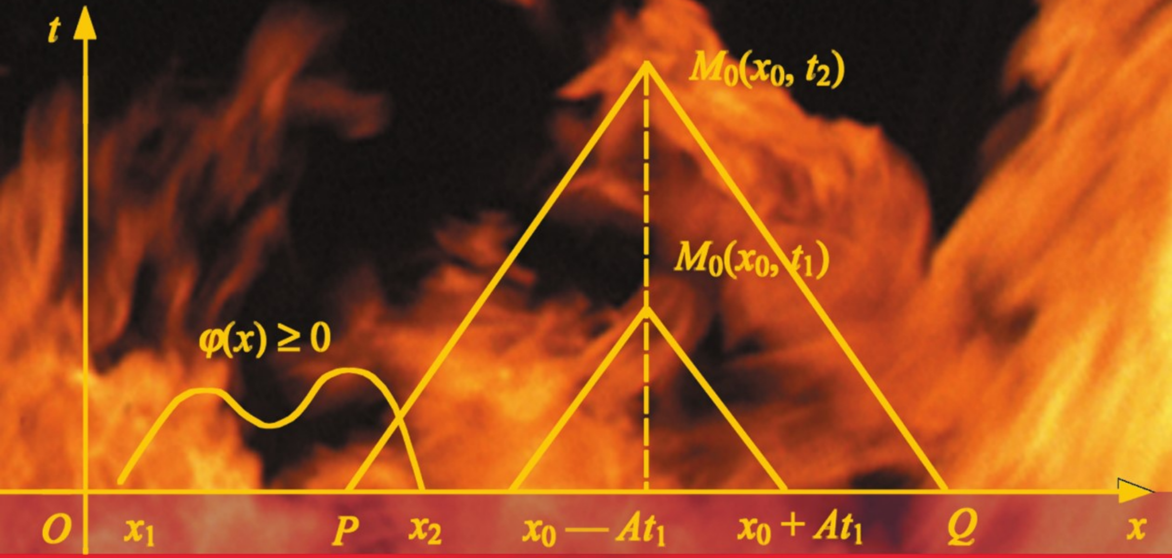

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Heat Conduction

Mathematical Models and Analytical Solutions

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Preface

As an important transport process, heat conduction is governed by the first law of thermodynamics, or the conservation of energy. The heat-conduction equation is its mathematical representation for heat conduction processes. However, we can have different types of heat-conduction equations depending on which constitutive relations of heat flux density, the relations between the heat flux density vector and the temperature gradient, are used. The dual-phase-lagging constitutive relation was first proposed by Professor D.Y. Tzou in 1995 and is used for describing heat conduction involving high heat flux and strong unsteadiness, where heat conduction is micro both in time and in space. Its two important special cases are the Cattaneo–Vernotte constitutive relation and the Fourier law of heat conduction. The former is appropriate for describing heat conduction that is micro in time and macro in space. The latter is a good representation of heat conduction that is macro both in time and in space.

By applying the dual-phase-lagging constitutive relation, the first law of thermodynamics leads to the dual-phase-lagging heat-conduction equation. Its two very important special cases are the hyperbolic heat-conduction equation and the classical parabolic heat-conduction equation. The former comes from the application of the Cattaneo–Vernotte constitutive relation. The latter is from the application of the Fourier law of heat conduction as the constitutive relation. For steady heat conduction, the dual-phase-lagging heat-conduction equation reduces into the potential equation.

Both the hyperbolic and the dual-phase-lagging heat-conduction equations share not only features of classical parabolic heat-conduction equations but also those of classical wave equations. Actually, the classical wave equation is mathematically a special case of dual-phase-lagging heat-conduction equation. The focus of the present monograph is on these equations: their solution structures, methods of finding their solutions under various supplementary conditions, as well as the physical implication and applications of their solutions.

Many phenomena in social, natural and engineering fields are also governed by wave equations, potential equations or one of three types of heat-conduction equations. These equations are the most important mathematical equations in physics.

Therefore, the present monograph can serve as a reference for researchers working on heat conduction of macro- and micro-scales as well as on mathematical methods of physics. It can also serve as a text for graduate courses on heat conduction or on mathematical equations in physics.

Features of the present monograph include: (1) interplay between mathematical theories and physical concepts. We address important concepts, methods and formulas from all angles of mathematical principles, physical implications, unit analyses and ideas behind the ways of tackling problems. In particular, unit analyses occur frequently throughout the monograph. This differs from other books and monographs published in the field; (2) comprehensive discussion of newly-developed hyperbolic and dual-phase-lagging heat-conduction equations. The heat-conduction equations of hyperbolic-type and dual-phase-lagging type were developed very recently, so their literature is very limited. The present monograph appears to be the first which offers a systematic and comprehensive discussion of them based on our research in the last ten years. The task of finding solutions of various equations under variety of supplementary equations has been tabulated for convenience and easy reference. This is also a fundamental difference from other books and monographs published in the field; (3) emphasis on tackling problems by using different methods and approaches. We address the problem from different angles and with different methods in order to gain deeper insight into the essence, nature and implications of the problem. The material that is presented adheres to the following procedure: translating physical problems into mathematical equations of physics, studying the equations systematically and rigorously and then verifying the results using examples; and (4) concise presentation in an easy-to-understand language. We describe concepts, methods and results in illuminating and concise language for the benefit of readability. We also pay special attention to the link with knowledge in advanced mathematics. The knowledge of undergraduate advanced mathematics and engineering mathematics is sufficient for understanding all materials in the monograph.

We owe much to our many colleagues in the fields of transport phenomena and mathematical equations in physics whose insights fill many of the pages of this monograph. We benefited immensely from the stimulating discussion with D. Y. Tzou and M. T. Xu. We are very grateful to K. Senechal who has provided us with valuable comments and suggestions for improving our manuscript. The support of our research program by the CRCG of the University of Hong Kong and the Research Grants Council of the Hong Kong Special Administrative Region is also greatly appreciated. We would also like to thank X. Lai for her professional typing of the manuscript. And, of course, we owe special thanks to our respective families for their tolerance of the obsession and the late nights that seemed necessary to bring this monograph to completion. Looking ahead, we will appreciate it very much if users of this monograph will write to call our attention to the imperfections that may have slipped into the final version.

Liqiu Wang
Xuesheng Zhou
Xiaohao Wei

Contents

Chapter 1
Introduction

In this chapter we discuss basic concepts and definitions of partial differential equations, general methods for developing equations of mathematical physics, the progress in heat-conduction theory and three types of heat-conduction equations. Also discussed are supplementary conditions and problems for determining solutions.

1.1 Partial Differential Equations

1.1.1 Partial Differential Equations and Their Orders

An *ordinary differential equation* (*ODE* for short) is a differential equation that contains one or more derivatives of the dependent variable (in addition to the dependent variable and the independent variable). The order of an ordinary differential equation is the order of the highest-ordered derivative appearing in the equation. Similarly, a partial differential equation and its order can be defined for the case of several independent variables.

Definition 1. A differential equation that contains the dependent variable, more than one independent variables and one or more partial derivatives of the dependent variable is called a *partial differential equation*, or *PDE* for short. The order of the highest-ordered partial derivative appearing in the equation is called the *order* of the PDE.

Independent and dependent variables may not appear in a PDE explicitly for some special cases. A PDE must contain, however, at least one partial derivative of the dependent variable. For an unknown function $u(x,y)$ of two independent variables,

$$u_x = 1, \quad u_{xy} = 0 \qquad \text{and} \qquad u_{xxy} + 2u_x = 0$$

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are the first-order, the second-order and the third-order PDE, respectively. Here, subscripts on dependent variables denote differentiations, e.g.

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}.$$

In general, the first- and the second-order PDE of $u(x,y)$ may be written in the form

$$F(u,x,y,u_x,u_y) = 0, \quad F(u,x,y,u_x,u_y,u_{xx},u_{xy},u_{yy}) = 0,$$

respectively. The k-th order PDE of an unknown function $u(x_1,x_2,\cdots,x_n)$ of n independent variables $x_1,x_2,\cdots,x_n$ can be written in a general form

$$F\left(u,x_1,x_2,\cdots,x_n,\frac{\partial u}{\partial x_1},\cdots,\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1^2},\frac{\partial^2 u}{\partial x_1 \partial x_2},\cdots,\frac{\partial^2 u}{\partial x_n^2},\cdots,\frac{\partial^k u}{\partial x_n^k}\right) = 0.$$

For example, the *three-dimensional wave equation*

$$u_{tt} = a^2 \Delta u + f(x,y,z,t), \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad a^2 = \text{constant}$$

is a second-order PDE of $u(x,y,z,t)$; the *two-dimensional heat-conduction equation*

$$u_t = a^2 \Delta u + f(x,y,t), \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad a^2 = \text{constant}$$

is a second-order PDE of $u(x,y,t)$. The second-order PDE

$$\Delta u(x,y) = 0, \quad \Delta u(x,y,z) = 0$$

are called the two-dimensional and the three-dimensional *Laplace equations*, respectively. *Equations of mathematical physics* are the PDE that comes from physical laws and describe physical processes or physical states. The wave equation, the heat-conduction equation and the Laplace equation are three typical equations of mathematical physics. They can be viewed as three special cases of *the dual-phase-lagging heat-conduction equation*. The latter has been recently developed in examining energy transport involving high-rate heating.

1.1.2 Linear, Nonlinear and Quasi-Linear Equations

A PDE can be linear or nonlinear, as is the case for an ordinary differential equation. The linear PDE has many good properties and it frequently arises in problems of mathematical physics. We shall primarily consider linear PDE in this book.

Definition 2. A PDE is said to be *linear* if it is linear in the unknown function and all its derivatives. An equation which is not linear is called a *nonlinear* equation. A nonlinear equation is said to be *quasi-linear* if it is linear in all highest-ordered derivatives of the unknown function.

For example, the above-mentioned three typical equations of mathematical physics and the equations

$$\frac{\partial u}{\partial x}+a(x,y)\frac{\partial u}{\partial x}=f(x,y), \quad \frac{\partial^2 u}{\partial x\partial y}=2y-x, \quad \frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}+u(x,y)$$

are linear, whereas

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}+a(x,y)\frac{\partial u}{\partial y}=u^2$$

are nonlinear. Equations

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+u^2=0 \qquad \text{and} \qquad u\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}=0$$

are quasi-linear.

The most general second-order linear PDE in n independent variables has the form

$$\sum_{i,j=1}^{n} a_{ij}u_{x_ix_j}+\sum_{i=1}^{n} b_iu_{x_i}+cu=f \tag{1.1}$$

where we assume $u_{x_ix_j}=u_{x_jx_i}$ and $a_{ij}=a_{ji}$ without loss of generality. We also assume that a_{ij}, b_i , c and f are known functions of the n independent variables x_i. If all coefficients a_{ij}, b_i and c are constants, the equation is called a *PDE with constant coefficients* ; otherwise it is a *PDE with variable coefficients*. As for ordinary differential equations, we can classify linear PDE into homogeneous and nonhomogeneous equations.

Definition 3. The *free term* in a PDE is the term that contains no unknown function and its partial derivatives. If the free term is identically zero, a linear equation is called a *homogeneous PDE*; otherwise it is called a *nonhomogeneous PDE*.

Equation (1.1) is *homogeneous* if $f\equiv 0$; otherwise it is *nonhomogeneous*. Note that the definition of homogeneity is only for linear PDE.

1.1.3 Solutions of Partial Differential Equations

Definition 4. A function u is called a *classical solution* of the PDE, a *solution* for short, if it has continuous partial derivatives of all orders appearing in a PDE and satisfies the equation.

The general solution of an ODE of n-th order is a family of functions depending on n independent arbitrary constants. In the case of a PDE, the general solution contains arbitrary functions. To illustrate this, consider the equation

$$u_{xy} = x^2 y, \quad u = u(x,y).$$

If we integrate this equation with respect to y, we obtain

$$u_x(x,y) = \frac{1}{2}x^2y^2 + f(x).$$

A second integration with respect to x yields

$$u = \frac{1}{6}x^3y^2 + \varphi_1(x) + \varphi_2(y),$$

where $\varphi_1(x)$ and $\varphi_2(y)$ are arbitrary functions.

Suppose that u is a function of three variables, x, y and z. Then for the equation $u_{yy} = 2$, we find the general solution

$$u(x,y,z) = y^2 + y\varphi_1(x,z) + \varphi_2(x,z),$$

where φ_1 and φ_2 are arbitrary functions of two variables x and z.

We recall that in the case of ordinary differential equations, the first task is to find its general solution, and then a particular solution is determined by finding the values of arbitrary constants from the prescribed conditions. But, for partial differential equations, selecting a particular solution satisfying supplementary conditions from the general solution of a PDE may be as difficult as, or even more difficult than, the problem of finding the general solution itself. This is so because the general solution of a partial differential equation involves arbitrary functions; the specialization of such a solution to the particular form which satisfies supplementary conditions requires the determination of these arbitrary functions, rather than merely the determination of constants.

For linear homogeneous ODE of order n, a linear combination of n linearly-independent solutions is a general solution. Unfortunately, this is not true for PDE in general. This is due to the fact that the solution space of every homogeneous linear partial differential equation is infinite dimensional. For example, the partial differential equations

$$u_x - u_y = 0 \qquad \text{and} \qquad u = u(x,y)$$

can be transformed into the equation $2u_\eta = 0$ by the transformation of variables

$$\xi = x + y, \quad \eta = x - y.$$

The general solution is

$$u(x,y) = f(x+y),$$

where $f(x+y)$ is an arbitrary function. Thus, we see that each of the functions

$$(x+y)^n, \quad \sin n(x+y), \quad \cos n(x+y), \quad e^{n(x+y)}, \quad n = 1,2,3,\cdots$$

is a solution. The fact that a simple equation such as $u_x - u_y = 0$ yields infinitely many solutions is an indication of an added difficulty which must be overcome in the study of partial differential equations. Thus, we generally prefer to directly determine the particular solution of a PDE satisfying prescribed supplementary conditions.

1.1.4 Classification of Linear Second-Order Equations

The classification of partial differential equations is suggested by the classification of quadratic equations in analytic geometry. The equation

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$$

is elliptic, parabolic, or hyperbolic accordingly as $\Delta = B^2 - AC$ is negative, zero, or positive.

Consider a second-order linear equation in the dependent variable u and the independent variables x and y,

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu + f = 0 \tag{1.2}$$

where the coefficients are functions of x and y. The equation is said to be elliptic, parabolic, or hyperbolic at a point (x_0, y_0) accordingly as

$$\Delta = a_{12}^2(x_0, y_0) - a_{11}(x_0, y_0)a_{22}(x_0, y_0)$$

is negative, zero, or positive. If this is true at all points in a domain D, the equation is said to be elliptic, parabolic, or hyperbolic in D.

It should be remarked here that a given PDE may be of a different type in a different domain. For example, *Tricomi equation*

$$yu_{xx} + u_{yy} = 0$$

is hyperbolic for $y < 0$, parabolic for $y = 0$, and elliptic for $y > 0$, since $\Delta = -y$.

To generalize the classification to the case of more than two independent variables, consider the quadratic form of the equation (1.2)

$$A(\lambda) = a_{11}\lambda_1^2 + 2a_{12}\lambda_1\lambda_2 + a_{22}\lambda_2^2 = \sum_{i,j=1}^{2} a_{ij}\lambda_i\lambda_j$$

whose matrix is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{12} = a_{21}\,.$$

The matrix has two characteristic roots which can be determined by its characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0\,.$$

The two characteristic roots have the same sign when $\Delta < 0$, and the opposite sign when $\Delta > 0$. When $\Delta = 0$, there is a vanished characteristic root. This can be easily verified. Therefore, the equation (1.2) is elliptic, parabolic, or hyperbolic accordingly as two characteristic roots have the same sign, there is a vanished characteristic root, or two characteristic roots have opposite signs.

An analogous classification can be made in the case of second-order linear equations with three independent variables for the three characteristic roots of the matrix of the quadratic form defined by

$$A(\lambda) = \sum_{i,j=1}^{3} a_{ij}\lambda_i\lambda_j\,.$$

For instance, the three-dimensional Laplace equation $u_{xx}+u_{yy}+u_{zz}=0$ is elliptic because the three characteristic roots $(\lambda_1=\lambda_2=\lambda_3=1)$ are with the same sign. The two-dimensional heat-conduction equation

$$u_t = a^2(u_{xx}+u_{yy}), \quad a^2 = \text{positive constant}$$

is parabolic because there is a vanished characteristic root $\left(\lambda_1=0\,,\lambda_2=\lambda_3=a^2\right)$. The three- dimensional nonhomogeneous wave equation

$$u_{tt} = a^2\Delta u + f(x,y,z,t), \quad a^2 = \text{positive constant}$$

contains four independent variables and is of hyperbolic type because there are both negative and positive characteristic roots $\left(\lambda_1=-1\,,\lambda_2=\lambda_3=\lambda_4=a^2\right)$ and three of them are with the same sign.

In general, the second-order linear partial differential equation (1.1) in n independent variables is elliptic, parabolic, or hyperbolic at a point P_0 accordingly as all n characteristic roots are with the same sign, there is a vanished characteristic root, or the n characteristic roots have different signs but $n-1$ of them share the same sign. Here the characteristic roots are those of the matrix of the quadratic form defined by

$$A(\lambda) \quad = \sum_{i,j=1}^{n} a_{ij}(p_0)\lambda_i\lambda_j\,.$$

If this is true at all points in a domain D, the equation is of elliptic, parabolic, or hyperbolic type in D.

It should be remarked here that this classification is only for the second-order linear equations. Whose types are fixed only by the coefficients of second-order partial derivatives.

1.1.5 Canonical Forms

To facilitate resolving Eq. (1.2), we may transform it into a canonical form by a transformation of independent variables.

Consider the transformation

$$\xi = \xi(x,y), \quad \eta = \eta(x,y). \tag{1.3}$$

Assume that ξ and η are twice continuously differentiable and the Jacobian

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$$

is nonzero in the region under consideration so that x and y can be determined uniquely from the system (1.3). By the chain rule, we have

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x\,, \quad u_y = u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}\,, \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}\,, \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}\,.
\end{aligned}$$

Substituting these into Eq. (1.2) leads to

$$A_{11}u_{\xi\xi} + 2A_{12}u_{\xi\eta} + A_{22}u_{\eta\eta} + B_1 u_\xi + B_2 u_\eta + Cu + F = 0\,, \tag{1.4}$$

where

$$\begin{aligned}
A_{11} &= a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2\,, \\
A_{12} &= a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y\,, \\
A_{22} &= a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2\,, \\
B_1 &= a_{11}\xi_{xx} + 2a_{12}\xi_{xy} + a_{22}\xi_{yy} + b_1\xi_x + b_2\xi_y\,, \\
B_2 &= a_{11}\eta_{xx} + 2a_{12}\eta_{xy} + a_{22}\eta_{yy} + b_1\eta_x + b_2\eta_y\,, \\
C &= c\,, \quad F = f\,.
\end{aligned}$$

The resulting equation (1.4) is in the same form as the original equation (1.2) under the general transformation (1.3). The nature of the equation remains invariant under such a transformation if the Jacobian does not vanish. This can be inferred from the fact that the sign of the discriminant Δ does not vary under the transformation, that is,

$$A_{12}^2 - A_{11}A_{22} = J^2\left(a_{12}^2 - a_{11}a_{22}\right)\,,$$

which can be easily verified. It should be noted here that the equation can be of a different type at different points in the domain, but for our purpose we shall assume that the equation under consideration is of a single type in a given domain.

We suppose first that a_{11}, a_{12} and a_{22} are non-zero. Let ξ and η be the new variables such that the coefficients A_{11} and A_{22} in Eq. (1.4) vanish. Thus

$$A_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 = 0, \quad A_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2 = 0\,.$$

These two equations are of the same type and hence we may write them in the form

$$a_{11}z_x^2 + 2a_{12}z_xz_y + a_{22}z_y^2 = 0\,, \tag{1.5}$$

in which z stands for either of the functions ξ or η. Dividing through by z_y^2, Eq. (1.5) becomes

$$a_{11}\left(\frac{z_x}{z_y}\right)^2 + 2a_{12}\left(\frac{z_x}{z_y}\right) + a_{22} = 0\,. \tag{1.6}$$

Along the curve $z = \text{constant}$, we have

$$\mathrm{d}z = z_x\,\mathrm{d}x + z_y\,\mathrm{d}y = 0\,.$$

Thus $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{z_x}{z_y}$, and therefore, Eq. (1.6) may be written in the form

$$a_{11}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2a_{12}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + a_{22} = 0\,, \tag{1.7}$$

the roots of which are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}\,, \tag{1.8}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_{12} - \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}\,. \tag{1.9}$$

These equations, which are called the *characteristic equations*, are the ordinary differential equations for families of curves in the xy-plane along which ξ and η are constant. The relation between x and y specified by Eq. (1.7) can be represented by a curve on the xy-plane, called a *characteristic curve*. Since the characteristic equations are first-order ordinary differential equations, their solutions may be written as

$$\xi(x,y) = c_1\,, \quad c_1 = \text{constant}\,,$$

$$\eta(x,y) = c_2\,, \quad c_2 = \text{constant}\,.$$

Hence the transformation

$$\xi = \xi(x,y), \quad \eta = \eta(x,y)$$

will transform Eq. (1.2) to a canonical form.

Hyperbolic Type

If $a_{12}^2 - a_{11}a_{22} > 0$, an integration of Eqs. (1.8) and (1.9) yield two real and distinct families of characteristic curves. Equation (1.2) reduces to

$$u_{\xi\eta} + \psi_1(\xi, \eta, u, u_\xi, u_\eta) = 0\,. \tag{1.10}$$

This is called the *first canonical form of the hyperbolic equation*. Now if the new independent variables

$$s = \xi + \eta, \quad t = \xi - \eta$$

are introduced, Eq. (1.10) is transformed into

$$u_{ss} - u_{tt} + \psi_2(s, t, u, u_s, u_t) = 0\,. \tag{1.11}$$

This form is called the *second canonical form of the hyperbolic equation*.

Parabolic Type

When $a_{12}^2 - a_{11}a_{22} = 0$, Eqs. (1.8) and (1.9) coincide. Thus, there exits only one real family of characteristic curves, and we obtain only a single integral $\xi =$ constant (or $\eta =$ constant). For any $\eta(x,y)$ that is linearly-independent of $\xi(x,y)$, Eq. (1.2) reduces to

$$u_{\eta\eta} + \psi_3(\xi, \eta, u, u_\xi, u_\eta) = 0\,. \tag{1.12}$$

This is called the *canonical form of the parabolic equation*.

Equation (1.2) may also assume the form

$$u_{\xi\xi} + \psi_4(\xi, \eta, u, u_\xi, u_\eta) = 0 \tag{1.13}$$

if we choose $\eta =$ constant as the integral of Eq. (1.8).

Elliptic Type

When $a_{12}^2 - a_{11}a_{22} < 0$, ξ and η in $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$ are complex conjugates. The variable transformation becomes $\xi = \xi(x,y)$, $\eta = \eta(x,y) = \bar{\xi}(x,y)$. We may introduce the new real variables

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{1}{2\mathrm{i}}(\xi - \eta)$$

so that $\xi = \alpha + \mathrm{i}\beta$, $\eta = \alpha - \mathrm{i}\beta$. Eq. (1.2) thus reduces to

$$u_{\alpha\alpha} + u_{\beta\beta} + \psi_5(\alpha, \beta, u, u_\alpha, u_\beta) = 0. \tag{1.14}$$

This is called the *canonical form of the elliptic equation*.

Remark 1. If the coefficients in the canonical forms are constants, a further function transformation $u(\xi,\eta) = e^{k_1\xi+k_2\eta}v(\xi,\eta)$ can be used to vanish terms of the first derivatives. Here k_1 and k_2 are undetermined coefficients.

Example. Simplify the second-order linear PDE with constant coefficients

$$u_{xx} + u_{xy} + u_{yy} + u_x = 0.$$

Solution. Since $\Delta = \left(\frac{1}{2}\right)^2 - 1 = -\frac{3}{4} < 0$, the equation is elliptic everywhere. The characteristic equation is

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{dy}{dx} + 1 = 0 \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 \pm \mathrm{i}\sqrt{3}}{2}.$$

Its integration gives

$$y - \frac{1+\mathrm{i}\sqrt{3}}{2}x = c_1, \quad y - \frac{1-\mathrm{i}\sqrt{3}}{2}x = c_2.$$

From a transformation of independent variables $\xi = y - \frac{x}{2}$, $\eta = -\frac{\sqrt{3}}{2}x$, we obtain the canonical form

$$u_{\xi\xi} + u_{\eta\eta} - \frac{2}{3}u_\xi - \frac{2\sqrt{3}}{3}u_\eta = 0.$$

To simplify further, we introduce the new dependent variable $v(\xi,\eta) = u(\xi,\eta)e^{-(k_1\xi+k_2\eta)}$, where k_1 and k_2 are undetermined coefficients. Substituting v into the canonical form yields

$$v_{\xi\xi} + v_{\eta\eta} + \left(2k_1 - \frac{2}{3}\right)v_\xi + \left(2k_2 - \frac{2\sqrt{3}}{3}\right)v_\eta + \left(k_1^2 + k_2^2 - \frac{2}{3}k_1 - \frac{2\sqrt{3}}{3}k_2\right)v = 0.$$

Set $k_1 = \frac{1}{3}$ and $k_2 = \frac{\sqrt{3}}{3}$ so that the terms involving the first derivatives vanish. Thus, the above equation is reduced to $v_{\xi\xi} + v_{\eta\eta} - \frac{4}{9}v = 0$.

Remark 2. *Principle of superposition of linear equations*. Linear equations have several remarkable properties that are very useful in finding their solutions. We discuss these properties here by using second-order partial differential equations as the example. Let L be a linear operator $L = \sum_{ij=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c$. The most general homogeneous or nonhomogeneous second-order linear partial differential equation in n independent variables [Eq. (1.1)] may respectively be written in the form

$$Lu = 0 \qquad \text{and} \qquad Lu = f\,.$$

The following properties can be readily shown.

Property 1. A linear combination of two solutions of a homogeneous equation is also a solution of the equation. That is $L(c_1u_1 + c_2u_2) = 0$, if $Lu_1 = 0$ and $Lu_2 = 0$. Here c_1 and c_2 are arbitrary constants.

Property 2. Let a sequence of functions $\{u_{\mathrm{i}}\}$, $i = 1, 2, \cdots$ be solutions of a homogeneous equation $Lu = 0$, and $u = \sum_{i=1}^{\infty} c_{\mathrm{i}} u_{\mathrm{i}}$, $c_{\mathrm{i}}\,(i = 1, 2, \cdots) =$ constants be uniformly convergent and twice differentiable with respect to the independent variables $x_1, x_2, \cdots, x_n$ term by term in a domain. Then u is also a solution of the equation, that is $Lu = L\left(\sum_{i=1}^{\infty} c_i u_i\right) = 0$.

Property 3. $u = u_1 + u_2$ is a solution of a nonhomogeneous equation if u_1 and u_2 are the solutions of the homogeneous and the nonhomogeneous equations, respectively. Therefore, we have $Lu = L(u_1 + u_2) = f$, if $Lu_1 = 0$ and $Lu_2 = f$.

These properties are collectively called the *principle of superposition*, which is important in finding solutions of linear equations. It should be remarked that the supplementary conditions must be linear as well in order to apply this principle.

1.2 Three Basic Equations of Mathematical Physics

1.2.1 Physical Laws and Equations of Mathematical Physics

In physics and engineering, we normally describe or characterize the state of a physical system and the variation of system states by physical variables that are functions of time and position in space. Physical laws govern and provide the fundamental relations among basic physical variables. Equations of mathematical physics come from and are the quantitative representation of physical laws. They are different from general partial differential equations.

The fundamental relations established by the physical laws can be local or global in space. They can also be for an instant or a period in time. The equations of mathematical physics contain partial derivatives of physical variables with respect to time and space coordinates. Therefore, the fundamental relation between physical variables in the equations of mathematical physics is local in space and time. The relation between physical variables in their solutions is global in space and time. Thus solving equations of mathematical physics under supplementary conditions is actually a mathematical method to find the *global relations for a process* from the *local relations at a time instant*.

1.2.2 Approaches of Developing Equations of Mathematical Physics

In this subsection, we discuss the methods of developing equations of mathematical physics from physical laws. As the equations represent physical relations at a point in space and at an instant in time, we should pay attention to instant states of physical points of the system.

Approach 1.

The equations are obtained by direct application of physical laws to a physical point and an instant in time. Since physical laws are, in general, applicable for a material system and a process, the physical point here refers to an infinitesimal special region; the time instant refers to an infinitesimal temporal period. Thus the one-dimensional point x stands actually for the region $(x,\ x+\Delta x)$ $(\Delta x \to 0)$; the time instant t denotes the time period from t to $t+\Delta t$ $(\Delta t \to 0)$. Using this approach we often use physical laws such as Newton's laws of motion and the conservation of mass, momentum and energy.

Approach 2.

Based on fundamental physical laws, this approach first develops relations among different physical variables for an arbitrary material system and process. Note that physical quantities for the system and the process are normally integrals. Using this approach, the physical laws first lead to equations in a form in which some integrals are equal to zero. Both the continuity of the integrands and the localization theorem are then used to conclude that the integrands must be zero. This leads to the desired differential equation governing the local and instant relations among variables.

The one-dimensional localization theorem states that

$$f(x) \equiv 0, \quad x \in [a,b],$$

if $f(x) \in C[a,b]$ and for any $(x, x+\Delta x) \in [a,b]$, $\displaystyle\int_x^{x+\Delta x} f(\xi)\,\mathrm{d}\xi = 0.$

This theorem can be shown by reduction to absurdity and can be extended to cases of multiple dimensions. Using this approach, we also use the Gauss formula

$$\oiint_S \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_\Omega \nabla\cdot\mathbf{a}\,\mathrm{d}v\,,$$

where $\nabla = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}$,

$$\mathbf{a} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}, \quad (x,y,z)\in\Omega\,,$$

Ω is the integration domain, S stands for the external surface of Ω and $\mathbf{n}$ is the outward unit normal of S.

Approach 3.

This approach develops the desired equations from some known equations through differential and integral operations. A typical example is that of developing the equation of electromagnetic waves from the Maxwell system of differential equations through differential operations.

1.2.3 Wave Equations

We develop wave equations and introduce some relevant concepts by considering the equation satisfied by the small transverse displacements of a vibration string.

Consider a homogenous and perfectly flexible thin string of length L fixed at the end points in Fig. 1.1 where $u(x,t)$ is the displacement of the string at point x and time t. For a small transverse vibration such that $u_x^2 \approx 0$, $\mathrm{d}s = \Delta x$, $\alpha \approx 0$ and $\beta \approx 0$ (Fig. 1.1), we determine what equation governs the motion of the string. By Approach 1, consider a small differential segment $\mathrm{d}s$ of the string in Fig. 1.1. Since $\alpha \approx 0$ and $\beta \approx 0$,

$$\sin\alpha \approx \tan\alpha = u_x(x,t)\,, \quad \sin\beta \approx \tan\beta = u_x(x+\Delta x,t)\,.$$

For a homogenous and perfectly flexible string, both its density ρ and the tension T are constants. Let $F(x,t)$ be the external force along the vertical direction per unit length of the string that is acting on the string, including the gravitational force. By Newton's second law of motion, for the vertical motion of the string we obtain

$$\begin{aligned} -T\sin\alpha + T\sin\beta + F(\xi_1,t)\Delta x &= T[u_x(x+\Delta x,t) - u_x(x,t)] + F(\xi_1,t)\Delta x \\ &= \rho\Delta x\frac{\partial^2 u(\xi_2,t)}{\partial t^2}\,, \quad \xi_1,\xi_2 \in (x,x+\Delta x)\,. \end{aligned}$$

Here $\rho\Delta x$ is the mass of string segment, $F(\xi_1,t)$ is the mean force density and $\dfrac{\partial^2 u(\xi_2,t)}{\partial t^2}$ is the mean acceleration of the segment. Dividing by $\rho\Delta x$ leads to

$$\frac{T}{\rho}\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x}+\frac{F(\xi_1,t)}{\rho}=\frac{\partial^2 u(\xi_2,t)}{\partial t^2}$$

By letting $\Delta x \to 0$, we arrive at

$$u_{tt}=a^2u_{xx}+f(x,t)\,, \tag{1.15}$$

where $a^2=T/\rho$ and $f(x,t)=F(x,t)/\rho$.

Equation (1.15) is the partial differential equation that is satisfied by the vertical motion $u(x,t)$ of the vibrating string under an external force (forced vibration). It is second-order, linear and nonhomogeneous. It is also called the *nonhomogeneous one-dimensional wave equation*. When $F(x,t)=0$, Eq. (1.15) reduces to

$$u_{tt}=a^2u_{xx}\,. \tag{1.16}$$

This is the partial differential equation satisfied by the vertical motion $u(x,t)$ of the vibrating string without external force (free vibration). It is second-order, linear and homogeneous. It is also called the *one-dimensional wave equation*.

Remark 1. Let L, M and T be the unit of length, mass and time, respectively. Thus, $[u_x]=1$, $[u_{xx}]=L^{-1}$, $[u_{tt}]=LT^{-2}$. For the wave equations, we obtain

$$[f]=[u_{tt}]=\mathrm{LT}^{-2},\quad [a^2]=[u_{tt}]/[u_{xx}]=\mathrm{L}^2\mathrm{T}^{-2}\mathrm{or}[a]=\mathrm{LT}^{-1}\,.$$

Hence a and f have the dimensions of velocity and acceleration, respectively. This can be further confirmed by their definitions $a^2=T/\rho$ and $f=F/\rho$. Therefore, a represents the *wave speed*. The nonhomogeneous term in Eq. (1.15) is not external force.

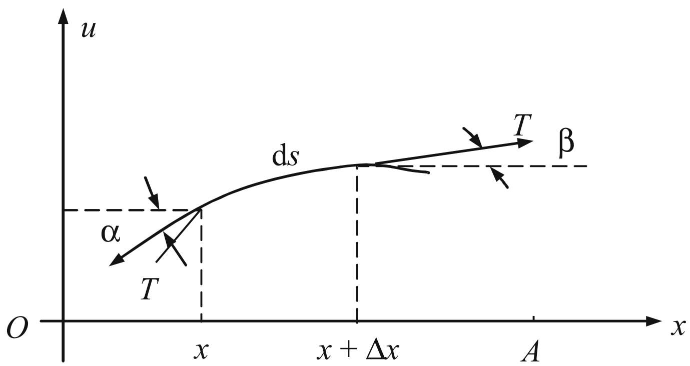

Fig. 1.1 Small transverse displacements of a vibration string

If an equation that represents some physical laws is correct, the unit of each of the terms in the equation must be the same. The same unit of each term can not, however, ensure the correctness of the equation. Therefore the same unit of each term is a necessary, but not a sufficient condition for the correctness of the equation.

Remark 2. Many other physical problems also lead to wave equations. In general, the wave equation describes the propagation of vibration. The dependent variable u is not necessarily the displacement and can be other physical quantities. For example, the electrical current and voltage in a high-frequency network also satisfy the wave equation.

Remark 3. Let M be a point in a plane or a three-dimensional space, and $u(M,t)$ be the dependent variable. Two- or three-dimensional homogeneous and nonhomogeneous equations are given by

$$u_{tt} = a^2 \Delta u, \quad u_{tt} = a^2 \Delta u + f(M,t),$$

where the Laplace operator Δ is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \text{(for the two-dimensional case)},$$
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad \text{(for the three-dimensional case)}.$$

1.2.4 Heat-Conduction Equations

Heat Flux

The basic law defining the relationship between the heat flow and the temperature gradient, based on experimental observations, is generally named after the French mathematical physicist Joseph Fourier who used it in his analytic theory of heat. For a homogeneous, isotropic solid (i.e. material in which thermal conductivity is independent of direction), the *Fourier law* is given in the form

$$\mathbf{q}(M,t) = -k\nabla u(M,t), \tag{1.17}$$

where the temperature gradient ∇u is a vector normal to the isothermal surface, the *heat flux vector* $\mathbf{q}(M,t)$ represents heat flow per unit time, per unit area of the isothermal surface in the direction of the decreasing temperature, and k is called the *thermal conductivity* of the material which is a positive, scalar quantity. In a Cartesian coordinate system, for example, Eq. (1.17) is written as

$$\mathbf{q}(x,y,z,t) = -k\left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}\right),$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit direction vectors along the x, y and z directions, respectively.

Heat-Conduction Equations

We now derive the differential equation of heat conduction for a stationary, homogeneous, isotropic solid with heat generation within the body. Heat generation may be due to unclear, electrical, chemical, gamma-ray, or other sources that may be a function of time and/or position. The heat generation rate in the medium, generally specified as heat generation per unit time, per unit volume, is denoted by the symbol $g(M,t)$.

Consider an arbitrary volume element V of boundary surface S in a heat conduction domain Ω and a heat conduction process within an arbitrary time period $[0,T]$. The first law of thermodynamics states that

$$\begin{bmatrix}\text{heat entering}\\ \text{through } S\end{bmatrix} + \begin{bmatrix}\text{energy generation}\\ \text{in } V\end{bmatrix} = \begin{bmatrix}\text{change in storage}\\ \text{energy in } V\end{bmatrix}.$$

Various terms in this equation are evaluated as

$$\begin{aligned}\text{First term in the left side} &= \int_0^T \iint\limits_S \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S\,\mathrm{d}t = -\int_0^T \iiint\limits_V \nabla\cdot\mathbf{q}\,\mathrm{d}V\,\mathrm{d}t \\ &= k\int_0^T \iiint\limits_V \Delta u\,\mathrm{d}V\,\mathrm{d}t\,, \end{aligned} \tag{1.18}$$

$$\text{Second term in the left side} = \int_0^T \iiint\limits_V g(M,t)\,\mathrm{d}V\,\mathrm{d}t\,, \tag{1.19}$$

$$\text{Term in the right side} = \int_0^T \iiint\limits_V \rho c\frac{\partial u(M,t)}{\partial t}\,\mathrm{d}V\,\mathrm{d}t\,. \tag{1.20}$$

Here $\mathbf{n}$ is the outward-drawn normal unit vector to the surface element $\mathrm{d}S$, and ρ and c are the density and the specific heat of material respectively. Note also that we have used the divergence theorem to convert the surface integral to volume integral.

Substituting those into the energy-balance equation yields

$$\int_0^T \iiint\limits_V \left[k\Delta u + g(M,t) - \rho c\frac{\partial u}{\partial t}\right]\mathrm{d}V\,\mathrm{d}t = 0\,.$$

By the continuity of the integrand and the localization theorem, we obtain

$$\rho c\frac{\partial u}{\partial t} = k\Delta u + g(M,t)\,, \quad M\in\Omega\,, \quad t>0$$

or

$$u_t = a^2\Delta u + f\,, \tag{1.21}$$

where $a^2 = k/(\rho c)$, $f = g/(\rho c)$.

Equation (1.21) is called the *three-dimensional heat-conduction equation with heat generation*. It is second-order, linear and nonhomogeneous.

For a medium with uniform thermal conductivity and no heat generation, Eq. (1.21) becomes

$$u_t = a^2 \Delta u\,. \tag{1.22}$$

It is called the *three-dimensional heat-conduction equation*. It is second-order, linear and homogeneous.

Here, the thermal diffusivity a^2 is a state property of the medium and has a dimension of L^2/T. The physical significance of thermal diffusivity is associated with the speed of propagation of heat into the solid during changes of temperature over time. The propagation rate of heat in the medium is proportional to the thermal diffusivity.

Remark 1. For heat conduction in planes, we have the *two-dimensional heat-conduction equations* $u_t = a^2\Delta u$ and $u_t = a^2\Delta u + f(x,y,t)$. For heat conduction in rods, we have the *one-dimensional heat-conduction equations* $u_t = a^2 u_{xx}$ and $u_t = a^2 u_{xx} + f(x,t)$. These are special cases of three-dimensional heat-conduction equations.

Remark 2. Many other physical problems also lead to heat-conduction equations. In general, the heat-conduction equation describes diffusion. The dependent variable u is not necessarily the temperature, and can represent other physical quantities.

1.2.5 Potential Equations

We have used Approach 1 to derive wave equations and Approach 2 to develop heat-conduction equations. Here we apply Approach 3 to derive potential equations by considering the temperature in steady-state heat conduction and the electric potential in an electrostatic field.

Temperature in Steady-State Heat Conduction

When the temperature is not dependent on the time any more, heat conduction becomes steady. The heat-conduction equations (1.21) and (1.22) reduce to

$$\Delta u(x,y,z) = -\frac{1}{a^2} f(x,y,z), \quad \Delta u(x,y,z) = 0\,.$$

These are both called *potential equations*. The former is second-order, linear and nonhomogeneous and is also called the *three-dimensional Poisson equation*. The latter is second-order, linear and homogeneous and is also called the *three-dimensional*

Laplace equation or the *harmonic equation*. For the two-dimensional case, $u = u(x,y)$, we have the corresponding two-dimensional potential equations.

Continuous solutions of Laplace equations are called *harmonic functions*. Harmonic functions of two variables have a close relation with analytical functions in function of complex variables. Both real and imaginary parts of an analytical function are harmonic. For any given harmonic function, we may construct an analytical function. These have very important applications in the theory of functions of complex variables and fluid mechanics.

Electric Potential in an Electrostatic Field

Let **E** be the electric field intensity, and ρ the charge density. By the Gauss theorem, we have

$$\nabla\cdot\mathbf{E} = 4\pi\rho, \quad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Since the electrostatic fields are irrotational, there exists an electric potential v such that $\mathbf{E} = -\nabla v$. Substituting it into $\nabla\cdot\mathbf{E} = 4\pi\rho$ yields $\Delta v = -4\pi\rho$. When $\rho = 0$, we have $\Delta v = 0$.

Therefore the electric potential in an electrostatic field is also called a *potential function*.

1.3 Theory of Heat Conduction And Three Types of Heat-Conduction Equations

1.3.1 Constitutive Relations of Heat Flux

By the second law of thermodynamics, there exists a physical quantity Q that is, at a given time instant, associated with each surface in a non-isothermal body. This quantity can be interpreted as the heat through the surface and has two fundamental properties: behaving additively on compatible material surfaces and satisfying the first law of thermodynamics (the conservation of energy). These two properties, when rendered precisely, imply the existence of the flux vector field **q** whose scalar product with the unit normal vector to the surface yields the surface density of the heat Q (Šilhavy 1985). **q** is therefore named the *heat-flux density vector*, or the *heat flux* for short.

The relation between the heat flux **q** and the temperature gradient ∇T is called the *constitutive relation of heat flux*, or the *constitutive relation* for short. It is the most fundamental and important relation in heat conduction, and is normally given by fundamental laws.

The Fourier Law

In deriving the classical heat-conduction equation (1.21), we have used the Fourier law of heat conduction. It was the first constitutive relation of heat flux and was proposed by the French mathematical physicist Joseph Fourier in 1807 based on experimentation and investigation (Wang 1994). For heat conduction in a homogeneous and isotropic medium, the Fourier law of heat conduction reads

$$\mathbf{q}(\mathbf{r},t) = -k\nabla T(\mathbf{r},t)\,, \tag{1.23}$$

where $\mathbf{r}$ stands for the material point, t the time, T the temperature and ∇ the gradient operator. k is the thermal conductivity of the material, which is a thermodynamic state property. By the state theorem of thermodynamics, k should be a function of two independent and intensive dynamic properties (normally pressure and temperature; Cengel and Boles 2006). The second law of thermodynamics requires that k is positive-definite (Wang 1994, 1995, 2001). In engineering applications, we often take k as a material constant because variations in pressure and temperature are normally sufficiently small. The value of k is material-dependent. If the material is not homogeneous or isotropic, k becomes a second order tensor (Wang 1994, 1995, 1996, 2001). Along with the first law of thermodynamics, this equation gives the classical *parabolic* heat-conduction equation (See Section 1.3.3 for the derivation)

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \Delta T + \frac{1}{k}F\,. \tag{1.24}$$

Here α is the thermal diffusivity of the material, F is the rate of internal energy generation per unit volume, and Δ is the Laplacian.

The Fourier law of heat conduction is an early empirical law. It assumes that $\mathbf{q}$ and ∇T appear at the same time instant t and consequently implies that thermal signals propagate with an infinite speed. If the material is subjected to a thermal disturbance, the effects of the disturbance will be felt instantaneously at distances infinitely far from its source. Although this result is physically unrealistic, it has been confirmed by many experiments that the Fourier law of heat conduction holds for many media in the usual range of heat flux $\mathbf{q}$ and temperature gradient (Wang 1994).

The CV Constitutive Relation

With the development of science and technology such as the application of ultrafast pulse-laser heating on metal films, heat conduction appears in the range of high heat flux and high unsteadiness. The drawback of infinite heat propagation speed in the Fourier law becomes unacceptable. This has inspired the work of searching for new constitutive relations. Among many proposed relations (Wang 1994), the constitutive relation proposed by Cattaneo (1958) and Vernotte (1958, 1961),

$$\mathbf{q}(\mathbf{r},t)+\tau_0\frac{\partial\mathbf{q}(\mathbf{r},t)}{\partial t}=-k\nabla T(\mathbf{r},t) \tag{1.25}$$

is the most widely accepted. This relation is named the CV constitutive relation after the names of the proposers. Here $\tau_0>0$ is a material property and is called the *relaxation time*. The corresponding heat-conduction equation is thus (See Section 1.3.3 for the derivation)

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}+\frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2}=\Delta T+\frac{1}{k}\left(F+\tau_0\frac{\partial F}{\partial t}\right). \tag{1.26}$$

Unlike its classical counterpart Eq. (1.24), this equation is of *hyperbolic* type, characterizes the combined diffusion and wave-like behavior of heat conduction, and predicts a *finite speed*

$$V_{CV}=\sqrt{\frac{k}{\rho c\tau_0}} \tag{1.27}$$

for heat propagation (Wang and Zhou 2000, 2001, Wang et al. 2007a).

Note that the CV constitutive relation is actually a first-order approximation of a more general constitutive relation (single-phase-lagging model; Tzou 1992a),

$$\mathbf{q}(\mathbf{r},t+\tau_0)=-k\nabla T(\mathbf{r},t). \tag{1.28}$$

according to which the temperature gradient established at a point $\mathbf{r}$ at time t gives rise to a heat flux vector at $\mathbf{r}$ at a *later* time $t+\tau_0$. There is a finite built-up time τ_0 for the onset of heat flux at $\mathbf{r}$ after a temperature gradient is imposed there. Thus the τ_0 represents the time lag needed to establish the heat flux (the result) when a temperature gradient (the cause) is suddenly imposed. The higher $\partial\mathbf{q}/\partial t$ corresponds to a larger derivation of the CV constitutive relation from the classical Fourier law.

The value of τ_0 is material-dependent (Chandrasekharaiah 1986, 1998, Tzou 1997). For most solid materials, τ_0 varies from 10^{-10} s to 10^{-14} s. For gases, τ_0 is normally in the range of $10^{-8}\sim10^{-10}$s. The value of τ_0 for some biological materials and materials with non-homogeneous inner structures can be up to 10^2 s (Beckert 2000, Kaminski 1990, Mitra et al. 1995, Peters 1999, Roetzel et al. 2003, Vedavarz et al. 1992). Therefore, the thermal relaxation effects can be of relevance even in common engineering applications where the time scales of interest are of the order of a fraction of a minute.

Three factors contribute to the significance of the second term in the hyperbolic heat-conduction equation (1.26): the value of τ_0, the rate of change of temperature, and the time scale involved. The wave nature of thermal signals will be over the diffusive behavior through this term when (Tzou 1992a)

$$\frac{\partial T}{\partial t}>>\frac{T_r}{2\tau_0}\exp(t/\tau_0) \tag{1.29}$$

where T_r is a reference temperature. Therefore, the wave-like features will become significant when: (1) τ_0 is large, (2) $\partial T/\partial t$ is high, or (3) t is small. Some typical situations where hyperbolic heat conduction differs from classical parabolic heat conduction include those concerned with a localized moving heat source with high intensity, a rapidly propagating crack tip, shock wave propagation, thermal resonance, interfacial effects between dissimilar materials, laser material processing, and laser surgery (Chandrasekharaiah 1986, 1998, Joseph and Preziosi 1989, 1990, Tzou 1992a, 1995a, 1997, Wang 1994, 2000a).

When $\tau_0 \to \infty$ but $k_{eff} = k/\tau_0$ is finite, the CV constitutive relation (1.25) and the hyperbolic heat-conduction equation (1.26) become (Joseph and Preziosi 1989)

$$\frac{\partial \mathbf{q}(\mathbf{r},t)}{\partial t} = -k_{\text{eff}} \nabla T(\mathbf{r},t) , \tag{1.30}$$

and

$$\frac{1}{\alpha_{\text{eff}}} \frac{\partial^2 T}{\partial t^2} = \Delta T + \frac{1}{k_{\text{eff}}} \frac{\partial F}{\partial t} , \tag{1.31}$$

where $\alpha_{\text{eff}} = k_{eff}/\rho c$, ρ and c are the density and the specific heat of the material, respectively. Therefore, when τ_0 is very large, a temperature gradient established at a point of the material results in an *instantaneous heat flux rate* at that point, and vice-versa. Eq. (1.31) is a classical wave equation that predicts thermal wave propagation with speed V_{CV}, like Eq. (1.26). A major difference exists, however, between Eqs. (1.26) and (1.31): the former allows damping of thermal waves, the latter does not (Wang and Zhou 2000, 2001).

The Dual-Phase-Lagging Constitutive Relation

It has been confirmed by many experiments that the CV constitutive relation generates a more accurate prediction than the classical Fourier law. However, some of its predictions do not agree with experimental results either (Tzou 1995a, 1997, Wang 1994). A thorough study shows that the CV constitutive relation has only taken account of the fast-transient effects, but not the micro-structural interactions. These two effects can be reasonably represented by the dual-phase-lag between $\mathbf{q}$ and ∇T, a further modification of Eq. (1.28) (Tzou 1995a, 1997),

$$\mathbf{q}(\mathbf{r},t+\tau_0) = -k\nabla T(\mathbf{r},t+\tau_T) . \tag{1.32}$$

According to this relation, the temperature gradient at a point $\mathbf{r}$ of the material at time $t+\tau_T$ corresponds to the heat flux density vector at $\mathbf{r}$ at time $t+\tau_0$. The delay time τ_T is interpreted as being caused by the micro-structural interactions (small-scale heat transport mechanisms occurring in the micro-scale, or small-scale effects of heat transport in space) such as phonon-electron interaction or phonon scattering, and is called the *phase-lag of the temperature gradient* (Tzou 1995a, 1997). The other delay time τ_0 is interpreted as the relaxation time due to the fast-transient effects of thermal inertia (or small-scale effects of heat transport in time) and is called the *phase-lag of the heat flux*. Both of the phase-lags are treated as intrinsic thermal or structural properties of the material. The corresponding heat-conduction equation reads (Xu and Wang 2005)

$$\frac{1}{\alpha}\frac{\partial T(\mathbf{r},\,t')}{\partial t}=\Delta T(\mathbf{r},\,t'-\tau)+\frac{1}{k}F(\mathbf{r},\,t'),\;t'=t+\tau_0,\;\tau=\tau_0-\tau_T\,,$$
$$\text{for}\quad \tau_0-\tau_T>0\quad\text{and}\quad t'>\tau_0\,, \tag{1.33}$$

or

$$\frac{1}{\alpha}\frac{\partial T(\mathbf{r},t'-\tau)}{\partial t}=\Delta T(\mathbf{r},t')+\frac{1}{k}F(\mathbf{r},t'-\tau),\;t'=t+\tau_T,\;\tau=\tau_T-\tau_0\,,$$
$$\text{for}\quad \tau_0-\tau_T<0\quad\text{and}\quad t'>\tau_T\,. \tag{1.34}$$

Unlike the relation (1.28) according to which the heat flux is the result of a temperature gradient in a transient process, the relation (1.32) allows either the temperature gradient or the heat flux to become the effect and the remaining one the cause. For materials with $\tau_0>\tau_T$, the heat flux density vector is the result of a temperature gradient. It is the other way around for materials with $\tau_T>\tau_0$. The relation (1.28) corresponds to the particular case where $\tau_0>0$ and $\tau_T=0$. If $\tau_0=\tau_T$ (not necessarily equal to zero), the response between the temperature gradient and the heat flux is instantaneous; in this case, the relation (1.32) is identical with the classical Fourier law (1.23). It may also be noted that while the classical Fourier law (1.23) is macroscopic in both space and time and the relation (1.28) is macroscopic in space but microscopic in time, the relation (1.32) is microscopic in both space and time. Also note that Eqs. (1.33) and (1.34) are of the delay and advance types, respectively. While the former has a wave-like solution and possibly resonance, the latter does not (Xu and Wang 2005). Both single-phase-lagging and dual-phase-lagging heat conduction have been shown to be admissible by the second law of extended irreversible thermodynamics (Tzou 1997) and by the Boltzmann transport equation (Xu and Wang 2005).

Expanding both sides of Eq. (1.32)by using the Taylor series and retaining only the first-order terms of τ_0 and τ_T, we obtain the following constitutive relation that is valid at point $\mathbf{r}$ and time t,

$$q(\mathbf{r},t)+\tau_0\frac{\partial q(\mathbf{r},t)}{\partial t}=-k\left\{\nabla T(\mathbf{r},t)+\tau_T\frac{\partial}{\partial t}[\nabla T(\mathbf{r},t)]\right\}, \tag{1.35}$$

which is known as the Jeffreys-type constitutive equation of heat flux (Joseph and Preziosi1989). In literature this relation is also called the *dual-phase-lagging constitutive relation*. When $\tau_0=\tau_T$, this relation reduces to the classicalFourier law (1.23), and it reduces to the CV constitutive relation (1.25) when $\tau_T=0$.

Eliminating $\mathbf{q}$ from Eq. (1.35) and the classical energy equation leads to the dual-phase-lagging heat conduction equation that reads, if all thermophysical material properties are assumed to be constant (see Section 1.3.3 for the derivation),

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}+\frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2}=\Delta T+\tau_T\frac{\partial}{\partial t}(\Delta T)+\frac{1}{k}\left(F+\tau_0\frac{\partial F}{\partial t}\right) \tag{1.36}$$

This equation is *parabolic* when $\tau_0 < \tau_T$ (Wang and Zhou2000, 2001). Although a wave term $(\tau_0/\alpha)\partial^2 T/\partial^2 t$ exists in the equation, the mixed derivative $\tau_T\partial(\Delta T)/\partial t$ completely destroys the wave structure. The equation, in this case, therefore predicts a nonwave-like heat conduction that differs from the usual diffusion predicted by the classical parabolic heat conduction (1.24). When $\tau_0 > \tau_T$, however, Eq. (1.36) can be approximated by Eq. (1.26) and then predominantly predicts wave-like thermal signals.

The dual-phase-lagging heat-conduction equation (1.36) forms a generalized, unified equation that reduces to the classical parabolic heat-conduction equation when $\tau_T = \tau_0$, the hyperbolic heat-conduction equation when $\tau_T = 0$ and $\tau_0 > 0$, the energy equation in the phonon scattering model (Joseph and Preziosi 1989, Guyer and Krumhansi 1966) when $\alpha = \frac{\tau_R c^2}{3}$, $\tau_T = \frac{9}{5}\tau_N$ and $\tau_0 = \tau_R$, and the energy equation in the phonon-electron interaction model (Kaganov et al. 1957, Anisimòv et al. 1974, Qiu and Tien 1993) when $\alpha = \frac{k}{c_e + c_l}$, $\tau_T = \frac{c_l}{G}$ and $\tau_0 = \frac{1}{G}\left[\frac{1}{c_e}+\frac{1}{c_l}\right]^{-1}$. In the phonon scattering model, c is the average speed of phonons (sound speed), τ_R is the relaxation time for the umklapp process in which momentum is lost from the phonon system, and τ_N isthe relaxation time for normal processes in which momentum is conserved in the phonon system. In the phonon-electron interaction model, k is the thermal conductivity of the electron gas, G is the phonon-electron coupling factor, and c_e and c_l are theheat capacity of the electron gas and the metal lattice, respectively. This, together with its success in describing and predicting phenomena such as ultra-fast pulse-laser heating, propagation of temperature pulses in superfluid liquid helium, nonhomogeneous lagging response in porous media, thermal lagging in amorphous materials, and effects of material defects and thermomechanical coupling, heat conduction in nanofluids, bi-composite media and two-phase systems (Tzou 1997, Tzou and Zhang 1995, Vadasz 2005a, 2005b, 2005c, 2006a, 2006b, Wang et al. 2007a, Wang and Wei 2007a, 2007b), has given rise to the research effort on various aspects of dual-phase-lagging heat conduction (Tzou 1997, Wang and Zhou 2000, 2001).

The dual-phase-lagging heat-conduction model that is based on Eqs. (1.36) has been shown to be well-posed in a finite region of n-dimensions ($n \geq 1$) under any linear boundary conditions including Dirichlet, Neumann and Robin types (Wang and Xu 2002, Wang et al. 2001). Solutions of one-dimensional (1D) heat conduction has been obtained for some specific initial and boundary conditions by Antaki (1998), Dai and Nassar (1999), Lin et al. (1997), Tang and Araki (1999), Tzou (1995a, 1995b, 1997), Tzou and Zhang (1995), Tzou and Chiu (2001). Wang and Zhou (2000, 2001) obtained analytical solutions for regular 1D, 2D and 3D heat-conduction domains under essentially arbitrary initial and boundary conditions. The solution structure theorems were also developed for both mixed and Cauchy problems of dual-phase-lagging heat-conduction equations (Wang and Zhou 2000, Wang et al. 2001) by extending those theorems for hyperbolic heat conduction (Wang 2000a). These theorems build relationships between the contributions (to the temperature field) by the initial temperature distribution, the source term and the initial time-rate of the temperature change, uncovering the structure of the temperature field and considerably simplifying the development of solutions. Xu and Wang (2002) addressed thermal features of dual-phase-lagging heat conduction (particularly conditions and features of thermal oscillation and resonance and their contrast with those of classical and hyperbolic heat conduction).

An experimental procedure for determining the value of τ_0 has been proposed by Mengi and Turhan (1978). The general problem of measuring short-time thermal transport effects has been discussed by Chester (1966). Wang and Zhou (2000, 2001) developed three methods of measuring τ_0. Tzou (1997) and Vadasz (2005a, 2005b, 2006a, 2006b) developed an *approximate* equivalence between Fourier heat conduction in porous media and dual-phase-lagging heat conduction, and applied the latter to examine features of the former. Based on that equivalence, Vadasz (2005a, 2005b, 2005c, 2006a, 2006b) showed that τ_T is always larger than τ_0 in porous-media heat conduction so that thermal waves cannot occur according to the necessary condition for thermal waves in dual-phase-lagging heat conduction (Xu and Wang 2002). However, such waves are observed in casting sand experiments by two independent groups (Tzou 1997). In an attempt to resolve this difference and to build the intrinsic relationship between the two heat-conduction processes, Wang and Wei (2007a, 2007b) developed an *exact* equivalence between dual-phase-lagging heat conduction and Fourier heat conduction in two-phase systems subject to a lack of local thermal equilibrium. Based on this new equivalence, Wang and Wei (2007a, 2007b) also show the possibility of and uncover the mechanism responsible for the thermal oscillation in two-phase-system heat conduction.

Tzou (1995b, 1997) also generalized Eq. (1.35), for $\tau_0 >> \tau_T$, by retaining terms up to the second order in τ_0 but only the term of the first order in τ_T in the Taylor expansions of Eq. (1.32) to obtain a τ_0-second-order dual-phase-lagging model

$$\mathbf{q} + \tau_0 \frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2}\tau_0^2 \frac{\partial^2 \mathbf{q}}{\partial t^2} = -k\left[\nabla T + \tau_T \frac{\partial}{\partial t}(\nabla T)\right] . \tag{1.37}$$

For this case the dual-phase-lagging heat conduction Eq. (1.36) is generalized into

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2} + \frac{\tau_0^2}{2\alpha}\frac{\alpha^3 T}{2t^3} = \Delta T + \tau_T \frac{\partial}{\partial t}(\Delta T) + \frac{1}{k}\left(F + \tau_0 \frac{\partial F}{\partial t} + \frac{\tau_0^2}{2}\frac{\partial^2 F}{\partial t^2}\right) , \tag{1.38}$$

whis is of hyperbolic type and thus predicts thermal wave propagation with a finite speed (Tzou 1995b, 1997)

$$V_T = \frac{1}{\tau_0}\sqrt{\frac{2k\tau_T}{\rho c}} . \tag{1.39}$$

The thermal wave from Eq. (1.26) is obviously different from that in Eq. (1.38).

While the former is caused only by the fast transient effects of thermal inertia, the latter comes from these effects as well as the delayed response due to the microstructural interaction. Tzou (1997) refers to the former wave as the CV-wave and thelatter wave as the T-wave. By Eqs. (1.27) and (1.39), we have

$$V_T = \sqrt{\frac{2\tau_T}{\tau_0}} V_{CV} \,. \tag{1.40}$$

Therefore, the T-wave is always slower than the CV-wave because Eqs. (1.37) and (1.38) are valid only for $\tau_0 >> \tau_T$. This has been shown by the heat propagation in superfluid helium at extremely low temperatures (Tzou 1997). It is interesting to note that Eq. (1.37) is the simplest constitutive relation that accounts for the dual-phase-lagging effects and yields a heat-conduction equation of hyperbolic type. If the second order term in τ_T is also retained, the resulting heat-conduction equation will no longer be hyperbolic (Tzou 1997). It is also of interest to note that Eq. (1.38) closely resembles the energy equation describing the ballistic behavior of heat transport in an electron gas (Qiu and Tien 1993, Tzou 1997).

In this section, we have presented a brief review of some valid models for heat conduction from macro- to micro-scales: the *Fourier model* based on Eqs. (1.23) and (1.24),the *CV model* based on Eqs. (1.25) and (1.26), the *wave model* based on

Eqs. (1.30) and (1.31), the *single-phase-lagging model* based onEq. (1.28), and the three *dual-phase-lagging models* the [dual-phase-lagging model based on Eqs. (1.32), (1.33) and (1.34); the first-order dual-phase-lagging model based on Eqs. (1.35) and (1.36); the second-order dual-phase-lagging model based on Eqs. (1.37) and (1.38)]. In literature, the *dual-phase-lagging model* usually refers to the first-order dual-phase-lagging model [Eqs. (1.35) and (1.36)], which is a generalized and unified model for heat conduction from macro- to micro-scales with the Fourier, wave and CV models as its special cases. A relatively comprehensive list of literature on these models can be found in the References.

1.3.2 The Boltzmann Transport Equation and Dual-Phase-Lagging Heat Conduction

The Boltzmann Transport Equation

Consider a classical system of N particles. Each particle has s degrees of freedom so that the number of coordinates needed to specify positions of all N particles is $l = Ns$. The classical mechanical state of the system can be completely described by l spatial coordinates q_i and l corresponding velocity coordinates υ_i. Introduce a conceptual Euclidean hyperspace of $2l$ dimensions, with a coordinate axis for each of the $2l$ spatial coordinates and velocities. This conceptual space is usually termed as the phase space for the system. The state of the classical N-particle or N-body system at any time t is completely specified by the location of one point in the phase space, referred to as a phase point. The evolution of the system state with time is completely described by the motion or trajectory of the phase point through phase space. The trajectory of the point is expressed by equations of motion of the N bodies. Integration of such a large system of equations is not feasible, and statistical methods are usually used.

Define the distribution function $f_{p\upsilon}$,

$$f_{p\upsilon}(\mathbf{r},\mathbf{v},t)\ \mathrm{d}\mathbf{r}\ \mathrm{d}\mathbf{v} \tag{1.41}$$

$=$ (the number of particles in the system that have phase points in $\mathrm{d}\mathbf{r}\ \mathrm{d}\mathbf{v}$ around $\mathbf{r}$ and $\mathbf{v}$ at time t).

Here $\mathbf{v}$ and $\mathbf{r}$ have components υ_i and $r_i (i = 1, 2, \ldots, l)$, respectively, $\mathrm{d}\mathbf{v} = \mathrm{d}\upsilon_1$, $\mathrm{d}\upsilon_2, \ldots, \mathrm{d}\upsilon_l$, and $\mathrm{d}\mathbf{r} = \mathrm{d}r_1, \mathrm{d}r_2, \ldots, \mathrm{d}r_l$. By this definition, we have

$$\iiint \cdots \int_{\text{all } \mathbf{r},\, \mathbf{v}} f_{p\upsilon}(\mathbf{r},\mathbf{v},t)\ \mathrm{d}\mathbf{r}\ \mathrm{d}\mathbf{v} = N. \tag{1.42}$$

The ensemble average of any function $\psi(\mathbf{r},\mathbf{v})$ of the position $\mathbf{r}$ andvelocity $\mathbf{v}$ of the system is defined by

$$\langle\psi\rangle = \frac{1}{N}\iiint \cdots \int_{\text{all}\mathbf{r},\, \mathbf{v}} \psi(\mathbf{r},\mathbf{v},t)\ f_{p\upsilon}\ \mathrm{d}\mathbf{r}\ \mathrm{d}\mathbf{v}. \tag{1.43}$$

The assumption that the particles do not interact with each other leads to (Carey 1999)

$$\frac{\mathrm{d}f_{p\upsilon}}{\mathrm{d}t} = 0, \tag{1.44}$$

whis is called the *Liouville equation*. It indicates that if we follow the particlesin a volume element along a flow line in phase space without collisions, the distribution is conserved.

$$f_{p\upsilon}(\mathbf{r} + \mathrm{d}\mathbf{r},\ \mathbf{v} + \mathrm{d}\mathbf{v},\ t + \mathrm{d}t) = f_{p\upsilon}(\mathbf{r},\mathbf{v},t). \tag{1.45}$$

If collisions occur, the distribution $f_{p\upsilon}$ will change over a time interval $\mathrm{d}t$ by an amount $(\partial f_{p\upsilon}/\partial t)_{\text{coll}}\,\mathrm{d}t$, and therefore

$$f_{p\upsilon}(\mathbf{r} + \mathrm{d}\mathbf{r},\ \mathbf{v} + \mathrm{d}\mathbf{v},\ t + \mathrm{d}t) - f_{p\upsilon}(\mathbf{r},\mathbf{v},t) = (\partial f_{p\upsilon}/\partial t)_{\text{coll}}\ \mathrm{d}t, \tag{1.46}$$

which is equivalent to

$$\frac{f_{p\upsilon}(\mathbf{r} + \mathrm{d}\mathbf{r},\ \mathbf{v} + \mathrm{d}\mathbf{v},\ t + \mathrm{d}t) - f_{p\upsilon}(\mathbf{r},\mathbf{v},t)}{\mathrm{d}t} = \left(\frac{\partial f_{p\upsilon}}{\partial t}\right)_{\text{coll}}. \tag{1.47}$$

The Taylor expansion of $f_{p\upsilon}(\mathbf{r} + \mathrm{d}\mathbf{r}, \mathbf{v} + \mathrm{d}\mathbf{v}, t + \mathrm{d}t)$ at the point $(\mathbf{r},\mathbf{v},t)$ yields

$$\frac{\sum_{j=1}^{l} \dfrac{\partial f_{p\upsilon}}{\partial r_j}\,\mathrm{d}r_j + \sum_{j=1}^{l} \dfrac{\partial f_{p\upsilon}}{\partial \upsilon_j}\,\mathrm{d}\upsilon_j + \dfrac{\partial f_{p\upsilon}}{\partial t}\,\mathrm{d}t + \text{higher order terms}}{\mathrm{d}t} = \left(\frac{\partial f_{p\upsilon}}{\partial t}\right)_{\text{coll}}, \tag{1.48}$$

or

$$\frac{\partial f_{p\upsilon}}{\partial t} + \sum_{j=1}^{l} v_j \frac{\partial f_{p\upsilon}}{\partial r_j} + \sum_{j=1}^{l} \frac{\partial \upsilon_j}{\partial t} \frac{\partial f_{p\upsilon}}{\partial \upsilon_j} = \left(\frac{\partial f_{p\upsilon}}{\partial t} \right)_{\text{coll}} . \tag{1.49}$$

This is the *Boltzmann transport equation*, which can also be written as

$$\frac{\partial f}{\partial t} + \sum_{j=1}^{l} v_j \frac{\partial f}{\partial r_j} + \sum_{j=1}^{l} \frac{\partial \upsilon_j}{\partial t} \frac{\partial f}{\partial \upsilon_j} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} , \tag{1.50}$$

where

$$f(\mathbf{r}, \mathbf{v}, t) = \frac{f_{p\upsilon}(\mathbf{r}, \mathbf{v}, t)}{N} .$$

The collision term $\left(\frac{\partial f}{\partial t} \right)_{\text{coll}}$ in the Boltzmann transport equation is usually written as (Carey 1999)

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = -\frac{f - f_0}{\tau_0} , \tag{1.51}$$

where f_0 is the equilibrium distribution for the system, and τ_0 is the relaxation time. Suppose that a non-equilibrium distribution of velocities is set up by external forces which are suddenly removed. Note that $\frac{\partial f_0}{\partial t} = 0$ by using the definition of the equilibrium distribution. The decay of the distribution towards the equilibrium is then obtained from (1.51) as,

$$\frac{\partial (f - f_0)}{\partial t} = -\frac{f - f_0}{\tau_0} .$$

This equation has the solution

$$(f = f_0)|_t = (f - f_0)|_{t=0} \exp(-t/\tau_0) .$$

By combining Eqs. (1.50) and (1.51), we obtain the Boltzmann transport equation with the relaxation time approximation:

$$\frac{\partial f}{\partial t} + \sum_{j=1}^{l} v_j \frac{\partial f}{\partial r_j} + \sum_{j=1}^{l} \frac{\partial \upsilon_j}{\partial t} \frac{\partial f}{\partial \upsilon_j} = -\frac{f - f_0}{\tau_0} . \tag{1.52}$$

Dual-Phase-Lagging Constitutive Relation

Consider a three-dimensional heat transfer problem. The position vector $\mathbf{r}$ has three components x, y and z, and the velocity vector $\mathbf{v}$ can be expressed as $(v_x, v_y, v_z)^T$. To study energy transport via particles, we must solve the Boltzmann transport equation to determine the distribution function $f(\mathbf{r}, \mathbf{v}, t)$. For most cases, however, we can only obtain an approximate distribution function. By using $f(\mathbf{r},\mathbf{v},t)$, the rate of energy flow per unit area (the energy flux) can be expressed as

$$\mathbf{q}(\mathbf{r},t) = \int_{\text{all}\,\mathbf{v}} \mathbf{v}(\mathbf{r},t) f(\mathbf{r},\mathbf{v},t)\varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{v}\,. \tag{1.53}$$

Here $\mathbf{q}(\mathbf{r},t)$ is the energy flux vector, $\mathbf{v}(\mathbf{r},t)$ is the velocity vector, and $\varepsilon(\mathbf{v})$ is the kinetic energy of the particle as a function of particle velocity. Note that $f(\mathbf{r},\mathbf{v},t)$ is the fraction of system particles in the ensemble per unit volume per unit velocity. Therefore $f(\mathbf{r},\mathbf{v},t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}$ is the fraction of system particles in the ensemble that have phase points in $\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}$ around $\mathbf{r}$ and $\mathbf{v}$.

In terms of an integral over momentum, Eq. (1.53) reads (Tien et al. 1998)

$$\mathbf{q}(\mathbf{r},t) = \int_{\text{all}\,\mathbf{p}} \mathbf{v}(\mathbf{r},t) f(\mathbf{r},\mathbf{p},t)\varepsilon(\mathbf{p})\,\mathrm{d}\mathbf{p}\,, \tag{1.54}$$

where the vector $\mathbf{p}$ is the momentum, and the distribution $f(\mathbf{r},\mathbf{p},t)$ is the fraction of system particles per unit volume per unit momentum. $f(\mathbf{r},\mathbf{p},t)\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p}$ is the fraction of system particles in the ensemble that have phase points in $\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{p}$ around $\mathbf{r}$ and $\mathbf{p}$. Because $\mathbf{p} = m\mathbf{v}$, we have

$$f(\mathbf{r},\mathbf{p},t)\,\mathrm{d}\mathbf{r}\ \mathrm{d}\mathbf{p} = mf(\mathbf{r},\mathbf{p},t)\,\mathrm{d}\mathbf{r}\ \mathrm{d}\mathbf{v}\,. \tag{1.55}$$

This implies that the fraction of system particles per unit volume and per unit velocity can also be expressed as $mf(\mathbf{r},\mathbf{p},t)$. Therefore we have

$$f(\mathbf{r},\mathbf{v},t) = mf(\mathbf{r},\mathbf{p},t)\ . \tag{1.56}$$

This equation enables us to rewrite Eq. (1.53) into Eq. (1.54).

By introducing the spherical coordinates for the integral in Eq. (1.54),

$$p_x = p\sin\theta\cos\varphi,\ \ p_y = p\sin\theta\sin\varphi,\ \ p_z = p\cos\theta,$$

where $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$, we have

$$\mathbf{q}(\mathbf{r},t) = \int_0^\infty \int_0^\pi \int_0^{2\pi} \mathbf{v}(\mathbf{r},t) f(\mathbf{r},\mathbf{p},t)\ \varepsilon(\mathbf{p}) p^2 \sin\theta\ \mathrm{d}p\ \mathrm{d}\theta\ \mathrm{d}\varphi\ . \tag{1.57}$$

Applying the relation between p and the kinetic energy ε yields

$$\mathbf{q}(\mathbf{r},t) = \int_0^\infty \int_0^\pi \int_0^{2\pi} \mathbf{v}(\mathbf{r},t) f(\mathbf{r},\mathbf{p},t)\ \varepsilon m\sqrt{2m\varepsilon}\sin\theta\ \mathrm{d}\varepsilon\ \mathrm{d}\theta\ \mathrm{d}\varphi\ . \tag{1.58}$$

For the case of no external forces acting on the heat transfer medium, the particle randomly accesses every direction with the same probability so that the distribution function $f(\mathbf{r},\mathbf{p},t)$ only depends on $\mathbf{r}$, ε and t. The density of states $D(\varepsilon)$ can then be defined as

$$D(\varepsilon) = \int_0^\pi \int_0^{2\pi} m\sqrt{2m\varepsilon}\sin\theta\ \mathrm{d}\theta\ \mathrm{d}\varphi = 4\pi m\sqrt{2m\varepsilon}\,. \tag{1.59}$$

This is the classical definition of the density of states. If the quantum effect is taken into account for the electron, we have

$$D(\varepsilon) = \frac{m\sqrt{2m\varepsilon}}{\hbar^3 \pi^2}\,, \tag{1.60}$$

where $\hbar$ is the Planck constant divided by 2π. By $D(\varepsilon)$,Eq. (1.58) can be written as (Xu and Wang 2005)

$$\mathbf{q}(\mathbf{r},t) = \int_{\varepsilon} \mathbf{v}(\mathbf{r},t)\, f(\mathbf{r},\varepsilon,t)\, \varepsilon D(\varepsilon)\, \mathrm{d}\varepsilon\,. \tag{1.61}$$

By Eqs. (1.47) and (1.51), we have

$$\frac{f(\mathbf{r}+\mathrm{d}\mathbf{r},\ \varepsilon(\mathbf{v}+\mathrm{d}\mathbf{v}),\ t+\mathrm{d}t) - f(\mathbf{r},\varepsilon(\mathbf{v}),t)}{\mathrm{d}t} = \frac{f_0 - f}{\tau}, \tag{1.62}$$

where $\mathrm{d}\mathbf{r}$ and $\mathrm{d}\mathbf{v}$ are the incremental of the position and velocity vectors, respectively. Note that it takes approximately time period τ_0 for $f(\mathbf{r},\varepsilon(\mathbf{v}),t)$ to decay to its equilibrium state f_0. Under the assumption that no external forces act on the heat transfer medium, the acceleration of this decaying process is zero (Xu and Wang 2005). Therefore Eq. (1.62) can be rewritten as

$$\begin{aligned} &\frac{f(\mathbf{r}+\mathrm{d}\mathbf{r},\ \varepsilon(\mathbf{v}),\ t+\mathrm{d}t) - f(\mathbf{r},\varepsilon(\mathbf{v}),\ t+\mathrm{d}t)}{\mathrm{d}t} \\ &+\frac{f(\mathbf{r},\ \varepsilon(\mathbf{v}),\ t+\mathrm{d}t) - f(\mathbf{r},\varepsilon(\mathbf{v}),t)}{\mathrm{d}t} = \frac{f_0 - f}{\tau}\,. \end{aligned} \tag{1.63}$$

By applying a Taylor expansion, we have

$$\begin{aligned} &\frac{f(\mathbf{r}+\mathrm{d}\mathbf{r},\varepsilon(\mathbf{v}),t+\mathrm{d}t) - f(\mathbf{r},\varepsilon(\mathbf{v}),t+\mathrm{d}t)}{\mathrm{d}t} \\ &= \frac{\mathrm{d}\mathbf{r}\cdot\nabla f(\mathbf{r},\varepsilon(\mathbf{v}),t+\mathrm{d}t) + \text{higher order terms}}{\mathrm{d}t}\,. \end{aligned} \tag{1.64}$$

Therefore, there exists a value τ_T such that $\mathbf{v}\cdot\nabla f(\mathbf{r},\varepsilon(\mathbf{v}),\ t+\tau_{\mathrm{T}})$ is the best approximation of the first term on the left side of Eq. (1.63). As $\mathrm{d}t \approx \tau_0$ (Tien et al. 1998, Xu and Wang 2005), Eq. (1.63) becomes

$$\tau\mathbf{v}\cdot\nabla f(\mathbf{r},\varepsilon(\mathbf{v}),\ t+\tau_T) + f(\mathbf{r},\varepsilon(\mathbf{v}),\ t+\tau) = f_0\,. \tag{1.65}$$

For the electron, f_0 is the Fermi-Dirac equilibrium distribution

$$f_0(\varepsilon) = \frac{1}{1+\exp\left(\dfrac{\varepsilon-\mu}{k_B T}\right)}\,, \tag{1.66}$$

where μ is the chemical potential, k_B is the Boltzmann constant and T is the temperature. For the phonon, f_0 is the Bose-Einstein equilibrium distribution

$$f_0(\varepsilon) = \frac{1}{\exp\left(\dfrac{\varepsilon}{k_B T}\right) - 1}\,, \tag{1.67}$$

where $\varepsilon = \hbar\omega$, and ω is the angular frequency of the quantum harmonic oscillator. Obviously, for both cases, f_0 is an even function of the velocity $\mathbf{v}$. Therefore (Xu and Wang 2005)

$$\int_\varepsilon f_0 \varepsilon D(\varepsilon)\mathbf{v}\,\mathrm{d}\varepsilon = \mathbf{0}\,. \tag{1.68}$$

Multiplying Eq. (1.65) by $\varepsilon D(\varepsilon)\mathbf{v}$ and integrating over all possible energies yields

$$\int_\varepsilon \tau\mathbf{v}\cdot\nabla f(\mathbf{r},\varepsilon(\mathbf{v}),\, t+\tau_T)\ \mathbf{v}\varepsilon D(\varepsilon)\ \mathrm{d}\varepsilon + \mathbf{q}(\mathbf{r},\mathbf{v},\, t+\tau) = \mathbf{0}, \tag{1.69}$$

in which we have used Eq. (1.68).

Under the assumption that the relaxation times τ_0 and τ_T are independent of the system energy and the system has achieved a quasi-equilibrium state, we have $\nabla f = (\mathrm{d}f_0/\mathrm{d}T)\nabla T$. Eq. (1.69) becomes

$$\mathbf{q}(\mathbf{r},t+\tau_0) = -\mathbf{k}\cdot\nabla T(\mathbf{r},t+\tau_T), \tag{1.70}$$

where $\mathbf{k}$ is the thermal conductivity tensor

$$\mathbf{k} = \int \tau\mathbf{v}\mathbf{v}\frac{\mathrm{d}f_0}{\mathrm{d}T}\varepsilon D(\varepsilon)\,\mathrm{d}D(\varepsilon)\,.$$

For isotropic materials, $\mathbf{k} = k\mathbf{I}$ with k and $\mathbf{I}$ being a constant and the unit tensor, respectively. Eq. (1.70) reduces to

$$\mathbf{q}(\mathbf{r},t+\tau_0) = -k\nabla T(\mathbf{r},t+\tau_T)\,, \tag{1.71}$$

which is the dual-phase-lagging constitutive relation [Eq. (1.32)].

1.3.3 Three Types of Heat-Conduction Equations

Heat-conduction equations come from the application of the first law of thermodynamics (also called the conservation of energy) to heat conduction. By the approaches for developing equations of mathematical physics outlined in Sect. 1.2.2

and the formula for calculating internal energy in thermodynamics, the first law of thermodynamics yields

$$\rho c \frac{\partial T(\mathbf{r},t)}{\partial t} + \nabla \cdot q - F(\mathbf{r},t) = 0\,, \tag{1.72}$$

where ρ and c are the density and the specific heat of the material, and F is the rate of internal energy generation per unit volume. Equation (1.72) is called the *energy equation*, and it contains two unknowns T and q. By usinga constitutive relation of heat flux density, we may eliminate q from Eq. (1.72) to obtain an equation of temperature T. Since both the Fourier law (1.23) and the CV constitutive relation (1.25) are the special cases of the dual-phase-lagging constitutive relation (1.35), we use Eq. (1.35) to derive the heat-conduction equations.

Assuming constant material properties, the divergence of Eq. (1.35) yields

$$\nabla \cdot q + \tau_0 \frac{\partial}{\partial t}[\nabla \cdot q] = -k\Delta T - k\tau_T \frac{\partial}{\partial t}[\Delta T]\,. \tag{1.73}$$

Substituting the expression of $\nabla \cdot q$ from Eq. (1.72)

$$\nabla \cdot q = F - \rho c \frac{\partial T}{\partial t}$$

into Eq. (1.73) and introducing the thermal diffusivity $\alpha = k/(\rho c)$ leads to

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2} = \Delta T + \tau_T \frac{\partial}{\partial t}(\Delta T) + \frac{1}{k}\left(F + \tau_0 \frac{\partial F}{\partial t}\right)\,. \tag{1.74}$$

This is called the *dual-phase-lagging heat-conduction equation*. When $\tau_T = 0$, it reduces to the *hyperbolic heat-conduction equation*

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2} = \Delta T + \frac{1}{k}\left(F + \tau_0 \frac{\partial F}{\partial t}\right)\,. \tag{1.75}$$

In the absence of two phase lags, i. e. when $\tau_0 = \tau_T = 0$, it reduces to the *classical parabolic heat-conduction equation*

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \Delta T + \frac{1}{k}F\,. \tag{1.76}$$

For steady-state heat conduction, both the first and the second derivatives of T with respect to t are zero. All three kinds of heat-conduction equations reduce to *potential equations*.

1.4 Conditions and Problems for Determining Solutions

Equations of mathematical physics are drawn from physical problems. In applications, their solutions always refer to particular solutions subjected to certain physical conditions. Such physical conditions are called the *conditions for determining solutions*, the *CDS* for short. The CDS is normally divided into initial conditions and boundary conditions. Finding solutions of equations of mathematical physics subjected to the CDS is called the *problem for determining solutions*, or the *PDS* for short.

1.4.1 Initial Conditions

For wave equations containing u_{tt}, the initial conditions refer to the initial values of u and u_t. The initial instant is normally $t=0$, but can also be $t=t_0$ for a known time instant t_0. Let M be a point x, (x,y) or (x,y,z) in one-, two- or three-dimensional space; the initial conditions are thus

$$u(M,0)=\varphi(M)\,,\quad u_t(M,0)=\psi(M)\,, \tag{1.77}$$

where φ and ψ are known functions of one, two or three variables. For the transverse displacement of a vibrating string, the initial conditions are the initial displacement and velocity of the string. The highest-order temporal derivative in the classical heat-conduction equation is u_t. The initial conditions for the classical heat-conduction equation thus give

$$u(M,0)=\varphi(M)\,,$$

which represents the initial temperature distribution. Both the hyperbolic and the dual-phase-lagging heat-conduction equations contain the term u_{tt}. The required initial conditions are the same as those in Eq. (1.77). Here $\varphi(M)$ and $\psi(M)$ are the initial temperature distribution and the initial rate of temperature changes in time, respectively. Note that the physical meaning of the initial conditions depends on physics and the nature of the dependent variable u.

In potential equations, the dependent variables are functions of position, but not of time; thus they require no initial conditions.

Remark 1. Initial conditions represent the initial state of a whole system, not just some parts or points of the system. Consider the vibration of a string of length l, fixed at the end points and subjected to an initial displacement h at the middle. The initial conditions are

$$u(x,0)=\begin{cases}\dfrac{2h}{l}x, & x\in\left[0,\dfrac{l}{2}\right],\\[2ex] \dfrac{2h}{l}(l-x), & x\in\left(\dfrac{l}{2},l\right],\end{cases} \quad \text{and} \quad u_t(x,0)=0\,.$$

Remark 2. Effects of initial conditions can sometimes be neglected. The initial-condition driven free vibration of a string will diminish progressively in reality due to unavoidable resistance. Therefore, when considering the vibration due to initial conditions and external non-decaying forces such as periodic forces, after a sufficiently long time t_0, we may neglect the effect of the real initial conditions and take $u(x,t_0) = u_t(x,t_0) = 0$ as the initial conditions. For heat conduction in a solid rod, the temperature due to an initial distribution $\varphi(x)$ must tend to a constant u_0 (some mean value of $\varphi(x)$ over the rod) as $t \to +\infty$. For the temperature distribution due to both the initial conditions and the heat source which is non-decaying with time, after a sufficiently long time t_0, we can take $u(x,t_0) = u_0$ or $u(x,0) = 0$ (without loss of generality) as the initial condition so that the temperature distribution depends only on the internal heat source.

1.4.2 Boundary Conditions

Boundary conditions describe situations of dependent variables on the system boundary orconstraints on the boundary. The system boundary is the end points, the boundary curve and the boundary surface in one-, two- and three-dimensional space, respectively. We normally have three types of boundary conditions.

Boundary Conditions of the First Kind

Consider the vibration of a string of length l, fixed at the two end points. The boundary conditions are $u(0,t) = u(l,t) = 0$. If the end $x = 0$ is fixed but the other end $x = l$ vibrates in the form of $u = \varphi(t)$, the boundary conditions become $u(0,t) = 0$, $u(l,t) = \varphi(t)$.

Consider heat conduction in a circle plate $D : x^2 + y^2 \leq R^2$. If the temperature distribution at the boundary ∂D is given as $\varphi(M,t)$, $M \in \partial D$, the boundary condition is

$$u(M,t)|_{\partial D} = \varphi(M,t), \text{ where } \partial D \text{ is the circle } x^2 + y^2 = R. \tag{1.78}$$

Consider heat conduction in a three-dimensional domain Ω of boundary surface $\partial\Omega$. If the temperature distribution on $\partial\Omega$ is given as $\varphi(M,t)$, $M \in \partial\Omega$, the boundary condition is

$$u(M,t)|_{\partial\Omega} = \varphi(M,t).$$

When heat conduction is steady, in particular, the temperature u satisfies the potential equation $\Delta u = 0$. If the temperature on the boundary S is given as $\varphi(M)$, $M \in S$, the boundary condition is

$$u(M)|_S = \varphi(M).$$

Let M and S be the point and the boundary of one-, two- or three-dimensional domain. When the dependent variable u is given on the S, the boundary condition reads

$$u(M,t)|_S = \varphi(M,t) \quad \text{or} \quad u(M)|_S = \varphi(M)\,, \tag{1.79}$$

where $\varphi(M,t)$ and $\varphi(M)$ are known functions. Such boundary conditions are called the *boundary conditions of the first kind*.

Boundary Conditions of the Second Kind

In many applications, boundary conditions are not given by values of the dependent variable on the boundary, but by its directional derivative along the boundary normal, the outward-drawn normal in particular. For one-dimensional cases, we can express a directional derivative by a partial derivative.

Consider the vibration of a string of length l. If the string can slide freely along the y-direction at $x=0$ without any force, a force balance at $x=0$ yields

$$T\frac{\partial u}{\partial x}\bigg|_{x=0} = 0 \quad \text{or} \quad u_x(0,t) = 0\,.$$

This is the boundary condition at $x=0$. If the string slides along the y-direction at $x=l$ with the action of a force $f(t)$, the boundary condition at $x=l$ is

$$T\frac{\partial u}{\partial x}\bigg|_{x=l} = f(t) \quad \text{or} \quad u_x(l,t) = f(t)/T\,,$$

where T is the tension of the vibrating string.

Consider heat conduction in a body. If the heat flux density on the boundary S is given, the normal derivative of temperature on S is known by the Fourier law of heat-conduction, say $\varphi(M,t)$. The boundary condition is thus

$$\frac{\partial u}{\partial n}\bigg|_S = \varphi(M,t)\,.$$

If the boundary is well insulated such that the heat flux vanishes on the boundary, the boundary condition reduces to

$$\frac{\partial u}{\partial n}\bigg|_S = 0\,.$$

If the two ends are well insulated for heat conduction in a solid rod of length l, the boundary conditions are

$$u_x(0,t) = u_x(l,t) = 0.$$

If the normal derivative of the dependent variable u is known on the boundary S, the boundary condition takes the form

$$\left.\frac{\partial u}{\partial n}\right|_S = \varphi(M,t) \quad \text{or} \quad \left.\frac{\partial u}{\partial n}\right|_S = \varphi(M). \tag{1.80}$$

Such boundary conditions are called *boundary conditions of the second kind*.

Boundary Conditions of the Third Kind

In some applications, boundary conditions are given by neither the value of dependent variable nor its derivative, but by a linear combination of the two.

Consider the longitudinal vibration of a rod of length $l(0 \leq x \leq l)$ and cross area S, fixed at $x = 0$ and connected to a spring of elasticity k at $x = l$. A force balance at $x = l$ yields the boundary condition

$$YS\frac{\partial u}{\partial x} = -ku \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{k}{YS}u = 0,$$

where Y is the Yung's modulus and $u = u(x,t)$ is the longitudinal displacement. Therefore a linear combination of the dependent variable u and its derivative is specified at the endpoint $x = l$.

Consider heat conduction in a solid body of boundary surfaces S surrounded by a fluid of temperature u_1 (Fig. 1.2). Let u and h_1 respectively be the body temperature and the convective heat transfer coefficient between S and the surrounding fluid. An energy balance for the differential element ΔS of S leads to the boundary condition, by using the Fourier law of heat conduction and the Newton law of cooling,

$$-k\Delta S \left.\frac{\partial u}{\partial n}\right|_S = h_1 \Delta S \left.(u - u_1)\right|_S \quad \text{or} \quad \left.\left(\frac{\partial u}{\partial n} + hu\right)\right|_S = \varphi(M,t),\; M \in S,$$

where n is the outward-drawn normal of S and k is the thermal conductivity of the body ($k > 0$), $h = h_1/k > 0$, $\varphi(M,t) = h_1 u_1/k$.

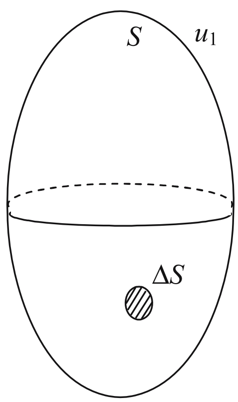

Fig. 1.2 Energy balance for differential element ΔS

Again, a linear combination of the dependent variable u and its normal derivative is given on the boundary S. Such a boundary condition is called a *convective boundary condition* in heat conduction. In particular, if the body is a rod of length $l(0 \leq x \leq l)$, then we have at the two endpoints

$$\left.\frac{\partial u}{\partial n}\right|_{x=l} = \left.\frac{\partial u}{\partial x}\right|_{x=l} \quad \text{and} \quad \left.\frac{\partial u}{\partial n}\right|_{x=0} = -\left.\frac{\partial u}{\partial x}\right|_{x=0}.$$

The above convective boundary condition reduces to

$$\left.\left(\frac{\partial u}{\partial x} - hu\right)\right|_{x=0} = \varphi_1(t), \quad \left.\left(\frac{\partial u}{\partial x} + hu\right)\right|_{x=l} = \varphi_2(t),$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are known functions.

When a linear combination of the dependent variable u and its normal derivative $\frac{\partial u}{\partial n}$ is known on the boundary, in general, the boundary condition reads

$$\left.\left(\frac{\partial u}{\partial n} + \sigma u\right)\right|_{S} = \varphi(M,t) \quad \text{or} \quad \left.\left(\frac{\partial u}{\partial n} + \sigma u\right)\right|_{S} = \varphi(M) \tag{1.81}$$

where $\varphi(M,t)$ and $\varphi(M)$ are known and the constant $\sigma > 0$. Such a boundary condition is called a *boundary condition of the third kind*.

All three kinds of boundary conditions discussed above are linear in the dependent variable u and its normal derivative. They are thus called *linear boundary conditions*. If a boundary condition is not linear in the dependent variable and/or its normal derivative, it is called a *nonlinear boundary condition*. If the free term (the right-hand side) in the three kinds of boundary conditions is zero, then we call them *homogeneous boundary conditions*. Otherwise, we call them *nonhomogeneous boundary conditions*.

Remark 1. Initial conditions describe the physical state of the whole system at an initial time instant, not just the initial state of one point. Boundary conditions represent the system state at the boundary over the whole process, not just the instant state at the boundary. Potential equations are independent of time; no initial conditions are required. While boundary conditions describe physical states on the system boundary, external forces or sources may act inside the system.

Remark 2. The effects of boundary conditions can sometimes be neglected. Consider, for example, the slow unidirectional diffusion of some molecules from $x = 0$ to $x = l$ in a rod of length $l(0 \leq x \leq l)$. If our attention is on diffusion in a sufficiently short time, we may neglect the effect of boundary conditions at $x = l$, replace the real boundary condition at $x = l$ by an infinite boundary $x \to +\infty$, and study the diffusion in a semi-infinite rod $0 \leq x < +\infty$.

Remark 3. Boundary conditions can be different on different parts of the boundary. For the longitudinal vibration of a rod of length $l(0 \leq x \leq l)$, we have boundary conditions of the first kind and the second kind at $x = 0$ and $x = l$ for the fixed end $x = 0$ and the free end $x = l$ such that $u(0,t) = 0$ and $u_x(l,t) = 0$. If the end $x = 0$

is well insulated and the end $x = l$ is exposed to an environment of convective heat transfer, for heat conduction in a bar of length $l \quad (0 \leq x \leq l)$ we have a boundary condition of the second kind at $x = 0$ $(u_x(0,t) = 0)$ and a boundary condition of the third kind at $x = l$ $(u_x(l,t) + hu(l,t) = \varphi(t))$. Nonlinear boundary conditions can also appear; a typical example comes from heat radiation from a solid body surface at a temperature u to the surrounding medium at temperature u_0. When we consider heat conduction in the body of boundary surface S exposed to a radiation environment, we arrive at a nonlinear boundary condition $\left.\left(\dfrac{\partial u}{\partial n} + hu^4\right)\right|_S = hu_0^4$, where h is a constant.

Remark 4. The CDS also may include periodic, bounded and linking conditions. At the interface between two media, for example, we should impose some linking conditions.

1.4.3 Problems for Determining Solutions

Finding solutions of equations of mathematical physics subjected to the CDS is called the *problem for determining solutions*, or the *PDS* for short. If some physical problems are defined in unbounded domains, then there exist no boundary conditions. A problem for determining solutions with initial conditions as the only CDS is called the *initial-value problem* or the *Cauchy problem*. A problem for determining solutions with boundary conditions as the only CDS is called the *boundary-value problem*. If the CDS contains both initial and boundary conditions, the PDS is called a *mixed problem*.

We have only boundary-value problems for potential equations. The PDS with boundary conditions of the first, the second and the third kind is called the *boundary-value problems of the first, the second and the third kind*, respectively. It is also called the *Dirichlet*, the *Neumann* and the *Robin problem*, respectively.

Unlike for wave and heat-conduction equations, we also classify the PDS of potential equations as an *internal problem* if the problem is studied inside of a boundary and as an *external problem* if it is studied outside of a boundary. For example, the steady-state temperature field outside a body Ω of boundary $\partial\Omega$ satisfies the external problem of the three-dimensional potential equation

$$\begin{cases} \Delta u = 0, \quad M \in R^3 \backslash \bar{\Omega}, \\ u|_{\partial\Omega} = \varphi(M), \\ \lim\limits_{r \to +\infty} u(M) = u_0, \quad r = \sqrt{x^2 + y^2 + z^2}\,. \end{cases}$$

Here $\bar{\Omega} = \Omega \cup \partial\Omega$, r is the distance of $M(x,y,z)$ from the origin, $\varphi(M)$ is known, u_0 is a given constant and $R^3 \backslash \overline{\Omega}$ is the region outside the closed domain $\overline{\Omega}$ in R^3.

Consider the irrotational flow of an incompressible fluid passing over a solid body Ω with boundary $\partial\Omega$. The velocity potential $\varphi(M)$ is the solution of

$$\begin{cases} \Delta\varphi = 0, \quad M \in \Omega', \\ \left.\dfrac{\partial\varphi}{\partial n}\right|_{\partial\Omega'} = \psi(M), \\ \lim\limits_{r\to+\infty} u(M) = 0, \quad r = \sqrt{x^2+y^2+z^2}. \end{cases}$$

This is an external boundary-value problem of the second kind of a potential equation. Here $\Omega' = R^3 \backslash \overline{\Omega}$ and $\psi(M)$ is a known function.

The constraint imposed at infinity in the above two external problems is for the uniqueness of the solution. In fact, we can review infinity as the boundary. From this point of view, we also expect some constraints in infinity to form a proper PDS.

1.4.4 Well-Posedness of PDS

The PDS is supposed to describe physical problems. Some assumptions are normally involved in deriving a PDS from specific physical problems. In order to have a reasonable approximation of physical reality by the PDS and be useful for application, the PDS must be well-posed.

Existence

A well-posed PDS must have solutions. In developing equations and their CDS from physical problems, we must normally make some idealized and simplifying assumptions. Many factors can lead to nonexistence of solutions. Examples are: (1) if the physically-dominant process or mechanism are not taken into account by the equations, and (2) the CDS is too restrictive or too many. In deriving the PDS, attention should also be given to the surrounding of physical systems as well as to the fundamental physical laws in order to ensure the existence of solutions.

Uniqueness

In applications, a PDE is used for describing a unique physical relation. A well-posed PDS should thus have a unique solution. Many factors can contribute to nonuniqueness; a typical example is that the PDS does not have enough CDS. The external problem of the potential equation

$$\begin{cases} \Delta u = 0, \quad x^2+y^2+z^2 > 1, \\ u|_{x^2+y^2+z^2=1} = 1 \end{cases}$$

has no constraint on u at infinity and has two solutions $u=1$ and $u = 1/\sqrt{x^2+y^2+z^2}$. If a further constraint $\lim\limits_{r\to\infty} u = 0$ $(r = \sqrt{x^2+y^2+z^2})$ is imposed, $u = 1/\sqrt{x^2+y^2+z^2}$ becomes the unique solution.

Stability

The CDS comes normally from experimental or field measurements with unavoidable errors. If the solution of a PDS is not stable with respect to the CDS such that a small error in obtaining the CDS can cause significant variations of the solution, then the PDS cannot be used to represent physical reality. A well-posed PDS thus must have a solution that is stable with respect to the CDS, so that the solution varies only a little for a small variation of data in the CDS. In other words, the solution of a well-posed PDS must depend continuously on its CDS. Since a nonhomogeneous boundary condition can normally be homogenized by some function transformations, stability often refers to the stability of solutions with respect to the initial conditions.

An analytical definition of stability can be made by introducing the size and norm of functions in some function space.

If a PDS has a unique and stable solution, it is called a *well-posed PDS*. Otherwise, it is ill-posed. While a well-posed PDS is normally desirable from the point of view of applications, some physical problems do lead to an ill-posed PDS. Therefore the study of ill-posed PDS is also an important branch of partial differential equations.

A study of well-posedness is normally mathematically-involved. We discuss the well-posedness only in a few places and focus our attention mainly on how to find solutions of various PDS in this book.

1.4.5 Example of Developing PDS

A PDS represents how physical variables distribute and evolve in the form of differential equations. The generation, distribution and evolution of a physical variable are the result of and a response to physical causes. The physical causes come normally from the following three mechanisms:

1. internal sources that are represented by the nonhomogeneous terms in the equations;
2. initial physical states of systems that are represented by the initial conditions;
3. system states at the boundary that are represented by the boundary conditions.

If all three mechanisms are absent, there would be no generation, distribution or evolution of physical variables. For example, the mixed problem describing the displacement $u(x,t)$ of a vibration string

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \quad 0 < x < l, 0 < t \\ u(0,t) = \mu_1(t), \, u(l,t) = \mu_2(t), \\ u(x,0) = \varphi(x), \, u_t(x,0) = \psi(x) \end{cases}$$

has $u(x,t) \equiv 0$ if $f(x,t) = \mu_1(t) = \mu_2(t) = \varphi(x) = \psi(x) \equiv 0$. This is physically-grounded and can be precisely proven. This simply shows that the string will be at rest if there is not any cause for vibration.

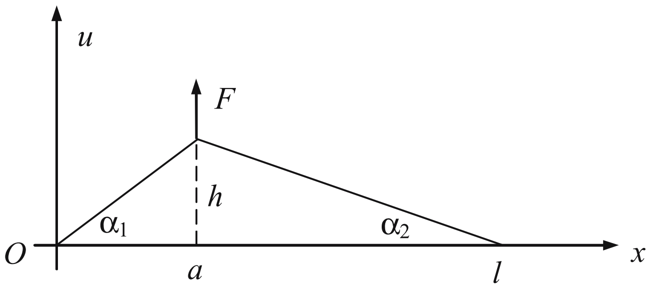

Fig. 1.3 Transverse vibration of a string due to the initial transverse force F at point $x = a$

For a specific problem, the cause may come from one, two or all three of the above-mentioned mechanisms. For example, the vibration of a string fixed at the two ends and with no initial displacement and no initial speed comes only from the internal sources.

Example. Consider a string of length l and line density ρ fixed at the two ends $x = 0$ and $x = l$. A transverse force F is initially acting at point $x = a$ of the string to pull the string to a small height h at $x = a$. The force is then removed, and the transverse vibration of the string begins. Find the PDS for describing the vibration of the string.

Solution. The free vibration of the string comes exclusively from its initial displacement at the instant when the force acts at $x = a$. Let T be the tension of the string. Since h is small, we have (Fig. 1.3)

$$\sin\alpha_1 \approx \tan\alpha_1, \quad \sin\alpha_2 \approx \tan\alpha_2.$$

A transverse force balance thus yields

$$T\frac{h}{a} + T\frac{h}{l-a} = F \quad \text{or} \quad h = \frac{F(l-a)a}{Tl}.$$

Thus the PDS is a mixed problem

$$\begin{cases} u_{tt} = a^2 u_{xx}, \quad 0 < x < l,\ 0 < t,\ a^2 = T/\rho, \\ u(x,0) = \begin{cases} \dfrac{F(l-a)}{Tl}x, & 0 \le x \le a, \\ \dfrac{Fa(l-x)}{Tl}, & a < x \le l, \end{cases} \qquad u_t(x,0) = 0, \\ u(0,t) = u(l,t) = 0, \quad t \ge 0. \end{cases}$$

Chapter 2
Wave Equations

We first develop the solution structure theorem for mixed problems of wave equations followed by methods of solving one-, two- and three-dimensional mixed problems. The solution structure theorem expresses contributions (to the solution of wave equations) of the initial distribution and the source term by using that from the initial rate of change of the solution, and hence considerably simplifies the development of solutions. For the two- or three-dimensional cases, some knowledge of special functions is required which can be found in Appendix A. The solution structure theorem is also valid for Cauchy problems. Finally, we discuss methods of solving one-, two- and three-dimensional Cauchy problems. The required knowledge of integral transformation is available in Appendix B.

2.1 The Solution Structure Theorem for Mixed Problems and its Application

Consider mixed problems of three-dimensional wave equations in a closed region $\overline{\Omega}$. Let $\overline{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega$ is the boundary surface. Three kinds of linear homogeneous boundary conditions can, therefore, be written in a unified form $L\left(u, \dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} = 0$. For example, for a one-dimensional bounded region $0 \leq x \leq l$, the boundary conditions at two ends are denoted by subscript 1 and 2 respectively and become $L_1(u, u_x)|_{x=0} = 0$ and $L_2(u, u_x)|_{x=l} = 0$.

The solution structure theorem is regarding the relation among the solutions of the following four PDS

$$\begin{cases} u_{tt} = a^2 \Delta u, & \Omega \times (0, +\infty), \\ L\left(u, \dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} = 0, \\ u(M,0) = \varphi(M), u_t(M,0) = 0, \end{cases} \tag{2.1}$$

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$$\begin{cases} u_{tt} = a^2\Delta u, & \Omega\times(0,+\infty), \\ L\left(u, \dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} = 0, \\ u(M,0) = 0, u_t(M,0) = \psi(M), \end{cases} \tag{2.2}$$

$$\begin{cases} u_{tt} = a^2\Delta u + f(M,t), & \Omega\times(0,+\infty), \\ L\left(u, \dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} = 0, \\ u(M,0) = 0, u_t(M,0) = 0, \end{cases} \tag{2.3}$$

$$\begin{cases} u_{tt} = a^2\Delta u + f(M,t), & \Omega\times(0,+\infty), \\ L\left(u, \dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} = 0, \\ u(M,0) = \varphi(M), u_t(M,0) = \psi(M), \end{cases} \tag{2.4}$$

where M represents the point x, (x,y) and (x,y,z) in one-, two- and three-dimensional space respectively. For the one-dimensional case, Δu is defined as u_{xx}.

Theorem. Suppose that $u_2 = W_\psi(M,t)$ is the solution of (2.2), then

1. $u_1 = \dfrac{\partial}{\partial t}W_\varphi(M,t)$ is the solution of (2.1).
2. $u_3 = \displaystyle\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau$ is the solution of (2.3) with $f_\tau = f(M,\tau)$.
3. $u = u_1 + u_2 + u_3 = \dfrac{\partial}{\partial t}W_\varphi + W_\psi + \displaystyle\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau$ is the solution of (2.4).

Proof.

1. Since $W_\varphi(M,t)$ satisfies

$$\begin{cases} \dfrac{\partial^2 W_\varphi}{\partial t^2} = a^2\Delta W_\varphi, & \Omega\times(0,+\infty), \\ L\left(W_\varphi, \dfrac{\partial W_\varphi}{\partial n}\right)\Big|_{\partial\Omega} = 0, \\ W_\varphi(M,0) = 0, \dfrac{\partial}{\partial t}W_\varphi(M,0) = \varphi(M), \end{cases}$$

thus

$$\frac{\partial^2 u_1}{\partial t^2} - a^2\Delta u_1 = \frac{\partial^2}{\partial t^2}\frac{\partial W_\varphi}{\partial t} - a^2\Delta\frac{\partial W_\varphi}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial^2 W_\varphi}{\partial t^2} - a^2\Delta W_\varphi\right) = 0.$$

Therefore, $u_1 = \dfrac{\partial}{\partial t}W_\varphi(M,t)$ satisfies the equation of (2.1).

Also,

$$L\left(u_1, \frac{\partial u_1}{\partial n}\right)\bigg|_{\partial\Omega} = L\left[\frac{\partial W_\varphi}{\partial t}, \frac{\partial}{\partial n}\left(\frac{\partial W_\varphi}{\partial t}\right)\right]\bigg|_{\partial\Omega} = \frac{\partial}{\partial t}\left[L\left(W_\varphi, \frac{\partial W_\varphi}{\partial n}\right)\bigg|_{\partial\Omega}\right] = 0,$$

$$u_1(M,0) = \frac{\partial}{\partial t}W_\varphi(M,0) = \varphi(M),$$

$$\frac{\partial}{\partial t}u_1(M,0) = \frac{\partial^2}{\partial t^2}W_\varphi(M,0) = a^2\Delta W_\varphi(M,0) = 0.$$

Hence, $u_1 = \dfrac{\partial}{\partial t}W_\varphi(M,t)$ also satisfies the boundary and initial conditions of (2.1), so that it is indeed the solution of (2.1).

2. As $W_{f_\tau}(M,t-\tau)$ satisfies

$$\begin{cases} \dfrac{\partial^2 W_{f_\tau}}{\partial t^2} = a^2\Delta W_{f_\tau}, \quad \Omega\times(0,+\infty), \\ L\left(W_{f_\tau}, \dfrac{\partial W_{f_\tau}}{\partial n}\right)\bigg|_{\partial\Omega} = 0, \\ W_{f_\tau}\big|_{t=\tau} = 0, \ \dfrac{\partial}{\partial t}W_{f_\tau}\bigg|_{t=\tau} = f(M,\tau), \end{cases}$$

then

$$\begin{aligned} \frac{\partial^2 u_3}{\partial t^2} - a^2\Delta u_3 &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau\right) - a^2\Delta\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau \\ &= \frac{\partial}{\partial t}\left(\int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}\big|_{\tau=t}\right) - a^2\int_0^t \Delta W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau \\ &= \int_0^t \frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau + \frac{\partial W_{f_\tau}}{\partial t}\bigg|_{\tau=t} - a^2\int_0^t \Delta W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau \\ &= \int_0^t \left(\frac{\partial^2 W_{f_\tau}}{\partial t^2} - a^2\Delta W_{f_\tau}\right)\mathrm{d}\tau + f(M,t) = f(M,t). \end{aligned}$$

Therefore, u_3 satisfies the equation of (2.3).
Also,

$$L\left(u_3, \frac{\partial u_3}{\partial n}\right)\bigg|_{\partial\Omega} = L\left(\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau, \frac{\partial}{\partial n}\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau\right)\bigg|_{\partial\Omega}$$

$$= \int_0^t L\left(W_{f_\tau}, \frac{\partial W_{f_\tau}}{\partial n}\right)\bigg|_{\partial\Omega}\mathrm{d}\tau = 0,$$

$$u_3|_{t=0} = 0, \qquad \frac{\partial u_3}{\partial t}\bigg|_{t=0} = \left(\int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}\big|_{\tau=t}\right)\bigg|_{t=0} = 0.$$

Therefore, u_3 also satisfies the boundary and initial conditions of (2.3), so that it is indeed the solution of (2.3).

3. Since PDS (2.4) is linear, the principle of superposition is valid. Applying this principle to (2.4) shows that $u = u_1 + u_2 + u_3$ is the solution of (2.4).

Remark 1. The solution structure theorem reduces the task of finding the solution of (2.4) to that of finding $W_\psi(M,t)$, the solution of (2.2). Problem (2.4) covers a large number of problems by noting that:

1. the spatial dimensions can be one, two, three or even higher;
2. one or two of the three functions $f(M,t)$(the source term),$\varphi(M)$and$\psi(M)$(the initial values) can be vanished; and
3. the boundary conditions may have variety of forms.

Remark 2. Mathematics is a discipline of high abstraction and wide application. It enjoys the beauty of conciseness. Here, $W_\square(M,t)$ can be viewed as a kind of function structure, similar to $\sin\square$ and $\ln\square$. Once $W_\psi(M,t)$ is available, $W_\varphi(M,t)$ and $W_{f_\tau}(M,t-\tau)$ should be regarded as known functions without the necessity of writing them explicitly. After $u = W_\psi(M,t)$, the solution of (2.2) is available, we may apply the solution structure theorem to readily write the solutions of (2.1), (2.3) and (2.4) by using $W_\square$. Therefore, it is of crucial importance to solve (2.2).

Remark 3. In developing the solution structure theorem, we assume that:

1. for all PDS there exist classical solutions of separable-variable type;
2. all derivatives are right-continuous so that all equations are also valid at $t = 0$; and
3. all the high-order mixed derivatives are continuous. When we apply the solution structure theorem, however, the solution sometimes only refers to the nominal solution.

This remark is valid for all solution structure theorems in the book unless otherwise stated.

Remark 4. Without loss of generality, PDS (2.1)–(2.4) are all with homogeneous boundary conditions. To apply the solution structure theorem for problems with nonhomogeneous boundary conditions, the homogenization of boundary conditions should first be made by some appropriate function transformations.

Remark 5. The solution structure can have other forms. Here, we emphasized the importance of solving (2.2). In fact, after any one of three PDS (2.1)–(2.3) is solved, the solutions of the other two and PDS (2.4) are readily available.

2.2 Fourier Method for One-Dimensional Mixed Problems

In this section, we discuss the Fourier method for finding solutions of

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \qquad (0,l)\times(0,+\infty) \\ L(u,u_x)|_{x=0,l} = 0, \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x). \end{cases} \tag{2.5}$$

By the solution structure theorem in Sect. 2.1, the solution of PDS (2.5) is readilyavailable if we have $u = W_\psi(x,t)$ solving

$$\begin{cases} u_{tt} = a^2 u_{xx}, \qquad (0,l)\times(0,+\infty) \\ L(u,u_x)|_{x=0,l} = 0, \\ u(x,0) = 0, u_t(x,0) = \psi(x). \end{cases} \tag{2.6}$$

Clearly, the structure of $u = W_\psi(x,t)$ depends on the boundary conditions at two ends. We apply the Fourier method here to find solutions of (2.6) with boundary conditions of the first kind and of the second kind at the ends.

2.2.1 Boundary Condition of the First Kind

When $u(0,t) = u(l,t) = 0$, by the theory of Fourier series, PDS (2.6) has a solution of type

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) \sin\frac{k\pi x}{l}, \tag{2.7}$$

where $T_k(t)$ is undetermined function.

Substituting Eq. (2.7) into the equation of (2.6) leads to

$$T_k''(t) + \left(\frac{k\pi a}{l}\right)^2 T_k(t) = 0,$$

which has the solution

$$T_k(t) = a_k \cos\frac{k\pi a t}{l} + b_k \sin\frac{k\pi a t}{l}, \tag{2.8}$$

where a_k and b_k are undetermined coefficients.

By substituting Eq. (2.8) into Eq. (2.7) and applying the initial conditions in (2.6), we arrive at

$$\sum_{k=1}^{\infty} a_k \sin\frac{k\pi x}{l} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} b_k \frac{k\pi a}{l} \sin\frac{k\pi x}{l} = \psi(x),$$

which yield

$$a_k = 0 \quad \text{and} \quad b_k = \frac{2}{k\pi a}\int_0^l \psi(x)\sin\frac{k\pi x}{l}\,\mathrm{d}x.$$

Therefore, the structure of $W_\psi(x,t)$, the solution of (2.6), is

$$\begin{cases} u = W_\psi(x,t) = \sum\limits_{k=1}^{\infty} b_k \sin\dfrac{k\pi at}{l}\sin\dfrac{k\pi x}{l}, \\ b_k = \dfrac{2}{k\pi a}\displaystyle\int_0^l \psi(x)\sin\frac{k\pi x}{l}\,\mathrm{d}x. \end{cases} \tag{2.9}$$

Finally, by the solution structure theorem, the solution of (2.5) is

$$u = \frac{\partial}{\partial t}W_\varphi + W_\psi(x,t) + \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau, \tag{2.10}$$

where the first term is the solution of (2.5) with $f = \psi = 0$ and the third term the solution of (2.5) for the case of $\varphi = \psi = 0$.

As $W_\psi(x,t)$ has been available, Eq. (2.10) can be regarded as the solution of PDS (2.5). If desirable, however, we can express Terms 1 and 3 similar to that of Eq. (2.9). Differentiating to $W_\varphi(x,t)$ yields

$$\begin{cases} u = \dfrac{\partial W_\varphi}{\partial t} = \sum\limits_{k=1}^{\infty} a_k \cos\dfrac{k\pi at}{l}\sin\dfrac{k\pi x}{l}, \\ a_k = \dfrac{2}{l}\displaystyle\int_0^l \varphi(x)\sin\frac{k\pi x}{l}\,\mathrm{d}x. \end{cases}$$

Note that $f_\tau = f(x,\tau)$, Term 3 in Eq.(2.9) can then be written as

$$\begin{cases} u = \displaystyle\int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau = \int_0^t \left(\sum_{k=1}^{+\infty} c_k \sin\frac{k\pi a(t-\tau)}{l}\sin\frac{k\pi x}{l}\right)\mathrm{d}x, \\ c_k = \dfrac{2}{k\pi a}\displaystyle\int_0^l f(x,\tau)\sin\frac{k\pi x}{l}\,\mathrm{d}x, \end{cases}$$

or

$$u(x,t) = \int_0^t \int_0^l G(x,\xi,t-\tau)f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{2.11}$$

Here, $G(x,\xi,t-\tau)$ is called the *Green function* of a one-dimensional wave equation subjected to the boundary condition of the first kind, and is defined by

$$G(x,\xi,t-\tau) = \sum_{k=1}^{+\infty} \frac{2}{k\pi a} \sin\frac{k\pi\xi}{l} \sin\frac{k\pi x}{l} \sin\frac{k\pi a(t-\tau)}{l}\,. \tag{2.12}$$

For the case $f=0$,

$$\begin{aligned} u &= \frac{\partial W_\varphi}{\partial t} + W_\psi(x,t) \\ &= \sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi at}{l} + b_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l} \\ &= \frac{1}{2}\sum_{k=1}^{+\infty} a_k\left[\sin\frac{k\pi(x-at)}{l} + \sin\frac{k\pi(x+at)}{l}\right] \\ &+ \frac{1}{2}\sum_{k=1}^{+\infty} b_k\left[\cos\frac{k\pi(x-at)}{l} - \cos\frac{k\pi(x+at)}{l}\right] \\ &= \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi\,, \end{aligned} \tag{2.13}$$

which depends on the initial values φ and ψ explicitly and is called the *D'Alembert formula* of wave equations. The D'Alembert formula shows that the solution u consists of two parts: the contribution of φ and of ψ. Note that both $\varphi(x)$ and $\psi(x)$ are defined in $[0,l]$. For a sufficiently large t, however, $x \pm at$ would be outside of $[0,l]$. In applying Eq. (2.13), therefore, an odd continuation of φ and ψ should be made, similarly to in Fourier series. In arriving at Eq. (2.13), the initial conditions are specified at $t=0$. If they are specified at $t=\tau$ Eq. (2.13) is still valid simply by replacing t by $t-\tau$.

Remark 1. The solution structure theorem is valid for PDS with homogeneous boundary conditions. Otherwise, the homogenization of boundary conditions should first be made by some appropriate function transformations. In the homogenization, the nonhomogeneous term (the source term) and the initial conditions are also varied. The function transformation is normally not unique.

For any function $w(x,t)$ with $w(0,t)=f_1(t)$ and $w(l,t)=f_2(t)$ such as $w(x,t)=f_1(t)+\frac{x}{l}[f_2(t)-f_1(t)]$, a function transformation of $u(x,t)=v(x,t)+w(x,t)$ always reduces

$$\begin{cases} u_{tt} = a^2 u_{xx}, & (0,l)\times(0,+\infty)\,, \\ u(0,t)=f_1(t),\ u(l,t)=f_2(t)\,, \\ u(x,0)=\varphi(x),\ u_t(x,0)=0 \end{cases}$$

into

$$\begin{cases} v_{tt} = a^2 v_{xx} + a^2 w_{xx} - w_{tt}, \quad (0,l) \times (0,+\infty), \\ v(0,t) = v(l,t) = 0, \\ v(x,0) = \varphi(x) - w(x,0), \; v_t(x,0) = -w_t(x,0). \end{cases}$$

Therefore, it is of crucial importance to have a function $w(x,t)$ satisfying the nonhomogeneous boundary conditions. Such kinds of auxiliary functions depend on the type of boundary conditions. Here we list such auxiliary functions for three typical kinds of boundary conditions

1. $w(x,t) = xf_2(t) + f_1(t) \quad \text{for} \quad u(0,t) = f_1(t), u_x(l,t) = f_2(t);$
2. $w(x,t) = (x-l)f_1(t) + f_2(t) \quad \text{for} \quad u_x(0,t) = f_1(t), u(l,t) = f_2(t);$
3. $w(x,t) = \dfrac{f_2(t) - f_1(t)}{2l}x^2 + f_1(t)x \quad \text{for} \quad u_x(0,t) = f_1(t), u_x(l,t) = f_2(t).$

For the case of $u(0,t) = 0$, $u(l,t) = A\sin\omega t$, both

$$w_1(x,t) = \frac{Ax}{l}\sin\omega t \quad \text{and} \quad w_2(x,t) = A\frac{\sin\dfrac{\omega x}{a}}{\sin\dfrac{\omega l}{a}}\sin\omega t$$

can serve as the auxiliary function of homogenization. As $a^2 w_{xx} - w_{tt} = 0$, however, the function transformation using $w_2(x,t)$ as the auxiliary function will not change the source term of the equation and is thus more desirable. Such a transformation can preserve the homogenenity of the original equation.

Remark 2. To understand the Green function, we must discuss the Dirac function, or δ-function for short. A precise mathematical discussion of the δ-function can be found in Appendix B. Here, we introduce it from the point of view of physics and engineering.

The δ-function is often called the *unit impulse function* and is defined by

$$\delta(x-x_0) = \begin{cases} 0, & x \neq x_0, \\ \infty, & x = x_0, \end{cases} \qquad \int_{-\infty}^{+\infty} \delta(x-x_0)\,\mathrm{d}x = 1.$$

Its introduction comes from the desire to describe some localized phenomena. For example, consider an infinite wire ($-\infty < x < +\infty$) with all the charges of unit quantity of electricity located at point x_0. The charge density $\rho(x-x_0)$ satisfies

$$\rho(x-x_0) = \begin{cases} 0, & x \neq x_0, \\ \infty, & x = x_0, \end{cases} \qquad \int_{-\infty}^{+\infty} \rho(x-x_0)\,\mathrm{d}x = 1.$$

For a force of unit impulse at time instant t_0, the force $F(t-t_0)$ can also be expressed by

$$F(t-t_0) = \begin{cases} 0, & t \neq t_0, \\ \infty, & t = t_0, \end{cases} \qquad \int_{-\infty}^{+\infty} F(t-t_0)\,\mathrm{d}t = 1\,.$$

In applications, the δ-function is often viewed as the limit of unit impulsive functions. For example, $\delta(x-x_0)$ can be regarded as the limit as $h \to 0$ of

$$\delta_h(x-x_0) = \begin{cases} 0, & \dfrac{h}{2} < |x-x_0|\,, \\ \dfrac{1}{h}, & |x-x_0| \leq \dfrac{h}{2}\,. \end{cases}$$

i.e.

$$\delta(x-x_0) = \lim_{h\to 0} \delta_h(x-x_0)\,.$$

Similarly,

$$\delta(x) = \lim_{\lambda\to+\infty} \frac{\lambda}{\pi(1+\lambda^2x^2)} \quad \text{and} \quad \delta(x) = \lim_{t\to 0}(4\pi\mu t)^{-\frac{1}{2}-\frac{|x|^2}{4\mu t}}\,, (\mu, t > 0)\,.$$

For any continuous function $\varphi(x)$, the δ-function can also be defined by

$$\delta(x-x_0) = \begin{cases} 0, & x \neq x_0\,, \\ \infty, & x = x_0\,. \end{cases} \qquad \int_{-\infty}^{+\infty} \varphi(x)\delta(x-x_0)\,\mathrm{d}x = \varphi(x_0)\,.$$

This can also be extended to the δ-function of multi-variables. For the case of two variables, for example, $\delta(x-x_0, y-y_0)$ satisfies

$$\delta(x-x_0, y-y_0) = \begin{cases} 0\,, & (x,y) \neq (x_0, y_0), \\ \infty\,, & (x,y) = (x_0, y_0), \end{cases}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x-x_0, y-y_0)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \varphi(x_0, y_0)\,,$$

where $\varphi(x,y)$ is a continuous function of x and y.

Letting $f(x,t) = \delta(x-x_0, t-t_0)$ in Eq. (2.11), we have $u(x,t) = G(x, x_0, t-t_0)$. Therefore, $u(x,t) = G(x, \xi, t-\tau)$ is the solution of

$$\begin{cases} G_{tt} = a^2 G_{xx} + \delta(x-\xi, t-\tau),\ 0 < x < l, 0 < \tau < t < +\infty \\ G|_{x=0} = G|_{x=l} = 0, \\ G|_{t=\tau} = G_t|_{t=\tau} = 0, \end{cases}$$

and thus comes from the source term of $\delta(x-\xi,t-\tau)$. Take the vibration of a string with two fixed ends as the example. The source term of $\delta(x-\xi,t-\tau)$ stands for a localized force action at spatial point ξ and time instant τ. The $G(x,\xi,t-\tau)$ is thus the displacement distribution due to such a force action.

Once the Green function is available, the solution of PDS (2.5) for the case of $\varphi=\psi=0$ can be expressed explicitly as a function of $f(x,t)$, simply by performing the integration in Eq. (2.11).

2.2.2 Boundary Condition of the Second Kind

For the boundary condition $u_x(0,t)=u_x(l,t)=0$, the solution of PDS (2.6) has the form

$$u(x,t)=\sum_{k=0}^{\infty} T_k(t)\cos\frac{k\pi x}{l},$$

where $T_k(t)$ is the undetermined function, and $k=0,1,2,\cdots$. By following the same procedure in Section 2.2.1, we can obtain $W_\psi(x,t)$, the solution of PDS (2.6) under the boundary condition of the second kind,

$$\begin{cases} u=W_\psi(x,t)=b_0 t+\sum\limits_{k=1}^{+\infty} b_k\sin\frac{k\pi at}{l}\cos\frac{k\pi x}{l}, \\ b_0=\frac{1}{l}\int_0^l \psi(x)\,\mathrm{d}x\,, b_k=\frac{2}{k\pi a}\int_0^l \psi(x)\cos\frac{k\pi x}{l}\,\mathrm{d}x\,. \end{cases} \tag{2.14}$$

For any nontrivial b_0, $b_0 t\to\infty$ as $t\to\infty$. There appears, therefore, an unbounded term $b_0 t$ in the solution. Take the string vibration again as the example. The boundary condition $u_x(0,t)=u_x(l,t)=0$ describes two free ends that can move freely in the vertical direction. This free movement leads, in combination with the effect of the initial velocity $\psi(x)$ of the string, to a uniform moment of velocity b_0 in addition to the vibration due to the initial velocity. An unit analysis yields

$$[b_0]=\left[\frac{1}{l}\right][\psi\,\mathrm{d}x]=L/T\,,\quad [b_k]=\left[\frac{1}{a}\right][\psi\,\mathrm{d}x]=L\,.$$

Therefore, b_0 has the units of a velocity and $b_k(k=1,2,\cdots)$ represent the displacement.

By the solution structure theorem, the solution of PDS (2.5) with $u_x(0,t)=u_x(l,t)=0$ is

$$u=\frac{\partial}{\partial t}W_\varphi+W_\psi(x,t)+\int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau\,,$$

where $f_\tau=f(x,\tau)$.

For the PDS

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), & (0,l) \times (0,+\infty) \\ u_x(0,t) = u_x(l,t) = 0, \\ u(x,0) = u_t(x,0) = 0, \end{cases}$$

we can also obtain its solution, similar to in Section 2.2.1,

$$u(x,t) = \int_0^t \int_0^l G(x,\xi,t-\tau) f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

Here $G(x,\xi,t-\tau)$ is the *Green function* of the one-dimensional wave equation under the boundary condition of the second kind and is defined by

$$G(x,\xi,t-\tau) = \frac{t-\tau}{l} + \sum_{k=1}^{+\infty} \frac{2}{k\pi a} \cos\frac{k\pi\xi}{l} \cos\frac{k\pi x}{l} \sin\frac{k\pi a(t-\tau)}{l}.$$

2.3 Method of Separation of Variables for One-Dimensional Mixed Problems

In the last section, we have applied the Fourier method to solve mixed problems under the first kind or the second kind of boundary conditions. The method uses the theory of Fourier series to obtain the solution of an odd or even continuation of the solution based on the characteristics of the boundary conditions. It involves the series expansion of the solution by orthogonal groups $\left\{\sin\frac{k\pi x}{l}\right\}$ and $\left\{\cos\frac{k\pi x}{l}\right\}$, respectively. However, it is not straightforward to know what is the type of series that should be used for the other boundary conditions, such as the third kind. For PDS with homogeneous boundary conditions, we may appeal to the method of separation of variables for the answer. In this section, we discuss this method by solving the following PDS with the mixed type of boundary conditions

$$\begin{cases} u_{tt} = a^2 u_{xx}, & (0,l) \times (0,+\infty) \\ u(0,t) = 0,\ u_x(l,t) = 0, \\ u(x,0) = 0,\ u_t(x,0) = \psi(x). \end{cases} \tag{2.15}$$

2.3.1 *Method of Separation of Variables*

To understand the method of separation of variables, it is helpful to elucidate the process of solving the PDS in Section 2.2.1 by using some physical results of a vibrating string. In the field of vibration, the PDS in Section 2.2.1 describes the free vibration of a string with two fixed ends being driven by an initial velocity. The propagation of the vibration forms waves. When these waves arrive at the endpoints, the reflected waves appear. The interaction between the waves and the reflected waves forms standing waves. For any individual standing wave, the displacement can be written in the form of the product of function $X(x)$of the spatial position x and function $T(t)$ of the time t. At any time instant, the string displacement is the sum of that of all standing waves, i.e.

$$u = \sum_k X_k(x) T_k(t)\,.$$

The Fourier method in Section 2.2.1 uses the characteristics of the boundary conditions to fix the type of $X_k(x)$, substitutes the series solution into the equation of the PDS, and applies the initial conditions to determine $T_k(t)$. Finally, we arrive at Eq.(2.9) in Section 2.2.1, the series solution of separable variables, where each term in the series satisfies the boundary conditions. Mathematically, it is a superposition to express the solution in a form of series. Note that the PDS indeed satisfies the conditions for applying the principle of superposition.

To apply the method of separation of variables to solve PDS (2.15), assume a solution of type

$$u(x,t) = X(x)T(t), X(0) = 0, X'(l) = 0$$

and substituting it to the equation of PDS (2.15) to obtain

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)}\,,$$

where the primes on the functions X and T represent differentiation with respect to the only variable present. Now the left-hand side of the above equation is independent of t and the right-hand side is independent of x. Since they are equal, their common value cannot be a function of x or t, and must be a constant, say $-\lambda$ called *separation constant*. Thus

$$X'' + \lambda X = 0\,,\; X(0) = 0\,,\; X'(l) = 0\,. \tag{2.16}$$

$$T'' + \lambda a^2 T = 0\,. \tag{2.17}$$

The partial differential equation of PDS (2.15) has thus been reduced to two ordinary differential equations.

The auxiliary problem defined by Eq.(2.16) is called an *eigenvalue problem*, because it has solutions only for certain values of the separation constant $\lambda = \lambda_k$, which are called the *eigenvalues*; the corresponding solution $X_k(x)$ are called the *eigenfunctions* of the problem. When λ is not an eigenvalue, that is, when $\lambda \neq \lambda_k$, the problem has trivial solutions (i.e. $X = 0$ if $\lambda \neq \lambda_k$).

The eigenvalue problem given by Eq.(2.16) is a special case of a more general eigenvalue problem called the *Sturm-Liouville problem*, the *S-L problem* for short. A discussion of the Sturm-Liouville problem is available in Appendix D.

The general solution of Eq. (2.16) is

$$X(x) = \begin{cases} A\mathrm{e}^{-\sqrt{-\lambda}x} + B\mathrm{e}^{\sqrt{-\lambda}x}, & \lambda < 0, \\ A + Bx, & \lambda = 0, \\ A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x, & \lambda > 0, \end{cases}$$

where A and B are constants. By applying the boundary conditions $X(0) = 0$ and $X'(l) = 0$, we know that both $\lambda < 0$ and $\lambda = 0$ lead to $X(x) \equiv 0$. Therefore $\lambda > 0$. Applying the boundary condition $X(0) = 0$ yields $A = 0$, thus $B \neq 0$. Applying the boundary condition $X'(l) = 0$ leads to $B\sqrt{\lambda}\cos\sqrt{\lambda}l = 0$ or $\cos\sqrt{\lambda}l = 0$. Therefore we obtain the eigenvalues λ_k and the eigenfunctions $X_k(x)$:

$$\lambda_k = \left[\frac{(2k+1)\pi}{2l}\right]^2, \quad X_k(x) = \sin\frac{(2k+1)\pi x}{2l}, \quad k = 0, 1, 2, \cdots.$$

It is easy to show both the completeness and the orthogonality of the eigenfunction group $\{X_k(x)\}$ in $[0, l]$.

Substituting λ_k into Eq. (2.17) leads to

$$T_k(t) = a_k\cos\frac{(2k+1)\pi at}{2l} + b_k\sin\frac{(2k+1)\pi at}{2l}, \quad k = 0, 1, 2, \cdots$$

where a_k and b_k are undetermined constants. A superposition of $u_k = X_k(x)T_k(t)$ yields the solution satisfying the equation and the boundary conditions, whatever the constants a_k and b_k,

$$u(x,t) = \sum_{k=0}^{\infty}\left(a_k\cos\frac{(2k+1)\pi at}{2l} + b_k\sin\frac{(2k+1)\pi at}{2l}\right)\sin\frac{(2k+1)\pi x}{2l}.$$

To satisfy the initial conditions a_k and b_k must be determined such that

$$\sum_{k=0}^{\infty} a_k\sin\frac{(2k+1)\pi x}{2l} = 0,$$
$$\sum_{k=0}^{\infty}\frac{(2k+1)\pi a}{2l}b_k\sin\frac{(2k+1)\pi x}{2l} = \psi(x).$$

Thus,

$$a_k = 0, \quad b_k = \frac{4}{(2k+1)\pi a}\int_0^l \psi(x)\sin\frac{(2k+1)\pi x}{2l}\,\mathrm{d}x,$$

where we have used the completeness and the orthogonality of the eigenfunction group and the normal square of the eigenfunction group

$$M_k = \int_0^l \sin^2\frac{(2k+1)\pi x}{2l}\,\mathrm{d}x = \frac{l}{2}.$$

Also, $\sqrt{M_k} = \sqrt{\frac{l}{2}}$ is called the *normal of the eigenfunction group*.

Finally, we have the solution of PDS (2.15),

$$\begin{cases} u = W_\psi(x,t) = \sum\limits_{k=0}^{\infty} b_k \sin\dfrac{(2k+1)\pi at}{2l}\sin\dfrac{(2k+1)\pi x}{2l}, \\ b_k = \dfrac{4}{(2k+1)\pi a}\displaystyle\int_0^l \psi(x)\sin\dfrac{(2k+1)\pi x}{2l}\,\mathrm{d}x. \end{cases} \tag{2.18}$$

The solution in Eq. (2.18) gives the structure of the solution of PDS (2.15). It is interesting to note that $\psi(x)$ only affects the constants b_k. By the solution structure theorem, the solution of

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t)\,,\ (0,l)\times(0,+\infty) \\ u(0,t) = u_x(l,t) = 0, \\ u(x,0) = \varphi(x)\,,\ u_t(x,0) = \psi(x) \end{cases}$$

is

$$u = \frac{\partial}{\partial t}W_\varphi + W_\psi(x,t) + \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau\,,$$

where $f_\tau = f(x,\tau)$.

A substitution of $W_{f_\tau}(x,t-\tau)$ into the last term of the above equation yields an integral expression of the solution for the case of $\varphi = \psi = 0$, i.e.

$$u(x,t) = \int_0^t \int_0^l G(x,\xi,t-\tau) f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau\,.$$

Here $G(x,\xi,t-\tau)$ is the *Green function* of the one-dimensional wave equation under the mixed boundary conditions in PDS (2.15) and is defined by

$$G(x,\xi,t-\tau) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi a}\sin\frac{(2k+1)\pi\xi}{2l}\sin\frac{(2k+1)\pi x}{2l}\sin\frac{(2k+1)\pi a(t-\tau)}{2l}\,.$$

2.3.2 Generalized Fourier Method of Expansion

The eigenvalue problem of second-order equations

$$\begin{cases} X''(x) + \lambda X(x) = 0, \quad (0,l) \\ L(X,X')|_{x=0,l} = 0 \end{cases} \tag{2.19}$$

plays a very important role in the method of separation of variables. If all combinations of the boundary conditions of the first, second and third kinds are considered, for a finite region $0 \le x \le l$, there exist nine combinations of boundary conditions. For three of them we have already detailed the finding of the eigenvalues, the eigenfunctions and their normal squares. The results for the remaining six combinations can be readily obtained using a similar approach and are summarized in Table 2.1.

It has been proven by the theory of eigenvalue problems that all eigenfunction groups in Table 2.1 are complete and orthogonal in $[0,l]$. Therefore, they can be used to expand the functions including solutions of PDS. The expanding series is called the *generalized Fourier series*. The generalized Fourier method of expansion solves PDS by using the generalized Fourier series to expand solutions. Using this method, the coefficients in the series can be easily determined by the completeness and the orthogonality of eigenfunction groups. We show this method here by solving

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \quad (0,l)\times(0,+\infty) \\ u_x(0,t) = 0\,,\ u_x(l,t) + hu(l,t) = 0, \\ u(x,0) = \varphi(x)\,,\ u_t(x,0) = \psi(x). \end{cases}$$

Solution. We first develop $W_\psi(x,t)$, the solution for the case of $f = \varphi = 0$. Based on the given boundary conditions, we should use the eigenfunctions in Row 6 in Table 2.1 to expand the solution so that

$$u = \sum_{m=1}^{\infty} T_m(t)\cos\frac{\mu_m x}{l}\,,$$

where $T_m(t)$ is the function to be determined later and μ_m is positive zero point of $f(x) = \cot x - x/lh$.

Substituting the above equation into the equation of the PDS leads to

$$T_m'' + \left(\frac{\mu_m a}{l}\right)^2 T_m = 0\,,$$

which has the general solution

$$T_m(t) = a_m\cos\frac{\mu_m a t}{l} + b_m\sin\frac{\mu_m a t}{l}\,.$$

Applying the initial condition $u(x,0) = 0$ yields $a_m = 0$, $m = 1,2,\cdots$. To satisfy the initial condition $u_t(x,0) = \psi(x)$, b_m must be determined such that

$$\sum_{m=1}^{\infty} b_m\frac{\mu_m a}{l}\cos\frac{\mu_m x}{l} = \psi(x)\,.$$

The b_m is thus, by the completeness and the orthogonality of the eigenfunction set,

$$b_m = \frac{l}{\mu_m M_m a}\int_0^l \psi(x)\cos\frac{\mu_m x}{l}\,\mathrm{d}x\,,$$

where $M_m = \dfrac{l}{2}\left(1 + \dfrac{\sin 2\mu_m}{2\mu_m}\right)$.

Finally, we have $W_\psi(x,t)$

$$\begin{cases} u = W_\psi(x,t) = \sum_{m=1}^{\infty} b_m \cos\frac{\mu_m x}{l} \sin\frac{\mu_m a t}{l}\,, \\ b_m = \frac{l}{\mu_m M_m a}\int_0^l \psi(x)\cos\frac{\mu_m x}{l}\,\mathrm{d}x\,. \end{cases}$$

The solution of the original PDS follows from the solution structure theorem

$$u = \frac{\partial}{\partial t}W_\varphi + W_\psi(x,t) + \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau\,.$$

Remark. The various mixed problems of one-dimensional wave equations can be solved very efficiently and concisely by using Table 2.1, the solution structure theorem and the structure function $W_\square$.

2.3.3 Important Properties of Eigenvalue Problems (2.19)

The method of separation of variables relies on the properties of eigenvalue problems (2.19). We list four important properties here and refer to Appendix D for a discussion of the theory of eigenvalue problems.

1. All eigenvalues are non-negative and real-valued for all combinations of boundary conditions. A vanished eigenvalue appears only when $X'(0) = X'(l) = 0$.
2. Eigenvalues form a sequence of numbers which is monotonically increasing towards infinity, whatever the boundary conditions, i.e.

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots\,, \quad \lim_{k\to\infty}\lambda_k = \infty\,.$$

3. All eigenfunction sets $\{X_k(x)\}$ are orthogonal in $[0,l]$, i.e.

$$(X_k, X_m) = \int_0^l X_k(x) X_m(x)\,\mathrm{d}x = 0,\ k \ne m\,.$$

4. Any function $f(x) \in L^2[a,b]$ can be expanded into a generalized Fourier series by an eigenfunction set, i.e.

$$\begin{cases} f(x) = \sum_{k=1}^{\infty} c_k X_k(x)\,, \\ c_k = \frac{1}{M_k}\int_a^b X_k(x) f(x)\,\mathrm{d}x\,,\ M_k = \|X_k\|^2_{L^2[a,b]} = \int_a^b X_k^2(x)\,\mathrm{d}x, \end{cases}$$

where $\sqrt{M_k}$ is called the normal of $\{X_k(x)\}$ and serves as the measure of function

Table 2.1 Eigenfunctions

$x=0$	$x=l$	Eigenvalues	Eigenfunctions	Normal square M_m	Notes
1)	$X=0$	$\left(\frac{m\pi}{l}\right)^2$	$\sin\frac{m\pi x}{l}$	$\frac{l}{2}$	$m=1,2,\cdots$
2) $X=0$	$X'=0$	$\left[\frac{(2m+1)\pi}{2l}\right]^2$	$\sin\frac{(2m+1)\pi x}{2l}$	$\frac{l}{2}$	$m=0,1,\cdots$
3)	$X'+h_2X=0$ $h_2>0$	$\left(\frac{\mu_m}{l}\right)^2$	$\sin\frac{\mu_m}{l}x$	$\frac{l}{2}\left(1-\frac{\sin 2\mu_m}{2\mu_m}\right)$	$\mu_m(m=1,2,\cdots)$ are positive zero-points of $f(x)=\tan x+\frac{x}{lh_2}$
4)	$X=0$	$\left[\frac{(2m+1)\pi}{2l}\right]^2$	$\cos\frac{(2m+1)\pi x}{2l}$	$\frac{l}{2}$	$m=0,1,\cdots$
5) $X'=0$	$X'=0$	$\left(\frac{m\pi}{l}\right)^2$	$\cos\frac{m\pi x}{l}$	$l,\frac{l}{2}$	$m=0,1,\cdots$
6)	$X'+h_2X=0$ $h_2>0$	$\left(\frac{\mu_m}{l}\right)^2$	$\cos\frac{\mu_m}{l}x$	$\frac{l}{2}\left(1+\frac{\sin 2\mu_m}{2\mu_m}\right)$	$\mu_m(m=1,2,\cdots)$ are positive zero-points of $g(x)=\cot x-\frac{x}{lh_2}$
7)	$X=0$	$\left(\frac{\mu_m}{l}\right)^2$	$\sin\left(\frac{\mu_m}{l}x+\varphi_m\right)$ $\tan\varphi_m=\mu_m/lh_1$	$\frac{l}{2}\left[1-\frac{\sin\mu_m}{\mu_m}\cdot\cos(\mu_m+2\varphi_m)\right]$	$\mu_m(m=1,2,\cdots)$ are positive zero-points of $f(x)=\tan x+\frac{x}{lh_1}$
8) $X'-h_1X=0$ $h_1>0$	$X'=0$	$\left(\frac{\mu_m}{l}\right)^2$	$\sin\left(\frac{\mu_m}{l}x+\varphi_m\right)$ $\tan\varphi_m=\mu_m/lh_1$	$\frac{l}{2}\left[1-\frac{\sin\mu_m}{\mu_m}\cdot\cos(\mu_m+2\varphi_m)\right]$	$\mu_m(m=1,2,\cdots)$ are positive zero-points of $g(x)=\cot x-\frac{x}{lh_1}$
9)	$X'+h_2X=0$ $h_2>0$	$\left(\frac{\mu_m}{l}\right)^2$	$\sin\left(\frac{\mu_m}{l}x+\varphi_m\right)$ $\tan\varphi_m=\mu_m/lh_1$	$\frac{l}{2}\left[1-\frac{\sin\mu_m}{\mu_m}\cdot\cos(\mu_m+2\varphi_m)\right]$	$\mu_m(m=1,2,\cdots)$ are positive zero-points of $F(x)=\cot x-\frac{1}{l(h_1+h_1)}\cdot\left(x-\frac{l^2h_1h_2}{x}\right)$

size. Therefore $\{X_k(x)\}$ forms a complete and orthogonal set in $[a,b]$, and

$$\lim_{N\to\infty}\left\|f(x)-\sum_{k=1}^{N}c_kX_k(x)\right\|_{L^2[a,b]}=0.$$

2.4 Well-Posedness and Generalized Solution

In this section, we first attempt to establish the existence, uniqueness and stability of the solution with respect to the initial conditions and followed by a discussion of the necessity, importance and relevance of introducing generalized solutions. In discussing well-posedness, we take the initial condition-driven free vibration of a string of two fixed ends as an example. Therefore, we will address the well-posedness of

$$\begin{cases} u &= \dfrac{\partial W_\varphi}{\partial t}+W_\psi(x,t) \\ &= \sum\limits_{k=1}^{\infty}(a_k\cos\dfrac{k\pi at}{l}+b_k\sin\dfrac{k\pi at}{l})\sin\dfrac{k\pi x}{l}\,, \\ a_k &= \dfrac{2}{l}\int_0^l\varphi(x)\sin\dfrac{k\pi x}{l}\,\mathrm{d}x\,, \\ b_k &= \dfrac{2}{k\pi a}\int_0^l\psi(x)\sin\dfrac{k\pi x}{l}\,\mathrm{d}x \end{cases} \tag{2.20}$$

for

$$\begin{cases} u_{tt}=a^2u_{xx}, \quad (0,l)\times(0,+\infty) \\ u(0,t)=0, u(l,t)=0, \\ u(x,0)=\varphi(x), u_t(x,0)=\psi(x). \end{cases} \tag{2.21}$$

2.4.1 Existence

Although we have obtained the solution (2.20) for PDS (2.21), we still need to prove its existence. In developing Eq. (2.20), we have taken the limit of the series solution and its first and second-order derivatives with respect to t and x term by term. This requires the uniform convergence of series (2.20) and the resulting series (from taking derivatives term by term) in $[0,l]\times[0,T]$, $t\in[0,T]$, were T is any positive number. Therefore, we must impose some constraints on $\varphi(x)$ and $\psi(x)$ and some consistency conditions at two ends.

Existence Theorem. Both series (2.20) and series u_x, u_{xx}, u_t, u_{tt}, constructed by taking derivatives of series (2.20) term by term, converge uniformly in $[0,l]\times[0,T]$, $t\in[0,T]$ with T as an arbitrary positive number if

1. $\varphi(x)\in C^4[0,l]$, $\psi(x)\in C^3[0,l]$ (constraints on φ and ψ);
2. $\varphi(x)$, $\varphi''(x)$, $\psi(x)$ and $\psi''(x)$ are all vanished at $x=0$ and $x=l$ (consistency conditions at two ends).

Proof. An application of integration by parts and the consistency conditions for φ yields

$$\begin{aligned} a_k &= \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi x}{l}\,\mathrm{d}x = \frac{2}{k\pi}\int_0^l \varphi'(x)\cos\frac{k\pi x}{l}\,\mathrm{d}x \\ &= -\frac{2l}{(k\pi)^2}\int_0^l \varphi''(x)\sin\frac{k\pi x}{l}\,\mathrm{d}x = \cdots \\ &= \frac{2l^3}{(k\pi)^4}\int_0^l \varphi^{(4)}(x)\sin\frac{k\pi x}{l}\,\mathrm{d}x = O\left(\frac{1}{k^4}\right), \end{aligned}$$

where $O\left(\frac{1}{k^4}\right)$ stands for the infinitesimal of higher or the same order of $\frac{1}{k^4}$ $(k\to\infty)$; $\int_0^l \varphi^{(4)}(x)\sin\frac{k\pi x}{l}\,\mathrm{d}x\to 0$ as $k\to\infty$ by Riemann lemma. Similarly, $b_k = O\left(\frac{1}{k^4}\right)$.

Thus, every term $u_k(x,t)$ in the series (2.20) satisfies

$$\begin{aligned} &|u_k(x,t)| \le |a_k|+|b_k| = O\left(\frac{1}{k^4}\right); \\ &\left|\frac{\partial u_k(x,t)}{\partial x}\right| \le \frac{k\pi}{l}\left(|a_k|+|b_k|\right) = O\left(\frac{1}{k^3}\right); \\ &\left|\frac{\partial^2 u_k(x,t)}{\partial x^2}\right| \le \left(\frac{k\pi}{l}\right)^2\left(|a_k|+|b_k|\right) = O\left(\frac{1}{k^2}\right). \end{aligned}$$

Similarly,

$$\left|\frac{\partial u_k(x,t)}{\partial t}\right| \le O\left(\frac{1}{k^3}\right), \left|\frac{\partial^2 u_k(x,t)}{\partial t^2}\right| \le O\left(\frac{1}{k^2}\right).$$

Also, the p-series $\sum\limits_{k=1}^{\infty}\frac{1}{k^p}$ converges for $p>1$. Therefore, all series u, u_x, u_{xx}, u_t and u_{tt} in the theorem converge uniformly, by the Weierstrass test, so that series (2.20) is the solution of PDS (2.21).

Remark 1. The conditions in the existence theorem are sufficient conditions for existence, but they are not necessary conditions.

2.4.2 Uniqueness

Suppose that $u_1(x,t)$ and $u_2(x,t)$ are two solutions of PDS (2.21). The difference between them

$$w(x,t) = u_1(x,t) - u_2(x,t)$$

must be the solution of the mixed problem

$$\begin{cases} w_{tt} = a^2 w_{xx}\,, & (0,l)\times(0,+\infty) \\ w(0,t) = w(l,t) = 0\,, \\ w(x,0) = w_t(x,0) = 0\,. \end{cases} \tag{2.22}$$

Its uniqueness will be established once we show $w(x,t) \equiv 0$.

From a physical point of view, $w(x,t)$ in PDS (2.22) represents the string displacement in the field of vibration. As $\varphi = \psi = f = 0$ and $w(0,t) = w(l,t) = 0$, there is no cause for vibration so $w(x,t) \equiv 0$. However, this kind of physical explanation cannot serve as the formal proof that $w(x,t) \equiv 0$.

The u in Eq. (2.20) is indeed the solution of PDS (2.21) if the conditions specified in the existence theorem hold. As $\varphi = 0$ and $\psi = 0$ satisfy these conditions, the solution $w(x,t)$ of PDS (2.22) can be obtained by applying $\varphi = 0$ and $\psi = 0$ into Eq. (2.20) so that $w(x,t) \equiv 0$. Note that the well-posedness of solution (2.20) has not been established yet. This can only serve as a demonstration of $w(x,t) \equiv 0$, not a formal proof.

To formally prove that $w(x,t) \equiv 0$, consider the energy of a vibrating string. Without loss of generality, let $\rho = 1$. The kinetic energy of string segment $\mathrm{d}s$ is

$$\mathrm{d}K = \frac{1}{2} w_t^2 \Delta x\,.$$

Its potential energy is, under the assumption of negligible w_x^4,

$$\mathrm{d}U = T(\mathrm{d}s - \Delta x) = T\left(\sqrt{1+w_x^2} - 1\right)\Delta x = \frac{1}{2} T w_x^2 \Delta x$$

where $T = a^2$ is the tension. The total energy of the whole string of length l at time instant t is thus

$$E(t) = \frac{1}{2}\int_0^l \left(w_t^2 + a^2 w_x^2\right)\mathrm{d}x\,.$$

This is called the *energy integral*.

Note that

$$\int_0^l w_x w_{xt}\,\mathrm{d}x = \int_0^l w_x\,\mathrm{d}w_t = w_x w_t|_0^l - \int_0^l w_t w_{xx}\,\mathrm{d}x$$
$$= -\int_0^l w_t w_{xx}\,\mathrm{d}x\,.$$

Therefore, $\dfrac{\mathrm{d}E}{\mathrm{d}t} = \int_0^l \left(w_t w_{tt} + a^2 w_x w_{xt}\right)\,\mathrm{d}x = \int_0^l \left(w_{tt} - a^2 w_{xx}\right) w_t\,\mathrm{d}x = 0$, so $E(t) = E(0)$. As $w(x,0) = 0$, we have $w_x(x,0) = 0$. Also, $w_t(x,0) = 0$, so

$$E(0) = \frac{1}{2}\int_0^l \left(w_t^2 + a^2 w_x^2\right)\big|_{t=0}\,\mathrm{d}x = 0\,.$$

Thus $E(t) = \dfrac{1}{2}\int_0^l \left(w_t^2 + a^2 w_x^2\right)\,\mathrm{d}x = 0$.
This and the fundamental lemma of variational method lead to $w_t(x,t) = w_x(x,t) = 0$ so $w(x,t)$ is independent of x and t. Thus $w(x,t) \equiv w(x,0) \equiv 0$, which establishes the uniqueness of the solution.

Remark 2. The auxiliary function $E(t)$ is the total energy of the string of unit density if $w(x,t)$ is viewed as the string displacement in the field of vibration. $\dfrac{\mathrm{d}E}{\mathrm{d}t} = 0$ is thus a mathematical expression of the conservation of energy, a fundamental physical law.

2.4.3 Stability

To establish the stability of solution (2.20) with respect to initial conditions, we first need to develop an important inequality for PDS (2.21).

Consider an auxiliary function

$$E_0(t) = \int_0^l u^2\,\mathrm{d}x\,,$$

where $u(x,t)$ is the solution of PDS (2.21). Thus

$$E_0'(t) = 2\int_0^l u u_t\,\mathrm{d}x \le \int_0^l u^2\,\mathrm{d}x + \int_0^l u_t^2\,\mathrm{d}x \le E_0(t) + 2E(t)\,,$$

where the energy integral $E(t) = \dfrac{1}{2}\int_0^l \left(u_t^2 + a^2 u_x^2\right)\,\mathrm{d}x\,.$

Using a similar approach to that in Section 2.4.2, we can readily show

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \quad \text{or} \quad E(t) = E(0)\,.$$

Multiplying the above inequality by e^{-t} and using $E(t) = E(0)$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_0(t)e^{-t}\right) \leq 2E(0)\mathrm{e}^{-t}.$$

Or, by integration with respect to t over $[0,t]$

$$E_0(t) \leq \mathrm{e}^t E_0(0) + 2E(0)(\mathrm{e}^t - 1)\,. \tag{2.23}$$

For stability, we substitute $\varphi(x)$ and $\psi(x)$ into the right hand side of (2.23) to obtain

$$\begin{aligned}\|u\|^2 &\leq \mathrm{e}^T\|\varphi\|^2 + \left(\|\psi\|^2 + a^2\|\varphi'\|^2\right)\left(e^T - 1\right)\\ &\leq e^T\left(\|\varphi\|^2 + \|\psi\|^2 + a^2\|\varphi'\|^2\right),\end{aligned}$$

where $\|f\| = \sqrt{\int_a^b f^2(x)\,\mathrm{d}x}$ is the normal of function $f(x)$ in $[a,b]$.

For $\|\varphi\| < \varepsilon$, $\|\varphi'\| < \varepsilon$ and $\|\psi\| < \varepsilon$, therefore,

$$\|u\|^2 \leq \mathrm{e}^T\left(2 + a^2\right)\varepsilon^2 \quad \text{or} \quad \|u\| \leq C\varepsilon\,.$$

Here, ε is a small positive constant, and C is a nonnegative constant. This shows that the solution (2.20) is stable with respect to the initial conditions.

