MATH 529 – Mathematical Methods for Physical Sciences II

Christoph Kirsch

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Contents

0 Overview of the lecture

This is the second half of a one year course on mathematical methods for physical sciences. The course web site is `http://www.unc.edu/~ckirsch/MATH529`, which contains the course syllabus, course material such as these lecture notes, as well as the homework assignments. Please check this web site regularly for updates.

The two topics covered in the lecture are

- partial differential equations
- complex analysis

The course is based on the textbook

> Erwin Kreyszig: *Advanced Engineering Mathematics.* 9th Edition; Wiley, 2006

We will basically go through chapters 12–18 of this textbook.

It is presumed that students are familiar with multivariable calculus. Fourier series, integrals and transforms will be reviewed as needed. This lecture will connect naturally to the first half, MATH 528. Students who have not attended that lecture may notice a larger gap in the first few weeks.

In addition to the weekly homework assignments, there will be two exams, a mid-term exam on March 3 (75 minutes, during regular class time), and the final exam on May 3, 8:00 AM (as set by the University Registrar). Classes of March 1 and April 26 will be review sessions in preparation of the respective exam. For the mid-term exam, you are allowed to use a summary of the lecture notes on three pages, and six pages for the final exam. Homework assignments and exams are graded and each counts for 1/3 of the total course grade.

Please ask questions as they occur, and take advantage of the office hours, too!

0.1 Partial differential equations

A *differential equation* is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. If the unknown function is of one variable, say $u(x)$, we have an *ordinary differential equation (ODE)* of the form

$$F(x, u, u', u'', \dots) = 0. \tag{1}$$

Solution techniques for these equations (series, Laplace transform, ...) have been discussed in MATH 528.

For an unknown function of several variables, $u(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, we have a *partial differential equation (PDE)* of the form

$$F(\boldsymbol{x}, u, \nabla u, \underline{\boldsymbol{Hu}}, \dots) = 0, \tag{2}$$

(with a possibly different function F than in (1), of course) with the partial derivatives

$$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{pmatrix}, \quad \underline{\boldsymbol{Hu}} = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_d \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_d^2} \end{pmatrix}, \tag{3}$$

and so on for higher order derivatives. Often, one of the variables is time, t.

PDEs often come up in the *mathematical modeling* of physical phenomena in continuous media, such as the propagation of sound, heat conduction, electrostatics, electrodynamics, fluid flow, and elasticity. In this context, we usually deal with *(initial-)boundary value problems,* where a PDE is to be solved in a region $\Omega \subset \mathbb{R}^d$ (typically $d \in \{1, 2, 3\}$), together with boundary conditions on $\partial\Omega$ and possibly initial conditions at time $t = 0$.

In this course, we will discuss *analytical techniques* for solving boundary value problems which involve three important *linear,* second order PDEs, namely the

- wave equation, $u_{tt} = \Delta u$,
- heat equation, $u_t = \Delta u$,
- Laplace equation, $\Delta u = 0$,

with the time derivatives

$$u_t := \frac{\partial u}{\partial t}, \quad u_{tt} := \frac{\partial^2 u}{\partial t^2}, \tag{4}$$

and with the Laplacian

$$\Delta u := \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2} \tag{5}$$

in d space dimensions. *Numerical methods* for PDEs, such as the finite element, finite volume, and finite difference methods, are covered in other courses, like MATH 566.

We will present techniques such as the *method of separation of variables,* which leads to a representation of the solution as a Fourier series.

Example: The solution of the one-dimensional initial-boundary value problem

$$\begin{aligned} u_t &= c^2 u_{xx}, \quad x \in (0, L),\ t > 0, && (6)\\ u(0,t) = u(L,t) &= 0, \quad t > 0, && (7)\\ u(x,0) &= f(x), \quad x \in [0, L], && (8) \end{aligned}$$

is given by the Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}, \quad \lambda_n := \frac{cn\pi}{L}, \tag{9}$$

where the coefficients B_n are Fourier sine coefficients of the initial data, f:

$$B_n = \frac{2}{L} \int_0^L f(y) \sin\left(\frac{n\pi y}{L}\right) \, dy. \tag{10}$$

Other techniques presented in this course are the *method of characteristics,* and *integral transform methods,* in particular Fourier and Laplace transforms. If the boundary value problem is given in a circular (2D) or spherical (3D) domain, a transformation to polar or spherical coordinates is useful. Solutions then involve special functions, such as Bessel and Legendre functions.

0.2 Complex analysis

We will introduce complex numbers $z \in \mathbb{C}$. Every non-constant polynomial in one variable with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$ (fundamental theorem of algebra), thus $\mathbb{C}$ is the algebraic closure of the field of real numbers, $\mathbb{R}$. Complex numbers may be represented geometrically in the complex plane, $z = (x, y)$, $x, y \in \mathbb{R}$. With the *imaginary unit,* $i \in \mathbb{C}$, $i^2 = -1$, we may write

$$z = x + iy = r\,(\cos\varphi + i\sin\varphi) = re^{i\varphi}, \tag{11}$$

with

$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z), \quad r = |z|, \quad \varphi = \arg(z). \tag{12}$$

We will discuss the elementary operations in the field of complex numbers (addition, subtraction, multiplication, division, complex conjugation, ...).

For *complex functions,* $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$, we will discuss what it means to be differentiable, leading to the notion of *holomorphic functions.* A function

$$z \mapsto f(z) := u(x, y) + iv(x, y), \quad x = \mathrm{Re}(z),\ y = \mathrm{Im}(z), \tag{13}$$

is holomorphic if and only if the partial derivatives of u and v satisfy the *Cauchy-Riemann equations,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{14}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{15}$$

We will look at line integrals of holomorphic functions, and study their behavior near *singularities.*

In *potential theory,* we study *harmonic functions* (solutions to Laplace's equation with continuous second-order derivatives). In two space dimensions, potential theory coincides with the investigation of holomorphic functions, because of the Cauchy-Riemann equations. Thus we establish a link between complex analysis and partial differential equations.

1 Review of Fourier analysis

Fourier analysis will be an important tool in our analysis of partial differential equations. Therefore it is reviewed here, although it has already been covered in MATH 528.

1.1 Fourier transform: definition

The *Fourier transform* of a function $f : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$\hat{f}(\boldsymbol{k}) \equiv \mathcal{F}[f](\boldsymbol{k}) := \frac{1}{(2\pi)^{d/2}} \int\limits_{\mathbb{R}^d} f(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \, d\boldsymbol{x}, \quad \boldsymbol{k} \in \mathbb{R}^d, \tag{16}$$

if the integral exists (!). The Fourier transform is a function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$, and the inverse Fourier transform is defined by

$$f(\boldsymbol{x}) \equiv \mathcal{F}^{-1}[\hat{f}](\boldsymbol{x}) := \frac{1}{(2\pi)^{d/2}} \int\limits_{\mathbb{R}^d} \hat{f}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, d\boldsymbol{k}, \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{17}$$

Here, the complex number $i \in \mathbb{C}$ denotes the imaginary unit, $i^2 = -1$.
<u>Remark:</u> The letter $\boldsymbol{\xi}$ is also used instead of $\boldsymbol{k}$ to denote the "Fourier space" variable.

1.2 Fourier transform: properties

Important properties of the Fourier transform include the *linearity,*

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g], \quad a, b \in \mathbb{C}, \tag{18}$$

and the *convolution theorem*

$$\mathcal{F}[f * g] = (2\pi)^{d/2} \mathcal{F}[f]\mathcal{F}[g], \tag{19}$$

where the *d-dimensional convolution* of the functions $f, g : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$(f * g)(\boldsymbol{x}) := \int\limits_{\mathbb{R}^d} f(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y}) \, d\boldsymbol{y} = \int\limits_{\mathbb{R}^d} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y}) \, d\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{20}$$

Writing $\hat{f} \equiv \mathcal{F}[f]$, $\hat{g} \equiv \mathcal{F}[g]$, and using the inverse Fourier transform (17) on both sides of (19), we obtain a different representation of the convolution (20):

$$(f * g)(\boldsymbol{x}) = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{k})\hat{g}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d\boldsymbol{k}, \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{21}$$

The formula (21) will help us in solving partial differential equations.

We continue in one space dimension ($d = 1$), where the Fourier transform $\hat{f}$ may be interpreted as the *spectral density* of a function f.
Remark: In one space dimension, ω is commonly used instead of k to denote the "Fourier space" variable.

If the n-th order derivative of the function f,

$$f^{(n)} \equiv \frac{d^n f}{dx^n}, \tag{22}$$

exists and has a Fourier transform (this requires f to decay sufficiently fast as $|x| \to \infty$), then we have the relation

$$\mathcal{F}[f^{(n)}](\omega) = (i\omega)^n \mathcal{F}[f](\omega). \tag{23}$$

The property (23) may be used to transform an ordinary differential equation into an algebraic equation (similar to the Laplace transform, cf. MATH 528), and will also useful to reduce the number of variables in a partial differential equation.

1.3 Periodic functions: Fourier series

If $f : \mathbb{R} \to \mathbb{C}$ is periodic, i. e. if

$$\exists\, L > 0 : \quad f(x + L) = f(x) \quad \forall\, x \in \mathbb{R}, \tag{24}$$

then the spectrum $\hat{f}$ is discrete. This can be seen by comparing the inverse Fourier transform of $\hat{f}$ with its shifted version,

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\, d\omega, \tag{25}$$

$$f(x + L) = \mathcal{F}^{-1}[\hat{f}](x + L) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega(x+L)}\, d\omega. \tag{26}$$

For the two expressions on the right-hand side of (25), (26) to be equal for all $x \in \mathbb{R}$, we require that

$$\hat{f}(\omega) = 0 \quad \forall\, \omega : e^{i\omega L} \neq 1 \quad \Leftrightarrow \quad \hat{f}(\omega) = 0 \quad \forall\, \omega : \frac{\omega L}{2\pi} \notin \mathbb{Z}. \tag{27}$$

With (27), the inverse Fourier transform reduces to a series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{L}x}, \quad c_n = \frac{1}{L}\int_0^L f(x) e^{-i\frac{2\pi n}{L}x}\, dx,\ n \in \mathbb{Z}. \tag{28}$$

Using Euler's formula

$$e^{it} = \cos t + i \sin t, \tag{29}$$

we may also write the Fourier series using sine and cosine functions:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{L}x\right) + b_n \sin\left(\frac{2\pi n}{L}x\right)\right), \tag{30}$$

with

$$a_0 = \frac{1}{L}\int_0^L f(x)\, dx, \quad a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{2\pi n}{L}x\right)\, dx,\ n \geq 1, \tag{31}$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{2\pi n}{L}x\right)\, dx, \quad n \geq 1. \tag{32}$$

Remark: Problem 1 in Problem Set 1.

1.4 Application: solution of ODEs

Consider the forced damped harmonic oscillator (mass-spring-dashpot system, RLC circuit)

$$\ddot{y}(t) + 2\kappa\dot{y}(t) + \Omega^2 y(t) = f(t), \quad \Omega, \kappa > 0, \tag{33}$$

for the unknown time-variant function $y(t)$. Taking the Fourier transform $\mathcal{F}$ (16) on both sides of (33), and using properties (18) and (23), we obtain the algebraic equation

$$\left(-\omega^2 + 2\kappa i\omega + \Omega^2\right) \hat{y}(\omega) = \hat{f}(\omega). \tag{34}$$

We define the function $\hat{g} : \mathbb{R} \to \mathbb{C}$,

$$\hat{g}(\omega) := \frac{1}{\sqrt{2\pi}} \frac{1}{-\omega^2 + 2\kappa i\omega + \Omega^2} = -\frac{1}{\sqrt{2\pi}} \frac{1}{(\omega - \omega_1)(\omega - \omega_2)}, \tag{35}$$

which has poles at

$$\omega_{1,2} = i\kappa \pm \sqrt{\Omega^2 - \kappa^2}, \tag{36}$$

and solve (34) for $\hat{y}(\omega)$:

$$\hat{y}(\omega) = \sqrt{2\pi} \hat{f}(\omega)\hat{g}(\omega). \tag{37}$$

Now we apply the inverse Fourier transform and use the convolution property to obtain

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}[\hat{y}](t) \overset{(17)}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{y}(\omega) e^{i\omega t}\, d\omega \overset{(37)}{=} \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega) e^{i\omega t}\, d\omega &(38)\\ &\overset{(21)}{=} (f * g)(t) \overset{(20)}{=} \int_{-\infty}^{\infty} f(\tau) g(t - \tau)\, d\tau. &(39) \end{aligned}$$

It remains to determine the function $g = \mathcal{F}^{-1}[\hat{g}]$ (which is also called a *Green's function* for this problem):

$$g(t) = \mathcal{F}^{-1}[\hat{g}](t) \overset{(17),(35)}{=} -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - \omega_1)(\omega - \omega_2)}\, d\omega \tag{40}$$

The integral in (40) may be evaluated as a contour integral in the complex plane, using *Cauchy's Residue Theorem.* We will learn about this in the Complex Analysis part of this course. The result is

$$g(t) = \begin{cases} \sin\left(\sqrt{\Omega^2 - \kappa^2} t\right) \frac{e^{-\kappa t}}{\sqrt{\Omega^2 - \kappa^2}}, & \kappa < \Omega \quad \text{(underdamped)} \\ t e^{-\kappa t}, & \kappa = \Omega \quad \text{(critically damped)} \\ \sinh\left(\sqrt{\kappa^2 - \Omega^2} t\right) \frac{e^{-\kappa t}}{\sqrt{\kappa^2 - \Omega^2}}, & \kappa > \Omega \quad \text{(overdamped)} \end{cases}. \tag{41}$$

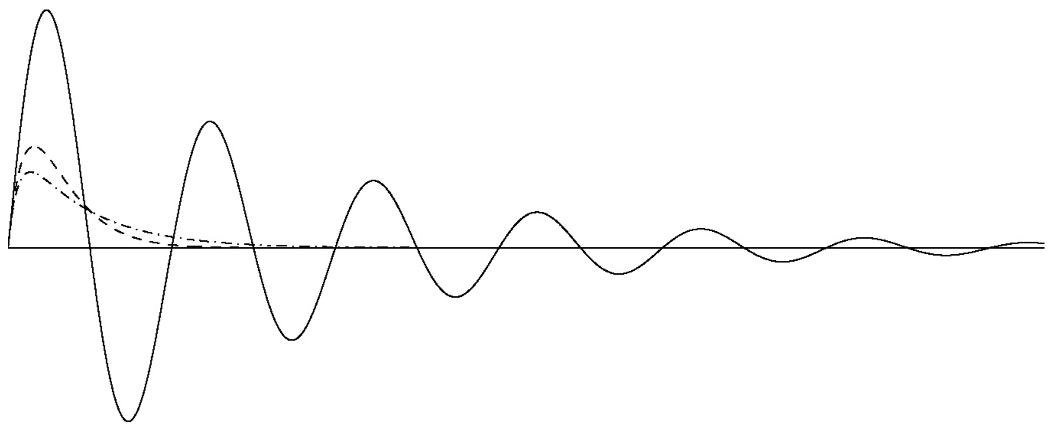


Figure 1: Qualitative behavior of a damped harmonic oscillator (33), for $\kappa < \Omega$ (solid line) $\kappa = \Omega$ (dashed line), and $\kappa > \Omega$ (dash-dot line)

12 Partial Differential Equations (PDEs)

As mentioned in the overview, PDEs often come up in the mathematical modeling of physical phenomena in continuous media, such as the propagation of sound, heat conduction, electrostatics, electrodynamics, fluid flow, and elasticity. There are several mathematical textbooks on the subject; I would recommend

Lawrence C. Evans: *Partial Differential Equations.* AMS, 1998

to students who like to learn more about the theory of PDEs.

12.1 Basic Concepts

In the Kreyszig textbook, this section is in words only. We shall follow a more formal approach here to describe the general concept. The special PDEs treated later in the lecture will not take advantage of the notation introduced here, but it is useful to be acquainted with it.

We consider a function $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^d$ is an open subset. As mentioned in the overview, one of the variables is often time, t. For the partial derivatives of u, we use the *multiindex notation:* $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multiindex of order

$$|\alpha| = \alpha_1 + \cdots + \alpha_d. \tag{42}$$

Given a multiindex $\alpha \in \mathbb{N}_0^d$, we define the partial derivative

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \equiv \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} u. \tag{43}$$

Example: $d = 3$, $\alpha = (2, 0, 1)$, $|\alpha| = 3$. We obtain the partial derivative

$$D^{(2,0,1)} u = \frac{\partial^3 u}{\partial x_1^2 \partial x_3}. \tag{44}$$

If the partial derivative exists, $D^\alpha u : \Omega \to \mathbb{R}$ is again a function. Partial derivatives of order $k \in \mathbb{N}_0$ (if they exist) are collected in a k-th order tensor,

$$D^k u := \{ D^\alpha u \mid \alpha \in \mathbb{N}_0^d, |\alpha| = k \}, \quad k \in \mathbb{N}_0. \tag{45}$$

The tensor $D^k u$ may be represented by a k-dimensional array.
Note: For $d > 1$, there should be no confusion between $D^k u$, $k \in \mathbb{N}_0$, and $D^\alpha u$, $\alpha \in \mathbb{N}_0^d$, because α is a multiindex and k is a scalar.
Examples: $D^2 u$ is a second-order tensor, which may be represented by a $d \times d$ matrix:

$$D^2 u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_d} \\ & \ddots & \\ \frac{\partial^2 u}{\partial x_d x_1} & \cdots & \frac{\partial^2 u}{\partial x_d^2} \end{pmatrix}. \tag{46}$$

This is the Hessian matrix, $\underline{\boldsymbol{Hu}}$, of u. Similarly, $D^1 u \equiv Du$ may be represented by the gradient vector, ∇u. Finally, $D^0 u \equiv u$.

Definition 1 *(partial differential equation) For $k \in \mathbb{N}$ ($k \neq 0$), an expression of the form*

$$F\big((D^k u)(x), (D^{k-1} u)(x), \ldots, (Du)(x), u(x), x\big) = 0, \quad x \in \Omega, \tag{47}$$

is called a k-th order partial differential equation (PDE), *where*

$$F : \mathbb{R}^{d^k} \times \mathbb{R}^{d^{k-1}} \times \cdots \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R} \tag{48}$$

is given, and where

$$u : \Omega \to \mathbb{R} \tag{49}$$

is unknown.

Remarks:

1. $x \in \Omega$ is called the *independent variable* in the equation, whereas $u : \Omega \to \mathbb{R}$ is called the *dependent variable* (because it depends on $x \in \Omega$).

2. A typical mathematical model is given by a PDE (47), together with auxiliary conditions on some part of $\Gamma \subset \partial\Omega$, such as *boundary conditions* or *initial conditions* (if time is involved). The problem is then called an *(initial-)boundary value problem ((I)BVP).*

3. A (classical) solution u of a k-th order PDE must, of course, have all partial derivatives up to order k. There is the notion of *weak solutions* too, but we will not go into that in this lecture.

Definition 2 *(classification of PDEs)*

1. *The PDE (47) is called* linear *if it has the form*

$$\sum_{|\alpha| \le k} a_\alpha(x) \left(D^\alpha u\right)(x) = f(x), \tag{50}$$

for given functions f and a_α, $|\alpha| \le k$. A linear PDE is called homogeneous *if $f \equiv 0$.*

2. *The PDE (47) is called* quasilinear *if it has the form*

$$\sum_{|\alpha| = k} a_\alpha(x) \left(D^\alpha u\right)(x) + a_0\left((D^{k-1}u)(x), \ldots, (Du)(x), u(x), x\right) = 0. \tag{51}$$

Remarks:

1. A linear PDE depends linearly on u and on all partial derivatives, i. e. the function F in (47) is linear in $D^k u, D^{k-1}u, \ldots, Du, u$, but is not necessarily linear in x. A linear PDE is homogeneous if each of its terms contains either u or one of its partial derivatives. Otherwise, (47) is called *nonhomogeneous.*

2. In a quasilinear PDE, the coefficient of the highest order derivative does not depend on the unknown function. Some authors call a PDE of the form (51) *semilinear,* and they use a weaker condition for quasilinear PDEs.

Examples of PDEs:

- linear, first order equation:
  - linear transport equation: $u_t + \boldsymbol{b} \cdot \nabla u = 0$
- linear, second order equations:
  - wave equation: $u_{tt} = \Delta u$
  - heat (diffusion) equation: $u_t = \Delta u$
  - Laplace's equation: $\Delta u = 0$
  - general wave equation: $u_{tt} - \operatorname{div}(\underline{\boldsymbol{A}} \nabla u) + \boldsymbol{b} \cdot \nabla u = 0$
- linear, higher order equations:
  - Airy's equation: $u_t + u_{xxx} = 0$
  - beam equation: $u_t + u_{xxxx} = 0$
- nonlinear equations:
  - minimal surface equation: $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0$
  - scalar conservation law: $u_t + \operatorname{div}(\boldsymbol{F}(u)) = 0$
  - scalar reaction-diffusion equation: $u_t - \Delta u = f(u)$
- linear systems (the unknown $\boldsymbol{u}$ is a vector-valued function):
  - evolution equations of linear elasticity:

    $$\boldsymbol{u}_{tt} - \mu \boldsymbol{\Delta u} - (\lambda + \mu) \nabla (\operatorname{div} \boldsymbol{u}) = \boldsymbol{0},$$

    where the *vector Laplacian* is defined by $\boldsymbol{\Delta} := \nabla \operatorname{div} - \operatorname{curl}\operatorname{curl}$
  - Maxwell's equations:

    $$\begin{aligned} \boldsymbol{E}_{tt} &= \operatorname{curl} \boldsymbol{B}, \\ \boldsymbol{B}_{tt} &= -\operatorname{curl} \boldsymbol{E}, \\ \operatorname{div} \boldsymbol{B} &= 0, \\ \operatorname{div} \boldsymbol{E} &= 0. \end{aligned}$$

- nonlinear systems:
  - reaction-diffusion system:
    $$\boldsymbol{u}_t - \boldsymbol{\Delta u} = \boldsymbol{f}(\boldsymbol{u}),$$
  - Navier-Stokes equations for incompressible, viscous flow:
    $$\begin{aligned} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} - \boldsymbol{\Delta u} &= -\nabla p, \\ \mathrm{div}\boldsymbol{u} &= 0. \end{aligned}$$

and many more!

A boundary value problem may not have a solution at all, or it may have more than one solutions. For a *well-posed* problem, we require both *existence* and *uniqueness* of the solution, and additionally that the solution *depends continuously on the data* given in the problem (such as boundary conditions).

Theorem 1 *(Superposition) If u_1, u_2 are solutions of a* homogeneous linear *PDE in some region Ω, then*

$$u = c_1 u_1 + c_2 u_2 \tag{52}$$

with any constants c_1, c_2 is also a solution of that PDE in Ω.

Proof: Use the linearity of both the PDE and of the partial derivatives (Problem Set 2).

12.2 1D Wave Equation: Vibrating String

We want to derive a mathematical model of small transverse vibrations of an elastic string, such as a violin string. The string is assumed to be fixed at the points $x = 0, L$. The deflection of the string at point $x \in [0, L]$ and time $t > 0$ is given by $u(x,t)$ [m]. For a fixed $t > 0$, the function $u(\cdot, t)$ describes the shape of the string at time t, whereas for a fixed $x \in [0, L]$, the function $u(x, \cdot)$ describes the vertical motion of this point on the string over time. We make the following simplifying physical assumptions (idealization!):

1. The mass per unit length of the string, ρ [kgm^{-1}], is constant (homogeneous material). The string is perfectly elastic and does not offer any resistance to bending.

2. The tension T [N] caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string can be neglected.

3. The string performs small transverse motions in a vertical plane.

To derive the model, we consider the forces acting on a small portion of the string (a line element $[x, x+\Delta x]$ of length $0 < \Delta x \ll 1$). Because of the model assumptions, these will be tensile forces, i. e. tangential to the curve of the string.

After restriction to the vertical plane where the transverse motion takes plane, we denote the tensile forces at the two endpoints of the portion by $\boldsymbol{F}_1, \boldsymbol{F}_2 \in \mathbb{R}^2$. The resultant $\boldsymbol{F}_1 + \boldsymbol{F}_2$ is acting on the line element. Because of *Newton's second law of motion* and due to the assumption that there is no horizontal motion, we have

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 = \begin{pmatrix} 0 \\ \rho \Delta x \frac{\partial^2 u}{\partial t^2}(x,t) \end{pmatrix}. \tag{53}$$

We write

$$\boldsymbol{F}_1 = \begin{pmatrix} -T_1 \cos\alpha \\ -T_1 \sin\alpha \end{pmatrix}, \quad \boldsymbol{F}_2 = \begin{pmatrix} T_2 \cos\beta \\ T_2 \sin\beta \end{pmatrix}, \tag{54}$$

and conclude that

$$T_2 \cos\beta - T_1 \cos\alpha = 0, \tag{55}$$

$$T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}(x,t). \tag{56}$$

Thus the magnitude of the horizontal component of each force is constant and must be equal to T: $T_1 \cos\alpha = T_2 \cos\beta = T$. After division of (56) by T we may write

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho}{T} \Delta x \frac{\partial^2 u}{\partial t^2}(x,t). \tag{57}$$

Now the tangent of the angles is given by the slope of the curve $u(\cdot, t)$, and can be written using spatial derivatives of u:

$$\tan\alpha = \frac{\partial u}{\partial x}(x,t), \quad \tan\beta = \frac{\partial u}{\partial x}(x+\Delta x, t). \tag{58}$$

Equation (57), after division by Δx, becomes

$$\frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}(x+\Delta x,t)-\frac{\partial u}{\partial x}(x,t)\right)=\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}(x,t). \tag{59}$$

As $\Delta x \to 0$, the left-hand side of (59) converges to the second spatial derivative of u evaluated at x, and because our considerations were independent of the horizontal position of the line element, we finally obtain the linear, second-order PDE

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \quad \text{in } (0,L), \quad c:=\sqrt{\frac{T}{\rho}} \quad [\mathrm{ms}^{-1}]. \tag{60}$$

Equation (60) is called the *one-dimensional wave equation.*

12.3 1D Wave Equation: Separation of Variables, Use of Fourier Series

In the previous section, we have derived the one-dimensional wave equation (60), which governs small transverse vibrations of an elastic string of length $L>0$. This PDE is given in $\Omega := (0,L)\times(0,\infty)\subset\mathbb{R}^2$, and according to our Definitions 1 and 2, it is a second-order, linear and homogeneous partial differential equation. In particular, Theorem 1 applies to the solutions of (60).

We complete the one-dimensional wave equation (60) by boundary and initial conditions, so that we obtain the following (well-posed) initial-boundary value problem:

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}, \quad x\in(0,L),\ t>0, \quad c:=\sqrt{\frac{T}{\rho}}, \tag{61}$$

$$u(0,t)=u(L,t)=0, \quad t>0, \tag{62}$$

$$u(x,0)=f(x),\ \frac{\partial u}{\partial t}(x,0)=g(x), \quad x\in[0,L]. \tag{63}$$

For compatibility of initial and boundary conditions, we require that $f(0)=f(L)=0$. The initial conditions (63) specify the *initial deflection,* $f(x)$, and the *initial velocity,* $g(x)$.

The method presented here to solve the initial-boundary value problem (61)–(63) consists of three steps:

1. *separation of variables* (*product method*): write the unknown function as a product of functions with fewer variables. From the PDE, derive separate differential equations for each factor.

2. Find solutions of these differential equations which satisfy the auxiliary conditions.

3. Using the superposition principle (Thm. 1), combine these solutions in a series ($\rightsquigarrow$ Fourier series) and determine coefficients from the data.

Separation of Variables We are looking for solutions $u \not\equiv 0$ of the one-dimensional wave equation (61) of the form

$$u(x,t) = F(x)G(t), \quad x \in (0,L),\, t > 0. \tag{64}$$

If we insert (64) into (61), we obtain

$$F\ddot{G} = c^2 F'' G, \quad x \in (0,L),\, t > 0. \tag{65}$$

where $\dot{}$ denotes time derivatives and where $'$ denotes spatial derivatives. Division by c^2FG leads to

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}, \quad x \in (0,L),\, t > 0. \tag{66}$$

The left-hand side of (66) depends only on t, whereas the right-hand side of (66) depends only on x, thus the variables have been separated. We conclude that both sides must be equal to a constant, the *separation constant* $k \in \mathbb{R}$. Then we obtain the following set of separate ODEs:

$$\begin{aligned} F'' &= kF, \quad x \in (0,L), & (67)\\ \ddot{G} &= c^2 kG, \quad t > 0. & (68) \end{aligned}$$

<u>Remark:</u> A PDE which can be broken down into a set of separate equations of lower dimensionality by a method of separating variables is called a *separable PDE.*

Auxiliary Conditions and Solution of Separate ODEs We solve the ODE (67) for F first.

Proposition 1 *The general solution of the ODE (67) is given by*

$$F(x;k) = \begin{cases} Ae^{\omega x} + Be^{-\omega x}, & k = \omega^2,\ \omega > 0 \\ Ax + B, & k = 0 \\ A\cos(\omega x) + B\sin(\omega x), & k = -\omega^2,\ \omega > 0 \end{cases}, \quad A, B \in \mathbb{R}. \tag{69}$$

From the boundary conditions (62) we infer with (64) that

$$F(0)G(t) = F(L)G(t) = 0, \quad t > 0. \tag{70}$$

Because $G \not\equiv 0$ (otherwise $u \equiv FG \equiv 0$), we obtain the following boundary conditions for F in (67):

$$F(0) = F(L) = 0. \tag{71}$$

Using Proposition 1, we verify that for $k \geq 0$, $F \equiv 0$ is the only solution of (67) which satisfies the boundary conditions (71). The case $k < 0$, or $k = -\omega^2$, $\omega > 0$, is more interesting: With Proposition 1, we obtain from the boundary conditions (71):

$$A = 0, \quad B\sin(\omega L) = 0. \tag{72}$$

$F \not\equiv 0$ requires

$$B \neq 0 \quad \overset{(72)}{\Rightarrow} \quad \sin(\omega L) = 0 \quad \overset{\omega, L > 0}{\Leftrightarrow} \quad \omega L = n\pi,\ n \in \mathbb{N}. \tag{73}$$

Setting $B = 1$ in Proposition 1 (move any constant factor over to the function G) leads to infinitely many solutions

$$F_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n \in \mathbb{N}, \tag{74}$$

of (67). The value of the separation constant is thus given by

$$k = -\omega^2 = -\frac{n^2\pi^2}{L^2}, \quad n \in \mathbb{N}. \tag{75}$$

We use (75) in the ODE (68) for G, which then becomes

$$\ddot{G} = -\lambda_n^2 G, \quad t > 0, \qquad \lambda_n := c\omega = \frac{cn\pi}{L} > 0, \quad n \in \mathbb{N}. \tag{76}$$

We use Proposition 1 again, and conclude that the general solutions for G_n are given by

$$G_n(t) = B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t), \quad n \in \mathbb{N}, \quad B_n, B_n^* \in \mathbb{R}. \tag{77}$$

Using the addition theorem for the cosine function, we may also write

$$G_n(t) = A_n \cos\left(\lambda_n t + \varphi_n\right), \quad B_n = A_n \cos\varphi_n,\ B_n^* = -A_n \sin\varphi_n, \tag{78}$$

for $n \in \mathbb{N}$. A_n and φ_n then have the interpretation of *amplitude* and *phase,* respectively. Now, solutions of the one-dimensional wave equation (61) satisfying the boundary conditions (62) are given by

$$u_n(x,t) = F_n(x)G_n(t) = (B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)) \sin\left(\frac{n\pi}{L}x\right), \tag{79}$$

for $n \in \mathbb{N}$. The functions u_n are called the *eigenfunctions* of the vibrating string, and the values λ_n are called the *eigenvalues,* $n \in \mathbb{N}$. The set $\{\lambda_1, \lambda_2, \dots\}$ is called the *spectrum.*

<u>Remark:</u> Notice that the functions u_n in (79) satisfy

$$\frac{\partial^2 u_n}{\partial t^2} = -\lambda_n^2 u_n, \quad \text{and} \quad \frac{\partial^2 u_n}{\partial x^2} = -\frac{n^2\pi^2}{L^2} u_n, \quad n \in \mathbb{N}. \tag{80}$$

Compare this with the eigenvalue problem $\underline{\boldsymbol{A}}\boldsymbol{v} = \lambda \boldsymbol{v}$, from linear algebra, where solutions $(\boldsymbol{v}, \lambda)$ consist of *eigenvectors* and *eigenvalues.* Instead of a matrix, we now have more general linear operators, ∂_t^2 and ∂_x^2. Spectra of operators are investigated in *functional analysis.*

Each u_n represents a harmonic motion with frequency $\lambda_n/(2\pi) = cn/(2L)$ [Hz]. This motion is called the *n-th normal mode* (or *n-th harmonic*) of the string. For $n = 1$, this is the *fundamental mode* ($\rightsquigarrow$ pitch), whereas higher modes ($n > 1$) are perceived as *overtones.* The n-th normal mode has $n-1$ *nodes,* that is, points on the string that do not move (in addition to the fixed endpoints):

$$x = \frac{k}{n}L, \quad k = 1, 2, \dots, n-1, \quad \Rightarrow \quad \sin\left(\frac{n\pi}{L}x\right) = 0. \tag{81}$$

The amplitudes of the normal modes determine the *timbre* of the sound.

Tuning of the string means changing its fundamental frequency,

$$\frac{c}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\rho}} \quad \text{[Hz]}. \tag{82}$$

For given values of ρ and L, this is achieved by varying the tension, T, in a certain range.

Superposition and Determination of Coefficients With Theorem 1, we conclude that the superposition,

$$u(x,t) := \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)\right) \sin\left(\frac{n\pi}{L}x\right), \quad (83)$$

is also a solution of (61) (again, we moved all scaling constants to the coefficients B_n, B_n^*). It is easily verified that the function u in (83) satisfies the boundary conditions (62). The remaining coefficients $B_n, B_n^* \in \mathbb{R}$, $n \in \mathbb{N}$, now need to be computed from the initial data in (63). We evaluate u in (83) as $t \to 0$ to obtain an expression for the initial displacement:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \stackrel{!}{=} f(x), \quad x \in [0,L]. \quad (84)$$

By differentiation of (83) with respect to t and evaluating as $t \to 0$, we also obtain an expression for the initial velocity:

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n^* \lambda_n \sin\left(\frac{n\pi}{L}x\right) \stackrel{!}{=} g(x), \quad x \in [0,L]. \quad (85)$$

We conclude that B_n, B_n^* must be the coefficients of the Fourier sine series of f and g, respectively:

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)\, dx, \quad (86)$$

$$B_n^* \lambda_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right)\, dx. \quad (87)$$

Summary The formal solution of the boundary value problem (61)–(63) may be written as a series,

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)\right) \sin\left(\frac{n\pi}{L}x\right), \quad x \in [0,L],\ t \geq 0, \quad (88)$$

where, for $n \in \mathbb{N}$, λ_n, B_n, and B_n^* are given by

$$\lambda_n = \frac{cn\pi}{L}, \tag{89}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx, \tag{90}$$

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx. \tag{91}$$

The function u is a classical solution of the boundary value problem (61)–(63), if the series in (88) converges, and if that is also true for all partial derivatives up to order 2 (differentiated termwise). This requires the data f, g to be sufficiently smooth.
Example: We consider a triangular initial deflection and zero initial velocity:

$$f(x) := 1 - \frac{2}{L}\left|x - \frac{L}{2}\right|, \quad g(x) := 0, \quad x \in [0, L]. \tag{92}$$

Then we obtain the coefficients

$$B_n = \frac{8}{\pi^2 n^2} \sin\left(n\frac{\pi}{2}\right), \quad B_n^* = 0, \quad n \in \mathbb{N}. \tag{93}$$

We compute

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0, & n \text{ even} \\ (-1)^k, & n = 2k+1, k \in \mathbb{N}_0 \quad (\Rightarrow n \text{ odd}) \end{cases}. \tag{94}$$

So we obtain the solution

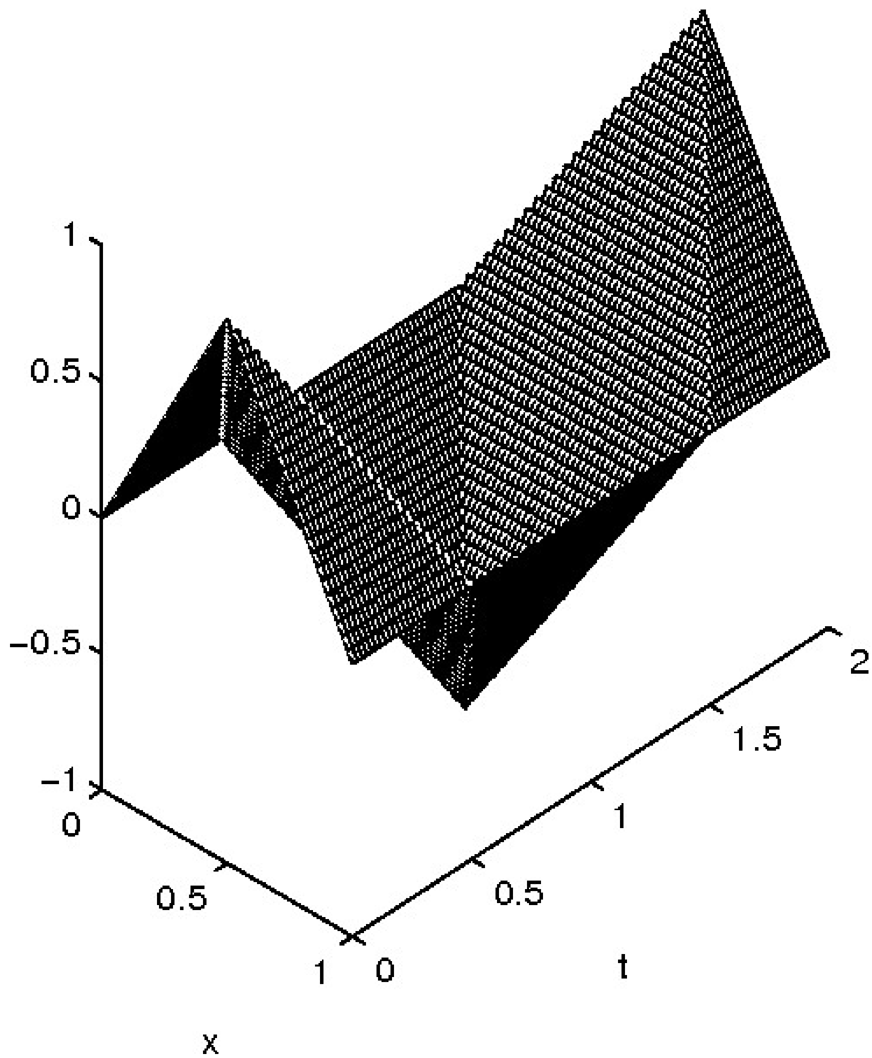
$$u(x,t) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos\left(\frac{(2k+1)\pi c}{L} t\right) \sin\left(\frac{(2k+1)\pi}{L} x\right), \tag{95}$$

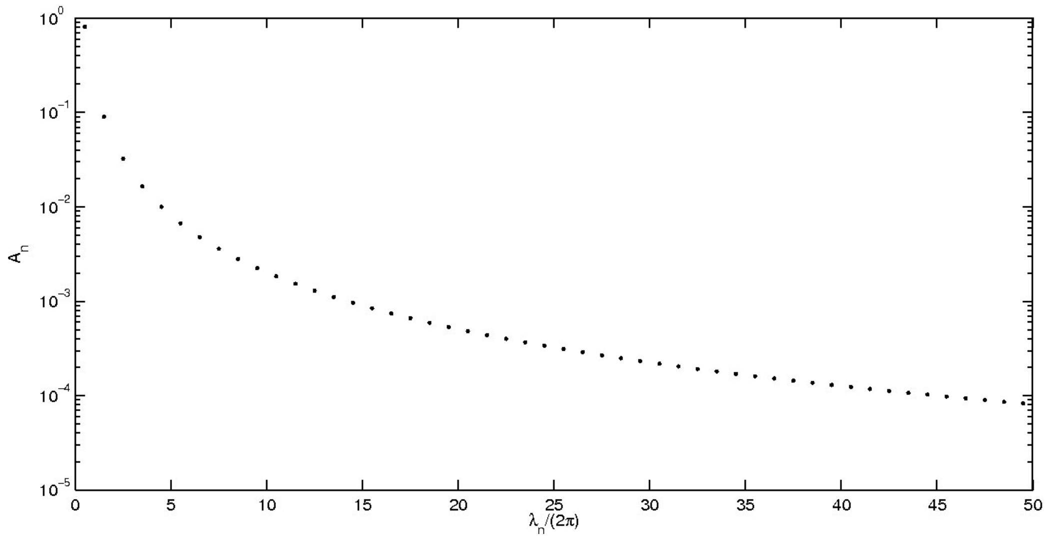
for $x \in [0, L]$, $t \geq 0$. The amplitudes in the alternative representation (78) are given by

$$A_n = \begin{cases} 0, & n \text{ even} \\ \frac{8}{\pi^2 n^2}, & n \text{ odd} \end{cases}. \tag{96}$$

Therefore, only odd harmonics are present in this wave, and their amplitude decays like n^{-2} as $n \to \infty$. There is a theorem from Fourier analysis which says that the smoother the function, the faster the decay of the Fourier coefficients with the index. Therefore, the decay is slower, for example $O(n^{-1})$, $n \to \infty$, for the sawtooth and square waves (Problem Set 1), which are not even continuous functions.

For $c = L = 1$, we plot the formal solution u (series (95) truncated at $k = 200$) and the amplitude spectrum up to $\lambda_n = 100\pi$:





Fourier series ansatz From what we have seen previously in this section, we may try to write the solution of (61)–(63) as a Fourier series in space with unknown time-variable coefficients. In other words, we make the ansatz

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} \left(a_n(t)\cos\left(\frac{n\pi}{L}x\right) + b_n(t)\sin\left(\frac{n\pi}{L}x\right)\right). \tag{97}$$

From the boundary conditions (62) we conclude that $a_n \equiv 0$, $n \geq 0$, so that the series becomes

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi}{L}x\right). \tag{98}$$

With formal differentiation, and with the PDE (61), we obtain

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = \sum_{n=1}^{\infty} \left(\ddot{b}_n(t) + c^2\frac{n^2\pi^2}{L^2} b_n(t)\right)\sin\left(\frac{n\pi}{L}x\right) = 0, \tag{99}$$

for $x \in (0,L)$, $t > 0$. This yields an ODE for each Fourier coefficient b_n:

$$\ddot{b}_n + \lambda_n^2 b_n = 0, \quad t > 0, \quad n \geq 1. \tag{100}$$

Again with Proposition 1, we obtain the solution

$$b_n(t) = b_n(0)\cos(\lambda_n t) + \frac{\dot{b}_n(0)}{\lambda_n}\sin(\lambda_n t), \quad n \geq 1. \tag{101}$$

From the initial conditions (63) we obtain, with termwise differentiation,

$$u(x,0) = \sum_{n=1}^{\infty} b_n(0)\sin\left(\frac{n\pi}{L}x\right) \stackrel{!}{=} f(x), \quad x \in [0,L], \tag{102}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \dot{b}_n(0)\sin\left(\frac{n\pi}{L}x\right) \stackrel{!}{=} g(x), \quad x \in [0,L]. \tag{103}$$

So the initial values of b_n are given as Fourier sine coefficients of the initial data:

$$b_n(0) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx, \quad n \geq 1, \tag{104}$$

$$\dot{b}_n(0) = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\,dx, \quad n \geq 1. \tag{105}$$

By comparison with the solution (83) derived before, we note that $b_n(0) = B_n$, $\dot{b}_n(0) = B_n^*\lambda_n$, as expected.

12.4 D'Alembert's Solution of the Wave Equation. Characteristics.

Recall the series solution of the wave equation (61) with homogeneous Dirichlet boundary conditions (62) (value of u prescribed as 0) and initial data f, g:

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n\cos\left(\frac{n\pi}{L}ct\right) + B_n^*\sin\left(\frac{n\pi}{L}ct\right)\right)\sin\left(\frac{n\pi}{L}x\right), \quad x \in [0,L],\ t \geq 0, \tag{106}$$

where, for $n \in \mathbb{N}$, λ_n, B_n, and B_n^* are given by

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx, \tag{107}$$

$$B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\,dx. \tag{108}$$

We assume for simplicity that $g \equiv 0$, i. e. $B_n^* = 0$, $n \in \mathbb{N}$. With the addition theorem for the sine function, we write

$$\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}ct\right) = \frac{1}{2}\left(\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right), \quad (109)$$

for $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}(x+ct)\right) + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}(x-ct)\right) && (110) \\ &= \frac{1}{2}\left(f^*(x+ct) + f^*(x-ct)\right), && (111) \end{aligned}$$

where f^* is defined as the odd periodic extension of the initial deflection f with period $2L$, i. e.

$$f^*(x) := \begin{cases} f(x), & x \in [0, L) \\ -f(2L - x), & x \in [L, 2L) \end{cases}, \quad x \in [0, 2L), \quad (112)$$

and extended periodically. Note that because of the compatibility condition $f(0) = f(L) = 0$, the function f^* is continuous.
<u>Remark:</u> In Problem Set 3, you will verify that the Fourier series of f^* is indeed given by

$$f^*(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right), \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx, \quad n \in \mathbb{N}. \quad (113)$$

Equation (111) motivates the ansatz

$$u(x,t) = \varphi(x+ct) + \psi(x-ct) \quad (114)$$

for the solution of the wave equation (61), with two sufficiently smooth functions φ and ψ (Problem 4 of Problem Set 2). Equation (114) is *d'Alembert's solution* of the wave equation. It states that the solution is the average of two waves, $u_\ell(x,t) := \varphi(x+ct)$ traveling to the left, and $u_r(x,t) := \psi(x-ct)$ traveling to the right. From the initial conditions (63) we conclude that

$$\begin{aligned} u(x,0) &= \varphi(x) + \psi(x) \stackrel{!}{=} f(x), && (115) \\ u_t(x,0) &= c\varphi'(x) - c\psi'(x) \stackrel{!}{=} g(x). && (116) \end{aligned}$$

Solutions are found to be

$$\varphi(x) = \frac{1}{2}(\varphi - \psi)(x_0) + \frac{1}{2}f(x) + \frac{1}{2c}\int\limits_{x_0}^{x} g(y)\,dy, \tag{117}$$

$$\psi(x) = -\frac{1}{2}(\varphi - \psi)(x_0) + \frac{1}{2}f(x) + \frac{1}{2c}\int\limits_{x}^{x_0} g(y)\,dy. \tag{118}$$

Then we obtain the following representation of the solution u:

$$u(x,t) = \varphi(x+ct) + \psi(x-ct) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(y)\,dy. \tag{119}$$

In the special case $g \equiv 0$, we conclude from the boundary conditions (62) that f must be an odd, $2L$-periodic function (Problem Set 3), and thus we obtain again (111).

Let's have a closer look at the left and right traveling waves $u_{\ell,r}$: apparently, they satisfy the first-order (!) transport equations

$$\left(u_{\ell,r}\right)_t \mp c\left(u_{\ell,r}\right)_x = 0, \tag{120}$$

and thus they are constant along the *characteristics*

$$x \pm ct = \text{const.} \tag{121}$$

The characteristics are solutions $x(t)$ of the *characteristic equation*

$$(\dot{x} + c)(\dot{x} - c) = \dot{x}^2 - c^2 = 0. \tag{122}$$

We define the transformed coordinates $v(x,t) := x + ct$, $w(x,t) := x - ct$ and write

$$u(x,t) = U(v(x,t), w(x,t)). \tag{123}$$

By the chain rule, we obtain the following expressions for the partial derivatives of u up to second order:

$$u_x = U_v v_x + U_w w_x, \tag{124}$$

$$u_{xx} = U_{vv}v_x^2 + U_{vw}v_x w_x + U_v v_{xx} + U_{wv}v_x w_x + U_{ww}w_x^2 + U_w w_{xx}, \tag{125}$$

$$u_{xt} = U_{vv}v_x v_t + U_{vw}v_x w_t + U_v v_{xt} + U_{wv}v_t w_x + U_{ww}w_x w_t + U_w w_{xt}, \tag{126}$$

$$u_t = U_v v_t + U_w w_t, \tag{127}$$

$$u_{tx} = U_{vv}v_x v_t + U_{vw}v_t w_x + U_v v_{tx} + U_{wv}v_x w_t + U_{ww}w_x w_t + U_w w_{tx}, \tag{128}$$

$$u_{tt} = U_{vv}v_t^2 + U_{vw}v_t w_t + U_v v_{tt} + U_{wv}v_t w_t + U_{ww}w_t^2 + U_w w_{tt}. \tag{129}$$

With our definitions of v, w, $v_x \equiv w_x \equiv 1$, $v_t = c$, $w_t = -c$, these expressions simplify considerably and the one-dimensional wave equation transforms to

$$u_{tt} - c^2 u_{xx} = -2c^2 \left(U_{vw} + U_{wv}\right) = 0. \tag{130}$$

If the second derivatives of u are continuous, we have $U_{wv} \equiv U_{vw}$ and we obtain the wave equation in *normal form:*

$$U_{vw} = 0. \tag{131}$$

Now the PDE (131) may be integrated in two steps to yield

$$U_v(v,w) = h(v), \quad U(v,w) = \underbrace{\int h(v)\, dv}_{=:\varphi(v)} + \psi(w). \tag{132}$$

Finally, we obtain

$$u(x,t) = U(v(x,t), w(x,t)) = \varphi(x+ct) + \psi(x-ct), \tag{133}$$

which is d'Alembert's solution, (114).

In the previous example for the wave equation, we have guessed the characteristics in the beginning, but the *method of characteristics* is more general. We shall illustrate it here for second-order quasilinear PDEs in two variables, which are of the form

$$\begin{aligned} A(x,y)u_{xx}(x,y) + 2B(x,y)u_{xy}(x,y) + C(x,y)u_{yy}(x,y) \\ = F(x,y,u(x,y),u_x(x,y),u_y(x,y)) \end{aligned} \tag{134}$$

(compare with Def. 2), with real-valued functions A, B, C, and F. Here we assume that second derivatives of u are continuous. Using new independent variables $v(x,y)$, $w(x,y)$ and writing

$$u(x,y) = U(v(x,y), w(x,y)), \tag{135}$$

we obtain expressions for the partial derivatives of u as before. Inserting into the PDE (134), we obtain

$$\begin{aligned} &\left(Av_x^2 + 2Bv_xv_y + Cv_y^2\right) U_{vv} + \left(Aw_x^2 + 2Bw_xw_y + Cw_y^2\right) U_{ww} + \\ &+2\left(Av_xw_x + B(v_xw_y + v_yw_x) + Cv_yw_y\right) U_{vw} = \tilde{F}(v,w,U,U_v,U_w). \end{aligned} \tag{136}$$

We now write y as a function of x: $y = y(x)$. Because characteristic curves are of the form $v(x, y) = \text{const.}$ or $w(x, y) = \text{const.}$, we know that the total derivatives must vanish:

$$\begin{aligned} \frac{dv}{dx}(x, y(x)) &= v_x(x, y(x)) + v_y(x, y(x))y'(x) = 0, & (137) \\ \frac{dw}{dx}(x, y(x)) &= w_x(x, y(x)) + w_y(x, y(x))y'(x) = 0. & (138) \end{aligned}$$

This allows us to eliminate v_x and w_x in the PDE for U above:

$$\left(Ay'^2 - 2By' + C\right)\left(v_y^2 U_{vv} + 2v_y w_y U_{vw} + w_y^2 U_{ww}\right) = \tilde{F}(v, w, U, U_v, U_w). \tag{139}$$

Now if $y(x)$ is a solution of the nonlinear ODE (the characteristic equation associated to PDE (134))

$$Ay'^2 - 2By' + C = 0, \tag{140}$$

then we obtain the first-order PDE

$$\tilde{F}(v, w, U, U_v, U_w) = 0, \tag{141}$$

to be solved for U.

We analyze the characteristic equation (140) in more detail: it is obviously quadratic in y', so that we may factorize

$$\left(y' - \frac{B - \sqrt{B^2 - AC}}{A}\right)\left(y' - \frac{B + \sqrt{B^2 - AC}}{A}\right) = 0. \tag{142}$$

There are three cases to distinguish, depending on the discriminant $B^2 - AC$:

1. If $B^2 - AC > 0$, then there are two real-valued solutions $y_{1,2}(x)$ of the characteristic ODE, corresponding to two characteristics. The associated PDE (134) is called *hyperbolic.*
   Example: wave equation, $u_{xx} - c^2 u_{yy} = 0$. We have $B^2 - AC = c^2 > 0$, and the characteristic ODE is given by $(y' + c)(y' - c) = 0$, as we have seen earlier.

2. If $B^2 - AC = 0$, then there is one solution $y(x)$ of the characteristic ODE, corresponding to one characteristic. The associated PDE (134) is called *parabolic.*
   Example: heat equation, $-c^2 u_{xx} = u_y$. We have $A = -c^2 < 0$ and $B = C = 0$, so that $B^2 - AC = 0$. The characteristic ODE is given by $y' = 0$.

3. If $B^2 - AC < 0$, then there are two complex-valued solutions $y_{1,2}(x)$. The associated PDE (134) is called *elliptic.*
   Example: Laplace equation, $u_{xx}+u_{yy} = 0$. We have $B^2-AC = -1 < 0$.

Because A, B, C are functions of x, y in general, the PDE (134) may be of *mixed type,* i. e. it may have different type depending on the location in the xy plane.
Example: The Euler-Tricomi equation (used in the study of transonic flow)

$$u_{xx} - xu_{yy} = 0 \tag{143}$$

is hyperbolic in the halfspace $x > 0$, parabolic at $x = 0$ and elliptic in the half space $x < 0$.

12.5 Heat Equation: Solution by Fourier Series

Mathematical model Consider a body of homogeneous material and denote its temperature by $u(\boldsymbol{x}, t)$ [K], $\boldsymbol{x} \in \Omega \subset \mathbb{R}^3$, $t \geq 0$. Let $V \subset \Omega$ denote a bounded control volume with surface $S := \partial V$. The amount of heat transferred per unit time from the control volume to the neighboring material is given by the integral of the heat flux $\boldsymbol{q}$ [Wm^{-2}] over the boundary,

$$\frac{\partial Q}{\partial t} = -\int\limits_S \boldsymbol{q} \cdot \boldsymbol{n}\, ds = -\int\limits_V \mathrm{div}\boldsymbol{q}\, d\boldsymbol{x}, \tag{144}$$

if no work is done and if there are no heat sources or sinks inside of V. In (144), $\boldsymbol{n}$ denotes the outward unit normal vector field on S. We used the divergence theorem in the second equation of (144). Quantities on both sides of (144) have the physical unit of power [W]. Fourier's law states that the heat flux $\boldsymbol{q}$ is proportional to the negative temperature gradient,

$$\boldsymbol{q} = -k\nabla u, \tag{145}$$

with the thermal conductivity k [$\mathrm{Wm}^{-1}\mathrm{K}^{-1}$], i. e. heat flows from warmer to colder regions. Assuming a constant k, the right-hand side of (144) becomes

$$-\int\limits_V \mathrm{div}\boldsymbol{q}\, d\boldsymbol{x} = \int\limits_V k\Delta u\, d\boldsymbol{x} \tag{146}$$

For the left-hand side of (144) we write

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial u}\frac{\partial u}{\partial t} = \int\limits_V c_p \rho \frac{\partial u}{\partial t}\, d\boldsymbol{x}, \tag{147}$$

with the specific heat capacity c_p [$\mathrm{Jkg^{-1}K^{-1}}$] and with the mass density ρ [$\mathrm{kgm^{-3}}$]. Therefore we obtain

$$\int\limits_V (c_p \rho u_t - k\Delta u)\, d\boldsymbol{x} = 0, \tag{148}$$

and because the control volume $V \subset \Omega$ was chosen arbitrarily, we obtain the PDE

$$u_t - c^2 \Delta u = 0, \quad \text{in } \Omega \times (0,\infty), \quad c^2 := \frac{k}{c_p \rho} \quad [\mathrm{m^2 s^{-1}}]. \tag{149}$$

Equation (149) is called the *heat equation,* with the thermal diffusivity c^2. It is a second-order, linear, parabolic PDE. The modeling approach used here is very common. You might have seen it already in a lecture on mathematical modeling, such as MATH 564.

We use separation of variables to obtain solutions in the one-dimensional case. For that purpose, we consider the initial-boundary value problem

$$u_t = c^2 u_{xx}, \quad \text{in } (0,L) \times (0,\infty), \tag{150}$$

$$u(0,t) = u(L,t) = 0, \quad t \geq 0, \tag{151}$$

$$u(x,0) = f(x), \quad x \in [0,L]. \tag{152}$$

A physical interpretation of the solution is the temperature distribution in a thin bar of length $L > 0$, which is insulated in the lateral direction, kept at a constant temperature at the endpoints, and has a prescribed initial temperature distribution.

Separation of Variables Again we write the solution as a product of two functions, $u(x,t) = F(x)G(t)$ (64). Then we obtain from (150):

$$F\dot{G} = c^2 F'' G, \quad \text{in } (0,L) \times (0,\infty), \tag{153}$$

and after division by $c^2 FG$ we have separated the variables:

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k \in \mathbb{R}, \tag{154}$$

because the left-hand side of (154) depends only on t, whereas the right-hand side of (154) depends only on x. We obtain the separate ODEs

$$
\begin{aligned}
F'' &= kF, \quad \text{in } (0, L), && (155)\\
\dot{G} &= c^2 kG, \quad \text{in } (0, \infty). && (156)
\end{aligned}
$$

Auxiliary Conditions and Solution of Separate ODEs From the boundary conditions (151), we conclude, with the assumption that $u \not\equiv 0$, that $F(0) = F(L) = 0$. Using Proposition 1, we infer that only the case $k < 0$ is interesting. As before, we obtain the following solutions for F:

$$
F_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n \in \mathbb{N}, \tag{157}
$$

and the values of the separation constant are given by $-(n\pi/L)^2$, $n \in \mathbb{N}$. The ODEs for G then become

$$
\dot{G} + \lambda_n^2 G = 0, \quad \lambda_n = \frac{cn\pi}{L}, \quad n \in \mathbb{N}. \tag{158}
$$

The general solution of (158) is given by

$$
G_n(t) = B_n e^{-\lambda_n^2 t}, \quad n \in \mathbb{N}, \quad B_n \in \mathbb{R}. \tag{159}
$$

Then we obtain the eigenfunctions

$$
u_n(x,t) = F_n(x)G_n(t) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}, \quad n \in \mathbb{N}, \tag{160}
$$

with associated eigenvalues λ_n.
<u>Remark:</u> Notice that

$$
\frac{\partial u_n}{\partial t} = -\lambda_n^2 u_n, \quad \frac{\partial^2 u_n}{\partial x^2} = -\left(\frac{\lambda_n}{c}\right)^2 u_n, \quad n \in \mathbb{N}. \tag{161}
$$

The n-th eigenfunction has $n-1$ nodes like for the wave equation. Unlike the normal modes of a vibrating string, however, these eigenfunctions decrease exponentially in time, and the rate of decrease increases with n.

Superposition and Determination of Coefficients With the superposition principle, Theorem 1, we may write the solution $u(x,t)$ as a Fourier series,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}. \tag{162}$$

As $t \to 0$, we obtain with the initial conditions (152)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = f(x), \tag{163}$$

so that the constants B_n must be the Fourier sine coefficients of f:

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)\, dx, \quad n \in \mathbb{N}. \tag{164}$$

Summary The formal solution of the boundary value problem (150)–(152) may be written as a series,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}, \quad x \in [0,L],\ t \geq 0, \tag{165}$$

where, for $n \in \mathbb{N}$, λ_n and B_n are given by

$$\lambda_n = \frac{cn\pi}{L}, \tag{166}$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)\, dx, \quad n \in \mathbb{N}. \tag{167}$$

Example: We consider a triangular initial temperature distribution:

$$f(x) := 1 - \frac{2}{L}\left|x - \frac{L}{2}\right|. \tag{168}$$

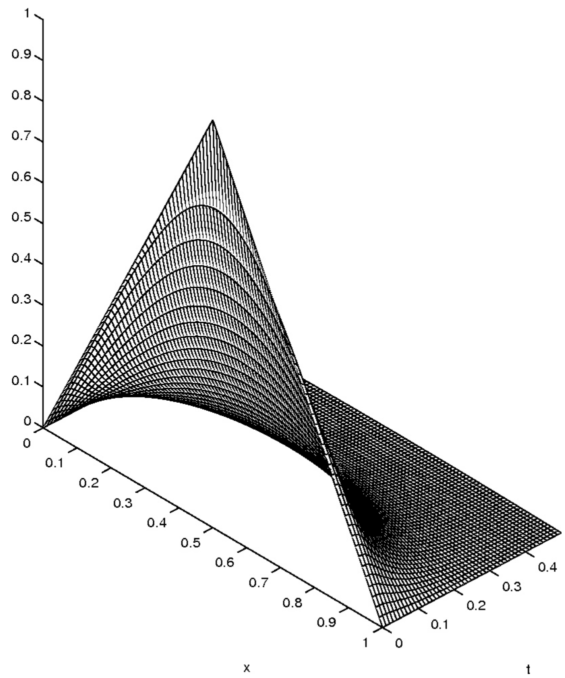
We already know from the example in Section 12.3 that the coefficients B_n are given by

$$B_n = \begin{cases} \frac{8}{\pi^2}\frac{(-1)^k}{(2k+1)^2}, & n = 2k+1,\ k \in \mathbb{N}_0 \\ 0, & n \text{ even} \end{cases}, \tag{169}$$

so that the solution of (150)–(152) with initial data f is given by

$$u(x,t) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin\left(\frac{(2k+1)\pi}{L} x\right) \exp\left(-\frac{c^2(2k+1)^2\pi^2}{L^2} t\right), \tag{170}$$

for $x \in [0, L]$, $t \geq 0$. We plot the solution for $L = c = 1$ in the xt plane in the following figure:



Steady 2D heat problems. Laplace equation. At steady state, we have $u_t \equiv 0$, so that the heat equation (149) simplifies to the Laplace equation,

$$\Delta u = 0, \quad \text{in } \Omega. \tag{171}$$

In two space dimensions, this yields again a PDE in two variables (notice how we try to avoid problems with more than two variables):

$$u_{xx} + u_{yy} = 0, \quad \text{in } \Omega \subset \mathbb{R}^2. \tag{172}$$

This second-order, linear, elliptic PDE needs to be completed with boundary conditions on $\partial\Omega$, so that we obtain a boundary value problem (BVP). We distinguish three different types of BVPs:

1. In a *Dirichlet problem,* the value of u is prescribed on $\partial\Omega$. This is called a *Dirichlet boundary condition.*

2. In a *Neumann problem,* the normal derivative of $\partial_n u$ is prescribed on $\partial\Omega$. This is called a *Neumann boundary condition.*

3. In a *mixed BVP,* the value of u is prescribed on Γ_D and the value of $\partial_n u$ is prescribed on Γ_N, where $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. This is a (special case of a) *Robin boundary condition.*

We consider a rectangle, $\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2$, $a, b > 0$, and Dirichlet boundary conditions

$$u = \begin{cases} f(x), & x \in [0,a],\ y = b \\ 0, & \text{otherwise} \end{cases}, \quad \text{on } \partial\Omega = ([0,a] \times \{0,b\}) \cup (\{0,a\} \times [0,b]) \,. \tag{173}$$

We need $f(0) = f(a) = 0$ for compatibility. Separation of variables, $u(x,y) = F(x)G(y)$, leads to

$$\frac{F''}{F} = -\frac{G''}{G} = k \in \mathbb{R}, \tag{174}$$

so that we obtain the separate ODEs

$$F'' = kF, \quad \text{in } (0,a), \tag{175}$$

$$G'' = -kG, \quad \text{in } (0,b). \tag{176}$$

We solve for F first. From the boundary conditions we conclude that $F(0) = F(a) = 0$. With Proposition 1 we obtain (once again) the solutions

$$F_n(x) = \sin\left(\frac{n\pi}{a}x\right), \quad n \in \mathbb{N}, \tag{177}$$

and the values of the separation constant are given by $-(n\pi/a)^2 < 0$. Therefore, the ODEs for the functions G_n, $n \in \mathbb{N}$, are given by

$$G_n'' = \left(\frac{n\pi}{a}\right)^2 G_n, \quad \text{in } (0,b). \tag{178}$$

With Proposition 1, we obtain the general solutions

$$G_n(y) = A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}, \quad A_n, B_n \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{179}$$

The boundary condition at $y = 0$ implies that $G_n(0) = A_n + B_n = 0$, $n \in \mathbb{N}$, so that we may eliminate the constants B_n, $n \in \mathbb{N}$, in (179). The functions G_n are then of the form

$$G_n(y) = A_n \left(e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}\right) = 2A_n \sinh\left(\frac{n\pi}{a}y\right). \tag{180}$$

Now the eigenfunctions of our Dirichlet problem are given by

$$u_n(x,y) = F_n(x)G_n(y) = A_n^* \sinh\left(\frac{n\pi}{a}y\right) \sin\left(\frac{n\pi}{a}x\right), \quad n \in \mathbb{N}. \tag{181}$$

With Theorem 1, we obtain a series representation for the solution u. Evaluated as $y \to b$ this becomes

$$u(x,b) = \sum_{n=1}^{\infty} u_n(x,b) = \sum_{n=1}^{\infty} A_n^* \sinh\left(\frac{n\pi}{a}b\right) \sin\left(\frac{n\pi}{a}x\right) \stackrel{!}{=} f(x), \tag{182}$$

because of the boundary conditions (173). We conclude that

$$A_n^* \sinh\left(\frac{n\pi}{a}b\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right)\, dx. \tag{183}$$

So we finally obtain the formal solution to the BVP (172), (173):

$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sinh\left(\frac{n\pi}{a}y\right) \sin\left(\frac{n\pi}{a}x\right), \quad x \in [0,a],\ y \in [0,b], \tag{184}$$

where the coefficients are determined from the boundary data f by

$$A_n^* = \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right)\, dx, \quad n \in \mathbb{N}. \tag{185}$$

<u>Example:</u> We consider triangular boundary data

$$f(x) := 1 - \frac{2}{a}\left|x - \frac{a}{2}\right|. \tag{186}$$

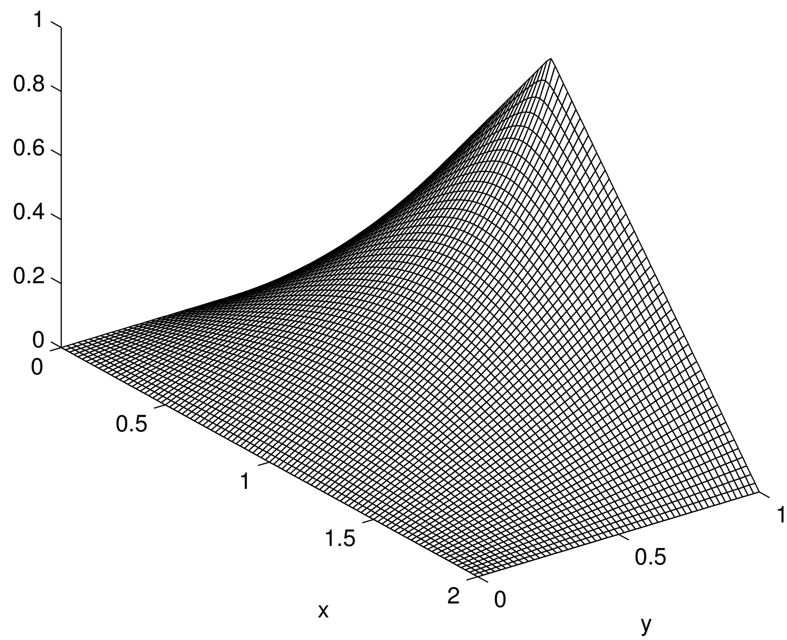
From the previous examples we know that

$$A_n^* = \begin{cases} \frac{1}{\sinh\left(\frac{n\pi}{a}b\right)} \frac{8}{\pi^2} \frac{(-1)^k}{(2k+1)^2}, & n = 2k+1,\ k \in \mathbb{N}_0 \\ 0, & n \text{ even} \end{cases}, \tag{187}$$

so that the solution is given by

$$u(x,y) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \frac{\sinh\left(\frac{(2k+1)\pi}{a} y\right)}{\sinh\left(\frac{(2k+1)\pi}{a} b\right)} \sin\left(\frac{(2k+1)\pi}{a} x\right). \tag{188}$$

In the following figure, we plot the solution for $a = 1$, $b = 2$:



12.6 1D Heat Equation: Solution by Fourier Integrals and Transforms

In the previous section, we stated the one-dimensional heat equation as a model for the heat conduction in a thin bar of length $L > 0$, which is laterally insulated. We may now consider the equation for an "infinitely long" bar. This leads to an *initial-value (or Cauchy) problem*

$$u_t = c^2 u_{xx}, \quad \text{in } \mathbb{R} \times (0,\infty), \tag{189}$$

$$u = f, \quad \text{on } \mathbb{R} \times \{0\}. \tag{190}$$

The Cauchy problem (189), (190) is a fairly good model for the heat conduction in a very long, thin bar, where the temperature values on the boundary become unimportant.

Separation of Variables We may still use separation of variables to solve the Cauchy problem (189), (190), so we write

$$u(x,t) = F(x)G(t), \quad x \in \mathbb{R},\, t > 0. \tag{191}$$

The separate ODEs are the same as before (155), (156):

$$\begin{aligned} F'' &= kF, \quad \text{in } \mathbb{R}, && (192)\\ \dot{G} &= c^2 kG, \quad \text{in } (0,\infty). && (193) \end{aligned}$$

Solution of Separate ODEs Although we do not impose any boundary conditions on u, we still require a physically meaningful solution, such as that it remains bounded:

$$\exists\, M > 0 : |u(x,t)| \le M, \quad \forall\, x \in \mathbb{R},\, t > 0. \tag{194}$$

This implies of course that both F and G must be bounded functions. With Proposition 1, we conclude that the only bounded solution of (192) for $k \ge 0$ is the zero function $F \equiv 0$, so that we may again restrict our considerations to the case of a negative separation constant, $k = -\omega^2$, $\omega > 0$. In that case, solutions of (192), (192) are given by

$$\begin{aligned} F(x;\omega) &= A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x), \quad x \in \mathbb{R}, && (195)\\ G(t;\omega) &= e^{-c^2\omega^2 t}, \quad t > 0. && (196) \end{aligned}$$

Note that any scaling factor in G may be moved over to A and B. Now the product

$$\tilde{u}(x,t;\omega) := F(x;\omega)G(t;\omega) = \left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right) e^{-c^2\omega^2 t} \tag{197}$$

satisfies

$$\begin{aligned} \tilde{u}_t &= F\dot{G} = -c^2\omega^2 FG = -c^2\omega^2 \tilde{u}, && (198)\\ c^2\tilde{u}_{xx} &= c^2 F'' G = -c^2\omega^2 FG = -c^2\omega^2 \tilde{u}, && (199) \end{aligned}$$

so that $\tilde{u}(\cdot,\cdot;\omega)$ is a solution of the heat equation (189), for any $\omega > 0$.

Superposition Unlike in the case of a bounded domain, the eigenvalues $-c^2\omega^2$, $\omega > 0$, do not form a countable set here. Therefore, we need a superposition of the functions $\tilde{u}(\cdot, \cdot; \omega)$ in the form of an integral instead of a series: the general solution of (189) is given by

$$u(x,t) = \int_0^\infty \tilde{u}(x,t;\omega)\, d\omega = \int_0^\infty \left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right) e^{-c^2\omega^2 t}\, d\omega \tag{200}$$

(any scaling factor in front of $\tilde{u}(x,t;\omega)$ may be moved over to A, B).

Determination of Coefficients The functions A and B now need to be determined from the initial data f (190). For that purpose, we evaluate (200) as $t \to 0$:

$$u(x,0) = \int_0^\infty A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\, d\omega \stackrel{!}{=} f(x), \quad x \in \mathbb{R}. \tag{201}$$

In complete analogy with Fourier series, we may also show that any function f may be written as a real *Fourier integral* of the form

$$f(x) = \int_0^\infty a(\omega)\cos(\omega x) + b(\omega)\sin(\omega x)\, d\omega, \tag{202}$$

where the coefficient functions a, b are given by

$$a(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos(\omega x)\, dx, \quad b(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin(\omega x)\, dx, \quad \omega > 0. \tag{203}$$

You will verify this in Problem Set 4. Therefore, we have determined our coefficients A, B to be

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(y)\cos(\omega y)\, dy, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(y)\sin(\omega y)\, dy, \quad \omega > 0. \tag{204}$$

We have chosen the integration variable y instead of x because we want to insert (204) into (200):

$$\begin{aligned}
u(x,t) &= \int_0^\infty \left(\frac{1}{\pi} \int_{-\infty}^\infty f(y)\cos(\omega y)\,dy\cos(\omega x) + \frac{1}{\pi}\int_{-\infty}^\infty f(y)\sin(\omega y)\,dy\sin(\omega x)\right) e^{-c^2\omega^2 t}\,d\omega \\
&= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(y)\left(\cos(\omega x)\cos(\omega y)+\sin(\omega x)\sin(\omega y)\right) e^{-c^2\omega^2 t}\,dy\,d\omega \qquad (205)\\
&= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(y)\cos\left(\omega(x-y)\right) e^{-c^2\omega^2 t}\,dy\,d\omega \qquad (206)\\
&= \int_{-\infty}^\infty f(y)\frac{1}{\pi}\int_0^\infty \cos\left(\omega(x-y)\right) e^{-c^2\omega^2 t}\,d\omega\,dy, \qquad (207)
\end{aligned}$$

where we have used an addition theorem for the cosine function as well as changed the order of integration. The inner integral of (207) may be evaluated with the formula

$$\int_0^\infty e^{-ax^2}\cos(bx)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(-\frac{b^2}{4a}\right). \qquad (208)$$

We obtain

$$\frac{1}{\pi}\int_0^\infty \cos\left(\omega(x-y)\right) e^{-c^2\omega^2 t}\,d\omega = \frac{1}{\sqrt{4\pi c^2 t}}\exp\left(-\frac{(x-y)^2}{4c^2 t}\right) = K(c^2 t, x, y), \qquad (209)$$

where the *heat kernel* K is defined by

$$K(t,x,y) := \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{(x-y)^2}{4t}\right), \quad t>0,\ x,y\in\mathbb{R}. \qquad (210)$$

So we may write the solution to the Cauchy problem (189), (190) in the form

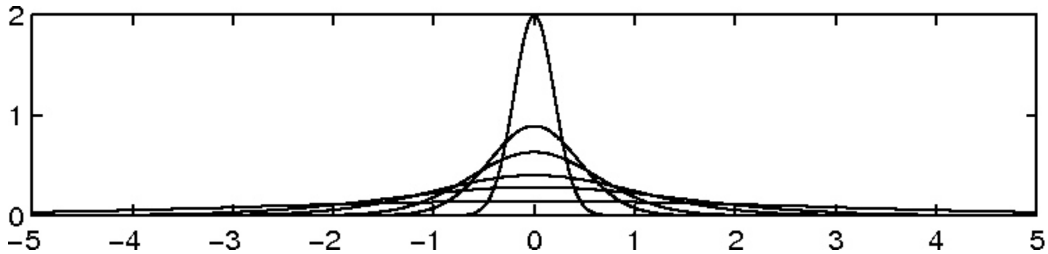
$$u(x,t) = \int_{-\infty}^\infty f(y)K(c^2 t, x, y)\,dy =: (Tf)(x,t), \qquad (211)$$

where T is an integral transform. Because K depends only on the difference $x-y$, it is actually a *convolution kernel,* and we may write (211) in the form

$$(Tf)(x,t) = \int_{-\infty}^{\infty} f(y)\varphi(x-y,c^2t)\,dy = \left(f * \varphi(\cdot,c^2t)\right)(x), \qquad (212)$$

with

$$\varphi(x,t) := \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right), \quad x \in \mathbb{R},\ t>0. \qquad (213)$$



Therefore, the solution $u(\cdot,t)$ to the Cauchy problem (189), (190) is given at any time $t>0$ by the convolution of the initial data with the *Gauss-Weierstrass kernel* $\varphi(\cdot,c^2t)$. Compare the kernel φ (213) with the probability density function of a normal distribution with mean $\mu=0$ and variance $\sigma^2=2t$. This kernel is also used in the (generalized) Weierstrass transform, $\mathcal{W}_t$: $u(x,t)=\mathcal{W}_{c^2t}[f](x)$ ($t>0$ is interpreted as a parameter in this context). The transformation $\mathcal{W}_t$ may also be inverted (this requires complex analysis), so that we can reconstruct the initial data from the solution (cf. image sharpening).
Example: We consider a triangular initial distribution

$$f(x) := \begin{cases} 1-2|x|, & x \in [-1/2,1/2] \\ 0, & \text{otherwise} \end{cases}, \quad x \in \mathbb{R}. \qquad (214)$$

The solution $u(x,t)$ of the Cauchy problem (189), (190) may be expressed using the *error function* erf, which is defined by

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\,dz, \quad x \in \mathbb{R}. \qquad (215)$$

We obtain

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(x-y)^2}{4c^2 t}\right) dy \tag{216}$$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \int_{-1/2}^{1/2} (1-2|y|) \exp\left(-\frac{(x-y)^2}{4c^2 t}\right) dy \tag{217}$$

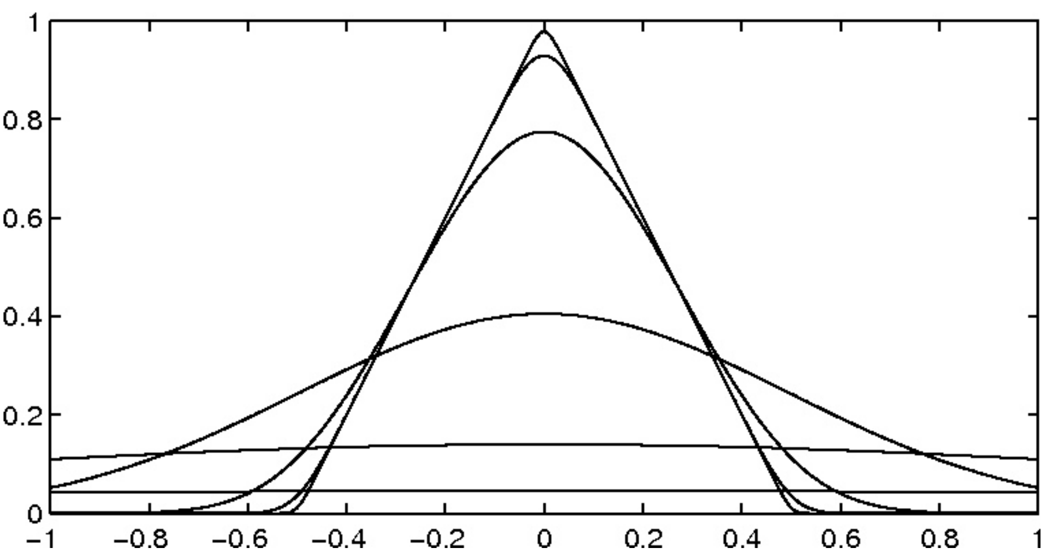
$$= (x+1/2)\left(\operatorname{erf}\left(\frac{x+1/2}{2c\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x}{2c\sqrt{t}}\right)\right) + \tag{218}$$

$$-(x-1/2)\left(\operatorname{erf}\left(\frac{x}{2c\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-1/2}{2c\sqrt{t}}\right)\right) + \tag{219}$$

$$+\sqrt{\frac{4c^2 t}{\pi}} \exp\left(-\frac{x^2}{4c^2 t}\right)\left(\exp\left(-\frac{x+1/4}{4c^2 t}\right) - 1\right) + \tag{220}$$

$$-\sqrt{\frac{4c^2 t}{\pi}} \exp\left(-\frac{x^2}{4c^2 t}\right)\left(1 - \exp\left(\frac{x-1/4}{4c^2 t}\right)\right), \tag{221}$$

for $x \in \mathbb{R}$, $t \geq 0$. Notice that although the initial data is compactly supported, $f(x) = 0$, $|x| \geq 1/2$, this is not true for the solution $u(x,t)$, $t > 0$. In fact, we have $|u(x,t)| > 0 \; \forall x \in \mathbb{R}$, for any $t > 0$!



Fourier transform We consider the spatial Fourier transform of the solution u,

$$\hat{u}(\omega,t) := \mathcal{F}[u(\cdot,t)](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx, \quad \omega \in \mathbb{R},\, t > 0. \tag{222}$$

We apply the Fourier transform $\mathcal{F}$ to both sides of the PDE (189) to obtain

$$\mathcal{F}[u_t(\cdot,t)] = c^2\mathcal{F}[u_{xx}(\cdot,t)], \quad \text{in } \mathbb{R},\, t > 0. \tag{223}$$

For the terms in (223) we obtain

$$\begin{aligned} \mathcal{F}[u_t(\cdot,t)](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,t)e^{-i\omega x}\,dx = \hat{u}_t(\omega,t), & (224)\\ \mathcal{F}[u_{xx}(\cdot,t)](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{xx}(x,t)e^{-i\omega x}\,dx = -\omega^2\hat{u}(\omega,t), & (225) \end{aligned}$$

if we assume sufficiently fast decay of $u(x,t)$ and $u_x(x,t)$ as $|x| \to \infty$. Applying the Fourier transform to the initial condition (190), we obtain

$$\mathcal{F}[u(\cdot,0)] = \mathcal{F}[f], \quad \text{in } \mathbb{R}. \tag{226}$$

We have

$$\begin{aligned} \mathcal{F}[u(\cdot,0)](\omega) &= \hat{u}(\omega,0), & (227)\\ \mathcal{F}[f](\omega) &= \hat{f}(\omega), & (228) \end{aligned}$$

where $\hat{f}$ denotes the Fourier transform of the initial data f. Now we have transformed the Cauchy problem (189), (190) into a new initial-value problem for $\hat{u}$ which involves only time derivatives:

$$\begin{aligned} \hat{u}_t &= -c^2\omega^2\hat{u}, \quad \text{in } \mathbb{R} \times (0,\infty), & (229)\\ \hat{u} &= \hat{f}, \quad \text{on } \mathbb{R} \times \{0\}. & (230) \end{aligned}$$

The general solution of (229) is given by

$$\hat{u}(\omega,t) = C(\omega)e^{-c^2\omega^2 t}, \quad \omega \in \mathbb{R},\, t > 0, \tag{231}$$

and from the initial condition (230) we obtain for the coefficient C:

$$\hat{u}(\omega, 0) = C(\omega) = \hat{f}(\omega), \quad \omega \in \mathbb{R}. \tag{232}$$

Now the solution u of the Cauchy problem (189), (190) is given by the inverse spatial Fourier transform of $\hat{u}$:

$$u(x,t) = \mathcal{F}^{-1}[\hat{u}(\cdot,t)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2\omega^2 t} e^{i\omega x}\, d\omega \tag{233}$$

We would like to use the convolution formula (21), and write

$$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega,t) e^{i\omega x}\, d\omega, \quad \hat{g}(\omega,t) := \frac{1}{\sqrt{2\pi}} e^{-c^2\omega^2 t}. \tag{234}$$

The Fourier transform of a Gaussian function is another Gaussian function, and we obtain

$$\mathcal{F}^{-1}[\hat{g}(\cdot,t)](x) = \frac{1}{\sqrt{4\pi c^2 t}} \exp\left(-\frac{x^2}{4c^2 t}\right) = \varphi(x, c^2 t), \tag{235}$$

with the Gauss-Weierstrass kernel φ (213). With the convolution formula (21), we obtain again the form (212) for the solution of the Cauchy problem (189), (190). We have used the Fourier transform method already in Section 1.4 (review part of this lecture) to solve an ODE. As we have seen there, the method also works for nonhomogeneous equations.

A related method is to assume that the solution u may be represented as an inverse Fourier transform,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega,t) e^{i\omega x}\, d\omega, \tag{236}$$

and to insert this ansatz into the problem (189), (190). This will lead to a new problem (229), (230) for $\hat{u}$, which is hopefully easier to solve.

Notice that for problems given in bounded domains, the integral in (236) reduces to a series. That leads to the *Fourier series ansatz* of the solution, which we have used several times in the previous sections. You have solved several initial-boundary value problems already with this ansatz in the Problem Sets, some of them involving nonhomogeneous PDEs.

12.7 2D Wave Equation: Vibrating Membrane

In a similar way as for the vibrating string (Section 12.2) we now want to derive a mathematical model of small transverse vibrations of an elastic membrane, such as a drumhead. The membrane at rest covers a bounded domain $\Omega \subset \mathbb{R}^2$ and is fixed on the boundary $\partial\Omega$. The deflection of the membrane at point $(x, y) \in \Omega$ and time $t > 0$ is given by $u(x, y, t)$ [m]. For a fixed $t > 0$, the function $u(\cdot, t)$ describes the shape of the membrane at time t, whereas for a fixed $(x, y) \in \Omega$, the function $u(x, y, \cdot)$ describes the vertical motion of this point on the membrane over time. We make the following simplifying physical assumptions (idealization!):

1. The mass per unit area of the membrane, ρ [kgm^{-2}] is constant (homogeneous membrane). The membrane is perfectly flexible and offers no resistance to bending.

2. The membrane is stretched and then fixed along its entire boundary $\partial\Omega$. The tension per unit length T [Nm^{-1}] caused by stretching the membrane is the same at all points and in all directions and does not change during the motion. The tension is so large that the action of the gravitational force on the membrane can be neglected.

3. The deflection $u(x, y, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

To derive the model, we consider the forces acting on a small portion of the membrane (an area element $[x, x + \Delta x] \times [y, y + \Delta y]$ of area $\Delta x \Delta y$, with $0 < \Delta x, \Delta y \ll 1$). Because of the model assumptions, these will be tensile forces, i. e. tangential to the membrane.

We consider the four forces acting on the edges of the area element, $\boldsymbol{F}_x^{1,2}$, $\boldsymbol{F}_y^{1,2}$. With the assumption of small deflections we conclude that the magnitudes of these forces are given by

$$|\boldsymbol{F}_x^{1,2}| = T\Delta y, \quad |\boldsymbol{F}_y^{1,2}| = T\Delta x. \tag{237}$$

We write $\boldsymbol{F}_y^{1,2}$ in polar coordinates in the yz plane:

$$\boldsymbol{F}_y^1 = \begin{pmatrix} 0 \\ -T\Delta x \cos\alpha \\ -T\Delta x \sin\alpha \end{pmatrix}, \quad \boldsymbol{F}_y^2 = \begin{pmatrix} 0 \\ T\Delta x \cos\beta \\ T\Delta x \sin\beta \end{pmatrix}. \tag{238}$$

The angles of inclination α, β are small by assumption, so that $\cos\alpha \simeq \cos\beta \simeq 1$. The y-component of the resulting force in y-direction, $\boldsymbol{F}_y^1 + \boldsymbol{F}_y^2$, is thus negligible and we have

$$T\Delta x \cos\alpha \simeq T\Delta x \cos\beta \simeq T\Delta x. \tag{239}$$

The z-component of the resulting force in y-direction is given by

$$\begin{aligned} T\Delta x\,(\sin\beta - \sin\alpha) &\simeq T\Delta x\,(\tan\beta - \tan\alpha) && (240)\\ &= T\Delta x\,(u_y(x_2, y+\Delta y, t) - u_y(x_1, y, t))\,, && (241) \end{aligned}$$

for any values $x_1, x_2 \in (x, x+\Delta x)$. In the same way, we write $\boldsymbol{F}_x^{1,2}$ in polar coordinates in the xz plane and obtain that the x-component of the resulting force in x-direction, $\boldsymbol{F}_x^1 + \boldsymbol{F}_x^2$, is negligible and that the z-component of the resulting force in x-direction is given by

$$T\Delta y\,(u_x(x+\Delta x, y_2, t) - u_x(x, y_1, t))\,, \tag{242}$$

for any values $y_1, y_2 \in (y, y+\Delta y)$. The only non-zero component of the total resulting force $\boldsymbol{F}_x^1 + \boldsymbol{F}_x^2 + \boldsymbol{F}_y^1 + \boldsymbol{F}_y^2$ is thus the z-component. It is given by

$$T\Delta x\Delta y\left(\frac{u_x(x+\Delta x, y_2, t) - u_x(x, y_1, t)}{\Delta x} + \frac{u_y(x_2, y+\Delta y, t) - u_y(x_1, y, t)}{\Delta y}\right). \tag{243}$$

By Newton's second law of motion, (243) must be equal to the mass of the area element times its vertical acceleration,

$$\rho\Delta x\Delta y\frac{\partial^2 u}{\partial t^2}(x, y, t). \tag{244}$$

If we let $\Delta x, \Delta y \to 0$, (243) converges to the sum of the second partial derivatives, so that we obtain

$$\frac{\partial^2 u}{\partial t^2}(x, y, t) = \frac{T}{\rho}\left(\frac{\partial^2 u}{\partial x^2}(x, y, t) + \frac{\partial^2 u}{\partial y^2}(x, y, t)\right). \tag{245}$$

The right-hand side of (245) may be written as the 2D Laplacian applied to u, and because the location of the area element within Ω was arbitrary, we finally obtain the *two-dimensional wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2\Delta u \quad \text{in } \Omega \times (0, \infty), \quad c := \sqrt{\frac{T}{\rho}} \quad [\text{ms}^{-1}]. \tag{246}$$

12.8 Rectangular Membrane. Double Fourier Series

In the previous section, we have derived the two-dimensional wave equation (246), which governs small vibrations of an elastic membrane. This PDE is given in $\Omega \times (0, \infty) \subset \mathbb{R}^3$, and according to our Definitions 1 and 2, it is a second-order, linear and homogeneous partial differential equation. In particular, Theorem 1 applies to the solutions of (246), which will allow us to write formal solutions as superpositions of eigenfunctions, just as we did for the one-dimensional PDEs in the previous sections.