2.4.4 Generalized Solution

A *classical solution* refers to a solution that satisfies the conditions in the existence theorem in Section 2.4.1. Such solutions must have continuous second-order derivatives and over-restricted initial functions $\varphi(x)$ and $\psi(x)$, thus their application is limited. Consider the string of two fixed ends discussed in Chapter 1, for example. If we raise the string at point x_0 and release it without initial velocity, the string will have a free vibration. The PDS governing the string displacement should have a unique solution. However, $\varphi(x)$ is not differentiable at x_0. Due to the reflection at the two ends, the discontinuity of $\varphi(x)$ at x_0 propagates in $(0,l) \times (0,+\infty)$. There exists no solution with continuous second-order derivatives in the whole domain $(0,l) \times (0,+\infty)$, thus no classical solution exists. By the D'Alembert formula, however, the solution is

$$u(x,t) = \frac{\varphi(x-at) + \varphi(x+at)}{2}\,,$$

which is well defined and indeed meaningful. It is therefore necessary to extend the concept of solutions from the classical one to the generalized one. Such an extension is not arbitrary and must follow a few rules. First, the classical solution must be, if it exists, the generalized solution. Second, the generalized solution must be unique and stable. Under these rules, we can define the generalized solution using different methods, for example by introducing conjugate operators. In solving PDS by using the generalized Fourier method of expansion, $\varphi(x)$ and $\psi(x)$ appear only in the integrand, so the demand for their smoothness is very weak. The solution obtained by this method is thus a *generalized solution*, the *solution* for short. All solutions of PDS in this book refer to this kind of solutions, unless otherwise stated. There is no essential distinction between the Fourier method and the method of separation of variables. The generalized solutions from these methods are called *nominal solutions*.

An analysis of the convergence of such nominal solutions will benefit our appreciation of their significance and relevance in applications. By normalizing eigenfunction set $\{X_k(x)\}$, we have a complete orthonormal basis $\{\mathrm{e}_k(x)\}$ in $[0,l]$, i.e.

$$e_k(x) = \frac{X_k(x)}{\|X_k\|}, \quad X_k(x) \in L^2[0,l] .$$

Consider an approximation of function $f(x) \in L^2[0,l]$ by a generalized polynomial

$$f_n(x) = \sum_{k=1}^{n} a_k e_k(x), \tag{2.24}$$

where a_k are constant coefficients. The error square of the approximation is

$$\begin{aligned}\Delta_n &= \int_0^l [f(x) - f_n(x)]^2\, \mathrm{d}x = (f - f_n, f - f_n) \\ &= (f,f) - 2(f,f_n) + (f_n,f_n),\end{aligned}$$

also,

$$\begin{aligned}(f,f_n) &= \left(f, \sum_{k=1}^{n} a_k e_k(x)\right) = \sum_{k=1}^{n} a_k c_k, \\ (f_n,f_n) &= \left(\sum_{k=1}^{n} a_k e_k(x), \sum_{k=1}^{n} a_k e_k(x)\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j (e_i(x), e_j(x)) = \sum_{k=1}^{n} a_k^2,\end{aligned}$$

where $c_k = \displaystyle\int_0^l f(x) e_k(x)\, \mathrm{d}x$ are the Fourier coefficients. Therefore,

$$\Delta_n = (f,f) - 2\sum_{k=1}^{n} a_k c_k + \sum_{k=1}^{n} a_k^2 = (f,f) + \sum_{k=1}^{n} (c_k - a_k)^2 - \sum_{k=1}^{n} c_k^2 .$$

As (f,f) and c_k are constant, the approximation error $\sqrt{\Delta_n}$ will reach its minimum at $a_k = c_k$,

$$\min \Delta_n = \|f\|^2 - \sum_{k=1}^{n} c_k^2 \geq 0\,, \tag{2.25}$$

which is called the *Bessel inequality*.

Therefore, the generalized polynomial constructed by the Fourier coefficient c_k

$$f_n(x) = \sum_{k=1}^{n} c_k e_k(x)\,,\ c_k = (f, e_k(x))$$

is the best approximation of $f(x)$ for fixed n and $\{e_k(x)\}$, i.e.

$$\min \Delta_n = \int_0^l \left[f(x) - \sum_{k=1}^{n} c_k e_k(x) \right]^2 \mathrm{d}x\,, \quad c_k = (f, e_k(x))\,.$$

When $\lim\limits_{n\to\infty} \min \Delta_n = 0$, $x \in [0,l]$, in particular, let

$$f(x) = \sum_{k=1}^{\infty} c_k e_k(x)\,, \tag{2.26}$$

which is called the *generalized Fourier expansion*. The right-hand side of Eq. (2.26) converges to $f(x)$ in an average sense. Also

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2\,. \tag{2.27}$$

It is this elegant convergence as well as its weak demand for the smoothness of $\varphi(x)$ and $\psi(x)$ that popularize the nominal solution and its application.

2.4.5 PDS with Variable Coefficients

When the coefficients in Eq. (1.2) are variable, it is very difficult to find analytical solutions of its PDS. Numerical methods are normally used to obtain numerical solutions. Here we attempt to find the analytical solution of a mixed problem of hyperbolic equations with variable coefficients by transforming the PDS to an eigenvalue problem of integral equations. We detail the procedure by considering the following example.

Example

Find the solution of

$$\begin{cases} u_{tt} = a^2 \dfrac{\partial}{\partial x}[w(x)u_x] + f(x,t), & (0,l) \times (0,+\infty), \\ u(0,t) = u(l,t) = 0, & u(x,0) = u_t(x,0) = 0. \end{cases} \tag{2.28}$$

where $a > 0$ is constant and $w(x) > 0$ is a differentiable function.

Solution

1. Homogenization of the Equation

The solution of PDS (2.28) can be expressed by $u(x,t) = \int_0^t v(x,t,\tau)\,\mathrm{d}\tau$, where $v(x,t,\tau)$ is the solution of

$$\begin{cases} v_{tt} = a^2 \dfrac{\partial}{\partial x}[w(x)v_x], & (0,l) \times (0,+\infty), \\ v(0,t,\tau) = v(l,t,\tau) = 0, & v|_{t=\tau} = 0,\ v_t|_{t=\tau} = f(x,\tau). \end{cases}$$

Considering the solution of type $v = X(x)T(t)$, we arrive at an eigenvalue problem

$$\frac{\mathrm{d}}{\mathrm{d}x}[w(x)X_x] + \lambda X = 0, \quad X(0) = X(l) = 0, \tag{2.29}$$

and

$$T_{tt} + \lambda a^2 T = 0, T(\tau) = 0, \tag{2.30}$$

where λ is the separation constant.

For a PDS with the δ-function as the nonhomogeneous term

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}[w(x)G_x] = \delta(x-s), \\ G(0,s) = G(l,s) = 0, \quad G(s^-,s) = G(s^+,s), \end{cases} \tag{2.31}$$

the solution is the Green function

$$G(x,s) = \begin{cases} a_1 + b_1 + (a_2 + b_2)u, & x \leq s, \\ a_1 - b_1 + (a_2 - b_2)u, & x \geq s. \end{cases}$$

Here $G(s^-,s) = \lim\limits_{x \to s-0} G(x,s)$, $G(s^+,s) = \lim\limits_{x \to s+0} G(x,s)$, $u(x) = \int \dfrac{\mathrm{d}x}{w(x)}$. By applying continuity and the boundary conditions of (2.31), we obtain

$$b_1 = \frac{u(s)}{2}, \quad b_2 = -\frac{1}{2},$$
$$a_1 = -b_1 - (a_2 + b_2)u(0), \quad a_2 = \frac{-2b_1 - b_2[u(0) + u(l)]}{u(0) - u(l)}.$$

Therefore, by the meaning of the Green function, (2.29) leads to an eigenvalue problem

$$X(x) = -\lambda \int_0^l G(x,s)X(s)\,\mathrm{d}s. \tag{2.32}$$

2. Method of Resolving (2.32)

Let $G(x,s) = \sum_{n=1}^{\infty} a_n(s)\sin\frac{n\pi x}{l}$; substituting it into (2.32) yields

$$X(x) = -\lambda \sum_{n=1}^{\infty} X_n \sin\frac{n\pi x}{l}, \quad X_n = -\lambda \int_0^l X(s)a_n(s)\,\mathrm{d}s. \tag{2.33}$$

Multiplying (2.33) by $a_m(x)$ and integrating over $[0,l]$ lead to

$$X_m = -\lambda \sum_{n=1}^{\infty} a_{mn}X_n, \quad a_{mn} = \int_0^l a_m(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x, \quad m = 1,2,\cdots$$

Therefore we obtain an eigenvalue problem

$$\mathbf{X} = -\lambda\mathbf{A}\mathbf{X}, \quad \mathbf{A} = (a_{mn})_{\infty\times\infty}, \quad \mathbf{X} = (X_1, X_2, \cdots)^T.$$

Let $X_1, X_2, \cdots$ be the eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots$. Substituting into (2.33) yields the eigenvector group of (2.32)

$$X_1(x), X_2(x), \cdots, X_n(x), \cdots$$

which is complete and orthogonal in $[0,l]$ by the theory of integral equations. Furthermore, $\lambda_i > 0$, $i = 1,2,\cdots$.

3. Analytical Solution of the Original PDS

$T_n(t)$ corresponding to $\lambda = \lambda_n$ can easily be obtained by Eq. (2.30) so that

$$v(x,t,\tau) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = \sum_{n=1}^{\infty} D_n X_n(x)\sin\sqrt{\lambda_n}a(t-\tau),$$
$$D_n = \int_0^l X_n(x)f(x,\tau)\,\mathrm{d}x \Big/ \sqrt{\lambda_n}\,a\int_0^l X_n^2(x)\,\mathrm{d}x.$$

Finally the solution of the original PDS (2.28) can be written as $u(x,t) = \int_0^t v(x,t,\tau)\,\mathrm{d}\tau$.

In practice, we can only solve eigenvalue problems where matrix **A** has a finite order to obtain an approximate analytical solution.

2.5 Two-Dimensional Mixed Problems

Separation of variables is the main method for solving mixed problems. It is applicable only for PDE with a separable equation and separable homogeneous boundary conditions. This is the case only for some regular domains.

2.5.1 Rectangular Domain

Consider

$$\begin{cases} u_{tt} = a^2\Delta u + f(x,y,t)\,, \qquad D\times(0,+\infty) \\ L(u,u_x,u_y)\big|_{\partial D} = 0\,, \\ u(x,y,0) = \varphi(x,y)\,,\ u_t(x,y,0) = \psi(x,y)\,, \end{cases} \tag{2.34}$$

where D is the domain $0<x<l_1, 0<y<l_2$, ∂D is the boundary of D, and $t\in(0,\infty)$. If all combinations of the boundary conditions of the first, second and third kinds are considered, for a finite rectangular domain D, there exist 81 combinations of linear boundary conditions $L(u,u_x,u_y)\big|_{\partial D}=0$. We detail the process of finding the solution of PDS (2.34) for the combination

$$\begin{aligned} &u(0,y,t) = u_x(l_1,y,t) + h_2 u(l_1,y,t) = 0\,, \\ &u_y(x,0,t) - h_1 u(x,0,t) = u_y(x,l_2,t) = 0\,. \end{aligned} \tag{2.35}$$

The results for the remaining 80 combinations are easily obtained by using a similar approach, Table 2.1 and the solution structure theorem.

By the solution structure theorem, we first develop $u = W_\psi(x,y,t)$, the solution for the case $f=\varphi=0$. Based on the given boundary conditions (2.35), we should use the eigenfunctions in Rows 3 and 8 in Table 2.1 to expand the solution,

$$u(x,y,t) = \sum_{m,n=1}^{\infty} T_{mn}(t)\sin\frac{\mu_m x}{l_1}\sin\left(\frac{\mu_n' y}{l_2}+\varphi_n\right),$$

where μ_m and μ_n' are positive zero-points of $f(x) = \tan x + \dfrac{x}{l_1 h_2}$ and $g(x) = \cot x - \dfrac{x}{l_2 h_1}$, respectively, and $\tan\varphi_n = \dfrac{\mu_n'}{l_2 h_1}$. Substituting this equation into the wave equation in PDS (2.34) yields

$$\sum_{m,n=1}^{\infty}\left\{T''_{mn}+a^2\left[\left(\frac{\mu_m}{l_1}\right)^2+\left(\frac{\mu'_n}{l_2}\right)^2\right]T_{mn}\right\}\sin\frac{\mu_m x}{l_1}\sin\left(\frac{\mu'_n y}{l_2}+\varphi_n\right)=0\,,$$

which leads to the equation of $T_{mn}(t)$ and its general solution

$$T''_{mn}+\omega_{mn}^2 T_{mn}=0,\quad T_{mn}(t)=a_{mn}\cos\omega_{mn}t+b_{mn}\sin\omega_{mn}t\,,$$

where

$$\omega_{mn}^2=a^2\left[\left(\frac{\mu_m}{l_1}\right)^2+\left(\frac{\mu'_n}{l_2}\right)^2\right].$$

Substituting $T_{mn}(t)$ and applying $u(x,y,0)=0$ leads to $a_{mn}=0$. To satisfy the initial condition $u_t(x,y,0)=\psi(x,y)$, b_{mn} must be determined such that

$$\sum_{m,n=1}^{\infty} b_{mn}\omega_{mn}\sin\frac{\mu_m x}{l_1}\sin\left(\frac{\mu'_n y}{l_2}+\varphi_n\right)=\psi(x,y)\,.$$

It is straightforward to obtain b_{mn} by orthogonality and the normal square of eigenfunction sets. Thus we have

$$\begin{cases} u=W_\psi(x,y,t)=\displaystyle\sum_{m,n=1}^{\infty} b_{mn}\sin\frac{\mu_m x}{l_1}\sin\left(\frac{\mu'_n y}{l_2}+\varphi_n\right)\sin\omega_{mn}t\,,\\ b_{mn}=\dfrac{1}{M_{mn}\omega_{mn}}\displaystyle\iint_D \psi(x,y)\sin\frac{\mu_m x}{l_1}\sin\left(\frac{\mu'_n y}{l_2}+\varphi_n\right)\mathrm{d}x\,\mathrm{d}y\,. \end{cases}$$

where $M_{mn}=M_m M_n$, M_m and M_n are the normal squares of the two eigenfunction sets, respectively.

Finally, the solution of PDS (2.34) under the boundary conditions (2.35) is, by the solution structure theorem,

$$u=\frac{\partial}{\partial t}W_\varphi+W_\psi(x,y,t)+\int_0^t W_{f_\tau}(x,y,t-\tau)\,\mathrm{d}\tau\,, \tag{2.36}$$

where $f_\tau=f(x,y,\tau)$.

We can also obtain the Green function by considering the case $f(x,y,t)=\delta(x-\xi,y-\eta,t-\tau)$, $\psi(x,y)=0$ and $\varphi(x,y)=0$.

The solution of PDS (2.34) for the other 80 combinations of boundary conditions can also be written in the form of Eq. (2.36). However, $W_\psi(x,y,t)$ differs from one to another.

2.5.2 Circular Domain

Consider the mixed problem

$$\begin{cases} u_{tt} = a^2 \Delta u + f(x,y,t), & D \times (0,+\infty) \\ L(u,u_n)|_{\partial D} = 0, \\ u(x,y,0) = \varphi(x,y),\ u_t(x,y,0) = \psi(x,y), \end{cases} \tag{2.37}$$

where D stands for the domain: $x^2+y^2<a_0^2$, ∂D is its boundary, $t \in (0,+\infty)$, and $L(u,u_n)|_{\partial D} = 0$ can be the first, the second or the third kind of boundary condition.

We first develop the solution for the case of $f = \varphi = 0$ by using the method of separation of variables. Considering a solution of the form $u = v(x,y)T(t)$, we substitute it into the wave equation in PDS (2.37) to obtain

$$\Delta v + \lambda v = 0, \quad L(v,v_n)|_{\partial D} = 0, \tag{2.38}$$

$$T''(t) + \lambda a^2 T(t) = 0,$$

where $-\lambda$ is the separation constant.

As the boundary condition in Eq. (2.38) is not separable for x and y, we cannot apply separation of variables to solve Eq. (2.38) in the Cartesian coordinate system. In order to solve Eq. (2.38), consider polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$. Equation (2.38) is thus transformed into

$$\begin{cases} \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} + k^2 v = 0 \quad 0<r<a_0 \\ L(v,v_r)|_{r=a_0} = 0,\ v(r,\theta+2\pi) = v(r,\theta) \end{cases} \tag{2.39}$$

where $k^2 = \lambda$. Now the boundary condition in Eq. (2.39) is separable for r and θ. If we assume a solution of the form $v = R(r)\Theta(\theta)$, Eq. (2.39) yields

$$\Theta'' + \mu\Theta = 0, \quad \Theta(\theta+2\pi) = \Theta(\theta), \text{(periodic condition)} \tag{2.40}$$

$$R'' + \frac{1}{r}R' + \left(k^2 - \frac{\mu}{r^2}\right)R = 0, \quad L(R,R_r)|_{r=a_0} = 0 \tag{2.41}$$

where μ is the separation constant. To satisfy the periodic condition in Eq. (2.40),

$$\mu = n^2,\ n = 0,1,2,\cdots$$

so $\quad \Theta_n(\theta) = A_n \cos n\theta + B_n \sin n\theta,$

where A_n and B_n are arbitrary constants.

Substituting $\mu = n^2$ into Eq. (2.41) yields a boundary-value problem with parameter $\lambda = k^2$

$$\begin{cases} R_n''(r) + \dfrac{1}{r}R_n'(r) + \left(k^2 - \dfrac{n^2}{r^2}\right)R_n(r) = 0, \quad 0<r<a_0 \\ L(R_n,R_n')|_{r=a_0} = 0,\ |R_n(0)|<\infty,\ |R_n'(0)|<\infty. \end{cases} \tag{2.42}$$

The periodic condition in Eq. (2.40) and the bounded condition in Eq. (2.42) are called the *natural boundary condition*, which is another kind of CDS.

The auxiliary problem defined by Eq. (2.42) is called an *eigenvalue problem of the Bessel equations*, because by letting $x = kr$, the equation in (2.42) becomes a Bessel equation of n-th order. The general solution of Eq. (2.42) is

$$R_n(r) = C_n J_n(kr) + D_n Y_n(kr),$$

where C_n and D_n are arbitrary constants, and J_n and Y_n are the n-th order Bessel functions of the first and the second kind, respectively. A discussion of Bessel functions is available in Appendix A.

Applying the bounded condition in Eq. (2.42) and using $\lim_{r\to 0} Y_n(kr) = \infty$ leads to $D_n = 0$. For the boundary condition of the first kind, $L(R_n, R_n')|_{r=a_0} = 0$ reduces to $R_n(a_0) = 0$; to satisfy it we have $J_n(ka_0) = 0$. Therefore we obtain the eigenvalues and the eigenfunctions of problem (2.42) subject to the boundary condition of the first kind

$$\begin{aligned} &\text{Eigenvalues} \qquad \lambda_m = k_{mn}^2 = \left(\mu_m^{(n)}/a_0\right)^2, \quad (m = 1, 2, \cdots) \\ &\text{Eigenfunctions} \quad J_n(k_{mn}r), \quad k_{mn} = \mu_m^{(n)}/a_0\ , \end{aligned} \tag{2.43}$$

where $\mu_m^{(n)} (m = 1, 2, \cdots)$ are the zero-points of $J_n(x)$.

Note. For $n = 0$, $J_0(0) = 1$ so that $x = 0$ is not a zero-point of $J_0(x)$. Although $x = 0$ is a zero-point of $J_n(x)$ for $n \geq 1$, $J_n(k_{mn}r)$ is not an eigenfunction because $J_n(k_{mn}r) = J_n(0) = 0$. When $k = 0$, Eq. (2.42) reduces to an Euler equation and a trivial solution of $R_n(r)$ when $k = 0$. By Eq. (2.38), on the other hand, we also arrive at a trivial solution of v when $k = 0$. Therefore $k_{mn}^2 \neq 0$. The distribution of zero-points of $J_n(x)$ is symmetric around the origin. We only account for their positive zero-points; therefore $\mu_m^{(n)}$ are the positive zero-points of $J_n(x)$ in Eq. (2.43): $\mu_1^{(n)} < \mu_2^{(n)} < \cdots < \mu_m^{(n)} < \cdots$.

With these eigenvalues λ_m, the solution of the equation for $T(t)$ yields

$$T_{mn}(t) = E_{mn}\cos\omega_{mn}t + F_{mn}\sin\omega_{mn}t\,,$$

where $\omega_{mn} = k_{mn}a$, E_{mn} and F_{mn} are arbitrary constants. Since PDS (2.37) is linear, by principle of superposition,

$$\begin{aligned} u &= \sum T_{mn}(t)\ R_n(r)\Theta_n(\theta) \\ &= \sum_{n=0,m=1}^{\infty} \quad (a_{mn}\cos\omega_{mn}t + b_{mn}\sin\omega_{mn}t)\,J_n(k_{mn}r)\cos n\theta \\ &\qquad\qquad + (c_{mn}\cos\omega_{mn}t + d_{mn}\sin\omega_{mn}t)\,J_n(k_{mn}r)\sin n\theta \end{aligned}$$

is also a solution of the wave equation. Here a_{mn}, b_{mn}, c_{mn} and d_{mn} are arbitrary constants. In order that u satisfies the initial condition $u(r, \theta, 0) = 0$, we must choose

a_{mn} and c_{mn} such that

$$\sum_{n=0,m=1}^{\infty}(a_{mn}\cos n\theta + c_{mn}\sin n\theta)J_n(k_{mn}r) = 0$$

and hence $a_{mn} = c_{mn} = 0$, by using (1) the completeness and the orthogonality of $\{1, \cos\theta, \sin\theta, \cdots, \cos n\theta, \sin n\theta, \cdots\}\,(n = 1, 2, \cdots)$ in $[-\pi, \pi]$, (2) the completeness of $\{J_n(k_{mn}r)\}$ in $[0, a_0]$ for every fixed value of n, and (3) the orthogonality of $\{J_n(k_{mn}r)\}$ in $[0, a_0]$ with respect to the weight function r for every fixed value of n, i.e.

$$\int_0^{a_0} J_n(k_{mn}r)J_n(k_{ln}r)r\,\mathrm{d}r = 0\,, \quad m \neq l\,.$$

To satisfy the initial condition $u_t(r, \theta, 0) = \Psi(r, \theta)$, b_{mn} and d_{mn} must be determined such that

$$\sum_{n=0,m=1}^{\infty}(b_{mn}\omega_{mn}\cos n\theta + d_{mn}\omega_{mn}\sin n\theta)J_n(k_{mn}r) = \Psi(r, \theta)\,.$$

Thus we obtain the solution of PDS (2.37) for the case $f = \varphi = 0$

$$\begin{cases} u = W_\Psi(r, \theta, t) = \displaystyle\sum_{n=0,m=1}^{\infty}(b_{mn}\cos n\theta + d_{mn}\sin n\theta)J_n(k_{mn}r)\sin\omega_{mn}t\,, \\ b_{m0} = \dfrac{1}{2\pi\omega_{m0}M_{m0}}\displaystyle\int_{-\pi}^{\pi}\mathrm{d}\theta\int_0^{a_0}\Psi(r, \theta)J_0(k_{m0}r)r\,\mathrm{d}r\,, \\ b_{mn} = \dfrac{1}{\pi\omega_{mn}M_{mn}}\displaystyle\int_{-\pi}^{\pi}\mathrm{d}\theta\int_0^{a_0}\Psi(r, \theta)J_n(k_{mn}r)r\cos n\theta\,\mathrm{d}r\,, \\ d_{mn} = \dfrac{1}{\pi\omega_{mn}M_{mn}}\displaystyle\int_{-\pi}^{\pi}\mathrm{d}\theta\int_0^{a_0}\Psi(r, \theta)J_n(k_{mn}r)r\sin n\theta\,\mathrm{d}r\,, \end{cases} \tag{2.44}$$

where $k_{mn} = \mu_m^{(n)}/a_0$, $M_{mn} = \displaystyle\int_0^{a_0} J_n^2(k_{mn}r)r\,\mathrm{d}r$ is the normal square of eigenfunction set $\{J_n(k_{mn}r)\}$.

Finally, the solution of PDS (2.37) follows from the solution structure theorem,

$$u = \frac{\partial}{\partial t}W_\Phi + W_\Psi(r, \theta, t) + \int_0^t W_{f_\tau}(r, \theta, t-\tau)\,\mathrm{d}\tau\,. \tag{2.45}$$

Remark 1. The solution of PDS (2.37) for the other two kinds of boundary conditions can also be written in the form of Eqs. (2.44) and (2.45). However, the $\mu_m^{(n)}$ are different and are the positive zero-points of $J_n(x)$, $J_n'(x)$ and $\dfrac{1}{a_0}xJ_n'(x) + hJ_n(x)$ for the first ($u|_{r=a_0} = 0$), the second ($u_r|_{r=a_0} = 0$) and the third ($(u_r + hu)|_{r=a_0} = 0$) kind of boundary conditions, respectively. The only exception is $\mu_1^{(0)} = 0$ for the case of a boundary condition of the second kind.

Remark 2. To apply Eqs. (2.44) and (2.45), we must determine M_{mn}, the normal square of eigenfunction sets. Here we develop expressions of M_{mn} for three cases of boundary conditions.

Rewrite Eq. (2.41) into

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}R}{\mathrm{d}r}\right]+\frac{n^2}{r}R=k^2 rR. \tag{2.46}$$

Multiplying Eq. (2.46) by rR' and integrating over $[0,a_0]$ yields

$$-\int_0^{a_0}\left(rR'\right)\left(rR'\right)'\,\mathrm{d}r+\int_0^{a_0}n^2RR'\,\mathrm{d}r=k^2\int_0^{a_0}r^2RR'\,\mathrm{d}r,$$

or

$$\begin{aligned}-\frac{1}{2}\left(rR'\right)^2\Big|_0^{a_0}+\frac{n^2}{2}R^2\Big|_0^{a_0}&=k^2\textstyle\int_0^{a_0}r^2\,\mathrm{d}\left(\frac{R^2}{2}\right)\\&=k^2\left(\frac{r^2R^2}{2}\Big|_0^{a_0}-M_{mn}\right).\end{aligned}$$

Thus

$$M_{mn}=\left(\frac{(rR)^2}{2}+\frac{(rR')^2}{2k^2}-\frac{(nR)^2}{2k^2}\right)\Bigg|_0^{a_0}. \tag{2.47}$$

1. Boundary Condition of the First Kind

Since $R(a_0)=0$ and $R(0)=J_n(0)=0\,(n=1,2,\cdots)$, Eq. (2.47) becomes

$$\begin{aligned}M_{mn}&=\frac{a_0^2}{2k_{mn}^2}\left[k_{mn}J_n'(k_{mn}a_0)\right]^2=\frac{a_0^2}{2}\left[J_n'\left(\mu_m^{(n)}\right)\right]^2\\&=\frac{a_0^2}{2}J_{n-1}^2\left(\mu_m^{(n)}\right)=\frac{a_0^2}{2}J_{n+1}^2\left(\mu_m^{(n)}\right),\end{aligned}$$

where $J_{n-1}(x)-J_{n+1}(x)=2J_n'(x)$ and $J_{n-1}(x)+J_{n+1}(x)=\frac{2n}{x}J_n(x)$ have been used.

2. Boundary Condition of the Second Kind

For this case, $R'(a_0)=0$ and $R(0)=J_n(0)=0\,(n=1,2,\cdots)$. Eq. (2.47) reduces to

$$M_{mn}=\frac{a_0^2}{2}\left[1-\left(\frac{n}{\mu_m^{(n)}}\right)^2\right]J_n^2\left(\mu_m^{(n)}\right).$$

3. Boundary Condition of the Third Kind

Since $R'(a_0) = -hR(a_0)$ and $R(0) = J_n(0) = 0\,(n = 1, 2, \cdots)$, Eq. (2.47) yields

$$M_{mn} = \frac{a_0^2}{2}\left[1 + \frac{(a_0 h)^2 - n^2}{\left(\mu_m^{(n)}\right)^2}\right] J_n^2\left(\mu_m^{(n)}\right).$$

When $n = 0$, M_{m0} is also available by using the above formula.

Remark 3. Here we prove the orthogonality of the Bessel function set in $[0, a_0]$ with respect to the weight function r used in developing Eq. (2.44).

Consider the S-L problem under some boundary conditions

$$L[y] - \lambda \rho(x) y = 0, \tag{2.48}$$

where λ is a parameter, L is an operator defined by

$$L = -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] + q(x), \qquad a < x < b.$$

By the definition of eigenvalues and eigenfunctions, we have

$$L[y_m(x)] = \lambda_m \rho(x) y_m(x), \quad L[y_n(x)] = \lambda_n \rho(x) y_n(x),$$

where λ_m and λ_n are two distinct eigenvalues of Eq. (2.48) ($\lambda_m \neq \lambda_n$), $y_m(x)$ and $y_n(x)$ are the corresponding eigenfunctions. Subtracting the latter from the former after multiplying the former by $y_n(x)$ and the latter by $y_m(x)$, we obtain

$$\begin{aligned} &\int_a^b y_n(x) L[y_m(x)]\,\mathrm{d}x - \int_a^b y_m(x) L[y_n(x)]\,\mathrm{d}x \\ &= (\lambda_m - \lambda_n)\int_a^b y_m(x) y_n(x) \rho(x)\,\mathrm{d}x. \end{aligned} \tag{2.49}$$

Also, the Lagrange equality requires, for all $u(x)$, $v(x) \in C^2(a,b) \cap L^2[a,b]$,

$$\begin{aligned} &\int_a^b u(x) L[v(x)]\,\mathrm{d}x - \int_a^b v(x) L[u(x)]\,\mathrm{d}x \\ &= \left\{p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\right\}\Big|_a^b. \end{aligned} \tag{2.50}$$

Here $L^2[a,b]$ stands for a function group which is both square integrable and square integrable with respect to a weight function $\rho(x)$. The Lagrange equality comes directly from an application of integration by parts and can be found in Appendix D.

Equation (2.42) (or Eq. (2.46)) is a special case of Eq. (2.48) at $p(r) = \rho(r) = r$, $(0 < r < a_0\ )$. Take two eigenfunctions $J_n(k_{mn} r)$ and $J_n(k_{ln} r)$ as $u(x)$ and $v(x)$ in

Eq. (2.50) ($\lambda_m \neq \lambda_l$). The right-hand side of Eq. (2.50) reduces to

$$\left\{ r \left[J_n'(k_{mn}r)J_n(k_{ln}r) - J_n(k_{mn}r)J_n'(k_{ln}r) \right] \right\} \Big|_0^{a_0}$$
$$= a_0 \left[k_{mn}J_n' \left(\mu_m^{(n)} \right) J_n \left(\mu_l^{(n)} \right) - J_n \left(\mu_m^{(n)} \right) k_{ln}J_n' \left(\mu_l^{(n)} \right) \right],$$

which is zero for the boundary condition of the first $[R(a_0) = 0]$, the second $[R'(a_0) = 0]$ and the third $[R(a_0) = cR_r(a_0)$ with c as a constant$]$ kinds. In arriving at the last case, we have used

$$J_n \left(\mu_m^{(n)} \right) = cJ_n' (k_{mn}r)\big|_{r=a_0} = ck_{mn}J_n' \left(\mu_m^{(n)} \right), \quad J_n \left(\mu_l^{(n)} \right) = ck_{ln}J_n' \left(\mu_l^{(n)} \right).$$

This, together with Eq. (2.49), shows that the Bessel function set is orthogonal in $[0, a_0]$ with respect to the weight function r.

An operator satisfying

$$\int_a^b u(x)L[v(x)]\,\mathrm{d}x - \int_a^b v(x)L[u(x)]\,\mathrm{d}x = 0, \quad \forall u(x), v(x) \in C^2(a,b) \cap L^2[a,b]$$

is called a *self-conjugate operator*.

A corresponding eigenvalue problem of $L[y] - \lambda\rho(x)y = 0$ that is subject to certain boundary conditions is called a *self-conjugate eigenvalue problem*.

Remark 4. Here we prove the solution structure theorem in a polar coordinate system, which was used in obtaining Eq. (2.45).

Theorem

The solution of

$$\begin{cases} u_{tt} = a^2\Delta u(r,\theta,t) + F(r,\theta,t), & 0 < r < R, 0 < t \\ L(u,u_r)|_{r=R} = 0, \lim\limits_{r\to 0} |u(r,\theta,t)| < \infty, & u(r,\theta + 2\pi, t) = u(r,\theta,t), \\ u(r,\theta,0) = \Phi(r,\theta), u_t(r,\theta,0) = \Psi(r,\theta), \; 0 < r < R \end{cases} \tag{2.51}$$

is

$$u = \frac{\partial}{\partial t}W_\Phi + W_\Psi(r,\theta,t) + \int_0^t W_{F_\tau}(r,\theta,t-\tau)\,\mathrm{d}\tau,$$

where $u_2 = W_\Psi(r,\theta,t)$ is the solution for the case of $\Phi = F = 0, F_\tau = F(r,\theta,\tau)$, and

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1\partial}{r\partial r} + \frac{1\partial^2}{r^2\partial\theta^2}.$$

Also note that $L(u,u_r)|_{r=R} = 0$ covers all three cases of boundary conditions.

Proof

1. Since $W_{\Phi}(r,\theta,t)$ satisfies

$$\begin{cases} \dfrac{\partial^2 W_{\Phi}}{\partial t^2} - a^2 \Delta W_{\Phi} = 0, \quad L\left(W_{\Phi}, \dfrac{\partial W_{\Phi}}{\partial r}\right)\Big|_{r=R} = 0, \\ W_{\Phi}(r,\theta,0) = 0, \dfrac{\partial}{\partial t} W_{\Phi}(r,\theta,0) = \Phi(r,\theta). \end{cases}$$

We have

$$\frac{\partial^2 u_1}{\partial t^2} - a^2 \Delta u_1 = \frac{\partial}{\partial t}\left(\frac{\partial^2 W_{\Phi}}{\partial t^2} - a^2 \Delta W_{\Phi}\right) = 0,$$

$$L(u_1, \frac{\partial u_1}{\partial r})\Big|_{r=R} = \frac{\partial}{\partial t}\left[L\left(W_{\Phi}, \frac{\partial W_{\Phi}}{\partial r}\right)\Big|_{r=R}\right] = 0,$$

$$u_1(r,\theta,0) = \frac{\partial}{\partial t} W_{\Phi}(r,\theta,0) = \Phi(r,\theta),$$

$$\frac{\partial}{\partial t} u_1(r,\theta,0) = \frac{\partial^2}{\partial t^2} W_{\Phi}(r,\theta,0) = a^2 \Delta W_{\Phi}(r,\theta,0) = 0.$$

Therefore $u_1 = \dfrac{\partial W_{\Phi}}{\partial t}$ is the solution for the case of $\Psi = F = 0$.

2. Since $W_{F_\tau}(r,\theta,t-\tau)$ satisfies

$$\begin{cases} \dfrac{\partial^2 W_{F_\tau}}{\partial t^2} - a^2 \Delta W_{F_\tau} = 0, \; L(W_{F_\tau}, \dfrac{\partial W_{F_\tau}}{\partial r})\Big|_{r=R} = 0, \\ W_{F_\tau}|_{t=\tau} = 0, \; \dfrac{\partial W_{F_\tau}}{\partial t}\Big|_{t=\tau} = F(r,\theta,\tau), \end{cases}$$

we have

$$\begin{aligned} &\frac{\partial^2 u_3}{\partial t^2} - a^2 \Delta u_3 \\ &= \frac{\partial}{\partial t}\left(\int_0^t \frac{\partial W_{F_\tau}}{\partial t} d\tau + W_{F_\tau}(r,\theta,t-\tau)|_{\tau=t}\right) - a^2 \int_0^t \Delta W_{F_\tau} d\tau \\ &= \int_0^t \frac{\partial^2 W_{F_\tau}}{\partial t^2}\,\mathrm{d}\tau + \frac{\partial W_{F_\tau}}{\partial t}\Big|_{\tau=t} - \int_0^t a^2 \Delta W_{F_\tau}\, d\tau \\ &= \int_0^t \left(\frac{\partial^2 W_{F_\tau}}{\partial t^2} - a^2 \Delta W_{F_\tau}\right) \mathrm{d}\tau + \frac{\partial W_{F_\tau}}{\partial t}\Big|_{\tau=t} = F(r,\theta,t), \end{aligned}$$

$$L\left(u_3, \frac{\partial u_3}{\partial r}\right)\bigg|_{r=R} = \int_0^t L\left(W_{F_\tau}, \frac{\partial W_{F_\tau}}{\partial r}\right)\bigg|_{r=R} \mathrm{d}\tau = 0\,,$$

$$u_3(r, \theta, 0) = 0\,,$$

$$\frac{\partial}{\partial t} u_3(r, \theta, 0) = \frac{\partial}{\partial t} \int_0^t W_{F_\tau} \mathrm{d}\tau \bigg|_{t=0} = \left(\int_0^t \frac{\partial W_{F_\tau}}{\partial t} \mathrm{d}\tau + W_{F_\tau}|_{\tau=t}\right)\bigg|_{t=0} = 0\,.$$

Thus $u_3 = \int_0^t W_{F_\tau}(r, \theta, t-\tau)\,\mathrm{d}\tau$ is the solution for the case of $\Phi = \Psi = 0$.

3. Since PDS (2.51) is linear, the principle of superposition is valid, so

$$u = \frac{\partial}{\partial t} W_\Phi + W_\Psi(r, \theta, t) + \int_0^t W_{F_\tau}(r, \theta, t-\tau)\,\mathrm{d}\tau$$

is the solution of PDS (2.51).

Therefore, the solution structure theorem is also valid in a polar coordinate system.

2.6 Three-Dimensional Mixed Problems

The Fourier method and the method of separation of variables work for mixed problems only in some regular domains. In this section, we apply them to solve mixed problems in cuboid and spherical domains.

2.6.1 Cuboid Domain

Consider

$$\begin{cases} u_{tt} = a^2 \Delta u + f(x, y, z, t), \quad \Omega \times (0, +\infty) \\ L(u, u_x, u_y, u_z)\big|_{\partial\Omega} = 0\,, \\ u(x, y, z, 0) = \varphi(x, y, z), \;\; u_t(x, y, z, 0) = \psi(x, y, z)\,, \end{cases} \tag{2.52}$$

where Ω stands for a cuboid domain: $0 < x < a_1,\ 0 < y < b_1,\ 0 < z < c_1,\ \partial\Omega$ is the boundary of Ω, and $t \in (0, \infty)$. If all combinations of the boundary conditions of the first, the second and the third kinds are considered, for a finite cuboid domain D, there exist 729 combinations of linear boundary conditions $L(u, u_x, u_y, u_z)\big|_{\partial\Omega} = 0$. We solve PDS (2.52) for the case of

$$\begin{cases} u(0, y, z, t) = u(a_1, y, z, t) = 0\,, \\ u_y(x, 0, z, t) = u_y(x, b_1, z, t) + h_2 u(x, b_1, z, t) = 0\,, \\ u_z(x, y, 0, t) - h_1 u(x, y, 0, t) = u(x, y, c_1, t) = 0\,. \end{cases} \tag{2.53}$$

The results for the remaining 728 combinations may be readily obtained using a similar approach, with Table 2.1and the solution structure theorem.

By the solution structure theorem, we first develop $u = W_\psi(x,y,z,t)$, the solution for the case of $f = \varphi = 0$. Based on the given boundary conditions (2.53), we should use the eigenfunctions in Rows 1,6 and 7 in Table 2.1 to expand the solution

$$u(x,y,z,t) = \sum_{m,n,l=1}^{+\infty} T_{mnl}(t) \sin\frac{m\pi x}{a_1} \cos\frac{\mu_n y}{b_1} \sin\left(\frac{\gamma_l z}{c_1} + \varphi_l\right),$$

where μ_n and γ_l are the positive zero points of $f(y) = \cot y - \dfrac{y}{b_1 h_2}$ and $g(z) = \tan z + \dfrac{z}{c_1 h_1}$, respectively, and $\tan\varphi_l = \dfrac{\gamma_l}{c_1 h_1}$. Substituting this into the wave equation in PDS (2.52) and comparing the coefficients yields

$$T''_{mnl}(t) + \omega_{mnl}^2 T_{mnl}(t) = 0,$$

where $\omega_{mnl}^2 = a^2\left[\left(\dfrac{m\pi}{a_1}\right)^2 + \left(\dfrac{\mu_n}{b_1}\right)^2 + \left(\dfrac{\gamma_l}{c_1}\right)^2\right]$. The general solution of this equation reads

$$T_{mnl}(t) = a_{mnl}\cos\omega_{mnl}t + b_{mnl}\sin\omega_{mnl}t.$$

Thus $\quad u = \sum\limits_{m,n,l=1}^{+\infty} (a_{mnl}\cos\omega_{mnl}t + b_{mnl}\sin\omega_{mnl}t)\sin\dfrac{m\pi x}{a_1}\cos\dfrac{\mu_n y}{b_1}\sin\left(\dfrac{\gamma_l z}{c_1} + \varphi_l\right)$.
Applying the initial condition $u(x,y,z,0) = 0$ yields $a_{mnl} = 0$. The b_{mnl} can be determined by the initial condition $u_t(x,y,z,0) = \psi(x,y,z)$. Finally, we have

$$\begin{cases} u = W_\psi(x,y,z,t) = \displaystyle\sum_{m,n,l=1}^{+\infty} b_{mnl}\sin\frac{m\pi x}{a_1}\cos\frac{\mu_n y}{b_1}\sin\left(\frac{\gamma_l z}{c_1} + \varphi_l\right)\sin\omega_{mnl}t, \\ b_{mnl} = \dfrac{1}{\omega_{mnl}M_{mnl}} \displaystyle\iiint_\Omega \psi(x,y,z)\sin\frac{m\pi x}{a_1}\cos\frac{\mu_n y}{b_1}\sin\left(\frac{\gamma_l z}{c_1} + \varphi_l\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \end{cases}$$

where $M_{mnl} = M_m M_n M_l$, M_m, M_n and M_l are the normal squares of the three eigenfunction sets, respectively.

The solution of PDS (2.52) under the boundary conditions (2.53) follows straightforwardly from the solution structure theorem

$$u = \frac{\partial}{\partial t}W_\varphi + W_\psi(x,y,z,t) + \int_0^t W_{f_\tau}(x,y,z,t-\tau)\,\mathrm{d}\tau,$$

where $f_\tau = f(x,y,z,\tau)$.

2.6.2 Spherical Domain

For a spherical domain, the boundary conditions are not separable for x, y and z in a Cartesian coordinate system, like in the case of a circular domain in Section 2.5.2. A spherical coordinate transformation of

$$\begin{cases} x = r\sin\theta\cos\varphi\,, & 0 \le \varphi \le 2\pi\,, \\ y = r\sin\theta\sin\varphi\,, & 0 \le \theta \le \pi\,, \\ z = r\cos\theta\,, & 0 < r \le a_0 \end{cases}$$

is required before applying the method of separation of variables. In the spherical coordinate system, PDS (2.52) reads

$$\begin{cases} u_{tt} = a^2 \Delta u(r,\theta,\varphi,t) + F(r,\theta,\varphi,t) \\ \qquad 0 < \theta < \pi\,, 0 < r < a_0, 0 < t\,, \\ \qquad |u(0,\theta,\varphi,t)| < \infty, \\ L(u,u_r)|_{r=a_0} = 0\,, u(r,\theta,\varphi+2\pi,t) = u(r,\theta,\varphi,t), \\ u(r,\theta,\varphi,0) = \Phi(r,\theta,\varphi), u_t(r,\theta,\varphi,0) = \Psi(r,\theta,\varphi), \end{cases} \tag{2.54}$$

where the Laplacian is

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\,.$$

It can be shown by following a similar approach to that in Section 2.5.2 that the solution structure theorem is also valid in a spherical coordinate system. We thus focus on seeking $u = W_\Psi(r,\theta,\varphi,t)$, the solution for the case $\Phi = F = 0$.

Separation of Variables

Assume a solution of type $u = v(r,\theta,\varphi)T(t)$. Substituting it into the wave equation in PDS (2.54) and denoting the separation constant $-k^2$, we obtain

$$\begin{cases} \Delta v + k^2 v = 0\,, \ v(r,\theta,\varphi+2\pi) = v(r,\theta,\varphi)\,, \\ |v(0,\theta,\varphi)| < \infty, \ 0 < r < a_0, \ 0 < \theta < \pi\,, \end{cases} \tag{2.55}$$

$$T''(t) + (ka)^2\,T(t) = 0\,.$$

Assume $v = R(r)Y(\theta,\varphi)$. Substituting it into (2.55) yields

$$\begin{cases} r^2R''(r)+2rR'(r)+\left[k^2r^2-l(l+1)\right]R(r)=0\,,|R(0)|<\infty\,, \\ \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial Y}{\partial\theta}\right)+\dfrac{1}{\sin^2\theta}\dfrac{\partial^2 Y}{\partial\varphi^2}+l(l+1)Y=0\,, \\ Y(\theta,\varphi+2\pi)=Y(\theta,\varphi)\,, \end{cases} \tag{2.56}$$

in which the separation constant is customarily denoted by $l(l+1)$.

Assume $Y = \Theta(\theta)\,\Phi(\varphi)$. Substituting it into Eq. (2.56) leads to

$$\begin{cases} \Phi''+\eta\Phi=0\,,\;\; \Phi(\varphi+2\pi)=\Phi(\varphi)\,, & (2.57) \\ \Theta''+(\cot\theta)\,\Theta'+\left[l(l+1)-\dfrac{\eta}{\sin^2\theta}\right]\Theta=0\,, & \\ \quad 0<\theta<\pi \quad |\Theta(\theta)|<\infty\,, & (2.58) \end{cases}$$

where η is the separation constant.

Solution of Eigenvalue Problems

1. The eigenvalue problem (2.57) can be very easily solved. The results are:

$$\begin{aligned} &\text{Eigenvalues} && \eta=m^2\,,\; m=0,1,2,\cdots; \\ &\text{Eigenfunctions} && \Phi_m(\varphi)=c_m^{(1)}\cos m\varphi+c_m^{(2)}\sin m\varphi\,. \end{aligned}$$

   Here $c_m^{(1)}$ and $c_m^{(2)}$ are arbitrary constants that are not all equal to zero. Also, $c_0^{(1)}\neq 0$. $\Phi_0(\varphi)=1$ for $\eta=0$.
2. With $\eta=m^2$, Eq. (2.58) forms an eigenvalue problem of the Legendre equation. Its solution is available in Appendix A.
   When $m=0$,

$$\begin{aligned} &\text{Eigenvalues} && l(l+1)=n(n+1)\,,\; n=0,1,2,\cdots, \\ &\text{Eigenfunctions} && \Theta_n(\theta)=P_n(\cos\theta)\,,\; P_n(x)\text{ is the } \textit{Legendre polynomial} \\ &&& \textit{of degree } n\,. \end{aligned}$$

When $m = 1, 2, \cdots$,

$$\text{Eigenvalues} \quad l(l+1) = n(n+1)\,,\ n = 1, 2, \cdots\,,\ m \leq n\,;$$

$$\text{Eigenfunctions} \quad \Theta_n^m(\theta) = P_n^m(\cos\theta)\,,\ P_n^m(x) \text{ is the } \textit{Associated Legendre polynomial of degree n and order } m\,.$$

3. With $l = n = 0, 1, 2, \cdots$, the equation of $R(r)$ in (2.56), together with $|R(0)| < \infty$, $|R'(0)| < \infty$ and $L(R(r), R'(r))|_{r=a_0} = 0$ forms another eigenvalue problem. To solve this problem, introduce a new variable x and a new function $y(x)$ defined by $x = kr$, $y(x) = x^{\frac{1}{2}}R$. The equation of $R(r)$ in (2.56) is thus transformed into

$$x^2y'' + xy' + \left[x^2 - \left(n + \frac{1}{2}\right)^2\right] y = 0\,,$$

which is a Bessel equation. Its solution reads

$$R_n(r) = \frac{1}{\sqrt{kr}}\left[c_n^{(1)} J_{n+\frac{1}{2}}(kr) + c_n^{(2)} J_{-\left(n+\frac{1}{2}\right)}(kr)\right]\,,$$

where $c_n^{(1)}$ and $c_n^{(2)}$ are constants that are not all zero. To satisfy $|R(0)| < \infty$, we have $c_n^{(2)} = 0$ and $c_n^{(1)} \neq 0$. Therefore, without taking account of a constant factor,

$$R_n(r) = \sqrt{\frac{\pi}{2kr}} J_{n+\frac{1}{2}}(kr) = j_n(kr)\,.$$

Here $j_l(x) = \sqrt{\dfrac{\pi}{2x}} J_{l+\frac{1}{2}}(x)$ is the *spherical Bessel function of the first kind*. The eigenvalues can be readily obtained by substituting the boundary conditions $R_n(a_0) = 0$ (first kind), $R_n'(a_0) = 0$ (second kind) or $R_n'(a_0) + hR_n(a_0) = 0$ (third kind) into it. The final results are:

$$\text{Eigenvalues} \quad \lambda = k_{nl}^2 = \left(\mu_l^{\left(n+\frac{1}{2}\right)} / a_0\right)^2\,,\ n = 0, 1, 2, \cdots\,,\ l = 1, 2, \cdots$$

$$\text{Eigenfunctions} \quad j_n(k_{nl}r) = j_n\left(\mu_l^{\left(n+\frac{1}{2}\right)} r / a_0\right)\,.$$

Here $\mu_l^{\left(n+\frac{1}{2}\right)}$ are the positive zero-points of $J_{n+\frac{1}{2}}(x)$, $xJ'_{n+\frac{1}{2}}(x) - \frac{1}{2}J_{n+\frac{1}{2}}(x)$ and $xJ'_{n+\frac{1}{2}}(x) + \left(ha_0 - \frac{1}{2}\right) J_{n+\frac{1}{2}}(x)$ for boundary conditions of the first kind, the second kind and the third kind, respectively.

Generalized Fourier Expansion of $\mathbf{u} = \mathbf{W}_{\Psi}(\mathbf{r}, \theta, \varphi, \mathbf{t})$

With $k_{nl} = \mu_l^{\left(n+\frac{1}{2}\right)}/a_0$, the $T(t)$-equation becomes

$$T''_{nl} + \omega_{nl}^2 T_{nl} = 0,$$

where $\omega_{nl}^2 = \left(a\mu_l^{\left(n+\frac{1}{2}\right)}/a_0\right)^2$. Its general solution reads

$$T_{nl}(t) = c_{nl}^{(1)} \cos \omega_{nl} t + c_{nl}^{(2)} \sin \omega_{nl} t\,.$$

Therefore

$$u = \sum_{m,n=0,l=1}^{\infty} \{(a_{mnl} \cos \omega_{nl} t + b_{mnl} \sin \omega_{nl} t) \cos m\varphi$$
$$+(c_{mnl} \cos \omega_{nl} t + d_{mnl} \sin \omega_{nl} t) \sin m\varphi\}\, P_n^m(\cos\theta)\, j_n\left(\frac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right)$$

satisfies the wave equation and the boundary conditions of PDS (2.54).

Rewriting Eqs. (2.56) and (2.58) in the form of Eq. (2.48) shows that the eigenfunction sets are orthogonal with respect to weight functions r^2 and $\sin\theta$, respectively.

Applying the initial condition $u(r,\theta,\varphi,0) = 0$ leads to $a_{mnl} = c_{mnl} = 0$. To satisfy the initial condition $u_t(r,\theta,\varphi,0) = \Psi(r,\theta,\varphi)$, b_{mnl} and d_{mnl} must be determined such that

$$\Psi(r,\theta,\varphi) = \sum_{m,n=0,l=1}^{\infty} (b_{mnl}\omega_{nl} \cos m\varphi + d_{mnl}\omega_{nl} \sin m\varphi)\, P_n^m(\cos\theta)\, j_n\left(\frac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right).$$

Thus, we obtain

$$\begin{cases} u = W_\Psi(r,\theta,\varphi,t) = \displaystyle\sum_{m,n=0,l=1}^{+\infty} (b_{mnl}\cos m\varphi + d_{mnl}\sin m\varphi) \\ \qquad \times P_n^m(\cos\theta)\, j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right)\sin\omega_{nl}t\,, \\ b_{0nl} = \dfrac{1}{2\pi\omega_{nl}M_{0nl}} \displaystyle\iiint\limits_{r\le a_0} \Psi(r,\theta,\varphi)P_n(\cos\theta)\, j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right) \\ \qquad \times r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,, \\ b_{mnl} = \dfrac{1}{\pi\omega_{nl}M_{mnl}} \displaystyle\iiint\limits_{r\le a_0} \Psi(r,\theta,\varphi)P_n^m(\cos\theta)\, j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right) \\ \qquad \times r^2 \cos m\varphi \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,, \\ d_{mnl} = \dfrac{1}{\pi\omega_{nl}M_{mnl}} \displaystyle\iiint\limits_{r\le a_0} \Psi(r,\theta,\varphi)P_n^m(\cos\theta)\, j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right) \\ \qquad \times r^2 \sin m\varphi \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,, \end{cases} \tag{2.59}$$

where $M_{mnl} = M_{mn}M_{nl}$, $M_{mn} = \displaystyle\int_0^\pi [P_n^m(\cos\theta)]^2 \sin\theta\,\mathrm{d}\theta = \dfrac{(n+m)!}{(n-m)!}\dfrac{2}{2n+1}$ is the normal square of $\left\{P_n^m(\cos\theta)\right\}$, M_{nl} is the normal square of $\left\{ j_n\left(\mu_l^{\left(n+\frac{1}{2}\right)} r/a_0\right)\right\}$ $(n \text{ fixed}\,,\ l = 1,2,\cdots)$ and can be determined by Eq. (2.47) in Section 2.5 as

$$M_{nl} = \frac{\pi a_0^3}{4\mu_l^{\left(n+\frac{1}{2}\right)}} \begin{cases} \left[J_{n+\frac{3}{2}}\left(\mu_l^{\left(n+\frac{1}{2}\right)}\right)\right]^2, & \text{Boundary condition of the first kind}\,, \\ \left[1 - \dfrac{n(n+1)}{\left(\mu_l^{\left(n+\frac{1}{2}\right)}\right)^2}\right] J_{n+\frac{1}{2}}^2\left(\mu_l^{\left(n+\frac{1}{2}\right)}\right), & \text{Boundary condition of the second kind}\,, \\ \left[1 + \dfrac{(ha_0+n)(ha_0-n-1)}{\left(\mu_l^{\left(n+\frac{1}{2}\right)}\right)^2}\right] J_{n+\frac{1}{2}}^2\left(\mu_l^{\left(n+\frac{1}{2}\right)}\right), & \text{Boundary condition of the third kind}\,. \end{cases}$$

Finally, the solution of PDS (2.54) follows from the solution structure theorem,

$$u = \frac{\partial}{\partial t} W_\Phi + W_\Psi(r,\theta,\varphi,t) + \int_0^t W_{F_\tau}(r,\theta,\varphi,t-\tau)\,\mathrm{d}\tau\,,$$

where $F_\tau = F(r,\theta,\varphi,\tau)$.

Remark 1. We can apply the Fourier method of expansion to easily solve 27 mixed problems in a cylindrical domain by using the eigenfunctions in Table 2.1 and in Sect. 2.5.2.

Remark 2. Mixed problems of wave equations can also be solved using the method of Laplace transformation, which is discussed in Appendix B.

2.7 Methods of Solving One-Dimensional Cauchy Problems

Clearly, the solution structure theorem is also valid for Cauchy problems. In this section, we consider the PDS

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \quad -\infty < x < +\infty, 0 < t\,, \\ u(x,0) = \varphi(x)\,,\ u_t(x,0) = \psi(x)\,. \end{cases} \tag{2.60}$$

In particular, we discuss methods of solving PDS (2.60) and examine properties of the solution.

2.7.1 Method of Fourier Transformation

The Fourier transformation and the Laplace transformation are two important integral transformations and find applications in a variety of fields. Readers are referred to Appendix B for a discussion of these two transformations. Here we apply the Fourier transformation to solve PDS. This approach is called the *method of Fourier transformation* and different from the Fourier method of expansion discussed before.

The solution structure theorem has reduced the problem of finding a solution of PDS (2.60) to seeking $W_\psi(x,t)$, the solution for the case $\varphi = f = 0$. Applying a Fourier transformation with respect to x to $u(x,t)$ in PDS (2.60) yields

$$F\left[a^2 u_{xx}\right] = a^2 (i\omega)^2 \bar{u}(\omega,t) = -(\omega a)^2 \bar{u}(\omega,t),$$

i.e.

$$\bar{u}_{tt}(\omega,t) + (\omega a)^2 \bar{u}(\omega,t) = 0\,,$$

where $F[u_{tt}] = \bar{u}_{tt}(\omega,t)$ has been used. The solution of the above equation is

$$\bar{u}(\omega,t) = A(\omega)\cos\omega at + B(\omega)\sin\omega at\,,$$

$A(\omega)$ and $B(\omega)$ can be determined by initial conditions $\bar{u}(\omega,0) = 0$ and $\bar{u}_t(\omega,0) = \bar{\psi}(\omega)$ as $A(\omega) = 0$, $B(\omega) = \dfrac{\bar{\psi}(\omega)}{\omega a}$. Thus

$$\bar{u}(\omega,t) = \frac{\bar{\psi}(\omega)}{\omega a}\sin\omega at = \frac{1}{2a}\left(\frac{\bar{\psi}(\omega)}{i\omega}\,\mathrm{e}^{\mathrm{i}\omega at} - \frac{\bar{\psi}(\omega)}{i\omega}\,\mathrm{e}^{-\mathrm{i}\omega at}\right).$$

Using the integral property and the shifting property of Fourier transformations and taking an inverse transformation leads to

$$u(x,t) = W_\psi(x,t) = \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi\,.$$

The solution of PDS (2.60) thus follows from the solution structure theorem

$$u = \frac{\partial}{\partial t}W_\varphi + W_\psi(x,t) + \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau\,, \tag{2.61}$$

where $f_\tau = f(x,\tau)$. Since

$$\frac{\partial W_\varphi}{\partial t} = \frac{1}{2a}\frac{\partial}{\partial t}\int_{x-at}^{x+at}\varphi(\xi)\,\mathrm{d}\xi = \frac{\varphi(x+at)+\varphi(x-at)}{2}\,,$$

the solution of

$$\begin{cases} u_{tt} = a^2u_{xx}, & -\infty < x < +\infty, 0 < t\,, \\ u(x,0) = \varphi(x)\,,\ u_t(x,0) = \psi(x) \end{cases}$$

reads

$$u(x,t) = \frac{\varphi(x+at)+\varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi\,. \tag{2.62}$$

This is called the *D'Alembert formula of one-dimensional wave equation*.

The solution of

$$\begin{cases} u_{tt} = a^2u_{xx} + f(x,t), & -\infty < x < +\infty, 0 < t\,, \\ u(x,0) = 0,\ u_t(x,0) = 0 \end{cases}$$

is

$$u(x,t) = \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau\,, \tag{2.63}$$

which is called the *Kirchhoff formula of one-dimensional wave equation*.

2.7.2 Method of Characteristics

The characteristic curves of the wave equation $u_{tt} = a^2 u_{xx}$ are (Sect. 1.1.5)

$$x + at = c_1 , \quad x - at = c_2 .$$

A variable transformation

$$\xi = x + at , \quad \eta = x - at$$

reduces $u_{tt} = a^2 u_{xx}$ into $u_{\xi\eta} = 0$. A twice integration with respect to ξ and η, respectively, leads to

$$u = f(\xi) + g(\eta) = f(x + at) + g(x - at) \tag{2.64}$$

where f and g are differentiable functions. Applying initial conditions $u(x,0) = 0$ and $u_t(x,0) = \psi(x)$ yields

$$\begin{cases} f(x) + g(x) = 0, \\ f'(x) - g'(x) = \dfrac{1}{a}\psi(x), \quad \text{or} \\ f(x) - g(x) = \dfrac{1}{a}\displaystyle\int_{x_0}^{x} \psi(\xi)\,\mathrm{d}\xi + C, \end{cases}$$

where $C = f(x_0) - g(x_0)$ and x_0 is an arbitrary point. Therefore

$$f(x) = \frac{1}{2a}\int_{x_0}^{x} \psi(\xi)\,\mathrm{d}\xi + \frac{C}{2}, \quad g(x) = -\frac{1}{2a}\int_{x_0}^{x} \psi(\xi)\,\mathrm{d}\xi - \frac{C}{2}.$$

Consequently, the solution of PDS (2.60) for the case of $\varphi = f = 0$ is

$$u(x,t) = W_{\psi}(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi .$$

Finally, the solution structure theorem shows that the solution of PDS (2.60) is Eq. (2.61).

Note. It can be shown that the $u(x,t)$ in Eq. (2.62) satisfies the equation $u_{tt} = a^2 u_{xx}$ and the initial conditions $\varphi(x) \in C^2(-\infty, +\infty)$, $\psi(x) \in C^1(-\infty, +\infty)$, so it is a solution. The demand for the smoothness of $\varphi(x)$ and $\psi(x)$ by Eq. (2.62) itself is quite weak, and the D'Alembert formula still works for those $\varphi(x)$ and $\psi(x)$ not satisfying the above conditions. Therefore, the solution in Eq. (2.62) is called a *generalized solution* or a *weak solution*. The former [the one satisfying $\varphi(x) \in C^2(-\infty, +\infty)$ and $\psi(x) \in C^1(-\infty, +\infty)$] is called the *classical solution*.

The solution (2.62) is also unique because it comes rigorously from the technique of integral transformation and characteristic curves based on the initial conditions. It is also stable with respect to initial conditions in the sense of uniform approximation.

To demonstrate this, consider two groups of initial values

$$\begin{cases} u_1(x,0) = \varphi_1(x), \\ \dfrac{\partial}{\partial t} u_1(x,0) = \psi_1(x) \end{cases} \quad \text{and} \quad \begin{cases} u_2(x,0) = \varphi_2(x), \\ \dfrac{\partial}{\partial t} u_2(x,0) = \psi_2(x), \end{cases}$$

where, for a small positive constant δ,

$$\max_{-\infty<x<+\infty} |\varphi_1(x) - \varphi_2(x)| < \delta, \quad \max_{-\infty<x<+\infty} |\psi_1(x) - \psi_2(x)| < \delta.$$

By Eq. (2.62), the difference between the two corresponding solutions u_1 and u_2 satisfies

$$\begin{aligned} |u_1(x,t) - u_2(x,t)| &\leq \frac{\delta}{2} + \frac{\delta}{2} + \frac{1}{2a} \int_{x-at}^{x+at} |\psi_1(\xi) - \psi_2(\xi)| \, \mathrm{d}\xi \\ &\leq \delta + \frac{1}{2a} \delta 2at = (1+t)\delta. \end{aligned}$$

Therefore, $\forall \varepsilon > 0$, we can always find a time instant $t = t_0$ satisfying $\delta = \varepsilon/(1+t_0)$ such that $|u_1 - u_2| < \varepsilon$.

2.7.3 Physical Meaning

To understand each term in Eq. (2.62), take $u(x,t)$ as the displacement of a vibrating string. For any fixed time instant t_0, $u = u(x,t_0) = g_1(x)$ represents the spatial distribution of displacement. Its graphic representation shows the string shape at t_0, i.e. the wave shape. For any fixed point x_0, $u = u(x_0,t) = g_2(t)$ illustrates the temporal distribution of displacement at x_0. Let

$$\Phi(x) = \frac{\varphi(x)}{2}, \quad \Psi'(x) = \frac{\psi(x)}{2a}. \tag{2.65}$$

Equation (2.62) becomes

$$\begin{aligned} u(x,t) &= [\Phi(x+at) + \Phi(x-at)] + [\Psi(x+at) + (-\Psi(x-at))] \\ &= u_1(x,t) + u_2(x,t). \end{aligned}$$

Here $\Phi(x \pm at)$ and $\Psi(x \pm at)$ represent the periodic prolongation of $\varphi(x)$ and $\psi(x)$, respectively. Since at has the dimension of a displacement, the function $u = \dfrac{\varphi(x-at)}{2}$ represents the $u = \dfrac{\varphi(x)}{2}$ (the wave shape at $t = 0$) displaced at units with a wave speed a to the right. $u = \dfrac{\varphi(x-at)}{2}$ is thus a forward wave. Similarly $\dfrac{\varphi(x+at)}{2}$ represents a backward wave. Therefore we have:

$\Phi(x+at)$: backward wave generated by initial displacement $\varphi(x)$,
$\Phi(x-at)$: forward wave generated by initial displacement $\varphi(x)$,
$\Psi(x+at)$: backward wave generated by initial velocity $\psi(x)$,
$-\Psi(x-at)$: forward wave generated by initial velocity $\psi(x)$,
$u_1(x,t)$: superposition of forward and backward waves generated by the initial displacement $\varphi(x)$,
$u_2(x,t)$: superposition of forward and backward waves generated by initial velocity $\psi(x)$,
$u(x,t)$: superposition of constant waves generated by initial displacement and velocity.

The general solution (2.64) from the method of characteristics is, in fact, a superposition of forward and backward waves that are fixed by initial conditions. This is the basis for also calling the method of characteristics the *method of traveling waves*.

Example. Consider

$$\begin{cases} u_{tt} = 4u_{xx}\,, \quad -\infty < x < +\infty, 0 < t\,, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)\,. \end{cases} \tag{2.66}$$

1. Draw the wave shape at $t = \frac{1}{2}$

$$\text{when } \varphi(x) = \begin{cases} 0, -\infty < x < 0\,, \\ x, 0 \leq x < 2\,, \\ 2, 2 \leq x < 3\,, \\ -x+5, 3 \leq x < 5\,, \\ 0, 5 \leq x < +\infty \end{cases} \quad \text{and } \psi(x) = 0\,.$$

2. For $\varphi(x) = 0$ and $\psi(x) = \begin{cases} \psi_0\,,\ x_1 \leq x \leq x_2\,, \\ 0, \quad x < x_1\,, x_2 < x \end{cases}$ show $\Psi(x)$ graphically and state the procedure of drawing wave shapes at $t = t_1\,, t_2\,, t_3$ and $t_4 (t_1 < t_2 < t_3 < t_4)$.

Solution.

1. Since the wave speed is 2, the sequence of drawing the wave shape is

$$\varphi(x) \Rightarrow \frac{\varphi(x)}{2} \Rightarrow \frac{\varphi(x-1)}{2}, \frac{\varphi(x+1)}{2} \Rightarrow u = \frac{\varphi(x+1)+\varphi(x-1)}{2}\,.$$

This is shown in Fig. 2.1. The wave shape at $t = \dfrac{1}{2}$ is shown in the last figure.

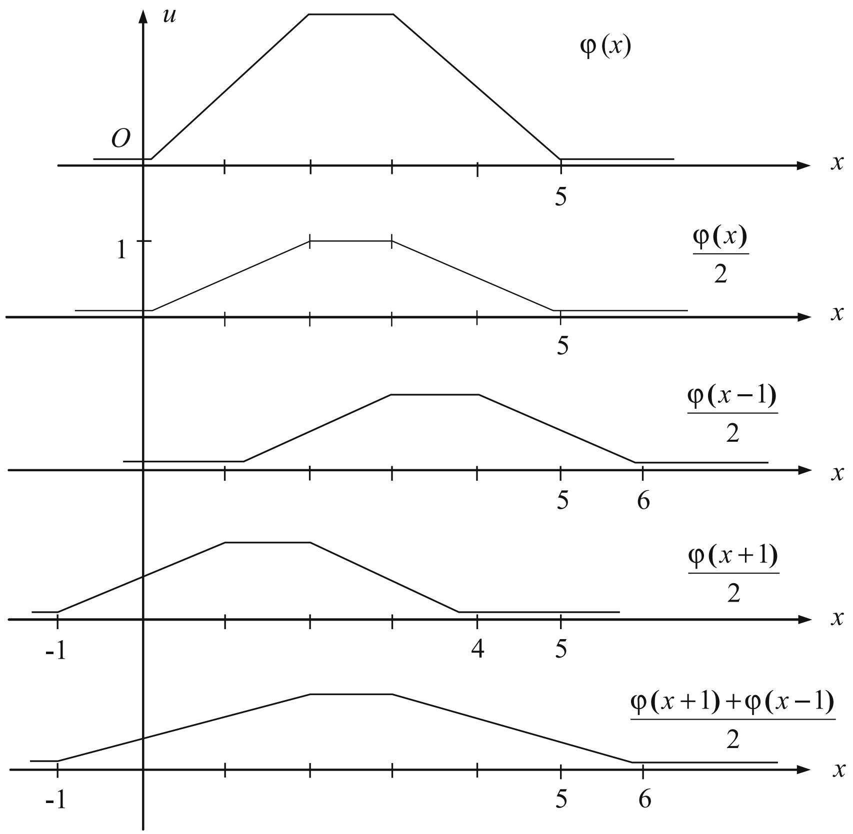

Fig. 2.1 Wave shapes

2. By (2.65), $\Psi(x) = \dfrac{1}{4}\displaystyle\int_{x_0}^{x} \psi(\xi)\,\mathrm{d}\xi\,, \quad x_0 \le x_1$. Thus

$$\Psi(x) = \begin{cases} 0\,, & x < x_1\,, \\ \frac{1}{4}(x - x_1)\psi_0\,, & x_1 < x < x_2\,, \\ \frac{1}{4}(x_2 - x_1)\psi_0\,, & x_2 \le x\,. \end{cases}$$

The solution of PDS (2.66) reads

$$u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} \psi(\xi)\,\mathrm{d}\xi = \Psi(x+2t) - \Psi(x-2t)\,.$$

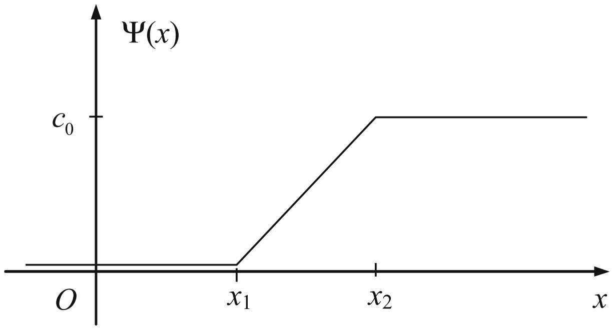

Fig. 2.2 Function $\Psi(x)$

The drawing procedure is

$$\begin{aligned}\Psi(x), -\Psi(x) &\Rightarrow u(x,t_1) = \Psi(x+2t_1) - \Psi(x-2t_1)\\ &\Rightarrow u(x,t_2) = \Psi(x+2t_2) - \Psi(x-2t_2)\\ &\Rightarrow u(x,t_3) = \Psi(x+2t_3) - \Psi(x-2t_3)\\ &\Rightarrow u(x,t_4) = \Psi(x+2t_4) - \Psi(x-2t_4).\end{aligned}$$

$\Psi(x)$ is graphically shown in Fig. 2.2, where $c_0 = \dfrac{1}{4}(x_2 - x_1)\psi_0$.

2.7.4 Domains of Dependence, Determinacy and Influence

Domain of Dependence

Let D be a semi-infinite plane: $-\infty < x < +\infty,\ 0 < t$. We note, by the D'Alembert formula, that at any point $(x_0,t_0) \in D$, the value $u(x_0,t_0)$ depends only on the initial values $\varphi(x)$ and $\psi(x)$ at points between $x_0 - at_0$ and $x_0 + at_0$ and not on initial values outside of this range. This interval $[x_0 - at_0, x_0 + at_0]$ is called the *domain of dependence* of point (x_0,t_0). $x_1 = x_0 - at_0$ and $x_2 = x_0 + at_0$ are the abscissas of two intersecting points of Ox-axis and two characteristic curves $x - x_0 = \pm a(t - t_0)$ that are passing though point (x_0,t_0).

Domain of Determinacy

For any interval $[x_1,x_2]$ on the Ox-axis, consider a triangle in D formed by $[x_1,x_2]$, the characteristic curve $x = x_1 + at$ passing though point $(x_1,0)$ and the characteristic curve $x = x_2 - at$ passing though point$(x_2,0)$, Fig. 2.3. For any point (x_0,t_0) inside the triangle, its domain of dependence is always in $[x_1,x_2]$, so that the value of $u(x,t)$

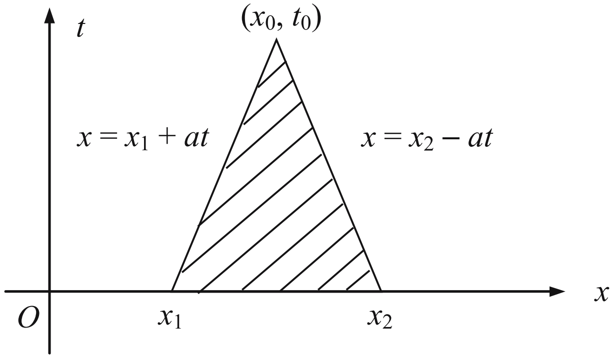

Fig. 2.3 Domain of determinacy

at any point inside the triangle depends only on the initial values in $[x_1, x_2]$ and not on initial values outside $[x_1, x_2]$. Therefore the initial values in $[x_1, x_2]$ completely determine values of $u(x,t)$ inside this triangle. Such a triangle region in D is called the *domain of determinacy* of interval $[x_1, x_2]$.

Domain of Influence

To illustrate the propagation in D of initial disturbances $\varphi(x)$ and $\psi(x)$ in $[x_1, x_2]$, consider initial disturbances such that both $\varphi(x)$ and $\psi(x)$ are vanished outside of $[x_1, x_2]$ and non-zero inside of $[x_1, x_2]$.

Let D' be the region: $x_1 - at \leq x \leq x_2 + at$, $0 < t$ (Fig. 2.4). For any point (x_0, t_0) outside of D', its domain of dependence is also outside of $[x_1, x_2]$ so that the effect of the initial disturbances has not propagated to this point yet. For any point (x_0, t_0) inside of D', however, its domain of dependence has always some parts in common

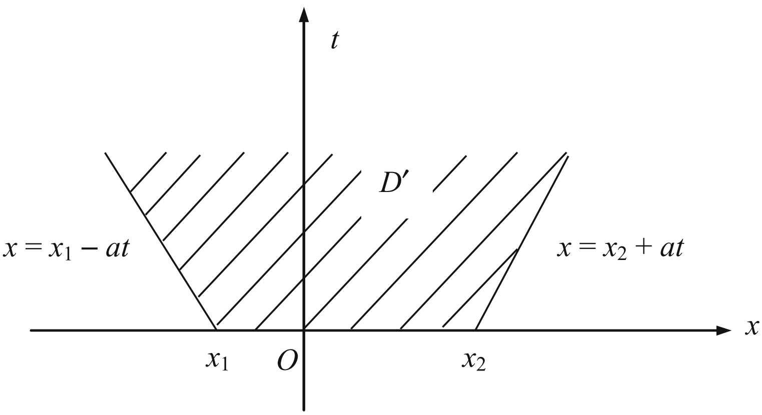

Fig. 2.4 Region D'

with $[x_1, x_2]$ so that the influence of the initial disturbances have already propagated to this point. The region D' is called *the domain of influence* of interval $[x_1, x_2]$. When the length of the interval $[x_1, x_2]$ tends to zero so that the interval tends to one point x^*, in particular, the domain of influence becomes (Fig. 2.5)

$$D^* : x^* - at \le x \le x^* + at\,,\ t > 0\,.$$

Remark. Characteristic curves $x \pm at = c$(constant) play a very important role in studying wave equations. The CDS can also be specified on those curves. The corresponding PDS is called the *Goursat Problem*. Such problems can also be solved by the method of traveling waves.

Example. Find the solution of the Goursat Problem

$$\begin{cases} u_{tt} = a^2 u_{xx}\,, \quad -\infty < x < +\infty,\ 0 < t\,, \\ u|_{x-at=0} = \varphi(x),\ \ u|_{x+at=0} = \psi(x),\ \ \varphi(0) = \psi(0)\,. \end{cases}$$

Solution. The general solution is, by Eq. (2.64),

$$u(x,t) = f(x+at) + g(x-at)\,.$$

To satisfy the initial conditions, f and g must be determined such that

$$f(2x) + g(0) = \varphi(x), \quad f(0) + g(2x) = \psi(x)$$

or

$$f(x) = \varphi\left(\frac{x}{2}\right) - g(0)\,, \quad g(x) = \psi\left(\frac{x}{2}\right) - f(0)\,.$$

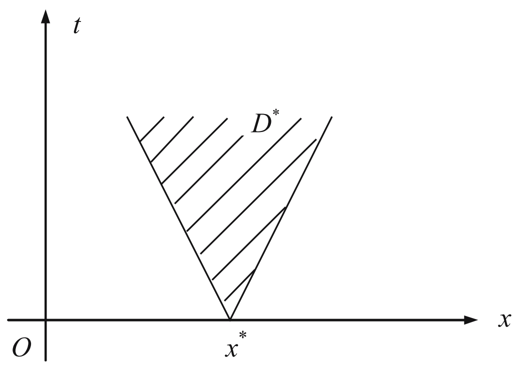

Fig. 2.5 Domain of influence

Adding these two equations yields

$$f(x)+g(x)=\varphi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{2}\right)-[f(0)+g(0)]$$

and

$$f(0)+g(0)=\varphi(0)\,.$$

Finally, we obtain

$$u(x,t)=\varphi\left(\frac{x+at}{2}\right)+\psi\left(\frac{x-at}{2}\right)-\varphi(0)\,.$$

2.7.5 *Problems in a Semi-Infinite Domain and the Method of Continuation*

Problems in a semi-infinite domain requires solving wave equations in the region: $0<x<+\infty$, $0<t$. We discuss the method of continuation for solving these problems by using homogeneous wave equations as an example. For problems in a semi-infinite domain, we require the boundary condition $x=0$ in addition to the initial conditions. Note that if an odd function of x has the definition at $x=0$, its value must be zero. Similarly, the derivative of an even function of x must be vanished at $x=0$ if it is differentiable at $x=0$. We should make an odd continuation when $u(0,t)=0$, and an even continuation when $u_x(0,t)=0$. Such a continuation of initial conditions does not change the CDS of the problem and reduces the problem into an auxiliary problem in an infinite domain which can be solved by the D'Alembert formula. The solution of the original problem in a semi-infinite domain can be obtained by using the solution of the auxiliary problem in $0<x<+\infty$, $0<t$.

Odd Continuation

Find the Solution of PDS

$$\begin{cases} u_{tt}=a^2u_{xx}\,, & 0<x<+\infty\,, 0<t\,,\\ u(0,t)=0\,, \\ u(x,0)=\varphi(x)\,,\ u_t(x,0)=\psi(x)\,. \end{cases} \tag{2.67}$$

Solution. Consider an auxiliary problem

$$\begin{cases} u_{tt}=a^2u_{xx}\,, & -\infty<x<+\infty\,, 0<t\,,\\ u(x,0)=\Phi(x)\,,\ u_t(x,0)=\Psi(x)\,, \end{cases} \tag{2.68}$$

where $\Phi(x)$ and $\Psi(x)$ come from an odd continuation of $\varphi(x)$ and $\psi(x)$, respectively,

$$\Phi(x)=\begin{cases}-\varphi(-x)\,, & x<0\,,\\ 0\,, & x=0\,,\\ \varphi(x)\,, & x>0\,,\end{cases}\qquad \Psi(x)=\begin{cases}-\psi(-x)\,, & x<0\,,\\ 0\,, & x=0\,,\\ \psi(x)\,, & x>0\,.\end{cases}$$

Clearly, PDS (2.68) is exactly the same as PDS (2.67) in $0<x<+\infty$, $0<t$. By the D'Alembert formula and focusing on the region of $x\geq 0$, we obtain the solution of PDS (2.67)

$$\begin{aligned}u(x,t)&=\frac{\Phi(x+at)+\Phi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(\xi)\,\mathrm{d}\xi\\ &=\begin{cases}\dfrac{\varphi(x+at)+\varphi(x-at)}{2}+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(\xi)d\xi, & \dfrac{x}{a}\geq t,\\[2ex] \dfrac{\varphi(x+at)-\varphi(at-x)}{2}+\dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at}\psi(\xi)d\xi, & \dfrac{x}{a}<t,\end{cases}\end{aligned}\tag{2.69}$$

in which we have used, for $x<at$,

$$\begin{aligned}\int_{x-at}^{x+at}\Psi(\xi)\,\mathrm{d}\xi&=\int_{x-at}^{0}\Psi(\xi)\,\mathrm{d}\xi+\int_{0}^{x+at}\Psi(\xi)\,\mathrm{d}\xi\\ &=-\int_{x-at}^{0}\psi(-\mu)\,\mathrm{d}\mu+\int_{0}^{x+at}\psi(\xi)\,\mathrm{d}\xi\\ &=\int_{at-x}^{x+at}\psi(\xi)\,\mathrm{d}\xi\,.\end{aligned}$$

The wave shape of PDS (2.67) at t_0 can be drawn by two approaches: using the part of the wave shape of PDS (2.68) in $[0,+\infty]$ and drawing directly based on Eq. (2.69). Although the detailed procedure differs for the two approaches, the final result is the same. It is helpful when using the latter to understand the meaning of terms in Eq. (2.69). When $t<\dfrac{x}{a}$, the effect of the end $x=0$ has not propagated to the point x yet, so the D'Alembert formula is valid and the superposition is that of initial forward and backward waves. When $t>\dfrac{x}{a}$, however, the effect of the end $x=0$ has already propagated to the point x. The forward waves $\dfrac{\varphi(x-at)}{2}$ and $\dfrac{1}{2a}\displaystyle\int_{x-at}^{0}\psi(\xi)\,\mathrm{d}\xi$ should thus be replaced by the reflected waves $\dfrac{-\varphi(at-x)}{2}$ and $\dfrac{1}{2a}\displaystyle\int_{at-x}^{0}\psi(\xi)\,\mathrm{d}\xi$, respectively. The reflected waves are also forward waves. For the end $x=0$, the backward wave is also called an *incoming wave*. The incoming and the reflected waves always have opposite phase. Such a phenomenon is called *semi-wave loss*. The forward, the backward and the reflected waves travel simultaneously with a common velocity a. The effect of reflected waves arrives exactly at the instant when the effect of forward waves disappears, regardless of the point x.

A better understanding of these phenomena can be achieved through drawing the wave shape at some instant t_0 for the case $\psi(x) \equiv 0$ and for some typical wave shape $\varphi(x)$ such as triangle wave in Fig. 2.6.

Even Continuation

Find the solution of PDS

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < +\infty, 0 < t, \\ u_x(0,t) = 0, \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x). \end{cases} \tag{2.70}$$

Solution. Consider an auxiliary problem

$$\begin{cases} u_{tt} = a^2 u_{xx}, & -\infty < x < +\infty, 0 < t, \\ u(x,0) = \Phi(x), u_t(x,0) = \Psi(x), \end{cases} \tag{2.71}$$

where $\Phi(x)$ and $\Psi(x)$ come from an even prolongation of $\varphi(x)$ and $\psi(x)$, respectively,

$$\Phi(x) = \begin{cases} \varphi(x), & x > 0, \\ \varphi(-x), & x \leq 0, \end{cases} \qquad \Psi(x) = \begin{cases} \psi(x), & x > 0, \\ \psi(-x), & x \leq 0. \end{cases}$$

The solution of PDS (2.71) can be expressed by the D'Alembert formula. It is straightforward to show that the D'Alembert solution of PDS (2.71) satisfies

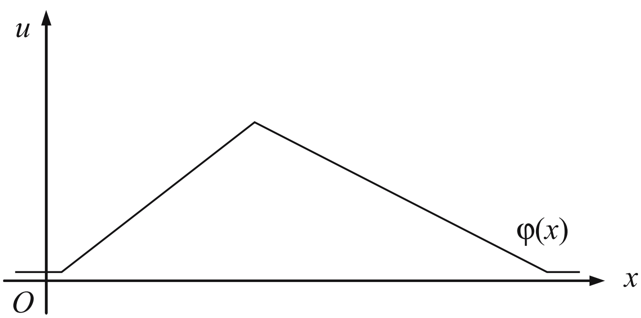

Fig. 2.6 Triangle wave

$u_x(0,t)=0$. PDS (2.71) is equivalent to PDS (2.70) in $0<x<+\infty$, $0<t$. Following the same approach as for the odd continuation, we obtain

$$u(x,t)=\begin{cases}\dfrac{\varphi(x+at)+\varphi(x-at)}{2}+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi\,,\ t\le\dfrac{x}{a}\,,\\ \dfrac{\varphi(x+at)+\varphi(at-x)}{2}+\dfrac{1}{2a}\displaystyle\int_{0}^{x+at}\psi(\xi)\,\mathrm{d}\xi\\ \qquad\qquad\qquad\qquad\quad+\dfrac{1}{2a}\displaystyle\int_{0}^{at-x}\psi(\xi)\,\mathrm{d}\xi\,,\ t>\dfrac{x}{a}\,,\end{cases} \tag{2.72}$$

which shows that the incoming and the reflected waves have the same phase at the end $x=0$; thus there is no semi-wave loss.

Method of Continuation for Mixed Problems in Finite Domains

When equations are not separable, we cannot use the Fourier method to solve mixed problems. While we may sometimes solve such problems by using integral transformations, it is often quite involved to perform the inverse transformations. Here we discuss the method of continuation for such problems by considering a PDS arising in mechanical machining

$$\begin{cases}u_{tt}+2vu_{tx}+(v^2-1)u_{xx}=f(x,t)\,, \quad 0<x<1,\ 0<t\,,\\ u(0,t)=u(1,t)=0\,,\\ u(x,0)=u_t(x,0)=0\,.\end{cases} \tag{2.73}$$

Based on the given boundary conditions, consider an odd continuation of the nonhomogeneous term $f(x,t)$. For convenience and conciseness, we still use $f(x,t)$ as the nonhomogeneous term after the continuation. This leads to the Cauchy problem

$$\begin{cases}u_{tt}+2vu_{tx}+(v^2-1)u_{xx}=f(x,t)\,, \quad R^1\times(0,+\infty),\ v>1\,,\\ u(x,0)=u_t(x,0)=0\,.\end{cases} \tag{2.74}$$

The equation in (2.74) is hyperbolic with characteristic curves $x=(v\pm 1)t+C$ (C is a constant). Consider a variable transformation

$$\xi=x-(v+1)t\,, \quad \eta=x-(v-1)t\,,$$

PDS (2.74) is thus reduced into

$$\begin{cases}u_{\xi\eta}=F(\xi,\eta)\,,\ F(\xi,\eta)=\dfrac{1}{4}f\left(-\dfrac{(v-1)\xi+(v+1)\eta}{2},-\dfrac{\xi-\eta}{2}\right)\,,\\ u_\eta\big|_{\xi=\eta}=u\big|_{\xi=\eta}=0\,.\end{cases} \tag{2.75}$$

Its solution can be obtained by integrating the equation twice with respect to ξ and η respectively and applying the initial conditions,

$$u(\xi,\eta)=-\int_{\xi}^{\eta}\mathrm{d}\eta'\int_{\xi}^{\eta'}F(\xi',\eta')\,\mathrm{d}\xi'\,. \tag{2.76}$$

To perform the double integration in Eq. (2.76), let

$$\xi' = x' - (v+1)t', \quad \eta' = x' - (v-1)t'.$$

The solution of the original PDS (2.73) can thus be obtained through the integration by substitution, in its integral representation,

$$u(x,t) = \frac{1}{2}\int_0^t \mathrm{d}\tau \int_{x-(v+1)(t-\tau)}^{x-(v-1)(t-\tau)} f(\xi,\tau)\,\mathrm{d}\xi\,.$$

2.8 Two- and Three-Dimensional Cauchy Problems

This section begins with two methods of solving three-dimensional Cauchy problems

$$\begin{cases} u_{tt} = a^2\Delta u + f(M,t), \quad \Omega \times (0,+\infty), \\ u(M,0) = \varphi(M),\ u_t(M,0) = \psi(M), \end{cases} \tag{2.77}$$

where Ω stands for: $-\infty < x < +\infty$, $-\infty < y < +\infty$, $-\infty < z < +\infty$. M represents point (x,y,z) in Ω. The solutions are then used to obtain the solutions of two-dimensional problems by using the method of descent. We conclude this section with an analysis of solution properties.

The solution of PDS (2.77) can be expressed by $W_\psi(M,t)$, the solution for the case $\varphi = f = 0$. In developing the solution of PDS (2.77), therefore, we focus on seeking this structure function.

2.8.1 Method of Fourier Transformation

We first develop an identity, which finds its application in seeking $W_\psi(M,t)$, using the method of Fourier transformation

$$I = \frac{1}{4\pi^2}\iiint\limits_{\Omega} \frac{1}{\mathrm{i}\omega}\left(\mathrm{e}^{\mathrm{i}\omega at} - \mathrm{e}^{-\mathrm{i}\omega at}\right)\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot r}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3 = \frac{1}{r}\left[\delta\left(r-at\right) - \delta\left(r+at\right)\right]. \tag{2.78}$$

Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is a vector, ω_1, ω_2 and ω_3 are variables of integration and Ω: $-\infty < \omega_1 < +\infty$, $-\infty < \omega_2 < +\infty$, $-\infty < \omega_3 < +\infty$, $|\mathbf{r}| = r$, $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$, $|\boldsymbol{\omega}| = \omega$.

Proof. To perform the triple integration over the whole space, take (x,y,z) as the origin and $\mathbf{r}$ as the polar axis. Regarding the whole space as a sphere of infinite radius, an integration in a spherical coordinate system yields

$$\begin{aligned} I &= \frac{1}{4\pi^2}\int_0^{+\infty}\frac{2\sin\omega at}{\omega}\,\mathrm{d}\omega\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{e}^{\mathrm{i}\omega r\cos\theta}\,\omega^2\sin\theta d\theta \\ &= \frac{1}{r\pi}\int_0^{+\infty}2\sin\omega at\sin\omega r\,\mathrm{d}\omega \\ &= \frac{1}{r\pi}\int_0^{+\infty}\left[\cos\omega\,(r-at)-\cos\omega\,(r+at)\right]\mathrm{d}\omega\,, \\ &= \frac{1}{r}\left[\delta(r+at)-\delta(r+at)\right], \end{aligned}$$

where $\delta(x)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega=\dfrac{1}{\pi}\displaystyle\int_0^{+\infty}cos\omega x\,\mathrm{d}\omega$ is the integral representation of the δ-function (see Appendix B).

Applying a Fourier transformation to $u(\mathbf{r},t)$, the solution of PDS (2.77) for the case $\varphi=f=0$ yields

$$u(\mathbf{r},t)=\frac{1}{(2\pi)^3}\iiint\limits_{\Omega}\bar{u}(\boldsymbol{\omega},t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot r}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3\,, \tag{2.79}$$

where $\bar{u}=F[u]$. Substituting it into PDS (2.77) with $\varphi=f=0$ leads to

$$\bar{u}_{tt}(\boldsymbol{\omega},t)+(\boldsymbol{\omega}a)^2\bar{u}(\boldsymbol{\omega},t)=0\,,\quad \bar{u}(\boldsymbol{\omega},0)=0\,,\quad \bar{u}_t(\boldsymbol{\omega},0)=\bar{\psi}(\boldsymbol{\omega})\,,$$

where $\bar{\psi}(\boldsymbol{\omega})=F[\psi(M)]$. Its solution reads

$$\bar{u}=\frac{\bar{\psi}(\boldsymbol{\omega})}{\omega a}\sin\omega at=\frac{\bar{\psi}(\boldsymbol{\omega})}{2a\mathrm{i}\omega}\left(\mathrm{e}^{\mathrm{i}\omega at}-\mathrm{e}^{-\mathrm{i}\omega at}\right), \tag{2.80}$$

where Ω': $-\infty<x'<+\infty, -\infty<y'<+\infty, -\infty<z'<+\infty$; $\mathbf{r}'=x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k}$; x', y' and z' are integral variables; and

$$\bar{\psi}(\boldsymbol{\omega})=\iiint\limits_{\Omega'}\psi(\mathbf{r}')\,\mathrm{e}^{-\mathrm{i}\boldsymbol{\omega}\cdot r'}\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'\,. \tag{2.81}$$

Equations (2.78)–(2.81) lead to

$$u(\mathbf{r},t)=\frac{1}{4\pi a}\iiint\limits_{\Omega'}\frac{\psi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\left[\delta\left(\left|\mathbf{r}-\mathbf{r}'\right|-at\right)-\delta\left(\left|\mathbf{r}-\mathbf{r}'\right|+at\right)\right]\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'\,. \tag{2.82}$$

Note that Ω' can be regarded as a sphere of center $\mathbf{r}$ and infinite radius; and, for $a>0$ and $t>0$, $\delta\left(|\mathbf{r}-\mathbf{r}\prime|+at\right)\equiv 0$. Also

$$\delta\left(\left|\mathbf{r}-\mathbf{r}'\right|-at\right)=\begin{cases}\neq 0\,,\ \mathbf{r}'\in S_{at}^{\mathbf{r}}\,,\\ =0\,,\ \mathbf{r}'\notin S_{at}^{\mathbf{r}}\,,\end{cases}$$

where $S^{\mathbf{r}}_{at}$ stands for the spherical surface of a sphere of center $\mathbf{r}$ and radius at. By performing the integration over Ω' in Eq. (2.82) first over $S^{\mathbf{r}}_{|\mathbf{r}-\mathbf{r}'|}$ and then over $|\mathbf{r}-\mathbf{r}'|$, we obtain

$$\begin{aligned} u(\mathbf{r},t) &= \frac{1}{4\pi a}\int_0^{+\infty}\left[\iint\limits_{S^r_{|\mathbf{r}-\mathbf{r}'|}} \frac{\psi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}S\right]\delta\left(|\mathbf{r}-\mathbf{r}'|-at\right)\mathrm{d}\left(|\mathbf{r}-\mathbf{r}'|\right) \\ &= \frac{1}{4\pi a}\iint\limits_{S^{\mathbf{r}}_{at}} \frac{\psi(\mathbf{r}')}{at}\,\mathrm{d}S. \end{aligned}$$

Finally

$$u(M,t) = W_\psi(M,t) = \frac{1}{4\pi a^2 t}\iint\limits_{S^{\mathbf{r}}_{at}} \psi(M')\,\mathrm{d}S, \tag{2.83}$$

where $M' \in S^{\mathbf{r}}_{at}$. The integration in Eq. (2.83) is a surface integration of the first kind over the spherical surface $S^{\mathbf{r}}_{at}$.

Remark 1. Consider the u in wave equations as representing the string displacement. The unit in Eq. (2.83) reads

$$[u] = \left[a^{-2}t^{-1}\right][\psi][\mathrm{d}S] = \frac{T^2}{L^2}\cdot\frac{1}{T}\cdot\frac{L}{T}\cdot L^2 = L,$$

which is correct.

Remark 2. By the solution structure theorem, the solution of

$$\begin{cases} u_{tt} = a^2\Delta u, & \Omega\times(0,+\infty), \\ u(M,0) = \varphi(M),\ u_t(M,0) = \psi(M) \end{cases}$$

is

$$u(M,t) = \frac{1}{4\pi a^2}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\iint\limits_{S^M_{at}} \psi(M')\,\mathrm{d}S\right) + \frac{1}{t}\iint\limits_{S^M_{at}} \psi(M')\,\mathrm{d}S\right], \tag{2.84}$$

which is called the *Poisson formula of three-dimensional wave equations*.

Remark 3. The solution of

$$\begin{cases} u_{tt} = a^2\Delta u + f(M,t), & \Omega\times(0,+\infty), \\ u(M,0) = u_t(M,0) = 0, \end{cases}$$

reads, by the solution structure theorem,

$$u(M,t) = \int_0^t W_{f_\tau}(M, t-\tau)\,\mathrm{d}\tau = \frac{1}{4\pi a^2}\int_0^t \left[\iint\limits_{S^M_{a(t-\tau)}} \frac{f(M',\tau)}{t-\tau}\,\mathrm{d}S\right]\mathrm{d}\tau$$

$$= \frac{1}{4\pi a^2}\int_0^{at}\left[\iint\limits_{S^M_r} \frac{f(M', t-\frac{r}{a})}{r}\,\mathrm{d}S\right]\mathrm{d}r$$

$$= \frac{1}{4\pi a^2}\iiint\limits_{V^M_{at}} \frac{f(M', t-\frac{r}{a})}{r}\,\mathrm{d}v\,, \tag{2.85}$$

where V^M_{at} stands for a sphere of center M and radius at. Equation (2.85) is called the *Kirchhoff formula of three-dimensional wave equations*.

2.8.2 Method of Spherical Means

Consider spherical surface $S^{M_0}_r$ of a sphere of center M_0 and radius r. The averaged value of $u(M,t)$ on $S^{M_0}_r$ is

$$\bar{u}(r,t) = \frac{1}{4\pi r^2}\iint\limits_{S^{M_0}_r} u(M,t)\,\mathrm{d}S\,.$$

Also

$$u(M_0,t_0) = \lim_{r\to 0}\bar{u}(r,t_0)\,.$$

Once $\bar{u}(r,t_0)$ is available, therefore, we can obtain $u(M_0,t_0)$ simply by taking the limit of $\bar{u}(r,t_0)$ as $r \to 0$. This is the essence of the method of spherical means. The method may be very useful for finding solutions of wave equations.

Let $V^{M_0}_r$ be a sphere of center M_0 and radius r. Integrating a wave equation over $V^{M_0}_r$ yields

$$\iiint\limits_{V^{M_0}_r} \frac{\partial^2 u}{\partial t^2}\,\mathrm{d}v = \iiint\limits_{V^{M_0}_r} a^2\Delta u\,\mathrm{d}v\,.$$

Since

$$\text{left-hand side} = \frac{\partial^2}{\partial t^2}\int_0^r \mathrm{d}\tau \iint\limits_{S_r^{M_0}} u\,\mathrm{d}S = 4\pi\frac{\partial^2}{\partial t^2}\int_0^r \bar{u}(\tau,t)\tau^2\,\mathrm{d}\tau\,,$$

$$\text{right-hand side} = a^2 \iint\limits_{S_r^{M_0}} \frac{\partial u}{\partial r}\,\mathrm{d}S = 4\pi a^2 r^2 \frac{\partial \bar{u}(r,t)}{\partial r}\,,$$

we arrive at

$$\frac{\partial^2}{\partial t^2}\int_0^r \bar{u}(\tau,t)\tau^2\,\mathrm{d}\tau = a^2 r^2 \frac{\partial \bar{u}(r,t)}{\partial r}\,.$$

Taking the derivative with respect to r yields

$$\frac{\partial^2(\bar{u}r^2)}{\partial t^2} = a^2\frac{\partial}{\partial r}\left[r^2\frac{\partial \bar{u}}{\partial r}\right]\,,$$

i.e.

$$\frac{\partial^2 v}{\partial t^2} = a^2\frac{\partial^2 v}{\partial r^2}\,, \tag{2.86}$$

where $v(r,t) = r\bar{u}(r,t)$. The general solution of Eq. (2.86) reads, by Sect. 2.7.2,

$$v(r,t) = f(r+at) + g(r-at)\,,$$

where f and g are two differentiable functions. Applying $v(0,t) = 0$ leads to $g(-at) = -f(at)$ so that

$$v(r,t) = r\bar{u} = f(r+at) - f(at-r)\,. \tag{2.87}$$

By taking derivatives of Eq. (2.87) with respect to r and t, respectively, we obtain

$$\bar{u} + r\bar{u}_r = f'(r+at) + f'(at-r)\,, \tag{2.88}$$

$$\frac{1}{a}\frac{\partial(r\bar{u})}{\partial t} = f'(r+at) - f'(at-r)\,. \tag{2.89}$$

Adding Eqs. (2.88) and (2.89) leads to

$$2f'(r+at) = \frac{\partial(r\bar{u})}{\partial r} + \frac{1}{a}\frac{\partial(r\bar{u})}{\partial t}\,,$$

and for $t = 0$ and $r = at_0$,

$$2f'(at_0) = \left.\left[\frac{\partial(r\bar{u})}{\partial r} + \frac{1}{a}\frac{\partial(r\bar{u})}{\partial t}\right]\right|_{r=at_0,t=0}\,.$$

Also, by Eq. (2.88), let $t = t_0$, $r \to 0$, i.e

$$u(M_0,t_0) = \bar{u}(0,t_0) = 2f'(at_0)\,.$$

Therefore

$$u(M_0,t_0) = \left[\frac{\partial(r\bar{u})}{\partial r} + \frac{1}{a}\frac{\partial(r\bar{u})}{\partial t}\right]\Bigg|_{r=at_0,t=0},$$

or, by the definition of $\bar{u}$

$$u(M_0,t_0) = \left[\frac{\partial}{\partial r}\left(\frac{1}{4}\pi r \iint\limits_{S_r^{M_0}} u\,\mathrm{d}S\right) + \frac{1}{a}\frac{1}{4}\pi r \iint\limits_{S_r^{M_0}} \frac{\partial u}{\partial t}\,\mathrm{d}S\right]\Bigg|_{r=at_0,t=0}. \tag{2.90}$$

Applying the initial conditions $u(M,0) = \varphi(M)$ and $u_t(M,0) = \psi(M)$ leads to

$$u(M_0,t_0) = \frac{1}{4\pi a^2}\left[\frac{\partial}{\partial t_0}\left(\frac{1}{t_0}\iint\limits_{S_{at_0}^{M_0}} \varphi(M)\,\mathrm{d}S + \frac{1}{t_0}\iint\limits_{S_{at_0}^{M_0}} \psi(M)\,\mathrm{d}S\right)\right],$$

or, as both M_0 and t_0 are arbitrary in Ω and $(0,+\infty)$, respectively,

$$u(M,t) = \frac{1}{4\pi a^2}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\iint\limits_{S_{at}^{M}} \varphi(M')\,\mathrm{d}S + \frac{1}{t}\iint\limits_{S_{at}^{M}} \psi(M')\,\mathrm{d}S\right)\right], \tag{2.91}$$

which is the same as the Poisson formula (2.84).

2.8.3 Method of Descent

Three-dimensional results such as Eqs. (2.84) and (2.85) can be used to obtain the corresponding results of the two- and one-dimensional cases by using the method of descent.

Two-dimensional Wave Equation

Expressing spherical surface integrals by double integrals in the Oxy-plane is the key for the reduction from three spatial dimensions to two dimensions. This can be achieved by noting that all functions $\varphi(x,y)$, $\psi(x,y)$, $f(x,y,t)$ and $u(x,y,t)$ of two spatial variables are even functions with respect to plane $z = k$(constant) because of their independency of z. We illustrate this by developing, from Eq. (2.84), the

Poisson formula of two-dimensional wave equations, the solution of

$$\begin{cases} u_{tt} = a^2\Delta u + f(x,y,t), \quad -\infty < x, y < +\infty, 0 < t\,, \\ u(x,y,0) = \varphi(x,y), u_t(x,y,0) = \psi(x,y)\,. \end{cases} \tag{2.92}$$

To find the solution of PDS (2.92) from Eqs. (2.84) or (2.91), consider a spherical surface S_{at}^M for a sphere of center $M(x,y)$ and radius at. Its projection onto plane-Oxy is a circle of center $M(x,y)$ and radius at denoted by $D_{at}^M : (\xi - x)^2 + (\eta - y)^2 \leq (at)^2$. Let γ be the angle between positive z-axis and normal of S_{at}^M. Thus

$$\cos\gamma = \pm\frac{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}{at}\,,$$

where $\pm$ is corresponding to points on the upper half S_1 or the lower half S_2 of S_{at}^M. Since $\varphi(x,y)$ and $\psi(x,y)$ are symmetric with respect to $z = 0$,

$$\begin{aligned} \iint\limits_{S_{at}^M} \frac{\varphi(M')}{at}\,\mathrm{d}S &= \iint\limits_{S_1} \frac{\varphi(M')}{at}\,\mathrm{d}S + \iint\limits_{S_2} \frac{\varphi(M')}{at}\,\mathrm{d}S \\ &= 2\iint\limits_{S_1} \frac{\varphi(M')}{at}\,\mathrm{d}S = 2\iint\limits_{D_{at}^M} \frac{\varphi(\xi,\eta)}{at}\frac{1}{\cos\gamma}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= 2\iint\limits_{D_{at}^M} \frac{\varphi(\xi,\eta)}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,, \end{aligned}$$

which reduces the spherical surface integral into a surface integral on plane-Oxy. Similarly, we can also express $\iint\limits_{S_{at}^M} \psi(M')\,\mathrm{d}S$ by using a double integral over D_{at}^M. Finally, we obtain the *Poisson formula* of two-dimensional wave equations

$$\begin{aligned} u(M,t) = \frac{1}{2\pi a}\Bigg[&\frac{\partial}{\partial t}\iint\limits_{D_{at}^M} \frac{\varphi(\xi,\eta)}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &+ \iint\limits_{D_{at}^M} \frac{\psi(\xi,\eta)}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\Bigg]\,. \end{aligned} \tag{2.93}$$

Remark 1. The two terms in the right-hand side of Eq. (2.93) clearly have the same unit. The unit of the first term is, by viewing u as the displacement,

$$[u] = \left[a^{-1}\right]\left[\frac{\partial}{\partial t}\right][\varphi]\left[\frac{1}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}\right][\mathrm{d}\xi\,\mathrm{d}\eta]$$
$$= \frac{T}{L}\cdot\frac{1}{T}\cdot L\cdot\frac{1}{L}\cdot L^2 = L,$$

which shows the correctness of the unit in Eq. (2.93).

Remark 2. In a polar coordinate system, Eq. (2.93) reads

$$u(x,y,t) = \frac{1}{2\pi a}\left[\frac{\partial}{\partial t}\int_0^{at} r\,\mathrm{d}r\int_0^{2\pi}\frac{\varphi(x+r\cos\theta, y+r\sin\theta)}{\sqrt{(at)^2-r^2}}\,\mathrm{d}\theta\right.$$
$$\left.+\int_0^{at} r\,\mathrm{d}r\int_0^{2\pi}\frac{\psi(x+r\cos\theta, y+r\sin\theta)}{\sqrt{(at)^2-r^2}}\,\mathrm{d}\theta\right]. \tag{2.94}$$

Example 1. Find the solution of

$$\begin{cases} u_{tt} = a^2\Delta u, & -\infty < x,\, y < +\infty,\, 0 < t\,, \\ u(x,y,0) = x^2(x+y), u_t(x,y,0) = 0\,. \end{cases}$$

Solution. By Eq. (2.94), we obtain

$$u(x,y,t) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int_0^{at} r\,\mathrm{d}r\int_0^{2\pi}\frac{(x+r\cos\theta)^2(x+y+r\cos\theta+r\sin\theta)}{\sqrt{(at)^2-r^2}}\,\mathrm{d}\theta\right]$$
$$= \frac{1}{2\pi a}\frac{\partial}{\partial t}\left\{\int_0^{at}\frac{r\,\mathrm{d}r}{\sqrt{(at)^2-r^2}}\left[\int_0^{2\pi} x^2(x+y)d\theta + \int_0^{2\pi}(3x+y)r^2\cos^2\theta\,\mathrm{d}\theta\right]\right\}$$
$$= x^2(x+y)+(3x+y)a^2t^2\,.$$

If x and y have the unit of length, $u(x,y,0)$ thus has unit L^3. From the solution, we have $[u(x,y,t)] = L^3$, which agrees with the unit of $u(x,y,0)$. Otherwise, there must exist some errors in derivation.

Remark 3. By a similar approach we can also obtain, from Eq. (2.85), the solution of

$$\begin{cases} u_{tt} = a^2\Delta u + f(x,y,t)\,, & -\infty < x,\, y < +\infty,\, 0 < t\,, \\ u(x,y,0) = 0,\, u_t(x,y,0) = 0\,. \end{cases}$$

The key is to perform the integration over V_{at}^M first over a spherical surface and then over the radius. The integration over the spherical surface can be reduced to one over D_{at}^M. The result is

$$u(x,y,t) = \frac{1}{4\pi a}\int_0^t \left[\iint\limits_{S_{a(t-\tau)}^M} \frac{f(M',\tau)}{a(t-\tau)}\,\mathrm{d}S \right] \mathrm{d}\tau$$
$$= \frac{1}{2\pi a}\int_0^t \left[\iint\limits_{D_{a(t-\tau)}^M} \frac{f(\xi,\eta,\tau)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta \right] \mathrm{d}\tau, \quad (2.95)$$

which is called the *Kirchhoff formula of two-dimensional wave equations.*

If $u(x,y,t)$ represents the displacement,

$$[u] = [a^{-1}]\,[f]\left[\frac{1}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}}\right][\mathrm{d}\xi\,\mathrm{d}\eta\, d\tau]$$
$$= \frac{T}{L}\cdot\frac{1}{T^2}\cdot\frac{1}{L}\cdot L^2 T = L,$$

which shows the unit correctness in Eq. (2.95). Such a unit check can help us to find some errors that may arise in derivation.

Remark 4. By the principle of superposition, the solution of

$$\begin{cases} u_{tt} = a^2\Delta u + f(x,y,t)\,, \quad -\infty < x,\, y < +\infty, 0 < t\,, \\ u(x,y,0) = \varphi(x,y),\, u_t(x,y,0) = \psi(x,y) \end{cases}$$

can be obtained by Eqs. (2.93) and (2.95),

$$u(M,t) = \frac{1}{2\pi a}\left[\frac{\partial}{\partial t}\iint\limits_{D_{at}^M}\frac{\varphi(\xi,\eta)}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\right.$$
$$\left. + \iint\limits_{D_{at}^M}\frac{\psi(\xi,\eta)}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\right]$$
$$+ \frac{1}{2\pi a}\int_0^t \left[\iint\limits_{D_{a(t-\tau)}^M} \frac{f(\xi,\eta,\tau)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}}\,d\xi\,\mathrm{d}\eta \right] \mathrm{d}\tau\,.$$

One-Dimensional Wave Equations

To solve, using the method of descent based on Eq. (2.84),

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \ -\infty < x < +\infty, 0 < t\,, \\ u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x)\,. \end{cases} \tag{2.96}$$

Consider the spherical surface S_{at}^M of center $M(x)$ and radius at. Its intersecting circumference with plane-Oxy is denoted by $C_{at}^M : (\xi - x)^2 + \eta^2 = (at)^2$, $x - at \leq \xi \leq x + at$. For a spherical zone on S_{at}^M,

$$\mathrm{d}S = 2\pi |\eta|\, \mathrm{d}s = \sqrt{(at)^2 + (\xi - x)^2}\sqrt{1 + \eta_\xi^2}\,\mathrm{d}\xi = 2\pi at\,\mathrm{d}\xi\,, \quad x - at \leq \xi \leq x + at\,.$$

Therefore,

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi a}\left[\frac{\partial}{\partial t}\iint\limits_{S_{at}^M} \frac{\varphi(M')}{at}\,\mathrm{d}S + \iint\limits_{S_{at}^M} \frac{\psi(M')}{at}\,\mathrm{d}S\right] \\ &= \frac{1}{4\pi a}\left[\frac{\partial}{\partial t}\int_{x-at}^{x+at} \varphi(\xi) 2\pi\,\mathrm{d}\xi + \int_{x-at}^{x+at} \psi(\xi) 2\pi\,\mathrm{d}\xi\right] \\ &= \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi\,, \end{aligned} \tag{2.97}$$

which is the D'Alembert formula of one-dimensional wave equations. Similarly, for the one-dimensional case, Eq. (2.85) reduces to

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi a}\int_0^t \left[\iint\limits_{S_{a(t-\tau)}^M} \frac{f(M',\tau)}{a(t-\tau)}\,\mathrm{d}S\right]\mathrm{d}\tau \\ &= \frac{1}{4\pi a}\int_0^t \left[\int_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{f(\xi,\tau)}{a\,(t-\tau)} \cdot 2\pi a\,(t-\tau)\,\mathrm{d}\xi\right]\mathrm{d}\tau \\ &= \frac{1}{2a}\int_0^t \mathrm{d}\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\,\mathrm{d}\xi\,. \end{aligned} \tag{2.98}$$

The superposition of Eqs. (2.97) and (2.98) is, by the principle of superposition, the solution of PDS (2.96).

Example 2. Prove that the solution of

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(t)\,, \quad -\infty < x < +\infty, 0 < t, \\ u(x,0) = u_t(x,0) = 0. \end{cases}$$

is

$$u(x,t) = \frac{1}{a^2}\int_0^{at} rf\left(t - \frac{r}{a}\right)\,\mathrm{d}r\,.$$

Proof. By Eq. (2.98),

$$\begin{aligned} u(x,t) &= \frac{1}{2a}\int_0^t \mathrm{d}\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau)\,\mathrm{d}\xi = \frac{1}{2a}\int_0^t f(\tau)2a(t-\tau)\,\mathrm{d}\tau \\ &= \frac{1}{a^2}\int_0^{at} rf\left(t - \frac{r}{a}\right)\,\mathrm{d}r\,. \end{aligned}$$

Introduce a new variable r such that $\tau = t - \dfrac{r}{a}$. Thus $\mathrm{d}\tau = -\dfrac{1}{a}\,\mathrm{d}r$, and r varies from at to 0 when τ changes from 0 to t. Therefore

$$u(x,t) = \frac{1}{a^2}\int_0^{at} rf\left(t - \frac{r}{a}\right)\,\mathrm{d}r\,.$$

Example 3. Find the solution of

$$\begin{cases} u_{tt} = a^2\Delta u\,, & -\infty < x,y,z < +\infty,\ 0 < t\,, \\ u(r,0) = \varphi(r),\ u_t(r,0) = \psi(r)\,,\ r = \sqrt{x^2+y^2+z^2}\,. \end{cases}$$

Solution. Since $\varphi(r)$ and $\psi(r)$ are functions of r only, they have spherical symmetry so that $\varphi(-r) = \varphi(r)$ and $\psi(-r) = \psi(r)$ for a generalized polar coordinate system in which points $(-r,\theta,\varphi)$ and (r,θ,φ) are symmetric with respect to the origin. Therefore the problem is a Cauchy problem in $-\infty < r < +\infty$, $0 < t$. Also, under a spherical symmetry,

$$\Delta u(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \quad -\infty < r < +\infty, \quad 0 < t\,.$$

The problem becomes,

$$\begin{cases} \dfrac{\partial^2(ru)}{\partial t^2} = a^2\dfrac{\partial^2(ru)}{\partial r^2}, -\infty < r < +\infty, 0 < t\,, \\ ru(r,0) = r\varphi(r),\ ru_t(r,0) = r\psi(r)\,. \end{cases}$$

By the D'Alembert formula (2.97) we obtain the solution

$$\begin{aligned} u(r,t) &= \frac{(r+at)\varphi(r+at) + (r-at)\varphi(r-at)}{2r} \\ &\quad + \frac{1}{2ar}\int_{r-at}^{r+at} \xi\,\psi(\xi)\,\mathrm{d}\xi\,. \end{aligned}$$

Remark 5. It is always useful to study a problem by using different methods. We have arrived at Eq. (2.98), the Kirchhoff formula of one-dimensional wave equations, using the method of descent and the solution structure theorem that shares the same essence as the homogenization of equations and the impulsive method. It can also be obtained by using other methods such as integral transformation. To demonstrate some mathematical techniques, we re-develop it here by using the method of characteristics.

Example 4. Using the method of characteristics, find the solution of

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), & -\infty < x < +\infty, 0 < t, \\ u(x,0) = u_t(x,0) = 0. \end{cases} \tag{2.99}$$

Solution. Introduce new variables ξ and η such that

$$\begin{cases} \xi = x + at, \\ \eta = x - at \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{\xi+\eta}{2}, \\ t = \frac{\xi-\eta}{2a}, \quad \xi > \eta. \end{cases}$$

The equation reduces to (Section 1.1.5)

$$u_{\xi\eta} = g(\xi,\eta), \quad g(\xi,\eta) = -\frac{1}{4a^2} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2a}\right).$$

Applying the initial condition $u(x,0) = 0$ yields $u(\xi,\eta)|_{\xi=\eta} = 0$. By the initial condition $u_t(x,0) = 0$, we have

$$u_t = u_\xi \cdot a + u_\eta \cdot (-a) \quad \text{or} \quad \left(u_\xi - u_\eta\right)\big|_{\xi=\eta} = 0. \tag{2.100}$$

We thus arrive at a Goursat problem

$$u_{\xi\eta} = g(\xi,\eta), \, u|_{\xi=\eta} = \left(u_\xi - u_\eta\right)\big|_{\xi=\eta} = 0. \tag{2.101}$$

i.e.

$$u_{\xi\eta} = g(\xi,\eta), \, u_\xi\big|_{\xi=\eta} = u_\eta\big|_{\xi=\eta} = 0. \tag{2.102}$$

Integrating the equation in (2.102) with respect to ξ from ξ to η and applying $u_\eta\big|_{\xi=\eta} = 0$ yields

$$u_\eta = -\int_\xi^\eta g(\xi',\eta)\,\mathrm{d}\xi'.$$

Integrating it with respect to η from ξ to η and using $u|_{\eta=\xi} = 0$ leads to

$$u(\xi,\eta) = -\int_\xi^\eta \left[\int_\xi^{\eta'} g(\xi',\eta')\,\mathrm{d}\xi'\right] \mathrm{d}\eta', \tag{2.103}$$

which is the solution of PDS (2.101). In Eq. (2.103), ξ and η are the parametric variables and ξ' and η' are the variables of integration. Introduce new variables x'

and t' such that

$$\xi' = x' + at', \quad \eta' = x' - at'.$$

To obtain the solution of PDS (2.99) in terms of x and t from Eq. (2.103), we must express the integral area $\mathrm{d}\xi'\mathrm{d}\eta'$ by $\mathrm{d}x'\mathrm{d}t'$; the integrand $g(\xi',\eta')$ by x' and t' and the integral limits in terms of x and t, respectively.

1. Transformation of area element: By the rule of variable transformation,

$$\mathrm{d}\xi'\mathrm{d}\eta' = |J|\,\mathrm{d}x'\,\mathrm{d}t' = 2a\,\mathrm{d}x'\,\mathrm{d}t', \quad \left(|J| = \left|\frac{D(\xi',\eta')}{D(x',t')}\right| = 2a\right),$$

   where J is the Jacobian determinant of the transformation, $|J|$ is the absolute value of J.
2. Transformation of the integrand: By the definition of $g(\xi,\eta)$,

$$g(\xi',\eta') = -\frac{1}{4a^2}f\left(\frac{\xi'+\eta'}{2}, \frac{\xi'-\eta'}{2a}\right) = -\frac{1}{4a^2}f\left(x',t'\right).$$

3. Transformation of the Integral Limit: Since $t>0$, $t'>0$ so that $\xi' > \eta'$. $\xi' = \xi$ is a straight line parallel to the η'-axis in $O\xi'\eta'$ coordinate system. Similarly, $\eta' = \eta$ is a straight line parallel to the ξ'-axis. Therefore, the domain of integration in Eq. (2.103) forms an isosceles right triangle on the $O\xi'\eta'$-plane [Fig. (2.7)].
   $\xi' = \eta'$ corresponds to $t' = 0$, therefore it is the x'-axis on $Ox't'$-plane. $\xi' = \xi$ implies $x' + at' = x + at$ ors $x' = x + a(t-t')$ so it is a straight line of slope

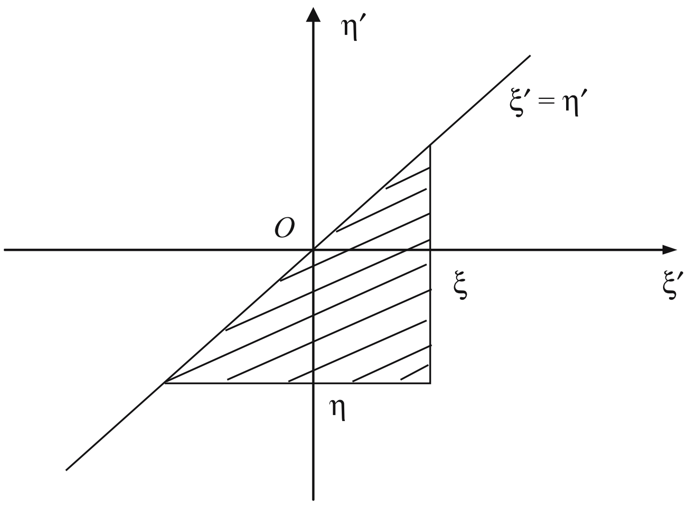

Fig. 2.7 Integration domain

$\left(-\frac{1}{a}\right)$ that passes through the point (x,t). $\eta' = \eta$ means $x' - at' = x - at$ or $x' = x - a(t - t')$ on the $Ox't'$-plane so it is a straight line that passes through the point (x,t) with a slope of $\dfrac{1}{a}$. On the $Ox't'$-plane, the integral domain is thus an isosceles triangle shown in Fig. 2.8.

Finally we obtain the solution of PDS (2.99)

$$u(x,t) = \frac{1}{2a}\int_0^t \mathrm{d}t' \int_{x-a(t-t')}^{x+a(t-t')} f(x',t')\,\mathrm{d}x',$$

which is the same as Eq. (2.98).

Similarly we may also obtain the Kirchhoff formula of two or three dimensional wave equations by using different methods.

2.8.4 Physical Meanings of the Poisson and Kirchhoff Formulas

Poisson Formula of Three-Dimensional Wave Equations

The Poisson formula (2.84) reveals that $u(M,t)$ depends only on initial disturbances $\varphi(M')$ and $\psi(M')$ on S_{at}^M, and not on the initial disturbance outside S_{at}^M. For an initial nonzero disturbance in a sphere Ω of radius R, i.e.

$$\varphi(M'), \psi(M') = \begin{cases} \neq 0, & M' \in \Omega\,, \\ = 0, & M' \notin \Omega\,, \end{cases}$$

generally $u(M,t) \neq 0$ if S_{at}^M intersects with Ω; Otherwise, $u(M,t) = 0$ (Fig. 2.9). Let

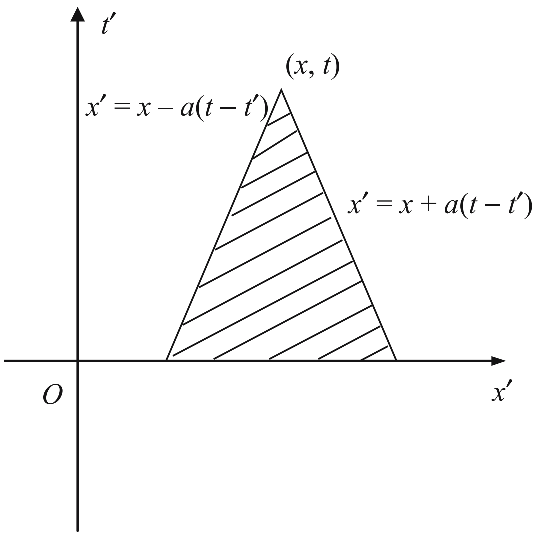

Fig. 2.8 Isosceles triangle

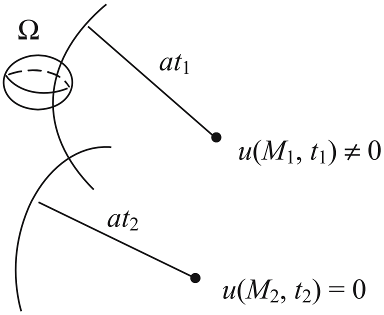

Fig. 2.9 u value and intersection of S_{at}^{M} with Ω

d and D be the minimum and the maximum distance of M_0 from Ω, respectively (Fig. 2.10). For a fixed point M_0, we have, as t and v are varied

$$u(M_0,t) = \begin{cases} = 0, \text{for } t < \dfrac{d}{a}, & \text{Effect of initial disturbances} \\ & \text{has not propagated to point } M_0 \text{ yet,} \\ \neq 0, \text{ for } \dfrac{d}{a} < t < \dfrac{D}{a}, & \text{Effect of initial disturbances has arrived} \\ & \text{at point } M_0, \\ = 0, \text{ for } t > \dfrac{D}{a}, & \text{Effect of initial disturbances has passed.} \end{cases}$$

For a sufficiently short or long time, S_{at}^{M} and Ω have no common part so $u(M,t) = 0$. Take initial disturbances as the wave source. The effect of the wave source always appears on a spherical surface of the sphere of origin M_0. Such a type of waves is thus called a *spherical wave*. Also, the effect of the wave source for point M_0 will disappear after a sufficiently long time. For example, the propagation of sound waves has this property.

Take the center of Ω as the origin O. For any specified time instant t_0 such that $at_0 - R > 0$, consider the region between the spherical surface $S_{at_0-R}^{O}$ and the sur-

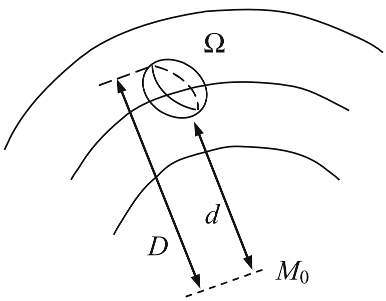

Fig. 2.10 Minimum and maximum distances of M_0 from Ω

face $S^O_{R+at_0}$. The $u(M,t_0)$ does not generally vanish inside this region and it is zero outside the region. Therefore, $S^O_{at_0-R}$ and $S^O_{R+at_0}$ are called the *wave rear* and the *wave front*, respectively (Fig. 2.11). After a time interval Δt, the wave front moves to $S^O_{R+a(t_0+\Delta t)}$ (Fig. 2.12). Thus any point on the $S^O_{R+at_0}$ may be viewed as the source of generating an individual wave. After the time period Δt, the *embracing surface* of all these individual waves forms the new wave front $S^O_{R+a(t_0+\Delta t)}$. The traveling of such waves is always characterized by a clear wave rear and a clear wave front. Such a phenomenon is called the *Huygens principle.*

Poisson Formula of Two-Dimensional Wave Equations

A similar analysis may be made for the Poisson formula of two-dimensional wave equations by replacing the Ω by a circle D and the S^M_{at} by D^M_{at}. However, a difference

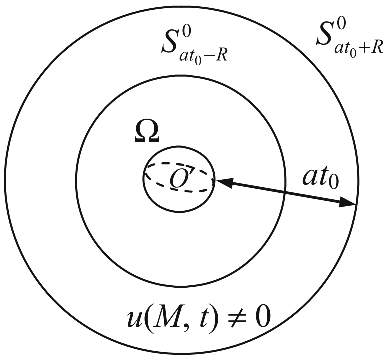

Fig. 2.11 Wave rear and wave front

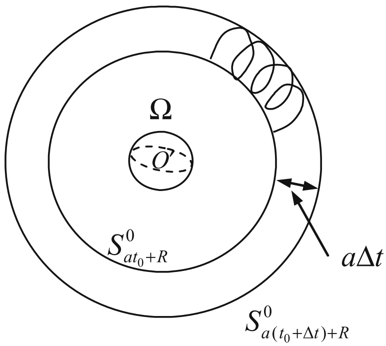

Fig. 2.12 Traveling of waves

exists. In three-dimensional space, $S^M_{a(t+\Delta t)}$ does not contain S^M_{at}. The D^M_{at} is however a part of $D^M_{a(t+\Delta t)}$ and

$$u(M_0,t) = \begin{cases} = 0\,, \text{for } t < \dfrac{d}{a}\,, & \text{Effect of initial disturbances} \\ & \text{has not yet propagated to point } M_0\,, \\ \neq 0\,,\ \text{for } \dfrac{d}{a} < t < \dfrac{D}{a}\,, & \text{Effect of initial disturbances has arrived} \\ & \text{at point } M_0, \\ \neq 0\,,\ \text{for } t > \dfrac{D}{a}\,, & \text{Wave front has passed, but left a lasting effect.} \end{cases}$$

Therefore, there is no wave rear. The phenomenon of having a lasting effect is called *wave dispersion*. The mechanism behind this difference can be understood by noting that a two-dimensional plane is a special case of three-dimensional space so $\varphi \neq 0$ and $\psi \neq 0$ in an infinitely long cylinder that is parallel to the z-axis and with D as the cross-plane. Therefore, waves propagate as cylindrical waves in two-dimensional cases. Similarly, they propagate as plane waves in one-dimensional cases.

Kirchhoff Formula of Three-Dimensional Wave Equations

The Kirchhoff formula (2.85) shows that the $u(M,t)$ comes from $f(M',t)$ in V^M_{at}. The structure of Eq. (2.85) is the same as that of potential functions. The Newton potential of a body Ω at point M is defined in mechanics as

$$v(M) = \iiint\limits_{\Omega} \frac{\rho(M')}{r}\,\mathrm{d}v\,, \quad r' = \overline{MM'}\,,$$

where $\rho(M')$ is the density of the body. Because of this similarity, $u(M,t)$ in Eq. (2.85) is also called a potential function. The value of u at time instant t depends on the value of f at time instant $\left(t - \frac{r}{a}\right)$. The effect of f is thus deferred by a period of $\dfrac{r}{a}$ in its propagation to point M. Therefore, $u(M,t)$ in Eq. (2.85) is called the *retarded potential*.

Chapter 3
Heat-Conduction Equations

We first develop the solution structure theorem for mixed problems of heat-conduction equations, followed by methods of solving one-, two- and three-dimensional mixed problems. For conciseness, we directly borrow the results in Chapter 2 for developing the solutions of heat-conduction equations. Emphasis is also placed on the difference between wave equations and heat-conduction equations. Finally, we discuss methods of solving one-, two- and three-dimensional Cauchy problems.

3.1 The Solution Structure Theorem For Mixed Problems

Consider mixed problems of three-dimensional heat-conduction equations in a closed region $\bar{\Omega}$. Let $\bar{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega$ is the boundary surface of Ω. Three kinds of linear homogeneous boundary conditions can, therefore, be written as

$$L\left(u, \frac{\partial u}{\partial n}\right)\bigg|_{\partial\Omega} = 0\,,$$

which refers to the boundary conditions at two ends for one-dimensional cases.

The solution structure theorem describes the relation among solutions of the following three PDS,

$$\begin{cases} u_t = a^2 \Delta u, \quad \Omega \times (0, +\infty)\,, \\ L\left(u, \dfrac{\partial u}{\partial n}\right)\bigg|_{\partial\Omega} = 0\,, \\ u(M,0) = \varphi(M)\,, \end{cases} \tag{3.1}$$

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$$\begin{cases} u_t = a^2 \Delta u + f(M,t), \quad \Omega \times (0, +\infty), \\ L\left(u, \dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} = 0, \\ u(M,0) = 0, \end{cases} \tag{3.2}$$

$$\begin{cases} u_t = a^2 \Delta u + f(M,t), \quad \Omega \times (0, +\infty), \\ L\left(u, \dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} = 0, \\ u(M,0) = \varphi(M), \end{cases} \tag{3.3}$$

where M represents the point x, (x,y) and (x,y,z) in one-, two- and three-dimensional space. For the one-dimensional case, Δu is defined as u_{xx}.

Theorem. Suppose that $u_1 = W_\varphi(M,t)$ is the solution of (3.1), then

1. $u_2 = \displaystyle\int_0^t W_{f_\tau}(M, t-\tau)\,\mathrm{d}\tau$ is the solution of (3.2), where $f_\tau = f(M,\tau)$;
2. $u_3 = W_\varphi(M,t) + \displaystyle\int_0^t W_{f_\tau}(M, t-\tau)\,\mathrm{d}\tau$ is the solution of (3.3).

Proof.

1. Since $W_{f_\tau}(M, t-\tau)$ satisfies

$$\begin{cases} \dfrac{\partial W_{f_\tau}}{\partial t} = a^2 \Delta W_{f_\tau}, \quad \Omega \times (0, +\infty), \\ L\left(W_{f_\tau}, \dfrac{\partial W_{f_\tau}}{\partial n}\right)\Big|_{\partial\Omega} = 0, \\ W_{f_\tau}\big|_{t=\tau} = f(M,\tau), \end{cases}$$

then

$$\begin{aligned} \frac{\partial^2 u_2}{\partial t^2} - a^2 \Delta u_2 &= \frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M, t-\tau)\,\mathrm{d}\tau - a^2\Delta \int_0^t W_{f_\tau}(M, t-\tau)\,\mathrm{d}\tau \\ &= \int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}\big|_{\tau=t} - a^2 \int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau \\ &= \int_0^t \left(\frac{\partial W_{f_\tau}}{\partial t} - a^2 \Delta W_{f_\tau}\right)\mathrm{d}\tau + f(M,t) = f(M,t). \end{aligned}$$

Therefore, u_2 satisfies the equation of (3.2).

Also,

$$L\left(u_2, \frac{\partial u_2}{\partial n}\right)\Big|_{\partial\Omega} = \int_0^t L\left(W_{f_\tau}, \frac{\partial W_{f_\tau}}{\partial n}\right)\Big|_{\partial\Omega}\mathrm{d}\tau = 0$$

and

$$u_2|_{t=0} = 0\,.$$

Hence, u_2 also satisfies the boundary and initial conditions of (3.2), so that u_2 is indeed the solution of (3.2).

2. Since PDS (3.3) is linear, the principle of superposition is valid. Applying this principle to (3.3) shows that u_3 is the solution of (3.3).

Remark 1. All Remarks made in Section 2.1 are also valid here.

Remark 2. The solution structure theorem reduces the development of solutions for mixed problems to solving PDS (3.1). Similar to the procedure in Chapter 2, PDS (3.1) can be solved by the Fourier method or the method of separation of variables. A comparison of PDS (3.1) with PDS (2.2) in Section 2.1 reveals the difference between the two: PDS (3.1) here only involves the first derivative of u with respect to t and requires only one initial condition. This only varies the $T(t)$-equation in the solution of separable variables and reduces the second-order ODE for PDS (2.2) in Section 2.1 to a first-order ODE. For PDS (2.2) in Section 2.1,

$$T''(t) + \lambda a^2 T(t) = 0$$

with the general solution

$$T(t) = C_1 \cos\sqrt{\lambda}at + C_2 \sin\sqrt{\lambda}at\,.$$

Here,

$$T'(t) + \lambda a^2 T(t) = 0$$

with a general solution

$$T(t) = Ce^{-\lambda a^2 t}\,.$$

Therefore, we can readily write out solutions of mixed problems of heat-conduction equations based on the solutions of wave equations in Chapter 2, simply by changing the trigonometric functions of t to an exponential functions of t.

Remark 3. The solution structure theorem is also valid for Cauchy problems. However, for Cauchy problems the structure of $W_\varphi(M,t)$ differs from that for the mixed problems.

3.2 Solutions of Mixed Problems

In this section, we follow Remark 2 in Section 3.1 to write out solutions of mixed problems of heat-conduction equations directly from those in Section 2.2 to Section 2.4. This is the Fourier method based on Table 2.1.

3.2.1 One-Dimensional Mixed Problems

Boundary Condition of the First Kind

Find the solution of PDS

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & (0,l)\times(0,+\infty), \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \varphi(x). \end{cases} \tag{3.4}$$

Solution. It follows from Section 2.2.1 that for the case $f(x,t)=0$, the solution of PDS (3.4) is

$$\begin{cases} u = W_\varphi(x,t) = \sum_{k=1}^{+\infty} b_k \, \mathrm{e}^{-\left(\frac{k\pi a}{l}\right)^2 t} \sin\frac{k\pi x}{l}, \\ b_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi x}{l}\,\mathrm{d}x. \end{cases} \tag{3.5}$$

Therefore, the solution of PDS (3.4) for the case $\varphi(x)=0$ would be, by the solution structure theorem,

$$\begin{aligned} u &= \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau = \int_0^t \sum_{k=1}^{+\infty} b_k \, \mathrm{e}^{-\left(\frac{k\pi a}{l}\right)^2 (t-\tau)} \sin\frac{k\pi x}{l}\,\mathrm{d}\tau \\ &= \int_0^t \int_0^l G(x,\xi,t-\tau) f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \end{aligned} \tag{3.6}$$

Here,

$$G(x,\xi,t-\tau) = \sum_{k=1}^{\infty} \frac{2}{l}\,\mathrm{e}^{-\left(\frac{k\pi a}{l}\right)^2 (t-\tau)} \sin\frac{k\pi\xi}{l}\sin\frac{k\pi x}{l}$$

is called the *Green function* of the mixed problem of the one-dimensional heat-conduction equation under boundary conditions of the first kind.

When $f(x,t) = \delta(x - x_0, t - t_0)$,

$$u = \int_0^t \int_0^l G(x,\xi,t-\tau)\delta(\xi - x_0, \tau - t_0)\,\mathrm{d}\xi\,\mathrm{d}\tau = G(x,x_0,t-t_0)$$

$G(x,x_0,t-t_0)$ is thus the temperature distribution in a heat-conduction rod that is due to a source term (the nonhomogeneous term $f(x,t)$ in the equation) of the unit $\delta-$function $\delta(x-x_0,t-t_0)$ or the temperature distribution caused by a unit point source at time instant t_0 and spatial point x_0. Therefore, $u = G(x,\xi,t-\tau)$ satisfies

$$\begin{cases} G_t = a^2 G_{xx} + \delta(x-\xi, t-\tau), & 0 < x < l,\ 0 < \tau < t < +\infty, \\ G|_{x=0} = G|_{x=l} = 0, \\ G|_{t=\tau} = 0. \end{cases}$$

The solution of PDS (3.4) is, by the principle of superposition, the sum of Eqs. (3.5) and (3.6).

Boundary Condition of the Second Kind

Find the solution of PDS

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & (0,l) \times (0,+\infty), \\ u_x(0,t) = u_x(l,t) = 0, \\ u(x,0) = \varphi(x). \end{cases} \tag{3.7}$$

Solution. It follows from the corresponding solution of the wave equation in Section 2.2.2 that the solution of PDS (3.7) is, for the case $f(x,t) = 0$,

$$\begin{cases} u = W_\varphi(x,t) = \dfrac{a_0}{2} + \sum\limits_{k=1}^{+\infty} a_k e^{-\left(\frac{k\pi a}{l}\right)^2 t} \cos\dfrac{k\pi x}{l}, \\ a_k = \dfrac{2}{l}\displaystyle\int_0^l \varphi(x)\cos\frac{k\pi x}{l}\,\mathrm{d}x, \quad k = 0,1,2,\cdots. \end{cases} \tag{3.8}$$

Therefore, the solution structure theorem yields the solution of PDS (3.7) for the case $\varphi(x) = 0$,

$$u = \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau = \int_0^t \int_0^l G(x,\xi,t-\tau) f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{3.9}$$

Here the *Green function* is

$$G(x,\xi,t-\tau) = \frac{1}{l} + \sum_{k=1}^{+\infty} \frac{2}{l} e^{-\left(\frac{k\pi a}{l}\right)^2 (t-\tau)} \cos\frac{k\pi\xi}{l} \cos\frac{k\pi x}{l}.$$

The solution of PDS (3.7) is thus the sum of Eqs. (3.8) and (3.9).

Remark. Solutions and Green functions can also be readily obtained for the other kinds of boundary conditions based on Table 2.1.

3.2.2 Two-Dimensional Mixed Problems

Rectangular Domain

The solutions can be readily obtained, based on Table 2.1, for mixed problems subjected to various boundary conditions. Here, we illustrate this by finding the solution of

$$\begin{cases} u_t = a^2 \Delta u + f(x,y,t), \quad 0 < x < l_1,\ 0 < y < l_2,\ 0 < t\,, \\ u(0,y,t) = u(l_1,y,t) + h_2 u_x(l_1,y,t) = 0\,, \\ u_y(x,0,t) - h_1 u(x,0,t) = u_y(x,l_2,t) = 0\,, \\ u(x,y,0) = \varphi(x,y)\,. \end{cases}$$

Solution. By the solution structure theorem, we first develop $W_\varphi(x,y,t)$, the solution for the case of $f(x,y,t) = 0$. Based on the given boundary conditions, we should use the eigenfunctions in Rows 3 and 8 in Table 2.1 to expand the solution. From Section 2.5.1, we have

$$\begin{cases} u = W_\varphi(x,y,t) = \displaystyle\sum_{m,n=1}^{+\infty} b_{mn}\, \mathrm{e}^{-(\omega_{mn} a\,)^2 t} \sin\frac{\mu_m x}{l_1} \sin\left(\frac{\mu_n' y}{l_2} + \varphi_n\right), \\ b_{mn} = \dfrac{1}{M_{mn}} \displaystyle\int_0^{l_2}\int_0^{l_1} \varphi(x,y) \sin\frac{\mu_m x}{l_1} \sin\left(\frac{\mu_n' y}{l_2} + \varphi_n\right) \mathrm{d}x\,\mathrm{d}y\,, \end{cases}$$

where μ_m, μ_n' are the positive zero points of $f(x) = \tan x + \dfrac{x}{l_1 h_2}$ and $g(x) = \cot x - \dfrac{x}{l_2 h_1}$, respectively; M_{mn} is the product of normal squares of two sets of eigenfunctions. Also,

$$\tan\varphi_n = \frac{\mu_n'}{l_2 h_1}, \quad \omega_{mn}^2 = a^2\left[\left(\frac{\mu_m}{l_1}\right)^2 + \left(\frac{\mu_n'}{l_2}\right)^2\right].$$

Therefore, the solution is, by the solution structure theorem,

$$u = W_\varphi(x,y,t) + \int_0^t W_{f_\tau}(x,y,t-\tau)\,\mathrm{d}\tau\,,$$

where $f_\tau = f(x,y,t-\tau)$.

Similar to the one-dimensional cases, we can also obtain the Green functions and their physical implications by considering the case $f(x,y,t) = \delta(x-\xi, y-\eta, t-\tau)$ and $\varphi(x,y) = 0$.

By using the corresponding eigenfunctions in Table 2.1 to expand the solution, we can also obtain the solutions for the other kinds of boundary conditions.

Circular Domain

Find the solution of PDS

$$\begin{cases} u_t = a^2 \Delta u + f(x,y,t) \quad x^2+y^2 < a_0^2, 0 < t, \\ L(u,u_n)|_{r=a_0} = 0, \\ u(x,y,0) = \varphi(x,y). \end{cases} \tag{3.10}$$

Solution. Note that the boundary conditions in PDS (3.10) contain the first, the second and the third kinds. It follows from Eq. (2.44) in Section 2.5.2 that the solution for the case of $f(x,y,t) = 0$ is

$$\begin{cases} u = W_\Phi(r,\theta,t) = \displaystyle\sum_{m=1,n=0}^{+\infty} (b_{mn}\cos n\theta + d_{mn}\sin n\theta) J_n(k_{mn}r)\, \mathrm{e}^{-\omega_{mn}^2 t}, \\ b_{m0} = \dfrac{1}{2\pi M_{m0}} \displaystyle\int_{-\pi}^{\pi}\int_0^{a_0} \Phi(r,\theta) J_0(k_{m0}r) r \,\mathrm{d}r\,\mathrm{d}\theta, \\ b_{mn} = \dfrac{1}{\pi M_{mn}} \displaystyle\int_{-\pi}^{\pi}\int_0^{a_0} \Phi(r,\theta) J_n(k_{mn}r) r\cos n\theta \,\mathrm{d}r\,\mathrm{d}\theta, \\ d_{mn} = \dfrac{1}{\pi M_{mn}} \displaystyle\int_{-\pi}^{\pi}\int_0^{a_0} \Phi(r,\theta) J_n(k_{mn}r) r\sin n\theta \,\mathrm{d}r\,\mathrm{d}\theta. \end{cases} \tag{3.11}$$

Here $\Phi(r,\theta) = \varphi(r\cos\theta, r\sin\theta)$, $k_{mn} = \mu_m^{(n)}/a_0$, $\omega_{mn} = k_{mn}a$, $\mu_m^{(n)}$ are the zero points of Bessel functions which depend on the boundary conditions. The normal squares M_{mn} depend on the boundary conditions and are available in Chapter 2 for the three kinds of boundary conditions. Therefore, the solution of PDS (3.10) is, by the solution structure theorem,

$$u = W_\Phi(r,\theta,t) + \int_0^t W_{F_\tau}(r,\theta,t-\tau)\,\mathrm{d}\tau,$$

where

$$f(r\cos\theta, r\sin\theta, t) = F(r,\theta,t), F_\tau = F(r,\theta,\tau).$$

3.2.3 Three-Dimensional Mixed Problems

Cuboid Domain

Consider finding the solution of

$$\begin{cases} u_t = a^2\Delta u + f(x,y,z,t)\,, \quad \Omega\times(0,+\infty)\,, \\ L(u,u_x,u_y,u_z)\big|_{\partial\Omega} = 0\,, \\ u(x,y,z,0) = \varphi(x,y,z)\,, \end{cases} \tag{3.12}$$

where Ω stands for the cuboid domain : $0<x<a_1, 0<y<b_1, 0<z<c_1$, and $\partial\Omega$ represents the six boundary surfaces of Ω. The boundary conditions can be different on each of the six surfaces. We can readily find the solutions for all combinations of different boundary conditions by following the procedure in Section 2.6, using Table 2.1 and the solution structure theorem.

To demonstrate, we consider

$$\begin{cases} u(0,y,z,t) = u_x(a_1,y,z,t) = 0\,, \\ u_y(x,0,z,t) = u_y(x,b_1,z,t) + h_2u(x,b_1,z,t) = 0\,, \\ u_z(x,y,0,t) - h_1u(x,y,0,t) = u_z(x,y,c_1,t) = 0\,. \end{cases} \tag{3.13}$$

By the solution structure theorem, we first develop $W_\varphi(x,y,z,t)$, the solution for the case $f=0$. Based on the given boundary conditions (3.13), we should use the eigenfunctions in Rows 2, 6 and 8 in Table 2.1 to expand the solution so that

$$u = \sum_{m=0,n,l=1}^{+\infty} T_{mnl}(t) \sin\frac{(2m+1)\pi x}{2a_1} \cos\frac{\mu_n y}{b_1} \sin\left(\frac{\mu_l' z}{c_1} + \varphi_l\right)\,. \tag{3.14}$$

Substituting Eq. (3.14) into the equation of PDS (3.12) leads to

$$T_{mnl}'(t) + \omega_{mnl}^2 T_{mnl}(t) = 0$$

with the general solution

$$T_{mnl}(t) = C_{mnl}\,\mathrm{e}^{-\omega_{mnl}^2 t}\,,$$

where

$$\omega_{mnl}^2 = a^2 \left[\left(\frac{(2m+1)\pi}{2a_1} \right)^2 + \left(\frac{\mu_n}{b_1} \right)^2 + \left(\frac{\mu_l'}{c_1} \right)^2 \right].$$

The generalized Fourier coefficients C_{mnl} can be obtained by using the initial conditions $u(x,y,z,0) = \varphi(x,y,z)$, the completeness and the orthogonality of the eigenfunction set. Finally, we have

$$\begin{cases} u = W_\varphi(x,y,z,t) = \sum\limits_{m=0,n,l=1}^{+\infty} C_{mnl}\, \mathrm{e}^{-\omega_{mnl}^2 t} \sin \dfrac{(2m+1)\pi x}{2a_1} \cos \dfrac{\mu_n y}{b_1} \sin \left(\dfrac{\mu_l' z}{c_1} + \varphi_l \right), \\ C_{mnl} = \dfrac{1}{M_{mnl}} \iiint\limits_{\Omega} \varphi(x,y,z) \sin \dfrac{(2m+1)\pi x}{2a_1} \cos \dfrac{\mu_n y}{b_1} \sin \left(\dfrac{\mu_l' z}{c_1} + \varphi_l \right) \mathrm{d}\Omega, \end{cases} \tag{3.15}$$

where $M_{mnl} = M_m M_n M_l$ is the product of normal squares of three sets of eigenfunctions, and μ_n, μ_l' and φ_l are referred to Table 2.1.

Therefore, the solution of PDS (3.12) subject to the boundary conditions (3.13) is, by Eq. (3.15) and the solution structure theorem,

$$u = W_\varphi(x,y,z,t) + \int_0^t W_{f_\tau}(x,y,z,t-\tau)\,\mathrm{d}\tau,$$

where

$$f_\tau = f(x,y,z,\tau).$$

Spherical Domain

Find the solution of PDS

$$\begin{cases} u_t = a^2 \Delta u + f(x,y,z,t), \quad x^2+y^2+z^2 < a_0^2, 0 < t, \\ L(u,u_n)|_{x^2+y^2+z^2=a_0^2} = 0, \\ u(x,y,z,0) = \varphi(x,y,z). \end{cases} \tag{3.16}$$

Solution. We first find the solution for the case $f = 0$. It can be readily written as, simply by replacing $\sin\omega_{nl}t$ in Eq. (2.59) by $\mathrm{e}^{-\omega_{nl}^2 t}$,

$$\begin{cases} u = W_\Phi(r,\theta,\varphi,t) = \displaystyle\sum_{m,n=0,l=1}^{+\infty} (b_{mnl}\cos m\varphi + d_{mnl}\sin m\varphi)\ P_n^m(\cos\theta) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0}r\right)\mathrm{e}^{-\omega_{nl}^2 t}\,, \\ b_{0nl} = \dfrac{1}{2\pi M_{0nl}} \displaystyle\iiint\limits_{r\le a_0} \Phi(r,\theta,\varphi)P_n(\cos\theta)\, j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0}r\right) r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,, \\ b_{mnl} = \dfrac{1}{\pi M_{mnl}} \displaystyle\iiint\limits_{r\le a_0} \Phi(r,\theta,\varphi)\cos m\varphi P_n^m(\cos\theta)\, j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0}r\right) r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,, \\ d_{mnl} = \dfrac{1}{\pi M_{mnl}} \displaystyle\iiint\limits_{r\le a_0} \Phi(r,\theta,\varphi)\sin m\varphi P_n^m(\cos\theta)\, j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0}r\right) r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,. \end{cases}$$

Thus the solution of PDS (3.16) is, by the solution structure theorem,

$$u = W_\Phi(r,\theta,\varphi,t) + \int_0^t W_{F_\tau}(r,\theta,\varphi,t-\tau)\,\mathrm{d}\tau\,,$$

where $F_\tau = F(r,\theta,\varphi,\tau)$, $f(x,y,z,t) = F(r,\theta,\varphi,t)$, $\varphi(x,y,z) = \Phi(r,\theta,\varphi)$, and the meanings of all parameters, functions and constants are the same as those for wave equations in Section 2.6.

3.3 Well-Posedness of PDS

We examine the well-posedness of PDS using the example

$$\begin{cases} u_t = a^2 u_{xx}\,, \quad 0 < x < l, 0 < t\,, \\ u(0,t) = u(l,t) = 0\,, \\ u(x,0) = \varphi(x)\,. \end{cases} \tag{3.17}$$

3.3.1 Existence

We have obtained the nominal solution of PDS (3.4) in Section 3.2 by using the method of separation of variables or the Fourier method,

$$\begin{cases} u = \sum_{k=1}^{+\infty} b_k \mathrm{e}^{-\left(\frac{k\pi a}{l}\right)^2 t} \sin \dfrac{k\pi x}{l}, \\ b_k = \dfrac{2}{l} \displaystyle\int_0^l \varphi(x) \sin \dfrac{k\pi x}{l} \, \mathrm{d}x. \end{cases} \tag{3.18}$$

To prove the series solution (3.18) satisfies PDS (3.17) so that it is the classical solution of PDS (3.17), we should show the convergence of series (3.18) and the uniform convergence of the resulting series after taking the derivative with respect to t and the second derivative with respect to x, term by term. Let D be the domain: $0 \le x \le l,\ \ 0 \le t \le T$ (T is an arbitary positive constant) and $u_k(x,t)$ is the general term of series (3.18). We have

$$\sum_{k=1}^{+\infty} \frac{\partial u_k}{\partial t} = \sum_{k=1}^{+\infty} -\left(\frac{k\pi a}{l}\right)^2 b_k \mathrm{e}^{-\left(\frac{k\pi a}{l}\right)^2 t} \sin \frac{k\pi x}{l}, \tag{3.19}$$

$$\sum_{k=1}^{+\infty} \frac{\partial^2 u_k}{\partial x^2} = \sum_{k=1}^{+\infty} -\left(\frac{k\pi}{l}\right)^2 b_k \mathrm{e}^{-\left(\frac{k\pi a}{l}\right)^2 t} \sin \frac{k\pi x}{l}. \tag{3.20}$$

For any natural number k,

$$\left|\sin \frac{k\pi x}{l}\right| \le 1, \quad |b_k| \le \frac{2}{l} \int_0^l |\varphi(x)| \, \mathrm{d}x = \text{constant},$$

and for any fixed natural number N, $\lim\limits_{k \to +\infty} \dfrac{k^N}{e^{k^2}} = 0$, there exists a constant C such that

$$|u_k(x,t)| \le \frac{C}{k^2}, \quad \left|\frac{\partial u_k}{\partial t}\right| \le \frac{C}{k^2}, \quad \left|\frac{\partial^2 u_k}{\partial x^2}\right| \le \frac{C}{k^2}.$$

$\sum\limits_{k=1}^{+\infty} \dfrac{C}{k^2}$ is also convergent. Therefore, series (3.18), (3.19) and (3.20) are all uniformly convergent so we can take derivatives of series (3.18) term by term. Thus the solution (3.18) can satisfy PDS (3.17).

Remark 1. The uniform convergence of a series with function terms is not sufficient for the term by term differentiation of the series. It is, however, ensured by the uniform convergence of the series itself and all the resulting series from term by term differentiation.

Remark 2. The nominal solution contains an exponentially decaying factor in each term. This relaxes the requirement for the smoothness of the initial distribution $\varphi(x)$ in the classical solution. The constraint on $\varphi(x)$ is dependent on the existence of the integral of the product of $\varphi(x)$ with the sine function over the domain $[0,l]$, i.e. the condition for using the Fourier series to expand.

3.3.2 Uniqueness

The proof of uniqueness and stability is based on the following important physical phenomenon. Consider one-dimensional heat conduction in a rod without any internal heat source. Let M be the maximum temperature of two ends and initial distribution. The temperature is then less than or equal to M at any point in the rod and at any time instant. This can be physically understood and can also be proven mathematically.

Theorem. Assume that $u(x,t)$ is continuous in the domain $D: 0 \leq x \leq l, 0 \leq t \leq T$ (T is any positive constant; Fig. 3.1) and satisfies $u_t = a^2 u_{xx}$ inside D. The $u(x,t)$ can then take its maximum or minimum value only at the boundary $\Gamma: x = 0, x = l$, $t = 0$. This is called the *extremum principle of heat conduction*.

Proof. Without loss of generality, we consider the case of maximum value only. We must then start from the given condition to arrive at

$$\max_D u(x,t) = \max_\Gamma u(x,t) \tag{3.21}$$

Let $M = \max_D u(x,t)$, $\bar{M} = \max_\Gamma u(x,t)$. Suppose $M > \bar{M}$. There would exist one point (x_0,t_0) inside D such that

$$\max_D u(x,t) = u(x_0,t_0) = M > \bar{M}\,.$$

Define a new function $v(x,t)$ by

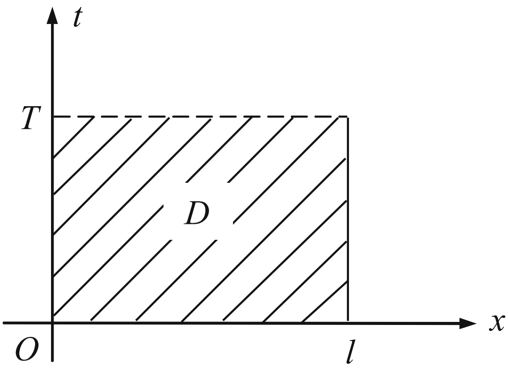

Fig. 3.1 Domain D

$$v(x,t) = u(x,t) + \frac{M - \bar{M}}{4l^2}(x - x_0)^2, \quad (x,t) \in D.$$

Then

$$v_t - a^2 v_{xx} = u_t - a^2 u_{xx} - a^2 \frac{M - \bar{M}}{2l^2} = -a^2 \frac{M - \bar{M}}{2l^2} < 0, \tag{3.22}$$

$$v(x_0, t_0) = u(x_0, t_0) = M. \tag{3.23}$$

Also, on the boundary Γ,

$$v(x,t) < \bar{M} + \frac{M - \bar{M}}{4} = \frac{M}{4} + \frac{3\bar{M}}{4} = \theta M, \tag{3.24}$$

where $0 < \theta < 1$.

Eqs. (3.23) and (3.24) show that $v(x,t)$ does not take its maximum value on Γ. Let (x_1, t_1) be the point inside D, where $v(x,t)$ takes its maximum value. At point (x_1, t_1), we have $v_{xx} \le 0$, $v_t \ge 0$ ($v_t = 0$ if $t_1 < T$, $v_t \ge 0$ if $t_1 = T$) such that $v_t - a^2 v_{xx} \ge 0$. This contradicts Eq. (3.22) so we arrive at Eq. (3.21).

Uniqueness follows from the extremum principle of heat conduction. Let u_1 and u_2 be two solutions of PDS (3.17). The function w defined by $u_1 - u_2$ satisfies

$$\begin{cases} w_t = a^2 w_{xx}, & (0,l) \times (0, +\infty) \\ w(0,t) = w(l,t) = 0, \\ w(x,0) = 0. \end{cases}$$

By the extremum principle of heat conduction, $w(x,t) \equiv 0$ so $u_1 = u_2$.

3.3.3 Stability

Consider any small variation of $\varphi(x)$ in $0 \le x \le l$ from $\varphi_1(x)$ to $\varphi_2(x)$,

$$\|u_1(x,0) - u_2(x,0)\| = \|\varphi_1(x) - \varphi_2(x)\| < \varepsilon,$$

where ε is any small constant, $\|\cdot\|$ is the norm in the sense of uniform approximation. By the extremum principle of heat conduction,

$$\|u_1(x,t) - u_2(x,t)\| \le \max_{0 \le x \le l} |\varphi_1(x) - \varphi_2(x)| < \varepsilon,$$

where $u_1(x,t)$ and $u_2(x,t)$ are the two solutions of PDS (3.17) corresponding to $\varphi_1(x)$ and $\varphi_2(x)$, respectively. Therefore, the solution of PDS (3.17) is stable.

Therefore, PDS (3.17) is well-posed. Well-posedness can also be established for the other PDS of heat-conduction equations in this book.

3.4 One-Dimensional Cauchy Problems: Fundamental Solution

In this section, we apply the Fourier transformation to develop solutions of Cauchy problems and discuss fundamental solutions. We also develop solutions of problem in a semi-infinite domain by using the method of continuation.

3.4.1 One-Dimensional Cauchy Problems

Using the Fourier transformation, find the solution of

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & -\infty < x < +\infty, 0 < t, \\ u(x,0) = \varphi(x). \end{cases} \tag{3.25}$$

Solution. We first develop $W_\varphi(x,t)$, the solution of PDS (3.25) with $f = 0$. Viewing $u(x,t)$ as a function of x, apply the Fourier transformation to the equation of PDS (3.25) to arrive at

$$\bar{u}_t(\omega,t) + (\omega a)^2 \bar{u}(\omega,t) = 0,$$

which yields $\bar{u}(\omega,t) = A(\omega)\,\mathrm{e}^{-(\omega a)^2 t}$. Here, $A(\omega)$ is the function to be determined by the initial condition. Applying the initial condition, we have $A(\omega) = \bar{\varphi}(\omega)$. Therefore,

$$\bar{u}(\omega,t) = \bar{\varphi}(\omega)\,\mathrm{e}^{-(\omega a)^2 t}, \quad F[\varphi(x)] = \bar{\varphi}(\omega). \tag{3.26}$$

The solution $u(x,t)$ follows from the inverse Fourier transformation of Eq. (3.26),

$$\begin{aligned} u = W_\varphi(x,t) &= F^{-1}\left[\bar{\varphi}(\omega)\,\mathrm{e}^{-(\omega a)^2 t}\right] = \varphi(x) \times \frac{1}{2a\sqrt{\pi t}}\,\mathrm{e}^{-\frac{x^2}{4a^2 t}} \\ &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\xi)\,\mathrm{e}^{-\dfrac{(x-\xi)^2}{4a^2 t}}\, d\xi. \end{aligned} \tag{3.27}$$

The solution of PDS (3.25) with $\varphi = 0$ is, by the solution structure theorem,

$$u = \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau = \frac{1}{2a\sqrt{\pi}} \int_0^t \mathrm{d}\tau \int_{-\infty}^{+\infty} \frac{f(\xi,\tau)}{\sqrt{t-\tau}}\,\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}\,\mathrm{d}\xi. \tag{3.28}$$

Finally, the solution of PDS (3.25) is

$$u = W_\varphi(x,t) + \int_0^t W_{f_\tau}(x,t-\tau)d\tau$$

$$= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\xi)\, \mathrm{e}^{-\frac{(x-\xi)^2}{4a^2 t}}\, \mathrm{d}\xi + \frac{1}{2a\sqrt{\pi}} \int_0^t \mathrm{d}\tau \int_{-\infty}^{+\infty} \frac{f(\xi,\tau)}{\sqrt{t-\tau}}\, \mathrm{e}^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}\, \mathrm{d}\xi\,.$$

Remark 1. When performing the inverse Fourier transformation, the following properties are applied.

Convolution theorem.
$$F\left[f_1(x) * f_2(x)\right] = F\left[\int_{-\infty}^{+\infty} f_1(\xi) f_2(x-\xi)\,\mathrm{d}\xi\right] = \bar{f}_1(\omega)\bar{f}_2(\omega)\,.$$

Similarity property.
$$F^{-1}\left[\bar{f}(a\omega)\right] = \frac{1}{|a|} f\left(\frac{x}{a}\right), \quad a \neq 0\,.$$

Transformation formula.
$$F\left[\mathrm{e}^{-px^2}\right] = \sqrt{\frac{\pi}{p}}\, \mathrm{e}^{-\frac{\omega^2}{4p}}, \quad \mathrm{Re}(p) > 0\,.$$

In particular, the similarity property and the transformation formula lead to

$$F^{-1}\left[\mathrm{e}^{-(\omega a)^2 t}\right] = \sqrt{\frac{a}{\pi}} F^{-1}\left[\sqrt{\frac{\pi}{a}}\, \mathrm{e}^{-\frac{1}{4a}(4a^3 t)\omega^2}\right] = \sqrt{\frac{a}{\pi}} F^{-1}\left[\sqrt{\frac{\pi}{a}}\, \mathrm{e}^{-\frac{\left(\sqrt{4a^3 t}\,\omega\right)^2}{4a}}\right]$$

$$= \frac{1}{\sqrt{4a^2 \pi t}}\, \mathrm{e}^{-\frac{x^2}{4a^2 t}} = \frac{1}{2a\sqrt{\pi t}}\, \mathrm{e}^{-\frac{x^2}{4a^2 t}}\,.$$

Remark 2. Let u in PDS (3.25) be temperature with Θ as its unit. Thus the unit of Eq. (3.28) is

$$[u] = \left[a^{-1}\right]\left[(t-\tau)^{-\frac{1}{2}}\right][\mathrm{d}\tau]\,[f(\xi,\tau)]\,[\mathrm{d}\xi] = \frac{\sqrt{T}}{L} \cdot \frac{1}{\sqrt{T}} \cdot T \cdot \frac{\Theta}{T} \cdot L = \Theta\,,$$

where

$$\left[-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right] = 1\,.$$

Therefore the unit in Eq. (3.28) is right. The unit is also correct in Eq. (3.27).

3.4.2 Fundamental Solution of the One-Dimensional Heat-Conduction Equation

Rewrite Eq. (3.27) into $u = \int_{-\infty}^{+\infty} \varphi(\xi)V(x,\xi,t)\,\mathrm{d}\xi$, where

$$V(x,\xi,t) = \frac{1}{2a\sqrt{\pi t}}\,\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2 t}}\,, \quad t > 0\,. \tag{3.29}$$

Consider a unit point source at $t = 0$ and $x = x_0$ such that $\varphi(x) = \delta(x - x_0)$ and Eq. (3.27) becomes

$$u = \int_{-\infty}^{+\infty} \delta(\xi - x_0)V(x,\xi,t)\,\mathrm{d}\xi = V(x,x_0,t)\,,$$

which is the temperature distribution due to the initial unit point source at $x = x_0$. Therefore, $V(x,\xi,t)$ in Eq. (3.29) must satisfy

$$\begin{cases} V_t = a^2 V_{xx}\,, & -\infty < x < +\infty, 0 < t\,, \\ V|_{t=0} = \delta(x-\xi)\,, \end{cases} \tag{3.30}$$

and is the temperature distribution due to $\varphi(x) = \delta(x - \xi)$, an initial unit point source at $x = \xi$.

The $V(x,\xi,t)$ defined by Eq. (3.29) is called the *fundamental solution of the one-dimensional heat-conduction equation*. By using V, we may rewrite the solution of PDS (3.25) by

$$u = \int_{-\infty}^{+\infty} \varphi(\xi)V(x,\xi,t)\,\mathrm{d}\xi + \int_0^t \mathrm{d}\tau \int_{-\infty}^{+\infty} f(\xi,\tau)V(x,\xi,t-\tau)\,\mathrm{d}\xi\,. \tag{3.31}$$

Therefore, $V(x,\xi,t)$ in Eq. (3.29) can also be regarded as the fundamental solution of PDS (3.25). Here

$$V(x,\xi,t-\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}}\,\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}\,, \quad t > \tau \tag{3.32}$$

satisfies

$$\begin{cases} V_t = a^2 V_{xx}\,, & -\infty < x < +\infty, 0 < \tau < t < +\infty\,, \\ V|_{t=\tau} = \delta(x-\xi) \end{cases} \tag{3.33}$$

or

$$\begin{cases} V_t = a^2 V_{xx} + \delta(x-\xi, t-\tau)\,, & -\infty < x < +\infty, 0 < \tau < t < +\infty\,, \\ V|_{t=\tau} = 0\,. \end{cases} \tag{3.34}$$

Remark 1. For the unit analysis of PDS (3.33) and (3.34), note that

$$[\delta(x-\xi)] = [1\cdot\delta(x-\xi)] = L\Theta\cdot\frac{1}{L} = \Theta\,,$$

$$[\delta(x-\xi,t-\tau)] = [1\cdot\delta(x-\xi,t-\tau)] = L\Theta\cdot\frac{1}{L\cdot T} = \frac{\Theta}{T}\,,$$

$$[V] = [1\cdot V] = L\Theta\cdot\frac{1}{L} = \Theta\,.$$

Remark 2. In developing Eq. (3.27) we did not consider whether $\varphi(x)$ satisfies the conditions for a Fourier transformation. Therefore, Eq. (3.27) is only a nominal solution. Here we show that it is indeed the solution of PDS (3.25) at $f=0$ [due to $\varphi(x)$] if $\varphi(x)\in C(-\infty,+\infty)$ and is bounded.

By taking the derivative of the fundamental solution (3.29), we obtain

$$V_t = a^2V_{xx}\,. \tag{3.35}$$

Note that the integrand in Eq. (3.27) contains a decaying factor of exponential order, which also exists in all derivatives of u with respect to x or t. This ensures the uniform convergence of integrals after differential operations inside the integration. Therefore, the integral in Eq. (3.27) satisfies

$$u_t - a^2u_{xx} = \int_{-\infty}^{+\infty}\left(V_t - a^2V_{xx}\right)\varphi(\xi)\,\mathrm{d}\xi = 0\,,$$

where we have used Eq. (3.35). This shows that Eq. (3.27) is a solution of the homogeneous equation in PDS (3.25) ($f=0$).

Also, the integral in Eq. (3.27) is uniformly convergent with respect to t. We can take the limit inside the integration,

$$\begin{aligned}\lim_{t\to+0} u(x,t) &= \int_{-\infty}^{+\infty}\lim_{t\to+0} V(x,\xi,t)\varphi(\xi)\,\mathrm{d}\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\eta^2}\varphi(x+2a\sqrt{t}\eta)\,\mathrm{d}\eta\\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\eta^2}\varphi(x)\,\mathrm{d}\eta = \varphi(x).\end{aligned}$$

Therefore, Eq. (3.27) also satisfies the initial condition $u(x,0)=\varphi(x)$.

Remark 3. Equation (3.35) shows that V satisfies the equation of PDS (3.30). Here we show that it also satisfies the initial condition. Note that $t=0$ means $t\to+0$. By Eq. (3.29),

$$V|_{t=0} = \begin{cases}\infty,\ x=\xi\,,\\ 0,\ x\neq\xi\,.\end{cases}$$

Also

$$\int_{-\infty}^{+\infty} V|_{t=0}\,\mathrm{d}\xi = \int_{-\infty}^{+\infty} \lim_{t\to+0} V\,\mathrm{d}\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\eta^2}\,\mathrm{d}\eta = 1\,.$$

Therefore,

$$V|_{t=0} = \delta(x-\xi)\,.$$

Remark 4. The structure of $W_\varphi(x,t)$ in Eq. (3.27) shows the following properties of the temperature distribution due to $\varphi(x)$.

1. $u(x,t)$ is odd or even in x if $\varphi(x)$ is an odd or even function of x;
2. $u(x,t)$ is periodic in x if $\varphi(x)$ is a periodic function of x;
3. The effect of a local, initial disturbance can propagate very quickly to a very long distance.

   To demonstrate this, consider

$$\varphi(x) = \begin{cases} k>0\,, & |x-x_0|\le\delta\,, \\ 0\,, & |x-x_0|>\delta\,, \end{cases}$$

where δ is an arbitrary small constant. For a very large $|x_1|$ and a very small t_1,

$$u(x_1,t_1) = \int_{-\infty}^{+\infty} V(x_1,\xi,t_1)\varphi(\xi)\,\mathrm{d}\xi = k\int_{x_0-\delta}^{x_0+\delta} V(x_1,\xi,t_1)\,\mathrm{d}\xi > 0\,.$$

This property is the signature of the Fourier law of heat conduction (see Section 1.3).

3.4.3 *Problems in Semi-Infinite Domain and the Method of Continuation*

Boundary Condition of the First Kind

Example 1. Find the solution of

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t)\,, & 0<x<+\infty, 0<t, \\ u(0,t)=0\,, \\ u(x,0)=\varphi(x)\,. \end{cases} \tag{3.36}$$

Solution. We first develop $W_\varphi(x,t)$, the solution of PDS (3.36) at $f(x,t)=0$. To satisfy the boundary condition at $x=0$, consider an odd continuation of $\varphi(x)$,

$$\Phi(x)=\begin{cases}-\varphi(-x)\,, & x<0\,,\\ 0\,, & x=0,\\ \varphi(x)\,, & x>0\,.\end{cases}$$

For this initial condition in the infinite domain, the solution is

$$u(x,t)=\int_{-\infty}^{+\infty}V(x,\xi,t)\Phi(\xi)\,\mathrm{d}\xi\,.$$

Therefore,

$$\begin{aligned}u=W_\varphi(x,t)&=\int_0^{+\infty}V(x,\xi,t)\Phi(\xi)\,\mathrm{d}\xi+\int_{-\infty}^{0}V(x,\xi,t)\Phi(\xi)\,\mathrm{d}\xi\\ &=\int_0^{+\infty}V(x,\xi,t)\varphi(\xi)\,\mathrm{d}\xi-\int_0^{+\infty}V(x,-\xi,t)\varphi(\xi)\,\mathrm{d}\xi\\ &=\int_0^{+\infty}\varphi(\xi)\left[V(x,\xi,t)-V(x,-\xi,t)\right]\mathrm{d}\xi\,.\end{aligned}$$

The solution of PDS (3.36) with $\varphi(x)=0$ is thus, by the solution structure theorem,

$$u=\int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau=\int_0^t\mathrm{d}\tau\int_0^{+\infty}f(\xi,\tau)\left[V(x,\xi,t-\tau)-V(x,-\xi,t-\tau)\right]\mathrm{d}\xi\,.$$

The solution of PDS (3.36) is, by the principle of superposition,

$$\begin{aligned}u&=W_\varphi(x,t)+\int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau\\ &=\int_0^{+\infty}\varphi(\xi)\left[V(x,\xi,t)-V(x,-\xi,t)\right]\mathrm{d}\xi\\ &\quad+\int_0^t\mathrm{d}\tau\int_0^{+\infty}f(\xi,\tau)\left[V(x,\xi,t-\tau)-V(x,-\xi,t-\tau)\right]\mathrm{d}\xi\,.\end{aligned}$$

Remark 1. The solution of PDS (3.36) at $\varphi=0$ can also be obtained by a continuation of nonhomogeneous term $f(x,t)$.

Boundary Condition of the Second Kind and the Third Kind

Example 2. Find the solution of

$$\begin{cases} u_t = a^2 u_{xx}\,, \quad 0 < x < +\infty, 0 < t\,, \\ (u_x - hu)|_{x=0} = 0, h > 0\,, \\ u(x,0) = \varphi(x)\,. \end{cases} \tag{3.37}$$

Solution. Let $v(x)$ be the initial distribution in $(-\infty, 0)$, which is the continuation of $\varphi(x)$. The solution of PDS (3.37) is thus, by Eq. (3.27),

$$\begin{aligned} u &= \int_{-\infty}^{0} v(\xi)V(x,\xi,t)\,\mathrm{d}\xi + \int_{0}^{+\infty} \varphi(\xi)V(x,\xi,t)\,\mathrm{d}\xi \\ &= \int_{0}^{+\infty} [\varphi(\xi)V(x,\xi,t) + v(-\xi)V(x,-\xi,t)]\,\mathrm{d}\xi \\ &= \int_{0}^{+\infty} \frac{1}{2a\sqrt{\pi t}} \left[\varphi(\xi)\,\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2 t}} + v(-\xi)\,\mathrm{e}^{-\frac{(x+\xi)^2}{4a^2 t}}\right] \mathrm{d}\xi\,. \end{aligned}$$

Applying the boundary condition leads to

$$(u_x - hu)|_{x=0} = \int_{0}^{+\infty} \frac{1}{2a\sqrt{\pi t}}\,\mathrm{e}^{-\frac{\xi^2}{4a^2 t}} \left[\frac{\xi\varphi(\xi) - \xi v(-\xi)}{2a^2 t} - h\varphi(\xi) - hv(-\xi)\right] \mathrm{d}\xi = 0\,,$$

which can be satisfied by

$$\frac{\xi\varphi(\xi) - \xi v(-\xi)}{2a^2 t} - h\varphi(\xi) - hv(-\xi) = 0 \quad \text{or} \quad v(-\xi) = \frac{\xi - 2a^2 ht}{\xi + 2a^2 ht}\varphi(\xi)\,.$$

Therefore, the solution of PDS (3.37) becomes

$$\begin{aligned} u &= W_\varphi(x,t) \\ &= \int_{0}^{+\infty} \varphi(\xi) \left[V(x,\xi,t) + \frac{\xi - 2a^2 ht}{\xi + 2a^2 ht} V(x,-\xi,t)\right] \mathrm{d}\xi\,. \end{aligned} \tag{3.38}$$

Remark 2. Consider

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t)\,, \quad 0 < x < +\infty, 0 < t\,, \\ (u_x - hu)|_{x=0} = 0\,, \\ u(x,0) = 0\,. \end{cases} \tag{3.39}$$

The solution of PDS (3.39) follows directly from the solution structure theorem as

$$\begin{aligned} u &= \int_0^t W_{f\tau}(x,t-\tau)\,\mathrm{d}\tau \\ &= \int_0^t \mathrm{d}\tau \int_0^{+\infty} f(\xi,\tau)\left[V(x,\xi,t-\tau) + \frac{\xi - 2a^2(t-\tau)h}{\xi + 2a^2(t-\tau)h} V(x,-\xi,t-\tau)\right] \mathrm{d}\xi \,. \end{aligned} \tag{3.40}$$

Remark 3. Solutions of PDS (3.37) and (3.39) that are subjected to boundary condition of the second kind can be obtained by letting $h = 0$ in Eqs. (3.38) and (3.40).

3.4.4 PDS with Variable Thermal Conductivity

When thermal conductivity is variable so that $k = k(x,t)$, the task of seeking PDS solutions of heat-conduction equations becomes difficult. There exists no well-developed method for developing analytical solutions. Numerical method are normally appealed for numerical solutions. Here we detail the procedure of developing analytic solutions of heat-conduction equations with variable thermal conductivity of type $k = k(x)$ by using the example

$$\begin{cases} \rho c u_t = \dfrac{\partial}{\partial x}[k(x)u_x]\,, \ (0,l)\times(0,+\infty)\,, \\ u(0,t) = u(l,t) = 0\,, \ u(x,0) = f(x)\,, \end{cases} \tag{3.41}$$

where ρ and c are the density and the specific heat of the material. The thermal conductivity $k(x) > 0$ is a differentiable function of x.

Transform into an Eigenvalue Problem of an Integral Equation

Consider a solution of type $u(x,t) = X(x)T(t)$. Substitute it into the equation of PDS (3.41) to obtain eigenvalue problems

$$\frac{\mathrm{d}}{\mathrm{d}x}[k(x)X_x] + \lambda X = 0\,, \quad X(0) = X(l) = 0 \tag{3.42}$$

and

$$T_t + \frac{\lambda}{\rho c} T = 0\,. \tag{3.43}$$

Here λ is the separation constant.

First find the Green function $G(x,s)$ that satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left[k(x)\dfrac{\mathrm{d}G(x,s)}{\mathrm{d}x}\right] = \delta(x-s)\,, \\ G(0,s) = G(l,s) = 0\,. \end{cases} \tag{3.44}$$

The general solution of $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[k(x)\dfrac{\mathrm{d}G(x,s)}{\mathrm{d}x}\right] = 0$ is

$$\bar{G} = c_1 u_1(x) + c_2 u_2(x)\,, \quad c_1, c_2 \text{ are constants}\,.$$

where

$$u_1(x) = 1\,, u_2(x) = \int \frac{\mathrm{d}x}{k(x)}\,.$$

Let

$$G(x,s) = \begin{cases} (a_1+b_1)u_1 + (a_2+b_2)u_2\,, x \le s\,, \\ (a_1-b_1)u_1 + (a_2-b_2)u_2\,, x \ge s\,. \end{cases} \tag{3.45}$$

Applying the property of G and $\dfrac{\mathrm{d}G}{\mathrm{d}x}$ at $x = s$,

$$G(s^+,s) - G(s^-,s) = 0\,, \quad \frac{\mathrm{d}G(s^+,s)}{\mathrm{d}x} - \frac{\mathrm{d}G(s^-,s)}{\mathrm{d}x} = \frac{1}{k(s)}$$

yields

$$b_1 = \frac{u_2(s)}{2u_2'(s)k(s)}\,, \quad b_2 = \frac{-1}{2u_2'(s)k(s)}\,.$$

Applying the boundary conditions $G(0,s) = G(l,s) = 0$ thus leads to

$$a_1 = -b_1 - (a_2+b_2)u_2(0)\,, \quad a_2 = \frac{-2b_1 - b_2[u_2(0)+u_2(l)]}{u_2(0)-u_2(l)}\,.$$

With the Green function $G(x,s)$, we transform PDS (3.42) into an eigenvalue problem of the integral equation

$$X(x) = -\lambda \int_0^l G(x,s)X(s)\,\mathrm{d}s\,. \tag{3.46}$$

Method of Solving (3.46)

Expand $G(x,s) = \sum_{n=1}^{\infty} a_n(s) \sin \frac{n\pi x}{l}$. Substitute it into the integral equation to obtain

$$X(x) = -\lambda \sum_{n=1}^{\infty} X_n \sin \frac{n\pi x}{l}, \quad X_n = -\lambda \int_0^l X(s) a_n(s)\,\mathrm{d}s \tag{3.47}$$

Multiplying the equation of (3.47) by $a_m(x)$ and then integrating over $[0,l]$ yields

$$X_m = -\lambda \sum_{n=1}^{\infty} a_{mn} X_n, \quad a_{mn} = \int_0^l a_m(x) \sin \frac{n\pi x}{l}\,\mathrm{d}x, \quad m = 1,2,\cdots \tag{3.48}$$

Let $\mathbf{A} = (a_{mn})_{\infty\times\infty}$, $\mathbf{X} = (X_1, X_2, \cdots, X_n, \cdots)^T$. Eq. (3.48) can then be written as an eigenvalue problem of $\mathbf{A}$

$$\mathbf{X} = -\lambda \mathbf{A}\mathbf{X}. \tag{3.49}$$

We denote its eigenvalues and the corresponding eigenvectors by

$$\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots,$$

$$\mathbf{X}_1, \mathbf{X}_2, \cdots, X_n, \cdots.$$

The eigenfunction set associated with the eigenvalues $\lambda_n (n = 1,2,\cdots)$

$$X_1(x), X_2(x), \cdots, X_n(x), \cdots$$

can thus be obtained by an substitution into Eq. (3.47). The $T(t)$ associated with $\lambda = \lambda_n$ can also be obtained by Eq. (3.43),

$$T_n(t) = \mathrm{e}^{-(\lambda_n t/\rho c)}, \quad n = 1,2,\cdots.$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} C_n T_n(t) X_n(x) = \sum_{n=1}^{\infty} C_n X_n(x)\,\mathrm{e}^{-(\lambda_n t/\rho c)},$$

which satisfies the equation and the boundary conditions of (3.41). Here C_n $(n = 1,2,\cdots)$ are constants and can be determined by applying the initial condition $u(x,0) = f(x)$ and using the completeness and orthogonality of $\{X_n(x)\}_{n=1}^{\infty}$ in $[0,l]$.

Analytical solution of PDS (3.41)

The analytical solution of PDS (3.41) follows from the above discussion,

$$\begin{cases} u(x,t) = \sum\limits_{n=1}^{\infty} C_n X_n(x)\,\mathrm{e}^{-(\lambda_n t/\rho c)}, \\ C_n = \int_0^l X_n(x) f(x)\,\mathrm{d}x \Big/ \int_0^l X_n^2(x)\,\mathrm{d}x. \end{cases}$$

3.5 Multiple Fourier Transformations for Two- and Three-Dimensional Cauchy Problems

3.5.1 Two-Dimensional Case

Find the solution of

$$\begin{cases} u_t = a^2 \Delta u\,, \quad R^2 \times (0,+\infty)\,, \\ u(x,y,0) = \varphi(x,y)\,, \end{cases} \tag{3.50}$$

where $R^2 \times (0,+\infty)$ stands for $-\infty < x,y < +\infty$. $t \in (0,+\infty)$.

Solution. Apply a double Fourier transformation with respect to x and y to PDS (3.50) to obtain

$$\bar{u}_t + (\omega a)^2 \bar{u} = 0\,, \bar{u}(\omega,0) = \bar{\varphi}(\omega)\,, \tag{3.51}$$

where $\bar{u}(\omega,t) = F\left[u(x,y,t)\right]$, $\omega^2 = \omega_1^2 + \omega_2^2$ (see Appendix B). The solution of Eq. (3.51) is

$$\bar{u} = A(\omega)\,\mathrm{e}^{-(\omega a)^2 t} = \bar{\varphi}(\omega)\,\mathrm{e}^{-(\omega a)^2 t}\,, \tag{3.52}$$

where $\bar{\varphi}(\omega) = F\left[\varphi(x,y)\right]$. It follows from Section 3.4.1 that

$$F^{-1}\left[\mathrm{e}^{-(\omega_1 a)^2 t}\right] = \frac{1}{2a\sqrt{\pi t}}\,\mathrm{e}^{-\frac{x^2}{4a^2 t}}\,, \quad F^{-1}\left[\mathrm{e}^{-(\omega_2 a)^2 t}\right] = \frac{1}{2a\sqrt{\pi t}}\,\mathrm{e}^{-\frac{y^2}{4a^2 t}}\,,$$

$$F^{-1}\left[\mathrm{e}^{-(\omega a)^2 t}\right] = F^{-1}\left[\mathrm{e}^{-(\omega_1^2+\omega_2^2)a^2 t}\right] = \frac{1}{4a^2 \pi t}\,\mathrm{e}^{-\frac{x^2+y^2}{4a^2 t}}\,.$$

Therefore, the solution of Eq. (3.50) is, by a double inverse Fourier transformation to Eq. (3.52) and the convolution theorem,

$$u = W_\varphi(x,y,t) = \iint\limits_{R^2} \varphi(\xi,\eta) V(x,\xi,t) V(y,\eta,t)\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \tag{3.53}$$

By the solution structure theorem,

$$u = \int_0^t W_{f_\tau}(x,y,t-\tau)\,\mathrm{d}\tau = \int_0^t \mathrm{d}\tau \iint\limits_{R^2} f(\xi,\eta,\tau) V(x,\xi,t-\tau) V(y,\eta,t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{3.54}$$

is the solution of

$$\begin{cases} u_t = a^2 \Delta u + f(x,y,t), & R^2 \times (0,+\infty)\,, \\ u(x,y,0) = 0\,. \end{cases}$$

The sum of Eqs. (3.53) and (3.54) is, by the principle of superposition, the solution of

$$\begin{cases} u_t = a^2 \Delta u + f(x,y,t)\,, & R^2 \times (0,+\infty)\,, \\ u(x,y,0) = \varphi(x,y)\,. \end{cases}$$

3.5.2 Three-Dimensional Case

We can apply a triple Fourier transformation followed by a triple inverse transformation to solve three-dimensional Cauchy problems. The method and the procedure are similar to those in Section 3.5.1. Here, we simply list the final result. For PDS

$$\begin{cases} u_t = a^2 \Delta u + f(x,y,z,t)\,, & R^3 \times (0,+\infty)\,, \\ u(x,y,z,0) = \varphi(x,y,z)\,. \end{cases}$$

The solution is

$$\begin{aligned} u &= W_\varphi(x,y,z,t) + \int_0^t W_{f_\tau}(x,y,z,t-\tau)\,\mathrm{d}\tau \\ &= \iiint\limits_{R^3} \varphi(\xi,\eta,\zeta) V(x,\xi,t) V(y,\eta,t) V(z,\zeta,t)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \\ &\quad + \int_0^t \mathrm{d}\tau \iiint\limits_{R^3} f(\xi,\eta,\zeta,\tau) V(x,\xi,t-\tau) V(y,\eta,t-\tau) V(z,\zeta,t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,. \end{aligned}$$

3.6 Typical PDS of Diffusion

The heat-conduction equation is used not only to represent the diffusion of heat, but also for other diffusions such as material diffusion. In this section, we discuss three typical PDS and their solutions arising from material diffusion in manufacturing silicon transistors to demonstrate the application of results in Section 3.4.

3.6.1 Fick's Law of Diffusion and Diffusion Equation

The diffusion of material is widely encountered in many diverse fields of science and technology. In manufacturing silicon transistors, for example, we use diffusion to mix silicon with other materials. The silicon plates are surrounded by steam containing the mixed material in a high-temperature oven. The mixed material diffuses into the plates like the diffusion of red ink in water. Using planographic technology, the diffusion of the mixed material is along one direction only. Such kinds of diffusion are termed as unidirectional diffusion. We refer to this kind of diffusion hereinafter in this section unless otherwise stated.

Diffusion always flows from a place with high density to a place of low density. The density of the mixed material is defined as the particle number per unit volume and is denoted by N. N is normally a function of position and time. In unidirectional diffusion, $N = N(x,t)$. $\dfrac{\partial N}{\partial x}$ is called the *density gradient* and plays the same role in material diffusion as the temperature gradient in heat conduction. The flux density is the particle number passing through the unit area in a unit time period and is denoted by j.

Fick's first law of diffusion. The flux density j is proportional to the density gradient. Therefore,

$$j = -D\frac{\partial N}{\partial x}, \tag{3.55}$$

where D is called the *diffusion coefficient*. The negative sign accounts for the fact that the diffusion is from an area with high density to another of low density. The unit of the diffusion coefficient is $\frac{m^2}{s}$. The unit of Eq. (3.55) is thus

$$\frac{Particle\,Number}{m^2 \cdot s} = \frac{m^2}{s} \cdot \frac{Particle\,Number}{m^4}.$$

Fick's law of diffusion is similar to the Fourier law of heat conduction. By the conservation of mass and Eq. (3.55), we can readily obtain the one-dimensional diffusion equation

$$\frac{\partial N}{\partial t} = D\frac{\partial^2 N}{\partial x^2} \quad \text{or} \quad N_t = DN_{xx}. \tag{3.56}$$

Similarly, the two- and the three-dimensional diffusion equation is

$$\frac{\partial N}{\partial t} = D\Delta N,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{or} \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

For the slow diffusion which we consider in this section, the domain of variation of x can be viewed as $-\infty < x < +\infty$ or $0 \le x < +\infty$. When manufacturing the silicon transistors, there is no source of mixed material inside the silicon plates. Therefore, we only consider Cauchy problems of homogeneous diffusion equations.

3.6.2 Diffusion from a Constant Source

Diffusion from a constant source refers to the particle diffusion in a semi-infinite domain without initial distribution of particles from a constant source of density N_0 at $x = 0$. Diffusion in closed tubes and boxes are all of this kind. The particle density $N(x,t)$ should thus satisfy the PDS

$$\begin{cases} N_t = DN_{xx}\,, \quad 0 < x < +\infty, 0 < t\,, \\ N(0,t) = N_0\,, \\ N(x,0) = 0. \end{cases} \tag{3.57}$$

Let $N(x,t) = v(x,t) + N_0$; thus $v(x,t)$ is the solution of a PDS with homogeneous boundary conditions

$$\begin{cases} v_t = Dv_{xx}\,, \quad 0 < x < +\infty, 0 < t\,, \\ v(0,t) = 0\,, \\ v(x,0) = -N_0\,. \end{cases} \tag{3.58}$$

By an odd continuation, PDS (3.58) becomes

$$\begin{cases} v_t = Dv_{xx}\,, \quad -\infty < x < +\infty, 0 < t, \\ v(0,t) = 0\,,\ v(x,0) = \begin{cases} -N_0\,, & x > 0\,, \\ 0\,, & x = 0\,, \\ N_0\,, & x < 0\,. \end{cases} \end{cases} \tag{3.59}$$

Note that the boundary homogenization and the continuation of the initial contribution indeed preserve the CDS in (3.57).

The solution of PDS (3.59) is, by the result in Section 3.4,

$$v(x,t) = \int_{-\infty}^{0} N_0 V(x,\xi,t)\,\mathrm{d}\xi + \int_{0}^{+\infty} (-N_0) V(x,\xi,t)\,\mathrm{d}\xi\,.$$

After performing the variable transformation of $\eta = \dfrac{x-\xi}{2\sqrt{Dt}}$ and $\eta = \dfrac{\xi - x}{2\sqrt{Dt}}$ for the integrand in the first and the second integrals, we obtain

$$v(x,t) = -\frac{N_0}{\sqrt{\pi}} \int_{-\frac{x}{2\sqrt{Dt}}}^{\frac{x}{2\sqrt{Dt}}} \mathrm{e}^{-\eta^2}\,\mathrm{d}\eta = -N_0 \mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right),$$

where the *Gauss error function* $\mathrm{erf}\,x$ is defined by

$$\mathrm{erf}\,x = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\,.$$

Finally, the solution of PDS (3.57) is

$$N(x,t) = N_0\left(1 - \mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right)\right) = N_0 \mathrm{erfc}\left(\frac{x}{2\sqrt{Dt}}\right).$$

3.6.3 Diffusion from an Instant Plane Source

We still use diffusion of mixed materials into a silicon plate to describe the physical model. The instant plane refers to the part of the plate surface where a very thin layer of mixed material is deposited. In a very short time period from the start of diffusion, all the mixed material on the plate is inside the silicon plate, and there is no further material being mixed into the plate surface afterwards. Therefore, the total amount of mixed material inside the silicon plate is constant with respect to time. As the thickness of the mixed material layer deposited on the plate surface is infinitesimal, such kind of diffusion is sometimes called the *diffusion from the infinitesimal layer*. For the one-dimensional case, the particle density $N(x,t)$ should then satisfy

$$\begin{cases} N_t = DN_{xx}\,, & -\infty < x < +\infty,\ 0 < t\,, \\ N(x,0) = \varphi(x) = \begin{cases} N_0\,, & |x| \le h\,, \\ 0\,, & |x| > h\,, \end{cases} \end{cases} \tag{3.60}$$

where h is a very small positive constant. The total particle number is $Q = 2hN_0$. The solution of PDS (3.60) is, by the result in Section 3.4,

$$N(x,t) = \int_{-\infty}^{+\infty} \varphi(\xi)V(x,\xi,t)\,\mathrm{d}\xi = \int_{-h}^{h} N_0 V(x,\xi,t)\,\mathrm{d}\xi \tag{3.61}$$

$$= \frac{Q}{2\sqrt{\pi Dt}}\frac{1}{2h}\int_{-h}^{h} \mathrm{e}^{-\frac{(x-\xi)^2}{4Dt}}\,\mathrm{d}\xi = \frac{Q}{2\sqrt{\pi Dt}}\mathrm{e}^{-\frac{(x-\bar{\xi})^2}{4Dt}}, \tag{3.62}$$

where the mean value theorem of integrals has been used, and $-h < \bar{\xi} < h$.

Now concentrate all particles Q to $x = 0$, so the initial density becomes

$$N(x,0) = \begin{cases} \infty\,, \; x = 0\,, \\ 0\,, \; x \neq 0\,, \end{cases}$$

and $h \to +0$ so that $\bar{\xi} \to 0$. The density distribution reduces to

$$N(x,t) = \lim_{h\to+0} \frac{Q}{2\sqrt{\pi D t}}\,\mathrm{e}^{-\frac{(x-\bar{\xi})^2}{4Dt}} = \frac{Q}{2\sqrt{\pi D t}}\,\mathrm{e}^{-\frac{x^2}{4Dt}}\,. \tag{3.63}$$

Remark. This method of arriving at Eq. (3.63) actually demonstrates again the physical background of the Dirac function, which is beneficial for understanding it. However, it would be more straightforward to obtain Eq. (3.63) if we apply the Dirac function directly to the initial distribution. By using the Dirac function, the PDS reads

$$\begin{cases} N_t = DN_{xx}\,, \quad -\infty < x < +\infty, 0 < t\,, \\ N(x,0) = Q\delta(x)\,. \end{cases}$$

Its solution is readily available from Section 3.4,

$$N(x,t) = \int_{-\infty}^{+\infty} Q\delta(\xi)V(x,\xi,t)\,\mathrm{d}\xi = \frac{Q}{2\sqrt{\pi D t}}\,\mathrm{e}^{-\frac{x^2}{4Dt}}\,,$$

which is the same as Eq. (3.63).

3.6.4 Diffusion Between Two Semi-Infinite Domains

The density of the mixed material is N_0 (constant) and zero in semi-infinite domains $x < 0$ and $x > 0$, respectively. This density difference drives the diffusion between the two domains. The density $N(x,t)$ should thus satisfy

$$\begin{cases} N_t = DN_{xx}\,, \; -\infty < x < +\infty\,, 0 < t\,, \\ N(x,0) = \begin{cases} N_0\,, \; x < 0\,, \\ 0\,, \quad x > 0\,. \end{cases} \end{cases}$$

Its solution is, by Eq. (3.27) in Section 3.4,

$$N(x,t) = \int_{-\infty}^{0} N_0 V(x,\xi,t)\,\mathrm{d}\xi = \frac{N_0}{2\sqrt{\pi D t}} \int_{-\infty}^{0} \mathrm{e}^{-\frac{(x-\xi)^2}{4Dt}}\,\mathrm{d}\xi$$

$$= \frac{N_0}{2\sqrt{\pi D t}} \int_{0}^{+\infty} \mathrm{e}^{-\frac{(x+\xi)^2}{4Dt}}\,\mathrm{d}\xi = \frac{N_0}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{Dt}}}^{+\infty} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi$$

$$= \frac{N_0}{\sqrt{\pi}} \left[\int_{0}^{+\infty} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi - \int_{0}^{\frac{x}{2\sqrt{Dt}}} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi \right]$$

$$= \frac{N_0}{2} \left(1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x}{2\sqrt{Dt}}} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi \right) = \frac{N_0}{2} \operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}} \right).$$

At $x = 0$, $N(0,t) = \frac{N_0}{2}$. Therefore, the density of the mixed material is constant at the interface $x = 0$ and is equal to the half of the initial density N_0.

Chapter 4
Mixed Problems of Hyperbolic Heat-Conduction Equations

We first develop the solution structure theorem for solving mixed problems of hyperbolic heat-conduction equations. We then apply the Fourier method of expansion to solve one-, two- and three-dimensional mixed problems.

4.1 Solution Structure Theorem

The hyperbolic heat-conduction equation reads

$$u_t + \tau_0 u_{tt} = a^2 \Delta u + F(M,t)\,, \tag{4.1}$$

or

$$\frac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u + f(M,t)\,, \tag{4.2}$$

where $A^2 = a^2/\tau_0$, $f(M,t) = F(M,t)/\tau_0$ and M stands for point x, (x,y) and (x,y,z) in one-, two- and three-dimensional space, respectively. The wave equation and the heat-conduction equation are two special cases of the hyperbolic heat-conduction equation at $\tau_0 \to \infty$ and $\tau_0 = 0$, respectively. A unit analysis shows that τ_0 and A have the dimensions of time and velocity, respectively. For most materials, τ_0 has a small positive value so u has some weak properties of waves.

Consider mixed problems of hyperbolic heat-conduction equations in a closed region $\overline{\Omega}$. Let $\overline{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega$ is the boundary surface of Ω . Three kinds of linear homogeneous boundary conditions may, therefore, be written as

$$L(u,u_n)|_{\partial\Omega} = 0\,,$$

where u_n stands for the normal derivative of u.

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Theorem. Let $u_2 = W_\psi(M,t)$ denote the solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2\Delta u\,, \quad \Omega\times(0,+\infty)\,, \\ L\left(u,\dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} = 0\,, \\ u(M,0)=0, u_t(M,0)=\psi(M) \end{cases} \tag{4.3}$$

The solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2\Delta u + f(M,t)\,, \quad \Omega\times(0,+\infty)\,, \\ L\left(u,\dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} = 0\,, \\ u(M,0)=\varphi(M),\ \ u_t(M,0)=\psi(M)\,. \end{cases} \tag{4.4}$$

can be written as

$$u = u_1+u_2+u_3 = \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t) + W_\psi(M,t) + \int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau\,, \tag{4.5}$$

where $f_\tau = f(M,\tau)$. Here $u_1 = \left(\dfrac{1}{\tau_0}+\dfrac{\partial}{\partial t}\right)W_\varphi(M,t)$ is the solution of (4.4) at $f=\psi=0$. $u_3 = \displaystyle\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau$ is the solution of (4.4) at $\varphi=\psi=0$.

Proof.

1. As $W_\varphi(M,t)$ satisfies

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_\varphi}{\partial t} + \dfrac{\partial^2 W_\varphi}{\partial t^2} = A^2\Delta W_\varphi\,, \quad \Omega\times(0,+\infty)\,, \\ L\left(W_\varphi,\dfrac{\partial W_\varphi}{\partial n}\right)\Big|_{\partial\Omega} = 0 \\ W_\varphi\big|_{t=0} = 0\,, \quad \dfrac{\partial W_\varphi}{\partial t}\Big|_{t=0} = \varphi(M)\,. \end{cases}$$

Hence

$$\frac{1}{\tau_0}\frac{\partial u_1}{\partial t} + \frac{\partial^2 u_1}{\partial t^2} - A^2\Delta u_1$$

$$= \frac{1}{\tau_0}\frac{\partial}{\partial t}\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi + \frac{\partial^2}{\partial t^2}\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi - A^2\Delta\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi$$

$$= \frac{1}{\tau_0}\left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t} + \frac{\partial^2 W_\varphi}{\partial t^2} - A^2 \Delta W_\varphi\right) + \frac{\partial}{\partial t}\left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t} + \frac{\partial^2 W_\varphi}{\partial t^2} - A^2 \Delta W_\varphi\right) = 0,$$

which indicates that u_1 satisfies the equation of PDS (4.4) at $f = 0$. Also

$$\begin{aligned} L\left(u_1, \frac{\partial u_1}{\partial n}\right)\Big|_{\partial\Omega} &= L\left[\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi, \frac{\partial}{\partial n}\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi\right]\Big|_{\partial\Omega} \\ &= \frac{1}{\tau_0} L\left(W_\varphi, \frac{\partial W_\varphi}{\partial n}\right)\Big|_{\partial\Omega} + \left[\frac{\partial}{\partial t} L\left(W_\varphi, \frac{\partial W_\varphi}{\partial n}\right)\right]\Big|_{\partial\Omega} \\ &= \frac{1}{\tau_0} L\left(W_\varphi, \frac{\partial}{\partial n} W_\varphi\right)\Big|_{\partial\Omega} + \frac{\partial}{\partial t}\left[L\left(W_\varphi, \frac{\partial W_\varphi}{\partial n}\right)\Big|_{\partial\Omega}\right] = 0. \end{aligned}$$

This shows that u_1 satisfies the boundary conditions of PDS (4.4).

Finally,

$$u_1(M,0) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi\Big|_{t=0} = \frac{1}{\tau_0} W_\varphi\Big|_{t=0} + \frac{\partial W_\varphi}{\partial t}\Big|_{t=0} = \varphi(M).$$

Also,

$$\begin{aligned} \frac{\partial u_1}{\partial t}\Big|_{t=0} &= \frac{\partial}{\partial t}\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi\Big|_{t=0} = \left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t} + \frac{\partial^2 W_\varphi}{\partial t^2}\right)\Big|_{t=0} \\ &= A^2 \Delta W_\varphi\big|_{t=0} = 0. \end{aligned}$$

Therefore, the $u_1 = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(M,t)$ is the solution of PDS (4.4) at $f = \psi = 0$.

2. Since $W_{f_\tau}(M, t-\tau)$ satisfies

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial W_{f_\tau}}{\partial t} + \frac{\partial^2 W_{f_\tau}}{\partial t^2} = A^2 \Delta W_{f_\tau}, & \Omega, 0 < \tau < t < +\infty, \\ L\left(W_{f_\tau}, \frac{\partial W_{f_\tau}}{\partial n}\right)\Big|_{\partial\Omega} = 0, \\ W_{f_\tau}\big|_{t=\tau} = 0, \ \frac{\partial}{\partial t} W_{f_\tau}\Big|_{t=\tau} = f(M,\tau), \end{cases}$$

therefore

$$\frac{1}{\tau_0}\frac{\partial u_3}{\partial t} + \frac{\partial^2 u_3}{\partial t^2} - A^2 \Delta u_3 = \frac{1}{\tau_0}\frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M, t-\tau)\,\mathrm{d}\tau$$

$$+ \frac{\partial^2}{\partial t^2}\int_0^t W_{f_\tau}(M, t-\tau)\,\mathrm{d}\tau - A^2 \Delta \int_0^t W_{f_\tau}(M, t-\tau)\,\mathrm{d}\tau$$

$$= \frac{1}{\tau_0}\int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + \int_0^t \frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau + \left.\frac{\partial W_{f_\tau}}{\partial t}\right|_{\tau=t} - A^2\int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau$$

$$= \int_0^t \left(\frac{1}{\tau_0}\frac{\partial W_{f_\tau}}{\partial t} + \frac{\partial^2 W_{f_\tau}}{\partial t^2} - A^2\Delta W_{f_\tau}\right)\mathrm{d}\tau + f(M,t) = f(M,t)\,,$$

which indicates that u_3 satisfies the equation of PDS (4.4).
Also,

$$L\left(u_3, \frac{\partial u_3}{\partial n}\right)\bigg|_{\partial\Omega} = L\left[\int_0^t W_{f_\tau}\,\mathrm{d}\tau, \frac{\partial}{\partial n}\int_0^t W_{f_\tau}\,\mathrm{d}\tau\right]\bigg|_{\partial\Omega}$$

$$= \int_0^t L\left(W_{f_\tau}, \frac{\partial W_{f_\tau}}{\partial n}\right)\bigg|_{\partial\Omega}\mathrm{d}\tau = 0\,,$$

which shows that the u_3 satisfies the boundary conditions of PDS (4.4).
Finally,

$$u_3(M,0) = 0 \text{ and } \left.\frac{\partial u_3}{\partial t}\right|_{t=0} = \left(\int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}\big|_{\tau=t}\right)\bigg|_{t=0} = 0\,.$$

Therefore, the u_3 also satisfies the initial conditions so $u_3 = \displaystyle\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau$ is indeed the solution of PDS (4.4) at $\varphi = \psi = 0$.

3. Since PDS (4.4) is linear, the principle of superposition is valid. Applying this principle to PDS (4.4) concludes that $u = u_1 + u_2 + u_3$ is the solution of PDS (4.4).

Remark 1. The solution of

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2\Delta u + f(M,t)\,, & \Omega\times(0,+\infty)\,, \\ L\left(u, \dfrac{\partial u}{\partial n}\right)\bigg|_{\partial\Omega} = 0\,, \\ u(M,0) = \varphi(M), \quad u_t(M,0) = \psi(M) \end{cases} \tag{4.6}$$

can still be written as

$$u = u_1 + u_2 + u_3 = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi(M,t) + W_\psi(M,t) + \int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau\,.$$

Here $u_2 = W_\psi(M,t)$ is the solution of PDS (4.6) at $\varphi = f = 0$; $u_1 = \left(\dfrac{1}{\tau_0} + \dfrac{\partial}{\partial t}\right)W_\varphi(M,t)$ is the solution of PDS (4.6) at $\psi = f = 0$; $u_3 = \displaystyle\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau$ is the solution of PDS (4.6) at $\varphi = \psi = 0$. The f_τ is defined by $f_\tau = \dfrac{1}{\tau_0} f(M,\tau)$.

$W_{f_\tau}(M,t-\tau)$ thus satisfies

$$\begin{cases} \dfrac{\partial W_{f_\tau}}{\partial t} + \tau_0 \dfrac{\partial^2 W_{f_\tau}}{\partial t^2} = a^2 \Delta W_{f_\tau}, \quad \Omega,\, 0<\tau<t<+\infty, \\ L\left(W_{f_\tau}, \dfrac{\partial W_{f_\tau}}{\partial n}\right)\Big|_{\partial\Omega} = 0, \\ W_{f_\tau}\big|_{t=\tau} = 0, \quad \dfrac{\partial}{\partial t} W_{f_\tau}\Big|_{t=\tau} = \dfrac{1}{\tau_0} f(M,\tau). \end{cases}$$

Remark 2. The solution structure theorem is also valid for Cauchy problems. However, the structure of $W_\psi(M,t)$ differs from that of mixed problems.

4.2 One-Dimensional Mixed Problems

In this section, we use Table 2.1, the solution structure theorem and the Fourier method of expansion to solve

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2 u_{xx} + f(x,t), \quad (0,l)\times(0,+\infty), \\ L_1(u,u_x)|_{x=0} = L_2(u,u_x)|_{x=l} = 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \end{cases} \tag{4.7}$$

If all combinations of linear homogeneous boundary conditions of the first, second and third kinds are considered at $x=0$ and $x=l$, there exist nine combinations of boundary conditions. We detail the process of seeking the solutions of PDS (4.7) for two combinations of boundary conditions. The results for the remaining seven combinations can be readily obtained by using a similar approach.

4.2.1 Mixed Boundary Conditions of the First and the Third Kind

Consider the boundary conditions of type

$$u(0,t) = u_x(l,t) + hu(l,t) = 0. \tag{4.8}$$

By the solution structure theorem, we first develop $W_\psi(x,t)$, the solution for the case of $\varphi = f = 0$. Based on the given boundary conditions (4.8), we should use the eigenfunctions in Row 3 in Table 2.1 to expand the solution

$$u(x,t) = \sum_{m=1}^{+\infty} T_m(t) \sin \frac{\mu_m x}{l}, \tag{4.9}$$

where $T_m(t)$ is a function to be determined later and μ_m are the positive zero-points of $f(x) = \tan x + \frac{x}{lh}$. Substituting Eq. (4.9) into the equation of (4.7) with $f = 0$ yields

$$\tau_0 T_m''(t) + T_m'(t) + \left(\frac{\mu_m a}{l}\right)^2 T_m(t) = 0. \tag{4.10}$$

Its general solution can be easily found as

$$T_m(t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \left(a_m \cos\gamma_m t + b_m \underline{\sin}\gamma_m t\right), \tag{4.11}$$

where a_m and b_m are undetermined constants,

$$\gamma_m = \frac{1}{2\tau_0}\sqrt{4\tau_0\left(\frac{\mu_m a}{l}\right)^2 - 1} \text{ and } \underline{\sin}\gamma_m t = \begin{cases} \sin\gamma_m t, & \gamma_m \neq 0, \\ t, & \gamma_m = 0. \end{cases} \tag{4.12}$$

We thus have

$$u(x,t) = \sum_{m=1}^{+\infty} \mathrm{e}^{-\frac{t}{2\tau_0}} \left(a_m \cos\gamma_m t + b_m \underline{\sin}\gamma_m t\right) \sin\frac{\mu_m x}{l}.$$

Applying the initial condition $u(x,0) = 0$ leads to $a_m = 0$. Applying the initial condition $u_t(x,0) = \psi(x)$ yields

$$\sum_{m=1}^{+\infty} b_m \underline{\gamma}_m \sin\frac{\mu_m x}{l} = \psi(x), \quad \underline{\gamma}_m = \begin{cases} \gamma_m, & \gamma_m \neq 0, \\ 1, & \gamma_m = 0. \end{cases}$$

which requires

$$b_m = \frac{1}{\underline{\gamma}_m M_m}\int_0^l \psi(x) \sin\frac{\mu_m x}{l}\, \mathrm{d}x.$$

Here M_m is the normal square of the eigenfunction set $\left\{\sin\frac{\mu_m x}{l}\right\}$.

Finally, we have

$$\begin{cases} u(x,t) = W_\psi(x,t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \displaystyle\sum_{m=1}^{+\infty} b_m \underline{\sin}\gamma_m t \sin\frac{\mu_m x}{l}, \\ b_m = \dfrac{1}{\underline{\gamma}_m M_m}\displaystyle\int_0^l \psi(x) \sin\frac{\mu_m x}{l}\, \mathrm{d}x. \end{cases} \tag{4.13}$$

Therefore the solution of PDS (4.7) under the boundary conditions (4.8) is, by the solution structure theorem,

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,t) + W_\psi(x,t) + \int_0^t W_{f_\tau}(x, t-\tau)\, d\tau. \tag{4.14}$$

Remark 1. Although there exist three cases of characteristic roots for Eq. (4.10), its general solution can be written in a unified form as Eq. (4.11) by using the relation between exponential and trigonometric functions. Since $\gamma_m = 0$ rarely occurs, $\underline{\sin}\gamma_m t = \sin\gamma_m t$ and $\underline{\gamma}_m = \gamma_m$ for most of cases.

Remark 2. By the solution structure theorem, the solution of

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2 u_{xx} + f(x,t), \quad (0,l)\times(0,+\infty), \\ u(0,t) = u_x(l,t) + hu(l,t) = 0, \\ u(x,0) = u_t(x,0) = 0 \end{cases} \tag{4.15}$$

is

$$\begin{aligned} u &= \int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau = \int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \sum_{m=1}^{+\infty} b_m \underline{\sin}\gamma_m(t-\tau)\sin\frac{\mu_m x}{l}\,\mathrm{d}\tau \\ &= \int_0^t \mathrm{d}\tau \int_0^l \left[\mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \sum_{m=1}^{+\infty} \frac{1}{\tau_0 \underline{\gamma}_m M_m} \sin\frac{\mu_m \xi}{l}\sin\frac{\mu_m x}{l}\underline{\sin}\gamma_m(t-\tau)\right] f(\xi,\tau)\,\mathrm{d}\xi \\ &= \int_0^t d\tau \int_0^l G(x,\xi,t-\tau) f(\xi,\tau)\,\mathrm{d}\xi\,. \end{aligned} \tag{4.16}$$

Here

$$G(x,\xi,t-\tau) = \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \sum_{m=1}^{+\infty} \frac{1}{\tau_0 \underline{\gamma}_m M_m} \sin\frac{\mu_m \xi}{l}\sin\frac{\mu_m x}{l}\underline{\sin}\gamma_m(t-\tau) \tag{4.17}$$

is called the *Green function* of the one-dimensional hyperbolic heat-conduction equation subject to boundary conditions (4.8).

Remark 3. Since $\left[\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\right] = 1$,

$$[G] = \left[\frac{1}{\tau_0}\right]\left[\frac{1}{\underline{\gamma}_m M_m}\right] = \frac{1}{L},$$

so that the unit of u in Eq. (4.16) is $[u] = [\mathrm{d}\tau]\,[G]\,[f]\,[\mathrm{d}\xi] = \Theta$. Also,

$$\left[a^2/\tau_0\right] = L^2 T^{-2}.$$

Therefore, $a\big/\sqrt{\tau_0}$ stands for the wave velocity. The temperature u at point x in the classical heat-conduction equation decreases exponentially as time t. Here the every term in the temperature (Eq. (4.17)) contains both an exponential factor $\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}$ and a factor $\underline{\sin}\gamma_m(t-\tau)$. Therefore, the temperature u may have wavelike properties.

Remark 4. Let $f(x,t) = \delta(x-x_0,t-t_0)$, $[\delta] = \Theta T^{-1}$ in (4.15), so that

$$u = \int_0^t \mathrm{d}\tau \int_0^l G(x,\xi,t-\tau)\delta(\xi - x_0, \tau - t_0)\,\mathrm{d}\xi = G(x,x_0,t-t_0), \quad [u] = \Theta$$

$G(x,x_0,t-t_0)$ is thus the temperature distribution in a rod of heat conduction due to a source term of the unit δ-function $\delta(x-\xi,t-\tau)$, or the temperature distribution caused by a point source with a unit changing rate of temperature at time instant t_0 and at spatial point x_0. Therefore, $G(x,\xi,t-\tau)$satisfies

$$\begin{cases} G_t + \tau_0 G_{tt} = a^2 G_{xx} + \delta(x-\xi,t-\tau), \quad 0<x<l,\, 0<\tau<t<+\infty, \\ G|_{x=0} = (G_x + hG)|_{x=l} = 0, \\ G|_{t=\tau} = G_t|_{t=\tau} = 0 \end{cases} \tag{4.18}$$

or

$$\begin{cases} G_t + \tau_0 G_{tt} = a^2 G_{xx}, \quad 0<x<l,\, 0<\tau<t<+\infty, \\ G|_{x=0} = (G_x + hG)|_{x=l} = 0, \\ G|_{t=\tau} = 0, \; G_t|_{t=\tau} = \dfrac{\delta(x-\xi)}{\tau_0}. \end{cases} \tag{4.19}$$

Remark 5. We can obtain the Green function $G(x,\xi,t-\tau)$ by using different methods.

1. We may obtain $G(x,\xi,t-\tau)$ by applying the generalized Fourier method of expansion to solve PDS (4.15) and expanding $f(x,t)$ using the eigenfunction set $\left\{\sin\dfrac{\mu_m x}{l}\right\}$.
2. We may apply the solution structure theorem to obtain $G(x,\xi,t-\tau)$ as in Remark 2.
3. We may obtain $G(x,\xi,t-\tau)$ by applying the generalized Fourier method of expansion to solve PDS (4.19). The $G(x,\xi,t-\tau)$ in Eq. (4.17) follows from Eq. (4.13) by substituting $\psi(x) = \dfrac{\delta(x-\xi)}{\tau_0}$ and replacing t by $t-\tau$.

4.2.2 *Mixed Boundary Conditions of the Second and the Third Kind*

Here we propose another way of using the solution structure theorem following a procedure of $u_3 \Rightarrow u_2 \Rightarrow u_1$instead of the normal procedure $u_2 \Rightarrow u_1 \Rightarrow u_3$. Find the solution of

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2 u_{xx} + f(x,t), \quad (0,l)\times(0,+\infty), \\ u_x(0,t) = u_x(l,t) + hu(l,t) = 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \end{cases} \tag{4.20}$$

Solution.

1. $u_3(x,t)$. To find $u_3(x,t)$, we first develop the Green function G satisfying

$$\begin{cases} G_t + \tau_0 G_{tt} = a^2 G_{xx}, \quad 0 < x < l, \quad 0 < \tau < t < +\infty, \\ G_x(0,t) = G_x(l,t) + hG(l,t) = 0, \\ G|_{t=\tau} = 0, G_t|_{t=\tau} = \delta(x-\xi)/\tau_0 \, . \end{cases} \tag{4.21}$$

By taking the boundary conditions into account, we use the eigenfunctions in Row 6 in Table 2.1 to expand G so that

$$G = \sum_{m=1}^{+\infty} T_m(t) \cos \beta_m x, \quad \beta_m = \mu_m / l \, ,$$

where the μ_m are the positive zero-points of $f(x) = \cot x - \dfrac{x}{lh}$. Substituting it into the equation of (4.21) yields

$$\tau_0 T_m'' + T_m' + (\beta_m a)^2 T_m = 0 \, .$$

Its general solution is

$$T_m(t) = \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \left[a_m \cos \gamma_m (t-\tau) + b_m \underline{\sin} \gamma_m (t-\tau) \right] ,$$

where a_m and b_m are undetermined constants, and

$$r_{1,2} = -\frac{1}{2\tau_0} \pm \frac{1}{2\tau_0} \sqrt{1 - 4\tau_0 (\beta_m a)^2} = -\frac{1}{2\tau_0} \pm \gamma_m \mathrm{i} \, .$$

Applying the initial condition $G|_{t=\tau} = 0$ yields $a_m = 0$. To satisfy the initial condition $G_t|_{t=\tau} = \delta(x-\xi)/\tau_0$ we have

$$\sum_{m=1}^{+\infty} b_m \underline{\gamma}_m \cos \beta_m x = \frac{\delta(x-\xi)}{\tau_0} \, ,$$

which requires

$$b_m = \frac{1}{\underline{\gamma}_m M_m} \int_0^l \frac{\delta(x-\xi)}{\tau_0} \cos \beta_m x \, \mathrm{d}x = \frac{1}{\tau_0 \underline{\gamma}_m M_m} \cos \beta_m \xi \, . \tag{4.22}$$

Finally, we have the Green function

$$G = \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \sum_{m=1}^{+\infty} \frac{1}{\tau_0 \underline{\gamma}_m M_m} \cos \beta_m \xi \cos \beta_m x \underline{\sin} \gamma_m (t-\tau) \, .$$

Thus

$$u_3 = \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau = \int_0^t \mathrm{d}\tau \int_0^l G(x,\xi,t-\tau)f(\xi,\tau)\,\mathrm{d}\xi\,. \tag{4.23}$$

2. $u_2(x,t)$. $u_2(x,t) = W_\psi(x,t)$ can be readily obtained by replacing $\dfrac{\delta(x-\xi)}{\tau_0}$ and τ in PDS (4.21) by $\psi(\xi)$ and 0, respectively.

$$\begin{cases} u_2 = W_\psi(x,t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \sum\limits_{m=1}^{+\infty} b_m \cos\beta_m x \underline{\sin}\gamma_m t\,, \\ b_m = \dfrac{1}{\underline{\gamma}_m M_m} \displaystyle\int_0^l \psi(x)\cos\beta_m x\,\mathrm{d}x. \end{cases} \tag{4.24}$$

3. $u_1(x,t)$. $W_\varphi(x,t)$ can be readily obtained by replacing $\psi(x)$ in Eq. (4.24) by $\varphi(x)$. The $u_1(x,t)$ is, thus, by the solution structure theorem,

$$u_1 = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,t)\,.$$

4. $u(x,t)$. The solution of PDS (4.20) is, by the principle of superposition,

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t)\,.$$

4.3 Two-Dimensional Mixed Problems

In this section we seek the solution of two-dimensional mixed problems in rectangular and circular domain. The methods and the Remarks made in Section 4.2 are all applicable in this section.

4.3.1 Rectangular Domain

We can solve mixed problems in a rectangular domain $D: 0 < x < l_1,\ 0 < y < l_2$ very efficiently by using Table 2.1 and the solution structure theorem. We demonstrate this with two examples.

Example 1. Solve

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2 \Delta u + f(x,y,t)\,, \quad D \times (0, +\infty)\,, \\ u(0,y,t) = u_x(l_1,y,t) + hu(l_1,y,t) = 0\,, \\ u_y(x,0,t) = u_y(x,l_2,t) + hu(x,l_2,t) = 0\,, \\ u(x,y,0) = \varphi(x,y), \quad u_t(x,y,0) = \psi(x,y)\,. \end{cases} \tag{4.25}$$

Solution. We first develop $W_\psi(x,y,t)$, the solution of (4.25) at $\varphi = f = 0$. By taking the boundary conditions into account, we use the eigenfunctions in Rows 3 and 6 in Table 2.1 to expand the solution

$$u = \sum_{m,n=1}^{+\infty} T_{mn}(t) \sin \alpha_m x \cos \beta_n y\,, \tag{4.26}$$

where $\alpha_m = \mu_m / l_1$, $\beta_n = \mu_n' / l_2$. $\mu_m$and μ_n' are the positive zero-points of $f(x) = \tan x + \dfrac{x}{l_1 h}$ and $g(x) = \cot y - \dfrac{y}{l_2 h}$, respectively.

Substituting Eq. (4.26) into the equation of (4.25) with $f = 0$ yields

$$\tau_0 T_{mn}'' + T_{mn}' + \left(\alpha_m^2 + \beta_n^2\right) a^2 T_{mn} = 0\,. \tag{4.27}$$

Its general solution is

$$T_{mn}(t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \left(a_{mn} \cos \gamma_{mn} t + b_{mn} \underline{\sin} \gamma_{mn} t\right)\,,$$

where a_{mn} and b_{mn} are undetermined constants, and

$$\gamma_{mn} = \frac{1}{2\tau_0} \sqrt{4\tau_0 \left(\alpha_m^2 + \beta_n^2\right) a^2 - 1}\,, \quad \underline{\sin} \gamma_{mn} t = \begin{cases} \sin \gamma_{mn} t\,, & \gamma_{mn} \neq 0\,, \\ t\,, & \gamma_{mn} = 0\,. \end{cases}$$

We thus obtain

$$u = \sum_{m,n=1}^{+\infty} \mathrm{e}^{-\frac{t}{2\tau_0}} \left(a_{mn} \cos \gamma_{mn} t + b_{mn} \underline{\sin} \gamma_{mn} t\right) \sin \alpha_m x \cos \beta_n y\,. \tag{4.28}$$

Applying the initial condition $u(x,y,0) = 0$ yields $a_{mn} = 0$. To satisfy the initial condition $u_t(x,y,0) = \psi(x,y)$, b_{mn} must be determined such that

$$\sum_{m,n=1}^{+\infty} b_{mn} \underline{\gamma}_{mn} \sin \alpha_m x \cos \beta_n y = \psi(x,y), \quad \underline{\gamma}_{mn} = \begin{cases} \gamma_{mn}\,, & \gamma_{mn} \neq 0\,, \\ 1\,, & \gamma_{mn} = 0\,. \end{cases}$$

Therefore

$$b_{mn} = \frac{1}{\underline{\gamma}_{mn} M_{mn}} \iint_D \psi(x,y) \sin \alpha_m x \cos \beta_n y \, \mathrm{d}\sigma\,,$$

where $M_{mn} = M_m M_n$, M_m and M_n are the normal square of the two eigenfunction sets $\{\sin \alpha_m x\}$ and $\{\cos \beta_n y\}$, respectively. Finally, the solution of PDS (4.25) is, by the solution structure theorem,

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_{\varphi}(x,y,t) + W_{\psi}(x,y,t) + \int_0^t W_{f_\tau}(x,y,t-\tau)\,\mathrm{d}\tau\,,$$

where $f_\tau = f(x,y,\tau)\,/\,\tau_0\,.$

Example 2. Find the solution of Eq. 4.28

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2 \Delta u + f(x,y,t), \quad D \times (0,+\infty)\,, \\ u(0,y,t) = u_x(l_1,y,t) = 0\,, \\ u_y(x,0,t) = u_y(x,l_2,t) = 0\,, \\ u(x,y,0) = \varphi(x,y), \quad u_t(x,y,0) = \psi(x,y)\,. \end{cases} \tag{4.29}$$

Solution. We first find the Green function $G(x,\xi;y,\eta;t-\tau)$ satisfying

$$\begin{cases} G_t + \tau_0 G_{tt} = a^2 \Delta G, \quad D, 0 < \tau < t < +\infty\,, \\ G\big|_{x=0} = G_x\big|_{x=l_1} = G_y\big|_{y=0} = G_y\big|_{y=l_2}\,, \qquad . \\ G\big|_{t=\tau} = 0,\, G_t\big|_{t=\tau} = \delta\,(x-\xi, y-\eta)\,/\,\tau_0\ . \end{cases} \tag{4.30}$$

Based on the given boundary conditions in (4.30), we use the eigenfunctions in Rows 2 and 5 in Table 2.1 to expand

$$G = \sum_{m,n=0}^{+\infty} T_{mn}(t) \sin\frac{(2m+1)\pi x}{2l_1} \cos\frac{n\pi y}{l_2}\,,$$

where $T_{mn}(t)$ is the undetermined function. Substituting it into the equation of (4.30) leads to

$$\tau_0 T''_{mn} + T'_{mn} + \left[\left(\frac{(2m+1)\pi}{2l_1}\right)^2 + \left(\frac{n\pi}{l_2}\right)^2\right] a^2 T_{mn} = 0\,.$$

Its general solution reads

$$T_{mn}(t) = \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \left[a_{mn} \cos \gamma_{mn}(t-\tau) + b_{mn} \underline{\sin} \gamma_{mn}(t-\tau)\right],$$

where a_{mn} and b_{mn} are the undertermined constants, and

$$\gamma_{mn} = \frac{1}{2\tau_0} \sqrt{4\tau_0 \left[\left(\frac{(2m+1)\pi}{2l_1}\right)^2 + \left(\frac{n\pi}{l_2}\right)^2\right] a^2 - 1}\,.$$

Therefore

$$G = \sum_{m,n=0}^{+\infty} \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \left[a_{mn} \cos \gamma_{mn}(t-\tau) + b_{mn} \underline{\sin} \gamma_{mn}(t-\tau)\right] \sin \frac{(2m+1)\pi x}{2l_1} \cos \frac{n\pi y}{l_2} .$$

Applying initial conditions leads to

$$\begin{aligned} a_{mn} &= 0 , \\ b_{mn} &= \frac{1}{\underline{\gamma}_{mn} M_{mn}} \iint_D \frac{1}{\tau_0} \delta(x-\xi, y-\eta) \sin \frac{(2m+1)\pi x}{2l_1} \cos \frac{n\pi y}{l_2} \, \mathrm{d}\sigma \\ &= \frac{1}{\tau_0 \underline{\gamma}_{mn} M_{mn}} \sin \frac{(2m+1)\pi \xi}{2l_1} \cos \frac{n\pi\eta}{l_2} , \end{aligned} \tag{4.31}$$

where $M_{mn} = M_m M_n$. M_m and M_n are the normal squares of the two eigenfunction sets $\left\{\sin \frac{(2m+1)\pi x}{2l_1}\right\}$ and $\left\{\cos \frac{n\pi y}{l_2}\right\}$, respectively. $M_{mn} = M_m M_n = \frac{l_1}{2}\frac{l_2}{2} = \frac{l_1 l_2}{4}$. Therefore,

$$\begin{aligned} G = \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \sum_{m,n=0}^{+\infty} \frac{1}{\tau_0 \underline{\gamma}_{mn} M_{mn}} \sin \frac{(2m+1)\pi\xi}{2l_1} \sin \frac{(2m+1)\pi x}{2l_1} \\ \cdot \cos \frac{n\pi\eta}{l_2} \cos \frac{n\pi y}{l_2} \underline{\sin} \gamma_{mn}(t-\tau) . \end{aligned} \tag{4.32}$$

$$u_3 = \int_0^t W_{f_\tau}(x,y,t-\tau)\,\mathrm{d}\tau = \int_0^t \mathrm{d}\tau \iint_D G(x,\xi;y,\eta;t-\tau) f(\xi,\eta,\tau)\,\mathrm{d}\sigma . \tag{4.33}$$

By replacing the integrand $\delta(x-\xi, y-\eta)/\tau_0$ in Eq. (4.31) by $\psi(x,y)$and letting $\tau = 0$ in Eq. (4.31), we obtain

$$\begin{cases} u_2 = W_\psi(x,y,t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \displaystyle\sum_{m,n=0}^{+\infty} b_{mn} \sin \frac{(2m+1)\pi x}{2l_1} \cos \frac{n\pi y}{l_2} \underline{\sin} \gamma_{mn} t , \\ b_{mn} = \dfrac{1}{\underline{\gamma}_{mn} M_{mn}} \displaystyle\iint_D \psi(x,y) \sin \frac{(2m+1)\pi x}{2l_1} \cos \frac{n\pi y}{l_2} \, \mathrm{d}\sigma . \end{cases} \tag{4.34}$$

$W_\varphi(x,y,t)$ can be obtained through replacing $\psi(x,y)$ in Eq. (4.34) by $\varphi(x,y)$. Also,

$$u_1 = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,y,t) . \tag{4.35}$$

The solution of PDS (4.29) is, by the principle of superposition,

$$u(x,y,t) = u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t) .$$

Remark 1. The $W_\psi(x,y,t)$ can be obtained from G simply by replacing the factor $\delta(x-\xi,y-\eta)/\tau_0$ in the integrand of Eq. (4.31) by $\psi(x,y)$ and letting $\tau=0$. $G(x,\xi;y,\eta;t-\tau)$ can be obtained from $W_\psi(x,y,t)$ in Eq. (4.28) simply by replacing t and the factor $\psi(x,y)$ of the integrand in Eq. (4.28) by $t-\tau$ and $\delta(x-\xi,y-\eta)/\tau_0$, respectively,

$$G(x,\xi;y,\eta;t-\tau)=\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\sum_{m,n=0}^{+\infty}\frac{1}{\tau_0\underline{\gamma}_{mn}M_{mn}}\sin\alpha_m\xi\sin\alpha_m \\ \cdot x\cos\beta_n\eta\cos\beta_n y\underline{\sin}\gamma_{mn}(t-\tau)\,. \tag{4.36}$$

Remark 2. The Green function $G(x,\xi;y,\eta;t-\tau)$ is the solution of

$$\begin{cases} u_t+\tau_0u_{tt}=a^2\Delta u+\delta(x-\xi,y-\eta,t-\tau)\,, & D, 0<\tau<t<+\infty, \\ L(u,u_x,u_y)\big|_{\partial D}=0\,, \\ u(x,y,t)\big|_{t=\tau}=u_t(x,y,t)\big|_{t=\tau}=0\,. \end{cases} \tag{4.37}$$

It is the temperature distribution due to a point source $\delta(x-\xi,y-\eta,t-\tau)$ at $t=\tau$ and point (ξ,η) that satisfies

$$\int_0^t\iint_D\delta(x-\xi,y-\eta,t-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau=1\,.$$

When all combinations of linear homogeneous boundary conditions of the first, the second and the third kinds are considered, on ∂D, there exist 81 combinations of boundary conditions in PDS (4.37). Clearly, the Green function G differs from one combination to another.

Remark 3. Example 2 shows that it is crucial to obtain G for solving

$$\begin{cases} u_t+\tau_0u_{tt}=a^2\Delta u+f(x,y,t)\,, & D\times(0,+\infty)\,, \\ L(u,u_x,u_y)\big|_{\partial D}=0\,, \\ u(x,y,0)=\varphi(x,y), \quad u_t(x,y,0)=\psi(x,y)\,. \end{cases} \tag{4.38}$$

Once the G is available, we can readily write out the solution u of PDS (4.38). The main steps consist of:

$$\text{Find G}\longrightarrow\text{write } u_3=\int_0^t W_{f_\tau}(x,y,t-\tau)\,\mathrm{d}\tau\,.$$
$$\text{Find G}\longrightarrow\text{write } u_2=W_\psi(x,y,t)\longrightarrow\text{write } u_1=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(x,y,t)\,.$$

The Green function (Eq. (4.32) and Eq. (4.36)) has an elegant, universal structure. It can be written out directly from the eigenfunctions in Table 2.1. The underlying rule will be demonstrated by examining how to construct the Green function G in Eq. (4.32) and Eq. (4.36) from the corresponding eigenfunction sets in Table 2.1.

1. **The Green function in Eq. (4.32).** The general term of the Green function can be obtained based on the eigenfunction sets $\left\{\sin\frac{(2m+1)\pi x}{2l_1}\right\}$ and $\left\{\cos\frac{n\pi y}{l_2}\right\}$ (Table 2.1) by the following procedure:

$$\sin\frac{(2m+1)\pi x}{2l_1}\cos\frac{n\pi y}{l_2} \overset{\substack{\text{multiplying factor}\\ \sin\frac{(2m+1)\pi\xi}{2l_1}\cos\frac{n\pi\eta}{l_2}}}{\Longrightarrow} \sin\frac{(2m+1)\pi x}{2l_1}\cos\frac{n\pi y}{l_2}$$

$$\cdot\sin\frac{(2m+1)\pi\xi}{2l_1}\cos\frac{n\pi\eta}{l_2} \overset{\substack{\text{multiplying another factor}\\ e^{-\frac{t-\tau}{2\tau_0}}\frac{1}{\tau_0\underline{\gamma}_{mn}M_{mn}}\underline{\sin}\gamma_{mn}(t-\tau)}}{\Longrightarrow} e^{-\frac{t-\tau}{2\tau_0}}\frac{1}{\tau_0\underline{\gamma}_{mn}M_{mn}}\sin\frac{(2m+1)\pi x}{2l_1}$$

$$\cdot\cos\frac{n\pi y}{l_2}\sin\frac{(2m+1)\pi\xi}{2l_1}\cos\frac{n\pi\eta}{l_2}\underline{\sin}\gamma_{mn}(t-\tau)\,.$$

The two operations of multiplication can also be combined together.

2. **The green function in Eq. (4.36)**. The general term of the Green function can be written out based on the eigenfunction sets $\{\sin\alpha_m x\}$ and $\{\cos\beta_n y\}$ (Table 2.1) by the following procedure:

$$\sin\alpha_m x\cos\beta_n y \overset{\substack{\text{multiplying factor}\\ e^{-\frac{t-\tau}{2\tau_0}}\frac{1}{\tau_0\underline{\gamma}_{mn}M_{mn}}\sin\alpha_m\xi\cos\beta_n\eta\underline{\sin}\gamma_{mn}(t-\tau)}}{\Longrightarrow} e^{-\frac{t-\tau}{2\tau_0}}\frac{1}{\tau_0\underline{\gamma}_{mn}M_{mn}}$$

$$\cdot\sin\alpha_m x\cos\beta_n y\sin\alpha_m\xi\cos\beta_n\eta\underline{\sin}\gamma_{mn}(t-\tau)\,.$$

We thus list the steps of writing Green functions from eigenfunctions in Table 2.1.

Step 1. Based on the given boundary conditions, find the eigenfunction sets from Table 2.1

$$X_i(x), Y_j(y), \quad i,j=1,2,\cdots,9\,.$$

When writing $Y_j(x)$, replace x, the domain of x and subscript index m in Table 2.1 by y, the domain of y and subscript index n, respectively. If applied, use μ_m and μ_n' (not μ_n) for the zero-points of the corresponding function occurring in $X_i(x)$ and $Y_j(y)$, respectively.

Step 2. Construct the complete and orthogonal group for expanding G**,**

$$u_{ij}(x,y)=X_i(x)Y_j(y), \quad i,j=1,2,\cdots,9\,.$$

$A(x,y)=(u_{ij})_{9\times 9}$ is a matrix of order 9. Each of its components corresponds to a complete and orthogonal group in D for one combination of boundary conditions.

Step 3. Construct the general term of G by multiplying the factor,

$$\mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \frac{1}{\tau_0 \underline{\gamma}_{mn} M_{mn}} u_{ij}(x,y) u_{ij}(\xi,\eta)\ \underline{\sin}\gamma_{mn}(t-\tau)\,.$$

Here $M_{mn} = M_m M_n$, the product of the normal squares of the two sets of eigenfunctions $\{X_i(x)\}$ and $\{Y_j(y)\}$,

$$\gamma_{mn} = \frac{1}{2\tau_0}\sqrt{4\tau_0 a^2 (\lambda_m + \lambda_n) - 1}\,,$$

λ_m and λ_n are the eigenvalues corresponding to $X_i(x)$ and $Y_j(y)$, respectively. Finally,

$$G(x,\xi;y,\eta;t-\tau) = \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \sum \frac{1}{\tau_0 \underline{\gamma}_{mn} M_{mn}} u_{ij}(x,y) u_{ij}(\xi,\eta)\underline{\sin}\gamma_{mn}(t-\tau)\,, \quad (4.39)$$

where $u_{ij}(x,y)u_{ij}(\xi,\eta) = X_i(\xi)X_i(x)Y_j(\eta)Y_j(y)$.

Remark 4. We can readily write out $W_\psi(x,y,t)$ from the Green function $G(x,\xi;y,\eta;t-\tau)$ in Eq. (4.39). This can be achieved simply by letting $\tau = 0$ and replacing $\dfrac{1}{\tau_0 \underline{\gamma}_{mn} M_{mn}} u_{ij}(\xi,\eta)$ by the Fourier coefficients

$$b_{mn} = \frac{1}{\underline{\gamma}_{mn} M_{mn}} \iint\limits_D \psi(x,y) u_{ij}(x,y) d\sigma\,. \quad (4.40)$$

Therefore

$$\begin{cases} W_\psi(x,y,t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \sum b_{mn} u_{ij}(x,y) \underline{\sin}\gamma_{mn} t\,, \\ b_{mn} = \dfrac{1}{\underline{\gamma}_{mn} M_{mn}} \displaystyle\iint\limits_D \psi(x,y) u_{ij}(x,y)\, \mathrm{d}\sigma\,. \end{cases} \quad (4.41)$$

Example 3. Solve

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2 \Delta u + f(x,y,t), \quad D \times (0,+\infty)\,, \\ u(0,y,t) = u_x(l_1,y,t) + h_2 u(l_1,y,t) = 0\,, \\ u_y(x,0,t) - h_1 u(x,0,t) = u(x,l_2,t) = 0\,, \\ u(x,y,0) = \varphi(x,y), \quad u_t(x,y,0) = \psi(x,y)\,. \end{cases} \quad (4.42)$$

Solution.

1. We first seek the solution due to $f(x,y,t)$, i.e. the solution of PDS (4.42) at $\varphi = \psi = 0$. Based on the given boundary conditions,

G can be expanded by using the complete and orthogonal set (Table 2.1)

$$u_{37} = \sin\frac{\mu_m x}{l_1}\sin\left(\frac{\mu_n' y}{l_2} + \varphi_n\right), \quad \tan\varphi_n = \frac{\mu_n'}{l_2 h_1}.$$

By multiplying the factor, we may obtain the general term of G, thus

$$\begin{aligned} G = \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \sum_{m,n=1}^{+\infty} \frac{1}{\tau_0 \underline{\gamma}_{mn} M_{mn}} \sin\frac{\mu_m \xi}{l_1}\sin\frac{\mu_m x}{l_1}\sin\left(\frac{\mu_n' \eta}{l_2} + \varphi_n\right) \\ \cdot \sin\left(\frac{\mu_n' y}{l_2} + \varphi_n\right) \underline{\sin}\gamma_{mn}(t-\tau). \end{aligned} \tag{4.43}$$

Finally,

$$u_3 = \int_0^t \mathrm{d}\tau \iint_D G(x,\xi;y,\eta;t-\tau) f(\xi,\eta,\tau)\,\mathrm{d}\sigma. \tag{4.44}$$

2. The solution of PDS (4.42) at $f = \varphi = 0$ is, by Eqs. (4.41) and (4.43),

$$\begin{cases} u_2 = W_\psi(x,y,t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \sum\limits_{m,n=1}^{\infty} b_{mn} \sin\frac{\mu_m x}{l_1}\sin\left(\frac{\mu_n' y}{l_2} + \varphi_n\right) \underline{\sin}\gamma_{mn} t, \\ b_{mn} = \dfrac{1}{\underline{\gamma}_{mn} M_{mn}} \iint\limits_D \psi(x,y) \sin\frac{\mu_m x}{l_1}\sin\left(\frac{\mu_n' y}{l_2} + \varphi_n\right) \mathrm{d}\sigma. \end{cases} \tag{4.45}$$

3. $W_\varphi(x,y,t)$ may be readily obtained simply by replacing $\psi(x,y)$ in Eq. (4.45) by $\varphi(x,y)$. The solution of PDS (4.42) at $f = \psi = 0$ is thus

$$u_1 = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,y,t). \tag{4.46}$$

4. The solution of PDS (4.42) is, by the principle of superposition,

$$u(x,y,t) = u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t).$$

Remark 5. If γ_{mn} has a purely imaginary value such as $\gamma_{mn} = \omega_{mn}\mathrm{i}$, the factor $\sin\gamma_{mn}t \big/ \gamma_{mn}$ can be transformed into a real exponential function.

$$\begin{aligned} \sin\gamma_{mn}t \big/ \gamma_{mn} &= \frac{1}{\omega_{mn}\mathrm{i}} \sin\mathrm{i}\omega_{mn}t = \frac{1}{\omega_{mn}} sh\omega_{mn}t \\ &= \frac{1}{2\omega_{mn}}\left(\mathrm{e}^{\omega_{mn}t} - \mathrm{e}^{-\omega_{mn}t}\right). \end{aligned}$$

For this case, the factor involving time t reads $\mathrm{e}^{-\frac{t}{2\tau_0}}\left(\mathrm{e}^{\omega_{mn}t} - \mathrm{e}^{-\omega_{mn}t}\right) = \mathrm{e}^{\left(\omega_{mn} - \frac{1}{2\tau_0}\right)t} - \mathrm{e}^{-\left(\frac{1}{2\tau_0} + \omega_{mn}\right)t}$, which is bounded as $t \to +\infty$ because $0 < \omega_{mn} < \dfrac{1}{2\tau_0}$.

Example 4. Let the boundary conditions $L(u,u_x,u_y)\big|_{\partial\Omega}=0$ in PDS (4.37) be

$$\begin{cases} u_x(0,y,t)=u_x(l,y,t)=0\,, \\ u_y(x,0,t)=u_y(x,l,t)=0\,. \end{cases} \tag{4.47}$$

Find the Green function and write out its first four terms.

Solution. Based the given boundary conditions (4.47), we use the eigenfunctions in Row 5 in Table 2.1 to expand G both in x and y,

$$u_{55}(x,y)=\cos\frac{m\pi x}{l}\cos\frac{n\pi y}{l}\,.$$

Therefore,

$$G=\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\sum_{m,n=0}^{+\infty}\frac{1}{\tau_0\underline{\gamma}_{mn}M_{mn}}\cos\frac{m\pi\xi}{l}\cos\frac{m\pi x}{l}\cos\frac{n\pi\eta}{l}\cos\frac{n\pi y}{l}\underline{\sin}\gamma_{mn}(t-\tau)\,,$$

where $M_{mn}=M_mM_n=\dfrac{l^2}{4}$, $\gamma_{mn}=\dfrac{1}{2\tau_0}\sqrt{4\tau_0a^2\left[\left(\dfrac{m\pi}{l}\right)^2+\left(\dfrac{n\pi}{l}\right)^2\right]-1}$.

The first four terms are

$$G_{00}=\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\frac{1}{\tau_0}\frac{2\tau_0}{\mathrm{i}}\frac{4}{l^2}\,i\frac{\mathrm{e}^{\frac{t-\tau}{2\tau_0}}-\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}}{2}=\frac{4}{l^2}\left(1-\mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right),$$

$$G_{01}=\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\frac{4}{\tau_0\underline{\gamma}_{01}l^2}\cos\frac{\pi\eta}{l}\cos\frac{\pi y}{l}\underline{\sin}\gamma_{01}(t-\tau)\,,$$

$$G_{10}=\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\frac{4}{\tau_0\underline{\gamma}_{10}l^2}\cos\frac{\pi\xi}{l}\cos\frac{\pi x}{l}\underline{\sin}\gamma_{10}(t-\tau)\,,$$

$$G_{11}=\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\frac{4}{\tau_0\underline{\gamma}_{11}l^2}\cos\frac{\pi\xi}{l}\cos\frac{\pi x}{l}\cos\frac{\pi\eta}{l}\cos\frac{\pi y}{l}\underline{\sin}\gamma_{11}(t-\tau)\,.$$

It is always useful to perform a unit analysis. Here $\left[\tau_0a^2\right]=L^2$ and $[\gamma_{mn}]=T^{-1}$ so $[G]=L^{-2}$. The u due to the nonhomogeneous term $f(x,y,t)$ reads

$$u=\int_0^t\mathrm{d}\tau\iint\limits_D Gf(\xi,\eta,\tau)\,\mathrm{d}\sigma\,,$$

whose unit is $[u]=[\mathrm{d}\tau]\,[G]\,[f]\,[\mathrm{d}\sigma]=\Theta$.

4.3.2 Circular Domain

Boundary conditions for mixed problems in a circular domain are separable in polar coordinate systems. This is similar to the wave equations in Chapter 2. Therefore, we consider the mixed problems here in a polar coordinate system:

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2 \Delta u(r,\theta,t) + F(r,\theta,t), \quad 0 < r < a_0,\ 0 < t\,, \\ L(u,u_r)|_{r=a_0} = 0\,,\ |u(0,\theta,t)| < \infty,\ u(r,\theta+2\pi,t) = u(r,\theta,t)\,, \\ u(r,\theta,0) = \Phi(r,\theta),\ u_t(r,\theta,0) = \Psi(r,\theta)\,. \end{cases} \tag{4.48}$$

where boundary condition includes all three kinds.

Preliminaries

1. By following the same approach of Remark 4 in 2.5.2, we can obtain the following solution structure theorem.

   Theorem. The solution of PDS (4.48) reads

   $$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\Phi(r,\theta,t) + W_\Psi(r,\theta,t) + \int_0^t W_{F_\tau}(r,\theta,t-\tau)\,\mathrm{d}\tau\,.$$

   Where $F_\tau = F(r,\theta,\tau)\big/\tau_0$, $W_\Psi(r,\theta,t)$ is the solution of PDS (4.48) at $\Phi = F = 0$.

2. Two S-L problems will follow from seeking $W_\Psi(r,\theta,t)$, the solution of PDS (4.48) at $\Phi = F = 0$. Consider $u = T(t)U(r,\theta)$. We have

   $$\begin{aligned} &\Delta U + \lambda U = 0\,, \\ &\tau_0 T'' + T' + \lambda a^2 T = 0\,, \end{aligned} \tag{4.49}$$

   where $-\lambda$ is the separation constant. Another separation of variables by letting $U = R(r)\Theta(\theta)$ and $\lambda = k^2$ leads to

   $$\text{(a)} \quad \Theta'' + \mu\Theta = 0,\ \Theta(\theta+2\pi) = \Theta(\theta)\,.$$

   The former has eigenvalues and eigenfunctions

   $$\begin{aligned} &Eigenvalues \qquad \mu = n^2, \quad n = 0,1,2,\cdots\,, \\ &Eigenfunctions \quad \Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad a_n^2 + b_n^2 \neq 0\,. \end{aligned}$$

   $$\text{(b)} \quad \begin{cases} R_n''(r) + \dfrac{1}{r} R_n'(r) + \left(k^2 - \dfrac{n^2}{r^2}\right) R_n(r) = 0\,, \\ L(R,R_r)|_{r=a_0} = 0,\ |R_n(0)| < \infty,\ |R_n'(0)| < \infty\,. \end{cases} \tag{4.50}$$

The latter is called the *eigenvalue problem of Bessel equations*. Its eigenvalues $\lambda = k^2$ are boundary-condition dependent, nonzero and real valued, and depend on the zero-points of Bessel functions (Sect. 2.5). Its eigenfunctions are Bessel functions and form an orthogonal set in $[0, a_0]$ with respect to the weight function $\rho(r) = r$. The normal square of the eigenfunction set is also available in Sect. 2.5 and is listed in Table 4.1.

While the form of eigenfunctions is boundary-condition independent, $\mu_m^{(n)}$ depends on boundary conditions. The general solution of Eq. (4.49) reads, with

$$\lambda_m = k_{mn}^2 = \left(\mu_m^{(n)} / a_0\right)^2,$$

$$T_{mn}(t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \left(a_{mn} \cos \gamma_{mn} t + b_{mn} \underline{\sin} \gamma_{mn} t\right),$$

where a_{mn} and b_{mn} are constants, and $\gamma_{mn} = \dfrac{1}{2\tau_0}\sqrt{4\tau_0 \left(k_{mn} a\right)^2 - 1}$.

Fourier Method of Expansion for PDS (4.48)

Consider the solution of PDS (4.48) at $\Phi = F = 0$,

$$\begin{aligned} u = \mathrm{e}^{-\frac{t}{2\tau_0}} \sum_{m=1,n=0}^{+\infty} & \left(a_{mn}^{(1)} \cos \gamma_{mn} t + b_{mn}^{(1)} \underline{\sin} \gamma_{mn} t\right) J_n(k_{mn} r) \cos n\theta \\ & + \left(a_{mn}^{(2)} \cos \gamma_{mn} t + b_{mn}^{(2)} \underline{\sin} \gamma_{mn} t\right) J_n(k_{mn} r) \sin n\theta \,. \end{aligned}$$

Applying the initial condition $u(r,\theta,0) = 0$ yields $a_{mn}^{(1)} = a_{mn}^{(2)} = 0$. We can also determine $b_{mn}^{(1)}$ and $b_{mn}^{(2)}$ by satisfying the initial condition $u_t(r,\theta,0) = \Psi(r,\theta)$.

Finally,

$$\begin{cases} u = W_\Psi(r,\theta,t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \displaystyle\sum_{m=1,n=0}^{+\infty} \left(b_{mn}^{(1)} \cos n\theta + b_{mn}^{(2)} \sin n\theta\right) J_n(k_{mn} r) \underline{\sin} \gamma_{mn} t \,, \\ b_{m0}^{(1)} = \dfrac{1}{2\pi \underline{\gamma}_{m0} M_{m0}} \displaystyle\int_{-\pi}^{\pi} \int_0^{a_0} \Psi(r,\theta) J_0(k_{m0} r) r \, \mathrm{d}r \, \mathrm{d}\theta \,, \\ b_{mn}^{(1)} = \dfrac{1}{\pi \underline{\gamma}_{mn} M_{mn}} \displaystyle\int_{-\pi}^{\pi} \int_0^{a_0} \Psi(r,\theta) J_n(k_{mn} r) \cos n\theta \cdot r \, \mathrm{d}r \, \mathrm{d}\theta \,, \\ b_{mn}^{(2)} = \dfrac{1}{\pi \underline{\gamma}_{mn} M_{mn}} \displaystyle\int_{-\pi}^{\pi} \int_0^{a_0} \Psi(r,\theta) J_n(k_{mn} r) \sin n\theta \cdot r \, \mathrm{d}r \, \mathrm{d}\theta \,. \end{cases} \tag{4.51}$$

The solution of PDS (4.48) is, thus, by the solution structure theorem

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\Phi(r,\theta,t) + W_\Psi(r,\theta,t) + \int_0^t W_{F_\tau}(r,\theta,t-\tau) \, \mathrm{d}\tau \,,$$

where $F_\tau = F(r,\theta,\tau) / \tau_0$.

4.4 Three-Dimensional Mixed Problems

We discuss mixed problems in cubic, cylinderical and spherical domains in this section.

4.4.1 Cuboid Domain

Consider the PDS of type

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2 \Delta u + f(x,y,z,t), \quad \Omega \times (0,+\infty), \\ L(u,u_x,u_y,u_z)\big|_{\partial\Omega} = 0, \\ u(x,y,z,0) = \varphi(x,y,z),\ u_t(x,y,z,0) = \psi(x,y,z), \end{cases} \tag{4.52}$$

where Ω stands for a cuboid domain: $0 < x < l_1, 0 < y < l_2, 0 < z < l_3$, $\partial\Omega$ the six boundary surfaces of Ω. If all combinations of the boundary conditions of all three kinds are considered, for a cuboid domain Ω, there exist 729 combinations.

The method of finding solutions of PDS (4.52) is the same as that in Example 3 in Section 4.3.1. It is crucial to find the Green function and $W_\psi(x,y,z,t)$, the solution of (4.52) at $\varphi = f = 0$. Depending on the boundary conditions, there are a total of 729 complete and orthogonal sets of eigenfunctions. All of them can be easily written out by using Table 2.1. Let the eigenfunction set corresponding to a com-

Table 4.1 Eigenvalues and Eigenfunctions of Bessel equations

Boundary Conditions	Eigenvalues	Eigen-functions	Normal square M_{mn} $n=0,1,2,\cdots$ $m=1,2,\cdots$
$u\|_{r=a_0}=0$	$\left(\mu_m^{(n)}/a_0\right)^2$ $\mu_m^{(n)}$ are the m-th positive zero-point of $J_n(x)$	$J_n(k_{mn}r)$ $k_{mn}=\mu_m^{(n)}/a_0$	$\frac{a_0^2}{2}J_{n+1}^2\left(\mu_m^{(n)}\right)$
$u_r\|_{r=a_0}=0$	$\left(\mu_m^{(n)}/a_0\right)^2$ $\mu_m^{(n)}$ are the positive zero-point of $J_n'(x)$ $\mu_1^{(0)}=0$	$J_n(k_{mn}r)$ $k_{mn}=\mu_m^{(n)}/a_0$	$\frac{a_0^2}{2}\left[1-\left(\frac{n}{\mu_m^{(n)}}\right)^2\right]J_n^2\left(\mu_m^{(n)}\right)$
$(u_r+hu)\|_{r=a_0}=0$	$\left(\mu_m^{(n)}/a_0\right)^2$ $\mu_m^{(n)}$ are the positive zero-point of $\frac{1}{a_0}xJ_n'(x)+hJ_n(x)$	$J_n(k_{mn}r)$ $k_{mn}=\mu_m^{(n)}/a_0$	$\frac{a_0^2}{2}\left[1+\frac{(a_0h)^2-n^2}{\left(\mu_m^{(n)}\right)^2}\right]J_n^2\left(\mu_m^{(n)}\right)$

bination of boundary conditions be $X_m(x)Y_n(y)Z_l(z)$. The solution of PDS (4.52) at $\varphi = f = 0$ can thus be written as

$$u = \sum T_{mnl}(t)X_m(x)Y_n(y)Z_l(z)\,.$$

Substituting it into the equation of PDS (4.52) leads to

$$\tau_0 T''_{mnl} + T'_{mnl} + a^2(\lambda_m + \lambda_n + \lambda_l)\,T_{mnl} = 0\,,$$

where λ_m, λ_n and λ_l are the corresponding eigenvalues of $\{X_m(x)\}$, $\{Y_n(y)\}$ and $\{Z_l(z)\}$, respectively. Its general solution is

$$T_{mnl}(t) = \mathrm{e}^{-\frac{t}{2\tau_0}}\,(a_{mnl}\cos\gamma_{mnl}t + b_{mnl}\underline{\sin}\gamma_{mnl}t)\,,$$

where

$$\gamma_{mnl} = \frac{1}{2\tau_0}\sqrt{4\tau_0 a^2(\lambda_m + \lambda_n + \lambda_l) - 1}\,. \tag{4.53}$$

The solution of PDS (4.52) can then be obtained using a similar approach to that in Example 3 in Section 4.3.1.

Example 1. Solve

$$\begin{cases} u_t + \tau_0 u_{tt} & = a^2\Delta u + f(x,y,z,t)\,, \quad \Omega\times(0,+\infty)\,, \\ u(0,y,z,t) & = u(l_1,y,z,t) = 0\,, \\ u(x,0,z,t) & = u_y(x,l_2,z,t) + hu(x,l_2,z,t) = 0\,, \\ u_z(x,y,0,t) & = u_z(x,y,l_3,t) + hu(x,y,l_3,z,t) = 0\,, \\ u(x,y,z,0) & = \varphi(x,y,z)\,,\ u_t(x,y,z,0) = \psi(x,y,z)\,. \end{cases} \tag{4.54}$$

Solution.

1. We first seek the Green function G. Based on the given boundary conditions, the appropriate eigenfunction sets are those in Rows 1, 3 and 6 in Table 2.1. Let $u_{ijk}(x,y,z)(i,j,k = 1,2,\cdots,9)$ denote a complete and orthogonal eigenfunction set of 729 in total. Here we use

$$u_{136}(x,y,z) = \sin\frac{m\pi x}{l_1}\sin\frac{\mu_n y}{l_2}\cos\frac{\mu'_l z}{l_3}\,,$$

where the eigenvalues $\lambda_m = \left(\frac{m\pi}{l_1}\right)^2$, $\lambda_n = \left(\frac{\mu_n}{l_2}\right)^2$, $\lambda_l = \left(\frac{\mu'_l}{l_3}\right)^2$. By following the rule of writing the Green function, we obtain

$$\begin{aligned} G(x,\xi;y,\eta;\, z,\zeta;t-\tau) &= \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \sum_{m,n,l=1}^{+\infty} \frac{1}{\tau_0 \underline{\gamma}_{mnl} M_{mnl}} \sin\frac{m\pi\xi}{l_1} \sin\frac{m\pi x}{l_1} \\ &\cdot \sin\frac{\mu_n \eta}{l_2} \sin\frac{\mu_n y}{l_2} \cos\frac{\mu_l' \zeta}{l_3} \cos\frac{\mu_l' z}{l_3} \underline{\sin}\gamma_{mnl}(t-\tau) \end{aligned} \tag{4.55}$$

where $M_{mnl} = M_m M_n M_l$, γ_{mnl} are determined by Eq. 4.53. Therefore, we have the solution of PDS (4.54) at $\varphi = \psi = 0$.

$$u_3 = \int_0^t \mathrm{d}\tau \iiint\limits_{\Omega} G(x,\xi;y,\eta;z,\zeta;t-\tau) f(\xi,\eta,\zeta,\tau)\,\mathrm{d}v\,. \tag{4.56}$$

2. By the rule for writing $W_\psi(x,y,z,t)$ from the G in Section 4.3.1 (Remark 1), we can obtain the solution of PDS (4.54) for the case $f = \varphi = 0$,

$$\begin{cases} u_2 &= W_\psi(x,y,z,t) \\ &= \mathrm{e}^{-\frac{t}{2\tau_0}} \displaystyle\sum_{m,n,l=1}^{+\infty} b_{mnl} \sin\frac{m\pi x}{l_1} \sin\frac{\mu_n y}{l_2} \cos\frac{\mu_l' z}{l_3} \underline{\sin}\gamma_{mnl} t\,, \\ b_{mnl} &= \dfrac{1}{\underline{\gamma}_{mnl} M_{mnl}} \displaystyle\iiint\limits_{\Omega} \psi(x,y,z) \sin\frac{m\pi x}{l_1} \sin\frac{\mu_n y}{l_2} \cos\frac{\mu_l' z}{l_3}\,\mathrm{d}v\,. \end{cases} \tag{4.57}$$

3. We can obtain $W_\varphi(x,y,z,t)$ by replacing ψ in Eq. (4.57) by φ. The solution of PDS (4.54) at $f = \psi = 0$ reads, by the solution structure theorem,

$$u_1 = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,y,z,t)\,. \tag{4.58}$$

4. The solution of PDS (4.54) is, by the principle of superposition

$$u(x,y,t) = u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t)\,.$$

Remark. By the multiple separation of variables for solving PDS (4.52), the first separation of variables separates t from x, y and z and yields

$$\tau_0 T'' + T' + \lambda a^2 T = 0\,, \tag{4.59}$$

where $-\lambda$ is the separation constant. Another two separations of variables lead to three eigenvalue groups λ_m, λ_n, λ_l and three eigenfunction sets $X_m(x)$, $Y_n(y)$ and $Z_l(z)$. The λ in Eq. 4.59 is $\lambda = \lambda_m + \lambda_n + \lambda_l$, which is true only for some special cases and is not universal.

4.4.2 Cylindrical Domain

We solve mixed problems in a cylindrical domain by using a cylindrical coordinate system in which boundary conditions are separable.

Consider mixed problems in a cylinder Ω of radius a_0 and height $H : x^2+y^2 < a_0^2,\ 0<z<H$. Its boundary $\partial\Omega$ consists of the cylindrical surface of radius a_0 and height H, the upper circle and the lower circle of radius a_0. If all combinations of boundary conditions are considered on $\partial\Omega$, there exist total 27 combinations. Their corresponding complete and orthogonal sets of eigenfunctions are denoted by

$$u_{ijk}(r,\theta,z) = R_i(r)\Theta_j(\theta)Z_k(z), \quad i=1,2,3,\ j=1,\ k=1,2,\cdots 9.$$

Here $\Theta_j(\theta)$ are available in Table 4.1, and $R_i(r)$ and $Z_k(z)$ can be found from Table 2.1.

Example 2. Solve

$$\begin{cases} u_t+\tau_0 u_{tt} = a^2\Delta u(r,\theta,z,t)+F(r,\theta,z,t), \quad \Omega\times(0,+\infty)\,, \\ L(u,u_r)|_{r=a_0}=0,\ u|_{z=0}=(u_z+hu)|_{z=H}=0\,, \\ u(r,\theta,z,0)=\Phi(r,\theta,z), \quad u_t(r,\theta,z,0)=\Psi(r,\theta,z)\,. \end{cases} \tag{4.60}$$

Solution. The boundary conditions on the cylindrical surface $L(u,u_r)|_{r=a_0}=0$ can be of three kinds; PDS (4.60) thus includes three PDS, each corresponding to one of three boundary conditions at $r=a_0$. Based on the given boundary conditions, we use the eigenfunction sets in Row 3 in Table 2.1 and expand the solution for the case $\Phi=F=0$,

$$u = e^{-\frac{t}{2\tau_0}} \sum_{n=0,m,l=1}^{+\infty} \left[\left(a_{mnl}^{(1)}\cos\gamma_{mnl}t + b_{mnl}^{(1)}\underline{\sin}\gamma_{mnl}t\right)\cos n\theta \right.$$
$$\left. + \left(a_{mnl}^{(2)}\cos\gamma_{mnl}t + b_{mnl}^{(2)}\underline{\sin}\gamma_{mnl}t\right)\sin n\theta\right] J_n(k_{mn}r)\sin\frac{\mu_l' z}{H}\,,$$

where μ_l' are the positive zero-points of $f(z)=\tan z + z\big/Hh$,

$$\gamma_{mnl} = \frac{1}{2\tau_0}\sqrt{4\tau_0 a^2\lambda - 1}, \quad \lambda = \left(\frac{\mu_m^{(n)}}{a_0}\right)^2 + \left(\frac{\mu_l'}{H}\right)^2, \quad k_{mn} = \frac{\mu_m^{(n)}}{a_0}\,.$$

Applying the initial condition $u|_{t=0}=0$ yields $a_{mnl}^{(1)}=a_{mnl}^{(2)}=0$. $b_{mnl}^{(1)}$ and $b_{mnl}^{(2)}$ can also be determined by applying the initial condition $u_t|_{t=0} = \Psi(r,\theta,z)$. Finally,

$$\begin{cases} u &= W_\Psi(r,\theta,z,t) \\ &= \mathrm{e}^{-\frac{t}{2\tau_0}} \displaystyle\sum_{n=0,m,l=1}^{\infty} \left(b_{mnl}^{(1)}\cos n\theta + b_{mnl}^{(2)}\sin n\theta\right) J_n(k_{mn}r)\sin\frac{\mu_l' z}{H}\,\underline{\sin}\gamma_{mnl}t\,, \\ b_{m0l}^{(1)} &= \dfrac{1}{2\pi\underline{\gamma}_{m0l}M_{m0l}} \displaystyle\iiint_\Omega \Psi(r,\theta,z)J_0(k_{m0}r)r\sin\frac{\mu_l' z}{H}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta\,, \\ b_{mnl}^{(1)} &= \dfrac{1}{\pi\underline{\gamma}_{mnl}M_{mnl}} \displaystyle\iiint_\Omega \Psi(r,\theta,z)J_n(k_{mn}r)r\sin\frac{\mu_l' z}{H}\cos n\theta\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta\,, \\ b_{mnl}^{(2)} &= \dfrac{1}{\pi\underline{\gamma}_{mnl}M_{mnl}} \displaystyle\iiint_\Omega \Psi(r,\theta,z)J_n(k_{mn}r)r\sin\frac{\mu_l' z}{H}\sin n\theta\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta\,. \end{cases}$$

Thus the solution of PDS (4.60) is, by the solution structure theorem,

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\Phi(r,\theta,z,t) + W_\Psi(r,\theta,z,t) + \int_0^t W_{F_\tau}(r,\theta,z,t-\tau)\,\mathrm{d}\tau\,,$$

where $F_\tau = F(r,\theta,z,\tau)/\tau_0$, $M_{mnl} = M_{mn}M_l$.

Remark. The eigenvalues $\lambda_{mn} = \left(\mu_m^{(n)}/a_0\right)^2$, $\lambda_n = n^2$ and $\lambda_l = \left(\mu_l'/H\right)^2$ correspond to the eigenfunctions $J_n\left(\mu_m^{(n)}r/a_0\right)$, $a_n\cos n\theta + b_n\sin n\theta$ $\left(a_n^2 + b_n^2 \neq 0\right)$ and $\sin\left(\mu_l' z/H\right)$, respectively. Similar to in a cube, they come from the triple application of separation of variables. Here, we also have the $T(t)$-equation [and Eq. (4.59)]. For the case of a cube, $\lambda = \lambda_m + \lambda_n + \lambda_l$. For the case of a cylinder, however, $\lambda = \lambda_{mn} + \lambda_l = \left(\mu_m^{(n)}/a_0\right)^2 + \left(\mu_l'/H\right)^2$. The former can be obtained either by separation of variables three times or by first expanding the solution into a series and then substituting it into the equation. However, the latter can be proven only by separation of variables three times.

4.4.3 Spherical Domain

As described in Section 2.6.2, for mixed problems in a sphere of radius a_0, boundary conditions are separable only in a spherical coordinate system. The difference from Section 2.6.2 lies in the $T(t)$-equation. After separating variables by $u = T(t)U(r,\theta,\varphi)$, we have, with $-\lambda$ standing for the separation constant,

$$\tau_0 T'' + T' + \lambda a^2 T = 0$$

and the boundary-value problem of Helmholtz equations

$$\begin{cases} \Delta U + \lambda U = 0, \quad L(U, U_r)|_{r=a_0} = 0, \\ U(r, \theta, \varphi + 2\pi) = U(r, \theta, \varphi), \quad |U(0, \theta, \varphi)| < \infty. \end{cases} \tag{4.61}$$

where $\lambda = k^2 > 0$. Applying separation of variables twice to Eq. 4.61 leads to three S-L problems

1. The S-L problem regarding $\Phi(\varphi)$,

$$\Phi'' + \eta \Phi = 0, \quad \Phi(\varphi + 2\pi) = \Phi(\varphi), \tag{4.62}$$

   where η is the parameter to be determined.
2. The S-L problem of $\Theta(\theta)$,

$$\begin{aligned} &\Theta''(\theta) + (\cot\theta)\,\Theta'(\theta) + \left[l(l+1) - \frac{\eta}{\sin^2\theta} \right] \Theta(\theta) = 0, \\ &0 < \theta < \pi, \quad |\Theta(\theta)| < \infty. \end{aligned} \tag{4.63}$$

   Here $l(l+1)$ are the undetermined parameters.
3. The S-L problem of $R(r)$

$$\begin{cases} r^2 R'' + 2rR' + \left[\lambda r^2 - l(l+1)\right] R = 0, \\ L(R, R_r)|_{r=a_0} = 0, \quad |R(0)| < \infty, \quad |R'(0)| < \infty. \end{cases} \tag{4.64}$$

   Here λ is the undetermined parameter, which is required by the $T(t)$-equation. The eigenfunctions of Eq. (4.64) have the same form for all three kinds of boundary conditions, but have zero-points of different functions and different normal squares (see Section 2.6.2).

   For convenience in applications, we list main results regarding these three S-L problems in Table 4.2.

Remark 1. The readers are referred to Section 2.6.2 for $\mu_l^{\left(n+\frac{1}{2}\right)}$ $(l = 1, 2, \cdots)$ and M_{nl}. It has also been discussed how to solve Eqs. (4.63) and (4.64) in Section 2.6.2.

Remark 2. Equation (4.61) has nontrivial bounded solutions only for certain values of separation constant λ. For $\lambda = \left(\mu_l^{\left(n+\frac{1}{2}\right)} / a_0\right)^2$, we have the corresponding nontrivial bounded solutions

$$(a_{mn} \cos m\varphi + b_{mn} \sin m\varphi)\, P_n^m(\cos\theta)\, j_n\left(\frac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right),$$

in which $a_{mn}^2 + b_{mn}^2 \neq 0$.

Table 4.2 Eigenfunctions of $\Delta U + k^2 U = 0$ in a sphere

S-L problem	Eigenvalues	Eigenfunctions	Normal square	Weight function
(4.62)	m^2 $m = 0, 1, \cdots$	$a_m \cos m\varphi + b_m \sin m\varphi$ $a_m^2 + b_m^2 \neq 0$	$\frac{1}{2\pi}, m = 0$ $\frac{1}{\pi}, m = 1, 2, \cdots$	1
(4.63)	$n(n+1)$ $n = 0, 1, 2, \cdots$	$P_n^m(\cos\theta), m \leq n,$ $P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$ $P_n^m(x) = (1-x^2)^{\frac{m}{2}} \frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m}$	$\frac{(n+m)!}{(n-m)!} \frac{2}{2n+1}$	$\sin\theta$
(4.64)	$\left(\mu_l^{(n+\frac{1}{2})}/a_0\right)^2$ $n = 0, 1, 2, \cdots$	$j_n\left(\mu_l^{(n+\frac{1}{2})} r/a_0\right),$ $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$	M_{nl}	r^2

Remark 3. It can be readily shown that the solution structure theorem is also valid in spherical coordinate systems.

Example 3. Solve

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2 \Delta u(r,\theta,\varphi,t) + F(r,\theta,\varphi,t)\,, \\ \qquad 0 < r < a_0,\ 0 < t\,, \\ L(u,u_r)|_{r=a_0} = 0, \quad |u(0,\theta,\varphi,t)| < \infty, \\ u(r,\theta,\varphi+2\pi,t) = u(r,\theta,\varphi,t)\,, \\ u(r,\theta,\varphi,0) = \Phi(r,\theta,\varphi),\ u_t(r,\theta,\varphi,0) = \Psi(r,\theta,\varphi)\,. \end{cases} \tag{4.65}$$

Solution. We first seek $u = W_\Psi(r,\theta,\varphi,t)$, the solution for the case $F = \Phi = 0$. After separating variables, the $T(t)$-equation reads

$$\tau_0 T'' + T' + (\lambda_{nl} a)^2 T = 0\,,$$

where $\lambda_{nl} = \mu_l^{(n+\frac{1}{2})}/a_0$. Thus the solution of PDS (4.65) for the case $F = \Phi = 0$ can be written as

$$u = \mathrm{e}^{-\frac{t}{2\tau_0}} \sum_{m,n=0,l=1}^{\infty} \left[\left(a_{mnl}^{(1)} \cos\gamma_{nl} t + b_{mnl}^{(1)} \underline{\sin}\gamma_{nl} t \right) \cos m\varphi \right.$$
$$\left. + \left(a_{mnl}^{(2)} \cos\gamma_{nl} t + b_{mnl}^{(2)} \underline{\sin}\gamma_{nl} t \right) \sin m\varphi \right] P_n^m(\cos\theta)\, j_n\left(\frac{\mu_l^{(n+\frac{1}{2})}}{a_0} r \right),$$

where $\gamma_{nl} = \dfrac{1}{2\tau_0}\sqrt{4\tau_0(\lambda_{nl} a)^2 - 1}$.

Applying the initial condition $u|_{t=0} = 0$ yields $a^{(1)}_{mnl} = a^{(2)}_{mnl} = 0$. $b^{(1)}_{mnl}$ and $b^{(2)}_{mnl}$ can be determined by applying the initial condition $u_t|_{t=0} = \Psi(r,\theta,\varphi)$. Finally,

$$\begin{cases} u = W_\Psi(r,\theta,\varphi,t) \\ \quad = \mathrm{e}^{-\frac{t}{2\tau_0}} \sum\limits_{m,n=0,l=1}^{+\infty} \left(b^{(1)}_{mnl}\cos m\varphi + b^{(2)}_{mnl}\sin m\varphi\right) P_n^m(\cos\theta) \\ \qquad \cdot j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right) \underline{\sin}\gamma_{nl} t\,, \\ b^{(1)}_{0nl} = \dfrac{1}{2\pi\underline{\gamma}_{nl} M_{0nl}} \iiint\limits_{r\le a_0} \Psi(r,\theta,\varphi) P_n(\cos\theta) \\ \qquad \cdot j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right) r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,, \\ b^{(1)}_{mnl} = \dfrac{1}{\pi\underline{\gamma}_{nl} M_{mnl}} \iiint\limits_{r\le a_0} \Psi(r,\theta,\varphi) P_n^m(\cos\theta) \\ \qquad \cdot j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right) r^2 \cos m\varphi \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,, \\ b^{(2)}_{mnl} = \dfrac{1}{\pi\underline{\gamma}_{nl} M_{mnl}} \iiint\limits_{r\le a_0} \Psi(r,\theta,\varphi) P_n^m(\cos\theta) \\ \qquad \cdot j_n\left(\dfrac{\mu_l^{\left(n+\frac{1}{2}\right)}}{a_0} r\right) r^2 \sin m\varphi \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,. \end{cases}$$

where M_{mnl} is the product of normal squares of $\{P_n^m(\cos\theta)\}$ and $\left\{ j_n\left(\dfrac{\mu_l^{n+\frac{1}{2}}}{a_0} r\right)\right\}$.

Thus the solution of PDS (4.65) is, by the solution structure theorem,

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\Phi(r,\theta,\varphi,t) + W_\Psi(r,\theta,\varphi,t) + \int_0^t W_{F_\tau}(r,\theta,\varphi,t-\tau)\,\mathrm{d}\tau\,,$$

where $F_\tau = F(r,\theta,\varphi,\tau)\big/\tau_0$.

Chapter 5
Cauchy Problems of Hyperbolic Heat-Conduction Equations

In this chapter we apply the Riemann method, integral transformations and the method of spherical means to solve Cauchy problems of hyperbolic heat-conduction equations of one-, two- and three-dimensions. The emphasis is placed on the physics and the methods of measuring τ_0 following the solutions of Cauchy problems. A comparison is also made with wave equations and classical heat-conduction equations.

5.1 Riemann Method for Cauchy Problems

In this section we introduce the Riemann function and the Riemann method for solving Cauchy problems of second-order equations.

5.1.1 Conjugate Operator and Green Formula

Let L be the linear differential operator of hyperbolic equations

$$L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + c(x,y)\,, \tag{5.1}$$

where a, b and c are all differentiable functions of x and y. A nonhomogeneous hyperbolic equations of second-order can thus be written as

$$L[u] = f(x,y)\,.$$

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The *conjugate operator* of L is defined by

$$M = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - a(x,y)\frac{\partial}{\partial x} - b(x,y)\frac{\partial}{\partial y} + c(x,y)\,. \tag{5.2}$$

The conjugate operator of M is clearly L. Therefore, the L and the M are *mutually conjugate operators*.

If $Lu = Mu$, in particular, the L (or the M) is called the *self-conjugate operator*. Such self-conjugate operators can be used to define generalized solutions and solve PDS.

For any twice differentiable functions $u(x,y)$ and $v(x,y)$, we have, by the rules of differentiation,

$$vLu - uMv = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\,,$$

where $P(x,y) = (uv)_x - (2v_x - av)u$, $Q(x,y) = -(uv)_y + (2v_y + bv)u$. Thus, for a plane domain D,

$$\begin{aligned} \iint\limits_D (vLu - uMv)\,\mathrm{d}\sigma &= \iint\limits_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}\sigma \\ &= \oint\limits_C [P\cos(\mathbf{n},x) + Q\cos(\mathbf{n},y)]\,\mathrm{d}s = \oint\limits_C -Qdx + Pdy\,, \end{aligned} \tag{5.3}$$

where C is the positive-directed boundary curve of D and $\mathbf{n}$ is the outer normal of C. This is called the *generalized Green formula*.

5.1.2 Cauchy Problems and Riemann Functions

We aim to solve the PDS with its CDS specified on curve c:

$$\begin{cases} \mathrm{L}u = f(x,y)\,, \\ u|_c = \varphi(x), \quad u_n|_c = \psi(x)\,, \end{cases} \tag{5.4}$$

where u_n is the normal derivative of u. It reduces to the normal Cauchy problem when $y = t$ and c is taken as the straight line $t = 0$.

To find the solution of (5.4) at any point $M_0(x_0,y_0)$, $u(x_0,y_0)$, construct two characteristic curves passing through point $M_0 : x+y = x_0+y_0, x-y = x_0-y_0$. The two characteristic curves intersect with curve c at points M_1 and M_2. The domain enclosed by $\overline{M_0M_1}$, $\overset{\frown}{M_1M_2}$ and $\overline{M_2M_0}$ is denoted by Δ_{M_0} (Fig. 5.1).

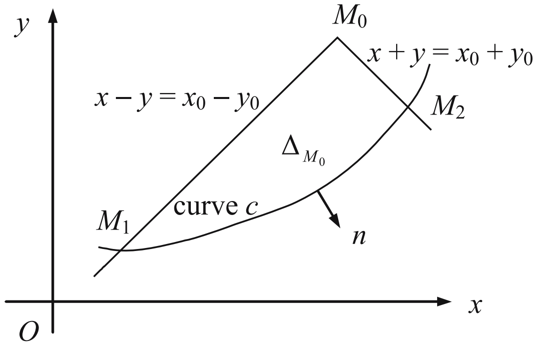

Fig. 5.1 Domain Δ_{M_0}

On $\overline{M_0M_1}$, $x-y=x_0-y_0$ so $\mathrm{d}y=\mathrm{d}x=-\mathrm{d}s\big/\sqrt{2}$ where $\mathrm{d}s$ is the differential of arclength s. Thus

$$\begin{aligned}\int_{\overline{M_0M_1}} -Q\,\mathrm{d}x+P\,\mathrm{d}y &= \int_{\overline{M_0M_1}} [(uv)_y-(2v_y+bv)u]\,\mathrm{d}x\\ &\qquad\qquad +[(uv)_x-(2v_x-av)u]\,\mathrm{d}y\\ &= \int_{\overline{M_0M_1}} \mathrm{d}(uv)-2\frac{\partial v}{\partial s}uds+\frac{b-a}{\sqrt{2}}uv\,\mathrm{d}s\\ &= uv|_{M_1}-uv|_{M_0}-\int_{\overline{M_0M_1}}\left(2\frac{\partial v}{\partial s}-\frac{b-a}{\sqrt{2}}v\right)u\,\mathrm{d}s.\end{aligned} \tag{5.5}$$

Similarly,

$$\int_{\overline{M_2M_0}} -Q\,\mathrm{d}x+P\,\mathrm{d}y = uv|_{M_2}-uv|_{M_0}+\int_{\overline{M_2M_0}}\left(2\frac{\partial v}{\partial s}+\frac{a+b}{\sqrt{2}}v\right)u\,\mathrm{d}s. \tag{5.6}$$

Substituting Eqs. (5.5) and (5.6) into Eq. (5.3) and solving for $(uv)|_{M_0}$ yields

$$\begin{aligned}(uv)|_{M_0} &= \frac{1}{2}\left[(uv)|_{M_1}+(uv)|_{M_2}\right]+\int_{\overline{M_2M_0}}\left(\frac{\partial v}{\partial s}+\frac{a+b}{2\sqrt{2}}v\right)u\,\mathrm{d}s\\ &\quad -\int_{\overline{M_0M_1}}\left(\frac{\partial v}{\partial s}-\frac{b-a}{2\sqrt{2}}v\right)u\,\mathrm{d}s+\frac{1}{2}\int_{\overset{\frown}{M_1M_2}} -Qdx+Pdy\\ &\quad -\frac{1}{2}\iint\limits_{\Delta_{M_0}}(vLu-uMv)\,\mathrm{d}\sigma,\end{aligned} \tag{5.7}$$

where $u|_{M_0}=u(x_0,y_0)$ is what we attempt to find.

Consider a function $v(x,y)$ such that

$$\begin{cases} Mv = 0,\ v|_{M_0} = 1\,, \\ \left(\dfrac{\partial v}{\partial s} - \dfrac{b-a}{2\sqrt{2}}v\right)\Bigg|_{\overline{M_0M_1}} = 0\,, \qquad \left(\dfrac{\partial v}{\partial s} + \dfrac{a+b}{2\sqrt{2}}v\right)\Bigg|_{\overline{M_2M_0}} = 0\,. \end{cases} \tag{5.8}$$

Clearly, such a function v depends on point M_0, and thus is denoted by $v(x,y;x_0,y_0)$ or $v(M,M_0)$. The function $v(x,y;x_0,y_0)$ or $v(M,M_0)$ defined by Eq. (5.8) is called the *Riemann function of operator* $\boldsymbol{L}$. For the Riemann function v defined by Eq. (5.8), Eq. (5.7) yields

$$\begin{aligned} u(x_0,y_0) = \frac{1}{2}\Bigg\{ & \left[(uv)|_{M_1} + (uv)|_{M_2}\right] + \int_{\overset{\frown}{M_1M_2}} -[-(uv)_y + (2v_y + bv)u]\,\mathrm{d}x \\ & + [(uv)_x - (2v_x - av)u]\,\mathrm{d}y - \iint\limits_{\Delta_{M_0}} v(x,y;x_0,y_0)f(x,y)\,\mathrm{d}\sigma \Bigg\}\,. \end{aligned} \tag{5.9}$$

Remark 1. Once the Riemann function $v(x,y;x_0,y_0)$ is available, the right-hand side of Eq. (5.9) becomes known. The task of solving PDS (5.4) reduces to that of finding the Riemann function from Eq. (5.8). Such a method of solving PDS is called the *Riemann method.*

Remark 2. On $\overset{\frown}{M_1M_2}$ of curve c,

$$\begin{aligned} u_x &= u_s\cos(x,s) + u_n\cos(x,\mathbf{n}) \\ &= u_x\frac{\mathrm{d}x}{\mathrm{d}s} + u_n[-\sin(x,s)] = \varphi'(x)\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 - \psi(x)\frac{\mathrm{d}y}{\mathrm{d}s} \\ &= \frac{\varphi'(x) - \psi(x)f'(x)\sqrt{1+[f'(x)]^2}}{1+[f'(x)]^2}\,. \end{aligned}$$

Similarly,

$$u_y = \frac{\varphi'(x)f'(x) + \psi(x)\sqrt{1+[f'(x)]^2}}{1+[f'(x)]^2}\,.$$

Remark 3. By using Eq. (5.9), the solution of the PDS

$$\begin{cases} Lu = -f(x,y) \\ u\big|_c = u_n\big|_c = 0 \end{cases} \tag{5.10}$$

reads
$$u(x,y) = \iint\limits_{\Delta_{M_0}} v(\xi,\eta;x,y)f(\xi,y)\,\mathrm{d}\xi\,\mathrm{d}\eta\,.$$

For $f(x,y) = \delta(x-x_0, y-y_0)$,

$$
\begin{aligned}
u(x,y) &= \iint\limits_{\Delta_{M_0}} v(\xi,\eta;x,y)\delta(\xi-x_0,\eta-y_0)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= v(x_0,y_0,x,y)\,.
\end{aligned}
$$

Therefore, the Riemann function is the solution of PDS (5.10) at $f(x,y) = \delta(x-x_0, y-y_0)$. The physical meaning of the Riemann function becomes known once the meanings of x, y and u in PDS (5.10) are available.

5.1.3 Example

Example. Use the Riemann method to solve

$$
\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \; -\infty < x < +\infty, \; 0 < t\,, \\ u(x,0) = \varphi(x), \; u_t(x,0) = \psi(x)\,. \end{cases} \tag{5.11}
$$

Solution. Step 1. Transform the equation into its standard form.
Let $y = at$, $x = x$. PDS (5.11) is transformed into

$$
\begin{cases} u_{xx} - u_{yy} = f_1\left(x, \dfrac{y}{a}\right), \; f_1 = -f/a^2\,, \quad -\infty < x < +\infty, 0 < y\,, \\ u(x,0) = \varphi(x)\,, \quad u_y(x,0) = \psi(x)/a\,. \end{cases} \tag{5.12}
$$

Since $Lu = Mu = u_{xx} - u_{yy}$, L is here a self-conjugate operator.

Step 2. Find the Riemann function.
To find the solution of (5.12) at any point $M(x,y)$, $u(x,y)$, construct two characteristic curves passing through point $M: \xi+\eta = x+y, \xi-\eta = x-y$, which intersect with the $O\xi$ axis at points P and Q. The domain enclosed by MP, PQ and $\overline{QM}$ is denoted by Δ_M (Fig. 5.2). By definition (5.8), the Riemann function $v(M'(\xi,\eta), M(x,y))$ satisfies, for any $M' \in \Delta_M$,

$$
\begin{cases} Mv = v_{\xi\xi} - v_{\eta\eta} = 0\,, \; v(M,M) = 1\,, \\ \left.\dfrac{\partial v}{\partial s}\right|_{\overline{MP}} = 0\;, \; \left.\dfrac{\partial v}{\partial s}\right|_{\overline{QM}} = 0\,. \end{cases} \tag{5.13}
$$

It reduces to, by a variable transformation of $x' = \xi - \eta$, $y' = \xi + \eta$,

$$
\begin{cases} v_{x'y'} = 0\,, \\ v|_{x'=x-y} = 1\,, \; v|_{y'=x+y} = 1\,. \end{cases} \tag{5.14}
$$

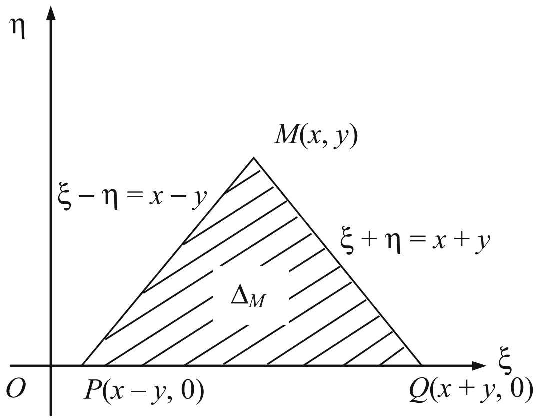

Fig. 5.2 Domain Δ_M

Its general solution is

$$v(x',y') = F(x') + G(y'),$$

where F and G are differentiable functions of x' and y', respectively. Applying $v|_{x'=x-y} = 1$ and $v|_{y'=x+y} = 1$ yields

$$\begin{aligned} v(x-y,y') &= F(x-y) + G(y') = 1, \\ v(x',x+y) &= F(x') + G(x+y) = 1, \end{aligned}$$

which requires $G(y') = C_1$ (constant), $F(x') = C_2$ (constant). Therefore

$$v(x',y') = C, \quad \forall (x',y') \in \Delta_M.$$

Also $$v|_{x'=x-y} = 1 \text{ and } v|_{y'=x+y} = 1.$$

Thus $$v(M',M) \equiv 1, \; M' \in \Delta_M.$$

Step 3. Solution of PDS (5.12).

Note that $v \equiv 1$, $b = 0$ and $d\eta = 0$ on PQ. Thus the solution of PDS (5.12) is, by Eq. (5.9),

$$u(x,y) = \frac{1}{2}[u(P) + u(Q)] + \frac{1}{2}\int_P^Q u_\eta \, d\xi - \frac{1}{2}\iint\limits_{\Delta_{MPQ}} f_1\left(\xi, \frac{\eta}{a}\right) d\xi \, d\eta.$$

Note that $u_\eta\big|_{\eta=0} = \dfrac{1}{a}\psi(\xi)$, $u(P) = \varphi(x-y)$, $u(Q) = \varphi(x+y)$. Also, $\overline{MP}$ and $\overline{QM}$ satisfy $\xi = x-(y-\eta)$, $\xi = x+(y-\eta)$, respectively. The solution of PDS (5.12) becomes

$$
\begin{aligned}
u(x,y) = {} & \frac{1}{2}\left[\varphi(x-y)+\varphi(x+y)\right] + \frac{1}{2a}\int_{x-y}^{x+y} \psi(\xi)\,\mathrm{d}\xi \\
& + \frac{1}{2a^2}\int_0^y \mathrm{d}\eta \int_{x-(y-\eta)}^{x+(y-\eta)} f\left(\xi, \frac{\eta}{a}\right)\mathrm{d}\xi\,.
\end{aligned}
$$

Step 4. Solution of PDS (5.11).
Since $y = at$, we have $\eta = a\tau$ and $\mathrm{d}\eta = a\,\mathrm{d}\tau$. Thus the solution of PDS (5.11) is

$$
\begin{aligned}
u(x,y) = {} & \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi \\
& + \frac{1}{2a}\int_0^t \mathrm{d}\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\,\mathrm{d}\xi\,.
\end{aligned}
$$

This is the same as what we obtained by using integral transformations, the method of characteristics or the method of descent.

5.2 Riemann Method and Method of Laplace Transformation for One-Dimensional Cauchy Problems

In this section we use the Riemann method and the method of Laplace transformation to solve

$$
\begin{cases}
u_t + \tau_0 u_{tt} = a^2 u_{xx} + f(x,t), & -\infty < x < +\infty,\ 0 < t\,, \\
u(x,0) = \varphi(x)\,, \quad u_t(x,0) = \psi(x)\,.
\end{cases} \tag{5.15}
$$

By the solution structure theorem, we can focus on solving

$$
\begin{cases}
u_t + \tau_0 u_{tt} = a^2 u_{xx}\,, & -\infty < x < +\infty,\ 0 < t\,, \\
u(x,0) = 0\,, \quad u_t(x,0) = \psi(x)\,.
\end{cases} \tag{5.16}
$$

5.2.1 Riemann Method

Transform the Equation into its Standard Form

Consider a variable transformation of $\xi = \sqrt{\tau_0}x/a$ and $t = t$. PDS (5.16) is thus transformed into

$$\begin{cases} u_{\xi\xi} - u_{tt} - \dfrac{u_t}{\tau_0} = 0\,, \quad -\infty < \xi < +\infty,\ 0 < t\,, \\ u(\xi\,,0) = 0\,, \quad u_t(\xi\,,0) = \psi(a\xi/\sqrt{\tau_0})\,. \end{cases} \tag{5.17}$$

To eliminate the term involving u_t, consider a function transformation of $u(\xi\,,t) = U(\xi\,,t)\,\mathrm{e}^{\lambda t}$. Substituting it into the equation of PDS (5.17) and vanishing the coefficient of u_t-term lead to $\lambda = -1/2\tau_0$. Therefore, a function transformation of

$$u(\xi\,,t) = U(\xi\,,t)\,\mathrm{e}^{-\frac{t}{2\tau_0}}$$

transforms PDS (5.17) into

$$\begin{cases} U_{\xi\xi} - U_{tt} + c^2 U = 0\,,\ c = \dfrac{1}{2\tau_0}\,, \quad -\infty < \xi < +\infty,\ 0 < t\,, \\ U(\xi,0) = 0\,, \quad U_t(\xi\,,0) = \psi\left(a\xi\left/\sqrt{\tau_0}\right.\right)\,. \end{cases} \tag{5.18}$$

Its equation can be written as $LU = 0$, where L is clearly a self-conjugate operator and is defined by

$$L = \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial t^2} + c^2\,.$$

Find the Riemann Function of L

By its definition, the Riemann function $v(M', M)$ of L satisfies

$$\begin{cases} Lv = 0\,,\ v(M,M) = 1\,,\ M' \in \Delta_M\,, \\ v_s|_{\overline{MP}} = v_s|_{\overline{QM}} = 0\,. \end{cases} \tag{5.19}$$

Here Δ_M is the region enclosed by the ξ'-axis and the two characteristic curves $\xi' - \tau = \xi - t$ and $\xi' + \tau = \xi + t$, respectively (Fig. 5.3). $P(\xi - t\,,0)$ and $Q(\xi + t\,,0)$ are the intersecting points of the two characteristic curves with the ξ'-axis. v_s stands for the directional derivative of v along $\overline{MP}$ and $\overline{QM}$. The CDS in PDS (5.19) implies that $v(M'\,,M') \equiv 1$, $\forall M' \in \overline{MP} \cup \overline{QM}$. PDS (5.19) reduces, thus, to

$$\begin{cases} v_{\xi'\xi'} - v_{\tau\tau} + c^2 v = 0\,, M' \in \Delta_M\,, \\ v|_{\overline{MP}\cup\overline{QM}} \equiv 1\,. \end{cases} \tag{5.20}$$

In order to transform (5.20) into an ordinary differential equation, consider a variable transformation

$$z = \sqrt{(t-\tau)^2 - (\xi - \xi')^2} \text{ or } z^2 = (t-\tau)^2 - \left(\xi - \xi'\right)^2, \left(\xi', \tau\right) \in \Delta_M . \quad (5.21)$$

Thus PDS (5.20) is transformed into

$$\begin{cases} \left(z_{\xi'}^2 - z_\tau^2\right) v''(z) + \left(z_{\xi'\xi'} - z_{\tau\tau}\right) v'(z) + c^2 v(z) = 0, \\ v(0) = 1. \end{cases} \quad (5.22)$$

Note that, by Eq. (5.21)

$$zz_\tau = -(t-\tau), \quad zz_{\xi'} = \xi - \xi',$$

which lead to, by subtracting the square of the former from the square of the latter,

$$z_{\xi'}^2 - z_\tau^2 = -1 . \quad (5.23)$$

Also, from Eq. (5.21)

$$z_\tau^2 + zz_{\tau\tau} = 1, \quad z_{\xi'}^2 + zz_{\xi'\xi'} = -1,$$

which yield, by subtracting the former from the latter,

$$z_{\xi'}^2 - z_\tau^2 + z\left(z_{\xi'\xi'} - z_{\tau\tau}\right) = -2 . \quad (5.24)$$

Substituting Eq. (5.23) into Eq. (5.24) yields

$$z_{\xi'\xi'} - z_{\tau\tau} = -\frac{1}{z} . \quad (5.25)$$

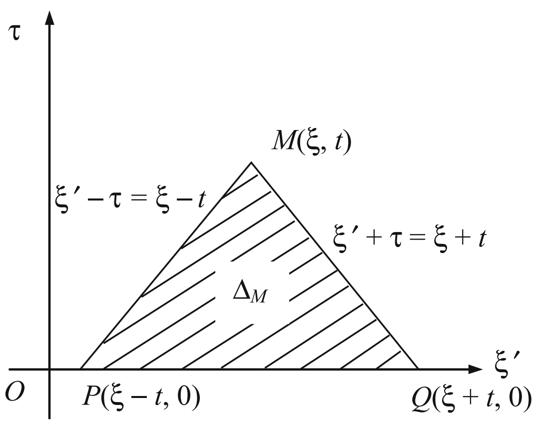

Fig. 5.3 Domain Δ_M

Substituting Eqs. (5.23) and (5.25) into (5.22) leads to

$$\begin{cases} v''(z)+\dfrac{1}{z}v'(z)-c^2v(z)=0\,, \\ v(0)=1\,. \end{cases}$$

Therefore, v depends on ξ', τ, ξ and t only through z. By another variable transformation $\eta = \mathrm{i}cz\,\left(\mathrm{i}=\sqrt{-1}\right)$, it becomes

$$\begin{cases} v''(\eta)+\dfrac{1}{\eta}v'(\eta)+v(\eta)=0\,, \\ v(0)=1\,, \end{cases}$$

where the equation is the Bessel equation of order zero. Its solution is (see Appendix A)

$$v\left(\xi,t;\xi',\tau\right)=J_0\left(\mathrm{i}c\sqrt{(t-\tau)^2-(\xi-\xi')^2}\right)=I_0\left(c\sqrt{(t-\tau)^2-(\xi-\xi')^2}\right),$$

where J_0 is the Bessel equation of order zero of the first kind. I_0 is the modified Bessel function of order zero of the first kind.

Find solutions of PDS (5.18) and (5.16)

The solution of PDS (5.18) follows from Eq. (5.9) in Section 5.1, by noting that $f(x,t)=0$, $\mathrm{d}\tau=0$ on $\overline{PQ}$, $u(\xi',0)=0$ and $u_\tau(\xi',0)=\psi(a\xi'/\sqrt{\tau_0})$,

$$U(\xi,t)=\frac{1}{2}\int_{\xi-t}^{\xi+t} I_0\left(c\sqrt{t^2-(\xi-\xi')^2}\right)\psi\left(a\xi'/\sqrt{\tau_0}\right)\mathrm{d}\xi'\,.$$

The solution of PDS (5.17) is

$$u(\xi,t)=\mathrm{e}^{-\frac{t}{2\tau_0}}U\left(\xi,t\right)\,.$$

Since $\xi=\sqrt{\tau_0}x/a$, the solution of PDS (5.16) is

$$\begin{aligned} u &= W_\psi(x,t) \\ &= \mathrm{e}^{-\frac{t}{2\tau_0}}\frac{1}{2}\int_{\frac{\sqrt{\tau_0}}{a}x-t}^{\frac{\sqrt{\tau_0}}{a}x+t} I_0\left(c\sqrt{t^2-\left(\frac{\sqrt{\tau_0}x}{a}-\xi'\right)^2}\right)\psi\left(\frac{a\xi'}{\sqrt{\tau_0}}\right)\mathrm{d}\xi'\,. \end{aligned} \tag{5.26}$$

Finally, the solution of PDS (5.15) is, by the solution structure theorem,

$$u=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(x,t)+W_\psi(x,t)+\int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau$$

where $f_\tau=f(x,\tau)/\tau_0$.

Remark 1. Note that $I_0'(x) = I_1(x)$. The solution of PDS (5.15) at $f = \psi = 0$ is thus

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \left\{ \frac{\varphi\left(x - at/\sqrt{\tau_0}\right) + \varphi\left(x + at/\sqrt{\tau_0}\right)}{2} \right.$$

$$\left. + \frac{1}{2}\int_{\frac{\sqrt{\tau_0}}{a}x-t}^{\frac{\sqrt{\tau_0}}{a}x+t} \left[\frac{1}{2\tau_0} I_0(y) + \frac{c^2 t}{y} I_1(y)\right] \varphi\left(\frac{a}{\sqrt{\tau_0}}\xi'\right) \mathrm{d}\xi' \right\}, \tag{5.27}$$

where $y = c\sqrt{t^2 - \left(\dfrac{\sqrt{\tau_0}}{a}x - \xi'\right)^2}$. Also, the solution of PDS (5.15) at $\psi = \varphi = 0$ reads

$$\begin{aligned} u &= \int_0^t W_{f\tau}(x, t-\tau)\,\mathrm{d}\tau \\ &= \frac{1}{2}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{\frac{\sqrt{\tau_0}}{a}x-t}^{\frac{\sqrt{\tau_0}}{a}x+t} \frac{I_0\left(c\sqrt{(t-\tau)^2 - \left(\frac{\sqrt{\tau_0}}{a}x - \xi'\right)^2}\right)}{\tau_0} f\left(\frac{a\xi'}{\sqrt{\tau_0}}, \tau\right) \mathrm{d}\xi'. \end{aligned} \tag{5.28}$$

Since $[\xi] = [\sqrt{\tau_0}x/a] = T$, we have $[\xi'] = T$, $[y] = 1$ and $[I_0] = 1$. Therefore,

Eq. (5.26): $[u] = [I_0][\psi][\mathrm{d}\xi'] = \Theta T^{-1} \cdot T = \Theta$.

Eq. (5.27): $[u] = \left[\dfrac{c^2 t}{y} I_1 \varphi\,\mathrm{d}\xi'\right] = T^{-1}\Theta T = \Theta$ (The unit of all the other terms is also clear.)

Eq. (5.28): $[u] = \left[\mathrm{d}\tau \dfrac{I_0}{\tau_0} f\,\mathrm{d}\xi'\right] = T \cdot T^{-1} \cdot \Theta T^{-1} T = \Theta$.

Remark 2. Rewrite PDS (5.15) into

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 u_{xx} + \dfrac{f(x,t)}{\tau_0}, & -\infty < x < +\infty,\ 0 < t, \\ u(x,0) = \varphi(x), \quad u_t(x0) = \psi(x), \end{cases} \tag{5.29}$$

where $A = a/\sqrt{\tau_0}$ $\left([A] = LT^{-1}\right)$ is the velocity of thermal waves. For a variable transformation $\xi' = \dfrac{\sqrt{\tau_0}}{a}\xi$,

$$\mathrm{d}\xi' = \frac{\sqrt{\tau_0}}{a}\mathrm{d}\xi = \frac{1}{A}\mathrm{d}\xi,$$

$$y = c\sqrt{t^2 - \left(\tfrac{\sqrt{\tau_0}}{a}x - \xi'\right)^2} = b\sqrt{(At)^2 - (x-\xi)^2}, \quad b = 1/2a\sqrt{\tau_0}.$$

The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 u_{xx}, & -\infty < x < +\infty,\ 0 < t, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{cases}$$

may thus be written as

$$\begin{aligned} u = \mathrm{e}^{-\frac{t}{2\tau_0}} \Bigg\{ & \frac{\varphi(x-At) + \varphi(x+At)}{2} \\ & + \frac{1}{2A} \int_{x-At}^{x+At} \left[\left[\frac{1}{2\tau_0} I_0 \left(b\sqrt{(At)^2 - (x-\xi)^2} \right) \right.\right. \\ & + \left. \frac{t}{4\tau_0^2 b\sqrt{(At)^2 - (x-\xi)^2}} I_1 \left(b\sqrt{(At)^2 - (x-\xi)^2} \right) \right] \varphi(\xi) \\ & \left. + I_0 \left(b\sqrt{(At)^2 - (x-\xi)^2} \right) \psi(\xi) \right] \mathrm{d}\xi \Bigg\}, \end{aligned} \tag{5.30}$$

which is similar to the D'Alembert formula of wave equations. The unit of the second term in the integrand reads

$$\left[\frac{1}{2A} \frac{t}{4\tau_0^2 b\sqrt{(At)^2 - (x-\xi)^2}} I_1 \varphi(\xi)\,\mathrm{d}\xi \right] = \frac{T}{L} \cdot \frac{1}{T} \cdot \Theta \cdot L = \Theta\,.$$

The unit of all the other terms is also Θ.

The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 u_{xx} + \dfrac{f(x,t)}{\tau_0}, & -\infty < x < +\infty,\ 0 < t, \\ u(x,0) = 0, \quad u_t(x,0) = 0 \end{cases} \tag{5.31}$$

is

$$u = \frac{1}{2A} \int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} \frac{I_0 \left(b\sqrt{A^2(t-\tau)^2 - (x-\xi)^2} \right)}{\tau_0} f(\xi,\tau)\,\mathrm{d}\xi\,, \tag{5.32}$$

which is similar to the Kirchhoff formula of wave equations.

For convenience, rewrite the equation of PDS (5.31) as

$$\frac{u_t}{\tau_0} + u_{tt} = A^2 u_{xx} + f(x,t), \quad [f] = \Theta T^{-2}\,.$$

The solution (5.32) reduces to

$$u = \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0\left(b\sqrt{A^2(t-\tau)^2-(x-\xi)^2}\right) f(\xi,\tau)\,\mathrm{d}\xi\,. \quad (5.33)$$

Remark 3. Verification of CDS.

1. Verify that Eq. (5.30) satisfies $u(x,0)=\varphi(x)$ and $u_t(x,0)=\psi(x)$.
   Equation (5.30) clearly satisfies $u(x,0)=\varphi(x)$ by integral properties. Taking the derivative of Eq. (5.30) with respect to t and using

$$\lim_{x\to 0}\frac{I_1(x)}{x}=\frac{1}{2},$$

$$\left.\frac{tI_1}{4\tau_0^2 b\sqrt{(At)^2-(x-\xi)^2}}\right|_{\xi=x\pm At} = \lim_{\xi\to x\pm At}\frac{tI_1}{4\tau_0^2 b\sqrt{(At)^2-(x-\xi)^2}}$$

$$= \lim_{\varepsilon\to 0}\frac{tI_1(\varepsilon)}{4\tau_0^2\varepsilon}=\frac{t}{8\tau_0^2},$$

   we obtain

$$u_t(x,0) = -\frac{1}{2\tau_0}\varphi(x)+\frac{1}{2\tau_0}\varphi(x)+\frac{1}{2A}2A\psi(x)=\psi(x)\,.$$

2. Verify that Eq. (5.32) satisfies $u(x,0)=0$ and $u_t(x,0)=0$. Clearly, $u(x,0)=0$. Also,

$$\begin{aligned} u_t = \frac{1}{2A\tau_0}\Bigg\{\int_0^t \frac{\partial}{\partial t}\Bigg[\mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \int_{x-A(t-\tau)}^{x+A(t-\tau)} \\ I_0\left(b\sqrt{A^2(t-\tau)^2-(x-\xi)^2}\right)\Bigg] f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\ + \int_x^x I_0\left(b\sqrt{-(x-\xi)^2}\right) f(\xi,\tau)\,\mathrm{d}\xi\cdot 1\Bigg\}\,. \end{aligned}$$

   Therefore $u_t(x,0)=0$.

Meaning of Riemann Function

Consider a source term $f(x,t) = \delta(x - x_0, t - t_0)$ in Eq. (5.32). We have

$$u(x,t) =$$

$$\frac{1}{2A\tau_0}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-At}^{x+At} I_0\left(b\sqrt{A^2(t-\tau)^2 - (x-\xi)^2}\right)\delta(\xi - x_0, \tau - t_0)\,\mathrm{d}\xi$$
$$= \frac{1}{2a\sqrt{\tau_0}} I_0\left(\frac{1}{2a\sqrt{\tau_0}}\sqrt{\frac{a^2}{\tau_0}(t-t_0)^2 - (x-x_0)^2}\right)\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}. \tag{5.34}$$

Therefore, the temperature distribution due to a unit impulsive source $\delta(x - x_0, t - t_0)$ at x_0 and t_0 can be represented by using the modified Bessel function of order zero of the first kind. This is also the meaning of the Riemann function.

Note that $I(0) = 1$ and $a^2 = k/\rho c$ (k-thermal conductivity, ρ-density, c-specific heat). By Eq. (5.34), we obtain

$$u(x_0,t_0) = \lim_{\substack{x \to x_0 \\ t \to t_0}} u(x,t) = \frac{1}{2}\sqrt{\frac{\rho c}{k\tau_0}}. \tag{5.35}$$

Therefore, the temperature $u(x_0,t_0)$ is proportional to $\sqrt{\rho c}$ and inversely proportional to $\sqrt{k\tau_0}$.

Remark 4. In Eq. (5.34), $[f(x,t)] = \Theta T^{-1}$ and $[\delta(\xi - x_0, \tau - t_0)] = \dfrac{L\Theta}{LT}$.

5.2.2 *Method of Laplace Transformation*

PDS (5.18) can easily be solved by the method of Laplace transformation due to the absence of first derivative terms in its equation. The readers are referred to Appendix B.2 for a discussion of Laplace transformations. By taking a Laplace transformation of (5.18), we obtain

$$\overline{U}_{\xi\xi} - s^2\overline{U} + \psi + c^2\overline{U} = 0 \quad \text{or} \quad \overline{U}_{\xi\xi} - \left(s^2 - c^2\right)\overline{U} = -\psi, \tag{5.36}$$

where $\overline{U}(\xi,s) = L[u]$. The general solution of the associated homogeneous equation is

$$\overline{U}(\xi,s) = c_1\,\mathrm{e}^{\xi\sqrt{s^2-c^2}} + c_2\,\mathrm{e}^{-\xi\sqrt{s^2-c^2}},$$

where c_1 and c_2 are constants. Since the Wronski determinant

$$\begin{vmatrix} e^{\xi\sqrt{s^2-c^2}} & e^{-\xi\sqrt{s^2-c^2}} \\ \left(e^{\xi\sqrt{s^2-c^2}}\right)_\xi & \left(e^{-\xi\sqrt{s^2-c^2}}\right)_\xi \end{vmatrix} = -2\sqrt{s^2-c^2},$$

the general solution of Eq. (5.36) is

$$\begin{aligned}\overline{U}(\xi,s) =& A\,e^{\xi\sqrt{s^2-c^2}} + B\,e^{-\xi\sqrt{s^2-c^2}} - \frac{1}{2}\int \frac{e^{-(\xi'-\xi)\sqrt{s^2-c^2}}}{\sqrt{s^2-c^2}}\psi\left(\frac{a\xi'}{\sqrt{\tau_0}}\right)\mathrm{d}\xi' \\ &+\frac{1}{2}\int \frac{e^{-(\xi-\xi')\sqrt{s^2-c^2}}}{\sqrt{s^2-c^2}}\psi\left(\frac{a\xi'}{\sqrt{\tau_0}}\right)\mathrm{d}\xi',\end{aligned}$$

where A and B are constants. Since $\overline{U}$ must be bounded as $\xi \to -\infty$ and $\xi \to +\infty$, $A = B = 0$. In order to ensure the convergence of the two integrals, let

$$\begin{aligned}\overline{U}(\xi,s) = \frac{1}{2}\Bigg[&-\int_{+\infty}^{\xi} \frac{e^{-(\xi'-\xi)\sqrt{s^2-c^2}}}{\sqrt{s^2-c^2}}\psi\left(\frac{a\xi'}{\sqrt{\tau_0}}\right)\mathrm{d}\xi' \\ &+\int_{-\infty}^{\xi} \frac{e^{-(\xi-\xi')\sqrt{s^2-c^2}}}{\sqrt{s^2-c^2}}\psi\left(\frac{a\xi'}{\sqrt{\tau_0}}\right)\mathrm{d}\xi'\Bigg].\end{aligned}$$

Since

$$L^{-1}\left[\frac{e^{-(\xi'-\xi)\sqrt{s^2-c^2}}}{\sqrt{s^2-c^2}}\right] = I_0\left(c\sqrt{t^2-(\xi-\xi')^2}\right)H\left[t-(\xi'-\xi)\right],$$

we have

$$\begin{aligned}&\frac{1}{2}\int_{\xi}^{+\infty} L^{-1}\left[\frac{e^{-(\xi'-\xi)\sqrt{s^2-c^2}}}{\sqrt{s^2-c^2}}\psi\left(\frac{a\xi'}{\sqrt{\tau_0}}\right)\right]\mathrm{d}\xi' \\ &= \frac{1}{2}\int_{\xi}^{\xi+t} I_0\left(c\sqrt{t^2-(\xi-\xi')^2}\right)\psi\left(\frac{a\xi'}{\sqrt{\tau_0}}\right)\mathrm{d}\xi'.\end{aligned}$$

Similarly,

$$\begin{aligned}&\frac{1}{2}\int_{-\infty}^{\xi} L^{-1}\left[\frac{e^{-(\xi-\xi')\sqrt{s^2-c^2}}}{\sqrt{s^2-c^2}}\psi\left(\frac{a\xi'}{\sqrt{\tau_0}}\right)\right]\mathrm{d}\xi' \\ &= \frac{1}{2}\int_{\xi-t}^{\xi} I_0\left(c\sqrt{t^2-(\xi-\xi')^2}\right)\psi\left(\frac{a\xi'}{\sqrt{\tau_0}}\right)\mathrm{d}\xi'.\end{aligned}$$

Here $H(x)$ is the Heaviside function defined by

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Finally, the solution of (5.18) is

$$U(\xi,t) = L^{-1}\left[\overline{U}(\xi,s)\right] = \frac{1}{2}\int_{\xi-t}^{\xi+t} I_0\left(c\sqrt{t^2-(\xi-\xi')^2}\right)\psi\left(\frac{a\xi'}{\sqrt{\tau_0}}\right)\mathrm{d}\xi',$$

which is the same as that obtained by the Riemann method.