We proceed as in Section 12.3: We complete the two-dimensional wave equation (246) by boundary and initial conditions, so that we obtain the following initial-boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \quad \text{in } \Omega \times (0, \infty), \quad c := \sqrt{\frac{T}{\rho}}, \tag{247}$$

$$u = 0, \quad \text{on } \partial\Omega \times (0, \infty), \tag{248}$$

$$u = f, \quad \frac{\partial u}{\partial t} = g, \quad \text{on } \Omega \times \{0\}, \tag{249}$$

with initial data $f, g : \Omega \to \mathbb{R}$. For compatibility of initial and boundary conditions, we require that $f = 0$ on $\partial\Omega$. The initial conditions (249) specify the initial deflection f and the initial velocity g.

The method presented here to solve the initial-boundary value problem (247)–(249) consists of three steps:

1. *separation of variables* (*product method*): write the unknown function as a product of functions with fewer variables. From the PDE, derive separate differential equations for each factor.

2. Find solutions of these differential equations which satisfy the boundary conditions.

3. Using the superposition principle (Thm. 1), combine these solutions in a series ($\rightsquigarrow$ Fourier series) and determine coefficients from the initial data.

Separation of Variables We are looking for solutions $u \not\equiv 0$ of the two-dimensional wave equation (247) of the form

$$u(x, y, t) = F(x, y)G(t), \quad (x, y) \in \Omega,\ t > 0. \tag{250}$$

If we insert (250) into (247), we obtain

$$F\ddot{G} = c^2 \left(F_{xx} + F_{yy}\right) G, \quad \text{in } \Omega \times (0, \infty), \tag{251}$$

and after division by $c^2 FG$ we have separated the time variable from the spatial variables:

$$\frac{\ddot{G}}{c^2 G} = \frac{F_{xx} + F_{yy}}{F} = k \in \mathbb{R}. \tag{252}$$

The separate differential equations are thus given by

$$F_{xx} + F_{yy} = kF, \quad \text{in } \Omega, \tag{253}$$
$$\ddot{G} = c^2 kG, \quad \text{in } (0, \infty). \tag{254}$$

Notice that the differential equation for F (253) is still a PDE. For a negative $k = -\lambda^2$, (253) becomes a *two-dimensional Helmholtz equation*.

We consider a rectangular membrane, $\Omega := (0, a) \times (0, b)$, with $a, b > 0$. Then we use the method of separating variables again and write $F(x, y) = H(x)Q(y)$, $(x, y) \in \Omega$. We obtain

$$H''Q + HQ'' - kHQ = 0, \tag{255}$$

which we can separate after division by HQ as

$$\frac{H''}{H} = -\frac{Q''}{Q} + k = p \in \mathbb{R}. \tag{256}$$

The separate ODEs for H and Q are then given by

$$H'' = pH, \quad \text{in } (0, a), \tag{257}$$
$$Q'' = (k - p)Q, \quad \text{in } (0, b). \tag{258}$$

We summarize our findings: after separation of variables we obtain three separate ODEs with two (!) separation constants $k, p \in \mathbb{R}$:

$$H'' = pH, \quad \text{in } (0, a), \tag{259}$$
$$Q'' = (k - p)Q, \quad \text{in } (0, b), \tag{260}$$
$$\ddot{G} = c^2 kG, \quad \text{in } (0, \infty). \tag{261}$$

After solutions for H, Q, G are found, the function $u(x, y, t) := H(x)Q(y)G(t)$ is a solution to the two-dimensional wave equation (247).

Auxiliary Conditions and Solution of Separate ODEs From the boundary condition (248) we conclude that

$$H(0) = H(a) = Q(0) = Q(b) = 0, \tag{262}$$

for a rectangular domain $\Omega = (0, a) \times (0, b)$. With Proposition 1 we find that solutions $H, Q \not\equiv 0$ exist for $p < 0$ and $k - p < 0$ only. For $p = -\omega^2$, $\omega > 0$, we have the general solution

$$H(x) = A\cos(\omega x) + B\sin(\omega x), \tag{263}$$

and with the boundary conditions, we obtain

$$A = 0, \quad B\sin(\omega a) = 0 \quad \Rightarrow \quad B = 0 \quad \forall\, \omega \neq \frac{m\pi}{a},\ m \in \mathbb{N}. \tag{264}$$

Therefore, solutions $H \not\equiv 0$ are given by

$$H_m(x) = \sin\left(\frac{m\pi}{a}x\right), \quad m \in \mathbb{N}, \tag{265}$$

where any scaling factor has been moved to the function G. For $k - p = -\nu^2$, $\nu > 0$, and with the same arguments as before, we have

$$Q_n(y) = \sin\left(\frac{n\pi}{b}y\right), \quad n \in \mathbb{N}. \tag{266}$$

Notice that the separation constant k is given by $k = p - \nu^2 = -\left(\omega^2 + \nu^2\right) < 0$. The functions $F_{mn} := H_m Q_n$, $m, n \in \mathbb{N}$,

$$F_{mn}(x, y) = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{267}$$

are solutions of two-dimensional Helmholtz equations:

$$\Delta F_{mn} + \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)F_{mn} = 0. \tag{268}$$

The ODEs for the time-dependent functions G_{mn} are given by

$$\ddot{G}_{mn} = -\lambda_{mn}^2 G_{mn}, \quad \text{in } (0, \infty), \quad m, n \in \mathbb{N}, \tag{269}$$

with the eigenvalues

$$\lambda_{mn} := c\sqrt{\omega^2 + \nu^2} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \quad m, n \in \mathbb{N}, \tag{270}$$

so that the general solutions are given by (Proposition 1)

$$G_{mn}(t) = B_{mn}\cos(\lambda_{mn}t) + B_{mn}^{*}\sin(\lambda_{mn}t), \quad B_{mn}, B_{mn}^{*} \in \mathbb{R}, \tag{271}$$

for $m, n \in \mathbb{N}$. The functions $u_{mn} := F_{mn}G_{mn}$,

$$u_{mn}(x,y,t) = (B_{mn}\cos(\lambda_{mn}t) + B_{mn}^{*}\sin(\lambda_{mn}t))\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), \tag{272}$$

$m, n \in \mathbb{N}$, are eigenfunctions of the second-order differential operators in both space and time:

$$\begin{aligned} \partial_t^2 u_{mn} &= F_{mn}\ddot{G}_{mn} = -\lambda_{mn}^2 F_{mn}G_{mn} = -\lambda_{mn}^2 u_{mn}, & (273)\\ c^2\Delta u_{mn} &= c^2\Delta F_{mn}G_{mn} = -\lambda_{mn}^2 F_{mn}G_{mn} = -\lambda_{mn}^2 u_{mn}, & (274) \end{aligned}$$

and therefore they are solutions of the two-dimensional wave equation (247) which satisfy the boundary condition (248). The frequency of u_{mn} is

$$\frac{\lambda_{mn}}{2\pi} = \frac{c}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \frac{c\sqrt{m^2b^2 + n^2a^2}}{2ab} \quad [\mathrm{Hz}], \quad m, n \in \mathbb{N}. \tag{275}$$

Compare this value with the frequency $cn/(2L)$ of the n-th normal mode of a vibrating string (Section 12.3). In the 2D case, it is possible that there are several modes with the same frequency, depending on the values of a and b!
Example: Consider a square membrane of area 1, i. e. $a = b = 1$. We obtain the modes

$$u_{mn}(x,y,t) = (B_{mn}\cos(\lambda_{mn}t) + B_{mn}^{*}\sin(\lambda_{mn}t))\sin(m\pi x)\sin(n\pi y), \tag{276}$$

with eigenvalues

$$\lambda_{mn} = c\pi\sqrt{m^2 + n^2}, \tag{277}$$

for $m, n \in \mathbb{N}$. Because $\lambda_{mn} = \lambda_{nm}$, The modes u_{mn} and u_{nm} have the same frequencies. However, these are two different functions if $m \neq n$. This can be seen from the location of the *nodal lines,* i. e. points $(x, y) \in \Omega$ with $F_{mn}(x,y) = \sin(m\pi x)\sin(n\pi y) = 0$:

$$x = \frac{k}{m},\ k = 1, \dots, m-1 \quad \text{or} \quad y = \frac{\ell}{n},\ \ell = 1, \dots, n-1. \tag{278}$$

For example, the modes u_{12} and u_{21} both have the same frequency $c\sqrt{5}/2$. Their nodal lines are located at $y = 1/2$ and $x = 1/2$, respectively. Any linear combination of u_{12} and u_{21} is again a solution of the two-dimensional wave equation (247) which satisfies the boundary condition (248), and it has the same frequency $c\sqrt{5}/2$. For example, by choosing $B_{12} = 1$, $B_{21} = p \in \mathbb{R}$, $B_{12}^* = B_{21}^* = 0$, we obtain the vibration

$$\cos(c\pi\sqrt{5}t)\,(F_{12}(x,y) + pF_{21}(x,y))\,, \quad p \in \mathbb{R}. \tag{279}$$

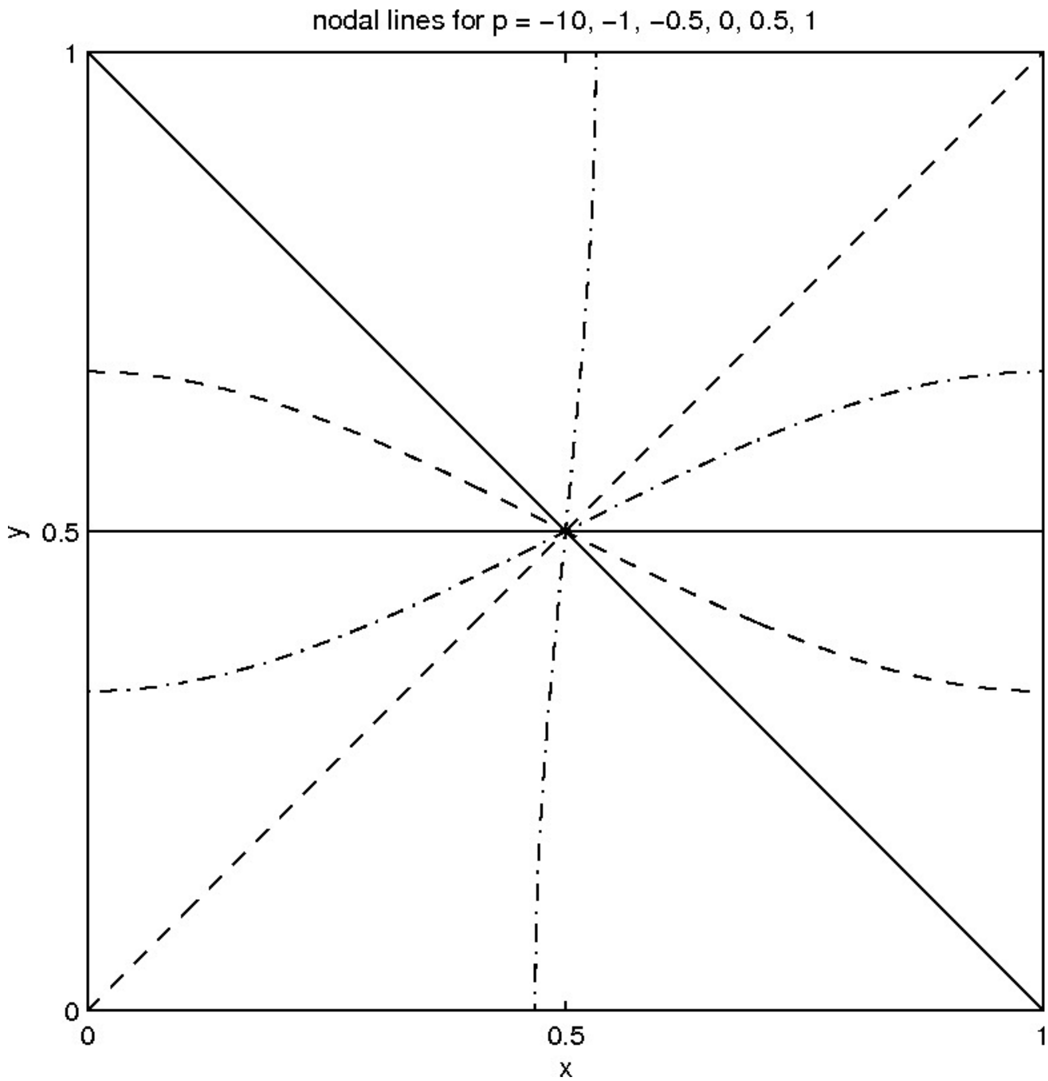
The nodal lines coincide with the zeros of $F_{12} + pF_{21}$, and so they are defined by the equation

$$\sin(\pi x)\sin(2\pi y) + p\sin(2\pi x)\sin(\pi y) = 0. \tag{280}$$

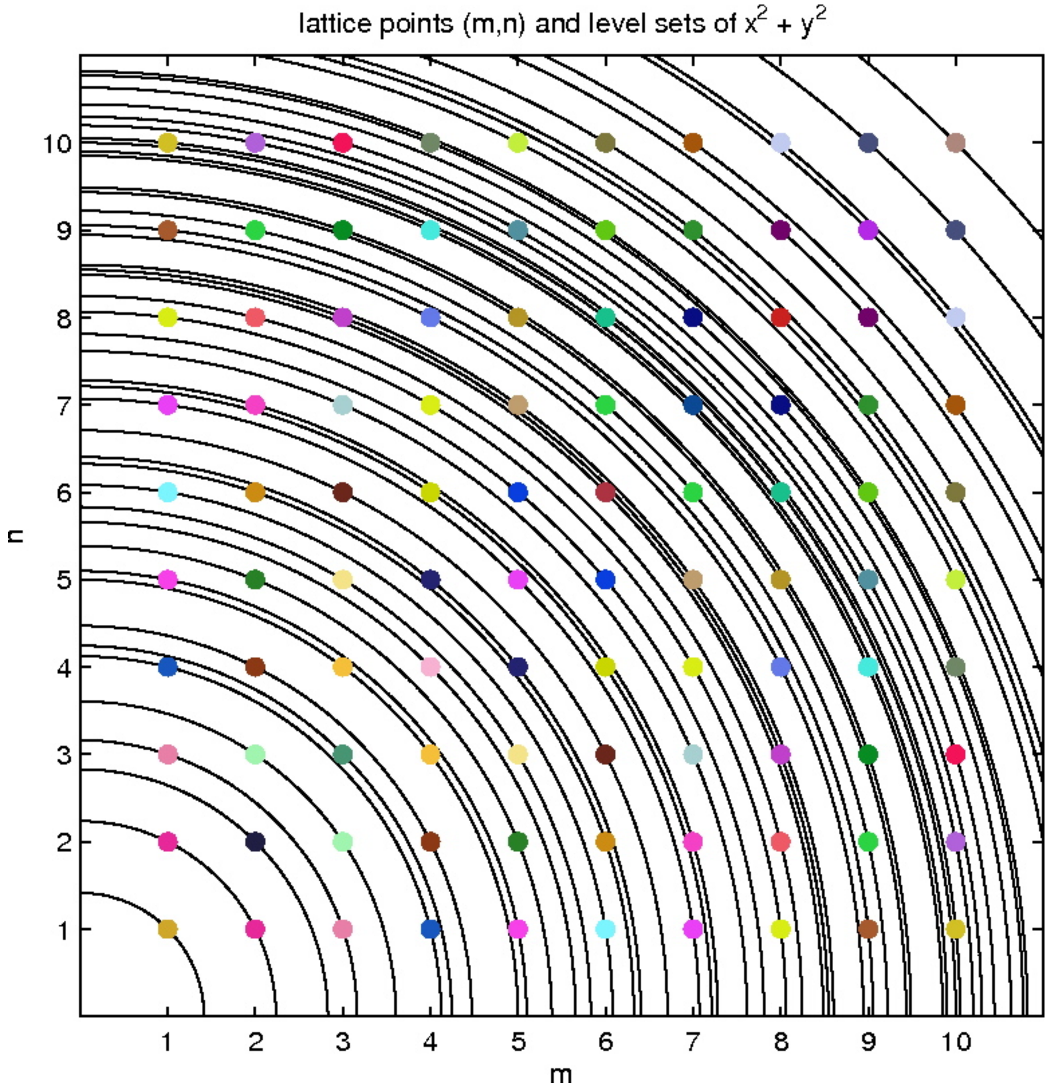
We find that solutions (x, y) of (280) satisfy

$$\cos(\pi y) + p\cos(\pi x) = 0. \tag{281}$$

For every value of $p \in \mathbb{R}$, we obtain different nodal lines.



There may also be more than two pairs (m, n) which lead to vibrations of the same frequency. It is not trivial to determine how many modes exist for a given frequency. In the following figure, we plot the lattice points $(m, n) \in \{1, \ldots, 10\}^2$ together with the level sets of the function $x^2 + y^2$. Each quarter circle corresponds to one frequency, and lattice points which lie on the same curve correspond to vibration modes which have the same frequency. Notice that there are curves with three or four lattice points on them.



In the following table, we write all possible values of $m^2 + n^2$, $m, n \in \{1, \ldots, 10\}^2$ together with the pairs (m, n) which correspond to these values.

m^2+n^2	2	5	8	10	13	17	18	20	25
(m,n)	$(1,1)$	$(1,2)$	$(2,2)$	$(1,3)$	$(2,3)$	$(1,4)$	$(3,3)$	$(2,4)$	$(3,4)$
(m,n)		$(2,1)$		$(3,1)$	$(3,2)$	$(4,1)$		$(4,2)$	$(4,3)$

m^2+n^2	26	29	32	34	37	40	41	45	50
(m,n)	$(1,5)$	$(2,5)$	$(4,4)$	$(3,5)$	$(1,6)$	$(2,6)$	$(4,5)$	$(3,6)$	$(1,7)$
(m,n)	$(5,1)$	$(5,2)$		$(5,3)$	$(6,1)$	$(6,2)$	$(5,4)$	$(6,3)$	$(5,5)$
(m,n)									$(7,1)$

m^2+n^2	52	53	58	61	65	68	72	73	74
(m,n)	$(4,6)$	$(2,7)$	$(3,7)$	$(5,6)$	$(1,8)$	$(2,8)$	$(3,8)$	$(6,6)$	$(5,7)$
(m,n)	$(6,4)$	$(7,2)$	$(7,3)$	$(6,5)$	$(4,7)$	$(8,2)$	$(8,3)$		$(7,5)$
(m,n)					$(7,4)$				
(m,n)					$(8,1)$				

m^2+n^2	80	82	85	89	90	97	98	100	101
(m,n)	$(4,8)$	$(1,9)$	$(2,9)$	$(5,8)$	$(3,9)$	$(4,9)$	$(7,7)$	$(6,8)$	$(1,10)$
(m,n)	$(8,4)$	$(9,1)$	$(6,7)$	$(8,5)$	$(9,3)$	$(9,4)$		$(8,6)$	$(10,1)$
(m,n)			$(7,6)$						
(m,n)			$(9,2)$						

m^2+n^2	104	106	109	113	116	117	125	128	130
(m,n)	$(2,10)$	$(5,9)$	$(3,10)$	$(7,8)$	$(4,10)$	$(6,9)$	$(5,10)$	$(8,8)$	$(7,9)$
(m,n)	$(10,2)$	$(9,5)$	$(10,3)$	$(8,7)$	$(10,4)$	$(9,6)$	$(10,5)$		$(9,7)$

m^2+n^2	136	145	149	162	164	181	200
(m,n)	$(6,10)$	$(8,9)$	$(7,10)$	$(9,9)$	$(8,10)$	$(9,10)$	$(10,10)$
(m,n)	$(10,6)$	$(9,8)$	$(10,7)$		$(10,8)$	$(10,9)$	

Also notice that the frequency ratios

$$\frac{\lambda_{mn}}{\lambda_{11}} = \sqrt{\frac{m^2+n^2}{2}}, \quad m,n \in \mathbb{N}, \tag{282}$$

are not integer values in general: the sound of most drums is characterized by strongly inharmonic spectra.

Superposition and Determination of Coefficients The superposition of the modes u_{mn} leads to the representation of the formal solution of (247),

(248) as a double Fourier series:

$$
\begin{aligned}
u(x, y, t) &= \sum_{m,n=1}^{\infty} u_{mn}(x, y, t) && (283)\\
&= \sum_{m,n=1}^{\infty} \left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right).
\end{aligned}
$$

Evaluated as $t \to 0$ we obtain

$$
\begin{aligned}
u(x, y, 0) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right) && (284)\\
&= \sum_{m=1}^{\infty} b_m(y)\sin\left(\frac{m\pi}{a}x\right) \overset{!}{=} f(x, y), && (285)\\
u_t(x, y, 0) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B^*_{mn}\lambda_{mn}\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right) && (286)\\
&= \sum_{m=1}^{\infty} b^*_m(y)\sin\left(\frac{m\pi}{a}x\right) \overset{!}{=} g(x, y), && (287)
\end{aligned}
$$

because of the initial conditions (249). We solve for the coefficients B_{mn} in two steps: first, we find that $b_m(y)$, $m \in \mathbb{N}$, are the Fourier sine coefficients of $f(\cdot, y)$, for any $y \in [0, b]$:

$$
b_m(y) = \sum_{n=1}^{\infty} B_{mn}\sin\left(\frac{n\pi}{b}y\right) = \frac{2}{a}\int_0^a f(x, y)\sin\left(\frac{m\pi}{a}x\right)\,dx, \tag{288}
$$

and second we observe that B_{mn}, $m, n \in \mathbb{N}$, are the Fourier sine coefficients of b_m, $m \in \mathbb{N}$:

$$
B_{mn} = \frac{2}{b}\int_0^b b_m(y)\sin\left(\frac{n\pi}{b}y\right)\,dy = \frac{2}{b}\int_0^b \frac{2}{a}\int_0^a f(x, y)\sin\left(\frac{m\pi}{a}x\right)\,dx\,\sin\left(\frac{n\pi}{b}y\right)\,dy, \tag{289}
$$

so that the coefficients B_{mn} are given by

$$
B_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x, y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\,dy\,dx, \quad m, n \in \mathbb{N}. \tag{290}
$$

In exactly the same way, we obtain for the coefficients B^*_{mn}:

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}} \int_0^a \int_0^b g(x,y) \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)\, dy\, dx, \quad m,n \in \mathbb{N}. \tag{291}$$

Summary The formal solution of the initial-boundary value problem (247)–(249) with a rectangular domain $\Omega = (0,a) \times (0,b)$, $a, b > 0$, may be written as a double Fourier series,

$$u(x,y,t) = \sum_{m,n=1}^{\infty} \left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right) \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right), \tag{292}$$

where, for $m, n \in \mathbb{N}$, λ_{mn}, B_{mn}, and B^*_{mn} are given by

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \tag{293}$$

$$B_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)\, dy\, dx, \tag{294}$$

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}} \int_0^a \int_0^b g(x,y) \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)\, dy\, dx. \tag{295}$$

<u>Example:</u> We consider a rectangular membrane with $a = 4$, $b = 2$, propagation speed $c = \sqrt{5}$, and initial data $g \equiv 0$ and

$$f(x,y) = 0.1(4x - x^2)(2y - y^2). \tag{296}$$

We have $B^*_{mn} = 0$, $m, n \in \mathbb{N}$, and for B_{mn} we obtain

$$B_{mn} = \frac{1}{20} \int_0^4 (4x - x^2) \sin\left(\frac{m\pi}{4}x\right)\, dx \int_0^2 (2y - y^2) \sin\left(\frac{n\pi}{2}y\right)\, dy. \tag{297}$$

The integrals over x and y may be evaluated using integration by parts. We obtain

$$\int_0^4 (4x - x^2) \sin\left(\frac{m\pi}{4}x\right)\, dx = \frac{8}{m\pi}\int_0^4 (2-x)\cos\left(\frac{m\pi}{4}x\right)\, dx \quad (298)$$

$$= \frac{32}{m^2\pi^2}\int_0^4 \sin\left(\frac{m\pi}{4}x\right)\, dx \quad (299)$$

$$= \frac{128}{m^3\pi^3}\left(1-(-1)^m\right) \quad (300)$$

$$= \begin{cases} \frac{256}{m^3\pi^3}, & m \text{ odd} \\ 0, & m \text{ even} \end{cases}, \quad (301)$$

and

$$\int_0^2 (2y - y^2) \sin\left(\frac{n\pi}{2}y\right)\, dy = \frac{4}{n\pi}\int_0^2 (1-y)\cos\left(\frac{n\pi}{2}y\right)\, dy \quad (302)$$

$$= \frac{8}{n^2\pi^2}\int_0^2 \sin\left(\frac{n\pi}{2}y\right)\, dy \quad (303)$$

$$= \frac{16}{n^3\pi^3}\left(1-(-1)^n\right) \quad (304)$$

$$= \begin{cases} \frac{32}{n^3\pi^3}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \quad (305)$$

Therefore we obtain

$$B_{mn} = \begin{cases} \frac{2048}{5m^3n^3\pi^6}, & m,n \text{ odd} \\ 0, & \text{otherwise} \end{cases}, \quad m,n \in \mathbb{N}, \quad (306)$$

and the solution of (247)–(249) is given by

$$u(x,y,t) = \frac{2048}{5\pi^6}\sum_{\substack{m,n=1 \\ m,n \text{ odd}}}^{\infty} \frac{1}{m^3n^3}\cos\left(\frac{\sqrt{5}\pi}{4}\sqrt{m^2+4n^2}t\right)\sin\left(\frac{m\pi}{4}x\right)\sin\left(\frac{n\pi}{2}y\right), \quad (307)$$

for $x, y \in [0,4]\times[0,2]$, $t \geq 0$.

12.9 Laplacian in Polar Coordinates. Circular Membrane. Fourier-Bessel Series.

Laplacian in Polar Coordinates We consider polar coordinates (r, φ), which are related to the cartesian coordinates (x, y) via

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad r \geq 0,\ \varphi \in [0, 2\pi). \tag{308}$$

These relations can be inverted and we obtain

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan\left(\frac{y}{x}\right), \quad (x, y) \in \mathbb{R}^2. \tag{309}$$

We write $u(x, y) = U(r(x, y), \varphi(x, y))$. With the chain rule we can express the partial derivatives of u using the partial derivatives of U as

$$\begin{aligned}
u_x &= U_r r_x + U_\varphi \varphi_x, && (310)\\
u_{xx} &= U_{rr} r_x^2 + U_{r\varphi} r_x \varphi_x + U_r r_{xx} + U_{\varphi r} \varphi_x r_x + U_{\varphi\varphi} \varphi_x^2 + U_\varphi \varphi_{xx}, && (311)\\
u_y &= U_r r_y + U_\varphi \varphi_y, && (312)\\
u_{yy} &= U_{rr} r_y^2 + U_{r\varphi} r_y \varphi_y + U_r r_{yy} + U_{\varphi r} \varphi_y r_y + U_{\varphi\varphi} \varphi_y^2 + U_\varphi \varphi_{yy}. && (313)
\end{aligned}$$

With $U_{r\varphi} \equiv U_{\varphi r}$, Δu may be written as

$$\begin{aligned}
u_{xx} + u_{yy} &= (r_x^2 + r_y^2) U_{rr} + 2(r_x \varphi_x + r_y \varphi_y) U_{r\varphi} + (\varphi_x^2 + \varphi_y^2) U_{\varphi\varphi} + \\
&\quad + (r_{xx} + r_{yy}) U_r + (\varphi_{xx} + \varphi_{yy}) U_\varphi. && (314)
\end{aligned}$$

The partial derivatives of $r(x, y)$ and $\varphi(x, y)$ are given by

$$\begin{aligned}
r_x &= \frac{x}{r}, \quad r_{xx} = \frac{y^2}{r^3}, \quad r_y = \frac{y}{r}, \quad r_{yy} = \frac{x^2}{r^3}, && (315)\\
\varphi_x &= -\frac{y}{r^2}, \quad \varphi_{xx} = \frac{2xy}{r^4}, \quad \varphi_y = \frac{x}{r^2}, \quad \varphi_{yy} = -\frac{2xy}{r^4}, && (316)
\end{aligned}$$

where we have used the quotient rule and

$$\arctan'(x) = \frac{1}{1 + x^2}, \quad x \in \mathbb{R}. \tag{317}$$

So we obtain

$$u_{xx} + u_{yy} = U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi}. \tag{318}$$

Therefore, the Laplacian in polar coordinates is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}. \tag{319}$$

For a circular membrane with radius $R > 0$,

$$\Omega := \{(r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 \mid r \in (0, R), \varphi \in [0, 2\pi)\}, \tag{320}$$

we obtain the following Dirichlet boundary value problem for $u = u(r, \varphi, t)$:

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}\right), \quad \text{in } \Omega \times (0, \infty), \tag{321}$$

$$u = 0, \quad \text{on } \partial\Omega \times (0, \infty), \tag{322}$$

$$u = f, \quad u_t = g, \quad \text{on } \Omega \times \{0\}. \tag{323}$$

Remark: Notice that in (r, φ)-coordinates, Ω is again a rectangle.

Separation of Variables We look for a solution of the form

$$u(r, \varphi, t) = F(r, \varphi)G(t), \quad \text{in } \Omega \times (0, \infty). \tag{324}$$

We already know the first separation step from Section 12.8:

$$\Delta F = F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\varphi\varphi} \;=\; -k^2 F, \quad \text{in } \Omega, \tag{325}$$

$$\ddot{G} \;=\; -c^2k^2G, \quad \text{in } (0, \infty), \tag{326}$$

with $k > 0$, where the Laplacian Δ is defined by (319).
Remark: Notice that we restrict ourselves to the case of a negative separation constant $-k^2$ here. This will turn out to be the right choice.
We separate the spatial PDE by assuming $F(r, \varphi) = H(r)Q(\varphi)$, which leads to

$$r^2\frac{H''}{H} + r\frac{H'}{H} + k^2r^2 = -\frac{Q''}{Q} = \nu^2, \tag{327}$$

for $\nu \geq 0$.
Remark: Notice the restriction to a non-negative separation constant ν^2.
The separate ODEs for H and Q are given by

$$r^2H'' + rH' + k^2r^2H \;=\; \nu^2 H, \quad \text{in } (0, R), \tag{328}$$

$$Q'' \;=\; -\nu^2 Q, \quad \text{in } [0, 2\pi). \tag{329}$$

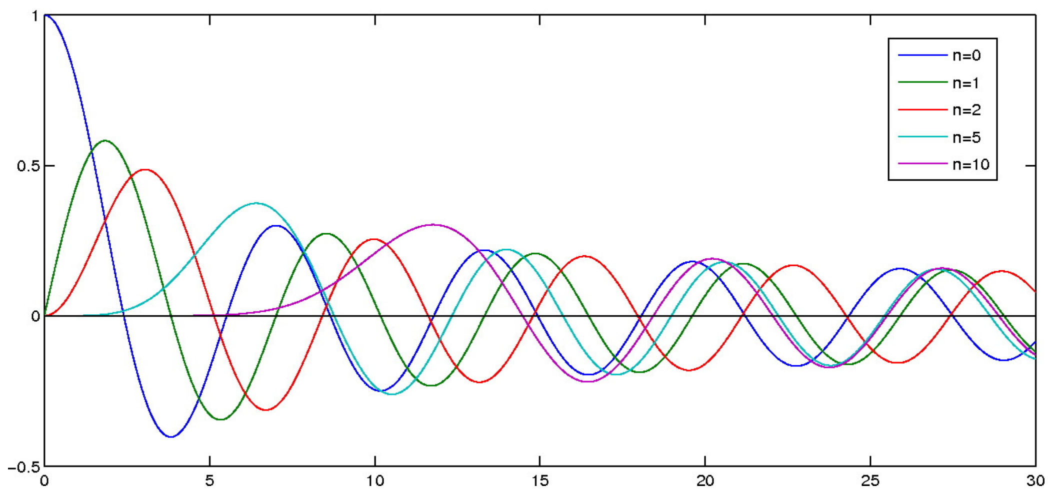
Auxiliary Conditions and Solution of Separate ODEs The angular functions Q must be continuous and 2π-periodic, and by Proposition 1, such solutions are found for $\nu = n \in \mathbb{N}_0$ (this also justifies our choice of a non-negative second separation constant):

$$Q_n^1(\varphi) = \cos(n\varphi), \quad Q_n^2(\varphi) = \sin(n\varphi), \quad n \in \mathbb{N}_0. \tag{330}$$

For the radial equation (328), we use the transformation $z := kr$ and write $w(z) := H(r)$. Then the ODE (328) transforms to

$$z^2 w'' + zw' + \left(z^2 - \nu^2\right) w = 0, \quad z > 0. \tag{331}$$

Equation (331) is *Bessel's differential equation* with parameter $\nu > 0$. For integer values of the parameter, $\nu = n \in \mathbb{N}_0$, a fundamental system of solutions is given by the *Bessel functions of the first and second kind,* $\{J_n, Y_n\}$, $n \in \mathbb{N}_0$. Only the Bessel functions of the first kind are bounded as $z \to 0$.



The boundary conditions (322) require that $J_n(kR) = 0$, $n \in \mathbb{N}_0$. We obtain infinitely many solutions which involve the values $k_{mn} > 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$. They satisfy

$$J_n(k_{mn}R) = 0. \tag{332}$$

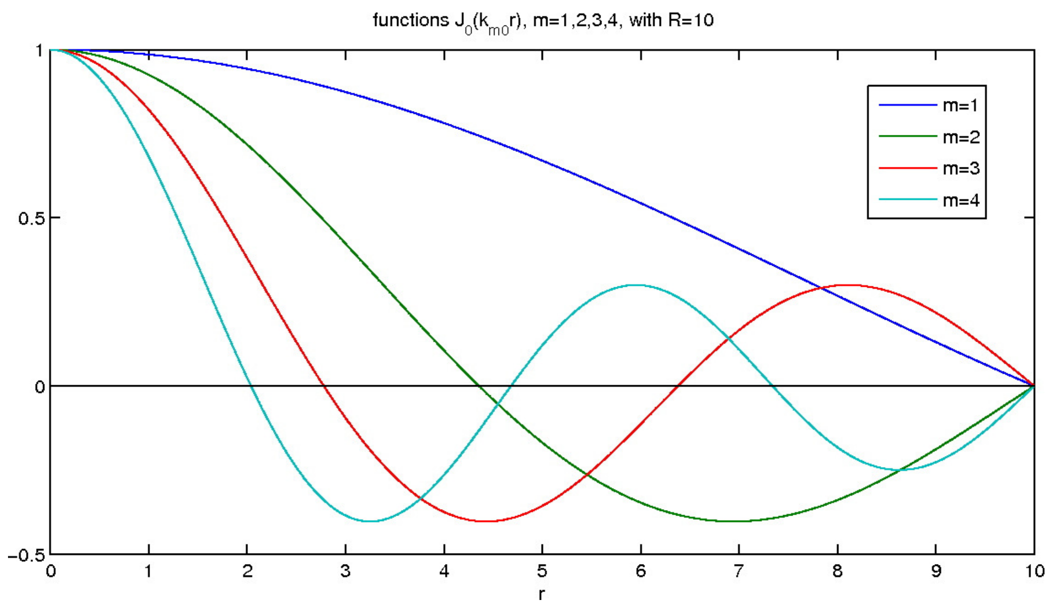
Notice that the values k_{mn} need to be approximated numerically (or found in a table) in practice.

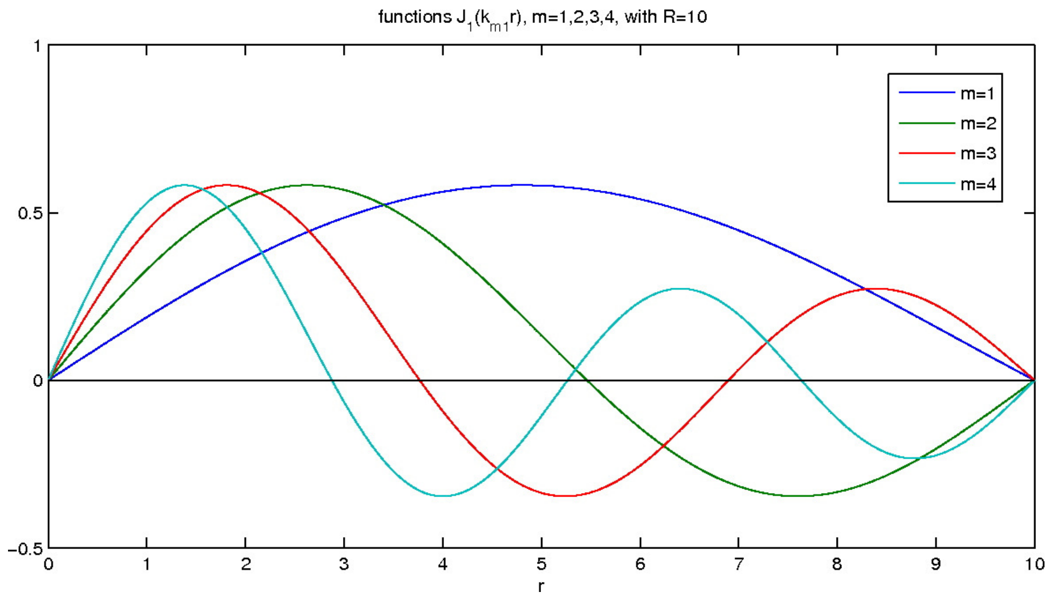
<u>Remark:</u> If we choose the first separation constant to be positive, we find the *modified Bessel functions of the first and second kind,* $\{I_n, K_n\}$, $n \in \mathbb{N}_0$.

Only the functions I_n are bounded as $z \to 0$. But they do not have any positive zeros, so they cannot satisfy the boundary condition at $r = R$. The case of a zero separation constant is left as an exercise (Problem Set 5).
We obtain the following solutions for the radial equation (328):

$$H_{mn}(r) = J_n(k_{mn} r), \quad m \in \mathbb{N},\, n \in \mathbb{N}_0. \tag{333}$$

Therefore, H_{mn} is the n-th order Bessel function of the first kind, scaled such that its m-th zero is located at $r = R$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$. This implies that H_{mn} has $m-1$ zeros inside the domain $r < R$. We illustrate this for $n = 0, 1$, $m = 1, \ldots, 4$, $R = 10$, in the following figures:





The functions $F^{1,2}_{mn} := H_{mn} Q^{1,2}_n$ are solutions of the two-dimensional Helmholtz equation in polar coordinates (325) which vanish at $r = R$:

$$\Delta F^{1,2}_{mn} + k^2_{mn} F^{1,2}_{mn} = 0, \quad m \in \mathbb{N},\, n \in \mathbb{N}_0. \tag{334}$$

With $\lambda_{mn} := c k_{mn}$, the ODE in time (326) becomes

$$\ddot{G} + \lambda^2_{mn} G = 0, \tag{335}$$

with solutions

$$G_{mn}(t) = A_{mn} \cos(\lambda_{mn} t) + B_{mn} \sin(\lambda_{mn} t), \quad m \in \mathbb{N},\, n \in \mathbb{N}_0. \tag{336}$$

We obtain the following eigenfunctions for the Dirichlet boundary value problem (321)–(323):

$$\begin{aligned} u_{mn}(r,\varphi,t) = G_{mn}(t) F^1_{mn}(r,\varphi) &= G_{mn}(t) J_n(k_{mn} r) \cos(n\varphi), & (337)\\ u^*_{mn}(r,\varphi,t) = G^*_{mn}(t) F^2_{mn}(r,\varphi) &= G^*_{mn}(t) J_n(k_{mn} r) \sin(n\varphi), & (338)\end{aligned}$$

for $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, with frequencies $\lambda_{mn}/(2\pi)$ and constants $A_{mn}, B_{mn} \in \mathbb{R}$ and $A^*_{mn}, B^*_{mn} \in \mathbb{R}$, respectively.

Superposition and Determination of Coefficients We use the superposition principle (Thm. 1) to obtain an expression for the general solution $u(r, \varphi, t)$. At $t = 0$, the solution has the form of a Fourier series in φ:

$$u(r, \varphi, 0) = a_0(r) + \sum_{n=1}^{\infty} (a_n(r)\cos(n\varphi) + a_n^*(r)\sin(n\varphi)) \stackrel{!}{=} f(r, \varphi), \qquad (339)$$

where the coefficients a_0 and a_n, a_n^*, $n \in \mathbb{N}$, are given by *Fourier-Bessel series* in r:

$$a_0(r) = \sum_{m=1}^{\infty} A_{m0} J_0(k_{m0} r), \qquad (340)$$

$$a_n(r) = \sum_{m=1}^{\infty} A_{mn} J_n(k_{mn} r), \quad a_n^*(r) = \sum_{m=1}^{\infty} A_{mn}^* J_n(k_{mn} r). \qquad (341)$$

As in Section 12.8, we determine the unknown coefficients A_{m0} and A_{mn}, A_{mn}^*, $m, n \in \mathbb{N}$, in two steps. First, the coefficients a_0, a_n, a_n^*, $n \in \mathbb{N}$, are determined as the Fourier coefficients of $f(r, \cdot)$, for any $r \in [0, R]$:

$$a_0(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r, \varphi)\, d\varphi, \quad a_n(r) = \frac{1}{\pi}\int_0^{2\pi} f(r, \varphi)\cos(n\varphi)\, d\varphi, \quad n \in \mathbb{N},$$

$$a_n^*(r) = \frac{1}{\pi}\int_0^{2\pi} f(r, \varphi)\sin(n\varphi)\, d\varphi, \quad n \in \mathbb{N}. \qquad (342)$$

Second, the coefficients A_{m0} and A_{mn}, A_{mn}^*, $m \in \mathbb{N}$, are the Fourier-Bessel coefficients of a_0, a_n, a_n^*, $n \in \mathbb{N}$, respectively.
<u>Remark:</u> The working principle behind the classical Fourier series are the orthogonality relations for the functions $\sin(nx)$ and $\cos(nx)$, $n \in \mathbb{N}_0$:

$$\int_0^{2\pi} \cos(mx)\cos(nx)\, dx = \begin{cases} 0, & m \neq n, \\ \pi, & m = n \geq 1 \\ 2\pi, & m = n = 0 \end{cases}, \quad m, n \in \mathbb{N}_0, \qquad (343)$$

$$\int_0^{2\pi} \sin(mx)\sin(nx)\, dx = \begin{cases} 0, & m \neq n, \\ \pi, & m = n \geq 1 \\ 0, & m = n = 0 \end{cases}, \quad m, n \in \mathbb{N}_0, \qquad (344)$$

$$\int_0^{2\pi} \sin(mx)\cos(nx)\, dx = 0, \quad m, n \in \mathbb{N}_0. \qquad (345)$$

Fourier-Bessel coefficients are found in the same way as the classical Fourier coefficients, but by using the following orthogonality relation for the Bessel functions instead:

$$\int_0^R r J_n(k_{\ell n} r) J_n(k_{mn} r)\, dr = \frac{R^2}{2} J_{n+1}(k_{mn}R)^2 \delta_{\ell m}, \quad n \in \mathbb{N}_0, \tag{346}$$

with the Kronecker delta

$$\delta_{\ell m} := \begin{cases} 1, & \ell = m \\ 0, & \ell \neq m \end{cases} . \tag{347}$$

12.10 Laplace's Equation in Cylindrical and Spherical Coordinates. Potential

Consider two point masses located at $\boldsymbol{x}_0 \in \mathbb{R}^3$ and $\boldsymbol{x} \in \mathbb{R}^3$. Define

$$\boldsymbol{r}(\boldsymbol{x}) := \boldsymbol{x} - \boldsymbol{x}_0, \tag{348}$$

then the distance between the two particles is given by $r(\boldsymbol{x}) := |\boldsymbol{r}(\boldsymbol{x})|$. The gravitational force on the particle at $\boldsymbol{x} \in \mathbb{R}^3$ is given by Newton's law of universal gravitation, and it is of the form

$$\boldsymbol{F}(\boldsymbol{x}) = \frac{c}{r(\boldsymbol{x})^2} \frac{\boldsymbol{x}_0 - \boldsymbol{x}}{|\boldsymbol{x}_0 - \boldsymbol{x}|} = -\frac{c\boldsymbol{r}(\boldsymbol{x})}{r(\boldsymbol{x})^3} \in \mathbb{R}^3, \tag{349}$$

or, in short, $\boldsymbol{F} = -c\boldsymbol{r}/r^3$.
Remark: In this example, we have $c = Gm_0m$, where $G \simeq 6.67{\cdot}10^{-11}\,\mathrm{Nm^2kg^{-2}}$ denotes the gravitational constant and where m_0 and m [kg] denote the mass of the particle at $\boldsymbol{x}_0$ and $\boldsymbol{x}$, respectively. We obtain the same form for $\boldsymbol{F}$ if we consider charged particles and use Coulomb's law instead.
With $\nabla r = \boldsymbol{r}/r$ we may also write

$$\boldsymbol{F} = -\frac{c\nabla r}{r^2} = \nabla\left(\frac{c}{r}\right). \tag{350}$$

Therefore the scalar field $f := -c/r$ [J] is called the *potential* of $\boldsymbol{F}$. So we have $\boldsymbol{F} = -\nabla f$, and we compute

$$\Delta f = \operatorname{div}(\nabla f) = -\operatorname{div}\boldsymbol{F} \overset{(349)}{=} c \operatorname{div}\left(\frac{\boldsymbol{r}}{r^3}\right) = c\left(\nabla\left(\frac{1}{r^3}\right)\cdot \boldsymbol{r} + \frac{1}{r^3}\operatorname{div}\boldsymbol{r}\right). \tag{351}$$

Furthermore, we have

$$\nabla\left(\frac{1}{r^3}\right) = -\frac{3r^2\nabla r}{r^6} = -\frac{3\boldsymbol{r}}{r^5} \quad\Rightarrow\quad \nabla\left(\frac{1}{r^3}\right)\cdot\boldsymbol{r} = -\frac{3}{r^3}, \tag{352}$$

$$\mathrm{div}\boldsymbol{r} = 3 \quad\Rightarrow\quad \frac{1}{r^3}\mathrm{div}\boldsymbol{r} = \frac{3}{r^3}, \tag{353}$$

so that the potential f satisfies the three-dimensional Laplace's equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0. \tag{354}$$

The theory of solutions of (354) is called *potential theory.* Solutions of (354) that have continuous second partial derivatives are known as *harmonic functions.* In Section 12.5, we had solved the two-dimensional Laplace's equation in cartesian coordinates with the method of separating variables. In this section, we shall see how the technique is used in three space dimensions.