5.2.3 Some Properties of Solutions

We have obtained solutions of wave, heat-conduction and hyperbolic heat-conduction equations due to initial disturbances $\varphi(x)$ and $\psi(x)$. They are, respectively,

$$u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,\mathrm{d}\xi, \tag{5.37}$$

$$u(x,t) = \int_{-\infty}^{+\infty} V(x,\xi,t)\varphi(\xi)\,\mathrm{d}\xi, \quad V = \frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2t}}, \tag{5.38}$$

$$\begin{aligned} u(x,t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \Bigg\{ & \frac{\varphi(x-At)+\varphi(x+At)}{2} \\ & + \frac{1}{2A}\int_{x-At}^{x+At}\Bigg\{\Bigg[\frac{1}{2\tau_0}I_0\left(b\sqrt{(At)^2-(x-\xi)^2}\right) \\ & + \frac{t}{4\tau_0^2 b\sqrt{(At)^2-(x-\xi)^2}} I_1\left(b\sqrt{(At)^2-(x-\xi)^2}\right)\Bigg]\varphi(\xi) \\ & + \left[I_0\left(b\sqrt{(At)^2-(x-\xi)^2}\right)\right]\psi(\xi)\Bigg\}\,\mathrm{d}\xi\Bigg\}. \end{aligned} \tag{5.39}$$

Equation (5.37) shows that vibration propagates in the form of superimposed forward and backward traveling waves that are created by initial disturbances. Waves travel at a speed a. The problems of semi-infinite domains can be solved by using the method of continuation. When the end is fixed, there is semi-wave loss. When the end is free, however, the semi-wave loss disappears.

The integrand in Eq. (5.38) contains a factor that is exponentially decreasing with respect to ξ. The demand for the smoothness of $\varphi(x)$ is thus very weak. Eq. (5.38) can also be written as

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}}\,\mathrm{e}^{-\frac{x^2}{4a^2t}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{\xi^2-2x\xi}{4a^2t}}\,\varphi(\xi)\,\mathrm{d}\xi\,.$$

Therefore, the temperature decreases as time increases, regardless of the value of point x, and $\lim\limits_{|x|\to+\infty} u(x,t)=0$ for all time instants.

Equation (5.39) has two terms involving $\varphi(x)$. The superimposed forward and backward waves and the integral limits in Eq. (5.39) are the signature of so-called thermal waves. The thermal waves in Eq. (5.39) differs, however, from the waves in Eq. (5.37) mainly on the appearance of an exponentially decaying factor $\mathrm{e}^{-\frac{t}{2\tau_0}}$ with respect to t. This is quite similar to Eq. (5.38) where a decaying factor $1\big/\sqrt{t}$ occurs.

Note that $\mathrm{e}^{-\frac{t}{2\tau_0}} = o\left(1\big/\sqrt{t}\right)$ as $t\to+\infty$ for all positive τ_0. The $u(x,t)$ in Eq. (5.39) decays with respect to t more quickly than in Eq. (5.38) . The smaller τ_0 is, the faster the decay.

One main difference between solutions in Eqs. (5.38) and (5.39) is the propagation of singularities of $\varphi(x)$ and $\psi(x)$ at x_0 along the characteristic curves $x-At=x_0$ and $x+At=x_0$ in Eq. (5.39), which is a typical wave property. While the integral in Eq. (5.39) shares the same format as that in Eq. (5.37), the integrand contains both $\varphi(x)$ and $\psi(x)$ as well as the modified Bessel function of the first kind. Hence the shape of thermal waves is intrinsically more complicated. However, the integral limit is still determined by the characteristic curves $\xi\in[x-At,x+At]$, which is a property of the traveling waves in Eq. (5.37).

Another striking difference between solutions in Eqs. (5.38) and (5.39) becomes visible by considering the solutions due to $\varphi(x)$ such that

$$\varphi(x)\begin{cases} >0\,,\ x\in[x_1,x_2]\,,\\ =0\,,\ x\notin[x_1,x_2]\,.\end{cases}$$

By Eq. (5.38),

$$u(x,t)=\int_{x_1}^{x_2}V(x,\xi,t)\varphi(\xi)\,\mathrm{d}\xi>0\,,\quad x\in(-\infty,+\infty)\,.$$

Therefore, the effect of $\varphi(x)$ in $[x_1,x_2]$ can be sensed instantly at all points including those towards $x\to-\infty$ and $x\to+\infty$. This is not the case, however, for the $u(x,t)$ in Eq. (5.39). Note that the domain of dependence of x_0 is $[x_0-At,x_0+At]$ (Fig. 5.4). For a sufficiently short time t_1, $[x_0-At_1,x_0+At_1]$ is outside of $[x_1,x_2]$, where the disturbance exists, so that $u(x_0,t_1)=0$. For a sufficiently long time t_2, however, $[x_0-At_2,x_0+At_2]$ has some part in common with $[x_1,x_2]$ so that $u(x_0,t_2)\neq 0$ in general. Since the propagation of any disturbance always requires some time, hyperbolic heat-conduction equation is thus a better representation of real heat conduction.

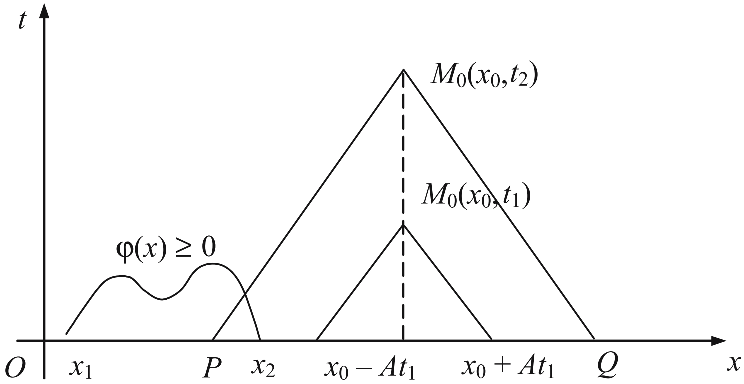

Fig. 5.4 Characteristic curves and domian of dependence

Note that the slope of characteristic curves $x \pm At = C$ is $k = \mp \frac{1}{A} = \mp \frac{\sqrt{\tau_0}}{a}$. As τ_0 increases, therefore, $|k|$ increases so that the speed of thermal waves gets smaller. As $\tau_0 \to 0$, on the other hand, $\lim\limits_{\tau_0 \to 0} A = \infty$. The speed of thermal waves tends to infinity; the hyperbolic heat-conduction equation reduces to the classical parabolic heat-conduction equation.

Therefore, the hyperbolic heat-conduction equation contains both the feature of traveling waves in wave equations and the property of decaying with time in classical heat-conduction equations. It is a hybrid of the two equations. The speed of temperature propagation in hyperbolic heat conduction has a finite value, which is a better reflection of reality.

5.3 Verification of Solutions, Physics and Measurement of τ_0

In this section we show that the solutions obtained in Section 5.2 satisfy the PDS for some special initial conditions. Such a verification further justifies the correctness of solutions obtained in Section 5.2, yields interesting results of some complicated integrals, uncovers the physics of τ_0 and provides the method of measuring τ_0.

5.3.1 Verify the Solution for $\mathbf{u}(\mathbf{x},\mathbf{0}) = \mathbf{0}$ *and* $\mathbf{u_t}(\mathbf{x},\mathbf{0}) = \mathbf{1}$

Verify that the solution of

$$\begin{cases} \frac{u_t}{\tau_0} + u_{tt} = A^2 u_{xx}, & -\infty < x < +\infty,\ 0 < t, \\ u(x,0) = 0, \quad u_t(x,0) = 1 \end{cases} \tag{5.40}$$

is

$$u = \mathrm{e}^{-\frac{t}{2\tau_0}} \frac{1}{2A} \int_{x-At}^{x+At} I_0\left(b\sqrt{(At)^2-(x-\xi)^2}\right) \mathrm{d}\xi \,. \tag{5.41}$$

The Initial Conditions

By Eq. (5.41), it is clear that $u(x,0) = 0$. Since

$$u_t = \frac{1}{2A}\left[-\frac{1}{2\tau_0}\mathrm{e}^{-\frac{t}{2\tau_0}} \int_{x-At}^{x+At} I_0\left(b\sqrt{(At)^2-(x-\xi)^2}\right) \mathrm{d}\xi \right.$$
$$\left. +\mathrm{e}^{-\frac{t}{2\tau_0}}\left(\int_{x-At}^{x+At} \frac{\partial}{\partial t} I_0 \,\mathrm{d}\xi + I_0|_{\xi=x+At} A - I_0|_{\xi=x-At}(-A)\right)\right],$$

we obtain $u_t(x,0) = 1$. Therefore $u(x,t)$ in Eq. (5.41) satisfies the two initial conditions of PDS (5.40).

The Equation

By using the series expansion of $I_0(x)$ (see Appendix A) and $b = \dfrac{1}{2A\tau_0}$, we obtain

$$\int_{x-At}^{x+At} I_0\left(b\sqrt{(At)^2-(x-\xi)^2}\right) \mathrm{d}\xi = 2At + \frac{1}{3}b^2A^3t^3 + \frac{1}{60}b^4A^5t^5 + \cdots \,. \tag{5.42}$$

This shows that the right-hand side of Eq. (5.41) is independent of x. Thus we only need to show that the u in Eq. (5.41) satisfies

$$\frac{u_t}{\tau_0} + u_{tt} = 0\,. \tag{5.43}$$

This can be achieved by using the formula $(uv)'' = u''v + 2u'v' + uv''$. But, the process is quite involved. Here we show it by using another method.

By expanding $\mathrm{e}^{-\frac{t}{2\tau_0}}$ into a series and using Eq. (5.42), we have

$$u(x,t) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{t^n}{n!\tau_0^{n-1}}\,. \tag{5.44}$$

Thus

$$u_t = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{t^{n-1}}{(n-1)!\tau_0^{n-1}}\,, \quad u_{tt} = -\sum_{n=2}^{+\infty} (-1)^{n-1} \frac{t^{n-2}}{(n-2)!\tau_0^{n-1}}\,, \tag{5.45}$$

which shows

$$\frac{u_t}{\tau_0} + u_{tt} = 0\,.$$

This can also be obtained by noting that $u_t = \mathrm{e}^{-\frac{t}{\tau_0}}$ and $\tau_0 u_{tt} = -\mathrm{e}^{-\frac{t}{\tau_0}}$ from the theory of series. Note that the temperature distribution $u(x,t)$ comes only from $u_t(x,0) = 1$. Since the initial condition $u_t(x,0) = 1$ is x-independent, $u(x,t)$ cannot depend on x either so $u_{xx} = 0$; consequently, Eq. (5.43) follows. From this point of view, the solution of PDS (5.40) should be that of an initial-value problem of an ordinary differential equation

$$\tau_0 \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \frac{\mathrm{d}u}{\mathrm{d}t} = 0\,, \quad u(0) = 0\,, \quad \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{t=0} = 1\,.$$

Its solution can be readily obtained

$$u(x,t) = \tau_0 \left(1 - \mathrm{e}^{-\frac{t}{\tau_0}}\right)\,, \tag{5.46}$$

which agrees with Eq. (5.44). This, with Eq. (5.41), leads to

$$\int_{x-At}^{x+At} I_0 \left(b\sqrt{(At)^2 - (x-\xi)^2}\right) \mathrm{d}\xi = 2A\tau_0 \,\mathrm{e}^{\frac{t}{2\tau_0}} \left(1 - \mathrm{e}^{-\frac{t}{\tau_0}}\right)\,, \tag{5.47}$$

where $b = \dfrac{1}{2A\tau_0}$.

Note. By Eq. (5.44), we have

$$\frac{u}{\tau_0} = \frac{t}{\tau_0} - \frac{1}{2!}\left(\frac{t}{\tau_0}\right)^2 + \frac{1}{3!}\left(\frac{t}{\tau_0}\right)^3 - \cdots\,.$$

Also, from series theory,

$$\mathrm{e}^{-\dfrac{t}{\tau_0}} = 1 - \frac{t}{\tau_0} + \frac{1}{2!}\left(\frac{t}{\tau_0}\right)^2 - \frac{1}{3!}\left(\frac{t}{\tau_0}\right)^3 + \cdots\,.$$

Adding these two equations together yields

$$u = \tau_0 \left(1 - \mathrm{e}^{-\frac{t}{\tau_0}}\right)\,,$$

which is the same as Eq. (5.46).

Consider a charging process of a RC-circuit. By replacing τ_0 in Eq. (5.46) by the time constant RC, the u in Eq. (5.46) will reflect how the electrical voltage varies.

5.3.2 *Verify the Solution for* $\mathbf{u}(\mathbf{x},\mathbf{0}) = \mathbf{1}$ *and* $\mathbf{u_t}(\mathbf{x},\mathbf{0}) = \mathbf{0}$

Verify that the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 u_{xx}\,, \quad -\infty < x < +\infty,\ 0 < t\,, \\ u(x,0) = 1\,, \quad u_t(x,0) = 0 \end{cases} \tag{5.48}$$

is

$$u = \mathrm{e}^{-\frac{t}{2\tau_0}} \left\{ \frac{\varphi(x-At) + \varphi(x+At)}{2} \right. + \frac{1}{2A} \int_{x-At}^{x+At} \left[\frac{1}{2\tau_0} I_0 \left(b\sqrt{(At)^2 - (x-\xi)^2} \right) + \frac{t}{4\tau_0^2 b \sqrt{(At)^2 - (x-\xi)^2}} \right. \left. \left. \cdot I_1 \left(b\sqrt{(At)^2 - (x-\xi)^2} \right) \right] \varphi(\xi)\,\mathrm{d}\xi \right\}. \tag{5.49}$$

The Initial Conditions

By Eq. (5.49), it is clear that $u(x,0) = 1$. Substituting $\varphi(x) = 1$ into Eq. (5.49) and taking the derivative of Eq. (5.49) with respect to t yields

$$u_t(x,0) = 0\,,$$

where we have used

$$\left(\mathrm{e}^{-\frac{t}{2\tau_0}} \cdot 1 \right)' \bigg|_{t=0} = -\frac{1}{2\tau_0}\,,$$

$$\left[\mathrm{e}^{-\frac{t}{2\tau_0}} \frac{1}{2A} \int_{x-At}^{x+At} \frac{1}{2\tau_0} I_0 \left(b\sqrt{(At)^2 - (x-\xi)^2} \right) \mathrm{d}\xi \right]' \bigg|_{t=0} = \frac{1}{2\tau_0}\,,$$

$$\left[\mathrm{e}^{-\frac{t}{2\tau_0}} \frac{1}{2A} \int_{x-At}^{x+At} \frac{t}{4\tau_0^2 b \sqrt{(At)^2 - (x-\xi)^2}}\, I_1 (b\sqrt{(At)^2 - (x-\xi)^2} \Big) \,\mathrm{d}\xi \right]' \Bigg|_{t=0}$$

$$= \left(\mathrm{e}^{-\frac{t}{2\tau_0}} \frac{\tau_0 t}{4} \right) \bigg|_{t=0} = 0\,.$$

The Equation

Substituting $\varphi(x) = 1$ into Eq. (5.49), we have

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t)\,,$$

where

$$\begin{aligned}
u_1(x,t) &= \mathrm{e}^{-\frac{t}{2\tau_0}}\,, \\
u_2(x,t) &= \mathrm{e}^{-\frac{t}{2\tau_0}} \frac{1}{2A} \int_{x-At}^{x+At} \frac{1}{2\tau_0} I_0\left(b\sqrt{(At)^2-(x-\xi)^2}\right) \mathrm{d}\xi\,, \\
u_3(x,t) &= \mathrm{e}^{-\frac{t}{2\tau_0}} I \\
&= \mathrm{e}^{-\frac{t}{2\tau_0}} \frac{1}{8A\tau_0^2} \int_{x-At}^{x+At} \frac{t}{b\sqrt{(At)^2-(x-\xi)^2}} I_1\left(b\sqrt{(At)^2-(x-\xi)^2}\right) \mathrm{d}\xi\,.
\end{aligned}$$

By Eq. (5.47), we have

$$u_2(x,t) = \frac{1}{2}\left(1 - \mathrm{e}^{-\frac{t}{\tau_0}}\right).$$

Also,

$$\begin{aligned}
I &= \frac{1}{8A\tau_0^2} \int_{x-At}^{x+At} \frac{t}{b\sqrt{(At)^2-(x-\xi)^2}} I_1\left(b\sqrt{(At)^2-(x-\xi)^2}\right) \mathrm{d}\xi \\
&= \frac{1}{8}\left(\frac{t}{\tau_0}\right)^2 + \frac{1}{384}\left(\frac{t}{\tau_0}\right)^4 + \frac{1}{46080}\left(\frac{t}{\tau_0}\right)^6 + \cdots \\
&= \frac{1}{2!}\left(\frac{t}{2\tau_0}\right)^2 + \frac{1}{4!}\left(\frac{t}{2\tau_0}\right)^4 + \frac{1}{6!}\left(\frac{t}{2\tau_0}\right)^6 + \cdots.
\end{aligned}$$

i.e.

$$\mathrm{ch}\frac{t}{2\tau_0} = \frac{1}{2}\left(\mathrm{e}^{\frac{t}{2\tau_0}} + \mathrm{e}^{-\frac{t}{2\tau_0}}\right) = 1 + I\,.$$

Hence,

$$u_3(x,t) = \mathrm{e}^{-\frac{t}{2\tau_0}}\left(ch\frac{t}{2\tau_0} - 1\right) = \frac{\mathrm{e}^{-\frac{t}{\tau_0}}}{2} + \frac{1}{2} - \mathrm{e}^{-\frac{t}{2\tau_0}}\,.$$

Finally,

$$
\begin{aligned}
u(x,t) &= u_1(x,t) + u_2(x,t) + u_3(x,t) \\
&= \mathrm{e}^{-\frac{t}{2\tau_0}} + \frac{1}{2} - \frac{\mathrm{e}^{-\frac{t}{\tau_0}}}{2} + \frac{\mathrm{e}^{-\frac{t}{\tau_0}}}{2} + \frac{1}{2} - \mathrm{e}^{-\frac{t}{2\tau_0}} = 1\,,
\end{aligned}
$$

so it satisfies the equation of PDS (5.48). By this verification, we find another integral

$$
\begin{aligned}
&\int_{x-At}^{x+At} \left[\frac{1}{2\tau_0} I_0\left(b\sqrt{(At)^2-(x-\xi)^2}\right) \right. \\
&\left. + \frac{t}{4\tau_0^2 b\sqrt{(At)^2-(x-\xi)^2}} I_1\left(b\sqrt{(At)^2-(x-\xi)^2}\right)\right] \mathrm{d}\xi \\
&= 2A\left(\mathrm{e}^{\frac{t}{2\tau_0}} - 1\right).
\end{aligned}
\tag{5.50}
$$

5.3.3 *Verify the Solution for* $\mathbf{f(x,t) = 1}$, $\mathbf{u(x,0) = 0}$ *and* $\mathbf{u_t(x,0) = 0}$

Verify that the solution of

$$
\begin{cases}
\dfrac{u_t}{\tau_0} + u_{tt} = A^2 u_{xx} + 1\,, \\
u(x,0) = 0\,, u_t(x,0) = 0
\end{cases}
\tag{5.51}
$$

is

$$
u(x,t) = \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0\left(b\sqrt{A^2(t-\tau)^2-(x-\xi)^2}\right)\mathrm{d}\xi\,. \tag{5.52}
$$

By the variable transformation $\tau' = t - \tau$, Eq. (5.52) becomes

$$
u(x,t) = \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{\tau'}{2\tau_0}}\,\mathrm{d}\tau' \int_{x-A\tau'}^{x+A\tau'} I_0\left(b\sqrt{(A\tau')^2-(x-\xi)^2}\right)\mathrm{d}\xi\,.
$$

It reduces to, using Eq. (5.47),

$$
u(x,t) = \int_0^t \tau_0\left(1-\mathrm{e}^{-\frac{\tau'}{\tau_0}}\right)\mathrm{d}\tau' = \tau_0 t + \tau_0^2\left(\mathrm{e}^{-\frac{t}{\tau_0}} - 1\right). \tag{5.53}
$$

It is straightforward to show that $u(x,t)$ in Eq. (5.53) satisfies PDS (5.51).

Thus, we find the integral

$$\begin{aligned}&\frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0\left(b\sqrt{A^2(t-\tau)^2-(x-\xi)^2}\right)\mathrm{d}\xi \\ &\quad = \tau_0 t + \tau_0^2\left(\mathrm{e}^{-\frac{t}{\tau_0}}-1\right).\end{aligned} \tag{5.54}$$

The unit of 1 in $u(x,0)=1$, $u_t(x,0)=1$ and $f(x,t)=1$ is Θ, ΘT^{-1} and ΘT^{-2}, respectively.

5.3.4 Physics and Measurement of τ_0

1. The non-zero τ_0 yields an additional term $\tau_0 u_{tt}$ in the hyperbolic heat-conduction equations compared with the classical heat-conduction equations. Since $[u_t] = [\tau_0 u_{tt}]$, τ_0 must be a time constant. While the hyperbolic heat-conduction equation is a better representation of real heat conduction processes, it has the same fundamental properties as the classical heat-conduction equation. The solution of PDS (5.48) is, for example, $u(x,t)\equiv 1$ which makes sense physically. For the classical heat-conduction equation, we also have

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2t}}\,\mathrm{d}\xi=\frac{2}{\sqrt{\pi}}\int_0^{+\infty}\mathrm{e}^{-\left(\frac{\xi'}{2a\sqrt{\pi t}}\right)^2}\mathrm{d}\left(\frac{\xi'}{2a\sqrt{\pi t}}\right)=1\,.$$

2. By Eq. (5.46), we have

$$\lim_{t\to+\infty} u = \tau_0\,. \tag{5.55}$$

   Therefore, the τ_0 is equal to the temperature value of an infinite rod as $t\to+\infty$ and due to $u_t(x,0)=1$. We may thus measure τ_0 by measuring the temperature of the rod due to $u_t(x,0)=1$ at $t\to+\infty$. The fact that $\mathrm{e}^{-\frac{t}{\tau_0}}$ decays very quickly as t, $t\to+\infty$ necessitates using only relatively large value of t. The value of τ_0 can also be obtained from Eq. (5.46) by measuring the temperature at different time instant.
3. The u_t and u_{tt} can be regarded as the velocity and acceleration of thermal waves, respectively. By Eq. (5.46), we have

$$u_t = \mathrm{e}^{-\frac{t}{\tau_0}} \tag{5.56}$$

   and

$$u_{tt} = -\frac{1}{\tau_0}\mathrm{e}^{-\frac{t}{\tau_0}}\,. \tag{5.57}$$

The former shows that velocity decays exponentially with increasing t, with a decaying constant $(-1/\tau_0)$. The latter yields

$$\lim_{t \to 0} u_{tt} = -1/\tau_0 \ . \tag{5.58}$$

Therefore, the decaying constant of the velocity is equal, in its value, to the initial value of the acceleration of thermal waves. This leads to another method of measuring τ_0 simply by measuring the initial value of u_{tt}. Under the CDS in PDS (5.40), we have, by Eqs. (5.55) -(5.58),

$$u|_{+\infty} \cdot u_{tt}|_{t=0} = -1 \text{ and } u_t = -\tau_0 u_{tt} \, .$$

4. By Eq. (5.53), we obtain

$$\lim_{t \to +\infty} u = \lim_{t \to +\infty} \left[\tau_0 t + \tau_0^2 \left(e^{-\frac{t}{\tau_0}} - 1 \right) \right] = +\infty .$$

Therefore, the temperature at all points increases as t tends to infinity. This is similar to mechanics, where the PDS governing the displacement $s(t)$

$$\begin{cases} s''(t) = 1 \, , \\ s(0) = s'(0) = 0 \end{cases} \tag{5.59}$$

is similar to PDS (5.51) in hyperbolic heat conduction. The solution of PDS (5.59) is $s = \dfrac{1}{2}t^2$. It also increases as t tends to infinity, but with a different speed.

5.3.5 *Measuring τ_0 by Characteristic Curves*

By the definition of thermal wave speed $A = \dfrac{a}{\sqrt{\tau_0}}$, we obtain

$$\tau_0 = \frac{a^2}{A^2} = k\frac{1}{A^2 \rho c} \, , \tag{5.60}$$

where k, ρ and c are thermal conductivity, density and specific heat, respectively. Since k, ρ and c are normally taken as material constants for most materials, we can obtain values of τ_0 of different materials by measuring A. The steps of measuring A can be summarized as follows.

Step 1. Mark three points O, A_1 and B on a slender rod (Fig. 5.5). The distances of A_1 and B from O are denoted by x_1 and x_0, respectively. Here point B is the test point.

Step 2. Apply an initial temperature $u(x,0) = K$ to the part of bar between O and A_1, i.e. in $[0, x_1]$. Here K is a positive constant. The larger the K, the higher the measuring accuracy.

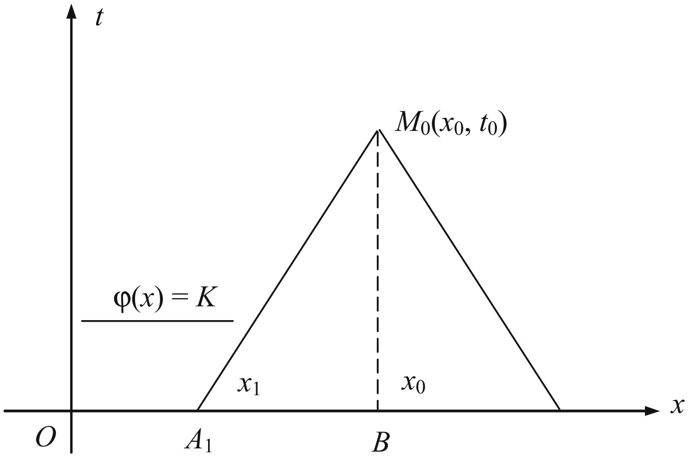

Fig. 5.5 Slender rod and points $0, A_1, B$

Mathematically, we consider

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 u_{xx}, & -\infty < x < +\infty\ , 0 < t\,, \\ u(x,0) = \begin{cases} K,\ x \in [0,x_1]\,, \\ 0,\ \ x \notin [0,x_1]\,, \end{cases} & u_t(x,0) = 0\,. \end{cases}$$

By Eq. (5.49), we know that the temperature $u(x,t)$ depends only on the K value in the domain of dependence.

Step 3. Observe the variation of temperature at point B and record the time instant t_0 when there is a reading of temperature at point $B(u(x_0,t_0) \neq 0$, no matter how small).

Step 4. Calculate $A = (x_0 - x_1)/t_0$ using known x_0, x_1 and t_0. This comes from the fact that the characteristic curve $x = x_0 + A(t - t_0)$ must cross the x-axis at point A_1 (Fig.5.5), Once A is available, we can obtain τ_0 by Eq. (5.60).

5.3.6 *Measuring τ_0 by a Unit Impulsive Source $\delta(\mathbf{x} - \mathbf{x_0}, \mathbf{t} - \mathbf{t_0})$*

The temperature due to a unit impulsive source $f(x,t) = \delta(x - x_0, t - t_0)$ has been found in Section 5.2 as Eq. (5.34). The temperature at x_0 and t_0 is, Eq. (5.35) in Section 5.2,

$$u(x_0, t_0) = \frac{1}{2}\sqrt{\frac{\rho c}{k\tau_0}}\,.$$

Therefore, τ_0 can also be obtained by measuring $u(x_0, t_0)$.

We have discussed several methods of measuring τ_0. For better accuracy, we may measure τ_0 by using different methods and take their mean value as its value.

5.4 Method of Descent for Two-Dimensional Problems and Discussion Of Solutions

In this section we first transform Cauchy problems of two-dimensional hyperbolic heat-conduction equations to Cauchy problems of three-dimensional wave equations by using a function transformation. We then apply the method of descent to obtain solutions of Cauchy problems of two-dimensional hyperbolic heat-conduction equations. Finally, we analyze the solutions for some special cases in order to obtain a better understanding of solutions and of τ_0.

5.4.1 Transform to Three-Dimensional Wave Equations

In this section, we attempt to solve

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u + f(x,y,t), & R^2 \times (0, +\infty), \\ u(x,y,0) = \varphi(x,y), \quad u_t(x,y,0) = \psi(x,y). \end{cases} \tag{5.61}$$

This can be achieved, by the solution structure theorem, if we can solve

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u, & R^2 \times (0, +\infty), \\ u(x,y,0) = 0, \quad u_t(x,y,0) = \psi(x,y). \end{cases} \tag{5.62}$$

By a function transformation $u = v(x,y,t)\mathrm{e}^{-\frac{t}{2\tau_0}}$ of eliminating the term of first derivative, we transform PDS (5.62) into

$$\begin{cases} v_{tt} = A^2 \Delta v + c^2 v, \quad c = \frac{1}{2\tau_0}, & R^2 \times (0, +\infty), \\ v(x,y,0) = 0, \quad v_t(x,y,0) = \psi(x,y). \end{cases} \tag{5.63}$$

Consider another function transformation of eliminating term $c^2 v$ in PDS (5.63),

$$v(x,y,t) = V(x,y,z,t)\mathrm{e}^{-\frac{c}{A}z}, \tag{5.64}$$

where $\left[\frac{c}{A}z\right] = 1$, $[V] = [v] = [u]$. The PDS (5.63) is thus transformed to

$$\begin{cases} V_{tt} = A^2 \Delta V\,, & R^3 \times (0, +\infty)\,, \\ V|_{t=0} = 0\,, & V_t|_{t=0} = \psi(x,y)\, \mathrm{e}^{\frac{c}{A} z}\,. \end{cases} \tag{5.65}$$

5.4.2 Solution of PDS (5.62)

The solution of PDS (5.65) is, by the Poisson formula (see Chapter 2)

$$V(x,y,z,t) = \frac{1}{4\pi \mathrm{A}^2 t} \iint\limits_{S_{At}^M} \psi(\xi,\eta)\, \mathrm{e}^{\frac{c}{A}\zeta}\, \mathrm{d}S\,,$$

where S_{At}^M stands for the spherical surface of a sphere of center M and radius At. Now consider the spherical surface S_{At}^M of a sphere of center $M(x,y,0)$ and radius At. Its projection on *plane-Oxy* is a circle of center $M(x,y,0)$ and radius At denoted by D_{At}^M : $(\xi - x)^2 + (\eta - y)^2 \leq (At)^2$. Let γ be the angle between Oz-axis and normal of S_{At}^M. The surface element $\mathrm{d}S$ on S_{At}^M is thus related to the area element in D_{At}^M by

$$\mathrm{d}S = \frac{1}{|\cos\gamma|}\, \mathrm{d}\xi\, \mathrm{d}\eta = \frac{At}{\sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}}\, \mathrm{d}\xi\, \mathrm{d}\eta\,. \tag{5.66}$$

Note also that the mathematical expression of S_{At}^M is

$$\zeta = \pm \sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}\,,$$

where the $\pm$ is corresponding to points on the upper or the lower half of S_{At}^M. The method of descent thus yields the solution of PDS (5.63)

$$\begin{aligned} v(x,y,t) = \frac{1}{4\pi A} & \left[\iint\limits_{D_{At}^M} \frac{\psi(\xi,\eta)\, \mathrm{e}^{\frac{c}{A}\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}}{\sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}}\, \mathrm{d}\xi\, \mathrm{d}\eta \right. \\ & \left. + \iint\limits_{D_{At}^M} \frac{\psi(\xi,\eta)\, \mathrm{e}^{-\frac{c}{A}\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}}{\sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}}\, \mathrm{d}\xi\, \mathrm{d}\eta \right] \\ = \frac{1}{2\pi A} & \iint\limits_{D_{At}^M} \frac{ch \frac{c}{A} \sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}}{\sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}}\, \psi(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta\,. \end{aligned}$$

Finally, the solution of PDS (5.62) is

$$
\begin{aligned}
u &= W_{\psi}(x,y,t) \\
&= \mathrm{e}^{-\frac{t}{2\tau_0}} \frac{1}{2\pi A} \iint\limits_{D_{At}^M} \frac{ch\frac{c}{A}\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}{\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}} \psi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,.
\end{aligned} \tag{5.67}
$$

5.4.3 Solution of PDS (5.61)

By the solution structure theorem, we obtain the solution of PDS (5.61)

$$
u = \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right) W_{\varphi}(x,y,t) + W_{\psi}(x,y,t) + \int_0^t W_{f_\tau}(x,y,t-\tau)\,\mathrm{d}\tau\,,
$$

where $f_\tau = f(x,y,\tau)$. Here the solution of

$$
\begin{cases}
\dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u\,, \quad R^2\times(0,+\infty)\,, \\
u(x,y,0) = \varphi(x,y)\,, \quad u_t(x,y,0) = 0
\end{cases} \tag{5.68}
$$

is

$$
\begin{aligned}
u &= \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right) W_{\varphi}(x,y,t) \\
&= \mathrm{e}^{-\frac{t}{2\tau_0}} \frac{1}{2\pi A} \left(\frac{1}{2\tau_0}+\frac{\partial}{\partial t}\right) \\
&\quad \cdot \iint\limits_{D_{At}^M} \frac{ch\frac{c}{A}\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}{\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}} \varphi(\xi,\eta)\,\mathrm{d}\xi\, d\eta\,.
\end{aligned} \tag{5.69}
$$

The solution of

$$
\begin{cases}
\dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + f(x,y,t)\,,\ R^2\times(0,+\infty)\,, \\
u(x,y,0) = 0\,,\ u_t(x,y,0) = 0
\end{cases}
$$

is

$$
\begin{aligned}
u &= \int_0^t W_{f_\tau}(x,y,t-\tau)\,\mathrm{d}\tau \\
&= \frac{1}{2\pi A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \\
&\quad \cdot \iint\limits_{D^M_{A(t-\tau)}} \frac{ch\frac{c}{A}\sqrt{A^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}}{\sqrt{A^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}} f(\xi,\eta,\tau)\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \qquad (5.70)
\end{aligned}
$$

A superposition of Eqs. (5.67) and (5.69) forms the counterpart of the Poisson formula of two-dimensional wave equations. Similarly, Eq. (5.70) is the counterpart of the Kirchhoff formula.

Remark. The unit of u is Θ in Eqs. (5.67), (5.69) and (5.70). In Eq. (5.69), for example

$$[c]=T^{-1}\,,\quad [A]=LT^{-1}\,,$$

$$\left[\frac{c}{A}\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}\right]=1$$

and

$$
\begin{aligned}
[u] &= \left[\frac{1}{A}\left(\frac{1}{2\tau_0}+\frac{\partial}{\partial t}\right)\frac{ch\square}{\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}\varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right] \\
&= TL^{-1}\cdot T^{-1}\cdot L^{-1}\cdot\Theta\cdot L^2=\Theta\,.
\end{aligned}
$$

5.4.4 Verification of CDS

We demonstrate here that the u in Eqs. (5.67), (5.69) and (5.70) satisfies the initial conditions.

1. **The u in Eq. (5.67)** By Eq. (5.67), it is clear that $u(x,y,0)=0$. For conciseness, denote the integrand of double integral in Eq. (5.67) by $K(\psi)$. After using the generalized mean value theorem for integral, we have

$$\iint\limits_{D^M_{At}} K(\psi)\,\mathrm{d}\sigma=\overline{\psi}\overline{\mathrm{ch}}\iint\limits_{D^M_{At}}\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}\,.$$

where

$$\overline{\psi} = \psi\left(\bar{\xi},\bar{\eta}\right),\overline{\mathrm{ch}} = ch\frac{c}{A}\sqrt{(At)^2-\left(x-\bar{\xi}\right)^2-(y-\bar{\eta})^2}\,,\left(\bar{\xi},\bar{\eta}\right)\in D_{At}^M\,.$$

Note that, by a polar coordinate transformation,

$$\iint\limits_{D_{At}^M}\frac{1}{\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta = 2\pi At\,.$$

Therefore

$$u(x,y,t) = \mathrm{e}^{-\frac{t}{2\tau_0}}\frac{1}{2\pi A}\overline{\psi}\cdot\overline{\mathrm{ch}}2\pi At = \mathrm{e}^{-\frac{t}{2\tau_0}}t\overline{\psi}\cdot\overline{\mathrm{ch}}\,.$$

Finally,

$$\begin{aligned}
&u_t(x,y,0) = \\
&\quad\frac{1}{2\pi A}\left\{-\frac{1}{2\tau_0}\mathrm{e}^{-\frac{t}{2\tau_0}}\cdot\iint\limits_{D_{At}^M}K(\psi)\,\mathrm{d}\xi\,\mathrm{d}\eta+\mathrm{e}^{-\frac{t}{2\tau_0}}\frac{\partial}{\partial t}\iint\limits_{D_{At}^M}K(\psi)\,\mathrm{d}\xi\,\mathrm{d}\eta\right\}\Bigg|_{t=0}\\
&= \frac{1}{2\pi A}\left\{-\frac{1}{2\tau_0}\mathrm{e}^{-\frac{t}{2\tau_0}}\iint\limits_{D_{At}^M}K(\psi)\,\mathrm{d}\xi\,\mathrm{d}\eta+\mathrm{e}^{-\frac{t}{2\tau_0}}\left[\frac{\partial}{\partial t}\left(\overline{\psi}\cdot\overline{\mathrm{ch}}\right)\cdot 2\pi At\right.\right.\\
&\quad\left.\left.+\ \overline{\psi}\cdot\overline{\mathrm{ch}}\cdot 2\pi A\right]\right\}\Bigg|_{t=0} = \psi(x,y)\,,
\end{aligned}$$

where we have used $\lim\limits_{t\to 0}\overline{\psi} = \psi(x,y)$ and $\lim\limits_{t\to 0}\overline{\mathrm{ch}} = 1$.

2. **The *u* in Eq. (5.69)**
Rewrite Eq. (5.69) into

$$u = \frac{1}{4\pi A\tau_0}\mathrm{e}^{-\frac{t}{2\tau_0}}\iint\limits_{D_{At}^M}K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta+\mathrm{e}^{-\frac{t}{2\tau_0}}\frac{1}{2\pi A}\frac{\partial}{\partial t}\iint\limits_{D_{At}^M}K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,, \tag{5.71}$$

where

$$K(\varphi) = \frac{ch\frac{c}{A}\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}{\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}\varphi(\xi,\eta)\,.$$

Note that

$$\frac{\partial}{\partial t}\iint\limits_{C_{At}^M}K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta = \frac{\partial}{\partial t}(\overline{\psi}\cdot\overline{\mathrm{ch}})\cdot 2\pi At+\overline{\psi}\cdot\overline{\mathrm{ch}}\cdot 2\pi A\,. \tag{5.72}$$

Therefore $u(x,y,0) = \varphi(x,y)$. By taking the derivative of Eq. (5.72) with respect to t, we obtain

$$\frac{\partial^2}{\partial t^2}\iint\limits_{D_{At}^M} K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta = \frac{\partial^2}{\partial t^2}(\overline{\psi}\overline{\mathrm{ch}})\cdot 2\pi At + \frac{\partial}{\partial t}(\overline{\psi}\overline{\mathrm{ch}})\cdot 4\pi A\,.$$

Thus, by noting that $\overline{\psi}\overline{\mathrm{ch}}$ is an even function of t so that $\frac{\partial}{\partial t}\,(\overline{\psi}\overline{\mathrm{ch}})\big|_{t=0} = 0$, $\frac{\partial^2}{\partial t^2}\iint\limits_{C_{At}^M} K(\varphi)\,\mathrm{d}\xi\ \ \mathrm{d}\eta\,|_{t=0} = 0$. Taking the derivative of Eq. (5.71) with respect to t yields

$$\begin{aligned} u_t = &-\frac{1}{8\pi A\tau_0^2}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\iint\limits_{D_{At}^M} K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta + \frac{1}{4\pi A\tau_0}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\frac{\partial}{\partial t}\iint\limits_{D_{At}^M} K(\varphi)\,\mathrm{d}\xi\ \mathrm{d}\eta \\ &-\frac{1}{4\pi A\tau_0}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\frac{\partial}{\partial t}\iint\limits_{D_{At}^M} K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta + \frac{1}{2\pi A}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\frac{\partial^2}{\partial t^2}\iint\limits_{D_{At}^M} K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ = &-\frac{1}{8\pi A\tau_0^2}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\iint\limits_{D_{At}^M} K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta + \frac{1}{2\pi A}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\frac{\partial^2}{\partial t^2}\iint\limits_{D_{At}^M} K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \end{aligned}$$

Therefore,

$$\begin{aligned} u_t|_{t=0} = \Bigg[&-\frac{1}{8\pi A\tau_0^2}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\iint\limits_{D_{At}^M} K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &+\frac{1}{2\pi A}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\frac{\partial^2}{\partial t^2}\iint\limits_{D_{At}^M} K(\varphi)\,\mathrm{d}\xi\,\mathrm{d}\eta\Bigg]\Bigg|_{t=0} = 0\,. \end{aligned}$$

3. **The u in Eq. (5.70)** It is straightforward to show $u(x,y,0) = 0$. By taking derivatives of Eq. (5.70) with respect to t, we obtain

$$u_t = \frac{1}{2\pi A}\left\{\int_0^t \frac{\partial}{\partial t}\left[\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\iint\limits_{D_{A(t-\tau)}^M} K(f)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]\mathrm{d}\tau + \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\iint\limits_{D_{A(t-\tau)}^M} K(f)\,\mathrm{d}\xi\,\mathrm{d}\eta\,|_{\tau=t}\right\}.$$

Therefore, $u_t(x,y,0) = 0$.

5.4.5 Special Cases

For conciseness, we define

$$L=\frac{1}{\tau_0}\frac{\partial}{\partial t}+\frac{\partial^2}{\partial t^2}-A^2\Delta\,,\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\,.$$

Example 1. Solve

$$\begin{cases} Lu=0\,, & R^2\times(0\,,+\infty)\,,\\ u(x,y,0)=0\,, & u_t(x,y,0)=1\,. \end{cases} \tag{5.73}$$

Solution. By Eq. (5.67), we obtain

$$\begin{aligned} u(x,y,t)&=\frac{1}{2\pi A}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\iint\limits_{D_{At}^M}\frac{ch\frac{c}{A}\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}{\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &=\frac{1}{2\pi A}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\int_0^{2\pi}\mathrm{d}\theta\int_0^{At}\frac{ch\frac{c}{A}\sqrt{(At)^2-r^2}}{\sqrt{(At)^2-r^2}}\,r\,\mathrm{d}r\\ &=\frac{1}{c}\,\mathrm{e}^{-ct}\int_0^{(At)^2}ch\frac{c}{A}\sqrt{v}\,\mathrm{d}\left(\frac{c}{A}\sqrt{v}\right)=\frac{1}{c}\,\mathrm{e}^{-ct}\;sh\frac{c}{A}\sqrt{v}\Big|_0^{(At)^2}\\ &=\frac{1}{2c}\,\mathrm{e}^{-ct}\left(\mathrm{e}^{ct}-\mathrm{e}^{-ct}\right)=\tau_0\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right), \end{aligned}$$

where $c=\dfrac{1}{2\tau_0}$. Therefore, the temperature is independent of spatial coordinates. This is the same as in the one-dimensional case. Also, $\lim\limits_{t\to\infty}u=\tau_0$, so that τ_0 is, in its value, equal to the temperature of any point at $t\to+\infty$. This can serve as one method of measuring τ_0.

From this special case, we obtain the integral

$$\begin{aligned} &\iint\limits_{D_{At}^M}\frac{ch\frac{c}{A}\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}{\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &=\int_0^{2\pi}\mathrm{d}\theta\int_0^{At}\frac{ch\frac{c}{A}\sqrt{(At)^2-r^2}}{\sqrt{(At)^2-r^2}}\,r\,\mathrm{d}r\\ &=2\pi A\tau_0\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right)\mathrm{e}^{\frac{t}{2\tau_0}}\,. \end{aligned}$$

Example 2. Solve

$$\begin{cases} Lu = 0, \quad R^2 \times (0, +\infty), \\ u(x, y, 0) = 1, \quad u_t(x, y, 0) = 0. \end{cases} \tag{5.74}$$

Solution. By Eq. (5.69) and the result in Example 1, we obtain

$$\begin{aligned} u(x,y,t) &= \mathrm{e}^{-\frac{t}{2\tau_0}} \frac{1}{2\pi A} \left(\frac{1}{2\tau_0} + \frac{\partial}{\partial t} \right) \iint\limits_{D_{At}^M} \frac{ch\frac{c}{A}\sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}}{\sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}} \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &= \mathrm{e}^{-ct} \left(c + \frac{\partial}{\partial t} \right) \frac{1}{2c} \left(\mathrm{e}^{ct} - \mathrm{e}^{-ct} \right) = 1 . \end{aligned}$$

This is also the same as its one-dimensional counterpart. It is also physically grounded by noting that the $u(x,y,t)$ comes only from $u(x,y,0) = 1$.

From this special case, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} \iint\limits_{D_{At}^M} \frac{ch\frac{c}{A}\sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}}{\sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}} \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &= 2\pi A \, \mathrm{e}^{\frac{t}{2\tau_0}} - \frac{1}{2\tau_0} \iint\limits_{D_{At}^M} \frac{ch\frac{c}{A}\sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}}{\sqrt{(At)^2 - (x-\xi)^2 - (y-\eta)^2}} \, \mathrm{d}\xi \, \mathrm{d}\eta . \end{aligned}$$

Example 3. Solve

$$\begin{cases} Lu = 1, \quad R^2 \times (0, +\infty), \\ u(x, y, 0) = 0, \quad u_t(x, y, 0) = 0. \end{cases} \tag{5.75}$$

Solution. By Eq. (5.70) and the result in Example 1, we obtain

$$\begin{aligned} u(x,y,t) &= \int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \\ &\quad \cdot \left[\frac{1}{2\pi A} \iint\limits_{D_{A(t-\tau)}^M} \frac{ch\frac{c}{A}\sqrt{A^2(t-\tau)^2 - (x-\xi)^2 - (y-\eta)^2}}{\sqrt{A^2(t-\tau)^2 - (x-\xi)^2 - (y-\eta)^2}} \, \mathrm{d}\xi \, \mathrm{d}\eta \right] \mathrm{d}\tau \\ &= \frac{1}{2c} \int_0^t \mathrm{e}^{-c(t-\tau)} \left(\mathrm{e}^{c(t-\tau)} - \mathrm{e}^{-c(t-\tau)} \right) \mathrm{d}\tau \\ &= \tau_0 \int_0^t \left(1 - \mathrm{e}^{-\frac{t-\tau}{\tau_0}} \right) d\tau = \tau_0 t + \tau_0^2 \left(\mathrm{e}^{-\frac{t}{\tau_0}} - 1 \right) . \end{aligned} \tag{5.76}$$

This is also the same as its one-dimensional counterpart (Eq. (5.53) in Section 5.3).

From this special case, we obtain the integral

$$\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \iint\limits_{D^M_{A(t-\tau)}} \frac{ch\frac{c}{A}\sqrt{A^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}}{\sqrt{A^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$= 2\pi A\tau_0\left[t+\tau_0\left(\mathrm{e}^{-\frac{t}{\tau_0}}-1\right)\right].$$

Remark 1. Since $[1]=\Theta T^{-2}$ in PDS (5.75), we have $[u]=\Theta T^{-2}$ in Eq. (5.76).

Remark 2. The $u(x,y,z,t)$ in last three examples is independent of x,y and z. This can be verified physically. The three PDS are thus equivalent to the following three initial-value problems of ordinary differential equations.

$$\begin{cases} \dfrac{\mathrm{d}^2u}{\mathrm{d}t^2}+\dfrac{1}{\tau_0}\dfrac{du}{dt}=0\,, \\ u(0)=0\,,\ u'(0)=1\,. \end{cases} \quad \text{Solution: } u=\tau_0\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right).$$

$$\begin{cases} \dfrac{\mathrm{d}^2u}{\mathrm{d}t^2}+\dfrac{1}{\tau_0}\dfrac{\mathrm{d}u}{\mathrm{d}t}=0\,, \\ u(0)=1\,,\ u'(0)=0\,. \end{cases} \quad \text{Solution: } u=1\,.$$

$$\begin{cases} \dfrac{\mathrm{d}^2u}{\mathrm{d}t^2}+\dfrac{1}{\tau_0}\dfrac{\mathrm{d}u}{\mathrm{d}t}=1\,, \\ u(0)=u'(0)=0\,. \end{cases} \quad \text{Solution: } u=\tau_0 t+\tau_0^2\left(\mathrm{e}^{-\frac{t}{\tau_0}}-1\right).$$

5.5 Domains of Dependence and Influence, Measuring τ_0 by Characteristic Cones

The hyperbolic heat-conduction equation shares features of wave equations including the domain of dependence and the domain of influence.

5.5.1 Domain of Dependence

Equations (5.67), (5.69) and (5.70) in Section 5.4 show that the effect of initial values φ, ψ and source term f is similar to that for wave equations. The solution due to $\varphi(xy)$ and $\psi(x,y)$ depends only on the initial values φ and ψ in D^M_{At}, but not on those outside D^M_{At}. The $u(x_0,y_0,t_0)$ in Eq. (5.67) is, for example, determined completely by $\psi(x,y)$ in $D^{M_0}_{At_0}$. In the three-dimensional space $Oxyt$, $D^{M_0}_{At_0}$ is the intersecting area between plane $t=0$ and the cone of top point $P_0(x_0\,,y_0\,,t_0)$ (Fig. 5.6), i.e.

$$D^{M_0}_{At_0}=\left\{(x,y,t)|\sqrt{(x-x_0)^2+(y-y_0)^2}\le A(t_0-t)\right\}\cap\{t|t=0\}\,.$$

The region $D^{M_0}_{At_0}$ is called the *domain of dependence* of point P_0 on the initial value.

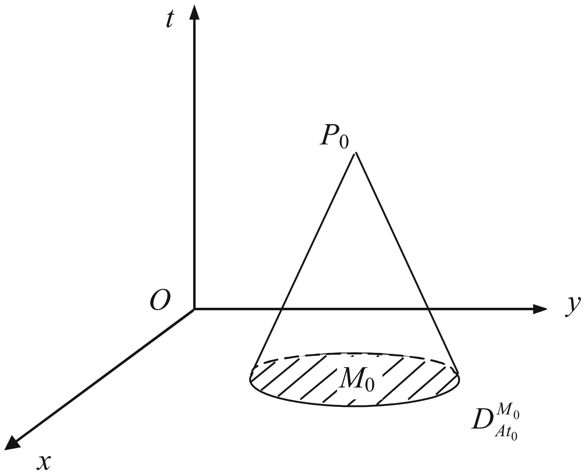

Fig. 5.6 Domain of dependence

As for the effect of the nonhomogeneous source term $f(x,y,t)$, the $u(x_0, y_0, t_0)$ in Eq. (5.70) depends only on $f(x,y,t)$ in a cone Ω_{P_0} of top point $P_0(x_0, y_0, t_0)$: $\sqrt{(x-x_0)^2+(y-y_0)^2} \leq A(t_0-t)$, $0 \leq t \leq t_0$. The Ω_{P_0} is thus called the *domain of dependence* of point P_0 on the source term. The Ω_{P_0} is also called the *characteristic cone* of passing point P_0. The surface $(x-x_0)^2+(y-y_0)^2 = A^2(t-t_0)^2$ is called the *characteristic cone surface*. For any point $P_1(x_1, y_1, t_1) \in \Omega_{P_0}$, its domain of dependence on the initial value $D_{At_1}^{M_1}$ is always in $D_{At_0}^{M_0}$, i.e. $D_{At_1}^{M_1} \subset D_{At_0}^{M_0}$. Hence the initial values in $D_{At_0}^{M_0}$ completely determine the solution due to the initial values at all points in Ω_{P_0}.

For the one-dimensional case, the $D_{At_0}^{M_0}$ reduces to the region $[x_0-At_0, x_0+At_0]$. The Ω_{P_0} becomes a triangle region formed by $t=0$ and characteristic curves $x-x_0 = \pm A(t-t_0)$. Characteristic curves $x \pm At = C$ (constant) are very important in studying one-dimensional hyperbolic heat-conduction equations. Similarly, the characteristic surface plays an important role in studying two-dimensional hyperbolic heat-conduction equations.

5.5.2 Domain of Influence

The structure of Eqs. (5.67) and (5.69) in Section 5.4 is similar to the Poisson formula of two-dimensional wave equations. The propagation of thermal waves shares those features of mechanical waves including the domain of influence of initial disturbances discussed in Section 2.8.4. Here the initial value of point $M_0(x_0, y_0, 0)$ can affect all points in an infinite cone region Ω_{M_0} : $\sqrt{(x-x_0)^2+(y-y_0)^2} \leq At$, $t > 0$ (Fig. 5.7). Point M_0 is always in the domain of dependence of all points $P_1(x_1, y_1, t_1) \in \Omega_{M_0}$. For any point $P_2(x_2, y_2, t_2)$ outside Ω_{M_0}, its domain of de-

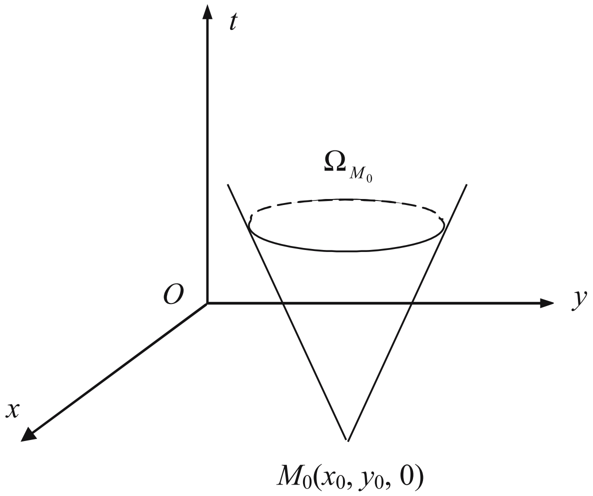

Fig. 5.7 Domain of influence

pendence never contains the point M_0. Therefore, the $\Omega_{M_0}$is called the *domain of influence* of the initial disturbance at point M_0. For $\psi(x,y) = \delta(x-x_0, y-y_0)$ in Eq. (5.67), in particular, we have

$$u(x,y,t) = \begin{cases} \dfrac{1}{2\pi A} e^{-\frac{t}{2\tau_0}} \dfrac{ch\frac{c}{A}\sqrt{(At)^2-(x-x_0)^2-(y-y_0)^2}}{\sqrt{(At)^2-(x-x_0)^2-(y-y_0)^2}}, & (x,y,t) \in \Omega_{M_0}, \\ 0, & (x,y,t) \notin \Omega_{M_0}. \end{cases}$$

5.5.3 Measuring τ_0 by Characteristic Cones

Characteristic curves and cones play an important role in studying hyperbolic heat-conduction equations. They have a close relation with the solutions of Cauchy problems and they reflect features of thermal wave propagation. In the one-dimensional case, for example, the singularity of $\varphi(x)$ at x_0 of type $\varphi(x_0-0) \neq \varphi(x_0+0)$ will at least lead to $\varphi'(x_0-0) \neq \varphi'(x_0+0)$ on both sides of the characteristic curves $x - At = x_0$ and $x + At = x_0$ so that u_{tt} and u_{xx} do not exist on them (Eq. (5.30) in Section 5.2). The characteristic curves at point $(x_0, 0)$ are in fact the discontinuous representation of the solution. Therefore, a singularity of initial values at a point will propagate along the characteristic curves. Solutions of two-dimensional Cauchy problems also possess similar features. Based on these observations, we may design a method of measuring τ_0 as follows.

Step 1. Choose a circle D_ε^O of center O and radius ε on a sufficiently large and thin plane (Fig. 5.8). The D_ε^O crosses $Oy-axis$ at $M_1(0, y_1, 0)$. Choose point $M_0(0, y_0, 0)$ sufficiently far away from D_ε^O as the test point.

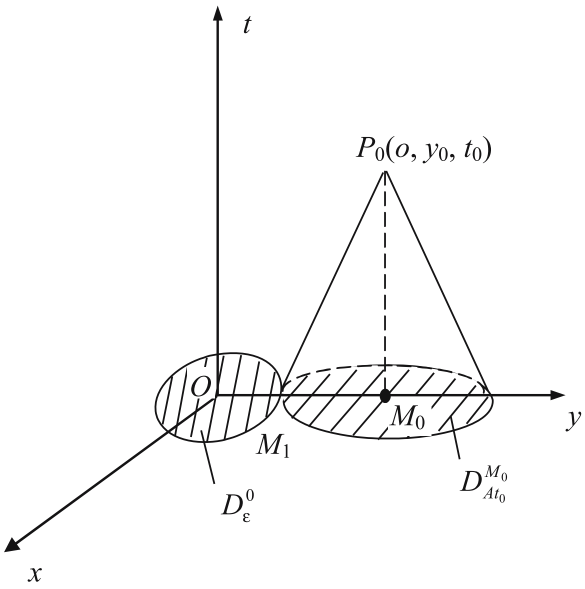

Fig. 5.8 D^0_ε and $D^{M_0}_{At_0}$

Step 2. Apply a constant initial temperature K on D^O_ε and start to record time. Mathematically, we consider

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u, \quad R^2 \times (0 + \infty), \\ u(x,y,0) = \varphi(x,y) = \begin{cases} K, & (x,y) \in D^O_\varepsilon, \\ 0, & (x,y) \notin D^O_\varepsilon, \end{cases} \quad u_t(x,y,0) = 0. \end{cases}$$

Its solution reads

$$u(x,y,t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \frac{1}{2\pi A} \left(\frac{1}{2\tau_0} + \frac{\partial}{\partial t}\right) \iint\limits_{D^M_{At}} K(\varphi)\, \mathrm{d}\xi\, \mathrm{d}\eta\,.$$

Step 3. Observe the variation of temperature at point M_0 and record the time instant t_0 when there is a reading of temperature at point M_0 ($u(0, y_0, t_0) \neq 0$, no matter how small).

Step 4. Calculate the wave speed $A = (y_0 - y_1)/t_0$ by using known y_0, y_1 and t_0 This comes from the fact that the characteristic cone surface $\sqrt{(x-x_0)^2 + (y-y_0)^2} = A(t_0 - t)$ must cross the y-axis at point M_1 (Fig. 5.8).

Once A is available, we can obtain τ_0 by the definition of A as $\tau_0 = a^2/A^2$.

5.6 Comparison of Fundamental Solutions of Classical and Hyperbolic Heat-Conduction Equations

In this section we analyze and compare fundamental solutions of classical and hyperbolic heat-conduction equations by using the two-dimensional case as the example. Such an analysis and comparison is useful for revealing what fundamental properties of the classical equation are preserved in its hyperbolic version, what the improvements from the classical equation to the hyperbolic one are and what are the undesirable features of hyperbolic heat-conduction equations.

5.6.1 Fundamental Solutions of Two Kinds of Heat-Conduction Equations

Classical Heat-Conduction Equation

By its definition, the fundamental solution of classical heat-conduction equations satisfies (Eq. (3.54) in Section 3.5),

$$\begin{cases} u_t = a^2 \Delta u + \delta(x-x_0, y-y_0, t-t_0), & R^2 \times (0, +\infty), \\ u(x,y,0) = 0. \end{cases}$$

Its solution reads

$$\begin{aligned} u_1(x,y,t) &= \int_0^t \mathrm{d}\tau \iint\limits_{R^2} V(x,\xi,t-\tau) V(y,\eta,t-\tau) \delta(\xi - x_0, \eta - y_0, \tau - t_0) \,\mathrm{d}\xi \,\mathrm{d}\eta \\ &= V(x,x_0,t-t_0) V(y,y_0,t-t_0) = \frac{1}{4a^2\pi(t-t_0)} \mathrm{e}^{-\frac{(x-x_0)^2+(y-y_0)^2}{4a^2(t-t_0)}} . \end{aligned} \tag{5.77}$$

Hyperbolic Heat-Conduction Equation

For hyperbolic heat-conduction equations, the fundamental solution is the solution of

$$\begin{cases} u_t + \tau_0 u_{tt} = a^2 \Delta u + \delta(x-x_0, y-y_0, t-t_0), & R^2 \times (0, +\infty), \\ u(x,y,0) = 0, \; u_t(x,y,0) = 0 \end{cases}$$

or

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u + \dfrac{\delta(x-x_0, y-y_0, t-t_0)}{\tau_0}, & R^2 \times (0, +\infty), \\ u(x,y,0) = 0, u_t(x,y,0) = 0. \end{cases}$$

By Eq. (5.70) in Section 5.4, we obtain the fundamental solution

$$u_2(x,y,t) = \frac{1}{2\pi A\tau_0}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \iint\limits_{D^M_{A(t-\tau)}} \frac{ch\frac{c}{A}\sqrt{A^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}}{\sqrt{A^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}}$$

$$\cdot\,\delta(\xi-x_0,\eta-y_0,\tau-t_0)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$= \frac{1}{2\pi A\tau_0}\,\mathrm{e}^{-\frac{t-t_0}{2\tau_0}}\,\frac{ch\frac{c}{A}\sqrt{A^2(t-t_0)^2-(x-x_0)^2-(y-y_0)^2}}{\sqrt{A^2(t-t_0)^2-(x-x_0)^2-(y-y_0)^2}}$$

$$= \frac{1}{2\pi a^2(t-t_0)}\,\mathrm{e}^{-\frac{t-t_0}{2\tau_0}}\,\frac{ch\frac{t-t_0}{2\tau_0}\sqrt{1-\frac{(x-x_0)^2+(y-y_0)^2}{A^2(t-t_0)^2}}}{\sqrt{1-\frac{(x-x_0)^2+(y-y_0)^2}{A^2(t-t_0)^2}}}\,, \tag{5.78}$$

where x, y and t satisfy $\sqrt{(x-x_0)^2+(y-y_0)^2}\le A(t-t_0)$ so that $M(x,y,t)\in\Omega_{P_0}$, the characteristic cone of $P_0(x_0,y_0,t_0)$. Outside Ω_{P_0}, $u(x,y,t)\equiv 0$. Therefore,

$$u_2(x,y,t) = \begin{cases} \text{Eq. (5.78)}(\neq 0)\,, & M(x,y,t)\in\Omega_{P_0}\,, \\ =\infty, & M(x,y,t)\quad \text{on the characteristic cone surface,} \\ =0, & M(x,y,t)\quad \text{outside } \Omega_{P_0}\,. \end{cases} \tag{5.79}$$

5.6.2 Common Properties

1. By using $\mathrm{ch}0=1$, we obtain from Eqs. (5.77) and (5.78)

$$\lim_{M\to P_0} u_1(x,y,t)=\infty, \quad \lim_{M\to P_0} u_2(x,y,t)=\infty\,.$$

   This is physically-grounded because $\delta(x-x_0,y-y_0,t-t_0)$ implies a very high temperature at point (x_0,y_0) and time instant t_0.

   Note here that $[\delta(x-x_0,y-y_0,t-t_0)]=\Theta T^{-1}$ and $u_2(x,y,t)$ is a higher-order infinity than $u_1(x,y,t)$ as $M\to P_0$.

2. By Eq. (5.77), we have, for any point $(x,y)\in R^2$,

$$\lim_{t\to+\infty} u_1(x,y,t)=0\,.$$

In Eq. (5.78),

$$c\frac{1}{2\tau_0}, \quad \lim_{t\to+\infty}\frac{(x-x_0)^2+(y-y_0)^2}{A^2(t-t_0)^2}=0,$$

$$\lim_{t\to+\infty} \mathrm{e}^{-\frac{t-t_0}{2\tau_0}}\, ch\frac{c}{A}\sqrt{A^2(t-t_0)^2-(x-x_0)^2-(y-y_0)^2}$$

$$=\frac{1}{2}+O\left(\mathrm{e}^{-\frac{t-t_0}{2\tau_0}}\right) \quad (t\to+\infty).$$

Thus

$$\lim_{t\to+\infty} u_2(x,y,t)=\lim_{t\to+\infty}\frac{1}{2\pi A\tau_0}\frac{\frac{1}{2}+O\left(\mathrm{e}^{-\frac{t-t_0}{2\tau_0}}\right)}{\sqrt{A^2(t-t_0)^2-(x-x_0)^2-(y-y_0)^2}}=0.$$

This is also physically correct because the temperature at any point due to $f(x,y,t)=\delta(x-x_0,y-y_0,t-t_0)$ should decay as t approaches ∞.

3. When $x_0=y_0=0$, both $u_1(x,y,t)$ and $u_2(x,y,t)$ are even functions of x and y so that they are symmetric around the origin $(0,0)$. This is again physically grounded because both u_1 and u_2 come exclusively from $f(x,y,t)=\delta(x-0,y-0,t-t_0)$ and because diffusion is direction-independent.

5.6.3 Different Properties

1. Solution (5.77) indicates that $u_1(x,y,t)\neq 0$ no matter how far point (x,y) is from the source (x_0,y_0) and no matter how short $(t-t_0)$ is. The disturbance at (x_0,y_0) can thus propagate over a very long distance in a very short time period, thus the speed of temperature propagation is infinite. This is clearly physically impossible.

   However, this is not the case for solution (5.78). When point (x,y) is sufficiently far away from source (x_0,y_0) and $(t-t_0)$ is sufficiently short such that $\sqrt{(x-x_0)^2+(y-y_0)^2}>A(t-t_0)$ for $M(x,y,t)$ outside of Ω_{P_0}, then $u_2(x,y,t)=0$. This implies a finite speed of temperature propagation. Thus the hyperbolic heat-conduction equation is a better representation of a real heat conduction process.

   While the classical heat-conduction equation assumes an infinite speed, $u_1(x,y,t)$ will quickly decay to zero for a large $(x-x_0)^2+(y-y_0)^2$ and a small $(t-t_0)$. This is the rationale behind its wide applications.

2. Equations (5.78) and (5.79) both show that $u_2(x,y,t) \to \infty$ as $M(x,y,t)$ tends to the characteristic cone surface. Therefore the singularity of the source point propagates along the characteristic surface, where the solution is consequently discontinuous. This is contrary to the physical reality. However, $u_1(x,y,t)$ does not have this drawback. For a point (x,y) such that $(x-x_0)^2+(y-y_0)^2 = A^2(t-t_0)^2$,

$$u_1(x,y,t) = \frac{1}{4\pi a^2(t-t_0)}\,\mathrm{e}^{-\frac{A^2(t-t_0)}{4a^2}}\,,$$

which still has a finite value. The $u_1(x,y,t) \to \infty$ occurs only when $t \to t_0$. The $t \to t_0$ implies that $(x-x_0)^2+(y-y_0)^2 \to 0$ so that (x,y) is in the neighborhood of the source point (x_0,y_0). The singularity is thus confined in the source point (x_0,y_0) without propagation.

5.7 Methods for Solving Axially Symmetric and Spherically-Symmetric Cauchy Problems

Axially symmetric and spherically-symmetric problems are special cases of two-dimensional and three-dimensional Cauchy problems. The solution structure theorem is valid for them. In this section we use the Hankel transformation and the spherical Bessel transformation of order zero to solve them.

5.7.1 The Hankel Transformation for Two-Dimensional Axially Symmetric Problems

An Integral Formula of Bessel Function of Order Zero

The expansion expression of the generating function for the Bessel function is (Appendix A)

$$\mathrm{e}^{\frac{x}{2}(t-t^{-1})} = \sum_{n=-\infty}^{+\infty} J_n(x)t^n\,.$$

By the formula for Laurent series coefficients, we have

$$J_0(x) = \frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{\mathrm{e}^{\frac{x}{2}(z-z^{-1})}}{z}\,\mathrm{d}z = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}x\cos\varphi}\,\mathrm{d}\varphi\,,$$

where $z = \mathrm{e}^{\mathrm{i}\left(\varphi+\frac{\pi}{2}\right)}$. It is transformed to, by a variable transformation $\varphi = \theta + \pi$,

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}x\cos\theta}\,\mathrm{d}\theta\,, \tag{5.80}$$

which is the integral formula of Bessel function of order zero.

Hankel Transformation

An axially-symmetric function $f(x,y)$ can be written as $f(x,y) = f(r) = f(r)$ with $r = \sqrt{x^2+y^2}$. Its double Fourier transformation is also axially symmetric, $\bar{f}(\omega) = \bar{f}(\omega)$ with $\omega = \sqrt{\omega_1^2+\omega_2^2}$, $0 \le \omega < +\infty$. Therefore, the double Fourier transformation of an axially symmetric function $f(x,y)$ and its inverse transformation have a special form

$$\begin{aligned}\bar{f}(\omega) &= \frac{1}{2\pi}\iint\limits_{R^2} f(x,y)\,\mathrm{e}^{-\mathrm{i}\omega\cdot r}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2\pi}\int_0^{+\infty} f(r)r\,\mathrm{d}r\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}\omega r\cos\theta}\,\mathrm{d}\theta \\ &= \int_0^{+\infty} rf(r)J_0(\omega r)\,\mathrm{d}r,\end{aligned} \tag{5.81}$$

$$f(r) = \frac{1}{2\pi}\iint\limits_{R^2}\bar{f}(\omega_1,\omega_2)\,\mathrm{e}^{\mathrm{i}\omega\cdot r}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2 = \int_0^{+\infty}\omega\bar{f}(\omega)J_0(\omega r)\,\mathrm{d}\omega, \tag{5.82}$$

in which we have applied Eq. (5.80). Denote Eqs. (5.81) and (5.82) by

$$\bar{f}(\omega) = H_0[f(r)], \quad f(r) = H_0^{-1}[\bar{f}(\omega)],$$

respectively. They are called the *Hankel transformation* and the *inverse Hankel transformation*. As a special case of Fourier transformation, the Hankel transformation shares some similar properties with the Fourier transformation.

Axially Symmetric Cauchy Problems

The axially symmetric Laplace operator reads, in a polar coordinate system

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right).$$

We now consider the axially symmetric Cauchy problem

$$\begin{cases}\dfrac{u_t}{\tau_0} + u_{tt} = A^2\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + f(r,t), \quad 0 < r < +\infty,\ 0 < t, \\ u(r,0) = \varphi(r),\ u_t(r,0) = \psi(r).\end{cases} \tag{5.83}$$

By the solution structure theorem, we can focus on seeking the solution of

$$\begin{cases}\dfrac{u_t}{\tau_0} + u_{tt} = A^2\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right), \quad 0 < r < +\infty,\ 0 < t, \\ u(r,0) = 0,\ u_t(r,0) = \psi(r).\end{cases} \tag{5.84}$$

By the function transformation $u(r,t) = v(r,t)\,\mathrm{e}^{-\frac{t}{2\tau_0}}$ for eliminating terms involving the first derivative u_t, PDS (5.84) reduces to

$$\begin{cases} v_{tt} = A^2 \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial v}{\partial r}\right) + c^2 v\,, \quad c = \frac{1}{2\tau_0}\,, \\ v(r,0) = 0\,, \quad v_t(r,0) = \psi(r)\,. \end{cases} \tag{5.85}$$

Applying a Hankel transformation to (5.85) and using

$$H_0\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)\right] = -\omega^2 \bar{v}(\omega,t)$$

leads to an initial-value problem of the ordinary differential equation

$$\begin{cases} \bar{v}_{tt}(\omega,t) + (A^2\omega^2 - c^2)\bar{v}(\omega,t) = 0\,, \\ \bar{v}(\omega,0) = 0, \quad \bar{v}_t(\omega,0) = \bar{\psi}(\omega)\,, \end{cases}$$

where $\bar{\psi}(\omega) = H_0[\psi(r)]$. Its solution can be readily obtained as

$$\bar{v}(\omega,t) = \frac{\bar{\psi}(\omega)\sin\left(\sqrt{A^2\omega^2 - c^2}\right)t}{\sqrt{A^2\omega^2 - c^2}}\,.$$

Its inverse Hankel transformation will yield the solution of PDS (5.85), and consequently, the solution of PDS (5.84)

$$u(r,t) = W_\psi(r,t) = \mathrm{e}^{-\frac{t}{2\tau_0}} \int_0^{+\infty} \frac{\bar{\psi}(\omega)\sin\left(\sqrt{A^2\omega^2 - c^2}\right)t}{\sqrt{A^2\omega^2 - c^2}}\,\omega J_0(\omega r)\,\mathrm{d}\omega\,.$$

Thus the solution of PDS (5.83) is, by the solution structure theorem,

$$u(r,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(r,t) + W_\psi(r,t) + \int_0^t W_{f_\tau}(r,t-\tau)\,\mathrm{d}\tau\,,$$

where $f_\tau = f(r,\tau)$.

5.7.2 *Spherical Bessel Transformation for Spherically-Symmetric Cauchy Problems*

Spherical Bessel Transformation of Order Zero

A spherically-symmetric function $f(x,y,z)$ can be written as $f(x,y,z) = f(r) = f(r)$ with $r = \sqrt{x^2+y^2+z^2}$. Its triple Fourier transformation is also spherically symmetric, $\bar{f}(\omega_1, \omega_2, \omega_3) = \bar{f}(\omega)$ with $\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$. Hence the triple Fourier transformation of a spherically symmetric function $f(x,y,z)$ and its inverse transformation have a special form

$$\begin{aligned}
\bar{f}(\omega) &= \frac{1}{(\sqrt{2\pi})^3} \iiint\limits_{R^3} f(r)\,\mathrm{e}^{-\mathrm{i}\omega\cdot r}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^3} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_0^{+\infty} f(r)\,\mathrm{e}^{-\mathrm{i}\omega r\cos\theta} r^2 \sin\theta\,\mathrm{d}r \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} r^2 f(r) \frac{\sin\omega r}{\omega r}\,\mathrm{d}r \\
&= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} r^2 f(r)\, j_0(\omega r)\,\mathrm{d}r, \qquad (5.86)
\end{aligned}$$

$$f(r) = \frac{1}{(\sqrt{2\pi})^3} \iiint\limits_{R^3} \bar{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega\cdot r}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3 = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \omega^2 \bar{f}(\omega)\, j_0(\omega r)\,\mathrm{d}\omega, \tag{5.87}$$

where we have applied (Appendix A)

$$j_0(x) = \sqrt{\frac{\pi}{2x}} J_{\frac{1}{2}}(x) = \frac{\sin x}{x}.$$

Denote Eqs. (5.86) and (5.87) by

$$\bar{f}(\omega) = F_{j_0}[f(r)], \quad f(r) = F_{j_0}^{-1}\left[\bar{f}(\omega)\right],$$

respectively. They are called the *spherical Bessel transformation of order zero* and the *inverse spherical Bessel transformation of order zero*.

Spherically Symmetric Cauchy Problems

The spherically-symmetric Laplace operator reads, in a spherical coordinate system,

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right).$$

We now consider the spherically-symmetric Cauchy problem

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \dfrac{1}{r^2} \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial}{\partial r}\right) + f(r,t), \quad 0 < r < +\infty,\ 0 < t, \\ u(r,0) = \varphi(r), \quad u_t(r,0) = \psi(r). \end{cases} \tag{5.88}$$

By the solution structure theorem, we can first seek the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u\,, \quad 0 < r < +\infty,\, 0 < t\,, \\ u(r,0) = 0\,, \quad u_t(r,0) = \psi(r)\,. \end{cases} \tag{5.89}$$

To eliminate the u_t-term, consider a function transformation

$$u(r,t) = v(r,t)\,\mathrm{e}^{-\frac{t}{2\tau_0}}\,,$$

PDS (5.89) is thus transformed to

$$\begin{cases} v_{tt} = A^2 \Delta v + c^2 v,\; c = \frac{1}{2\tau_0}\,, \quad 0 < r < +\infty,\, 0 < t\,, \\ v(r,0) = 0\,, \quad v_t(r,0) = \psi(r)\,. \end{cases} \tag{5.90}$$

Applying a spherical Bessel transformation to (5.90) and using $F_{j_0}[\Delta v] = -\omega^2 \bar{v}(\omega,t)$ yields an initial-value problem of the ordinary differential equation

$$\begin{cases} \bar{v}_{tt}(\omega,t) + \left(A^2\omega^2 - c^2\right)\bar{v}(\omega,t) = 0\,, \\ \bar{v}(\omega,0) = 0\,, \quad \bar{v}_t(\omega,0) = \bar{\psi}(\omega)\,. \end{cases}$$

Its solution can be readily obtained as

$$\bar{v}(\omega,t) = \frac{\bar{\psi}(\omega)\sin\left(\sqrt{A^2\omega^2 - c^2}\right)t}{\sqrt{A^2\omega^2 - c^2}}\,. \tag{5.91}$$

Its inverse spherical Bessel transformation will lead to the solution of PDS (5.90), and consequently, the solution of PDS (5.89)

$$\begin{aligned} u(r,t) = W_\psi(r,t) &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \omega^2 \bar{v}(\omega,t) j_0(\omega r) d\omega \\ &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{\bar{\psi}(\omega)\sin\left(\sqrt{A^2\omega^2 - c^2}\right)t}{\sqrt{A^2\omega^2 - c^2}} \frac{\omega \sin \omega r}{r}\, \mathrm{d}\omega\,. \end{aligned} \tag{5.92}$$

Finally, the solution of PDS (5.88) is, by the solution structure theorem,

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(r,t) + W_\psi(r,t) + \int_0^t W_{f_\tau}(r,t-\tau)\,\mathrm{d}\tau\,,$$

where $f_\tau = f(r,\tau)$.

5.7.3 Method of Continuation for Spherically-Symmetric Problems

We now apply the method of continuation to solve

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \dfrac{1}{r^2} \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r}\right) + f(r,t), & 0 < r < +\infty,\ 0 < t, \\ u_r(0,t) = 0, \\ u(r,0) = \varphi(r), \quad u_t(r,0) = \psi(r). \end{cases} \tag{5.93}$$

By the solution structure theorem, we can first seek the solution of

$$\begin{cases} u_t / \tau_0 + u_{tt} = A^2 \Delta u, \quad 0 < r < +\infty,\ 0 < t, \\ u_r(0,t) = 0, \\ u(r,0) = 0, \quad u_t(r,0) = \psi(r). \end{cases} \tag{5.94}$$

Based on the given boundary condition $u_r(0,t) = 0$, the $u(r,t)$ should be an even function of r. To obtain the solution of PDS (5.94), consider an auxiliary problem

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u, \quad -\infty < r < +\infty,\ 0 < t, \\ u(r,0) = 0, \quad u_t(r,0) = \Psi(r) = \begin{cases} \psi(r), & 0 \le r, \\ \psi(-r), & r < 0, \end{cases} \end{cases} \tag{5.95}$$

where $\Psi(r)$ comes from an even prolongation of $\psi(r)$.

By function transformation $v(r,t) = ru(r,t)$, PDS (5.95) is transformed to

$$\begin{cases} \dfrac{v_t}{\tau_0} + v_{tt} = A^2 v_{rr}, \quad -\infty < r < +\infty,\ 0 < t, \\ v(r,0) = 0, \quad v_t(r,0) = \begin{cases} r\psi(r), & 0 \le r, \\ r\psi(-r), & r < 0. \end{cases} \end{cases} \tag{5.96}$$

Its solution is available in Eq. (5.30). Thus the solution of PDS (5.94) is,

$$\begin{aligned} u(r,t) &= W_\psi(r,t) \\ &= \frac{1}{2Ar} \mathrm{e}^{-\frac{t}{2\tau_0}} \int_{r-At}^{r+At} I_0\left(b\sqrt{(At)^2 - (r-\rho)^2}\right) \rho\psi(\rho)\,\mathrm{d}\rho, \end{aligned} \tag{5.97}$$

where $\psi(\rho)$ should be $\psi(-\rho)$ if $\rho < 0$.

Finally, the solution of PDS (5.93) reads, by the solution structure theorem,

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(r,t) + W_\psi(r,t) + \int_0^t W_{f_\tau}(r, t-\tau)\,\mathrm{d}\tau, \tag{5.98}$$

where $f_\tau = f(r,\tau)$, $b = 1/2a\sqrt{\tau_0}$, $\varphi(-r) = \varphi(r)$, $f(-r,t) = f(r,t)$.

Remark 1. An odd continuation should be applied if the boundary condition is $u(0,t) = 0$.

Remark 2. In Eq. (5.97),

$$[b] = L^{-1}, \ \left[b\sqrt{(At)^2 - (r-\rho)^2}\right] = 1, \ [I_0] = 1,$$
$$[u] = \left[\frac{1}{Ar}\right][\rho]\,[\psi(\rho)]\,[\mathrm{d}\rho] = TL^{-2} \cdot L \cdot \Theta T^{-1} \cdot L = \Theta$$

Remark 3. It is straightforward to show that the u in Eq. (5.97) satisfied $u(r,0) = 0$. Note that $I_0(0) = 1$. Also

$$u_t(r,t) = \frac{1}{2Ar}\left\{-\frac{1}{2\tau_0}\mathrm{e}^{-\dfrac{t}{2\tau_0}}\int_{r-At}^{r+At} I_0\left(b\sqrt{(At)^2-(r-\rho)^2}\right)\rho\,\psi(\rho)\,\mathrm{d}\rho\right.$$
$$+\,\mathrm{e}^{-\dfrac{t}{2\tau_0}}\left[\int_{r-At}^{r+At}\frac{\partial I_0}{\partial t}\rho\,\psi(\rho)\,\mathrm{d}\rho + (r+At)\psi(r+At)A\right.$$
$$\left.\left.+(r-At)\psi(r-At)A\right]\right\}.$$

Thus $u_t(r,0) = \psi(r)$.

Remark 4. An examination of the right-hand side of Eq. (5.97) shows that the $u(r,t)$ in Eq. (5.97) is indeed an even function of r. By L'Hôpital's rule, we have

$$\lim_{r\to 0} u(r,t) = \frac{1}{2A}\mathrm{e}^{-\frac{t}{2\tau_0}}\left[\lim_{r\to 0}\int_{r-At}^{r+At}\frac{I_1(y)}{y}(\rho-r)\rho\,\psi(\rho)\,\mathrm{d}\rho\right.$$
$$\left.+\,(r+At)\psi(r+At) - (r-At)\psi(r-At)\right]$$
$$= \frac{1}{2A}\mathrm{e}^{-\frac{t}{2\tau_0}}\left(\int_{-At}^{At}\frac{I_1(y_1)}{y_1}\rho^2\psi(\rho)\,\mathrm{d}\rho + 2At\,\psi(At)\right),$$

where $y = b\sqrt{(At)^2-(r-\rho)^2}$ and $y_1 = b\sqrt{(At)^2-\rho^2}$. By the mean value theorem of integrals,

$$\int_{-At}^{At}\frac{I_1(y_1)}{y_1}\rho^2\psi(\rho)\,\mathrm{d}\rho = I_1(\bar{y}_1)\,\bar{\rho}^2\psi(\bar{\rho})\int_{-At}^{At}\frac{\mathrm{d}\rho}{b\sqrt{(At)^2-\rho^2}}$$
$$= \frac{I_1(\bar{y}_1)\,\bar{\rho}^2\psi(\bar{\rho})\pi}{b},$$

where $\bar{y}_1 = b\sqrt{(At)^2-\bar{\rho}^2}$, $-At < \bar{\rho} < At$. Thus $\lim_{r\to 0} u(r,t)$ does exist and $r = 0$ is a removable discontinuous point.

Remark 5. If $\psi(r) = 1$ in Eq. (5.94), Eq. (5.97) yields

$$\begin{aligned} u(r,t) &= \frac{1}{2Ar} \mathrm{e}^{-\frac{t}{2\tau_0}} \int_{r-At}^{r+At} I_0\left(b\sqrt{(At)^2-(r-\rho)^2}\right)\rho\,\mathrm{d}\rho \\ &= \frac{1}{2Ar} \mathrm{e}^{-\frac{t}{2\tau_0}} \int_{-At}^{At} I_0\left(b\sqrt{(At)^2-\xi^2}\right)(r-\xi)\,\mathrm{d}\xi \\ &= \frac{1}{2A} \mathrm{e}^{-\frac{t}{2\tau_0}} \int_{-At}^{At} I_0\left(b\sqrt{(At)^2-\xi^2}\right)\mathrm{d}\xi = \tau_0\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right), \end{aligned}$$

which is the same as Eq. (5.46).

5.7.4 Discussion of Solution (5.98)

1. By Eq. (5.98), the solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2\Delta u(r,t), \quad 0<r<+\infty,\ 0<t, \\ u_r(0,t) = 0, \\ u(r,0) = \varphi(r), \quad u_t(r,0) = 0 \end{cases} \tag{5.99}$$

is

$$\begin{aligned} u &= \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(r,t) \\ &= \mathrm{e}^{-\frac{t}{2\tau_0}} \left\{ \frac{(r-At)\varphi(r-At)+(r+At)\varphi(r+At)}{2r} \right. \\ &\quad + \frac{1}{2Ar}\int_{r-At}^{r+At}\left[\frac{1}{2\tau_0} I_0\left(b\sqrt{(At)^2-(r-\rho)^2}\right)\right. \\ &\quad \left.\left. + \frac{I_1\left(b\sqrt{(At)^2-(r-\rho)^2}\right)}{4\tau_0^2 b\sqrt{(At)^2-(r-\rho)^2}}t\right]\rho\varphi(\rho)\,\mathrm{d}\rho \right\}. \end{aligned} \tag{5.100}$$

1. It is straightforward to show that the $u(r,t)$ in Eq. (5.100) satisfies the initial conditions $u(r,0) = \varphi(r)$ and $u_t(r,0) = 0$.

2. In Eq. (5.100), $[b] = L^{-1}$, $\left[b\sqrt{(At)^2-(r-\rho)^2}\right] = 1$. The unit of the second term in the integrand of the right-hand side of Eq. (5.100) is thus

$$\left[\frac{1}{Ar}\frac{t}{\tau_0^2}\rho\psi(\rho)\,\mathrm{d}\rho\right] = TL^{-2}\cdot T^{-1}\cdot L\cdot\Theta\cdot L = [u] = \Theta\,.$$

Similarly, the unit of the other two terms is also Θ.

3. When $\varphi(r) = 1$, we have

$$u_1(r,t) = \mathrm{e}^{-\frac{t}{2\tau_0}}\frac{(r-At)+(r+At)}{2r} = \mathrm{e}^{-\frac{t}{2\tau_0}}\,,$$

$$u_2(r,t) = \frac{\tau_0}{2\tau_0}\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right) = \frac{1}{2}\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right),$$

$$\begin{aligned} u_3(r,t) &= \frac{1}{2Ar}\mathrm{e}^{-\frac{t}{2\tau_0}}\int_{r-At}^{r+At}\frac{I_1\left(b\sqrt{(At)^2-(r-\rho)^2}\right)t}{4\tau_0^2 b\sqrt{(At)^2-(r-\rho)^2}}\rho\,\mathrm{d}\rho \\ &= \mathrm{e}^{-\frac{t}{2\tau_0}}\frac{t}{8\tau_0^2 A}\left(At+\frac{b^2}{12}A^3t^3+\frac{b^4}{360}A^5t^5+\cdots\right) \\ &= \mathrm{e}^{-\frac{t}{2\tau_0}}\left[\frac{1}{8}\left(\frac{t}{\tau_0}\right)^2+\frac{1}{384}\left(\frac{t}{\tau_0}\right)^4+\frac{1}{46080}\left(\frac{t}{\tau_0}\right)^6+\cdots\right] \\ &= \mathrm{e}^{-\frac{t}{2\tau_0}}\left[\frac{1}{2!}\left(\frac{t}{2\tau_0}\right)^2+\frac{1}{4!}\left(\frac{t}{2\tau_0}\right)^4+\frac{1}{6!}\left(\frac{t}{2\tau_0}\right)^6+\cdots\right] \\ &= \mathrm{e}^{-\frac{t}{2\tau_0}}\left(ch\frac{t}{2\tau_0}-1\right) = \frac{1}{2}\mathrm{e}^{-\frac{t}{\tau_0}}+\frac{1}{2}-\mathrm{e}^{-\frac{t}{2\tau_0}}\,, \end{aligned}$$

and

$$u(r,t) = u_1+u_2+u_3 = 1\,, \tag{5.101}$$

which is physically correct because the $u(r,t)$ comes here exclusively from $u(r,0) = \varphi(r) = 1$.

From the above expression of $u_3(r,t)$, we obtain an integral

$$\begin{aligned} &\int_{r-At}^{r+At}\frac{I_1\left(b\sqrt{(At)^2-(r-\rho)^2}\right)t}{\sqrt{(At)^2-(r-\rho)^2}}\rho\,\mathrm{d}\rho \\ &= 8rbA\tau_0^2\,\mathrm{e}^{\frac{t}{2\tau_0}}\left(\frac{1}{2}\mathrm{e}^{-\frac{t}{\tau_0}}+\frac{1}{2}-\mathrm{e}^{-\frac{t}{2\tau_0}}\right). \end{aligned}$$

The counterpart of PDS (5.99) in classical heat-conduction equations is

$$\begin{cases} u_t = a^2 \Delta u(r,t), \quad 0 < r < +\infty,\ 0 < t, \\ u_r(0,t) = 0, \quad u(r,0) = 1. \end{cases} \tag{5.102}$$

After an even continuation and a function transformation $v(r,t) = ru(r,t)$, it reduces to

$$\begin{cases} v_t = a^2 \Delta v(r,t), \quad -\infty < r < +\infty,\ 0 < t, \\ v(r,0) = r. \end{cases} \tag{5.103}$$

Its solution (also the solution of PDS (5.102)) is available in Chapter 3 as

$$\begin{aligned} u(r,t) &= \frac{1}{2a\sqrt{\pi t}r} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(r-\rho)^2}{4a^2 t}} \rho \,\mathrm{d}\rho = \frac{1}{2a\sqrt{\pi t}r} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\xi^2}{4a^2 t}} (\xi - r)\,\mathrm{d}\xi \\ &= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\xi^2}{4a^2 t}} \,\mathrm{d}\xi = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\eta^2} d\eta = 1. \end{aligned}$$

Therefore, this property is also preserved by the hyperbolic heat-conduction equation.

2. By Eq. (5.98), the solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 \Delta u(r,t) + f(r,t)/\tau_0, \quad 0 < r < +\infty,\ 0 < t, \\ u_r(0,t) = 0, \\ u(r,0) = 0, \quad u_t(r,0) = 0 \end{cases} \tag{5.104}$$

is

$$\begin{aligned} u &= \int_0^t W_{f_\tau}(r,t-\tau)\,\mathrm{d}\tau \\ &= \frac{1}{2A\tau_0 r} \int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} \,\mathrm{d}\tau \int_{r-A(t-\tau)}^{r+A(t-\tau)} I_0\left(b\sqrt{(At)^2 - (r-\rho)^2}\right) \rho f(\rho,\tau)\,\mathrm{d}\rho. \end{aligned} \tag{5.105}$$

When $f(r,t) = \delta(r - r_0, t - t_0)$, we have

$$u(r,t) = \frac{r_0}{2A\tau_0 r} I_0\left(b\sqrt{A^2(t-t_0)^2 - (r-r_0)^2}\right) \mathrm{e}^{-\frac{t-t_0}{2\tau_0}}, \tag{5.106}$$

which shows that under the effect of a point source $\delta(r - r_0, t - t_0)/\tau_0$, the temperature is higher inside the spherical surface $r = r_0$ than that outside the surface, and is relatively lower on the surface $r = r_0$. Furthermore,

$$\lim_{r\to 0} u(r,t) = \infty, \quad \lim_{r\to\infty} u(r,t) = 0, \quad \lim_{\substack{r\to r_0 \\ t\to t_0}} u(r,t) = \frac{1}{2}\sqrt{\frac{\rho c}{k\tau_0}}. \tag{5.107}$$

For classical heat-conduction equations, the counterpart of Eq. (5.106) is

$$
\begin{aligned}
u(r,t) &= \frac{1}{2a\sqrt{\pi}r}\int_0^t \frac{1}{\sqrt{t-\tau}}\,\mathrm{d}\tau \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(r-\rho)^2}{4a^2(t-\tau)}}\rho\delta(\rho-r_0,\tau-t_0)\,\mathrm{d}\rho \\
&= \frac{r_0}{2a\sqrt{\pi(t-t_0)}r}\,\mathrm{e}^{-\frac{(r-r_0)^2}{4a^2(t-t_0)}}\,. \qquad (5.108)
\end{aligned}
$$

It shows that: (1) $\lim\limits_{r\to 0} u(r,t) = \infty$ and $\lim\limits_{r\to\infty} u(r,0) = 0$, the same as those in Eq. (5.107), but (2) $\lim\limits_{\substack{r\to r_0\\ t\to t_0}} u(r,t) = \infty$. Hence Eq. (5.107) appears more reasonable than Eq. (5.108) because of its finite limit of $u(r,t)$ as $t \to t_0$ and $r \to r_0$.

5.8 Methods of Fourier Transformation and Spherical Means for Three-Dimensional Cauchy Problems

In this section we apply the Fourier transformation and the method of spherical means to solve three-dimensional Cauchy problems. We also make a comparison with three-dimensional wave equations to demonstrate some features of thermal waves.

5.8.1 An Integral Formula of Bessel Function

Let

$$
I(r,\varphi) = \frac{1}{2}\int_0^{\pi} J_0(r\sin\varphi\sin\theta)\,\mathrm{e}^{ir\cos\varphi\cos\theta}\sin\theta\,\mathrm{d}\theta\,, \qquad (5.109)
$$

where (r,θ,φ) are the coordinates of a spherical coordinate system. We attempt to prove

$$
I(r,\varphi) = \frac{\sin r}{r}\,.
$$

Note that $J_0'(x) = -J_1(x)$. We have

$$
\frac{\mathrm{d}I}{\mathrm{d}\varphi} = I_1 + I_2\,,
$$

where

$$I_1 = \frac{1}{2}\int_0^{\pi} (-1)J_1(r\sin\varphi\sin\theta)\,\mathrm{e}^{\mathrm{i}r\cos\varphi\cos\theta} r\cos\varphi\sin^2\theta\,\mathrm{d}\theta\,,$$

$$I_2 = \frac{1}{2}\int_0^{\pi} (-\mathrm{i})J_0(r\sin\varphi\sin\theta)\,\mathrm{e}^{\mathrm{i}r\cos\varphi\cos\theta} r\cos\theta\sin\theta\sin\varphi\,\mathrm{d}\theta\,.$$

Since $$\mathrm{d}(xJ_1(x)) = xJ_0(x)\,\mathrm{d}x\,,$$

we obtain

$$\begin{aligned}&\frac{1}{r\sin\varphi}\,\mathrm{d}\left[(r\sin\varphi\sin\theta)J_1(r\sin\varphi\sin\theta)\right]\\ &\qquad = r\sin\varphi\sin\theta J_0\left(r\sin\varphi\sin\theta\right)\cos\theta\,\mathrm{d}\theta\,.\end{aligned}$$

Thus

$$\begin{aligned} I_2 &= -\frac{\mathrm{i}}{2r\sin\varphi}\int_0^{\pi} \mathrm{e}^{\mathrm{i}r\cos\varphi\cos\theta}\,\mathrm{d}\left[(r\sin\varphi\sin\theta)J_1(r\sin\varphi\sin\theta)\right]\\ &= -\frac{\mathrm{i}}{2r\sin\varphi}\left[\ \mathrm{e}^{\mathrm{i}r\cos\varphi\cos\theta}(r\sin\varphi\sin\theta)J_1(r\sin\varphi\sin\theta)\Big|_0^{\pi}\right.\\ &\quad\left. -\int_0^{\pi}(r\sin\varphi\sin\theta)J_1(r\sin\varphi\sin\theta)\,\mathrm{e}^{\mathrm{i}r\cos\varphi\cos\theta}(-\mathrm{i}r\cos\varphi\sin\theta)\,\mathrm{d}\theta\right]\\ &= \frac{1}{2}\int_0^{\pi} J_1(r\sin\varphi\sin\theta)\,\mathrm{e}^{\mathrm{i}r\cos\varphi\cos\theta} r\cos\varphi\sin^2\theta\,\mathrm{d}\theta = -I_1\,.\end{aligned}$$

Hence $\dfrac{\mathrm{d}I}{\mathrm{d}\varphi} = 0$, so that I is independent of φ. Also,

$$I(r,\varphi) = I(r,0) = \frac{1}{2}\int_0^{\pi} \mathrm{e}^{\mathrm{i}r\cos\theta}\sin\theta\,\mathrm{d}\theta = \frac{\sin r}{r}\,.$$

Finally, we obtain the integral formula of Bessel function of order zero

$$\frac{1}{2}\int_0^{\pi} J_0(r\sin\varphi\sin\theta)\,\mathrm{e}^{\mathrm{i}r\cos\varphi\cos\theta}\sin\theta\,\mathrm{d}\theta = \frac{\sin r}{r}\,. \tag{5.110}$$

Consider now the variable transformation $r\cos\varphi = -A\omega t$, $r\sin\varphi = \mathrm{i}ct$, $At\cos\theta = \beta$, where $\mathrm{i} = \sqrt{-1}$.

Thus

$$r^2 = (-A\omega t)^2 + (\mathrm{i}ct)^2 = t^2\left(A^2\omega^2 - c^2\right) \quad \text{or} \quad r = t\sqrt{(A\omega)^2 - c^2}\,,$$

$$\sin\theta\,\mathrm{d}\theta = -\frac{\mathrm{d}\beta}{At}\,, \quad \mathrm{e}^{\mathrm{i}r\cos\varphi\cos\theta} = \mathrm{e}^{-\mathrm{i}A\omega t\cos\theta} = \mathrm{e}^{-\mathrm{i}\omega\beta}\,,$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{\beta}{At}\right)^2}\,, (\sin\theta \geq 0\,,\ \theta \in [0,\pi])$$

Substituting these and $J_0(\mathrm{i}x) = I_0(x)$ into Eq. (5.110) yields a very important formula for Fourier transformation

$$\frac{\sin\left(t\sqrt{(A\omega)^2 - c^2}\right)}{\sqrt{(A\omega)^2 - c^2}} = \frac{1}{2A}\int_{-At}^{At} I_0\left(c\sqrt{t^2 - \left(\frac{\beta}{A}\right)^2}\right)\mathrm{e}^{-\mathrm{i}\omega\beta}\,\mathrm{d}\beta\,. \tag{5.111}$$

5.8.2 Fourier Transformation for Three-Dimensional Problems

Consider the PDS

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u, & R^3 \times (0,+\infty)\,, \\ u(M,0) = 0, \quad u_t(M,0) = \psi(M)\,, \end{cases} \tag{5.112}$$

where M stands for point $(x,y,z) \in R^3$. It is transformed to, by the function transformation $u(M,t) = v(M,t)\,\mathrm{e}^{-\frac{t}{2\tau_0}}$,

$$\begin{cases} v_{tt} = A^2\Delta v + c^2 v\,,\ c = \frac{1}{2\tau_0}\,, & R^3 \times (0,+\infty)\,, \\ v(M,0) = 0\,, \quad v_t(M,0) = \psi(M)\,. \end{cases} \tag{5.113}$$

Applying a triple Fourier transformation to PDS (5.113) yields an initial-value problem of the ordinary differential equation

$$\begin{cases} \bar{v}_{tt}(\boldsymbol{\omega},t) + (A^2\omega^2 - c^2)\bar{v}(\boldsymbol{\omega},t) = 0\,, \\ \bar{v}(\boldsymbol{\omega},0) = 0, \quad \bar{v}_t(\boldsymbol{\omega},0) = \bar{\psi}(\boldsymbol{\omega})\,, \end{cases}$$

where $\bar{\psi}(\boldsymbol{\omega}) = F[\psi(M)]$ and $\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$. Its solution is

$$\bar{v}(\boldsymbol{\omega},t) = \frac{\sin\left(t\sqrt{(A\omega)^2 - c^2}\right)}{\sqrt{(A\omega)^2 - c^2}}\bar{\psi}(\boldsymbol{\omega})\,.$$

Note that $\quad F^{-1}[\bar{\psi}(\boldsymbol{\omega})] = \psi(M)\,,$

$$
\begin{aligned}
&F^{-1}\left[\frac{\sin\left(t\sqrt{(A\omega)^2-c^2}\right)}{\sqrt{(A\omega)^2-c^2}}\right]\\
&= \frac{1}{(2\pi)^3}\iiint\limits_{R^3}\frac{\sin\left(t\sqrt{(A\omega)^2-c^2}\right)}{\sqrt{(A\omega)^2-c^2}}\,\mathrm{e}^{\mathrm{i}\omega\cdot r}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3\\
&= \frac{1}{(2\pi)^2}\int_0^{+\infty}\frac{\omega}{2A}\int_{-At}^{At} I_0\left(c\sqrt{t^2-\left(\frac{\beta}{A}\right)^2}\right)\mathrm{e}^{-\mathrm{i}\omega\beta}\,\mathrm{d}\beta\left[-\frac{1}{\mathrm{i}r}\,\mathrm{e}^{\mathrm{i}\omega r\cos\theta}\right]\Bigg|_0^{\pi}\mathrm{d}\omega\\
&= \frac{1}{(2\pi)^2}\frac{1}{Ar}\int_0^{+\infty}\omega\left[\int_{-At}^{At} I_0\left(c\sqrt{t^2-\left(\frac{\beta}{A}\right)^2}\right)\cos\omega\beta\,\mathrm{d}\beta\right]\sin\omega r\,\mathrm{d}\omega\\
&= \frac{1}{4\pi^2 Ar}\int_0^{+\infty}\left[\int_{r-At}^{r+At} I_0\left(c\sqrt{t^2-\frac{(r-\rho)^2}{A^2}}\right)\omega\sin\omega\rho\,\mathrm{d}\rho\right]\mathrm{d}\omega\\
&= v_1(M,t)\,,
\end{aligned}
\tag{5.114}
$$

where we have used Eq. (5.111). Therefore, by the convolution theorem

$$
v(M,t) = v_1(M,t) * \psi(M)\,.
$$

Finally, the solution of

$$
\begin{cases}
\dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + f(x,y,z,t)\,, \quad R^3\times(0,+\infty)\,,\\
u(x,y,z,0) = \varphi(x,y,z)\,, \quad u_t(x,y,z,0) = \psi(x,y,z)
\end{cases}
\tag{5.115}
$$

is, by the solution structure theorem,

$$
\begin{aligned}
u(x,y,z,t) &= \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t) + W_\psi(M,t) + \int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau\\
&= \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)\left[\mathrm{e}^{-\frac{t}{2\tau_0}}\iiint\limits_{R^3} v_1(\xi,\eta,\zeta,t)\varphi(x-\xi,y-\eta,z-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\right.
\end{aligned}
$$

$$+ \mathrm{e}^{-\frac{t}{2\tau_0}} \iiint_{R^3} v_1(\xi,\eta,\zeta,t)\psi(x-\xi,y-\eta,z-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$

$$\left. + \int_0^t \mathrm{e}^{-\frac{t}{2\tau_0}}\,\mathrm{d}\tau \iiint_{R^3} v_1(\xi,\eta,\zeta,t) f(x-\xi,y-\eta,z-\zeta,t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \right] . \tag{5.116}$$

Remark. By the definition of Fourier transformation, we have

$$[\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z] = L^3 , \quad [\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3] = L^{-3} ,$$

Thus, in Eq. (5.114) ,

$$[v_1(M,t)] = \left[\frac{\sin\left(t\sqrt{(A\omega)^2 - c^2}\right)}{\sqrt{(A\omega)^2 - c^2}} \right] [\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3] = TL^{-3} .$$

The unit of the first term in Eq. (5.116) is

$$\left[\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right] [v_1]\,[\varphi]\,[\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta] = \Theta = [u] .$$

It is straightforward to show that the unit of the second term in Eq. (5.116) is also Θ. Note that $[f] = \Theta T^{-2}$. The unit of the third term in Eq. (5.116) is thus

$$[\mathrm{d}\tau]\,[v_1]\,[f]\,[\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta] = \Theta = [u] .$$

5.8.3 Method of Spherical Means for PDS (5.115)

Since the equation in PDS (5.115) is hyperbolic, its solution has wavelike properties. We may obtain its solution by the method of spherical means which we have used to obtain the Poisson formula of three-dimensional wave equations in Section 2.8. By the solution structure theorem, we can focus our attention on PDS (5.113) to find the solution of PDS (5.115).

Consider now a sphere V_r^M of center M and radius r. Its spherical surface is denoted by S_r^M. The mean value of $V(M,t)$ on S_r^M is

$$\bar{v}(r,t) = \frac{1}{4\pi r^2} \iint_{S_r^M} v(M',t)\,\mathrm{d}S . \tag{5.117}$$

Also $v(M,t) = \lim\limits_{r\to+0} \bar{v}(r,t)$.

Let $\mathrm{d}S$ and $\mathrm{d}\omega$ be the infinitesimal spherical area on S_r^M and the corresponding infinitesimal solid angle. Thus

$$\mathrm{d}S = r^2\,\mathrm{d}\omega \quad \text{and} \quad \bar{v}(r,t) = \frac{1}{4\pi}\iint\limits_{S_r^M} v(M',t)\,\mathrm{d}\omega\,.$$

Integrating the equation in PDS (5.113) over V_r^M yields

$$\iiint\limits_{V_r^M} \frac{\partial^2 v}{\partial t^2}\,\mathrm{d}V = A^2\iiint\limits_{V_r^M} \Delta v\,\mathrm{d}V + c^2\iiint\limits_{V_r^M} v\,\mathrm{d}V\,.$$

Since

$$\iiint\limits_{V_r^M} \frac{\partial^2 v}{\partial t^2}\,\mathrm{d}V = \frac{\partial^2}{\partial t^2}\int_0^r \left(\iint\limits_{S_\rho^M} v\,\mathrm{d}S\right)\mathrm{d}\rho\,,$$

$$\begin{aligned}
A^2\iiint\limits_{V_r^M} \Delta v\,dV &= A^2\iint\limits_{S_r^M} \frac{\partial v}{\partial n}\,\mathrm{d}S \\
= A^2 r^2\iint\limits_{S_r^M} \frac{\partial v}{\partial n}\,\mathrm{d}\omega &= 4\pi A^2 r^2 \frac{1}{4\pi}\iint\limits_{S_r^M} \frac{\partial v}{\partial r}\,d\omega \\
&= 4\pi A^2\left(r^2\frac{\partial \bar{v}}{\partial r}\right)\,,
\end{aligned}$$

$$c^2\iiint\limits_{V_r^M} v\,\mathrm{d}V = c^2\int_0^r \left(\iint\limits_{S_\rho^M} v\,\mathrm{d}S\right)\mathrm{d}\rho\,,$$

we arrive at

$$\frac{\partial^2}{\partial t^2}\int_0^r \left(\iint\limits_{S_\rho^M} v\,\mathrm{d}S\right)\mathrm{d}\rho = 4\pi A^2\left(r^2\frac{\partial \bar{v}}{\partial r}\right) + c^2\int_0^r \left(\iint\limits_{S_\rho^M} v\,\mathrm{d}S\right)\mathrm{d}\rho\,.$$

Taking the derivative with respect to r yields

$$\frac{\partial^2 \bar{v}}{\partial t^2} = A^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \bar{v}}{\partial r}\right) + c^2\bar{v}\,.$$

By following the function transformation from Eq. (5.95) to Eq. (5.96) in Section 5.7, we obtain

$$\begin{cases} \dfrac{\partial^2 (r\bar{v})}{\partial t^2} = A^2 \dfrac{\partial^2 (r\bar{v})}{\partial r^2} + c^2 (r\bar{v}), \quad 0 < r < +\infty,\ 0 < t, \\ r\bar{v}\big|_{t=0} = 0, \quad (r\bar{v})_t\big|_{t=0} = r\bar{\psi}(r), \end{cases} \tag{5.118}$$

where $\bar{\psi}(r) = \dfrac{1}{4\pi r^2} \displaystyle\iint\limits_{S_r^M} \psi(M')\,\mathrm{d}S$, $\lim\limits_{r\to+0} \bar{\psi}(r) = \psi(M)$.

PDS (5.118) is one-dimensional; hence the method of spherical means reduces the dimensions of the problem. To apply the results for one-dimensional Cauchy problems, we should make a continuation of initial values. Since $r\bar{\psi}(r) = 0$ at $r = 0$, we should make an odd continuation, i.e.

$$(r\bar{v})_t\big|_{t=0} = \begin{cases} r\bar{\psi}(r), & r \geq 0 \\ r\bar{\psi}(-r), & r < 0, \end{cases} \qquad -\infty < r < +\infty$$

By Eq. (5.30), we obtain the solution of PDS (5.118)

$$r\bar{v} = \frac{1}{2A} \int_{r-At}^{r+At} I_0\left(b\sqrt{(At)^2 - (r-r')^2}\right) r'\,\bar{\psi}(r')\,\mathrm{d}r'$$

or

$$\bar{v} = \frac{1}{2Ar} \int_{r-At}^{r+At} I_0\left(b\sqrt{(At)^2 - (r-r')^2}\right) r'\,\bar{\psi}(r')\,\mathrm{d}r',$$

where $b = 1\big/2a\sqrt{\tau_0} = \dfrac{1}{2A\tau_0} = \dfrac{c}{A}$. By L'Hôpital's rule, we obtain

$$\begin{aligned} v(M,t) &= \lim_{r\to+0} \bar{v} \\ &= \lim_{r\to+0} \frac{1}{2A} \frac{d}{dr} \int_{r-At}^{r+At} I_0\left(b\sqrt{(At)^2 - (r-r')^2}\right) r'\,\bar{\psi}(r')\,\mathrm{d}r' \\ &= \frac{1}{2A} \lim_{r\to+0} \left[\int_{r-At}^{r+At} \frac{b(r-r')I_1\left(b\sqrt{(At)^2-(r-r')^2}\right)}{\sqrt{(At)^2-(r-r')^2}} r'\,\bar{\psi}(r')\,\mathrm{d}r' \right. \\ &\qquad \left. + (r+At)\bar{\psi}(r+At) - (r-At)\bar{\psi}(r-At) \right] \end{aligned}$$

$$= \frac{1}{2A}\left[\int_{-At}^{At} \frac{br' I_1\left(b\sqrt{(At)^2 - r'^2}\right)}{\sqrt{(At)^2 - r'^2}} r'\,\bar{\psi}(r')\,\mathrm{d}r' + 2At\,\bar{\psi}(At)\right]$$

$$= \frac{1}{8\pi A}\int_{-At}^{At} \frac{bI_1\left(b\sqrt{(At)^2 - r'^2}\right)}{\sqrt{(At)^2 - r'^2}} \left(\iint\limits_{S_{r'}^M} \psi(M')\,\mathrm{d}S\right)\mathrm{d}r'$$

$$+ \frac{1}{4\pi A^2 t}\iint\limits_{S_r^M} \psi(M')\,\mathrm{d}S$$

$$= \frac{1}{4\pi A}\left[\frac{1}{2}\int_{-At}^{At} \frac{bI_1\left(b\sqrt{(At)^2 - r'^2}\right)}{\sqrt{(At)^2 - r'^2}} \left(\iint\limits_{S_{r'}^M} \psi(M')\,\mathrm{d}S\right)\mathrm{d}r'\right.$$

$$\left. + \frac{1}{At}\iint\limits_{S_r^M} \psi(M')\,\mathrm{d}S\right].$$

Thus the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u, \quad R^3 \times (0, +\infty), \\ u(M,0) = 0, \quad u_t(M,0) = \psi(M) \end{cases} \tag{5.119}$$

is

$$u(M,t) = W_\psi(M,t) = \frac{1}{4\pi A}\,\mathrm{e}^{-\frac{t}{2\tau_0}}\left[\frac{1}{At}\iint\limits_{S_{At}^M} \psi(M')\,\mathrm{d}S\right.$$
$$\left. + \frac{1}{2}\int_{-At}^{At} \frac{bI_1\left(b\sqrt{(At)^2 - r'^2}\right)}{\sqrt{(At)^2 - r'^2}} \left(\iint\limits_{S_{r'}^M} \psi(M')\,\mathrm{d}S\right)\mathrm{d}r'\right]. \tag{5.120}$$

Finally, the solution of PDS (5.115) comes from the solution structure theorem,

$$u(M,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(M,t) + W_\psi(M,t) + \int_0^t W_{f_\tau}(M, t-\tau)\,\mathrm{d}\tau, \tag{5.121}$$

where $f_\tau = f(M,\tau)$.

5.8.4 Discussion

Unit Analysis

In Eq. (5.120), $[b] = \left[\dfrac{c}{A}\right] = \dfrac{T^{-1}}{LT^{-1}} = \dfrac{1}{L}$, $\left[\mathrm{e}^{-\frac{t}{2\tau_0}}\right] = 1$, $[I_1] = 1$. Therefore, the unit of the first term in the right-hand side of Eq. (5.120)

$$= \left[\frac{1}{A^2}\right]\left[\frac{1}{t}\right][\psi]\,[\mathrm{d}S] = \frac{T^2}{L^2}\cdot\frac{1}{T}\cdot\frac{\Theta}{T}\cdot L^2 = \Theta\,.$$

The unit of the second term

$$= \left[\frac{1}{A}\right][b]\left[\frac{1}{\sqrt{(At)^2 - r'^2}}\right][\psi]\,[\mathrm{d}S]\,\left[dr'\right]$$

$$= \frac{T}{L}\cdot\frac{1}{L}\cdot\frac{1}{L}\cdot\frac{\Theta}{T}\cdot L^2\cdot L = \Theta\,.$$

Therefore, the unit of Eq. (5.120) is correct.

Initial Conditions

Note that

$$\frac{1}{At}\iint\limits_{S_{At}^M}\psi(M')\,\mathrm{d}S = 4\pi At\cdot\bar{\psi}(At) \quad \text{and} \quad \lim_{t\to+0}\bar{\psi}(At) = \psi(M)\,.$$

The $u(M,t)$ in Eq. (5.120) thus satisfies the initial condition $u(M,0) = 0$. Note also that

$$\lim_{t\to+0}\frac{I_1(x)}{x} = \frac{1}{2}\,.$$

Hence

$$u_t(M,t) = -\frac{1}{2\tau_0}\mathrm{e}^{-\frac{t}{2\tau_0}}\frac{1}{4\pi A}\left[4\pi At\cdot\bar{\psi}(At)\right.$$
$$\left. +\frac{1}{2}\int_{-At}^{At}\frac{bI_1\left(b\sqrt{(At)^2 - r'^2}\right)}{\sqrt{(At)^2 - r'^2}}\left(\iint\limits_{S_{r'}^M}\psi(M')dS\right)\mathrm{d}r'\right]$$

$$+\frac{1}{4\pi A}\mathrm{e}^{-\frac{t}{2\tau_0}}\left\{\left[4\pi A\bar{\psi}(At)+4\pi At\bar{\psi}_t(At)\right.\right.$$

$$+\frac{1}{2}\int_{-At}^{At}\left[\frac{bI_1\left(b\sqrt{(At)^2-r'^2}\right)}{\sqrt{(At)^2-r'^2}}\right]'\left(\iint\limits_{S_{r'}^M}\psi(M')\,\mathrm{d}S\right)\mathrm{d}r'\Bigg]$$

$$+\frac{1}{2}\left[\frac{b^2I_1\left(b\sqrt{(At)^2-r'^2}\right)}{b\sqrt{(At)^2-r'^2}}\iint\limits_{S_{r'}^M}\psi(M')\ \mathrm{d}S|_{r'=At}\cdot A\right.$$

$$\left.\left.-\frac{b^2I_1\left(b\sqrt{(At)^2-r'^2}\right)}{b\sqrt{(At)^2-r'^2}}\iint\limits_{S_{r'}^M}\psi(M')\ \mathrm{d}S|_{r'=-At}\cdot(-A)\right]\right\}.$$

From the point view of continuation,

$$\iint\limits_{S_{r'}^M}\psi(M')\ \mathrm{d}S|_{r'=-At}=\iint\limits_{S_{r'}^M}\psi(M')\ \mathrm{d}S|_{r'=At}\,.$$

Also

$$\lim_{t\to+0}\iint\limits_{S_{At}^M}\psi(M')\,\mathrm{d}S=0\,.$$

Therefore, the $u(M,t)$ in Eq. (5.120) also satisfies the initial condition $u_t(M,0)=\psi(M)$.

Comparison with Three-Dimensional Wave Equations

The counterpart of PDS (5.119) in wave equations reads

$$\begin{cases} u_{tt}=A^2\Delta u,\ R^3\times(0,+\infty)\,,\\ u(M,0)=0,\ u_t(M,0)=\psi(M)\,. \end{cases} \tag{5.122}$$

Its solution is, by the Poisson formula

$$u(M,t)=\frac{1}{4\pi A^2t}\iint\limits_{S_{At}^M}\psi(M')\,\mathrm{d}S\,, \tag{5.123}$$

which is the first term of the right-hand side of Eq. (5.120) without the decaying factor $\mathrm{e}^{-\frac{t}{2\tau_0}}$. Since the solution is a surface integral of the first kind, there exist a wave front and rear in the propagation of waves from initial disturbance $\psi(M)$ in some regions (Section 2.8). The sound wave is a typical example of such a wave.

The solution (5.120) of PDS (5.119) differs from that in Eq. (5.123) not only on the appearance of factor $\mathrm{e}^{-\frac{t}{2\tau_0}}$ but also on its second term in the right-hand side. Since

$$\int_{-At}^{At}\left(\iint\limits_{S_{r'}^M} dS\right)\mathrm{d}r' = \text{Volume of } V_{At}^M\,,$$

the integral of $\psi(M)$ in the second term is, in fact, a volume integral. The thermal waves from the initial disturbances $\psi(M)$ thus have a wave front, but not a wave rear. The factor $\mathrm{e}^{-\frac{t}{2\tau_0}}$ in Eq. (5.120) causes the temperature to decay as time increases. This agrees with the physical reality of heat conduction.

The wave front will not appear if there exist global initial disturbances $\psi(M)$, $\varphi(M)$ or global source disturbances $f(x,y,z,t)$. The distribution and strength of initial disturbances $\psi(M)$, $\varphi(M)$ and source disturbances $f(x,y,z,t)$ will determine whether the temperature at a given point decays or not and the degree of decaying.

Chapter 6
Dual-Phase-Lagging Heat-Conduction Equations

In this chapter we first develop the solution structure theorem for mixed problems of dual-phase-lagging heat-conduction equations. We then apply it to solve the problems under some boundary conditions. We also develop the solution structure theorem for Cauchy problems and discuss the methods of solving Cauchy problems. Finally we examine thermal waves and resonance and develop equivalence between dual-phase-lagging heat conduction and heat conduction in two-phase systems.

6.1 Solution Structure Theorem for Mixed Problems

In this section we develop the solution structure theorem for mixed problems of dual-phase-lagging heat-conduction equations

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}+\frac{\tau_q}{\alpha}\frac{\partial^2 T}{\partial t^2}=\Delta T+\tau_T\frac{\partial}{\partial t}\Delta T+F(M,t)\,. \tag{6.1}$$

Here, α, τ_q and τ_T are all positive constants. $T=T(M,t)$ stands for the temperature at spatial point M and time instant t.

6.1.1 Notes on Dual-Phase-Lagging Heat-Conduction Equations

For the benefit of developing the solution structure theorem and solutions and for comparing with the results in Chapters 3– 5, we rewrite the dual-phase-lagging heat-conduction equations (6.1) into

$$\frac{u_t}{\tau_0}+u_{tt}=A^2\Delta u+B^2\frac{\partial}{\partial t}\Delta u+f(M,t)\,, \tag{6.2}$$

where $u(M,t)\equiv T(M,t)$, $A^2=\dfrac{\alpha}{\tau_q}$, $B^2=\dfrac{\alpha\tau_T}{\tau_q}$, $f(M,t)=\dfrac{\alpha}{\tau_q}F(M,t)$ and $\tau_0=\tau_q$.

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1. M stands for a spatial point in one-, two- and three-dimensional space. Its coordinates are x, (x,y) and (x,y,z), respectively. The Δ is the Laplace operator, which reduces to u_{xx} for the one-dimensional case. Equation (6.1) or (6.2) is a third-order linear nonhomogeneous partial differential equation with constant coefficients.
2. Mixed problems are always specified in certain spatial domains denoted by Ω, D and a interval $[x_1,x_2]$, respectively, for three-, two- and one-dimensional cases. The boundary of Ω and D are denoted by $\partial\Omega$ and ∂D, respectively. The boundary of $[x_1,x_2]$ consists of two end points x_1 and x_2. We develop the solution structure theorem under linear homogeneous boundary conditions of all three kinds

$$L(u,u_n)|_{\partial\Omega} = 0\,,$$

   where the u_n stands for the normal derivative on the boundary. The solution structure theorem developed for the three-dimensional case is clearly also valid for the one- and two-dimensional cases.
3. In Eqs. (6.1) and (6.2),

$$\left[\frac{1}{\alpha}\frac{\partial T}{\partial t}\right] = [\Delta T] \Rightarrow [\alpha] = \left[\frac{\partial T}{\partial t}\right] \Big/ [\Delta T] = \frac{L^2}{T}\,,$$

$$[\Delta T] = \left[\tau_T \frac{\partial}{\partial t}\Delta T\right] \Rightarrow [\tau_T] = T\,,$$

$$\left[\frac{1}{\alpha}\frac{\partial T}{\partial t}\right] = \left[\frac{\tau_q}{\alpha}\frac{\partial^2 T}{\partial t^2}\right] \Rightarrow [\tau_q] = T\,, (\tau_q \text{ (or } \tau_0) \quad \text{is also a time constant})$$

$$[\Delta T] = [F(M,t)] \Rightarrow [F] = \frac{\Theta}{L^2}\,, \quad [f] = [u_{tt}] = \frac{\Theta}{\mathrm{T}^2}\,,$$

$$[A] = \left[\sqrt{\frac{\alpha}{\tau_q}}\right] = \frac{L}{T}\,, \quad \text{the speed of thermal waves}\,,$$

$$[B] = \left[\sqrt{\frac{\alpha\tau_T}{\tau_q}}\right] = \left[\sqrt{\alpha}\right] = \frac{L}{\sqrt{T}} \Rightarrow [B^2] = \frac{L^2}{T} = [\alpha]\,.$$

6.1.2 Solution Structure Theorem

Theorem 1. Let $u = W_\psi(M,t)$ be the solution of well-posed PDS

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + B^2\dfrac{\partial}{\partial t}\Delta u\,, \\ \qquad\qquad\qquad\qquad \Omega \times (0,+\infty)\,, \\ L(u,u_n)|_{\partial\Omega} = 0\,, \\ u(M,0) = 0\,, \qquad u_t(M,0) = \psi(M)\,. \end{cases} \tag{6.3}$$

The solution of the well-posed PDS

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + B^2 \dfrac{\partial}{\partial t}\Delta u\,, \\ \qquad\qquad\qquad\qquad \Omega \times (0,+\infty)\,, \\ L(u,u_n)|_{\partial\Omega} = 0\,, \\ u(M,0) = \varphi(M)\,, \quad u_t(M,0) = 0\,. \end{cases} \tag{6.4}$$

is

$$u = \left[\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right] W_\varphi(M,t) + W_{\varphi_1}(M,t)\,, \tag{6.5}$$

where $\varphi_1 = -B^2\Delta\varphi$.

Proof. By the definition of $W_\varphi(M,t)$ and $W_{\varphi_1}(M,t)$, we have

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_\varphi}{\partial t} + \dfrac{\partial^2 W_\varphi}{\partial t^2} - A^2\Delta W_\varphi - B^2\dfrac{\partial}{\partial t}\Delta W_\varphi = 0\,, & (6.6a) \\ L\left(W_\varphi, \dfrac{\partial W_\varphi}{\partial n}\right)\bigg|_{\partial\Omega} = 0\,, & (6.6b) \\ W_\varphi\big|_{t=0} = 0\,, \quad \dfrac{\partial W_\varphi}{\partial t}\bigg|_{t=0} = \varphi(M)\,. & (6.6c) \end{cases}$$

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_{\varphi_1}}{\partial t} + \dfrac{\partial^2 W_{\varphi_1}}{\partial t^2} - A^2\Delta W_{\varphi_1} - B^2\dfrac{\partial}{\partial t}\Delta W_{\varphi_1} = 0\,, & (6.7a) \\ L\left(W_{\varphi_1}, \dfrac{\partial W_{\varphi_1}}{\partial n}\right)\bigg|_{\partial\Omega} = 0\,, & (6.7b) \\ W_{\varphi_1}\big|_{t=0} = 0\,, \quad \dfrac{\partial W_{\varphi_1}}{\partial t}\bigg|_{t=0} = \varphi_1(M)\,. & (6.7c) \end{cases}$$

Satisfaction of Equation

Substituting Eq. (6.5) into the equation of PDS (6.4) yields

$$\begin{aligned} \frac{u_t}{\tau_0} + u_{tt} - A^2\Delta u - B^2\frac{\partial}{\partial t}\Delta u &= \frac{1}{\tau_0}\frac{\partial}{\partial t}\left[\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi + W_{\varphi_1}\right] \\ &+ \frac{\partial^2}{\partial t^2}\left[\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi + W_{\varphi_1}\right] - A^2\Delta\left[\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi + W_{\varphi_1}\right] \\ &- B^2\frac{\partial}{\partial t}\Delta\left[\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi + W_{\varphi_1}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\tau_0}\left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t} + \frac{\partial^2 W_\varphi}{\partial t^2} - A^2\Delta W_\varphi - B^2\frac{\partial}{\partial t}\Delta W_\varphi\right) \\
&\quad + \frac{\partial}{\partial t}\left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t} + \frac{\partial^2 W_\varphi}{\partial t^2} - A^2\Delta W_\varphi - B^2\frac{\partial}{\partial t}\Delta W_\varphi\right) \\
&\quad + \left(\frac{1}{\tau_0}\frac{\partial W_{\varphi_1}}{\partial t} + \frac{\partial^2 W_{\varphi_1}}{\partial t^2} - A^2\Delta W_{\varphi_1} - B^2\frac{\partial}{\partial t}\Delta W_{\varphi_1}\right) = 0\,,
\end{aligned}
$$

in which we have used Eqs. (6.6a) and (6.7a).

Satisfaction of Boundary Conditions

Substituting Eq. (6.5) into the boundary conditions of PDS (6.4) leads to

$$
\begin{aligned}
&L\left(u, \frac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} \\
&= L\left\{\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi + W_{\varphi_1}, \frac{\partial}{\partial n}\left[\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi + W_{\varphi_1}\right]\right\}\Big|_{\partial\Omega} \\
&= \frac{1}{\tau_0}L\left(W_\varphi, \frac{\partial W_\varphi}{\partial n}\right)\Big|_{\partial\Omega} + \frac{\partial}{\partial t}L\left(W_\varphi, \frac{\partial W_\varphi}{\partial n}\right)\Big|_{\partial\Omega} + L\left(W_{\varphi_1}, \frac{\partial W_{\varphi_1}}{\partial n}\right)\Big|_{\partial\Omega} = 0\,,
\end{aligned}
$$

where we have used Eqs. (6.6b) and (6.7b).

Satisfaction of Initial Conditions

By using Eqs. (6.6c) and (6.7c), we have

$$
\begin{aligned}
u(M,0) &= \left[\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi(M,t) + W_{\varphi_1}(M,t)\right]\Big|_{t=0} \\
&= \frac{1}{\tau_0}W_\varphi(M,0) + \frac{\partial W_\varphi}{\partial t}\Big|_{t=0} + W_{\varphi_1}(M,0) = \varphi(M)\,.
\end{aligned}
$$

Also

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left[\left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi(M,t) + W_{\varphi_1}(M,t)\right] \\
&= \frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t} + \frac{\partial^2 W_\varphi}{\partial t^2} + \frac{\partial W_{\varphi_1}}{\partial t} \\
&= A^2\Delta W_\varphi + B^2\frac{\partial}{\partial t}\Delta W_\varphi + \frac{\partial W_{\varphi_1}}{\partial t} \\
&= A^2\Delta W_\varphi + B^2\Delta\left(\frac{\partial W_\varphi}{\partial t}\right) + \frac{\partial W_{\varphi_1}}{\partial t}\,.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\left.\frac{\partial u}{\partial t}\right|_{t=0} &= A^2\Delta W_\varphi\big|_{t=0} + B^2\Delta\left(\frac{\partial W_\varphi}{\partial t}\right)\bigg|_{t=0} + \left.\frac{\partial W_{\varphi_1}}{\partial t}\right|_{t=0} \\ &= B^2\Delta\varphi + \varphi_1(M) = B^2\Delta\varphi - B^2\Delta\varphi = 0\,.\end{aligned}$$

Theorem 2. Let $u = W_\psi(M,t)$ be the solution of the well-posed PDS

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + B^2\dfrac{\partial}{\partial t}\Delta u\,, \\ \qquad\qquad\qquad\qquad \Omega\times(0,+\infty)\,, \\ L(u,u_n)|_{\partial\Omega} = 0\,, \\ u(M,0) = 0, u_t(M,0) = \psi(M)\,. \end{cases} \tag{6.8}$$

The solution of the well-posed PDS

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + B^2\dfrac{\partial}{\partial t}\Delta u + f(M,t)\,, \\ \qquad\qquad\qquad\qquad \Omega\times(0,+\infty)\,, \\ L(u,u_n)|_{\partial\Omega} = 0\,, \\ u(M,0) = 0, u_t(M,0) = 0 \end{cases} \tag{6.9}$$

is

$$u = \int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau\,, \tag{6.10}$$

where $f_\tau = f(M,\tau)$.

Proof. By the definition of $W_{f_\tau}(M,t-\tau)$, we have

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_{f_\tau}}{\partial t} + \dfrac{\partial^2 W_{f_\tau}}{\partial t^2} - A^2\Delta W_{f_\tau} - B^2\frac{\partial}{\partial t}\Delta W_{f_\tau} = 0\,, \\ L\left(W_{f_\tau}, \dfrac{\partial W_{f_\tau}}{\partial n}\right)\bigg|_{\partial\Omega} = 0\,, \\ W_{f_\tau}(M,t-\tau)\big|_{t=\tau} = 0\,, \quad \left.\dfrac{\partial}{\partial t}W_{f_\tau}(M,t-\tau)\right|_{t=\tau} = f(M,\tau)\,. \end{cases} \tag{6.11}$$

Satisfaction of Equation

By substituting Eq. (6.10) into the equation of PDS (6.9) and applying Eq. (6.11), we obtain

$$\frac{u_t}{\tau_0} + u_{tt} - A^2\Delta u - B^2\frac{\partial}{\partial t}\Delta u$$

$$
\begin{aligned}
&= \frac{1}{\tau_0}\frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau + \frac{\partial^2}{\partial t^2}\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau \\
&\quad - A^2\Delta\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau - B^2\frac{\partial}{\partial t}\Delta\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau \\
&= \frac{1}{\tau_0}\left[\int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}(M,t-\tau)\big|_{\tau=t}\right] + \int_0^t \frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau \\
&\quad + \left.\frac{\partial W_{f_\tau}}{\partial t}\right|_{\tau=t} - A^2\int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau - B^2\frac{\partial}{\partial t}\int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau \\
&= \frac{1}{\tau_0}\int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + \int_0^t \frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau + f(M,t) - A^2\int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau \\
&\quad - B^2\left[\int_0^t \frac{\partial}{\partial t}\Delta W_{f_\tau}\,\mathrm{d}\tau + \Delta W_{f_\tau}\big|_{\tau=t}\right] \\
&= \int_0^t \left(\frac{1}{\tau_0}\frac{\partial W_{f_\tau}}{\partial t} + \frac{\partial^2 W_{f_\tau}}{\partial t^2} - A^2\Delta W_{f_\tau} - B^2\frac{\partial}{\partial t}\Delta W_{f_\tau}\right)\mathrm{d}\tau + f(M,t) \\
&= \int_0^t 0\,\mathrm{d}\tau + f(M,t) = f(M,t)\,.
\end{aligned}
$$

Satisfaction of Boundary Conditions

By substituting Eq. (6.10) into the boundary conditions of PDS (6.9) and applying the boundary conditions of PDS (6.11), we have

$$
\begin{aligned}
L\left(u,\frac{\partial u}{\partial n}\right)\bigg|_{\partial\Omega} &= L\left(\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau,\ \frac{\partial}{\partial n}\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau\right)\bigg|_{\partial\Omega} \\
&= L\left(\int_0^t W_{f_\tau}\,\mathrm{d}\tau,\ \int_0^t \frac{\partial}{\partial n}W_{f_\tau}\,\mathrm{d}\tau\right)\bigg|_{\partial\Omega} \\
&= \int_0^t L\left(W_{f_\tau},\ \frac{\partial}{\partial n}W_{f_\tau}\right)\bigg|_{\partial\Omega}\mathrm{d}\tau = 0\,.
\end{aligned}
$$

Satisfaction of Initial Conditions

It is straightforward to show that the u in Eq. (6.10) satisfies the initial condition $u(M,0)=0$. Also,

$$
\begin{aligned}
u_t(M,t) &= \frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau \\
&= \int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}(M,t-\tau)\big|_{\tau=t}\,.
\end{aligned}
$$

Thus,

$$u_t(M,0) = 0\,.$$

Remark. By the principle of superposition, the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u + B^2 \dfrac{\partial}{\partial t} \Delta u + f(M,t)\,, \\ \qquad\qquad\qquad \Omega \times (0,+\infty), \\ L(u,u_n)|_{\partial\Omega} = 0\,, \\ u(M,0) = \varphi(M), u_t(M,0) = \psi(M) \end{cases} \tag{6.12}$$

is

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(M,t) + W_{\psi_1}(M,t) + \int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau\,, \tag{6.13}$$

where $\psi_1 = \psi(M) - B^2\Delta\varphi$, $f_\tau = f(M,\tau)$. $u = W_\psi(M,t)$ is the solution of PDS (6.3).

6.2 Fourier Method of Expansion for One-Dimensional Mixed Problems

In this section we apply both the Fourier method of expansion and the solution structure theorem to solve

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 u_{xx} + B^2 u_{txx} + f(x,t)\,, \\ \qquad\qquad\qquad (0,l) \times (0,+\infty)\,, \\ u(0,t) = u(l,t) = 0\,, \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x)\,. \end{cases} \tag{6.14}$$

We also compare the results obtained by the two methods to further demonstrate the correctness of the latter.

6.2.1 Fourier Method of Expansion

Solution from Initial Thermal Speed $\psi(\mathbf{x})$

Solve

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx}\,, \quad (0,l) \times (0,+\infty)\,, \\ u(0,t) = u(l,t) = 0\,, \\ u(x,0) = 0, u_t(x,0) = \psi(x)\,. \end{cases} \tag{6.15}$$

Solution. Based on the given boundary conditions, let

$$u(x,t) = \sum_{n=1}^{+\infty} T_n(t) \sin\frac{n\pi x}{l}. \tag{6.16}$$

Substituting it into the equation in PDS (6.15) yields

$$\sum_{n=1}^{+\infty}\left[\frac{T_n'(t)}{\tau_0} + T_n''(t) + \left(\frac{n\pi A}{l}\right)^2 T_n(t) + \left(\frac{n\pi B}{l}\right)^2 T_n'(t)\right]\sin\frac{n\pi x}{l} = 0.$$

Thus

$$T_n''(t) + \left[\frac{1}{\tau_0} + \left(\frac{n\pi B}{l}\right)^2\right] T_n'(t) + \left(\frac{n\pi A}{l}\right)^2 T_n(t) = 0. \tag{6.17}$$

Its general solution reads

$$T_n(t) = \mathrm{e}^{\alpha_n t}\left(A_n \cos\beta_n t + B_n \underline{\sin}\beta_n t\right),$$

where A_n and B_n are constants,

$$\alpha_n = -\frac{1}{2}\left[\frac{1}{\tau_0} + \left(\frac{n\pi B}{l}\right)^2\right],$$

$$\beta_n = \frac{1}{2}\sqrt{4\left(\frac{n\pi A}{l}\right)^2 - \left[\frac{1}{\tau_0} + \left(\frac{n\pi B}{l}\right)^2\right]^2},$$

$$\underline{\sin}\beta_n t = \begin{cases} \sin\beta_n t, & \text{if } \beta_n \neq 0, \\ t, & \text{if } \beta_n = 0. \end{cases}$$

Substituting $T_n(t)$ into Eq. (6.16) leads to

$$u(x,t) = \sum_{n=1}^{+\infty} \mathrm{e}^{\alpha_n t}\left(A_n \cos\beta_n t + B_n \underline{\sin}\beta_n t\right)\sin\frac{n\pi x}{l}.$$

Applying the initial condition $u(x,0) = 0$ yields

$$\sum_{n=1}^{+\infty} A_n \sin\frac{n\pi x}{l} = 0.$$

Thus

$$A_n = 0, \quad n = 1, 2, \ldots,$$

$$u(x,t) = \sum_{n=1}^{+\infty} \mathrm{e}^{\alpha_n t} B_n \underline{\sin}\beta_n t \sin\frac{n\pi x}{l},$$

$$u_t(x,t) = \sum_{n=1}^{+\infty} B_n\left(\alpha_n \underline{\sin}\beta_n t + \beta_n \cos\beta_n t\right)\mathrm{e}^{\alpha_n t}\sin\frac{n\pi x}{l},$$

where $\underline{\beta}_n = \begin{cases} \beta_n\,, \text{ if } \beta_n \neq 0\,, \\ 1\,, \quad \text{if } \beta_n = 0\,. \end{cases}$ Applying the initial condition $u_t(x,0) = \psi(x)$ yields

$$\sum_{n=1}^{+\infty} B_n \underline{\beta}_n \sin\frac{n\pi x}{l} = \psi(x)\,.$$

Thus

$$B_n = \frac{2}{l\underline{\beta}_n}\int_0^l \psi(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\,.$$

Finally, the solution of PDS (6.15) is

$$\begin{cases} u(x,t) = \displaystyle\sum_{n=1}^{+\infty} B_n\,\mathrm{e}^{\alpha_n t}\,\underline{\sin}\beta_n t\sin\frac{n\pi x}{l}\,, \\ B_n = \dfrac{2}{l\underline{\beta}_n}\displaystyle\int_0^l \psi(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\,. \end{cases} \tag{6.18}$$

Solution from Initial Temperature $\varphi(\mathbf{x})$

Solve

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx}\,, \; (0,l)\times(0,+\infty)\,, \\ u(0,t) = u(l,t) = 0\,, \\ u(x,0) = \varphi(x)\,, \qquad\qquad u_t(x,0) = 0\,. \end{cases} \tag{6.19}$$

Solution. Based on the given boundary conditions, let

$$u(x,t) = \sum_{n=1}^{+\infty} T_n(t)\sin\frac{n\pi x}{l}\,.$$

The $T_n(t)$ can be determined by substituting it into the equation in PDS (6.19). Thus

$$u(x,t) = \sum_{n=1}^{+\infty} \mathrm{e}^{\alpha_n t}\left(A_n\cos\beta_n t + B_n\underline{\sin}\beta_n t\right)\sin\frac{n\pi x}{l}\,. \tag{6.20}$$

Applying the initial condition $u(x,0) = \varphi(x)$ leads to

$$A_n = \frac{2}{l}\int_0^l \psi(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\,.$$

By taking the derivative of Eq. (6.20) with respect to t, we obtain

$$\begin{aligned} u_t(x,t) = \sum_{n=1}^{+\infty} &\left\{\alpha_n\,\mathrm{e}^{\alpha_n t}\left(A_n\cos\beta_n t + B_n\underline{\sin}\beta_n t\right)\right. \\ &\left.+\,\mathrm{e}^{\alpha_n t}\left(-A_n\beta_n\sin\beta_n t + B_n\beta_n\cos\beta_n t\right)\right\}\sin\frac{n\pi x}{l}\,. \end{aligned}$$

Applying the initial condition $u_t(x,0)=0$ yields

$$\sum_{n=1}^{+\infty}(A_n\alpha_n+B_n\beta_n)\sin\frac{n\pi x}{l}=0\,.$$

Thus

$$A_n\alpha_n+B_n\beta_n=0 \quad \text{or} \quad B_n=-\frac{A_n\alpha_n}{\beta_n}\,.$$

Finally, the solution of PDS (6.19) is

$$\begin{cases} u(x,t)=\displaystyle\sum_{n=1}^{+\infty}\mathrm{e}^{\alpha_n t}\left(A_n\cos\beta_n t+B_n\underline{\sin}\beta_n t\right)\sin\frac{n\pi x}{l}\,,\\ A_n=\displaystyle\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\,,\\ B_n=\displaystyle\frac{-2\alpha_n}{l\underline{\beta}_n}\int_0^l\varphi(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\,. \end{cases} \tag{6.21}$$

Solution from Source Term $\mathbf{f(x,t)}$

Solve

$$\begin{cases} u_t/\tau_0+u_{tt}=A^2u_{xx}+B^2u_{txx}+f(x,t)\,, \quad (0,l)\times(0,+\infty)\,,\\ u(0,t)=u(l,t)=0\,,\\ u(x,0)=0, u_t(x,0)=0\,. \end{cases} \tag{6.22}$$

Solution. Based on the given boundary conditions, let

$$u(x,t)=\sum_{n=1}^{+\infty}T_n(t)\sin\frac{n\pi x}{l}\,, \quad T_n(0)=T_n'(0)=0\,. \tag{6.23}$$

$$f(x,t)=\sum_{n=1}^{+\infty}f_n(t)\sin\frac{n\pi x}{l}\,, \quad f_n(t)=\frac{2}{l}\int_0^l f(\xi,t)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\,.$$

Substituting them into the equation of PDS (6.22) yields

$$T_n''(t)+\left[\frac{1}{\tau_0}+\left(\frac{n\pi B}{l}\right)^2\right]T_n'(t)+\left(\frac{n\pi A}{l}\right)^2T_n(t)=f_n(t)\,. \tag{6.24}$$

Its corresponding homogeneous equation is Eq. (6.17). The solution of Eq. (6.24) subject to $T_n(0)=T_n'(0)=0$ can be written as, by variation of constants,

$$T_n(t)=c_1^{(n)}(t)\,\mathrm{e}^{\alpha_n t}\cos\beta_n t+c_2^{(n)}(t)\,\mathrm{e}^{\alpha_n t}\underline{\sin}\beta_n t\,, \tag{6.25}$$

where $c_1^{(n)}$ and $c_2^{(n)}$ are constants. Let

$$\Delta = \begin{vmatrix} e^{\alpha_n t}\cos\beta_n t & e^{\alpha_n t}\underline{\sin}\beta_n t \\ (e^{\alpha_n t}\cos\beta_n t)_t & (e^{\alpha_n t}\underline{\sin}\beta_n t)_t \end{vmatrix},$$

thus

$$\begin{aligned} c_1^{(n)}(t) &= \int \frac{-e^{\alpha_n t}\underline{\sin}\beta_n t f_n(t)}{\Delta}\,dt + c_1 = \int \frac{-f_n(t)e^{\alpha_n t}\underline{\sin}\beta_n t}{\beta_n e^{2\alpha_n t}}\,dt + c_1 \\ &= -\frac{1}{\beta_n}\int f_n(t)e^{-\alpha_n t}\underline{\sin}\beta_n t\,dt + c_1 = -\frac{1}{\beta_n}\int_0^t f_n(\tau)e^{-\alpha_n \tau}\underline{\sin}\beta_n\tau\,d\tau\,. \end{aligned}$$

Note that

$$\begin{aligned} T_n'(t) &= \left[c_1^{(n)}(t)\right]' e^{\alpha_n t}\cos\beta_n t + c_1^{(n)}(t)\left(e^{\alpha_n t}\cos\beta_n t\right)_t' \\ &\quad + \left[c_2^{(n)}(t)\right]' e^{\alpha_n t}\underline{\sin}\beta_n t + c_2^{(n)}(t)\left(e^{\alpha_n t}\underline{\sin}\beta_n t\right)_t' \\ &= \left(-\frac{1}{\beta_n}f_n(t)e^{-\alpha_n t}\underline{\sin}\beta_n t\right) e^{\alpha_n t}\cos\beta_n t + c_1^{(n)}(t)\left(e^{\alpha_n t}\cos\beta_n t\right)_t' \\ &\quad + \left[c_2^{(n)}(t)\right]' e^{\alpha_n t}\underline{\sin}\beta_n t + c_2^{(n)}(t)\left(\alpha_n e^{\alpha_n t}\underline{\sin}\beta_n t + e^{\alpha_n t}\beta_n\cos\beta_n t\right)\,. \end{aligned}$$

Thus $c_2^{(n)}(0)\underline{\beta}_n = 0$ so that $c_2^{(n)}(0) = 0$. We then have

$$\begin{aligned} c_2^{(n)}(t) &= \int \frac{e^{\alpha_n t}\cos\beta_n t f_n(t)}{\Delta}\,dt + c_2 \\ &= \int \frac{e^{\alpha_n t}\cos\beta_n t \cdot f_n(t)}{\underline{\beta}_n e^{2\alpha_n t}}\,dt + c_2 \\ &= \frac{1}{\underline{\beta}_n}\int_0^t f_n(\tau)e^{-\alpha_n\tau}\cos\beta_n\tau\,d\tau\,. \end{aligned}$$

Substituting $c_1^{(n)}(t)$ and $c_2^{(n)}(t)$ into Eq. (6.25) yields

$$\begin{aligned} T_n(t) &= \left(-\frac{1}{\underline{\beta}_n}\int_0^t f_n(\tau)e^{-\alpha_n\tau}\underline{\sin}\beta_n\tau\,d\tau\right) e^{\alpha_n t}\cos\beta_n t \\ &\quad + \left(\frac{1}{\underline{\beta}_n}\int_0^t f_n(\tau)e^{-\alpha_n\tau}\cos\beta_n\tau\,d\tau\right) e^{\alpha_n t}\underline{\sin}\beta_n t \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\underline{\beta}_n}\int_0^t \mathrm{e}^{\alpha_n(t-\tau)}\frac{1}{2}\left[\underline{\sin}\beta_n(\tau+t)+\underline{\sin}\beta_n(\tau-t)\right]f_n(\tau)\,\mathrm{d}\tau \\
&\quad+\frac{1}{\underline{\beta}_n}\int_0^t \mathrm{e}^{\alpha_n(t-\tau)}\frac{1}{2}\left[\underline{\sin}\beta_n(t+\tau)+\underline{\sin}\beta_n(t-\tau)\right]f_n(\tau)\,\mathrm{d}\tau \\
&= \frac{1}{\underline{\beta}_n}\int_0^t \mathrm{e}^{\alpha_n(t-\tau)}\underline{\sin}\beta_n(t-\tau)f_n(\tau)\,\mathrm{d}\tau \\
&= \frac{2}{l\underline{\beta}_n}\int_0^l \mathrm{d}\xi\int_0^t \mathrm{e}^{\alpha_n(t-\tau)}\underline{\sin}\beta_n(t-\tau)f(\xi,\tau)\,\mathrm{d}\tau\,.
\end{aligned}
\tag{6.26}
$$

A substitution of Eq. (6.26) into Eq. (6.23) yields the solution of PDS (6.22),

$$
u(x,t)=\int_0^l \mathrm{d}\xi\int_0^t\sum_{n=1}^{+\infty}\frac{2}{l\underline{\beta}_n}\mathrm{e}^{\alpha_n(t-\tau)}\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}\underline{\sin}\beta_n(t-\tau)f(\xi,\tau)\,\mathrm{d}\tau\,.
$$

Let

$$
G(x,\xi,t-\tau)=\frac{2}{l}\sum_{n=1}^{+\infty}\frac{1}{\underline{\beta}_n}\mathrm{e}^{\alpha_n(t-\tau)}\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}\underline{\sin}\beta_n(t-\tau)\,. \tag{6.27}
$$

The solution of PDS (6.22) becomes

$$
u(x,t)=\int_0^l \mathrm{d}\xi\int_0^t G(x,\xi,t-\tau)f(\xi,\tau)\,\mathrm{d}\tau\,. \tag{6.28}
$$

The $G(x,\xi,t-\tau)$ in Eq. (6.27) can be called the *Green function of one-dimensional dual-phase-lagging heat-conduction equations*. It is clearly boundary-condition dependent. When $f(x,t)=\delta(x-x_0,t-t_0)$, Eq. (6.28) reduces to

$$
u=G(x,x_0,t-t_0)\,,
$$

so that the Green function $G(x,\xi,t-\tau)$ is the temperature distribution exclusively from the source term $\delta(x-\xi,t-\tau)$.

The solution of PDS (6.14) is, by the principle of superposition, the sum of those in Eqs. (6.18), (6.21) and (6.28).

Solve PDS (6.19) and verify its solution

$u=W_\psi(x,t)$ is available in Eq. (6.18). By Theorem 1 in Section 6.1, we have the solution of PDS (6.19)

$$
u=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(x,t)+W_{-B^2\Delta\varphi}(x,t)\,, \tag{6.29}
$$

which is the sum of three terms

$$\begin{cases} u_1(x,t) = \sum_{n=1}^{+\infty} B_n^{(1)} \mathrm{e}^{\alpha_n t} \underline{\sin}\beta_n t \sin \frac{n\pi x}{l}, \\ B_n^{(1)} \quad = \frac{2}{\tau_0 l \underline{\beta}_n} \int_0^l \varphi(\xi) \sin \frac{n\pi\xi}{l} \,\mathrm{d}\xi. \end{cases} \tag{6.30a}$$

$$\begin{cases} u_2(x,t) = \sum_{n=1}^{+\infty} B_n^{(2)} \left(\alpha_n \underline{\sin}\beta_n t + \underline{\beta}_n \cos \beta_n t\right) \mathrm{e}^{\alpha_n t} \sin \frac{n\pi x}{l} \\ \qquad = \sum_{n=1}^{+\infty} B_n^{(2)} \alpha_n \mathrm{e}^{\alpha_n t} \underline{\sin}\beta_n t \sin \frac{n\pi x}{l} \\ \qquad\quad + \sum_{n=1}^{+\infty} B_n^{(2)} \underline{\beta}_n \cos \beta_n t \cdot \mathrm{e}^{\alpha_n t} \sin \frac{n\pi x}{l}, \\ B_n^{(2)} \quad = \frac{2}{l \underline{\beta}_n} \int_0^l \varphi(\xi) \sin \frac{n\pi\xi}{l} \,\mathrm{d}\xi. \end{cases} \tag{6.30b}$$

$$\begin{cases} u_3(x,t) = \sum_{n=1}^{+\infty} B_n^{(3)} \mathrm{e}^{\alpha_n t} \underline{\sin}\beta_n t \sin \frac{n\pi x}{l}, \\ B_n^{(3)} \quad = -\frac{2B^2}{l \underline{\beta}_n} \int_0^l \varphi''(\xi) \sin \frac{n\pi\xi}{l} \,\mathrm{d}\xi. \end{cases} \tag{6.30c}$$

i.e.

$$u = u_1(x,t) + u_2(x,t) + u_3(x,t). \tag{6.31}$$

Note that $\alpha_n = -\left[\frac{1}{\tau_0} + \left(\frac{n\pi B}{l}\right)^2\right] \Big/ 2$; thus the coefficient B_n in Eq. (6.21) is

$$B_n = \frac{1}{l \underline{\beta}_n} \left[\frac{1}{\tau_0} + \left(\frac{n\pi B}{l}\right)^2\right] \int_0^l \varphi(\xi) \sin \frac{n\pi\xi}{l} \,\mathrm{d}\xi.$$

Also,

$$\begin{aligned} \left(\frac{n\pi B}{l}\right)^2 \int_0^l \varphi(\xi) \sin \frac{n\pi\xi}{l} \,\mathrm{d}\xi &= B^2 \left(-\frac{n\pi}{l}\right) \int_0^l \varphi(\xi) \,\mathrm{d}\left(\cos \frac{n\pi\xi}{l}\right) \\ &= -B^2 \left(\frac{n\pi}{l}\right) \left[\varphi(\xi) \cos \frac{n\pi\xi}{l}\Big|_0^l \right. \\ &\quad \left. - \int_0^l \varphi'(\xi) \cos \frac{n\pi\xi}{l} \,\mathrm{d}\xi\right] \\ &= B^2 \int_0^l \varphi'(\xi) \,\mathrm{d}\left(\sin \frac{n\pi\xi}{l}\right) \end{aligned}$$

$$= B^2\left[\varphi'(\xi)\sin\frac{n\pi\xi}{l}\Big|_0^l - \int_0^l \varphi''(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\right]$$
$$= -B^2\int_0^l \varphi''(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\,(\varphi(0)=\varphi(l)=0)\,.$$

Therefore, the B_n in Eq. (6.21) reads

$$B_n = \frac{1}{\tau_0 l\underline{\beta}_n}\int_0^l \varphi(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi + \frac{-B^2}{l\underline{\beta}_n}\int_0^l \varphi''(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\,.$$

Also, in Eq. (6.30b)

$$B_n^{(2)}\alpha_n = -\frac{1}{l\underline{\beta}_n}\left[\frac{1}{\tau_0}+\left(\frac{n\pi B}{l}\right)^2\right]\int_0^l \varphi(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi$$
$$= -\frac{1}{\tau_0 l\underline{\beta}_n}\int_0^l \varphi(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi + \frac{B^2}{l\underline{\beta}_n}\int_0^l \varphi''(\xi)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\,.$$

Therefore, Eqs. (6.21) and (6.31) are the same although they have different forms.

Solve PDS (6.22) and verify its solution

By Theorem 2 in Section 6.1, the solution of PDS (6.22) is

$$u(x,t) = \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau$$
$$= \int_0^t \left[\sum_{n=1}^{+\infty}\left(\frac{2}{l\underline{\beta}_n}\int_0^l f(\xi,t)\sin\frac{n\pi\xi}{l}\,\mathrm{d}\xi\right)\mathrm{e}^{\alpha_n(t-\tau)}\underline{\sin}\beta_n(t-\tau)\sin\frac{n\pi x}{l}\,\mathrm{d}\tau\right]$$
$$= \int_0^l \mathrm{d}\xi\int_0^t G(x,\xi,t-\tau)f(\xi,\tau)\,\mathrm{d}\tau\,,$$

where

$$G(x,\xi,t-\tau) = \frac{2}{l}\sum_{n=1}^{+\infty}\frac{1}{\underline{\beta}_n}\mathrm{e}^{\alpha_n(t-\tau)}\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}\underline{\sin}\beta_n(t-\tau)\,.$$

It is the same as Eq. (6.28) obtained by the Fourier method of expansion.

6.2.2 Existence

The nominal solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx}\,, & (0,l)\times(0,+\infty)\,,\\ u(0,t)=u(l,t)=0,\, t>0\,, \\ u(x,0)=\varphi(x)\,, \quad u_t(x,0)=\psi(x) \end{cases} \tag{6.32}$$

can be obtained by the superposition of Eqs. (6.18) and (6.21)

$$\begin{cases} u(x,t) = \sum_{n=1}^{+\infty} (A_n \cos \beta_n t + B_n \underline{\sin} \beta_n t)\, \mathrm{e}^{\alpha_n t} \sin \frac{n\pi x}{l}, \\ A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l}\, \mathrm{d}x, \\ B_n = \frac{2}{l\underline{\beta}_n} \int_0^l \psi(x) \sin \frac{n\pi x}{l}\, \mathrm{d}x - \frac{2\alpha_n}{l\underline{\beta}_n} \int_0^l \varphi(x) \sin \frac{n\pi x}{l}\, \mathrm{d}x. \end{cases} \tag{6.33}$$

To prove that the $u(x,t)$in Eq. (6.33) is indeed a solution of PDS (6.32), we must show that we can obtain u_{tt}, u_{xx} and u_{xxt} from Eq. (6.33) by taking derivatives term by term. Thus we must prove that the series $u(x,t)$, $u_{tt}(x,t)$, $u_{xx}(x,t)$ and $u_{xxt}(x,t)$ are all uniformly convergent in $(0,l) \times (0,T)$, where T is an arbitrary positive constant and $t \in [0,T]$.

Let the general term of series $u(x,t)$(Eq. (6.33)) be $u_n(x,t)$, i.e.

$$u_n(x,t) = (A_n \cos \beta_n t + B_n \underline{\sin} \beta_n t)\, \mathrm{e}^{\alpha_n t} \sin \frac{n\pi x}{l},$$

thus

$$\begin{aligned} \frac{\partial^2}{\partial x^2} u_n(x,t) &= -\left(\frac{n\pi}{l}\right)^2 (A_n \cos \beta_n t + B_n \underline{\sin} \beta_n t)\, \mathrm{e}^{\alpha_n t} \sin \frac{n\pi x}{l}, \\ \frac{\partial^2}{\partial t^2} u_n(x,t) &= \frac{\partial}{\partial t} \Big[(-A_n \beta_n \sin \beta_n t + B_n \underline{\beta}_n \cos \beta_n t + A_n \alpha_n \cos \beta_n t \\ &\quad + B_n \alpha_n \underline{\sin} \beta_n t)\, \mathrm{e}^{\alpha_n t} \Big] \sin \frac{n\pi x}{l} \\ &= \frac{\partial}{\partial t} \Big\{ \Big[\left(A_n \alpha_n + B_n \underline{\beta}_n\right) \cos \beta_n t \\ &\quad + (B_n \alpha_n - A_n \beta_n) \underline{\sin} \beta_n t \Big]\, \mathrm{e}^{\alpha_n t} \Big\} \sin \frac{n\pi x}{l} \\ &= \Big[-\left(A_n \alpha_n \beta_n + B_n \beta_n \underline{\beta}_n\right) \sin \beta_n t + \left(B_n \alpha_n \underline{\beta}_n - A_n \beta_n \underline{\beta}_n\right) \cos \beta_n t \\ &\quad + (A_n \alpha_n^2 + B_n \alpha_n \beta_n) \cos \beta_n t \\ &\quad + (B_n \alpha_n^2 - A_n \alpha_n \beta_n) \underline{\sin} \beta_n t \Big]\, \mathrm{e}^{\alpha_n t} \sin \frac{n\pi x}{l} \\ &= \Big\{ \Big[A_n \left(\alpha_n^2 - \beta_n \underline{\beta}_n\right) + 2 B_n \alpha_n \beta_n \Big] \cos \beta_n t \\ &\quad + (-1) \left(A_n \alpha_n \beta_n + B_n \underline{\beta}_n \beta_n\right) \sin \beta_n t \\ &\quad + (B_n \alpha_n^2 - A_n \alpha_n \beta_n) \underline{\sin} \beta_n t \Big\}\, \mathrm{e}^{\alpha_n t} \sin \frac{n\pi x}{l}, \\ \frac{\partial^3}{\partial t \partial x^2} u_n(x,t) &= -\left(\frac{n\pi}{l}\right)^2 \Big[\left(A_n \alpha_n + B_n \underline{\beta}_n\right) \cos \beta_n t \\ &\quad (B_n \alpha_n - A_n \beta_n) \underline{\sin} \beta_n t \Big]\, \mathrm{e}^{\alpha_n t} \sin \frac{n\pi x}{l}. \end{aligned}$$

Note that $\alpha_n < 0$, $\alpha_n = 0(n^2)$ and $\beta_n = 0(n^2)$ as $n \to +\infty$. For any natural number N and $t > 0$, we thus have

$$e^{\alpha_n t} = O\left(\frac{1}{n^N}\right)(n \to +\infty).$$

Since $e^{\alpha_n t}$ occurs in all $u(x,t)$, $\dfrac{\partial^2 u(x,t)}{\partial x^2}$, $\dfrac{\partial^2 u(x,t)}{\partial t^2}$ and $\dfrac{\partial^3 u(x,t)}{\partial t \partial x^2}$, the demand for the smoothness of $\varphi(x)$ and $\psi(x)$ is very weak. Provided that their Fourier coefficients exist, we always have

$$|u_n(x,t)| \le O\left(\frac{1}{n^2}\right), \left|\frac{\partial^2 u_n}{\partial x^2}\right| \le O\left(\frac{1}{n^2}\right),$$
$$\left|\frac{\partial^2 u_n}{\partial t^2}\right| \le O\left(\frac{1}{n^2}\right), \left|\frac{\partial^3 u_n}{\partial t \partial x^2}\right| \le O\left(\frac{1}{n^2}\right).$$

Note that the series $\sum \dfrac{1}{n^2}$ is convergent. Thus, the series $u(x,t)$, $\dfrac{\partial^2 u(x,t)}{\partial x^2}$, $\dfrac{\partial^2 u(x,t)}{\partial t^2}$ and $\dfrac{\partial^3 u(x,t)}{\partial t \partial x^2}$ are all uniformly convergent so that the u in Eq. (6.33) is indeed the solution of PDS (6.22).

Remark. *Variation of constants for second-order nonhomogeneous ordinary differential equations*: Consider a nonhomogeneous ODE

$$y'' + p(x)y' + q(x)y = f(x). \tag{6.34}$$

Let the solution of its corresponding homogeneous ODE be

$$y = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are constants.

Consider the solution of Eq. (6.34) of type

$$y = c_1(x)y_1(x) + c_2(x)y_2(x), \tag{6.35}$$

where $c_1(x)$ and $c_2(x)$ are functions to be determined. We have

$$y' = c_1'(x)y_1 + c_1(x)y_1' + c_2'(x)y_1 + c_1(x)y_2'. \tag{6.36}$$

Let

$$c_1'(x)y_1 + c_2'(x)y_2 = 0. \tag{6.37}$$

By substituting Eq. (6.37) into Eq. (6.36), we can obtain y' and consequently y''. By substituting y'', y' and y into Eq. (6.34) and using

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \quad y_2'' + p(x)y_2' + q(x)y_2 = 0,$$

we arrive at

$$c_1'(x)y_1 + c_2'(x)y_2 = f(x)\,. \tag{6.38}$$

The solution of Eq. (6.37) and Eq. (6.38) is

$$c_1(x) = \int \frac{-y_2(x)f(x)}{\Delta}\,\mathrm{d}x\,, \quad c_2(x) = \int \frac{y_1(x)f(x)}{\Delta}\,\mathrm{d}x\,,$$

where $\Delta = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$ is the *Wronskian* of $y_1(x)$ and $y_2(x)$. A substitution of this into Eq. (6.35) will yield the general solution of Eq. (6.34).

6.3 Separation of Variables for One-Dimensional Mixed Problems

A general form of the one-dimensional mixed problem reads

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx} + f(x,t), & (0,l)\times(0,+\infty)\,, \\ -b_1 u_x(0,t) + k_1 u(0,t) = 0,\ b_2 u_x(l,t) + k_2 u(l,t) = 0\,, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x)\,, \end{cases} \tag{6.39}$$

where b_i and $k_i (i = 1,2)$ are nonnegative real constants satisfying $b_i + k_i \neq 0\,, (i = 1,2)$. We can first find the solution with $f = \varphi = 0$, i.e.

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx}\,, & (0,l)\times(0,+\infty)\,, \\ -b_1 u_x(0,t) + k_1 u(0,t) = 0\,,\ b_2 u_x(l,t) + k_2 u(l,t) = 0\,, \\ u(x,0) = 0\,,\ u_t(x,0) = \psi(x)\,. \end{cases} \tag{6.40}$$

Once the solution $W_\psi(x,t)$ of PDS (6.40) is available, we can easily write out the solution of PDS (6.39) by using the solution structure theorem.

6.3.1 Eigenvalue Problems

Consider the nontrivial solution of separation of variables of PDS (6.40)

$$u = X(x)T(t)\,,$$

where $X(x)$ and $T(t)$ are functions of the only variables present to be determined. Substituting this into the equation of PDS (6.40) yields

$$\frac{1}{\tau_0}X(x)T'(t) + X(x)T''(t) = A^2 X'' T(t) + B^2 X'' T'(t)\,,$$

or, by separation of variables,

$$\frac{\frac{1}{\tau_0}T'(t)+T''(t)}{A^2T(t)+B^2T'(t)}=\frac{X''(x)}{X(x)},$$

where the primes on the function X and T represent differentiation with respect to the only variable present. Therefore, we obtain the separation equation for the temporal part $T(t)$ with $-\lambda$ as the separation constant,

$$T''(t)+\left(\frac{1}{\tau_0}+\lambda B^2\right)T'(t)+\lambda A^2T(t)=0 \tag{6.41}$$

and the homogeneous system for the spatial part $X(x)$

$$\begin{cases} X''(x)+\lambda X(x)=0\,, \\ -b_1X'(0)+k_1X(0)=0,\, b_2X'(l)+k_2X(l)=0\,. \end{cases} \tag{6.42}$$

The problem (6.42) is called an *eigenvalue problem* because it has solutions only for certain values of the separation constant $\lambda=\lambda_k$, $k=1,2,3,\cdots$, which are called the *eigenvalues*; the corresponding solutions $X_k(x)$ are called the *eigenfunctions* of the problem. The eigenvalue problem (6.42) is a special case of a more general eigenvalue problem called the Sturm-Liouville problem, which is discussed in Appendix D.

The equation in the eigenvalue problem generally depends on the equation of the original PDS. If after substituting $u(x,t)=X(x)T(t)$, it is impossible to have an equation with terms involving x and t on two separate sides, then we cannot have an eigenvalue problem.

6.3.2 Eigenvalues and Eigenfunctions

The boundary condition at $x=0$ in Eq. (6.42) contains three cases: $b_1=0$, $k_1\neq 0$; $b_1\neq 0$, $k_1=0$; $k_1\neq 0$, $b_1\neq 0$. Similarly, there also exist three kinds of boundary condition at $x=l$. Hence the eigenvalue problem (6.42) is a general form encompassing nine problems, each corresponding to nine combinations of boundary conditions. We consider the case of all nonzero b_1, k_1, b_2 and k_2, i.e. the problem

$$\begin{cases} X''(x)+\lambda X(x)=0\,, \\ X'(0)-h_1X(0)=0\,,\, X'(l)+h_2X(l)=0\,, \\ h_1=k_1/b_1\,, h_2=k_2/b_2\,. \end{cases} \tag{6.43}$$

If $\lambda=0$, its general solution is $X(x)=c_1x+c_2$. Applying the boundary conditions yields

$$\begin{cases} c_1 - h_1 c_2 = 0\,, \\ c_1(1+h_2 l) + h_2 c_2 = 0\,. \end{cases}$$

Its solution is $c_1 = c_2 = 0$, because $\Delta = \begin{vmatrix} 1 & -h_1 \\ 1+h_2 l & h_2 \end{vmatrix} = h_2 + h_1(1+h_2 l) \neq 0$. We thus obtain the trivial solution $X(x) \equiv 0$; therefore λ cannot be zero.

If $\lambda < 0$, the general solution of the equation in (6.43) reads

$$X(x) = c_1\,\mathrm{e}^{-ax} + c_2\,\mathrm{e}^{ax}\,,$$

where $-a^2 = \lambda$ and $a > 0$. Applying the boundary conditions leads to

$$\begin{cases} -ac_1 + ac_2 = 0\,, \\ -a\mathrm{e}^{-la} + a\mathrm{e}^{la} = 0\,. \end{cases}$$

Its solution is $c_1 = c_2 = 0$, because $\Delta = a^2\left(\mathrm{e}^{-la} - \mathrm{e}^{la}\right) \neq 0$. We thus again have the trivial solution $X(x) \equiv 0$; therefore the eigenvalues of (6.43) must be positive.

For positive $\lambda = \beta^2 > 0$, the general solution of the equation in (6.43) reads

$$X(x) = c_1 \cos\beta x + c_2 \sin\beta x\,.$$

Applying the boundary conditions yields

$$\begin{cases} \beta c_2 - h_1 c_1 = 0\,, \text{or}\, c_2 = h_1 c_1/\beta \\ -c_1\beta\sin\beta l + c_2\beta\cos\beta l + h_2 c_1\cos\beta l + h_2 c_2 \sin\beta l = 0\,, \end{cases}$$

so that $c_1 \neq 0$, $c_2 \neq 0$ to have a nontrivial solution. Its solution is, by noting that h_1 and h_2 are physically positive values,

$$\cos\beta l = \frac{1}{h_1+h_2}\left(\beta - \frac{h_1 h_2}{\beta}\right) = \frac{1}{(h_1+h_2)\,l}\left(\beta l - \frac{l^2 h_1 h_2}{\beta l}\right).$$

Let

$$f(x) = \cot x - \frac{1}{l(h_1+h_2)}\left(x - \frac{l^2 h_1 h_2}{x}\right), \tag{6.44}$$

βl thus represent the zero points of $f(x)$. Since $f(x)$ is an odd function and $\lambda = \beta^2$, we wish to find the positive zero points of $f(x)$ only. Let μ_m be the m-th positive zero point of $f(x)$. Hence, we have eigenvalues

$$\lambda_m = \beta_m^2 = \left(\frac{\mu_m}{l}\right)^2, \quad m = 1, 2, \ldots.$$

Without taking account of the arbitrary constant c_2, the corresponding eigenfunctions can be written as

$$X_m(x) = \frac{\mu_m}{lh_1}\cos\frac{\mu_m x}{l} + \sin\frac{\mu_m x}{l} = \sqrt{1+\left(\frac{\mu_m}{lh_1}\right)^2}\sin\left(\frac{\mu_m x}{l}+\varphi_m\right),$$

where $\tan\varphi_m = \dfrac{\mu_m}{lh_1}$.

Finally, we have eigenvalues λ_m and eigenfunctions $X_m(x)$

Eigenvalues $\lambda_m = \left(\dfrac{\mu_m}{l}\right)^2$, μ_m are the positive zero points of $f(x)$ in Eq. (6.44).

Eigenfunctions $X_m(x) = \sin\left(\dfrac{\mu_m x}{l}+\varphi_m\right)$, $\tan\varphi_m = \dfrac{\mu_m}{lh_1}$. Normal square of eigenfunction set

$$\begin{aligned}
\|X_m(x)\|^2 = (X_m(x), X_m(x)) &= \int_0^l \sin^2\left(\frac{\mu_m x}{l}+\varphi_m\right)\mathrm{d}x \\
&= \frac{1}{\beta_m}\int_0^l \sin^2(\beta_m x+\varphi_m)\,\mathrm{d}(\beta_m x+\varphi_m) \\
&= \frac{1}{\beta_m}\int_0^l \frac{1-\cos 2(\beta_m x+\varphi_m)}{2}\,\mathrm{d}(\beta_m x+\varphi_m) \\
&= \frac{1}{\beta_m}\left[\frac{1}{2}(\beta_m x+\varphi_m) - \frac{1}{4}\sin 2(\beta_m x+\varphi_m)\right]\Bigg|_0^l \\
&= \frac{1}{\beta_m}\left[\frac{1}{2}l\beta_m - \frac{1}{4}\sin 2(l\beta_m+\varphi_m) + \frac{1}{4}\sin 2\varphi_m\right] \\
&= \frac{l}{2}\left[1 - \frac{1}{2\mu_m}[\sin 2(\mu_m+\varphi_m) - \sin 2\varphi_m]\right] \\
&= \frac{l}{2}\left[1 - \frac{\sin\mu_m}{\mu_m}\cos(\mu_m+2\varphi_m)\right].
\end{aligned} \tag{6.45}$$

Remark 1. Eigenvalues and eigenfunctions of the other eight combinations can also be obtained using a similar approach. The results for all nine combinations are listed in Table 2.1. Therefore we can use Table 2.1 to solve the mixed problems of dual-phase-lagging heat-conduction equations subject to different boundary conditions.

Remark 2. We can also apply integration by parts to prove that the eigenvalues of Eq. (6.43) cannot be negative. Integrating the equation in Eq. (6.43) from $x=0$ to $x=l$ after multiplying $X(x)$ yields $\int_0^l X(x)\left[X''(x)+\lambda X(x)\right]\mathrm{d}x = 0$, which becomes, by making use of integration by parts, to $\int_0^l X(x)X''(x)\,\mathrm{d}x$,

$$\lambda\int_0^l X^2(x)\,\mathrm{d}x = -X(x)X'(x)\big|_0^l + \int_0^l \left[X'(x)\right]^2\mathrm{d}x.$$

It becomes, after applying the boundary conditions in Eq. (6.43),

$$\lambda\int_0^l X^2(x)\,\mathrm{d}x = h_1X^2(0) + h_2X^2(l) + \int_0^l \left[X'(x)\right]^2\mathrm{d}x,$$

which implies

$$\lambda = \begin{cases} > 0\,, \; h_1 + h_2 \neq 0\,, \\ 0\,, \quad\; h_1 + h_2 = 0\,. \end{cases}$$

However, the latter will lead to a trivial solution for PDS (6.43), which cannot in fact occur because physically $h_1 > 0$ and $h_2 > 0$ for boundary conditions of the third kind (if $h_1 = h_2 = 0$, the case reduces to the boundary conditions of the second kind at both ends).

6.3.3 Solve Mixed Problems with Table 2.1

We can use the eigenfunctions in Table 2.1 and the solution structure theorem to effectively solve mixed problems of dual-phase-lagging heat-conduction equations for nine combinations of boundary conditions.

Example 1. Solve

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx} + f(x,t)\,, \quad (0,l)\times(0,+\infty)\,, \\ u_x(0,t) = 0\,,\; u(l,t) = 0\,, \\ u(x,0) = \varphi(x)\,,\; u_t(x,0) = \psi(x)\,. \end{cases} \tag{6.46}$$

Solution. To obtain the solution of PDS (6.46), by the solution structure theorem we first seek its solution at $f(x,t) = 0$ and $\varphi(x) = 0$. Based on the given boundary conditions, we should use the eigenfunctions in Row 4 in Table 2.1 to expand the solution,

$$u(x,t) = \sum_{m=0}^{+\infty} T_m(t) \cos\frac{(2m+1)\pi x}{2l}\,.$$

Substituting it into the equation yields

$$\sum_{m=0}^{+\infty}\left[T_m''(t) + \left(\frac{1}{\tau_0} + \left(\frac{(2m+1)\pi B}{2l}\right)^2\right) T_m'(t) + \left(\frac{(2m+1)\pi A}{2l}\right)^2 T_m(t)\right]\cos\frac{(2m+1)\pi x}{2l} = 0\,,$$

which leads to, by the completeness and the orthogonality of $\left\{\cos\frac{(2m+1)\pi x}{2l}\right\}$,

$$T_m''(t) + \left(\frac{1}{\tau_0} + \left(\frac{(2m+1)\pi B}{2l}\right)^2\right) T_m'(t) + \left(\frac{(2m+1)\pi A}{2l}\right)^2 T_m(t) = 0\,.$$

Thus

$$u(x,t) = \sum_{m=0}^{+\infty} \mathrm{e}^{\alpha_m t} \left(A_m \cos\beta_m t + B_m \underline{\sin}\beta_m t\right) \cos\frac{(2m+1)\pi x}{2l},$$

$$\begin{aligned} u_t(x,t) = \sum_{m=0}^{+\infty} \{&\alpha_m \mathrm{e}^{\alpha_m t} (A_m \cos\beta_m t + B_m \underline{\sin}\beta_m t) \\ &+ \mathrm{e}^{\alpha_m t}(-A_m\beta_m \sin\beta_m t + B_m \underline{\beta}_m \cos\beta_m t)\} \cos\frac{(2m+1)\pi x}{2l} \end{aligned}$$

where

$$\alpha_m = -\frac{1}{2}\left(\frac{1}{\tau_0} + \lambda_m B^2\right),\ \beta_m = \frac{1}{2}\sqrt{4\lambda_m A^2 - \left(\frac{1}{\tau_0} + \lambda_m B^2\right)^2},\ \lambda_m = \left(\frac{(m+\frac{1}{2})\pi}{l}\right)^2.$$

Applying the initial condition $u(x,0) = 0$ yields $A_m = 0$. To satisfy the initial condition $u_t(x,0) = \varphi(x)$, B_m must be determined such that

$$\sum_{m=0}^{+\infty} B_m \underline{\beta}_m \cos\frac{(2m+1)\pi x}{2l} = \psi(x).$$

Thus

$$B_m = \frac{1}{M_m \underline{\beta}_m} \int_0^l \psi(x) \cos\frac{(2m+1)\pi x}{2l}\,\mathrm{d}x,$$

where $M_m = \frac{l}{2}$ is the normal square of $\left\{\cos\frac{(2m+1)\pi x}{2l}\right\}$.

Finally, we have

$$\begin{cases} u(x,t) = W_\psi(x,t) = \displaystyle\sum_{m=0}^{+\infty} B_m \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t \cos\frac{(2m+1)\pi x}{2l}, \\ B_m = \dfrac{1}{M_m \beta_m} \int_0^l \psi(x) \cos\dfrac{(2m+1)\pi x}{2l}\,\mathrm{d}x \end{cases} \tag{6.47}$$

and the solution of PDS (6.46), by the solution structure theorem, is

$$u(x,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,t) + W_{\psi - B^2\varphi''}(x,t) + \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau.$$

Remark 3. $u(x,t) = W_\psi(x,t)$ actually enjoys a very elegant structure. We may use this structure and Table 2.1 to write out $W_\psi(x,t)$ directly. Let λ_m and $X_m(x)$ be the eigenvalues and eigenfunctions from Table 2.1 based on given boundary conditions. The structure of $W_\psi(x,t)$ is thus

$$\begin{cases} u(x,t) = W_\psi(x,t) = \displaystyle\sum_{m=0\,\mathrm{or}\,1}^{+\infty} B_m \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t \cdot X_m(x), \\ B_m = \dfrac{1}{M_m \underline{\beta}_m} \displaystyle\int_0^l \psi(x) X_m(x)\,\mathrm{d}x. \end{cases} \tag{6.48}$$

where $\alpha_m = -\frac{1}{2}\left(\frac{1}{\tau_0}+\lambda_m B^2\right)$, $\beta_m = \frac{1}{2}\sqrt{4\lambda_m A^2 - \left(\frac{1}{\tau_0}+\lambda_m B^2\right)^2}$. λ_m and the normal square M_m are available in Table 2.1.

Example 2. Solve

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx} + f(x,t), \quad (0,l)\times(0,+\infty), \\ u_x(0,t) = 0, \ u_x(l,t) + hu(l,t) = 0, \\ u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x). \end{cases} \tag{6.49}$$

Solution. Based on the given boundary conditions, eigenvalues, eigenfunctions and the normal square must be those in Row 6 in Table 2.1, i.e.

$$\lambda_m = \left(\frac{\mu_m}{l}\right)^2, \ \mu_m \text{ is the } m\text{-th positive zero point of } f(x) = \cot x - \frac{x}{lh}.$$
$$X_m(x) = \cos\frac{\mu_m x}{l}, M_m = \frac{l}{2}\left(1+\frac{\sin 2\mu_m}{2\mu_m}\right).$$

Thus, we have

$$\begin{cases} u(x,t) = W_\psi(x,t) = \sum_{m=1}^{+\infty} B_m e^{\alpha_m t}\underline{\sin}\beta_m t\cos\frac{\mu_m x}{l}, \\ B_m = \frac{1}{M_m \underline{\beta}_m}\int_0^l \psi(x)\cos\frac{\mu_m x}{l}\,dx \end{cases}$$

and the solution of PDS (6.49)

$$u(x,t) = \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(x,t) + W_{\psi - B^2\varphi''}(x,t) + \int_0^t W_{f_\tau}(x,t-\tau)\,d\tau.$$

Example 3. Solve

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx} + f(x,t), \quad (0,l)\times(0,+\infty), \\ u_x(0,t) - hu(0,t) = 0, \ u_x(l,t) + hu(l,t) = 0, \\ u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x). \end{cases} \tag{6.50}$$

Solution. The given boundary condition is that in Row 9 in Table 2.1 with $h_1 = h_2 = h$. Therefore,

$$\begin{cases} u(x,t) = W_\psi(x,t) = \sum_{m=1}^{+\infty} B_m e^{\alpha_m t}\underline{\sin}\beta_m t\sin\left(\frac{\mu_m x}{l}+\varphi_m\right), \\ B_m = \frac{1}{M_m \underline{\beta}_m}\int_0^l \psi(x)\sin\left(\frac{\mu_m x}{l}+\varphi_m\right)dx, \end{cases} \tag{6.51}$$

where $M_m = \frac{l}{2}\left[1 - \frac{\sin\mu_m}{\mu_m}\cos(\mu_m + 2\varphi_m)\right]$. Also, the solution of PDS (6.50) is, by the solution structure theorem,

$$u(x,t)=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(x,t)+W_{\psi-B^2\varphi''}(x,t)+\int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau$$

Remark 4.

1. The structure of $W_\psi(x,t)$ is invariant with the combination of boundary conditions. The detailed equations of λ_m, $X_m(x)$, and β_m depend, however, on the boundary conditions. The μ_m also have different meanings in Rows 3, 6 and 9 of Table 2.1.
2. We may follow the process in Section 6.2.2 to obtain the final detailed series solution if so required. Take the solution in Example 3 as an example. We can obtain detailed expressions of $W_\varphi(x,t)$, $W_{-B^2\varphi''}(x,t)$ and $W_{f_\tau}(x,t)$ simply through replacing $\psi(x)$ in coefficients B_m of (6.51) by $\varphi(x)\,,-B^2\varphi''(x)$ and $f(x,\tau)$, respectively. Also, $W_{\psi-B^2\varphi''}(x,t)=W_\psi(x,t)-B^2W_{\varphi''}(x,t)$.
3. The solution structure theorem developed in Section 6.1 is valid only for well-posed CDS. The $\varphi(x)$ and $\psi(x)$ must thus satisfy consistency conditions. In PDS (6.50), for example, the boundary conditions must be satisfied for all $t\geq 0$. At $t=0$, we have $\varphi'(0)-h\varphi(0)=\varphi'(l)+h\varphi(l)=0$. By taking the derivative of boundary conditions with respect to t, we arrive at $u_{xt}(0,t)-hu_t(0,t)=0\,,u_{xt}(l,t)+hu_t(l,t)=0$, so $\psi(x)$ must satisfy $\psi'(0)-h\psi(0)=\psi'(l)+h\psi(l)=0$.
4. If β_m is purely imaginary for some m, we can change $\sin\beta_m t$ into $\dfrac{e^{i\beta_m t}-e^{-i\beta_m t}}{2i}$ by using the formula $\sin z=\dfrac{e^{iz}-e^{-iz}}{2i}$ for any imaginary z.
5. The general term of the series solution decays very quickly toward zero. This facilitates its applications of taking only the first few terms and is deduced from the following observations: (a) the appearance of β_m in the denominator B_m of which increases very quickly with m ($\beta_m=O(m^2)$ as $m\to+\infty$); (b) the integral $\int_0^l\psi(x)X_m(x)\,\mathrm{d}x$ in Eq. (6.48) tends to zero, by the Riemann Lemma, as $m\to+\infty$ because the $X_m(x)$ are either the sin or cos functions; (c) the appearance of $e^{\alpha_m t}$ where $\alpha_m<0$ and $\alpha_m=O(m^2)$ as $m\to+\infty$, and (d) the $\dfrac{1}{\underline{\beta}_m}e^{\alpha_m t}\underline{\sin}\beta_m t$ in the general term. Depending on the characteristic roots of Eq. (6.41), we have

$$\frac{1}{\underline{\beta}_m}e^{\alpha_m t}\underline{\sin}\beta_m t=\begin{cases}\frac{1}{\beta_m}e^{\alpha_m t}\sin\beta_m t\,, & \text{when }\Delta<0\,,\\ t\,e^{\alpha_m t}\,, & \text{when }\Delta=0\,,\\ \frac{1}{2\gamma_m}\left(e^{-r_1 t}-e^{-r_2 t}\right), & \text{when }\Delta>0\,.\end{cases}$$

where

$$\Delta = \sqrt{\left(\frac{1}{\tau_0} + \lambda_m B^2\right)^2 - 4\lambda_m A^2}\,, \; \gamma_m = \frac{1}{2}\left(\frac{1}{\tau_0} + \lambda_m B^2\right)\sqrt{1 - \frac{4\lambda_m A^2}{\frac{1}{\tau_0} + \lambda_m B^2}}\,,$$
$$r_1 = \alpha_m + \gamma_m < 0\,, \quad r_2 = \alpha_m - \gamma_m < 0\,.$$

Note that $|\sin\beta_m t| \leq 1$ and $\gamma_m = O(m^2)$ as $m \to +\infty$. The $\frac{1}{\underline{\beta}_m} \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t$ thus decays quickly for all cases.

6.4 Solution Structure Theorem: Another Form and Application

The solution structure theorem developed in Section 6.1 is valid only for well-posed PDS. It requires that initial values $\varphi(x)$ and $\psi(x)$ must satisfy consistency conditions. This limits its applications because we only need nominal solutions for applications so that the given $\varphi(x)$ and $\psi(x)$ do not normally satisfy the consistency conditions. In this section we improve the solution structure theorem in Cartesian coordinates to find the structural relations among nominal solutions due to φ, ψ and f. We also use examples to show some applications of the modified solution structure theorem.

6.4.1 One-Dimensional Mixed Problems

Let Ω be an one-dimensional region: $0 < x < l$. Its boundary $\partial\Omega$ thus consists of the two ends point $x = 0$ and $x = l$. Nine combinations of boundary conditions can be written in a general form as

$$L(u, u_n)|_{\partial\Omega} = 0\,,$$

where the u_n are $\pm u_x$. The corresponding eigenvalues and the eigenfunction set are denoted by λ_m and $\{X_m(x)\}$, respectively.

Theorem 1. Let $u(x,t) = W_\psi(x,t)$ be the solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx}\,, & (0,l) \times (0,+\infty)\,, \\ L(u, u_n)|_{\partial\Omega} = 0\,, \\ u(x,0) = 0\,, \quad u_t(x,0) = \psi(x)\,. \end{cases} \tag{6.52}$$

The solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx} + f(x,t)\,, & (0,l) \times (0,+\infty)\,, \\ L(u, u_n)|_{\partial\Omega} = 0\,, \\ u(x,0) = \varphi(x)\,, \quad u_t(x,0) = \psi(x) \end{cases} \tag{6.53}$$

is

$$u(x,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_{\varphi}(x,t) + B^2 W_{\lambda_m \varphi}(x,t) \tag{6.54}$$

$$+ W_{\psi}(x,t) + \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau\,, \tag{6.55}$$

where $f_\tau = f(x,\tau)$, λ_m are the eigenvalues in Table 2.1 corresponding to the boundary conditions in PDS (6.53).

Proof. By Eq. (6.13) in Section 6.1, we only need to prove that the solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx}\,, \quad (0,l)\times(0,+\infty)\,, \\ L(u,u_n)|_{\partial\Omega} = 0\,, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = 0 \end{cases} \tag{6.56}$$

is

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_{\varphi}(x,t) + B^2 W_{\lambda_m \varphi}(x,t)\,. \tag{6.57}$$

Let the solution of PDS (6.52) be

$$u = \sum_m T_m(t) X_m(x)\,,$$

where $\sum\limits_m$ stands for $\sum\limits_{m=0}^{\infty}$ or $\sum\limits_{m=1}^{\infty}$ depending on $\{X_m(x)\}$. Substituting this into the equation in PDS (6.52) yields the $T_m(t)$-equation

$$T''(t) + \left(\frac{1}{\tau_0} + \lambda_m B^2\right) T_m'(t) + \lambda_m A^2 t_m(t) = 0\,.$$

Its two characteristic roots are

$$r_{1,2} = \frac{1}{2}\left[-\left(\frac{1}{\tau_0} + \lambda_m B^2\right) \pm \sqrt{\left(\frac{1}{\tau_0} + \lambda_m B^2\right)^2 - 4\lambda_m A^2}\right]$$

$$= \alpha_m \pm \beta_m \mathrm{i}\,.$$

Thus the solution of PDS (6.52) reads

$$u(x,t) = \sum_m \mathrm{e}^{\alpha_m t}\,(A_m \cos\beta_m t + B_m \underline{\sin}\beta_m t)\, X_m(x)\,, \tag{6.58}$$

where A_m and B_m are both constants. Applying the initial condition $u(x,0) = 0$ yields $A_m = 0$. Clearly we can differentiate the series (6.58) term by term. The B_m can be determined by the initial condition $u_t(x,0) = \psi(x)$. Thus we have

$$\begin{cases} u(x,t) = W_\psi(x,t) = \sum\limits_m B_m \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t \cdot X_m(x)\,, \\ B_m = \dfrac{1}{M_m \underline{\beta}_m} \displaystyle\int_0^l \psi(x) X_m(x)\,\mathrm{d}x\,, \end{cases} \tag{6.59}$$

where M_m is the normal square of $\{X_m(x)\}$ available in Table 2.1, i.e. $M_m = (X_m(x), X_m(x))$.

Similarly, consider the solution of PDS (6.56)

$$u(x,t) = \sum_m \mathrm{e}^{\alpha_m t} \left(C_m \cos\beta_m t + D_m \underline{\sin}\beta_m t\right) X_m(x)\,. \tag{6.60}$$

Thus

$$\begin{aligned} u_t(x,t) = \sum_m \mathrm{e}^{\alpha_m t} [&\alpha_m (C_m \cos\beta_m t + D_m \underline{\sin}\beta_m t) \\ &+ (-C_m \beta_m \sin\beta_m t + D_m \underline{\beta}_m \cos\beta_m t)] X_m(x)\,. \end{aligned}$$

To satisfy the two initial conditions the C_m and the D_m must be determined such that

$$C_m = \frac{1}{M_m} \int_0^l \varphi(x) X_m(x)\,\mathrm{d}x\,, \quad \alpha_m C_m + D_m \underline{\beta}_m = 0\,.$$

Thus

$$D_m = -\frac{\alpha_m}{\beta_m} C_m = \frac{\frac{1}{\tau_0} + \lambda_m B^2}{2} \frac{1}{M_m \underline{\beta}_m} \int_0^l \varphi(x) X_m(x)\,\mathrm{d}x\,. \tag{6.61}$$

By Eq. (6.59), we have

$$\begin{cases} \dfrac{\partial}{\partial t} W_\varphi(x,t) = \sum\limits_m D_m^* \left(\alpha_m \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t + \mathrm{e}^{\alpha_m t} \underline{\beta}_m \cos\beta_m t\right) X_m(x) \\ D_m^* = \dfrac{1}{M_m \underline{\beta}_m} \displaystyle\int_0^l \varphi(x) X_m(x)\,\mathrm{d}x\,. \end{cases}$$

Thus

$$\sum_m C_m \mathrm{e}^{\alpha_m t} \cos\beta_m t \cdot X_m(x) = \frac{\partial}{\partial t} W_\varphi(x,t)$$

$$-\sum_m \left(\frac{\alpha_m}{M_m \underline{\beta}_m} \int_0^l \varphi(x) X_m(x)\,\mathrm{d}x \right) \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t \cdot X_m(x)$$

$$= \frac{\partial}{\partial t} W_\varphi(x,t) + \sum_m \left[\frac{\frac{1}{\tau_0} + \lambda_m B^2}{2} \frac{1}{M_m \underline{\beta}_m} \right.$$

$$\left. \cdot \int_0^l \varphi(x) X_m(x)\,\mathrm{d}x \right] \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t \cdot X_m(x) . \tag{6.62}$$

Substituting Eqs. (6.61) and (6.62) into Eq. (6.60) leads to the solution of PDS (6.56), by using the structure of $W_\psi(x,t)$ in Eq. (6.59),

$$u(x,t) = \frac{\partial W_\varphi}{\partial t} + \sum_m \left[\frac{\frac{1}{\tau_0} + \lambda_m B^2}{2} \frac{1}{M_m \underline{\beta}_m} \int_0^l \varphi(x) X_m(x)\,\mathrm{d}x \right] \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t \cdot X_m(x)$$

$$+\sum_m \left[\frac{\frac{1}{\tau_0} + \lambda_m B^2}{2} \frac{1}{M_m \underline{\beta}_m} \int_0^l \varphi(x) X_m(x)\,\mathrm{d}x \right] \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t \cdot X_m(x)$$

$$= \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t} \right) W_\varphi(x,t)$$

$$+ B^2 \sum_m \lambda_m \left[\left(\frac{1}{M_m \underline{\beta}_m} \int_0^l \varphi(x) X_m(x)\,\mathrm{d}x \right) \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t \cdot X_m(x) \right]$$

$$= \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t} \right) W_\varphi(x,t) + B^2 W_{\lambda_m \varphi}(x,t) .$$

Example 1. Solve

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + B^2 u_{txx} + f(x,t), & (0,l)\times(0,+\infty), \\ u_x(0,t) = 0,\ u(l,t) = 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) . \end{cases} \tag{6.63}$$

Solution. Based on the given boundary conditions, we have, from Row 4 in Table 2.1,

$$\lambda_m = \left(\frac{(2m+1)\pi}{2l} \right)^2, \quad X_m(x) = \cos\frac{(2m+1)\pi x}{2l}, \quad M_m = (X_m(x), X_m(x)) = \frac{l}{2} .$$

Thus the solution for the case of $f = \varphi = 0$ reads, by Eq. (6.59),

$$\begin{cases} u(x,t) = W_\psi(x,t) = \sum_{m=0}^{+\infty} B_m \mathrm{e}^{\alpha_m t} \underline{\sin}\beta_m t \cdot \cos\dfrac{(2m+1)\pi x}{2l}, \\ B_m = \dfrac{2}{l\underline{\beta}_m} \displaystyle\int_0^l \psi(x)\cos\dfrac{(2m+1)\pi x}{2l}\,\mathrm{d}x. \end{cases}$$

where

$$\alpha_m = -\frac{1}{2}\left(\frac{1}{\tau_0} + \lambda_m B^2\right), \quad \beta_m = \frac{1}{2}\sqrt{4\lambda_m A^2 - \left(\frac{1}{\tau_0} + \lambda_m B^2\right)^2}.$$

Finally, the solution of PDS (6.63) is , by the solution structure theorem,

$$u(x,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,t) + B^2 W_{\lambda_m\varphi}(x,t) + W_\psi(x,t) + \int_0^t W_{f_\tau}(x,t-\tau)\,\mathrm{d}\tau,$$

where $f_\tau = f(x,\tau)$.

6.4.2 Two-Dimensional Mixed Problems

Let D be the rectangular domain: $0 < x < l_1$, $0 < y < l_2$; its boundary ∂D consists of four boundary lines. 81 combinations of linear homogeneous boundary conditions can be written in a general form as

$$L(u,u_n)|_{\partial D} = 0,$$

where the u_n, the normal derivative, are $\pm u_x$ or $\pm u_y$. The corresponding eigenvalues and the eigenfunction set are denoted by λ_m, $X_m(x)$; λ_n, $Y_n(y)$, respectively.

Theorem 2. Let $u = W_\psi(x,y,t)$ be the solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2\Delta_2 u + B^2\dfrac{\partial}{\partial t}\Delta_2 u, \\ \qquad\qquad D\times(0,+\infty), \\ L(u,u_n)|_{\partial D} = 0, \\ u(x,y,0) = 0, \qquad u_t(x,y,0) = \psi(x,y). \end{cases} \tag{6.64}$$

The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta_2 u + B^2 \dfrac{\partial}{\partial t}\Delta_2 u + f(x,y,t)\,, \\ \hfill D\times(0,+\infty)\,, \\ L(u,u_n)|_{\partial D} = 0\,, \\ u(x,y,0) = \varphi(x,y), \quad u_t(x,y,0) = \psi(x,y) \end{cases} \tag{6.65}$$

is

$$u = \left[\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right] W_{\varphi}(x,y,t) + B^2 W_{(\lambda_m+\lambda_n)\varphi}(x,y,t) \tag{6.66}$$

$$+ W_{\psi}(x,y,t) + \int_0^t W_{f_\tau}(x,y,t-\tau)\,\mathrm{d}\tau \tag{6.67}$$

Proof. By Theorem 2 in Section 6.1, we only need to prove that the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta_2 u + B^2 \dfrac{\partial}{\partial t}\Delta_2 u\,, \\ \hfill D\times(0,+\infty)\,, \\ L(u,u_n)|_{\partial D} = 0\,, \\ u(x,y,0) = \varphi(x,y)\,,\ u_t(x,y,0) = 0 \end{cases} \tag{6.68}$$

is

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_{\varphi}(x,y,t) + B^2 W_{(\lambda_m+\lambda_n)\varphi}(x,y,t)\,. \tag{6.69}$$

From the given boundary conditions in PDS (6.64), we may obtain, from Table 2.1, the complete and orthogonal set of eigenfunctions and their corresponding eigenvalues

$$\lambda_m\,, X_m(x)\,;\, \lambda_n\,, Y_n(y)\,.$$

Consider now the solution of PDS (6.64) of the form

$$u = \sum_{m,n} T_{mn}(t) X_m(x) Y_n(y)\,. \tag{6.70}$$

Substituting it into the equation in (6.64) leads to the $T_{mn}(t)$-equation

$$T_{mn}''(t) + \left[\frac{1}{\tau_0} + (\lambda_m+\lambda_n)B^2\right] T_{mn}'(t) + (\lambda_m+\lambda_n)A^2 T_{mn}(t) = 0\,.$$

Its characteristic roots are $r_{1,2} = \alpha_{mn} \pm \beta_{mn}\mathrm{i}$, where

$$\alpha_{mn} = -\frac{1}{2}\left[\frac{1}{\tau_0} + (\lambda_m + \lambda_n)B^2\right], \quad \beta_{mn} = \frac{1}{2}\sqrt{4(\lambda_m + \lambda_n)A^2 - \left[\frac{1}{\tau_0} + (\lambda_m + \lambda_n)B^2\right]^2}.$$

Thus the equation (6.70) becomes

$$u(x,y,t) = \sum_{m,n} \mathrm{e}^{\alpha_{mn}t}\,(A_{mn}\cos\beta_{mn}t + B_{mn}\underline{\sin}\beta_{mn}t)X_m(x)Y_n(y) \tag{6.71}$$

where $\underline{\sin}\beta_{mn}t = \begin{cases} \sin\beta_{mn}t\,, & \text{when } \beta_{mn} \neq 0\,, \\ t\,, & \text{when } \beta_{mn} = 0\,. \end{cases}$ A_m and B_m are constants to be determined from the initial conditions. Applying the initial condition $u(x,y,0) = 0$ yields $A_{mn} = 0$. Also

$$u_t(x,y,t) = \sum_{m,n} B_{mn}\,\mathrm{e}^{\alpha_{mn}t}\,(\alpha_{mn}\underline{\sin}\beta_{mn}t + \beta_{mn}\cos\beta_{mn}t)X_m(x)Y_n(y)\,,$$

where

$$\underline{\beta}_{mn} = \begin{cases} \beta_{mn}\,, & \text{when } \beta_{mn} \neq 0\,, \\ 1\,, & \text{when } \beta_{mn} = 0\,. \end{cases}$$

To satisfy the initial condition $u_t(x,y,0) = \psi(x,y)$, B_{mn} must be determined such that

$$\sum_{m,n} B_{mn}\underline{\beta}_{mn}X_m(x)Y_n(y) = \psi(x,y)\,.$$

Thus we obtain $W_\psi(x,y,t)$, the solution of PDS (6.64)

$$\begin{cases} u = W_\psi(x,y,t) = \sum\limits_{m,n} B_{mn}\,\mathrm{e}^{\alpha_{mn}t}\,\underline{\sin}\beta_{mn}t \cdot X_m(x)Y_n(y)\,, \\ B_{mn} = \dfrac{1}{M_{mn}\underline{\beta}_{mn}} \iint\limits_D \psi(x,y)X_m(x)Y_n(y)\,\mathrm{d}x\,\mathrm{d}y\,. \end{cases} \tag{6.72}$$

Similarly, the solution of PDS (6.68) can be written as

$$u = \sum_{mn} \mathrm{e}^{\alpha_{mn}t}\,(C_{mn}\cos\beta_{mn}t + D_{mn}\underline{\sin}\beta_{mn}t)\,X_m(x)Y_n(y)\,. \tag{6.73}$$

Applying the initial condition $u(x,y,0) = \varphi(x,y)$ yields

$$\sum_{m,n} C_{mn}X_m(x)Y_n(y) = \varphi(x,y)\,.$$

Thus

$$C_{mn} = \frac{1}{M_{mn}} \iint\limits_D \varphi(x,y) X_m(x) Y_n(y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Also,

$$u_t(x,y,t) = \sum_{m,n} \mathrm{e}^{\alpha_{mn}t} \Big[\alpha_{mn}\left(C_{mn}\cos\beta_{mn}t + D_{mn}\underline{\sin}\beta_{mn}t\right)$$
$$+ \Big(-C_{mn}\beta_{mn}\sin\beta_{mn}t + D_{mn}\underline{\beta}_{mn}\cos\beta_{mn}t \Big) \Big] X_m(x)Y_n(y)\,.$$

Applying the initial condition $u_t(x,y,0) = 0$ yields

$$\alpha_{mn}C_{mn} + D_{mn}\underline{\beta}_{mn} = 0\,,$$

which leads to

$$\begin{aligned} D_{mn} &= -\frac{\alpha_{mn}}{\underline{\beta}_{mn}} C_{mn} \\ &= \frac{\frac{1}{\tau_0} + (\lambda_m + \lambda_n)B^2}{2} \frac{1}{M_{mn}\underline{\beta}_{mn}} \iint\limits_D \varphi(x,y)X_m(x)Y_n(y)\,\mathrm{d}x\,\mathrm{d}y\,. \end{aligned} \tag{6.74}$$

By Eq. (6.72), we have

$$\begin{cases} \dfrac{\partial}{\partial t} W_\varphi(x,y,t) = \sum\limits_{m,n} D^*_{mn}\, \mathrm{e}^{\alpha_{mn}t}\Big(\alpha_{mn}\underline{\sin}\beta_{mn}t + \underline{\beta}_{mn}\cos\beta_{mn}t\Big) X_m(x)Y_n(y)\,, \\ D^*_{mn} = \dfrac{1}{M_{mn}\underline{\beta}_{mn}} \iint\limits_D \varphi(x,y)X_m(x)Y_n(y)\,\mathrm{d}x\,\mathrm{d}y\,. \end{cases}$$

Thus

$$\begin{aligned} \sum_{m,n} C_{mn}\,\mathrm{e}^{\alpha_{mn}t}\cos\beta_{mn}t \cdot X_m(x)Y_n(y) &= \frac{\partial}{\partial t} W_\varphi(x,y,t) \\ &+ \sum_{m,n}\left(\frac{\frac{1}{\tau_0} + (\lambda_m+\lambda_n)B^2}{2} \frac{1}{M_{mn}\underline{\beta}_{mn}} \iint\limits_D \varphi(x,y)X_m(x)Y_n(y)\,\mathrm{d}x\,\mathrm{d}y \right) \\ &\cdot \mathrm{e}^{\alpha_{mn}t}\underline{\sin}\beta_{mn}t \cdot X_m(x)Y_n(y)\,. \end{aligned} \tag{6.75}$$

Substituting Eqs. (6.74) and (6.75) into Eq. (6.73) and using the structure of $W_\psi(x,y,t)$ in Eq. (6.72) leads to the solution of PDS (6.68)

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,y,t) + B^2 W_{(\lambda_m+\lambda_n)\varphi}(x,y,t)\,.$$

Example 2. Seek the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta_2 u + B^2 \dfrac{\partial}{\partial t} \Delta_2 u + f(x,y,t), \ \ 0 < x < l_1, \ \ 0 < y < l_2, 0 < t, \\ u_x(0,y,t) = 0, \, u_x(l_1,y,t) + hu(l_1,y,t) = 0, \\ u_y(x,0,t) = 0, \, u_y(x,l_2,t) = 0, \\ u(x,y,0) = \varphi(x,y), \, u_t(x,y,0) = \psi(x,y). \end{cases} \tag{6.76}$$

Solutions. Based on the given boundary conditions in PDS (6.76), we read, from Table 2.1, the eigenvalues and the eigenfunctions,

$$\lambda_m = \left(\frac{\mu_m}{l_1}\right)^2, \ \cos\frac{\mu_m x}{l_1}, \ \mu_m \text{ are the positive zero points of } f(x) = \cot x - \frac{x}{l_1 h},$$

$$\lambda_n = \left(\frac{n\pi}{l_2}\right)^2, \ \cos\frac{n\pi y}{l_2}, \ n = 0,1,2\cdots.$$

Also

$$M_{mn} = M_m M_n = \frac{l_1}{2}\left(1 + \frac{\sin 2\mu_m}{2\mu_m}\right)\frac{l_2}{2} = \frac{l_1 l_2}{4}\left(1 + \frac{\sin 2\mu_m}{2\mu_m}\right).$$

Therefore, by Eq. (6.72),

$$\begin{cases} u(x,y,t) = W_\psi(x,y,t) = \displaystyle\sum_{m=1,n=0}^{\infty} B_{mn}\, \mathrm{e}^{\alpha_{mn} t}\, \underline{\sin}\beta_{mn} t \cdot \cos\frac{\mu_m x}{l_1} \cos\frac{n\pi y}{l_2}, \\ B_{mn} = \dfrac{4}{l_1 l_2 \left(1 + \frac{\sin 2\mu_m}{2\mu_m}\right) \underline{\beta}_{mn}} \displaystyle\int_0^{l_1} \mathrm{d}x \int_0^{l_2} \psi(x,y) \cos\frac{\mu_m x}{l_1} \cos\frac{n\pi y}{l_2}\, \mathrm{d}y, \end{cases}$$

where

$$\alpha_{mn} = -\frac{1}{2}\left\{\frac{1}{\tau_0} + \left[\left(\frac{\mu_m}{l_1}\right)^2 + \left(\frac{n\pi}{l_2}\right)^2\right] B^2\right\},$$

$$\beta_{mn} = \frac{1}{2}\sqrt{4\left[\left(\frac{\mu_m}{l_1}\right)^2 + \left(\frac{n\pi}{l_2}\right)^2\right] A^2 - 4\alpha_{mn}^2}.$$

Finally, the solution of PDS (6.76) is, by the solution structure theorem,

$$u(x,y,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,y,t) + B^2 W_{(\lambda_m+\lambda_n)\varphi}(x,y,t)$$

$$+ W_\psi(x,y,t) + \int_0^t W_{f_\tau}(x,y,t-\tau)\,\mathrm{d}\tau\,,$$

where $\quad f_\tau = f(x,y,\tau)\,.$

6.4.3 Three-Dimensional Mixed Problems

Let Ω be the cubic region: $0 < x < l_1\,, 0 < y < l_2, 0 < z < l_3$. Its boundary $\partial\Omega$ thus consists of six boundary surfaces. 729 combinations of linear homogeneous boundary conditions can be written in a general form as

$$L(u,u_n)|_{\partial\Omega} = 0\,.$$

where the normal derivative u_n are $\pm u_x$, $\pm u_y$ and $\pm u_z$, respectively. The corresponding eigenvalues and eigenfunctions are available in Table 2.1 and are denoted by

$$\lambda_m\,,\ X_m(x)\,;\ \lambda_n\,,\ Y_n(y)\,;\ \lambda_k\,,\ Z_k(z)\,.$$

Theorem 3. Let $u = W_\psi(x,y,z,t)$ be the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + B^2\dfrac{\partial}{\partial t}\Delta u\,, \\ \qquad\qquad\qquad\qquad \Omega\times(0,+\infty)\,, \\ L(u,u_n)|_{\partial\Omega} = 0\,, \\ u(x,y,z,0) = 0\,, \quad u_t(x,y,z,0) = \psi(x,y,z)\,. \end{cases} \tag{6.77}$$

The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + B^2\dfrac{\partial}{\partial t}\Delta u + f(x,y,z,t)\,, \\ \qquad\qquad\qquad\qquad \Omega\times(0,+\infty)\,, \\ L(u,u_n)|_{\partial\Omega} = 0\,, \\ u(x,y,z,0) = \varphi(x,y,z),\ u_t(x,y,z,0) = \psi(x,y,z) \end{cases} \tag{6.78}$$

is

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_{\varphi}(x,y,z,t) + B^2 W_{(\lambda_m+\lambda_n+\lambda_k)\varphi}(x,y,z,t) \tag{6.79}$$

$$+ W_{\psi}(x,y,z,t) + \int_0^t W_{f_\tau}(x,y,z,t-\tau)\,\mathrm{d}\tau\,, \tag{6.80}$$

where $f_\tau = f(x,y,z,\tau)$.

Proof. By Theorem 2 in Section 6.1, we only need to prove that the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u + B^2 \dfrac{\partial}{\partial t} \Delta u\,, \\ \qquad\qquad\qquad \Omega \times (0,+\infty)\,, \\ L(u,u_n)|_{\partial\Omega} = 0\,, \\ u(x,y,z,0) = \varphi(x,y,z),\ u_t(x,y,z,0) = 0 \end{cases} \tag{6.81}$$

is

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_{\varphi}(x,y,z,t) + B^2 W_{(\lambda_m+\lambda_n+\lambda_k)\varphi}(x,y,z,t)\,. \tag{6.82}$$

Similar to the two-dimensional case, we expand the solution of PDS (6.77) into

$$u = \sum_{m,n,k} T_{mnk}(t) X_m(x) Y_n(y) Z_k(z)\,. \tag{6.83}$$

Substituting this into the equation yields the $T_{mnk}(t)$-equation

$$T''_{mnk}(t) + \left[\frac{1}{\tau_0} + (\lambda_m+\lambda_n+\lambda_k)B^2\right] T'_{mnk}(t) + (\lambda_m+\lambda_n+\lambda_k)A^2 T_{mnk}(t) = 0\,.$$

Its characteristic roots read

$$r_{1,2} = \alpha_{mnk} \pm \beta_{mnk}\mathrm{i}\,,$$

where

$$\alpha_{mnk} = -\frac{1}{2}\left[\frac{1}{\tau_0} + (\lambda_m+\lambda_n+\lambda_k)B^2\right],\ \beta_{mnk} = \frac{1}{2}\sqrt{4(\lambda_m+\lambda_n+\lambda_k)A^2 - 4\alpha_{mnk}^2}\,.$$

Thus the solution (6.83) becomes

$$u = \sum_{m,n,k} \mathrm{e}^{\alpha_{mnk}t}\left(A_{mnk}\cos\beta_{mnk}t + B_{mnk}\underline{\sin}\beta_{mnk}t\right) X_m(x) Y_n(y) Z_k(z)\,,$$

where

$$\underline{\sin}\beta_{mnk}t = \begin{cases} \sin\beta_{mnk}t\,, & \text{when } \beta_{mnk} \neq 0\,, \\ t\,, & \text{when } \beta_{mnk} = 0\,. \end{cases}$$

Applying the initial condition $u(x,y,z,0) = 0$ yields $A_{mnk} = 0$. Also

$$u_t(x,y,z,t) = \sum_{m,n,k} \mathrm{e}^{\alpha_{mnk}t}\,(\alpha_{mnk}B_{mnk}\underline{\sin}\beta_{mnk}t + B_{mnk}\underline{\beta}_{mnk}\cos\beta_{mnk}t)X_m(x)Y_n(y)Z_k(z)\,,$$

where

$$\underline{\beta}_{mnk} = \begin{cases} \beta_{mnk}\,, & \text{when } \beta_{mnk} \neq 0\,, \\ 1\,, & \text{when } \beta_{mnk} = 0\,. \end{cases}$$

B_{mnk} can be determined by applying the initial condition $u_t(x,y,z,0) = \psi(x,y,z)$. Finally, we have $W_\psi(x,y,z,t)$

$$\begin{cases} u = W_\psi(x,y,z,t) = \displaystyle\sum_{m,n,k} B_{mnk}\,\mathrm{e}^{\alpha_{mnk}t}\,\underline{\sin}\beta_{mnk}t \cdot X_m(x)Y_n(y)Z_k(z)\,, \\ B_{mnk} = \dfrac{1}{M_{mnk}\underline{\beta}_{mnk}} \displaystyle\iiint_\Omega \psi(x,y,z)X_m(x)Y_n(y)Z_k(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,, \end{cases} \tag{6.84}$$

where $M_{mnk} = M_m M_n M_k$.

Similarly, consider the solution of PDS (6.81) satisfying the equation and the boundary conditions

$$u(x,y,z,t) = \sum_{m,n,k} \mathrm{e}^{\alpha_{mnk}t}\,(C_{mnk}\cos\beta_{mnk}t + D_{mnk}\underline{\sin}\beta_{mnk}t)\,X_m(x)Y_n(y)Z_k(z)\,. \tag{6.85}$$

Applying the initial condition $u(x,y,z,0) = \varphi(x,y,z)$ yields

$$C_{mnk} = \frac{1}{M_{mnk}} \iiint_\Omega \varphi(x,y,z)X_m(x)Y_n(y)Z_k(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

Also,

$$\begin{aligned} u_t(x,y,t) = \sum_{m,n,k} \mathrm{e}^{\alpha_{mnk}t} \Big[&\alpha_{mnk}\,(C_{mnk}\cos\beta_{mnk}t + D_{mnk}\underline{\sin}\beta_{mnk}t) \\ &+ \Big(-C_{mnk}\beta_{mnk}\sin\beta_{mnk}t + D_{mnk}\underline{\beta}_{mnk}\cos\beta_{mnk}t\Big)\Big]\,X_m(x)Y_n(y)Z_k(z)\,. \end{aligned}$$

Applying the initial condition $u_t(x,y,z,0) = 0$ leads to

$$\alpha_{mnk}C_{mnk} + D_{mnk}\underline{\beta}_{mnk} = 0\,,$$

so that

$$D_{mnk} = -\frac{\alpha_{mnk}}{\underline{\beta}_{mnk}} C_{mnk} = \frac{\frac{1}{\tau_0} + (\lambda_m + \lambda_n + \lambda_k)B^2}{2} \cdot \frac{1}{M_{mnk}\underline{\beta}_{mnk}} \iiint\limits_{\Omega} \varphi(x,y,z) X_m(x) Y_n(y) Z_k(z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,. \tag{6.86}$$

By Eq. (6.84), we have

$$\begin{cases} \dfrac{\partial W_\varphi(x,y,z,t)}{\partial t} = \sum\limits_{m,n,k} D^*_{mnk}\, \mathrm{e}^{\alpha_{mnk}t} \left(\alpha_{mnk} \underline{\sin}\beta_{mnk}t + \underline{\beta}_{mnk} \cos\beta_{mnk}t\right) \\ \qquad \cdot X_m(x) Y_n(y) Z_k(z)\,, \\ D^*_{mnk} = \dfrac{1}{M_{mnk}\underline{\beta}_{mnk}} \iiint\limits_{\Omega} \varphi(x,y,z) X_m(x) Y_n(y) Z_k(z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,. \end{cases}$$

Thus

$$\sum_{m,n,k} C_{mnk}\, \mathrm{e}^{\alpha_{mnk}t} \cos\beta_{mnk}t \cdot X_m(x) Y_n(y) Z_k(z)$$

$$= \frac{\partial}{\partial t} W_\varphi(x,y,z,t) + \sum_{m,n,k} \left[\frac{\frac{1}{\tau_0} + (\lambda_m + \lambda_n + \lambda_k) B^2}{2} \frac{1}{M_{mnk}\beta_{mnk}} \right.$$

$$\left. \cdot \iiint\limits_{\Omega} \varphi(x,y,z) X_m(x) Y_n(y) Z_k(z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \right] \mathrm{e}^{\alpha_{mnk}t} \underline{\sin}\beta_{mnk}t \cdot X_m(x) Y_n(y) Z_k(z)\,. \tag{6.87}$$

Substituting Eqs. (6.86) and (6.84) into Eq. (6.85) and using the structure of $W_\psi(x,y,z,t)$ in Eq. (6.84) yields the solution of PDS (6.81)

$$u(x,y,z,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,y,z,t) + B^2 W_{(\lambda_m+\lambda_n+\lambda_k)\varphi}(x,y,z,t)\,.$$

Example 3. Solve

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u + B^2 \dfrac{\partial}{\partial t} \Delta u + f(x,y,z,t)\,, \\ \qquad 0 < x < l, 0 < y < a, 0 < z < b, 0 < t\,, \\ u(0,y,z,t) = u(l,y,z,t) = 0\,, \\ u(x,0,z,t) = u_y(x,a,z,t) + h_1 u(x,a,z,t) = 0\,, \\ u_z(x,y,0,t) - h_2 u(x,y,0,t) = u(x,y,b,t) = 0\,, \\ u(x,y,z,0) = \varphi(x,y,z),\ u_t(x,y,z,0) = \psi(x,y,z)\,. \end{cases} \tag{6.88}$$

Solution. Based on the given boundary conditions, we have from Rows 1, 3 and 7 in Table 2.1,

$$\lambda_m = \left(\frac{m\pi}{l}\right)^2, X_m(x) = \sin\frac{m\pi x}{l}, M_m = \frac{l}{2}, m = 1,2,\cdots;$$

$$\lambda_n = \left(\frac{\mu_n}{a}\right)^2, X_n(y) = \sin\frac{\mu_n y}{a}, M_n = \frac{a}{2}\left(1+\frac{\sin 2\mu_n}{2\mu_n}\right),$$

the $\mu_n$are the positive zero points of $f_1(y) = \tan y + \dfrac{y}{ah_1}$, $n = 1,2,\cdots;$

$$\lambda_k = \left(\frac{\mu_k'}{b}\right)^2, X_k(z) = \sin\left(\frac{\mu_k' z}{b}+\varphi_k\right), M_k = \frac{b}{2}\left[1-\frac{\sin 2\mu_k'}{2\mu_k'}\cos(\mu_k'+2\varphi_k)\right],$$

the μ_k' are the positive zero points of $f_2(z) = \tan z + \dfrac{z}{bh_2}$, $k = 1,2,\cdots.$
Thus the $W_\psi(x,y,z,t)$ is, by Eq. (6.84)

$$\begin{cases} u = W_\psi(x,y,z,t) = \displaystyle\sum_{m,n,k=1}^{\infty} B_{mnk}\,\mathrm{e}^{\alpha_{mnk}t}\,\underline{\sin}\beta_{mnk}t \\ \qquad \cdot\sin\dfrac{m\pi x}{l}\sin\dfrac{\mu_n y}{a}\sin\left(\dfrac{\mu_k' z}{b}+\varphi_k\right), \\ B_{mnk} = \dfrac{1}{M_m M_n M_k \underline{\beta}_{mnk}}\displaystyle\int_0^l \mathrm{d}x\int_0^a \mathrm{d}y\int_0^b \psi(x,y,z) \\ \qquad \cdot\sin\dfrac{m\pi x}{l}\sin\dfrac{\mu_n y}{a}\sin\left(\dfrac{\mu_k' z}{b}+\varphi_k\right)\mathrm{d}z, \end{cases} \tag{6.89}$$

where

$$\alpha_{mnk} = -\frac{1}{2}\left\{\frac{1}{\tau_0}+\left[\left(\frac{m\pi}{l}\right)^2+\left(\frac{\mu_n}{a}\right)^2+\left(\frac{\mu_k'}{b}\right)^2\right]B^2\right\},$$

$$\beta_{mnk} = \frac{1}{2}\sqrt{4\left[\left(\frac{m\pi}{l}\right)^2+\left(\frac{\mu_n}{a}\right)^2+\left(\frac{\mu_k'}{b}\right)^2\right]A^2-4\alpha_{mnk}^2}.$$

Finally, the solution of PDS (6.88) is

$$u = \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(x,y,z,t) + B^2 W_{(\lambda_m+\lambda_n+\lambda_k)\varphi}(x,y,z,t)$$

$$+W_\psi(x,y,z,t)+\int_0^t W_{f_\tau}(x,y,z,t-\tau)\,\mathrm{d}\tau.$$

6.4.4 Summary and Remarks

Summary

The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u + B^2 \dfrac{\partial}{\partial t} \Delta u + f(M,t)\,, \\ \qquad\qquad\qquad \Omega \times (0, +\infty)\,, \\ L(u, u_n)|_{\partial \Omega} = 0\,, \\ u(M,0) = \varphi(M), u_t(M,0) = \psi(M) \end{cases} \tag{6.90}$$

is

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_{\varphi}(M,t) + B^2 W_{\lambda_{mnk}\varphi}(M,t)$$

$$+ W_{\psi}(M,t) + \int_0^t W_{f_\tau(M,t-\tau)}\, \mathrm{d}\tau \tag{6.91}$$

where

$$\Omega : \begin{cases} 0 < x < l_1\,, & \text{one-dimensional case}\,, \\ 0 < x < l_1\,, 0 < y < l_2\,, & \text{two-dimensional case}\,, \\ 0 < x < l_1\,, 0 < y < l_2\,, 0 < z < l_3\,, & \text{three-dimensional case}\,. \end{cases}$$

$\partial \Omega$: the boundary of Ω.

$$\lambda_{mnk} = \begin{cases} \lambda_m\,, & \text{one-dimensional case}\,, \\ \lambda_m + \lambda_n\,, & \text{two-dimensional case}\,, \\ \lambda_m + \lambda_n + \lambda_k\,, & \text{three-dimensional case}\,. \end{cases}$$

$$M_{mnk} = \begin{cases} M_m\,, & \text{one-dimensional case}\,, \\ M_m M_n\,, & \text{two-dimensional case}\,, \\ M_m M_n M_k\,, & \text{three-dimensional case}\,. \end{cases}$$

$$F_{mnk}(M) = \begin{cases} X_m(x)\,, & \text{one-dimensional case}\,, \\ X_m(x) Y_n(y)\,, & \text{two-dimensional case}\,, \\ X_m(x) Y_n(y) Z_k(z)\,, & \text{three-dimensional case}\,. \end{cases}$$

$$\alpha_{mnk} = \begin{cases} -\dfrac{1}{2}\left(\dfrac{1}{\tau_0} + \lambda_m B^2\right), & \text{one-dimensional case}, \\ -\dfrac{1}{2}\left(\dfrac{1}{\tau_0} + \lambda_{mn} B^2\right), & \text{two-dimensional case}, \\ -\dfrac{1}{2}\left(\dfrac{1}{\tau_0} + \lambda_{mnk} B^2\right), & \text{three-dimensional case}. \end{cases}$$

$$\beta_{mnk} = \begin{cases} \dfrac{1}{2}\sqrt{4\lambda_m A^2 - 4\alpha_m^2}, & \text{one-dimensional case}, \\ \dfrac{1}{2}\sqrt{4\lambda_{mn} A^2 - 4\alpha_{mn}^2}, & \text{two-dimensional case}, \\ \dfrac{1}{2}\sqrt{4\lambda_{mnk} A^2 - 4\alpha_{mnk}^2}, & \text{three-dimensional case}. \end{cases}$$

$$f_\tau = f(M,\tau) = \begin{cases} f(x,\tau), & \text{one-dimensional case}, \\ f(x,y,\tau), & \text{two-dimensional case}, \\ f(x,y,z,\tau), & \text{three-dimensional case}. \end{cases}$$

$W_\psi(M,t)$ is the solution at $f = \varphi = 0$, i.e.

$$\begin{cases} u = W_\psi(M,t) = \sum\limits_{m,n,k} B_{mnk} F_{mnk}(M)\, \mathrm{e}^{\alpha_{mnk} t} \underline{\sin} \beta_{mnk} t, \\ B_{mnk} = \dfrac{1}{M_{mnk}\beta_{mnk}} \displaystyle\int_\Omega \psi(M) F_{mnk}(M)\, \mathrm{d}\Omega. \end{cases} \tag{6.92}$$

Here the integral $\int_\Omega$ is the definite integral, the double integral and the triple integral depending on the dimensions of Ω. The $\mathrm{d}\Omega$ stands for $\mathrm{d}x$, $\mathrm{d}x\,\mathrm{d}y$ and $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, respectively.

Remarks

1. To apply the solution structure theorem and the Fourier method of expansion, the boundary conditions must be linear, homogeneous, separable and with constant coefficients. The ordinals m, n and k cannot be confused. Their starting value (0 or 1) depends on the boundary conditions. We should also pay attention to the symbol change of h_1 and h_2 etc.
2. The μ_m, μ_n and μ_k carry different meanings. We should use μ_n' and μ_k'' to distinguish if necessary. Note also that the μ_m have different definitions in Rows 3, 6 and 9 in Table 2.1.
3. $\sum\limits_{m,n}$ is a double summation in the two-dimensional case. Taking the starting values of m and n equal to 1 as the example, coefficients B_{mn} read

n	m 1	2	3	4	5	...
1	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	...
2	B_{21}	B_{22}	B_{23}	B_{24}	B_{25}	...
3	B_{31}	B_{32}	B_{33}	B_{34}	B_{35}	...
4	B_{41}	B_{42}	B_{43}	B_{44}	B_{45}	...
5	B_{51}	B_{52}	B_{53}	B_{54}	B_{55}	...
...	...	...	...	...	...	

which is a matrix.

$$B = (B_{mn})_{\infty\times\infty}\,.$$

If the first 3, 6 or 10 terms are used, we normally use those in the upper triangular matrix.

$\sum\limits_{m,n,k}$ is a triple summation at the three-dimensional case. For any fixed value of m, we have a coefficient matrix $(B_{mnk})_{\infty\times\infty}$. Take the starting values of m, n and k as equal to 1 as an example. Calculating the summation, the first eight terms consists of the four terms $(B_{1nk})_{2\times 2}$ at $m=1$ and the four terms $(B_{2mk})_{2\times 2}$ at $m=2$. If taking $m,n,k=1,2,3$, we have a total of 27 terms.

4. If β_{mnk} is purely imaginary for some m, n and k, we can change $\sin\beta_{mnk}t$ into $\dfrac{e^{i\beta_{mnk}t}-e^{-i\beta_{mnk}t}}{2i}$ by using the formula $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$ for any imaginary variable z. If $\beta_{132}=ir_{132}$ with r_{132} taking a positive real value, for example, $\sin\beta_{132}t$ can be changed to an exponential function such that

$$\frac{1}{\underline{\beta}_{132}}e^{\alpha_{132}t}\underline{\sin}\beta_{132}t = \frac{e^{\alpha_{132}t}}{ir_{132}}\frac{e^{-r_{132}t}-e^{r_{132}t}}{2i} = \frac{e^{\alpha_{132}t}}{2r_{132}}\left(e^{r_{132}t}-e^{-r_{132}t}\right).$$

Since $|\alpha_{132}| > r_{132}$ and $\alpha_{132}<0$, the term $\left(\dfrac{1}{\underline{\beta}_{132}}e^{\alpha_{132}t}\underline{\sin}\beta_{132}t\right)$ decays as $t\to\infty$.

5. If the $\varphi(M)$ and the $\psi(M)$ satisfy consistency conditions, we can also use Theorem 1 in Section 6.1 to express the φ-contribution to the solution by the ψ-contribution.

6.5 Mixed Problems in a Circular Domain

Boundary conditions of all three kinds for mixed problems in a circular domain become separable with respect to the spatial variables in a polar coordinate system. In this section we use separation of variables to solve mixed problems in a polar coordinate system

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u(r,\theta,t) + B^2 \dfrac{\partial}{\partial t} \Delta u(r,\theta,t) + f(r,\theta,t)\,, \\ \qquad\qquad\qquad\qquad 0 < r < a\,,\ 0 < \theta < 2\pi,\ 0 < t\,, \\ L(u,u_r)|_{r=a} = 0\,, \\ u(r,\theta,0) = \varphi(r,\theta)\,, \quad u_t(r,\theta,0) = \psi(r,\theta)\,, \end{cases} \tag{6.93}$$

where the linear homogeneous boundary condition $L(u,u_r)|_{r=a}$ encompasses all three kinds. Here u_r stands for the normal derivative on the circle $r = a$. We will also examine the relation among solutions due to $\varphi(r,\theta)$, $\psi(r,\theta)$ and $f(r,\theta,t)$, respectively.

The linearity of PDS (6.93) ensures that its solution is the superposition of three solutions from $\varphi(r,\theta)$, $\psi(r,\theta)$ and $f(r,\theta,t)$, respectively.

6.5.1 Solution from $\psi(\mathbf{r},\theta)$

The solution from $\psi(r,\theta)$ satisfies

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u(r,\theta,t) + B^2 \dfrac{\partial}{\partial t} \Delta u(r,\theta,t)\,, \\ \qquad\qquad 0 < r < a\,,\ 0 < \theta < 2\pi\,,\ 0 < t\,, \\ L(u,u_r)|_{r=a} = 0\,, \\ u(r,\theta,0) = 0\,, \quad u_t(r,\theta,0) = \psi(r,\theta)\,. \end{cases} \tag{6.94}$$

Assume that $u(r,\theta,t) = v(r,\theta)T(t)$. Substituting it into the equation in PDS (6.94) yields, with λ as the separation constant,

$$\frac{T''(t) + \frac{1}{\tau_0}T'(t)}{A^2 T(t) + B^2 T'(t)} = \frac{\Delta v(r,\theta)}{v(r,\theta)} = -\lambda\,.$$

Thus we arrive at

$$T''(t) + \left(\frac{1}{\tau_0} + \lambda B^2\right) T'(t) + \lambda A^2 T(t) = 0 \tag{6.95}$$

and

$$\begin{cases} \Delta v(r,\theta) + \lambda v(r,\theta) = 0, \quad 0 < r < a, 0 < \theta < 2\pi\,, \\ L(v,v_r)|_{r=a} = 0\,. \end{cases} \tag{6.96}$$

Let $v(r,\theta) = R(r)\Theta(\theta)$. We can thus obtain the solution of Eq. (6.96) (see Section 4.3.2).

$$v_{mn}(r,\theta) = (a_{mn}\cos n\theta + b_{mn}\sin n\theta)J_n(k_{mn}r),$$

where the a_{mn} and the b_{mn} are not all zero, $\lambda_m = k_{mn}^2$, $k_{mn} = \mu_m^{(n)}/a$, $m = 1,2,\cdots$, $n = 0,1,2,\cdots$. The $\mu_m^{(n)}$ $(m = 1,2,\cdots)$ depend on the boundary conditions and are the positive zero points of

$$f_n(x) = \begin{cases} J_n(x), & \text{Boundary condition of the first kind} \\ & u|_{r=a} = 0, \\ J_n'(x), & \text{Boundary condition of the second kind} \\ & u_r|_{r=a} = 0,\ \mu_1^{(0)} = 0, \\ \frac{1}{a}xJ_n'(x) + hJ_n(x), & \text{Boundary condition of the third kind} \\ & (u_r + hu)|_{r=a} = 0. \end{cases} \tag{6.97}$$

Substituting $\lambda = k_{mn}^2$ into Eq. (6.95) yields

$$T_{mn}''(t) + \left(\frac{1}{\tau_0} + k_{mn}^2 B^2\right) T_{mn}'(t) + k_{mn}^2 A^2 T_m(t) = 0.$$

Its characteristic roots are

$$r_{1,2} = \frac{-\left(\frac{1}{\tau_0} + k_{mn}^2 B^2\right) \pm \sqrt{\left(\frac{1}{\tau_0} + k_{mn}^2 B^2\right)^2 - 4k_{mn}^2 A^2}}{2} = \alpha_{mn} + \beta_{mn}\mathrm{i}. \tag{6.98}$$

Therefore, the $u(r,\theta,t)$ that satisfies the equation and the boundary conditions of PDS (6.94) reads

$$\begin{aligned} u(r,\theta,t) = \sum_{m=1,n=0}^{+\infty} & \mathrm{e}^{\alpha_{mn}t}[(A_{mn}\cos\beta_{mn}t + B_{mn}\underline{\sin}\beta_{mn}t)\cos n\theta \\ & + (C_{mn}\cos\beta_{mn}t + D_{mn}\underline{\sin}\beta_{mn}t)\sin n\theta]J_n(k_{mn}r). \end{aligned}$$

Note that $\{1, \cos\theta, \sin\theta, \cdots, \cos n\theta, \sin n\theta, \cdots\}$ is orthogonal in $[-\pi,\pi]$ and $\{J_n(k_{mn}r)\}$ is orthogonal in $[0,a]$ with respect to the weight function r,

$$\int_0^a J_n(k_{mn}r)J_n(k_{ln}r)r\,\mathrm{d}r = 0, \quad m \neq l.$$

Applying the initial condition $u(r,\theta,0) = 0$ yields $A_{mn} = C_{mn} = 0$. B_{mn} and D_{mn} can be determined to satisfy the initial condition $u_t(r,\theta,0) = \psi(r,\theta)$. Finally, we obtain the solution of PDS (6.94)

$$\begin{cases} u = W_\psi(r,\theta,t) = \sum_{m,n} (B_{mn}\cos n\theta + D_{mn}\sin n\theta)\, J_n(k_{mn}r)\, \mathrm{e}^{\alpha_{mn}t}\, \underline{\sin}\beta_{mn}t\,, \\ B_{mn} = \dfrac{1}{M_n M_{mn} \underline{\beta}_{mn}} \displaystyle\int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^a \psi(r,\theta) J_n(k_{mn}r) r \cos n\theta \, \mathrm{d}r, \\ D_{mn} = \dfrac{1}{M_n M_{mn} \underline{\beta}_{mn}} \displaystyle\int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^a \psi(r,\theta) J_n(k_{mn}r) r \sin n\theta \, \mathrm{d}r, \end{cases} \tag{6.99}$$

where M_n is the normal square of $\{1, \cos\theta, \sin\theta, \cdots, \cos n\theta, \sin n\theta, \cdots\}$

$$M_n = \begin{cases} 2\pi\,, & \text{when } n = 0\,, \\ \pi\,, & \text{otherwise}\,. \end{cases}$$

$M_{mn} = \int_0^a J_n^2(k_{mn}r)\, r\, \mathrm{d}r$ is the normal square of $\{J_n(k_{mn}r)\}$ and is available in Table 4.1.

6.5.2 *Solution from* $\varphi(\mathbf{r},\theta)$

Theorem 1. Let $u(r,\theta,t) = W_\psi(r,\theta,t)$ be the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u(r,\theta,t) + B^2 \dfrac{\partial}{\partial t} \Delta u(r,\theta,t) \\ \qquad 0 < r < a\,, 0 < \theta < 2\pi\,,\ 0 < t\,, \\ L(u,u_r)|_{r=a} = 0\,, \\ u(r,\theta,0) = 0\,, \quad u_t(r,\theta,0) = \psi(r,\theta)\,. \end{cases} \tag{6.100}$$

The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u(r,\theta,t) + B^2 \dfrac{\partial}{\partial t} \Delta u(r,\theta,t)\,, \\ \qquad 0 < r < a, 0 < \theta < 2\pi, 0 < t\,, \\ L(u,u_r)|_{r=a} = 0\,, \\ u(r,\theta,0) = \varphi(r,\theta)\,, \quad u_t(r,\theta,0) = 0 \end{cases} \tag{6.101}$$

is

$$u(r,\theta,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(r,\theta,t) + B^2 W_{k_{mn}^2\varphi}(r,\theta,t)\,,$$

where $k_{mn} = \mu_m^{(n)}/a$, the $\mu_m^{(n)}$ are the non-negative zero points of $f_n(x)$ in Eq. (6.97).

Proof. By following a similar approach in Section 6.5.1, we obtain $u(r,\theta,t)$, satisfying the equation and the boundary conditions of PDS (6.101),

$$\begin{aligned} u(r,\theta,t) = \sum_{m,n} \mathrm{e}^{\alpha_{mn}t}[(A_{mn}\cos\beta_{mn}t + B_{mn}\underline{\sin}\beta_{mn}t)\cos n\theta \\ + (C_{mn}\cos\beta_{mn}t + D_{mn}\underline{\sin}\beta_{mn}t)\sin n\theta]J_n(k_{mn}r)\,. \end{aligned} \tag{6.102}$$

Applying the initial condition $u(r,\theta,0) = \varphi(r,\theta)$ yields

$$\begin{cases} A_{mn} = \dfrac{1}{M_n M_{mn}} \displaystyle\int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^a \varphi(r,\theta) r J_n(k_{mn}r)\cos n\theta\,\mathrm{d}r\,, \\ C_{mn} = \dfrac{1}{M_n M_{mn}} \displaystyle\int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^a \varphi(r,\theta) r J_n(k_{mn}r)\sin n\theta\,\mathrm{d}r\,. \end{cases} \tag{6.103}$$

Applying the initial condition $u_t(r,\theta,0) = 0$ leads to

$$\begin{cases} \alpha_{mn}A_{mn} + \underline{\beta}_{mn}B_{mn} = 0\,, \\ \alpha_{mn}C_{mn} + \underline{\beta}_{mn}D_{mn} = 0\,, \end{cases}$$

so that

$$\begin{cases} B_{mn} &= -\dfrac{\alpha_{mn}}{\underline{\beta}_{mn}}A_{mn} \\ &= -\dfrac{\alpha_{mn}}{\underline{\beta}_{mn}}\dfrac{1}{M_n M_{mn}} \displaystyle\int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^a \varphi(r,\theta) J_n(k_{mn}r) r\cos n\theta\,\mathrm{d}r\,, \\ D_{mn} &= -\dfrac{\alpha_{mn}}{\underline{\beta}_{mn}}C_{mn} \\ &= -\dfrac{\alpha_{mn}}{\underline{\beta}_{mn}}\dfrac{1}{M_n M_{mn}} \displaystyle\int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^a \varphi(r,\theta) J_n(k_{mn}r) r\sin n\theta\,\mathrm{d}r\,. \end{cases} \tag{6.104}$$

By $W_\psi(r,\theta,t)$ in Eq. (6.99), we have

$$\begin{aligned} \frac{\partial}{\partial t}W_\varphi(r,\theta,t) = \sum_{m,n}\left(\alpha_{mn}\underline{\sin}\beta_{mn}t + \underline{\beta}_{mn}\cos\beta_{mn}t\right) \\ \cdot (B^*_{mn}\cos n\theta + D^*_{mn}\sin n\theta)\,\mathrm{e}^{\alpha_{mn}t}J_n(k_{mn}r)\,, \end{aligned} \tag{6.105}$$

where

$$\begin{cases} B^*_{mn} = \dfrac{1}{M_n M_{mn}\underline{\beta}_{mn}} \displaystyle\int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^a \varphi(r,\theta) J_n(k_{mn}r) r\cos n\theta\,\mathrm{d}r\,, \\ D^*_{mn} = \dfrac{1}{M_n M_{mn}\underline{\beta}_{mn}} \displaystyle\int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^a \varphi(r,\theta) J_n(k_{mn}r) r\sin n\theta\,\mathrm{d}r\,. \end{cases} \tag{6.106}$$

Substituting Eqs. (6.103), (6.104) and (6.106) into (6.105) yields

$$\frac{\partial}{\partial t}W_{\varphi}(r,\theta,t)=\sum_{m,n}\{[(-B_{mn}\,\mathrm{e}^{\alpha_{mn}t}\,\underline{\sin}\beta_{mn}t)\cos n\theta + (-D_{mn}\,\mathrm{e}^{\alpha_{mn}t}\,\underline{\sin}\beta_{mn}t)\sin n\theta]+[(A_{mn}\,\mathrm{e}^{\alpha_{mn}t}\cos\beta_{mn}t)\cos n\theta + \left(C_{mn}\,\mathrm{e}^{\alpha_{mn}t}\cos\beta_{mn}t\right)\sin n\theta]\}J_n(k_{mn}r)\,.$$

Therefore,

$$\begin{aligned}&\sum_{m,n}[(A_{mn}\,\mathrm{e}^{\alpha_{mn}t}\cos\beta_{mn}t)\cos n\theta+\left(C_{mn}\,\mathrm{e}^{\alpha_{mn}t}\cos\beta_{mn}t\right)\sin n\theta]J_n(k_{mn}r)\\&=\frac{\partial}{\partial t}W_{\varphi}(r,\theta,t)+\sum_{m,n}[(\mathrm{e}^{\alpha_{mn}t}B_{mn}\underline{\sin}\beta_{mn}t)\cos n\theta\\&\quad+(\mathrm{e}^{\alpha_{mn}t}D_{mn}\underline{\sin}\beta_{mn}t)\sin n\theta]J_n(k_{mn}r)\,.\end{aligned}\tag{6.107}$$

Note that

$$-2\alpha_{mn}=\frac{1}{\tau_0}+k_{mn}^2B^2\,,\quad k_{mn}=\mu_m^{(n)}/a\,.$$

A substitution of Eqs. (6.104) and (6.107) into Eq. (6.102) yields the solution of PDS (6.101),

$$u(r,\theta,t)=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_{\varphi}(r,\theta,t)+B^2W_{k_{mn}^2\varphi}(r,\theta,t)\,.$$

6.5.3 Solution from $\mathbf{f}(\mathbf{r},\theta,\mathbf{t})$

Theorem 2. Let $u(r,\theta,t)=W_{\psi}(r,\theta,t)$ be the solution of

$$\begin{cases}\dfrac{u_t}{\tau_0}+u_{tt}=A^2\Delta u(r,\theta,t)+B^2\dfrac{\partial}{\partial t}\Delta u(r,\theta,t),\\ \qquad\qquad 0<r<a,\ 0<\theta<2\pi,\ 0<t,\\ L(u,u_r)|_{r=a}=0\,,\\ u(r,\theta,0)=0\,,\quad u_t(r,\theta,0)=\psi(r,\theta)\,.\end{cases}\tag{6.108}$$

The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u(r,\theta,t) + B^2 \dfrac{\partial}{\partial t} \Delta u(r,\theta,t) + f(r,\theta,t)\,, \\ \qquad\qquad\qquad\qquad 0 < r < a, 0 < \theta < 2\pi, 0 < t\,, \\ L(u,u_r)|_{r=a} = 0\,, \\ u(r,\theta,0) = 0\,, \quad u_t(r,\theta,0) = 0 \end{cases} \tag{6.109}$$

is

$$u(r,\theta,t) = \int_0^t W_{f_\tau}(r,\theta,t-\tau)\,\mathrm{d}\tau\,, \tag{6.110}$$

where $f_\tau = f(r,\theta,\tau)$. Therefore, Theorem 2 in Section 6.1 is also valid in polar coordinate systems.

Proof. By the definition of $W_\psi(r,\theta,t)$, the $W_{f_\tau} = W_{f_\tau}(r,\theta,t-\tau)$ satisfies

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_{f_\tau}}{\partial t} + \dfrac{\partial^2 W_{f_\tau}}{\partial t^2} = A^2 \Delta W_{f_\tau} + B^2 \dfrac{\partial}{\partial t} \Delta W_{f_\tau}\,, & \text{(6.111a)} \\ L\left(W_{f_\tau}, \dfrac{\partial W_{f_\tau}}{\partial r}\right)\bigg|_{r=a} = 0\,, & \text{(6.111b)} \\ W_{f_\tau}\big|_{t=\tau} = 0\,, \quad \dfrac{\partial}{\partial t} W_{f_\tau}\bigg|_{t=\tau} = f(r,\theta,\tau)\,. & \text{(6.111c)} \end{cases}$$

By Eq. (6.111b), we have

$$\begin{aligned} L(u,u_r)|_{r=a} &= L\left(\int_0^t W_{f_\tau}\,\mathrm{d}\tau, \frac{\partial}{\partial r}\int_0^t W_{f_\tau}\,\mathrm{d}\tau\right)\bigg|_{r=a} \\ &= \int_0^t L\left(W_{f_\tau}, \frac{\partial}{\partial r} W_{f_\tau}\right)\bigg|_{r=a}\,\mathrm{d}\tau = 0\,, \end{aligned}$$

so that the $u(r,\theta,t)$ in Eq. (6.110) satisfies the boundary conditions of PDS (6.109).

Clearly, the $u(r,\theta,t)$ in Eq. (6.110) satisfies $u(r,\theta,0) = 0$. Also,

$$u_t(r,\theta,t) = \int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}\big|_{\tau=t}\,,$$

which is zero at $t = 0$, by Eq. (6.111c). Therefore the $u(r,\theta,t)$ in Eq. (6.110) also satisfies the two initial conditions of PDS (6.109).

Since for the $u(r,\theta,t)$ in Eq. (6.110),

$$u_t = \int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}\big|_{\tau=t} = \int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau,$$

$$u_{tt} = \int_0^t \frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau + \left.\frac{\partial W_{f_\tau}}{\partial t}\right|_{\tau=t} = \int_0^t \frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau + f(r,\theta,t),$$

$$\Delta u = \int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau,$$

$$\frac{\partial}{\partial t}\Delta u = \frac{\partial}{\partial t}\int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau = \int_0^t \frac{\partial}{\partial t}\Delta W_{f_\tau}\,\mathrm{d}\tau + \Delta W_{f_\tau}\big|_{\tau=t} = \int_0^t \frac{\partial}{\partial t}\Delta W_{f_\tau}\,\mathrm{d}\tau,$$

a substitution of Eq. (6.110) into the equation of PDS (6.109) yields

$$\begin{aligned}&\frac{u_t}{\tau_0} + u_{tt} - A^2\Delta u(r,\theta,t) - B^2\frac{\partial}{\partial t}\Delta u(r,\theta,t)\\ &\qquad = \int_0^t \left(\frac{1}{\tau_0}\frac{\partial W_{f_\tau}}{\partial t} + \frac{\partial^2 W_{f_\tau}}{\partial t^2} - A^2\Delta W_{f_\tau} - B^2\frac{\partial}{\partial t}\Delta W_{f_\tau}\right)\mathrm{d}\tau\\ &\qquad\quad + f(r,\theta,t) = f(r,\theta,t).\end{aligned}$$

Therefore the $u(r,\theta,t)$ in Eq. (6.110) is indeed the solution of PDS (6.109).

By the principle of superposition, the solution of

$$\begin{cases}\dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u(r,\theta,t) + B^2\dfrac{\partial}{\partial t}\Delta u(r,\theta,t) + f(r,\theta,t),\\ \qquad\qquad\qquad\qquad 0<r<a,\ 0<\theta<2\pi,\ 0<t,\\ L(u,u_r)\big|_{r=a} = 0,\\ u(r,\theta,0) = \varphi(r,\theta),\ u_t(r,\theta,0) = \psi(r,\theta)\end{cases} \tag{6.112}$$

is

$$\begin{aligned}u =& \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi(r,\theta,t) + B^2 W_{k_{mn}^2\varphi}(r,\theta,t)\\ &+ W_\psi(r,\theta,t) + \int_0^t W_{f_\tau}(r,\theta,t-\tau)\,\mathrm{d}\tau,\end{aligned} \tag{6.113}$$

where $f_\tau = f(r,\theta,\tau)$. The $W_\psi(r,\theta,t)$ is the solution of PDS (6.94) and is available in Eq. (6.99). Note that Eqs. (6.99) and (6.113) are valid for all three kinds of boundary conditions. However, the $\mu_m^{(n)}$ are boundary-condition dependent and are determined by Eq. (6.97).

6.6 Mixed Problems in a Cylindrical Domain

Boundary conditions of all the three kinds for mixed problems in a cylinder become separable with respect to the spatial variables in a cylindrical coordinate system. In this section we apply separation of variables to seek solutions to mixed problems in a cylindrical coordinate system,

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u(r,\theta,z,t) + B^2 \dfrac{\partial}{\partial t} \Delta u(r,\theta,z,t) \\ \qquad\qquad + f(r,\theta,z,t), \quad \Omega \times (0,+\infty)\,, \\ L(u,u_r,u_z)|_{\partial\Omega} = 0\,, \\ u(r,\theta,z,0) = \varphi(r,\theta,z), \quad u_t(r,\theta,z,0) = \psi(r,\theta,z)\,, \end{cases} \tag{6.114}$$

where the Ω stands for a cylindrical domain: $0<r<a, 0<z<H$, the $\partial\Omega$ is the boundary of Ω. If all combinations of boundary conditions of all three kinds are considered on $\partial\Omega$, there exist 27 combinations. We will also examine the relation among solutions from $\varphi(r,\theta,z)$, $\psi(r,\theta,z)$ and $f(r,\theta,z,t)$, respectively.

6.6.1 *Solution from* $\psi(\mathbf{r},\theta,\mathbf{z})$

The solution due to $\psi(r,\theta,z)$ satisfies

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u(r,\theta,z,t) + B^2 \dfrac{\partial}{\partial t} \Delta u(r,\theta,z,t), \\ \qquad\qquad\qquad\qquad \Omega \times (0,+\infty)\,, \\ L(u,u_r,u_z)|_{\partial\Omega} = 0\,, \\ u(r,\theta,z,0) = 0, \quad u_t(r,\theta,z,0) = \psi(r,\theta,z)\,. \end{cases} \tag{6.115}$$

Assume $u = T(t)v(r,\theta,z)$. Substituting this into the equation of PDS (6.115) yields, with $\lambda^{(1)}$ as the separation constant,

$$\frac{T'' + \frac{1}{\tau_0}T'}{A^2T + B^2T'} = \frac{\Delta v}{v} = -\lambda^{(1)}\,.$$

Thus we arrive at

$$T'' + \left(\frac{1}{\tau_0} + \lambda^{(1)}B^2\right)T' + \lambda^{(1)}A^2T = 0 \tag{6.116}$$

and

$$\begin{cases} \Delta v(r,\theta,z) + \lambda^{(1)} v(r,\theta,z) = 0, \quad (r,\theta,z) \in \Omega, \\ L(v, v_r, v_z)|_{\partial\Omega} = 0, \quad v(r,\theta+2\pi,z) = v(r,\theta,z). \end{cases} \tag{6.117}$$

Let $v = V(r,\theta)Z(z)$. Substituting this into Eq. (6.117) yields, with $\lambda^{(2)}$ as the separation constant,

$$\frac{\Delta V(r,\theta)}{V(r,\theta)} = -\frac{Z''(z)}{Z(z)} - \lambda^{(1)} = -\lambda^{(2)}.$$

Thus we have

$$\begin{cases} \Delta V(r,\theta) + \lambda^{(2)} V(r,\theta) = 0, \quad 0 < r < a, \\ L(V, V_r)|_{r=a} = 0, \quad V(r,\theta+2\pi) = V(r,\theta). \end{cases} \tag{6.118}$$

and

$$\begin{cases} Z''(z) + \lambda^{(3)} Z(z) = 0,\ \lambda^{(3)} = \lambda^{(1)} - \lambda^{(2)}, \\ L(Z, Z_z)|_{z=0} = 0,\ \ L(Z, Z_z)|_{z=H} = 0. \end{cases} \tag{6.119}$$

The solution of the former is available in Section 4.3.2

$$V_{mn}(r,\theta) = J_n(k_{mn} r)(a_{mn} \cos n\theta + b_{mn} \sin n\theta),$$

where a_{mn} and b_{mn} are constants, the $\mu_m^{(n)}$ are determined by Eq. (6.97),

$$\lambda^{(2)} = k_{mn}^2 = \left(\mu_m^{(n)}/a_0\right)^2, \quad m = 1, 2, \cdots, \quad n = 0, 1, 2, \cdots. \tag{6.120}$$

There are a total of nine combinations of boundary conditions in Eq. (6.119). Let $\lambda^{(3)} = \lambda_k^2$ and $Z_k(z)$ stand for the eigenvalues and eigenfunctions of Eq. (6.119), which are available in Table 2.1. Substituting $\lambda^{(1)} = \lambda^{(2)} + \lambda^{(3)} = k_{mn}^2 + \lambda_k^2$ into Eq. (6.116) leads to the $T(t)$-equation

$$T''_{mnk}(t) + \left(\frac{1}{\tau_0} + \left(k_{mn}^2 + \lambda_k^2\right) B^2\right) T'_{mnk}(t) + \left(k_{mn}^2 + \lambda_k^2\right) A^2 T_{mnk}(t) = 0. \tag{6.121}$$

Its characteristic roots read

$$\begin{aligned} r_{1,2} &= \frac{-\left[\frac{1}{\tau_0} + (k_{mn}^2 + \lambda_k^2)B^2\right] \pm \sqrt{\left[\frac{1}{\tau_0} + (k_{mn}^2 + \lambda_k^2)B^2\right]^2 - 4(k_{mn}^2 + \lambda_k^2)A^2}}{2} \\ &= \alpha_{mnk} \pm \beta_{mnk} \mathrm{i}. \end{aligned} \tag{6.122}$$

Thus $T_{mnk}(t) = \mathrm{e}^{\alpha_{mnk} t}(c_{mnk} \cos \beta_{mnk} t + d_{mnk} \underline{\sin} \beta_{mnk} t)$.

The $u(r,\theta,t)$ that satisfies the equation and the boundary conditions of PDS (6.115) is thus

$$
\begin{aligned}
u(r,\theta,t) = \sum_{m,n,k} &\mathrm{e}^{\alpha_{mnk}t}[(c_{mnk}\cos\beta_{mnk}t + d_{mnk}\underline{\sin}\beta_{mnk}t) \\
&\cdot(a_{mn}\cos n\theta + b_{mn}\sin n\theta)]J_n(k_{mn}r)Z_k(z) \\
= \sum_{m,n,k} &\mathrm{e}^{\alpha_{mnk}t}[(A_{mnk}\cos\beta_{mnk}t + B_{mnk}\underline{\sin}\beta_{mnk}t)\cos n\theta \\
&+ (C_{mnk}\cos\beta_{mnk}t + D_{mnk}\underline{\sin}\beta_{mnk}t)\sin n\theta]J_n(k_{mn}r)Z_k(z)\,,
\end{aligned}
$$

where $\sum\limits_{m,n,k}$ stands for a triple summation, $m = 1,2,\cdots$, $n = 0,1,2,\cdots$, $k = 0,1,2,\cdots$ for the cases of Rows 2, 4 and 5 in Table 2.1, and $k = 1,2,\cdots$ for the other six cases.

Note that $\{1,\cos\theta,\sin\theta,\cdots,\cos n\theta,\sin n\theta,\cdots\}$, $\{J_n(k_{mn}r)\}$ and $Z_k(z)$ are orthogonal in $[-\pi,\pi]$, in $[0,a]$ with respect to the weight function r and in $[0,H]$, respectively. Applying the initial condition $u(r,\theta,z,0) = 0$ yields $A_{mnk} = C_{mnk} = 0$. Thus

$$
\begin{aligned}
u_t(r,\theta,z,t) = \sum_{m,n,k} &\mathrm{e}^{\alpha_{mnk}t}[(\alpha_{mnk}B_{mnk}\underline{\sin}\beta_{mnk}t \\
&+ \underline{\beta}_{mnk}B_{mnk}\cos\beta_{mnk}t)\cos n\theta + (\alpha_{mnk}D_{mnk}\underline{\sin}\beta_{mnk}t \\
&+ \underline{\beta}_{mnk}D_{mnk}\cos\beta_{mnk}t)\sin n\theta]J_n(k_{mn}r)Z_k(z)\,.
\end{aligned}
$$

B_{mnk} and D_{mnk} can thus be determined by applying the initial condition $u_t(r,\theta,z,0) = \psi(r,\theta,z)$. Finally, we have the solution of PDS (6.115)

$$
\begin{cases}
u = W_\psi(r,\theta,z,t) = \sum\limits_{m,n,k}(B_{mnk}\cos n\theta \\
\qquad + D_{mnk}\sin n\theta)\,J_n(k_{mn}r)Z_k(z)\,\mathrm{e}^{\alpha_{mnk}t}\,\underline{\sin}\beta_{mnk}t\,, \\
B_{mnk} = \dfrac{1}{M_{mnk}\underline{\beta}_{mnk}}\iiint\limits_{\Omega}\psi(r,\theta,z)rJ_n(k_{mn}r)Z_k(z)\cos n\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z\,, \\
D_{mnk} = \dfrac{1}{M_{mnk}\underline{\beta}_{mnk}}\iiint\limits_{\Omega}\psi(r,\theta,z)rJ_n(k_{mn}r)Z_k(z)\sin n\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z,
\end{cases}
\tag{6.123}
$$

where M_{mnk} is the product of normal squares of $\{1,\cos\theta,\sin\theta,\cdots,\cos n\theta,\sin n\theta,\cdots\}$, $\{J_n(k_{mn}r)\}$ and $Z_k(z)$.

6.6.2 Solution from $\varphi(\mathbf{r},\theta,\mathbf{z})$

Theorem 1. Let $u(r,\theta,z,t) = W_\psi(r,\theta,z,t)$ be the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u(r,\theta,z,t) + B^2\dfrac{\partial}{\partial t}\Delta u(r,\theta,z,t) \\ \qquad\qquad \Omega\times(0,+\infty)\,, \\ L(u,u_r,u_z)|_{\partial\Omega} = 0\,, \\ u(r,\theta,z,0) = 0, u_t(r,\theta,z,0) = \psi(r,\theta,z)\,. \end{cases} \tag{6.124}$$

The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u(r,\theta,z,t) + B^2\dfrac{\partial}{\partial t}\Delta u(r,\theta,z,t) \\ \qquad\qquad \Omega\times(0,+\infty)\,, \\ L(u,u_r,u_z)|_{\partial\Omega} = 0\,, \\ u(r,\theta,z,0) = \varphi(r,\theta,z), u_t(r,\theta,z,0) = 0 \end{cases} \tag{6.125}$$

is

$$u(r,\theta,z,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(r,\theta,z,t) + B^2 W_{(k_{mn}^2+\lambda_k^2)\varphi}(r,\theta,z,t)\,,$$

where k_{mn}^2 and λ_k^2 are the eigenvalues $\lambda^{(2)}$ and $\lambda^{(3)}$ in Section 6.6.1.

Proof. By following a similar approach in Section 6.6.1, we have $u(r,\theta,z,t)$ satisfying the equation and the boundary conditions of PDS (6.125)

$$\begin{aligned} u(r,\theta,z,t) = \sum_{m,n,k} \mathrm{e}^{\alpha_{mnk}t} [(A_{mnk}\cos\beta_{mnk}t + B_{mnk}\underline{\sin}\beta_{mnk}t)\cos n\theta \\ + (C_{mnk}\cos\beta_{mnk}t + D_{mnk}\underline{\sin}\beta_{mnk}t)\sin n\theta] J_n(k_{mn}r) Z_k(z)\,. \end{aligned} \tag{6.126}$$

Applying the initial condition $u(r,\theta,z,0) = \varphi(r,\theta,z)$ yields

$$\begin{cases} A_{mnk} = \dfrac{1}{M_{mnk}} \displaystyle\iiint_\Omega \varphi(r,\theta,z) J_n(k_{mn}r) Z_k(z) r\cos n\theta \,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z, \\ C_{mnk} = \dfrac{1}{M_{mnk}} \displaystyle\iiint_\Omega \varphi(r,\theta,z) J_n(k_{mn}r) Z_k(z) r\sin n\theta \,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z\,. \end{cases} \tag{6.127}$$

Applying the initial condition $u_t(r,\theta,z,0) = 0$ leads to

$$\begin{cases} \alpha_{mnk}A_{mnk} + \underline{\beta}_{mnk}B_{mnk} = 0\,, \\ \alpha_{mnk}C_{mnk} + \underline{\beta}_{mnk}D_{mnk} = 0\,, \end{cases}$$

so that

$$\begin{cases} B_{mnk} = -\dfrac{\alpha_{mnk}}{\underline{\beta}_{mnk}} \dfrac{1}{M_{mnk}} \displaystyle\iiint_{\Omega} \varphi(r,\theta,z) J_n(k_{mn}r) Z_k(z) r \cos n\theta \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z\,, \\ D_{mnk} = -\dfrac{\alpha_{mnk}}{\underline{\beta}_{mnk}} \dfrac{1}{M_{mnk}} \displaystyle\iiint_{\Omega} \varphi(r,\theta,z) J_n(k_{mn}r) Z_k(z) r \sin n\theta \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z\,. \end{cases} \tag{6.128}$$

On the other hand, by the structure of $W_{\psi}(r,\theta,z,t)$(Eq. (6.123)),

$$\frac{\partial}{\partial t} W_{\varphi}(r,\theta,z,t) = \sum_{m,n,k} (\alpha_{mnk} \underline{\sin} \beta_{mnk} t + \underline{\beta}_{mnk} \cos \beta_{mnk} t)$$

$$\cdot (B^*_{mnk} \cos n\theta + D^*_{mnk} \sin n\theta)\, \mathrm{e}^{\alpha_{mnk}t} J_n(k_{mn}r) Z_k(z)\,, \tag{6.129}$$

$$\begin{cases} B^*_{mnk} = \dfrac{1}{M_{mnk}\underline{\beta}_{mnk}} \displaystyle\iiint_{\Omega} \varphi(r,\theta,z) J_n(k_{mn}r) Z_k(z) r \cos n\theta \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z\,, \\ D^*_{mnk} = \dfrac{1}{M_{mnk}\underline{\beta}_{mnk}} \displaystyle\iiint_{\Omega} \varphi(r,\theta,z) J_n(k_{mn}r) Z_k(z) r \sin n\theta \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z\,. \end{cases} \tag{6.130}$$

Substituting Eqs. (6.127), (6.128) and (6.130) into Eq. (6.129) yields

$$\frac{\partial}{\partial t} W_{\varphi}(r,\theta,z,t) = \sum_{m,n,k} \{[(-B_{mnk}\, \mathrm{e}^{\alpha_{mnk}t} \underline{\sin} \beta_{mnk} t) \cos n\theta$$

$$+ (-D_{mnk}\, \mathrm{e}^{\alpha_{mnk}t} \underline{\sin} \beta_{mnk} t) \sin n\theta] + [(A_{mnk}\, \mathrm{e}^{\alpha_{mnk}t} \cos \beta_{mnk} t) \cos n\theta$$

$$+ \left(C_{mnk}\, \mathrm{e}^{\alpha_{mnk}t} \cos \beta_{mnk} t\right) \sin n\theta]\} J_n(k_{mn}r) Z_k(z)\,. \tag{6.131}$$

Thus

$$\sum_{m,n,k} [(A_{mnk}\, \mathrm{e}^{\alpha_{mnk}t} \cos \beta_{mnk} t) \cos n\theta + \left(C_{mnk}\, \mathrm{e}^{\alpha_{mnk}t} \cos \beta_{mnk} t\right) \sin n\theta]$$

$$\cdot J_n(k_{mn}r) Z_k(z) = \frac{\partial}{\partial t} W_{\varphi}(r,\theta,z,t) + \sum_{m,n,k} [(B_{mnk}\, \mathrm{e}^{\alpha_{mnk}t} \underline{\sin} \beta_{mnk} t) \cos n\theta$$

$$+ (D_{mnk}\, \mathrm{e}^{\alpha_{mnk}t} \underline{\sin} \beta_{mnk} t) \sin n\theta] J_n(k_{mn}r) Z_k(z). \tag{6.132}$$

Note that

$$\alpha_{mnk} = -\frac{\frac{1}{\tau_0} + (k_{mn}^2 + \lambda_k^2) B^2}{2}\,.$$

Substituting Eqs. (6.127), (6.128) and (6.132) into Eq. (6.126) thus yields the solution of PDS (6.125)

$$u(r,\theta,z,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(r,\theta,z,t) + B^2 W_{\left(k_{mn}^2+\lambda_k^2\right)\varphi}(r,\theta,z,t). \tag{6.133}$$

6.6.3 Solution from $\mathbf{f}(\mathbf{r},\theta,\mathbf{z},\mathbf{t})$

Theorem 2. Let $u(r,\theta,z,t) = W_\psi(r,\theta,z,t)$ be the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u(r,\theta,z,t) + B^2\dfrac{\partial}{\partial t}\Delta u(r,\theta,z,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \Omega\times(0,+\infty), \\ L(u,u_r,u_z)|_{\partial\Omega} = 0, \\ u(r,\theta,z,0) = 0,\ u_t(r,\theta,z,0) = \psi(r,\theta,z). \end{cases}$$

The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u(r,\theta,z,t) + B^2\dfrac{\partial}{\partial t}\Delta u(r,\theta,z,t) \\ \qquad\qquad\qquad\qquad +f(r,\theta,z,t), \quad \Omega\times(0,+\infty), \\ L(u,u_r,u_z)|_{\partial\Omega} = 0, \\ u(r,\theta,z,0) = u_t(r,\theta,z,0) = 0 \end{cases} \tag{6.134}$$

is

$$u = \int_0^t W_{f_\tau}(r,\theta,z,t-\tau)\,\mathrm{d}\tau, \tag{6.135}$$

where $f_\tau = f(r,\theta,z,\tau)$. Therefore, Theorem 2 in Section 6.1 is also valid in cylindrical coordinate systems.

Proof. By the definition of $W_\psi(r,\theta,z,t)$, the $W_{f_\tau}(r,\theta,z,t-\tau)$ satisfies

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_{f_\tau}}{\partial t} + \dfrac{\partial^2 W_{f_\tau}}{\partial t^2} = A^2\Delta W_{f_\tau} + B^2\dfrac{\partial}{\partial t}\Delta W_{f_\tau}, & (6.136\text{a}) \\ L\left(W_{f_\tau}, \dfrac{\partial W_{f_\tau}}{\partial r}, \dfrac{\partial W_{f_\tau}}{\partial z}\right)\Bigg|_{\partial\Omega} = 0, & (6.136\text{b}) \\ W_{f_\tau}\big|_{t=\tau} = 0, \quad \dfrac{\partial W_{f_\tau}}{\partial t}\Bigg|_{t=\tau} = f(r,\theta,z,\tau). & (6.136\text{c}) \end{cases}$$

By Eq. (6.136b)

$$L(u,u_r,u_z)|_{\partial\Omega} = L\left[\int_0^t W_{f_\tau}\,\mathrm{d}\tau, \frac{\partial}{\partial r}\int_0^t W_{f_\tau}\,\mathrm{d}\tau, \frac{\partial}{\partial z}\int_0^t W_{f_\tau}\,\mathrm{d}\tau\right]\bigg|_{\partial\Omega}$$

$$= \int_0^t L\left(W_{f_\tau}, \frac{\partial}{\partial r}W_{f_\tau}, \frac{\partial}{\partial z}W_{f_\tau}\right)\bigg|_{\partial\Omega}\mathrm{d}\tau = 0,$$

so the $u(r,\theta,z,t)$ in Eq. (6.135) satisfies the boundary conditions of PDS (6.134).

Clearly, the $u(r,\theta,z,t)$ in Eq. (6.135) satisfies $u(r,\theta,z,0) = 0$. Also,

$$u_t(r,\theta,z,t) = \int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}\big|_{\tau=t},$$

which shows that $u_t(r,\theta,z,0) = 0$ by Eq. (6.136c). Thus the $u(r,\theta,z,t)$ in Eq. (6.135) also satisfies the two initial conditions of PDS (6.134).

Since for the $u(r,\theta,z,t)$ in Eq. (6.135)

$$u_t = \frac{\partial}{\partial t}\int_0^t W_{f_\tau}\,\mathrm{d}\tau = \int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}\big|_{\tau=t} = \int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau,$$

$$u_{tt} = \frac{\partial}{\partial t}\int_0^t \frac{\partial}{\partial t}W_{f_\tau}\,\mathrm{d}\tau = \int_0^t \frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau + \frac{\partial W_{f_\tau}}{\partial t}\bigg|_{\tau=t}$$

$$= \int_0^t \frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau + f(r,\theta,z,t),$$

$$\Delta u = \Delta\int_0^t W_{f_\tau}\,\mathrm{d}\tau = \int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau,$$

$$\frac{\partial}{\partial t}\Delta u = \frac{\partial}{\partial t}\Delta\int_0^t W_{f_\tau}\,\mathrm{d}\tau = \frac{\partial}{\partial t}\int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau$$

$$= \int_0^t \frac{\partial}{\partial t}\Delta W_{f_\tau}\,\mathrm{d}\tau + \Delta W_{f_\tau}\big|_{\tau=t} = \int_0^t \frac{\partial}{\partial t}\Delta W_{f_\tau}\,\mathrm{d}\tau,$$

a substitution of Eq. (6.135) into the equation of PDS (6.134) yields

$$\frac{u_t}{\tau_0} + u_{tt} - A^2\Delta u(r,\theta,z,t) - B^2\frac{\partial}{\partial t}\Delta u(r,\theta,z,t)$$

$$= \int_0^t\left(\frac{1}{\tau_0}\frac{\partial W_{f_\tau}}{\partial t} + \frac{\partial^2 W_{f_\tau}}{\partial t^2} - A^2\Delta W_{f_\tau} - B^2\frac{\partial}{\partial t}\Delta W_{f_\tau}\right)\mathrm{d}\tau + f(r,\theta,z,t)$$

$$= f(r,\theta,z,t).$$

Thus the $u(r,\theta,z,t)$ in Eq. (6.135) is indeed the solution of PDS (6.134).

6.6.4 Green Function of the Dual-Phase-Lagging Heat-Conduction Equation

By using Eq. (6.123), the $u(r,\theta,z,t)$ in Eq. (6.135) reads

$$u = \int_0^t W_{f_\tau}(r,\theta,z,t-\tau)\,\mathrm{d}\tau = \int_0^t \iiint\limits_{\Omega} \sum_{m,n,k} \frac{1}{M_{mnk}\underline{\beta}_{mnk}} \mathrm{e}^{\alpha_{mnk}(t-\tau)}$$

$$\cdot f(r',\theta',z',\tau)(\cos n\theta \cos n\theta' + \sin n\theta \sin n\theta')J_n(k_{mn}r)r'$$

$$\cdot J_n(k_{mn}r')Z_k(z)Z_k(z')\underline{\sin}\beta_{mnk}(t-\tau)\,\mathrm{d}\theta'\,\mathrm{d}r'\,\mathrm{d}z'\,\mathrm{d}\tau$$

$$= \int_0^t \iiint\limits_{\Omega} G(r,r';\theta,\theta';z,z';t-\tau)f(r',\theta',z',\tau)\,\mathrm{d}\Omega\,\mathrm{d}\tau, \tag{6.137}$$

where

$$G = \sum_{m,n,k} \frac{\mathrm{e}^{\alpha_{mnk}(t-\tau)}}{M_{mnk}\underline{\beta}_{mnk}} J_n(k_{mn}r)J_n(k_{mn}r')Z_k(z)Z_k(z')\cos n(\theta-\theta')\underline{\sin}\beta_{mnk}(t-\tau) \tag{6.138}$$

is called the *Green function of the dual-phase-lagging heat-conduction equation in a cylindrical domain*. The Green function is clearly boundary-condition dependent. When $f(r,\theta,z,t) = \delta(\mathbf{r}-\mathbf{r}_0, t-t_0)$, the solution of PDS (6.134) reduces to

$$u = G(r,r_0;\theta,\theta_0;z,z_0;t-t_0),$$

where $\mathbf{r} = (r,\theta,z)$, $\mathbf{r}_0 = (r_0,\theta_0,z_0)$. Therefore the Green function $G(r,r_0;\theta,\theta_0;z,z_0;t,t_0)$ is the solution from the source term $\delta(\mathbf{r}-\mathbf{r}_0,t-t_0)$.

Remark 1.The boundary condition $L(u,u_r,u_z)|_{\partial\Omega} = 0$ in PDS (6.114) encompasses 27 combinations. Since PDS (6.114) has nontrivial solutions for any nontrivial φ, ψ or f, there exist a total of seven cases $\left(C_3^1 + C_3^2 + C_3^3 = 7\right)$ of nontrivial solutions. Therefore, PDS (6.114) actually contains $27 \times 7 = 189$ PDS. Their solutions can be written in the general form

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(r,\theta,z,t) + B^2 W_{(k_{mn}^2+\lambda_k^2)\varphi}(r,\theta,z,t)$$

$$+ W_\psi(r,\theta,z,t) + \int_0^t W_{f_\tau}(r,\theta,z,t-\tau)\,\mathrm{d}\tau, \tag{6.139}$$

where the structure of $W_\psi(r,\theta,z,t)$ is available in Eq. (6.123). Since Eq. (6.123) exhibits the structure of $W_\psi(r,\theta,z,t)$, we can write $W_\varphi(r,\theta,z,t)$ and $W_{f_\tau}(r,\theta,z,t-\tau)$ using the structure in Eq. (6.123) even for PDS with $\psi(r,\theta,z) = 0$. Therefore, in applications, we can directly write out the solutions of the 189 PDS based on Eqs. (6.123) and (6.139) without going through the individual details.

Remark 2. If β_{mnk} is purely imaginary for some m, n and k such that $\beta_{mnk} = \mathrm{i}\gamma_{mnk}$ (γ_{mnk} is real), we can change $\sin\beta_{mnk}t$ into $\dfrac{\mathrm{e}^{\mathrm{i}\beta_{mnk}t} - \mathrm{e}^{-\mathrm{i}\beta_{mnk}t}}{2\mathrm{i}}$ by using the formula $\sin z = \dfrac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}$ for any imaginary variable z. We thus have the term

$$\frac{1}{\beta_{mnk}}\sin\beta_{mnk}t = \frac{1}{2\gamma_{mnk}}\left(\mathrm{e}^{\gamma_{mnk}t} - \mathrm{e}^{-\gamma_{mnk}t}\right) = \frac{1}{\gamma_{mnk}}\mathrm{sh}\gamma_{mnk}t\,.$$

6.7 Mixed Problems in a Spherical Domain

Boundary conditions of all the three kinds for mixed problems in a spherical domain are separable with respect to the spatial variables in a spherical coordinate system. In this section we apply the separation of variables to find solutions of mixed problems in a spherical coordinate system

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u(r,\theta,\varphi,t) + B^2\dfrac{\partial}{\partial t}\Delta u(r,\theta,\varphi,t) \\ \qquad + f(r,\theta,\varphi,t), \quad 0<r<a,\ 0<\theta<\pi,\ 0<\varphi<2\pi,\ 0<t\,, \\ L(u,u_r)\big|_{r=a} = 0, \\ u(r,\theta,\varphi,0) = \Phi(r,\theta,\varphi),\ u_t(r,\theta,\varphi,0) = \psi(r,\theta,\varphi)\,, \end{cases} \tag{6.140}$$

where Ω stands for a sphere of radius a, with $\partial\Omega$ as its boundary. The boundary condition $L(u,u_r)|_{r=a} = 0$ contains all three kinds. We will also examine the relation among solutions from $\Phi(r,\theta,\varphi)$, $\psi(r,\theta,\varphi)$ and $f(r,\theta,\varphi,t)$, respectively.

6.7.1 Solution from $\psi(\mathbf{r},\theta,\varphi)$

The solution due to $\psi(r,\theta,\varphi)$ satisfies

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u(r,\theta,\varphi,t) + B^2\dfrac{\partial}{\partial t}\Delta u(r,\theta,\varphi,t)\,, \\ \qquad 0<r<a, 0<\theta<\pi, 0\le\varphi\le 2\pi, 0<t\,, \\ L(u,u_r)\big|_{r=a} = 0\,, \\ u(r,\theta,\varphi,0) = 0, \quad u_t(r,\theta,\varphi,0) = \psi(r,\theta,\varphi)\,. \end{cases} \tag{6.141}$$

Assume $u = T(t)v(r,\theta,\varphi)$. Substituting it into the equation of PDS (6.141) yields, with $-\lambda = -k^2$ as the separation constant,

$$\frac{T'' + \frac{1}{\tau_0}T'}{A^2T + B^2T'} = \frac{\Delta v(r,\theta,\varphi)}{v(r,\theta,\varphi)} = -\lambda = -k^2 .$$

Thus we arrive at

$$T''(t) + \left(\frac{1}{\tau_0} + \lambda B^2\right) T'(t) + \lambda A^2 T(t) = 0 \tag{6.142}$$

and

$$\begin{cases} \Delta v(r,\theta,\varphi) + k^2 v(r,\theta,\varphi) = 0,\ 0 < r < a,\ 0 < \theta < \pi , \\ L(v, v_r)|_{r=a} = 0 . \end{cases} \tag{6.143}$$

The solution of the latter is available in Section 2.6.2

$$v_{mnl}(r,\theta,\varphi) = (a_{mnl}\cos m\varphi + b_{mnl}\sin m\varphi)P_n^m(\cos\theta)\, j_n(k_{nl}r) , \quad m \le n ,$$

where constants a_{mnl} and b_{mnl} are not all zero, $\lambda = k_{nl}^2 = \left(\mu_l^{\left(n+\frac{1}{2}\right)} / a\right)^2$, $n = 0,1,2,\cdots$, $l = 1,2,\cdots$ and the $\mu_l^{\left(n+\frac{1}{2}\right)}$ are available in Section 2.6.2. Substituting $\lambda = k_{nl}^2 = \left(\mu_l^{\left(n+\frac{1}{2}\right)} /a\right)^2$ into Eq. (6.142) yields

$$T_{nl}''(t) + \left(\frac{1}{\tau_0} + (k_{nl}B)^2\right) T_{nl}'(t) + (k_{nl}A)^2 T_{nl}(t) = 0 .$$

Its characteristic roots read

$$\begin{aligned} r_{1,2} &= \frac{-\left[\frac{1}{\tau_0} + (k_{nl}B)^2\right] \pm \sqrt{\left[\frac{1}{\tau_0} + (k_{nl}B)^2\right]^2 - 4(k_{nl}A)^2}}{2} \\ &= \alpha_{nl} \pm \beta_{nl}\mathrm{i} . \end{aligned} \tag{6.144}$$

Thus

$$T_{nl}(t) = \mathrm{e}^{\alpha_{nl}t}(c_{nl}\cos\beta_{nl}t + d_{nl}\underline{\sin}\beta_{nl}t) .$$

Thus the $u(r,\theta,\varphi,t)$ that satisfies the equation and the boundary conditions of PDS (6.141) is

$$u(r,\theta,\varphi,t)=\sum_{m,n,l}\mathrm{e}^{\alpha_{nl}t}[(A_{mnl}\cos\beta_{nl}t+B_{mnl}\underline{\sin}\beta_{nl}t)\cos m\varphi$$
$$+(C_{mnl}\cos\beta_{nl}t+D_{mnl}\underline{\sin}\beta_{nl}t)\sin m\varphi]P_n^m(\cos\theta)j_n(k_{nl}r)\,. \quad (6.145)$$

Applying the initial condition $u(r,\theta,\varphi,0)=0$ yields $A_{mnl}=C_{mnl}=0$. B_{mnl} and D_{mnl} can also be determined by applying the initial condition $u_t(r,\theta,\varphi,0)=\psi(r,\theta,\varphi)$. Finally we obtain the solution of PDS (6.141)

$$\begin{cases} u=W_\psi(r,\theta,\varphi,t)=\sum\limits_{m,n,l}\mathrm{e}^{\alpha_{nl}t}(B_{mnl}\cos m\varphi+D_{mnl}\sin m\varphi) \\ \qquad\cdot P_n^m(\cos\theta)j_n(k_{nl}r)\underline{\sin}\beta_{nl}t\,, \\ B_{mnl}=\dfrac{1}{M_{mnl}\underline{\beta}_{nl}}\iiint\limits_{r\le a}\psi(r,\theta,\varphi)P_n^m(\cos\theta)j_n(k_{nl}r)r^2\cos m\varphi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,, \\ D_{mnl}=\dfrac{1}{M_{mnl}\underline{\beta}_{nl}}\iiint\limits_{r\le a}\psi(r,\theta,\varphi)P_n^m(\cos\theta)j_n(k_{nl}r)r^2\sin m\varphi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,, \end{cases} \quad (6.146)$$

where M_{mnl} is the product of three normal squares.

6.7.2 Solution from $\Phi(\mathbf{r},\theta,\varphi)$

Theorem 1. Let $u=W_\psi(r,\theta,\varphi,t)$ be the solution of PDS (6.141). The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0}+u_{tt}=A^2\Delta u(r,\theta,\varphi,t)+B^2\dfrac{\partial}{\partial t}\Delta u(r,\theta,\varphi,t) \\ \qquad 0<r<a,\,0<\theta<\pi,\,0\le\varphi\le 2\pi,\,0<t\,, \\ L(u,u_r)|_{r=a}=0, \\ u(r,\theta,\varphi,0)=\Phi(r,\theta,\varphi),\,u_t(r,\theta,\varphi,0)=0 \end{cases} \quad (6.147)$$

is

$$u(r,\theta,\varphi,t)=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\Phi(r,\theta,\varphi,t)+B^2W_{k_{nl}^2\Phi}(r,\theta,\varphi,t)\,.$$

Proof. Following a similar approach as that in Section 6.7.1, we obtain the $u(r,\theta,\varphi,t)$ that satisfies the equation and the boundary condition of PDS (6.147)

$$u(r,\theta,\varphi,t)=\sum_{m,n,l}\mathrm{e}^{\alpha_{nl}t}[(A_{mnl}\cos\beta_{nl}t+B_{mnl}\underline{\sin}\beta_{nl}t)\cos m\varphi$$
$$+(C_{mnl}\cos\beta_{nl}t+D_{mnl}\underline{\sin}\beta_{nl}t)\sin m\varphi]P_n^m(\cos\theta)j_n(k_{nl}r)\,. \quad (6.148)$$

Applying the initial condition $u(r,\theta,\varphi,0)=\Phi(r,\theta,\varphi)$ yields

$$\begin{cases} A_{mnl}=\dfrac{1}{M_{mnl}}\displaystyle\iiint\limits_{r\le a}\Phi(r,\theta,\varphi)P_n^m(\cos\theta)j_n(k_{nl}r)r^2\cos m\varphi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,,\\ C_{mnl}=\dfrac{1}{M_{mnl}}\displaystyle\iiint\limits_{r\le a}\Phi(r,\theta,\varphi)P_n^m(\cos\theta)j_n(k_{nl}r)r^2\sin m\varphi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,. \end{cases} \tag{6.149}$$

Applying the initial condition $u_t(r,\theta,\varphi,0)=0$ yields

$$\begin{cases} \alpha_{nl}A_{mnl}+\underline{\beta}_{nl}B_{mnl}=0\,,\\ \alpha_{nl}C_{mnl}+\underline{\beta}_{nl}D_{mnl}=0\,, \end{cases}$$

so that

$$\begin{cases} B_{mnl}=-\dfrac{\alpha_{nl}}{\underline{\beta}_{nl}}A_{mnl}=-\dfrac{\alpha_{nl}}{M_{mnl}\underline{\beta}_{nl}}\\ \qquad\cdot\displaystyle\iiint\limits_{r\le a}\Phi(r,\theta,\varphi)P_n^m(\cos\theta)j_n(k_{nl}r)r^2\cos m\varphi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,,\\ D_{mnl}=-\dfrac{\alpha_{nl}}{\underline{\beta}_{nl}}C_{mnl}=-\dfrac{\alpha_{nl}}{M_{mnl}\underline{\beta}_{nl}}\\ \qquad\cdot\displaystyle\iiint\limits_{r\le a}\Phi(r,\theta,\varphi)P_n^m(\cos\theta)j_n(k_{nl}r)r^2\sin m\varphi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,. \end{cases}$$

By the structure of $W_\psi(r,\theta,\varphi,t)$ in Eq. (6.146), we have

$$\begin{cases} \dfrac{\partial}{\partial t}W_\Phi(r,\theta,\varphi,t)=\displaystyle\sum_{m,n,l}\left(\alpha_{nl}\underline{\sin}\beta_{nl}t+\underline{\beta}_{nl}\cos\beta_{nl}t\right)\\ \qquad\cdot(B^*_{mnl}\cos m\varphi+D^*_{mnl}\sin m\varphi)\,\mathrm{e}^{\alpha_{nl}t}P_n^m(\cos\theta)j_n(k_{nl}r),\\ B^*_{mnl}=\dfrac{A_{mnl}}{\underline{\beta}_{nl}}=\dfrac{1}{M_{mnl}\underline{\beta}_{nl}}\\ \qquad\cdot\displaystyle\iiint\limits_{r\le a}\Phi(r,\theta,\varphi)P_n^m(\cos\theta)j_n(k_{nl}r)r^2\cos m\varphi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,,\\ D^*_{mnl}=\dfrac{C_{mnl}}{\beta_{nl}}=\dfrac{1}{M_{mnl}\underline{\beta}_{nl}}\\ \qquad\cdot\displaystyle\iiint\limits_{r\le a}\Phi(r,\theta,\varphi)P_n^m(\cos\theta)j_n(k_{nl}r)r^2\sin m\varphi\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,. \end{cases} \tag{6.150}$$

Note that

$$B^*_{mnl}\underline{\beta}_{nl} = A_{mnl},\, D^*_{mnl}\underline{\beta}_{nl} = C_{mnl}\,,$$
$$\alpha_{nl} B^*_{mnl} = -B_{mnl},\, \alpha_{nl} D^*_{mnl} = -D_{mnl}\,.$$

Thus

$$\begin{aligned}\frac{\partial}{\partial t} W_{\Phi}(r,\theta,\varphi,t) = \sum_{m,n,l} \{&[(-B_{mnl}\underline{\sin}\beta_{nl}t)\cos m\varphi \\ &+ (-D_{mnl}\underline{\sin}\beta_{nl}t)\sin m\varphi] + [(A_{mnl}\cos\beta_{nl}t)\cos m\varphi \\ &+ (C_{mnl}\cos\beta_{nl}t)\sin m\varphi]\}\, \mathrm{e}^{\alpha_{nl}t} P_n^m(\cos\theta)\, j_n(k_{nl}r)\,. \end{aligned} \tag{6.151}$$

Substituting Eqs. (6.149) and (6.151) into Eq. (6.148) and using the structure of $W_{\psi}(r,\theta,\varphi,t)$ in Eq. (6.146) thus yields the solution of PDS (6.147)

$$u(r,\theta,\varphi,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_{\Phi}(r,\theta,\varphi,t) + B^2 W_{k^2_{nl}\Phi}(r,\theta,\varphi,t)\,.$$

6.7.3 *Solution from* $\mathbf{f}(\mathbf{r},\theta,\varphi,\mathbf{t})$

Theorem 2. Let $u = W_{\psi}(r,\theta,\varphi,t)$ be the solution of PDS (6.141). The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2 \Delta u(r,\theta,\varphi,t) + B^2 \dfrac{\partial}{\partial t}\Delta u(r,\theta,\varphi,t) \\ \quad + f(r,\theta,\varphi,t),\, 0<r<a,\, 0<\theta<\pi,\, 0<\varphi<2\pi,\, 0<t\,, \\ L(u,u_r)|_{r=a} = 0\,, \\ u(r,\theta,\varphi,0) = 0, \quad u_t(r,\theta,\varphi,0) = 0 \end{cases} \tag{6.152}$$

is

$$u(r,\theta,\varphi,t) = \int_0^t W_{f_\tau}(r,\theta,\varphi,t-\tau)\,\mathrm{d}\tau\,, \tag{6.153}$$

where $f_\tau = f(r,\theta,\varphi,\tau)$. Thus Theorem 2 in Section 6.1 is also valid for spherical coordinate systems.

Proof. By the definition of $W_\psi(r,\theta,\varphi,t)$, the $W_{f_\tau}(r,\theta,\varphi,t-\tau)$ satisfies

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_{f_\tau}}{\partial t}+\dfrac{\partial^2 W_{f_\tau}}{\partial t^2}=A^2\Delta W_{f_\tau}+B^2\dfrac{\partial}{\partial t}\Delta W_{f_\tau} & \text{(6.154a)}\\ L\left(W_{f_\tau},\dfrac{\partial W_{f_\tau}}{\partial r}\right)\Big|_{r=a}=0, & \text{(6.154b)}\\ W_{f_\tau}\big|_{t=\tau}=0,\ \dfrac{\partial W_{f_\tau}}{\partial t}\Big|_{t=\tau}=f(r,\theta,\varphi,\tau). & \text{(6.154c)} \end{cases}$$

By Eq. (6.154b)

$$L\left(\int_0^t W_{f_\tau}\,\mathrm{d}\tau,\frac{\partial}{\partial r}\int_0^t W_{f_\tau}\,\mathrm{d}\tau\right)\Big|_{r=a}=\int_0^t L\left(W_{f_\tau},\frac{\partial}{\partial r}W_{f_\tau}\right)\Big|_{r=a}\mathrm{d}\tau=0,$$

so that the $u(r,\theta,\varphi,t)$ in Eq. (6.153) satisfies the boundary conditions of PDS (6.152).

Clearly, the $u(r,\theta,\varphi,t)$ in Eq. (6.153) satisfies the initial condition $u(r,\theta,\varphi,0)=0$. Also

$$\frac{\partial u}{\partial t}=\int_0^t\frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau+W_{f_\tau}\big|_{\tau=t},$$

which shows that $u_t(r,\theta,\varphi,0)=0$ by Eq. (6.154c). Thus the $u(r,\theta,\varphi,t)$ in Eq. (6.153) also satisfies the two initial conditions of PDS (6.152).

Since for the $u(r,\theta,\varphi,t)$ in Eq. (6.153),

$$\frac{\partial u}{\partial t}=\int_0^t\frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau+W_{f_\tau}\big|_{\tau=t}=\int_0^t\frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau,$$

$$\frac{\partial^2 u}{\partial t^2}=\int_0^t\frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau+\frac{\partial W_{f_\tau}}{\partial t}\Big|_{\tau=t}=\int_0^t\frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau+f(r,\theta,\varphi,t),$$

$$\Delta u=\int_0^t\Delta W_{f_\tau}\,\mathrm{d}\tau,$$

$$\begin{aligned}\frac{\partial}{\partial t}\Delta u&=\frac{\partial}{\partial t}\int_0^t\Delta W_{f_\tau}\,\mathrm{d}\tau=\int_0^t\frac{\partial}{\partial t}\Delta W_{f_\tau}\,\mathrm{d}\tau+\Delta W_{f_\tau}\big|_{\tau=t}\\&=\int_0^t\frac{\partial}{\partial t}\Delta W_{f_\tau}\,\mathrm{d}\tau,\end{aligned}$$

we have

$$
\begin{aligned}
&\frac{1}{\tau_0}\frac{\partial}{\partial t}\int_0^t W_{f_\tau}\,\mathrm{d}\tau+\frac{\partial^2}{\partial t^2}\int_0^t W_{f_\tau}\,\mathrm{d}\tau-A^2\Delta\int_0^t W_{f_\tau}\,\mathrm{d}\tau-B^2\frac{\partial}{\partial t}\Delta\int_0^t W_{f_\tau}\,\mathrm{d}\tau\\
&=\int_0^t\left(\frac{1}{\tau_0}\frac{\partial W_{f_\tau}}{\partial t}+\frac{\partial^2 W_{f_\tau}}{\partial t^2}-A^2\Delta W_{f_\tau}-B^2\frac{\partial}{\partial t}\Delta W_{f_\tau}\right)\mathrm{d}\tau+f(r,\theta,\varphi,t)\\
&=f(r,\theta,\varphi,t)\,.
\end{aligned}
$$

Therefore, the $u(r,\theta,\varphi,t)$in Eq. (6.153) is indeed the solution of PDS (6.152).

Remark 1. Let $u=W_\psi(r,\theta,\varphi,t)$ be the solution of PDS (6.141) (Eq. (6.146)). The solution of PDS (6.140) is, by the principle of superposition,

$$
\begin{aligned}
u=&\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\Phi(r,\theta,\varphi,t)+B^2W_{k_{nl}^2\Phi}(r,\theta,\varphi,t)\\
&+W_\psi(r,\theta,\varphi,t)+\int_0^t W_{f_\tau}(r,\theta,\varphi,t-\tau)\,\mathrm{d}\tau\,,
\end{aligned}
\tag{6.155}
$$

where $f_\tau=f(r,\theta,\varphi,\tau)$, $k_{nl}=\mu_l^{(n+\frac{1}{2})}/a$. The $\mu_l^{(n+\frac{1}{2})}$ are available in Section 2.6.2.

Remark 2. All results for the hyperbolic heat-conduction equations are recovered as the special case of dual-phase-lagging heat-conduction equations at $B=0$.

Remark 3. By using the structure of W_ψ in Eq. (6.146), we obtain the $u(r,\theta,\varphi,t)$ in Eq. (6.153).

$$
\begin{aligned}
u&=\int_0^t W_{f_\tau}(r,\theta,\varphi,t-\tau)\,\mathrm{d}\tau\\
&=\int_0^t\iiint\limits_{r\le a}G(r,r';\theta,\theta';\varphi,\varphi';t-\tau)f(r',\theta',\varphi',\tau)\,\mathrm{d}\Omega\,\mathrm{d}\tau\,.
\end{aligned}
$$

This is called the *integral expression* of the solution of PDS (6.152). The triple series $G(r,r';\theta,\theta';\varphi,\varphi';t-\tau)$ is called the *Green function of dual-phase-lagging heat-conduction equations in a spherical domain*. When $f(r,\theta,\varphi,t)=\delta(\mathbf{r}-\mathbf{r}_0,t-t_0)$, in particular, the solution of PDS (6.152) reduces

$$
u=G(r,r_0;\theta,\theta_0;\varphi,\varphi_0;t-t_0)\,.
$$

where $\mathbf{r}=(\mathbf{r},\theta,\varphi)$, $\mathbf{r}_0=(r_0,\theta_0,\varphi_0)$. Thus the Green function $G(r,r_0;\theta,\theta_0;\varphi,\varphi_0;t,t_0)$ is the solution due to a source term $\delta(\mathbf{r}-\mathbf{r}_0,t-t_0)$.

6.8 Cauchy Problems

In this section we develop the solution structure theorem for Cauchy problems of dual-phase-lagging heat-conduction equations. We also briefly discuss the methods of solving Cauchy problems without going into the details.

In this section we let Ω be R^1, R^2 or R^3. Δ stands for one-, two- or three-dimensional Laplace operator. M represents a point in R^1, R^2 or R^3 and $u(M,t)$ is the temperature at point M and time instant t.

Theorem 1. Let $u = W_\psi(M,t)$ be the solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + B^2\dfrac{\partial}{\partial t}\Delta u, \quad \Omega\times(0,+\infty)\,, \\ u(M,0) = 0, \quad u_t(M,0) = \psi(M)\,. \end{cases} \tag{6.156}$$

The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + B^2\dfrac{\partial}{\partial t}\Delta u, \quad \Omega\times(0,+\infty), \\ u(M,0) = \varphi(M), \quad u_t(M,0) = 0 \end{cases} \tag{6.157}$$

is

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t} - B^2\Delta\right) W_\varphi(M,t)\,. \tag{6.158}$$

Proof. By its definition, the $W_\varphi(M,t)$ satisfies

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_\varphi}{\partial t} + \dfrac{\partial^2 W_\varphi}{\partial t^2} - A^2\Delta W_\varphi - B^2\dfrac{\partial}{\partial t}\Delta W_\varphi = 0\,, & \text{(6.159a)} \\ W_\varphi(M,0) = 0, \quad \left.\dfrac{\partial W_\varphi}{\partial t}\right|_{t=0} = \varphi(M)\,. & \text{(6.159b)} \end{cases}$$

Substituting Eq. (6.158) into the equation of PDS (6.157) and using Eq. (6.159a) yields

$$
\begin{aligned}
&\frac{u_t}{\tau_0}+u_{tt}-A^2\Delta u-B^2\frac{\partial}{\partial t}\Delta u\\
=&\frac{1}{\tau_0}\frac{\partial}{\partial t}\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi+\frac{\partial^2}{\partial t^2}\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi\\
&-A^2\Delta\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi-B^2\frac{\partial}{\partial t}\Delta\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi\\
=&\frac{1}{\tau_0}\left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t}+\frac{\partial^2 W_\varphi}{\partial t^2}-A^2\Delta W_\varphi-B^2\frac{\partial}{\partial t}\Delta W_\varphi\right)\\
&+\frac{\partial}{\partial t}\left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t}+\frac{\partial^2 W_\varphi}{\partial t^2}-A^2\Delta W_\varphi-B^2\frac{\partial}{\partial t}\Delta W_\varphi\right)\\
&-B^2\Delta\left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t}+\frac{\partial^2 W_\varphi}{\partial t^2}-A^2\Delta W_\varphi-B^2\frac{\partial}{\partial t}\Delta W_\varphi\right)=0\,.
\end{aligned}
$$

Therefore, the $u(M,t)$ in Eq. (6.158) satisfies the equation of PDS (6.157).

By Eq. (6.159b), we have

$$
\begin{aligned}
u(M,0)=&\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi\Big|_{t=0}\\
&=\frac{1}{\tau_0}W_\varphi(M,0)+\frac{\partial W_\varphi}{\partial t}\Big|_{t=0}-B^2\Delta W_\varphi\big|_{t=0}=\varphi(M)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial u}{\partial t}&=\frac{\partial}{\partial t}\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi(M,t)\\
&=\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t}+\frac{\partial^2 W_\varphi}{\partial t^2}-B^2\frac{\partial}{\partial t}\Delta W_\varphi=A^2\Delta W_\varphi\,,
\end{aligned}
$$

such that

$$
\frac{\partial u}{\partial t}\Big|_{t=0}=A^2\Delta W_\varphi\big|_{t=0}=0\,.
$$

Therefore, the $u(M,t)$ in Eq. (6.158) also satisfies the two initial conditions of PDS (6.157), thus verifying that it is the solution of Eq. (6.157).

Theorem 2. Let $u = W_\psi(M,t)$ be the solution of PDS (6.156). The solution of

$$\begin{cases} \dfrac{u_t}{\tau_0} + u_{tt} = A^2\Delta u + B^2 \dfrac{\partial}{\partial t}\Delta u + f(M,t), \quad \Omega \times (0,+\infty), \\ u(M,0) = 0, \quad u_t(M,0) = 0 \end{cases} \tag{6.160}$$

is

$$u = \int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau, \tag{6.161}$$

where $\quad f_\tau = f(M,\tau)$.

Proof. By the definition of $W_\psi(M,t)$, the $W_{f_\tau}(M,t-\tau)$ satisfies

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_{f_\tau}}{\partial t} + \dfrac{\partial^2 W_{f_\tau}}{\partial t^2} - A^2\Delta W_{f_\tau} - B^2\dfrac{\partial}{\partial t}\Delta W_{f_\tau} = 0, \\ W_{f_\tau}(M,t-\tau)\big|_{t=\tau} = 0, \quad \dfrac{\partial}{\partial t}W_{f_\tau}(M,t-\tau)\Big|_{t=\tau} = f(M,\tau). \end{cases} \tag{6.162}$$

Thus

$$\begin{aligned}
&\frac{u_t}{\tau_0} + u_{tt} - A^2\Delta u - B^2\frac{\partial}{\partial t}\Delta u \\
&= \frac{1}{\tau_0}\frac{\partial}{\partial t}\int_0^t W_{f_\tau}\,\mathrm{d}\tau + \frac{\partial^2}{\partial t^2}\int_0^t W_{f_\tau}\,\mathrm{d}\tau - A^2\Delta\int_0^t W_{f_\tau}\,\mathrm{d}\tau - B^2\frac{\partial}{\partial t}\Delta\int_0^t W_{f_\tau}\,\mathrm{d}\tau \\
&= \frac{1}{\tau_0}\left(\int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}\big|_{\tau=t}\right) + \frac{\partial}{\partial t}\left(\int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + W_{f_\tau}\big|_{\tau=t}\right) \\
&\quad - A^2\int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau - B^2\frac{\partial}{\partial t}\int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau \\
&= \frac{1}{\tau_0}\int_0^t \frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau + \int_0^t \frac{\partial^2 W_{f_\tau}}{\partial t^2}\,\mathrm{d}\tau + \frac{\partial W_{f_\tau}}{\partial t}\bigg|_{\tau=t} \\
&\quad - A^2\int_0^t \Delta W_{f_\tau}\,\mathrm{d}\tau - B^2\left[\int_0^t \frac{\partial}{\partial t}\Delta W_{f_\tau}\,\mathrm{d}\tau + \Delta W_{f_\tau}\big|_{\tau=t}\right] \\
&= \int_0^t \left(\frac{1}{\tau_0}\frac{\partial W_{f_\tau}}{\partial t} + \frac{\partial^2 W_{f_\tau}}{\partial t^2} - A^2\Delta W_{f_\tau} - B^2\frac{\partial}{\partial t}\Delta W_{f_\tau}\right)\mathrm{d}\tau + f(M,t) \\
&= f(M,t).
\end{aligned}$$

Hence the $u(M,t)$ in Eq. (6.161) satisfies the equation of PDS (6.160). Clearly, the $u(M,t)$ in Eq. (6.161) also satisfies the initial condition $u(M,0)=0$. Also,

$$u_t(M,t)=\frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M,t-\tau)\,\mathrm{d}\tau=\int_0^t\frac{\partial W_{f_\tau}}{\partial t}\,\mathrm{d}\tau+W_{f_\tau}\big|_{\tau=t}\,,$$

which shows that $u_t(M,0)=0$ by the initial condition of PDS (6.162). Therefore the $u(M,t)$ in Eq. (6.161) is indeed the solution of PDS (6.160).

Remark 1. Let $u=W_\psi(M,t)$ be the solution of PDS (6.156). The solution of

$$\begin{cases}\dfrac{u_t}{\tau_0}+u_{tt}=A^2\Delta u+B^2\dfrac{\partial}{\partial t}\Delta u+f(M,t)\,, & \Omega\times(0,+\infty),\\ u(M,0)=\varphi(M)\,, \quad u_t(M,0)=\psi(M)\,. \end{cases}\tag{6.163}$$

is, thus by the principle of superposition,

$$u=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi(M,t)+W_\psi(M,t)+\int_0^t W_{f_\tau(M,t-\tau)}\,\mathrm{d}\tau\,,$$

where

$$f_\tau=f(M,\tau)\,.$$

Remark 2. The $W_\psi(M,t)$ can be readily obtained by integral transformations. The one-dimensional $W_\psi(M,t)$ can be obtained by using either the Laplace transformation with respect to the temporal variable t or the Fourier transformation with respect to the spatial variable x. The two- or three-dimensional $W_\psi(M,t)$ can be found by using multiple Fourier transformations. For the three-dimensional case, for example, the $W_\psi(M,t)$ is the solution of

$$\begin{cases}\dfrac{u_t}{\tau_0}+u_{tt}=A^2\Delta u+B^2\dfrac{\partial}{\partial t}\Delta u, & R^3\times(0,+\infty)\,,\\ u(M,0)=0, \quad u_t(M,0)=\psi(M)\,. \end{cases}\tag{6.164}$$

Its triple Fourier transformation yields

$$\begin{cases}\bar{u}_{tt}(\boldsymbol{\omega},t)+\left(\dfrac{1}{\tau_0}+B^2\omega^2\right)\bar{u}_t(\boldsymbol{\omega},t)+A^2\omega^2\bar{u}_t(\boldsymbol{\omega},t)=0\,,\\ \bar{u}(\boldsymbol{\omega},0)=0, \quad \bar{u}_t(\boldsymbol{\omega},0)=\bar{\psi}(\boldsymbol{\omega})\,. \end{cases}\tag{6.165}$$

Let $\alpha(\boldsymbol{\omega})\pm\mathrm{i}\beta(\omega)$ be the characteristic roots of the equation of PDS (6.165). Then

$$\bar{u}(\boldsymbol{\omega},t)=\mathrm{e}^{\alpha(\omega)t}[A(\omega)\cos\beta(\omega)t+B(\omega)\underline{\sin}\beta(\omega)t]\,,$$

where the $A(\omega)$ and the $B(\omega)$ are functions of ω to be determined. Applying the initial conditions in (6.165) leads to

$$A(\omega)=0\,, \quad B(\omega)=\frac{\bar{\psi}(\boldsymbol{\omega})}{\beta(\omega)}\,.$$

Thus

$$\bar{u}(\boldsymbol{\omega},t)=\frac{\bar{\psi}(\boldsymbol{\omega})}{\beta(\omega)}\,\mathrm{e}^{\alpha(\omega)t}\,\underline{\sin}\beta(\omega)t\,.$$

Thus its inverse Fourier transformation yields the solution of PDS (6.164)

$$u(M,t)=\frac{1}{(2\pi)^3}\iiint\limits_{R^3}\bar{u}(\boldsymbol{\omega},t)\,\mathrm{e}^{\mathrm{i}(\omega_1x+\omega_2y+\omega_3z)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3\,.$$

The one-dimensional $W_\psi(M,t)$ satisfies

$$\begin{cases}\dfrac{u_t}{\tau_0}+u_{tt}=A^2u_{xx}+B^2u_{txx}\,, & R^1\times(0,+\infty)\,,\\ u(x,0)=0, \quad u_t(x,0)=\psi(x)\,.\end{cases}\tag{6.166}$$

Its Laplace transformation with respect to t yields

$$\frac{s}{\tau_0}\bar{u}(x,s)+s^2\bar{u}(x,s)-\psi(x)=A^2\bar{u}_{xx}(x,s)+B^2s\bar{u}_{xx}$$

or

$$\left(B^2s+A^2\right)\bar{u}_{xx}(x,s)+\left(s^2+\frac{s}{\tau_0}\right)\bar{u}(x,s)=-\psi(x)\,,$$

so that

$$\bar{u}_{xx}(x,s)+\frac{s^2+\frac{s}{\tau_0}}{B^2s+A^2}\bar{u}(x,s)=\frac{-\psi(x)}{B^2s+A^2}\,.$$

An integral expression of $W_\psi(M,t)$ can thus be found by inverse Laplace transformation (see Section 5.2.2).

Remark 3. In PDS (6.164), $A^2=\dfrac{\alpha}{\tau_0}$ and $B^2=\dfrac{\alpha}{\tau_0}\cdot\tau_T$. Since α, τ_0 and τ_T are all normally very small, we have $B^2\ll A^2$ and $0<B^2\ll 1$. Also the solution of PDS (6.166) is known at $B=0$ (see Section 5.2). Hence we can obtain an approximate analytical solution of PDS (6.166) by using a perturbation method with respect to B^2.

6.9 Perturbation Method for Cauchy Problems

While PDS (6.164) can be solved by methods of integral transformation, the inverse transformations are normally quite involved. The perturbation method is an effective method of obtaining approximate analytical solutions and can also be used to find exact solutions for some special cases. We apply the perturbation method here mainly for one-dimensional problems. A similar approach can be followed for two- and three-dimensional problems.

6.9.1 Introduction

The perturbation method aims to find approximate solutions of mathematical problems involving a small parameter ε; for example: algebraic equations, initial-value problems of ODE and PDE. Let P_ε be the problem involving such a small parameter ε and P_0 be the corresponding problem when $\varepsilon = 0$.

Once the exact solution of P_0 is available, we can obtain an approximate solution of P_ε by expanding the solution of P_ε in terms of power series of ε and keeping the first few terms of the series. The ε^0 term in the expansion is the exact solution of P_0.

Therefore, the perturbation method is a method of obtaining an approximate solution of P_ε that is based on the exact solution of P_0 and corrected by a few terms of the power function of ε.

In the case $B^2 = \varepsilon$, consider the one-dimensional version of PDS (6.164)

$$P_\varepsilon : \begin{cases} u_t/\tau_0 + u_{tt} = A^2\Delta u + \varepsilon u_{txx}\,, & R^1 \times (0,+\infty)\,, \\ u(x,0) = 0\,, \quad u_t(x,0) = \psi(x)\,,\ 0 < \varepsilon \ll 1\,. \end{cases} \tag{6.167}$$

The solution of P_0 is available in Section 5.2,

$$\begin{aligned} u(x,t) &= \frac{1}{2A}\,\mathrm{e}^{-\frac{t}{2\tau_0}} \int_{x-At}^{x+At} I_0\left(b\sqrt{(At)^2-(\xi-x)^2}\right)\psi(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{2A}\,\mathrm{e}^{-\frac{t}{2\tau_0}} \int_{-At}^{At} I_0\left(b\sqrt{(At)^2-u^2}\right)\psi(u+x)\,\mathrm{d}u\,, \end{aligned} \tag{6.168}$$

where $b = 1/2A\tau_0$.

The perturbation method for P_ε is to correct the $u(x,t)$ in Eq. (6.168) by a polynomial of power terms of ε to obtain an approximate analytical solution of (6.167). We focus our discussion only on regular perturbation. When $\psi(x)$ is a polynomial of x, in particular, the perturbation method can lead to the exact solution of P_ε. Note that elementary functions can normally be approximated by Taylor polynomials. It is thus very useful to discuss solutions of PDS (6.167) with a polynomial $\psi(x)$, i.e.

$$\psi(x) = P_N(x) = \sum_{n=0}^{N} a_n x^n\,.$$

6.9.2 The Perturbation Method for Solving Hyperbolic Heat-Conduction Equations

Consider

$$P_0 : \begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx}, & R^1 \times (0, +\infty), \\ u(x,0) = 0, & u_t(x,0) = P_N(x). \end{cases} \tag{6.169}$$

1. When N is even such that $N = 2m$, the solution of PDS (6.169) reads, by Eq. (6.168),

$$u = \frac{1}{2A} \mathrm{e}^{-\frac{t}{2\tau_0}} \int_{-At}^{+At} I_0\left(b\sqrt{(At)^2 - u^2}\right) \left[\sum_{n=0}^{N} a_n (u+x)^n\right] \mathrm{d}u \tag{6.170}$$

$$= \frac{1}{2A} \mathrm{e}^{-\frac{t}{2\tau_0}} \int_{-At}^{+At} I_0\left(b\sqrt{(At)^2 - u^2}\right) \left[\sum_{n=0}^{N} a_n \left(\sum_{k=0}^{n} C_n^k u^k x^{n-k}\right)\right] \mathrm{d}u. \tag{6.171}$$

For convenience, define

$$G_i(t) = \int_{-At}^{+At} I_0\left(b\sqrt{(At)^2 - u^2}\right) u^i \,\mathrm{d}u, \quad i = 0, 1, \cdots n, \tag{6.172}$$

where $I_0(x)$ is the modified Bessel function of order zero and the first kind. Since $I_0(x)$ is defined by a power series that converges very quickly, we can easily obtain $G_i(t)$ by integration term by term. The $G_i(t)$ can sometimes be expressed by elementary functions; for example,

$$G_0(t) = \int_{-At}^{+At} I_0\left(b\sqrt{(At)^2 - u^2}\right) \mathrm{d}u = 2A\tau_0 \mathrm{e}^{-\frac{t}{2\tau_0}} \left(1 - \mathrm{e}^{-\frac{t}{\tau_0}}\right). \tag{6.173}$$

When i is odd, $G_i(t) = 0$.