Cylindrical Coordinates Cylindrical coordinates (r, φ, z) are related to the cartesian coordinates (x, y, z) via

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad r \geq 0,\ \varphi \in [0, 2\pi). \tag{355}$$

These are just polar coordinates used in the xy plane; the height is unchanged. Therefore, the Laplacian is very easy to compute in these coordinates, and we obtain a similar expression as in Section 12.9 (319):

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}. \tag{356}$$

The unknown function u may be written as $u(r, \varphi, z) = F(r, \varphi)G(z)$ and we may proceed exactly as in Section 12.9.

Spherical Coordinates The spherical coordinates (r, φ, ϑ) are related to the cartesian coordinates (x, y, z) via

$$x = r\cos\varphi\sin\vartheta, \quad y = r\sin\varphi\sin\vartheta, \quad z = r\cos\vartheta, \tag{357}$$

with $r \geq 0$, $\varphi \in [0, 2\pi)$ and $\vartheta \in [0, \pi]$ ($\Rightarrow \sin\vartheta \geq 0$). These relations can be inverted as

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \varphi = \arctan\left(\frac{y}{x}\right), \quad \vartheta = \arccos\left(\frac{z}{r}\right), \tag{358}$$

for $(x, y, z) \in \mathbb{R}^3$. With the chain rule and using the partial derivatives of r, φ, ϑ with respect to x, y, z, we obtain for the Laplacian in spherical coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2}, \tag{359}$$

where the *Laplace-Beltrami operator* (spherical Laplacian) Δ_{S^2} is given by

$$\Delta_{S^2} = \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right). \tag{360}$$

We consider Dirichlet boundary value problems which are posed either inside or outside of a sphere with radius $R > 0$: our domain Ω is either

$$\Omega^- \quad := \quad \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, |\boldsymbol{x}| < R\} \quad (\rightsquigarrow \text{“interior problem”}), \text{ or} \tag{361}$$

$$\Omega^+ \quad := \quad \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, |\boldsymbol{x}| > R\} \quad (\rightsquigarrow \text{“exterior problem”}). \tag{362}$$

Notice that Ω^- is a bounded domain, whereas Ω^+ is not. In both cases, the boundary value problem has the form

$$\Delta u = 0 \quad \text{in } \Omega, \tag{363}$$

$$u(R, \varphi, \vartheta) = f(\varphi, \vartheta), \quad \varphi \in [0, 2\pi),\, \vartheta \in [0, \pi], \tag{364}$$

$$\lim_{r\to\infty} u(r, \varphi, \vartheta) = 0, \quad \varphi \in [0, 2\pi),\, \vartheta \in [0, \pi]. \tag{365}$$

Separation of Variables We write the unknown function in the form $u(r, \varphi, \vartheta) = F(r)G(\varphi, \vartheta)$. We separate the radial variable from the angular variables by

$$r^2\frac{F''}{F} + 2r\frac{F'}{F} = -\frac{1}{G}\Delta_{S^2}G = k \in \mathbb{R}, \tag{366}$$

and the separate differential equations are given by

$$r^2F'' + 2rF' - kF \quad = \quad 0, \quad r < R \quad \text{or} \quad r > R, \tag{367}$$

$$\Delta_{S^2}G + kG \quad = \quad 0, \quad \varphi \in [0, 2\pi),\, \vartheta \in [0, \pi]. \tag{368}$$

We solve for G first. For that purpose, we separate again, $G(\varphi, \vartheta) = H(\varphi)Q(\vartheta)$, and we obtain

$$-\frac{H''}{H} = \frac{\sin\vartheta}{Q}\left(\sin\vartheta Q'\right)' + k\sin^2\vartheta = \mu^2 \in \mathbb{R}. \tag{369}$$

The separate ODEs are now given by

$$\begin{aligned} H'' + \mu^2 H &= 0, \quad \varphi \in [0, 2\pi), & (370) \\ \sin^2\vartheta Q'' + \sin\vartheta\cos\vartheta Q' + \left(k\sin^2\vartheta - \mu^2\right) Q &= 0, \quad \vartheta \in [0, \pi]. & (371) \end{aligned}$$

Auxiliary Conditions and Solution of Separate ODEs 2π-periodic solutions of (370) are found for $\mu = m \in \mathbb{Z}$, and they are given by

$$H_m(\varphi) = e^{im\varphi}. \tag{372}$$

Remark: We prefer to use the complex exponentials here, instead of $\cos(m\varphi)$ and $\sin(m\varphi)$, $m \in \mathbb{Z}$.
In (371) we use $z := \cos\vartheta \in [-1, 1]$ to obtain

$$\frac{d}{d\vartheta} = -\sin\vartheta \frac{d}{dz}, \quad \frac{d^2}{d\vartheta^2} = (1 - z^2)\frac{d^2}{dz^2} - z\frac{d}{dz}. \tag{373}$$

Then the ODE (371) transforms to

$$(1 - z^2)w'' - 2zw' + \left(k - \frac{\mu^2}{1 - z^2}\right) w = 0, \tag{374}$$

for the function $w(z) = Q(\vartheta)$. For $k = \nu(\nu + 1)$, this becomes *Legendre's differential equation* with parameters ν and μ.
Remark: Every number $k \geq -1/4$ can be written in the form $k = \nu(\nu + 1)$, with $\nu \in \mathbb{R}$. For $k < 1/4$, we have $\nu \in \mathbb{C}$.
Periodic and continuous solutions are found for $\nu = n \in \mathbb{N}_0$, $0 \leq |m| \leq n$, and they are given by the *associated Legendre functions*

$$Q_{mn}(\vartheta) = P_n^{|m|}(\cos\vartheta), \quad 0 \leq |m| \leq n. \tag{375}$$

For integer values of the parameters, these functions are polynomials (of degree n). For $m = 0$, they reduce to the *Legendre polynomials,* $P_n^0 \equiv P_n$.
Remark: The polynomial $P_n^{|m|}$ has $n - |m|$ real zeros in $[-1, 1]$, $0 \leq |m| \leq n$.
We may normalize the product $G_{mn} := H_m Q_{mn}$, and so we define the *spherical harmonics* $Y_{nm} : [0, 2\pi) \times [0, \pi] \to \mathbb{C}$ (!)

$$Y_{nm}(\varphi, \vartheta) := \sqrt{\frac{2n + 1}{4\pi}\frac{(n - |m|)!}{(n + |m|)!}} P_n^{|m|}(\cos\vartheta)e^{im\varphi}, \quad 0 \leq |m| \leq n. \tag{376}$$

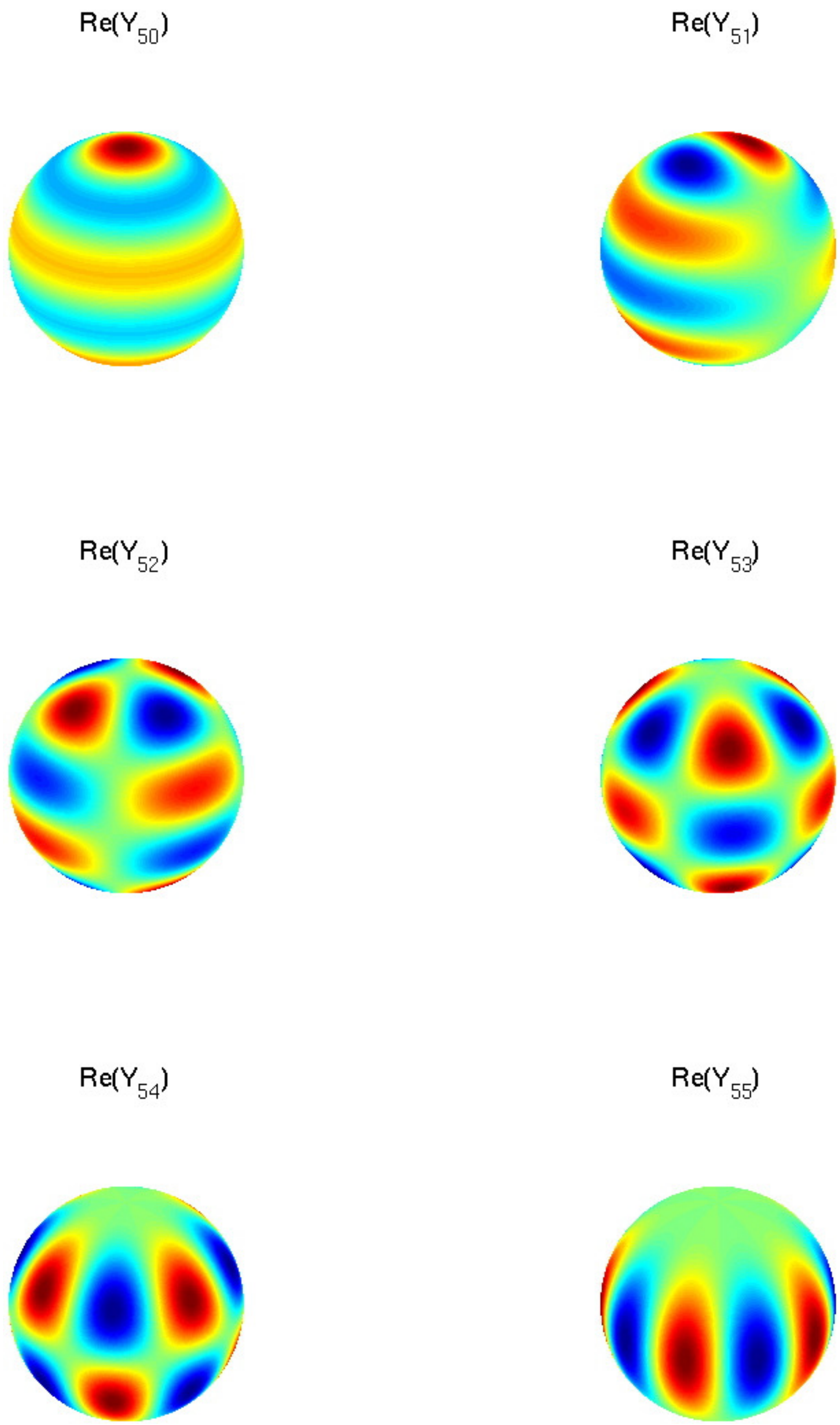
Remark: For any $n \in \mathbb{N}_0$, there are $2n+1$ spherical harmonics Y_{nm}. The first index is often denoted by ℓ instead of n. Y_ℓ^m is also a common notation.
The spherical harmonics Y_{nm} are eigenfunctions of the spherical Laplacian,

$$\Delta_{S^2} Y_{nm} + n(n+1) Y_{nm} = 0, \tag{377}$$

and they satisfy the orthogonality relations

$$\int_0^{2\pi} \int_0^{\pi} Y_{nm}(\varphi, \vartheta) \overline{Y_{n'm'}(\varphi, \vartheta)} \sin\vartheta \, d\vartheta \, d\varphi = \delta_{nn'} \delta_{mm'}, \tag{378}$$

where the bar denotes complex conjugation.
Remark: Notice that $\overline{Y_{nm}} = Y_{n,-m}$, so that the real part of Y_{nm} is given by $\mathrm{Re}(Y_{nm}) = (Y_{nm} + Y_{n,-m})/2$, and therefore $\mathrm{Re}(Y_{n,-m}) = \mathrm{Re}(Y_{n,m})$, $0 \le |m| \le n$. Spherical harmonics Y_{n0} are called *zonal,* and $Y_{n,\pm n}$ are called *sectoral.* All other spherical harmonics are called *tesseral.*
Nodal lines of the real and imaginary parts of Y_{nm} are located at the zeros of the associated Legendre polynomials and of the cos and sin functions. Visualization on the unit sphere S^2 (red: positive, blue: negative), for $n = 5$:


Re(Y_{50})
Re(Y_{51})
Re(Y_{52})
Re(Y_{53})
Re(Y_{54})
Re(Y_{55})

With the separation constant $k = n(n+1)$, the ODE for the radial functions F becomes

$$r^2 F'' + 2rF' - n(n+1)F = 0. \tag{379}$$

This is a *Cauchy-Euler differential equation,* in which powers of r are equal to the order of the derivative in each term. Solutions are given by

$$F_n(r) = r^n, \quad F_n^*(r) = \frac{1}{r^{n+1}}, \quad n \in \mathbb{N}_0. \tag{380}$$

Solutions of the three-dimensional Laplace equation (171) in spherical coordinates are given by

$$u_{mn}(r,\varphi,\vartheta) = A_{mn} r^n Y_{nm}(\varphi,\vartheta), \quad u_{mn}^*(r,\varphi,\vartheta) = \frac{A_{mn}^*}{r^{n+1}} Y_{nm}(\varphi,\vartheta), \tag{381}$$

for $n \in \mathbb{N}_0$, $0 \le |m| \le n$.

Superposition and Determination of Coefficients *Interior Problem:* We must have $A_{mn}^* = 0$, because the solution must be bounded as $r \to 0$. Therefore, the solution in this case is given by

$$u^-(r,\varphi,\vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn} r^n Y_{nm}(\varphi,\vartheta), \quad r \le R. \tag{382}$$

Evaluated as $r \to R$, we obtain

$$u^-(R,\varphi,\vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn} R^n Y_{nm}(\varphi,\vartheta) \stackrel{!}{=} f(\varphi,\vartheta). \tag{383}$$

Using the orthogonality relations for the spherical harmonics, the coefficients A_{mn} may be computed as

$$A_{mn} = \frac{1}{R^n} \int_0^{2\pi} \int_0^{\pi} f(\varphi,\vartheta) \overline{Y_{nm}(\varphi,\vartheta)} \sin\vartheta \, d\vartheta \, d\varphi. \tag{384}$$

Exterior Problem: We must have $A_{mn} = 0$, because the solution must vanish in the limit as $r \to \infty$. Therefore, the solution for this case is given by

$$u^+(r,\varphi,\vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{A_{mn}^*}{r^{n+1}} Y_{nm}(\varphi,\vartheta), \quad r \ge R. \tag{385}$$

Evaluated as $r \to R$, we obtain

$$u^+(R, \varphi, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{A^*_{mn}}{R^{n+1}} Y_{nm}(\varphi, \vartheta) \stackrel{!}{=} f(\varphi, \vartheta). \tag{386}$$

The coefficients A^*_{mn} are given by

$$A^*_{mn} = R^{n+1} \int_0^{2\pi} \int_0^{\pi} f(\varphi, \vartheta) \overline{Y_{nm}(\varphi, \vartheta)} \sin\vartheta \, d\vartheta \, d\varphi. \tag{387}$$

Notice that $A^*_{mn} = R^{2n+1} A_{mn}$. If we want to study the behavior of the solution u as $r \to \infty$, it is useful to write

$$u^+(r, \varphi, \vartheta) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} f_n(\varphi, \vartheta), \quad f_n(\varphi, \vartheta) := \sum_{m=-n}^{n} \frac{A^*_{mn}}{R^{n+1}} Y_{nm}(\varphi, \vartheta). \tag{388}$$

A Taylor expansion in $1/r$ of $u(\cdot, \varphi, \vartheta)$ yields

$$u^+(r, \varphi, \vartheta) = \frac{R}{r} f_0(\varphi, \vartheta) + O(r^{-2}), \quad r \to \infty, \quad \forall\, \varphi, \vartheta. \tag{389}$$

It turns out that $f_0(\varphi, \vartheta)$ is actually constant, and equal to the average of the boundary data f over the unit sphere S^2:

$$f_0 = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(\varphi, \vartheta) \sin\vartheta \, d\vartheta \, d\varphi. \tag{390}$$

12.11 Solution of PDEs by Laplace Transforms

We may transform a PDE involving both space and time into a family of PDEs involving spatial variables only. This is accomplished by the Laplace transform, which for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$f^{\#}(s) \equiv \mathcal{L}[f](s) := \int_0^{\infty} f(t) e^{-st} \, dt, \quad s \in \mathbb{C}. \tag{391}$$

if the integral exists (it does for a large class of functions, for $\mathrm{Re}(s)$ large enough). $\mathcal{L}$ is a linear transform and has an inverse, $\mathcal{L}^{-1}$. Using integration by parts, we find that the Laplace transform of the n-th derivative of F is given by

$$\mathcal{L}[f^{(n)}](s) = s\mathcal{L}[f^{(n-1)}](s) - f^{(n-1)}(0). \tag{392}$$

By induction we conclude that

$$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{393}$$

Remark: Because of property (393), the Laplace transform is particularly suited to initial value problems, which model physical systems that are at rest for all times $t < 0$ and suddenly change at $t = 0$. If, on the other hand, the system is time-invariant, it is better to use the Fourier transform instead. We may use (393) to eliminate the time derivatives in a PDE. Consider the following initial value problem for the wave equation:

$$U_{tt} = c^2\Delta U, \quad \text{in } \Omega \times (0, \infty), \quad c > 0, \tag{394}$$

$$U = 0, \quad U_t = 0, \quad \text{in } \Omega \times \{0\}. \tag{395}$$

This problem is not well posed, but we will take care of the boundary conditions later. The Laplace transform $u(\boldsymbol{x}; s) := \mathcal{L}[U(\boldsymbol{x}, \cdot)](s)$ of the unknown function $U(\boldsymbol{x}, t)$ satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad \text{in } \Omega, \quad k := \frac{is}{c} \in \mathbb{C}. \tag{396}$$

Any boundary condition imposed on U on $\partial\Omega \times (0, \infty)$ also needs to be Laplace transformed, which yields a new boundary condition for u on $\partial\Omega$. Thus we end up with a boundary value problem in Ω. For simple shapes of Ω, we can solve this problem analytically, as we have seen for a rectangle in 2D (Section 12.8 $\rightsquigarrow$ double Fourier series), a disk in 2D (Section 12.9 $\rightsquigarrow$ Fourier-Bessel series), as well as in the case of a spherical boundary in 3D (Section 12.10 $\rightsquigarrow$ spherical harmonics).

12.12 Application: hydrogen-like atomic orbitals

A hydrogen-like atom is one with a single electron (charge $-e$, where $e \simeq 1.60 \cdot 10^{-19}$ C denotes the elementary charge) and a nucleus of charge Ze, with the atomic number $Z \in \mathbb{N}$. The quantum state of the electron is described by the wave function $\Psi : \mathbb{R}^3 \times (0, \infty) \to \mathbb{C}$: the function $|\Psi|^2$ is the probability density for the position of the electron at time $t > 0$. The time evolution of the wave function Ψ is described by the *Schrödinger equation*

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m_e} \Delta \Psi + V\Psi, \tag{397}$$

where $i^2 = -1$ and where $\hbar \simeq 1.05 \cdot 10^{-34}\,\mathrm{Js}$ denotes the reduced Planck constant, $m_e \simeq 9.11 \cdot 10^{-31}\,\mathrm{kg}$ denotes the electron mass and $V(\boldsymbol{x})$ [J] denotes the potential energy at position $\boldsymbol{x} \in \mathbb{R}^3$. A Fourier transform in time leads to an expansion into standing waves of the form

$$\Psi(\boldsymbol{x}, t; \omega) = \psi(\boldsymbol{x}; \omega) e^{-i\omega t}, \tag{398}$$

with $\psi : \mathbb{R}^3 \to \mathbb{C}$, $|\Psi|^2 = |\psi|^2$. The functions ψ satisfy the time independent Schrödinger equation

$$E\psi = -\frac{\hbar^2}{2m_e} \Delta \psi + V\psi, \tag{399}$$

where $E = E(\omega) = \hbar\omega$ denotes the total energy of the electron. We wish to solve the eigenvalue problem associated to (399), i. e. find all energy levels E for which (399) has a solution, as well as the solution ψ itself. An analytic solution is possible for simple cases, such as the one considered here, where the potential is given by

$$V(\boldsymbol{x}) = -\frac{1}{4\pi\varepsilon_0} \frac{Ze^2}{|\boldsymbol{x}|}, \tag{400}$$

with the vacuum permittivity $\varepsilon_0 \simeq 8.85 \cdot 10^{-12}\,\mathrm{Fm}^{-1}$. Here we have assumed without loss of generality that the nucleus is located at the origin $\mathbf{0} \in \mathbb{R}^3$. We write the stationary Schrödinger equation (399) in spherical coordinates:

$$\left(-\frac{\hbar^2}{2m_e} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^2} \right) - \frac{1}{4\pi\varepsilon_0} \frac{Ze^2}{r} \right) \psi = E\psi. \tag{401}$$

Expanding the solution ψ into spherical harmonics,

$$\psi(r, \varphi, \vartheta; E) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} R_{\ell m}(r; E) Y_{\ell m}(\varphi, \vartheta), \tag{402}$$

we obtain, with $\Delta_{S^2} Y_{\ell m} + \ell(\ell+1) Y_{\ell m} = 0$, for $0 \le |m| \le \ell$, $\ell \in \mathbb{N}_0$:

$$\left(-\frac{\hbar^2}{2m_e} \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} \right) - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r} \right) R_\ell(r;E) = E R_\ell(r;E), \tag{403}$$

for $r > 0$. We have dropped the index m from R_ℓ because the radial ODE (403) is independent of that index. We keep $\ell \in \mathbb{N}_0$ fixed for the moment and define the function $u(r) := r R_\ell(r;E)$.

$$R_\ell(r;E) = \frac{u(r)}{r}, \quad R'_\ell(r;E) = \frac{r u'(r) - u(r)}{r^2}, \tag{404}$$

$$R''_\ell(r;E) = \frac{r^2 u''(r) - 2r u'(r) + 2u(r)}{r^3}. \tag{405}$$

$$-\frac{\hbar^2}{2m_e} u''(r) + \left(\frac{\hbar^2}{2m_e} \frac{\ell(\ell+1)}{r^2} - \frac{Ze^2}{4\pi\varepsilon_0 r} \right) u(r) = E u(r), \tag{406}$$

for $r > 0$. We consider the ODE (406) in the limit as $r \to \infty$:

$$-\frac{\hbar^2}{2m_e} u''(r) = E u(r), \quad r \to \infty. \tag{407}$$

Bound states are characterized by a negative energy $E < 0$, and according to Proposition 1, the asymptotic behavior of u is given by

$$u \sim e^{-\kappa r}, \ r \to \infty, \quad \kappa := \sqrt{-\frac{2m_e E}{\hbar^2}} > 0 \quad [\mathrm{m}^{-1}]. \tag{408}$$

We transform variables: $\rho := \kappa r$ (notice that this scaling depends on E) and obtain the following ODE for $v(\rho(r)) = u(r)$, from (406):

$$\rho^2 v''(\rho) = \big(\rho^2 - 2\nu\rho + \ell(\ell+1)\big) v(\rho), \ \rho > 0, \quad \nu := -\frac{Ze^2\kappa}{8\pi\varepsilon_0 E} > 0. \tag{409}$$

We consider the ODE (409) as $\rho \to 0$:

$$\rho^2 v''(\rho) \sim \ell(\ell+1) v(\rho), \quad \rho \to 0. \tag{410}$$

This is a Cauchy-Euler equation, with solutions $\rho^{-\ell}$ and $\rho^{\ell+1}$. Because v must be bounded as $\rho \to 0$, its asymptotic behavior must be given by

$$v \sim \rho^{\ell+1}, \quad \rho \to 0. \tag{411}$$

We factor out both asymptotics and write $v(\rho) := e^{-\rho}\rho^{\ell+1}w(\rho)$, with a new unknown function w. It satisfies the second-order ODE

$$\rho w''(\rho) + ((2\ell+1) + 1 - 2\rho)\, w'(\rho) + 2\,(\nu - \ell - 1)\, w(\rho) = 0, \quad \rho > 0. \tag{412}$$

with $z := 2\rho$, $\tilde{w}(z) = w(\rho)$, we obtain

$$z\tilde{w}''(z) + ((2\ell+1) + 1 - z)\, \tilde{w}'(z) + (\nu - \ell - 1)\, \tilde{w}(z) = 0. \tag{413}$$

Solutions are found for $\nu = n \in \mathbb{N}$, $\ell < n$, and they are given by the *associated Laguerre polynomials*

$$\tilde{w}(z) = L_{n-\ell-1}^{2\ell+1}(z) = \sum_{k=0}^{n-\ell-1} (-1)^k \begin{pmatrix} n+\ell \\ n-\ell-1-k \end{pmatrix} \frac{z^k}{k!}. \tag{414}$$

We can now also express the admissible energy states as

$$E_n = -\frac{Z^2}{n^2}\frac{m_e e^4}{32\pi^2\varepsilon_0^2\hbar^2}, \quad n \in \mathbb{N}. \tag{415}$$

The radial solutions are then given by

$$R_{n\ell}(r) := R_\ell(r; E_n) = \kappa e^{-\kappa r}(\kappa r)^\ell L_{n-\ell-1}^{2\ell+1}(2\kappa r) \tag{416}$$

The solution can now be given as a superposition of the products

$$\psi_{n\ell m}(r, \varphi, \vartheta) := c_{n\ell} R_{n\ell}(r) Y_{\ell m}(\varphi, \vartheta), \quad 0 \le |m| \le \ell < n, \tag{417}$$

with some normalization constants $c_{n\ell} > 0$. So the wave function for the n-th energy state is of the form

$$\psi(r, \varphi, \vartheta; E_n) = \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} c_{n\ell} R_{n\ell}(r) Y_{\ell m}(\varphi, \vartheta), \quad n \in \mathbb{N}. \tag{418}$$

12.13 Application: Unbounded domains and artificial boundaries. Dirichlet-to-Neumann map.

Many practical problems, such as scattering problems, are posed in unbounded or at least very large domains. In these problems, the scatterer is typically a region of space with complicated properties, such as inhomogeneities, nonlinearities, obstacles and sources. Analytical methods are thus not applicable to the problem, and we need to use numerical methods for its solution.

However, with numerical methods we immediately run into a conflict with the unbounded domain. Typical grid-based numerical methods (such as the finite difference, finite element, and finite volume method) cannot handle infinite domains, so the problem is not (yet) solvable. One approach (there are others!) is to introduce an artificial boundary B, which separates space into a bounded domain Ω^- and an unbounded region Ω^+. The domain Ω^- should be chosen such that it contains all the complicated features of the scatterer. Ω^+ will contain the "uninteresting region", where the medium is supposed to be simple. The idea is then to solve the problem numerically in Ω^- only. We still need to impose a boundary condition on $B \subseteq \Omega^-$ in order to obtain a well-posed problem. In the context of wave propagation, this boundary must be such that waves impinging on B can pass that interface without any spurious reflection. Such reflection would be non-physical (because the boundary itself is only artificial), but it would spoil the numerical solution throughout Ω^-. A boundary condition on B with the desired property is therefore called a *non-reflecting boundary condition* (NRBC, NBC).

The idea is that because the problem is supposed to be simple in Ω^+, we can solve it analytically and represent the solution in Ω^+ using the boundary values on $B = \partial\Omega^+$. We can then get an analytic expression for the *Dirichlet-to-Neumann (DtN) map M*,

$$M : u\big|_B \mapsto \left.\frac{\partial u}{\partial n}\right|_B . \tag{419}$$

Then the boundary condition

$$\partial_n u = Mu, \quad \text{on } B, \tag{420}$$

(Robin type) is an exact non-reflecting boundary condition for the complicated problem in Ω^-. To find an analytic expression for M, we usually assume a simple shape for B, such as a sphere with radius $R > 0$ (then $\partial_n \equiv \partial_r$).

After Laplace transform in time, if necessary, an acoustic scattering problem satisfies a Helmholtz equation in $\Omega^+ \subset \mathbb{R}^3$, together with the Rellich-

Sommerfeld condition (which ensures uniqueness of the solution):

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega^+, \tag{421}$$

$$\lim_{a\to\infty} \int\limits_{|\boldsymbol{x}|=a} \left| \frac{\partial u}{\partial r} - iku \right|^2 ds = 0. \tag{422}$$

We may solve this problem using separation of variables. It is natural for this geometry to use spherical coordinates. We expand the unknown function u into spherical harmonics,

$$u(r,\varphi,\vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} u_{mn}(r) Y_{nm}(\varphi,\vartheta), \quad r \geq R. \tag{423}$$

Then we obtain the following ODE for the coefficients u_{mn}, $0 \leq |m| \leq n$:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{n(n+1)}{r^2} + k^2 \right) u_{mn}(r) = 0, \quad r \geq R, \tag{424}$$

$$\lim_{r\to\infty} \left(\frac{d}{dr} - ik \right) u_{mn}(r) = 0. \tag{425}$$

A transformation $z := kr$ leads to a Bessel differential equation for $u_{mn}(r) = w(z)/\sqrt{z}$. Together with the radiation condition, we find that solutions are given by the *Hankel functions of the first kind* of order $n + 1/2$, $n \in \mathbb{N}_0$:

$$u_{mn}(r) = A_{mn} \frac{H^{(1)}_{n+1/2}(kr)}{\sqrt{r}}, \quad r \geq R, \tag{426}$$

The coefficients $A_{mn} \in \mathbb{C}$ can be determined from the boundary values of u_{mn} at $r = R$:

$$u_{mn}(R) = A_{mn} \frac{H^{(1)}_{n+1/2}(kR)}{\sqrt{R}} \quad \Rightarrow \quad A_{mn} = \frac{\sqrt{R}}{H^{(1)}_{n+1/2}(kR)} u_{mn}(R), \tag{427}$$

so that the coefficients u_{mn} may also be written as

$$u_{mn}(r) = \sqrt{\frac{R}{r}} \frac{H^{(1)}_{n+1/2}(kr)}{H^{(1)}_{n+1/2}(kR)} u_{mn}(R). \tag{428}$$

We compute the derivative with respect to r and evaluate at $r = R$ to obtain

$$u'_{mn}(R) = \frac{1}{R} E_n(kR) u_{mn}(R), \tag{429}$$

with the *DtN kernel*

$$E_n(z) = \frac{z {H^{(1)}_{n+1/2}}'(z)}{H^{(1)}_{n+1/2}(z)} - \frac{1}{2}, \quad n \in \mathbb{N}_0. \tag{430}$$

Now we may write the DtN map M as a spherical harmonic expansion

$$\begin{aligned}
(Mu)(\varphi,\vartheta) &= \left.\frac{\partial u}{\partial n}\right|_B (\varphi,\vartheta) = \frac{\partial u}{\partial r}(R,\varphi,\vartheta) \\
&= \frac{1}{R}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} E_n(kR) u_{mn}(R) Y_{nm}(\varphi,\vartheta) \qquad (431)\\
&= \frac{1}{R}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} E_n(kR) \int_B u|_B \overline{Y_{nm}}\, ds Y_{nm}(\varphi,\vartheta).
\end{aligned}$$

In a numerical scheme, the exact non-reflecting boundary condition will need to be approximated by truncating the series in M. In 3D, the efficient evaluation of M is another issue: the application of the non-reflecting boundary condition must not be more costly than the numerical method used inside of Ω^-.

12.14 Review of Chapter 12

12.14.1 Basic Concepts (12.1)

A *partial differential equation* of order $k \in \mathbb{N}$ for an unknown function $u : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^d$ open, $d > 1$ (the *total* number of variables), is an expression of the form

$$F\left((D^k u)(x), (D^{k-1}u)(x), \ldots, (Du)(x), u(x), x\right) = 0, \quad x \in \Omega, \tag{432}$$

with a function $F : \mathbb{R}^{d^k} \times \mathbb{R}^{d^{k-1}} \times \cdots \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$. A typical mathematical model is given by (432), together with auxiliary conditions on $\Gamma \subseteq \partial\Omega$, such as initial and/or boundary conditions. We have mainly looked at *homogeneous, linear* PDEs, which are of the form

$$\sum_{|\alpha|\le k} a_\alpha(x)(D^\alpha u)(x) = 0, \tag{433}$$

with coefficients $a_\alpha : \Omega \to \mathbb{R}$, $|\alpha| \le k$. For these PDEs, the *superposition principle* (Theorem 1) is valid.

12.14.2 Examples

We have considered

- problems in two variables ($d = 2$):
  - 1 spatial variable + time: 1D wave equation, $u_{tt} = c^2 u_{xx}$, $c > 0$ (12.3, 12.4), 1D heat equation $u_t = c^2 u_{xx}$, $c > 0$ (12.5, 12.6)
  - 2 spatial variables: 2D Laplace equation, $\Delta u = 0$ (steady heat problems, 12.5)
- problems in three variables ($d = 3$):
  - 2 spatial variables + time: 2D wave equation, $u_{tt} = c^2 \Delta u$, $c > 0$ (12.8, 12.9)
  - 3 spatial variables: 3D Laplace equation, $\Delta u = 0$ (12.10)

12.14.3 Separation of Variables (12.3, 12.5, 12.8, 12.9, 12.10)

The method of separating variables aims at transforming a PDE into a system of differential equations involving fewer independent variables, which may be solved more easily than the original PDE.

1. Separation of Variables For problems involving time, we write the dependent variable in the form $u(\boldsymbol{x}, t) = F(\boldsymbol{x})G(t)$. In the case of wave and heat equations, respectively, this leads to

$$\begin{aligned} u_{tt} - c^2\Delta u &= F\ddot{G} - c^2\Delta F G = 0, \quad \text{in } \Omega \times (0,\infty), && (434)\\ u_t - c^2\Delta u &= F\dot{G} - c^2\Delta F G = 0, \quad \text{in } \Omega \times (0,\infty), && (435) \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^{d-1}$ denotes the spatial domain only, from now on. After division by c^2FG we obtain

$$\frac{\ddot{G}}{c^2G} = \frac{\Delta F}{F}, \quad \text{in } \Omega \times (0,\infty), \tag{436}$$
$$\frac{\dot{G}}{c^2G} = \frac{\Delta F}{F}, \quad \text{in } \Omega \times (0,\infty). \tag{437}$$

Because the left-hand side of (436), (437) depends only on the time variable $t > 0$, and the right-hand side of (436), (437) depends only on the spatial variable(s) $\boldsymbol{x} \in \Omega$, both sides must be equal to a constant, the separation constant $k \in \mathbb{R}$. So we obtain two separate equations, one in space and one in time (which is an ODE):

$$\Delta F = kF, \quad \text{in } \Omega, \quad (+\text{ boundary conditions}), \tag{438}$$
$$\text{and either} \quad \ddot{G} = c^2kG, \quad \text{in } (0,\infty), \quad (+\text{ initial conditions}), \tag{439}$$
$$\text{or} \quad \dot{G} = c^2kG, \quad \text{in } (0,\infty), \quad (+\text{ initial conditions}). \tag{440}$$

The separation constant $k \in \mathbb{R}$ is part of the solution of the spatial *eigenvalue problem* (438), which involves $d-1$ variables. The Laplace equation, $\Delta u = 0$, is just a special case of (438), with $k = 0$. Therefore, the same solution strategies apply.

2. Solution of the spatial eigenvalue problem

$d = 2$: In this case, the spatial eigenvalue problem is given by

$$F'' = kF, \quad \text{in } \Omega \subseteq \mathbb{R}, \tag{441}$$

together with boundary conditions. The general solution of (441) was given in Proposition 1:

$$F(x;k) = \begin{cases} Ae^{\omega x} + Be^{-\omega x}, & k = \omega^2,\ \omega > 0 \\ Ax + B, & k = 0 \\ A\cos(\omega x) + B\sin(\omega x), & k = -\omega^2,\ \omega > 0 \end{cases}, \quad A, B \in \mathbb{R}. \tag{442}$$

Let the domain be given by $\Omega = (0, L)$, $L > 0$. The boundary conditions on $\partial\Omega = \{0, L\}$ will determine which of these solutions are feasible; in general we need to check each case for the separation constant k separately.

With *Dirichlet boundary conditions* $F(0) = F(L) = 0$, we find only the trivial solution $F \equiv 0$ for $k \geq 0$. However, we find non-trivial solutions for $k = -(n\pi/L)^2$, $n \in \mathbb{N}$:

$$F_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n \geq 1, \tag{443}$$

where we have moved any constant factor over to the time-dependent functions $G_n(t)$.

With *Neumann boundary conditions* $F'(0) = F'(L) = 0$, we find only the trivial solution $F \equiv 0$ for $k > 0$. However, we find non-trivial solutions for $k = -(n\pi/L)^2$, $n \in \mathbb{N}$:

$$F_n(x) = \cos\left(\frac{n\pi}{L}x\right), \quad n \geq 0. \tag{444}$$

Notice the constant solution, F_0!

With *mixed boundary conditions* $F(0) = F'(L) = 0$, we find only the trivial solution $F \equiv 0$ for $k \geq 0$. However, we find non-trivial solutions for $k = -((n + 1/2)\pi/L)^2$, $n \geq 0$:

$$F_n(x) = \sin\left(\left(n + \frac{1}{2}\right)\frac{\pi}{L}x\right), \quad n \geq 0. \tag{445}$$

$d = 3$: The spatial differential equation for F will be a PDE in two space dimensions. We choose coordinates *such that the boundary of* Ω *becomes as simple as possible.* So, if $\Omega = (0, a) \times (0, b)$, $a, b > 0$, is a rectangle, we use cartesian coordinates (x, y), and if Ω is a disk of radius $R > 0$ (or an annulus), we use polar coordinates (r, φ). The Laplacian in these two coordinate systems is given by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}. \tag{446}$$

We separate the spatial variables by writing $u(x, y) = H(x)Q(y)$, or $u(r, \varphi) = H(r)Q(\varphi)$, and we obtain

$$\frac{H''}{H} = -\frac{Q''}{Q} + k \quad = \quad p \in \mathbb{R}, \tag{447}$$

$$r^2\frac{H''}{H} + r\frac{H'}{H} - kr^2 = -\frac{Q''}{Q} \quad = \quad p \in \mathbb{R}. \tag{448}$$

The two separate ODEs are thus given by either

$$
\begin{aligned}
H'' &= pH, \quad \text{in } (0,a), && (449)\\
Q'' &= (k-p)Q, \quad \text{in } (0,b), && (450)\\
\text{or} \quad r^2H'' + rH' - (kr^2+p)H &= 0, \quad \text{in } [0,R), && (451)\\
Q'' &= -pQ, \quad \text{in } [0,2\pi). && (452)
\end{aligned}
$$

Again, to find feasible values of the separation constants as well as the solutions themselves, we require the boundary conditions for u on $\partial\Omega$. We assume homogeneous Dirichlet boundary conditions here.

(x,y): As in the 1D case, we find solutions for $p = -(m\pi/a)^2$, $m \in \mathbb{N}$, $k - p = -(n\pi/b)^2$, $n \in \mathbb{N}$. They are given by

$$
\begin{aligned}
H_m(x) &= \sin\left(\frac{m\pi}{a}x\right), \quad m \in \mathbb{N}, && (453)\\
Q_n(y) &= \sin\left(\frac{n\pi}{b}y\right), \quad n \in \mathbb{N}. && (454)
\end{aligned}
$$

The spatial eigenfunctions are thus given by

$$F_{mn}(x,y) = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), \quad m,n \in \mathbb{N}, \tag{455}$$

with

$$\Delta F_{mn} = kF_{mn} = -\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)F_{mn}, \quad m,n \in \mathbb{N}. \tag{456}$$

(r,φ): With the condition that the angular functions Q be 2π-periodic, we obtain the following solutions for $p = n^2$, $n \in \mathbb{N}_0$:

$$Q_n^{(1)}(\varphi) = \cos(n\varphi), \quad Q_n^{(2)}(\varphi) = \sin(n\varphi), \quad n \in \mathbb{N}_0. \tag{457}$$

For the radial functions H, we need to go through the possible cases for the first separation constant $k \in \mathbb{R}$.

$k = \omega^2$, $\omega > 0$: The radial equation becomes

$$r^2H'' + rH' - \left(\omega^2r^2 + n^2\right)H = 0, \quad \text{in } [0,R). \tag{458}$$

After transformation $z := \omega r$, $w(z) := H(r)$, we obtain the modified Bessel's differential equation:

$$z^2w'' + zw' - \left(z^2 + n^2\right)w = 0, \quad \text{in } [0,\omega R). \tag{459}$$

The general solutions are given by the modified Bessel functions $\{I_n, K_n\}_{n\geq 0}$. If we require the solution to be bounded as $r \to 0$, and with the homogeneous Dirichlet boundary conditions at $r = R$, we find only the trivial solution $H \equiv 0$ in this case.

$k = 0$: The radial equation becomes a Cauchy-Euler ODE:

$$r^2 H'' + rH' - n^2 H = 0, \quad \text{in } [0, R), \tag{460}$$

with solutions $H_n(r) = A_n r^n + B_n r^{-n}$, $n \geq 0$ (Problem Set 5). Again, boundedness of the solution as $r \to 0$ and the homogeneous Dirichlet boundary condition at $r = R$ yields only the trivial solution $H \equiv 0$.

$k = -\omega^2$, $\omega > 0$: The radial equation becomes

$$r^2 H'' + rH' + \left(\omega^2 r^2 - n^2\right) H = 0, \quad \text{in } [0, R), \tag{461}$$

and after transformation $z := \omega r$, $w(z) := H(r)$, we obtain

$$z^2 w'' + zw' + \left(z^2 - n^2\right) H = 0, \quad \text{in } [0, kR). \tag{462}$$

General solutions are given by the Bessel functions $\{J_n, Y_n\}_{n\geq 0}$. Only the Bessel functions of the first kind are bounded as $z \to 0$. From the homogeneous Dirichlet boundary conditions, we obtain the equations

$$J_n(\omega R) = 0, \quad n \in \mathbb{N}_0. \tag{463}$$

There are infinitely many positive solutions ω_{mn}, $m \in \mathbb{N}$, for every $n \in \mathbb{N}_0$. So the radial eigenfunctions are given by $H_{mn}(r) = J_n(\omega_{mn} r)$, $r \in [0, R)$. Now the spatial eigenfunctions are given by

$$F^{(1)}_{mn}(r, \varphi) = J_n(\omega_{mn} r) \cos(n\varphi), \tag{464}$$
$$F^{(2)}_{mn}(r, \varphi) = J_n(\omega_{mn} r) \sin(n\varphi), \tag{465}$$

for $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, with

$$\Delta F^{(1,2)}_{mn} = -\omega^2_{mn} F^{(1,2)}_{mn}, \quad m \in \mathbb{N},\ n \in \mathbb{N}_0. \tag{466}$$

$d = 4$: For this case, we did not consider a time variable in the lecture, so we will only consider the case $k = 0$, for which we obtain the Laplace equation in 3D. We have considered three coordinate systems, depending on the shape of the spatial domain Ω: cartesian, cylindrical, and spherical.