Substituting (6.172) into (6.171) and applying (6.172) yield the solution of (6.169),

$$\begin{aligned}
u &= \frac{1}{2A} e^{-\frac{t}{2\tau_0}} \sum_{k=0}^{N} \sum_{n=0}^{N} C_n^k a_n x^{n-k} G_k(t) \\
&= \frac{1}{2A} e^{-\frac{t}{2\tau_0}} \left(\sum_{n=0}^{N} a_n x^n \right) G_0(t) + \frac{1}{2A} e^{-\frac{t}{2\tau_0}} \sum_{k=1}^{N} \sum_{n=0}^{N} a_n \frac{(x^n)^{(k)}}{k!} G_k(t) \\
&= \tau_0 P_N(x) \left(1 - e^{-\frac{t}{\tau_0}}\right) + \frac{1}{2A} e^{-\frac{t}{2\tau_0}} \sum_{k=1}^{N} \frac{P_N^{(k)}(x)}{k!} G_k(t) \\
&= \tau_0 P_N(x) \left(1 - e^{-\frac{t}{\tau_0}}\right) + \frac{1}{2A} e^{-\frac{t}{2\tau_0}} \left(\sum_{k=1}^{m} \frac{P_N^{(2k)}(x)}{(2k)!} G_{2k} \right).
\end{aligned} \tag{6.174}$$

2. When N is odd such that $N = 2m+1$, the solution of PDS (6.169) can be formed by the sum of that in Eq. (6.174) and the solution due to the last term of $P_N(x) = \sum_{n=0}^{N} a_n x^n$, i.e. $a_{2m+1} x^{2m+1}$. The latter can be obtained by Eq. (6.168),

$$\begin{aligned}
u &= \frac{1}{2A} e^{-\frac{t}{2\tau_0}} \int_{x-At}^{x+At} I_0\left(b\sqrt{(At)^2 - (x-\xi)^2}\right) a_{2m+1} \xi^{2m+1} \, d\xi \\
&= \frac{1}{2A} e^{-\frac{t}{2\tau_0}} \int_{-At}^{+At} I_0\left(b\sqrt{(At)^2 - u^2}\right) \left(a_{2m+1} \sum_{k=0}^{2m+1} C_{2m+1}^k u^k x^{2m+1-k} \right) du \\
&= \frac{1}{2A} e^{-\frac{t}{2\tau_0}} a_{2m+1} x^{2m+1} G_0(t) + \frac{1}{2A} e^{-\frac{t}{2\tau_0}} \sum_{k=1}^{2m+1} \frac{a_{2m+1} \left(x^{2m+1}\right)^{(k)}}{k!} G_k(t) \\
&= \tau_0 a_{2m+1} x^{2m+1} \left(1 - e^{-\frac{t}{\tau_0}}\right) + \frac{1}{2A} e^{-\frac{t}{2\tau_0}} \sum_{k=1}^{m} \frac{a_{2m+1} \left(x^{2m+1}\right)^{(2k)}}{(2k)!} G_{2k}(t).
\end{aligned} \tag{6.175}$$

Thus the solution of PDS (6.169) is, by adding Eqs. (6.174) and (6.175),

$$u = \tau_0 P_N(x) \left(1 - e^{-\frac{t}{\tau_0}}\right) + \frac{1}{2A} e^{-\frac{t}{2\tau_0}} \left(\sum_{k=1}^{[N/2]} \frac{P_N^{(2k)}(x)}{(2k)!} G_{2k}(t) \right), \tag{6.176}$$

where $[N/2]$ stands for the maximum positive integer not larger than $N/2$. Equation (6.176) shows that the solution is a sum of $[N/2]+1$ terms. All the terms are in the form of separation of variables.

3. Denote the solution (6.176) of PDS (6.169) by $u = W_{P_N}(x,t)$. The solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + P_N(x), & R^1 \times (0, +\infty), \\ u(x,0) = P_m(x), \; u_t(x,0) = P_l(x) \end{cases} \tag{6.177}$$

is, by the solution structure theorem,

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_{P_m}(x,t) + W_{P_l}(x,t) + \int_0^t W_{P_N}(x,t-\tau)\,\mathrm{d}\tau\,. \tag{6.178}$$

Here

$$P_m(x) = \sum_{n=0}^{m} b_n x^n\,,\ P_l(x) = \sum_{n=0}^{l} a_n x^n\,,\ P_N(x) = \sum_{n=0}^{N} c_n x^n\,.$$

Consider

$$f(x,t) = P_N(x,t) = \sum_{n=0}^{N} c_n(t) x^n\,, \quad c_n(t) \text{ are functions of } t\,.$$

The solution of PDS (6.177) for the case of $P_m(x) = P_l(x) = 0$ is, by Eq. (5.33),

$$\begin{aligned} u &= \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0\left(b\sqrt{A^2(t-\tau)^2-(x-\xi)^2}\right)\left(\sum_{n=0}^{N} c_n(\tau)\xi^n\right)\mathrm{d}\xi \\ &= \tau_0\int_0^t P_N(x,\tau)\left(1-\mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right)\mathrm{d}\tau + \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\left(\sum_{k=1}^{[N/2]}\frac{P_N^{(2k)}(x,\tau)}{(2k)!}G_{2k}(t-\tau)\right)\mathrm{d}\tau \end{aligned} \tag{6.179}$$

or

$$u = \int_0^t W_{P_{N_\tau}}(x,t-\tau)\,\mathrm{d}\tau\,,\ P_{N_\tau} = P_N(x,\tau)\,. \tag{6.180}$$

Here $P_N^{(2k)}(x,\tau)$ is the $2k$-th derivative of $P_N(x,\tau)$ with respect to x. It can be shown that the unit in the above solutions is correct. For example, the unit of the second term on the right side of Eq. (6.179) is

$$[u] = \left[A^{-1}\right]\left[P_N^{(2k)}(x,\tau)\right][G_{2k}(t-\tau)]\,[\mathrm{d}\tau] = \frac{T}{L}\cdot\frac{\Theta}{T^2L^{2k}}\cdot L^{2k+1}\cdot T = \Theta\,.$$

6.9.3 Perturbation Solutions of Dual-Phase-Lagging Heat-Conduction Equations

Consider

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + \varepsilon u_{txx}\,, & R^1 \times (0,+\infty)\,, \\ u(x,0) = 0\,, \quad u_t(x,0) = P_N(x)\,. \end{cases} \tag{6.181}$$

Let the solution of PDS (6.181) be

$$u = u_0(x,t) + u_1(x,t)\varepsilon + u_2(x,t)\varepsilon^2 + \cdots + u_n(x,t)\varepsilon^n + \cdots \quad , \tag{6.182}$$

where $u_n(x,t)(n=0,1,2,\cdots)$ are functions to be determined. Substituting Eq. (6.182) into (6.181) and comparing the coefficients of $\varepsilon^n(n=0,1,2,\cdots)$ yields the PDS of $u_n(x,t)(n=0,1,2,\cdots)$

$$\varepsilon^0: \begin{cases} u_{0t}/\tau_0 + u_{0tt} = A^2 u_{0xx}, & R^1 \times (0,+\infty) \\ u_0(x,0) = 0, \quad u_{0t}(x,0) = P_N(x). \end{cases} \tag{6.183}$$

$$\varepsilon^1: \begin{cases} u_{1t}/\tau_0 + u_{1tt} = A^2 u_{1xx} + u_{0txx}, & R^1 \times (0,+\infty) \\ u_1(x,0) = 0, \quad u_{1t}(x,0) = 0. \end{cases} \tag{6.184}$$

$$\varepsilon^2: \begin{cases} u_{2t}/\tau_0 + u_{2tt} = A^2 u_{2xx} + u_{1txx}, & R^1 \times (0,+\infty) \\ u_2(x,0) = 0, \quad u_{2t}(x,0) = 0. \end{cases} \tag{6.185}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

The solution $u_0(x,t)$ of (6.183) is the same as that in Eq. (6.176), which is a $(N-0)$-th polynomial of x. The nonhomogeneous term u_{0txx} in PDS (6.184) can be obtained and is a $(N-2)$-th polynomial of x. The $u_1(x,t)$ can be obtained by Eq. (5.33) by replacing $f(x,t)$ by u_{0txx} and is a $(N-2)$-th polynomial of x. The u_{1txx} is thus a $(N-4)$-th polynomial of x. Therefore, $u_n(x,t)(n=0,1,2,\cdots)$ is a $(N-2n)$-th polynomial of x and $u_n(x,t)=0$ $(n=[N/2]+1,[N/2]+2,\cdots)$. Therefore, the analytical solution of PDS (6.181) can be expressed by

$$u = u_0(x,t) + u_1(x,t)\varepsilon + u_2(x,t)\varepsilon^2 + \cdots + u_{[N/2]}(x,t)\varepsilon^{[N/2]}. \tag{6.186}$$

Let

$$S(\square) = \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0\left(b\sqrt{A^2(t-\tau)^2-(x-\xi)^2}\right) \left.\frac{\partial^3\square}{\partial t\partial x^2}\right|_{\substack{x=\xi \\ t=\tau}} \mathrm{d}\xi\,, \tag{6.187}$$

so that $S(u_0)=u_1$, $S(u_1)=S^2(u_0)=u_2$, $\cdots$, $S^{[N/2]}(u_0)=u_{[N/2]}$. Thus the solution of PDS (6.181) is

$$u = u_0 + S(u_0)\varepsilon + S^2(u_0)\varepsilon^2 + \cdots + S^{[N/2]}(u_0)\varepsilon^{[N/2]}. \tag{6.188}$$

Here u_0 is the solution of PDS (6.169). The terms containing powers of ε come from the effect of the term of third order derivative u_{txx}. This property of solutions is useful for examining heat conduction processes. By Eq. (6.188), the task of finding solutions is reduced to that of applying the operator S to u_0. In applications, the order of polynomials is normally not larger than 5 so that $[N/2] \le 2$. For $N = 4$ or 5, for example, Eq. (6.188) becomes

$$u = u_0 + S(u_0)\varepsilon + S^2(u_0)\varepsilon^2 .$$

Without loss of generality, consider an even N such that $N = 2m$ and calculate the terms in the right side of Eq. (6.188). By Eq. (6.176), we have

$$u_{0txx} = P_N''(x)\,\mathrm{e}^{-\frac{t}{\tau_0}} + \frac{1}{2A}\sum_{k=1}^{m-1}\frac{P_N^{(2k+2)}(x)}{(2k)!}g_{2k}(t)\,, \tag{6.189}$$

where $g_{2k}(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-\frac{t}{2\tau_0}}G_{2k}(t)\right]$, $k = 1, 2, \cdots, m-1$, which can be readily obtained by the definition of $G_{2k}(t)$. By the operator S defined in Eq. (6.187),

$$\begin{aligned} u_1 = S(u_0) = \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0 \\ \cdot\left[P_N''(\xi)\,\mathrm{e}^{-\frac{t}{\tau_0}} + \frac{1}{2A}\sum_{k=1}^{m-1}\frac{P_N^{(2k+2)}(\xi)}{(2k)!}g_{2k}(\tau)\right]\mathrm{d}\xi\,, \end{aligned} \tag{6.190}$$

whose integrand is the sum of two parts u_{11} and u_{12},

$$\begin{aligned} u_{11} &= \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0 \cdot P_N''(\xi)\,\mathrm{e}^{-\frac{\tau}{\tau_0}}\,\mathrm{d}\xi \\ &= \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t+\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{-A(t-\tau)}^{A(t-\tau)} I_0 \cdot \left(\sum_{k=2}^{N} k(k-1)a_k(u+x)^{k+2}\right)\mathrm{d}u \\ &= \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t+\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{-A(t-\tau)}^{A(t-\tau)} I_0 \cdot \left(\sum_{k=2}^{N} k(k-1)\sum_{i=0}^{k-2} a_k C_{k-2}^i u^i x^{k-2-i}\right)\mathrm{d}u \\ &= \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t+\tau}{2\tau_0}}\left(\sum_{k=2}^{N} k(k-1)a_k x^{k-2}\right)2A\tau_0\,\mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\left(1-\mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right)\mathrm{d}\tau \\ &\quad + \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t+\tau}{2\tau_0}}\left[\sum_{k=2}^{N}\sum_{i=1}^{k-2} k(k-1)a_k\frac{(x^{k-2})^{(i)}}{i!}G_i(t-\tau)\right]\mathrm{d}\tau \end{aligned}$$

$$= \tau_0 P_N''(x)\left[\tau_0 - (\tau_0 + t)\,\mathrm{e}^{-\frac{t}{\tau_0}}\right] + \frac{1}{2A}\sum_{j=1}^{m-1}\frac{P_N^{(2j+2)}(x)}{(2j)!}\int_0^t \mathrm{e}^{-\frac{t+\tau}{2\tau_0}} G_{2j}(t-\tau)\,\mathrm{d}\tau\,, \tag{6.191}$$

$$u_{12} = \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0\cdot\left[\frac{1}{2A}\sum_{k=1}^{m-1}\frac{P_N^{(2k+2)}(\xi)}{(2k)!} g_{2k}(\tau)\right]\mathrm{d}\xi$$

$$= \frac{1}{4A^2}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{-A(t-\tau)}^{A(t-\tau)} I_0\cdot\left[\sum_{k=1}^{m-1}\frac{g_{2k}(\tau)}{(2k)!}\left(\sum_{n=0}^{N} a_n \sum_{i=0}^{N} C_n^i u^i x^{n-i}\right)^{(2k+2)}\mathrm{d}u\right]$$

$$= \frac{\tau_0}{2A}\int_0^t \sum_{k=1}^{m-1}\frac{P_N^{(2k+2)}(x)}{(2k)!} g_{2k}(\tau)\left(1 - \mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right)\mathrm{d}\tau$$

$$+ \frac{1}{4A^2}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{-A(t-\tau)}^{A(t-\tau)} I_0\cdot\left[\sum_{k=1}^{m-2}\frac{g_{2k}(\tau)}{(2k)!}\sum_{i=1}^{m-2}\frac{P_N^{(2k+2+i)}(x)}{i!}u^i\right]\mathrm{d}u$$

$$= \frac{1}{2A}\sum_{k=1}^{m-1}\frac{P_N^{(2k+2)}(x)}{(2k)!}\int_0^t \tau_0 g_{2k}(\tau)\left(1 - \mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right)\mathrm{d}\tau$$

$$+ \frac{1}{4A^2}\sum_{\substack{i,k=1\\(i+k\le m-1)}}^{m-2}\frac{P_N^{(2i+2k+2)}(x)}{(2i)!(2k)!}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}} G_{2i}(t-\tau) g_{2k}(\tau)\,\mathrm{d}\tau\,. \tag{6.192}$$

Remark 1. In the second term of the right side of Eq. (6.192), define a matrix of order $(m-2): A = (a_{ik}) = \left(P_N^{(2i+2k+2)}(x)\right)$. $(m-2)(m-3)/2$ elements of A in the lower triangular matrix under the auxiliary diagonal are zero so that the sum in the second term of the right side of Eq. (6.192) is actually over $(m-1)(m-2)/2$ terms. While Eq. (6.192) appears complicated, it is often very simple for specific problems. If $P_N(x)$ is a polynomial of order not larger than 5, for example, the second term of the right side of Eq. (6.192) is zero.

Remark 2. Eq. (6.176) shows that the $u_0(x,t)$ has the same form for both even and odd N. For an odd N such that $P_N(x) = \sum_{n=0}^{2m+1} a_n x^n$, the $u_1(x,t)$ can be obtained by adding the solution for an even N and that from the initial value $a_{2m+1}x^{2m+1}$. Let u_{00} and u_{10} be the u_0 and u_1 for an odd N. By Eq. (6.176), we have

$$u_{00} = \tau_0 a_{2m+1} x^{2m+1}\left(1 - \mathrm{e}^{-\frac{t}{\tau_0}}\right) + \frac{1}{2A}\sum_{k=1}^{[N/2]}\frac{(a_{2m+1}x^{2m+1})^{(2k)}}{(2k)!} G_{2k}(t)\,\mathrm{e}^{-\frac{t}{2\tau_0}}\,.$$

Since $u_{10} = S(u_{00})$, we have

$$u_{10} = \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0 \cdot \frac{\partial^3 u_{00}}{\partial t \partial x^2}\bigg|_{\substack{x=\xi \\ t=\tau}} \mathrm{d}\xi \,. \tag{6.193}$$

Since $G_{2m+1}(t) = 0$, $a_{2m+1}x^{2m+1}g_l(t)$ and $a_{2m}x^{2m}g_l(t)$ share the same form for any nonnegative integer l and any differentiable function $g_l(t)$. Therefore, the right side of Eq. (6.193) share the same form of u_1 in Eq. (6.190) after expansion. We can obtain u_{10} simply by replacing $P_{2m}(x)$ in Eqs. (6.191) and (6.192) by $P_{2m+1}(x)$. The $P_N(x)$ in Eq. (6.190) can be any even or odd polynomial with $m = [N/2]$. This can also be extended to u_2, u_3, $\cdots$, $u_{[N/2]}$. In finding solutions of PDS (6.181) by Eq. (6.188), the solutions corresponding to initial values $P_2(x)$ and $P_3(x)$ have the same form. This is also true for the solutions from $P_4(x)$ and $P_5(x)$, etc. From the point of view of approximation, we should use the odd polynomial to approximate $\psi(x)$, the principle of superposition implies that the solution corresponding to a higher order polynomial can be formed by adding that for a lower order polynomial and that due to additional terms in the higher order polynomial. Therefore, the results for a lower order polynomial are still useful even for the case of a higher order polynomial.

Remark 3. It can be shown that the unit u_1 is correct. For the second term in the right side of Eqs. (6.191) and (6.192), for example,

$$[\varepsilon u_{11}] = \frac{L^2}{T}\cdot\frac{T}{L}\cdot\frac{\Theta}{TL^{2j+2}}\cdot L^{2j+1}\cdot T = \Theta\,,$$

$$[\varepsilon u_{12}] = \frac{L^2}{T}\cdot\frac{T^2}{L^2}\cdot\frac{\Theta}{TL^{2i+2k+2}}\cdot L^{2i+1}\cdot L^{2k+1}\cdot T = \Theta\,.$$

Thus $[\varepsilon u_1] = \Theta$. Since

$$[S] = \frac{T}{L}\cdot T\cdot\frac{1}{TL^2}\cdot L = \frac{T}{L^2}\,,\quad [u_1] = \frac{T}{L^2}\,,$$

we have

$$[u_2] = [S]\,[u_1] = \frac{T^2}{L^4} \quad \text{and} \quad \left[u_2(x,t)\varepsilon^2\right] = \Theta\,.$$

In general,

$$\left[u_k(x,t)\varepsilon^k\right] = \Theta\,, k = 0,1,2,\cdots[N/2]\,.$$

Therefore, the units of Eq. (6.186) and (6.188) are correct.

Remark 4. Note that

$$S^m(u_0)\varepsilon^m = P_N^{(2m)}(x)q(t)\varepsilon^m\,, \tag{6.194}$$

where $N = 2m$ or $2m+1$, $m = 0,1,2,\cdots$, $q(t)$ is a function to be determined. By introducing an operator

$$\begin{aligned} B &= \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{-A(t-\tau)}^{A(t-\tau)} I_0\left(b\sqrt{A^2(t-\tau)^2-u^2}\right)\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=\tau}\mathrm{d}u \\ &= \tau_0\int_0^t\left(1-\mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right)\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=\tau}\mathrm{d}\tau\,, \end{aligned}$$

we have

$$q(t) = B^m\left[\tau_0\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right)\right]. \tag{6.195}$$

Therefore,

$$\begin{aligned} B\left[\tau_0\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right)\right] &= \tau_0\int_0^t\left(1-\mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right)\mathrm{e}^{-\frac{\tau}{\tau_0}}\,\mathrm{d}\tau \\ &= \tau_0\left[\tau_0-(\tau_0+t)\,\mathrm{e}^{-\frac{t}{\tau_0}}\right], \\ B^2\left[\tau_0\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right)\right] &= \tau_0\int_0^t\left(1-\mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right)\tau\,\mathrm{e}^{-\frac{\tau}{\tau_0}}\,\mathrm{d}\tau \\ &= \tau_0^2\left[\tau_0-\left(\tau_0+t+\frac{t^2}{2\tau_0}\right)\mathrm{e}^{-\frac{t}{\tau_0}}\right], \\ B^3\left[\tau_0\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right)\right] &= \tau_0\int_0^t\left(1-\mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right)\frac{\tau^2}{2!}\,\mathrm{e}^{-\frac{\tau}{\tau_0}}\,\mathrm{d}\tau \\ &= \tau_0^3\left[\tau_0-\left(\tau_0+t+\frac{t^2}{2!\tau_0}+\frac{t^3}{3!\tau_0^2}\right)\mathrm{e}^{-\frac{t}{\tau_0}}\right]. \end{aligned}$$

For the real computation, m is normally not larger than 4. By the recurrence method, we have

$$\begin{aligned} q(t) = B^m\left[\tau_0\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right)\right] &= \tau_0\int_0^t\left(1-\mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right)\frac{\tau^{m-1}}{(m-1)!}\,\mathrm{e}^{-\frac{\tau}{\tau_0}}\,\mathrm{d}\tau \\ &= \tau_0^m\left[\tau_0-\left(\tau_0+t+\frac{t^2}{2!\tau_0}+\cdots+\frac{t^m}{m!\tau_0^{m-1}}\right)\mathrm{e}^{-\frac{t}{\tau_0}}\right]. \end{aligned}$$

The unit of the last term is $[S^m(u_0)\varepsilon^m] = \dfrac{\Theta}{TL^{2m}}\cdot T^{m+1}\cdot\left(\dfrac{L^2}{T}\right)^m = \Theta$, which is correct. Therefore, the last term of the solution is independent of x if $P_N(x)$ is even. It increases toward τ_0^{m+1} as time increases since $q'(t) > 0$. When $P_N(x)$ is odd, however, the last term depends not only on t but also on a function of first degree of x.

Remark 5. Solution (6.188) of PDS (6.181) is a function of x and t expressed by the polynomial of initial value $P_N(x)$. It is the $W_{P_N}(x,t)$ by the solution structure theorem. The solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + \varepsilon u_{txx} + P_m(x)\,, & R^1 \times (0,+\infty)\,, \\ u(x,0) = P_l(x)\,, \quad u_t(x,0) = P_N(x) \end{cases} \tag{6.196}$$

is thus

$$u = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t} - \varepsilon \frac{\partial^2}{\partial x^2}\right) W_{P_l}(x,t) + W_{P_N}(x,t) + \int_0^t W_{P_m}(x,t-\tau)\,\mathrm{d}\tau\,, \tag{6.197}$$

where $P_N(x)$, $P_l(x)$ and $P_m(x)$ are the N-th, l-th and m-th polynomials respectively.

All terms in $W_{P_N}(x,t)$ are separable regarding x and t. This is the same as that for hyperbolic heat-conduction equations and is very useful for examining features of heat conduction. When $P_m(x) = \sum\limits_{n=0}^{m} a_n(t)x^n$, the third term in Eq. (6.197) reads

$$\int_0^t W_{P_{m\tau}}(x,t-\tau)\,\mathrm{d}\tau\,, \quad P_{m_\tau} = P_m(x,\tau)\,.$$

6.9.4 Solutions for Initial-Value of Lower-Order Polynomials

The lower-order polynomials find their applications in many fields. For $n \le 5$, we discuss $W_{P_N}(x,t)$ of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 u_{xx} + \varepsilon u_{txx}\,, & R^1 \times (0,+\infty)\,, \\ u(x,0) = 0\,, \quad u_t(x,0) = P_n(x)\,, & n = 0,1,2,3,4,5\,. \end{cases} \tag{6.198}$$

Define

$$G_{2k}(t) = \int_{-A(t-\tau)}^{A(t-\tau)} I_0\left(b\sqrt{A^2(t-\tau)^2 - u^2}\right) u^{2k}\,\mathrm{d}u\,, k = 1,2,3\,, \tag{6.199}$$

where

$$I_0(x) = \sum_{m=0}^{\infty} \frac{1}{(m!)^2}\left(\frac{x}{2}\right)^{2m}, \; b = \frac{1}{2A\tau_0}\,, g_{2k}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(G_{2k}(t)\,\mathrm{e}^{-\frac{t}{2\tau_0}}\right). \tag{6.200}$$

The integrand of (6.199) contains power series that are quickly convergent. The integral (6.199) can thus be obtained very easily by using integration term by term. The $g_{2k}(t)$ in Eq. (6.200) can also be obtained quite easily.

Polynomials of Zero- and First-Order

By Eq. (6.176), for $P_1(x) = a_0 + a_1 x$, $u_0(x,t) = \tau_0 P_1(x)\left(1 - \mathrm{e}^{-\frac{t}{\tau_0}}\right)$. Thus

$$u_1 = S(u_0) = 0\,.$$

The solution of PDS (6.198) is thus

$$u = W_{P_1}(x,t) = u_0 = \tau_0(a_0 + a_1 x)\left(1 - \mathrm{e}^{-\frac{t}{\tau_0}}\right). \tag{6.201}$$

This shows that u_{txx}-term in (6.198) has no effect on the solution for an initial value of type $P_1(x)$. Its solution is the same as that of PDS (6.169).

Polynomials of Second- and Third-Order

For polynomials of second- or third- order, $P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ ($P_3(x)$ reduces to $P_2(x) = a_0 + a_1 x + a_2 x^2$ when $a_3 = 0$). By Eq. (6.176), we have

$$u_0(x,t) = \tau_0 P_3(x)\left(1 - \mathrm{e}^{-\frac{t}{\tau_0}}\right) + \frac{1}{2A}\frac{P_3''(x)}{2!}G_2(t)\,\mathrm{e}^{-\frac{t}{2\tau_0}}\,,$$

$$\begin{aligned}
u_1 = S(u_0) &= \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0 \cdot (2a_2 + 6a_3\xi)\,\mathrm{e}^{-\frac{\tau}{\tau_0}}\,\mathrm{d}\xi \\
&= \tau_0 P_3''(x)\left[\tau_0 - (\tau_0 + t)\,\mathrm{e}^{-\frac{t}{\tau_0}}\right] \\
&= \tau_0(2a_2 + 6a_3 x)\left[\tau_0 - (\tau_0 + t)\,\mathrm{e}^{-\frac{t}{\tau_0}}\right].
\end{aligned}$$

Thus, the solution of (6.198) is

$$\begin{aligned}
u &= u_0 + S(u_0)\varepsilon \\
&= \tau_0\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)\left(1 - \mathrm{e}^{-\frac{t}{\tau_0}}\right) + \frac{1}{2A}\frac{2a_2 + 6a_3 x}{2!}G_2(t)\,\mathrm{e}^{-\frac{t}{2\tau_0}} \\
&\quad + \tau_0(2a_2 + 6a_3 x)\left[\tau_0 - (\tau_0 + t)\,\mathrm{e}^{-\frac{t}{\tau_0}}\right]\varepsilon\,.
\end{aligned} \tag{6.202}$$

This shows that the effect of u_{txx}-term is x-independent for the case of $a_3 = 0$ and increases towards $2a_2\tau_0^2$ as $t \to \infty$.

Polynomials of Fourth- and Fifth-Order

For polynomials of fourth- and fifth- order, $P_5(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ ($P_5(x)$ reduces to $P_4(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ when $a_5 = 0$). By Eqs. (6.176), (6.191) and (6.192), we have

$$u_0(x,t) = \tau_0 P_5(x)\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right) + \frac{1}{2A}\mathrm{e}^{-\frac{t}{2\tau_0}}\left[\frac{P_5''(x)}{2!}G_2(t) + \frac{P_5^{(4)}(x)}{4!}G_4(t)\right],$$

$$\begin{aligned}
u_1 = S(u_0) &= \frac{1}{2A}\int_0^t \mathrm{e}^{-\frac{t-\tau}{2\tau_0}}\,\mathrm{d}\tau \int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0 \\
&\quad \cdot\left[P_5''(\xi)\,\mathrm{e}^{-\frac{\tau}{\tau_0}} + \frac{1}{2A}P_5^{(4)}(\xi)g_2(\tau)\right]\mathrm{d}\xi \\
&= \tau_0 P_5''(x)\left[\tau_0 - (\tau_0+t)\,\mathrm{e}^{-\frac{t}{\tau_0}}\right] + \frac{1}{2A}\frac{P_5^{(4)}(x)}{2}\int_0^t \mathrm{e}^{-\frac{t+\tau}{2\tau_0}}G_2(t-\tau)\,\mathrm{d}\tau \\
&\quad + \frac{1}{2A}\frac{P_5^{(4)}(x)}{2}\int_0^t \tau_0 g_2(\tau)\left(1-\mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right)\mathrm{d}\tau.
\end{aligned}$$

By Eq. (6.195), we obtain

$$\begin{aligned}
u_2 = S(u_1) = S^2(u_0) &= P_5^{(4)}(x)B^2\left[\tau_0\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right)\right] \\
&= \tau_0^2 P_5^{(4)}(x)\left[\tau_0 - \left(\tau_0 + t + \frac{t^2}{2\tau_0}\right)\mathrm{e}^{-\frac{t}{\tau_0}}\right].
\end{aligned}$$

The solution of PDS (6.198) is thus

$$\begin{aligned}
u &= u_0 + S(u_0)\varepsilon + S^2(u_0)\varepsilon^2 \\
&= \tau_0\left(\sum_{n=0}^{5} a_n x^n\right)\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right) \\
&\quad + \frac{1}{2A}\mathrm{e}^{-\frac{t}{2\tau_0}}\left[\frac{2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3}{2!}G_2(t) + \frac{24a_4 + 120a_5x}{4!}G_4(t)\right] \\
&\quad + \left\{\tau_0\left(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3\right)\left[\tau_0 - (\tau_0+t)\,\mathrm{e}^{-\frac{t}{\tau_0}}\right] + \frac{24a_4 + 120a_5x}{4A}\right.
\end{aligned}$$

$$\cdot \left[\int_0^t \mathrm{e}^{-\frac{t+\tau}{2\tau_0}} G_2(t-\tau)\,\mathrm{d}\tau + \int_0^t \tau_0 g_2(\tau) \left(1 - \mathrm{e}^{-\frac{t-\tau}{\tau_0}}\right) \mathrm{d}\tau \right] \Bigg\} \varepsilon$$

$$+ \tau_0^2 (24a_4 + 120a_5 x) \left[\tau_0 - \left(\tau_0 + t + \frac{t^2}{2\tau_0} \right) \mathrm{e}^{-\frac{t}{\tau_0}} \right] \varepsilon^2 . \tag{6.203}$$

The solution due to $P_4(x)$ can be obtained by letting $a_5 = 0$ in Eq. (6.203).

6.9.5 Perturbation Method for Two- and Three-dimensional Problems

The perturbation method can also be applied to solve two- and three-dimensional Cauchy problems. By the solution structure theorem, we can focus our attention only on

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2 \Delta u + \varepsilon \dfrac{\partial}{\partial t} \Delta u, \quad R^2 \times (0, +\infty) \text{ or } R^3 \times (0, +\infty)\,, \\ u(M, 0) = 0\,, \quad u_t(M, 0) = \psi(M), \quad 0 < \varepsilon \ll 1\,. \end{cases} \tag{6.204}$$

Consider the perturbation solution

$$u(M, t, \varepsilon) = \sum_{n=0}^{\infty} u_n(M, t) \varepsilon^n\,, \tag{6.205}$$

where $u_n(M,t)$ are undetermined functions.

Substituting Eq. (6.205) into the equation of PDS (6.204) and comparing coefficients of ε^n $(n = 0, 1, 2, \cdots)$ yield

$$\varepsilon^0: \quad \begin{cases} u_{0t}/\tau_0 + u_{0tt} = A^2 \Delta u_0\,, \\ u_0(M, 0) = 0,\, u_{0_t}(M, 0) = \psi(M)\,. \end{cases} \tag{6.206}$$

$$\varepsilon^1: \quad \begin{cases} u_{1t}/\tau_0 + u_{1tt} = A^2 \Delta u_1 + \dfrac{\partial}{\partial t} \Delta u_0\,, \\ u_1(M, 0) = 0,\, u_{1_t}(M, 0) = \psi(M)\,. \end{cases} \tag{6.207}$$

$$\varepsilon^2: \quad \begin{cases} u_{2t}/\tau_0 + u_{2tt} = A^2 \Delta u_2 + \dfrac{\partial}{\partial t} \Delta u_1\,, \\ u_2(M, 0) = 0,\, u_{2_t}(M, 0) = \psi(M)\,. \end{cases} \tag{6.208}$$

$$\cdots \quad \cdots \quad \cdots$$

For two-dimensional cases, let $u_t(M,0) = \psi(x,y)$. By Eq. (5.67), we obtain

$$u_0(M,t) = W_\psi(x,y,t)$$

$$= e^{-\frac{t}{2\tau_0}} \frac{1}{2\pi A} \iint\limits_{D_{At}^M} \frac{\operatorname{ch}\frac{c}{A}\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}}{\sqrt{(At)^2-(x-\xi)^2-(y-\eta)^2}} \psi(\xi,\eta)\,d\xi\,d\eta\,.$$

Denote $\dfrac{\partial}{\partial t}\Delta u_0 = f_1(x,y,t)$. By Eq. (6.207), we have

$$u_1(M,t) = \int_0^t W_{f_{1\tau}}(M,t-\tau)\,d\tau\,, \quad f_{1\tau} = f_1(x,y,\tau)\,.$$

Using this approach, we can also obtain $u_2(M,t), u_3(M,t), \cdots$.

For the three-dimensional case, let $u_t(M,0) = \psi(x,y,z)$, $u_0(M,t) = W_\psi(M,t)$ is available in Eq. (5.120). Similar to the two-dimensional case, we can obtain the perturbation solution of PDS (6.204)

$$u(M,t,\varepsilon) = u_0(M,t) + \varepsilon u_1(M,t) + \varepsilon^2 u_2(M,t) + \cdots + \varepsilon^n u_n(M,t) + O\left(\varepsilon^{n+1}\right)\,.$$

In applications, we normally take the first- or second-order perturbation solution as the approximate solution of PDS (6.204). Once the approximate solution of PDS (6.204) is available, an approximate solution of

$$\begin{cases} u_t/\tau_0 + u_{tt} = A^2\Delta u + \varepsilon\dfrac{\partial}{\partial t}\Delta u + f(M,t), & R^2\times(0,+\infty) \text{ or } R^3\times(0,+\infty), \\ u(M,0) = \varphi(M), \quad u_t(M,0) = \psi(M) & \end{cases}$$

can be obtained by the solution structure theorem,

$$u(M,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t} - \varepsilon\Delta\right) W_\varphi(M,t) + W_\psi(M,t) + \int_0^t W_{f_\tau}(M,t-\tau)\,d\tau\,,$$

where $f_\tau = f(M,\tau)$.

6.10 Thermal Waves and Resonance

In this section we examine thermal oscillation and resonance described by the dual-phase-lagging heat-conduction equations. Conditions and features of underdamped, critically-damped and overdamped oscillations are obtained and compared with those described by the classical parabolic heat-conduction equation and the hyperbolic heat-conduction equation. Also derived is the condition for thermal resonance. Both underdamped oscillation and critically-damped oscillation cannot appear if τ_T is larger than τ_0. The modes of underdamped thermal oscillation are limited to a region fixed by two relaxation distances for the case $\tau_T > 0$, and by one relaxation distance for the case $\tau_T = 0$.

6.10.1 Thermal Waves

Without loss of generality, Xu and Wang (2002) consider the one-dimensional initial-boundary value problem of dual-phase-lagging heat conduction

$$\begin{cases} \frac{1}{\alpha}\left(\frac{\partial T}{\partial t}+\tau_0\frac{\partial^2 T}{\partial t^2}\right)=\frac{\partial^2 T}{\partial x^2}+\tau_{\mathrm{T}}\frac{\partial^3 T}{\partial t\partial x^2}+\frac{1}{k}\left(F+\tau_0\frac{\partial F}{\partial t}\right)\,,\ (0,l)\times(0,\,+\infty)\,,\\ T(0,t)=T(l,t)=0\,,\\ T(x,0)=\phi(x)\,,\ T_t(x,0)=\psi(x)\,, \end{cases} \tag{6.209}$$

whose solution represents the temperature distribution in an infinitely-wide slab of thickness l. Here t is the time, T is the temperature, α is the thermal diffusivity of the medium, F is the volumetric heat source, ϕ and ψ are two given functions, and τ_T and τ_0 are the phase-lags of the temperature gradient and heat flux vector, respectively.

For a free thermal oscillation, $F=0$. By taking the boundary conditions into account, let

$$T(x,t)=\sum_{m=1}^{\infty}\Gamma_m(t)\sin\beta_m x\,, \tag{6.210}$$

where

$$\beta_m=\frac{m\pi}{l}\,.$$

Using the Fourier sine series to express ϕ and ψ as

$$\phi(x)=\sum_{m=1}^{\infty}\phi_m\sin\beta_m x\,, \tag{6.211}$$

and

$$\psi(x)=\sum_{m=1}^{\infty}\psi_m\sin\beta_m x\,, \tag{6.212}$$

where

$$\phi_m=\frac{2}{l}\int_0^l\phi(\xi)\sin\beta_m\xi\,\mathrm{d}\xi\,,$$

and

$$\psi_m=\frac{2}{l}\int_0^l\psi(\xi)\sin\beta_m\xi\,\mathrm{d}\xi\,.$$

A substitution of Eqs. (6.210)–(6.212) into (6.209) yields, by making use of the orthogonality of $\sin\beta_m x$ $(m = 1, 2, \ldots.)$,

$$\tau_q \ddot{\Gamma}_m + (1 + \alpha\tau_T\beta_m^2)\,\dot{\Gamma}_m + \alpha\beta_m^2\Gamma_m = 0\,, \tag{6.213}$$

$$\Gamma_m(0) = \phi_m\,,\ \dot{\Gamma}_m(0) = \psi_m\,. \tag{6.214}$$

Introduce the damping coefficient f_m by

$$f_m = \frac{1}{\tau_0} + \tau_T\,\omega_m^2$$

and the natural frequency coefficient ω_m by

$$\omega_m^2 = \frac{\alpha\beta_m^2}{\tau_0}\,.$$

Eq. (6.213) reduces to

$$\ddot{\Gamma}_m + f_m\dot{\Gamma}_m + \omega_m^2\Gamma_m = 0\,. \tag{6.215}$$

The solution of Eq. (6.215) can be readily obtained by the method of undetermined coefficients as

$$\Gamma_m(t) = be^{\lambda t} \tag{6.216}$$

with λ as a coefficient to be determined. Substituting Eq. (6.216) into Eq. (6.215) leads to

$$\lambda^2 + f_m\lambda + \omega_m^2 = 0\,, \tag{6.217}$$

which has solution λ_1, λ_2

$$\lambda_{1,2} = -\frac{f_m}{2} \pm \sqrt{\Lambda}\,. \tag{6.218}$$

Here Λ is the discriminate of Eq. (6.217) and is defined by

$$\Lambda = \left(\frac{f_m}{2}\right)^2 - \omega_m^2\,.$$

Therefore, a positive, negative and vanished discriminate yields two distinct real numbers λ_1 and λ_2, two complex conjugates λ_1 and λ_2, and two equal real numbers λ_1 and λ_2, respectively. The critical damping coefficient f_{mc} is the damping coefficient at a fixed ω_m and $\Lambda = 0$. Therefore,

$$f_{mc} = 2\omega_m\,. \tag{6.219}$$

The nondimensional damping ratio, ζ_m, is defined as the ratio of f_m over f_{mc},

$$\zeta_m = \frac{f_m}{f_{mc}} = \frac{f_m}{2\omega_m} = \frac{1}{2\tau_0\omega_m} + \frac{\tau_T\,\omega_m}{2}\,. \tag{6.220}$$

The system is at underdamped oscillation, critically-damped oscillation or overdamped oscillation, respectively, when $\zeta_m < 1$, $\zeta_m = 1$ or $\zeta_m > 1$. By Eqs. (6.218) and (6.220), Xu and Wang (2002) obtain an expression of $\lambda_{1,2}$ in terms of ζ_m and ω_m,

$$\lambda_{1,2} = \omega_m \left(-\zeta_m \pm \sqrt{\zeta_m^2 - 1}\right). \tag{6.221}$$

Underdamped Oscillation

For the case $\zeta_m < 1$, there are two complex conjugates λ_1 and λ_2,

$$\lambda_{1,2} = \omega_m \left(-\zeta_m \pm i\sqrt{1 - \zeta_m^2}\right). \tag{6.222}$$

Therefore

$$\Gamma_m(t) = \mathrm{e}^{-\zeta_m \omega_m t} \left(a_m \cos \omega_m t \sqrt{1-\zeta_m^2} + b_m \sin \omega_m t \sqrt{1-\zeta_m^2}\right). \tag{6.223}$$

After the determination of integration constants a_m and b_m by the initial conditions [Eq. (6.214)], Xu and Wang (2002) obtain

$$\Gamma_m(t) = \mathrm{e}^{-\zeta_m \omega_m t} \left(\phi_m \cos \omega_m t \sqrt{1-\zeta_m^2} + \frac{\psi_m + \zeta_m \omega_m \phi_m}{\omega_m \sqrt{1-\zeta_m^2}} \sin \omega_m t \sqrt{1-\zeta_m^2}\right) \tag{6.224}$$

which may be rewritten as

$$\Gamma_m(t) = A_m \mathrm{e}^{-\zeta_m \omega_m t} \sin(\omega_{dm} t + \varphi_{dm}). \tag{6.225}$$

Here,

$$A_m = \sqrt{\phi_m^2 + \left(\frac{\psi_m + \zeta_m \omega_m \phi_m}{\omega_{dm}}\right)^2}, \tag{6.226}$$

$$\omega_{dm} = \omega_m \sqrt{1-\zeta_m^2}, \tag{6.227}$$

$$\varphi_{dm} = \tan^{-1}\left(\frac{\phi_m \omega_{dm}}{\psi_m + \zeta_m \omega_m \phi_m}\right). \tag{6.228}$$

Therefore, the system is oscillating with frequency ω_{dm} and an exponentially decaying amplitude $A_m \mathrm{e}^{-\zeta_m \omega_m t}$ [Eq. (6.215)]. Fig. 6.1 typifies the oscillatory pattern, for $\zeta_m = 0.1$, $\omega_m = 1.0$, $\phi_m = 1.0$ and $\psi_m = 0.0$. The wave behavior is still observed in dual-phase-lagging heat conduction. However, the amplitude decays exponentially due to the damping of thermal diffusion. This differs enormously from classical heat conduction. $\zeta_m < 1$ forms the condition for thermal oscillation of this kind.

Figure 6.2 illustrates the variation of $\Gamma_m(t)$ with the time t when ψ_m is changed to 1.0 from 0. Γ_m is observed sometimes to be capable of surpassing ϕ_m. This phe-

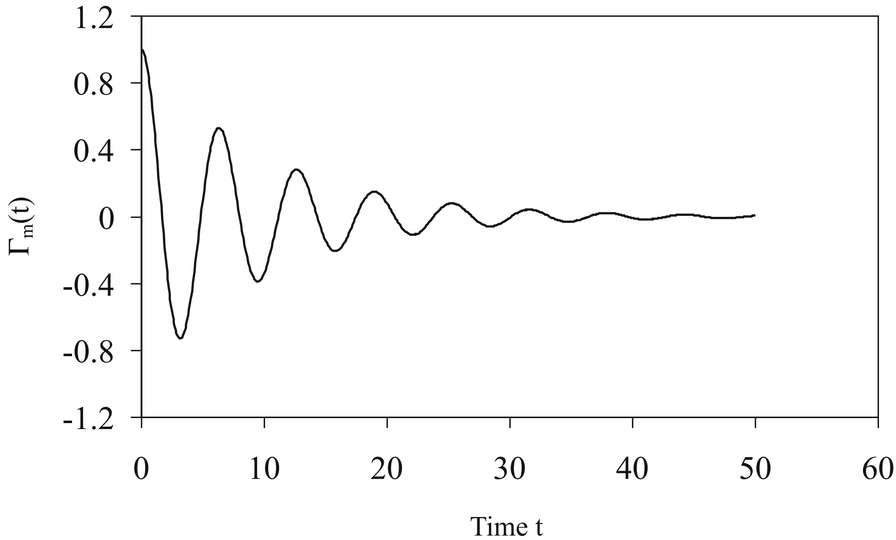

Fig. 6.1 Variation of $\Gamma_m(t)$ with the time t: $\zeta_m = 0.1$, $\omega_m = 1.0$, $\phi_m = 1.0$, $\psi_m = 0.0$ (after Xu and Wang 2002)

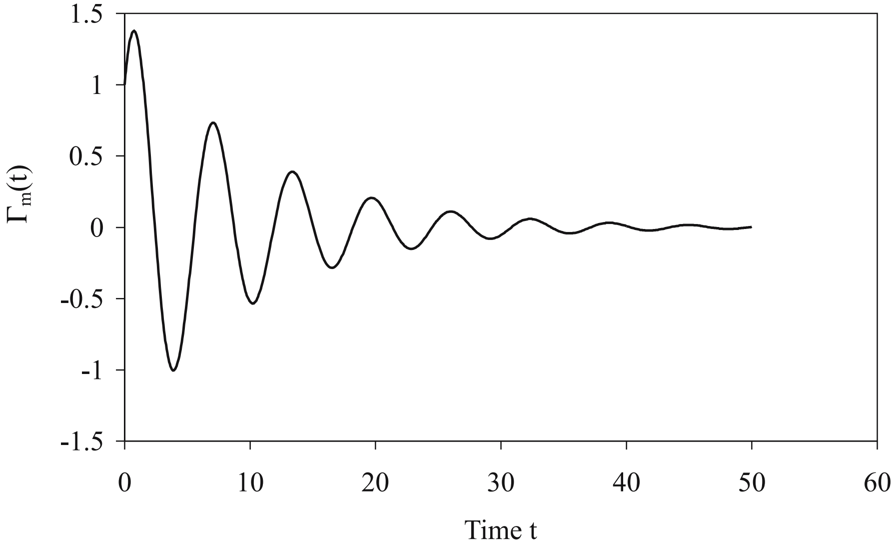

Fig. 6.2 Variation of $\Gamma_m(t)$ with the time t: $\zeta_m = 0.1$, $\omega_m = 1.0$, $\phi_m = 1.0$, $\psi_m = 1.0$ (after Xu and Wang 2002)

nomenon is caused by the non-vanishing initial time-rate of the temperature change and cannot appear in classical heat conduction. The classical maximum and minimum principle is, therefore, not valid in dual-phase-lagging heat conduction.

While $\Gamma_m(t)$ is oscillatory, it is not periodic because of the decaying amplitude. $\Gamma_m(t)$ oscillates in time with a fixed damped period T_{dm} given by

$$T_{dm} = \frac{2\pi}{\omega_{dm}}. \tag{6.229}$$

Critically-Damped Oscillation

For this case, $\zeta_m = 1$. This requires, by Eq. (6.220),

$$\omega_m = \frac{1 \pm \sqrt{1 - \frac{\tau_T}{\tau_0}}}{\tau_T}. \tag{6.230}$$

Therefore, critically damped oscillation appears only when $\tau_T \leq \tau_0$. When the system is in critically damped oscillation, there are two equal numbers λ_1 and λ_2. Therefore,

$$\Gamma_m(t) = a_m \mathrm{e}^{-\omega_m t} + b_m t\, \mathrm{e}^{-\omega_m t},$$

which becomes, after determining the integration constants a_m and b_m by the initial conditions [Eq. (6.214)],

$$\Gamma_m(t) = \mathrm{e}^{-\omega_m t}\left[\phi_m + (\psi_m + \omega_m \phi_m)\, t\right]. \tag{6.231}$$

By letting $\dfrac{\mathrm{d}\,|\Gamma_m(t)|}{\mathrm{d}t} = 0$ and analyzing the sign of $\dfrac{\mathrm{d}^2\,|\Gamma_m(t)|}{\mathrm{d}t^2}$, Xu and Wang (2002) obtain the maximal value of $|\Gamma_m(t)|$

$$\mathrm{Max}\left[|\Gamma_m(t)|\right] = \mathrm{e}^{-\frac{\psi_m}{\psi_m + \omega_m \phi_m}} \left|\phi_m + \frac{\psi_m}{\omega_m}\right| \tag{6.232}$$

at

$$t_m = \frac{\psi_m}{\omega_m(\psi_m + \omega_m \phi_m)}, \tag{6.233}$$

that is positive if

$$\psi_m^2 > -\omega_m \phi_m \psi_m.$$

This clearly requires that $\psi_m \neq 0$. Therefore, $|\Gamma_m(t)|$ decreases monotonically as t increases from 0 when

$$\psi_m^2 \leq -\omega_m \phi_m \psi_m.$$

This is very similar to the classical heat-conduction equation. When

$$\psi_m^2 > -\omega_m \phi_m \psi_m,$$

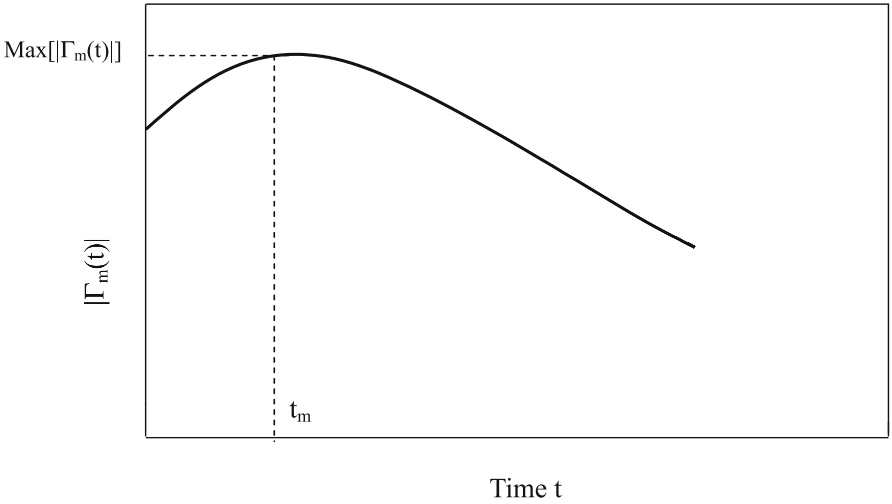

Fig. 6.3 $|\Gamma_m(t)|$ at the critically-damped oscillation and $\psi_m^2 > -\omega_m\phi_m\psi_m$ (after Xu and Wang 2002)

however, $|\Gamma_m(t)|$ first increases from ϕ_m to $\mathrm{Max}\left[|\Gamma_m(t)|\right]$ as t increases from 0 to t_m and then decreases monotonically (Fig. 6.3). Therefore, although the temperature field does not oscillate, its absolute value reaches the maximum value at $t = t_m > 0$ rather than at the initial time instant $t = 0$.

Overdamped Oscillation

For this case ($\zeta_m > 1$),

$$\lambda_{1,2} = \omega_m\left(-\zeta_m \pm \sqrt{\zeta_m^2 - 1}\right). \tag{6.234}$$

Thus the solution of Eq. (6.215) subject to Eq. (6.214) is

$$\begin{aligned}\Gamma_m(t) = \frac{e^{-\zeta_m\omega_m t}}{2\sqrt{\zeta_m^2-1}}&\left[\left(\frac{\psi_m}{\omega_m} + \phi_m\left(\zeta_m + \sqrt{\zeta_m^2-1}\right)\right)\, e^{\omega_m t\sqrt{\zeta_m^2}}\right.\\ &\left.+\left(-\frac{\psi_m}{\omega_m} + \phi_m\left(-\zeta_m + \sqrt{\zeta_m^2-1}\right)\right)\, e^{-\omega_m t\sqrt{\zeta_m^2-1}}\right].\end{aligned} \tag{6.235}$$

Letting $\frac{\mathrm{d}|\Gamma_m(t)|}{\mathrm{d}t} = 0$ leads to two extreme points,

$$t_{m1} = 0 \tag{6.236}$$

and

$$t_{m2} = -\frac{1}{2\omega_m\sqrt{\zeta_m^2 - 1}} \ln\left[\frac{\zeta_m - \sqrt{\zeta_m^2 - 1}}{\zeta_m + \sqrt{\zeta_m^2 - 1}} \frac{\frac{\psi_m}{\omega_m} + \phi_m\left(\zeta_m + \sqrt{\zeta_m^2 - 1}\right)}{\frac{\psi_m}{\omega_m} + \phi_m(\zeta_m - \sqrt{\zeta_m^2 - 1})}\right], \tag{6.237}$$

with $\text{Max1}\,[|\Gamma_m(t)|] = |\phi_m|$ and $\text{Max2}\,[|\Gamma_m(t)|] = |\Gamma_m(t_{m2})|$, respectively. Therefore, $|\Gamma_m(t)|$ decreases monotonically from $t = 0$ when $t_{m2} = 0$ (very like in classical heat conduction). When $t_{m2} > 0$, however, $|\Gamma_m(t)|$ first increases from $|\phi_m|$ to a maximal value $\text{Max2}\,[|\Gamma_m(t)|]$ as t increases from 0 to t_{m2} and then decreases for $t \geq t_m$ (Fig.6.4). There is no oscillation if $\zeta_m > 1$.

When $\tau_T > \tau_0$,

$$1 + \alpha\tau_T\frac{m^2\pi^2}{l^2} > 1 + \alpha\tau_0\frac{m^2\pi^2}{l^2} \geq 2\sqrt{\alpha\tau_0}\frac{m\pi}{l}\,.$$

This, with Eq. (6.220), yields

$$\zeta_m = \frac{1 + \tau_T\alpha\frac{m^2\pi^2}{l^2}}{2\sqrt{\alpha\tau_0}\frac{m\pi}{l}} > 1\,.$$

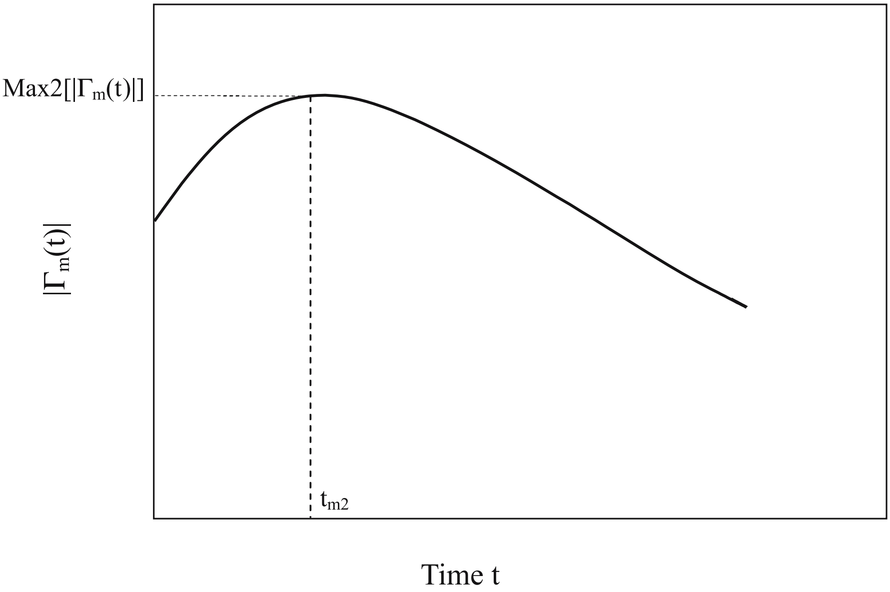

Fig. 6.4 $|\Gamma_m(t)|$ at the overdamped oscillation and $t_{m2} > 0$ (after Xu and Wang 2002)

Therefore the system is always at overdamped oscillation if $\tau_T > \tau_0$. Consequently, there are no thermal waves.

$\zeta_m = 1.0$ separates the underdamped modes from the overdamped modes. Applying $\zeta_m < 1$ in Eq. (6.220) yields the region of m where the underdamped modes can occur

$$\frac{1}{\pi C_1} < m < \frac{1}{\pi C_2}, \qquad \text{if } \tau_0 > \tau_T > 0; \tag{6.238}$$

$$m > \frac{1}{\pi C}, \qquad \text{if } \tau_0 > \tau_T = 0. \tag{6.239}$$

Hence C_1 and C_2 are the relaxation distances defined by (Xu and Wang 2002)

$$C_1 = \sqrt{\alpha\tau_T}\left(\sqrt{\frac{\tau_0}{\tau_T}} + \sqrt{\frac{\tau_0}{\tau_T} - 1}\right),$$

$$C_2 = \sqrt{\alpha\tau_T}\left(\sqrt{\frac{\tau_0}{\tau_T}} - \sqrt{\frac{\tau_0}{\tau_T} - 1}\right),$$

$$C = 2\sqrt{\alpha\tau_0}.$$

Therefore, thermal oscillation occurs only for the modes between $\dfrac{1}{\pi C_1}$ and $\dfrac{1}{\pi C_2}$ for the case of $\tau_0 > \tau_T > 0$. This is different from thermal waves in hyperbolic heat conduction where oscillation always appears for the high order modes (Tzou 1992a).

The behavior of an individual temperature mode discussed above also represents the entire thermal response if $\phi(x) = A \sin \dfrac{m\pi x}{l}$ and $\psi(x) = B \sin \dfrac{m\pi x}{l}$ with A and B as constants. In general, a change $\Delta\Gamma_m(t)$ in the m-th mode would lead to a change $\Delta\Gamma_m(t) \sin \frac{m\pi x}{l}$ in $T(x, t)$ because

$$T(x,t) = \sum_{m=1}^{\infty} \Gamma_m(t) \sin \frac{m\pi x}{l}.$$

6.10.2 Resonance

For dual-phase-lagging heat conduction, the amplitude of the thermal wave may become exaggerated if the oscillating frequency of an externally applied heat source is at the resonance frequency.

Consider a heat source in the system (6.209) in the form of

$$F(x,t) = Qg(x)\mathrm{e}^{\mathrm{i}\Omega t}.$$

Here Q, independent of x and t, is the strength, $g(x)$ is the spanwise distribution, and Ω is the oscillating frequency. Expand $T(x, t)$ and $g(x)$ by the Fourier sine series,

$$T(x,t) = \sum_{m=1}^{\infty} \Gamma_m(t) \sin(\beta_m x), \tag{6.240}$$

$$g(x) = \sum_{m=1}^{\infty} D_m \sin(\beta_m x), \tag{6.241}$$

where $\Gamma_m(t)$ and β_m are defined in Section 6.10.1, and

$$D_m = \frac{2}{l} \int_0^l g(x) \sin(\beta_m x) \, \mathrm{d}x. \tag{6.242}$$

$T(x,t)$ in Eq. (6.240) automatically satisfies the boundary conditions in Eq. (6.209). Substituting Eqs. (6.240) and (6.241) into the equation of (6.209) and making use of the orthogonality of the set $\{\sin(\beta_m x)\}$ yields

$$\ddot{\Gamma}_m(t) + 2\zeta_m \omega_m \dot{\Gamma}_m(t) + \omega_m^2 \Gamma_m(t) = \frac{QD_m \alpha}{k\tau_q}(1 + \mathrm{i}\Omega\tau_q)\,\mathrm{e}^{\mathrm{i}\Omega t}, \tag{6.243}$$

whose solution is readily obtained as

$$\Gamma_m(t) = B_m \, \mathrm{e}^{(\Omega t + \varphi_m)\mathrm{i}}. \tag{6.244}$$

Here,

$$B_m = \frac{QD_m \alpha}{k\omega_m} B_{\Omega_m^*}, \tag{6.245}$$

$$B_{\Omega_m^*} = \frac{\eta_m + \mathrm{i}\Omega_m^*}{\sqrt{(1-\Omega_m^{*2})^2 + 4\zeta_m^2 \Omega_m^{*2}}}, \tag{6.246}$$

$$\tan^{-1}(\varphi_m) = -\frac{2\zeta_m \Omega_m^*}{1-\Omega_m^{*2}}, \tag{6.247}$$

$$\eta_m = \frac{1}{\sqrt{\alpha\tau_q}\beta_m}, \tag{6.248}$$

$$\Omega_m^* = \frac{\Omega}{\omega_m}. \tag{6.249}$$

For resonance, $\left|B_{\Omega_m^*}\right|^2$ reaches its maximum value. Note that, by Eq. (6.246),

$$\left|B_{\Omega_m^*}\right|^2 = \frac{\eta_m^2 + \Omega_m^{*2}}{(1-\Omega_m^{*2})^2 + 4\zeta_m^2 \Omega_m^{*2}}. \tag{6.250}$$

Therefore, resonance requires

$$\frac{\partial \left|B_{\Omega_m^*}\right|^2}{\partial \Omega_m^{*2}} = 0,$$

which yields, by noting also that $\Omega^*{}_m \geq 0$,

$$\Omega_{mr}^{*2} = -\eta_m^2 + \sqrt{(1+\eta_m^2)^2 - 4\zeta_m^2\eta_m^4}, \tag{6.251}$$

where Ω_{mr}^* stands for the external source frequency at resonance. Since Ω_{mr}^* must be real, Xu and Wang (2002) obtain another condition for resonance in addition to Eq. (6.251),

$$\left(1+\eta_m^2\right)^2 - 4\zeta_m^2\eta_m^2 > \eta_m^4. \tag{6.252}$$

The variation of Ω_{mr}^* with ζ_m and η_m is shown in Figs. 6.5 and 6.6. It is observed that Ω_{mr}^* decreases as the damping parameter ζ_m and the phase lagging parameter η_m increase. Fig. 6.7 illustrates the variation of $\left|B_{\Omega_m^*}\right|$ with Ω_m^* and ζ_m at $\eta_m = 1$. For $\zeta_m = 0.9$, Eq. (6.252) cannot be satisfied. Therefore, there is no resonance when $\zeta_m = 0.9$ at $\eta_m = 1$ (Fig. 6.7).

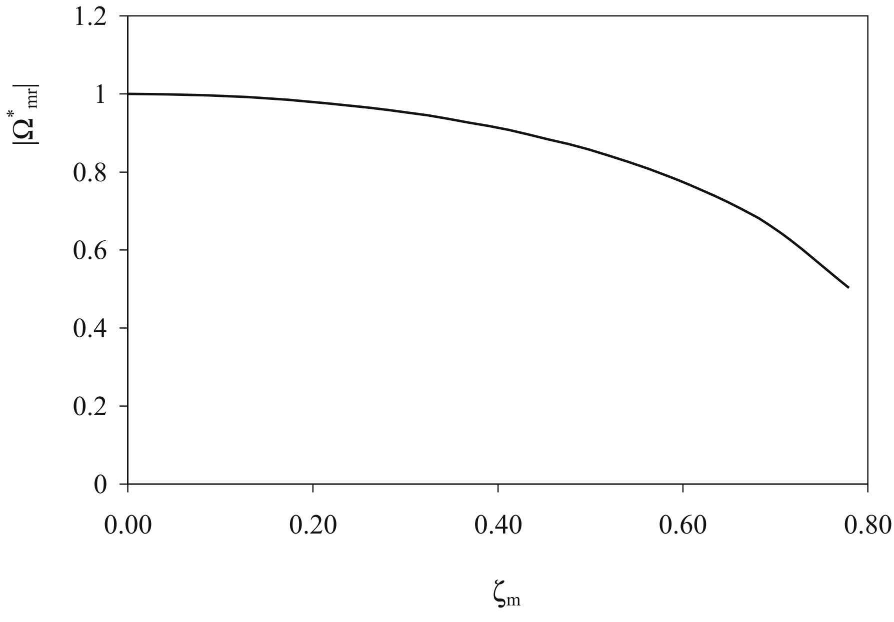

Fig. 6.5 Variation of $|\Omega_{mr}^*|$ with ζ_m at $\eta_m = 1.0$ (after Xu and Wang, 2002)

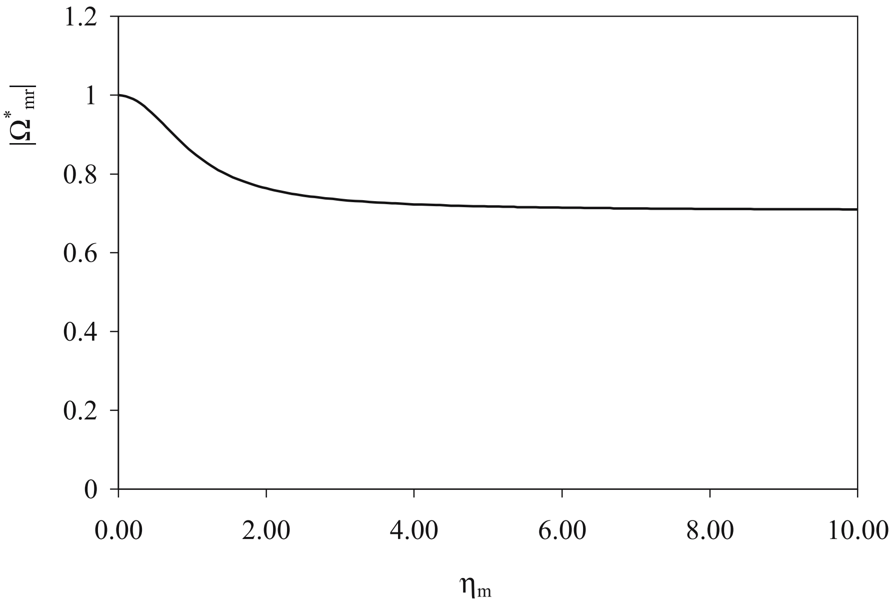

Fig. 6.6 Variation of $|\Omega^*_{mr}|$ with ζ_m at $\eta_m = 0.5$ (after Xu and Wang, 2002)

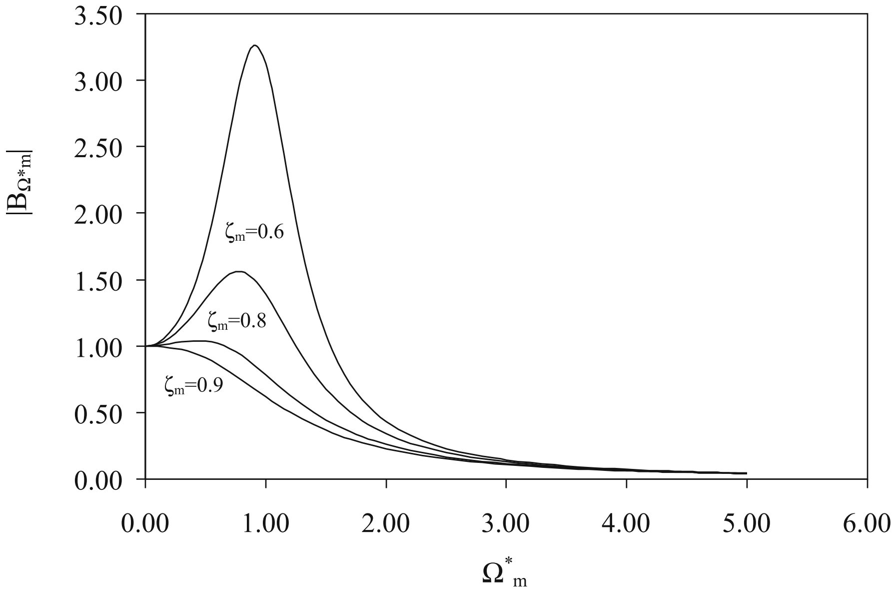

Fig. 6.7 Variation of $|B_{\Omega^* m}|$ with Ω^*_m and ζ_m : $\eta_m = 1$ (after Xu and Wang, 2002)

6.11 Heat Conduction in Two-Phase Systems

In this section we develop an *exact* equivalence between dual-phase-lagging heat conduction and Fourier heat conduction in two-phase systems. This equivalence builds the intrinsic relationship between the two heat-conduction processes, provides an additional tool for studying the two heat-conduction processes, and demonstrates the possibility of the presence of thermal waves and resonance in two-phase-system heat conduction, a phenomenon that has been observed experimentally. We also discuss the mechanism responsible for such thermal waves and resonance.

6.11.1 One- and Two-Equation Models

The microscale model for heat conduction in two-phase systems is well-known. It consists of the field equation and the constitutive equation. The field equation comes from the conservation of energy (the first law of thermodynamics). The commonly-used constitutive equation is the Fourier law of heat conduction for the relation between the temperature gradient ∇T and the heat flux density vector $\mathbf{q}$ (Wang 1994).

For transport in two-phase systems, the macroscale is a phenomenological scale that is much larger than the microscale of pores and grains and much smaller than the system length scale. Interest in the macroscale rather than the microscale comes from the fact that a prediction at the microscale is complicated due to the complex microscale geometry of two-phase systems such as porous media, and also because we are usually more interested in large scales of transport for practical applications. Existence of such a macroscale description equivalent to the microscale behavior requires a good separation of length scales and has been well discussed by Auriault (1991).

To develop a macroscale model of transport in two-phase systems, the method of volume averaging starts with a microscale description (Wang 2000b, Whitaker 1999). Both conservation and constitutive equations are introduced at the microscale. The resulting microscale field equations are then averaged over a representative elementary volume (REV), the smallest differential volume resulting in statistically meaningful local averaging properties, to obtain the macroscale field equations. In the process of averaging, the *multiscale theorems* are used to convert integrals of gradient, divergence, curl, and partial time derivatives of a function into some combination of gradient, divergence, curl, and partial time derivatives of integrals of the function and integrals over the boundary of the REV (Wang 2000b, Wang et al. 2007, Whitaker 1999). The readers are referred to Wang (2000b), Wang et al. (2007b) and Whitaker (1999) for the details of the method of volume averaging and to Wang (2000b) and Wang et al. (2007b) for a summary of the other methods of obtaining macroscale models.

Quintard and Whitaker (1993) use the method of volume averaging to develop one- and two-equation macroscale models for heat conduction in two-phase systems. First, they define the microscale problem by the first law of thermodynamics and the Fourier law of heat conduction (Fig. 6.8)

$$(\rho c)_\beta \frac{\partial T_\beta}{\partial t} = \nabla \cdot (k_\beta \nabla T_\beta), \text{ in the } \beta\text{-phase} \tag{6.253}$$

$$(\rho c)_\sigma \frac{\partial T_\sigma}{\partial t} = \nabla \cdot (k_\sigma \nabla T_\sigma), \text{ in the } \sigma\text{-phase} \tag{6.254}$$

$$T_\beta = T_\sigma, \text{ at the } \beta - \sigma \text{ interface } A_{\beta\sigma} \tag{6.255}$$

$$\mathbf{n}_{\beta\sigma} \cdot k_\beta = \mathbf{n}_{\beta\sigma} \cdot k_\sigma \nabla T_\sigma, \text{ at the } \beta - \sigma \text{ interface } A_{\beta\sigma} \tag{6.256}$$

Here ρ, c and k are the density, specific heat and thermal conductivity, respectively. Subscripts β and σ refer to the β- and σ-phases, respectively. $A_{\beta\sigma}$ represents the area of the β–σ interface contained in the REV, $\mathbf{n}_{\beta\sigma}$ is the outward-directed surface normal from the β-phase toward the σ-phase, and $\mathbf{n}_{\beta\sigma} = -\mathbf{n}_{\sigma\beta}$ (Fig. 6.8). To be thorough, Quintard and Whitaker (1993) have also specified the initial conditions and the boundary conditions at the entrances and exits of the REV; however, we need not do so for our discussion.

Next Quintard and Whitaker (1993) apply the superficial averaging process to Eqs. (6.253) and (6.254) to obtain,

$$\frac{1}{V_{\text{REV}}} \int_{V_\beta} (\rho c)_\beta \frac{\partial T_\beta}{\partial t} \mathrm{d}V = \frac{1}{V_{\text{REV}}} \int_{V_\beta} \nabla (k_\beta \nabla T_\beta) \mathrm{d}V, \tag{6.257}$$

and

$$\frac{1}{V_{\text{REV}}} \int_{V_\sigma} (\rho c)_\sigma \frac{\partial T_\sigma}{\partial t} \mathrm{d}V = \frac{1}{V_{\text{REV}}} \int_{V_\sigma} \nabla \cdot (k_\sigma \nabla T_\sigma) \mathrm{d}V, \tag{6.258}$$

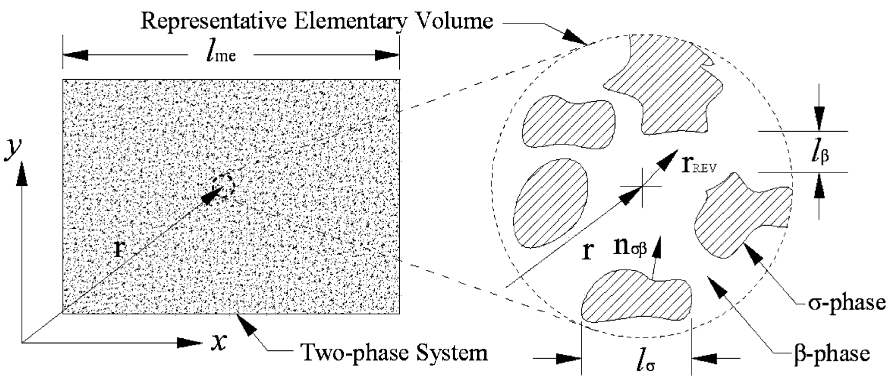

Fig. 6.8 Rigid two-phase system

where V_{REV}, V_β and V_σ are the volumes of the REV, β-phase in REV and σ-phase in REV, respectively. We should note that the superficial temperature is evaluated at the centroid of the REV, whereas the phase temperature is evaluated throughout the REV. Neglecting variations of (ρc) within the REV and considering the system to be rigid so that V_β and V_σ are not functions of time, the volume-averaged form of Eqs. (6.253) and (6.254) are

$$(\rho c)_\beta \frac{\partial \left\langle T_\beta \right\rangle}{\partial t} = \left\langle \nabla \cdot (k_\beta \nabla T_\beta) \right\rangle , \tag{6.259}$$

and

$$(\rho c)_\sigma \frac{\partial \langle T_\sigma \rangle}{\partial t} = \langle \nabla \cdot (k_\sigma \nabla T_\sigma) \rangle , \tag{6.260}$$

where angle brackets indicate superficial quantities such as

$$\langle T_B \rangle = \frac{1}{V_{\mathrm{REV}}} \int_{V_\beta} T_\beta \, \mathrm{d}V ,$$

and

$$\langle T_\sigma \rangle = \frac{1}{V_{\mathrm{REV}}} \int_{V_\sigma} T_\sigma \, \mathrm{d}V .$$

The superficial average, however, is an unsuitable variable because it can yield erroneous results. For example, if the temperature of the β-phase were constant, the superficial average would differ from it (Quintard and Whitaker 1993). On the other hand, intrinsic phase averages do not have this shortcoming. These averages are defined by

$$\left\langle T_\beta \right\rangle^\beta = \frac{1}{V_\beta} \int_{V_\beta} T_\beta \, \mathrm{d}V , \tag{6.261}$$

and

$$\langle T_\sigma \rangle^\sigma = \frac{1}{V_\sigma} \int_{V_\sigma} T_\sigma \, \mathrm{d}V . \tag{6.262}$$

Also, intrinsic averages are related to superficial averages by

$$\left\langle T_\beta \right\rangle = \in_\beta \left\langle T_\beta \right\rangle^\beta , \tag{6.263}$$

and

$$\langle T_\sigma \rangle = \in_\sigma \langle T_\sigma \rangle^\sigma , \tag{6.264}$$

where $\in_\beta$ and $\in_\sigma$ are the volume fractions of the β- and σ-phases with $\in_\beta = \varphi$, $\in_\sigma = 1 - \varphi$ with a constant porosity φ for a rigid two-phase system.

Quintard and Whitaker (1993) substitute Eqs. (6.263) and (6.264) into Eqs. (6.259) and (6.260) to obtain

$$\in_\beta (\rho c)_\beta \frac{\partial \left\langle T_\beta \right\rangle^\beta}{\partial t} = \left\langle \nabla \cdot (k_\beta \nabla T_\beta) \right\rangle , \tag{6.265}$$

and

$$\in_\sigma (\rho c)_\sigma \frac{\partial \left\langle T_\sigma \right\rangle^\sigma}{\partial t} = \left\langle \nabla \cdot (k_\sigma \nabla T_\sigma) \right\rangle . \tag{6.266}$$

Next Quintard and Whitaker (1993) apply the spatial averaging theorem (Theorem 5.2 in Wang et al. 2007b) to Eqs. (6.265) and (6.266) and neglect variations of physical properties within the REV. The result is

$$\begin{aligned} \underbrace{\in_\beta (\rho c)_\beta \frac{\partial \left\langle T_\beta \right\rangle^\beta}{\partial t}}_{\text{accumulation}} &= \underbrace{\nabla \cdot \left\{ k_\beta \left[\in_\beta \nabla \left\langle T_\beta \right\rangle^\beta + \left\langle T_\beta \right\rangle^\beta \nabla \in_\beta + \frac{1}{V_{\text{REV}}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} T_\beta \, \mathrm{d}A \right] \right\}}_{\text{conduction}} \\ &\quad + \underbrace{\frac{1}{V_{\text{REV}}} \int_{A_{\beta\sigma}} n_{\beta\sigma} \cdot k_\beta \nabla T_\beta \, \mathrm{d}A}_{\text{interfacial flux}} , \end{aligned} \tag{6.267}$$

and

$$\begin{aligned} \underbrace{\in_\sigma (\rho c)_\sigma \frac{\partial \left\langle T_\sigma \right\rangle^\sigma}{\partial t}}_{\text{accumulation}} &= \underbrace{\nabla \cdot \left\{ k_\sigma \left[\in_\sigma \nabla \left\langle T_\sigma \right\rangle^\sigma + \left\langle T_\sigma \right\rangle^\sigma \nabla \in_\sigma + \frac{1}{V_{\text{REV}}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} T_\sigma \, \mathrm{d}A \right] \right\}}_{\text{conduction}} \\ &\quad + \underbrace{\frac{1}{V_{\text{REV}}} \int_{A_{\beta\sigma}} n_{\beta\sigma} \cdot k_\sigma \nabla T_\sigma \, \mathrm{d}A}_{\text{interfacial flux}} . \end{aligned} \tag{6.268}$$

By introducing the spatial decompositions $T_\beta = \left\langle T_\beta \right\rangle^\beta + \tilde{T}_\beta$ and $T_\sigma = \left\langle T_\sigma \right\rangle^\sigma + \tilde{T}_\sigma$ and by applying scaling arguments and Theorem 5.2 in Wang et al. 2007b, Eqs. (6.267) and (6.268) are simplified into (Quintard and Whitaker 1993)

$$\in_\beta (\rho c)_\beta \frac{\partial \left\langle T_\beta \right\rangle^\beta}{\partial t} = \nabla \cdot \left\{ k_\beta \left[\in_\beta \nabla \left\langle T_\beta \right\rangle^\beta + \frac{1}{V_{\mathrm{REV}}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \tilde{T}_\beta \, \mathrm{d}A \right] \right\}$$
$$+ \frac{1}{V_{\mathrm{REV}}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot k_\beta \nabla \left\langle T_\beta \right\rangle^\beta \mathrm{d}A$$
$$+ \frac{1}{V_{\mathrm{REV}}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot k_\beta \nabla \tilde{T}_\beta \, \mathrm{d}A \,, \tag{6.269}$$

and

$$\in_\sigma (\rho c)_\sigma \frac{\partial \left\langle T_\sigma \right\rangle^\sigma}{\partial t} = \nabla \cdot \left\{ k_\sigma \left[\in_\sigma \nabla \left\langle T_\sigma \right\rangle^\sigma + \frac{1}{V_{\mathrm{REV}}} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \tilde{T}_\sigma \, \mathrm{d}A \right] \right\}$$
$$+ \frac{1}{V_{\mathrm{REV}}} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot k_\sigma \nabla \left\langle T_\sigma \right\rangle^\sigma \mathrm{d}A$$
$$+ \frac{1}{V_{\mathrm{REV}}} \int_{A_{\beta\sigma}} \mathbf{n}_{\sigma\beta} \cdot k_\sigma \nabla \tilde{T}_\sigma \, \mathrm{d}A \,. \tag{6.270}$$

After developing the closure for $\tilde{T}_\beta$ and $\tilde{T}_\sigma$, Quintard and Whitaker (1993) obtain a two-equation model

$$\in_\beta (\rho c)_\beta \frac{\partial \left\langle T_\beta \right\rangle^\beta}{\partial t} = \nabla \cdot \left\{ \mathbf{K}_{\beta\beta} \cdot \nabla \left\langle T_\beta \right\rangle^\beta + \mathbf{K}_{\beta\sigma} \cdot \nabla \left\langle T_\sigma \right\rangle^\sigma \right\}$$
$$+ h a_\upsilon \left(\left\langle T_\sigma \right\rangle^\sigma - \left\langle T_\beta \right\rangle^\beta \right) , \tag{6.271}$$

and

$$\in_\sigma (\rho c)_\sigma \frac{\partial \left\langle T_\sigma \right\rangle^\sigma}{\partial t} = \nabla \cdot \left\{ \mathbf{K}_{\sigma\sigma} \cdot \nabla \left\langle T_\sigma \right\rangle^\sigma + \mathbf{K}_{\sigma\beta} \cdot \nabla \left\langle T_\beta \right\rangle^\beta \right\}$$
$$- h a_\upsilon \left(\left\langle T_\sigma \right\rangle^\sigma - \left\langle T_\beta \right\rangle^\beta \right) , \tag{6.272}$$

where h and a_υ come from modeling of the interfacial flux and are the film heat transfer coefficient and the interfacial area per unit volume, respectively. $\mathbf{K}_{\beta\beta}$, $\mathbf{K}_{\sigma\sigma}$, $\mathbf{K}_{\beta\sigma}$ and $\mathbf{K}_{\sigma\beta}$ are the effective thermal conductivity tensors, and the coupled thermal conductivity tensors are equal

$$\mathbf{K}_{\beta\sigma} = \mathbf{K}_{\sigma\beta} \,.$$

When the system is isotropic and the physical properties of the two phases are constant, Eqs. (6.271) and (6.272) reduce to

$$\gamma_\beta \frac{\partial \left\langle T_\beta \right\rangle^\beta}{\partial t} = k_\beta \Delta \left\langle T_\beta \right\rangle^\beta + k_{\beta\sigma}\, \Delta \left\langle T_\sigma \right\rangle^\sigma + ha_\upsilon \left(\left\langle T_\sigma \right\rangle^\sigma - \left\langle T_\beta \right\rangle^\beta \right), \tag{6.273}$$

and

$$\gamma_\sigma \frac{\partial \left\langle T_\sigma \right\rangle^\sigma}{\partial t} = k_\sigma \Delta \left\langle T_\sigma \right\rangle^\sigma + k_{\sigma\beta}\, \Delta \left\langle T_\beta \right\rangle^\beta + ha_\upsilon \left(\left\langle T_\sigma \right\rangle^\sigma - \left\langle T_\beta \right\rangle^\beta \right), \tag{6.274}$$

where $\gamma_\beta = \varphi(\rho c)_\beta$ and $\gamma_\sigma = (1-\varphi)(\rho c)_\sigma$ are the β-phase and σ-phase effective thermal capacities, respectively, φ is the porosity, k_β and k_σ are the effective thermal conductivities of the β- and σ-phases, respectively, and $k_{\beta\sigma} = k_{\sigma\beta}$ is the cross effective thermal conductivity of the two phases.

The one-equation model is valid whenever the two temperatures $\left\langle T_\beta \right\rangle^\beta$ and $\left\langle T_\sigma \right\rangle^\sigma$ are sufficiently close to each other so that

$$\left\langle T_\beta \right\rangle^\beta = \left\langle T_\sigma \right\rangle^\sigma = \left\langle T \right\rangle . \tag{6.275}$$

This *local thermal equilibrium* is valid when any one of the following three conditions occurs (Quintard and Whitaker 1993, Whitaker 1999): (1) either $\in_\beta$ or $\in_\sigma$ tends to zero, (2) the difference in the β-phase and σ-phase physical properties tends to zero, (3) the square of the ratio of length scales $(l_{\beta\sigma}/L)^2$ tends to zero (e.g. for steady, one-dimensional heat conduction). Here $l^2_{\beta\sigma} = \left[\in_\beta \in_\sigma \left(\in_\beta k_\sigma + \in_\sigma k_\beta\right)\right] / (ha_\upsilon)$, and $L = L_T L_{T1}$ with L_T and L_{T1} as the characteristic lengths of $\nabla \left\langle T \right\rangle$ and $\nabla\nabla \left\langle T \right\rangle$, respectively, such that $\nabla \left\langle T \right\rangle = O(\Delta \left\langle T \right\rangle / L_T)$ and $\nabla\nabla \left\langle T \right\rangle = O(\Delta \left\langle T \right\rangle / L_{T1} L_T)$.

When the local thermal equilibrium is valid, Quintard and Whitaker (1993) add Eqs. (6.271) and (6.272) to obtain a one-equation model

$$\langle \rho \rangle C \frac{\partial \langle T \rangle}{\partial t} = \nabla \cdot \left[\mathbf{K}_{eff} \cdot \nabla \langle T \rangle \right] . \tag{6.276}$$

Here $\langle \rho \rangle$ is the spatial average density defined by

$$\langle \rho \rangle = \in_\beta \rho_\beta + \in_\sigma \rho_\sigma , \tag{6.277}$$

and C is the mass-fraction-weighted thermal capacity given by

$$C = \frac{\in_\beta (\rho c)_\beta + \varepsilon_\sigma (\rho c) \sigma}{\in_\beta \rho_\beta + \in_\sigma \rho_\sigma} . \tag{6.278}$$

The effective thermal conductivity tenor is

$$\mathbf{K}_{eff} = \mathbf{K}_{\beta\beta} + 2\mathbf{K}_{\beta\sigma} + \mathbf{K}_{\sigma\sigma} . \tag{6.279}$$

The choice between the one-equation model and the two-equation model has been well discussed by Quintard and Whitaker (1993) and Whitaker (1999). They have also developed methods of determining the effective thermal conductivity tensor $\mathbf{K}_{eff}$ in the one-equation model and the four coefficients $\mathbf{K}_{\beta\beta}$, $\mathbf{K}_{\beta\sigma} = \mathbf{K}_{\sigma\beta}$, $\mathbf{K}_{\sigma\sigma}$, and ha_υ in the two-equation model. Their studies suggest that the coupling coefficients are on the order of the smaller of $\mathbf{K}_{\beta\beta}$ and $\mathbf{K}_{\sigma\sigma}$. Therefore, the coupled conductive terms should not be omitted in any detailed two-equation model of heat conduction processes. When the principle of local thermal equilibrium is not valid, the commonly-used two-equation model in the literature is the one without the coupled conductive terms (Glatzmaier and Ramirez 1988)

$$\in_\beta (\rho c)_\beta \frac{\partial \langle T_\beta \rangle^\beta}{\partial t} = \nabla \cdot \left(\mathbf{K}_{\beta\beta} \cdot \nabla \langle T_\beta \rangle^\beta \right) + ha_\upsilon \left(\langle T_\sigma \rangle^\sigma - \langle T_\beta \rangle^\beta \right), \tag{6.280}$$

and

$$\in_\sigma (\rho c)_\sigma \frac{\partial \langle T_\sigma \rangle^\sigma}{\partial t} = \nabla \cdot \left(\mathbf{K}_{\sigma\sigma} \cdot \nabla \langle T_\sigma \rangle^\sigma \right) - ha_\upsilon \left(\langle T_\sigma \rangle^\sigma - \langle T_\beta \rangle^\beta \right). \tag{6.281}$$

On the basis of the above analysis, we now know that the coupled conductive terms $\mathbf{K}_{\beta\sigma} \cdot \nabla \langle T_\sigma \rangle^\sigma$ and $\mathbf{K}_{\sigma\beta} \cdot \nabla \langle T_\beta \rangle^\beta$ cannot be discarded in the exact representation of the two-equation model. However, we could argue that Eqs. (6.280) and (6.281) represent a reasonable approximation of Eqs. (6.271) and (6.272) for a heat conduction process in which $\nabla \langle T_\beta \rangle^\beta$ and $\nabla \langle T_\sigma \rangle^\sigma$ are *sufficiently close* to each other. Under these circumstances $\mathbf{K}_{\beta\beta}$ in Eq. (6.280) would be given by $\mathbf{K}_{\beta\beta} + \mathbf{K}_{\beta\sigma}$ while $\mathbf{K}_{\sigma\sigma}$ in Eq. (6.281) should be interpreted as $\mathbf{K}_{\sigma\beta} + \mathbf{k}_{\sigma\sigma}$. This limitation of Eqs. (6.280) and (6.281) is believed to be the reason behind the paradox of heat conduction in porous media subject to lack of local thermal equilibrium analyzed by Vadasz 2005a, b, c. For an isotropic system with constant physical properties of the two phases, Eqs. (6.280) and (6.281) reduce to the traditional formulation of heat conduction in two-phase systems (Vadasz 2005a, b, c, Bejan 2004, Bejan et al. 2004, Nield and Bejan 2006)

$$\gamma_\beta \frac{\partial \langle T_\beta \rangle^\beta}{\partial t} = k_{e\beta} \Delta \langle T_\beta \rangle^\beta + ha_\upsilon \left(\langle T_\sigma \rangle^\sigma - \langle T_\beta \rangle^\beta \right), \tag{6.282}$$

and

$$\gamma_\sigma \frac{\partial \langle T_\sigma \rangle^\sigma}{\partial t} = k_{e\sigma}\, \Delta \langle T_\sigma \rangle^\beta - ha_\upsilon \left(\langle T_\sigma \rangle^\sigma - \langle T_\beta \rangle^\beta \right), \tag{6.283}$$

where we introduce the *equivalent* effective thermal conductivities $k_{e\beta} = k_\beta + k_{\beta\sigma}$ and $k_{e\sigma} = k_\sigma + k_{\sigma\beta}$ for the β- and σ-phases, respectively, to take the above note into account. To describe the thermal energy exchange between solid and gas phases in casting sand, Tzou (1997) has also directly postulated Eqs. (6.282) and (6.283) (using k_β and k_σ rather than $k_{e\beta}$ and $k_{e\sigma}$) as a two-step model, parallel to the two-step equations in the microscopic phonon-electron interaction model (Qiu and Tien 1993, Kaganov et al. 1957, Anisimòv et al. 1974).

6.11.2 Equivalence with Dual-Phase-Lagging Heat Conduction

The two-equation model can be used to establish equivalence between the dual-phase lagging and two-phase-system heat conduction (Wang and Wei 2007a, 2007b, Wang et al. 2007b). We first rewrite Eqs. (6.273) and (6.274) in their operator form

$$\begin{bmatrix} \gamma_\beta \frac{\partial}{\partial t} - k_\beta \Delta + h & -k_{\beta\sigma}\Delta - ha_\upsilon \\ -k_{\beta\sigma}\Delta - ha_\upsilon & \gamma_\sigma \frac{\partial}{\partial t} - k_\sigma \Delta + ha_\upsilon \end{bmatrix} \begin{bmatrix} \langle T_\beta \rangle^\beta \\ \langle T_\sigma \rangle^\sigma \end{bmatrix} = 0. \tag{6.284}$$

We then obtain a uncoupled form by evaluating the operator determinant such that

$$\left[\left(\gamma_\beta \frac{\partial}{\partial t} - k_\beta \Delta + ha_\upsilon\right)\left(\gamma_\sigma \frac{\partial}{\partial t} - k_\sigma \Delta + ha_\upsilon\right) - \left(k_{\beta\sigma}\Delta - ha_\upsilon\right)^2\right] \langle T_i \rangle^i = 0, \tag{6.285}$$

where the index i can take β or σ. Its explicit form reads, after dividing by $ha_\upsilon(\gamma_\beta + \gamma_\sigma)$

$$\frac{\partial \langle T_i \rangle^i}{\partial t} + \tau_0 \frac{\partial^2 \langle T_i \rangle^i}{\partial t^2} = \alpha \Delta \langle T_i \rangle^i + \alpha \tau_T \frac{\partial}{\partial t}\left(\Delta \langle T_i \rangle^i\right) + \frac{\alpha}{k}\left[F(\mathbf{r},t) + \tau_0 \frac{\partial F(\mathbf{r},t)}{\partial t}\right], \tag{6.286}$$

where

$$\begin{aligned} &\tau_0 = \frac{\gamma_\beta \gamma_\sigma}{ha_\upsilon(\gamma_\beta + \gamma_\sigma)}, \quad \tau_T = \frac{\gamma_\beta k_\sigma + \gamma_a k_\beta}{ha_\upsilon(k_\beta + k_\sigma + 2k_{\beta\sigma})}, \\ &k = k_\beta + k_\sigma + 2k_{\beta\sigma}, \quad \alpha = \frac{k_\beta + k_\sigma + 2k_{\beta\sigma}}{\gamma_\beta + \gamma_\sigma}, \\ &F(\mathbf{r},t) + \tau_0 \frac{\partial F(\mathbf{r},t)}{\partial t} = \frac{k_{\beta\sigma}^2 - k_\beta k_\sigma}{ha_\upsilon} \Delta^2 \langle T_i \rangle^i . \end{aligned} \tag{6.287}$$

This is the dual-phase-lagging heat-conduction equation with τ_0 and τ_T as the phase lags of the heat flux and the temperature gradient, respectively. k, ρc and α are the effective thermal conductivity, capacity and diffusivity of two-phase system, respectively. $F(\mathbf{r},t)$ is the volumetric heat source. The reported conductivity and diffusivity data of two-phase systems (nanofluids, bi-composite media, porous media etc.) in the literature were based on the Fourier heat conduction and should be reexamined.

Note that Eqs. (6.273) and (6.274) are the mathematical representation of the first law of thermodynamics and the Fourier law of heat conduction for heat conduction processes in two-phase systems at the macroscale. Therefore, we have an *exact* equivalence between dual-phase-lagging heat conduction and Fourier heat conduction in two-phase systems. This is significant because all results in these two fields become mutually applicable. In particular, all analytical methods and results (such as the solution structure theorems) in the present monograph can be applied to study heat conduction in two-phase systems such as nanofluids, bi-composite media and porous media (Wang and Wei 2007a, 2007b, Wang et al. 2007b).

By Eq. (6.287), we can readily obtain that, in two-phase-system heat conduction

$$\frac{\tau_T}{\tau_0} = 1 + \frac{\gamma_\beta^2 k_\sigma + \gamma_\sigma^2 k_\beta - 2\gamma_\beta \gamma_\sigma k_{\beta\sigma}}{\gamma_\beta \gamma_\sigma (k_\beta + k_\sigma + 2k_{\beta\sigma})}. \tag{6.288}$$

It can be large, equal or smaller than 1 depending on the sign of $\gamma_\beta^2 k_\sigma + \gamma_\sigma^2 k_\beta - 2\gamma_\beta \gamma_\sigma k_{\beta\sigma}$. Therefore, by the condition for the existence of thermal waves that requires $\tau_T/\tau_0 < 1$ (Section 6.10, Xu and Wang 2002), we may have thermal waves in two-phase-system heat conduction when

$$\gamma_\beta^2 k_\sigma + \gamma_\sigma^2 k_\beta - 2\gamma_\beta \gamma_\sigma k_{\beta\sigma} < 0.$$

Note also that for heat conduction in two-phase systems there is a time-dependent source term $F(\mathbf{r},t)$ in the dual-phase-lagging heat conduction [Eqs. (6.286) and (6.287)]. Therefore, the resonance can also occur. This agrees with the experimental data of casting sand tests in Tzou (1997). Discarding the coupled conductive terms in Eqs. (6.273) and (6.274) assumes $k_{\beta\sigma} = 0$ so that τ_T/τ_0 is always larger than 1, which leads to the exclusion of thermal oscillation and resonance (Vadasz 2005a, 2005c, 2006a) and generates an inconsistency between theoretical and experimental results in the literature regarding the possibility of thermal waves and resonance in two-phase-system heat conduction (Tzou 1997, Vadasz 2005a, 2005c, 2006a). The coupled conductive terms in Eqs. (6.273) and (6.274) are thus responsible for the thermal waves and resonance in two-phase-system heat conduction. These thermal waves and possibly resonance are believed to be the driving force for the extraordinary conductivity enhancement reported in nanofluids (Assael et al. 2006, Choi et al. 2001, 2004, Das 2006, Das et al. 2006, Eastman et al. 2001, 2004 Jang and Choi 2004, Kumar et al. 2004, Murshed et al. 2006, Peterson and Li 2006, Phelan 2005, Putnam et al. 2006, Strauss and Pober 2006, Wang and Mujumdar 2007, Yu and Choi 2006).

Although each τ_0 and τ_T is ha_v-dependent, the ratio τ_T/τ_0 is not; this makes its evaluation much simpler.

Chapter 7
Potential Equations

When transients have died away and initial values have been forgotten, systems become steady and processes are driven by boundary conditions and the source term. Equations of wave, classical heat-conduction, hyperbolic heat-conduction and dual-phase-lagging heat- conduction all reduce to potential equations. Their mixed problems also reduce to the boundary-value problems of potential equations. In this chapter we mainly discuss methods of solving boundary-value problems of potential equations.

7.1 Fourier Method of Expansion

In this section we use examples to show the Fourier method of expansion in solving boundary-value problems of potential equations.

Example 1. Solve the PDS in a rectangular domain:

$$\begin{cases} \Delta u = 0\,, & 0 < x < a, 0 < y < b\,, \\ u(0,y) = \varphi_1(y)\,, & u(a,y) = \varphi_2(y)\,, \\ u(x,0) = \psi_1(x),\ u(x,b) = \psi_2(x)\,. \end{cases} \tag{7.1}$$

Solution. Let $u_1(x,y)$ and $u_2(x,y)$ be the solutions for the case of $\psi_1(x) = \psi_2(x) = 0$ and for the case of $\varphi_1(y) = \varphi_2(y) = 0$, respectively. Since PDS (7.1) is linear, the solution of PDS (7.1) is thus, by the principle of superposition,

$$u(x,y) = u_1(x,y) + u_2(x,y)\,.$$

Based on the given boundary conditions, consider

$$u_1(x,y) = \sum_{n=1}^{\infty} X_n(x) \sin \frac{n\pi y}{b}\,, \tag{7.2}$$

where $X_n(x)$ is a undetermined function. Substituting it into the equation in PDS (7.1)

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yields

$$X_n''(x) - \left(\frac{n\pi}{b}\right)^2 X_n(x) = 0\,.$$

Its general solution is

$$X_n(x) = a_n\,\mathrm{e}^{\frac{n\pi x}{b}} + b_n\,\mathrm{e}^{-\frac{n\pi x}{b}}\,,$$

where the a_n and the b_n are undetermined constants. Thus

$$u_1(x,y) = \sum_{n=1}^{\infty}\left(a_n\,\mathrm{e}^{\frac{n\pi x}{b}} + b_n\,\mathrm{e}^{-\frac{n\pi x}{b}}\right)\sin\frac{n\pi y}{b}\,. \tag{7.3}$$

By applying boundary conditions $u_1(0,y) = \varphi_1(y)$ and $u_1(a,y) = \varphi_2(y)$, we obtain

$$\begin{cases} a_n + b_n = \dfrac{2}{b}\displaystyle\int_0^b \varphi_1(y)\sin\frac{n\pi y}{b}\,\mathrm{d}y\,, \\ \mathrm{e}^{\frac{n\pi a}{b}}a_n + \mathrm{e}^{-\frac{n\pi a}{b}}b_n = \dfrac{2}{b}\displaystyle\int_0^b \varphi_2(y)\sin\frac{n\pi y}{b}\,\mathrm{d}y\,, \end{cases} \tag{7.4}$$

so that the a_n and the b_n can be determined. Similarly, we can also obtain $u_2(x,y)$.

Example 2. Solve the PDS in a cube

$$\begin{cases} \Delta u = 0\,, \quad 0 < x,y,z < a\,, \\ u(0,y,z) = u(a,y,z) = 0\,, \\ u(x,0,z) = u(x,a,z) = 0\,, \\ u(x,y,0) = \varphi(x,y)\,,\ u(x,y,a) = 0\,. \end{cases} \tag{7.5}$$

Solution. Based on the given boundary conditions, consider

$$u(x,y,z) = \sum_{m,n=1}^{\infty} Z_n(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,. \tag{7.6}$$

Substituting it into the equation in PDS (7.5) and using the orthogonality of $\left\{\sin\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{a}\right\}$ in $0 < x,y < a$ yield

$$Z_{mn}''(z) - \lambda_{mn}Z_{mn}(z) = 0\,,\ \lambda_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2\,,\ Z_{mn}(a) = 0\,. \tag{7.7}$$

Its two linearly-independent particular solutions are $\mathrm{e}^{\sqrt{\lambda_{mn}}z}$, $\mathrm{e}^{-\sqrt{\lambda_{mn}}z}$. Thus its general solution can be written as

$$Z_{mn}(z) = a_{mn}\mathrm{ch}\sqrt{\lambda_{mn}}(a-z) + b_{mn}\mathrm{sh}\sqrt{\lambda_{mn}}(a-z)\,,$$

where the a_{mn} and the b_{mn} are constants. Applying $Z_{mn}(a) = 0$ yields $a_{mn} = 0$. Therefore,

$$u(x,y,z) = \sum_{m,n=1}^{\infty} b_{mn}\mathrm{sh}\sqrt{\lambda_{mn}}(a-z)\cdot\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,.$$

Applying the boundary condition $u(x,y,0)=\varphi(x,y)$ leads to

$$\sum_{m,n=1}^{\infty} b_{mn}\mathrm{sh}\sqrt{\lambda_{mn}}a\cdot\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}=\varphi(x,y)\,,$$

so that the b_{mn} can be determined. Finally, we obtain the solution of PDS (7.5)

$$\begin{cases} u(x,y,z)=\displaystyle\sum_{m,n=1}^{\infty} b_{mn}\mathrm{sh}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{a}\right)^2}(a-z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,,\\ b_{mn}=\dfrac{1}{\mathrm{sh}\left(a\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{a}\right)^2}\right)}\dfrac{4}{a^2}\displaystyle\int_0^a \mathrm{d}x\int_0^a \varphi(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,\mathrm{d}y\,. \end{cases}$$

Example 3. Solve

$$\begin{cases} \Delta T=0\,, \quad 0<x,y<a,\\ T(0,y)=T(a,y)=0\,,\\ T(x,0)=0\,, \quad T(x,a)=100\sin\frac{\pi x}{a}\,. \end{cases} \tag{7.8}$$

Solution. Based on the given boundary conditions of x-side, consider

$$T(x,y)=\sum_{m=1}^{\infty} Y_m(y)\sin\frac{m\pi x}{a}\,,$$

where the $Y_m(y)$ is a undetermined function. Substituting it into the equation in PDS (7.8) yields

$$Y_m''(y)-\left(\frac{m\pi}{a}\right)^2 Y_m(y)=0\,, \quad Y_m(0)=0\,.$$

Its general solution can be written as

$$Y_m(y)=a_m\mathrm{ch}\frac{m\pi y}{a}+b_m\mathrm{sh}\frac{m\pi y}{a}\,,$$

where the a_m and the b_m are constants. Applying $Y_m(0)=0$ yields $a_m=0$. Thus

$$T(x,y)=\sum_{m=1}^{\infty} b_m\mathrm{sh}\frac{m\pi y}{a}\sin\frac{m\pi x}{a}\,.$$

Applying the boundary condition $T(x,a)=100\sin\frac{\pi x}{a}$ yields

$$\sum_{m=1}^{\infty}(b_m\mathrm{sh}m\pi)\sin\frac{m\pi x}{a}=100\sin\frac{\pi x}{a}\,.$$

Since $\left\{\sin\dfrac{m\pi x}{a}\right\}$ is orthogonal in $[0,a]$, we have $b_m = \begin{cases} 0, & m \neq 1\,, \\ 100/\mathrm{sh}\pi\,, & m = 1\,. \end{cases}$

Finally, we obtain the solution

$$T(x,y) = \frac{100}{\mathrm{sh}\pi}\mathrm{sh}\frac{\pi y}{a}\sin\frac{\pi x}{a}\,.$$

Example 4. Solve

$$\begin{cases} \Delta u = 0\,, & 0 < x < a,\ 0 < y < +\infty\,, \\ u(x,0) = A\left(1 - \dfrac{x}{a}\right)\,,\ u(x,+\infty) = 0\,, \\ u(0,y) = u(a,y) = 0\,. \end{cases} \tag{7.9}$$

Solution. The $u(x,y)$ that satisfies $u(0,y) = u(a,y) = 0$ can be written as

$$u(x,y) = \sum_{m=1}^{\infty}\left(a_m\,\mathrm{e}^{\frac{m\pi y}{a}} + b_m\,\mathrm{e}^{-\frac{m\pi y}{a}}\right)\sin\frac{m\pi x}{a}\,.$$

Applying $u(x,+\infty) = 0$ yields $a_m = 0$. Substituting the $u(x,y)$ into the equation in PDS (7.9) leads to

$$\sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{a} = A\left(1 - \frac{x}{a}\right)\,,$$

so that the b_m can be determined by the integration by parts,

$$b_m = \frac{2}{a}\int_0^a A\left(1 - \frac{x}{a}\right)\sin\frac{m\pi x}{a}\,\mathrm{d}x = \frac{2A}{m\pi}\,.$$

Finally, the solution of PDS (7.9) is

$$u(x,y) = \frac{2A}{\pi}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-\frac{m\pi y}{a}}}{m}\sin\frac{m\pi x}{a}\,. \tag{7.10}$$

The presence of $\mathrm{e}^{-\frac{m\pi y}{a}}$ in the general term of $u(x,y)$ indicates that the series general term decays quickly as $m \to \infty$. In calculation, we can take first a few terms as an approximation of $u(x,y)$. If the first three terms are taken, for example, we have

$$u(x,y) \approx \frac{2A}{\pi}\left(\mathrm{e}^{-\frac{\pi y}{a}}\sin\frac{\pi x}{a} + \frac{\mathrm{e}^{-\frac{2\pi y}{a}}}{2}\sin\frac{2\pi x}{a} + \frac{\mathrm{e}^{-\frac{3\pi y}{a}}}{3}\sin\frac{3\pi x}{a}\right)\,.$$

Remark. Examples 1–4 show applications of eigenvalues and eigenfunctions in Row 1 in Table 2.1. We can follow a similar approach to find solutions subject to the other boundary conditions simply by using corresponding eigenvalues and eigenfunctions.

7.2 Separation of Variables and Fourier Sin/Cos Transformation

The Fourier method of expansion is effective once the function base for expanding the solution is available. It is not, however, straightforward to know such a base for some boundary conditions. We must first apply separation of variables, in general, to find eigenvalues and eigenfunctions which form a function base for solution expansion. For some problems, it is more convenient to solve by a Fourier sin/cos transformation in a finite region.

7.2.1 Separation of Variables

Example 1. Solve the internal Dirichlet problem:

$$\begin{cases} u_{xx}+u_{yy}=0\,, \quad x^2+y^2<a^2\,, \\ u|_{x^2+y^2=a^2}=F(x,y)\,. \end{cases} \tag{7.11}$$

Solution. Consider $x=r\cos\theta$, $y=r\sin\theta$. The PDS (7.11) becomes, in a polar coordinate system

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0\,, \quad 0<r<a\,, \\ u|_{r=a}=f(\theta),\, f(\theta+2\pi)=f(\theta)\,. \end{cases} \tag{7.12}$$

Let $u=R(r)\Theta(\theta)$. Substituting it into the equation in PDS (7.12) yields

$$\begin{cases} \Theta''(\theta)+\lambda\Theta(\theta)=0\,, \\ \Theta(\theta+2\pi)=\Theta(\theta) \end{cases} \tag{7.13}$$

and

$$r^2R''(r)+rR'(r)-\lambda R(r)=0\,, |R(0)|<\infty\,, \tag{7.14}$$

where λ is the separation constant.

By Eq. (7.13), we obtain the eigenvalues and the eigenfunctions

$$\lambda=n^2\,, \quad n=0,1,2,\cdots\,,$$

$$\Theta_n(\theta)=C_n\cos n\theta+D_n\sin n\theta\,,$$

where constants C_n and D_n are not all equal to zero.

Substituting $\lambda=n^2$ into Eq. (7.14) yields a homogeneous Euler equation

$$r^2R''(r)+rR'(r)-n^2R(r)=0\,.$$

It can be transformed into an equation with constant coefficients by a variable transformation of $r = \mathrm{e}^t$. Its general solution can thus be readily obtained as

$$R_n(r) = \begin{cases} c_0 + d_0 \ln r\,, & n = 0\,, \\ c_n r^n + d_n r^{-n}\,, & n = 1, 2, \cdots\,, \end{cases}$$

where the c_n and the d_n are constants. Applying $|R(0)| < \infty$ yields $d_n = 0$, so that

$$R_n(r) = c_n r^n\,, \quad n = 0, 1, 2, \cdots\,.$$

Thus

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)\,. \tag{7.15}$$

Applying the boundary condition $u|_{r=a} = f(\theta)$ yields

$$\begin{cases} a_n = \dfrac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \cos n\theta \,\mathrm{d}\theta\,, \\ b_n = \dfrac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta\,, \quad n = 0, 1, 2, \cdots\,. \end{cases} \tag{7.16}$$

Remark 1. Note that

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty} z^n \cos n(\theta - \theta') &= -1 + 2\sum_{n=0}^{\infty} z^n \cos n(\theta - \theta') \\ &= -1 + 2\mathrm{Re}\left[\sum_{n=0}^{\infty} z^n \,\mathrm{e}^{\mathrm{i}n(\theta-\theta')}\right] = -1 + 2\mathrm{Re}\left[\frac{1}{1 - z\mathrm{e}^{\mathrm{i}(\theta-\theta')}}\right] \\ &= -1 + 2\mathrm{Re}\left[\frac{1 - z\cos(\theta-\theta') + \mathrm{i}z\sin(\theta-\theta')}{1 - 2z\cos(\theta-\theta') + z^2}\right] \\ &= -1 + \frac{2 - 2z\cos(\theta-\theta')}{1 - 2z\cos(\theta-\theta') + z^2} \\ &= \frac{1 - z^2}{1 + z^2 - 2z\cos(\theta-\theta')}\,, \quad |z| < 1\,. \end{aligned}$$

The $u(r,\theta)$ in Eq. (7.15) becomes

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta') \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta-\theta')} \,\mathrm{d}\theta'\,. \tag{7.17}$$

This is called the *Poisson formula of internal problems in a circle*.

Remark 2. An external problem in a circle reads

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0\,, \quad a < r < +\infty\,, \\ u|_{r=a} = f(\theta)\,, \;\; f(\theta + 2\pi) = f(\theta)\,. \end{cases}$$

It can be solved by following the same approach in Example 1.

Here $d_0 = 0$ and $c_n = 0$ in

$$R_n(r) = \begin{cases} c_0 + d_0 \ln r, & n = 0, \\ c_n r^n + d_n r^{-n}, & n = 1, 2, \cdots. \end{cases}$$

Thus the r^n in Eq. (7.15) should be replaced by r^{-n}. Finally, the solution is

$$u(r,\theta) = -\frac{1}{2\pi}\int_0^{2\pi} f(\theta')\frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \theta')}\,\mathrm{d}\theta'.$$

This is called the *Poisson formula of external problems in a circle*.

Remark 3. The general form of homogeneous Euler equations of second order is

$$x^2 y'' + pxy' + qy = 0, \tag{7.18}$$

where p and q are constants. It reduces to an equation with constant coefficients by a variable transformation of $x = \mathrm{e}^t$

$$D(D-1)y + pDy + qy = 0, \tag{7.19}$$

where $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Its characteristic equation can be obtained by replacing D and y in Eq. (7.19) by the characteristic roots r and 1, respectively.

$$r(r-1) + pr + q = 0.$$

Remark 4. The result in Example 1 can also be used to solve problems in a semi-circular domain by using the method of continuation. Consider two-dimensional steady heat conduction, for example, in a semi-circular plate of radius a. The temperature is $u(a,\theta) = T_0\theta(\pi - \theta)$ with T_0 as a constant on the semi-circle of the boundary and zero on the other part of the boundary. Thus, by an odd continuation of boundary value, the temperature $u(r,\theta)$ satisfies

$$\begin{cases} \Delta u(r,\theta) = 0, & 0 < r < a, -\pi < \theta < \pi, \\ u(a,\theta) = f(\theta), \end{cases}$$

where $f(\theta) = \begin{cases} T_0\theta(\pi - \theta), & 0 < \theta < \pi, \\ T_0\theta(\pi + \theta), & -\pi < \theta < 0. \end{cases}$ Note that $f(\theta)$ is an odd function of θ so that $a_n = 0$ in Eq. (7.16). Thus

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta.$$

Applying the boundary condition at $r = a$ yields

$$\sum_{n=1}^{\infty} b_n a^n \sin n\theta = T_0\theta(\pi - \theta),$$

so

$$b_n = \frac{2}{a^n \pi} \int_0^{\pi} T_0 \theta (\pi - \theta) \sin n\theta \, d\theta = \frac{4T_0}{a^n n^3 \pi} [1 - (-1)^n] .$$

Thus

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{4T_0}{a^n n^3 \pi} [1 - (-1)^n] r^n \sin n\theta .$$

Example 2. Consider heat conduction in a sphere Ω of center O and radius a. Its boundary is denoted by $\partial\Omega$. There is no heat source/sink inside the sphere. The temperature on S_1 and S_2 is T_0(constant) and 0, respectively. Here S_1 is the part of $\partial\Omega$ inside a cone of vertex O and vertex angle 2α, and S_2 is the part of $\partial\Omega$ outside the cone. Find the steady temperature distribution in Ω.

Solution. Take the center of the sphere (also the cone vertex) as the origin of the coordinate system. Let u be the temperature; it must satisfy

$$\begin{cases} \Delta u = 0\,, x^2 + y^2 + z^2 < a^2\,, \\ u = \begin{cases} T_0\,, \ (x,y,z) \in S_1\,, \\ 0\,, \ \ (x,y,z) \in S_2\,. \end{cases} \end{cases} \tag{7.20}$$

Without loss of generality, take the positive Oz-axis as the axis of the cone. In a spherical coordinate system (r, θ, φ), u is independent of φ based on a symmetry of the boundary conditions. Therefore, u satisfies

$$\begin{cases} \Delta u = \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r}\right) + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta \dfrac{\partial u}{\partial\theta}\right) = 0\,, \quad 0 < r < a, 0 < \theta < \pi\,, \\ u(a,\theta) = \begin{cases} T_0\,, \ 0 \le \theta \le \alpha\,, \\ 0\,, \ \ \alpha < \theta \le \pi\,. \end{cases} \end{cases} \tag{7.21}$$

Assume $u = R(r)\Theta(\theta)$. Substituting it into the equation in PDS (7.21) yields

$$\frac{\left(r^2 R'(r)\right)'}{R(r)} = -\frac{1}{\sin\theta} \frac{(\Theta'(\theta)\sin\theta)'}{\Theta(\theta)} = \lambda\,,$$

where λ is the separation constant. Thus we have

$$\begin{cases} \dfrac{1}{\sin\theta}\,(\Theta'(\theta)\sin\theta)' + \lambda\Theta(\theta) = 0\,, 0 < \theta < \pi\,, \\ |\Theta(0)| < \infty, \theta \in (0,\pi)\,. \end{cases} \tag{7.22}$$

and

$$r^2 R''(r) + 2rR'(r) - \lambda R(r) = 0\,, \quad |R(0)| < \infty\,. \tag{7.23}$$

By a variable transformation $x = \cos\theta$, the equation in (7.22) is transformed into a Legendre equation

$$(1-x^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}\Theta}{\mathrm{d}x} + \lambda\Theta = 0\,.$$

Its eigenvalues and eigenfunctions are available in Section 2.6,

$$\lambda = n(n+1)\,, \quad n = 0,1,2,\cdots,$$
$$\Theta_n(\theta) = P_n(\cos\theta)\,.$$

Substituting $\lambda = n(n+1)(n = 0,1,2,\cdots)$ into the Euler equation (7.23) yields

$$r^2R''(r) + 2rR'(r) - n(n+1)R(r) = 0\,.$$

Its general solution is

$$R_n(r) = a_n r^n + b_n r^{-(n+1)}\,, \quad n = 0,1,2,\cdots.$$

Since $|R_n(0)| < \infty$, we obtain $b_n = 0$. Thus

$$u(r,\theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta)\,, \quad 0 \le \theta \le \pi\,. \tag{7.24}$$

Note that $\{P_n(\cos\theta)\}$ is orthogonal in $[0,\pi]$ with respect to the weight function $\sin\theta$. Applying the boundary condition in PDS (7.21) yields

$$\begin{aligned} a_n &= \frac{2n+1}{2a^n}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\,\mathrm{d}\theta \\ &= \frac{2n+1}{2a^n}T_0\int_0^{\alpha} P_n(\cos\theta)\sin\theta\,\mathrm{d}\theta \\ &= \frac{2n+1}{2a^n}T_0\int_{\cos\alpha}^{1} P_n(x)\,\mathrm{d}x\,. \end{aligned}$$

By using $P_n(1) = 1$ and $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$, we arrive at

$$a_0 = \frac{T_0}{2}\int_{\cos\alpha}^{1}\mathrm{d}x = \frac{T_0}{2}(1-\cos\alpha)\,,$$

$$a_n = \frac{T_0}{2a^n}[P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)]\,, \quad n = 1,2,\cdots.$$

Example 3. Consider heat conduction in a cylinder Ω of radius a and height h. Its boundary $\partial\Omega$ consists of the cylindrical surface, the upper and the lower circles of radius a. There is no heat source/sink inside the cylinder. The temperature is $f(r)$ (axially symmetric) on the upper circle of $\partial\Omega$ and is zero on the other parts of $\partial\Omega$. Find the steady temperature distribution in Ω.

Solution. In the interest of specifying the boundary conditions, we consider this problem in a cylindrical coordinate system. Let u be the temperature in Ω. Based on the given boundary conditions, u should be independent of θ so u must satisfy

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial z^2}=0, 0<r<a, 0<z<h, \\ |u(0,z)|<\infty, u(a,z)=0, \\ u(r,0)=0, \quad u(r,h)=f(r). \end{cases} \tag{7.25}$$

Assume $u=R(r)Z(z)$. Substituting it into PDS (7.25) yields the eigenvalue problem of the Bessel equation

$$\begin{cases} R''(r)+\dfrac{1}{r}R'(r)+\lambda^2 R(r)=0, \\ |R(0)|<\infty,\ |R'(0)|<\infty,\ R(a)=0. \end{cases} \tag{7.26}$$

where λ is the separation constant. Its eigenvalues and eigenfunctions are available in Table 4.1 and Section 4.4.2,

$$\lambda=\lambda_n=\frac{\mu_n}{a}, \quad R_n(r)=J_0\left(\frac{\mu_n r}{a}\right),$$

where μ_n are the positive zero points of $J_0(x)$.

The $Z(z)$ satisfies

$$Z''(z)-\lambda_n^2 Z(z)=0.$$

Its general solution reads

$$Z_n(z)=A_n \mathrm{ch}\lambda_n z+B_n \mathrm{sh}\lambda_n z,$$

where A_n andB_n are constants. Applying $Z_n(0)=0$ yields $A_n=0$, so the solution of PDS (7.25) can be written as

$$u(r,z)=\sum_{n=1}^{\infty} B_n J_0(\lambda_n r)\mathrm{sh}\lambda_n z. \tag{7.27}$$

Note that $\{J_0(\lambda_n r)\}$ is orthogonal in $[0,a]$ with respect to the weight function r. Thus applying the boundary condition $u(r,h)=f(r)$ yields

$$B_n=\frac{1}{M_n \mathrm{sh}\lambda_n h}\int_0^a \rho f(\rho) J_0(\lambda_n \rho)\,\mathrm{d}\rho, \tag{7.28}$$

where the normal square $M_n=\displaystyle\int_0^a r J_0^2(\lambda_n r)\,\mathrm{d}r=\frac{a^2}{2}J_1^2(\lambda_n a)$ (see Section 2.5).

Equations (7.27) and (7.28) form the solution of PDS (7.25).

Remark 1. The boundary conditions in Examples 1–3 are all of the first kind. For Neumann problems with boundary conditions of the second kind, there exist some constraints on the boundary values to ensure the existence of solutions. Take the two-dimensional case as an example. By the Gauss formula,

$$\iint\limits_D \Delta u \, \mathrm{d}\sigma = \iint\limits_D \nabla \cdot \nabla u \, \mathrm{d}s = \oint_C \nabla u \cdot \mathbf{n} \, \mathrm{d}s = \oint_C \frac{\partial u}{\partial n} \, \mathrm{d}s,$$

where C is the positive-directed boundary curve of plane domain D and $\mathbf{n}$ is the external unit normal of C. This shows that $\oint_C \frac{\partial u}{\partial n} \, \mathrm{d}s$ must vanish when u is a solution of the Laplace equation. Thus the boundary value $u_n|_C = g(x)$ in a Neumann problem must satisfy the necessary condition for the existence of solution

$$\oint_C g(x) \, \mathrm{d}s = 0.$$

In a circular domain, it reduces to

$$\oint_{r=a} \frac{\partial u}{\partial r} \, \mathrm{d}s = a \oint_{r=a} f(\theta) \, \mathrm{d}\theta = 0.$$

Example 4. Find the solution of the Neumann problem in a circular domain

$$\begin{cases} \Delta u(r,\theta) = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \; < r < a, 0 < \theta < 2\pi, \\ \left.\dfrac{\partial u}{\partial r}\right|_{r=a} = f(\theta), \; |u(0,\theta)| < \infty, \qquad u(r,\theta+2\pi) = u(r,\theta). \end{cases} \tag{7.29}$$

Solution. By a separation of variables similar to that in Example 1, we obtain

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta). \tag{7.30}$$

Thus

$$u_r(a,\theta) = \sum_{n=1}^{\infty} n a^{n-1} (a_n \cos n\theta + b_n \sin n\theta) = f(\theta).$$

Using Fourier coefficients, we have

$$a_n = \frac{1}{n a^{n-1} \pi} \int_0^{2\pi} f(\theta) \cos n\theta \, \mathrm{d}\theta, \quad b_n = \frac{1}{n a^{n-1} \pi} \int_0^{2\pi} f(\theta) \sin n\theta \, \mathrm{d}\theta.$$

Since A_0 can take any value, the solution of the Neumann problem is not unique. It can be proven that subject to the necessary condition for existence, the solution is unique up to an arbitrary constant term.

Let $A_n = na^{n-1}a_n$, $B_n = na^{n-1}b_n$. Equation (7.29) becomes

$$\begin{aligned}
u(r,\theta) &= A_0 + \sum_{n=1}^{\infty} \frac{a}{n}\left(\frac{r}{a}\right)^n (A_n \cos n\theta + B_n \sin n\theta) \\
&= A_0 + \sum_{n=1}^{\infty} \frac{a}{n}\left(\frac{r}{a}\right)^n \frac{1}{\pi}\int_0^{2\pi} f(\theta')[\cos n\theta' \cos n\theta + \sin n\theta' \sin n\theta]\,\mathrm{d}\theta' \\
&= A_0 + \frac{a}{\pi}\int_0^{2\pi} f(\theta')\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \cos n(\theta'-\theta)\right]\mathrm{d}\theta' \\
&= A_0 + \frac{a}{\pi}\int_0^{2\pi} f(\theta')\left\{-\frac{1}{2}\ln\left[1-\frac{2r}{a}\cos(\theta'-\theta)+\left(\frac{r}{a}\right)^2\right]\right\}\mathrm{d}\theta' \\
&= A_0 + \frac{a}{2\pi}\ln a^2 \textstyle\int_0^{2\pi} f(\theta')\,\mathrm{d}\theta' - \frac{a}{2\pi}\int_0^{2\pi} f(\theta') \\
&\qquad \cdot \ln\left[a^2 - 2ar\cos(\theta'-\theta)+r^2\right]\mathrm{d}\theta' \\
&= A_0 - \frac{a}{2\pi}\int_0^{2\pi} f(\theta')\ln\left[a^2-2ar\cos(\theta'-\theta)+r^2\right]\mathrm{d}\theta' . \qquad (7.31)
\end{aligned}$$

This is called the *integral formula of internal Neumann problems in a circular domain*. It can be readily shown that the integral formula of external Neumann problems in a circular domain is

$$\begin{aligned}
u(r,\theta) &= A_0 - \sum_{n=1}^{\infty} \frac{a}{n}\left(\frac{r}{a}\right)^{-n} (A_n \cos n\theta + B_n \sin n\theta) \\
&= A_0 + \frac{a}{2\pi}\int_0^{2\pi} f(\theta')\ln\left[a^2-2ar\cos(\theta'-\theta)+r^2\right]\mathrm{d}\theta' . \qquad (7.32)
\end{aligned}$$

Remark 2. For the case of steady heat conduction in a domain without an internal source/sink, the net heat flowing into the domain from the boundary must be zero by the first law of the thermodynamics. This agrees with the necessary condition for the existence of solutions of the Neumann problems. Under these conditions, the system can be at different steady-states (depending on the initial conditions of the system) with a constant temperature difference for all points. This agrees with the presence of constant A_0 in Eq. (7.31).

7.2.2 Fourier Sine/Cosine Transformation in a Finite Region

For a function $f(x)$ that satisfies the conditions for Fourier sine expansion in $[0,\pi]$, we have

$$\begin{cases}
\bar{f}(k) = \int_0^{\pi} f(x)\sin kx\,\mathrm{d}x , & (7.33a) \\
f(x) = \frac{2}{\pi}\sum_{k=1}^{\infty} \bar{f}(k)\sin kx . & (7.33b)
\end{cases}$$

where the former is called the *Fourier sine transformation* and the latter is called the *inverse Fourier sine transformation*. Similarly, we also have the *Fourier cosine transformation* and the *inverse Fourier cosine transformation*

$$\begin{cases} \bar{f}(k) = \displaystyle\int_0^\pi f(x)\cos kx\,\mathrm{d}x\,, & (7.34\text{a}) \\ f(x) = \dfrac{\bar{f}(0)}{\pi} + \dfrac{2}{\pi}\displaystyle\sum_{k=1}^{\infty}\bar{f}(k)\cos kx\,. & (7.34\text{b}) \end{cases}$$

We can apply the Fourier sine/cosine transformation to solve some boundary-value problems of potential equations. If the boundary conditions are all of the first kind at two ends along one spatial direction, we should use the Fourier sine transformation. When the boundary conditions are all of the second kind at two ends along one spatial direction, the Fourier cosine transformation becomes appropriate. After a Fourier sine/cosine transformation, some terms involving second derivatives disappear and the problem reduces to a problem of ordinary differential equations. To illustrate this, consider $u = u(x,t)$. The Fourier sine transformation of u_{xx} is

$$\begin{aligned} \int_0^\pi \frac{\partial^2 u}{\partial x^2}\sin kx\,\mathrm{d}x &= \frac{\partial u}{\partial x}\sin kx\bigg|_0^\pi - k\int_0^\pi \frac{\partial u}{\partial x}\cos kx\,\mathrm{d}x \\ &= -(ku\cos kx)|_0^\pi - k^2\int_0^\pi u\sin kx\,\mathrm{d}x \\ &= k\left[u(0,t) - (-1)^k u(\pi,t)\right] - k^2\bar{u}(k,t)\,, \end{aligned}$$

where $\bar{u}(k,t) = \displaystyle\int_0^\pi u(x,t)\sin kx\,\mathrm{d}x$. The Fourier cosine transformation of u_{xx} is

$$\int_0^\pi \frac{\partial^2 u}{\partial x^2}\cos kx\,\mathrm{d}x = (-1)^k u_x(\pi,t) - u_x(0,t) - k^2\bar{u}(k,t)\,,$$

where $\bar{u}(k,t) = \displaystyle\int_0^\pi u(x,t)\cos kx\,\mathrm{d}x$.

Note that the Fourier sine/cosine transformation of u_{xx} contains boundary values of the first kind and the second kind.

Example 5. Find the solution of

$$\begin{cases} u_{xx} + u_{yy} = 0\,, < x,y < \pi\,, \\ u(0,y) = u(\pi,y) = 0\,, \\ u(x,0) = 0, \quad u(x,\pi) = u_0 \neq 0\,. \end{cases} \tag{7.35}$$

It governs the steady temperature distribution in a square plate of side length π, with three sides having a temperature of zero and the fourth side a temperature of u_0.

Solution. The solution of PDS (7.35) can be obtained by using a Fourier method of expansion or the method of separation of variables. Here we attempt to use a Fourier transformation to solve it.

Note that the boundary conditions in PDS (7.35) are all of the first kind along both the x- and y-directions, thus we should use the Fourier sine transformation. Also, the boundary conditions are homogeneous along the x-direction. A Fourier sine transformation appears convenient with respect to x.

Let $\bar{u} = \int_0^\pi u(x,y)\sin kx\,\mathrm{d}x$. Thus

$$\int_0^\pi \frac{\partial^2 u}{\partial x^2}\sin kx\,\mathrm{d}x = -k^2\bar{u}, \quad \int_0^\pi \frac{\partial^2 u}{\partial y^2}\sin kx\,\mathrm{d}x = \frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}.$$

Also

$$\int_0^\pi u_0 \sin kx\,\mathrm{d}x = \frac{u_0}{k}\left[1+(-1)^{k+1}\right].$$

Thus PDS (7.35) reduces to

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2} - k^2\bar{u} = 0.$$

Its general solution can be readily obtained as

$$\bar{u} = c_1(k)\mathrm{sh}ky + c_2(k)\mathrm{ch}ky,$$

where $c_1(k)$ and $c_2(k)$ are constants with respect to y. Applying the boundary condition $\bar{u}|_{y=0} = 0$ yields $c_2(k) = 0$. $c_1(k)$ can be determined by applying another boundary condition $\bar{u}|_{y=\pi} = \frac{u_0}{k}\left[1+(-1)^{k+1}\right]$ and using $\mathrm{csch}k\pi = 1/\mathrm{sh}k\pi$. Finally, we have

$$\bar{u} = \begin{cases} \dfrac{2u_0}{k}(\mathrm{csch}k\pi)\mathrm{sh}ky, & k = 2n+1,\ n = 0,1,2,\cdots, \\ 0, & k = 2n. \end{cases}$$

Its inverse Fourier sine transformation yields the solution of PDS (7.35)

$$u(x,y) = \frac{4u_0}{\pi}\sum_{n=0}^{\infty}\frac{\mathrm{csch}(2n+1)\pi\cdot\mathrm{sh}(2n+1)y\cdot\sin(2n+1)x}{2n+1}. \tag{7.36}$$

Example 6. Consider the steady heat conduction in a cube of side length π. The temperature is kept at zero on five boundary surfaces. The temperature is kept at constant u_0 on the sixth boundary surface. Find the temperature distribution in the cube.

Solution. Let $u(x,y,z)$ be the temperature. It must satisfy

$$\begin{cases} \Delta u = 0, \quad 0 < x,y,z < \pi, \\ u(0,y,z) = u(\pi,y,z) = 0, \\ u(x,y,0) = u(x,y,\pi) = 0, \\ u(x,0,z) = 0, \quad u(x,\pi,z) = u_0. \end{cases} \tag{7.37}$$

Consider a Fourier sine transformation of u with respect to x

$$\bar{u} = \int_0^\pi u(x,y,z) \sin kx \, \mathrm{d}x.$$

The PDS (7.37) is thus transformed into

$$\begin{cases} \bar{u}_{yy} + \bar{u}_{zz} = k^2 \bar{u}, \\ \bar{u} = \bar{u}(k,y,z), 0 < y,z < \pi, \\ \bar{u}\big|_{z=0} = \bar{u}\big|_{z=\pi} = 0, \\ \bar{u}\big|_{y=0} = 0, \quad \bar{u}\big|_{y=\pi} = \begin{cases} \dfrac{2u_0}{k}, & \text{for odd } k, \\ 0, & \text{for even } k. \end{cases} \end{cases}$$

Consider a Fourier sine transformation of $\bar{u}$ with respect to z

$$\bar{\bar{u}} = \int_0^\pi \bar{u} \sin k' z \, \mathrm{d}z.$$

The PDS (7.37) is hence transformed into

$$\frac{d^2 \bar{\bar{u}}}{\mathrm{d}y^2} = (k^2 + k'^2)\bar{\bar{u}}, \quad \bar{\bar{u}}\big|_{y=0} = 0, \quad \bar{\bar{u}}\big|_{y=\pi} = \begin{cases} 0, & k \text{ or } k' \text{ is even}, \\ \dfrac{4u_0}{kk'}, & \text{both } k \text{ and } k' \text{ are odd}. \end{cases}$$

Its general solution is

$$\bar{\bar{u}} = c_1(k,k')\mathrm{sh} ly + c_2(k,k')\mathrm{ch} ly,$$

where $l^2 = k^2 + k'^2 = (2m+1)^2 + (2n+1)^2$, $m,n = 0,1,2,\cdots$, c_1 and c_2 are constants with respect to y and can be determined by applying the boundary conditions of $\bar{\bar{u}}$.

Finally, we obtain

$$\bar{\bar{u}} = \begin{cases} 0, & k \text{ or } k' \text{ is even}, \\ \dfrac{4u_0}{kk'} \dfrac{\mathrm{sh} ly}{\mathrm{sh} l\pi}, & k' = 2m+1, \ k = 2n+1. \end{cases}$$

Its inverse transformation reads

$$\bar{u} = \frac{8u_0}{(2n+1)\pi} \sum_{m=0}^{\infty} \frac{\mathrm{sh} ly}{\mathrm{sh} l\pi} \frac{\sin(2m+1)z}{2m+1}.$$

Its inverse transformation thus yields the solution of PDS (7.37)

$$u(x,y,z) = \frac{16u_0}{\pi^2} \sum_{m,n=0}^{\infty} \frac{\mathrm{sh} ly}{\mathrm{sh} l\pi} \frac{\sin(2n+1)x}{2n+1} \frac{\sin(2m+1)z}{2m+1}. \tag{7.38}$$

Remark 1. If the region is $[0,l]$, the Fourier sine/cosine transformation of $f(x)$ becomes

$$\begin{cases} \bar{f}(k) = \int_0^l f(x) \sin\frac{k\pi x}{l}\,\mathrm{d}x, \\ f(x) = \frac{2}{l}\sum_{k=1}^{\infty} \bar{f}(k) \sin\frac{k\pi x}{l}. \end{cases}$$

$$\begin{cases} \bar{f}(k) = \int_0^l f(x) \cos\frac{k\pi x}{l}\,\mathrm{d}x, \\ f(x) = \frac{\bar{f}(0)}{\pi} + \frac{2}{l}\sum_{k=1}^{\infty} \bar{f}(k) \cos\frac{k\pi x}{l}. \end{cases}$$

Such a Fourier transformation works only for the case that the boundary conditions at the two ends along one spatial direction are of the same type (either the first or the second kind). Its application can remove the terms involving u_{xx} and thus reduce the problem to one of ordinary differential equations. Such a transformation fails, however, to work if the equation involves both a u_{xx}-term and a u_x-term.

Remark 2. Consider a function $v = f(x)$ that must satisfy the conditions

$$v_x|_{x=0} = 0, \quad (v_x + hv)|_{x=l} = 0, h > 0.$$

Based on the condition $v_x|_{x=0} = 0$, we consider a cosine expansion of $f(x)$

$$f(x) = \sum_n a_n \cos p_n x, \tag{7.39}$$

where the a_n are constants and the p_n are determined to satisfy $(v_x + hv)|_{x=l} = 0$ so that

$$\sum_n (-p_n \sin p_n l + h\cos p_n l) a_n = 0.$$

Thus the p_n are the positive zero-roots of $p\tan pl - h = 0$. The function set $\{\cos p_n x\}$ is complete and orthogonal in $[0,l]$. Also note that

$$\int_0^l \cos p_m x \cos p_n x\,\mathrm{d}x = \begin{cases} 0, & p_m \neq p_n, \\ \frac{l(p_n^2+h^2)+h}{2(p_n^2+h^2)}, & p_m = p_n. \end{cases}$$

Thus the a_n can be determined by Eq. (7.39)

$$a_n = \frac{2(p_n^2+h^2)}{l(p_n^2+h^2)+h} \int_0^l f(x) \cos p_n x\,\mathrm{d}x.$$

Finally, we have

$$\begin{cases} \bar{f}(p_n) = \int_0^l f(x)\cos p_n x\,\mathrm{d}x\,, \\ f(x) = 2\sum_n \dfrac{(p_n^2+h^2)}{l(p_n^2+h^2)+h}\bar{f}(p_n)\cos p_n x\,. \end{cases}$$

If the conditions are $v|_{x=0} = 0$ and $(v_x + hv)|_{x=l} = 0, (h > 0)$, we consider

$$f(x) = \sum_n b_n \sin q_n x\,,$$

where the q_n and the b_n can be determined by following a similar approach.

Remark 3. The Fourier sine/cosine transformation can also be used to solve unsteady problems. We discuss this by considering the following PDS arising from the heat transfer enhancement in energy, chemical engineering and aerospace engineering

$$\begin{cases} u_t = a^2\Delta u\,, \quad 0 < x < +\infty,\ 0 < y < b,\ t > 0\,, \\ \left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0\,, \quad \left.\left(\dfrac{\partial u}{\partial y} + \sigma u\right)\right|_{y=b} = 0\,, \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = f(t) = \begin{cases} A\sin t\,,\ t \in (2n\pi, (2n+1)\pi)\,, \\ 0\,, \qquad \text{otherwise}\,, \end{cases} \\ u(x,y,0) = v_0\,. \end{cases} \tag{7.40}$$

where v_0 is a constant.

Consider a Fourier cosine transformation in $(0, +\infty)$ with respect to x so that

$$\bar{u}(x,y,t) = \int_0^{+\infty} u(x,y,t)\cos\omega x\,\mathrm{d}x$$

Since, with $\delta(\omega)$ as the δ-function,

$$\int_0^{+\infty} u_{xx}\cos\omega x\,\mathrm{d}x = -u_x(0,y,t) - \omega^2\bar{u} = -f(t) - \omega^2\bar{u}\,,$$

$$\int_0^{+\infty} v_0\cos\omega x\,\mathrm{d}x = \frac{v_0}{2}\int_{-\infty}^{+\infty}\cos\omega x\,\mathrm{d}x = v_0\pi\delta(\omega)\,.$$

PDS (7.40) is transformed into a mixed problem in $[0, b]$

$$\begin{cases} \bar{u}_t - a^2\bar{u}_{yy} + (\omega a)^2\bar{u} = -a^2 f(t)\,, \quad (0,b)\times(0,+\infty)\,, \\ \left.\dfrac{\partial\bar{u}}{\partial y}\right|_{y=0} = 0\,, \quad \left.\left(\frac{\partial\bar{u}}{\partial y} + \sigma\bar{u}\right)\right|_{y=b} = 0\,, \\ \bar{u}(x,y,0) = v_0\pi\delta(\omega)\,. \end{cases} \tag{7.41}$$

By the principle of superposition, its solution is a superposition of two parts: $\bar{u}_1$ from $v_0\pi\delta(\omega)$ and $\bar{u}_2$ from $-a^2 f(t)$. The former satisfies

$$\begin{cases} \bar{\bar{u}} + a^2(\omega^2 + p_n^2)\bar{\bar{u}} = 0\,, \\ \bar{\bar{u}}\big|_{t=0} = v_0\pi\delta(\omega)\dfrac{\sin p_n b}{p_n}\,. \end{cases}$$

Its solution is $\bar{\bar{u}}(\omega, p_n, t) = v_0\pi\delta(\omega)\dfrac{\sin p_n b}{p_n}\,\mathrm{e}^{-a^2(\omega^2+p_n^2)t}$. Its inverse transformation leads to

$$\bar{u}_1(\omega,y,t) = 2v_0\pi\delta(\omega)\sum_n \frac{p_n^2+\sigma^2}{b(p_n^2+\sigma^2)+\sigma}\,\frac{\sin p_n b\cos p_n y}{p_n}\,\mathrm{e}^{-a^2(\omega^2+p_n^2)t}\,.$$

In order to find $\bar{u}_2$, by the solution structure theorem, we consider the PDS

$$\begin{cases} v_t - a^2 v_{yy} + (\omega a)^2 v = 0\,, \quad (0,b)\times(0,+\infty)\,, \\ \left.\dfrac{\partial v}{\partial y}\right|_{y=0} = 0\,, \quad \left.\left(\dfrac{\partial v}{\partial y} + \sigma v\right)\right|_{y=b} = 0\,, \\ v\big|_{t=\tau} = -a^2 f(\tau). \end{cases}$$

Its solution is

$$v = -2a^2 f(\tau)\sum_n \frac{p_n^2+\sigma^2}{b(p_n^2+\sigma^2)+\sigma}\,\frac{\sin p_n b\cos p_n y}{p_n}\,\mathrm{e}^{-a^2(\omega^2+p_n^2)(t-\tau)}\,.$$

Therefore,

$$\begin{aligned} \bar{u}_2(\omega,y,t) &= \int_0^t v\,\mathrm{d}\tau \\ &= -2a^2\int_0^t f(\tau)\sum_n \frac{p_n^2+\sigma^2}{b(p_n^2+\sigma^2)+\sigma}\,\frac{\sin p_n b\cos p_n y}{p_n}\,\mathrm{e}^{-a^2(\omega^2+p_n^2)(t-\tau)}\,\mathrm{d}\tau\,. \end{aligned}$$

Thus the solution of PDS (7.41) is

$$\bar{u}(\omega,y,t) = \bar{u}_1(\omega,y,t) + \bar{u}_2(\omega,y,t)\,.$$

Its inverse transformation yields the solution of PDS (7.40)

$$\begin{aligned} u(x,y,t) &= 2v_0\sum_n \frac{p_n^2+\sigma^2}{b(p_n^2+\sigma^2)+\sigma}\,\frac{\sin p_n b\cos p_n y}{p_n}\,\mathrm{e}^{-(ap_n)^2 t} \\ &- \frac{2a}{\sqrt{\pi}}\int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\sum_n \frac{p_n^2+\sigma^2}{b(p_n^2+\sigma^2)+\sigma}\,\frac{\sin p_n b\cos p_n y}{p_n}\,\mathrm{e}^{-(ap_n)^2(t-\tau)}\,\mathrm{e}^{-x^2/4a^2(t-\tau)}\,\mathrm{d}\tau \\ &= \left[2v_0 - \frac{2a}{\sqrt{\pi}}\int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\,\mathrm{e}^{\frac{4a^4(t-\tau)p_n^2\tau - x^2}{4a^2(t-\tau)}}\,\mathrm{d}\tau\right] \\ &\quad\cdot\sum_n \frac{p_n^2+\sigma^2}{b(p_n^2+\sigma^2)+\sigma}\,\frac{\sin p_n b\cos p_n y}{p_n}\,\mathrm{e}^{-(ap_n)^2 t}\,, \end{aligned}$$

where we have used

$$F^{-1}\left[\mathrm{e}^{-(a\omega)^2(t-\tau)}\right] = \frac{1}{2a\sqrt{t-\tau}}\,\mathrm{e}^{-\frac{x^2}{4a^2(t-\tau)}}\,.$$

7.3 Methods for Solving Nonhomogeneous Potential Equations

We have discussed several methods for obtaining solutions of homogeneous potential equations (i.e. the Laplace equations) in Section 7.1 and Section 7.2. In this section, we discuss the two methods for solving nonhomogeneous potential equations, i.e. the Poisson equations.

7.3.1 Equation Homogenization by Function Transformation

Some problems of Poisson equations can be transformed into those of Laplace equations by some proper function transformations. We demonstrate this with two examples.

Example 1. Find the solution of

$$\begin{cases} \Delta u = py+q, \quad 0<x<a, 0<y<b\,, \\ u_x(0,y) = u(a,y) = 0\,, \\ u(x,0) = u(x,b) = 0\,. \end{cases} \tag{7.42}$$

where p and q are constants.

Solution. Consider a function transformation

$$v(x,y) = u(x,y) + (x^2 - a^2)(c_1 y + c_2)\,.$$

Such a transformation preserves the homogeneity of boundary conditions with respect to x in PDS (7.42), regardless of the values of constants c_1 and c_2, i.e.

$$v_x(0,y) = v(a,y) = 0\,.$$

To homogenize the equation, substitute $v(x,y)$ into the Poisson equation in PDS (7.42) yields

$$c_1 = -p/2\,, \quad c_2 = -q/2\,.$$

Thus the PDS (7.42) is transformed into

$$\begin{cases} \Delta v = 0, \quad 0 < x < a, 0 < y < b, \\ v_x(0,y) = v(a,y) = 0, \\ v(x,0) = f(x), \, v(x,b) = g(x). \end{cases} \tag{7.43}$$

where

$$f(x) = -\frac{q}{2}(x^2 - a^2), \quad g(x) = -\frac{1}{2}(x^2 - a^2)(pb + q). \tag{7.44}$$

By Table 2.1 , we can expand $v(x,y)$ by

$$v = \sum_{k=0}^{\infty} Y_k(y) \cos \lambda_k x,$$

where $\lambda_k = \dfrac{(2k+1)\pi}{2a}$. Substituting this into the Laplace equation in PDS (7.43) yields the ordinary differential equation $Y_k(y)$, whose general solution can be readily determined. Finally, we obtain $v(x,y)$ so that it satisfies the Laplace equation and the boundary conditions with respect to x in PDS (7.43).

$$v = \sum_{k=0}^{\infty} (a_k \mathrm{e}^{\lambda_k y} + b_k \mathrm{e}^{-\lambda_k y}) \cos \lambda_k x, \tag{7.45}$$

where constants a_k and b_k can be determined by applying the boundary conditions with respect to y in PDS (7.43)

$$a_k = \frac{\beta_k \mathrm{e}^{-\lambda_k b} - \alpha_k}{-2\mathrm{sh}\lambda_k b}, \quad b_k = \frac{\alpha_k - \beta_k \mathrm{e}^{\lambda_k b}}{-2\mathrm{sh}\lambda_k b}. \tag{7.46}$$

Here

$$\alpha_k = \frac{2}{a} \int_0^a -\frac{q}{2}(x^2 - a^2) \cos \lambda_k x \, \mathrm{d}x = \frac{2(-1)^{k+1} q}{a \lambda_k^3},$$

$$\beta_k = \frac{2}{a} \int_0^a -\frac{1}{2}(x^2 - a^2)(pb + q) \cos \lambda_k x \, \mathrm{d}x = \frac{2(-1)^{k+1}(pb + q)}{a \lambda_k^3}.$$

Finally, we obtain the solution of PDS (7.42) by substituting Eq. (7.46) into Eq. (7.45) and using the relation between $v(x,y)$ and $u(x,y)$

$$\begin{aligned} u &= v(x,y) + (x^2 - a^2)\left(\frac{p}{2}y + \frac{q}{2}\right) \\ &= 2\sum_{k=0}^{\infty} \frac{(-1)^k}{\lambda_k^3 \mathrm{sh}\lambda_k b} [(pb + q)\mathrm{sh}\lambda_k(y - b) - q\mathrm{sh}\lambda_k y] \cos \lambda_k x \\ &\quad + \frac{1}{2}(py + q)(x^2 - a^2). \end{aligned}$$

Suppose that u^* is a particular solution of the nonhomogeneous equation $\Delta u = f(x,y)$. A function transformation $u(x,y) = v(x,y) + u^*$ will transform $\Delta u = f(x,y)$ into $\Delta v = 0$.

Example 2. Solve the internal problem in a circular domain

$$\begin{cases} \Delta u(r,\theta) = -r^2 \sin\theta\cos\theta,\ 0 < r < a, \\ u(a,\theta) = 0\,. \end{cases} \tag{7.47}$$

Solution. In a Cartesian coordinate system, the equation in PDS (7.47) reads

$$u_{xx} + u_{yy} = -xy\,. \tag{7.48}$$

It is clear that $u^* = -\frac{1}{12}xy(x^2+y^2)$ can satisfy Eq. (7.48) so $u^* = -\frac{r^4}{12}\sin\theta\cos\theta$ is a particular solution of the equation in PDS (7.47). Now consider the function transformation

$$u(r,\theta) = v(r,\theta) + u^*(r,\theta)\,.$$

The PDS (7.47) is thus transformed into

$$\begin{cases} \Delta v(r,\theta) = 0\,, \quad 0 < r < a, \\ v(a,\theta) = -\frac{a^4}{12}\sin\theta\cos\theta\,. \end{cases} \tag{7.49}$$

The solution of PDS (7.47) thus follows from the Poisson formula of internal problems in a circular domain (Eq. (7.17))

$$u = -\frac{r^4}{24}\sin 2\theta + \frac{a^4}{48}\int_0^{2\pi} \frac{(a^2-r^2)\sin 2\theta'}{a^2 - 2ar\cos(\theta-\theta') + r^2}\,\mathrm{d}\theta'\,.$$

7.3.2 Extremum Principle

Laplace equations can be used to describe the steady temperature field in a domain Ω without any heat source or sink inside. Therefore they satisfy the extremum principle of heat conduction discussed in Section 3.3.2; their solutions take the minimum and the maximum values only on the boundary $\partial\Omega$ of the domain. If $u|_{\partial\Omega} = C$ (constant) and $u(M)$ is continuous on $\bar{\Omega}(\bar{\Omega} = \Omega \cup \partial\Omega)$, the principle concludes that $u(M) = C$ for all $M \in \bar{\Omega}$.

Example 1. Solve

$$\begin{cases} \Delta u = 1\,, \quad x^2 + y^2 < 1\,, \\ u|_{x^2+y^2=1} = 4\,. \end{cases} \tag{7.50}$$

Solution. It is clear that the Poisson equation in PDS (7.50) has a particular solution

$$u^* = \frac{1}{4}(x^2+y^2).$$

Consider a function transformation $u(x,y) = v(x,y) + u^*$. The PDS (7.50) is thus transformed into

$$\begin{cases} \Delta v = 0\,, \quad x^2+y^2<1\,, \\ v|_{x^2+y^2=1} = \frac{15}{4}\,. \end{cases}$$

By the extremum principle, we have

$$v(x,y) = \frac{15}{4}, \quad x^2+y^2 \le 1\,,$$

so that

$$u = \frac{15}{4} + \frac{x^2+y^2}{4}\,.$$

Example 2. Find the solution of

$$\begin{cases} \Delta u = \sqrt{x^2+y^2}\,, \quad x^2+y^2<1\,, \\ u|_{x^2+y^2=1} = 2\,. \end{cases} \tag{7.51}$$

Solution. It is not straightforward to find u^*, a particular solution of the Poisson equation in PDS (7.51). In a polar coordinate system, the Poisson equation in PDS (7.51) reads

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = r\,,$$

whose nonhomogeneous term is independent of θ. We can thus consider a particular solution independent of θ. A simple test shows that $u^* = \frac{r^3}{9}$ is a particular solution. By the function transformation $u(r,\theta) = v(r,\theta) + \frac{r^3}{9}$, we obtain

$$\begin{cases} \Delta v(r,\theta) = 0\,, \quad 0<r<1\,, \\ v(1,\theta) = \dfrac{17}{9}\,. \end{cases}$$

Its solution is, by the extremum principle,

$$v = \frac{17}{9}, \quad r \le 1\,,$$

so the solution of PDS (7.51) is

$$u = \frac{17}{9} + \frac{\sqrt{(x^2+y^2)^3}}{9}\,.$$

Example 3. Find the solution of

$$\begin{cases} u_{xx}+u_{yy}+u_{zz} = 1\,, \quad x^2+y^2+z^2<1\,, \\ u|_{x^2+y^2+z^2=1} = 9\,. \end{cases} \tag{7.52}$$

Solution. It is straightforward to show that the Poisson equation in PDS (7.52) has a particular solution

$$u^* = \frac{1}{6}(x^2+y^2+z^2)\,.$$

Consider the function transformation

$$u(x,y,z) = v(x,y,z) + u^*.$$

The PDS (7.52) is thus transformed into

$$\begin{cases} v_{xx}+v_{yy}+v_{zz} = 0\,, & x^2+y^2+z^2<1\,, \\ v|_{x^2+y^2+z^2=1} = \dfrac{53}{6}\,. \end{cases}$$

Its solution is, by the extremum principle,

$$v = \frac{53}{6}\,, \quad x^2+y^2+z^2 \le 1\,.$$

Thus the solution of PDS (7.52) is

$$u = \frac{53}{6} + \frac{x^2+y^2+z^2}{6}\,.$$

7.3.3 Four Examples of Applications

Example 1. Consider two-dimensional heat conduction in a rectangular plate. Inside the plate, there exists a uniform heat source of strength q. The heat can flow out through the central part of one side of the plate boundary. All the other parts of plate boundary are well insulated. Find the steady temperature distribution in the plate.

Solution. (1) Coordinate system: Consider the Cartesian coordinate system in Fig. 7.1. Let $2a$ and b be the length and the width of plate. $2c$ is the length of the part of the plate side where heat can be conducted to the outside.

(2) PDS: Let k be the thermal conductivity of the plate material. The steady temperature $T(x,y,t)$ at point (x,y) satisfies

$$T_{xx}+T_{yy} = -q/k\,.$$

The heat generated by the heat source inside the plate per unit of time is $2abq$ by the definition of q. The heat conducted to the outside via the boundary is $-2kc\left.\dfrac{\partial T}{\partial y}\right|_{y=b}$ per unit of time. At a steady state, we have

$$-2kc\left.\frac{\partial T}{\partial y}\right|_{y=b} = 2abq \quad \text{or} \quad \left.\frac{\partial T}{\partial y}\right|_{y=b} = -\frac{abq}{kc}\,, |x|<c$$

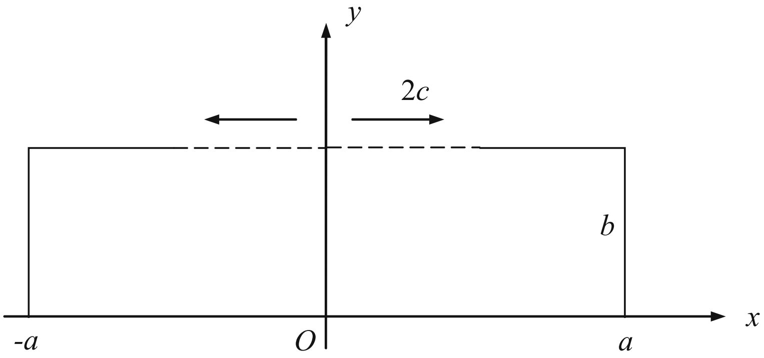

Fig. 7.1 Physical problem and coordinate system

Finally, we obtain the PDS

$$\begin{cases} T_{xx} + T_{yy} = -\frac{q}{k}\,, \quad -a < x < a, 0 < y < b\,, \\ T_x|_{x=\pm a} = T_y\big|_{y=0} = 0\,, \\ T_y\big|_{y=b} = f(x) = \begin{cases} -abq/kc\,, & |x| < c\,, \\ 0\,, & |x| \geq c\,. \end{cases} \end{cases} \tag{7.53}$$

(3) Solution of PDS (7.53): It is clear that the Poisson equation in PDS (7.53) has a particular solution

$$T^* = -\frac{qy^2}{2k}\,.$$

By the function transformation $T(x,y) = v(x,y) + T^*$, the PDS (7.53)is transformed into

$$\begin{cases} v_{xx} + v_{yy} = 0, -a < x < a, 0 < y < b \\ v_x|_{x=\pm a} = v_y\big|_{y=0} = 0 \\ v_y\big|_{y=b} = g(x)\,. \end{cases} \tag{7.54}$$

where

$$g(x) = \begin{cases} bq/k - abq/kc\,, & |x| < c\,, \\ bq/k\,, & |x| \geq c\,. \end{cases}$$

Note that the problem is symmetric with respect to the Oy-axis, so we can focus our attention on the region: $0 \leq x \leq a, 0 \leq y \leq b$ by replacing the boundary conditions $v_x|_{x=\pm a} = 0$ with $v_x|_{x=0} = v_x|_{x=a} = 0$. Based on the given boundary conditions in the x-direction, we use the eigenfunction set from Row 5 in Table 2.1 to expand $v(x,y)$

$$v(x,y) = \sum_{m=0}^{\infty} Y_m(y) \cos \frac{m\pi x}{a}\,. \tag{7.55}$$

Substituting it into the Laplace equation in PDS (7.54) yields the ordinary differential equation of $Y_m(y)$, whose solution can be readily obtained. We thus obtain the $v(x,y)$ that satisfies the Laplace equation in PDS (7.54) and x-direction boundary conditions

$$v(x,y) = \sum_{m=0}^{\infty} \left(a_m \mathrm{ch}\frac{m\pi y}{a} + b_m \mathrm{sh}\frac{m\pi y}{a} \right) \cos\frac{m\pi x}{a}, \tag{7.56}$$

where a_m and b_m are constants. Applying the boundary condition $v_y\big|_{y=0} = 0$ yields $b_m = 0$. Thus

$$v = \sum_{m=0}^{\infty} a_m \mathrm{ch}\frac{m\pi y}{a} \cos\frac{m\pi x}{a}, \quad v_y = \sum_{m=1}^{\infty} \frac{m\pi a_m}{a} \mathrm{sh}\frac{m\pi y}{a} \cos\frac{m\pi x}{a}.$$

Note that the normal square of $\left\{\cos\frac{m\pi x}{a}\right\}$ is $\frac{a}{2}$. Applying the boundary condition $v_y\big|_{y=b} = g(x)$ yields

$$\begin{aligned}
\frac{m\pi a_m}{a}\mathrm{sh}\frac{m\pi b}{a} &= \frac{2}{a}\int_0^a g(x)\cos\frac{m\pi x}{a}\,\mathrm{d}x \\
&= \frac{2}{a}\int_0^c -\frac{abq}{kc}\cos\frac{m\pi x}{a}\,\mathrm{d}x \\
&= -\frac{2bq}{kc}\frac{a}{m\pi}\sin\frac{m\pi x}{a}\Big|_0^c \\
&= -\frac{2abq}{m\pi kc}\sin\frac{m\pi c}{a},
\end{aligned}$$

so that

$$a_m = -\frac{2a^2bq}{k\pi^2 c}\sin\frac{m\pi c}{a} \Big/ m^2 \mathrm{sh}\frac{m\pi b}{a}. \tag{7.57}$$

Finally, we obtain the solution of PDS (7.53)

$$T(x,y) = -\frac{q}{k}\left[\frac{y^2}{2} + \frac{2a^2b}{\pi^2 c}\sum_{m=1}^{\infty}\frac{1}{m^2}\frac{\sin\frac{m\pi c}{a}}{\mathrm{sh}\frac{m\pi b}{a}}\mathrm{ch}\frac{m\pi y}{a}\cos\frac{m\pi x}{a}\right]. \tag{7.58}$$

Example 2. Consider the distribution of electric potential in a hollow sphere of inner radius R_1 and outer radius R_2 $(R_1 < R_2)$. The electric potential is kept at a constant v_0 on the inner surface and at zero on the outer surface. Find the steady distribution of the electric potential inside of the hollow sphere.

Solution. The electric potential v satisfies the Laplace equation at a steady state (see Section 1.2). By the given boundary conditions, v cannot depend on θ and φ in a sphere coordinate system (r,θ,φ), so it varies only along the radial direction. Thus the $v(r)$ must satisfy

$$\begin{cases} \dfrac{d^2 v}{\mathrm{d}r^2} + \dfrac{2}{r}\dfrac{\mathrm{d}v}{\mathrm{d}r} = 0, \ R_1 < r < R_2, \\ v(R_1) = v_0, \qquad v(R_2) = 0. \end{cases} \tag{7.59}$$

Note that the equation in PDS (7.59) is a second-order homogeneous Euler equation. Its general solution can be readily obtained

$$v = \frac{C_1}{r} + C_2\,,$$

where C_1 and C_2 are constants. Applying the boundary conditions $v(R_1) = v_0$ and $v(R_2) = 0$ yields

$$C_1 = \frac{R_1 R_2}{R_2 - R_1} v_0\,, \quad C_2 = \frac{-R_1}{R_2 - R_1} v_0\,.$$

Thus the solution of (7.59) is

$$v(r) = \frac{R_1 R_2}{R_2 - R_1}\left(\frac{1}{r} - \frac{1}{R_1}\right) v_0\,.$$

Example 3. Consider the distribution of electric potential u in a sphere of radius 1. The potential distribution on its boundary surface is $u|_{r=1} = \cos^2\theta$. Find the potential distribution in the sphere.

Solution. By the given conditions, the electric potential u cannot depend on φ in a spherical coordinate system $(r\,,\theta\,,\varphi)(0 \le \varphi \le 2\pi)$. Thus the u must satisfy

$$\begin{cases} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial u}{\partial\theta}\right) = 0\,, \quad 0 < \theta < \pi\,, 0 < r < 1\,, \\ u|_{r=1} = \cos^2\theta\,. \end{cases} \tag{7.60}$$

Assume $u = R(r)\Theta(\theta)$. Substituting it into the Laplace equation in PDS (7.60) yields

$$\frac{r^2R'' + 2rR'}{R} = -\frac{\Theta'' + (\cot\theta)\Theta'}{\Theta} = \lambda\,,$$

where λ is the separation constant. Let $\lambda = n(n+1)$. Thus we have the Euler and the Legendre equations

$$r^2R'' + 2rR' - n(n+1)R = 0\,,$$
$$\Theta'' + (\cot\theta)\Theta' + n(n+1)\Theta = 0\,.$$

They have the bounded solutions $C_n r^n$ in $r \le 1$ and $P_n(\cos\theta)(n = 0, 1, 2, \cdots)$ in $[0, \pi]$, respectively. Here the C_n are constants. Thus we obtain the solution of the Laplace equation in PDS (7.60)

$$u = \sum_{n=0}^{\infty} C_n r^n P_n(\cos\theta)\,.$$

Applying the boundary condition $u|_{r=1} = \cos^2\theta$ yields

$$x^2 = \sum_{n=0}^{\infty} C_n P_n(x)\,, \quad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n\,,$$

where $x = \cos\theta$, $-1 \le x \le 1$. Note that

$$\int_{-1}^{1} x^2 P_n(x)\,\mathrm{d}x = \frac{1}{2^n n!}\left[x^2 \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left(x^2-1\right)^n\Big|_{-1}^{1} - 2\int_{-1}^{1} x \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left(x^2-1\right)^n \mathrm{d}x\right]$$

$$= -\frac{2}{2^n n!}\left[x \frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}\left(x^2-1\right)^n\Big|_{-1}^{1} - \int_{-1}^{1} \frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}\left(x^2-1\right)^n \mathrm{d}x\right]$$

$$= \frac{2}{2^n n!}\frac{\mathrm{d}^{n-3}}{\mathrm{d}x^{n-3}}\left(x^2-1\right)^n\Big|_{-1}^{1} = 0, \quad \text{when } n \ge 3.$$

Thus we obtain

$$C_n = \frac{2n+1}{2}\int_{-1}^{1} x^2 P_n(x)\,\mathrm{d}x = \begin{cases} \frac{1}{3}, & n = 0, \\ \frac{2}{3}, & n = 2, \\ 0, & n \ne 0, 2. \end{cases}$$

Finally, we have

$$u(r,\theta) = \frac{1}{3} + \frac{2}{3} r^2 P_2(x) = \frac{1}{3} + r^2\left(\cos^2\theta - \frac{1}{3}\right).$$

Example 4. Consider steady heat conduction in a sphere of radius R. The temperature is kept at a constant v_0 on the upper half of the spherical boundary surface and at zero on the lower half. Find the steady temperature distribution in the sphere.

Solution. By the given conditions, the temperature T cannot depend on φ in a spherical coordinate system (r, θ, φ). It must satisfy

$$\begin{cases} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial T}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial T}{\partial\theta}\right) = 0, \\ \qquad\qquad 0 < r < R, 0 < \theta < \pi, \\ T(R,\theta) = \begin{cases} v_0, & 0 \le \theta < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < \theta \le \pi. \end{cases} \end{cases} \tag{7.61}$$

By following a similar approach as that in Example 3, we can obtain the solution of the Laplace equation in PDS (7.61)

$$T(r,\theta) = \sum_{n=0}^{\infty} C_n r^n P_n(\cos\theta),$$

where the C_n are constants. Applying the boundary condition at $r = R$ yields

$$\sum_{n=0}^{\infty} C_n R^n P_n(x) = \begin{cases} v_0, 0 < x \le 1, \\ 0, -1 \le x < 0, \end{cases}$$

where $x=\cos\theta$, $-1\le x\le 1$. Note that $\{P_n(x)\}$ is orthogonal in $[-1,1]$. Thus we have

$$\int_{-1}^{1} C_0 P_0^2(x)\,\mathrm{d}x=\int_0^1 v_0\,\mathrm{d}x\,,\quad C_0=\frac{v_0}{2}\,.$$

Also

$$C_n=\frac{2n+1}{2R^n}\int_0^1 v_0 P_n(x)\,\mathrm{d}x=\frac{v_0}{2R^n}\int_0^1 [P'_{n+1}(x)-P'_{n-1}(x)]\,\mathrm{d}x$$

$$=\frac{v_0}{2R^n}[P_{n-1}(0)-P_{n+1}(0)]\,.$$

Note that $P_0(0)=0, P_2(0)=-\frac{1}{2}$. Therefore $C_1=\frac{v_0}{2R}\left(1+\frac{1}{2}\right)=\frac{3v_0}{4R}$.

Since

$$P_{2n+1}(0)=0\,,\quad P_{2n}(0)=(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\,,$$

we obtain

$$C_{2k}=0\,, k=1,2,\cdots\,.$$

When $n=2k+1$ $(k=1,2,\cdots)$,

$$C_{2k+1}=\frac{v_0}{2R^{2k+1}}[P_{2k}(0)-P_{2k+2}(0)]$$

$$=\frac{v_0}{2R^{2k+1}}\left[(-1)^k\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2k}-(-1)^{k+1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k+1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k+2)}\right]$$

$$=\frac{(-1)^k v_0}{2R^{2k+1}}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2k}\frac{4k+3}{2k+2}\,.$$

Therefore the solution of PDS (7.61) is

$$T(r,\theta)=\frac{v_0}{2}\left[1+\frac{3}{2}\left(\frac{r}{R}\right)P_1(\cos\theta)\right]+\frac{v_0}{2}\sum_{k=1}^{\infty}(-1)^k$$

$$\cdot\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2k}\frac{4k+3}{2k+2}\left(\frac{r}{R}\right)^{2k+1}P_{2k+1}(\cos\theta)\,.$$

Remark. The Laplace operator of two and three dimensions plays an important role in potential equations. Its form depends on the coordinate system. Table 7.1 lists the Laplace operator in some typical coordinate systems.

Table 7.1 Laplace Operator Δ

Two-dimensional	Cartisian coordinate system	$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$
	Polar coordinate system	$\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$
Three-dimensional	Cartisian coordinate system	$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$
	Cylindrical coordinate system	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$
	Spherical coordinate system	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)$ $+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$

7.4 Fundamental Solution and the Harmonic Function

In this section we discuss fundamental solutions of homogeneous potential equations and analyze features of harmonic functions.

7.4.1 Fundamental Solution

A two-dimensional harmonic equation in a polar coordinate system reads

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0\,.$$

It reduces to a Euler equation when u is axis-symmetric (independent of θ)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}=0\,.$$

Its general solution is thus

$$u=c_1\ln\frac{1}{r}+c_2\,,$$

where c_1 and c_2 are constants, and $r=\sqrt{(x-x_0)^2+(y-y_0)^2}$.

A *harmonic function* in a domain Ω refers to the solution of harmonic equations that has continuous derivatives of second order in Ω. The particular solution $\ln\frac{1}{r}$ is a harmonic function in a two-dimensional plane with a single discontinuity at the point $(0,0)$. It is called the *fundamental solution of two-dimensional harmonic equations*. It can be shown that $u=\ln\frac{1}{r}$ satisfies the two-dimensional harmonic equation except at the point $(0,0)$,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Similarly, a spherically-symmetric three-dimensional harmonic equation reads

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}u}{\mathrm{d}r}\right)=0.$$

Its general solution is

$$u=c_1\frac{1}{r}+c_2,$$

where c_1 and c_2 are constants, and $r=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$.

The particular solution $\frac{1}{r}$ is thus a harmonic function in all three-dimensional space except at the point $(0,0,0)$. It is called the *fundamental solution of three-dimensional harmonic equations*. It can also be shown that $u=\frac{1}{r}$ satisfies the three-dimensional harmonic equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

Fundamental solutions play a very important role in examining harmonic functions and in seeking solutions of PDS of potential equations. Consider the electric field generated by an infinite electric wire passing perpendicularly through the point $M_0(x_0,y_0,0)$ on the Oxy-plane and with a uniform electric-charge density $2\pi\varepsilon$. Here ε is the dielectric constant. Since the field is uniform at all planes perpendicular to the wire, it is a typical model of plane fields. The x- and y-components of electric field intensity at $M(x,y,0)$ are

$$E_1=\int_{-\infty}^{+\infty}\frac{x-x_0}{2(r^2+z^2)^{3/2}}\,\mathrm{d}z=\frac{x-x_0}{2r^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\alpha\,\mathrm{d}\alpha=\frac{x-x_0}{r^2},$$

$$E_2=\int_{-\infty}^{+\infty}\frac{y-y_0}{2\sqrt{r^2+z^2}}\,\mathrm{d}z=\frac{y-y_0}{r^2},$$

where $z=r\tan\alpha$, $r=\sqrt{(x-x_0)^2+(y-y_0)^2}$. Thus the electric field intensity can be written as

$$\mathbf{E}=-\nabla\left(\ln\frac{1}{r}\right).$$

The fundamental solution $\ln\frac{1}{r}$ thus represents the potential of the electric field generated by the wire.

Similarly, consider the electric field generated by an electric charge of total electricity $4\pi\varepsilon$ at point $M_0(x_0, y_0, z_0)$ in three-dimensional space. The electric field intensity at $M(x,y,z)$ is

$$\mathbf{E} = \frac{x-x_0}{r^3}\mathbf{i} + \frac{y-y_0}{r^3}\mathbf{j} + \frac{z-z_0}{r^3}\mathbf{k} = -\nabla\frac{1}{r},$$

where $r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$. Thus the fundamental solution $\frac{1}{r}$ is the electric potential at M.

7.4.2 Green Function

Let Ω be a domain in three-dimensional space, and $\partial\Omega$ is the boundary of Ω. $\bar{\Omega} = \Omega \cup \partial\Omega$ is thus a closed domain. $C^n(\Omega)$ stands for the set of functions with n-th continuous partial derivatives in Ω. $C(\bar{\Omega})$ is the set of continuous functions in $\bar{\Omega}$.

Theorem 1. Let Ω be a three-dimensional bounded domain. Its boundary $\partial\Omega$ is assumed to be piecewise smooth. If $u, v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$, we have

$$\iiint\limits_{\Omega} u\Delta v \,\mathrm{d}\Omega = \iint\limits_{\partial\Omega} u\frac{\partial v}{\partial n}\,\mathrm{d}S - \iiint\limits_{\Omega} \nabla u\cdot\nabla v\,\mathrm{d}\Omega\,, \tag{7.62}$$

$$\iiint\limits_{\Omega} (u\Delta v - v\Delta u)\,\mathrm{d}\Omega = \iint\limits_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S\,, \tag{7.63}$$

where n stands for the external normal of $\partial\Omega$. Equations (7.62) and (7.63) are called *the first and second Green formulas*.

Proof. Let $P = u\dfrac{\partial v}{\partial x}$, $Q = u\dfrac{\partial v}{\partial y}$, $R = u\dfrac{\partial v}{\partial z}$. The first Green formula thus follows from the Gauss formula

$$\iiint\limits_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}\Omega = \iint\limits_{\partial\Omega}[P\cos(n,x)+Q\cos(n,y)+R\cos(n,z)]\,\mathrm{d}S$$

and

$$\frac{\partial v}{\partial x}\cos(n,x)+\frac{\partial v}{\partial y}\cos(n,y)+\frac{\partial v}{\partial z}\cos(n,z) = \frac{\partial v}{\partial n}\,.$$

By interchanging u and v in Eq. (7.62), we have

$$\iiint\limits_{\Omega} v\Delta u\,\mathrm{d}\Omega = \iint\limits_{\partial\Omega} v\frac{\partial u}{\partial n}\,\mathrm{d}S - \iiint\limits_{\Omega}\nabla u\cdot\nabla v\,\mathrm{d}\Omega\,. \tag{7.64}$$

The second Green formula follows from subtracting Eq. (7.64) from Eq. (7.62).

Theorem 2. Let Ω be a bounded domain with a piecewise smooth boundary $\partial\Omega$. If $u(M) = u(x,y,z) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$, at any point $M_0(x_0,y_0,z_0) \in \Omega$, we have

$$u(M_0) = -\frac{1}{4\pi}\iint\limits_{\partial\Omega}\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u(M)}{\partial n}\right]\mathrm{d}S - \frac{1}{4\pi}\iiint\limits_{\Omega}\frac{\Delta u(M)}{r}\,\mathrm{d}\Omega\,, \quad (7.65)$$

where the n stands for the external normal of $\partial\Omega$, $M_0 \in \Omega$ and

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}.$$

Equation (7.65) is called *the third Green formula* and is a fundamental integral formula in studying harmonic functions.

Proof. Consider two functions $u = u(M) = u(x,y,z)$ and $v = v(x,y,z) = \frac{1}{r}$. Here

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}\,.$$

Both the two functions satisfy the conditions in Theorem 1 for $\Omega\backslash\bar{\Omega}_\varepsilon^{\mathrm{M}_0}$. Here $\bar{\Omega}_\varepsilon^{\mathrm{M}_0} = \Omega_\varepsilon^{\mathrm{M}_0} \cup \partial\Omega_\varepsilon^{\mathrm{M}_0}$, the $\Omega_\varepsilon^{\mathrm{M}_0}$ is a sphere of center M_0 and radius ε and the $\partial\Omega_\varepsilon^{\mathrm{M}_0}$ is the boundary of $\Omega_\varepsilon^{\mathrm{M}_0}$. Applying the second Green formula to $\Omega\backslash\bar{\Omega}_\varepsilon^{\mathrm{M}_0}$ yields

$$\iiint\limits_{\Omega\backslash\bar{\Omega}_\varepsilon^{\mathrm{M}_0}}\left(u\Delta\frac{1}{r} - \frac{1}{r}\Delta u\right)\mathrm{d}\Omega$$

$$= \iint\limits_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right]\mathrm{d}S + \iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}}\left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right]\mathrm{d}S\,.$$

Note that the external normal of the inner boundary surface $\partial\Omega_\varepsilon^{\mathrm{M}_0}$ of $\Omega\backslash\bar{\Omega}_\varepsilon^{\mathrm{M}_0}$ is actually the internal normal of the boundary surface of $\Omega_\varepsilon^{\mathrm{M}_0}$ so $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$. Also $\Delta\frac{1}{r} = 0$. Thus

$$\iiint\limits_{\Omega\backslash\bar{\Omega}_\varepsilon^{\mathrm{M}_0}}\frac{\Delta u}{r}\,\mathrm{d}\Omega + \iint\limits_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right]\mathrm{d}S + \iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}}\frac{u}{r^2}\,\mathrm{d}S + \iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}}\frac{1}{r}\frac{\partial u}{\partial r}\,\mathrm{d}S = 0\,.$$

Note that, by the mean value theorem of integrals,

$$\iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}}\frac{u}{r^2}\,\mathrm{d}S = \frac{1}{\varepsilon^2}u(M_{\xi_1})\cdot 4\pi\varepsilon^2 = 4\pi u(M_{\xi_1})\,, \qquad M_{\xi_1} \in \partial\Omega_\varepsilon^{\mathrm{M}_0}\,, \quad (7.66)$$

$$\iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}}\frac{1}{r}\frac{\partial u}{\partial r}\,\mathrm{d}S = \frac{1}{\varepsilon}\left.\frac{\partial u}{\partial r}\right|_{M_{\xi_2}}\cdot 4\pi\varepsilon^2 = \left.\frac{\partial u}{\partial r}\right|_{M_{\xi_2}}\cdot 4\pi\varepsilon\,, \qquad M_{\xi_2} \in \partial\Omega_\varepsilon^{\mathrm{M}_0}\,. \quad (7.67)$$

Therefore, we obtain, as $\varepsilon \to 0$

$$u(M_0) = \frac{1}{4\pi} \iint\limits_{\partial\Omega} \left[\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \right] \mathrm{d}S - \frac{1}{4\pi} \iiint\limits_{\Omega} \frac{\Delta u}{r} \mathrm{d}\Omega\,.$$

The third Green formula plays an important role in examining harmonic functions and in seeking solutions of boundary-value problems of potential equations. Here we list some of its corollaries.

1. If u is a solution of the Poisson equation $\Delta u = F$, we have

$$u(M_0) = \frac{1}{4\pi} \iint\limits_{\partial\Omega} \left[\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \right] \mathrm{d}S - \frac{1}{4\pi} \iiint\limits_{\Omega} \frac{F}{r} \mathrm{d}\Omega\,. \tag{7.68}$$

2. If u is a harmonic function, we have, by Eq. (7.65) and for an internal point M_0 of Ω,

$$u(M_0) = \frac{1}{4\pi} \iint\limits_{\partial\Omega} \left[\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \right] \mathrm{d}S\,, \tag{7.69}$$

   which is called the *fundamental integral formula of harmonic functions*. The value of a harmonic function at any internal point of Ω can be thus expressed by its values and its normal derivatives on the boundary $\partial\Omega$.

3. If M_0 is outside of Ω such that $\Delta\frac{1}{r} = 0$ in Ω, the second Green formula leads to

$$\iint\limits_{\partial\Omega} \left[\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \right] \mathrm{d}S - \iiint\limits_{\Omega} \frac{\Delta u}{r} \mathrm{d}\Omega = 0\,. \tag{7.70}$$

   If M_0 is on $\partial\Omega$, $\Omega_\varepsilon^{\mathrm{M}_0}$ refers to the part inside Ω. $\partial\Omega_\varepsilon^{\mathrm{M}_0}$ also refers to the part inside Ω. Note that the relation between the area element $\mathrm{d}S$ on $\partial\Omega_\varepsilon^{\mathrm{M}_0}$ and its corresponding solid angle $\mathrm{d}\omega$ with respect to the sphere center is

$$\mathrm{d}S = \varepsilon^2\,\mathrm{d}\omega\,.$$

   Equations (7.66) and (7.67) thus reduce to

$$\iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}} \frac{u}{r^2} \mathrm{d}S = \iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}} u\frac{\mathrm{d}S}{\varepsilon^2} = \iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}} u\,\mathrm{d}\omega = u(M_{\xi_1}) \iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}} \mathrm{d}\omega\,, \qquad M_{\xi_1} \in \partial\Omega_\varepsilon^{\mathrm{M}_0}\,, \tag{7.71}$$

$$\iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}} \frac{1}{r}\frac{\partial u}{\partial r} \mathrm{d}S = \varepsilon \iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}} \frac{\partial u}{\partial r}\frac{\mathrm{d}S}{\varepsilon^2} = \varepsilon \left.\frac{\partial u}{\partial r}\right|_{M_{\xi_2}} \iint\limits_{\partial\Omega_\varepsilon^{\mathrm{M}_0}} \mathrm{d}\omega\,, \qquad M_{\xi_2} \in \partial\Omega_\varepsilon^{\mathrm{M}_0}\,, \tag{7.72}$$

which tend to $2\pi u(M_0)$ and zero, respectively, as $\varepsilon \to 0$. because as $\varepsilon \to 0$,

$$\iint\limits_{\partial\Omega_\varepsilon^{M_0}} d\omega \to 2\pi\,, \quad u(M_\xi) \to u(M_0)\,, \quad \left.\frac{\partial u}{\partial r}\right|_{M_\xi} \to \left.\frac{\partial u}{\partial r}\right|_{M_0}.$$

Equation (7.65) becomes

$$u(M_0) = -\frac{1}{2\pi}\iint\limits_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right] dS - \frac{1}{2\pi}\iiint\limits_{\Omega}\frac{\Delta u}{r}\,d\Omega\,. \tag{7.73}$$

Equations (7.68), (7.69) and (7.73) can be summarized as

$$-\iint\limits_{\partial\Omega}\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u(M)}{\partial n}\right] dS - \iiint\limits_{\Omega}\frac{\Delta u}{r}\,d\Omega = \begin{cases} 4\pi u(M_0)\,, & M_0 \in \Omega\,, \\ 2\pi u(M_0)\,, & M_0 \in \partial\Omega\,, \\ 0\,, & M_0 \notin \bar{\Omega}\,. \end{cases} \tag{7.74}$$

where $r = \overline{M_0 M} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$. If u satisfies $\Delta u = 0$, in particular, the triple integral on the left-hand side of Eq. (7.74) vanishes.

The counterpart of Eq. (7.74) in a two-dimensional case reads

$$-\int\limits_{\partial D}\left[u(M)\frac{\partial}{\partial n}\left(\ln\frac{1}{r}\right) - \left(\ln\frac{1}{r}\right)\frac{\partial u}{\partial n}\right] dS - \iint\limits_{D}\frac{\Delta u}{r}\,d\sigma = \begin{cases} 2\pi u(M_0)\,, & M_0 \in D\,, \\ \pi u(M_0)\,, & M_0 \in \partial D\,, \\ 0\,, & M_0 \notin \bar{D}\,. \end{cases} \tag{7.75}$$

This can be obtained by following a similar approach to that used in developing Eq. (7.74), simply replacing Ω, $\partial\Omega$, $\overline{M_0 M}$ and the solid angle 4π of a sphere by a plane domain D, the piecewise smooth boundary curve ∂D, $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$ and the central angle 2π of a circle, and using the two-dimensional fundamental solution $\ln\frac{1}{r}$. When u satisfies $\Delta u = 0$, the double integral in the left-hand side of Eq. (7.75) reduces to zero.

Remark. For a fixed point $M_0(x_0, y_0, z_0) \in R^3$ and a variable point $M(x,y,z) \in R^3$, the function $v(M) = \frac{1}{r}$ satisfies the harmonic equation in R^3 except at the point M_0 so

$$\Delta\frac{1}{r} = 0\,, \quad r \neq 0\,.$$

Thus

$$\iiint\limits_{R^3}\Delta\left(\frac{1}{r}\right) d\Omega = \iiint\limits_{r\le\varepsilon}\Delta\left(\frac{1}{r}\right) d\Omega = \iint\limits_{r=\varepsilon}\Delta\left(\frac{1}{r}\right)\cdot\mathbf{n}\,dS = \iint\limits_{r=\varepsilon}\frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\,dS$$

$$= -\frac{1}{\varepsilon^2}\iint\limits_{r=\varepsilon} \mathbf{r}_1 \cdot \mathbf{n}\,\mathrm{d}S = -\frac{1}{\varepsilon^2}\iint\limits_{r=\varepsilon} \mathrm{d}S = -4\pi\,, \tag{7.76}$$

or

$$-\iiint\limits_{R^3} \Delta\left(\frac{1}{4\pi r}\right)\mathrm{d}\Omega = 1\,,$$

where the $\mathbf{r}_1$ is the unit normal vector of $\mathbf{r} = \overline{M_0 M}$.

Equation (7.76) shows that $\Delta\frac{1}{r}$ does not change its sign in the neighborhood of M_0. Therefore we have, for any $\varphi(x,y,z) \in C(R^3)$,

$$\iiint\limits_{R^3} -\Delta\left(\frac{1}{4\pi r}\right)\varphi(x,y,z)\,\mathrm{d}\Omega = \lim_{\varepsilon\to 0}\iiint\limits_{r\le\varepsilon} -\Delta\left(\frac{1}{4\pi r}\right)\varphi(x,y,z)\,\mathrm{d}\Omega$$

$$= \lim_{\varepsilon\to 0}\varphi(\xi,\eta,\zeta)\iiint\limits_{r\le\varepsilon} -\Delta\left(\frac{1}{4\pi r}\right)\mathrm{d}\Omega = \lim_{\varepsilon\to 0}\varphi(\xi,\eta,\zeta) = \varphi(x_0,y_0,z_0)\,.$$

Thus, by the definition of the Dirac function, $-\Delta\left(\frac{1}{4\pi r}\right) = \delta(M-M_0)$. Similarly, for the two-dimensional case, $-\Delta\left(\frac{1}{2\pi}\ln\frac{1}{r}\right) = \delta(M-M_0)$. Therefore, $\frac{1}{2\pi}\ln\frac{1}{r}$ and $\frac{1}{4\pi r}$ are also called the fundamental solutions of two- and three-dimensional harmonic equations, respectively.

7.4.3 Harmonic Functions

Five Theorems

Theorem 1. Let $u \in C^2(\Omega)$. u is a harmonic function in Ω if and only if, for any closed sub-domain $\bar{\Omega}^*$ with a smooth boundary surface $\partial\Omega^*$,

$$\iint\limits_{\partial\Omega^*} \frac{\partial u}{\partial n}\,\mathrm{d}S = 0\,, \quad \bar{\Omega}^* = \Omega^* \cup \partial\Omega^*\,, \tag{7.77}$$

where $\frac{\partial u}{\partial n}$ is the normal derivative.

Proof. With $v = 1$ (thus $\Delta v = 0$, $\frac{\partial v}{\partial n} = 0$), applying the second Green formula yields

$$\iiint\limits_{\Omega^*} \Delta u\,\mathrm{d}\Omega = \iint\limits_{\partial\Omega^*} \frac{\partial u}{\partial n}\,\mathrm{d}S\,. \tag{7.78}$$

If u is a harmonic function in Ω such that $\Delta u = 0$, Eq. (7.78) leads to Eq. (7.77). If Eq. (7.77) holds for any $\bar{\Omega}^*$, Eq. (7.78) yields

$$\iiint\limits_{\Omega^*} \Delta u \, \mathrm{d}\Omega = 0\,, \quad \forall \Omega^*\,.$$

This shows that, by the continuity of Δu and the arbitrariness of Ω^*, $\Delta u = 0$.

Remark 1. Take Ω^* in Theorem 1 to be Ω. For a harmonic function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, an application of Theorem 1 yields

$$\iint\limits_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}S = 0\,.$$

Therefore a necessary condition for the existence of solutions of the Neumann problem

$$\begin{cases} \Delta u = 0\,, \quad \Omega\,, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = f\,. \end{cases} \tag{7.79}$$

is

$$\iint\limits_{\partial\Omega} f \, \mathrm{d}S = 0\,.$$

It can be readily proven that $\iint\limits_{\partial\Omega} f \, \mathrm{d}S = 0$ is also a sufficient condition for the existence of solutions of PDS (7.79).

For example, the problem

$$\begin{cases} \Delta u = 0\,, \quad r < 1 \left(r = \sqrt{x^2 + y^2 + z^2} \right)\,, \\ \left.\dfrac{\partial u}{\partial n}\right|_{r=1} = 1\,. \end{cases}$$

has no solution because $\iint\limits_{r=1} f \, \mathrm{d}S = \iint\limits_{r=1} \mathrm{d}S = 4\pi \neq 0$.

Remark 2. Take the u in Theorem 1 as the steady temperature in a domain Ω without any source or sink inside so that $\Delta u = 0$. The $-k \iint\limits_{\partial\Omega^*} \dfrac{\partial u}{\partial n} \, \mathrm{d}S$ is, by the Fourier law of heat conduction, the net heat across the boundary $\partial\Omega^*$. Theorem 1 shows that at a steady state, both the necessary and sufficient condition for non-existence of an internal heat source/sink is zero net heat flux across any closed surface in the domain. This clearly agrees with the physical reality.

Theorem 2. Let $u(M)$ be a harmonic function in Ω. For any $M_0 \in \Omega$, $u(M_0)$ must be

$$u(M_0) = \frac{1}{4\pi R^2} \iint\limits_{S_R^{M_0}} u(M)\,\mathrm{d}S\,, \tag{7.80}$$

where the $S_R^{M_0}$ stands for the spherical surface of a sphere (inside Ω) of center M_0 and radius R.

Proof. Take $\partial\Omega$ in the fundamental integral formula of harmonic functions (Eq. (7.69)) to be $S_R^{M_0}$. Equation (7.69) and theorem 1 thus yield

$$\begin{aligned} u(M_0) &= \frac{1}{4\pi} \iint\limits_{S_R^{M_0}} \left[\frac{1}{R}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right] \mathrm{d}S \\ &= \frac{1}{4\pi} \iint\limits_{S_R^{M_0}} \frac{u}{R^2}\,\mathrm{d}S = \frac{1}{4\pi R^2} \iint\limits_{S_R^{M_0}} u\,\mathrm{d}S\,. \end{aligned}$$

Remark 1. For a two-dimensional harmonic function $u = u(x,y)$, its value at point $M_0(x_0,y_0)$ is

$$u(M_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\,\mathrm{d}\theta\,,$$

which is its mean value over the circumference of a circle of center M_0 and radius R.

Remark 2. If u is an harmonic function in a open domain Ω, we have

$$u(M_0) = \frac{1}{\frac{4}{3}\pi R^3} \iiint\limits_{V_R^{M_0}} u(M)\,\mathrm{d}\Omega\,, \quad V_R^{M_0} \subset \Omega\,, \tag{7.81}$$

where $V_R^{M_0}$ denotes a sphere inside Ω of center M_0 and radius R. Note that for any $S_r^{M_0}$ with $0 < r \leq R$, we have

$$u(M_0)4\pi r^2 = \iint_{S_r^{M_0}} u(M)\,\mathrm{d}S\,.$$

Thus an integration from $r = 0$ to $r = R$ will arrive at Eq. (7.81).

Theorem 3 (Liouville Theorem). Let u be a harmonic function in the entire domain R^3. If u is bounded from below or above, it must be a constant.

Proof. If u is a harmonic function, $-u$ is also a harmonic function. Therefore, we need to prove the theorem only for the case $u \geq c$ (bounded from below).

Since $u \geq c$ and u is a harmonic function, $u - c$ is nonnegative and is also a harmonic function. Now consider two spheres $V_{R_1}^{M_1}$ and $V_{R_2}^{M_2}$. Here M_1 and M_2 are two arbitrary points in R^3, $R_1 = R_2 + r_{M_1M_2}$, and $r_{M_1M_2}$ is the distance between M_1 and M_2. Thus

$$V_{R_2}^{M_2} \subset V_{R_1}^{M_1}.$$

This, with $u - c \geq 0$, yields

$$\iiint\limits_{V_{R_2}^{M_2}} [u(M) - c]\,\mathrm{d}\Omega \leq \iiint\limits_{V_{R_1}^{M_1}} [u(M) - c]\,\mathrm{d}\Omega .$$

By using Eq. (7.81), we obtain

$$u(M_2) - c \leq \left(\frac{R_1}{R_2}\right)^3 [u(M_1) - c].$$

Note that $\lim\limits_{R_2 \to \infty} \left(\frac{R_1}{R_2}\right)^3 = 1$. Thus

$$u(M_2) - c \leq u(M_1) - c \quad \text{or} \quad u(M_2) \leq u(M_1).$$

By interchanging the rule for M_1 and M_2, we can also obtain

$$u(M_1) \leq u(M_2).$$

Therefore,

$$u(M_2) = u(M_1).$$

The u thus must be a constant due to the arbitrariness of M_1 and M_2.

Theorem 4 (Extremum Principle). Assume that $u(M)$ is a harmonic function in Ω, continuous in a closed domain $\bar{\Omega} = \Omega \cup \partial\Omega$ and non-constant in $\bar{\Omega}$. Then $u(M)$ can take its maximum and minimum values only on the boundary $\partial\Omega$.

Proof. Since a maximum of u is the minimum of $-u$, we need to prove the theorem only for the case of maximum value.

In contradiction to the theorem, suppose that u takes its maximum value at point M_0 inside Ω. This assumption would lead to $u(M) \equiv u(M_0)$ for all points M on $S_R^{M_0}$, which is the spherical surface of a sphere $V_R^{M_0}$ of center M_0 and radius R inside Ω. Additionally, suppose that $u(M_1) < u(M_0)$ for a point M_1 on $S_R^{M_0}$. Because of the continuity of u in $\bar{\Omega}$, there is a region σ around M_1 on $S_R^{M_0}$ where $u(M) < u(M_0)$ for all M on σ. Here we use σ to represent both region and its area. We thus have, by Theorem 2

$$u(M_0) = \frac{1}{4\pi R^2} \iint\limits_{S_R^{M_0}} u\,\mathrm{d}S = \frac{1}{4\pi R^2} \left[\iint\limits_{S_R^{M_0} \setminus \sigma} u\,\mathrm{d}S + \iint\limits_{\sigma} u\,\mathrm{d}S \right]$$

$$< \frac{1}{4\pi R^2}[u(M_0)(4\pi R^2 - \sigma) + u(M_0)\sigma] = u(M_0),$$

which is absurd. Also note that R is arbitrary. Therefore $u(M) \equiv u(M_0)$ for all points in $V_R^{M_0}$, if the u takes its maximum value inside of Ω.

Now consider another point M^* inside of Ω. We can always find a broken line inside of Ω that connects M_0 and M^*. This broken line can be always covered by a finite number of spheres $K_0, K_1, \cdots K_n, \cdots K_N$ that are also inside of Ω. Here K_0 is a sphere with its center located at M_0, K_n is a sphere with its center in K_{n-1} $(n = 1, 2, \cdots, N)$ and K_N contains point M^*. In all these spheres, $u(M) \equiv u(M_0)$ by the above results. In particular, $u(M^*) = u(M_0)$. Therefore, $u(M) \equiv u(M_0)$ for all $M \in \Omega$ by the arbitrariness of M^*.

Note also that u is continuous in $\bar{\Omega}$. Therefore $u(M) \equiv u(M_0)$ for all $M \in \bar{\Omega}$. This is a contradiction to the condition in Theorem 4, so u can take its maximum value only on the boundary $\partial\Omega$.

In heat conduction, the Laplace equation governs the steady temperature distribution in a domain without an internal heat source or sink. Since the temperature takes its minimum value on the domain boundary (say, point M_0 on the boundary) by the extremum principle, the heat will be conducted inside the domain towards this point and finally to the outside of the domain. Let n be the external normal of the boundary. The normal derivative of the temperature $\dfrac{\partial u}{\partial n}$ thus must be negative semi-definite at the point M_0. In effect, we can prove that $\dfrac{\partial u}{\partial n}$ is negative definite at point M_0. Similarly, $\dfrac{\partial u}{\partial n}$ is positive definite at the boundary point where u takes its maximum value. All these results are summarized in the following strong extremum principle.

Theorem 5 (Strong Extremum Principle). Assume that $u(M)$ is a harmonic function in a closed domain Ω, is continuous and non-constant in $\bar{\Omega}$, and takes its minimum (maximum) value at point M_0 on the boundary $\partial\Omega$. If the normal derivative $\dfrac{\partial u}{\partial n}$ exists at M_0, it must be negative (positive) definite.

Proof. If u is a harmonic function, $-u$ is also a harmonic function. Therefore without loss of generality, we prove the theorem only for the case of minimum value.

Consider a closed sphere K_R of radius R in Ω with its spherical surface ∂K_R tangential at point M_0. By the extremum principle,

$$u(M_0) < u(M), \quad M \in K_R \backslash M_0.$$

Therefore

$$\left.\frac{\partial u}{\partial n}\right|_{M_0} \leq 0.$$

In order to prove the theorem, we must show that this equality is impossible. Without loss of generality and for convenience, take the origin as the center of K_R. Consider another sphere $K_{R/2}$ of center origin and radius $R/2$, and an auxiliary function in the region D between ∂K_R and $\partial K_{R/2}$,

$$v(r) = \frac{1}{r} + \frac{r}{2R^2} - \frac{3}{2R}\,,$$

where $r = \sqrt{x^2+y^2+z^2}$, $M(x,y,z) \in D$. In D, we have

$$v(r)|_{\partial K_R} = v(R) = 0\,, \quad \left.\frac{\partial v}{\partial r}\right|_{\partial K_R} = \left.\frac{\partial v}{\partial r}\right|_{r=R} = -\frac{1}{2R^2} < 0\,,$$

$$\Delta v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{2R^2} - 1\right) = \frac{1}{n^2 r} > 0\,, \quad R/2 < r < R\,.$$

To analyze the normal derivative of $u(M)$ at M_0, consider another new function

$$w(M) = u(M) - \varepsilon v(M)\,, \quad \varepsilon > 0\,, M \in D\,,$$

we attempt to prove that $w(M)$ takes its minimum value at M_0. For $w(M) = u(M) - \varepsilon v(M)$, we have

$$\begin{aligned} w(M) &= u(M)\,, \quad M \in \partial K_R\,,\\ w(M) - w(M_0) &= u(M) - \varepsilon v(M) - u(M_0)\\ &= u(M) - u(M_0) - \varepsilon v(R/2)\,, \quad M \in \partial K_{R/2}\,. \end{aligned}$$

Define $d = \min\limits_{r=R/2}[u(M) - u(M_0)] > 0$ and consider $0 < \varepsilon < d/v(R/2)$, we have

$$w(M) \geq w(M_0)\,, \quad M \in \partial K_{R/2}\,.$$

Thus

$$w(M) \geq w(M_0)\,, M \in \partial K_{R/2} \cup 2K_r\,.$$

Also, $w(M)$ cannot take its minimum value in D. Otherwise, at the point of minimum value,

$$w_{xx} \geq 0\,, \quad w_{yy} \geq 0\,, \quad w_{zz} \geq 0\,,$$

so $\Delta w \geq 0$. Or, alternately,

$$\Delta w = \Delta u - \varepsilon \Delta v = -\varepsilon \Delta v < 0\,.$$

Therefore, $w(M)$, $M \in D$, must take its minimum value at M_0, so

$$\frac{\partial w}{\partial n} = \frac{\partial u}{\partial n} - \varepsilon\frac{\partial v}{\partial n} \leq 0 \quad \text{or} \quad \frac{\partial u}{\partial n} \leq \varepsilon\frac{\partial v}{\partial n} < 0\,.$$

Harmonic Functions and Analytical Functions

Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$ be an analytical function in a plane domain D. By the theory of complex functions, $u(x,y)$ and $v(x,y)$ must be differentiable at any point

$z = x + \mathrm{i}y$ in D and must satisfy the Cauchy-Riemann equation

$$u_x = v_y\,, \quad u_y = -v_x \tag{7.82}$$

Thus $\Delta u = 0$, $\Delta v = 0$, so both u and v are harmonic functions. $v(x,y)$ is called the *conjugate harmonic function of* $\mathbf{u}(\mathbf{x},\mathbf{y})$. Note that $u(x,y)$ is not necessarily the conjugate harmonic function of $v(x,y)$. Once u is available, we can use Eq. (7.82) to find the harmonic function $v(x,y)$ and consequently the analytical function $f(z) = u + \mathrm{i}v$. This can help us to find the Poisson formula of internal problems of two-dimensional harmonic equations.

Example. Find the solution of

$$\begin{cases} \Delta u(r,\theta) = 0, \quad 0 < r < R\,, \\ u(R,\theta) = f(\theta)\,. \end{cases} \tag{7.83}$$

Solution. Let v be the conjugate harmonic function of u, then $g(z) = u + \mathrm{i}v$ is an analytical function. By the Cauchy integral formula we have

$$g(z) = \frac{1}{2\pi\mathrm{i}} \oint_{|\xi|=R} \frac{g(\xi)}{\xi - z}\,\mathrm{d}\xi\,,$$

$$0 = \frac{1}{2\pi\mathrm{i}} \oint_{|\xi|=R} \frac{g(\xi)}{\xi - z'}\,\mathrm{d}\xi\,,$$

where $z = r\mathrm{e}^{\mathrm{i}\theta}$ is any point in a circle $0 < r < R$. $z' = \frac{R^2}{r}\mathrm{e}^{\mathrm{i}\theta}$ is the symmetric point of z with respect to the circle $r = R$ so that $|z||z'| = R^2$. Let $\xi = R\mathrm{e}^{\mathrm{i}\theta'}$. A subtraction of the latter from the former yields

$$\begin{aligned} g(z) = u + \mathrm{i}v &= \frac{1}{2\pi\mathrm{i}} \oint_{|\xi|=R} g(\xi) \frac{z - z'}{(\xi - z)(\xi - z')}\,\mathrm{d}\xi \\ &= \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} (u + \mathrm{i}v) \frac{r\mathrm{e}^{\mathrm{i}\theta} - \frac{R^2}{r}\mathrm{e}^{\mathrm{i}\theta}}{(R\mathrm{e}^{\mathrm{i}\theta'} - r\mathrm{e}^{\mathrm{i}\theta})(R\mathrm{e}^{\mathrm{i}\theta'} - \frac{R^2}{r}\mathrm{e}^{\mathrm{i}\theta})} R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta'}\,\mathrm{d}\theta' \\ &= \frac{1}{2\pi} \int_0^{2\pi} (u + \mathrm{i}v) \frac{\frac{R}{r}(r^2 - R^2)}{(R\mathrm{e}^{\mathrm{i}(\theta'-\theta)} - r)(R - \frac{R^2}{r}\mathrm{e}^{\mathrm{i}(\theta-\theta')})}\,\mathrm{d}\theta' \\ &= \frac{1}{2\pi} \int_0^{2\pi} (u + \mathrm{i}v) \frac{r^2 - R^2}{(Rr\mathrm{e}^{\mathrm{i}(\theta'-\theta)} - r^2)(1 - \frac{R}{r}\mathrm{e}^{\mathrm{i}(\theta-\theta')})}\,\mathrm{d}\theta' \\ &= \frac{1}{2\pi} \int_0^{2\pi} (u + \mathrm{i}v) \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \theta')}\,\mathrm{d}\theta'\,. \end{aligned}$$

Extracting its real part leads to the solution of PDS (7.83), by noting that $u|_{r=R} = f(\theta)$,

$$u = \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \theta')}\,\mathrm{d}\theta',$$

which is the same as Eq. (7.17).

7.5 Well-Posedness of Boundary-Value Problems

Since boundary-value problems of nonhomogeneous potential equations can be transformed to those of homogeneous equations, we only discuss the well-posedness of boundary-value problems of Laplace equations in this section.

Theorem 1. If a Dirichlet problem has a solution, the solution must be unique and stable.

Proof. Uniqueness: Let u_1 and u_2 be two solutions of a Dirichlet problem of the Laplace equation. Then $u = u_1 - u_2$ must satisfy

$$\begin{cases} \Delta u = 0, & \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{7.84}$$

By the extremum principle, we obtain

$$u \equiv 0, \quad \Omega,$$

so that $u_1 \equiv u_2$. By the arbitrariness of u_1 and u_2, we establish the uniqueness of the solution.

Stability: Let u_1 and u_2 be the solutions of

$$\begin{cases} \Delta u = 0, & \Omega, \\ u|_{\partial\Omega} = f_1 \end{cases} \quad \text{and} \quad \begin{cases} \Delta u = 0, & \Omega, \\ u|_{\partial\Omega} = f_2, \end{cases}$$

respectively. $u = u_1 - u_2$ thus must satisfy

$$\begin{cases} \Delta u = 0, & \Omega, \\ u|_{\partial\Omega} = f_1 - f_2. \end{cases}$$

By the extremum principle,

$$\sup_{\bar{\Omega}} |u| = \sup_{\partial\Omega} |f_1 - f_2| = \sup_{\bar{\Omega}} |u_1 - u_2|.$$

Thus, if $\sup\limits_{\partial\Omega} |f_1 - f_2| < \varepsilon$, $\sup\limits_{\Omega} |u_1 - u_2| < \varepsilon$, so the solution is stable with respect to the boundary values.

Theorem 2. If an external Dirichlet problem has a solution, the solution must be unique and stable.

Proof. Uniqueness: Let u_1 and u_2 be two solutions of the problem, then $u = u_1 - u_2$ must satisfy

$$\begin{cases} \Delta u = 0\,, \quad \Omega'\,, \\ u|_{\partial\Omega'} = 0\,, \\ \lim\limits_{r\to\infty} u = 0\,, r = \sqrt{x^2+y^2+z^2}\,. \end{cases} \tag{7.85}$$

Uniqueness will be established once we show that $u(M) \equiv 0$, $M \in \Omega'$. Suppose $u(M_0) \neq 0$ at M_0. Without loss of generality, consider $u(M_0) > 0$. We can always find a spherical surface S_R with a sufficiently large radius $R : x^2+y^2+z^2 = R^2$ such that M_0 is inside the region Ω_R of boundary $\partial\Omega'$ and S_R, and $u|_{S_R} < u(M_0)$ by the condition $\lim\limits_{r\to\infty} u = 0$. The harmonic function u in Ω_R thus does not take its maximum value on its boundary $\partial\Omega'$ and S_R, which contradicts the extremum principle. Therefore, $u \equiv 0$ so the solution is unique.

Stability: Let u_1 and u_2 be solutions of

$$\begin{cases} \Delta u = 0\,, \quad \Omega'\,, \\ u|_{\partial\Omega'} = f_1\,, \\ \lim\limits_{r\to\infty} u = 0 \end{cases} \quad \text{and} \quad \begin{cases} \Delta u = 0\,, \quad \Omega'\,, \\ u|_{\partial\Omega'} = f_2\,, \\ \lim\limits_{r\to\infty} u = 0\,, \end{cases} \tag{7.86}$$

respectively. Then $u = u_1 - u_2$ must satisfy

$$\begin{cases} \Delta u = 0\,, \quad \Omega'\,, \\ u|_{\partial\Omega'} = f_1 - f_2\,, \\ \lim\limits_{r\to\infty} u = 0\,. \end{cases}$$

Let $\sup\limits_{\partial\Omega} |f_1 - f_2| < \varepsilon$. For any point M in Ω', by the condition $\lim\limits_{r\to\infty} u = 0$, we can always find a spherical surface S_R with a sufficiently large radius $R : x^2+y^2+z^2 = R^2$ such that M is inside the region Ω_R of boundary $\partial\Omega'$ and S_R and $\sup\limits_{S_R} |u| < \varepsilon$. By the extremum principle, the harmonic function u satisfies

$$\sup_{\Omega_R} |u| = \sup_{\partial\Omega_R} |u|\,, \quad \partial\Omega_R = \partial\Omega \cup S_R\,.$$

Thus $\sup\limits_{\Omega_R} |u| < \varepsilon$. By the arbitrariness of M, we thus obtain

$$\sup_{\Omega'} |u| < \varepsilon \quad \text{or} \quad \sup_{\Omega'} |u_1 - u_2| < \varepsilon\,.$$

Therefore, the solution is stable with respect to the boundary values.

Theorem 3. If an internal Neumann problem has a solution, the solution is unique up to a constant.

Proof. Let u_1 and u_2 be two solutions of an internal Neumann problem

$$\begin{cases} \Delta u = 0\,, & \Omega\,, \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = f\,. \end{cases} \tag{7.87}$$

The $v = u_1 - u_2$ thus must satisfy

$$\begin{cases} \Delta v = 0\,, & \Omega\,, \\ \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0\,. \end{cases}$$

The first Green formula yields

$$\iiint\limits_{\Omega} \nabla v \cdot \nabla v \,\mathrm{d}\Omega = \iiint\limits_{\Omega} \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right] \mathrm{d}\Omega$$

$$= \iint\limits_{\partial\Omega} v \frac{\partial v}{\partial n} \,\mathrm{d}S - \iiint\limits_{\Omega} v \Delta v \,\mathrm{d}\Omega = 0\,.$$

Therefore $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial z} = 0$, so that

$$u_1 = u_2 + c.$$

Here c is a constant.

Theorem 4. If an external Neumann problem has a solution, the solution must be unique.

Proof. Let u_1 and u_2 be two solutions of the external problem

$$\begin{cases} \Delta u = 0\,, & \Omega'\,, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega'} = f\,, \ \lim\limits_{r\to\infty} u = 0\,, \ \ r = \sqrt{x^2+y^2+z^2}\,. \end{cases} \tag{7.88}$$

Thus $v = u_1 - u_2$ must satisfy

$$\begin{cases} \Delta v = 0\,, & \Omega'\,, \\ \left.\dfrac{\partial v}{\partial n}\right|_{\partial\Omega'} = 0\,, \ \lim\limits_{r\to\infty} v = 0\,, \end{cases}$$

where the Ω' is the region outside of $\partial\Omega'$ and the n is the external normal of Ω' on $\partial\Omega'$. Once we show that $v(M) \equiv 0$, $M \in \Omega'$, we establish uniqueness. Suppose

$v(M_0) \neq 0$ at $M_0 \in \Omega'$. Without loss of generality, consider $v(M_0) > 0$. By the condition $\lim\limits_{r\to\infty} v = 0$, there always exists a spherical surface S_R with a sufficiently large radius $R : x^2 + y^2 + z^2 = R^2$ such that M_0 is inside the region Ω_R of boundary $\partial\Omega'$ and S_R and $\sup\limits_{S_R} |v| < v(M_0)$. By the extremum principle, the harmonic function in Ω_R must take its maximum value at the boundary $\partial\Omega'$. This requires a strictly positive $\dfrac{\partial v}{\partial n}$ somewhere on the boundary by the strong extremum principle. However, $\left.\dfrac{\partial v}{\partial n}\right|_{\partial\Omega'} = 0$ and $\sup\limits_{S_R} |v| < v(M_0)$ so v does not take its maximum on the boundary of region Ω_R, which is contrary to the extremum principle. Thus we obtain $v \equiv 0$ so the solution is unique.

Theorem 5. If a Robin problem

$$\begin{cases} \Delta u = 0\,, \partial\Omega'\,, \\ \left.\left(\dfrac{\partial u}{\partial n} + \sigma u\right)\right|_{\partial\Omega} = f\,, \sigma > 0\,. \end{cases} \tag{7.89}$$

has a solution, the solution must be unique.

Proof. Let u_1 and u_2 be two solutions of PDS (7.89). $v = u_1 - u_2$ must thus satisfy

$$\begin{cases} \Delta v = 0\,, \\ \left.\left(\dfrac{\partial v}{\partial n} + \sigma v\right)\right|_{\partial\Omega} = 0\,, \sigma > 0\,. \end{cases}$$

First prove $v \equiv c$(constant). Suppose that $v \not\equiv c$(constant) in Ω. By the extremum principle, v can take its minimum and maximum values only on the boundary $\partial\Omega$, say at points M_0 and M_1, respectively. By the strong extremum principle, we have $\dfrac{\partial v(M_0)}{\partial n} < 0$, so that, by the boundary condition $\left.\left(\dfrac{\partial v}{\partial n} + \sigma v\right)\right|_{\partial\Omega} = 0$, $v(M_0) = -\dfrac{1}{\sigma}\dfrac{\partial v(M_0)}{\partial n} > 0$. Thus the minimum value of the harmonic function v is larger than zero in $\bar{\Omega}$. Similarly, we can show $v(M_1) < 0$, so that the maximum value of v is smaller than zero. We thus arrive at a contradiction. Therefore, $v \equiv c$ in $\bar{\Omega}$. Also, by the boundary condition,

$$\left.\left(\frac{\partial v}{\partial n} + \sigma v\right)\right|_{\partial\Omega} = \left.\left(\frac{\partial c}{\partial n} + \sigma c\right)\right|_{\partial} \Omega = \sigma v|_{\partial\Omega} = 0\,,$$

so that $c = 0$ and $v \equiv 0$. Therefore the solution is unique.

Remark. Let $v = u$ in the first Green formula. We can readily use $C^2(\Omega) \cap C^1(\bar{\Omega})$ to show the uniqueness of boundary-value problems of $\Delta u = 0$ of the first and the third kinds and the uniqueness up to a constant of the second boundary-value problems of $\Delta u = 0$.

Example 1. Demonstrate that the extremum principle does not hold for

$$u_{xx} + u_{yy} + cu = 0 \ \ (c > 0)\,.$$

Solution. Note that $\Delta u + cu = 0$ is not a Laplace equation, hence the extremum principle is not valid. To demonstrate this, consider a function

$$u = \sin\sqrt{\frac{c}{2}}x \cdot \sin\sqrt{\frac{c}{2}}y\,.$$

u satisfies $\Delta u + cu = 0$ and is equal to zero on the boundary of the square domain $\bar{\Omega} : -\sqrt{2/c}\pi \le x \le \sqrt{2/c}\pi\,, -\sqrt{2/c}\pi \le y \le \sqrt{2/c}\pi$. However, $u(x,y) = \pm 1$ when $x = \dfrac{1}{2}\sqrt{2/c}\pi$ and $y = \pm\frac{1}{2}\sqrt{2/c}\pi$. Also, $u(x,y) \le 1$ in $\bar{\Omega}$. Therefore the u solving the equation $\Delta u + cu = 0$ does not take its minimum and maximum values on the boundary.

Example 2. Let Ω be a n-dimensional bounded region whose boundary is denoted by $\partial\Omega$. Show that the necessary condition for the existence of a solution of

$$\begin{cases} \Delta u = f(x)\,,\ x = (x_1, x_2, \cdots, x_n) \in \Omega\,, \\ u|_{\partial\Omega} = c\,,\quad \int_{\partial\Omega} \frac{\partial u}{\partial n}\,\mathrm{d}S = A\,, \end{cases} \tag{7.90}$$

is

$$\int_{\Omega} f\,\mathrm{d}x = A\,.$$

Here c and A are constants, f is a function of n variables, the integral is n-multiple and

$$\mathrm{d}x = \mathrm{d}x_1,\, \mathrm{d}x_2, \cdots, \mathrm{d}x_n$$

Proof. Let $v = 1$ in the second Green formula. An application of the second Green formula yields

$$\int_{\Omega} \Delta u\,\mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,\mathrm{d}S\,.$$

Since $\Delta u = f$, we arrive at $\int_{\Omega} f\,\mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,\mathrm{d}S = A$.

Example 3. Suppose that $u^* \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution of

$$\begin{cases} \Delta u = f(x)\,,\ x = (x_1, x_2, \cdots, x_n) \in \Omega\,, \\ u|_{\partial\Omega} = c\,,\quad \int_{\partial\Omega} \dfrac{\partial u}{\partial n}\,\mathrm{d}S = A\,, \end{cases} \tag{7.91}$$

where c is a undetermined constant and A is a given constant. Show that all solutions of PDS (7.91) can be written as

$$u = u^* + a\,,\quad a \text{ is a constant}.$$

Proof. Substituting $u = u^* + a$ into PDS (7.91) yields

$$\begin{cases} \Delta(u^* + a) = \Delta u^* = f(x), \quad x \in \Omega, \\ (u^* + a)|_{\partial\Omega} = u^*|_{\partial\Omega} + a = c + a, \\ \int_{\partial\Omega} \frac{\partial(u^*+a)}{\partial n} \mathrm{d}S = \int_{\partial\Omega} \frac{\partial u}{\partial n} \mathrm{d}S = A, \end{cases}$$

so that $u = u^* + a$ are indeed solutions of PDS (7.91).

Let u_1 be a solution of PDS (7.91). The $v = u_1 - u^*$ must thus satisfy

$$\begin{cases} \Delta v = 0, \quad x \in \Omega, \\ v|_{\partial\Omega} = c', \int_{\partial\Omega} \frac{\partial v}{\partial n} \mathrm{d}S = 0, \end{cases}$$

where c' is a undetermined constant. By the first Green formula in n-dimensional space,

$$\int_\Omega u \Delta v \, \mathrm{d}x = \int_{\partial\Omega} u \frac{\partial v}{\partial n} \mathrm{d}S - \int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) \mathrm{d}x.$$

Therefore

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} \mathrm{d}S = \int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) \mathrm{d}x.$$

Let $u = v$. Thus, by applying the boundary condition

$$\int_\Omega \left[\sum_{i=1}^n \left(\frac{\partial v_i}{\partial x_i} \right)^2 \right] \mathrm{d}x = \int_{\partial\Omega} v \frac{\partial v}{\partial n} \mathrm{d}S = \int_{\partial\Omega} c' \frac{\partial v}{\partial n} \mathrm{d}S = c' \int_{\partial\Omega} \frac{\partial v}{\partial n} \mathrm{d}S = 0,$$

so that $\dfrac{\partial v_i}{\partial x_i} = 0$, $i = 1, 2, \cdots, n$. Therefore $v = u_1 - u^*$ must be equivalent to a constant a.

7.6 Green Functions

In this section we introduce Green functions of the Laplace operator and discuss their properties.

7.6.1 Green Function

The third Green formula reads (see Eq. (7.65))

$$u(M_0) = -\frac{1}{4\pi}\iint\limits_{\partial\Omega}\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u(M)}{\partial n}\right]\mathrm{d}S - \frac{1}{4\pi}\iiint\limits_{\Omega}\frac{\Delta u}{r}\,\mathrm{d}\Omega\,, \tag{7.92}$$

where M_0 and M are two points in Ω, and their distance r is

$$r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\,.$$

If $u(M)$ is a harmonic function, Eq. (7.92) reduces to

$$u(M_0) = -\frac{1}{4\pi}\iint\limits_{\partial\Omega}\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u(M)}{\partial n}\right]\mathrm{d}S\,. \tag{7.93}$$

Note that the right-hand side of Eq. (7.93) contains both $u(M)|_{\partial\Omega}$ and $\left.\dfrac{\partial u(M)}{\partial n}\right|_{\partial\Omega}$.
In order to use Eq. (7.93) for expressing solutions of Dirichlet or Neumann problems of potential equations, we must eliminate $\left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega}$ or $u|_{\partial\Omega}$ in Eq. (7.93).

Consider a harmonic function $g(M,M_0)$ in Ω with $M_0 \in \Omega$ as the parameter and $M \in \Omega$ as the variable. By letting $v = g$, the second Green formula yields

$$\iint\limits_{\partial\Omega}\left(g\frac{\partial u}{\partial n} - u\frac{\partial g}{\partial n}\right)\mathrm{d}S = \iiint\limits_{\Omega} g\Delta u\,\mathrm{d}\Omega\,. \tag{7.94a}$$

For a harmonic function $u(M)$, Eq. (7.94a) reduces to

$$\iint\limits_{\partial\Omega}\left(g\frac{\partial u}{\partial n} - u\frac{\partial g}{\partial n}\right)\mathrm{d}S = 0\,. \tag{7.94b}$$

Subtracting Eq. (7.94b) from Eq. (7.93) yields

$$u(M_0) = \iint\limits_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(g - \frac{1}{4\pi r}\right) + \left(\frac{1}{4\pi r} - g\right)\frac{\partial u}{\partial n}\right]\mathrm{d}S\,. \tag{7.95}$$

Consider a function g that satisfies

$$\begin{cases} \Delta g = 0\,, \quad M \in \Omega\,, \\ g|_{\partial\Omega} = \dfrac{1}{4\pi r}\,, \end{cases}$$

and removes $\left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega}$ in Eq. (7.95). Eq. (7.93) reduces to

$$u(M_0) = -\iint\limits_{\partial\Omega} u\frac{\partial G}{\partial n}\,\mathrm{d}S\,, \tag{7.96}$$

where

$$G(M,M_0) = \frac{1}{4\pi r} - g(M,M_0) \tag{7.97}$$

is called the *Green function of the Laplace operator in Ω for the first kind of boundary-value problems*.

Let $\Delta u = F(M)(M \in \Omega)$ so u is the solution of Poisson equations. Subtracting Eq. (7.94a) from Eq. (7.92) yields

$$u(M_0) = -\iint\limits_{\partial\Omega} u\frac{\partial G}{\partial n}\,\mathrm{d}S - \iiint\limits_{\Omega} G(M,M_0)F(M)\,\mathrm{d}\Omega\,. \tag{7.98}$$

Once the Green function $G(M,M_0)$ is available, the solutions of

$$\begin{cases} \Delta u = 0\,, \quad M \in \Omega\,, \\ u|_{\partial\Omega} = f(M)\,, \end{cases} \tag{7.99}$$

$$\begin{cases} \Delta u = F(M)\,, \quad M \in \Omega\,, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{7.100}$$

and

$$\begin{cases} \Delta u = F(M)\,, \quad M \in \Omega\,, \\ u|_{\partial\Omega} = f(M) \end{cases} \tag{7.101}$$

are thus

$$u(M_0) = -\iint\limits_{\partial\Omega} f(M)\frac{\partial G}{\partial n}\,\mathrm{d}S\,, \quad u(M_0) = -\iiint\limits_{\Omega} G(M,M_0)F(M)\,\mathrm{d}\Omega\,,$$

and

$$u(M_0) = -\iint\limits_{\partial\Omega} f(M)\frac{\partial G}{\partial n}\,\mathrm{d}S - \iiint\limits_{\Omega} G(M,M_0)F(M)\,\mathrm{d}\Omega\,,$$

respectively. Therefore, the Green function plays a critical role in solving Dirichlet problems of potential equations. Here we list analytical definitions of Green functions for three kinds of boundary conditions.

Definition 1. Let Ω be a bounded domain with a piecewise smooth boundary $\partial\Omega$. $g(M,M_0)$ is the solution of

$$\begin{cases} \Delta g = 0\,, & M \in \Omega\,, \\ g|_{\partial\Omega} = \dfrac{1}{4\pi r}\,, & r \text{ is distance between } M_0 \text{ and } M\,. \end{cases} \tag{7.102}$$

$G(M,M_0)$ is defined by

$$G(M,M_0) = \frac{1}{4\pi r} - g(M,M_0) \tag{7.103}$$

and is called the *Green function of the three-dimensional Laplace operator in Ω for the first kind of boundary-value problems*.

Note that the Green function is independent of $f(M)$ and $F(M)$ in Eq. (7.101). Equations (7.102) and (7.103) are equivalent to

$$\begin{cases} -\Delta G(M,M_0) = \delta(M-M_0)\,, & M \in \Omega\,, \\ G(M,M_0)|_{\partial\Omega} = 0\,. \end{cases} \tag{7.104}$$

Thus $G(M,M_0)$ is also called the fundamental solution of Laplace equations for the first kind of boundary-value problems. It reflects the effect of the point source at M_0 on the solution at $M \in \Omega$.

Definition 2. Let Ω be a bounded domain with a piecewise smooth boundary $\partial\Omega$, and $g(M,M_0)$ is the solution of

$$\begin{cases} \Delta g = 0\,, & M \in \Omega\,, \\ \left.\dfrac{\partial g}{\partial n}\right|_{\partial\Omega} = \dfrac{1}{4\pi}\dfrac{\partial}{\partial n}\left(\dfrac{1}{r}\right)\,. \end{cases}$$

Then $G(M,M_0)$ is defined by

$$G(M,M_0) = \frac{1}{4\pi r} - g(M,M_0)$$

or

$$\begin{cases} -\Delta G(M,M_0) = \delta(M,M_0)\,, & M \in \Omega\,, \\ \left.\dfrac{\partial G}{\partial n}\right|_{\partial\Omega} = 0 \end{cases} \tag{7.105}$$

and is called the *Green function of the three-dimensional Laplace operator in Ω for the second kind of boundary-value problems*.

Definition 2 comes from a similar argument to that used in Definition 1, by eliminating $u|_{\partial\Omega}$ in Eq. (7.92). However, PDS (7.105) has no solution. In heat conduction, the boundary condition $\frac{\partial G}{\partial n}\Big|_{\partial\Omega} = 0$ means that the boundary is well-insulated. The heat generated by the internal source $\delta(M,M_0)$ cannot flow through the boundary. Thus we cannot have a steady temperature field so the $G(M,M_0)$ satisfying PDS (7.105) does not exist.

Definition 3. Let Ω be a bounded domain with a piecewise smooth boundary $\partial\Omega$, and $g(M,M_0)$satisfies

$$\begin{cases} \Delta g = 0, \quad M \in \Omega\,, \\ \left(\dfrac{\partial g}{\partial n} + \sigma g\right)\Big|_{\partial\Omega} = \dfrac{1}{4\pi}\left[\dfrac{\partial}{\partial n}\left(\frac{1}{r}\right) + \sigma\dfrac{1}{r}\right], \quad \sigma > 0\,. \end{cases}$$

Then $G(M,M_0)$ is defined by

$$G(M,M_0) = \frac{1}{4\pi r} - g(M,M_0)$$

or

$$\begin{cases} -\Delta G(M,M_0) = \delta(M - M_0)\,, \quad M \in \Omega\,, \\ \left(\dfrac{\partial G}{\partial n} + \sigma G\right)\Big|_{\partial\Omega} = 0\,. \end{cases} \tag{7.106}$$

and is called the *Green function of the three-dimensional Laplace operator in Ω for the third kind of boundary-value problems.*

The Green functions defined by Eq. (7.104) can be obtained for some simple regular domain Ω, and can be used to solve Dirichlet problems of potential equations. However, the Green functions defined by Eq. (7.105) and (7.106) are either non-existent or very difficult to find. Therefore, we use the Green functions only for solving the Dirichlet problems in some simple regular domains, and we use the other methods such as the transformation of potential equations into integral equations for the Neumann and Robin problems.

7.6.2 Properties of Green Functions of the Dirichlet Problems

1. $G(M,M_0)$ tends to $+\infty$ with the same order of $\dfrac{1}{r}$ as $M \to M_0$.

   Proof. Note that $r \to 0$ as $M \to M_0$. Since $g(M,M_0)$ is a harmonic function in Ω and $\lim\limits_{M\to M_0} \frac{1}{4\pi r} = +\infty$, $g(M,M_0)$ is continuous at M_0. Thus

$$\lim_{M\to M_0} G/(1/r) = \lim_{M\to M_0}\left(\frac{1}{4\pi} - rg\right) = \frac{1}{4\pi}\,.$$

2. In Ω

$$0 < G(M,M_0) < \frac{1}{4\pi r}\,. \tag{7.107}$$

Proof. Applying the extremum principle to PDS (7.102) yields

$$g(M,M_0) > 0 \quad \text{or} \quad G(M,M_0) < \frac{1}{4\pi r}\,, \quad M \in \Omega\,.$$

By the definition of $G(M,M_0)$ (Eq. (7.103)), we have

$$\begin{cases} \Delta G(M,M_0) = 0\,, \quad M \in \Omega \backslash V_\varepsilon^{M_0}\,, \\ G(M,M_0)|_{\partial\Omega} = 0\,, \\ G(M,M_0)|_{S_\varepsilon^{M_0}} = \frac{1}{4\pi r} - g(M,M_0) > 0\,, \end{cases}$$

where $V_\varepsilon^{M_0}$ is a sphere of center M_0 and radius ε, ε is a sufficiently small constant and $S_\varepsilon^{M_0}$ is the boundary of the $V_\varepsilon^{M_0}$. By the extremum principle, for $M \in \Omega \backslash V_\varepsilon^{M_0}$ we have

$$0 < G(M,M_0)\,.$$

Since ε can be arbitrarily small, for $M \in \Omega$ we obtain

$$0 < G(M,M_0) < \frac{1}{4\pi r}\,.$$

3.

$$\iint\limits_{\partial\Omega} \frac{\partial G}{\partial n}\,\mathrm{d}S = -1\,.$$

Proof. Consider the domain enclosed by $\partial\Omega$ and $S_\varepsilon^{M_0}$ in Ω, where $G(M,M_0)$ is a harmonic function. By Theorem 1 in Section 7.4.3, we have

$$\iint\limits_{\partial\Omega \cup S_\varepsilon^{M_0}} \frac{\partial G}{\partial n}\,\mathrm{d}S = 0\,,$$

where n is the external normal on the boundary surface. Thus

$$\iint\limits_{\partial\Omega} \frac{\partial G}{\partial n}\,\mathrm{d}S = -\iint\limits_{S_\varepsilon^{M_0}} \frac{\partial G}{\partial n}\,\mathrm{d}S = -\iint\limits_{S_\varepsilon^{M_0}} \frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right)\mathrm{d}S + \iint\limits_{S_\varepsilon^{M_0}} \frac{\partial g}{\partial n}\,\mathrm{d}S$$

$$= \frac{1}{4\pi}\iint\limits_{S_R^{M_0}} \frac{\partial(\frac{1}{r})}{\partial r}\,\mathrm{d}S = \frac{1}{4\pi}\iint\limits_{S_R^{M_0}} -\left(\frac{1}{r^2}\right)\mathrm{d}S$$

$$= \frac{1}{4\pi} \cdot \left(-\frac{1}{\varepsilon^2}\right) \cdot 4\pi\varepsilon^2 = -1\,.$$

4. For any two points $M_1, M_2 \in \Omega$, $G(M_1,M_2) = G(M_2,M_1)$ so the Green function is symmetric. While this property can be rigorously proven, we demonstrate it here physically. Consider an electric field generated by the point electric charge of capacity ε at M_0. The electric potential is $\frac{1}{4\pi r}$ at M, where r is the distance between M_0 and M. If M_0 is located in a grounded hollow conductor, the electric potential at M will be a superposition of two parts: $\frac{1}{4\pi r}$ from the point electric charge at M_0 and $-g(M,M_0)$ from the induced electric charges on the inner wall of the conductor. Since this additional field has no electric charge inside the hollow space of the conductor, we have

$$\Delta g = 0\,.$$

Since the conductor is grounded, the electric potential is zero on the surface $\partial\Omega$ of the conductor. Therefore, the $g(M,M_0)$ satisfies

$$\left.\left(\frac{1}{4\pi r} - g\right)\right|_{\partial\Omega} = 0 \quad \text{or} \quad g|_{\partial\Omega} = \frac{1}{4\pi r}\,.$$

Note that $G(M,M_0) = \frac{1}{4\pi r} - g(M,M_0)$ is the Green function. Therefore, it represents the electrical potential at M of the electric field generated by a point electric charge at M_0 inside a grounded hollow conductor. The symmetry of the Green function agrees with the principle of reciprocity in physics, which states that the electrical potential at M_2 of an electric field generated by a point electric charge at M_1 is the same as that at M_1 of the electric field due to a point electric charge of the same capacity ε at M_2.

Remark 1. If the additional potential due to the induced electric charges is denoted by $g(M,M_0)$, the Green function becomes

$$G(M,M_0) = \frac{1}{4\pi r} + g(M,M_0)\,.$$

Remark 2. In a two-dimensional plane domain, the Green function reads

$$G(M,M_0) = \frac{1}{2\pi}\ln\frac{1}{r} - g(M,M_0)\,,$$

where $g(M,M_0)$ satisfies

$$\begin{cases} \Delta g = 0\,, \quad M \in \Omega \subset R^2\,, \\ g|_{\partial\Omega} = \dfrac{1}{2\pi}\ln\frac{1}{r}\,. \end{cases}$$

Similar to its three-dimensional counterpart, the two-dimensional Green function can also be defined by

$$G(M,M_0) = \frac{1}{2\pi}\ln\frac{1}{r} + g(M,M_0)\,.$$

7.7 Method of Green Functions for Boundary-Value Problems of the First Kind

Once the Green function is available in domain Ω, the solution of

$$\begin{cases} \Delta u = F(M)\,, \quad M \in \Omega\,, \\ u|_{\partial\Omega} = f(M)\,. \end{cases} \tag{7.108}$$

is thus

$$u(M_0) = -\iint\limits_{\partial\Omega} f(M)\frac{\partial G}{\partial n}\,\mathrm{d}S - \iiint\limits_{\Omega} G(M,M_0)F(M)\,\mathrm{d}\Omega\,. \tag{7.109}$$

In this section we develop the Green functions for some domains Ω.

7.7.1 *Mirror Image Method for Finding Green Functions*

For some domains with certain kinds of symmetry, we can find the electric potential $g(M,M_0)$ of the electric field generated by induced electric charges. Thus we may find the Green function, using their physical implications discussed in Section 7.6.

The mirror image method attempts to locate a mirror (symmetric) point M_1 of M_0 outside the hollow conductor. Image that there is a point electric charge with capacity not necessarily equal to ε at M_1 such that its induced electric field completely neutralizes the $\frac{1}{4\pi r}$ due to the charge at M_0, and the electric potential becomes zero on $\partial\Omega$. Once the electric potential g (inside Ω) of electric field due to the charge at M_1 is known, we can obtain the Green function

$$G(M,M_0) = \frac{1}{4\pi r} - g(M,M_0)\,. \tag{7.110}$$

7.7.2 *Examples Using the Method of Green Functions*

Example 1. Find the Green function in a spherical domain $\Omega : x^2+y^2+z^2 < R^2$ and the solution of the Dirichlet problem in Ω.

Solution. Put a point electric charge of capacity ε at $M_0(x_0, y_0, z_0)$ inside Ω. Locate the symmetric point M_1 of M_0 with respect to the spherical surface $\partial\Omega(r = R)$ along the ray OM_0 and outside the spherical surface such that

$$r_0 r_1 = r_{OM_0} r_{OM_1} = R^2. \tag{7.111}$$

Here $r_0 = r_{OM_0}$ and $r_1 = r_{OM_1}$ are the distances of M_0 and M_1 from the center O of the sphere, respectively. (Fig. 7.2)

For any point P on $\partial\Omega$, the ΔOPM_0 and the ΔOPM_1 have a common angle $\angle POM_1$. Its sides are also, by Eq. (7.111), proportional to each other. Therefore, $\Delta OPM_0 \backsim U\Delta OM_1P$. Thus

$$r_{M_1P} = \frac{R}{r_0} r_{M_0P}$$

or

$$\frac{1}{r_{M_0P}} - \frac{R}{r_0}\frac{1}{r_{M_1P}} = 0.$$

Therefore, we have

$$\frac{1}{4\pi r_{M_0P}} - \frac{R}{4\pi r_0}\frac{1}{r_{M_1P}} = 0. \tag{7.112}$$

This holds for all P on $\partial\Omega$, and thus the induced electric field by a point electric charge of capacity $\frac{R}{r_0}\varepsilon$ at M_1 can completely neutralize that due to the charge at M_0 on $\partial\Omega$. The electric potential generated by such a charge is, for $M \in \Omega$,

$$g(M, M_0) = \frac{R}{4\pi r_0}\frac{1}{r_{M_1M}}.$$

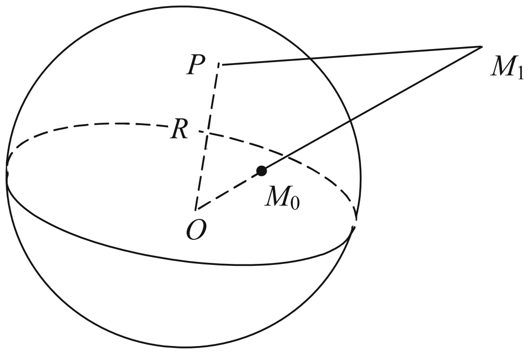

Fig. 7.2 Symmetric point M_1 of M_0

It is clear that $\Delta g = 0$ and $g|_{\partial\Omega} = \dfrac{1}{4\pi r_{M_0M}}$ by Eq. (7.112). Finally, the Green function is

$$G(M,M_0) = \frac{1}{4\pi}\left[\frac{1}{r_{M_0M}} - \frac{R}{r_0}\frac{1}{r_{M_1M}}\right]. \tag{7.113}$$

Let $(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$ be the spherical coordinates of point M in Ω. Here $r < R$, $0 \le \theta \le \pi, 0 \le \varphi \le 2\pi$. Note that

$$r_{M_0M} = \sqrt{r_0^2 + r^2 - 2r_0 r\cos\psi}, \quad r_{M_1M} = \sqrt{r_1^2 + r^2 - 2r_1 r\cos\psi}, \quad r_0 r_1 = R^2,$$

where $r = r_{OM}$, ψ is the angle between OM_0 (or OM_1) and OM. Thus

$$G(M,M_0) = \frac{1}{4\pi}\left[\frac{1}{\sqrt{r_0^2 + r^2 - 2r_0 r\cos\psi}} - \frac{R}{\sqrt{r_0^2 r^2 - 2R^2 r_0 r\cos\psi + R^4}}\right],$$

and, by noting that the external normal of $\partial\Omega$ is along $r = r_{OM}$,

$$\begin{aligned}
\left.\frac{\partial G}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial G}{\partial r}\right|_{r=R} &= -\frac{1}{4\pi}\left[\frac{r - r_0\cos\psi}{(r_0^2 + r^2 - 2r_0 r\cos\psi)^{3/2}}\right.\\
&\quad \left.\left. - \frac{(r_0^2 r - R^2 r_0\cos\psi)R}{(r_0^2 r^2 - 2R^2 r_0 r\cos\psi + R^4)^{3/2}}\right]\right|_{r=R}\\
&= -\frac{1}{4\pi R}\frac{R^2 - r_0^2}{(r_0^2 + R^2 - 2r_0 R\cos\psi)^{3/2}}.
\end{aligned}$$

Hence the solution of

$$\begin{cases} \Delta u(r,\theta,\varphi) = F(r,\theta,\varphi), \quad 0 < r < R, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi, \\ u|_{r=R} = f(R,\theta,\varphi). \end{cases} \tag{7.114}$$

is

$$u(M_0) = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{R^2 - r_0^2}{(r_0^2 + R^2 - 2r_0 R\cos\psi)^{3/2}} f(R,\theta,\varphi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

$$-\frac{1}{4\pi}\int_0^R\int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{\sqrt{r_0^2 + r^2 - 2r_0 r\cos\psi}}\right.$$

$$-\frac{R}{\sqrt{r_0^2r^2-2R^2r_0r\cos\psi+R^4}}\Bigg]F(r,\theta,\varphi)r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r\,.$$

Here the coordinates of M_0 are (r_0,θ_0,φ_0) and the area element $\mathrm{d}S$ on surface $r=R$ is

$$\mathrm{d}S=R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,.$$

Remark 1. When $F(M)=0$, the solution reduces to

$$u(r_0,\theta_0,\varphi_0)=\frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{R^2-r_0^2}{(r_0^2+R^2-2r_0R\cos\psi)^{3/2}}$$

$$\cdot f(R,\theta,\varphi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{7.115}$$

which is called the *Poisson formula for problems in a sphere*.

Remark 2. Let $\mathbf{e}_0$ and $\mathbf{e}$ be the unit vectors of OM_0 and OM, respectively. Since

$$\mathbf{e}_0=(\sin\theta_0\cos\varphi_0,\sin\theta_0\sin\varphi_0,\cos\theta_0)\,,$$
$$\mathbf{e}=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta),$$

we have

$$\begin{aligned}\cos\psi=\mathbf{e}_0\cdot\mathbf{e}&=\sin\theta\sin\theta_0(\cos\varphi\cos\varphi_0+\sin\varphi\sin\varphi_0)+\cos\theta\cos\theta_0\\&=\sin\theta\sin\theta_0\cos(\varphi-\varphi_0)+\cos\theta\cos\theta_0\,.\end{aligned}$$

Remark 3. For the external problems in the domain $x^2+y^2+z^2>R^2$,

$$\frac{\partial G}{\partial n}=-\frac{\partial G}{\partial r}\,.$$

The solution of external problems thus differs from that in Eq. (7.115) only by a sign, i.e.

$$u(r_0,\theta_0,\varphi_0)=\frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{r_0^2-R^2}{(r_0^2+R^2-2r_0R\cos\psi)^{3/2}}f(R,\theta,\varphi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,,$$

where $M_0(r_0,\theta_0,\varphi_0)$ is a point outside the sphere $\Omega: x^2+y^2+z^2<R^2$.

Example 2. Find the Green function in the upper half space $\Omega : z > 0$ and solve the boundary-value problem

$$\begin{cases} \Delta u = F(M)\,, \quad M \in \Omega\,, \\ u|_{z=0} = f(M)\,. \end{cases} \tag{7.116}$$

Solution. For any point $M_0(x_0, y_0, z_)$ $\in \Omega$, its symmetric point with respect to the boundary $z = 0$ is $M_1(x_0, y_0, -z_0)$. The Green function can be readily obtained as

$$G(M, M_0) = \frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right.$$

$$\left. - \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right],$$

where $M(x,y,z) \in \Omega$. Clearly, the G satisfies $G|_{z=0} = 0$. Also,

$$\left.\frac{\partial G}{\partial n}\right|_{z=0} = -\left.\frac{\partial G}{\partial z}\right|_{z=0} = -\frac{z_0}{2\pi}\frac{1}{[(x-x_0)^2+(y-y_0)^2+z_0^2]^{3/2}}\,.$$

Thus the solution of PDS (7.116) is

$$u(M_0) = -\iint\limits_{z=0} f(x,y,0)\frac{\partial G}{\partial n}\,\mathrm{d}S - \iiint\limits_{z>0} G(M, M_0)F(x,y,z)\,\mathrm{d}\Omega$$

$$= \frac{z_0}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(x,y,0)}{\left[(x-x_0)^2+(y-y_0)^2+z_0^2\right]^{3/2}}\,\mathrm{d}x\,\mathrm{d}y$$

$$-\frac{1}{4\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right.$$

$$\left. - \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right]F(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

Example 3. Find the Green function in the quarter space $\Omega : x > 0, y > 0, -\infty < z < +\infty$.

Solution. Consider a point electric charge of capacity ε at $M_0 \in \Omega$. The electric potential at $M(x,y,z) \in \Omega$ is thus $\frac{1}{4\pi r_{M_0M}}$. Here

$$r_{M_0M} = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\,.$$

The electrical potential on the boundary $y=0$ and $x=0$ can be completely neutralized by the electric fields due to the point electric charges of capacity ε, ε and $-\varepsilon$ at $M_1(x_0,-y_0,z_0)$, $M_2(-x_0,y_0,z_0)$ and $M_3(-x_0,-y_0,z_0)$, respectively, which are the symmetric points of M_0. The electric potential of the additional field at point M is

$$g(M,M_0)=\frac{1}{4\pi}\left[\frac{1}{r_{M_1M}}+\frac{1}{r_{M_2M}}-\frac{1}{r_{M_3M}}\right].$$

Clearly, $\Delta g=0,\ g|_{\partial\Omega}=\frac{1}{4\pi}\frac{1}{r_{M_0M}}$. Thus the Green function is

$$\begin{aligned}G(M,M_0)=&\frac{1}{4\pi}\frac{1}{r_{M_0M}}-g(M,M_0)\\ =&\frac{1}{4\pi}\left[\frac{1}{r_{M_0M}}-\frac{1}{r_{M_1M}}-\frac{1}{r_{M_2M}}+\frac{1}{r_{M_3M}}\right].\end{aligned}$$

It can be used to obtain the solution of

$$\begin{cases}\Delta u=F(M), & M\in\Omega, x>0, y>0, -\infty<z<+\infty,\\ u|_{\partial\Omega}=f(M).\end{cases}$$

Example 4. By using the method of Green functions, find the solution of

$$\begin{cases}\Delta u=0, & x^2+y^2<R^2,\\ u|_{x^2+y^2=R^2}=F(x,y).\end{cases}\tag{7.117}$$

Solution. In a two-dimensional plane domain, from Section 7.6 the Green function is

$$G(M,M_0)=\frac{1}{2\pi}\ln\frac{1}{r_{M_0M}}-g(M,M_0),$$

where r_{M_0M} is the distance between M_0 and M and the $g(M,M_0)$ satisfies

$$\begin{cases}\Delta g=0, & M \text{ and } M_0 \text{ are inside the circle } r=R,\\ g|_{r=R}=\dfrac{1}{2\pi}\ln\dfrac{1}{r_{M_0M}}.\end{cases}\tag{7.118}$$

To obtain the solution of PDS (7.118), consider the symmetric point $M_1(r_1\cos\theta_0,r_1\sin\theta_0)$ of $M_0(r_0\cos\theta_0,r_0\sin\theta_0)$ with respect to the circle $r=R$ (Fig. 7.3) such that $r_0r_1=R^2$. Note that

$$r_{M_0M}=\sqrt{r_0^2+r^2-2rr_0\cos(\theta-\theta_0)},\quad r_{M_1M}=\sqrt{r_1^2+r^2-2rr_1\cos(\theta-\theta_0)}.$$

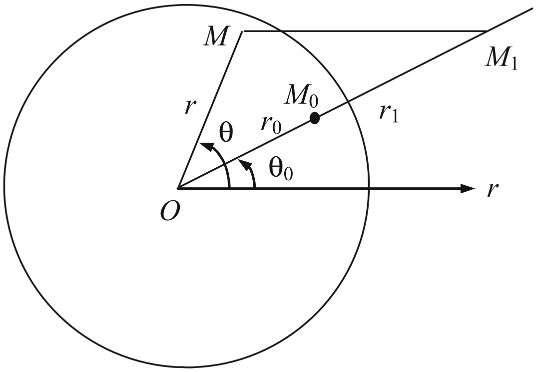

Fig. 7.3 Symmetric point M_1 $(r_1 \cos\theta_0\,,\ r_1\theta_0)$ of M_0 $(r_0 \cos\theta_0\,,\ r_0\theta_0)$

For any point P on the circle $r = R$, we have $\Delta OM_0P \backsim \Delta OPM_1$ so that

$$\frac{r_0}{R} = \frac{R}{r_1} = \frac{r_{M_0P}}{r_{M_1P}}\,.$$

Therefore

$$\frac{1}{r_{M_0P}} = \frac{R}{r_0}\frac{1}{r_{M_1P}} \quad \text{or} \quad \frac{1}{2\pi}\ln\frac{1}{r_{M_0P}} = \frac{1}{2\pi}\ln\left[\frac{R}{r_0}\frac{1}{r_{M_1P}}\right],$$

which yields

$$g(M, M_0) = \frac{1}{2\pi}\ln\left[\frac{R}{r_0}\frac{1}{r_{M_1M}}\right].$$

Finally, the Green function is $G(M, M_0) = \dfrac{1}{2\pi}\left[\ln\dfrac{1}{r_{M_0P}} - \ln\left(\dfrac{R}{r_0}\dfrac{1}{r_{M_1M}}\right)\right]$.

Also

$$\left.\frac{\partial G}{\partial n}\right|_{r=R} = \left.\frac{\partial G}{\partial r}\right|_{r=R} = -\frac{1}{2\pi R}\frac{R^2 - r_0^2}{R^2 - 2Rr_0\cos(\theta - \theta_0) + r_0^2}\,.$$

Thus the solution of PDS (7.117) in a polar coordinate system is

$$u(r_0, \theta_0) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0\cos(\theta - \theta_0)}\,\mathrm{d}\theta\,. \tag{7.119}$$

This is the same as that obtained in Section 7.2 by the method of separation of variables and in Section 7.4 by the relation between harmonic functions and analytical functions.

Example 5. Find the harmonic function in a circle of radius R and with its boundary value $k_0 \cos\theta$ on the circle $r = R$. Here k_0 is a given constant.

Solution. The required harmonic function is the solution of

$$\begin{cases} \Delta u(r,\theta) = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}, & 0 < r < R, \\ u(R,\theta) = k_0 \cos\theta. \end{cases} \tag{7.120}$$

By the Poisson formula (7.119), the solution of PDS (7.120) is

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} k_0 \cos\theta' \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta' - \theta)}\,\mathrm{d}\theta' \\ &= \frac{k_0}{2\pi}\int_0^{2\pi} \cos\theta' \left[1 + 2\sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k \cos k\left(\theta' - \theta\right)\right]\mathrm{d}\theta' \\ &= \frac{k_0}{\pi}\int_0^{2\pi} \cos\theta' \left[\sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k \left(\cos k\theta' \cos k\theta + \sin k\theta' \sin k\theta\right)\right]\mathrm{d}\theta' \\ &= \frac{k_0}{\pi}\int_0^{2\pi} \frac{r}{R}\cos\theta \cos^2\theta'\,\mathrm{d}\theta' = \frac{k_0}{R} r\cos\theta, \end{aligned}$$

where we have used Remark 1 in Example 1 in Section 7.2.

Remark 1. We can also obtain the solution of PDS (7.120) by using the method of undetermined coefficients. It is straightforward to show that $u(r,\theta) = Ar\cos\theta$, where A is a constant, satisfies the Laplace equation in PDS (7.120). Applying the boundary condition $u(r,\theta) = AR\cos\theta = k_0\cos\theta$ yields $A = \frac{k_0}{R}$. The solution of PDS (7.120) is thus

$$u(r,\theta) = \frac{k_0}{R} r\cos\theta.$$

Remark 2. In a Cartesian coordinate system, PDS (7.120) reads

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & x^2 + y^2 < R^2, \\ u|_{x^2+y^2=R^2} = k_0 \dfrac{x}{R}. \end{cases}$$

Consider the solution $u = Ax$. Applying the boundary condition yields $A = \frac{k_0}{R}$. Therefore the solution is

$$u = \frac{k_0}{R}x = \frac{k_0}{R} r\cos\theta.$$

Example 6. Find the Green function in the upper half plane $\Omega : y > 0$ and the solution of

$$\begin{cases} \Delta u = F(M), \quad M(x,y) \in \Omega, \\ u|_{y=0} = f(x). \end{cases} \tag{7.121}$$

Solution. Consider the electric field in a medium of unit dielectric constant generated by an infinite electric wire passing perpendicularly through point $M_0(x_0,y_0) \in \Omega$ and with a density of unit positive electric charge. Thus the electric potential at $M(x,y) \in \Omega$ is, by the theory of static electric fields, $\frac{1}{2\pi}\ln\frac{1}{r_{M_0M}}$. To have a vanished potential at the boundary $y = 0$, we require an additional electrical field generated from an infinite wire passing perpendicularly through point $M_1(x_0,-y_0)$ (the symmetric point of M_0 with respect to Ox-axis) and with a density of unit negative electric charge. The potential of this additional field is $-\frac{1}{2\pi}\ln\frac{1}{r_{M_1M}}$ at point M. Thus, the Green function is

$$G(M,M_0) = \frac{1}{2\pi}\left[\ln\frac{1}{r_{M_0M}} - \ln\frac{1}{r_{M_1M}}\right] = \frac{1}{2\pi}\ln\frac{r_{M_1M}}{r_{M_0M}},$$

where

$$r_{M_0M} = \sqrt{(x-x_0)^2+(y-y_0)^2}, \; r_{M_1M} = \sqrt{(x-x_0)^2+(y+y_0)^2}.$$

Also

$$\begin{aligned} \left.\frac{\partial G}{\partial n}\right|_{y=0} = -\left.\frac{\partial G}{\partial y}\right|_{y=0} &= -\frac{1}{2\pi}\left.\left[-\frac{\partial}{\partial y}\ln r_{M_0M} + \frac{\partial}{\partial y}\ln r_{M_1M}\right]\right|_{y=0} \\ &= \frac{1}{2\pi}\left.\left[-\frac{1}{r_{M_0M}}\frac{y-y_0}{r_{M_0M}} + \frac{1}{r_{M_1M}}\frac{y+y_0}{r_{M_1M}}\right]\right|_{y=0} \\ &= -\frac{y_0}{\pi}\frac{1}{(x-x_0)^2+y_0^2}. \end{aligned}$$

Applying Eq. (7.109) in the plane domain Ω yields

$$\begin{aligned} u(x_0,y_0) = &\frac{y_0}{\pi}\int_{-\infty}^{+\infty}\frac{f(x)}{(x-x_0)^2+y_0^2}\,\mathrm{d}x \\ &+ \frac{1}{2\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty}F(x,y)\ln\sqrt{\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}}\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

or

$$\begin{aligned} u(x,y) = &\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(\xi)}{(x-\xi)^2+y_0^2}\,\mathrm{d}\xi \\ &+ \frac{1}{2\pi}\int_0^{+\infty}\mathrm{d}\eta\int_{-\infty}^{+\infty}F(\xi,\eta)\ln\sqrt{\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}}\,\mathrm{d}\xi. \end{aligned}$$

7.7.3 Boundary-Value Problems in Unbounded Domains

We have proven the third Green formula (Eq. (7.65)) and the formula (7.109) for a bounded domain Ω, and used them in some examples with unbounded domains in Section 7.7.2. Here we discuss the conditions for their validation for unbounded domains.

Fundamental Integral Formula

Theorem 1. Let Ω' be an unbounded domain with a piecewise smooth boundary $\partial\Omega'$. Suppose that $u(M) = u(x,y,z) \in C^1(\bar{\Omega}') \cap C^2(\Omega')$, and as $r_{OM} \to \infty$

$$u(M) = O\left(\frac{1}{r_{OM}}\right), \quad \frac{\partial u}{\partial n} = O\left(\frac{1}{r_{OM}^2}\right). \tag{7.122}$$

Here the r_{OM} is the distance between the origin O and point M, and

$$\bar{\Omega}' = \Omega' \cup \partial\Omega'.$$

Thus the third Green formula is valid,

$$u(M_0) = -\frac{1}{4\pi}\iint\limits_{\partial\Omega'}\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right]\mathrm{d}S - \frac{1}{4\pi}\iiint\limits_{\Omega'}\frac{\Delta u}{r}\,\mathrm{d}\Omega\,, \tag{7.123}$$

where $M_0 \in \Omega'$ and r is the distance between M_0 and M.

Proof. Consider two spherical surfaces $S_\varepsilon^{M_0}$ and $S_R^{M_0}$ with sufficiently small ε and sufficiently large R such that the $\partial\Omega'$ is contained in the region of boundary $S_\varepsilon^{M_0}$ and $S_R^{M_0}$ (Fig. 7.4). Let Ω^* be the region formed by $S_\varepsilon^{M_0}$, $\partial\Omega'$ and $S_R^{M_0}$. The Ω^* reduces to Ω' as $\varepsilon \to 0$ and $R \to \infty$. With $v = \dfrac{1}{r_{M_0M}} = \dfrac{1}{r}$, applying the second Green formula yields

$$\iint\limits_{\partial\Omega'\cup S_\varepsilon^{M_0}\cup S_R^{M_0}}\left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right]\mathrm{d}S = -\iiint\limits_{\Omega^*}\frac{\Delta u}{r}\,\mathrm{d}\Omega\,. \tag{7.124}$$

Note that

$$\iint\limits_{S_\varepsilon^{M_0}}\left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right]\mathrm{d}S = \iint\limits_{S_\varepsilon^{M_0}} -u\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\mathrm{d}S - \frac{1}{\varepsilon}\iint\limits_{S_\varepsilon^{M_0}}\frac{\partial u}{\partial n}\,\mathrm{d}S$$

$$= \frac{1}{\varepsilon^2}\iint\limits_{S_\varepsilon^{M_0}} u\,\mathrm{d}S - \overline{\frac{\partial u}{\partial n}}4\pi\varepsilon = 4\pi\bar{u}(M_\xi) - \overline{\frac{\partial u}{\partial n}}4\pi\varepsilon\,,$$

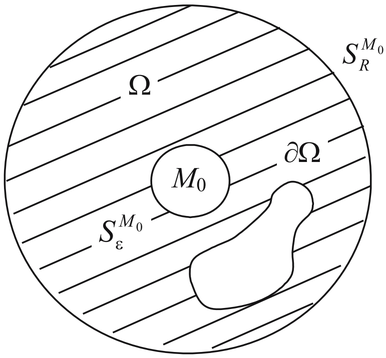

Fig. 7.4 Spherical surfaces $S_\varepsilon^{M_0}$ and $S_R^{M_0}$

where $\bar{u}$ and $\overline{\frac{\partial u}{\partial n}}$ are the mean-values of u and $\frac{\partial u}{\partial n}$ on $S_\varepsilon^{M_0}$, respectively. Since $u(M_\xi) \to u(M_0)$ as $\varepsilon \to 0$, we obtain

$$\iint\limits_{S_\varepsilon^{M_0}} \left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n} \right] \mathrm{d}S \to 4\pi u(M_0), \quad \text{as} \quad \varepsilon \to 0.$$

Also,

$$\iint\limits_{S_R^{M_0}} \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right] \mathrm{d}S = \iint\limits_{S_R^{M_0}} \left[u\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial r} \right] \mathrm{d}S$$

$$= -\frac{1}{R^2}\iint\limits_{S_R^{M_0}} u\,\mathrm{d}S - \frac{1}{R}\iint\limits_{S_R^{M_0}} \frac{\partial u}{\partial r}\,\mathrm{d}S.$$

Consider now ΔOMM_0 where the O is the origin of the coordinate system, M_0 is a fixed point inside Ω', and M is a point on $S_R^{M_0}$. We have

$$r_{OM} + r_{OM_0} > r_{M_0M} \quad \text{or} \quad 1 + \frac{r_{OM_0}}{r_{OM}} > \frac{r_{M_0M}}{r_{OM}}. \tag{7.125}$$

Therefore, as $r_{OM} \to \infty$, $1 \geq \lim \frac{r_{M_0M}}{r_{OM}}$, $r_{M_0M} = R$. Since, on the other hand, $r_{M_0M} > r_{OM} - r_{OM_0}$, we have $\lim \frac{r_{M_0M}}{r_{OM}} \geq 1$. Therefore we obtain

$$\lim_{r_{OM}\to\infty} \frac{R}{r_{OM}} = 1.$$

Therefore, there always exists a constant K for a sufficiently large R such that, by Eq. (7.122),

$$|u(M)| \leq \frac{K}{R}, \quad \left|\frac{\partial u}{\partial n}\right| \leq \frac{K}{R^2}.$$

Thus

$$\left|\iint\limits_{S_R^{M_0}} \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right] \mathrm{d}S\right| \leq \frac{1}{R^2}\iint\limits_{S_R^{M_0}} \frac{K}{R}\,\mathrm{d}S + \frac{1}{R}\iint\limits_{S_R^{M_0}} \frac{K}{R^2}\,\mathrm{d}S = \frac{8\pi K}{R}$$

Or

$$\lim_{R\to\infty}\iint\limits_{S_R^{M_0}} \left[u\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial r}\right] \mathrm{d}S = 0.$$

Finally, Eq. (7.124) leads to Eq. (7.123) by letting $\varepsilon \to 0$ and $R \to \infty$. Equation (7.123) is called the *fundamental integral formula of harmonic functions in unbounded domains*.

Green Functions in Unbounded Domains

For a harmonic function u, Eq. (7.123) reduces to

$$u(M_0) = -\frac{1}{4\pi}\iint\limits_{\partial\Omega'} \left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right] \mathrm{d}S. \tag{7.126}$$

This shows that the value of a harmonic function at any internal point of Ω' can be expressed by its values and normal derivatives on the boundary $\partial\Omega'$. It is the counterpart of Eq. (7.69) for the case of an unbounded domain. Equation (7.126) is also called the *fundamental integral formula of harmonic functions in unbounded domains*.

Let $v = g(M, M_0)$ be a harmonic function. The second Green formula thus yields

$$\iint\limits_{\partial\Omega'} \left[g\frac{\partial u}{\partial n} - u\frac{\partial g}{\partial n}\right] \mathrm{d}S - \iiint\limits_{\Omega'} g\Delta u\,\mathrm{d}\Omega = 0. \tag{7.127}$$

Subtracting Eq. (7.127) from Eq. (7.123) leads to

$$u(M_0) = \iint\limits_{\partial\Omega'} \left[u(M)\frac{\partial}{\partial n}\left(g - \frac{1}{4\pi r}\right) + \left(\frac{1}{4\pi r} - g\right)\frac{\partial u}{\partial n}\right] \mathrm{d}S$$
$$+ \iiint\limits_{\Omega'} \left(g - \frac{1}{4\pi r}\right)\Delta u\,\mathrm{d}\Omega.$$

Similar to introducing the Green function for a bounded domain Ω, let $g(M,M_0)$ be the solution of

$$\begin{cases} \Delta g = 0\,, & M_0 \in \partial\Omega'\,, \\ g|_{\partial\Omega'} = \dfrac{1}{4\pi r}\,, & r = \overline{MM_0} \text{ is the distance between } M \text{ and } M_0\,. \end{cases} \tag{7.128}$$

The solution of

$$\begin{cases} \Delta u = F(M)\,, & M \in \Omega\,, \\ u|_{\partial\Omega'} = f(M)\,. \end{cases} \tag{7.129}$$

is

$$u(M_0) = -\iint\limits_{\partial\Omega'} f(M)\frac{\partial G}{\partial n}\,\mathrm{d}S - \iiint\limits_{\Omega'} G(M,M_0)F(M)\,\mathrm{d}\Omega\,, \tag{7.130}$$

where n is the external normal of the domain boundary, and

$$G(M,M_0) = \frac{1}{4\pi r} - g(M,M_0)\,. \tag{7.131}$$

$G(M,M_0)$ is called the *Green function of the three-dimensional Laplace operators in unbounded domains for the first kind of boundary value problems*. Once it is available, we can readily write out the solution of PDS (7.129).

Green Functions in Unbounded Plane Domains

Theorem 2. Let D' be an unbounded plane domain with a piecewise smooth boundary $\partial D'$. Suppose that $u(M) = u(x,y) \in C^1(\bar{D}') \cap C^2(D')$, and as $r_{OM} \to \infty$,

$$u(M) = O\left(\frac{1}{r_{OM}}\right)\,, \quad \frac{\partial u}{\partial n} = O\left(\frac{1}{r_{OM}^2}\right)\,. \tag{7.132}$$

Here the r_{OM} is the distance between the origin O and point M, and $\bar{D}' = D' \cup \partial D'$. Then

$$u(M_0) = -\frac{1}{2\pi}\oint\limits_{\partial D'}\left[u(M)\frac{\partial}{\partial n}\left(\ln\frac{1}{r}\right) - \left(\ln\frac{1}{r}\right)\frac{\partial u}{\partial n}\right]\mathrm{d}s - \frac{1}{2\pi}\iint\limits_{D'}\Delta u \ln\frac{1}{r}\,\mathrm{d}\sigma\,, \tag{7.133}$$

where $M_0 \in D'$ and r is the distance between M_0 and M. Equation (7.133) is called the *fundamental integral formula of two-dimensional harmonic functions in unbounded domains*.

Proof. Consider two circles $C_\varepsilon^{M_0}$ of center M_0 and radius ε and $C_R^{M_0}$ with sufficiently small ε and sufficiently large R such that the $\partial D'$ is contained in the region of boundary $C_\varepsilon^{M_0}$ and $C_R^{M_0}$. Let D^* be the region enclosed by $C_\varepsilon^{M_0}$, $\partial D'$ and $C_R^{M_0}$. D^* reduces to D' as $\varepsilon \to 0$ and $R \to \infty$. With the harmonic function $v = \ln \frac{1}{r}$, applying the second Green formula yields

$$\iint_{\partial D' \cup C_\varepsilon^{M_0} \cup C_R^{M_0}} \left[u \frac{\partial}{\partial n}\left(\ln \frac{1}{r}\right) - \left(\ln \frac{1}{r}\right) \frac{\partial u}{\partial n} \right] \mathrm{d}s = -\iint_{D^*} \Delta u \ln \frac{1}{r} \,\mathrm{d}\sigma .$$

Note that

$$\begin{aligned}
&\oint_{C_\varepsilon^{M_0}} \left[u \frac{\partial}{\partial n}\left(\ln \frac{1}{r}\right) - \left(\ln \frac{1}{r}\right) \frac{\partial u}{\partial n} \right] \mathrm{d}s \\
&= -\oint_{C_\varepsilon^{M_0}} u \frac{\partial}{\partial r}(-\ln r)\,\mathrm{d}s - \ln \varepsilon \oint_{C_\varepsilon^{M_0}} \frac{\partial u}{\partial r}\,\mathrm{d}s \\
&= \frac{1}{\varepsilon} \oint_{C_\varepsilon^{M_0}} u\,\mathrm{d}s - (\ln \varepsilon) \oint_{C_\varepsilon^{M_0}} \frac{\partial u}{\partial r}\,\mathrm{d}s \\
&= 2\pi \bar{u} - \overline{\frac{\partial u}{\partial r}} 2\pi \varepsilon \ln \varepsilon ,
\end{aligned}$$

where $\bar{u}$ and $\overline{\dfrac{\partial u}{\partial r}}$ are the mean-values of u and $\dfrac{\partial u}{\partial r}$ on $C_\varepsilon^{M_0}$, respectively. By the L'Hôpital's rule, $\lim\limits_{\varepsilon \to 0} (\varepsilon \ln \varepsilon) = 0$. Also, $\lim\limits_{\varepsilon \to 0} \bar{u} = u(M_0)$. Thus

$$\lim_{\varepsilon \to 0} \oint_{C_\varepsilon^{M_0}} \left[u \frac{\partial}{\partial n}\left(\ln \frac{1}{r}\right) - \left(\ln \frac{1}{r}\right) \frac{\partial u}{\partial n} \right] \mathrm{d}s = 2\pi u(M_0) .$$

Similar to in Theorem 1, there always exists a constant k for a sufficiently large R such that, by Eq. (7.132)

$$\begin{aligned}
&\left| \oint_{C_R^{M_0}} \left[u \frac{\partial}{\partial n}\left(\ln \frac{1}{r}\right) - \left(\ln \frac{1}{r}\right) \frac{\partial u}{\partial n} \right] \mathrm{d}s \right| \\
&= \left| \frac{1}{R} \oint_{C_R^{M_0}} u\,\mathrm{d}s - (\ln R) \oint_{C_R^{M_0}} \frac{\partial u}{\partial r}\,\mathrm{d}s \right| \\
&\le \frac{1}{R} \frac{k}{R} 2\pi R + (\ln R) \frac{k}{R^2} 2\pi R \\
&= \frac{2k\pi}{R} + \frac{2k\pi \ln R}{R} .
\end{aligned}$$

By the L'Hôpital's rule, $\lim\limits_{R \to \infty} \dfrac{\ln R}{R} = 0$. Therefore

$$\lim_{R \to \infty} \oint_{C_R^{M_0}} \left[u \frac{\partial}{\partial n}\left(\ln \frac{1}{r}\right) - \left(\ln \frac{1}{r}\right) \frac{\partial u}{\partial n} \right] \mathrm{d}s = 0 .$$

Finally, we obtain Eq. (7.133) by letting $\varepsilon \to 0$ and $R \to \infty$. When $u(M)$ is a harmonic function in D', in particular, we obtain the *fundamental integral formula of harmonic functions in unbounded plane domains*

$$u(M_0) = \frac{1}{2\pi} \oint_{\partial D'} \left[u \frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) - \left(\ln \frac{1}{r} \right) \frac{\partial u}{\partial n} \right] \mathrm{d}s \,, \tag{7.134}$$

where $r = r_{M_0 M}$ is the distance between M_0 and M. Therefore, we may express the value of harmonic functions at any point in unbounded domains by using a line integral over the domain boundary of the function and its normal derivative.

Let $v = g(M, M_0)$ be a harmonic function of two variables. The second Green formula in a plane leads to

$$\oint_{\partial D'} \left[g \frac{\partial u}{\partial n} - u \frac{\partial g}{\partial n} \right] \mathrm{d}s - \iint\limits_{D'} g \Delta u \, \mathrm{d}\sigma = 0 \,. \tag{7.135}$$

Subtracting Eq. (7.135) from Eq. (7.133) yields

$$u(M_0) = \oint_{\partial D'} \left[u(M) \frac{\partial}{\partial n} \left(g - \frac{1}{2\pi} \ln \frac{1}{r} \right) + \left(\frac{1}{2\pi} \ln \frac{1}{r} - g \right) \frac{\partial u}{\partial n} \right] \mathrm{d}s$$

$$+ \iint\limits_{D'} \left(g - \frac{1}{2\pi} \ln \frac{1}{r} \right) \Delta u \, \mathrm{d}\sigma \,.$$

Similar to the case of a three-dimensional unbounded domain, let $g(M, M_0)$ be the solution of

$$\begin{cases} \Delta g = 0 \,, & M_0 \in \Omega' \,, \\ g|_{\partial \Omega'} = \dfrac{1}{2\pi} \ln \dfrac{1}{r} \,, \; r = r_{MM_0} & \text{is distance between } M \text{ and } M_0 \,. \end{cases} \tag{7.136}$$

The solution of the external plane Dirichlet problem

$$\begin{cases} \Delta u = F(M) \,, & M \in D' \,, \\ u|_{\partial D'} = f(M) \,. \end{cases} \tag{7.137}$$

is thus

$$u(M_0) = - \oint_{\partial D'} f(M) \frac{\partial G}{\partial n} \mathrm{d}s - \iint\limits_{D'} G(M, M_0) F(M) \, \mathrm{d}\sigma \,, \tag{7.138}$$

where n is the external normal of the domain boundary and

$$G(M, M_0) = \frac{1}{2\pi} \ln \frac{1}{r} - g(M, M_0) \,. \tag{7.139}$$

$G(M,M_0)$ is called the *Green function of the two-dimensional Laplace operators in unbounded domains for the first kind of boundary-value problems*. Once it is available, we can readily write out the solution of PDS (7.137) as Eq. (7.138).

Remark 1. Solutions in Eqs. (7.130) and (7.138) are the superposition of two parts: the one from the boundary value f and the one from the nonhomogeneous term F. The former (latter) will be zero if $f = 0 (F = 0)$.

Remark 2. The Green functions of Laplace operators depend only on the domain, but not on $f(M)$. If the boundary is a closed curve, the Green functions depend only on the domain boundary. The Green functions are also the same for the bounded domain inside a boundary and the unbounded domain outside the boundary.

Remark 3. The method of Green functions is first proposed for Laplace operators. The idea can however be extended to other operators. This also requires, of course, an extension of Green formulas.

7.8 Potential Theory

Potential theory has very important applications in seeking solutions of boundary-value problems of elliptic equations. In this section we introduce various potentials and examine their properties.

7.8.1 Potentials

Consider the static electric field generated by a point electric charge of capacity q at $P(x_0,y_0,z_0)$. Its electric potential at $M(x,y,z)$ is, without taking account of dielectric constant ε,

$$u(M) = \frac{q}{r_{PM}},$$

where $r_{PM} = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$ is the distance between P and M. By the principle of superposition, the potential at a point due to a finite number of charges is the superposition of the individual potentials. The potential due to a continuous distribution of charges is an integral of individual potentials. The potential due to an electrified body Ω of electric-charge density $\rho(P)$ is thus

$$u(M) = \iiint\limits_{\Omega} \frac{\rho(P)}{r_{PM}}\,\mathrm{d}\Omega\,, \tag{7.140}$$

where r_{PM} is the distance between M and $P \in \Omega$. The integral in Eq. (7.140) is called the *volume potential* or the *Newton potential*.

Similarly, consider the electrical field due to an electrified surface of electric-charge surface density $\mu(P)$. The electric potential reads

$$u(M) = \iint_S \frac{\mu(P)}{r_{PM}} \,\mathrm{d}S\,, \tag{7.141}$$

where r_{PM} is the distance between M and $P \in S$. The integral in Eq. (7.141) is called the *single-layer potential*.

Consider a dipole formed by two point electric charges of capacity $+q$ and $-q$ at P_1 and P_2 along the l-axis, respectively (Fig. 7.5). The l-axis is called the dipole axis. Let Δl be the distance between P_1 and P_2. The constant $p(= q\Delta l)$ is called the *dipole moment*. The electric potential at M due to a dipole reads

$$u(M) = \frac{q}{r_2} - \frac{q}{r_1} = p\frac{1}{\Delta l}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = p\frac{\Delta(\frac{1}{r})}{\Delta l}\,,$$

which is called the *dipole potential*. Now let $\Delta l \to 0$ and vary q so that P_1 and P_2 tend to P and p is kept as a constant. The electric potential at M becomes

$$\begin{aligned} u(M) &= \lim_{q\to+\infty}\left(\frac{q}{r_2} - \frac{q}{r_1}\right) = \lim_{\Delta l \to 0} p\frac{\Delta(\frac{1}{r})}{\Delta l} \\ &= p\frac{\partial(\frac{1}{r})}{\partial l} = -\frac{p}{r^2}\frac{\partial r}{\partial l} \\ &= -\frac{p}{r^2}\cos(MP, l) = \frac{p}{r^2}\cos(PM, l)\,, \end{aligned} \tag{7.142}$$

where $\mathbf{r} = MP$ and, for $M(x,y,z)$ and $P(x',y',z')$

$$r = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}\,, \quad \frac{\partial r}{\partial x} = \frac{x'-x}{r} = \cos(MP, Ox)\,,$$

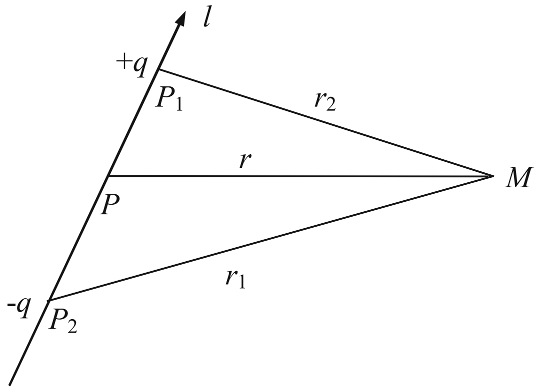

Fig. 7.5 A dipole formed by two point electric charges

$$\frac{\partial r}{\partial y} = \cos(MP, Oy)\,, \quad \frac{\partial r}{\partial z} = \cos(MP, Oz)\,,$$

$$\begin{aligned}\cos(MP, \mathbf{l}) &= \cos(MP, Ox)\cos(\mathbf{l}, Ox) \\ &\quad + \cos(MP, Oy)\cos(\mathbf{l}, Oy) + \cos(MP, Oz)\cos(\mathbf{l}, Oz) \\ &= \tfrac{\partial r}{\partial x}\cos(\mathbf{l}, Ox) + \tfrac{\partial r}{\partial y}\cos(\mathbf{l}, Oy) + \tfrac{\partial r}{\partial z}\cos(\mathbf{l}, Oz) = \tfrac{\partial r}{\partial l}\,.\end{aligned}$$

The $u(M)$ in Eq. (7.142) is called the *dipole potential* at M due to the dipole of moment p at P.

Now consider two parallel and very close surfaces S and S' (Fig. 7.6). Their distance δ along the normal is very small. The number of electric charges at any point on S' is the same as that at its counterpart point on S, but with a negative sign. Let $\mathbf{n}$ be the common normal of the two surfaces from the negative charge to the positive charge. Since δ is very small, we may regard S and S' as a surface with two sides. For a area element $\mathrm{d}S$ on the surface around point P, let $\tau(P)$ be its surface density of the dipole moment. Note that the normal is the dipole-axis. The electric potential at any point M outside the surface due to dipoles on $\mathrm{d}S$ is thus

$$\mathrm{d}u(M) = \tau(P)\frac{\partial}{\partial n}\left(\frac{1}{r_{PM}}\right)\mathrm{d}S = \frac{\tau(P)\cos(PM, \mathbf{n})}{r_{PM}^2}\,\mathrm{d}S\,.$$

The potential at M due to all dipoles on the surface is,

$$u(M) = \iint_S \tau(P)\frac{\partial}{\partial n}\left(\frac{1}{r_{PM}}\right)\mathrm{d}S = \iint_S \frac{\tau(P)\cos(PM, \mathbf{n})}{r_{PM}^2}\,\mathrm{d}S\,. \tag{7.143}$$

The integral in Eq. (7.143) is called the *double-layer potential*. The single-layer and double-layer potentials are both called the *surface potential*.

The fundamental solution of Laplace equations is $\ln\frac{1}{r_{PM}}$ in two-dimensional cases. It is the electrical potential of the electric field generated by an infinitely-long electrified wire with a uniform linear density of electric charges (Section 7.4.1).

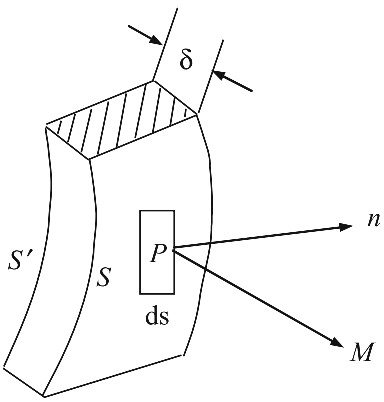

Fig. 7.6 Surfaces S and S'

Thus the counterparts of Eqs. (7.140), (7.141) and (7.143) in two-dimensional cases are

$$u(M) = \iint\limits_D \rho(P) \ln \frac{1}{r_{PM}} \,\mathrm{d}\sigma, \tag{7.144}$$

$$u(M) = \int_C M(p) \ln \frac{1}{r_{PM}} \,\mathrm{d}s, \tag{7.145}$$

$$u(M) = \int_C \tau(p) \frac{\partial}{\partial n} \left(\ln \frac{1}{r_{PM}} \right) \mathrm{d}s, \tag{7.146}$$

where D is a plane domain, C is a plane curve, and n is the normal of curve C from the negative charge side to the positive charge side.

The integral in Eq. (7.144) is called the *logarithmic potential*. The integrals in Eqs. (7.145) and (7.146) are called the *single-layer and the double-layer potential in a plane*, respectively.

7.8.2 Generalized Integrals with Parameters

The point M is a parameter in the potentials that is defined by the integrals. If M is outside the integral domain, the integral is a normal integration provided that all densities of electric-charges or dipole moments are continuous. When M is inside the integral domain, however, the integral is a generalized one because

$$\lim_{P \to M} (1/r_{PM}) = \infty.$$

To examine properties of potentials, we must first discuss some properties of generalized integrals with parameters, which we do here by using surface integrals as examples.

Let M_0 be a point on surface S. $F(M,P)$ is a function of three variables (spatial coordinates of P) with coordinates of M as the parameter. Let $\mathrm{d}S$ be the area element at point P. Consider a surface integral

$$v(M) = \iint\limits_S F(M,P) \,\mathrm{d}S. \tag{7.147}$$

When M is outside of S, $v(M)$ is continuous at M provided that $F(M,P)$ is a continuous function of P. When $M = M_0 \in S$ and $\lim\limits_{P \to M} F(M,P) = \infty$, however, the $v(M)$ in Eq. (7.147) is a generalized integral.

Definition. Let M_0 be a point on S. If, for any $\varepsilon > 0$, there always exists a neighboring region V of M_0 and a surface S_δ containing M_0 on S such that, for all $M \in V$, $\iint\limits_{S_\delta} F(M,P) \,\mathrm{d}S$ (spatial coordinates of P are the integral variables, M is a parameter) is convergent and

$$\left|\iint\limits_{S_\delta} F(M,P)\,\mathrm{d}S\right| < \varepsilon\,.$$

The generalized integral in Eq. (7.147) is called *uniformly convergent* at point M_0 on S.

Theorem 1. If the integral in Eq. (7.147) is uniformly convergent at the point M_0 on the surface S, then $v(M)$ is continuous at M_0.

Proof. By the definition of uniform convergence, we can choose a sufficiently small neighboring region V of M_0 such that its intersecting part with S is on S_δ. The integral in Eq. (7.147) thus converges at all points M in V that are sufficiently close to M_0. By the definition of the uniform convergence, we have, for M_0 and $M \in V$,

$$\left|\iint\limits_{S_\delta} F(M_0,P)\,\mathrm{d}S\right| < \frac{\varepsilon}{3}\,, \quad \left|\iint\limits_{S_\delta} F(M,P)\,\mathrm{d}S\right| < \frac{\varepsilon}{3}\,.$$

Since $F(M,P)$ is a uniformly continuous function of $M \in V$ for $P \in S\backslash S_\delta$, we have, for M sufficiently close to M_0,

$$\iint\limits_{S\backslash S_\delta} |F(M,P) - F(M_0,P)|\,\mathrm{d}S < \frac{\varepsilon}{3}\,.$$

Thus

$$|v(M) - v(M_0)| = \left|\iint\limits_{S} F(M,P)\,\mathrm{d}S - \iint\limits_{S} F(M_0,P)\,\mathrm{d}S\right|$$

$$\leq \left|\iint\limits_{S_\delta} F(M,P)\,\mathrm{d}S\right| + \left|\iint\limits_{S_\delta} F(M_0,P)\,\mathrm{d}S\right|$$

$$+ \left|\iint\limits_{S\backslash S_\delta} [F(M,P) - F(M_0,P)]\,\mathrm{d}S\right| < \varepsilon\,,$$

so $v(M)$ is continuous at M_0.

By a similar approach, we can show that this result is also valid for triple integrals and line integrals.

Theorem 2. Let S be a bounded smooth surface. Suppose that $F(M,P)$ is continuous when $M \neq P$. If, for a point M_0 on S, there exists a neighboring region V of M_0 and a surface S_δ of containing M_0 on S such that, for all $M \in V$, $P \in S_\delta$,

$$|F(M,P)| \leq \frac{C}{r_{PM}^{2-\delta}}\,, \tag{7.148}$$

the integral in Eq. (7.147) converges uniformly at M_0 so $v(M)$ is continuous at M_0. In Eq. (7.148), C is a constant and $0 < \delta \leq 1$.

Proof. Without loss of generality and for convenience, take M_0 as the origin of the coordinate system (ξ, η, ζ), and the tangent plane of S at M_0 as the $\xi\eta$-plane. The surface S can thus be expressed by, in the neighborhood of M_0,

$$\zeta = \varphi(\xi, \eta)\,.$$

Also,

$$\varphi(0,0) = 0\,,\ \left.\frac{\partial\varphi}{\partial\xi}\right|_{\substack{\xi=0\\ \eta=0}} = 0\,,\ \left.\frac{\partial\varphi}{\partial\eta}\right|_{\substack{\xi=0\\ \eta=0}} = 0\,.$$

Consider a sufficiently small circle on $\xi\eta$-plane: $\xi^2+\eta^2 \leq h^2$, and let S_h be its corresponding surface on S. On S_h, both $\left|\frac{\partial\varphi}{\partial\xi}\right|$ and $\left|\frac{\partial\varphi}{\partial\eta}\right|$ are smaller than a constant and Eq. (7.148) holds. Let the coordinates of M and P be (x,y,z) and (ξ,η,ζ), respectively. Thus,

$$\begin{aligned}\left|\iint\limits_{S_h} F(M,P)\,\mathrm{d}S\right| &\leq C \iint\limits_{\xi^2+\eta^2\leq h^2} \frac{\sqrt{1+(\frac{\partial\varphi}{\partial\xi})^2+(\frac{\partial\varphi}{\partial\eta})^2}}{[(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2]^{\frac{2-\delta}{2}}}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leq C_1 \iint\limits_{\xi^2+\eta^2\leq h^2} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{[(\xi-x)^2+(\eta-y)^2]^{\frac{2-\delta}{2}}} \\ &= C_1 \iint\limits_{(\mu-x)^2+(\gamma-y)^2\leq h^2} \frac{\mathrm{d}\mu\,\mathrm{d}\gamma}{[\mu^2+\gamma^2]^{\frac{2-\delta}{2}}}\,,\end{aligned} \tag{7.149}$$

where C and C_1 are constants.

For a sufficiently small V, $M(x,y,z) \in V$ satisfies $x^2+y^2 \leq h^2$ so the circular region $(\mu-x)^2+(\gamma-y)^2 \leq h^2$ is inside the circular region $\mu^2+\gamma^2 \leq (2h)^2$. Thus we have

$$\begin{aligned}&\iint\limits_{(\mu-x)^2+(\gamma-y)^2\leq h^2} \frac{\mathrm{d}\mu\,\mathrm{d}\gamma}{(\mu^2+\gamma^2)^{\frac{2-\delta}{2}}} \leq \iint\limits_{\mu^2+\gamma^2\leq(2h)^2} \frac{\mathrm{d}\mu\,\mathrm{d}\gamma}{(\mu^2+\gamma^2)^{\frac{2-\delta}{2}}} \\ &= 2\pi\int_0^{2h} \frac{\mathrm{d}r}{r^{1-\delta}} = \begin{cases} 4\pi h\,, \delta = 1\,, \\ 2\pi\frac{1}{\delta}(2h)^{\delta}\,,\ \ 0<\delta<1\,. \end{cases}\end{aligned} \tag{7.150}$$

The integral on the left-hand side of Eq. (7.150) tends to zero as $h \to 0$ such that the integral in the right-hand side of Eq. (7.149) also tends to zero. Therefore, for any small $\varepsilon > 0$, there always exists a neighboring region V of M_0 and a surface S_h containing M_0 on S such that, for all $M \in V$,

$$\left| \iint_{S_h} F(M,P)\,\mathrm{d}S \right| < \varepsilon\,.$$

Thus the integral in Eq. (7.147) converges uniformly at M_0, and $v(M)$ is continuous at M_0.

Remark 1. The uniform convergence of the integral in Eq. (7.147) at M_0 is a sufficient condition for the continuity of $v(M)$ at M_0. But, it is not a necessary condition.

Remark 2. If the condition in Eq. (7.148) is only valid at M_0, i.e.

$$|F(M_0,P)| \leq \frac{C}{r_{M_0P}^{2-\delta}}\,, \quad 0 < \delta \leq 1\,.$$

we can only have the convergence of the integral in Eq. (7.147) at M_0, but not necessarily the continuity of $v(M)$ at M_0.

7.8.3 *Solid Angle and Russin Surface*

Consider a surface S with fixed normal direction. Its normal $\mathbf{n}$ is shown in Fig. 7.7. The solid angle of surface S viewing from point O refers to the angle occupied by its projection on the spherical surface K of a sphere of center O and unit radius. If the viewing direction is along $\mathbf{n}(-\mathbf{n})$, the solid angle is defined as positive (negative). For the case shown in Fig. 7.7, the solid angle is positive. If we reverse the normal, the solid angle becomes negative. For a closed surface, we always refer the external normal as the normal of surface. Therefore, the solid angle of a closed surface is 4π if the O is inside S, 0 if the O is outside S and 2π if the O is on the S.

If we use r^2 to measure the area of a spherical surface of radius r, we obtain 4π so that the spherical area is $4\pi r^2$. The 4π is called the *solid angle of spherical surfaces*. Let $d\omega$ be the solid angle of area $\mathrm{d}\sigma$ on a spherical surface of radius r. Then

$$\mathrm{d}\sigma = r^2\,\mathrm{d}\omega \quad \text{or} \quad \mathrm{d}\omega = \mathrm{d}\sigma/r^2\,.$$

Consider an area element $\mathrm{d}S$ at P on a general surface S. Let $\mathrm{d}\omega$ be the solid angle of $\mathrm{d}S$ viewing from point O. The projection of $\mathrm{d}S$ on the spherical surface $S^O_{r_{OP}}$ is denoted by $\mathrm{d}\sigma$. Thus

$$\mathrm{d}S\cos(OP,\mathbf{n}) = \mathrm{d}\sigma\,,$$

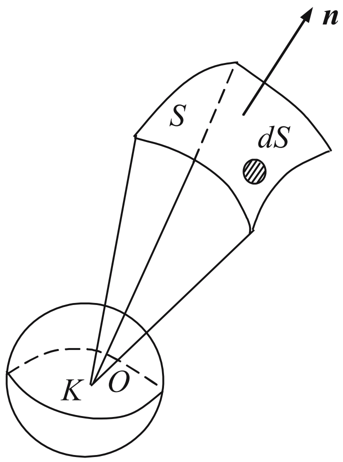

Fig. 7.7 Positive solid angle

where $\mathbf{n}$ is the normal of the surface S at P. OP is the normal of $S^O_{r_{OP}}$ at P. Therefore the solid angle of $\mathrm{d}S$ viewing from point O reads

$$\mathrm{d}\omega = \frac{\mathrm{d}\sigma}{r^2_{OP}} = \frac{\cos(OP,\mathbf{n})}{r^2_{OP}}\,\mathrm{d}S\,.$$

Here r_{OP} is the distance between O and P. The solid angle of surface S viewing from O is thus

$$\omega = \iint\limits_S \frac{\cos(OP,\mathbf{n})}{r^2_{OP}}\,\mathrm{d}S\,.$$

$\iint\limits_S \left|\frac{\cos(OP,\mathbf{n})}{r^2_{OP}}\right| \mathrm{d}S$ is called the *absolute solid angle* of surface S viewing from O.

There are some constraints on the boundary surface S in studying boundary value problems of elliptic equations. It is normally required to be a Ляпунов surface defined as follows.

Definition. A surface S is called a Ляпунов *(Russin) surface* if it satisfies:

1. There exists a tangent plane at any point on S;
2. For any point P_0 on S, there exists a sphere $V_R^{P_0}$ of center P_0 and radius R such that the part S_{P_0} of S inside $V_R^{P_0}$ can be expressed by a unique-valued function $z=\varphi(x,y)$. Here R normally depends on the position of P_0;
3. The unit normals $\mathbf{n}_1$ and $\mathbf{n}_2$ at any two points P_1 and P_2 on S must satisfy

$$(\mathbf{n}_1,\mathbf{n}_2) \le A r^{\delta}_{P_1P_2}\,,$$

where A is a constant, constant δ satisfies $0 < \delta \le 1$, and $r_{P_1P_2}$ is the distance between P_1 and P_2;

4. The absolute solid angle of any part S_Δ of S viewing from any point O in the space is bounded.

Theorem 3. Let S be a Ляпунов surface. For any two points P and M on S, we have

$$\left|\frac{\cos(PM,\mathbf{n})}{r_{PM}^2}\right| \le \frac{C}{r_{PM}^{2-\delta}},$$

where constant C does not depend on points on S, constant δ satisfies $0 < \delta \le 1$, r_{PM} is the distance between P and M, and $\mathbf{n}$ is the external normal of S at P.

This theorem plays an important role in potential theory. Hereafter, surfaces in single-layer and double-layer potentials are all assumed to be Ляпунов surfaces. The surfaces in real applications normally satisfy the four conditions for Ляпунов surfaces.

7.8.4 *Properties of Surface Potentials*

Theorem 4. If density functions $\mu(P)$ and $\tau(P)$ in single-layer potentials (Eq. (7.141)) and double-layer potentials (Eq. (7.143)) are continuous on S, the potentials must be harmonic functions for all M not on S.

Proof. When M is not on S, integrands in Eqs. (7.141) and (7.143) are continuous and differentiable with respect to M up to any order. We can thus take derivatives with respect to $M(x,y,z)$ inside the integration. For the single-layer potential (Eq. (7.141)), we have

$$\Delta u(M) = \iint_S \mu(P)\Delta\left(\frac{1}{r_{PM}}\right)\mathrm{d}S = 0,$$

because $\Delta\left(\dfrac{1}{r_{PM}}\right) = 0$. For the double-layer potential Eq. (7.143), we also have

$$\Delta u(M) = \iint_S \tau(P)\frac{\partial}{\partial n}\left[\Delta\frac{1}{r_{PM}}\right]\mathrm{d}S = 0,$$

because

$$\frac{\cos(PM,n)}{r_{PM}^2} = \frac{\partial}{\partial n}\left(\frac{1}{r_{PM}}\right), \quad \Delta\frac{\partial}{\partial n} = \frac{\partial}{\partial n}\Delta.$$

Theorem 5. Let S be a bounded smooth surface. If $\mu(P)$ is continuous on S, the single-layer potential (Eq. (7.141)) is convergent when $M \in S$ and continuous on S.

Proof. For any point M on S, the integrand in the single-layer potential $F(M,P)\left(=\dfrac{\mu(P)}{r_{PM}^2}\right)$ satisfies condition (7.148) in Theorem 2. Thus the single-layer potential converges uniformly at all points $M \in S$.

Therefore the single-layer potential exists at all points M in space, is a continuous function of M, and is a harmonic function outside S provided that S is smooth, bounded and with a continuous density function $\mu(P)$.

Theorem 6. Let S be a Ляпунов surface. If the density function $\tau(P)$ in the double-layer potential is continuous, the potential (Eq. (7.143)) is convergent for all $M \in S$.

Proof. The integrand $\dfrac{\tau(P)\cos(PM,\mathbf{n})}{r_{PM}^2}$ in the double-layer potential satisfies, by Theorem 3, the condition in Remark 2 of Theorem 2. Therefore the double-layer potential $u(M)$ is convergent for $M \in S$.

However, the double-layer potential $u(M)$ is not continuous on S in general.

Theorem 7. Let S be a closed Ляпунов surface. Suppose that density function $\tau(P)$ is continuous on S. The double-layer potential $u(M)$ in Eq. (7.143) has a discontinuity of the first kind at $M \in S$, i.e.

$$\begin{aligned} \underline{u}(M) &= u(M) - 2\pi\tau(M)\,, \\ \bar{u}(M) &= u(M) + 2\pi\tau(M)\,, \end{aligned} \qquad M \in S \tag{7.151}$$

where $\underline{u}(M)$ is the limit of $u(M)$ as M tends to S from the inside and $\bar{u}(M)$ is the limit of $u(M)$ as M tends to S from the outside.

Proof. First we consider the case $\tau(P) = \tau_0$ (constant). Note that the solid angle of area element $\mathrm{d}S$ at P on S viewing from M is

$$\mathrm{d}\omega = \frac{\cos(MP,\mathbf{n})}{r_{PM}^2}\,\mathrm{d}S = -\frac{\cos(PM,\mathbf{n})}{r_{PM}^2}\,\mathrm{d}S\,.$$

The double-layer potential is thus

$$u_1(M) = \iint\limits_S \frac{\tau_0\cos(PM,\mathbf{n})}{r_{PM}^2}\,\mathrm{d}S = \begin{cases} -4\pi\tau_0\,, & M \in \Omega\,, \\ -2\pi\tau_0\,, & M \in S\,, \\ 0\,, & M \in \Omega'\,. \end{cases}$$

where Ω and Ω' are the domains inside S and outside S, respectively. For a point P on S, we have

$$\begin{cases} \underline{u_1}(P) = -4\pi\tau_0\,, \\ u_1(P) = -2\pi\tau_0\,, P \in S\,, \\ \overline{u_1}(P) = 0\,. \end{cases} \tag{7.152}$$

We now move to consider the case of a general $\tau(P)$. Consider a point P_0 on S with $\tau_0 = \tau(P_0)$ and

$$u(M) - u_1(M) = \iint_S [\tau(P) - \tau(P_0)] \frac{\cos(PM, \mathbf{n})}{r_{PM}^2} \, \mathrm{d}S .$$

The integral is uniformly convergent at $M = P_0$. For a surface S_δ containing P_0 on S, we have, using condition (4) for a Ляпунов surface,

$$\iint_{S_\delta} \left| \frac{\cos(PM, \mathbf{n})}{r_{PM}^2} \right| \mathrm{d}S \leq k \text{ (bounded)}.$$

Thus

$$\begin{aligned} &\left| \iint_{S_\delta} [\tau(P) - \tau(P_0)] \frac{\cos(PM, \mathbf{n})}{r_{PM}^2} \, \mathrm{d}S \right| \\ &\leq \max_{P \in S_\delta} |\tau(P) - \tau(P_0)| \iint_{S_\delta} \left| \frac{\cos(PM, \mathbf{n})}{r_{PM}^2} \right| \mathrm{d}S \\ &\leq k \max_{P \in S_\delta} |\tau(P) - \tau(P_0)| \, . \end{aligned}$$

By the continuity of $\tau(P)$, we thus have for any $\varepsilon > 0$ and all M in a neighboring region V of P_0,

$$\left| \iint_{S_\delta} [\tau(P) - \tau(P_0)] \frac{\cos(PM, \mathbf{n})}{r_{PM}^2} \, \mathrm{d}S \right| < \varepsilon$$

provided that S_δ is sufficiently small such that all points on S_δ are sufficiently close to P_0. Therefore, $u(M) - u_1(M)$ converges at P_0 uniformly. By Theorem 1, $u(M) - u_1(M)$ must be continuous at P_0. By the definition of continuity, we have

$$\underline{u}(P_0) - \underline{u}_1(P_0) = u(P_0) - u_1(P_0) = \bar{u}(P_0) - \bar{u}_1(P_0) .$$

Therefore, by Eq. (7.152),

$$\underline{u}(P_0) = u(P_0) + \underline{u}_1(P_0) - u_1(P_0) = u(P_0) - 2\pi\tau(P_0) ,$$

$$\bar{u}(P_0) = u(P_0) + \bar{u}_1(P_0) - u_1(P_0) = u(P_0) + 2\pi\tau(P_0) .$$

Since P_0 is an arbitrary point on S, we arrive at Eq. (7.151).

This theorem is also valid for two-dimensional cases. However 2π in Eq. (7.151) should be replaced by π.

Theorem 8. Let S be a closed Ляпунов surface with $\mathbf{n}$ as its external normal. Suppose that density function $\mu(P)$ is continuous on S. The $u(M)$ is the single-layer potential in Eq. (7.141). The normal derivative $\frac{\partial u}{\partial n}$ has a discontinuity of the first kind at $M \in S$, i.e.

$$\frac{\partial u}{\partial n^+} = \frac{\partial u}{\partial n} - 2\pi\mu(M), \quad \frac{\partial u}{\partial n^-} = \frac{\partial u}{\partial n} + 2\pi\mu(M), \quad M \in S, \tag{7.153}$$

where $\frac{\partial u}{\partial n^-}$ and $\frac{\partial u}{\partial n^+}$ are the limits of $\frac{\partial u}{\partial n}$ as M tends to S along the normal from inside and outside of S, respectively.

For two-dimensional cases, the 2π in Eq. (7.153) should be replaced by π.

7.9 Transformation of Boundary-Value Problems of Laplace Equations to Integral Equations

This section starts with a brief discussion of integral equations. Potential theory is thus applied to transform boundary-value problems of Laplace equations of the first and the second kind into integral equations. Readers are referred to books on integral equations for methods of solving integral equations.

7.9.1 Integral Equations

An equation that contains an unknown function in the integrand is called an *integral equation*. If the unknown function occurs in a linear form, the equation is called a *linear integral equation*. For example, in the one-dimensional case, the equation

$$\varphi(x) = \lambda \int_a^b K(x,\xi)\varphi(\xi)\,\mathrm{d}\xi + f(x) \tag{7.154}$$

is a linear integral equation. Here λ is a real-valued or complex-valued parameter. $f(x)$ is a given function. $K(x,\xi)$ is also a known function and is called the *kernel of integral equations*. Two independent variables x and ξ vary in the region $[a,b]$. $\varphi(x)$ is an unknown function.

If $f(x) \equiv 0$, we have

$$\varphi(x) = \lambda \int_a^b K(x,\xi)\varphi(\xi)\,\mathrm{d}\xi\,, \tag{7.155}$$

which is called the *associated homogeneous equation of Eq. (7.154)*. Equation (7.154) is called a *nonhomogeneous equation*, and $f(x)$ is the *nonhomogeneous term*.

In n-dimensional space, integrals in the integral equations are n-multiple, the unknown function is a function of independent variables and the integration domain is a finite region. For example, counterparts of Eq. (7.154) and (7.155) are, in the three-dimensional case,

$$\varphi(M) = \lambda \iiint\limits_{\Omega} K(M,P)\varphi(P)\,\mathrm{d}\Omega + f(M)\,, \tag{7.156}$$

$$\varphi(M) = \lambda \iiint\limits_{\Omega} K(M,P)\varphi(P)\,\mathrm{d}\Omega\,, \tag{7.157}$$

respectively. Here Ω stands for a three-dimensional finite region, and M and P are points in Ω.

With known kernel $K(M,P)$ and nonhomogeneous term $g(M)$, the integral equation

$$\psi(M) = \lambda \iiint\limits_{\Omega} K(M,P)\psi(P)\,\mathrm{d}\Omega + g(M)\,, \tag{7.158}$$

is called the *transpose equation of Eq. (7.156).*

When the unknown function occurs only in the integrands, the equation is called the *Fredholm integral equation of the first kind*.

Examples are

$$\begin{aligned} &\int_a^b K(x,\xi)\varphi(\xi)\,\mathrm{d}\xi = f(x)\,, \\ &\iiint\limits_{\Omega} K(M,P)\varphi(P)\,\mathrm{d}\Omega = f(M)\,. \end{aligned} \tag{7.159}$$

An integral equation that involves the unknown function outside of the integrands is called a *Fredholm integral equation of the second kind*. Examples are Eqs. (7.154), (7.156) and (7.158).

If the kernel is symmetric such that $K(M,P) = K(P,M)$, the equation is called a *symmetric equation*. If the kernel satisfies

$$K(M,P) = \frac{H(M,P)}{r_{PM}^{\alpha}}\,, \quad 0 < \alpha < n\,,$$

it is called a *weakly-singular kernel*. The corresponding equation is called an *integral equation with a weakly-singular kernel*. Here $H(M,P)$ is a continuous function. r_{PM} is the distance between P and M and n is the dimension of the integration domain.

An equivalent definition of a weakly-singular kernel is

$$|K(M,P)| \le \frac{C}{r_{PM}^{n-\alpha}}\,, \quad 0 < \alpha < n\,,$$

where C is a constant. In three-dimensional space, the dimension of the integration domain is two instead of three if the integrals involved are surface integrals.

7.9.2 Transformation of Boundary-Value Problems into Integral Equations

For boundary-value problems of Laplace equations in a simple, regular domains we can obtain their solutions by using the Fourier method of expansion, separation of variables, the integral transformation or the method of Green functions. These methods do not work, however, for problems in domains that are not simple and regular. We normally use either of the following two methods for those problems:

1. Find an analytical expression that contains the undetermined function and satisfies the Laplace equation. The undetermined function is then determined by imposing the boundary conditions.
2. Find the function set of functions satisfying the CDS. The solution is then determined by constructing harmonic functions from the set. This method belongs to the direct category in mathematical equations of physics.

A typical example of the former is seeking solutions of boundary-value problems using the potential theory, i.e. by transforming boundary-value problems into integral equations. If S is a Ляпунов surface and density functions $\tau(P)$ and $\mu(P)$ are continuous on S, the single-layer and the double-layer potentials are harmonic functions in both Ω (the region inside S) and Ω' (the region outside S). In particular, the single-layer potential is continuous in both $\Omega + S$ and $\Omega' + S$; its normal derivative also has the limit from both inside and outside of S. Therefore, we may use the potentials as solutions of boundary value problems of Laplace equations. Their density functions are undetermined functions and can be determined by imposing the boundary conditions.

Here we discuss this method for solving boundary-value problems of Laplace equations of the first and the second kind

$$\begin{cases} \Delta u = 0\,, \\ u|_S = f(M)\,. \end{cases} \quad \text{or} \quad \begin{cases} \Delta u = 0\,, \\ \left.\dfrac{\partial u}{\partial n}\right|_S = f(M)\,. \end{cases}$$

where the S is a Ляпунов surface and $f(M)$ is a continuous function.

First Internal Boundary-Value Problems (Dirichlet Internal problems)

Assume that the double-layer potential with the undetermined density function $\tau(P)$

$$u(M) = \iint_S \frac{\tau(P)\cos(PM,\mathbf{n})}{r_{PM}^2}\,\mathrm{d}S \tag{7.160a}$$

is a function that satisfies $\Delta u = 0$, $M \in \Omega$ and $u|_S = f(M)$, and is continuous in $\Omega + S$. By Theorem 7 in Section 7.8, we have, for $M \in S$

$$\underline{u}(M) = \iint_S \frac{\tau(P)\cos(PM,\mathbf{n})}{r_{PM}^2}\,\mathrm{d}S - 2\pi\tau(M).$$

To obtain a solution u that is continuous on $\Omega + S$ and satisfies $u|_S = f(M)$, we should impose

$$\underline{u}(M) = f(M),$$

so that the boundary-value problem is transformed into the problem of seeking the solution $\tau(P)$ of

$$f(M) = \iint_S \frac{\tau(P)\cos(PM,\mathbf{n})}{r_{PM}^2}\,\mathrm{d}S - 2\pi\tau(M)$$

or

$$\tau(M) = \iint_S K(M,P)\tau(P)\,\mathrm{d}S - \frac{1}{2\pi}f(M), \tag{7.160b}$$

where $K(M,P) = \dfrac{\cos(PM,\mathbf{n})}{2\pi r_{PM}^2}$. This is a Fredholm integral equation of the second kind regarding $\tau(P)$. Once $\tau(P)$ is available from Eq. (7.160b), the solution of the Dirichlet internal problems can thus readily be obtained from the double-layer potential (7.160a).

First External Boundary-Value Problems (Dirichlet External Problems)

Similar to the Dirichlet internal problems, we can use the double-layer potential as a function that satisfies $\Delta u = 0$, $M \in \Omega'$ and $u|_S = f(M)$, and is continuous in $\Omega' + S$. The density function $\tau(P)$ is determined such that

$$\tau(M) = \iint_S -K(M,P)\tau(P)\,\mathrm{d}S + \frac{1}{2\pi}f(M). \tag{7.161}$$

Equation (7.161) is also a Fredholm integral equation of the second kind.

Second Internal Boundary-Value Problems (Neumann Internal Problems)

Assume that the single-layer potential with the undetermined density function $\mu(P)$

$$u(M) = \iint_S \frac{\mu(P)}{r_{PM}}\,\mathrm{d}S$$

is a function that satisfies $\Delta u = 0$, $M \in \Omega$ and $\left.\dfrac{\partial u}{\partial n}\right|_S = f(M)$, and is continuous on

$\Omega + S$. By Theorem 8 in Section 7.8, we have, for $M \in S$

$$\frac{\partial u}{\partial n^-} = \iint\limits_S \mu(P) \frac{\partial}{\partial n}\left(\frac{1}{r_{PM}}\right) \mathrm{d}S + 2\pi\mu(M)$$

$$= -\iint\limits_S \frac{\mu(P)\cos(PM,\mathbf{n})}{r_{PM}^2} \mathrm{d}S + 2\pi\mu(M).$$

Thus the Neumann internal problems are transformed into Fredholm integral equations of the second kind regarding

$$\mu(M) = \iint\limits_S K(M,P)\mu(P)\,\mathrm{d}S + \frac{1}{2\pi} f(M), \tag{7.162}$$

where $K(M,P)$ is the same as in the Dirichlet problems.

Second External Boundary-Value Problems (Neumann External Problems)

By following a similar approach as that used in obtaining Eq. (7.162), we can transform the Neumann external problems into Fredholm integral equations of the second kind regarding

$$\mu(M) = \iint\limits_S -K(M,P)\mu(P)\,\mathrm{d}S + \frac{1}{2\pi} f(M), \tag{7.163}$$

where $f(M) = \left.\frac{\partial u}{\partial n}\right|_S$, n is the outer normal for Ω' that is also the inner normal of the closed surface S.

Remark 1. In integral equations (7.160)–(7.163), the kernel $K(M,P)$ and its transpose $K(P,M)$ satisfy

$$K(M,P) = -K(P,M).$$

By using these relations, the four equations become

$$\tau(M) = \iint\limits_S K(M,P)\tau(P)\,\mathrm{d}S - \frac{1}{2\pi} f(M), \tag{7.160'}$$

$$\tau(M) = -\iint\limits_S K(M,P)\tau(P)\,\mathrm{d}S + \frac{1}{2\pi} f(M), \tag{7.161'}$$

$$\mu(M) = -\iint\limits_S K(P,M)\mu(P)\,\mathrm{d}S + \frac{1}{2\pi} f(M), \tag{7.162'}$$

$$\mu(M) = \iint\limits_S K(P,M)\mu(P)\,\mathrm{d}S + \frac{1}{2\pi} f(M). \tag{7.163'}$$

Therefore, Eqs. (7.160') and (7.163') are transpose equations of each other. Equations (7.161') and (7.162') are also transpose equations of each other.

Remark 2. By Theorem 3 in Section 7.8, the kernel $K(M,P)$ satisfies

$$|K(M,P)| = \left|\frac{\cos(PM,\mathbf{n})}{2\pi r_{PM}^2}\right| \leq \frac{c}{r_{PM}^{2-\delta}}, \quad 0 < \delta < 1 .$$

Thus the kernel $K(M,P)$ is weakly singular, and the four integral equations all have a weakly-singular kernel.

7.9.3 Boundary-Value Problems of Poisson Equations

In studying steady heat conduction or steady electric fields in domains with an internal source or sink, we arrive at the Poisson equations

$$\Delta u = F(M), \quad M \in \Omega . \tag{7.164}$$

The key for seeking solutions of boundary-value problems of Eq. (7.164) is to find a particular solution u^* of Eq. (7.164). Once u^* is available, a function transformation of $u = \omega + u^*$ will transform the boundary-value problems of Poisson equations into Laplace equations. For example, Dirichlet problems of the Poisson equation

$$\begin{cases} \Delta u = F(M), \quad M \in \Omega , \\ u|_S = f(M) . \end{cases} \tag{7.165}$$

will be transformed into

$$\begin{cases} \Delta \omega = 0, \quad M \in \Omega, \\ \omega|_S = f(M) - u^*|_S \end{cases} \tag{7.166}$$

by a function transformation of $u = \omega + u^*$.

By the third Green formula (Eq. (7.65)), the solution of a Poisson equation satisfies

$$u(M) = -\frac{1}{4\pi}\iint\limits_S \left[u(P)\frac{\partial}{\partial n}\left(\frac{1}{r_{PM}}\right) - \frac{1}{r_{PM}}\frac{\partial u(P)}{\partial n}\right] \mathrm{d}S - \frac{1}{4\pi}\iiint\limits_\Omega \frac{F(P)}{r_{PM}} \mathrm{d}\Omega . \tag{7.167}$$

The two surface integrals in the right-hand side are the single-layer potential and the double-layer potential, respectively. They satisfy the Laplace equation. The volume integral in the right-hand side thus must be a particular solution of the Poisson equation (7.164) such that

$$\Delta\left(-\frac{1}{4\pi}\iiint\limits_{\Omega}\frac{F(P)}{r_{PM}}\,\mathrm{d}\Omega\right)=F(M)\,.$$

Therefore, the particular solution is

$$u^*=-\frac{1}{4\pi}\iiint\limits_{\Omega}\frac{F(P)}{r_{PM}}\,\mathrm{d}\Omega\,. \tag{7.168}$$

Note that the $u^*(M)$ is obtained on the assumption that the solution $u(M)$ of the Poisson equation exists and has continuous second derivatives on $\Omega+S$. A verification by substituting into the equation can serve as the justification that Eq. (7.168) is indeed the particular solution of the Poisson equation.

Therefore boundary-value problems of Poisson equations can be transformed into boundary-value problems of Laplace equations, which are then reduced into the four integrals equations (7.160')–(7.163'). These integral equations have been studied extensively in literature.

7.9.4 Two-Dimensional Potential Equations

Dirichlet internal and external problems of Laplace equations in two-dimensional space read

$$\begin{cases}\Delta u=0\,, & M\in D\,,\\ u|_C=f(M)\end{cases}\quad\text{and}\quad\begin{cases}\Delta u=0\,, & M\in D'\,,\\ u|_C=f(M)\,,\end{cases} \tag{7.169}$$

respectively. Here D and D' are the plane domains inside of closed boundary curve C and outside of C, respectively. The Neumann internal and external problems of Laplace equations can also be written out simply by replacing $u|_C=f(M)$ in Eq. (7.169) by $\left.\dfrac{\partial u}{\partial n}\right|_C=f(M)$.