(x, y, z): The Laplacian in cartesian coordinates is given by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{467}$$

The separation of variables is like in cartesian coordinates in 2D, and leads to a triple Fourier series in x, y, and z.

(r, φ, z): The Laplacian in cylindrical coordinates is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}. \tag{468}$$

The separation of variables leads to a Fourier series in z and a Fourier-Bessel series in r and φ, similar to polar coordinates in 2D.

(r, φ, ϑ): The Laplacian in spherical coordinates is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2}, \tag{469}$$

where the Laplace-Beltrami operator on the unit sphere $S^2 \subset \mathbb{R}^3$ is given by

$$\Delta_{S^2} = \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial}{\partial \vartheta}\right). \tag{470}$$

The eigenfunctions of Δ_{S^2} are given by the spherical harmonics

$$Y_{nm}(\varphi, \vartheta) := \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\vartheta)e^{im\varphi}, \quad 0 \le |m| \le n. \tag{471}$$

The functions $P_n^{|m|}$ are the associated Legendre polynomials, which reduce to the Legendre polynomials for $m = 0$. The spherical harmonics satisfy

$$\Delta_{S^2} Y_{nm} = -n(n+1)Y_{nm}, \quad 0 \le |m| \le n, \tag{472}$$

and the orthogonality relations for the spherical harmonics are given by

$$\int\limits_{S^2} Y_{nm}\overline{Y_{n'm'}}\,ds = \delta_{nn'}\delta_{mm'}, \quad 0 \le |m| \le n. \tag{473}$$

The radial functions $F_n(r)$ then satisfy the ODE

$$r^2 F_n'' + 2rF_n' - n(n+1)F_n = 0, \quad n \in \mathbb{N}_0. \tag{474}$$

This is a Cauchy-Euler ODE, with solutions given by

$$F_n(r) = r^n, \quad F_n^*(r) = r^{-(n+1)}, \quad n \in \mathbb{N}_0. \tag{475}$$

Eigenfunctions are now given by

$$u_{mn}(r,\varphi,\vartheta) = F_n(r)Y_{nm}(\varphi,\vartheta), \quad u_{mn}^*(r,\varphi,\vartheta) = F_n^*(r)Y_{nm}(\varphi,\vartheta), \tag{476}$$

for $0 \le |m| \le n$.

3. Superposition and Determination of Coefficients

$d = 2$: The general solution of the one-dimensional heat or wave equation is given by a superposition of the products $u_n(x,t) = F_n(x)G_n(t)$, where G_n satisfy the appropriate ODE. Evaluated as $t \to 0$, we obtain Fourier series in x, with coefficients to be determined from the initial data.

If the boundary conditions are *not mixed,* it is *faster* to use the Fourier series ansatz

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty}\left(a_n(t)\cos\left(\frac{n\pi}{L}x\right) + b_n(t)\sin\left(\frac{n\pi}{L}x\right)\right). \tag{477}$$

Depending on the boundary conditions, either the coefficients a_n or b_n will vanish, and we obtain ODEs in time for the remaining coefficients, by plugging into the PDE.

For the *wave equation with Dirichlet boundary conditions,* we obtain $a_n \equiv 0$, $n \ge 0$, and

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = \sum_{n=1}^{\infty}\left(\ddot{b}_n(t) + \left(\frac{cn\pi}{L}\right)^2 b_n(t)\right)\sin\left(\frac{n\pi}{L}x\right) = 0. \tag{478}$$

For the *heat equation with Neumann boundary conditions,* we obtain $b_n \equiv 0$, $n \geq 1$, and

$$\begin{aligned} u_t(x,t) - c^2 u_{xx}(x,t) &= \dot{a}_0(t) + \sum_{n=1}^{\infty} \left(\dot{a}_n(t) + \left(\frac{cn\pi}{L}\right)^2 a_n(t) \right) \cos\left(\frac{n\pi}{L}x\right) \\ &= 0. \end{aligned} \tag{479}$$

The initial values of the coefficients a_n, b_n are given as the Fourier coefficients of the initial data:

$$u(x,0) = a_0(0) + \sum_{n=1}^{\infty} \left(a_n(0) \cos\left(\frac{n\pi}{L}x\right) + b_n(0) \sin\left(\frac{n\pi}{L}x\right)\right), \tag{480}$$

$$u_t(x,0) = \dot{a}_0(0) + \sum_{n=1}^{\infty} \left(\dot{a}_n(0) \cos\left(\frac{n\pi}{L}x\right) + \dot{b}_n(0) \sin\left(\frac{n\pi}{L}x\right)\right), \tag{481}$$

where again either the coefficients a_n or b_n vanish due to the boundary conditions.

$d = 3$: Again, we consider both cartesian and polar coordinates.

(x, y): The general solution is given by a superposition of products

$$u_{mn}(x,y,t) = F_{mn}(x,y)G_{mn}(t). \tag{482}$$

Evaluated as $t \to 0$, we obtain double Fourier series in x and y, with coefficients determined from the initial data, similar to the $d = 2$ case.

Again, if the boundary conditions in, say, the x-variable are *not mixed,* it is faster to use a Fourier series ansatz of the form

$$u(x,y,t) = a_0(y,t) + \sum_{m=1}^{\infty} \left(a_n(y,t) \cos\left(\frac{m\pi}{a}x\right) + b_n(y,t) \sin\left(\frac{m\pi}{a}x\right)\right). \tag{483}$$

The same ansatz may be used for steady problems, i. e. the 2D Laplace equation; in that case of course, the coefficients a_n, b_n will depend only on y.

(r, φ): The general solution is given by a superposition of the products $u_{mn}(r, \varphi, t) = F^{(1)}_{mn}(r, \varphi) G_{mn}(t)$ and $u^*_{mn}(r, \varphi, t) = F^{(2)}_{mn}(r, \varphi) G^*_{mn}(t)$. Evaluated as $t \to 0$, we obtain Fourier-Bessel series in r and φ, the coefficients of which can again be evaluated from the initial data.

$d = 4$: The general solution of the Laplace equation in 3D is given by the superposition

$$u(r, \varphi, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(A_{mn} r^n + \frac{B_{mn}}{r^{n+1}} \right) Y_{mn}(\varphi, \vartheta), \tag{484}$$

where the coefficients are determined from the boundary data. These may also be given by conditions as $r \to 0$ or $r \to \infty$. For example, for the *interior problem* in the sphere $r < R$, we must have $B_{mn} = 0$, $0 \le |m| \le n$, so that the solution is bounded as $r \to 0$. In that case, we have

$$u^-(r, \varphi, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn} r^n Y_{nm}(\varphi, \vartheta), \quad r \le R,\ \varphi \in [0, 2\pi),\ \vartheta \in [0, \pi], \tag{485}$$

and the coefficients A_{mn} are determined from the boundary data:

$$A_{mn} = \frac{1}{R^n} \int_{S^2} u|_{r=R} \overline{Y_{nm}} \, ds. \tag{486}$$

12.14.4 Method of Characteristics (12.4)

A second-order quasilinear PDE in two variables is of the form

$$A(x, y) u_{xx}(x, y) + 2B(x, y) u_{xy}(x, y) + C(x, y) u_{yy}(x, y) \tag{487}$$
$$= F(x, y, u(x, y), u_x(x, y), u_y(x, y)). \tag{488}$$

The type of this PDE is determined by the discriminant $B^2 - AC$:

$$B^2 - AC > 0: \quad \text{hyperbolic} \tag{489}$$
$$B^2 - AC = 0: \quad \text{parabolic} \tag{490}$$
$$B^2 - AC < 0: \quad \text{elliptic} \tag{491}$$

we write down the characteristic ODE

$$A{y'}^2 - 2By' + C = 0. \tag{492}$$

With the solutions $y(x)$, we define new variables $v(x,y)$ and $w(x,y)$. If there is only one characteristic, we typically set $v(x,y) := x$ and define w using the characteristic.

For the new variables $v(x,y)$ and $w(x,y)$, the second partial derivatives of $u(x,y) = U(v(x,y), w(x,y))$ are obtained from the chain rule as

$$\begin{aligned} u_{xx} &= U_{vv}v_x^2 + 2U_{vw}v_xw_x + U_vv_{xx} + U_{ww}w_x^2 + U_ww_{xx}, & (493)\\ u_{xy} &= U_{vv}v_xv_y + U_{vw}\left(v_xw_y + v_yw_x\right) + U_vv_{xy} + U_{ww}w_xw_y + U_ww_{xy}, & (494)\\ u_{yy} &= U_{vv}v_y^2 + 2U_{vw}v_yw_y + U_vv_{yy} + U_{ww}w_y^2 + U_ww_{yy} & (495)\end{aligned}$$

Plugging this into the PDE will lead to its normal form, which should be easier to solve than the original PDE.
Example: Consider the PDE

$$u_{xx} + 2u_{xy} - u_{yy} = 0. \tag{496}$$

We have $A = B = 1$, $C = -1$, which implies $B^2 - AC = 2 > 0$, so this PDE is hyperbolic. The characteristic ODE is given by

$${y'}^2 - 2y' - 1 = \left(y' - (1+\sqrt{2})\right)\left(y' - (1-\sqrt{2})\right) = 0, \tag{497}$$

with solutions $y_{1,2}(x) = (1 \pm \sqrt{2})x + c_{1,2}$. We define the new variables v, w as

$$v(x,y) = -(1+\sqrt{2})x + y, \quad w(x,y) = -(1-\sqrt{2})x + y, \tag{498}$$

with partial derivatives

$$\begin{aligned} v_x &= -(1+\sqrt{2}),\ v_y = 1, \quad v_{xx} = v_{xy} = v_{yy} = 0, & (499)\\ w_x &= -(1-\sqrt{2}),\ w_y = 1, \quad w_{xx} = w_{xy} = w_{yy} = 0. & (500)\end{aligned}$$

We obtain from the PDE:

$$\begin{aligned} u_{xx} + 2u_{xy} - u_{yy} &= (3+2\sqrt{2})U_{vv} - 2U_{vw} + (3-2\sqrt{2})U_{ww} + & (501)\\ &\quad -(2+2\sqrt{2})U_{vv} - 4U_{vw} - (2-2\sqrt{2})U_{ww} + & (502)\\ &\quad -U_{vv} - 2U_{vw} - U_{ww} = -8U_{vw}, & (503)\end{aligned}$$

so that the normal form of the PDE is given by $U_{vw} = 0$. We integrate twice to obtain

$$U_v(v, w) = f(v), \quad U(v, w) = F(v) + G(w). \tag{504}$$

The general solution of the PDE is now given by

$$\begin{aligned} u(x, y) &= U(v(x, y), w(x, y)) = F(v(x, y)) + G(w(x, y)) & (505) \\ &= F(-(1 + \sqrt{2})x + y) + G(-(1 - \sqrt{2})x + y). & (506) \end{aligned}$$

12.14.5 Fourier Integrals (12.6)

For unbounded domains, we may need Fourier integrals instead of Fourier series, because the eigenvalues may not form a countable set anymore. We had looked at the Cauchy problem for the one-dimensional heat equation:

$$u_t = c^2 u_{xx}, \quad \text{in } \mathbb{R} \times (0, \infty), \tag{507}$$

$$u = f, \quad \text{on } \mathbb{R} \times \{0\}. \tag{508}$$

Separation of variables leads to the separate ODEs

$$F'' = kF, \quad \text{in } \mathbb{R}, \tag{509}$$

$$\dot{G} = c^2 kG, \quad \text{in } (0, \infty). \tag{510}$$

In this case, we require a bounded solution as $|x| \to \infty$, which excludes all but the trivial solution for $k \geq 0$. For $k = -\omega^2$, $\omega > 0$, on the other hand, we find uncountably many oscillatory solutions in space:

$$\begin{aligned} F(x; \omega) &= A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x), \quad x \in \mathbb{R}, & (511) \\ G(t; \omega) &= e^{-c^2 \omega t}, \quad t > 0. & (512) \end{aligned}$$

Eigenfunctions are now given by

$$\tilde{u}(x, t; \omega) = F(x; \omega) G(t; \omega), \quad \omega > 0, \tag{513}$$

and the superposition is an integral:

$$u(x, t) = \int_0^\infty (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x))\, e^{-c^2 \omega t}\, d\omega. \tag{514}$$

The coefficients A, B are determined from the initial data, f:

$$u(x,0) = \int_0^\infty \left(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right)\, d\omega \stackrel{!}{=} f(x), \quad x \in \mathbb{R}, \tag{515}$$

from which we conclude that

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\cos(\omega y)\, dy, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\sin(\omega y)\, dy. \tag{516}$$

After some manipulation we were able to write the solution u as an integral transform of the initial data f,

$$u(x,t) = \int_{-\infty}^{\infty} f(y) K(c^2 t, x, y)\, dy, \tag{517}$$

with the heat kernel

$$K(t,x,y) := \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right), \quad t > 0,\ x, y \in \mathbb{R}. \tag{518}$$

It turns out that $K(t,\cdot,\cdot)$ is a convolution kernel, $K(t,x,y) = \varphi(x-y,t)$, where φ denotes the Gauss-Weierstrass kernel.

13 Complex Numbers and Functions

Complex analysis is the theory of functions of a complex variable. Holomorphic functions are of particular interest; these are complex differentiable in every point $z \in U$ in a domain $U \subseteq \mathbb{C}$. If $f = u + iv$ is holomorphic, then the real and imaginary parts u, v, understood as functions of two real variables $(x, y) \in \mathbb{R}^2$, satisfy the Cauchy-Riemann equations, which is a system of two first-order partial differential equations. If, furthermore, u and v have continuous second partial derivatives, they are harmonic functions and thus solutions to Laplace's differential equation (cf. Chapter 12) in two space dimensions. Therefore, potential theory in 2D (to be treated in Chapter 18) is related to complex analysis, and this shall be the main motivation for its study in this lecture.

There are other useful results from complex analysis, such as that Fourier series may be written down in a somewhat simpler way by using complex exponentials, or that certain complicated integrals can be evaluated by the elegant method of complex integration, the calculus of residues (Chapter 16).

13.1 Complex Numbers. Complex Plane.

We follow the *Cayley-Dickson construction* to obtain the field of complex numbers, $\mathbb{C}$, from the field of real numbers, $\mathbb{R}$.

Ordered pairs of real numbers We consider ordered pairs of real numbers,

$$(x, y) \in \mathbb{R} \times \mathbb{R}. \tag{519}$$

Addition We define the addition $+$ of ordered pairs componentwise, i. e. by

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2). \tag{520}$$

The set of ordered pairs together with the addition just defined forms a commutative group:

1. Closure is obvious from the definition.

2. Associativity:

$$\begin{aligned}((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (x_1 + x_2, y_1 + y_2) + (x_3, y_3) &(521)\\ &= (x_1 + x_2 + x_3, y_1 + y_2 + y_3) &(522)\\ &= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) &(523)\\ &= (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &(524)\end{aligned}$$

3. Identity element: the pair $(0, 0) \in \mathbb{R} \times \mathbb{R}$ is the additive identity:

$$(0, 0) + (x, y) = (x, y) = (x, y) + (0, 0). \tag{525}$$

4. Inverse element: for a pair $(x, y) \in \mathbb{R} \times \mathbb{R}$, the additive inverse is given by $(-x, -y) \in \mathbb{R} \times \mathbb{R}$:

$$(x, y) + (-x, -y) = (0, 0) = (-x, -y) + (x, y). \tag{526}$$

5. Commutativity:

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) &(527)\\ &= (x_2, y_2) + (x_1, y_1). &(528)\end{aligned}$$

Remark: If we add a scalar product $\lambda(x, y) := (\lambda x, \lambda y)$, $\lambda \in \mathbb{R}$, we obtain a vector space, which is of course $\mathbb{R}^2$ with the standard basis $(1, 0)$, $(0, 1)$. Addition and subtraction of ordered pairs can thus be interpreted as addition and subtraction of vectors in $\mathbb{R}^2$.

Multiplication We define the multiplication $\cdot$ of ordered pairs by

$$(x_1, y_1) \cdot (x_2, y_2) \equiv (x_1, y_1)(x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2). \tag{529}$$

Field of Complex Numbers The set of ordered pairs together with the multiplication and addition defined above form a field:

1. Closure with respect to both addition and multiplication is obvious from the definitions.

2. Associativity: proven already for the addition. For the multiplication we obtain

$$
\begin{aligned}
&\left((x_1, y_1)(x_2, y_2)\right)(x_3, y_3) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)(x_3, y_3) && (530)\\
&= ((x_1x_2 - y_1y_2)x_3 - (x_1y_2 + y_1x_2)y_3, (x_1x_2 - y_1y_2)y_3 + (x_1y_2 + y_1x_2)x_3)\\
&= (x_1x_2x_3 - y_1y_2x_3 - x_1y_2y_3 - y_1x_2y_3, x_1x_2y_3 - y_1y_2y_3 + x_1y_2x_3 + y_1x_2x_3)\\
&= (x_1(x_2x_3 - y_2y_3) - y_1(x_2y_3 + y_2x_3), x_1(x_2y_3 + y_2x_3) + y_1(x_2x_3 - y_2y_3))\\
&= (x_1, y_1)(x_2x_3 - y_2y_3, x_2y_3 + y_2x_3) = (x_1, y_1)\left((x_2, y_2)(x_3, y_3)\right) && (531)
\end{aligned}
$$

3. Commutativity: already shown for the addition. For the multiplication we obtain

$$
\begin{aligned}
(x_1, y_1)(x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) && (532)\\
&= (x_2x_1 - y_2y_1, x_2y_1 + y_2x_1) && (533)\\
&= (x_2, y_2)(x_1, y_1) && (534)
\end{aligned}
$$

4. Identity elements: we already know the additive identity $(0, 0) \in \mathbb{R}\times\mathbb{R}$. The multiplicative identity is given by the pair $(1, 0) \in \mathbb{R} \times \mathbb{R}$:

$$(1, 0)(x, y) = (x, y) = (x, y)(1, 0). \qquad (535)$$

5. Inverse elements: we already know the additive inverse $(-x, -y) \in \mathbb{R}\times\mathbb{R}$ for a pair $(x, y) \in \mathbb{R}\times\mathbb{R}$. For $(x, y) \neq (0, 0)$ ("we can't divide by 0!"), the multiplicative inverse is given by $(x, y)^{-1} = (x/(x^2 + y^2), -y/(x^2 + y^2))$:

$$
\begin{aligned}
(x, y)\left(x/(x^2 + y^2), -y/(x^2 + y^2)\right) &= (1, 0) && (536)\\
&= \left(x/(x^2 + y^2), -y/(x^2 + y^2)\right)(x, y).
\end{aligned}
$$

6. Distributivity of multiplication over addition:

$$
\begin{aligned}
&(x_1, y_1)\left((x_2, y_2) + (x_3, y_3)\right) = (x_1, y_1)(x_2 + x_3, y_2 + y_3) && (537)\\
&= (x_1(x_2 + x_3) - y_1(y_2 + y_3), x_1(y_2 + y_3) + y_1(x_2 + x_3)) && (538)\\
&= (x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3, x_1y_2 + x_1y_3 + y_1x_2 + y_1x_3) && (539)\\
&= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) + (x_1x_3 - y_1y_3, x_1y_3 + y_1x_3) && (540)\\
&= (x_1, y_1)(x_2, y_2) + (x_1, y_1)(x_3, y_3) && (541)
\end{aligned}
$$

The *field of complex numbers* is denoted by $\mathbb{C}$. The fundamental theorem of algebra states that every non-constant polynomial in one variable with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$, i. e. $\mathbb{C}$ is *algebraically closed.*
Remark: The field of real numbers, $\mathbb{R}$, is not algebraically closed, because the polynomial x^2+1 (which is non-constant and has real coefficients) has no root in $\mathbb{R}$.

By construction, any complex number $z \in \mathbb{C}$ may be written as an ordered pair, $z = (x, y)$, and we call $x := \mathrm{Re}z \in \mathbb{R}$ the *real part* of z and $y := \mathrm{Im}z \in \mathbb{R}$ the *imaginary part* of z.

Complex Plane We go back to the vector space notion with the scalar product: if we identify the standard basis vectors with complex numbers,

$$(1,0) \equiv 1, \quad (0,1) \equiv i, \tag{542}$$

with the *imaginary unit* $i \in \mathbb{C}$, we may also write

$$z = (\mathrm{Re}z, \mathrm{Im}z) = (x, y) = x(1,0)+y(0,1) = x\cdot 1+y\cdot i = x+iy = \mathrm{Re}z+i\mathrm{Im}z. \tag{543}$$

We can visualize complex numbers as points in the complex plane, which is isomorphic to $\mathbb{R}^2$. By the definition of multiplication, we have

$$i^2 = i\cdot i = (0,1)(0,1) = (-1,0) = -(1,0) = -1. \tag{544}$$

Now we may compute sums and products of complex numbers just as we would for real numbers:

$$\begin{aligned}
z_1+z_2 &= (\mathrm{Re}z_1+i\mathrm{Im}z_1)+(\mathrm{Re}z_2+i\mathrm{Im}z_2) && (545)\\
&= (x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2), && (546)\\
&= (\mathrm{Re}z_1+\mathrm{Re}z_2)+i(\mathrm{Im}z_1+\mathrm{Im}z_2), && (547)\\
z_1z_2 &= (\mathrm{Re}z_1+i\mathrm{Im}z_1)(\mathrm{Re}z_2+i\mathrm{Im}z_2) && (548)\\
&= (x_1+iy_1)(x_2+iy_2) = (x_1x_2+ix_1y_2+iy_1x_2+i^2y_1y_2)\\
&= (x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2) && (549)\\
&= (\mathrm{Re}z_1\mathrm{Re}z_2-\mathrm{Im}z_1\mathrm{Im}z_2)+i(\mathrm{Re}z_1\mathrm{Im}z_2+\mathrm{Im}z_1\mathrm{Re}z_2). && (550)
\end{aligned}$$

We proceed in a similar way for the additive identity and inverse:

$$\begin{aligned}
(0,0) &= 0(1,0)+0(0,1) = 0\cdot 1+0\cdot i = 0, && (551)\\
-z &= -(\mathrm{Re}z, \mathrm{Im}z) = (-\mathrm{Re}z, -\mathrm{Im}z) = -\mathrm{Re}z - i\mathrm{Im}z. && (552)
\end{aligned}$$

Complex conjugate For a complex number $z \in \mathbb{C}$, we define the *complex conjugate* $z^* \equiv \overline{z} \in \mathbb{C}$ by

$$\overline{z} = \overline{(x,y)} := (x,-y) = x - iy = \mathrm{Re}z - i\mathrm{Im}z. \tag{553}$$

Now we may express the real and imaginary parts of a complex number $z \in \mathbb{C}$ by

$$\begin{aligned} \frac{1}{2}(z+\overline{z}) &= \frac{1}{2}(\mathrm{Re}z + i\mathrm{Im}z + \mathrm{Re}z - i\mathrm{Im}) = \mathrm{Re}z, & (554)\\ \frac{1}{2i}(z-\overline{z}) &= \frac{1}{2i}(\mathrm{Re}z + i\mathrm{Im}z - \mathrm{Re}z + i\mathrm{Im}z) = \mathrm{Im}z. & (555)\end{aligned}$$

Complex conjugation has the properties

$$\begin{aligned} \overline{z_1+z_2} &= (\mathrm{Re}z_1 + \mathrm{Re}z_2) - i(\mathrm{Im}z_1 + \mathrm{Im}z_2) & (556)\\ &= \mathrm{Re}z_1 - i\mathrm{Im}z_1 + \mathrm{Re}z_2 - i\mathrm{Im}z_2 = \overline{z_1} + \overline{z_2}, & (557)\\ \overline{z_1 z_2} &= (\mathrm{Re}z_1\mathrm{Re}z_2 - \mathrm{Im}z_1\mathrm{Im}z_2) - i(\mathrm{Re}z_1\mathrm{Im}z_2 + \mathrm{Im}z_1\mathrm{Re}z_2) & (558)\\ &= (\mathrm{Re}z_1 - i\mathrm{Im}z_1)(\mathrm{Re}z_2 - i\mathrm{Im}z_2) = \overline{z_1}\,\overline{z_2}, & (559)\\ \overline{\overline{z}} &= \overline{\mathrm{Re}z - i\mathrm{Im}z} = \mathrm{Re}z + i\mathrm{Im}z = z. & (560)\end{aligned}$$

Complex conjugation is simply identity for real numbers, which satisfy $\mathrm{Im}z = 0$. The following product is a non-negative number:

$$\overline{z}z = (x-iy)(x+iy) = x^2 - i^2y^2 = x^2 + y^2 = (\mathrm{Re}z)^2 + (\mathrm{Im}z)^2 \geq 0. \tag{561}$$

The number

$$|z| := \sqrt{\overline{z}z} = \sqrt{(\mathrm{Re}z)^2 + (\mathrm{Im}z)^2} \tag{562}$$

is the *absolute value* or *modulus* of $z \in \mathbb{C}$.

<u>Remark:</u> The vector space mentioned before becomes a normed vector space with the norm $|(x,y)| = \sqrt{x^2+y^2}$. This is, of course, the Euclidean norm of the vector $(x,y) \in \mathbb{R}^2$.

With the complex conjugate, we may write the multiplicative inverse in a simpler way:

$$z^{-1} = (x,y)^{-1} = \big(x/(x^2+y^2), -y/(x^2+y^2)\big) = \frac{(x,-y)}{|z|^2} = \frac{\overline{z}}{|z|^2}. \tag{563}$$

As with any group, the inverse operations *subtraction* and *division* of complex numbers are defined using addition and multiplication by

$$\begin{aligned} z_1 - z_2 &:= z_1 + (-z_2), \quad z_1, z_2 \in \mathbb{C} & (564)\\ \frac{z_1}{z_2} &:= z_1 z_2^{-1} = \frac{z_1\overline{z_2}}{|z_2|^2}, \quad z_1, z_2 \in \mathbb{C},\ z_2 \neq 0. & (565)\end{aligned}$$

13.2 Polar Form of Complex Numbers. Powers and Roots

The polar form of a complex number $z \in \mathbb{C}$ is motivated from using polar coordinates in the complex plane:

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad r \geq 0,\ \varphi \in [0, 2\pi). \tag{566}$$

Then we have

$$z = x + iy = r\left(\cos\varphi + i\sin\varphi\right), \tag{567}$$

with

$$|z| = \sqrt{x^2 + y^2} = r, \tag{568}$$

so the absolute value $|z|$ is just the distance of the point z from the origin in the complex plane. The *argument* of a complex number $z \in \mathbb{C} \setminus \{0\}$, $\arg z = \varphi$, is determined by the equations:

$$\cos\varphi = \frac{x}{r}, \quad \sin\varphi = \frac{y}{r}. \tag{569}$$

Notice that the argument φ is not uniquely determined by the equation $\tan\varphi = y/x$.

Triangle Inequality Complex numbers cannot be ordered, but we have the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|, \quad z_1, z_2 \in \mathbb{C}. \tag{570}$$

By induction, we may also show that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|, \quad z_k \in \mathbb{C},\ k = 1, \ldots, n. \tag{571}$$

These equations are, of course, the same as the triangle inequalities in $\mathbb{R}^2$.

Multiplication and Division in Polar Form The product of two complex numbers $z_1, z_2 \in \mathbb{C}$ in polar form is given by

$$\begin{aligned} z_1 z_2 &= r_1(\cos\varphi_1 + i\sin\varphi_1) r_2(\cos\varphi_2 + i\sin\varphi_2) && (572)\\ &= r_1 r_2(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + \\ &\quad + i r_1 r_2(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2) && (573)\\ &= r_1 r_2\left(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)\right), && (574)\end{aligned}$$

where we have used addition theorems for trigonometric functions. Therefore, the absolute value and angle of the product $z_1 z_2$ are given by

$$\begin{aligned} |z_1 z_2| &= r_1 r_2 = |z_1||z_2|, && (575)\\ \arg(z_1 z_2) &= \varphi_1 + \varphi_2 = \arg z_1 + \arg z_2. && (576)\end{aligned}$$

With $z_1 = z_2 = z$, and by induction, we conclude that

$$\begin{aligned} |z^n| &= |z|^n, && (577)\\ \arg(z^n) &= n\arg z. && (578)\end{aligned}$$

By comparing the two formulas for z^n, we obtain *De Moivre's formula:*

$$(\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi). \tag{579}$$

To obtain the absolute value and argument for the quotient, we write

$$z_1 = \frac{z_1}{z_2} z_2 \quad \Rightarrow \quad |z_1| = \left|\frac{z_1}{z_2}\right| |z_2| \tag{580}$$

and therefore we have

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}. \tag{581}$$

In a similar way we obtain for the argument:

$$\arg z_1 = \arg\left(\frac{z_1}{z_2}\right) + \arg z_2, \tag{582}$$

and thus

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{583}$$

Therefore, we may write the quotient z_1/z_2 in polar form as

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)\right). \tag{584}$$

Roots The n-th root of $z \in \mathbb{C}$, $n \in \mathbb{N}$, is the solution of the polynomial equation $w^n = z$. We write the numbers $w, z \in \mathbb{C}$ in polar form as

$$z = r(\cos\varphi + i\sin\varphi), \quad w = R(\cos\vartheta + i\sin\vartheta). \tag{585}$$

Then, with De Moivre's formula, we obtain

$$w^n = R^n(\cos(n\vartheta) + i\sin(n\vartheta)) = r(\cos\varphi + i\sin\varphi), \tag{586}$$

and thus

$$R^n = r, \quad \cos(n\vartheta) = \cos\varphi, \quad \sin(n\vartheta) = \sin\varphi. \tag{587}$$

There are n solutions, given by

$$R = r^{1/n} = \sqrt[n]{r}, \quad \vartheta_k = \frac{\varphi + 2k\pi}{n}, \; k = 0, \ldots, n-1. \tag{588}$$

Therefore, the n-th root of z is n-valued:

$$\begin{aligned} \sqrt[n]{z} &= w_k = \sqrt[n]{|z|}\left(\cos\left(\frac{\arg z + 2k\pi}{n}\right) + i\sin\left(\frac{\arg z + 2k\pi}{n}\right)\right) \qquad (589) \\ &= \sqrt[n]{|z|}\left(\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right)\left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right), \\ &= \sqrt[n]{|z|}\left(\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right)\left(\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right)^k, \end{aligned}$$

for $k = 0, \ldots, n-1$. In particular, for the *n-th roots of unity,* $\sqrt[n]{1}$, we obtain, with $|1| = 1$, $\arg 1 = 0$:

$$\sqrt[n]{1} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) = \left(\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right)^k, \tag{590}$$

for $k = 0, \ldots, n-1$. These points lie on the *unit circle* in the complex plane, and they form the vertices of a regular n-sided polygon.
Remark: It is a common mistake to just write down the solution for $k = 0$, and to neglect the other values of k, for example to give the solution of

$$x^2 = 5 \quad (n = 2) \text{ as } \quad x = \sqrt{5} \quad \text{instead of} \quad x = \pm\sqrt{5}. \tag{591}$$

13.3 Derivative. Holomorphic Function.

We want to consider functions $f : \mathbb{C} \to \mathbb{C}$. In order to analyze these functions, we need a few topological definitions. First of all, we notice that the function

$$d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}, \quad d(z_1, z_2) := |z_1 - z_2|, \tag{592}$$

with the absolute value $|\cdot|$ of a complex number defined as in (562), is a *metric* on $\mathbb{C}$, namely the metric induced by the norm $|\cdot|$ (Problem Set 7). Therefore, $(\mathbb{C}, d)$ is a metric space. We define the *open balls* (disks) with center $a \in \mathbb{C}$ and radius $\rho > 0$ by

$$B_\rho(a) := \{z \in \mathbb{C} \,|\, d(z, a) < \rho\}. \tag{593}$$

The open balls generate a *topology* on $\mathbb{C}$, making it a topological space. Any subset $U \subseteq \mathbb{C}$ inherits the topology from $\mathbb{C}$ and becomes a topological space itself. Standard topological definitions apply:

Definition 3 *Let $U \subseteq \mathbb{C}$ denote a subset with the topology inherited from $\mathbb{C}$.*

1. *U is* open *if every point $z \in U$ has an open neighborhood contained in U:*

$$\forall\, z \in U \;\exists\, \varepsilon > 0 : \quad B_\varepsilon(z) \subseteq U. \tag{594}$$

   Remark: *ε will depend on z in general.*

2. *U is* closed *if its complement $\mathbb{C} \setminus U$ is open.*

3. *U is* disconnected *if it is the union of two disjoint nonempty open sets:*

$$\exists\, \emptyset \neq V, W \subseteq \mathbb{C},\ V, W \text{ open},\ V \cap W = \emptyset : \quad U = V \cup W. \tag{595}$$

   Otherwise, U is connected.

4. *If U is both open and connected, it is called a* domain.

5. *The* closure *$\overline{U}$ of U consists of the points $z \in \mathbb{C}$ which are "close" to U:*

$$z \in \overline{U} \quad :\Leftrightarrow \quad \forall\, \varepsilon > 0 \;\exists\, w \in U : \quad w \in B_\varepsilon(z). \tag{596}$$

   Remark: *w will depend on ε in general. If $z \in U$, we may have $w = z$.*

6. *The* interior $\mathring{U}$ *of* U *consists of the points* $z \in U$ *which are "not on the edge" of* U*:*

$$z \in \mathring{U} \quad :\Leftrightarrow \quad \exists\, \varepsilon > 0 : \quad B_\varepsilon(z) \subseteq U. \tag{597}$$

<u>*Remark:*</u> *U open $\Leftrightarrow$ $U = \mathring{U}$ (compare Definitions 1 and 6). We also have the duality $\mathring{U} = \mathbb{C} \setminus \overline{\mathbb{C} \setminus U}$ and $\overline{U} = \mathbb{C} \setminus (\mathbb{C} \mathring{\setminus} U)$.*

7. *The* boundary *of* U*,* ∂U*, is the set of points which are in the closure of* U*, but not in the interior:* $\partial U := \overline{U} \setminus \mathring{U} = \overline{U} \cap \overline{\mathbb{C} \setminus U}$*. In other words, every open ball around a boundary point* $z \in \partial U$ *contains points from both* U *and* $\mathbb{C} \setminus U$*.*

Complex function For a function $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto w = f(z)$, we may split both the argument and the value into real and imaginary parts:

$$f(x + iy) = u(x, y) + iv(x, y), \tag{598}$$

with real-valued functions $u, v : \mathbb{R}^2 \to \mathbb{R}$.
<u>Examples:</u>

1. $f(z) := z^2 + 3z$. Then we have

$$f(x+iy) = (x + iy)^2 + 3\,(x + iy) = x^2 - y^2 + 3x + i\,(2xy + 3y)\,, \tag{599}$$

so that we find $u(x, y) = x^2 - y^2 + 3x$, $v(x, y) = 2xy + 3y$.

2. $f(z) := 2iz + 6\overline{z}$. Then we have

$$f(z) = 2i(x + iy) + 6(x - iy) = 6x - 2y + i\,(2x - 6y)\,, \tag{600}$$

and so we find $u(x, y) = 6x - 2y$, $v(x, y) = 2x - 6y$.

Limit, Continuity

Definition 4 *Let $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$, denote a complex function.*

1. *The* limit of f *as z approaches z_0 is L if $f(z)$ is "close" to L for all z "close" to z_0:*

$$\forall\, \varepsilon > 0\; \exists\, \delta > 0 : \quad f(z) \in B_\varepsilon(L) \quad \forall\, z \in B_\delta(z_0),\; z \neq z_0. \tag{601}$$

In this case we write

$$\lim_{z \to z_0} f(z) = L. \tag{602}$$

Remark: z_0 *need not be in the domain of* f*, nor does* L *need to be in the range of* f*.* z *may approach* z_0 *from any direction in the complex plane.*

2. f *is* continuous at $z_0 \in \mathbb{C}$ *if* $f(z_0)$ *is defined and*

$$\lim_{z \to z_0} f(z) = f(z_0). \tag{603}$$

3. f *is* continuous in the domain $U \subseteq \mathbb{C}$ *if it is continuous at all points* $z_0 \in U$*.*

Derivative

Definition 5 *Let* f *denote a complex function. The* derivative of f at $z_0 \in \mathbb{C}$ *is defined by*

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{604}$$

provided this limit exists. In this case, f *is said to be* differentiable at z_0.

Complex differentiability is a strong condition, because it means that on whatever path in the complex plane the value z approaches z_0, the quotient $(f(z) - f(z_0))/(z - z_0)$ always approaches the same value.
Examples:

1. The function $f(z) := z^2$ is differentiable for all $z_0 \in \mathbb{C}$ with derivative $f'(z_0) = 2z_0$:

$$\lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{(z + z_0)(z - z_0)}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0. \tag{605}$$

2. The function $f(z) := \overline{z}$ is not differentiable at any point $z_0 \in \mathbb{C}$: for any two points $z, z_0 \in \mathbb{C}$, we have

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\overline{z} - \overline{z_0}}{z - z_0} = \frac{\overline{z - z_0}}{z - z_0} = \frac{\mathrm{Re}(z - z_0) - i\mathrm{Im}(z - z_0)}{\mathrm{Re}(z - z_0) + i\mathrm{Im}(z - z_0)} =: w. \tag{606}$$

If $\mathrm{Re}(z-z_0) = 0$, we have $w = -1$, and if $\mathrm{Im}(z-z_0) = 0$, we have $w = 1$. Therefore, we may construct a sequence $\{z_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} z_n \to z_0$, so that the sequence $\{w_n\}_{n\in\mathbb{N}}$ oscillates between 1 and -1, i. e. is not convergent.

If f is differentiable at z_0, it must be continuous at z_0, but the opposite is not true. The rules for differentiation are the same as for real functions (product rule, quotient rule, chain rule).

Holomorphic functions

Definition 6 *A function f which is defined and differentiable at every point $z \in U \subseteq \mathbb{C}$ is called* holomorphic in U*. A* holomorphic function *is a function that is holomorphic in some domain $U \subseteq C$. A function which is holomorphic in all of $\mathbb{C}$ is an* entire function.

Example:

1. Monomials z^n, $n \in \mathbb{N}_0$, are entire functions and so are polynomials, i. e. functions of the form

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n, \quad c_0, \ldots, c_n \in \mathbb{C}. \tag{607}$$

2. Rational functions

$$f(z) := \frac{p(z)}{q(z)}, \tag{608}$$

where p, q are polynomials, are differentiable everywhere except at the zeros of q, assuming that common factors have been canceled.

13.4 Cauchy-Riemann Equations. Laplace's Equation.

The Cauchy-Riemann equations are a system of two linear first-order partial differential equations which provide a criterion for differentiability of a complex function:

Theorem 2 *If a complex function $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic in $U \subseteq \mathbb{C}$, then the first partial derivatives of u and v exist and they satisfy the Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x, \quad in\ U. \tag{609}$$

Theorem 3 *If two real-valued continuous functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ have* continuous *first partial derivatives that satisfy the Cauchy-Riemann equations (609) in some domain $U \subseteq \mathbb{C}$, then the complex function $f(x+iy) := u(x,y) + iv(x,y)$ is holomorphic in U.*

Remark: valid under weaker conditions (Looman, 1923; Menchoff, 1936)
Sketch of Proofs:

- Theorem 2: For a point $z_0 \in U$, we know that the derivative of f is given by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{610}$$

  (and that this limit exists). We choose two particular paths $z \to z_0$ such that

$$\begin{aligned} f'(x_0 + iy_0) &= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}, \\ f'(x_0 + iy_0) &= -i \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{y - y_0} + \lim_{y \to y_0} \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0}. \end{aligned}$$

  From these equations, we conclude that the first partial derivatives of u, v exist at (x_0, y_0) and also that

$$u_x(x_0, y_0) = v_y(x_0, y_0), \quad u_y(x_0, y_0) = -v_x(x_0, y_0). \tag{611}$$

  This is true for any point $z_0 = x_0 + iy_0 \in U$.
  Remark: We also conclude from this proof that

$$f'(x_0 + iy_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0). \tag{612}$$

- Theorem 3: Choose $z_0 \in U$. Because U is open, it contains an open ball around $z_0 = (x_0, y_0)$, and therefore we may find a second point $z = (x, y) \in U$ such that the line segment which connects the two points lies entirely in U. Because of the differentiability of u and v, we may apply the mean value theorem in $\mathbb{R}^2$. It guarantees the existence of points $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathbb{R}^2$ on the line segment connecting the points (x_0, y_0) and (x, y), such that

$$\begin{aligned} u(x,y) - u(x_0, y_0) &= (x - x_0) u_x(\boldsymbol{m}_1) + (y - y_0) u_y(\boldsymbol{m}_1), && (613) \\ v(x,y) - v(x_0, y_0) &= (x - x_0) v_x(\boldsymbol{m}_2) + (y - y_0) v_y(\boldsymbol{m}_2). && (614) \end{aligned}$$

Then we have, with the Cauchy-Riemann equations (609),

$$
\begin{aligned}
f(z) - f(z_0) &= u(x,y) - u(x_0,y_0) + i\left(v(x,y) - v(x_0,y_0)\right) && (615) \\
&= (x-x_0)u_x(\boldsymbol{m}_1) + (y-y_0)u_y(\boldsymbol{m}_1) + \\
&\quad +i\left((x-x_0)v_x(\boldsymbol{m}_2) + (y-y_0)v_y(\boldsymbol{m}_2)\right) && (616) \\
&\overset{(609)}{=} (x-x_0)u_x(\boldsymbol{m}_1) - (y-y_0)v_x(\boldsymbol{m}_1) + \\
&\quad +i\left((x-x_0)v_x(\boldsymbol{m}_2) + (y-y_0)u_x(\boldsymbol{m}_2)\right). && (617)
\end{aligned}
$$

With $z - z_0 = x - x_0 + i(y - y_0)$ we obtain

$$
\begin{aligned}
\frac{f(z)-f(z_0)}{z-z_0} &= u_x(\boldsymbol{m}_1) + iv_x(\boldsymbol{m}_1) + \\
&\quad -i\frac{y-y_0}{z-z_0}\left(u_x(\boldsymbol{m}_1) - u_x(\boldsymbol{m}_2)\right) + \\
&\quad -i\frac{x-x_0}{z-z_0}\left(v_x(\boldsymbol{m}_1) - v_x(\boldsymbol{m}_2)\right). && (618)
\end{aligned}
$$

We let $z \to z_0$ and with the continuity of the partial derivatives of u and v we obtain

$$
\begin{aligned}
\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} &= u_x(x_0,y_0) + iv_x(x_0,y_0) && (619) \\
&= \lim_{x\to x_0} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i \lim_{x\to x_0} \frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}.
\end{aligned}
$$

We have shown existence of the limit for a path along the line connecting z and z_0. Because z was arbitrarily chosen in an open ball around z_0, f is differentiable at z_0. This is true for any $z_0 \in U$, and therefore f is holomorphic in U. $\square$

Examples:

1. $f(z) = z^2$ is an entire function. We have $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$. The partial derivatives are given by

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x, \qquad (620)$$

so u, v satisfy the Cauchy-Riemann equations (609) everywhere in $\mathbb{R}^2$

2. $f(z) = \overline{z} = x - iy$, so that $u(x,y) = x$, $v(x,y) = -y$. We obtain

$$u_x = 1 = -v_y, \quad u_y = 0 = -v_x, \tag{621}$$

so the first Cauchy-Riemann equation is not satisfied. By Theorem 2, f cannot be differentiable. We have verified that before, in Section 13.3.

3. $f(z) = e^x(\cos y + i\sin y)$. We find $u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$ and therefore

$$u_x = e^x \cos y = v_y, \quad u_y = -e^x \sin y = -v_x. \tag{622}$$

This is true in any subset $U \subseteq \mathbb{C}$. By Theorem 3, f is an entire function.

4. Assume that a function f is holomorphic in $U \subseteq \mathbb{C}$ and satisfies $|f(z)| = k = \text{const.}$, $z \in U$. We want to show that f must be a constant function.

We know that $|f(x+iy)|^2 = u(x,y)^2 + v(x,y)^2 = k^2$. We differentiate with respect to x and y to obtain

$$uu_x + vv_x = 0, \quad uu_y + vv_y = 0. \tag{623}$$

By Theorem 2, u and v must satisfy the Cauchy-Riemann equations (609), so that

$$uu_x - vu_y = 0, \quad uu_y + vu_x = 0. \tag{624}$$

We may eliminate u_y from the first equation and u_x from the second and obtain

$$(u^2 + v^2)u_x = 0, \quad (u^2 + v^2)u_y = 0. \tag{625}$$

Now if $u^2 + v^2 = k^2 = 0$, we have $u \equiv v \equiv 0$ and thus $f \equiv 0$. If $k^2 \neq 0$, we conclude that $u_x \equiv u_y \equiv 0$. With the Cauchy-Riemann equations, we also obtain that $v_x \equiv v_y \equiv 0$. Therefore $u \equiv \text{const.}$, $v \equiv \text{const.}$ and hence $f = u + iv \equiv \text{const.}$.

In polar form, $f(r(\cos\varphi + i\sin\varphi)) = u(r,\varphi) + iv(r,\varphi)$, and the Cauchy-Riemann equations are given by (Problem Set 8)

$$u_r = \frac{1}{r}v_\varphi, \quad v_r = -\frac{1}{r}u_\varphi. \tag{626}$$

Laplace's Equation. Harmonic Functions The real and imaginary parts of a holomorphic function satisfy Laplace's equation:

Theorem 4 *If $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic in a domain $U \subseteq \mathbb{C}$, then both u and v satisfy Laplace's equation:*

$$\Delta u = u_{xx} + u_{yy} = 0, \quad \Delta v = v_{xx} + v_{yy} = 0, \quad \textit{in } U, \tag{627}$$

and they have continuous second partial derivatives in U.

Sketch of proof: Taking second partial derivatives of u, we obtain

$$u_{xx} = (v_y)_x = v_{yx}, \quad u_{yy} = -(v_x)_y = -v_{xy}. \tag{628}$$

We shall prove in Chapter 14 that the derivative of a holomorphic function is again holomorphic, which implies that u and v have continuous partial derivatives of any order. This means that we can exchange the order of differentiation: $v_{yx} = v_{xy}$, and therefore $u_{xx} = -u_{yy}$. Similar for v. $\square$

A *harmonic function* is a twice continuously differentiable function which satisfies Laplace's equation. Harmonic functions are studied in *potential theory.* Because the real and imaginary parts of a holomorphic function in $\mathbb{C}$ are harmonic functions in $\mathbb{R}^2$, complex analysis is useful for potential theory in two space dimensions (to be treated in Chapter 18).

Let a function u be defined in some set $\Omega \subset \mathbb{R}^2$. The function v is a *harmonic conjugate* of u if $f(x+iy) := u(x,y) + iv(x,y)$ is holomorphic.
Example: $u(x,y) := x^2 - y^2 - y$. We have $u_x = 2x$, $u_y = -2y - 1$, and $u_{xx} = 2 = -u_{yy}$, so that u is a harmonic function. We want to solve the Cauchy-Riemann equations (609) for the partial derivatives of v:

$$v_x \stackrel{!}{=} -u_y = 2y + 1, \quad v_y \stackrel{!}{=} u_x = 2x. \tag{629}$$

We obtain by integration of the second equation:

$$v(x,y) = 2xy + h(x) \quad \Rightarrow \quad v_x(x,y) = 2y + h'(x), \tag{630}$$

and from the first equation we conclude that $h'(x) \stackrel{!}{=} 1$. Therefore, $h(x) = x + c$, $c \in \mathbb{R}$, and $v(x,y) = 2xy + x + c$. The corresponding holomorphic function is given by

$$f(x+iy) = x^2 - y^2 - y + i\,(2xy + x + c) = (x+iy)^2 + i(x + iy + c), \tag{631}$$

or $f(z) = z^2 + i(z + c)$, $z \in \mathbb{C}$. The conjugate of a given harmonic function is uniquely determined up to an arbitrary additive constant.

13.5 Exponential Function

The *exponential function* $\exp : \mathbb{C} \to \mathbb{C}$ is defined by

$$\exp(x+iy) \equiv e^{x+iy} := e^x(\cos y + i \sin y) = e^x \cos y + ie^x \sin y. \qquad (632)$$

We shall now list several important properties of this function:

- extension of real exponential function: $z = x \in \mathbb{R} \Rightarrow e^z = e^x$
- exp is an entire function (Section 13.4, Example 3)
- derivative: $\exp'(z) = \exp(z)$

$$\begin{aligned}\exp'(x+iy) &= (e^x \cos y)_x + i\,(e^x \sin y)_x = e^x \cos y + ie^x \sin y \\ &= e^x(\cos y + i \sin y) = \exp(x+iy) \qquad (633)\end{aligned}$$

- $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$, $z_1, z_2 \in \mathbb{C}$
  - Special case $z_1 := x \in \mathbb{R}$, $z_2 := iy$, $y \in \mathbb{R}$, then $e^{x+iy} = e^x e^{iy}$. Compare with the definition (632) $\Rightarrow$ *Euler's formula*

$$e^{iy} = \cos y + i \sin y, \quad y \in \mathbb{R}. \qquad (634)$$

  Then the polar form of $z \in \mathbb{C}$ may also be written as

$$z = r(\cos\varphi + i \sin\varphi) = re^{i\varphi}. \qquad (635)$$

  - Absolute value

$$|e^{iy}| = \sqrt{\cos^2 y + \sin^2 y} = 1, \quad \left|e^{x+iy}\right| = e^x > 0. \qquad (636)$$

  This implies that the function exp never vanishes.
  - Set $r = 1$ and choose specific values for φ:

$$e^{i\frac{\pi}{2}} = i, \quad e^{i\pi} = -1, \quad e^{i\frac{3\pi}{2}} = -i, \quad e^{2\pi i} = 1. \qquad (637)$$

  - Periodicity with period $2\pi i$:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z, \quad z \in \mathbb{C}. \qquad (638)$$

  All the values that $\exp(z)$ can assume are already assumed in the *fundamental region* Imz $\in (-\pi, \pi]$ (or any other horizontal strip of width 2π in the complex plane). This implies that exp is not an injective (one-to-one) function.

13.6 Trigonometric and Hyperbolic Functions

With the complex exponential function, trigonometric and hyperbolic functions may also be generalized to the complex numbers: for $z \in \mathbb{C}$, we define

$$\cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z := \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \tag{639}$$

as well as

$$\cosh z := \frac{1}{2}\left(e^{z} + e^{-z}\right), \quad \sinh z := \frac{1}{2}\left(e^{z} - e^{-z}\right). \tag{640}$$

Since the complex exponential is entire, so are the functions cos, sin, cosh, and sinh. We may also consider quotients such as

$$\tan z := \frac{\sin z}{\cos z}, \quad \tanh z := \frac{\sinh z}{\cosh z}. \tag{641}$$

The derivatives of these functions are given like for the real functions, namely by

$$\cos' z = -\sin z, \qquad \sin' z = \cos z, \tag{642}$$
$$\cosh' z = \sinh z, \qquad \sinh' z = \cosh z. \tag{643}$$

We determine the real and imaginary parts from the definitions:

$$\begin{aligned}
\cos(x+iy) &= \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) = \frac{1}{2}\left(e^{ix}e^{-y} + e^{-ix}e^{y}\right) && (644)\\
&= \frac{1}{2}e^{-y}\left(\cos x + i\sin x\right) + \frac{1}{2}e^{y}\left(\cos x - i\sin x\right) && (645)\\
&= \cos x\frac{1}{2}\left(e^{y} + e^{-y}\right) - i\sin x\frac{1}{2}\left(e^{y} - e^{-y}\right) && (646)\\
&= \cos x\cosh y - i\sin x\sinh y. && (647)
\end{aligned}$$

In the same way, we find

$$\begin{aligned}
\sin(x+iy) &= \sin x\cosh y + i\cos x\sinh y, && (648)\\
\cosh(x+iy) &= \cosh x\cos y + i\sinh x\sin y, && (649)\\
\sinh(x+iy) &= \sinh x\cos y + i\cosh x\sin y. && (650)
\end{aligned}$$

Absolute values are given by

$$
\begin{aligned}
|\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y && (651)\\
&\overset{\cosh^2 y - \sinh^2 y = 1}{=} \cos^2 x(1 + \sinh^2 y) + \sin^2 x \sinh^2 y && (652)\\
&= \cos^2 x + (\cos^2 x + \sin^2 x)\sinh^2 y && (653)\\
&= \cos^2 x + \sinh^2 y, && (654)
\end{aligned}
$$

and in the same way

$$
\begin{aligned}
|\sin z|^2 &= \sin^2 x + \sinh^2 y && (655)\\
|\cosh z|^2 &= \cosh^2 x - \sin^2 y && (656)\\
|\sinh z|^2 &= \sinh^2 x + \sin^2 y && (657)
\end{aligned}
$$

Notice that the complex trigonometric functions sin, cos are not bounded. All zeros of sin, cos lie on the real axis, because the only solution of $\sinh y = 0$ is $y = 0$. The following formulas are also true for the complex trigonometric functions:

$$
\begin{aligned}
\cos(z_1 \pm z_2) &= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2, && (658)\\
\sin(z_1 \pm z_2) &= \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2, && (659)
\end{aligned}
$$

and

$$\cos^2 z + \sin^2 z = 1. \qquad (660)$$

The following relations between the complex trigonometric and hyperbolic functions do not exist for their real counterparts:

$$
\begin{aligned}
\cosh(iz) &= \cos z, \qquad \sinh(iz) = i \sin z, && (661)\\
\cos(iz) &= \cosh z, \qquad \sin(iz) = i \sinh z. && (662)
\end{aligned}
$$

13.7 Logarithm. General Power

The *natural logarithm* $w = \log z$ (sometimes denoted by $\ln z$) is defined as the inverse of the exponential function: $e^w = z$. We already know that $e^w \neq 0$

$\forall\, w \in \mathbb{C}$, and therefore log is undefined at $z = 0$. For $w = u + iv$, $z = re^{i\varphi}$, $r > 0$, we obtain the equations

$$e^u e^{iv} = re^{i\varphi}. \tag{663}$$

The solutions u, v are given by

$$u = \log r, \quad v = \varphi + 2\pi n,\ n \in \mathbb{Z}. \tag{664}$$

The natural logarithm of z is multivalued: $w = \log r + i(\varphi + 2\pi n)$, $n \in \mathbb{Z}$. The *principal value,* $\mathrm{Log}z$, is the one which lies in the fundamental region of the exponential function: $\mathrm{Im}(\mathrm{Log}z) \in (-\pi, \pi]$. Then the other values of $\log z$ are given by

$$\log z = \mathrm{Log}z + 2\pi i n, \quad n \in \mathbb{Z}. \tag{665}$$

Theorem 5 *The natural logarithm* log *(665) is holomorphic except for* $z \leq 0$*. Its derivative is given by*

$$\log' z = \frac{1}{z}, \quad z \neq 0,\ \arg z \neq \pi. \tag{666}$$

The negative real axis $z < 0$ *is called the* branch cut *of* log*.*

Proof: We know that the logarithm cannot be defined for $z = 0$. For all other values $z = re^{i\varphi} \in \mathbb{C}$, $r > 0$, $\varphi \in [0, 2\pi)$, the principal value of the logarithm is given by

$$\mathrm{Log}z = \begin{cases} \log r + i\varphi, & \varphi \in [0, \pi] \\ \log r + i(\varphi - 2\pi), & \varphi \in (\pi, 2\pi) \end{cases}. \tag{667}$$

Therefore, log is not continuous at $\varphi = \pi$, and thus not holomorphic for $z < 0$. For values $z \in \mathbb{C} \setminus \{z \leq 0\}$ we have, in polar form

$$\log\left(re^{i\varphi}\right) = \log r + i\left(\varphi + 2\pi n\right) = u(r, \varphi) + iv(r, \varphi), \quad n \in \mathbb{Z}, \tag{668}$$

with $u(r, \varphi) := \log r$, $v(r, \varphi) := \varphi + 2\pi n$. We verify the Cauchy-Riemann equations

$$u_r = \frac{1}{r} = \frac{1}{r} v_\varphi, \quad v_r = 0 = -\frac{1}{r} u_\varphi, \tag{669}$$

so log is holomorphic in $\mathbb{C} \setminus \{z \leq 0\}$, by Theorem 3. For the derivative, we find

$$\begin{aligned} \log'\left(re^{i\varphi}\right) &= \cos\varphi u_r(r,\varphi) - \frac{1}{r}\sin\varphi u_\varphi(r,\varphi) + i\cos\varphi v_r(r,\varphi) - i\frac{1}{r}\sin\varphi v_\varphi(r,\varphi) \\ &= \frac{1}{r}\cos\varphi - i\frac{1}{r}\sin\varphi = \frac{1}{r}e^{-i\varphi}, \end{aligned} \tag{670}$$

so that $\log' z = 1/z$. $\square$

The *general power* of $z \in \mathbb{C}$ is defined by

$$z^c := e^{c \log z}, \quad z, c \in \mathbb{C},\ z \neq 0. \tag{671}$$

For $c = n$ and $c = 1/n$, $n \in \mathbb{N}$, we obtain the n-th power and n-th root of $z \in \mathbb{C}$, respectively (Section 13.2).

14 Complex Integration

For functions of real numbers $f : \mathbb{R} \to \mathbb{R}$, we know *indefinite integrals* (or antiderivatives)

$$F(x) = \int f(x)\,dx, \quad F'(x) = f(x), \tag{672}$$

and *definite integrals* over an interval $[a, b] \subset \mathbb{R}$:

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{673}$$

(fundamental theorem of calculus). For complex functions $f : \mathbb{C} \to \mathbb{C}$, indefinite integrals are defined in the same way, and definite integrals generalize to line integrals in the complex plane.

14.1 Line Integral in the Complex Plane

Curves in $\mathbb{C}$ A curve C in $\mathbb{C}$ is represented by a continuous mapping (path) $\gamma : [a, b] \to \mathbb{C}$, $a, b \in \mathbb{R}$, as

$$C = \{\gamma(t) \,|\, t \in [a, b]\}. \tag{674}$$

Examples:

- $\gamma(t) := t+3it$, $t \in [0, 2]$, parametrizes a segment of the line $\mathrm{Im}z = 3\mathrm{Re}z$.
- $\gamma(t) := 4\cos t + 4i \sin t$, $t \in [0, 2\pi)$, parametrizes the circle $|z| = 4$.

Notice that the parametrization for a given curve C is not unique. We denote by $-C$ the curve oriented in the opposite direction:

$$-C := \{\gamma(a + b - t) \,|\, t \in [a, b]\}. \tag{675}$$

The *arc length* of a curve C is defined by

$$L(C) := \sup_{\substack{a=t_0<t_1<\cdots<t_{n-1}<t_n=b \\ n\in\mathbb{N}}} \sum_{m=1}^{n} \left|\gamma(t_m) - \gamma(t_{m-1})\right|. \tag{676}$$

The arc length is an intrinsic property of the curve C, i. e. it does not depend on the choice of the parametrization γ. The curve C is *rectifiable* if $L(C) < \infty$. This is true for Lipschitz-continuous functions γ, for example.

Definition of the line integral For a given $n \in \mathbb{N}$, we choose $n+1$ points in the interval $[a, b]$,

$$a = t_0 < t_1 < \cdots < t_n = b. \tag{677}$$

With the mapping γ we obtain $n+1$ points $z_m := \gamma(t_m) \in \mathbb{C}$, $m = 0, \ldots, n$. Now we can divide the curve C into n segments:

$$C = \bigcup_{m=1}^{n} C_m, \quad C_m = \{\gamma(t) \,|\, t \in [t_{m-1}, t_m]\},\ m = 1, \ldots, n. \tag{678}$$

Consider a rectifiabe curve C. For a continuous complex function f which is defined (at least) at each point $z \in C$, and for any choice of points $\zeta_m \in C_m$, $m = 1, \ldots, n$, we define the sums

$$S_n := \sum_{m=1}^{n} f(\zeta_m)\,(z_m - z_{m-1})\,. \tag{679}$$

Remark: S_n depends on the partition of the interval $[a, b]$ and on the choices of $\zeta_1, \ldots, \zeta_n$.
The line integral of f over the path of integration C is defined by

$$\int_C f(z)\,dz := \lim_{n\to\infty} S_n. \tag{680}$$

Under the assumptions that f is continuous and C is rectifiable, the limit on the right-hand side of (680) exists. It turns out that it is independent of the choice of points ζ_m, $m = 1, \ldots, n$, and also that the line integral is independent of the choice of the parametrization γ of C. If the curve C is *closed,* $\gamma(b) = \gamma(a)$, then we also write

$$\oint_C f(z)\,dz. \tag{681}$$

Basic Properties

1. Linearity:

$$\int_C \left(k_1 f_1(z) + k_2 f_2(z)\right)\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz. \tag{682}$$

2. Sense reversal:
$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz. \tag{683}$$

3. Partitioning of path:
$$\int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \tag{684}$$

Evaluation via antiderivative This evaluation method requires a complex function which is holomorphic on a simply connected domain.

Definition 7 *1. A subset $U \subseteq \mathbb{C}$ is* path-connected (0-connected) *if for any two points $z_0, z_1 \in U$ there is a (continuous) path $\gamma : [0,1] \to U$ with $\gamma(0) = z_0$, $\gamma(1) = z_1$.*

2. A path-connected subset $U \subseteq \mathbb{C}$ is simply connected (1-connected) *if any path from $z_0 \in U$ to $z_1 \in U$ can be continuously transformed into any other such path, while staying within U, for any $z_0, z_1 \in U$.*

Remarks:

- Path-connected sets are connected, but not every connected set is path-connected.
- Simply connected sets are necessarily path-connected, and therefore also connected.
- Any subset of $\mathbb{C}$ with a hole is not simply connected.

A complex function which is holomorphic on a simply connected, open set $U \subseteq \mathbb{C}$ has an antiderivative:

Theorem 6 *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic in a simply connected domain U. Then the antiderivative F of f exists ($F'(z) = f(z)\ \forall\, z \in U$) and is also holomorphic. Furthermore, for any path in U joining two points $z_0, z_1 \in U$, we have*
$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0). \tag{685}$$

Proof: Section 14.2 (Cauchy's Integral Theorem)

Remark: We write $\int\limits_{z_0}^{z_1}$ instead of $\int\limits_C$ in this case, because the value of the integral is the same for any curve C from z_0 to z_1.

Examples:

1. $U = \mathbb{C}$, $f(z) = z^2$:

$$\int\limits_0^{1+i} z^2 \, dz = \left. \frac{z^3}{3} \right|_0^{1+i} = \frac{1}{3}(i+1)^3 = -\frac{2}{3} + \frac{2}{3} i. \tag{686}$$

2. $U = \mathbb{C} \setminus \{z \leq 0\}$, $f(z) = 1/z$:

$$\int\limits_{-i}^{i} \frac{1}{z} \, dz = \mathrm{Log} i - \mathrm{Log}(-i) = \frac{i\pi}{2} - \left(-\frac{i\pi}{2} \right) = i\pi. \tag{687}$$

Evaluation via parametrization This method is not restricted to holomorphic functions, nor does it require special topological properties.

Theorem 7 *Let $C = \{\gamma(t) \,|\, t \in [a,b]\}$ be a regular piecewise C^1-curve (i. e. γ continuous in $[a,b]$, piecewise continuously differentiable in (a,b) and $\dot{\gamma}(t) \neq 0 \; \forall \, t \in (a,b)$), and let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function on C. Then*

$$\int\limits_C f(z) \, dz = \int\limits_a^b f(\gamma(t)) \, \dot{\gamma}(t) \, dt. \tag{688}$$

Proof: We write $f(x+iy) = u(x,y) + iv(x,y)$, $z_m = x_m + iy_m$, $m = 0, \ldots, n$, $\zeta_m = \xi_m + i\eta_m$, $m = 1, \ldots, n$. Then, for any $n \in \mathbb{N}$:

$$\begin{aligned} S_n &= \sum_{m=1}^{n} \left(u(\xi_m, \eta_m) + iv(\xi_m, \eta_m)\right)\left(x_m - x_{m-1} + i(y_m - y_{m-1})\right) && (689) \\ &= \sum_{m=1}^{n} \left(u(\xi_m, \eta_m)(x_m - x_{m-1}) - v(\xi_m, \eta_m)(y_m - y_{m-1})\right) + \\ &\quad + i \sum_{m=1}^{n} \left(v(\xi_m, \eta_m)(x_m - x_{m-1}) + u(\xi_m, \eta_m)(y_m - y_{m-1})\right) && (690) \\ &= \sum_{m=1}^{n} \begin{pmatrix} u(\xi_m, \eta_m) \\ -v(\xi_m, \eta_m) \end{pmatrix} \cdot \begin{pmatrix} x_m - x_{m-1} \\ y_m - y_{m-1} \end{pmatrix} + \\ &\quad + i \sum_{m=1}^{n} \begin{pmatrix} v(\xi_m, \eta_m) \\ u(\xi_m, \eta_m) \end{pmatrix} \cdot \begin{pmatrix} x_m - x_{m-1} \\ y_m - y_{m-1} \end{pmatrix}. && (691) \end{aligned}$$

With $\gamma = \lambda + i\mu$, we define $\boldsymbol{r}(t) := (\lambda(t), \mu(t))^\top$. The function $\boldsymbol{r}$ parametrizes a curve in $\mathbb{R}^2$, which contains the points $(\xi_m, \eta_m)^\top$, $m = 1, \ldots, n$, and $(x_m, y_m)^\top$, $m = 0, \ldots, n$. We also denote this curve by C. We also define the vector fields

$$\boldsymbol{F}_1(x,y) := \begin{pmatrix} u(x,y) \\ -v(x,y) \end{pmatrix}, \quad \boldsymbol{F}_2(x,y) := \begin{pmatrix} v(x,y) \\ u(x,y) \end{pmatrix} \tag{692}$$

Then the sum S_n (691) converges to a line integral in $\mathbb{R}^2$, as $n \to \infty$:

$$\lim_{n\to\infty} S_n = \int_C \boldsymbol{F}_1(\boldsymbol{r}) \cdot d\boldsymbol{r} + i \int_C \boldsymbol{F}_2(\boldsymbol{r}) \cdot d\boldsymbol{r}. \tag{693}$$

Because $\boldsymbol{r}$ is also a regular piecewise C^1 function, we may write

$$\lim_{n\to\infty} S_n = \int_C f(z)\, dz = \int_a^b \boldsymbol{F}_1(\boldsymbol{r}(t)) \cdot \dot{\boldsymbol{r}}(t)\, dt + i \int_a^b \boldsymbol{F}_2(\boldsymbol{r}(t)) \cdot \dot{\boldsymbol{r}}(t)\, dt. \tag{694}$$

Using the definitions, we write

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b u(\lambda(t),\mu(t))\dot{\lambda}(t)\,dt - \int_a^b v(\lambda(t),\mu(t))\dot{\mu}(t)\,dt + \\
&\quad + i\int_a^b v(\lambda(t),\mu(t))\dot{\lambda}(t)\,dt + i\int_a^b u(\lambda(t),\mu(t))\dot{\mu}(t)\,dt \quad (695)\\
&= \int_a^b \left(u(\lambda(t),\mu(t)) + iv(\lambda(t),\mu(t))\right)\left(\dot{\lambda}(t)+i\dot{\mu}(t)\right)\,dt \quad (696)\\
&= \int_a^b f(\gamma(t))\dot{\gamma}(t)\,dt \qquad \Box \quad (697)
\end{aligned}$$

<u>Examples:</u>

1. $f(z) = 1/z$, $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$ (C is the unit circle in the complex plane, oriented counterclockwise). Then $\dot{\gamma}(t) = ie^{it}$, and therefore

$$\oint_C \frac{1}{z}\,dz = \int_0^{2\pi} e^{-it} i e^{it}\,dt = i\int_0^{2\pi} 1\,dt = 2\pi i. \qquad (698)$$

   <u>Remark:</u> $1/z$ is holomorphic in $D := \mathbb{C}\setminus\{0\}$, but D is not simply connected. Therefore, Theorem 6 (which yields zero for the integral over any closed path) is not applicable!

2. $z_0 \in \mathbb{C}$, $f(z) = (z-z_0)^m$, $m \in \mathbb{Z}$, $\gamma(t) = z_0 + \rho e^{it}$, $\rho > 0$, $t \in [0, 2\pi]$. That is, $C = \{z \in \mathbb{C}\,|\,|z - z_0| = \rho\}$ is the circle around z_0 with radius ρ. We obtain

$$\begin{aligned}
f(\gamma(t)) &= \left(z_0 + \rho e^{it} - z_0\right)^m = \rho^m e^{imt}, \quad (699)\\
\dot{\gamma}(t) &= i\rho e^{it}. \quad (700)
\end{aligned}$$

Then

$$\oint_C (z - z_0)^m \, dz = \int_0^{2\pi} \rho^m e^{imt} i\rho e^{it} \, dt = i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t} \, dt \tag{701}$$

$$= i\rho^{m+1} \left(\int_0^{2\pi} \cos\left((m+1)t\right) \, dt + i \int_0^{2\pi} \sin\left((m+1)t\right) \, dt \right)$$

$$= \left\{ \begin{array}{ll} 2\pi i, & m = -1 \\ 0, & m \in \mathbb{Z} \setminus \{-1\} \end{array} \right. . \tag{702}$$

3. If Theorem 6 is applicable, then the value of the integral is independent of the path connecting the points $z_0, z_1 \in \mathbb{C}$. This is not true in general: Consider the function $f(z) = \mathrm{Re} z$, which is continuous but not holomorphic (Thm. 2), and two curves C_1, C_2 represented by

$$\gamma_1(t) := t + 2it, \quad t \in [0,1], \tag{703}$$

$$\gamma_2(t) := \left\{ \begin{array}{ll} t, & t \in [0,1) \\ 1 + 2i(t-1), & t \in [1,2] \end{array} \right. . \tag{704}$$

Then

$$f(\gamma_1(t)) = t, \ t \in [0,1], \quad f(\gamma_2(t)) = \left\{ \begin{array}{ll} t, & t \in [0,1) \\ 1, & t \in [1,2] \end{array} \right. . \tag{705}$$

Both curves connect the points 0 and $1 + 2i$ in the complex plane, and both are regular and piecewise C^1:

$$\dot\gamma_1(t) = 1 + 2i, \ t \in [0,1], \quad \dot\gamma_2(t) = \left\{ \begin{array}{ll} 1, & t \in [0,1) \\ 2i, & t \in [1,2] \end{array} \right. . \tag{706}$$

Then, by Theorem 7:

$$\int\limits_{C_1} f(z)\,dz = \int\limits_0^1 f(\gamma_1(t))\dot{\gamma}_1(t)\,dt = \int\limits_0^1 t(1+2i)\,dt, \tag{707}$$

$$= (1+2i)\,\frac{t^2}{2}\bigg|_0^1 = \frac{1}{2} + i, \tag{708}$$

$$\int\limits_{C_2} f(z)\,dz = \int\limits_0^2 f(\gamma_2(t))\dot{\gamma}_2(t)\,dt = \int\limits_0^1 t\,dt + \int\limits_1^2 2i\,dt \tag{709}$$

$$= \frac{t^2}{2}\bigg|_0^1 + 2i\,t|_1^2 = \frac{1}{2} + 2i, \tag{710}$$

so we have

$$\int\limits_{C_1} f(z)\,dz \neq \int\limits_{C_2} f(z)\,dz. \tag{711}$$

Bounds for Integrals. ML-Inequality

Proposition 2 *If f is bounded on C, i. e. $\exists\, M > 0$: $|f(z)| \leq M\ \forall\, z \in C$, then*

$$\left|\int\limits_C f(z)\,dz\right| \leq ML, \tag{712}$$

where $L = L(C)$ denotes the arc length of the curve C.

Proof: For the sums (679), we obtain ($z_m = \gamma(t_m)$)

$$|S_n| \leq \sum_{m=1}^{n} |f(\zeta_m)|\,|\gamma(t_m) - \gamma(t_{m-1})| \leq M \sum_{m=1}^{n} |\gamma(t_m) - \gamma(t_{m-1})| \tag{713}$$

$$\leq ML(C), \tag{714}$$

by the definition of the arc length (676). $\square$

Example: $f(z) = z^2$, $\gamma(t) = t + it$, $t \in [0,1]$. $L(C) = |1+i| = \sqrt{2}$, and

$$|f(\gamma(t))| = |(1+i)t|^2 = 2t^2 \leq 2, \quad t \in [0,1]. \tag{715}$$

Therefore the line integral of f over C can be bounded as

$$\left| \int_C z^2 \, dz \right| \leq 2\sqrt{2}. \tag{716}$$

With Theorem 6, we obtain the exact value

$$\int_C z^2 \, dz = \left. \frac{z^3}{3} \right|_0^{1+i} = \frac{(1+i)^3}{3} = -\frac{2}{3} + \frac{2}{3} i, \tag{717}$$

and therefore

$$\left| \int_C z^2 \, dz \right| = \frac{2\sqrt{2}}{3}. \tag{718}$$

14.2 Cauchy's Integral Theorem

This is the main theorem in this Chapter, and it will allow us to prove Thm. 6.

Theorem 8 *Let $U \subseteq \mathbb{C}$ be open and simply connected, let the complex function f be holomorphic in U, and let C be a rectifiable, closed curve in U. Then*

$$\oint_C f(z) \, dz = 0. \tag{719}$$

Proof: We write $f(x+iy) = u(x,y) + iv(x,y)$. This proof uses Green's theorem, which requires that the partial derivatives of u and v are continuous. Note that Thm. 2 implies only existence of these partial derivatives, but not continuity (Goursat's proof works without assuming continuity of f', but it's more complicated)! We know from the proof of Thm. 7 that

$$\oint_C f(z) \, dz = \int_C (u \, dx - v \, dy) + i \int_C (v \, dx + u \, dy) \tag{720}$$

By Green's theorem

$$\int_C (u\,dx - v\,dy) = \int_U (-v_x - u_y)\; dx\,dy, \tag{721}$$

$$\int_C (v\,dx + u\,dy) = \int_U (u_x - v_y)\; dx\,dy. \tag{722}$$

From the Cauchy-Riemann equations (Thm. 2) $u_x = v_y$, $u_y = -v_x$, we conclude that (721), (722) vanish. $\square$

<u>Examples</u>

1. If f is an entire function, the integral over any closed curve C vanishes:

$$\oint_C e^z\,dz = 0, \quad \oint_C \cos z\,dz = 0, \quad \oint_C z^n\,dz = 0,\; n \in \mathbb{N}_0. \tag{723}$$

2. Unit circle $C = \{z \in \mathbb{C}\,|\,|z| = 1\}$. Functions which are not entire but holomorphic on and inside of C: $\tan z$ is holomorphic except at $(2k+1)\pi/2$, $k \in \mathbb{Z}$, $(z^2+4)^{-1}$ is holomorphic except at $z = \pm 2i$. These points lie outside of C, and therefore

$$\oint_C \tan z\,dz = 0, \quad \oint_C \frac{1}{z^2+4}\,dz = 0. \tag{724}$$

3. $\gamma(t) = e^{it}$, $f(z) = \overline{z}$ is not holomorphic, and therefore Thm. 8 is not applicable. We compute, with Thm. 7:

$$\oint_C \overline{z}\,dz = \int_0^{2\pi} e^{-it} i e^{it}\,dt = 2\pi i \neq 0. \tag{725}$$

4. Domains which are not simply connected: $f(z) := z^{-1}$ is holomorphic in $\mathbb{C} \setminus \{0\}$, but this domain is not simply connected. We have (Example 1 after Thm. 7, Section 14.1), for $\gamma(t) = e^{it}$:

$$\oint_C \frac{1}{z}\,dz = 2\pi i \neq 0. \tag{726}$$

Independence of Path

Theorem 9 *If the complex function f is holomorphic in a simply connected domain $U \subseteq \mathbb{C}$, then the integral of f is* independent of path *in U, i. e. its value depends only on the two endpoints of C.*

Proof: Let z_1, z_2 be any points in U. Consider two curves C_1, C_2 from z_1 to z_2 which do not intersect each other (so z_1, z_2 are the only common points of C_1 and C_2). The path from z_1 to z_2 on C_1 and back from z_2 to z_1 on $-C_2$ is closed, and from Thm. 8 we know that

$$\int_{C_1} f\,dz + \int_{-C_2} f\,dz = 0 \quad \Rightarrow \quad \int_{C_1} f\,dz = -\int_{-C_2} f\,dz = \int_{C_2} f\,dz. \tag{727}$$

For paths with a finite number of common points, we may integrate over loops between these points. □

Principle of Deformation of Path From the path independence (Thm. 9), we conclude that the integral of f over any continuous transformation of a curve C is the same as the integral of f over C, as long as the deforming path contains only points at which f is holomorphic.
Example: We know from Example 2 after Thm. 7 (Section 14.1) that

$$\oint_C (z - z_0)^m\,dz = \begin{cases} 2\pi i, & m = -1 \\ 0, & m \in \mathbb{Z} \setminus \{-1\} \end{cases}, \tag{728}$$

where C is a circle in the complex plane with center z_0 and radius $\rho > 0$. This is, in fact, true for any closed curve enclosing z_0.

Existence of Indefinite Integral We may now prove Thm. 6: The conditions of Thm. 8 are satisfied. The line integral of f from any $z_0 \in U$ to any $z \in U$ is independent of path in U (Thm. 9). For a fixed $z_0 \in U$, we define

$$F(z) := \int_{z_0}^{z} f(\zeta)\,d\zeta. \tag{729}$$

We show that F is holomorphic in U and that $F'(z) = f(z)$. We form the difference quotient

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left(\int\limits_{z_0}^{z+\Delta z} f(\zeta)\,d\zeta - \int\limits_{z_0}^{z} f(\zeta)\,d\zeta\right) = \frac{1}{\Delta z}\int\limits_{z}^{z+\Delta z} f(\zeta)\,d\zeta. \tag{730}$$

Because U is open, we may choose Δz such that the curve segment from z to $z + \Delta z$ lies inside of U. With

$$\frac{1}{\Delta z}\int\limits_{z}^{z+\Delta z} f(z)\,d\zeta = f(z)\frac{1}{\Delta z}\int\limits_{z}^{z+\Delta z} d\zeta = \frac{z+\Delta z - z}{\Delta z} f(z) = f(z), \tag{731}$$

we obtain

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int\limits_{z}^{z+\Delta z} \left(f(\zeta) - f(z)\right)\,d\zeta. \tag{732}$$

f is holomorphic, and therefore continuous, at z (Problem Set 7). Let $\varepsilon > 0$ be given, then

$$\exists\,\delta > 0: \quad |f(\zeta) - f(z)| < \varepsilon \quad \forall\,|\zeta - z| < \delta. \tag{733}$$

We choose Δz such that $|\Delta z| < \delta$, and with the ML-inequality (Prop. 2), we obtain

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int\limits_{z}^{z+\Delta z} \left(f(\zeta) - f(z)\right)\,d\zeta\right| \le \frac{1}{|\Delta z|}\varepsilon|\Delta z| = \varepsilon. \tag{734}$$

By the definition of the limit and of the derivative, this proves that

$$F'(z) = \lim_{\Delta z\to 0}\frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z). \tag{735}$$

$z \in U$ was arbitrary, and therefore F is holomorphic in U and is an antiderivative of f. It remains to show that we may use any antiderivative to evaluate (685). Indeed, if we have another holomorphic function G with $G'(z) = f(z)\ \forall\, z \in U$, then $F'(z) - G'(z) = 0$ in U, hence $F - G$ is constant in U. Now

$$F(z_1) - F(z_0) - (G(z_1) - G(z_0)) = F(z_1) - G(z_1) - (F(z_0) - G(z_0)) = 0, \tag{736}$$

for two points $z_0, z_1 \in U$, and so (685) holds for any antiderivative of f. $\square$

Cauchy's Integral Theorem for Multiply Connected Domains If a complex function f is holomorphic in a domain U^* containing a doubly connected domain U with boundary curves C_1, C_2 (both with the same orientation), then U can be cut into two simply connected domains U_1 and U_2, in each of which Thm. 8 is applicable:

$$\oint_{\partial U_1} f(z)\,dz = \oint_{\partial U_2} f(z)\,dz = 0, \tag{737}$$

where both integrals are taken counterclockwise, for example. If we take the sum of these two integrals, the integrals over the cuts will cancel, and we obtain

$$0 = \oint_{C_1} f(z)\,dz + \oint_{-C_2} f(z)\,dz = \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz, \tag{738}$$

so that

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz. \tag{739}$$

This procedure can be generalized to any multiply connected domain.

14.3 Cauchy's Integral Formula

This is the most important consequence from Cauchy's Integral Theorem (Thm. 8), and it is useful for evaluating integrals. Furthermore, it will help us prove that holomorphic functions have derivatives of all orders (Section 14.4), and that they are analytic (Section 15.4).

Theorem 10 *Let the complex function f be holomorphic in a simply connected domain U. Then for any point z_0 in U and any closed curve C in U that encloses z_0,*

$$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0), \tag{740}$$

the integration being taken counterclockwise.

Remark: The domain U does not need to be simply connected, as long as the curve C can be continuously transformed into an arbitrarily small circle

around z_0, while staying inside of U.
<u>Proof:</u> We write

$$\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)\oint_C \frac{1}{z-z_0}\,dz + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz \tag{741}$$

$$= 2\pi i f(z_0) + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{742}$$

It remains to prove that

$$\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0. \tag{743}$$

The integrand is holomorphic in $U \setminus \{z_0\}$. We define the closed curve $K_\rho := \{z \in \mathbb{C} \,|\, |z-z_0| = \rho\}$, $\rho > 0$, oriented counterclockwise. For $\rho > 0$ small enough, we have, with the principle of deformation of path:

$$\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = \oint_{K_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{744}$$

The function f is holomorphic, and therefore continuous, at z_0. Let $\varepsilon > 0$ be given. Then

$$\exists\,\delta > 0: \quad |f(z)-f(z_0)| < \varepsilon \quad \forall\,|z-z_0| < \delta,\, z \neq z_0. \tag{745}$$

If we choose $0 < \rho < \delta$, we obtain

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\varepsilon}{\rho}, \quad \text{on } K_\rho. \tag{746}$$

From the ML-Inequality (Prop. 2), we obtain

$$\left|\oint_{K_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \frac{\varepsilon}{\rho}2\pi\rho = 2\pi\varepsilon. \tag{747}$$

$\varepsilon > 0$ was arbitrary, so the integral must vanish. $\square$
<u>Remark:</u> We already know from Cauchy's Integral Theorem (Thm. 8) that the integral in (740) vanishes if C does not enclose z_0.
<u>Examples:</u>

1. $z_0 = 2$, $f(z) = e^z$:

$$\oint\limits_C \frac{e^z}{z-2}\,dz = 2\pi i e^z\big|_{z=2} = 2\pi i e^2 \simeq 46.4i, \tag{748}$$

for any closed curve C enclosing $z = 2$.

2. $z_0 = i/2$, $f(z) = (z^3/2 - 3)/(z - i/2)$:

$$\oint\limits_C \frac{z^3-6}{2z-i}\,dz = \oint\limits_C \frac{\frac{1}{2}z^3 - 3}{z - \frac{i}{2}}\,dz = 2\pi i \left(\frac{1}{2}z^3 - 3\right)\bigg|_{z=i/2} = \frac{\pi}{8} - 6\pi i, \tag{749}$$

for any closed curve C enclosing $z = i/2$.

3.

$$\oint\limits_C \frac{z^2+1}{z^2-1}\,dz = \oint\limits_C \frac{z^2+1}{(z+1)(z-1)}\,dz \tag{750}$$

Four cases:

a) C encloses $z = 1$ but not $z = -1$:

$$\oint\limits_C \frac{z^2+1}{z^2-1}\,dz = \oint\limits_C \frac{\frac{z^2+1}{z+1}}{z-1}\,dz = 2\pi i \left.\frac{z^2+1}{z+1}\right|_{z=1} = 2\pi i, \tag{751}$$

because $f(z) := (z^2+1)/(z+1)$ is holomorphic in any set U with $-1 \notin U$.

b) C encloses $z = -1$ but not $z = 1$:

$$\oint\limits_C \frac{z^2+1}{z^2-1}\,dz = \oint\limits_C \frac{\frac{z^2+1}{z-1}}{z+1}\,dz = 2\pi i \left.\frac{z^2+1}{z-1}\right|_{z=-1} = -2\pi i, \tag{752}$$

because $f(z) := (z^2+1)/(z-1)$ is holomorphic in any set U with $1 \notin U$.

c) Both points $z = 1$ and $z = -1$ lie outside of C:

$$\oint\limits_C \frac{z^2+1}{z^2-1}\,dz = 0, \tag{753}$$

because $f(z) := (z^2+1)/(z^2-1)$ is holomorphic in any set U with $\{-1, 1\} \not\subseteq U$.

d) Both points $z = 1$ and $z = -1$ lie inside of C: We draw two closed curves C_2, C_3 around $z = 1$ and $z = -1$, respectively. With Cauchy's integral theorem we obtain

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = \oint_{C_2} \frac{z^2+1}{z^2-1}\,dz + \oint_{C_3} \frac{z^2+1}{z^2-1}\,dz \qquad (754)$$

$$= \oint_{C_2} \frac{\frac{z^2+1}{z+1}}{z-1}\,dz + \oint_{C_3} \frac{\frac{z^2+1}{z-1}}{z+1}\,dz \qquad (755)$$

$$= 2\pi i - 2\pi i = 0, \qquad (756)$$

from cases a) and b).

14.4 Derivatives of Holomorphic Functions

Cauchy's Integral Formula (Thm. 10) may be used to show that holomorphic functions have derivatives of any order, which is different from the situation in real calculus. This is Cauchy's differentiation formula:

Theorem 11 *If the complex function f is holomorphic in a domain $U \subseteq \mathbb{C}$, then it has derivatives of all orders in U, which are then also holomorphic functions in U. The values of these derivatives at a point $z_0 \in U$ are given by*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad n \in \mathbb{N}. \qquad (757)$$

Here, C is any closed curve in U that encloses z_0 and whose full interior belongs to U; and we integrate counterclockwise around C.

Remark: For $n = 0$, this is Cauchy's Integral Formula (Thm. 10). Derivatives for $n \geq 1$ are obtained by formally differentiating under the integral sign, with respect to z_0.
Proof: We know from Thm. 10 that we have a contour integral representation of f at any point $z_0 \in U$:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\,dz. \qquad (758)$$

Choose a point $z_0 \in U$. The derivative of f at z_0 is given by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}, \tag{759}$$

if the limit exists. With (758) we have

$$\begin{aligned}
\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{1}{2\pi i \Delta z} \left(\oint_C \frac{f(z)}{z - (z_0 + \Delta z)}\, dz - \oint_C \frac{f(z)}{z - z_0}\, dz \right) \\
&= \frac{1}{2\pi i \Delta z} \oint_C \frac{(z - z_0) f(z) - (z - z_0 - \Delta z) f(z)}{(z - z_0 - \Delta z)(z - z_0)}\, dz \\
&= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}\, dz. \qquad (760)
\end{aligned}$$

Now we compute

$$\oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}\, dz - \oint_C \frac{f(z)}{(z - z_0)^2}\, dz = \oint_C \frac{\Delta z f(z)}{(z - z_0 - \Delta z)(z - z_0)^2}\, dz. \tag{761}$$

We now show that this integral approaches 0 as $\Delta z \to 0$. f is continuous on C, and therefore bounded: $\exists\, K > 0 : |f(z)| \leq K\ \forall\, z \in C$. Let d be the miminum distance from z_0 to any point on C:

$$d := \min_{z \in C} |z - z_0| \quad \Rightarrow \quad \frac{1}{|z - z_0|^2} \leq \frac{1}{d^2} \quad \forall\, z \in C \tag{762}$$

Furthermore, with the triangle inequality,

$$d \leq |z - z_0| \leq |z - z_0 - \Delta z| + |\Delta z|, \quad \forall\, z \in C. \tag{763}$$

For $|\Delta z| \leq d/2$ we have

$$\frac{d}{2} \leq d - |\Delta z| \leq |z - z_0 - \Delta z| \quad \Rightarrow \quad \frac{1}{|z - z_0 - \Delta z|} \leq \frac{2}{d}, \quad \forall\, z \in C. \tag{764}$$

Let $L := L(C)$ denote the arc length of the curve C. For $|\Delta z| \leq d/2$, we use the ML-inequality to obtain

$$\left| \oint_C \frac{\Delta z f(z)}{(z - z_0 - \Delta z)(z - z_0)^2}\, dz \right| \leq |\Delta z| K \frac{2}{d} \frac{1}{d^2} = \frac{2K}{d^3} |\Delta z| \to 0, \quad \Delta z \to 0. \tag{765}$$

This proves that $f'(z_0)$ exists, and is given by

$$f'(z_0) = \frac{1}{2\pi i} \oint\limits_C \frac{f(z)}{(z-z_0)^2}\, dz. \tag{766}$$

Therefore f' may be represented at z_0 as a contour integral. This is true for any $z_0 \in U$. We repeat the previous argument with f replaced by f' and (758) replaced by (766) to obtain a contour integral representation of f'' at any $z_0 \in U$. Thus we can prove Cauchy's differentiation formula for any $n \in \mathbb{N}$. $\square$

Examples:

1. For any contour enclosing the point πi (counterclockwise):

$$\oint\limits_C \frac{\cos z}{(z-\pi i)^2}\, dz = 2\pi i\, (\cos z)'\big|_{z=\pi i} = -2\pi i \sin(\pi i) = 2\pi \sinh \pi. \tag{767}$$

2. For any countour which encloses 1 but $\pm 2i$ lie outside (counterclockwise):

$$\begin{aligned} \oint\limits_C \frac{e^z}{(z-1)^2(z^2+4)}\, dz &= 2\pi i \left(\frac{e^z}{z^2+4}\right)'\Bigg|_{z=1} = 2\pi i\, \frac{e^z(z^2+4) - e^z 2z}{(z^2+4)^2}\Bigg|_{z=1} \\ &= \frac{6e\pi}{25} i \simeq 2.05i. \end{aligned} \tag{768}$$

Cauchy's Inequality. Liouville's and Morera's Theorems Cauchy's differentiation formula (Thm. 11) has several interesting corollaries. We choose $C := \{z \in \mathbb{C} \,|\, |z - z_0| = r\}$, for an $r > 0$. With $|f(z)| \leq M$, $z \in C$, we obtain using the ML-inequality

$$\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi} \left| \oint\limits_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right| \leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n! M}{r^n}. \tag{769}$$

Equation (769) is Cauchy's inequality.