Following a similar approach for three-dimensional cases, we can transform these four problems into four integral equations regarding $\tau(M)$ or $\mu(M)$

$$\begin{aligned}
&\tau(M)=\oint_C K(M,P)\tau(P)\,\mathrm{d}s-\frac{1}{\pi}f(M)\,,\\
&\tau(M)=-\oint_C K(M,P)\tau(P)\,\mathrm{d}s+\frac{1}{\pi}f(M)\,,\\
&\mu(M)=-\oint_C K(P,M)\mu(P)\,\mathrm{d}s+\frac{1}{\pi}f(M)\,,\\
&\mu(M)=\oint_C K(P,M)\mu(P)\,\mathrm{d}s+\frac{1}{\pi}f(M)\,,
\end{aligned} \tag{7.170}$$

where, with r_{PM} as the distance between P and M,

$$K(M,P) = \frac{\cos(PM,\mathbf{n})}{\pi r_{PM}} \tag{7.171}$$

and thus,

$$K(P,M) = -K(M,P).$$

Remark 1. We demonstrate the transformation from Eq. (7.169) to Eq. (7.170) by using Dirichlet internal problems as an example. By Theorem 7 in Section 7.8 (Eq. (7.151)), for two-dimensional cases we have

$$\underline{u}(M) = u(M) - \pi\tau(M),$$

where the plane double-layer potential

$$u(M) = \oint_C \tau(P)\frac{\partial}{\partial n}\left(\ln\frac{1}{r_{PM}}\right)\mathrm{d}s \tag{7.172}$$

is a harmonic function in the domain both inside and outside of C. The $\underline{u}(M)$ stands for the limit of $u(M)$ as M tends to C from inside of C. By Eq. (7.169), we have

$$\underline{u}(M) = u(M)|_C = f(M).$$

Thus

$$f(M) = \oint_C \tau(P)\frac{\partial}{\partial n}\left(\ln\frac{1}{r_{PM}}\right)\mathrm{d}s - \pi\tau(M), \tag{7.173}$$

where r_{PM} is the normal of vector PM or the distance between P and M. Also,

$$\begin{aligned}
\frac{\partial}{\partial n}\left(\ln\frac{1}{r_{PM}}\right) &= \frac{\partial}{\partial r_{PM}}(-\ln r_{PM})\frac{\partial r_{PM}}{\partial n} \\
&= -\frac{1}{r_{PM}}\left[\frac{\partial r_{PM}}{\partial x}\cos(\mathbf{n},x) + \frac{\partial r_{PM}}{\partial y}\cos(\mathbf{n},y)\right] \\
&= \frac{1}{r_{PM}}\left[\cos(MP,x)\cos(\mathbf{n},x) + \cos(MP,y)\cos(\mathbf{n},y)\right] \\
&= -\frac{1}{r_{PM}}(MP)_1 \cdot \mathbf{n}_1 = \frac{\cos(PM,\mathbf{n})}{r_{PM}}
\end{aligned}$$

where $(MP)_1$ and $(\mathbf{n})_1$ are the unit vectors of MP and $\mathbf{n}$, respectively. Therefore, Eq. (7.172) becomes

$$f(M) = \oint_C \frac{\cos(PM,\mathbf{n})}{\pi r_{PM}}\tau(P)\,\mathrm{d}s - \pi\tau(M)$$

or

$$\tau(M) = \oint_C K(M,P)\tau(P)\,\mathrm{d}s - \frac{1}{\pi}f(M)\,, \tag{7.174}$$

where the $K(M,P)$ is defined in Eq. (7.171).

Remark 2. Similar to three-dimensional Poisson equations, two-dimensional Poisson equations $\Delta u = F(x,y)$, $(x,y) \in D$ have a particular solution

$$u^*(x,y) = -\frac{1}{2\pi}\iint_D F(\xi,\eta)\ln\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \tag{7.175}$$

Thus a function transformation of $u(M) = \omega(M) + u^*(M)$ will transform Dirichlet internal problems of Poisson equations

$$\begin{cases} \Delta u = F(M)\,, & M \in D\,, \\ u|_C = f(M)\,. \end{cases} \tag{7.176}$$

into Dirichlet internal problems of Laplace equations

$$\begin{cases} \Delta\omega = 0\,, & M \in D\,, \\ \omega|_C = f(M) - u^*|_C\,. \end{cases} \tag{7.177}$$

Similarly, the other three problems of Poisson equations can also be transformed into those of Laplace equations.

Remark 3. Neumann problems in a plane can also be transformed into the Dirichlet problems by using the relation between analytical and harmonic functions. Suppose that a Neumann problem of the Laplace equation has solution $u(x,y)$ in a simply-connected plane domain D. The $u(x,y)$ and its first partial derivatives are assumed to be continuous in a closed domain $\bar{D} = D \cup \partial D$. There must exist a harmonic function $v(x,y)$, conjugate to u, such that u and v satisfy the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,. \tag{7.178}$$

The $v(x,y)$ that satisfies Eq. (7.178) is also unique up to an arbitrary constant.

Let $\mathbf{n}$ and $\boldsymbol{\tau}$ be the external normal and positive tangent of ∂D, respectively. A counter-clockwise rotation of $\mathbf{n}$ by 90° will thus arrive at $\boldsymbol{\tau}$. By the relation between directional and partial derivatives, we have, for $(x,y) \in \partial D$,

$$\begin{aligned} \frac{\partial u}{\partial n} &= \frac{\partial u}{\partial x}\cos(\mathbf{n},x) + \frac{\partial u}{\partial y}\cos(\mathbf{n},y) \\ &= \frac{\partial v}{\partial y}\cos(\mathbf{n},x) - \frac{\partial v}{\partial x}\cos(\mathbf{n},y) \\ &= \frac{\partial v}{\partial x}\cos(\boldsymbol{\tau},x) + \frac{\partial v}{\partial y}\cos(\boldsymbol{\tau},y) = \frac{\partial v}{\partial \tau} \end{aligned}$$

Consider Neumann problems of Laplace equations

$$\begin{cases} \Delta u = 0\,, (x,y) \in D\,, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial D} = f(x,y)\,. \end{cases} \tag{7.179}$$

By the necessary condition for the existence of solutions,

$$\oint\limits_{\partial D} f(x,y)\,\mathrm{d}s = 0\,,$$

where s is the arclength. Since $\left.\dfrac{\partial v}{\partial \tau}\right|_{\partial D} = f(x,y)$, the monotropic continuous function defined on ∂D

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} f(x,y)\,\mathrm{d}s + v(x_0,y_0) \tag{7.180}$$

can be determined for fixed point $A(x_0,y_o) \in \partial D$. Thus $v(x,y)$ must satisfy

$$\begin{cases} \Delta v = 0\,, \quad (x,y) \in D\,, \\ v|_{\partial D} = F(x,y)\,. \end{cases} \tag{7.181}$$

where $F(x,y) = \displaystyle\int_{(x_0,y_0)}^{(x,y)} f(x,y)\,\mathrm{d}s + v(x_0,y_0)$. Once the solution of Dirichlet problem (7.181) is available, we may obtain the solution $u(x,y)$ of PDS (7.179) by Eq. (7.178). The $u(x,y)$ so obtained is unique up to an arbitrary constant.

Remark 4. Green functions for Dirichlet problems of two-dimensional potential equations can be found by using a mirror image method if the domain has some symmetry. If the domain is lacking such symmetry, we can first apply a conformal transformation to transform the domain into a symmetrical one. Let be a simply-connected plane domain with a smooth boundary. Suppose that $w = w(z)$ ($z = x + \mathrm{i}y$ is a complex-valued variable) is a conformal transformation that is able to transform D into a unit circle $|w| < 1$. Let $z_0 = x_0 + \mathrm{i}y_0$ be an internal point of D. The conformal transformation

$$w = w(z,z_0) = \frac{w(z) - w(z_0)}{1 - w(z)\overline{w(z_0)}}$$

will map D into the unit circle $|w| < 1$ and z_0 into the center of $|w| < 1$. The solution of

$$\begin{cases} -\Delta G(x,y;x_0,y_0) = \delta(x - x_0, y - y_0), (x,y) \in D\,, \\ G|_{\partial D} = 0\,, \end{cases}$$

is

$$G(x,y;x_0,y_0)=\frac{1}{2\pi}\ln\frac{1}{|w(z,z_0)|}\,.$$

$G(x,y;x_0,y_0)$ is the required Green function.

Remark 5. Two-dimensional problems can also sometimes be solved directly by using a conformal transformation. For example, consider

$$\begin{cases}\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0\,,\\ T|_{y=0}=\begin{cases}T_0\,, & |x|<a\,,\\ 0\,, & |x|>a\,.\end{cases}\end{cases}$$

Using the conformal transformation $w=\ln\frac{z-a}{z+a}=\xi+\mathrm{i}\eta$, it is transformed into a boundary-value problem of an ordinary differential equation

$$\begin{cases}\frac{\mathrm{d}^2 T}{\mathrm{d}\eta^2}=0\\ T|_{\eta=0}=0\,, T|_{\eta=\pi}=T_0\,.\end{cases}$$

Its solution is

$$\begin{aligned}T(x,y)&=\frac{T_0}{\pi}\eta=\frac{T_0}{\pi}\mathrm{Im}\left(\ln\frac{z-a}{z+a}\right)\\ &=\frac{T_0}{\pi}\mathrm{Im}[\ln(z-a)-\ln(z+a)]=\frac{T_0}{\pi}\arctan\frac{2ay}{x^2+y^2-a^2}\,.\end{aligned}$$

Appendix A
Special Functions

In solving PDS, separation of variables sometimes leads to some special kinds of linear ordinary differential equations. Two typical examples are the Bessel equation and the Legendre equation. Particular solutions of these equations are called *special functions*. They play the same role as the orthogonal set of trigonometric functions in Fourier series and serve as the function bases for expanding solutions of PDS. The series of function terms so obtained is a generalized Fourier series.

A.1 Bessel and Legendre Equations

Consider a mixed problem of two-dimensional heat-conduction equations

$$\begin{cases} u_t = a^2 \Delta u\,, \quad x^2+y^2<R^2, 0<t\,, \\ u(x,y,0)=\varphi(x,y)\,, \\ u|_{x^2+y^2=R^2}=0\,. \end{cases}$$

Let $u=U(x,y)T(t)$. Substituting it into the equation yields, with $-\lambda$ as the separation constant,

$$\Delta U + \lambda U = 0\,, \quad U|_{x^2+y^2=R^2}=0\,. \tag{A.1}$$

$$T' + \lambda T = 0\,. \tag{A.2}$$

If $\lambda=0$, $U(x,y)\equiv 0$. Therefore $\lambda \neq 0$ by Eq. (A.1). By Eq. (A.2), we have $T(t)=c\,\mathrm{e}^{-\lambda t}$. Since $T(t)$ must be bounded, we obtain $\lambda>0$. The equation in (A.1) is called the *Helmholtz equation*. In a polar coordinate system, Eq. (A.1) reads

$$\begin{cases} \dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial \theta^2} + \lambda U = 0, \quad 0<r<R\,, \\ U|_{r=R}=0\,, \end{cases} \tag{A.3}$$

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which has a homogeneous boundary condition. Assume $U(r,\theta) = R(r)\Theta(\theta)$. Substituting it into Eq. (A.3) yields

$$\frac{r^2R'' + rR' + \lambda r^2 R}{R} = -\frac{\Theta''}{\Theta}\,.$$

Thus, with μ as the separation constant,

$$r^2R'' + rR' + (\lambda r^2 - \mu)R = 0 \tag{A.4}$$

and

$$\Theta''(\theta) + \mu\Theta(\theta) = 0\,.$$

Since $\Theta(\theta + 2\pi) = \Theta(\theta)$, μ_n must be

$$\mu_n = n^2, \quad n = 0, 1, 2, \cdots.$$

Therefore Eq. (A.4) becomes

$$x^2F''(x) + xF'(x) + \left(x^2 - n^2\right)F(x) = 0\,, \tag{A.5}$$

where $x = \sqrt{\lambda}r$, $F(x) = R\left(\dfrac{x}{\sqrt{\lambda}}\right)$. Equation (A.5) is a linear homogeneous ordinary differential equation of second order with variable coefficients. It is called the *Bessel equation of n-th order* and also appears in solving two-dimensional wave equations and Laplace equations in a cylindrical coordinate system.

The three-dimensional Laplace equation in a spherical coordinate system reads

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0\,. \tag{A.6}$$

Let $u = R(r)\Theta(\theta)\Phi(\varphi)$. Substituting it into Eq. (A.6) yields

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = -\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) - \frac{1}{\Phi\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}\,.$$

Thus with $n(n+1)$ as the separation constant,

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + 2r\frac{\mathrm{d}R}{\mathrm{d}r} - n(n+1)R = 0\,,$$

$$\frac{1}{\Theta}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + n(n+1)\sin^2\theta = -\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}\,.$$

The former is the Euler equation. The latter leads to, with η as the separation constant,

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) - \frac{\eta}{\sin^2\theta}\Theta + n(n+1)\Theta = 0\,,$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} + \eta\Phi = 0\,, \quad \Phi(\varphi + 2\pi) = \Phi(\varphi)\,.$$

By $\Phi(\varphi)$-equation, we obtain $\eta = m^2$, $m = 0, 1, 2, \cdots$. Therefore,

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0. \tag{A.7}$$

It is called the *associated Legendre equation*. Let $x = \cos\theta \, (-1 < x < 1)$ and $P(x) = \Theta(\theta)$. Equation (A.7) becomes

$$\left(1 - x^2\right)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]P = 0. \tag{A.8}$$

It is called *another form of the associated Legendre equation*.

When $u(r, \theta, \varphi)$ is independent of φ, in particular, $\eta = 0$ and $m = 0$. Equation (A.8) thus reduces to

$$\left(1 - x^2\right)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + n(n+1)P = 0, \tag{A.9}$$

which is called the *Legendre equation*.

A.2 Bessel Functions

The Bessel function is the series solution of the Bessel equation. We have the following theorem regarding series solutions.

Theorem. Suppose that $a(x)$ and $b(x)$ are expandable into power series at $x = 0$. The equation

$$y'' + \frac{a(x)}{x}y' + \frac{b(x)}{x^2}y = 0$$

has at least one series solution $y = x^r \sum_{k=0}^{+\infty} c_k x^k$, where r is a constant.

Let x and y be the independent and dependent variables, respectively. A general form of Bessel equations of γ-th order is

$$x^2y'' + xy' + (x^2 - \gamma^2)y = 0, \quad \gamma \geq 0 \tag{A.10}$$

or

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\gamma^2}{x^2}\right)y = 0,$$

where γ can be any real or complex constant. Suppose that the solution of Eq. (A.10) is

$$y = x^c \sum_{k=0}^{+\infty} a_k x^k = \sum_{k=0}^{+\infty} a_k x^{c+k}, \quad a_0 \neq 0. \tag{A.11}$$

Note that by the theory of power series, we can take derivatives of y with respect to x term by term in the convergent domain of $\sum_{k=0}^{+\infty} a_k x^k$. Thus

$$y' = \sum_{k=0}^{+\infty} (c+k) a_k x^{c+k-1}, \quad y'' = \sum_{k=0}^{+\infty} (c+k)(c+k-1) a_k x^{c+k-2}.$$

Substituting y, y' and y'' into Eq. (A.10) yields

$$\left(c^2-\gamma^2\right) a_0 + \left[(c+1)^2-\gamma^2\right] a_1 x + \sum_{k=2}^{+\infty} \left\{\left[(c+k)^2-\gamma^2\right] a_k + a_{k-2}\right\} x^k = 0.$$

By the uniqueness of expansion of power series, we thus have

$$\begin{aligned} a_0\left(c^2-\gamma^2\right) = 0, \quad a_1\left[(c+1)^2-\gamma^2\right] = 0, \\ \left[(c+k)^2-\gamma^2\right] a_k + a_{k-2} = 0, \quad k = 2, 3, \cdots. \end{aligned}$$

From the first equation, we obtain $c^2 - \gamma^2 = 0$, so $c = \pm\gamma$. Substituting it into the second equation yields $a_1 = 0$. A substitution of $c = \gamma$ into the third equation leads to $a_k = \dfrac{-a_{k-2}}{k(2\gamma+k)}$. Since $a_1 = 0$, we have $a_1 = a_3 = a_5 = \cdots = 0$.

When k is even,

$$\begin{aligned} a_2 &= \frac{-a_0}{2(2\gamma+2)}, \quad a_4 = \frac{a_0}{2\cdot 4(2\gamma+2)(2\gamma+4)}, \\ a_6 &= \frac{-a_0}{2\cdot 4\cdot 6(2\gamma+2)(2\gamma+4)(2\gamma+6)} \cdots, \\ a_{2m} &= (-1)^m \frac{a_0}{2^{2m} m!(\gamma+1)(\gamma+2)\cdots(\gamma+m)}. \end{aligned}$$

Thus Eq. (A.11) becomes

$$y = \sum_{m=0}^{+\infty} (-1)^m \frac{a_0 x^{n+2m}}{2^{2m} m!(\gamma+1)(\gamma+2)\cdots(\gamma+m)},$$

where a_0 can be any constant. For a specified a_0, we have a particular solution of Eq. (A.10). For convenience, let

$$a_0 = \frac{1}{2^n \Gamma(\gamma+1)},$$

where the $\Gamma-$function satisfies $\Gamma(x+1) = x\Gamma(x)$. The particular solution $J_\gamma(x)$ of Eq. (A.10) is thus

$$J_\gamma(x) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\Gamma(\gamma+m+1)} \left(\frac{x}{2}\right)^{\gamma+2m}. \tag{A.12}$$

It is called the *Bessel function of γ-th order of the first kind*. It converges, by the ratio test, over the whole real-axis.

For $c=-\gamma$, similarly, we obtain another particular solution of Eq. (A.10)

$$J_{-\gamma}(x)=\sum_{m=0}^{+\infty}\frac{(-1)^m}{m!\Gamma(-\gamma+m+1)}\left(\frac{x}{2}\right)^{-\gamma+2m}. \tag{A.13}$$

When $\gamma=n=0,1,2,\cdots$, in particular,

$$J_n(x)=\sum_{m=0}^{+\infty}\frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{n+2m},$$

and consequently,

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^4(2!)^2}-\cdots+(-1)^k\frac{x^{2k}}{2^{2k}(k!)^2}+\cdots,$$
$$J_1(x)=\frac{x}{2}-\frac{x^3}{2^3 2!}+\frac{x^5}{2^5 2!3!}-\cdots+(-1)^k\frac{x^{2k+1}}{2^{2k+1}k!(k+1)!}+\cdots.$$

By the last remark in the present appendix, for any natural number N,

$$\frac{1}{\Gamma(-N+m+1)}=0, \quad m=0,1,2,\cdots,N-1.$$

Thus

$$\begin{aligned}J_{-N}(x)&=\sum_{m=N}^{+\infty}(-1)^m\frac{1}{m!\Gamma(-N+m+1)}\left(\frac{x}{2}\right)^{-N+2m}\\&=(-1)^N\left[\frac{x^N}{2^N N!}-\frac{x^{N+2}}{2^{N+2}(N+1)!}+\frac{x^{N+4}}{2^{N+4}(N+2)!2!}+\cdots\right]\\&=(-1)^N J_N(x).\end{aligned}$$

Therefore, $J_n(x)$ and $J_{-n}(x)$ are linearly dependent for any natural number n. When γ is a positive but not natural number, $J_\gamma(0)=0$ and $J_{-\gamma}(0)=\infty$. Therefore, $J_\gamma(x)$ and $J_{-\gamma}(x)$ are linearly independent. Thus we obtain the general solution of Eq. (A.10)

$$y=C_1J_\gamma(x)+C_2J_{-\gamma}(x). \tag{A.14}$$

By letting $C_1=\cot\gamma\pi$ and $C_2=-\csc\gamma\pi$ in Eq. (A.14), we obtain another solution

$$Y_\gamma(x)=\frac{J_\gamma(x)\cos\gamma\pi-J_{-\gamma}(x)}{\sin\gamma\pi}. \tag{A.15}$$

It is called the *Bessel function of γ-th order of the second kind.*

Since $J_\gamma(0)=0$ and $Y_\gamma(0)=\infty$, $Y_\gamma(x)$ and $J_\gamma(x)$ are linearly independent. When γ is not an integer, the general solution of Eq. (A.10) can also be written as

$$y=C_1J_\gamma(x)+C_2Y_\gamma(x).$$

When $\gamma=n$ (0 or natural numbers), the right-hand side of Eq. (A.15) becomes $\dfrac{0}{0}$.

Thus we define

$$Y_n(x) = \lim_{\gamma\to n} Y_\gamma(x) = \lim_{\gamma\to n} \frac{J_\gamma(x)\cos\gamma\pi - J_{-\gamma}(x)}{\sin\gamma\pi}\,.$$

This is called the *Bessel function of integer-order of the second kind* and can be found by using the L'Hôpital's rule. It can also be proven that $Y_n(x)$ and $J_n(x)$ are linearly independent. Thus the general solution of Eq. (A.10) is for all γ ($\gamma \geq 0$ without loss of generality),

$$y = C_1 J_\gamma(x) + C_2 Y_\gamma(x)\,.$$

Remark 1. $J_\gamma(x)$ and $Y_\gamma(x)$, the symbols of Bessel functions, can be used like trigonometric and logarithmic functions. We also have tables of Bessel functions. By using these symbols we may concisely express general solutions of Bessel equations. For example, the general solution of the Bessel equation

$$x^2y'' + xy' + \left(x^2 - \frac{9}{25}\right)y = 0$$

is

$$y = C_1 J_{\frac{3}{5}}(x) + C_2 Y_{\frac{3}{5}}(x)$$

or

$$y = C_1 J_{\frac{3}{5}}(x) + C_2 J_{-\frac{3}{5}}(x)\,,$$

where C_1 and C_2 are arbitrary constants. The general solution of

$$y'' + \frac{1}{x}y' + \left(1 - \frac{16}{x^2}\right)y = 0$$

is $y = C_1 J_4(x) + C_2 Y_4(x)$, where C_1 and C_2 are arbitrary constants.

Remark 2. By a variable transformation of $t = mx$, the equation

$$x^2y'' + xy' + \left(m^2x^2 - n^2\right)y = 0$$

is transformed into a Bessel equation of n-th order

$$t^2y'' + ty' + \left(t^2 - n^2\right)y = 0\,.$$

Example 1. Find the general solution of $y'' + \frac{1}{x}y' + \left(9 - \frac{4}{x^2}\right)y = 0$.

Solution. Consider a variable transformation $t = 3x$. The equation is transformed into a Bessel equation of second order

$$t^2y'' + ty' + (t^2 - 4)y = 0\,.$$

Its general solution is

$$y = C_1 J_2(t) + C_2 Y_2(t) = C_1 J_2(3x) + C_2 Y_2(3x)\,.$$

Example 2. Find the solution of

$$\begin{cases} x^2 y'' + xy' + \left(4x^2 - \dfrac{1}{9}\right) y = 0\,, \\ y(0.3) = 2\,. \end{cases}$$

where $y(x)$ is continuous at $x = 0$.

Solution. Consider a variable transformation $t = 2x$. The equation is transformed into a Bessel equation of $\frac{1}{3}$-order. Therefore the general solution is

$$y = C_1 J_{\frac{1}{3}}(2x) + C_2 J_{-\frac{1}{3}}(2x)\,.$$

The continuity of $y(x)$ at $x = 0$ thus yields $C_2 = 0$. Applying $y(0.3) = 2$ leads to

$$C_1 = \frac{2}{J_{\frac{1}{3}}(0.6)} = 2.857\,.$$

Thus $y = 2.857 J_{\frac{1}{3}}(2x)$.

Remark 3. The linear combination of Bessel functions of the first and second kinds

$$H_\gamma^{(1)}(x) = J_\gamma(x) + \mathrm{i} Y_\gamma(x)\,, \quad H_\gamma^{(2)}(x) = J_\gamma(x) - \mathrm{i} Y_\gamma(x)$$

are called the *Bessel function of the third kind* or the *Hankel function*. Here $\mathrm{i} = \sqrt{-1}$.

In solving some PDS we arrive at

$$y'' + \frac{1}{x} y' - \left(1 + \frac{\gamma^2}{x^2}\right) y = 0\,. \tag{A.16}$$

By a variable transformation $t = \mathrm{i}x$, it is transformed into

$$y'' + \frac{1}{t} y' + \left(1 - \frac{\gamma^2}{t^2}\right) y = 0\,.$$

Its general solution reads $y = C_1 J_\gamma(\mathrm{i}x) + C_2 Y_\gamma(\mathrm{i}x)$.

Note that $(\mathrm{i})^{\gamma+2m} = (-1)^m \mathrm{i}^\gamma$. Thus

$$J_\gamma(\mathrm{i}x) = \mathrm{i}^\gamma \sum_{m=0}^{+\infty} \frac{1}{m!\Gamma(\gamma+m+1)} \left(\frac{x}{2}\right)^{\gamma+2m}$$

and

$$I_\gamma(x) = \mathrm{i}^{-\gamma} J_\gamma(\mathrm{i}x) = \sum_{m=0}^{+\infty} \frac{1}{m!\Gamma(\gamma+m+1)} \left(\frac{x}{2}\right)^{\gamma+2m}$$

is also a particular solution of Eq. (A.16) and is called the *modified Bessel function of the first kind*. When γ is not an integer, the particular solution

$$K_\gamma(x) = \frac{\frac{1}{2}\pi\left[I_{-\gamma}(x) - I_\gamma(x)\right]}{\sin\gamma\pi}$$

is linearly independent of $I_\gamma(x)$ and is called the *modified Bessel function of the second kind*. To demonstrate its linear independence with $I_\gamma(x)$, consider $\gamma > 0$ (γ is not an integral) so that $I_\gamma(0) = 0$, $I_{-\gamma}(0) = \infty$ and $K_\gamma(0) = \infty$. Therefore $K_\gamma(x)$ is linearly independent of $I_\gamma(x)$. When γ is an integer n, define

$$K_n(x) = \lim_{\gamma\to n} K_\gamma(x) = \lim_{\gamma\to n} \frac{\frac{1}{2}\pi\left[I_{-\gamma}(x) - I_\gamma(x)\right]}{\sin\gamma\pi}\,.$$

It can be shown that $K_n(x)$ so defined is linearly independent of $I_n(x)$. Therefore the general solution of Eq. (A.6) is, regardless of the value of γ

$$y = C_1 I_\gamma(x) + C_2 K_\gamma(x)\,,$$

where C_1 and C_2 are constants.

The Kelvin function of n-th order of the first kind has two forms: the real part and the imaginary part of $J_n\left(x\sqrt{-\mathrm{i}}\right)$ termed by $ber_n(x)$ and $bei_n(x)$, respectively.

$$ber_n(x) = Re\left[J_n\left(x\sqrt{-\mathrm{i}}\right)\right]\,, \quad bei_n(x) = Im\left[J_n\left(x\sqrt{-\mathrm{i}}\right)\right]\,.$$

The $ber_0(x)$, $bei_0(x)$, $ber_1(x)$ and $bei_1(x)$ appear frequently in applications.

A.3 Properties of Bessel Functions

Differential Property and Recurrence Formula

Based on the series expression of $J_1(x)$, the Bessel function of the first kind, a differentiation term by term yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\gamma J_\gamma(x)\right] = x^\gamma J_{\gamma-1}(x)\,, \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\gamma} J_\gamma(x)\right] = -x^{-\gamma} J_{\gamma+1}(x)\,,$$

$$J_{\gamma-1}(x) + J_{\gamma+1}(x) = \frac{2}{x}\gamma J_\gamma(x)\,, \quad J_{\gamma-1}(x) - J_{\gamma+1}(x) = 2J'_\gamma(x)\,.$$

When $\gamma = n$ (natural numbers), in particular, we have the recurrence formula

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2}{x} n J_n(x)\,.$$

Therefore we may obtain values of $J_N(x)$ based on the function tables of $J_0(x)$ and $J_1(x)$ for any natural number $N > 1$. We may also obtain $J_{-N}(x)$ by

$$J_{-N}(x) = (-1)^N J_N(x)\,.$$

Since $Y_\gamma(x)$, $H_\gamma^{(1)}(x)$ and $H_\gamma^{(2)}(x)$ are formed by linear combinations of $J_\gamma(x)$, they also have corresponding differential properties. The differential properties of Bessel functions can also be expressed in an integral form.

Bessel Functions of Semi-Odd Order

Note that

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}+m\right)=\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2m+1)}{2^{m+1}}\sqrt{\pi}, \quad m=0,1,2,\cdots.$$

Thus

$$J_{\frac{1}{2}}(x)=\sum_{m=0}^{+\infty}\frac{(-1)^m}{m!\Gamma\left(\frac{3}{2}+m\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2m}=\sqrt{\frac{2}{\pi x}}\sin x.$$

Similarly, $J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos x$. For $\gamma=n-\dfrac{1}{2}$, the formula

$$J_{\gamma+1}(x)=\frac{2}{x}\gamma J_\gamma(x)-J_{\gamma-1}(x)$$

yields

$$J_{n+\frac{1}{2}}(x)=\frac{2}{x}\left(n-\frac{1}{2}\right)J_{n-\frac{1}{2}}(x)-J_{n-\frac{3}{2}}(x). \tag{A.17}$$

For $\gamma-1=-\left(n+\dfrac{1}{2}\right)$ the formula $J_{\gamma-1}(x)=\dfrac{2}{x}\gamma J_\gamma(x)-J_{\gamma+1}(x)$ leads to

$$J_{-\left(n+\frac{1}{2}\right)}(x)=\frac{2}{x}\left(\frac{1}{2}-n\right)J_{-\left(n-\frac{1}{2}\right)}(x)-J_{-n+\frac{3}{2}}(x). \tag{A.18}$$

A repeated application of Eqs. (A.17) and (A.18) thus yields

$$J_{n+\frac{1}{2}}(x)=(-1)^n\sqrt{\frac{2}{\pi}}x^{n+\frac{1}{2}}\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\frac{\sin x}{x}\right),$$

$$J_{-\left(n+\frac{1}{2}\right)}(x)=\sqrt{\frac{2}{\pi}}x^{n+\frac{1}{2}}\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\frac{\cos x}{x}\right),$$

where $\left(\dfrac{1}{x}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^n$ stands for the $\dfrac{1}{x}\dfrac{\mathrm{d}}{\mathrm{d}x}$ operation on a function n-times so that $\left(\dfrac{1}{x}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^n\neq\dfrac{1}{x^n}\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}$.

Therefore we may obtain Bessel functions of semi-odd order by a finite number of the fundamental arithmetic operations of sine, cosine and power functions. For this reason they are called the elementary functions.

3. Generating Function and Integral Formula

The generating function of Bessel functions is the function of two variables whose power series expansion has the Bessel functions as its coefficients. The generating function of Bessel functions of the first kind is

$$w(x,t) = \mathrm{e}^{\frac{x}{2}\left(t-\frac{1}{t}\right)}.$$

Its power series expansion in t and $-t^{-1}$ is

$$\begin{aligned}
w(x,t) &= \mathrm{e}^{\frac{x}{2}\left(t-\frac{1}{t}\right)} \\
&= \left[1+\frac{\frac{x}{2}t}{1!}+\frac{\left(\frac{x}{2}\right)^2 t^2}{2!}+\frac{\left(\frac{x}{2}\right)^3 t^3}{3!}+\cdots\right] \\
&\quad \cdot \left[1-\frac{\frac{x}{2}t^{-1}}{1!}+\frac{\left(\frac{x}{2}\right)^2 t^{-2}}{2!}-\frac{\left(\frac{x}{2}\right)^3 t^{-3}}{3!}+\cdots\right] \\
&= \cdots + t^{-2}J_{-2}(x)+t^{-1}J_{-1}(x)+J_0(x)+tJ_1(x)+t^2J_2(x)+\cdots \\
&= \sum_{n=-\infty}^{+\infty} J_n(x)\,t^n .
\end{aligned}$$

To obtain an integral formula of Bessel functions, consider a variable transformation

$$t = \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta \quad \text{or} \quad t^{-1} = \cos\theta - \mathrm{i}\sin\theta .$$

Note also that $J_{-N}(x) = (-1)^N J_N(x)$. Thus

$$\mathrm{e}^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \mathrm{e}^{\mathrm{i}x\sin\theta} = \cos(x\sin\theta) + \mathrm{i}\sin(x\sin\theta) .$$

Also

$$\begin{aligned}
J_1(x)t + J_{-1}(x)t^{-1} &= J_1(x)\left(t-\frac{1}{t}\right) = 2\mathrm{i}J_1(x)\sin\theta , \\
J_2(x)t^2 + J_{-2}(x)t^{-2} &= J_2(x)\left(t^2+\frac{1}{t^2}\right) = 2J_2(x)\cos 2\theta , \\
&\cdots\cdots .
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{e}^{\frac{x}{2}\left(t-\frac{1}{t}\right)} &= J_0(x) + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right] \\
&\quad + 2\mathrm{i}\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\right] .
\end{aligned}$$

A comparison of real and imaginary parts yields

$$\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots],$$

$$\sin(x\sin\theta) = 2[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots].$$

Therefore, by the theory of Fourier series,

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos n\theta\,\mathrm{d}\theta = \begin{cases} J_n(x), & n = 0,2,4,\cdots, \\ 0, & n = 1,3,5,\cdots, \end{cases}$$

$$\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin n\theta\,\mathrm{d}\theta = \begin{cases} J_n(x), & n = 1,3,5,\cdots, \\ 0, & n = 0,2,4,\cdots. \end{cases}$$

Finally, we obtain an integral formula of Bessel functions by adding these two equations,

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\,\mathrm{d}\theta, \quad n = 0,1,2,\cdots.$$

Similarity with Sine and Cosine Functions

A comparison of power series expansions of $\cos x$ and $J_0(x)$ shows that $J_0(x)$ is quite similar to $\cos x$. Similarly, $J_1(x)$ is similar to $\sin x$. For example, $J_0(x)$ is a even function, while $J_1(x)$ is an odd function. $J_0(x) = 0$ has no complex root but an infinite number of distinct real roots. The approximate values of its positive real roots are,

$$2.405, \quad 5.520, \quad 8.654, \quad 11.792, \quad \cdots$$

Similarly, $J_1(x) = 0$ has no complex root but an infinite number of real roots. The approximate values of its positive real roots are

$$3.832, \quad 7.061, \quad 10.173, \quad 13.324, \quad \cdots$$

Thus, the zero points of $J_0(x)$ and $J_1(x)$ occur alternately. Also, the distance between two adjoining zero points of $J_0(x)$ or $J_1(x)$ tends to π as $|x| \to +\infty$. $J_0(x)$ and $J_1(x)$ are thus called *periodic functions with a period of almost* 2π. The graphs of $J_0(x)$ and $J_1(x)$ are also quite similar to those of $\cos x$ and $\sin x$.

Similarly, we have the following properties for $J_n(x)$ with n as an integer: (1) $J_n(x)$ has no complex zero-points, but an infinite number of real zero-points. All zero points are symmetrically distributed with respect to $x = 0$. All zero-points except $x = 0$ are single zero-points; (2) the zero-points of $J_n(x)$ and $J_{n+1}(x)$ occur alternately; (3) the $\mu_{m+1}^{(n)} - \mu_m^{(n)}$ tends to π as $m \to +\infty$. Here $\mu_m^{(n)}$ stands for the m-th positive zero-point of $J_n(x)$. Therefore, $J_n(x)$ is a *periodic function with a period of almost* 2π.

5. Approximate Formulas

When x is sufficiently large, we have

$$J_\gamma(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{\gamma}{2}\pi\right), \quad Y_\gamma(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{\gamma}{2}\pi\right),$$
$$H_\gamma^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} \mathrm{e}^{\mathrm{i}\left(x - \frac{\pi}{4} - \frac{\gamma}{2}\pi\right)}, \quad H_\gamma^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} \mathrm{e}^{-\mathrm{i}\left(x - \frac{\pi}{4} - \frac{\gamma}{2}\pi\right)}.$$

A.4 Legendre Polynomials

Consider the Legendre equation

$$\left(1 - x^2\right) y'' - 2xy' + n(n+1)y = 0, \tag{A.19}$$

where n is a real constant. Its series solution takes the form

$$y = x^c \sum_{k=0}^{+\infty} a_k x^k = \sum_{k=0}^{+\infty} a_k x^{c+k},$$

where $a_0 \neq 0$. Substituting it into Eq. (A.19) leads to

$$\begin{aligned} &- \sum_{k=0}^{+\infty} \left[(k+c)(k+c+1) - n(n+1)\right] a_k x^{k+c} \\ &+ \sum_{k=0}^{+\infty} (k+c)(k+c-1) a_k x^{k+c-2} = 0. \end{aligned} \tag{A.20}$$

Therefore all the coefficients must be zero. From the coefficients of x^{c-2} and x^{c-1}, we obtain

$$c(c-1)a_0 = 0, \quad c(c+1)a_1 = 0.$$

Since $a_0 \neq 0$, $c = 0$ or $c = 1$, Eq. (A.20) can also be rewritten as

$$\begin{aligned} &- \sum_{k=0}^{+\infty} \left[(k+c)(k+c+1) - n(n+1)\right] a_k x^{k+c} \\ &+ \sum_{k=-2}^{+\infty} (k+c+2)(k+c+1) a_{k+2} x^{k+c} = 0. \end{aligned}$$

By the coefficient of the general term, we have

$$a_{k+2} = \frac{(k+c)(k+c+1) - n(n+1)}{(k+c+1)(k+c+2)} a_k, \quad k = 0, 1, 2, \cdots. \tag{A.21}$$

When $c=0$, a_0 and a_1 can take any value. A repeated application of Eq. (A.21) yields a_2, a_4, a_6, $\cdots$ in terms of a_0 and a_3, a_5, a_7, $\cdots$ in terms of a_1. Finally, we obtain the solution of Eq. (A.19)

$$y = a_0 y_1 + a_1 y_2 \,,$$

where

$$y_1 = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots \,,$$

$$y_2 = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots \,.$$

Note that a_0 and a_1 are two arbitrary constants. Also, the y_1 and the y_2 are linearly independent. Thus y is the general solution of Eq. (A.19). By the solution structure of linear homogeneous equations, $Y_1 = a_0 y_1$ and $Y_2 = a_1 y_2$ are also solutions of Eq. (A.19). The convergence radius of series Y_1 and Y_2 is 1. The Y_1 and the Y_2 are, however, divergent at $x = \pm 1$.

When $c=1$, $a_1 = 0$ so that $a_3 = a_5 = \cdots = 0$. The xy_1 is thus a particular solution. For an integer n, in particular, Y_1 or Y_2 becomes a polynomial. When n is positive and even or negative and odd, Y_1 reduces into a polynomial of degree not larger than n. When n is positive and odd or negative and even, Y_2 also reduces into a polynomial of degree not larger than n. When $c=0$, to express such polynomials, rewrite Eq. (A.21) as

$$a_k = -\frac{(k+2)(k+1)}{(n-k)(k+n+1)}a_{k+2} \,,$$

where $k+2 \le n$ so that $k \le n-2$. A repeated application of the recurrence formula thus yields a_{n-2}, a_{n-4}, $\cdots$ in terms of a_n. For example,

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)}a_n \,.$$

Therefore, a_n becomes an arbitrary constant. Take $a_n = \dfrac{(2n)!}{2^n (n!)^2}$, we have

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} \,, \qquad n-2m \ge 0 \,.$$

The Y_1 or the Y_2 thus reduces into a fixed polynomial denoted by

$$P_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \,, \tag{A.22}$$

where $\left[n/2\right]$ stands for the maximum integer less than or equal to $n/2$. The $P_n(x)$ in Eq. (A.22) is called the *Legendre polynomial of n-th degree* or the *Legendre function of the first kind*.

When $n = 0, 1, 2, 3, 4, 5$, we have

$$P_0(x) = 1\,, \quad P_1(x) = x\,, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)\,, \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)\,,$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)\,, \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)\,.$$

Another particular solution $Q_n(x)$ is linearly independent of $P_n(x)$ and is a series of infinite terms. It is called the *Legendre function of the second kind*. When n is an integer, therefore, the general solutions of Legendre equations can be expressed by the Legendre functions of the first and the second kind, i.e.

$$y = C_1 P_n(x) + C_2 Q_n(x)\,.$$

It can be shown that the convergence radius of $Q_n(x)$ is also 1. However, $Q_n(x)$ is divergent at $x = \pm 1$; thus we discuss the Legendre polynomials always in $(-1, 1)$.

A.5 Properties of Legendre Polynomials

1. The Rodrigue Expression of $\mathbf{P_n(x)}$

Let u and v be the n-th differentiable functions. By using the binomial theorem to expand

$$(x^2 - 1)^n \quad \text{or} \quad (uv)^{(n)} = \sum_{k=0}^{n} c_n^k u^{(k)} v^{(n-k)}\,,$$

we can obtain

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n\,,$$

which is called the *Rodrigue expression of the Legendre polynomials*.

2. Generating Function and Recurrence Formula

Consider a function of the complex variable t

$$w(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}}\,.$$

Let r be the root with a smaller normal of two roots of the quadratic equation $1 - 2xt + t^2 = 0$. Thus the $w(x,t)$ is analytical in the circle $|t| < r$. By the theory of

Laurent series, we have

$$w(x,t)=\sum_{n=0}^{+\infty}c_n(x)t^n\,,\quad |t|<r\,,$$

$$c_n(x)=\frac{1}{2\pi\mathrm{i}}\oint_C\frac{\left(1-2xt+t^2\right)^{-\frac{1}{2}}}{t^{n+1}}\,\mathrm{d}t\,.$$

Here C is a closed curve containing $t=0$ in $|t|<r$.

Consider the Euler transformation

$$\left(1-2xt+t^2\right)^{\frac{1}{2}}=1-tu\quad\text{or}\quad u=\frac{1-\left(1-2xt+t^2\right)^{\frac{1}{2}}}{t}\,,$$

when $t\to 0$, $u\to x$ and curve C becomes a closed curve C' that contains $u=x$. Therefore

$$\begin{aligned}c_n(x)&=\frac{1}{2\pi\mathrm{i}}\oint_{C'}\frac{\left(u^2-1\right)^n}{2^n(u-x)^{n+1}}\,\mathrm{d}u\\&=\frac{1}{2^n n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}u^n}(u^2-1)^n\right]\bigg|_{u=x}=P_n(x)\,.\end{aligned}$$

Thus

$$w(x,t)=\left(1-2xt+t^2\right)^{-\frac{1}{2}}=\sum_{n=0}^{+\infty}P_n(x)t^n\,;\tag{A.23}$$

and $w(x,t)$ is called the *generating function* of the Legendre polynomials $P_n(x)$.

When $x=1$,

$$w(x,t)=\sum_{n=0}^{+\infty}P_n(x)t^n=\frac{1}{\sqrt{t^2-2t+1}}=\frac{1}{1-t}=1+t+t^2+\cdots+t^{n-1}+\cdots\,.$$

Thus $P_n(1)=1$. Similarly, we have

$$P_n(-1)=(-1)^n\,,\quad P_{2n-1}(0)=0\,,\quad P_{2n}(0)=\frac{(-1)^n\,(2n)!}{2^{2n}(n!)^2}\,.$$

By taking derivatives of

$$w(x,t)=\left(1-2xt+t^2\right)^{\frac{1}{2}}$$

with respect to t and x, we obtain

$$\left(1-2xt+t^2\right)\frac{\partial w}{\partial t}+(t-x)w=0\,,\tag{A.24}$$

$$\left(1-2xt+t^2\right)\frac{\partial w}{\partial x}-tw=0\,.\tag{A.25}$$

Substituting Eq. (A.23) into Eq. (A.24) yields

$$\left(1-2xt+t^2\right)\sum_{n=0}^{+\infty} nP_n(x)t^{n-1} + (t-x)\sum_{n=0}^{+\infty} P_n(x)t^n = 0\,.$$

Since the coefficients of t^n must vanish, we have, for $n \geq 1$,

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0\,. \tag{A.26}$$

A repeated application of this recurrence formula will yield the Legendre polynomial $P_N(x)$ of any arbitrary degree N in terms of $P_0(x)$ and $P_1(x)$.

Similarly, substituting Eq. (A.23) into Eq. (A.25) yields

$$P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) - P_n(x) = 0\,. \tag{A.27}$$

By taking derivatives of Eq. (A.26) with respect to x, we obtain

$$(n+1)P'_{n+1}(x) - (2n+1)\left[P_n(x) + xP'_n(x)\right] + nP'_{n-1}(x) = 0\,.$$

This together with Eq. (A.27) leads to, by eliminating $P'_{n-1}(x)$ and $P'_{n+1}(x)$,

$$P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x)\,,$$

$$xP'_n(x) - P'_{n-1}(x) = nP_n(x)\,.$$

Thus, by adding the two equations,

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)\,. \tag{A.28}$$

A.6 Associated Legendre Polynomials

The associated Legendre equation is

$$\left(1-x^2\right)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0\,, \tag{A.29}$$

where m is a natural number. To express its particular solutions for any natural number n by $P_n(x)$, the particular solutions of Legendre equations, consider the Legendre equation,

$$\left(1-x^2\right)v'' - 2xv' + n(n+1)v = 0\,. \tag{A.30}$$

By taking m-th derivatives with respect to x, we obtain

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\left(1-x^2\right)v''\right] - \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(2xv'\right) + n(n+1)\frac{\mathrm{d}^m v}{\mathrm{d}x^m} = 0\,.$$

Note that

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\left(1-x^2\right)v''\right] = \frac{\mathrm{d}^{m+2}}{\mathrm{d}x^{m+2}}\left(1-x^2\right) + \frac{m}{1!}\frac{\mathrm{d}^{m+1}v}{\mathrm{d}x^{m+1}}(-2x)$$

$$+\frac{m(m-1)}{2}\frac{\mathrm{d}^m v}{\mathrm{d}x^m}(-2),$$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(2xv'\right) = \frac{\mathrm{d}^{m+1}v}{\mathrm{d}x^{m+1}}2x + \frac{m}{1!}\frac{\mathrm{d}^m v}{\mathrm{d}x^m}2.$$

Thus

$$\left(1-x^2\right)u'' - 2x(m+1)u' + [n(n+1)-m(m+1)]u = 0,\ u = \frac{\mathrm{d}^m v}{\mathrm{d}x^m}. \tag{A.31}$$

Now consider the function transformation

$$w = \left(1-x^2\right)^{\frac{m}{2}}u \quad \text{or} \quad u = (1-x^2)^{-\frac{m}{2}}w.$$

Thus

$$\frac{\mathrm{d}u}{\mathrm{d}x} = mx(1-x^2)^{-\frac{m}{2}-1}w + (1-x^2)^{-\frac{m}{2}}\frac{\mathrm{d}w}{\mathrm{d}x},$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = m(1-x^2)^{-\frac{m}{2}-2}\left[\left(1-x^2\right)+(m+2)x^2\right]w$$

$$+2mx\left(1-x^2\right)^{-\frac{m}{2}-1}\frac{\mathrm{d}w}{\mathrm{d}x} + \left(1-x^2\right)^{-\frac{m}{2}}\frac{\mathrm{d}^2w}{\mathrm{d}x^2}.$$

Substituting them into Eq. (A.31) yields the associated Legendre equations

$$\left(1-x^2\right)w'' - 2xw' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]w = 0.$$

When n is a natural number, therefore, the solutions of Eq. (A.29) are

$$w = \left(1-x^2\right)^{\frac{m}{2}}u = \left(1-x^2\right)^{\frac{m}{2}}\frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m}.$$

The solution of the associated Legendre equations

$$P_n^m(x) = \left(1-x^2\right)^{\frac{m}{2}}\frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m}, \quad m \le n, |x| < 1$$

is called the *associated Legendre polynomial of degree n and order m* (see Table A.6. By substituting the Rodrigue expression of $P_n(x)$ into the definition of $P_n^m(x)$, we obtain the Rodrigue expression of $P_n^m(x)$

$$P_n^m(x) = \frac{1}{2^n n!}\left(1-x^2\right)^{\frac{m}{2}}\frac{\mathrm{d}^{n+m}}{\mathrm{d}x^{n+m}}\left(x^2-1\right)^n.$$

It is clear that if $m > n$, $P_n^m(x) \equiv 0$.

Remark 1. Consider the Laplace equation in a spherical coordinate system

$$\Delta u(r, \theta, \varphi) = 0.$$

Assuming that $u = R(r)Y(\theta, \varphi)$, by separation of variables, we have an equation of spherical functions.

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + n(n+1)Y = 0.$$

Assume that $Y(\theta, \varphi) = \Theta(\theta)\,\Phi(\varphi)$; by another separation of variables, we arrive at the associated Legendre equation regarding $\Theta(\theta)$. Its solution is $P_n^m(\cos\theta)$. The $\Phi(\varphi)$-equation

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} + m^2\Phi = 0$$

has the solution $\Phi(\varphi) = \mathrm{e}^{im\varphi}$. Therefore, the equation of spherical functions has the solution $P_n^m(\cos\theta)\,\mathrm{e}^{im\varphi}$. It is called the *spherical harmonic function* and is independent of radius r. Its real and imaginary parts are

$$\begin{aligned}
&P_n^m(\cos\theta)\cos m\varphi\,, \quad n = 0, 1, 2, \cdots, \quad m = 0, 1, 2, \cdots, \quad m \le n\,,\\
&P_n^m(\cos\theta)\sin m\varphi\,, \quad n = 0, 1, 2, \cdots, \quad m = 0, 1, 2, \cdots, \quad m \le n\,.
\end{aligned}$$

They are called the *spherical functions of order n*.

Remark 2. Special functions can also be introduced by an expansion of generating functions. The generating function for a Bessel function is

$$\mathrm{e}^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{+\infty} J_n(x)t^n\,.$$

Let $t = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}$ and $x = kr$, thus we obtain

$$\mathrm{e}^{\mathrm{i}kr\cos\theta} = J_0(kr) + 2\sum_{n=1}^{+\infty} \mathrm{i}^n J_n(kr)\cos n\theta\,, \tag{A.32}$$

in which every Bessel function represents an amplitude factor of cylindrical waves (Section 2.5.2 and Section 2.8).

Consider the solution of the one-dimensional wave equation in the form

$$u(x,t) = v(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\,.$$

The $v(x)$ satisfies $\dfrac{\mathrm{d}^2 v}{\mathrm{d}x^2} + k^2 v = 0$, where $k = \omega/a$ is called the *phase constant*. Its solution can be expressed by

$$v(x) = v_0\,\mathrm{e}^{\mathrm{i}kx} = v_0\,\mathrm{e}^{\mathrm{i}kr\cos\theta}\,, \tag{A.33}$$

where v_0 is the amplitude of plane waves and $e^{ikr\cos\theta}$ is their amplitude factor. Therefore Eq. (A.32) shows that plane waves can be expanded by those of cylindrical waves.

We may also explain the physical meaning of the generating function of Legendre polynomials. Consider two particles of unit mass located at the origin O and at a point P of distance R from the origin. When the particle at point P is moved infinitely far away, the work done by gravitation is

$$W = \int_R^{+\infty} \frac{1}{r^2}\,\mathrm{d}r = -\frac{1}{r}\bigg|_R^{+\infty} = \frac{1}{R}.$$

This is called the *Newton potential* at point P that is due to the particle at the origin. The Newton potential at point P' due to the particle of unit mass at point P is thus

$$\frac{1}{R} = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}},$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of P and P' respectively, $|\mathbf{r}| = r$, $|\mathbf{r}'| = r'$ and θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Let $t = \frac{r'}{r}$, $x = \cos\theta$. Thus $\frac{1}{R} = \frac{1}{r\sqrt{1-2xt+t^2}}$. When $r = 1$, in particular, we have $\frac{1}{R} = \frac{1}{\sqrt{1-2xt+t^2}}$. Therefore, the generating function of the Legendre poly-nomial represents the Newton potential.

Remark 3. The Euler integral of the first kind is a generalized integral containing two positive parameters p and q,

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x, \quad p>0,\, q>0, \tag{A.34}$$

which is called the *Beta function*, the *B-function* for short, and is continuous when $p > 0$ and

$$q > 0.$$

By a variable transformation $x = 1-t$, we may show the symmetry of the $B-$function, i.e.

$$B(p,q) = B(q,p).$$

We may also obtain the recurrence formula by using the integration by parts for $p > 0$ and $q > 1$

$$\begin{aligned} B(p,q) &= \frac{1}{p}\int_0^1 (1-x)^{q-1}\,\mathrm{d}x^p \\ &= \frac{q-1}{p}\int_0^1 x^p(1-x)^{q-2}\,\mathrm{d}x \\ &= \frac{q-1}{p}B(p,q-1) - \frac{q-1}{p}B(p,q) \end{aligned}$$

or

$$B(p,q)=\frac{q-1}{p+q-1}B(p,q-1)\,. \tag{A.35}$$

When $q>0$ and $p>1$, by the symmetry of the B-function we have

$$B(p,q)=\frac{p-1}{p+q-1}B(p-1,q). \tag{A.36}$$

Equations (A.35) and (A.36) lead to

$$B(p,q)=\frac{(p-1)(q-1)}{(p+q-1)(p+q-2)}B(p-1,q-1)\,. \tag{A.37}$$

Let $x=\cos^2\theta$; we obtain another form of the B-function

$$B(p,q)=2\int_0^{\frac{\pi}{2}} cos^{2p-1}\theta sin^{2q-1}\theta\,\mathrm{d}\theta\,.$$

Thus $B\left(\frac{1}{2},\frac{1}{2}\right)=\pi$. Let $x=\frac{1}{1+t}$. Thus

$$B(p,q)=\int_0^1\frac{t^{q-1}}{(1+t)^{p+q}}\,\mathrm{d}t+\int_1^{+\infty}\frac{t^{q-1}}{(1+t)^{p+q}}\,\mathrm{d}t\,,$$

which reduces into, by a variable transformation $t=1/u$ in the second integral,

$$B(p,q)=\int_0^1\frac{t^{p-1}+t^{q-1}}{(1+t)^{p+q}}\,\mathrm{d}t\,.$$

The Euler integral of the second kind is a generalized integral that contains positive parameter x,

$$\Gamma(x)=\int_0^{+\infty}\mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t\,,\quad x>0\,. \tag{A.38}$$

It is called the *Gamma function*, the *Γ-function* for short, and satisfies

$$\Gamma(x+1)=x\Gamma(x)\,.$$

When $x\le 0$, the integral in Eq. (A.38) is divergent. We thus define

$$\Gamma(x)=\Gamma(x+1)/x\,,\quad -1<x<0\,. \tag{A.39}$$

This definition is also valid for $-2<x<-1$ etc. However, $\Gamma(x)\to\infty$ as x tends to 0, -1, -2, $\cdots$. Therefore, the Γ-function is defined by

$$\Gamma(x)=\begin{cases}\displaystyle\int_0^{+\infty}\mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t\,,\ x>0\,,\\ \Gamma(x+1)\big/x\,,\qquad x<0,\ x\neq -1,-2,\cdots.\end{cases}$$

Table A.1 Associated legendre polynomials of degree n and order m

m / 1	0	1	2	3	4
0	$P_0(x)$				
1	$P_1(x)$	$P_1^1(x)$ $=(1-x^2)^{\frac{1}{2}}$ $=\sin\theta$			
2	$P_2(x)$	$P_2^1(x)$ $=3(1-x^2)^{\frac{1}{2}}x$ $=\frac{3}{2}\sin 2\theta$	$P_2^2(x)$ $=3(1-x^2)$ $=\frac{3}{2}(1-\cos 2\theta)$		
3	$P_3(x)$	$P_3^1(x)$ $=\frac{3}{2}(1-x^2)^{\frac{1}{2}}(5x^2-1)$ $=\frac{3}{8}(\sin\theta+5\sin 3\theta)$	$P_3^2(x)$ $=15(1-x^2)x$ $=\frac{15}{4}(\cos\theta-\cos 3\theta)$	$P_3^3(x)$ $=15(1-x^2)^{\frac{3}{2}}$ $=\frac{15}{4}(3\sin\theta$ $-\sin 3\theta)$	
4	$P_4(x)$	$P_4^1(x)$ $=\frac{5}{2}(1-x^2)^{\frac{1}{2}}(7x^3-3x^2)$ $=\frac{5}{16}(2\sin 2\theta+7\sin 4\theta)$	$P_4^2(x)$ $=\frac{15}{2}(1-x^2)(7x^2-1)$ $=\frac{15}{16}(3+4\cos 2\theta$ $-7\cos 4\theta)$	$P_4^3(x)$ $=105(1-x^2)^{\frac{3}{2}}x$ $=\frac{105}{8}(2\sin 2\theta$ $-\sin 4\theta)$	$P_4^4(x)$

For a natural number n, $\Gamma(n+1)=n!$. Also

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\,.$$

We may show, by the ratio test, that the convergence radius of the Bessel function $J_\gamma(x)$ $(\gamma>0)$ is $R=+\infty$.

There is an important relation between the two Euler integrals

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\,,\quad p>0\,,\quad q>0\,. \tag{A.40}$$

Appendix B
Integral Transformations

B.1 Fourier Integral Transformation

B.1.1 Fourier Integral

A periodic function can be expanded into a Fourier series. We show here that a non-periodic function can be expressed by using a Fourier integral.

Fourier Series of Exponential Form

If a periodic function $g(t)$ of period T satisfies the Dirichlet condition, it can be expanded into a Fourier series. Therefore, at any continuous point of $g(t)$ we have

$$\begin{cases} g(t) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{+\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \\ a_n = \dfrac{2}{T} \displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) \cos n\omega t \, \mathrm{d}t \,, \quad n = 0, 1, 2, \cdots \,, \\ b_n = \dfrac{2}{T} \displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) \sin n\omega t \, \mathrm{d}t \,, \quad n = 1, 2, \cdots \,, \end{cases}$$

where $\omega = \dfrac{2\pi}{T}$. By applying the Euler formula $\mathrm{e}^{\pm \mathrm{i} n\omega t} = \cos n\omega t \pm \mathrm{i} \sin n\omega t$, we have

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(\frac{a_n - \mathrm{i} b_n}{2} \mathrm{e}^{\mathrm{i} n\omega t} + \frac{a_n + \mathrm{i} b_n}{2} \mathrm{e}^{-\mathrm{i} n\omega t} \right).$$

Let

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - \mathrm{i} b_n}{2}, \quad c_{-n} = \frac{a_n + \mathrm{i} b_n}{2}, \quad n = 1, 2, \cdots .$$

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Thus

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)\,\mathrm{d}t\,, \quad c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)\,\mathrm{e}^{-\mathrm{i}n\omega t}\,\mathrm{d}t\,,$$

$$c_{-n} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)\,\mathrm{e}^{\mathrm{i}n\omega t}\,\mathrm{d}t\,,$$

or

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)\,\mathrm{e}^{-\mathrm{i}n\omega t}\,\mathrm{d}t\,, n = 0\,, \pm 1\,, \pm 2\,, \cdots\,.$$

Finally we obtain the Fourier series of exponential form

$$\begin{aligned} g(t) &= c_0 + \sum_{n=-\infty}^{-1} c_n\,\mathrm{e}^{\mathrm{i}n\omega t} + \sum_{n=1}^{+\infty} c_{-n}\,\mathrm{e}^{\mathrm{i}n\omega t} \\ &= \sum_{n=-\infty}^{+\infty} c_n\,\mathrm{e}^{\mathrm{i}n\omega t} = \frac{1}{T}\sum_{n=-\infty}^{+\infty}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)\,\mathrm{e}^{-\mathrm{i}\omega_n t}\,\mathrm{d}t\right)\mathrm{e}^{\mathrm{i}\omega_n t}\,, \end{aligned} \tag{B.1}$$

where $\omega_n = n\omega$.

Fourier Integral

Consider a non-periodic function $f(t)$. Introduce a periodic function $f_T(t)$ of period T such that

$$f_T(t) = f(t)\,, \quad t \in \left[-\frac{T}{2}, \frac{T}{2}\right].$$

Therefore $\qquad f(t) = \lim_{T\to+\infty} f_T(t)\,.$

Let $\Delta\omega = \omega_n - \omega_{n-1}$. Thus

$$\Delta\omega = n\omega - (n-1)\omega = \frac{2\pi}{T} \quad \text{or} \quad \frac{1}{T} = \frac{1}{2\pi}\Delta\omega\,.$$

Since $\Delta\omega \to 0$ as $T \to +\infty$, we have, by Eq. (B.1) and when $f(t)$ satisfies some conditions,

$$\begin{aligned} f(t) &= \lim_{T\to+\infty} f_T(t) = \lim_{\substack{\Delta\omega\to 0 \\ (T\to+\infty)}} \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(\tau)\,\mathrm{e}^{-\mathrm{i}\omega_n\tau}\,\mathrm{d}\tau\right)\mathrm{e}^{\mathrm{i}\omega_n t}\Delta\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau\right]\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\,. \end{aligned} \tag{B.2}$$

This is the *Fourier integral of exponential form* for the non-periodic function $f(t)$, i.e.

$$\begin{cases} f(t) = \int_{-\infty}^{+\infty} g(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\,, \\ g(\omega) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty} f(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau\,, \end{cases} \tag{B.3}$$

where the generalized integrals are defined under the meaning of principal value.

Remark 1. (Fourier integral theorem). Let function $f(t)$ be defined in the infinite region $(-\infty,+\infty)$. If it satisfies the Dirichlet condition in any finite region and the integral $\int_{-\infty}^{+\infty} |f(t)|\,\mathrm{d}t$ converges,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau\right]\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega$$

$$= \begin{cases} f(t)\,, & \text{for continuous point t}\,, \\ \dfrac{f(t+0)+f(t-0)}{2}\,, & \text{for discontinuous point t}\,. \end{cases}$$

Remark 2. (Other forms of the Fourier integral). By the Euler formula, we may transform the exponential form into a trigonometric form

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)\,\mathrm{e}^{\mathrm{i}\omega(t-\tau)}\,\mathrm{d}\tau\right]\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)\cos\omega(t-\tau)\,\mathrm{d}\tau\right]\mathrm{d}\omega \\ &\quad + \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)\sin\omega(t-\tau)\,\mathrm{d}\tau\right]\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)\cos\omega(t-\tau)\,\mathrm{d}\tau\right]\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_{0}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)\cos\omega(t-\tau)\,\mathrm{d}\tau\right]\mathrm{d}\omega\,. \end{aligned}$$

By expanding $\cos\omega(t-\tau)$, we may obtain a form similar to the Fourier series

$$\begin{cases} f(t) = \int_{0}^{+\infty} [a(\omega)\cos\omega t + b(\omega)\sin\omega t]\,\mathrm{d}\omega\,, \\ a(\omega) = \dfrac{1}{\pi}\int_{-\infty}^{+\infty} f(\tau)\cos\omega\tau d\tau\,, \\ b(\omega) = \dfrac{1}{\pi}\int_{-\infty}^{+\infty} f(\tau)\sin\omega\tau\,\mathrm{d}\tau\,. \end{cases}$$

When $f(t)$ is an odd or even function, in particular, we have

$$\begin{cases} f(t) = \int_0^{+\infty} b(\omega)\sin\omega t\,\mathrm{d}\omega\,, \\ b(\omega) = \dfrac{2}{\pi}\int_0^{+\infty} f(\tau)\sin\omega\tau\,\mathrm{d}\tau\,, \end{cases} \quad \text{and} \quad \begin{cases} f(t) = \int_0^{+\infty} a(\omega)\cos\omega t\,\mathrm{d}\omega\,, \\ a(\omega) = \dfrac{2}{\pi}\int_0^{+\infty} f(\tau)\cos\omega\tau\,\mathrm{d}\tau\,, \end{cases}$$

respectively.

B.1.2 Fourier Transformation

Definition

If the function $f(t)$ satisfies certain conditions, we may express it by using a Fourier integral. Let

$$\bar{f}(\omega) = \int_{-\infty}^{+\infty} f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\,. \tag{B.4}$$

By Eq. (B.3) we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\,. \tag{B.5}$$

Therefore we may express $\bar{f}(\omega)(f(t))$ by the integral of $f(t)(\bar{f}(\omega))$. Equations (B.4) and (B.5) are called the *Fourier transformation* and the *inverse Fourier transformation*, respectively. The $\bar{f}(\omega)$ is called the *image function* of $f(t)$. The $f(t)$ is called the *inverse image function* of $\bar{f}(\omega)$. For convenience in applications, Eqs. (B.4) and (B.5) are written as

$$\bar{f}(\omega) = F\left[f(t)\right]\,, \quad f(t) = F^{-1}\left[\bar{f}(\omega)\right]\,.$$

Example 1. Find the Fourier transformation and the integral expression of the exponentially decaying function

$$f(t) = \begin{cases} 0\,, & t < 0\,, \\ \mathrm{e}^{-\beta t}\,, & 0 \le t\,,\ \beta > 0\,. \end{cases}$$

Solution. By Eq. (B.4), we obtain the image function

$$\bar{f}(\omega) = F\left[f(t)\right] = \int_0^{+\infty} \mathrm{e}^{-(\beta+\mathrm{i}\omega)t}\,\mathrm{d}t = \frac{1}{\beta+\mathrm{i}\omega} = \frac{\beta-\mathrm{i}\omega}{\beta^2+\omega^2}\,,$$

which is a complex-valued function of ω. By Eq. (B.5), we obtain an integral expression of $f(t)$

$$f(t) = F^{-1}\left[\bar{f}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\beta - \mathrm{i}\omega}{\beta^2+\omega^2}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\beta\cos\omega t + \omega\sin\omega t}{\beta^2+\omega^2}\,\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_{0}^{+\infty} \frac{\beta\cos\omega t + \omega\sin\omega t}{\beta^2+\omega^2}\,\mathrm{d}\omega\,.$$

Thus we can also obtain the integral

$$\int_{0}^{+\infty} \frac{\beta\cos\omega t + \omega\sin\omega t}{\beta^2+\omega^2}\,\mathrm{d}\omega = \begin{cases} 0, & t<0\,, \\ \dfrac{\pi}{2}, & t=0\,, \\ \pi\,\mathrm{e}^{-\beta t}, & t>0\,,\ \beta>0\,. \end{cases}$$

Example 2. Find the image function and the integral expression of the impulse function

$$f(t) = \begin{cases} A, & |t|<a\,, \\ 0, & |t|\geq a\,. \end{cases}$$

Solution. By the definition of an image function, we have

$$\bar{f}(\omega) = F\left[f(t)\right] = \int_{-a}^{a} A\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \begin{cases} \dfrac{2A\sin\omega a}{\omega}, & \omega\neq 0\,, \\ 2aA, & \omega = 0\,. \end{cases}$$

Since $\dfrac{2A\sin\omega a}{\omega} \to 2aA$ as $\omega\to 0$, $\omega = 0$ is a removable discontinuous point. Thus we may write

$$\bar{f}(\omega) = F\left[f(t)\right] = \frac{2A\sin\omega a}{\omega}\,.$$

The integral expression of $f(t)$ can be obtained by taking the inverse Fourier transformation

$$f(t) = F^{-1}\left[\bar{f}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2A\sin\omega a}{\omega}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega$$

$$= \frac{A}{\pi}\int_{-\infty}^{+\infty} \frac{\sin\omega a\cos\omega t}{\omega}\,\mathrm{d}\omega\,.$$

When $A = 1$ and $a = 1$, in particular, we have

$$\frac{2}{\pi}\int_0^{+\infty}\frac{\sin\omega\cos\omega t}{\omega}\,\mathrm{d}\omega = \begin{cases} 1\,, & |t| < 1\,, \\ \dfrac{1}{2}\,, & |t| = 1\,, \\ 0\,, & |t| > 1\,. \end{cases}$$

This is called the *Dirichlet discontinuous factor*.

The commonly used inverse image function and image functions can be found from the Table of Fourier transformations (Appendix C).

Properties

In discussing the properties of Fourier transformations, all functions are assumed to satisfy the conditions for the Fourier transformation.

1. **Linearity**
   If $F\left[f_1(t)\right] = \bar{f}_1(\omega)\,, F\left[f_2(t)\right] = \bar{f}_2(\omega)$, for any two constants α and β,
   $$F\left[\alpha f_1(t) + \beta f_2(t)\right] = \alpha\bar{f}_1(\omega) + \beta\bar{f}_2(\omega)$$
   or
   $$F^{-1}\left[\alpha\bar{f}_1(\omega) + \beta\bar{f}_2(\omega)\right] = \alpha f_1(t) + \beta f_2(t)\,.$$
2. **Shifting Property**
   $$F\left[f(t \pm t_0)\right] = \mathrm{e}^{\pm \mathrm{i}\omega t_0}F\left[f(t)\right]\,, \quad F^{-1}\left[\bar{f}(\omega \mp \omega_0)\right] = f(t)\,\mathrm{e}^{\pm \mathrm{i}\omega_0 t}\,.$$
   The above two properties follows directly from the properties of integration.
3. **Differential Property**
   $F\left[f'(t)\right] = \mathrm{i}\omega F\left[f(t)\right]$.

   Proof. Since $f(t)$ is continuous in $(-\infty, +\infty)$ and $\int_{-\infty}^{+\infty}|f(t)|\,\mathrm{d}t$ is convergent, $\lim\limits_{|t|\to+\infty} f(t) = 0$. Thus
   $$\begin{aligned} F\left[f'(t)\right] &= \int_{-\infty}^{+\infty} f'(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\ &= f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\Big|_{-\infty}^{+\infty} + \mathrm{i}\omega\int_{-\infty}^{+\infty} f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \mathrm{i}\omega F\left[f(t)\right]\,. \end{aligned}$$
   Similarly, if $f(t)$ has n-th continuous derivatives in $(-\infty, +\infty)$ and $\lim\limits_{|t|\to+\infty} f^{(j)}(t) = 0$, $(j = 0, 1, 2, \cdots, n-1)$,
   $$F\left[f^{(n)}(t)\right] = (\mathrm{i}\omega)^n F\left[f(t)\right].$$

This property plays an important role in solving PDS by integral transformation.

4. **Integral property**

$$F\left[\int_{-\infty}^{t} f(t)\,\mathrm{d}t\right] = \frac{1}{\mathrm{i}\omega} F\left[f(t)\right] \,.$$

Proof. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{-\infty}^{t} f(t)\,\mathrm{d}t\right] = f(t)\,,$$

we have, by the differential property,

$$F\left[f(t)\right] = F\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t} f(t)\,\mathrm{d}t\right] = \mathrm{i}\omega F\left[\int_{-\infty}^{t} f(t)\,\mathrm{d}t\right]$$

or

$$F\left[\int_{-\infty}^{t} f(t)\,\mathrm{d}t\right] = \frac{1}{\mathrm{i}\omega} F\left[f(t)\right] .$$

5. **Multiplying Property**

If $\bar{f}_1(\omega) = F\left[f_1(t)\right]$ and $\bar{f}_2(\omega) = F\left[f_2(t)\right]$,

$$\int_{-\infty}^{+\infty} f_1(t) f_2(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{\bar{f}_1(\omega)}\,\bar{f}_2(\omega) d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{f}_1(\omega)\,\overline{\bar{f}_2(\omega)}\,\mathrm{d}\omega\,,$$

where $\overline{\bar{f}_1(\omega)}$ and $\overline{\bar{f}_2(\omega)}$ are the conjugate function of $\bar{f}_1(\omega)$ and $\bar{f}_2(\omega)$, respectively.

Proof. Note that: 1) the conjugate of the product of two functions is the product of their conjugates, 2) the conjugate of a real-value function is itself, and 3) $\overline{\mathrm{e}^{-\mathrm{i}\omega t}} = \mathrm{e}^{\mathrm{i}\omega t}$. Thus

$$f_1(t)\,\mathrm{e}^{-\mathrm{i}\omega t} = \overline{f_1(t)\,\mathrm{e}^{\mathrm{i}\omega t}}\,,$$

$$\begin{aligned}
\int_{-\infty}^{+\infty} f_1(t) f_2(t)\,\mathrm{d}t &= \int_{-\infty}^{+\infty} f_1(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{f}_2(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\right]\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{f_1(t)\,\mathrm{e}^{-\mathrm{i}\omega t}}\,\bar{f}_2(\omega)\,\mathrm{d}t\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{\bar{f}_1(\omega)}\,\bar{f}_2(\omega)\,\mathrm{d}\omega\,.
\end{aligned}$$

Since $\bar{f}_1(\omega) = \int_{-\infty}^{+\infty} f_1(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$, we have

$$\overline{\bar{f}_1(\omega)} = \int_{-\infty}^{+\infty} \overline{f_1(t)\,\mathrm{e}^{-\mathrm{i}\omega t}}\,\mathrm{d}t\,.$$

Similarly, we can show that

$$\int_{-\infty}^{+\infty} f_1(t) f_2(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{f}_1(\omega)\overline{\bar{f}_2(\omega)}\,\mathrm{d}\omega\,.$$

This establishes the relation between the integral of the product of two inverse image functions and the integral of image functions, and plays an important role in calculating various forms of energies. For example, in electricity,

$$w = R\int_{-\infty}^{+\infty} \mathrm{i}^2(t)\,\mathrm{d}t = \frac{1}{R}\int_{-\infty}^{+\infty} v^2(t)\,\mathrm{d}t$$

is the total energy passing through an electric resistance R. Here $i(t)$ and $v(t)$ are the electric current passing through R and the electric voltage acting on R, respectively. The integral of $\int_{-\infty}^{+\infty} f^2(t)\,\mathrm{d}t$ is often called the *energy integral*.

Corollary 1. If $\bar{f}(\omega) = F\,[f(t)]$, the energy integral is

$$\int_{-\infty}^{+\infty} f^2(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\bar{f}(\omega)\right|^2 \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega)\,\mathrm{d}\omega\,,$$

where $S(\omega) = \left|\bar{f}(\omega)\right|^2$ is called the *density of the energy spectrum*. It can be shown that the $S(\omega)$ is a real-valued function of ω and is an even function such that $S(-\omega) = S(\omega)$. Thus we may obtain the total energy by integrating the density of the energy spectrum with respect to the frequency ω.

Corollary 2. Define $R(t) = \int_{-\infty}^{+\infty} f(\tau)f(\tau+t)\,\mathrm{d}\tau$. Thus

$$R(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\,, \quad S(\omega) = \int_{-\infty}^{+\infty} R(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\,,$$

so that the $R(t)$ and the $S(\omega)$ form a Fourier transformation couple. Here $R(t)$ is called the *self-correlation function* of $f(t)$.

Proof. Let $\bar{f}(\omega) = F\,[f(t)]$. By the shifting and multiplying properties, we have

$$R(t) = \int_{-\infty}^{+\infty} f(\tau)f(\tau+t)\,\mathrm{d}\tau = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\bar{f}(\omega)\right|^2 \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\,.$$

Thus, by the Fourier integral,

$$S(\omega) = \int_{-\infty}^{+\infty} R(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\,.$$

Convolution Theorem

Consider two known functions $f_1(t)$ and $f_2(t)$. The integral $\int_{-\infty}^{+\infty} f_1(\tau)f_2(t-\tau)\,\mathrm{d}\tau$ is called the *convolution* of functions $f_1(t)$ and $f_2(t)$ and is denoted by $f_1(t) * f_2(t)$, i.e.

$$f_1(t) * f_2(t) = \int_{-\infty}^{+\infty} f_1(\tau)f_2(t-\tau)\,\mathrm{d}\tau\,.$$

It is clear that the convolution satisfies

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)\,,$$
$$f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)\,.$$

Convolution theorem. If $\bar{f}_1(\omega) = F[f_1(t)]$ and $\bar{f}_2(\omega) = F[f_2(t)]$, then

$$F[f_1(t) * f_2(t)] = \bar{f}_1(\omega)\bar{f}_2(\omega) \quad \text{or} \quad F^{-1}\left[\bar{f}_1(\omega)\bar{f}_2(\omega)\right] = f_1(t) * f_2(t)\,.$$

Proof. By the definition of convolution,

$$\begin{aligned}
F[f_1(t) * f_2(t)] &= \int_{-\infty}^{+\infty} [f_1(t) * f_2(t)]\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \\
&= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau)\, \mathrm{d}\tau\right] \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \\
&= \int_{-\infty}^{+\infty} f_1(\tau)\, \mathrm{e}^{-\mathrm{i}\omega\tau}\, \mathrm{d}\tau \int_{-\infty}^{+\infty} f_2(t-\tau)\, \mathrm{e}^{-\mathrm{i}\omega(t-\tau)}\, \mathrm{d}t = \bar{f}_1(\omega)\bar{f}_2(\omega)\,,
\end{aligned}$$

where

$$\int_{-\infty}^{+\infty} f_2(t-\tau)\, \mathrm{e}^{-\mathrm{i}\omega(t-\tau)}\, \mathrm{d}t = \int_{-\infty}^{+\infty} f_2(u)\, \mathrm{e}^{-\mathrm{i}\omega u}\, \mathrm{d}u = \bar{f}_2(\omega)\,.$$

B.1.3 Generalized Functions and the δ-function

Generalized Functions Defined by the Functional

The functional is defined in function spaces to possess some good properties. The commonly-used spaces are: (1) the *K-space* or $C_0^\infty(a,b)$ where functions are infinitely differentiable in (a,b) and vanished outside a finite interval, (2) $C(a,b)$ where functions are continuous in (a,b), (3) $L^2(a,b)$ where functions are quadratically integrable in (a,b), and (4) $C^\infty(a,b)$ where functions have continuous derivatives of any order up to infinity in (a,b). Here the a and the b can be $-\infty$ and $+\infty$. For example,

$$\varphi(x) = \begin{cases} \mathrm{e}^{-\frac{c^2}{c^2-|x|^2}}\,, & |x| < c,\ \varphi(x) \in C_0^\infty(-\infty,+\infty)\,, \\ 0\,, & |x| \geq c\,. \end{cases} \tag{B.6}$$

Definition 1. Suppose that there exists a real value corresponding to every function $y(x)$ in the function space $\{y(x)\}$ according to a certain rule; such a corresponding relation is called the *functional* and is denoted by $F = F(y(x))$.

If F represents a functional from K-space to R^1 and satisfies $\lim_{n\to\infty} F(\varphi_n) = F(\varphi)$, $\forall\{\varphi_n\} \subset K$ when $\lim_{n\to\infty} \varphi_n = \varphi$, in particular, then F is said to be continuous at point φ. If F is continuous at all points in K, it is called a *continuous functional* in K. If, for all $\varphi_1, \varphi_2 \in K$ and $k_1, k_2 \in R^1$,

$$F(k_1\varphi_1 + k_2\varphi_2) = k_1F(\varphi_1) + k_2F(\varphi_2),$$

the F is called a *linear functional*.

Definition 2. A linear continuous functional in K is called a *generalized function in* K and is denoted by $F(\varphi)$ or (F,φ), $\varphi \in K$.

Remark. Generalized functions depend on function spaces and differ from classical functions. Once its value corresponding to every element in a function space is known, it is regarded as fixed. In a specified function space, we may express the generalized functions in various formats. The commonly used one is the integral of an inner product. If $f(x)$ is integrable in R^1,

$$F_f(\varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,\mathrm{d}x, \quad \forall\varphi(x) \in K \tag{B.7}$$

is a generalized function in K and is often denoted by (f,φ) or $\langle f,\varphi\rangle$, $\forall\varphi(x) \in K$.

Generalized Functions Defined by Equivalent Classes of Basic Sequences

Definition 3. Suppose that the function sequences $\{f_n(x)\} \subset C_0^\infty(a,b)$.
If $\lim_{n\to\infty} \int_a^b f_n(x)\varphi(x)\,\mathrm{d}x$ exists $\forall\varphi \in K$, the $\{f_n\}$ is called a *basic sequence of* K. If two basic sequences $\{f_n\}$ and $\{g_n\}$ of K satisfy

$$\lim_{n\to\infty} \int_a^b f_n(x)\varphi(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b g_n(x)\varphi(x)\,\mathrm{d}x,$$

the $\{f_n\}$ and the $\{g_n\}$ are called *equivalent* and denoted by

$$\{f_n\} \sim \{g_n\}, \quad K.$$

Definition 4. The equivalent classes of basic sequences of K-space are called *generalized functions* and denoted by $f = \{f_n\}$ or $f_n \to f$. Then $\{f_n\}$ is said to be *weakly convergent* to f. Also, denote

$$\langle f,\varphi\rangle = \int_a^b f(x)\varphi(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n(x)\varphi(x)\,\mathrm{d}x, \quad \forall\varphi \in K. \tag{B.8}$$

Remark. An irrational number is regarded as the limit of a sequence of rational numbers. Similarly, a generalized function is defined as the weak limit of a sequence of classical functions in a specified function space. Equation (B.8) shows that $\{f_n\}$ is weakly convergent to f. If a sequence $\{r_n\}$ of rational numbers satisfies

$$\lim_{n\to\infty} r_n = \alpha \quad \text{(irrational number)},$$

we define $\alpha = \{r_n\}$. Similarly, we define $f = \{f_n\}$ if $\{f_n\}$ is weakly convergent to f. We can also write $f_n \to f$ as

$$\lim_{n\to\infty} f_n(x) \overset{\text{weak}}{=\!=} f(x).$$

The $f(x)$ is called the *weak limit* of $\{f_n(x)\}$ as $n \to +\infty$. For a fixed α, $\{r_n\}$ is not unique. Similarly, $\{f_n\}$ is not unique either for a fixed f.

Analytical Definition of the δ-Function

Definition 5. The *Dirac function* (or δ-function) is a functional in K whose values are $\varphi(0)$ for all $\varphi \in K$ such that $\delta(\varphi) = \varphi(0)$. The $\delta(\varphi)$ is denoted by $\delta(x)$.

It can be shown that δ-function is a continuous linear functional; therefore it is a generalized function. Let the functional in K defined by Eq. (B.7) be $\varphi(0)$ such that

$$(f,\varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,\mathrm{d}x = \varphi(0), \quad \forall\varphi \in K. \tag{B.9}$$

Clearly, the f in Eq. (B.9) cannot be a classical integrable function. The generalized function f in Eq. (B.9) is in fact the δ-function. Equation (B.9) is often written as

$$(\delta,\varphi) = \int_{-\infty}^{+\infty} \delta(x)\varphi(x)\,\mathrm{d}x = \varphi(0), \quad \forall\varphi \in K.$$

Similarly, $\delta(x-x_0)$ represents $\delta(\varphi) = \varphi(x_0)$ such that

$$(\delta,\varphi) = \int_{-\infty}^{+\infty} \delta(x-x_0)\,\varphi(x)\,\mathrm{d}x = \varphi(x_0), \quad \forall\varphi \in K.$$

Also, $\delta(x-x_0, y-y_0)$ and $\delta(x-x_0, y-y_0, z-z_0)$ represent

$$\delta(\varphi) = \varphi(x_0,y_0), \quad \delta(\varphi) = \varphi(x_0,y_0,z_0),$$

respectively.

Definition 6. If the equivalent classes $f = \{f_n\}$ of basic sequences in K satisfy

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f_n(x)\varphi(x)\,\mathrm{d}x = \varphi(0), \quad \forall\varphi \in K, \tag{B.10}$$

the generalized function $f = \{f_n\}$ is called the δ-*function*. Denote $\delta = \{f_n\}$, i.e.

$$(\delta,\varphi) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} f_n(x)\varphi(x)\,\mathrm{d}x = \varphi(0), \quad \forall\varphi \in K. \tag{B.11}$$

The function sequence $\{f_n\}$ that satisfies Eq. (B.11) is called the δ-*function sequence*. The $\{f_n\}$ is called to be *weakly convergent* to $\delta(x)$. Also denote

$$f_n(x) \to \delta(x)(n \to \infty) \quad \text{or} \quad \lim_{n\to\infty} f_n(x) \overset{\text{weak}}{=\!=} \delta(x)\,.$$

Note that δ-function sequence is not unique.

Example 1. Let $\rho(x) = \begin{cases} k\,\mathrm{e}^{-\frac{1}{1-|x|^2}}\,, & |x| < 1\,, \\ 0, & |x| \geq 1\,, \end{cases}$ where k is a constant such that

$$\int_{-\infty}^{+\infty} \rho(x)\,\mathrm{d}x = 1\,.$$

Consider now $\rho_n(x) = n\rho(nx)$. We have, for all $\varphi(x) \in K$,

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \rho_n(x)\varphi(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{-\frac{1}{n}}^{\frac{1}{n}} n\rho(nx)\varphi(x)\,\mathrm{d}x = \lim_{n\to\infty} \varphi(\xi) = \varphi(0)\,.$$

Therefore, $\rho_n(x) \to \delta(x)$.

Example 2. $f(x,t) = \dfrac{1}{2\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4t}} \to \delta(x)$, $t \to +0$.

Example 3. The Kelvin function for a thermal source is defined by

$$f(x,t) = (4\pi\mu t)^{-\frac{1}{2}-\frac{|x|^2}{4\mu t}} \to \delta(x)\,, \quad t \to +0\,, \mu > 0\,.$$

B.1.4 Generalized Fourier Transformation

The foundation of Fourier transformations is the Fourier integral. Some commonly-used functions such as 1, $\sin ax$ and $\cos ax$ are not, however, absolutely integrable in the domain $(-\infty, +\infty)$ as required by the Fourier integral theorem. To have a wide application, therefore, we need to extend the definition of the Fourier transformation.

Fast-Decreasing Functions and Weak Limits

If for the function $u(x) \in C^\infty(-\infty, +\infty)$,

$$\lim_{|x|\to\infty} \left| x^n \frac{\mathrm{d}^k u(x)}{\mathrm{d}x^k} \right| = 0$$

holds for any integer $n \geq 0$ and $k \geq 0$, then $u(x)$ is called a *fast-decreasing function*.

An example is the exponentially-decreasing function $f(t) = \begin{cases} 0, t < 0, \\ \mathrm{e}^{-\beta t}, 0 \leq t, \ \beta > 0 \end{cases}$

that we discussed before. The $\left\{\sqrt{\frac{n}{\pi}}\,\mathrm{e}^{-nx^2}\right\}$ $(n = 1, 2, \cdots)$ is a sequence of fast-decreasing functions.

By Definition 4, for all $\varphi(x) \in C_0^\infty(-\infty, +\infty)$ we have

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} u_n(x)\varphi(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} u(x)\varphi(x)\,\mathrm{d}x. \tag{B.12}$$

Similarly,

$$\lim_{\varepsilon\to+0} \int_{-\infty}^{+\infty} u_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} u(x)\varphi(x)\,\mathrm{d}x. \tag{B.13}$$

We call the $u(x)$ the weak limit of $\{u_n(x)\}$ or $\{u_\varepsilon(x)\}$, and denote

$$\lim_{n\to\infty} u_n(x) \overset{weak}{=} u(x) \quad \text{or} \quad \lim_{\varepsilon\to+0} u_\varepsilon(x) \overset{weak}{=} u(x).$$

Equations (B.12) and (B.13) show that we can actually interchange the order of limits and integration. We regard the above-mentioned functions, which cannot undergo the Fourier transformation, as the weak limits of some fast-decreasing functions. It can easily be shown that a classical limit must also be a weak limit, but a weak limit is not necessarily a classical limit.

We can use the weak limits of fast-decreasing functions to extend the definition of the Fourier transformation.

Examples of Generalized Fourier Transformation

Example 1. Find the image function of unit function

$$I(x) = \begin{cases} 0, \ x < 0, \\ 1, \ x \geq 0. \end{cases}$$

Solution. The Fourier transformation $I(x)$ does not exist under the classical definition. Consider the product of $I(x)$ and an exponentially-decreasing function $f(x)$; we have

$$\lim_{\beta\to 0} I(x) f(x) = \lim_{\beta\to 0} I(x)\,\mathrm{e}^{-\beta x} = I(x). \tag{B.14}$$

Also,

$$F[I(x)f(x)] = \frac{1}{\beta + \mathrm{i}\omega}\,, \qquad \lim_{\beta\to 0} F[I(x)f(x)] = \frac{1}{\mathrm{i}\omega}\,. \tag{B.15}$$

Thus we obtain, by Eq. (B.14) and (B.15), as $\beta \to 0$,

$$I(x)f(x) \to I(x)\,, \quad F[I(x)f(x)] \to \frac{1}{\mathrm{i}\omega}\,. \tag{B.16}$$

This shows that the weak limits of $I(x)f(x)$ and the image function are $I(x)$ and $\frac{1}{\mathrm{i}\omega}$, respectively. Thus, Eq. (B.16) forms a Fourier transformation couple and is denoted by

$$F[I(x)] = \frac{1}{\mathrm{i}\omega} \quad \text{or} \quad F^{-1}\left[\frac{1}{\mathrm{i}\omega}\right] = I(x)\,.$$

Example 2. Find the generalized Fourier transformation of $\delta(x)$.

Solution. It can be shown that the fast-decreasing function sequence $\left\{\sqrt{\frac{n}{\pi}}\,\mathrm{e}^{-nx^2}\right\}$ is a δ-function sequence such that

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}\sqrt{\frac{n}{\pi}}\,\mathrm{e}^{-nx^2}\varphi(x)\,\mathrm{d}x = \varphi(0) = \int_{-\infty}^{+\infty}\delta(x)\varphi(x)\,\mathrm{d}x\,, \quad \forall\varphi\in K\,.$$

Also,

$$F\left[\sqrt{\frac{n}{\pi}}\,\mathrm{e}^{-nx^2}\right] = \mathrm{e}^{-\frac{\omega^2}{4n}}\,, \quad \lim_{n\to\infty} F\left[\sqrt{\frac{n}{\pi}}\,\mathrm{e}^{-nx^2}\right] = 1\,.$$

Thus, as $n \longrightarrow \infty$,

$$\sqrt{\frac{n}{\pi}}\,\mathrm{e}^{-nx^2} \longrightarrow \delta(x)\,, \quad F\left[\sqrt{\frac{n}{\pi}}\,\mathrm{e}^{-nx^2}\right]\ 1\,. \tag{B.17}$$

Therefore the weak limits $\delta(x)$ and 1 form a Fourier transformation couple such that

$$F[\delta(x)] = 1 \quad \text{or} \quad F^{-1}[1] = \delta(x)\,. \tag{B.18}$$

Remark. We can also obtain Eq. (B.18) directly by applying definitions of the δ-function and the Fourier transformation, i.e.

$$F[\delta(x)] = \int_{-\infty}^{+\infty}\delta(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x = \left.\mathrm{e}^{-\mathrm{i}\omega x}\right|_{x=0} = 1 \quad \text{or} \quad F^{-1}[1] = \delta(x)\,.$$

Similarly,

$$F^{-1}\left[\delta(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\delta(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega = \frac{1}{2\pi}\left.\mathrm{e}^{\mathrm{i}\omega x}\right|_{\omega=0}$$
$$= \frac{1}{2\pi}\,.$$

Thus

$$F^{-1}\left[2\pi\delta(\omega)\right] = 1 \quad \text{or} \quad F\left[1\right] = 2\pi\delta(\omega)\,.$$

The Generalized Fourier Transformation and its Properties

Suppose that there exist weak limits of the function sequences $\{u_n(x)\}$ and $\{F\left[u_n(x)\right]\}$ such that

$$u_n(x) \to u(x) \quad \text{and} \quad F\left[u_n(x)\right] \to \bar{u}(\omega)\,.$$

The $\bar{u}(\omega)$ is called the *generalized Fourier transformation* of $u(x)$. As before, we denote this as

$$F\left[u(x)\right] = \bar{u}(\omega) \quad \text{or} \quad F^{-1}\left[\bar{u}(\omega)\right] = u(x)\,.$$

By using the notation of Fourier transformations, we have

$$F\left[u(x)\right] = \int_{-\infty}^{+\infty} u(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x\,, \tag{B.19}$$

$$F^{-1}\left[\bar{u}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\bar{u}(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega\,. \tag{B.20}$$

Note that the integrals in Eq. (B.20) are not the integrations of the classical Fourier transformation.

It can be shown that the generalized Fourier transformation shares fundamental properties with the classical Fourier transformation except that the integral property now reads

$$F\left[\int_{-\infty}^{x} f(\xi)\,\mathrm{d}\xi\right] = \frac{\bar{f}(\omega)}{\mathrm{i}\omega} + \pi\bar{f}(0)\delta(\omega)\,.$$

Example 3. Find the generalized Fourier transformation of the Heaviside function

$$H(x) = \begin{cases} 0\,, & x < 0\,, \\ 1\,, & 0 < x\,. \end{cases}$$

Solution. By the Dirichlet integral $\int_0^{+\infty} \frac{sinx}{x}\,\mathrm{d}x = \frac{\pi}{2}$, we have

$$\int_0^{+\infty} \frac{sin\omega x}{\omega}\,\mathrm{d}\omega = \begin{cases} -\frac{\pi}{2}, & x<0, \\ 0, & x=0, \\ \frac{\pi}{2}, & x>0. \end{cases}$$

Thus

$$F^{-1}\left[\frac{1}{\mathrm{i}\omega}\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\mathrm{i}\omega}\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega = \frac{1}{\pi}\int_0^{+\infty} \frac{sin\omega x}{\omega}\,\mathrm{d}\omega = \begin{cases} -\frac{1}{2}, & x<0, \\ 0, & x=0, \\ \frac{1}{2}, & x>0. \end{cases}$$

Since

$$F^{-1}\left[\pi\delta(\omega)+\frac{1}{\mathrm{i}\omega}\right] = F^{-1}\left[\pi\delta(\omega)\right] + F^{-1}\left[\frac{1}{\mathrm{i}\omega}\right]$$

$$= \frac{1}{2} + \begin{cases} -\frac{1}{2}, & x<0 \\ \frac{1}{2}, & x>0 \end{cases} = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases} = H(x),$$

we finally obtain $F\left[H(x)\right] = \pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}$.

Remark. The Fourier transformations of some functions in the tables of Fourier transformation are the generalized Fourier transformation.

B.1.5 The Multiple Fourier Transformation

Consider a function $f(x,y,z)$ of three variables x, y and z. By taking a Fourier transformation with respect to x, we have

$$\bar{f}(\omega_1,y,z) = \int_{-\infty}^{+\infty} f(x,y,z)\,\mathrm{e}^{-\mathrm{i}\omega_1 x}\,\mathrm{d}x,$$

$$f(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{f}(\omega_1,y,z)\,\mathrm{e}^{\mathrm{i}\omega_1 x}\,\mathrm{d}\omega_1.$$

Taking a Fourier transformation with respect to y leads to

$$\bar{f}(\omega_1,\omega_2,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y,z)\,\mathrm{e}^{-\mathrm{i}(\omega_1 x+\omega_2 y)}\,\mathrm{d}x\,\mathrm{d}y\,,$$

$$f(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \bar{f}(\omega_1,\omega_2,z)\,\mathrm{e}^{\mathrm{i}(\omega_1 x+\omega_2 y)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,.$$

Finally, a Fourier transformation with respect to z yields

$$\bar{f}(\omega_1,\omega_2,\omega_3) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y,z)\,\mathrm{e}^{-\mathrm{i}(\omega_1 x+\omega_2 y+\omega_3 z)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,, \tag{B.21}$$

$$f(x,y,z) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \bar{f}(\omega_1,\omega_2,\omega_3)\,\mathrm{e}^{\mathrm{i}(\omega_1 x+\omega_2 y+\omega_3 z)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3\,. \tag{B.22}$$

The former is called the *triple Fourier transformation* of $f(x,y,z)$. The latter is called the *inverse triple Fourier transformation*. The $\bar{f}(\omega_1,\omega_2,\omega_3)$ is called the *image function* of $f(x,y,z)$, and $f(x,y,z)$ is the *inverse image function* of $\bar{f}(\omega_1,\omega_2,\omega_3)$. Denote the points (x,y,z) and $(\omega_1,\omega_2,\omega_3)$ as $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $\boldsymbol{\omega} = \omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k}$, respectively. Equations (B.21) and (B.22) can thus be written as

$$\bar{f}(\boldsymbol{\omega}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\mathbf{r})\,\mathrm{e}^{\mathbf{i}\boldsymbol{\omega}\cdot\mathbf{r}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,,$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \bar{f}(\boldsymbol{\omega})\,\mathrm{e}^{\mathbf{i}\boldsymbol{\omega}\cdot\mathbf{r}}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3\,,$$

which are denoted as

$$\bar{f}(\boldsymbol{\omega}) = F\left[f(\mathbf{r})\right]\,, \quad f(\mathbf{r}) = F^{-1}\left[\bar{f}(\boldsymbol{\omega})\right]\,.$$

Remark. By following a similar approach, we can also define other multiple Fourier transformations such as the double Fourier transformation. The multiple transformation also shares the same properties as the Fourier transformation. For a function $f(x,y,z)$ of three variables, for example, we have

$$F\left[\frac{\partial f}{\partial x}\right] = \mathrm{i}\omega_1 F[f]\,, \quad F\left[\frac{\partial f}{\partial y}\right] = \mathrm{i}\omega_2 F[f]\,, \quad F\left[\frac{\partial f}{\partial z}\right] = \mathrm{i}\omega_3 F[f]\,.$$

For two functions $f_1(x,y,z)$ and $f_2(x,y,z)$ of three variables x, y and z, their convolution is defined by

$$f_1(x,y,z) * f_2(x,y,z)$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_1(t_1,t_2,t_3)f_2(x-t_1,y-t_2,z-t_3)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,.$$

Also, we have the convolution theorem

$$F^{-1} = \left[\bar{f}_1 \bar{f}_2\right] = f_1 * f_2 \,.$$

B.2 Laplace Transformation

The Laplace transformation draws from an extension of the Fourier transformation. When the conditions for the Fourier transformation cannot be satisfied, we can sometimes apply a Laplace transformation.

B.2.1 Laplace Transformation

Functions must be absolutely integrable in $(-\infty,+\infty)$ to have a Fourier transformation under the classical definition. Many functions, some of which are important, cannot however satisfy this condition. Since the Fourier transformation can only be applied to functions defined in $(-\infty,+\infty)$, its application is limited. In order to relax these limits, we need to introduce a new integral transformation.

Consider the function $\varphi(t)$, $t \in (-\infty,+\infty)$,

$$g(t) = \varphi(t)\mathbf{I}(t)e^{-\beta t}, \ \beta > 0 \,,$$

where $\mathbf{I}(t)$ is a unit function. Through multiplying $\varphi(t)$ by $\mathbf{I}(t)$, we reduce the domain $(-\infty,+\infty)$ to $(0,+\infty)$. Through multiplying $\varphi(t)$ by $\mathrm{e}^{-\beta t}$, we increase the speed of tending to zero of $\varphi(t)$ such that the absolutely integrable condition can be satisfied.

Consider the Fourier transformation of $g(t)$,

$$\bar{g}(\omega) = \int_{-\infty}^{+\infty} \varphi(t)\mathbf{I}(t)\,\mathrm{e}^{-\beta t}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \int_0^{+\infty} f(t)\,\mathrm{e}^{-(\beta+\mathrm{i}\omega)t}\,\mathrm{d}t = \int_0^{+\infty} f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t \,,$$

where $f(t) = \varphi(t)\mathbf{I}(t)$, $s = \beta + \mathrm{i}\omega$. Let $\bar{g}(\omega) = \bar{g}\left(\dfrac{s-\beta}{\mathrm{i}}\right) = \bar{f}(s)$. The complex-valued function $\bar{f}(s)$ thus comes from the integral transformation of $f(t)$, $\displaystyle\int_0^{+\infty} f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$. This can be used to introduce the Laplace transformation.

Suppose that $f(t)$ is defined for $t \geq 0$. When $t > 0$, $f(t) \equiv 0$. Assume that the integral $\displaystyle\int_0^{+\infty} f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$ is convergent in a certain region of complex variables. Let

$$\bar{f}(s) = \int_0^{+\infty} f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t \,, \tag{B.23}$$

This is called the *Laplace transformation* of $f(t)$ and denoted by

$$\bar{f}(s) = L[f(t)].$$

$f(s)$ is called the *image function* of $f(t)$.

The Laplace transformation of $f(t)$ with $t \in (0, +\infty)$ as defined by Eq. (B.23) is actually the Fourier transformation of $f(t)\mathbf{I}(t)\,\mathrm{e}^{-\beta t}$ (for $\beta > 0$). Thus we have, by the Fourier integral,

$$f(t)\mathbf{I}(t)\,\mathrm{e}^{-\beta t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)\mathbf{I}(\tau)\,\mathrm{e}^{-\beta\tau}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\tau\right]\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\omega t}\left[\int_{0}^{+\infty} f(\tau)\,\mathrm{e}^{-(\beta+\mathrm{i}\omega)\tau}\,\mathrm{d}\tau\right]\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\bar{f}(s)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\,, \quad t > 0\,,$$

where $s = \beta + \mathrm{i}\omega$. Thus

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\bar{f}(s)\,\mathrm{e}^{(\mathrm{i}\omega+\beta)t}\,\mathrm{d}\omega = \frac{1}{2\pi\mathrm{i}}\int_{\beta-i\infty}^{\beta+i\infty}\bar{f}(s)\,\mathrm{e}^{st}\,\mathrm{d}s\,, \tag{B.24}$$

which is called the *inverse Laplace transformation* and denoted by

$$f(t) = L^{-1}\left[\bar{f}(s)\right]\,.$$

The $f(t)$ is called the *inverse image function* of $\bar{f}(s)$.

Therefore we may find $\bar{f}(s)$ by Eq. (B.23) and $f(t)$ by Eq. (B.24) from the known $f(t)$ or $\bar{f}(s)$. The integral in Eq. (B.23) is a regular integral. The integral in Eq. (B.24) is however an integral of a complex-valued function and is normally obtained by using the residue theorem which will be given in Section B.2.3.

Example 1. Find the Laplace transformation of the unit function $\mathbf{I}(t)$.

Solution. By Eq. (B.23),

$$L[\mathbf{I}(t)] = \int_0^{+\infty}\mathrm{e}^{-st}\,\mathrm{d}t = -\frac{1}{s}\,\mathrm{e}^{-st}\Big|_0^{+\infty} = \frac{1}{s}\,.$$

Its convergence domain is $Re(s) > 0$. Thus

$$L[\mathbf{I}(t)] = \frac{1}{s}\,, \quad Re(s) > 0\,.$$

Example 2. Find the Laplace transformation of the exponential function $\mathrm{e}^{\alpha t}$, where α is a real constant.

Solution. When $Re(s) > 0$, by Eq. (B.23) we have

$$L\left[\mathrm{e}^{\alpha t}\right] = \int_0^{+\infty}\mathrm{e}^{\alpha t}\,\mathrm{e}^{-st}\,\mathrm{d}t = \int_0^{+\infty}\mathrm{e}^{-(s-\alpha)t}\,\mathrm{d}t = \frac{1}{s-\alpha}\,.$$

Example 3. Find the Laplace transformation of $\delta(t)$.

Solution. By the properties of the δ-function,

$$L[\delta(t)] = \int_0^{+\infty} \delta(t)\,\mathrm{e}^{-st}\,\mathrm{d}t = \mathrm{e}^{-st}\big|_{t=0} = 1\,.$$

In applications, we can find corresponding relations between $\bar{f}(s)$ and $f(t)$ from the tables of Laplace transformations.

Similar to the Fourier transformation, $f(t)$ must satisfy certain conditions for the existence of $\bar{f}(s)$. It can be shown that the Laplace transformation

$$\bar{f}(s) = L[f(t)] = \int_0^{+\infty} f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$$

exists in $Re(s) > k$ and is an analytical function if: (1) $f(t) \equiv 0$ for $t < 0$, (2) for $t \geq 0$, $f(t)$ is continuous in any finite region except a finite number of discontinuous points of the first kind, (3) the increasing speed of $f(t)$ is less than an exponential function as $t \to +\infty$ so that there exist constants c and k such that

$$|f(t)| \leq c\,\mathrm{e}^{kt}\,, \quad 0 < t < +\infty\,.$$

Here $k \geq 0$ is called the *increasing exponent* of $f(t)$.

The integral $\bar{f}(s)$ is, in fact, absolutely and uniformly convergent in $Re(s) = \beta > k$, i.e.

$$\int_0^{+\infty} \left|f(t)\,\mathrm{e}^{-st}\right| \mathrm{d}t \leq \int_0^{+\infty} c\,\mathrm{e}^{-(\beta-k)t}\,\mathrm{d}t = \frac{c}{\beta - k}\,.$$

Also

$$\left[\bar{f}(s)\right]' = \int_0^{+\infty} -t f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$$

exists in $Re(s) = \beta > k$ because

$$\int_0^{+\infty} \left|-t f(t)\,\mathrm{e}^{-st}\right| \mathrm{d}t \leq c\int_0^{+\infty} t\,\mathrm{e}^{-(\beta-k)t}\,\mathrm{d}t = \frac{c}{(\beta - k)^2}\,.$$

Therefore, $\bar{f}(s)$ not only exists in $Re(s) = \beta > k$ but also is an analytical function.

B.2.2 Properties of Laplace Transformation

The Laplace transformation and its inverse transformation are defined by integrals. By the properties of integration, we can easily obtain the following properties of Laplace transformations.

Linearity

Let $L[f_1(t)] = \bar{f}_1(s)$, $L[f_2(t)] = \bar{f}_2(s)$. For any two constants α and β, we have

$$L[\alpha f_1(t) + \beta f_2(t)] = \alpha L[f_1(t)] + \beta L[f_2(t)]$$

or

$$L^{-1}\left[\alpha\bar{f}_1(s)+\beta\bar{f}_2(s)\right]=\alpha L^{-1}\left[\bar{f}_1(s)\right]+\beta L^{-1}\left[\bar{f}_2(s)\right].$$

Differential Property

Let $L[f(t)]=\bar{f}(s)$. Applying integration by parts leads to

$$L\left[f'(t)\right]=s\bar{f}(s)-f(0), \quad Re(s)>k.$$

In general,

$$L\left[f^{(n)}(t)\right]=s^n\bar{f}(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0), \quad Re(s)>k.$$

In particular, if $f^{(k)}(0)=0$, $k=0,1,2,\cdots,n-1$ we have

$$L\left[f'(t)\right]=s\bar{f}(s), \quad L\left[f''(t)\right]=s^2\bar{f}(s), \quad \cdots, \quad L\left[f^{(n)}(t)\right]=s^n\bar{f}(s).$$

Integral Property

Suppose that $L[f(t)]=\bar{f}(s)$. Thus

$$L\left[\int_0^t f(t)\,\mathrm{d}t\right]=\frac{1}{s}\bar{f}(s).$$

Proof. Let $h(t)=\int_0^t f(t)\,\mathrm{d}t$. Thus

$$h'(t)=f(t), \quad h(0)=0.$$

By applying the differential property we obtain $L[h'(t)]=sL[h(t)]$ so that

$$L\left[\int_0^t f(t)\,\mathrm{d}t\right]=\frac{1}{s}\bar{f}(s).$$

In general, $$L\underbrace{\left[\int_0^t \mathrm{d}t\int_0^t \mathrm{d}t\cdots\int_0^t f(t)\,\mathrm{d}t\right]}_{n \quad times}=\frac{1}{s^n}\bar{f}(s).$$

Similarly, we can prove that, if $L[f(t)]=\bar{f}(s)$ and $\lim\limits_{t\to 0}\frac{f(t)}{t}$ exist,

$$L\left[\frac{f(t)}{t}\right]=\int_s^\infty \bar{f}(s)\,\mathrm{d}s \quad \text{or} \quad f(t)=tL^{-1}\left[\int_s^\infty \bar{f}(s)\,\mathrm{d}s\right].$$

Shifting Property

Let $L[f(t)]=\bar{f}(s)$. Then $L[\mathrm{e}^{\alpha t}f(t)]=\bar{f}(s-\alpha)$, $Re(s-\alpha)>k$.

Delay Property Let $L[f(t)] = \bar{f}(s)$. Then $L[f(t-\tau)] = \mathrm{e}^{-st}\bar{f}(s)$, where the real constant $\tau > 0$.

Proof. By the definition of the Laplace transformation,

$$\begin{aligned}
L[f(t-\tau)] &= \int_0^{+\infty} f(t-\tau)\,\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \int_0^{\tau} f(t-\tau)\,\mathrm{e}^{-st}\,\mathrm{d}t + \int_{\tau}^{+\infty} f(t-\tau)\,\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \int_{\tau}^{+\infty} f(t-\tau)\,\mathrm{e}^{-st}\,\mathrm{d}t = \int_0^{+\infty} f(u)\,\mathrm{e}^{-s(u+\tau)}\,\mathrm{d}u \\
&= \mathrm{e}^{-st}\bar{f}(s)\,, \quad Re(s) > k\,.
\end{aligned}$$

When $\tau < 0$, $\int_0^{\tau} f(t-\tau)\,\mathrm{e}^{-st}\,\mathrm{d}t = 0$ is not valid.

Theorems of Initial Value and Final Value

Theorem of initial value. If $L[f(t)] = \bar{f}(s)$ and $\lim\limits_{s\to\infty} s\bar{f}(s)$ exist, then

$$\lim_{t\to+0} f(t) = \lim_{s\to\infty} s\bar{f}(s) \quad \text{or} \quad f(0) = \lim_{s\to\infty} s\bar{f}(s)\,.$$

Proof. Since $\lim\limits_{s\to\infty} s\bar{f}(s)$ exists, we have

$$\lim_{s\to\infty} L\left[f'(t)\right] = \lim_{s\to\infty}\left[s\bar{f}(s) - f(0)\right] = \lim_{s\to\infty} s\bar{f}(s) - f(0)\,.$$

Also, by the existence theorem of Laplace transformations,

$$\int_0^{+\infty} \left|f'(t)\,\mathrm{e}^{-st}\right|\,\mathrm{d}t \le \frac{c}{\beta - k}\,,$$

where $|f'(t)| \le c\,\mathrm{e}^{-kt}$, $Re(s) = \beta$. As $s \to \infty$, $\beta \to +\infty$ so that

$$\lim_{s\to\infty} L\left[f'(t)\right] = 0\,.$$

Thus $\qquad f(0) = \lim\limits_{s\to\infty} s\bar{f}(s)$.

Therefore we may obtain the initial value of $f(t)$ by taking the limit of $s\bar{f}(s)$ as $s \to \infty$.

Theorem of final value. If $L[f(t)] = \bar{f}(s)$ and $\lim\limits_{s\to\infty} s\bar{f}(s)$ exist, then

$\lim\limits_{t\to+\infty} f(t) = \lim\limits_{s\to 0} s\bar{f}(s)$ or $f(\infty) = \lim\limits_{s\to 0} s\bar{f}(s)$.

Proof. By the differential property of the Laplace transformation,

$$\lim_{s\to 0} L\left[f'(t)\right] = \lim_{s\to 0}\int_0^{+\infty} f'(t)\,\mathrm{e}^{-st}\,\mathrm{d}t = \lim_{s\to 0} s\bar{f}(s) - f(0)\,,$$

on the other hand,

$$\lim_{s\to 0}\int_0^{+\infty} f'(t)\,\mathrm{e}^{-st}\,\mathrm{d}t = \int_0^{+\infty} f'(t)\,\mathrm{d}t = f(+\infty) - f(0)\,,$$

where taking the limit inside the integral is allowed by the existence theorem. A comparison of the two equations yields

$$f(+\infty) = \lim_{s\to 0} s\bar{f}(s)\,.$$

Therefore the final value of $f(t)$ is the limit of $s\bar{f}(s)$ as $s \to 0$.

Theorems of initial value and final value play an important role in finding $f(0)$ and $f(+\infty)$ from the image function $\bar{f}(s)$.

If $L[f(t)] = \frac{1}{s+\alpha}$, for example, then

$$f(0) = \lim_{s\to\infty}\frac{s}{s+\alpha} = 1\,, \quad f(+\infty) = \lim_{s\to 0}\frac{s}{s+\alpha} = 0\,.$$

B.2.3 Determine Inverse Image Functions by Calculating Residues

Theorem. Let s_1, s_2, $\cdots$, s_n be all singular points of $\bar{f}(s)$. Take a real constant β such that all the s_k are in the half plane $Re(s) < \beta$. Suppose that $\lim\limits_{s\to\infty}\bar{f}(s) = 0$. The inverse image function is thus

$$f(t) = \sum_{j=1}^{n} Res\left[\bar{f}(s)\,\mathrm{e}^{st}, s_j\right],$$

where $Res\left[\bar{f}(s)\,\mathrm{e}^{st}, s_j\right]$ stands for the residue of $\bar{f}(s)\,\mathrm{e}^{st}$ at $s = s_j$.

Proof. Consider the half circle $C = L + C_R$ in Fig. B.1 where all singular points s_j in C $(j = 1, 2, \cdots, n)$ are also singular points of $\bar{f}(s)\,\mathrm{e}^{st}$. By the residue theorem,

$$\oint_C \bar{f}(s)\,\mathrm{e}^{st}\,\mathrm{d}s = 2\pi\mathrm{i}\sum_{j=1}^{n} Res\left[\bar{f}(s)\,\mathrm{e}^{st}, s_j\right],$$

i.e.,

$$\frac{1}{2\pi\mathrm{i}}\left[\int_{\beta-\mathrm{i}R}^{\beta+\mathrm{i}R}\bar{f}(s)\,\mathrm{e}^{st}\,\mathrm{d}s + \int_{C_R}\bar{f}(s)\,\mathrm{e}^{st}\,\mathrm{d}s\right] = \sum_{j=1}^{n} Res\left[\bar{f}(s)\,\mathrm{e}^{st}, s_j\right].$$

As $R \to +\infty$, by applying the Jordan lemma from the theory of functions of complex variables, we have

$$\lim_{R\to+\infty}\int_{C_R}\bar{f}(s)\,\mathrm{e}^{st}\,\mathrm{d}s = 0\,.$$

Therefore

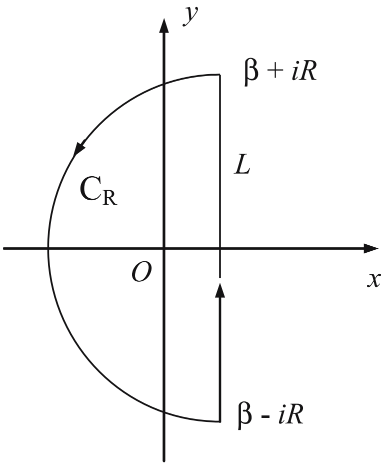

Fig. B.1 A half circle $C = L + C_R$

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty} \bar{f}(s)\,\mathrm{e}^{st}\,\mathrm{d}s = \sum_{j=1}^{n} Res\left[\bar{f}(s)\,\mathrm{e}^{st}, s_j\right]. \tag{B.25}$$

Corollary. Consider the irreducible rational function $\bar{f}(s) = \frac{A(s)}{B(s)}$ where the degree n of $B(s)$ is higher than that of $A(s)$. We have:

1. When $B(s)$ has n distinct zero points s_j of order 1, $j = 1, 2, \cdots, n$,

$$f(t) = \sum_{j=1}^{n} \frac{A(s_j)}{B'(s_j)}\,\mathrm{e}^{s_j t} \quad t > 0. \tag{B.26}$$

2. When $B(s)$ has the zero point s_1 of order m and zero points s_{m+1}, s_{m+2}, $\cdots$, s_n of order 1,

$$f(t) = \sum_{j=m+1}^{n} \frac{A(s_j)}{B'(s_j)}\,\mathrm{e}^{s_j t} + \frac{1}{(m-1)!} \lim_{s \to s_1} \frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}} \left[(s - s_1)^m \frac{A(s)}{B(s)}\,\mathrm{e}^{st}\right], \quad t > 0. \tag{B.27}$$

These two equations together are called the *Heaviside expansion* and play an important role in solving equations using the Laplace transformation.

In calculating $\lim\limits_{s \to s_1} \frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}} \left[(s - s_1)^m \frac{A(s)}{B(s)}\,\mathrm{e}^{st}\right]$ in Eq. (B.27), common factors in $(s - s_1)^m$ and $B(s)$ are reducible so that $\left[(s - s_1)^m \frac{A(s)}{B(s)}\,\mathrm{e}^{st}\right]$ becomes an analytical function. Note that the $(m-1)$-th derivative of an analytical function is still an analytical function. To find $\lim\limits_{s \to s_1} \frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}} \left[(s - s_1)^m \frac{A(s)}{B(s)}\,\mathrm{e}^{st}\right]$ is thus reduced to the problem of finding the value of an analytical function at s_1.

If $B(s)$ also has a zero point s_2 of order l, the Heaviside expansion (B.27) will have another form

$$\frac{1}{(l-1)!}\lim_{s\to s_2}\frac{\mathrm{d}^{l-1}}{\mathrm{d}s^{l-1}}\left[(s-s_2)^l\frac{A(s)}{B(s)}\,\mathrm{e}^{st}\right].$$

In the expansion, of course, there are only $n-(m+l)$ terms left corresponding to the zero points of order 1.

Example 4.

Find $f(t)$ for $\bar{f}(s)=\dfrac{s}{s^2+1}$. (2) Find $f(t)$ for $\bar{f}(s)=\dfrac{1}{s(s-1)^2}$.

Solution.

1. s^2+1 has two zero points of order 1: $\pm\mathrm{i}$. Thus,

$$f(t)=\frac{s}{2s}\,\mathrm{e}^{st}\Big|_{s=\mathrm{i}}+\frac{s}{2s}\,\mathrm{e}^{st}\Big|_{s=-\mathrm{i}}=\frac{1}{2}(\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t})=\cos t\,.$$

2. $s(s-1)^2$ has a zero point of order 1 and a zero point of order 2: $s_1=0$ and $s_2=1$. Thus

$$f(t)=\frac{1}{[s(s-1)^2]'}\,\mathrm{e}^{st}\Big|_{s=0}+\lim_{s\to 1}\left(\frac{\mathrm{e}^{st}}{s}\right)'=1+\mathrm{e}^t(t-1)\,.$$

Remark. If $B(s)$ has only zero points of order 1, then

$$B(s)=(s-s_1)(s-s_2)\cdots(s-s_n)\,,$$
$$B'(s_k)=(s-s_1)(s-s_2)\cdots(s-s_{k-1})(s-s_{k+1})\cdots(s-s_n)|_{s=s_k}\,,$$

where $s_i(i=1,2,\cdots n)$ are the zero points of $B(s)$. This is quite useful for finding some inverse image functions. For example,

If $\bar{f}(s)=\dfrac{1}{(s+1)(s-3)}$, then $f(t)=-\dfrac{1}{4}\mathrm{e}^{-t}+\dfrac{1}{4}\mathrm{e}^{3t}$.

If $\bar{f}(s)=\dfrac{1}{(s+1)(s-2)(s+3)}$, then $f(t)=-\dfrac{\mathrm{e}^{-t}}{6}+\dfrac{\mathrm{e}^{2t}}{15}+\dfrac{\mathrm{e}^{-3t}}{10}$.

B.2.4 Convolution Theorem

The convolution of two functions $f_1(t)$ and $f_2(t)$ is

$$f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)\,\mathrm{d}\tau\,.$$

Since $f_1(t)\equiv 0$ and $f_2(t)\equiv 0$ for $t<0$, we have

$$\begin{aligned}f_1(t)*f_2(t)&=\int_{-\infty}^{0}f_1(\tau)f_2(t-\tau)\,\mathrm{d}\tau+\int_0^t f_1(\tau)f_2(t-\tau)\,\mathrm{d}\tau\\&\quad+\int_t^{+\infty}f_1(\tau)f_2(t-\tau)d\tau=\int_0^t f_1(\tau)f_2(t-\tau)\,\mathrm{d}\tau\,.\end{aligned}$$

If $f_1(t) = t$ and $f_2(t) = \sin t$, for example, we have

$$\begin{aligned} t * \sin t &= \int_0^t \tau \sin(t-\tau)\,\mathrm{d}\tau \\ &= \tau\cos(t-\tau)|_0^t - \int_0^t \cos(t-\tau)\,\mathrm{d}\tau \\ &= t - \sin t\,. \end{aligned}$$

Convolution Theorem. Suppose that $f_1(t)$ and $f_2(t)$ satisfy the condition for the existence of the Laplace transformation. Let

$$L[f_1(t)] = \bar{f}_1(s)\,, \quad L[f_2(t)] = \bar{f}_2(s)\,.$$

Thus $\quad L[f_1(t) * f_2(t)] = \bar{f}_1(s)\bar{f}_2(s)$ or $L^{-1}\left[\bar{f}_1(s)\bar{f}_2(s)\right] = f_1(t) * f_2(t)$.

Proof. Clearly, $f_1(t) * f_2(t)$ also satisfies the condition for the existence of the Laplace transformation. Thus

$$\begin{aligned} L[f_1(t) * f_2(t)] &= \int_0^{+\infty} [f_1(t) * f_2(t)]\,\mathrm{e}^{-st}\,\mathrm{d}t \\ &= \int_0^{+\infty} \left[\int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau\right] \mathrm{e}^{-st}\,\mathrm{d}t \\ &= \int_0^{+\infty} f_1(\tau)\,\mathrm{d}\tau \int_\tau^{+\infty} f_2(t-\tau)\,\mathrm{e}^{-st}\,\mathrm{d}t\,. \end{aligned}$$

Let $t - \tau = u$. Thus

$$\int_0^{+\infty} f_2(t-\tau)\,\mathrm{e}^{-st}\,\mathrm{d}t = \int_0^{+\infty} f_2(u)\,\mathrm{e}^{-(u+\tau)s}\,\mathrm{d}u = \mathrm{e}^{-st}\bar{f}_2(s)\,.$$

Finally, $\quad L[f_1(t) * f_2(t)] = \int_0^{+\infty} f_1(\tau)\,\mathrm{e}^{-s\tau}\bar{f}_2(s)\,\mathrm{d}\tau = \bar{f}_1(s)\bar{f}_2(s)$.

Example 5. Find $f(t)$ for $\bar{f}(s) = \dfrac{s^2}{(s^2+1)^2}$.

Solution. By direct calculation or the table of Laplace transformations (Appendix C), we have

$$L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t.$$

Thus

$$\begin{aligned} f(t) &= L^{-1}\left[\frac{s^2}{(s^2+1)^2}\right] = L^{-1}\left[\frac{s}{s^2+1}\cdot\frac{s}{s^2+1}\right] \\ &= \cos t * \cos t = \int_0^t \cos\tau\cos(t-\tau)\,\mathrm{d}\tau \\ &= \int_0^t \frac{1}{2}\left[\cos t + \cos(2\tau - t)\right]\mathrm{d}\tau = \frac{1}{2}(t\cos t + \sin t)\,. \end{aligned}$$

Example 6. Find $f(t) = L^{-1}\left[\dfrac{1}{s^2(1+s^2)}\right]$ by using different methods.

Solution.

1. By the linearity of Laplace transformation,

$$f(t) = L^{-1}\left[\frac{1}{s^2(1+s^2)}\right] = L^{-1}\left[\frac{1}{s^2}\right] - L^{-1}\left[\frac{1}{1+s^2}\right] = t - \sin t\,.$$

2. By the Heaviside expansion,

$$f(t) = \left.\frac{1}{[s^2(1+s^2)]'}\,\mathrm{e}^{st}\right|_{s=\mathrm{i}} + \left.\frac{1}{[s^2(1+s^2)]'}\,\mathrm{e}^{st}\right|_{s=-\mathrm{i}} + \lim_{s\to 0}\left[\frac{1}{1+s^2}\,\mathrm{e}^{st}\right]'$$

$$= -\frac{\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}}{2\mathrm{i}} + \left.\frac{t\,\mathrm{e}^{st}}{1+s^2}\right|_{s=0} = t - \sin t\,.$$

3. By the convolution theorem,

$$f(t) = L^{-1}\left[\frac{1}{s^2}\,\frac{1}{1+s^2}\right] = t * \sin t = \int_0^t (t-\tau)\sin\tau\,\mathrm{d}\tau$$

$$= (-t\cos\tau)\big|_0^t - (-\tau\cos\tau + \sin\tau)\big|_0^t = t - \sin t\,.$$

Appendix C
Tables of Integral Transformations

Table C.1 Fourier transformations

Inverse Image Functions	Image Functions
$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\bar{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega$	$\bar{f}(\omega)=\int_{-\infty}^{+\infty}f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x$
$\frac{\sin\alpha x}{x},\alpha>0$	$\begin{cases}\pi, & \lvert\omega\rvert<\alpha\\ 0, & \lvert\omega\rvert>\alpha\end{cases}$
$\begin{cases}\mathrm{e}^{\mathrm{i}kx} & a<x<b\\ 0, & x<a \quad \text{or} \quad x>b\end{cases}$	$\frac{\mathrm{i}}{k-\omega}\left(\mathrm{e}^{\mathrm{i}(k-\omega)a}-\mathrm{e}^{\mathrm{i}(k-\omega)b}\right)$
$\begin{cases}\mathrm{e}^{-cx+\mathrm{i}kx}, & x>0\\ 0, & x<0\end{cases}, \qquad c>0$	$\frac{\mathrm{i}}{k+\omega+\mathrm{i}c}$
$\mathrm{e}^{-\eta x^2},\eta>0$	$\sqrt{\frac{\pi}{\eta}}\,\mathrm{e}^{-\frac{\omega^2}{4\eta}}$
$\cos\eta x^2, \quad \eta>0$	$\sqrt{\frac{\pi}{\eta}}\cos\left(\frac{\omega^2}{4\eta}-\frac{\pi}{4}\right)$
$\sin\eta x^2, \quad \eta>0$	$\sqrt{\frac{\pi}{\eta}}\cos\left(\frac{\omega^2}{4\eta}+\frac{\pi}{4}\right)$
$\lvert x\rvert^{-s}, \quad 0<\mathrm{Re}(s)<1$	$\frac{2}{\lvert\omega\rvert^{1-s}}\Gamma(1-s)\sin\frac{1}{2}\pi s$
$\frac{1}{\sqrt{\lvert x\rvert}}\mathrm{e}^{-\alpha\lvert x\rvert}, \quad \alpha>0$	$\sqrt{\frac{2\pi}{\alpha^2+\omega^2}}\sqrt{\sqrt{\alpha^2+\omega^2}+\alpha}$

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Table C.1 continued

Inverse Image Functions	Image Functions
$\dfrac{1}{\|x\|}$	$\dfrac{\sqrt{2\pi}}{\|\omega\|}$
$\dfrac{\mathrm{ch}\alpha x}{\mathrm{ch}\pi x}, \quad -\pi < \alpha < \pi$	$\dfrac{2\cos\dfrac{\alpha}{2}\mathrm{ch}\dfrac{\omega}{2}}{\mathrm{ch}\omega + cos\alpha}$
$\dfrac{\mathrm{sh}\alpha x}{\mathrm{sh}\pi x}, \quad -\pi < \alpha < \pi$	$\dfrac{\sin\alpha}{\mathrm{ch}\omega + \cos\alpha}$
$\dfrac{\mathrm{sh}\alpha x}{\mathrm{ch}\pi x}, \quad \|\alpha\| < \pi$	$2\mathrm{i}\dfrac{\sin\frac{\alpha}{2}\mathrm{sh}\frac{\omega}{2}}{\mathrm{ch}\omega + cos\alpha}$
$\begin{cases} \dfrac{1}{\sqrt{\alpha^2 - x^2}}, & \|x\| < \alpha \\ 0, & \|x\| > \alpha \end{cases}$	$\pi J_0(\alpha\,\|\omega\|)$
$\delta(x)$	1
1	$2\pi\delta(\omega)$
Polynomial $P(x)$	$2\pi P\left(\mathrm{i}\dfrac{\mathrm{d}}{\mathrm{d}\omega}\right)\delta(\omega)$
$\delta^{(m)}(x)$	$(\mathrm{i}\omega)^m$
e^{bx}	$2\pi\delta(\omega + \mathrm{i}b)$
$\sin bx$	$\mathrm{i}\pi[\delta(\omega + b) - \delta(\omega - b)]$
$\cos bx$	$\pi[\delta(\omega + b) + \delta(\omega - b)]$
$\mathrm{sh}bx$	$\pi[\delta(\omega + \mathrm{i}b) - \delta(\omega - \mathrm{i}b)]$
$\mathrm{ch}bx$	$\pi[\delta(\omega + \mathrm{i}b) + \delta(\omega - \mathrm{i}b)]$
$\dfrac{1}{\alpha^2 + x^2}, \quad \mathrm{Re}(\alpha) < 0$	$\dfrac{\pi}{\alpha}\,\mathrm{e}^{-\alpha\|\omega\|}$
$\dfrac{x}{(\alpha^2 + x^2)^2}, \quad \mathrm{Re}(\alpha) < 0$	$\dfrac{\mathrm{i}\omega\pi}{2\alpha}\,\mathrm{e}^{\alpha\|\omega\|}$

Table C.1 continued

Inverse Image Functions	Image Functions
$\dfrac{e^{\mathrm{i}bx}}{\alpha^2+x^2}$, $\mathrm{Re}(\alpha)<0$, b is a real number	$-\dfrac{\pi}{\alpha}\,\mathrm{e}^{\lvert\omega-b\rvert}$
$\dfrac{\cos bx}{\alpha^2+x^2}$, $\mathrm{Re}(\alpha)<0$, b is a real number	$-\dfrac{\pi}{2\alpha}\left[\mathrm{e}^{\alpha\lvert\omega-b\rvert}+e^{\alpha\lvert\omega+b\rvert}\right]$
$\dfrac{\sin bx}{\alpha^2+x^2}$, $\mathrm{Re}(\alpha)<0, b$ is a real number	$-\dfrac{\pi}{2\alpha\mathrm{i}}\left[\mathrm{e}^{\alpha\lvert\omega-b\rvert}-\mathrm{e}^{\alpha\lvert\omega+b\rvert}\right]$
$\dfrac{\sin\left(b\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}}$, $a,b>0$	$\begin{cases}0, & \lvert\omega\rvert>b\\ \pi J_0\left(a\sqrt{b^2-\omega^2}\right), & \lvert\omega\rvert<b\end{cases}$
$\dfrac{\sin\left(b\sqrt{x^2-a^2}\right)}{\sqrt{x^2-a^2}}$, $\lvert x\rvert>a$	$\begin{cases}0, & \lvert\omega\rvert>b\\ \pi I_0\left(a\sqrt{b^2-\omega^2}\right), & \lvert\omega\rvert<b\end{cases}$
$\begin{cases}\cos\left(b\sqrt{a^2-x^2}\right), & \lvert x\rvert<\alpha\\ 0, & \lvert x\rvert>\alpha\end{cases}$	$\pi J_0\left(a\sqrt{\omega^2-b^2}\right)$
$\begin{cases}\mathrm{ch}\left(b\sqrt{a^2-x^2}\right), & \lvert x\rvert<\alpha\\ 0, & \lvert x\rvert>\alpha\end{cases}$	$\begin{cases}\pi J_0\left(a\sqrt{\omega^2-b^2}\right), & \lvert\omega\rvert>b\\ 0, & \lvert\omega\rvert<b\end{cases}$

Table C.2 Laplace transfomations

Inverse Image Functions	Image Functions
$f(t)=\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty}\bar{f}(s)\,\mathrm{e}^{st}\,ds$	$\bar{f}(s)=\displaystyle\int_0^{+\infty}f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$
1	$\dfrac{1}{s}$
t^n, $n=1,2,3,\cdots$	$\dfrac{n!}{s^{n+1}}$
$\mathrm{e}^{\alpha t}$	$\dfrac{1}{s-\alpha}$
$t\,\mathrm{e}^{\alpha t}$	$\dfrac{1}{(s-\alpha)^2}$

Table C.2 countinued

Inverse Image Functions	Image Functions
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$\mathrm{sh} kt$	$\dfrac{k}{s^2-k^2}$
$\mathrm{ch} kt$	$\dfrac{s}{s^2-k^2}$
$\dfrac{1}{\sqrt{t}}$	$\sqrt{\dfrac{\pi}{s}}$
$\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2\sqrt{s^3}}$
$t^n \mathrm{e}^{\alpha t}$	$\dfrac{\Gamma(n+1)}{(s-\alpha)^{n+1}}$
$\mathrm{e}^{\alpha t} \sin \omega t$	$\dfrac{\omega}{(s-\alpha)^2+\omega^2}$
$\mathrm{e}^{\alpha t} \cos \omega t$	$\dfrac{s-\alpha}{(s-\alpha)^2+\omega^2}$
$\delta(t)$	1
$t \sin \alpha t$	$\dfrac{2\alpha s}{(s^2+\alpha^2)^2}$
$t \cos \alpha t$	$\dfrac{s^2-\alpha^2}{(s^2+\alpha^2)^2}$
$\dfrac{1}{t} \sin \alpha t$	$\arctan\left(\dfrac{\alpha}{s}\right)$
$t \mathrm{sh} \alpha t$	$\dfrac{2\alpha s}{(s^2-\alpha^2)^2}$
$t \mathrm{ch} \alpha t$	$\dfrac{s^2+\alpha^2}{(s^2-\alpha^2)^2}$

Table C.2 countinued

Inverse Image Functions	Image Functions
$\sin^2 t$	$\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$
$\cos^2 t$	$\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$
$\sin at \sin bt$	$\frac{2abs}{[s^2+(a+b)^2]\,[s^2+(a-b)^2]}$
$\mathrm{e}^{at}-\mathrm{e}^{bt}$	$\frac{a-b}{(s-a)(s-b)}$
$a\,\mathrm{e}^{at}-b\,\mathrm{e}^{bt}$	$\frac{(a-b)s}{(s-a)(s-b)}$
$J_0(\alpha t)$	$\frac{1}{\sqrt{s^2+\alpha^2}}$
$I_0(\alpha t)$	$\frac{1}{\sqrt{s^2-\alpha^2}}$
$J_\gamma(\alpha t)\,, \quad \mathrm{Re}(\gamma)>-1$	$\frac{\alpha^\gamma}{\sqrt{\alpha^2+s^2}}\left(\frac{1}{s^2+\sqrt{\alpha^2+s^2}}\right)^\gamma$
$\mathrm{erf}(\sqrt{\alpha t})\,, \quad \alpha>0$	$\frac{\sqrt{\alpha}}{s\sqrt{s+\alpha}}$
$\mathrm{erfc}\left(\frac{\alpha}{2\sqrt{t}}\right), \quad \alpha>0$	$\frac{1}{s}\,\mathrm{e}^{-\alpha\sqrt{s}}$
$\mathrm{e}^t\,\mathrm{erfc}\sqrt{t}$	$\frac{1}{s+\sqrt{s}}$
$\frac{1}{\sqrt{\pi t}}\,\mathrm{e}^{-\frac{\alpha^2}{4t}}\,, \quad \alpha\geq 0$	$\frac{1}{\sqrt{s}}\,\mathrm{e}^{-\alpha\sqrt{s}}$
$\frac{1}{\sqrt{\pi t}}\sin 2\sqrt{\alpha t}$	$\frac{1}{s\sqrt{s}}\,\mathrm{e}^{-\frac{\alpha}{s}}$
$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{\alpha t}$	$\frac{1}{\sqrt{s}}\,\mathrm{e}^{-\frac{\alpha}{s}}$

Table C.2 countinued

Inverse Image Functions	Image Functions
$\frac{1}{\sqrt{\pi t}} \sin \frac{1}{2t}$	$\frac{1}{\sqrt{s}} \mathrm{e}^{-\sqrt{s}} \sin \sqrt{s}$
$\frac{1}{\sqrt{\pi t}} \cos \frac{1}{2t}$	$\frac{1}{\sqrt{s}} \mathrm{e}^{-\sqrt{s}} \cos \sqrt{s}$
$\frac{1}{\pi t} \sin(2\alpha\sqrt{t}) I_0 \left(c\sqrt{t^2-(\xi-\xi')^2}\right) H\left[t-(\xi-\xi')\right]$	$\operatorname{erf}\left(\frac{\alpha}{\sqrt{s}}\right) \frac{\mathrm{e}^{-(\xi'-\xi)\sqrt{s^2-c^2}}}{\sqrt{s^2-c^2}}$

Table C.3 Hankel Transformations

Inverse Image Functions	Image Functions
$f(r) = \int_0^{+\infty} \omega \bar{f}(\omega) J_0(\omega r)\, \mathrm{d}\omega$	$\bar{f}(\omega) = \int_0^{+\infty} r f(r) J_0(\omega r)\, \mathrm{d}r$
$f(r) = \begin{cases} 1, & r < \alpha \\ 0, & r > \alpha \end{cases}$	$\frac{\alpha}{\omega} J_1(\alpha\omega)$
$\begin{cases} \alpha^2 - r^2, & r < \alpha \\ 0, & r > \alpha \end{cases}$	$\frac{4\alpha}{\omega^3} J_1(\alpha\omega) - \frac{2\alpha^2}{\omega^2} J_0(\alpha\omega)$
$\frac{\mathrm{e}^{-\alpha r}}{r}$	$\frac{1}{\sqrt{\omega^2+\alpha^2}}$
$\frac{\sin \alpha r}{r}, \quad \alpha > 0$	$\begin{cases} \frac{1}{\sqrt{\alpha^2-\omega^2}}, & 0 < \omega < \alpha \\ 0, & \omega > \alpha \end{cases}$
$\frac{\cos \alpha r}{r}, \quad \alpha > 0$	$\begin{cases} \frac{1}{\sqrt{\omega^2-\alpha^2}}, & \omega > \alpha \\ 0, & 0 < \omega < \alpha \end{cases}$
$\frac{1}{2\pi r} \delta(r-\alpha)$	$\frac{1}{2\pi} J_0(\alpha\omega)$

Table C.3 continued

Inverse Image Functions	Image Functions
$\mathrm{e}^{-\alpha r}$	$\dfrac{\alpha}{(\omega^2+\alpha^2)^{3/2}}$
$\mathrm{e}^{-\alpha r^2}$	$\dfrac{1}{2\alpha}\,\mathrm{e}^{-\omega^2/4\alpha}$
$\dfrac{1}{(r^2+\alpha^2)^{1/2}}$	$\dfrac{1}{\omega}\,\mathrm{e}^{-\alpha\omega}$
$\dfrac{1}{(r^2+\alpha^2)^{3/2}}$	$\dfrac{1}{\alpha}\,\mathrm{e}^{-\alpha\omega}$
$\dfrac{1}{r^2+\alpha^2}$	$K_0(\alpha\omega)$
$\dfrac{2\alpha^2}{(r^2+\alpha^2)^2}$	$\alpha\omega K_1(\alpha\omega)$
$\dfrac{4\alpha^4}{(r^2+\alpha^2)^3}$	$\alpha\omega K_1(\alpha\omega)+\dfrac{\alpha^2\omega^2}{2}K_0(\alpha\omega)$

Table C.4 Spherical Bessel Transformations

Inverse Image Functions	Image Functions
$f(r)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{+\infty}\omega^2\bar{f}(\omega)\,j_0(\omega r)\,\mathrm{d}\omega$	$\bar{f}(\omega)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{+\infty}r^2 f(r)J_0(\omega r)\,\mathrm{d}r$
$\dfrac{\mathrm{e}^{-\alpha r}}{r}$	$\sqrt{\dfrac{2}{\pi}}\dfrac{1}{\alpha^2+\omega^2}$
$\mathrm{e}^{-\alpha r}$	$2\sqrt{\dfrac{2}{\pi}}\dfrac{\alpha}{(\alpha^2+\omega^2)^2}$
$\mathrm{e}^{-\alpha r^2}$	$\left(\dfrac{1}{\sqrt{2\alpha}}\right)^3\mathrm{e}^{-\omega^2/4\alpha}$

Table C.4 continued

Inverse Image Functions	Image Functions
$\frac{1}{4\pi r^2}\delta(r-\alpha), \quad \alpha > 0$	$\left(\frac{1}{\sqrt{2\pi}}\right)^3 \frac{\sin\alpha\omega}{\alpha\omega}$
$\frac{\sin\alpha r}{r}$	$\sqrt{\frac{\pi}{2}}\frac{1}{\omega}\left[\delta(\omega-\alpha)-\delta(\omega+\alpha)\right]$
$\frac{1}{r}$	$\sqrt{\frac{2}{\pi}}\frac{1}{\omega^2}$
$r^{-3/2}$	$\omega^{-3/2}$
$r^{n-1}\mathrm{e}^{-\alpha r}, \quad \alpha > 0,\ n \geq 1$	$\sqrt{\frac{2}{\pi}}\frac{n!}{\omega}\left(\frac{\alpha}{\alpha^2+\omega^2}\right)^{n+1} \cdot \sum_{m=0}^{\left[\frac{n}{2}\right]}(-1)^m C_{n+1}^{2m+1}\left(\frac{\omega}{\alpha}\right)^{2m+1}$
$\begin{cases}(1-r^2)^\gamma, & 0 < r < 1\\ 0, & r > 1\end{cases}, \quad \mathrm{Re}\gamma > -1,$	$2^\gamma\Gamma(\gamma+1)\omega^{-\gamma-3/2}J_{\gamma+3/2}(\omega)$
$\begin{cases}0, & 0 < r < \alpha\\ \frac{1}{r\sqrt{r^2-\alpha^2}}, & r > \alpha\end{cases}$	$\sqrt{\frac{\pi}{2}}\frac{1}{\omega}J_0(\alpha\omega)$

Appendix D
Eigenvalue Problems

In this appendix, we discuss eigenvalue problems of second-order equations

$$a(x)y''(x)+b(x)y'+c(x)y(x)+\lambda y(x)=0\,. \tag{D.1}$$

Here λ is a parameter, $a(x)$, $b(x)$ and $c(x)$ are functions of x.

D.1 Regular Sturm-Liouville Problems

Let

$$p(x)=\mathrm{e}^{\int \frac{b(x)}{a(x)}\,\mathrm{d}x}\,, \qquad q(x)=-\frac{c(x)}{a(x)}p(x)\,,$$

$$\rho(x)=\frac{p(x)}{a(x)}\,, \qquad L=-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right]+q(x)\,.$$

Equation (D.1) is thus reduced to

$$Ly-\lambda\rho(x)y=0\,. \tag{D.2}$$

It is called the *Sturm-Liouville equation*, the *S-L* equation for short.

In order for it to have nontrivial solutions of Eq. (D.2) in interval $[a,b]$, $p(x)$, $q(x)$ and $\rho(x)$ should satisfy

$$(1) \qquad p(x)\,,p'(x)\,,q(x) \quad \rho(x)\in C\,[a,b]\,. \tag{D.3}$$

$$(2) \qquad p(x)>0\,,q(x)\geq 0\,, \quad \rho(x)>0 \text{ in } [a,b]\,. \tag{D.4}$$

The boundary conditions are homogeneous and separable at both ends, i.e.,

$$\alpha_1 y(a)+\alpha_2 y'(a)=0\,, \quad \alpha_1^2+\alpha_2^2\neq 0\,. \tag{D.5}$$

$$\beta_1 y(b)+\beta_2 y'(b)=0\,, \quad \beta_1^2+\beta_2^2\neq 0\,. \tag{D.6}$$

where α_i and β_i $(i=1,2)$ are known constants.

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Equation (D.2) is called the *regular S-L equation* if conditions (D.3) and (D.4) are satisfied. The problem of finding its nontrivial solutions under boundary conditions (D.5) and (D.6) is called the *regular S-L problem*. It can be shown that the problem has solutions only for certain values of parameter $\lambda = \lambda_k$, $k = 1, 2, \cdots$. The λ_k are called the *eigenvalues*. The corresponding solutions are called the *eigenfunctions* of the problem.

D.2 The Lagrange Equality and Self-Conjugate Boundary-Value Problems

Suppose that $u(x)$, $v(x) \in C^2[a,b]$. Integration by parts leads to

$$\begin{aligned}\int_a^b vLu\,\mathrm{d}x &= \int_a^b \left[-v(pu')' + vqu\right] \mathrm{d}x \\ &= -v(x)p(x)u'(x)\big|_a^b + \int_a^b \left(v'pu' + vqu\right) \mathrm{d}x \\ &= -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\big|_a^b + \int_a^b uLv\,\mathrm{d}x\,.\end{aligned}$$

Thus

$$\int_a^b (vLu - uLv)\,\mathrm{d}x = -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\big|_a^b\,. \tag{D.7}$$

This is called the *Lagrange equality*.

If $u(x)$ and $v(x)$ satisfy boundary conditions (D.5) and (D.6), when $\alpha_2 \neq 0$ and $\beta_2 \neq 0$ the right-hand side of Eq. (D.7) becomes

$$\begin{aligned}&-p(b)\left[u'(b)v(b) - u(b)v'(b)\right] + p(a)\left[u'(a)v(a) - u(a)v'(a)\right] \\ &= -p(b)\left[-\frac{\beta_1}{\beta_2}u(b)v(b) + \frac{\beta_1}{\beta_2}u(b)v(b)\right] \\ &\quad + p(a)\left[-\frac{\alpha_1}{\alpha_2}u(a)v(a) + \frac{\alpha_1}{\alpha_2}u(a)v(a)\right] = 0\,.\end{aligned}$$

Thus

$$\int_a^b (vLu - uLv)\,\mathrm{d}x = 0\,. \tag{D.8}$$

This is called the *self-conjugate relation*. It can be shown that Eq. (D.8) is also valid when one or all of α_2 and β_2 are vanished. A boundary-value problem is called the *self-conjugate boundary-value problem* if it satisfies the self-conjugate relation.

When the boundary conditions (D.5) and (D.6) are replaced by periodic boundary conditions

$$y(a) = y(b), \quad y'(a) = y'(b), \tag{D.9}$$

the corresponding eigenvalue problem is called the *periodic S-L problem*. This is also a self-conjugate boundary-value problem.

D.3 Properties of S-L Problems

Property 1. The eigenvalues $\lambda_k (k = 1, 2, \cdots)$ of S-L problems are countable, real, progressively increasing and tending towards infinity

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k \to \infty} \lambda_k = \infty.$$

Proof. Here we prove that $\lambda_k (k = 1, 2, \cdots)$ are real. Suppose that the eigenvalue λ is complex such that

$$\lambda = \alpha + \mathrm{i}\beta, \quad \beta \neq 0.$$

Denote the corresponding eigenfunctions by

$$y(x) = u(x) + \mathrm{i}v(x),$$

where $u(x)$ and $v(x)$ are not all zero. By the definition, $y(x)$ must satisfy

$$Ly = \lambda \rho(x) y$$

i.e.

$$[p(x)y]' - q(x)y + \lambda \rho(x) y = 0. \tag{D.10}$$

Thus by taking the conjugate

$$\left(p(x)\bar{y}'\right)' - q(x)\bar{y} + \bar{\lambda}\rho(x)\bar{y} = 0$$

or

$$L\bar{y} = \bar{\lambda}\rho(x)\bar{y}. \tag{D.11}$$

Therefore $\bar{\lambda}$ is also an eigenvalue. Its corresponding eigenfunction is $\bar{y}$. Multiplying Eqs. (D.10) and (D.11) by $\bar{y}$ and y respectively and then subtracting each other leads to

$$\int_a^b (\bar{y}Ly - yL\bar{y})\,\mathrm{d}x = \left(\lambda - \bar{\lambda}\right) \int_a^b \rho(x) y(x) \bar{y}(x)\,\mathrm{d}x$$

$$= \left(\lambda - \bar{\lambda}\right) \int_a^b \rho(x) \left[u^2(x) + v^2(x)\right] \mathrm{d}x = 0,$$

in which the self-conjugate relation (D.8) has been used. Since $\rho(x) > 0$,

$$\int_a^b \rho(x)\left[u^2(x)+v^2(x)\right]\mathrm{d}x > 0$$

so that $\lambda = \bar{\lambda}$. Thus λ must be real-valued.

Property 2. Let $y_k(x)$ and $y_i(x)$ be the eigenfunctions corresponding to λ_k and λ_i respectively. If $\lambda_k \neq \lambda_i$, then

$$\int_a^b \rho(x)y_k(x)y_i(x)\,\mathrm{d}x = 0\,. \tag{D.12}$$

Proof. By the definition, we have

$$Ly_k(x) = \lambda_k\rho(x)y_k(x)\,, \quad Ly_i(x) = \lambda_i\rho(x)y_i(x)\,.$$

By the self-conjugate relation (D.8)

$$(\lambda_k-\lambda_i)\int_a^b \rho(x)y_k(x)y_i(x)\,\mathrm{d}x = \int_a^b [y_i(x)Ly_k(x)-y_k(x)Ly_i(x)]\,\mathrm{d}x = 0\,.$$

Since

$$\lambda_k \neq \lambda_i\,, \quad \text{we arrive at} \quad \int_a^b \rho(x)y_k(x)y_i(x)\,\mathrm{d}x = 0\,.$$

Note that properties 1 and 2 are valid for both regular and periodic S-L problems.

Property 3. Every eigenvalue of a regular S-L problem has a unique corresponding eigenfunction up to a constant factor.

Proof. Suppose that an eigenvalue has two linearly independent corresponding eigenfunctions $y_1(x)$ and $y_2(x)$. The Wronski determinant is thus

$$W(y_1,y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x)-y_2(x)y_1'(x)\,.$$

By the theory of differential equations, $W(y_1,y_2) \neq 0$ at every point in $[a,b]$. At the end point $x = a$, however, we have, by the boundary condition (D.5) (assuming $\alpha_2 \neq 0$ without loss of the generality),

$$W(y_1,y_2)|_{x=a} = -\frac{\alpha_1}{\alpha_2}y_1(a)y_2(a)+\frac{\alpha_1}{\alpha_2}y_1(a)y_2(a) = 0\,.$$

Therefore $y_1(x)$ and $y_2(x)$ must be linearly dependent.

Note that this property is not valid for periodic S-L problems. In the eigenvalue problem (2.40) in Section 2.5, for example, one eigenvalue has two linearly independent corresponding eigenfunctions.

Property 4. The eigenvalues of regular S-L problems are positive semi-definite if the coefficients in Eqs. (D.5) and (D.6) satisfy $-\dfrac{\alpha_1}{\alpha_2} \geq 0$ and $\dfrac{\beta_1}{\beta_2} \geq 0$.

Proof. Let $y(x)$ be the eigenfunction corresponding to eigenvalue λ, i.e.,

$$\left(py'\right)' - qy + \lambda\rho(x)y = 0\,.$$

Multiplying it by y and integrating over $[a,b]$ leads to

$$\lambda\int_a^b \rho(x)y^2\,\mathrm{d}x = -\int_a^b y\,\mathrm{d}(py') + \int_a^b qy^2\,\mathrm{d}x = -ypy'\big|_a^b + \int_a^b \left(py'^2 + qy^2\right)\mathrm{d}x$$

$$= -\frac{\alpha_1}{\alpha_2}p(a)y^2(a) + \frac{\beta_1}{\beta_2}p(b)y^2(b) + \int_a^b \left(py'^2 + qy^2\right)\mathrm{d}x \geq 0\,,$$

in which Eqs. (D.3)-(D.6) have been used. Note that

$$\int_a^b \rho(x)y^2\,\mathrm{d}x > 0\,.$$

Thus $\lambda \geq 0$. When $\alpha_2 = 0$ or $\beta_2 = 0$, $y(a) = 0$ or $y(b) = 0$. We also have $\lambda \geq 0$. By following a similar approach, we can also show that the eigenvalues of periodic S-L problems are also positive semi-definite.

Property 5 (Стклов Expansion Theorem). If $f(x)$ has continuous first derivative and piece-wise continuous second derivative in $[a,b]$ and satisfies the boundary conditions of S-L problems, it can be expanded into an absolutely and uniformly convergent function series by using the eigenfunction set $\{y_n(x)\}$, i.e.

$$f(x) = \sum_{k=1}^{\infty} c_k y_k(x)\,, \quad c_k = \int_a^b f(x)y_k(x)\rho(x)\,\mathrm{d}x \Big/ \int_a^b y_k^2(x)\rho(x)\,\mathrm{d}x\ .$$

This is called the *generalized Fourier series*. The c_k are called the *generalized Fourier coefficients.*

Therefore the $\{y_n(x)\}$ form a complete and orthogonal base in $[a,b]$.

Remark. If $f(x)$ only satisfies the Dirichlet condition, the generalized Fourier series converges to $f(x)$ at a continuous point x and it converges to $[f(x_0-0) + f(x_0+0)]/2$ at a discontinuous point x_0.

D.4 Singular S-L Problems

Conditions (D.3) and (D.4) are often not satisfied by the coefficients of second-order linear equations which arise in applications. For example, the Legendre equation in Chapter 2

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \lambda y = 0\,,\ p(x) = 1-x^2\,, \tag{D.13}$$

$$q(x) = 0\,, \quad \rho(x) = 1$$

cannot satisfy $p(x) > 0$ in $[-1,1]$. The associated Legendre equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] - \frac{m^2}{1-x^2}y + \lambda y = 0\,, \tag{D.14}$$

$$p(x) = 1-x^2\,, \quad q(x) = \frac{m^2}{1-x^2}\,, \quad \rho(x) = 1$$

also cannot satisfy $p(x) > 0$ or the continuous requirement for $q(x)$ in $[-1,1]$. The Bessel equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}R}{\mathrm{d}r}\right] - \frac{\gamma^2}{r}R + \lambda rR = 0\,, \quad p(r) = r\,, \tag{D.15}$$

$$q(r) = \gamma^2/r\,, \quad \rho(r) = r$$

does not satisfy conditions (D.3) and (D.4) either because, in $[0,a_0]$, $p(0) = 0$, $\lim\limits_{r\to 0} q(r) = \infty$ and $\rho(0) = 0$.

Equation (D.2) is called the *singular S-L equation* if one of following three conditions is valid:

1. the domain is semi-infinite or infinite,
2. $p(x)$ or $q(x)$ is vanished at one or two ends of the finite domain $[a,b]$,
3. the pole of $q(x)$ or $\rho(x)$ appears at the end point of finite domain $[a,b]$; and
   a. $p(x) \in C^1[a,b]$, $q(x), \rho(x) \in C(a,b)$,
   b. in (a,b), $p(x) > 0$, $q(x) \geq 0$ and $\rho(x) > 0$.

Equation (D.13) satisfies condition 2. Equation (D.14) satisfies both conditions 2 and 3. Equation (D.15) satisfies conditions 2 and 3 when $\gamma \neq 0$. Therefore, they are all singular S-L equations.

A singular S-L equation with a boundary condition that satisfies the self-conjugate relation is called a *singular S-L problem*.

Boundary conditions that satisfies the self-conjugate relation are singularity-dependent. Here we briefly discuss three commonly-used singular S-L equations.

In Eq. (D.13), $p(x) = 0$ at the end points $x = \pm 1$. The Lagrange equality (D.7) shows that the self-conjugate relation (D.8) is valid in $[-1,1]$ if u, v and their derivatives at $x = \pm 1$ are bounded. Therefore Eq. (D.12) with bounded $y(\pm 1)$ and $y'(\pm 1)$ forms a singular S-L problem.

In Eq.(D.14), $p(x) = 0$ at the end points $x = \pm 1$. $x = \pm 1$ are also the poles of $q(x)$. The Lagrange equality (D.7) reads, in generalized integrals,

$$\lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \int_{a+\varepsilon_1}^{b-\varepsilon_2} (vLu - uLv)\,\mathrm{d}x = \lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \left\{-p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\right\}\Big|_{-1+\varepsilon_1}^{1-\varepsilon_2}\,.$$

Once the limits of u, v and their derivatives at $x = \pm 1$ are finite or bounded we have

$$\lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \int_{a+\varepsilon_1}^{b-\varepsilon_2} (vLu - uLv)\,\mathrm{d}x = 0 \quad \text{or} \quad \int_a^b (vLu - uLv)\,\mathrm{d}x = 0\,,$$

so that the self-conjugate relation (D.8) is valid. Therefore, Eq. (D.14) with bounded $y(\pm 1)$ and $y'(\pm 1)$ forms a singular S-L problem.

When $\gamma = 0$, Eq. (D.15) is similar to Eq. (D.12). When $\gamma \neq 0$, it is similar to Eq. (D.14). Here the singular point is the left end of the interval $[0, a_0]$. Thus the Lagrange equality reads

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{a_0} (vLu - uLv)\,\mathrm{d}r = \lim_{\varepsilon \to 0} p(\varepsilon)\left[u'(\varepsilon)v(\varepsilon) - u(\varepsilon)v'(\varepsilon)\right]\,,$$

where u and v are assumed to satisfy boundary condition (D.6) at $r = a_0$ so that

$$\beta_1 u(a_0) + \beta_2 u'(a_0) = 0\,, \quad \beta_1 v(a_0) + \beta_2 v'(a_0) = 0\,.$$

Note that

$$\lim_{\varepsilon \to 0} p(\varepsilon) = 0\,.$$

Therefore if u, v and their derivatives are bounded at $r = 0$, then the self-conjugate relation holds, i.e.

$$\int_0^{a_0} (vLu - uLv)\,\mathrm{d}r = 0\,.$$

Thus, under the boundary condition

$$|R(0)| < \infty\,, \quad \left|R'(0)\right| < \infty\,, \quad \beta_1 R(a_0) + \beta_2 R'(a_0) = 0\,,$$

Eq.(D.15) forms a singular S-L problem.

The self-conjugate relation can be used to show properties of eigenvalues of singular S-L problems. Examples of such properties are: (1) all eigenvalues are real, and (2) eigenfunction sets are orthogonal with respect to the weight function $\rho(x)$.

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