Theorem 12 *(Liouville) If an entire function is bounded in absolute value in the whole complex plane, then this function must be constant.*

Proof: Assume that $|f(z)| \leq K$ for all $z \in \mathbb{C}$. For a $z_0 \in \mathbb{C}$, we conclude from Cauchy's inequality that $|f'(z_0)| \leq K/r$. This holds for every $r > 0$, because f is entire. It follows that $f'(z_0) = 0$. Since z_0 is arbitrary, we have $f'(z) = 0$ for all $z \in \mathbb{C}$, hence $u_x = v_x = 0$, and $u_y = v_y = 0$, by Thm. 2. Thus $u \equiv$ const. and $v \equiv$ const., and $f = u + iv \equiv$ const.. $\square$

Theorem 13 *(Morera) If a complex function f is continuous in a simply connected domain U and if*

$$\oint\limits_C f(z)\,dz = 0 \tag{770}$$

for every closed curve C in U, then f is holomorphic in U.

Remark: This is the converse of Cauchy's Integral Theorem (Thm. 8).
Proof: In Section 14.2, we had derived the antiderivative

$$F(z) = \int\limits_{z_0}^{z} f(\zeta)\,d\zeta, \quad z \in U, \tag{771}$$

which is holomorphic in U and satisfies $F'(z) = f(z)$. By Thm. 11, F' is holomorphic in U, which implies that f is holomorphic in U. $\square$

15 Power Series, Taylor Series

Chapters 15 and 16 deviate from the "main road", which leads us from PDEs via Complex Analysis to Potential Theory in 2D (Chapter 18). This separate path will lead to the technique of Residue Integration (Section 16.3), which is useful, for example, to evaluate certain complicated real integrals (Section 16.4).

In this chapter, we will introduce series and (complex) analytic functions, and we will prove that every holomorphic function is analytic.

15.1 Sequences, Series, Convergence Tests

These concepts are very similar as in real calculus. We will go through them rather quickly and not present all of the proofs in detail.

Definition 8 1. *A* sequence $\{z_n\}_{n\geq 1}$ *is a function from* $\mathbb{N}$ *to* $\mathbb{C}$. *The numbers* $z_n \in \mathbb{C}$, $n \in \mathbb{N}$, *are the* terms *of the sequence.*

2. *A sequence* $\{z_n\}_{n\geq 1}$ *is* convergent *if it has a limit, i. e. there is a* $c \in \mathbb{C}$ *so that* $\lim_{n\to\infty} z_n = c$, *or*

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} : \quad |z_n - c| < \varepsilon \quad \forall n > N. \tag{772}$$

Otherwise the sequence is called divergent.

3. *A* Cauchy sequence *is a sequence* $\{z_n\}_{n\geq 1}$ *with the property that the terms are getting closer and closer together:*

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} : \quad |z_m - z_n| < \varepsilon \quad \forall m, n > N. \tag{773}$$

Remark: Every convergent sequence is a Cauchy sequence. In fact, the space $(\mathbb{C}, d)$, with the metric d induced by the absolute value $|\cdot|$, is a *complete* metric space, so that the opposite is also true!
Examples:

1. The sequence $\{i^n/n\}_{n\geq 1}$ is convergent with limit 0.
2. The sequences $\{i^n\}_{n\geq 1}$ and $\{(1+i)^n\}_{n\geq 1}$ are divergent.

Definition 9 *1. Given a sequence $\{z_n\}_{n\geq 1}$, we call the numbers*

$$s_n := \sum_{m=1}^{n} z_m = z_1 + z_2 + \cdots + z_n, \quad n \in \mathbb{N}, \tag{774}$$

the n-th partial sums *of the* series

$$\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots \tag{775}$$

The complex numbers $z_1, z_2, \ldots$ are the terms *of the series. The partial sums $\{s_n\}_{n\geq 1}$ form a new sequence.*

2. *A series is* convergent *if its sequence of partial sums $\{s_n\}_{n\geq 1}$ converges, say $\lim_{n\to\infty} s_n = s \in \mathbb{C}$. In that case, we call s the* sum *of the series, and we write*

$$s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots \tag{776}$$

Otherwise the series is called divergent.

3. *A series $z_1 + z_2 + \cdots$ is called* absolutely convergent *if the series*

$$\sum_{m=1}^{\infty} |z_m| \tag{777}$$

is convergent. A series which is convergent but not absolutely convergent is called conditionally convergent.

4. *The series*

$$R_n := \sum_{m=n+1}^{\infty} z_m = z_{n+1} + z_{n+2} + \cdots, \quad n \in \mathbb{N}, \tag{778}$$

is called the remainder *of the series (775) after the term z_n.*

Remarks:

1. For convergent series, we have $R_n = s - s_n \to 0$, $n \to 0$.
2. Due to completeness, every absolutely convergent series is convergent (triangle inequality and Cauchy criterion).

Example: The alternating harmonic series $\sum_{n\geq 1} ((-1)^{n+1}/n)$ converges, but not absolutely.

Tests for Convergence and Divergence of Series The following criterion is commonly used to establish divergence of a series:

Theorem 14 *If a series $z_1 + z_2 + \cdots$ converges, then* $\lim_{n\to\infty} z_n = 0$.

Proof: Since $z_n = s_n - s_{n-1}$, $n \geq 2$, we have

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (s_n - s_{n-1}) = 0, \tag{779}$$

because the sequence $\{s_n\}_{n\geq 1}$ is convergent by assumption, and therefore a Cauchy sequence. $\Box$

Remark: The harmonic series $\sum_{n\geq 1}(1/n)$, satisfies $\lim_{n\to\infty}(1/n) = 0$, but is divergent. So $\lim_{n\to\infty} z_n = 0$ is necessary, but not sufficient, for convergence.

The Cauchy convergence principle allows us to establish convergence of a series without knowing its sum:

Theorem 15 *A series $z_1 + z_2 + \cdots$ is convergent if and only if the sequence of partial sums $\{s_n\}_{n\geq 1}$ is a Cauchy sequence:*

$$\forall\, \varepsilon > 0\; \exists\, N \in \mathbb{N} : \quad |s_{n+p} - s_n| = \left| \sum_{m=n+1}^{n+p} z_m \right| < \varepsilon \quad \forall\, n > N,\, p \in \mathbb{N}. \tag{780}$$

From Cauchy's convergence principle, we may derive several convergence tests.

Theorem 16 *(Comparison Test) If a series $z_1 + z_2 + \cdots$ is given and we can find a convergent (real) series $b_1 + b_2 + \cdots$ with $b_n \geq 0$, $n \in \mathbb{N}$, such that $|z_n| \leq b_n$, $n \in \mathbb{N}$, then the series $z_1 + z_2 + \cdots$ converges absolutely.*

Proof: Let $\varepsilon > 0$ be given. We can find an $N \in \mathbb{N}$ such that

$$b_{n+1} + \cdots + b_{n+p} < \varepsilon \quad \forall\, n > N,\, p \in \mathbb{N}. \tag{781}$$

For those values of n and p, we have

$$|z_{n+1}| + \cdots + |z_{n+p}| \leq b_{n+1} + \cdots + b_{n+p} < \varepsilon. \tag{782}$$

Then by Thm. 15, the series $|z_1| + |z_2| + \cdots$ converges, i. e. the series $z_1 + z_2 + \cdots$ converges absolutely. $\Box$

A series often used in a comparison test is the geometric series.

Proposition 3 *The geometric series*

$$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \cdots \tag{783}$$

converges with the sum $1/(1-q)$ *if* $|q| < 1$ *and diverges if* $|q| \geq 1$.

Proof: For $|q| \geq 1$ we have $|q^m| \geq 1$, $m \geq 0$, and Thm. 14 implies divergence. Let $|q| < 1$. The n-th partial sum is given by

$$s_n = 1 + q + \cdots + q^n, \quad n \in \mathbb{N}, \tag{784}$$

so that

$$qs_n = q + \cdots + q^n + q^{n+1}. \tag{785}$$

After subtraction, we are left with

$$s_n - qs_n = (1-q)s_n = 1 - q^{n+1}. \tag{786}$$

Since $q \neq 1$, we have $1 - q \neq 0$, and we may solve (786) for s_n:

$$s_n = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q} \quad \to \quad \frac{1}{1-q}, \quad n \to \infty, \tag{787}$$

since $|q| < 1$. $\square$

In the following sections, we will mostly use the ratio test to establish convergence of a series.

Theorem 17 *If a series* $z_1 + z_2 + \cdots$ *with* $z_n \neq 0$, $n \in \mathbb{N}$, *has the property that for some* $N \in \mathbb{N}$ *and for some* $q < 1$

$$\left| \frac{z_{n+1}}{z_n} \right| \leq q, \quad \forall\, n > N, \tag{788}$$

then this series converges absolutely. If

$$\left| \frac{z_{n+1}}{z_n} \right| \geq 1, \quad \forall\, n > N, \tag{789}$$

then the series $z_1 + z_2 + \cdots$ *diverges.*

Remark: $q = 1$ is not sufficient (cf. harmonic series)!
Proof: From (789), we have that $|z_{n+1}| \geq |z_n|$, $n > N$, so divergence of the series follows from Thm. 14. If (788) holds, then $|z_{n+1}| \leq q|z_n|$, $n > N$. In particular, $|z_{N+p}| \leq q|z_{N+p-1}| \leq q^2|z_{N+p-2}| \leq \cdots \leq q^{p-1}|z_{N+1}|$, $p \in \mathbb{N}$. With Prop. 3, we find that

$$|z_{N+1}| + |z_{N+2}| + \cdots \leq |z_{N+1}| \left(1 + q + q^2 + \cdots\right) \leq |z_{N+1}| \frac{1}{1-q}. \tag{790}$$

The comparison test (Thm. 16) yields absolute convergence of the series $z_1 + z_2 + \cdots$. $\square$

If the sequence of ratios in Thm. 17 converges, we get the more convenient

Theorem 18 *If a series $z_1 + z_2 + \cdots$ with $z_n \neq 0$, $n \in \mathbb{N}$, is such that*

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L, \tag{791}$$

then

a) If $L < 1$, the series converges absolutely.

b) If $L > 1$, the series diverges.

c) If $L = 1$, the series may converge or diverge, i. e. the test fails and permits no conclusion.

Proof:

a) We define $k_n := |z_{n+1}/z_n|$ and let $L = 1 - b < 1$. By definition of the limit, k_n must eventually get close to $1 - b$, say, $k_n \leq q := 1 - b/2 < 1$, $n > N$, for some N. Convergence of $z_1 + z_2 + \cdots$ now follows from Thm. 17.

b) For $L = 1 + c > 1$ we have $k_n \geq 1 + c/2 > 1$, $n > N^*$, for some N^* sufficiently largte, which implies divergence of $z_1 + z_2 + \cdots$ by Thm. 17.

c) The (divergent) harmonic series $\sum_{n\geq 1}(1/n)$ satisfies

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1. \tag{792}$$

On the other hand, the series $\sum_{n\geq 1}(1/n^2)$ also satisfies

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1. \tag{793}$$

This series is convergent, however, as can be seen from the inequality

$$s_n = \sum_{m=1}^{n}\frac{1}{m^2} \leq 1 + \int_1^n \frac{1}{x^2}\,dx = 2 - \frac{1}{n}. \tag{794}$$

The (real) sequence $\{s_n\}_{n\geq 1}$ is bounded, and it is also monotone increasing, and thus convergent (integral test for a real series).

<u>Examples:</u>

1. The series

$$\sum_{n=0}^{\infty}\frac{(100+75i)^n}{n!} \tag{795}$$

is convergent: We have

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|100+75i|^{n+1}n!}{|100+75i|^n(n+1)!} = \frac{|100+75i|}{n+1} = \frac{125}{n+1} \to 0, \quad n\to\infty. \tag{796}$$

Absolute convergence of the series now follows from Thm. 18 with $L = 0$. The sum of this series is $\exp(100+75i)$.

2. Consider the two sequences $\{a_n\}_{n\geq 0}$ with $a_n := 2^{-3n}i$, $n \geq 0$, and $\{b_n\}_{n\geq 0}$ with $b_n := 2^{-(3n+1)}$, $n \geq 0$. Consider the series

$$a_0 + b_0 + a_1 + b_1 + \cdots = i + \frac{1}{2} + \frac{i}{8} + \frac{1}{16} + \frac{i}{64} + \frac{1}{128} + \cdots \tag{797}$$

The sequence of ratios of absolute values of successive terms are given by

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \ldots \tag{798}$$

so that Thm. 18 is not applicable. From Thm. 17, however, we infer absolute convergence of the series.

Another important convergence test is the root test, and there are several others.

15.2 Power Series

Power series are the most important series in complex analysis because we shall see that their sums are holomorphic functions, and every holomorphic function can be represented by power series.

Definition 10 *The* power series *with* center $z_0 \in \mathbb{C}$ *and coefficients* $a_n \in \mathbb{C}$, $n \in \mathbb{N}_0$, *is given by*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots . \tag{799}$$

Remark: Notice that a power series is a function of $z \in \mathbb{C}$. For z fixed, all the concepts for series with constant terms in the last section apply. Usually a series with variable terms will converge for some z and diverge for others.
Examples:

1. Convergence in a disk: the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \tag{800}$$

   converges absolutely if $|z| < 1$ and diverges if $|z| \geq 1$ (Prop. 3).

2. Convergence for every z: the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \tag{801}$$

   is absolutely convergent for every $z \in \mathbb{C}$. In fact, by the ratio test (Thm. 18)

$$\left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right| = \frac{|z|}{n+1} \to 0, \quad n \to \infty. \tag{802}$$

3. Convergence only at z_0: the power series

$$\sum_{n=0}^{\infty} n! z^n = 1 + z + 2z^2 + 6z^3 + \cdots \tag{803}$$

converges only at $z = 0$, but diverges for every $z \neq 0$. In fact, from the ratio test (Thm. 17)

$$\left|\frac{(n+1)!z^{n+1}}{n!z^n}\right| = (n+1)|z| \geq 1 \quad \forall\, n \geq \frac{1}{|z|} - 1. \tag{804}$$

Theorem 19 *a) Every power series (799) converges at the center z_0.*

b) If (799) converges at $z = z_1 \neq z_0$, it converges absolutely for every z closer to z_0 than z_1, $|z - z_0| < |z_1 - z_0|$.

c) If (799) diverges at $z = z_2$, it diverges for every z farther away from z_0 than z_2, $|z - z_0| > |z_2 - z_0|$.

Proof:

a) For $z = z_0$, the series reduces to the single term a_0.

b) Convergence at $z = z_1$ implies by Thm. 14 that $a_n(z_1 - z_0)^n \to 0$, $n \to \infty$. Therefore, the sequence $\{a_n(z_1 - z_0)^n\}_{n\geq 0}$ must be bounded:

$$|a_n(z_1 - z_0)^n| < M \quad \forall\, n \in \mathbb{N}_0. \tag{805}$$

Then we have

$$|a_n(z - z_0)^n| = \left|a_n(z_1 - z_0)^n \left(\frac{z - z_0}{z_1 - z_0}\right)^n\right| \leq M \left|\frac{z - z_0}{z_1 - z_0}\right|^n, \quad n \in \mathbb{N}_0. \tag{806}$$

Summation over n gives

$$\sum_{n=0}^{\infty} |a_n(z - z_0)^n| \leq M \sum_{n=1}^{\infty} \left|\frac{z - z_0}{z_1 - z_0}\right|^n. \tag{807}$$

For $|z - z_0| < |z_1 - z_0|$, the series on the right-hand side is a converging geometric series (Prop. 3). Absolute convergence of (799) now follows from the comparison test (Thm. 16).

c) If we had convergence at a point z_3 with $|z_3 - z_0| > |z_2 - z_0|$, then we would have convergence at z_2 by b), a contradiction.

□

Radius of Convergence of a Power Series

Definition 11 *Consider circles with center z_0, which include all the points at which a given power series (799) converges. The smallest such circle,*

$$\{z \in \mathbb{C} \,|\, |z - z_0| = R\}, \tag{808}$$

is called the circle of convergence *and its radius R the* radius of convergence *of (799). We write $R = \infty$ if the series (799) converges for all z, and $R = 0$ if it converges only at the center.*

Remark: From Thm. 19, we conclude that (799) converges for every $|z - z_0| < R$. Because R is as small as possible, we also have that the series (799) diverges for all z with $|z - z_0| > R$. No general statements can be made for the points on the circle, $|z - z_0| = R$.
Example: The following three series all have convergence radius $R = 1$. On the circle of convergence,

- $\sum_n \frac{z^n}{n^2}$ converges: $\sum_n \frac{|z|^n}{n^2} = \sum_n \frac{1}{n^2}$,
- $\sum_n \frac{z^n}{n}$ converges at $z = -1$, but diverges at $z = 1$ (harmonic series),
- $\sum_n z^n$ diverges: $\sum_n |z|^n = \sum_n 1^n$

The Cauchy-Hadamard formula allows us to determine the radius of convergence from the coefficients of the power series:

Theorem 20 *Suppose that the sequence $|a_{n+1}/a_n|$, $n \in \mathbb{N}$, converges with limit L^*. If $L^* = 0$, then $R = \infty$ (convergence everywhere). If $L^* > 0$, then the convergence radius is given by*

$$R = \frac{1}{L^*} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|. \tag{809}$$

If $|a_{n+1}/a_n| \to \infty$, then $R = 0$ (convergence only at the center).

Proof: We consider the absolute values of the ratios

$$\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right| = \left|\frac{a_{n+1}}{a_n}\right||z-z_0| \to L^*|z-z_0| =: L, \quad n\to\infty. \tag{810}$$

We want to use the ratio test, Thm. 18. For $L^* > 0$, we have convergence if $L = L^*|z-z_0| < 1$, i. e. if $|z-z_0| < 1/L^*$, and divergence if $|z-z_0| > 1/L^*$. Therefore, $R = 1/L^*$ is the convergence radius of the power series. $L^* = 0$ implies $L = 0$ which gives convergence for all z by the ratio test. If $|a_{n+1}/a_n| \to \infty$, then $|a_{n+1}/a_n||z-z_0| > 1$ for any $z \neq z_0$ and all sufficiently large n. This implies divergence for all $z \neq z_0$ by the ratio test, Thm. 17. □

Example: For the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n \tag{811}$$

with center $z_0 = 3i$, we compute the convergence radius

$$R = \lim_{n\to\infty}\left(\frac{\frac{(2n)!}{(n!)^2}}{\frac{(2n+2)!}{((n+1)!)^2}}\right) = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}. \tag{812}$$

Remark: Thm. 20 can be extended to cases where the sequence of ratios does not converge: we have

$$R = \frac{1}{\tilde{L}}, \quad \tilde{L} = \lim_{n\to\infty}|a_n|^{1/n}, \tag{813}$$

and even

$$R = \frac{1}{\tilde{\ell}}, \quad \tilde{\ell} \text{ the greatest limit point of the sequence } \left\{|a_n|^{1/n}\right\}_{n\geq 0}. \tag{814}$$

15.3 Functions Given by Power Series

For a power series with center $z_0 \in \mathbb{C}$ and with positive convergence radius $R > 0$, we consider the function

$$f(z) := \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots, \quad z \in B_R(z_0) \tag{815}$$

(we use the notation with open balls again, cf. Section 13.3). We say that f is *represented* by the power series of that it is *developed* in the power series near z_0.

Example: The function $f(z) := 1/(1-z)$ is represented by the geometric series in the interior of the unit circle:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad z \in B_1(0), \tag{816}$$

by Prop. 3.

Definition 12 *A complex function f is* analytic at $z_0 \in \mathbb{C}$ *if it can be represented as a power series centered at z_0 with positive convergence radius:*

$$\exists\, R > 0,\ \{a_n\}_{n\geq 0}: \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \forall\, z \in B_R(z_0). \tag{817}$$

We want to prove uniqueness of this representation first.

Theorem 21 *If a complex function f is analytic at z_0, then f is continuous at z_0.*

Proof: From (817), we know that $f(z_0) = a_0$, so we must prove that $\lim_{z\to z_0} f(z) = a_0$. For any $r \in (0, R)$, the power series (817) converges absolutely if $|z - z_0| \leq r$, by Thm. 19. Hence the series

$$S := \sum_{n=1}^{\infty} |a_n| r^{n-1} = \frac{1}{r}\sum_{n=1}^{\infty} |a_n| r^n \tag{818}$$

converges. For $0 < |z - z_0| \leq r$ we obtain

$$|f(z) - a_0| = \left|\sum_{n=1}^{\infty} a_n (z-z_0)^n\right| \leq |z-z_0| \sum_{n=1}^{\infty} |a_n||z-z_0|^{n-1} \tag{819}$$

$$\leq |z-z_0| \sum_{n=1}^{\infty} |a_n| r^{n-1} = |z-z_0| S. \tag{820}$$

If $S = 0$, we are done. Otherwise, let $\varepsilon > 0$ be given, and set

$$\delta := \min\left\{r, \frac{\varepsilon}{S}\right\}. \tag{821}$$

Then we have $|z - z_0| S < \varepsilon$, for all $z \in B_\delta(z_0)$, and therefore $f(z) \to a_0$, $z \to z_0$. $\square$

Theorem 22 *Let the power series*

$$\sum_{m=0}^{\infty} a_m(z-z_0)^m, \quad \sum_{m=0}^{\infty} b_m(z-z_0)^m \tag{822}$$

both be convergent in $B_R(z_0)$, for an $R>0$, and let them both have the same sum in $B_R(z_0)$. Then the series are identical, i. e. $a_n = b_n$, $n \in \mathbb{N}_0$.

Remark: This implies that if a complex function f is analytic at z_0, then its power series representation near z_0 is unique.

Proof: We proceed by induction with respect to n. By assumption,

$$\sum_{m=0}^{\infty} a_m(z-z_0)^m = \sum_{m=0}^{\infty} b_m(z-z_0)^m, \quad z \in B_R(z_0). \tag{823}$$

By Theorem 21, the sums of these power series are continuous at z_0. Consider $z_0 \neq z \in B_R(z_0)$ and let $z \to z_0$ on both sides; this yields $a_0 = b_0$, i. e. the assertion is true for $n = 0$. Now assume that $a_m = b_m$ for all $m \leq n$. We subtract equal terms from both sides of (823) and divide by $(z-z_0)^{n+1} \neq 0$:

$$\sum_{m=n+1}^{\infty} a_m(z-z_0)^{m-(n+1)} = \sum_{m=n+1}^{\infty} b_m(z-z_0)^{m-(n+1)}. \tag{824}$$

Like before we let $z \to z_0$ and conclude that $a_{n+1} = b_{n+1}$. $\square$

Operations on Power Series This discussion will prepare for our main goal in this section, namely, to show that a function which is analytic at z_0 is holomorphic in $B_R(z_0)$.

- *Termwise addition or subtraction* of two power series with the same center $z_0 \in \mathbb{R}$ and radii of convergence R_1, R_2 yields a power series with center z_0 and radius of convergence $R \geq \min\{R_1, R_2\}$. This is because we may add/subtract the partial sums s_n and s_n^*, $n \in \mathbb{N}$, term by term and sue $\lim_{n\to\infty}(s_n \pm s_n^*) = \lim_{n\to\infty} s_n \pm \lim_{n\to\infty} s_n^*$.

- *Termwise multiplication* of two power series with the same center $z_0 \in \mathbb{C}$,

$$f(z) := \sum_{k=0}^{\infty} a_k(z-z_0)^k, \quad z \in B_{R_1}(z_0), \tag{825}$$

$$g(z) := \sum_{m=0}^{\infty} b_m(z-z_0)^m, \quad z \in B_{R_2}(z_0), \tag{826}$$

leads to the *Cauchy product*

$$s(z) = f(z)g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad z \in B_R(z_0), \tag{827}$$

with $R = \min\{R_1, R_2\}$ and

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n \in \mathbb{N}_0. \tag{828}$$

- *Termwise differentiation and integration* of power series is permissible:

Theorem 23 *The power series*

$$\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} \tag{829}$$

obtained by differentiating the power series $\sum_{n\geq 0} a_n (z - z_0)^n$ term by term has the same radius of convergence as the original series.

Example: Differentiating the geometric series ($R = 1$) twice term by term and multiplying with $z^2/2$ yields

$$\sum_{n=2}^{\infty} \binom{n}{2} z^n, \tag{830}$$

which also has radius of convergence $R = 1$, by Thm. 23.

Theorem 24 *The power series*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} \tag{831}$$

obtained by integrating the power series $\sum_{n\geq 0} a_n (z - z_0)^n$ term by term has the same radius of convergence as the original series.

Both theorems can be proved with the Cauchy-Hadamard formula (Thm. 20) or with the generalizations (813) or (814).

Power Series Represent Holomorphic Functions

Theorem 25 *A power series with center $z_0 \in \mathbb{C}$ and radius of convergence $R > 0$ represents a holomorphic function in $B_R(z_0)$. The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence R. Hence by the first statement, each of them represents a holomorphic function.*

Corollary 1 *A complex function f which is analytic at $z_0 \in \mathbb{C}$ is holomorphic in $B_R(z_0)$, where $R > 0$ denotes the radius of convergence of the power series representation of f near z_0.*

Remark: In the next section, we will show that the opposite is also true (Taylor's theorem).
Proof of the Theorem: The proof is in three steps:

1. We write the power series representation of f and the derived series by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad f_1(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}, \quad z \in B_R(z_0). \tag{832}$$

   We want to show that f is holomorphic in $B_R(z_0)$ and $f' = f_1$. With termwise addition, we obtain

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z) = \sum_{n=2}^{\infty} a_n \left(\frac{(z + \Delta z - z_0)^n - (z - z_0)^n}{\Delta z} - n(z - z_0)^{n-1} \right), \tag{833}$$

   for $z \in B_R(z_0)$.

2. With $b := z + \Delta z - z_0$, $a := z - z_0$, $\Delta z = b - a$, we have

$$\frac{(z + \Delta z - z_0)^n - (z - z_0)^n}{\Delta z} - n(z - z_0)^{n-1} = \frac{b^n - a^n}{b - a} - n a^{n-1}, \quad n \geq 2. \tag{834}$$

   We prove by induction that

$$\frac{b^n - a^n}{b - a} - n a^{n-1} = (b - a) \underbrace{\sum_{k=0}^{n-2} (k+1) a^k b^{n-2-k}}_{=: A_n}, \quad n \geq 2. \tag{835}$$

- $n = 2$:

$$\frac{b^2 - a^2}{b - a} - 2a = \frac{(b + a)(b - a)}{b - a} - 2a = b - a = (b - a)A_2 \quad (836)$$

- $m \mapsto m + 1$: Assume the formula is true for $n = m$. We write

$$\begin{aligned} \frac{b^{m+1} - a^{m+1}}{b - a} &= \frac{b^{m+1} - ba^m + ba^m - a^{m+1}}{b - a} && (837) \\ &= \frac{b(b^m - a^m) + (b - a)a^m}{b - a} = b\frac{b^m - a^m}{b - a} + a^m \\ &= b\left((b - a)A_m + ma^{m-1}\right) + a^m && (838) \\ &= (b - a)\left(bA_m + ma^{m-1}\right) + (m + 1)a^m. && (839) \end{aligned}$$

From the definition of A_m we obtain

$$bA_m + ma^{m-1} = \sum_{k=0}^{m-2}(k+1)a^k b^{m-1-k} + ma^{m-1} = \sum_{k=0}^{m-1}(k+1)a^k b^{m-1-k} = A_{m+1}, \quad (840)$$

so that

$$\frac{b^{m+1} - a^{m+1}}{b - a} - (m + 1)a^m = A_{m+1}. \quad (841)$$

Therefore we have

$$\frac{(z + \Delta z - z_0)^n - (z - z_0)^n}{\Delta z} - n(z - z_0)^{n-1} = \Delta z \sum_{k=0}^{n-2}(k+1)(z - z_0)^k (z + \Delta z - z_0)^{n-2-k}, \quad (842)$$

for $n \geq 2$.

3. We use this to obtain

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z) = \sum_{n=2}^{\infty} a_n \Delta z \sum_{k=0}^{n-2}(k+1)(z - z_0)^k (z + \Delta z - z_0)^{n-2-k}. \quad (843)$$

For $|z| \leq R_0$ and $|z + \Delta z| \leq R_0$, $R_0 < R$, we estimate

$$\left|\sum_{k=0}^{n-2}(k + 1)(z - z_0)^k (z + \Delta z - z_0)^{n-2-k}\right| \leq \sum_{k=0}^{n-2}(k+1)R_0^{n-2} = \frac{n(n - 1)}{2}R_0^{n-2}. \quad (844)$$

Therefore we have

$$\left|\frac{f(z+\Delta z)-f(z)}{\Delta z}-f_1(z)\right| \leq |\Delta z|\frac{1}{2}\sum_{n=2}^{\infty}|a_n|n(n-1)R_0^{n-2}. \qquad (845)$$

The second derived series of f at $z = z_0 + R_0$ is given by

$$\sum_{n=2}^{\infty} a_n n(n-1)R_0^{n-2}, \qquad (846)$$

and because of $R_0 < R$ it converges absolutely by Thms. 23 and 19. Thus, finally

$$\left|\frac{f(z+\Delta z)-f(z)}{\Delta z}-f_1(z)\right| \leq |\Delta z|K(R_0), \quad K(R_0) := \frac{1}{2}\sum_{n=2}^{\infty}|a_n|n(n-1)R_0^{n-2}. \qquad (847)$$

Because $R_0 < R$ is arbitrary, and letting $\Delta z \to 0$, we conclude that f is differentiable at any point $z \in B_R(z_0)$ (i. e. f holomorphic in $B_R(z_0)$), and that its derivative is represented by the derived series.

The statements about the higher derivatives follow by induction. □

15.4 Taylor and Maclaurin Series

The *Taylor series* of a complex function f is given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad (848)$$

where the coefficients $\{a_n\}_{n\geq 0}$ are given by

$$a_n = \frac{1}{n!}f^{(n)}(z_0) \overset{\text{Thm. 11}}{=} \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta, \quad n \in \mathbb{N}_0. \qquad (849)$$

A *Maclaurin series* is a Taylor series with center $z_0 = 0$. The *remainder* of the Taylor series (848) after the term $a_n(z-z_0)^n$ is

$$R_n(z) = \sum_{k=n+1}^{\infty} a_k(z-z_0)^k = \frac{(z-z_0)^{n+1}}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}(\zeta-z)}\,d\zeta, \qquad (850)$$

$n \geq 0$. Thus we obtain *Taylor's formula* with remainder:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + R_n(z), \qquad (851)$$

$n \geq 0$. Taylor's theorem states that every holomorphic function can be represented by Taylor series (with various centers):

Theorem 26 *Let the complex function f be holomorphic in a domain $U \subseteq \mathbb{C}$, and let $z_0 \in U$. Then there exists precisely one Taylor series (848) with center z_0 that represents f. This representation is valid in the largest open disk with center z_0 in which f is holomorphic. The remainders R_n, $n \geq 0$, of (848) can be represented in the form (850). The coefficients $\{a_n\}_{n \geq 0}$ satisfy the inequality*

$$|a_n| \leq \frac{M}{r^n}, \quad n \geq 0, \qquad M := \max_{|z-z_0|=r} |f(z)|, \qquad (852)$$

where $r > 0$ is such that $\overline{B_r(z_0)} \subseteq U$.

Corollary 2 *A complex function f which is holomorphic in $U \subseteq \mathbb{C}$ is analytic at every point $z_0 \in U$.*

Remark: Now we understand why *holomorphic* and *analytic* are often used synonymously. Keep in mind, however, that this equivalence did not follow immediately from the definition of a holomorphic function (Section 13.3)!
Proof of the theorem: From Cauchy's Integral Formula (Thm. 10), we know that

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta, \qquad (853)$$

if z is enclosed by C. We choose $C = \{z \in \mathbb{C} \,|\, |z - z_0| = r\}$, and we develop $1/(\zeta - z)$ in powers of $z - z_0$:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)} \qquad (854)$$

Because $\zeta \in C$ and $|z - z_0| < r$, we have

$$\left|\frac{z - z_0}{\zeta - z_0}\right| < 1. \qquad (855)$$

From the finite geometric sum

$$1 + q + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q}, \quad q \neq 1, \tag{856}$$

we conclude that

$$\frac{1}{1-q} = 1 + q + \cdots + q^n + \frac{q^{n+1}}{1-q}, \tag{857}$$

for any $n \in \mathbb{N}_0$. Using this with $q := \frac{z - z_0}{\zeta - z_0}$ we obtain

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\left(1 + \frac{z - z_0}{\zeta - z_0} + \cdots + \left(\frac{z - z_0}{\zeta - z_0}\right)^n\right) + \frac{1}{\zeta - z}\left(\frac{z - z_0}{\zeta - z_0}\right)^{n+1}. \tag{858}$$

We insert this into (853):

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z_0}\, d\zeta + \frac{z - z_0}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^2}\, d\zeta + \cdots \\ &\quad + \frac{(z - z_0)^n}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta + R_n(z), \end{aligned} \tag{859}$$

with R_n given by (850). The integrals are those in (849) related to the derivatives, so that we have proved the Taylor formula (851). Because the holomorphic function f has derivatives of all orders (Thm. 11), we may take n in (851) arbitrarily large. For $n \to \infty$, we obtain (848). This series will converge and represent f if and only if $\lim_{n\to\infty} R_n(z) = 0$, which we prove as follows. Because z lies inside of C and $\zeta \in C$, we have $|\zeta - z| > 0$. Because f is holomorphic inside and on C (by the choice of r), f is bounded on C, and so is the function $f(\zeta)/(\zeta - z)$, say,

$$\left|\frac{f(\zeta)}{\zeta - z}\right| \leq \tilde{M}, \quad \forall\, \zeta \in C. \tag{860}$$

Also, C has the radius $r = |\zeta - z_0|$ and the length $2\pi r$. By the ML-inequality (Prop. 2), we obtain from (850):

$$|R_n(z)| = \frac{|z - z_0|^{n+1}}{2\pi}\left|\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}(\zeta - z)}\, d\zeta\right| \tag{861}$$

$$\leq \frac{|z - z_0|^{n+1}}{2\pi}\tilde{M}\frac{1}{r^{n+1}}2\pi r = \tilde{M} r \left|\frac{z - z_0}{r}\right|^{n+1}. \tag{862}$$

Because z lies inside of C we have $|z-z_0| < r$, or $|(z-z_0)/r| < 1$, and therefore $R_n(z) \to 0$, $n \to \infty$. This proves that the Taylor series converges and has the sum $f(z)$. Uniqueness follows from Thm. 22. Finally, the inequality for the coefficients (852) follows from (849) and from Cauchy's inequality (769). $\square$

Remarks:

1. We may achieve any prescribed accuracy in approximating $f(z)$ by a partial sum of (848) by choosing n large enough.
2. On the circle of convergence of the Taylor series (848) there is at least one *singular point* of f, i. e. a point c at which f is not holomorphic (otherwise the radius of convergence would be larger). We also say that f *is singular* at c or *has a singularity* at c. Hence the radius of convergence R of (848) is usually equal to the distance from z_0 to the nearest singular point of f.

Power Series as Taylor Series The relation to Section 15.3 is given by

Theorem 27 *A power series with a nonzero radius of convergence is the Taylor series of its sum.*

Proof: Given the power series

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^m, \quad z \in B_R(z_0), \tag{863}$$

we have $f(z_0) = a_0$. By Thm. 25, we obtain by termwise differentiation

$$f^{(n)}(z) = \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} a_m (z-z_0)^{m-n}, \quad z \in B_R(z_0), \quad n \in \mathbb{N}, \tag{864}$$

and therefore $f^{(n)}(z_0) = n! a_n$. Therefore we have

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^m = \sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(z_0)(z-z_0)^m, \tag{865}$$

which is the Taylor series of f with center z_0. $\square$

Remark: We know that holomorphic functions have derivatives of all orders (Thm. 11), and that they can always be represented by Taylor series

(Thm. 26). This is not true in general for *real functions.* As an example, consider the function

$$f(x) := \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & \text{otherwise} \end{cases}, \quad x \in \mathbb{R}. \tag{866}$$

This function has derivatives of all orders, with $f^{(n)}(0) = 0$, $n \in \mathbb{N}$, and therefore it cannot be represented by a Maclaurin series in an open disk with center 0.

Important Special Taylor Series These are as in calculus, with x replaced by a complex number z.

1. $f(z) = 1/(1-z)$. We have $f^{(n)}(z) = n!/(1-z)^{n+1}$, $n \in \mathbb{N}_0$. For $z_0 = 0$, we obtain $f^{(n)}(0) = n!$, $n \in \mathbb{N}_0$ and thus the coefficients of the Maclaurin series are given by $a_n = f^{(n)}(0)/n! = 1$, $n \in \mathbb{N}_0$. Hence the Maclaurin series of f near 0 is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad z \in B_1(0). \tag{867}$$

   f is singular at $z = 1$; this point lies on the circle of convergence.

2. $f(z) = \exp z$. We have $f^{(n)}(z) = \exp(z)$, $n \in \mathbb{N}_0$ (Section 13.5). For $z_0 = 0$, we obtain $f^{(n)}(0) = 1$, $n \in \mathbb{N}_0$ and thus the coefficients of the Maclaurin series are given by $a_n = 1/n!$, $n \in \mathbb{N}_0$. Hence the Maclaurin series of f near 0 is

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}. \tag{868}$$

3. Trigonometric and Hyperbolic functions: We use (868) in the definitions of cos, sin, cosh, sinh (Section 13.6) to obtain Maclaurin series

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \tag{869}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \tag{870}$$

   for $z \in \mathbb{C}$.

4. The function $f(z) = \mathrm{Log}(1+z)$ has derivatives $f^{(n)}(z) = (-1)^{n-1}(n-1)!/(1+z)^n$, $n \in \mathbb{N}$ (Thm. 5). For $z_0 = 0$ we obtain $f(0) = 0$ and $f^{(n)}(0) = (-1)^{n-1}(n-1)!$, $n \in \mathbb{N}$ and thus the coefficients of the Maclaurin series are given by $a_0 = 0$, $a_n = (-1)^{n-1}/n$, $n \in \mathbb{N}$. Hence the Maclaurin series of f near 0 is

$$\mathrm{Log}(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1}, \quad z \in B_1(0). \tag{871}$$

We may also write the Maclaurin series of $g(z) := -f(-z) = -\mathrm{Log}(1-z)$ as

$$\mathrm{Log}\left(\frac{1}{1-z}\right) = -\mathrm{Log}(1-z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}, \quad z \in B_1(0). \tag{872}$$

By adding both series, we obtain

$$\mathrm{Log}\left(\frac{1+z}{1-z}\right) = 2\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}, \quad z \in B_1(0). \tag{873}$$

Practical Methods There are several methods which allow us to obtain Taylor series more quickly than by the use of coefficient formulas. Because of uniqueness, they will be the same regardless of the method used.

1. Substitution:

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} \overset{(867)}{=} \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}, \quad z \in B_1(0). \tag{874}$$

2. Integration: For $f(z) = \arctan z$ we have $f'(z) = 1/(1+z^2)$. We may integrate (874) termwise to obtain

$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}, \quad z \in B_1(0). \tag{875}$$

arctan is a multi-valued function. This series represents the values which satisfy $|\mathrm{Re}(\arctan z)| < \pi/2$.

3. Development by Using the Geometric Series: we had used this in the proof of Thm. 26. For $R := |c - z_0| > 0$ we have

$$\frac{1}{c-z} = \frac{1}{c-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{c-z_0}\right)^n, \quad z \in B_R(z_0). \tag{876}$$

4. Binomial Series, Reduction by Partial Fractions: For the rational function

$$f(z) = \frac{2z^2+9z+5}{z^3+z^2-8z-12} = \frac{1}{(z+2)^2} + \frac{2}{z-3}, \tag{877}$$

we use the binomial series

$$\frac{1}{(1-z)^m} = \sum_{n=0}^{\infty}\binom{n+m-1}{n} z^n, \quad z \in B_1(0), \quad m \in \mathbb{N} \tag{878}$$

(for $m = 1$, we obtain the geometric series, as expected). We look for a development at $z_0 = 1$ and therefore write

$$\frac{1}{(z+2)^2} = \frac{1}{(3+(z-1))^2} = \frac{1}{9}\frac{1}{\left(1-\left(-\frac{z-1}{3}\right)\right)^2} \tag{879}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-1}{3}\right)^n, \quad z \in B_3(1), \tag{880}$$

$$\frac{2}{z-3} = -\frac{2}{2-(z-1)} = -\frac{1}{1-\frac{z-1}{2}} \tag{881}$$

$$= -\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n, \quad z \in B_2(1). \tag{882}$$

Addition yields the Taylor series of f near 1:

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{(-1)^n(n+1)}{3^{n+2}} - \frac{1}{2^n}\right)(z-1)^n, \quad z \in B_2(1). \tag{883}$$

16 Laurent Series. Residue Integration

Laurent series generalize Taylor series, because they also allow for negative integer powers of $z - z_0$. Therefore, they will converge in an annulus with center z_0. Hence by a Laurent series we can represent a given function f that is holomorphic in an annulus and may have singularities outside the ring as well as in the "hole" of the annulus. Laurent series will help us in classifying singularities of complex functions (16.2), and will also lead to a powerful and elegant integration method, called residue integration (16.3, 16.4).

16.1 Laurent Series

If in an application we want to develop a function f in powers of $z - z_0$ when f is singular at z_0, we cannot use a Taylor series. Instead we may use a new kind of series, called Laurent series.

Theorem 28 *Let the complex function f be holomorphic in a domain containing two concentric circles C_1, C_2 with center z_0 and the annulus between them. Then f can be represented by the* Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n + \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{884}$$

consisting of all integer powers of $z - z_0$. The coefficients of this Laurent series are given by the integrals

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta, \quad n \in \mathbb{Z}, \tag{885}$$

taken counterclockwise around any closed curve C that lies in the annulus and encloses the inner circle C_2.

This series converges and represents f in the enlarged open annulus obtained from the given annulus by continuously increasing the outer circle C_1 and decreasing C_2 until each of the two circles reaches a point where f is singular.

In the important special case that z_0 is the only singular point of f inside C_2, this circle can be shrunk to the point z_0, giving convergence in a disk *except at the center:* $\{z \in \mathbb{C} \,|\, 0 < |z - z_0| < R\}$. *In this case (!) the series (or finite sum) of the negative powers of (884),*

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} \tag{886}$$

(the notation b_n *instead of* a_{-n}, $n \in \mathbb{N}$, *is also used), is called the* principal part *of* f *at* z_0 *(or of that Laurent series (884)).*

Proof: The proof is in several steps:

1. The nonnegative powers are those of a Taylor series. We use Cauchy's Integral Formula (Thm. 10), which in our annulus (doubly connected) is given by

$$f(z) = \underbrace{\frac{1}{2\pi i} \oint\limits_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{=:g(z)} + \underbrace{\left(-\frac{1}{2\pi i} \oint\limits_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right)}_{=:h(z)}, \tag{887}$$

   where the integration is taken counterclockwise over both C_1 and C_2. The first integral is transformed exactly as in the proof of Taylor's theorem (Thm. 26), which yields

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{2\pi i} \oint\limits_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta, \tag{888}$$

   for $n \in \mathbb{N}_0$. By the principle of deformation of path we may replace C_1 by C, which proves the formula for the coefficients (885) for $n \in \mathbb{N}_0$.

2. For the negative powers, we consider the second term in (887). Because z lies outside of the circle C_2, we have $|z - z_0| > |\zeta - z_0|$, for $\zeta \in C_2$. We do a similar development as in the proof of Taylor's theorem (Thm. 26):

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{-1}{(z - z_0)\left(1 - \frac{\zeta - z_0}{z - z_0}\right)}. \tag{889}$$

   We use (857) to obtain

$$\frac{1}{\zeta - z} = -\frac{1}{z - z_0}\left(1 + \frac{\zeta - z_0}{z - z_0} + \cdots + \left(\frac{\zeta - z_0}{z - z_0}\right)^n\right) - \frac{1}{z - \zeta}\left(\frac{\zeta - z_0}{z - z_0}\right)^{n+1}, \tag{890}$$

$n \in \mathbb{N}$. Multiplication by $-f(\zeta)/(2\pi i)$ and integration over C_2 on both sides now yield

$$\begin{aligned} h(z) &= \frac{1}{(z-z_0)}\frac{1}{2\pi i}\oint_{C_2} f(\zeta)\,d\zeta + \frac{1}{(z-z_0)^2}\frac{1}{2\pi i}\oint_{C_2}(\zeta-z_0)f(\zeta)\,d\zeta + \cdots \\ &\quad + \frac{1}{(z-z_0)^{n+1}}\frac{1}{2\pi i}\oint_{C_2}(\zeta-z_0)^n f(\zeta)\,d\zeta + R_n^*(z), \end{aligned}$$

with

$$R_n^*(z) := \frac{1}{(z-z_0)^{n+1}}\frac{1}{2\pi i}\oint_{C_2}(\zeta-z_0)^{n+1}\frac{f(\zeta)}{z-\zeta}\,d\zeta, \tag{891}$$

for $n \in \mathbb{N}$. We may again replace C_2 by C. Therefore

$$h(z) = \sum_{m=1}^{n}\frac{a_{-m}}{(z-z_0)^m} + R_n^*(z), \quad n \in \mathbb{N}, \tag{892}$$

with coefficients as given in (885), and it remains to show that $R_n^*(z) \to 0$, $n \to \infty$.

3. Because f is bounded on C_2 and $z \notin C_2$ we have

$$\left|\frac{f(\zeta)}{z-\zeta}\right| < \tilde{M}, \quad \forall\, \zeta \in C_2. \tag{893}$$

With the radius of the circle C_2, $|\zeta - z_0| = r_2 > 0$, and $L = 2\pi r_2$, and we obtain from the ML-inequality (Prop. 2)

$$|R_n^*(z)| \leq \frac{1}{2\pi|z-z_0|^{n+1}}r_2^{n+1}\tilde{M}L = \tilde{M}r_2\left(\frac{r_2}{|z-z_0|}\right)^{n+1}, \quad n \in \mathbb{N}. \tag{894}$$

Because z lies outside of the circle C_2 we have $|z-z_0| > r_2$, so that the right-hand side approaches 0 as $n \to \infty$. Therefore, we have established convergence of (884) with coefficients (885) in the given annulus bounded by C_1 and C_2.

4. It remains to show that the annulus can be enlarged as stated in the theorem. The part of the Laurent series with the nonnegative powers (representing g) is a Taylor series; hence it converges in the disk D

with center z_0 whose radius equals the distance from z_0 to the closest singularity of g. Also g must be singular at all points outside C_1 where f is singular.

The part with the negative powers (representing h) is a power series in $1/(z-z_0)$. Let the given annulus be $r_2 < |z-z_0| < r_1$, where r_1 and r_2 are the radii of C_1 and C_2, respectively. Then we have $1/r_2 > 1/|z-z_0| > 1/r_1$, so that this power series in $1/(z-z_0)$ must converge at least in the disk $1/|z-z_0| < 1/r_2$. This implies that h is holomorphic outside of C_2 (Thm. 25). Also h must be singular inside C_2 where f is singular, and the series of the negative powers of (884) converges in a region E exterior to the circle with center z_0 and radius equal to the maximum distance from z_0 to the singularities of f inside C_2. The intersection $D \cap E$ is the enlarged open annulus characterized in Laurent's theorem.

□

Remark: The Laurent series of a given holomorphic function f in its annulus of convergence is unique. However, f may have different Laurent series in two annuli with the same center; see the examples below.

As for Taylor series, we may use other methods than the integral formula (885) to obtain the Laurent series of f. Due to uniqueness, they will be the same regardless of the method used.

Examples:

1. Use of Maclaurin series: $f(z) = z^{-5}\sin z$, $z_0 = 0$. With (869) we obtain

$$f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n-4}}{(2n+1)!} = \sum_{n=-2}^{\infty}(-1)^n \frac{z^{2n}}{(2n+5)!}, \quad |z| > 0. \tag{895}$$

   This is the Laurent series of f at 0, with principal part $z^{-4} - (6z^2)^{-1}$.

2. Substitution: $f(z) = z^2\exp(1/z)$, $z_0 = 0$. With (868) with z replaced by $1/z$ we obtain

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2-n}}{n!} = \sum_{n=-2}^{\infty} \frac{z^{-n}}{(n+2)!} = \sum_{n=-\infty}^{2} \frac{z^n}{(2-n)!}, \quad |z| > 0. \tag{896}$$

   This is the Laurent series of f at 0, and the principal part is an infinite series.

3. Development of $1/(1-z)$

   a) in nonnegative powers of z: $z_0 = 0$. We have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1, \tag{897}$$

   b) in negative powers of z: $z_0 = 0$. We write

$$\frac{1}{1-z} = \frac{-1}{z\left(1-\frac{1}{z}\right)} \overset{(867)}{=} -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty} z^{-(n+1)} \tag{898}$$

$$= -\sum_{n=1}^{\infty} z^{-n} = -\sum_{n=-\infty}^{-1} z^n, \quad |z| > 1. \tag{899}$$

4. Laurent Expansions in Different Concentric Annuli: $f(z) = 1/(z^3 - z^4)$, $z_0 = 0$. With the previous example, we obtain

   - $\displaystyle \frac{1}{z^3 - z^4} = \sum_{n=0}^{\infty} z^{n-3} = \sum_{n=-3}^{\infty} z^n, \quad 0 < |z| < 1,$
   - $\displaystyle \frac{1}{z^3 - z^4} = -\sum_{n=-\infty}^{-1} z^{n-3} = -\sum_{n=-\infty}^{-4} z^n, \quad |z| > 1.$

5. Use of Partial Fractions: $z_0 = 0$,

$$f(z) = \frac{-2z+3}{z^2 - 3z + 2} = \frac{1}{1-z} - \frac{1}{z-2}. \tag{900}$$

   We already know the Laurent series for the first fraction from Example 3. For the second fraction, we obtain

$$-\frac{1}{z-2} = \frac{1}{2\left(1-\frac{z}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}, \quad |z| < 2, \tag{901}$$

$$-\frac{1}{z-2} = -\frac{1}{z\left(1-\frac{2}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} \tag{902}$$

$$= -\sum_{n=-\infty}^{-1}\frac{z^n}{2^{n+1}} \quad |z| > 2. \tag{903}$$

   Now we may add the series:

a) $f(z) = \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n, \quad |z| < 1$

b) $f(z) = -\sum_{n=-\infty}^{-1} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \quad 1 < |z| < 2,$

c) $f(z) = -\sum_{n=-\infty}^{-1} z^n - \sum_{n=-\infty}^{-1} \frac{z^n}{2^{n+1}} = -\sum_{n=-\infty}^{-1} \left(1 + \frac{1}{2^{n+1}}\right) z^n, \ |z| > 2.$

Remark: If f in Thm. 28 is holomorphic inside C_2, then the coefficients in front of the negative powers in (884) are zero by Cauchy's Integral Theorem (Thm. 8), so that the Laurent series reduces to a Taylor series. Examples 3a) and 5a) illustrate this.

16.2 Singularities and Zeros. Infinity

Definition 13 *A complex function f* is singular at $z_0 \in \mathbb{C}$ *or* has a singularity at z_0 *if f is not differentiable (perhaps not even defined) at $z_0 \in \mathbb{C}$, but every open ball $B_r(z_0) \subseteq \mathbb{C}$, $r > 0$ contains points at which f is differentiable. Such a point $z_0 \in \mathbb{C}$ is called a* singular point *of f. $z_0 \in \mathbb{C}$ is an* isolated singularity *if there is an open ball $B_r(z_0)$ which contains no further singularities of f, i. e. if f is holomorphic on $B_r(z_0) \setminus \{z_0\}$. An isolated singularity z_0 of f is*

1. removable, *if f can be made differentiable at z_0 by assigning a suitable value $f(z_0)$.*

2. *a* pole of order $m \in \mathbb{N}$, *if the principal part of the Laurent series of f at z_0 (886) is of the form*

$$\sum_{n=-m}^{-1} a_n (z - z_0)^n = \sum_{n=1}^{m} \frac{a_{-n}}{(z - z_0)^n}, \quad a_{-m} \neq 0. \tag{904}$$

   If $m = 1$, z_0 is called a simple pole.

3. *an* (isolated) essential singularity, *if it is neither removable nor a pole.*

Remarks:

- The function $f(z) = \tan(1/z)$ has a *nonisolated singularity* at $z_0 = 0$. We will not discuss this type of singularity here.

- f has an essential singularity $z_0 \in \mathbb{C}$ if and only if the principal part of the Laurent series of f at z_0 contains infinitely many terms.

- Be careful to consider the right Laurent series when analyzing singularities! It must be the one which is valid in the immediate neighborhood of z_0.

Examples:

1. The function $f(z) = \sin z/z$ has a removable singularity at $z = 0$. It is removed by assigning $f(0) = 1$. This can be seen from the Laurent series of f at 0, which we obtain from (869):

$$f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - + \cdots, \quad z \in \mathbb{C}. \tag{905}$$

2. The function

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2} \tag{906}$$

has a simple pole at $z = 0$ and a pole of order 5 at $z = 2$. From the examples in Section 16.1 we know that $f(z) = z^{-5}\sin z$ has a fourth-order pole at $z = 0$ and that $f(z) = 1/(z^3 - z^4)$ has a third order pole at $z = 0$

3. The functions

$$\exp\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}, \quad |z| > 0, \tag{907}$$

$$\sin\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{0} (-1)^{-n} \frac{z^{2n-1}}{(-2n+1)!}, \quad |z| > 0, \tag{908}$$

have an (isolated) essential singularity at $z = 0$.

The behavior of complex functions near singularities is characterized by the following theorems.

Theorem 29 *If a holomorphic function f has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.*

Example: The function $f(z) = 1/z^2$ hs a second-order pole at $z = 0$ We have $|f(z)| = |z|^{-2} \to \infty$ as $z \to 0$ in any manner.

Theorem 30 *(Picard) A holomorphic function f with an isolated singularity at a point z_0 takes on every value, with at most one exceptional value, in $B_\varepsilon(z_0)$, for any $\varepsilon > 0$.*

Example: Consider the function $f(z) = \exp(1/z)$, which has an essential singularity at $z = 0$. For a given complex number $c = c_0 \exp(i\alpha) \neq 0$, we want to find $z \in \mathbb{C}$ with $f(z) = c$. We set $z = r\exp(i\varphi)$ and obtain the equation

$$\exp(1/z) = \exp\left(\frac{\cos\varphi}{r}\right)\exp\left(-i\frac{\sin\varphi}{r}\right) = c_0 \exp(i\alpha), \qquad (909)$$

so that we need to solve

$$\cos\varphi = r\log c_0, \quad \sin\varphi = -r\alpha. \qquad (910)$$

We have $1 = \cos^2\varphi + \sin^2\varphi = r^2\left(\log^2 c_0 + \alpha^2\right)$, so that

$$r = \frac{1}{\sqrt{\log^2 c_0 + \alpha^2}}. \qquad (911)$$

Now we may replace α by $\alpha + 2\pi n$, $n \in \mathbb{Z}$, without changing the given number c. This allows us to make $r > 0$ arbitrarily small, and so a complex number z with $f(z) = c$ can be found in $B_\varepsilon(0)$, for any $\varepsilon > 0$.

Zeros of Holomorphic Functions

Definition 14 *A complex function f which is holomorphic in a domain $U \subseteq \mathbb{C}$ has a* zero *at $z_0 \in U$ if $f(z_0) = 0$. A zero of f has* order $n \in \mathbb{N}$ *if $f^{(m)}(z_0) = 0$, $m < n$ and $f^{(n)}(z_0) \neq 0$. If $n = 1$, z_0 is called a* simple zero.

Remark: With Taylor's theorem (Thm. 26) we conclude that the first $n-1$ terms of the Taylor series of f at z_0 vanish, if z_0 is a zero of order n of f, i. e. the Taylor series of f at z_0 has the form

$$f(z) = \sum_{m=n}^{\infty} a_m (z - z_0)^m = (z - z_0)^n \underbrace{\sum_{m=0}^{\infty} a_{m+n}(z - z_0)^m}_{=:g(z)}, \quad z \in B_R(z_0), \qquad (912)$$

with $a_n \neq 0$. Unlike singularities, zeros are always isolated:

Theorem 31 *The zeros of a holomorphic function $f \not\equiv 0$ are isolated; that is, each of them has a neighborhood that contains no further zeros of f.*

Proof: The factor $(z - z_0)^n$ in (912) is zero only at $z = z_0$. Because $a_n \neq 0$ for a zero of order $n \in \mathbb{N}$, we have $g(z_0) \neq 0$. Because g is holomorphic in $B_R(z_0)$ (Thm. 25), it must be continuous in $B_R(z_0)$ and therefore $g(z) \neq 0$ in some open neighborhood $B_\varepsilon(z_0)$. Hence the same holds for f. $\square$

Poles are often caused by zeros in the denominator. For example, the poles of tan are located at the zeros of cos.

Theorem 32 *Let the complex function f be holomorphic in a domain $U \subseteq \mathbb{C}$ and assume that it has a zero of order $n \in \mathbb{N}$ at $z = z_0 \in U$. Then the function $1/f(z)$ has a pole of order n at z_0; and so does the function $h(z)/f(z)$, if h is analytic at z_0 and $h(z_0) \neq 0$.*

Riemann Sphere. Point at Infinity For the study of the behavior of complex functions as $|z| \to \infty$, it is useful to use a representation of complex numbers on the so-called *Riemann sphere.* This is a sphere $S \subset \mathbb{R}^3$ with diameter 1. We let its "north pole" be located at $(0, 0, 1)$. Then the "south pole" of the sphere is located at $(0, 0, 0)$. Now we represent any complex number $z \in \mathbb{C}$ by $(\mathrm{Re}z, \mathrm{Im}z, 0) \in \mathbb{R}^3$. The *stereographic projection* of z on S is given by the intersection point of the line from $(\mathrm{Re}z, \mathrm{Im}z, 0)$ to $(0, 0, 1)$ with S. There is exactly one such point on S for any $z \in \mathbb{C}$. Conversely, each point on S represents a complex number, except for the north pole $(0, 0, 1)$. This suggests that we introduce an additional point, called the *point at infinity* and denoted by ∞ ("infinity"), and let its image be $(0, 0, 1)$. The complex plane together with ∞ is called the *extended complex plane.*

Analytic or Singular at Infinity Instead of investigating f for large $|z|$, we may set $z = 1/w$ and investigate $f(z) = f(1/w) =: g(w)$ in a neighborhood of $w = 0$. We say that f is *analytic* or *singular*, respectively, *at infinity* if g is analytic or singular, respectively, at 0. We also define

$$g(0) = \lim_{w \to 0} g(w), \tag{913}$$

if this limit exists. We say that f has a *zero of order $n \in \mathbb{N}$ at inifinity* if g has a zero of order n at 0. Similarly for poles and essential singularities.
Examples:

1. $f(z) = 1/z^2$. Then we have $g(w) = w^2$, and so we say that f is analytic at ∞ and has a second-order zero at ∞.

2. $f(z) = z^3$. We have $g(w) = 1/w^3$, and so we say that f is singular at ∞ and has a third-order pole at ∞.

3. $f(z) = \exp(z)$. We have $g(w) = \exp(1/w)$, and so we say that f has an essential singularity at ∞. Similarly, cos and sin have an essential singularity at ∞.

We know from Liouville's theorem (Thm. 12) that a nonconstant entire function must be unbounded. Hence it has a singularity at ∞, which is a pole if the function is a polynomial or an essential singularity if it is not.

A holomorphic function whose only singularities in the complex plane (without ∞) are poles is called a *meromorphic function.* Examples are rational functions with nonconstant denominator, tan, cot, sec, and csc.

16.3 Residue Integration Method

The purpose of Cauchy's residual integration method is the evaluation of integrals

$$\oint_C f(z)\, dz \tag{914}$$

taken around a closed curve C. There are two cases:

- if f is holomorphic everywhere on C and inside C, such an integral (914) is zero by Cauchy's Integral Theorem (Thm. 8), and we are done.

- if f is singular at a point z_0 inside of C, but is otherwise holomorphic on C and inside of C, then f has a Laurent series representation near z_0 which converges in $\{z \in \mathbb{C} \,|\, 0 < |z - z_0| < R\}$, for some $R > 0$ (Thm. 28). By the coefficient formula (885), we have

  $$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz. \tag{915}$$

  If we can find the coefficient a_{-1} of the Laurent series representation of f at z_0 without using the integral formula (885), then we may use (915) to evaluate the integral. The coefficient with index -1 of the Laurent series of f at a singular point z_0 is called the *residue* of f at z_0, and we denote it by

$$a_{-1} = \mathrm{Res}_{z=z_0} f(z) \equiv \mathrm{Res}(f, z_0) \tag{916}$$

Example: $f(z) = z^{-4} \sin z$, $C = \{z \in \mathbb{C} \,|\, |z| = 1\}$ (counterclockwise). f is singular at 0 and 0 lies inside of C. From (869) we obtain the Laurent series of f at 0:

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-3}}{(2n+1)!} = \sum_{n=-2}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+5)!}, \quad |z| > 0, \tag{917}$$

so that the residue of f at 0 is given by $\mathrm{Res}(f, 0) = a_{-1} = -1/3! = -1/6$. Then we obtain

$$\oint_C \frac{\sin z}{z^4}\, dz = 2\pi i \mathrm{Res}(f, 0) = -\frac{\pi}{3} i. \tag{918}$$

Again, be careful to consider the right Laurent series: it must be the one which is valid in the immediate neighborhood of z_0.

Formulas for Residues If the singularity of f at z_0 is a pole of order $m \in \mathbb{N}$, then the residue of f at z_0 is given by

$$\mathrm{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left(\frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z) \right) \right). \tag{919}$$

Notice that if the function in the large brackets is continuous at z_0, then we may just evaluate it at z_0. To prove this, we consider the Laurent series of f valid near z_0:

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n, \quad z \in B_R(z_0) \setminus \{z_0\}, \tag{920}$$

with $a_{-m} \neq 0$. We multiply both sides by $(z - z_0)^m$ to obtain

$$g(z) := (z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_{n-m} (z - z_0)^n, \quad z \in B_R(z_0), \tag{921}$$

which is the Taylor series of g with center z_0. From Taylor's theorem (Thm. 26) we obtain for the coefficient $a_{-1} = a_{m-1-m}$:

$$a_{-1} = \frac{1}{(m-1)!} g^{(m-1)}(z_0). \tag{922}$$

If $f(z) = p(z)/q(z)$ with $p(z_0) \neq 0$ and where q has a simple zero at z_0, then f has a simple pole at z_0 (Thm. 32). The Taylor series of q at z_0 is given by

$$q(z) = (z - z_0) \sum_{n=1}^{\infty} \frac{1}{n!} q^{(n)}(z_0)(z - z_0)^{n-1}. \tag{923}$$

We also have, using the formula derived before for $m = 1$:

$$\begin{aligned} \operatorname{Res}(f, z_0) &= \lim_{z \to z_0} ((z - z_0) f(z)) && (924) \\ &= \lim_{z \to z_0} \left(\frac{p(z)}{\sum_{n=1}^{\infty} \frac{1}{n!} q^{(n)}(z_0)(z - z_0)^{n-1}} \right) = \frac{p(z_0)}{q'(z_0)}. && (925) \end{aligned}$$

Example: The function

$$f(z) = \frac{50z}{z^3 + 2z^2 - 7z + 4} = \frac{50z}{(z+4)(z-1)^2} \tag{926}$$

has a second-order pole at $z = 1$. With (919) we obtain the residue

$$\operatorname{Res}(f, 1) = \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 f(z) \right) = \lim_{z \to 1} \frac{d}{dz} \left(\frac{50z}{z+4} \right) = \lim_{z \to 1} \frac{200}{(z+4)^2} = 8. \tag{927}$$

Several Singularities Inside the Contour. Residue Theorem Residue integration can be extended from the case of a single singularity to the case of several singularities within the curve C. This is the purpose of the residue theorem.

Theorem 33 *Let the complex function f be holmorphic on a closed curve C and inside of C, except for finitely many singular points $z_1, \ldots, z_k$, $k \in \mathbb{N}$, inside of C. Then the integral of f taken counterclockwise around C is given by*

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, z_j). \tag{928}$$

Proof: We enclose each of the singular points z_j in a circle C_j with radius small enough that those k circles and C are all separated. Then f is holomorphic in the multiply connected domain bounded by $C, C_1, \ldots, C_k$, as well as on the boundary of this domain. With Cauchy's Integral Theorem (Thm. 8) we have

$$\oint_C f(z)\,dz = \sum_{j=1}^{k} \oint_{C_j} f(z)\,dz, \tag{929}$$

where all integrals are taken counterclockwise. Now we use (915) and (916) to obtain

$$\oint_{C_j} f(z)\,dz = 2\pi i \mathrm{Res}(f, z_j), \quad j = 1, \ldots, k. \tag{930}$$

Now just take the sum over j. □

Remark: For $k = 1$ and $f(z) = g(z)/(z - z_0)^{n+1}$, $n \in \mathbb{N}_0$, with g holomorphic on and inside of C, we obtain Cauchy's Integral Formula and Cauchy's Differentiation Formula (Thms. 10 & 11) for g. So Thm. 33 is a generalization of these.

Examples:

1. We consider the function

$$f(z) := \frac{4 - 3z}{z^2 - z} = \frac{4 - 3z}{z(z-1)}, \tag{931}$$

which is singular at $z \in \{0, 1\}$. Each singularity is a simple pole. We compute the residues

$$\mathrm{Res}(f, 0) = \lim_{z \to z_0} (z f(z)) = \left.\frac{4 - 3z}{z - 1}\right|_{z=0} = -4, \tag{932}$$

$$\mathrm{Res}(f, 1) = \lim_{z \to z_0} ((z-1) f(z)) = \left.\frac{4 - 3z}{z}\right|_{z=1} = 1. \tag{933}$$

Now we may consider four different curves C:

- both 0 and 1 lie inside of C. Then we obtain with Thm. 33

$$\oint_C f(z)\,dz = 2\pi i \left(\mathrm{Res}(f, 0) + \mathrm{Res}(f, 1)\right) = -6\pi i. \tag{934}$$

- 0 lies inside of C, 1 lies outside. Then we obtain

$$\oint_C f(z)\,dz = 2\pi i \mathrm{Res}(f, 0) = -8\pi i. \tag{935}$$

- 1 lies inside of C, 0 lies outside. We obtain

$$\oint_C f(z)\,dz = 2\pi i \mathrm{Res}(f,1) = 2\pi i. \tag{936}$$

- both 0 and 1 lie outside of C. Then we obtain with Thm. 8

$$\oint_C f(z)\,dz = 0. \tag{937}$$

2. $f(z) = (\tan z)/(z^2-1)$, $C = \{z \in \mathbb{C} \,|\, |z| = 3/2\}$. tan is holomorphic except at the zeros of cos, which lie at $(2k+1)\pi/2$, $k \in \mathbb{Z}$, i. e. outside of C. The zeros of the denominator lie at $z = \pm 1$, i. e. inside of C. f has simple poles at $z = \pm 1$. We compute the residues of f at ± 1:

$$\mathrm{Res}(f,\pm 1) = \left.\frac{\tan z}{2z}\right|_{z=\pm 1} = \pm\frac{1}{2}\tan(\pm 1) = \frac{1}{2}\tan 1. \tag{938}$$

With the residue theorem (Thm. 33) we obtain

$$\oint_C f(z)\,dz = 2\pi i\,(\mathrm{Res}(f,-1) + \mathrm{Res}(f,1)) = 2\pi i \tan 1. \tag{939}$$

3. $C = \{z \in \mathbb{C} \,|\, 9(\mathrm{Re}z)^2 + (\mathrm{Im}z)^2 = 9\}$ (counterclockwise),

$$f(z) = \frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}. \tag{940}$$

By the linearity of the integral, we may integrate each term of f separately. The first term of f has simple poles at $z \in \{\pm 2, \pm 2i\}$, but only the points $\pm 2i$ lie inside of C. We compute the residuals

$$\mathrm{Res}\left(\frac{ze^{\pi z}}{z^4-16}, \pm 2i\right) = \left.\frac{ze^{\pi z}}{4z^3}\right|_{z=\pm 2i} = \frac{\pm 2ie^{\pm 2\pi i}}{4(\pm 2i)^3} = \frac{\pm 2i}{\mp 32 i} = -\frac{1}{16}. \tag{941}$$

The second term of f has an essential singularity at 0, which lies inside of C. Its Laurent series at 0 is given by

$$ze^{\pi/z} = z\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\pi}{z}\right)^n = \sum_{n=0}^{\infty}\frac{\pi^n}{n!}z^{-n+1} = \sum_{n=-\infty}^{-1}\frac{\pi^{1-n}}{(1-n)!}z^n \tag{942}$$

so that

$$\operatorname{Res}\left(ze^{\pi/z},0\right)=\frac{\pi^2}{2}. \tag{943}$$

Now from the residue theorem (Thm. 33), we obtain

$$\begin{aligned}\oint_C f(z)\,dz &= 2\pi i\operatorname{Res}\left(\frac{ze^{\pi z}}{z^4-16},-2i\right)+2\pi i\operatorname{Res}\left(\frac{ze^{\pi z}}{z^4-16},2i\right)+\\ &\quad +2\pi i\operatorname{Res}\left(ze^{\pi/z},0\right)=\pi\left(\pi^2-\frac{1}{4}\right)i\simeq 30.2i. \qquad (944)\end{aligned}$$

16.4 Residue Integration of Real Integrals

Integrals of Rational Functions of $\cos\vartheta$ and $\sin\vartheta$ We first consider integrals of the type

$$J=\int_0^{2\pi}F(\cos\vartheta,\sin\vartheta)\,d\vartheta, \tag{945}$$

where F is a real rational function of $\cos\vartheta$ and $\sin\vartheta$ with finite values on the interval of integration.

We set $z=e^{i\vartheta}$ and obtain

$$\cos\vartheta = \frac{1}{2}\left(e^{i\vartheta}+e^{-i\vartheta}\right)=\frac{1}{2}\left(z+\frac{1}{z}\right), \tag{946}$$

$$\sin\vartheta = \frac{1}{2i}\left(e^{i\vartheta}-e^{-i\vartheta}\right)=\frac{1}{2i}\left(z-\frac{1}{z}\right). \tag{947}$$

So we obtain a complex rational function f with $f(e^{i\vartheta})=F(\cos\vartheta,\sin\vartheta)$. We also find that $d\vartheta=dz/(iz)$, and therefore the integral (945) can be evaluated by the complex line integral

$$J=\oint_C\frac{f(z)}{iz}\,dz, \tag{948}$$

with $C=\{z\in\mathbb{C}\,|\,|z|=1\}$, oriented counterclockwise.
Example: $F(\cos\vartheta,\sin\vartheta)=1/(\sqrt{2}-\cos\vartheta)$. We use $z=e^{i\vartheta}$ to obtain

$$F(\cos\vartheta,\sin\vartheta)=\frac{1}{\sqrt{2}-\frac{1}{2}\left(e^{i\vartheta}+e^{-i\vartheta}\right)}=\frac{-2e^{i\vartheta}}{e^{2i\vartheta}-2\sqrt{2}e^{i\vartheta}+1}\overset{!}{=}f(e^{i\vartheta}), \tag{949}$$

so that we obtain

$$f(z) = \frac{-2z}{z^2 - 2\sqrt{2}z + 1} = \frac{-2z}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}. \tag{950}$$

Now

$$\int_0^{2\pi} \frac{1}{\sqrt{2} - \cos\vartheta}\, d\vartheta = -\frac{2}{i} \oint_C \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}\, dz. \tag{951}$$

The integrand has simple poles at $z = \sqrt{2} \pm 1$; only the pole at $z = \sqrt{2} - 1$ lies inside the unit circle. By the residue theorem (Thm. 33) we have

$$\begin{aligned} -\frac{2}{i} \oint_C \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}\, dz &= -4\pi \mathrm{Res}\left(\frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}, \sqrt{2} - 1\right) \\ &= -4\pi \left. \frac{1}{z - \sqrt{2} - 1} \right|_{z = \sqrt{2} - 1} = 2\pi. \end{aligned} \tag{952}$$

Improper Integral of the First Kind We consider real integrals over the whole real axis:

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{a \to -\infty} \int_a^c f(x)\, dx + \lim_{b \to \infty} \int_c^b f(x)\, dx, \tag{953}$$

for any $c \in \mathbb{R}$. The *Cauchy principal value* of the integral (953) is given by

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\, dx := \lim_{R \to \infty} \int_{-R}^{R} f(x)\, dx. \tag{954}$$

The principal value of the integral may exist even if the integral does not. We now consider a closed curve $C = [-R, R] \cup S_R$ (oriented counterclockwise), which coincides with the interval $[-R, R]$ on the real axis, and where S_R is a curve in the complex plane such that C is closed and encloses all singularities of f in the upper half-plane of the complex plane (we could also choose the lower half-plane). By the residue theorem (Thm. 33) we know that

$$\oint_C f(z)\,dz = \int_{S_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{\mathrm{Im} z_j > 0} \mathrm{Res}(f, z_j). \tag{955}$$

We want to show that the integral over S_R vanishes under the following assumptions: We assume that f is a rational function $f(z) = p(z)/q(z)$ (meromorphic), with $q(x) \neq 0$, $x \in \mathbb{R}$ (i. e. no poles on the real axis), and $\deg q \geq \deg p + 2$. We also assume that S_R is a semicircle in the upper half-plane with center 0 and radius R, where $R > 0$ has to be chosen large enough so that $C = [-R, R] \cup S_R$ encloses all poles of f (zeros of q) in the upper half-plane. The length of $S_R = \{z \in \mathbb{C} \,|\, |z| = R, \mathrm{Im} z \geq 0\}$ is πR. With the assumption on the degrees of p and q, we may find two constants $k, R_0 > 0$ so that

$$|f(z)| < \frac{k}{|z|^2}, \quad |z| > R_0. \tag{956}$$

With the ML-inequality (Prop. 2) we obtain

$$\left| \int_{S_R} f(z)\,dz \right| < \frac{k}{R^2} \pi R = \frac{k\pi}{R}, \quad R > R_0, \tag{957}$$

and so the integral over S_R approaches 0 as $R \to \infty$. We obtain

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\mathrm{Im} z_j > 0} \mathrm{Res}(f, z_j). \tag{958}$$

Example: The function $f(x) := 1/(1 + x^4)$ satisfies the degree requirement. (Simple) Poles of f are located at the zeros of $z^4 + 1$, i. e. in $\{e^{i(2k+1)\pi/4}\}_{k=0}^{3}$; only the first two of these poles lie in the upper half-plane. With $\mathrm{Res}(f, z_0) = p(z_0)/q(z_0)$ (Section 16.3) we compute

$$\mathrm{Res}\left(f, e^{i\pi/4}\right) = \left.\frac{1}{4z^3}\right|_{z=e^{i\pi/4}} = \frac{1}{4} e^{-i3\pi/4} = -\frac{1}{4} e^{i\pi/4}, \tag{959}$$

$$\mathrm{Res}\left(f, e^{3i\pi/4}\right) = \left.\frac{1}{4z^3}\right|_{z=e^{3i\pi/4}} = \frac{1}{4} e^{-9i\pi/4} = \frac{1}{4} e^{-i\pi/4}. \tag{960}$$

With the formula derived above, we obtain

$$\int_{-\infty}^{\infty} f(x)\,dx = -\frac{2\pi i}{4}\left(e^{i\pi/4} - e^{-i\pi/4}\right) = -\frac{\pi i}{2} 2i \sin\left(\frac{\pi}{4}\right) = \frac{\pi}{\sqrt{2}}. \tag{961}$$

Because $f(x) = f(-x)$ is an even function, we also obtain

$$\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2\sqrt{2}}. \tag{962}$$

Fourier Integrals We may extend the method described before to integrals of the form

$$\int_{-\infty}^{\infty} f(x)\cos(sx)\,dx, \quad \int_{-\infty}^{\infty} f(x)\sin(sx)\,dx, \quad s \in \mathbb{R}. \tag{963}$$

We consider the corresponding integral

$$\oint_C f(z)e^{isz}\,dz, \quad s > 0, \tag{964}$$

over the same contour C as before (if $s < 0$ we would go via the lower half-plane). Now the integrand is the function $f(z)e^{isz}$, where we assume again that $f = p/q$ is a rational function with $q(x) \neq 0$, $x \in \mathbb{R}$, and that $\deg q \geq \deg p + 2$. The estimate used before works again because

$$|e^{isz}| = e^{-s\mathrm{Im}z} \leq 1, \quad s > 0,\ \mathrm{Im}z \geq 0, \tag{965}$$

which implies that $|f(z)e^{isz}| \leq |f(z)|$, and we may proceed as before. So we have

$$\int_{-\infty}^{\infty} f(x)e^{isx}\,dx = 2\pi i \sum_{\mathrm{Im}z_j > 0} \mathrm{Res}(f(z)e^{isz}, z_j), \tag{966}$$

and if we take the real and imaginary parts on both sides, we obtain

$$\int_{-\infty}^{\infty} f(x)\cos(sx)\,dx = -2\pi \sum_{\mathrm{Im} z_j > 0} \mathrm{Im}\left(\mathrm{Res}(f(z)e^{isz}, z_j)\right), \tag{967}$$

$$\int_{-\infty}^{\infty} f(x)\sin(sx)\,dx = 2\pi \sum_{\mathrm{Im} z_j > 0} \mathrm{Re}\left(\mathrm{Res}(f(z)e^{isz}, z_j)\right). \tag{968}$$

Example: $f(x) = 1/(k^2 + x^2)$, $k > 0$. The integrand $e^{isz}/(k^2 + z^2)$ has two simple poles at $z = \pm ik$, and only $z = ik$ lies in the upper half-plane. We compute

$$\mathrm{Res}\left(f(z)e^{isz}, ik\right) = \left.\frac{e^{isz}}{2z}\right|_{z=ik} = \frac{e^{-sk}}{2ik}, \tag{969}$$

and

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{k^2 + x^2}\,dx = 2\pi i \frac{e^{-sk}}{2ik} = \frac{\pi}{k} e^{-ks} \in \mathbb{R}. \tag{970}$$

The imaginary part of this number is 0, so that the Fourier sine coefficient of f is also 0.

Improper Integral from Section 12.6 In Section 12.6 we had used the integral (208)

$$\int_{0}^{\infty} e^{-ax^2}\cos(bx)\,dx, \quad a, b > 0. \tag{971}$$

We will now show how to compute its value. The function $f(z) := e^{-z^2}$ is entire. We define the closed curve

$$\begin{aligned} C_s := \; & \{\mathrm{Re} z \in [-R, R], \mathrm{Im} z = 0\} \cup \{\mathrm{Re} z = R, \mathrm{Im} z \in [0, s]\} \cup \\ & \cup \{\mathrm{Re} z \in [-R, R], \mathrm{Im} z = s\} \cup \{\mathrm{Re} z = -R, \mathrm{Im} z = [0, s]\}, \end{aligned} \tag{972}$$

for $s > 0$ (oriented counterclockwise). With $z = x + iy$ we obtain, with Thm. 8

$$\begin{aligned} 0 \;=\; & \oint_{C_s} e^{-z^2}\, dz = \int_{-R}^{R} e^{-x^2}\, dx + ie^{-R^2} \int_0^s e^{y^2-2iRy}\, dy + \\ & +e^{s^2} \int_R^{-R} e^{-x^2-2isx}\, dx - ie^{-R^2} \int_s^0 e^{y^2-2iRy}\, dy \end{aligned} \tag{973}$$

As $R \to \infty$, the terms with e^{-R^2} vanish. We obtain for the real part

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(2sx)\, dx = e^{-s^2} \int_{-\infty}^{\infty} e^{-x^2}\, dx \tag{974}$$

Because the integrand is an even function of x we obtain

$$\int_0^{\infty} e^{-x^2} \cos(2sx)\, dx = \frac{e^{-s^2}}{2} \int_{-\infty}^{\infty} e^{-x^2}\, dx. \tag{975}$$

After transformation of the original integral, we have with $s := b/(2\sqrt{a})$

$$\int_0^{\infty} e^{-ax^2} \cos(bx)\, dx \;=\; \frac{1}{\sqrt{a}} \int_0^{\infty} e^{-x^2} \cos\left(2\frac{b}{2\sqrt{a}}x\right) dx \tag{976}$$

$$= \frac{1}{2} \exp\left(-\frac{b^2}{4a}\right) \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-x^2}\, dx \tag{977}$$

The remaining integral can be evaluated in various ways, and yields $\sqrt{\pi}$, for example like this:

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 \;&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\varphi = 2\pi \int_0^{\infty} re^{-r^2}\, dr \\ &=\; -\pi\, e^{-r^2}\Big|_0^{\infty} = \pi. \end{aligned} \tag{978}$$

So we obtain (208).

17 Conformal Mapping

We now go back to the "main road", which leads from Chapter 12 (PDEs), via Chapters 13, 14 & 17, to Chapter 18 (Potential Theory in 2D).

In this chapter, we interpret complex functions $f : \mathbb{C} \to \mathbb{C}$ as mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$, and thus consider a geometric approach to complex analysis. This new approach gives new insight on holomorphic functions; its importance is similar to the study of curves $\{(x, f(x)) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}\}$ for real functions.

Conformal mapping will also yield a standard method for solving boundary value problems in two-dimensional potential theory, by transforming a complicated region into a simple one.

17.1 Geometry of Holomorphic Functions: Conformal Mapping

We interpret a complex function f as a mapping of its domain of definition (in $\mathbb{C} \simeq \mathbb{R}^2$) onto its image (in $\mathbb{C} \simeq \mathbb{R}^2$):

$$w = f(z) = u(x, y) + iv(x, y), \quad z = x + iy. \tag{979}$$

We shall refer to such a mapping as "the mapping $w = f(z)$", and we refer to the "z-plane" and the "w-plane" to distinguish between the spaces containing the domain and image of f. We use cartesian coordinates (x, y) and (u, v) or polar coordinates (r, ϑ) and (R, φ) to represent points in the z- and w-planes, respectively.

Example: The mapping $w = z^2$. Using polar forms $z = re^{i\vartheta}$ in the z-plane and $w = Re^{i\varphi}$ in the w-plane, we have $w = z^2 = r^2 e^{2i\vartheta}$, and therefore $R = r^2$, $\varphi = 2\vartheta$ (cf. Section 13.2). Hence circles $|z| = r = r_0$ in the z-plane are mapped onto circles $|w| = R = r_0^2$ in the w-plane and rays $\arg z = \vartheta = \vartheta_0$ in the z-plane onto rays $\arg w = \varphi = 2\vartheta_0$ in the w-plane.

In Cartesian coordinates we have $z = x + iy$ and

$$u = \mathrm{Re} w = \mathrm{Re}(z^2) = x^2 - y^2, \quad v = \mathrm{Im} w = \mathrm{Im}(z^2) = 2xy. \tag{980}$$

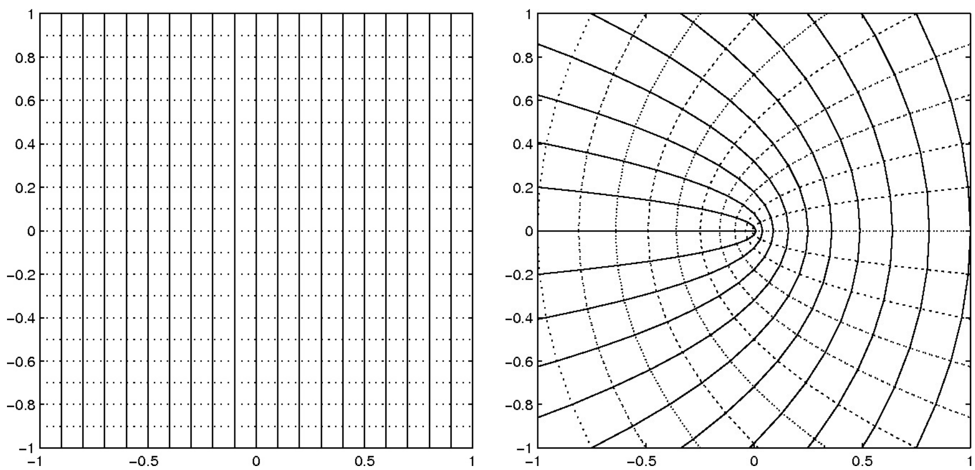
Hence vertical lines $x = c = $ const. in the z-plane are mapped onto $u = c^2 - y^2$, $v = 2cy$. So we obtain

$$v^2 = 4c^2 y^2 = 4c^2 \left(c^2 - u\right), \tag{981}$$

which defines for each $c \in \mathbb{R}$ a parabola in the w-plane that opens to the left. Similarly, horizontal lines $y = k =$ const. in the z-plane are mapped onto parabolas in the w-plane,

$$v^2 = 4k^2 \left(k^2 + u\right), \tag{982}$$

which are parabolas opening to the right, for each $k \in \mathbb{R}$.



Conformal Mapping A mapping $w = f(z)$ is called *conformal* if it preserves angles between oriented curves, in magnitude as well as in sense.

Theorem 34 *The mapping $w = f(z)$ by a holomorphic function f is conformal, except at critical points, that is, points at which the derivative of f vanishes.*

<u>Proof:</u> We consider a curve $C = \{\gamma(t) \,|\, t \in [a, b]\}$ in the z-plane, with $\gamma : [a, b] \to \mathbb{C}$ (cf. Section 14.1). The tangent to C at $z_0 = \gamma(t_0) \in C$ is given by $\dot{\gamma}(t_0) \in \mathbb{C}$, which is to be understood as a vector in the z-plane, attached to z_0. The image of C under f is given by the curve $C^* = \{f(\gamma(t)) \,|\, t \in [a, b]\}$ in the w-plane, and the tangent vector to C^* at $w_0 = f(z_0) = f(\gamma(t_0)) \in C^*$ is given by the chain rule: $f'(z_0)\dot{\gamma}(t_0) \in \mathbb{C}$ (because f is holomorphic, it is differentiable at z_0). Now the angle between the two tangent vectors is given by the argument of the quotient, $\arg(f'(z_0))$ (cf. Section 13.2), which is well-defined if $f'(z_0) \neq 0$. We have found that the tangent vector at z_0 to any curve C with $z_0 \in C$ is rotated by the angle $\arg(f'(z_0))$ under f. This implies conformality of the mapping, if $f'(z_0) \neq 0$. $\square$

<u>Examples:</u>

1. The mapping $w = z^n$, $n = 2, 3, \ldots$, is conformal, except at $z = 0$, where $f'(0) = 0$.

2. The mapping $w = f(z) = z + 1/z$. We obtain

$$w = u + iv = r(\cos\vartheta + i\sin\vartheta) + \frac{1}{r}(\cos\vartheta - i\sin\vartheta), \tag{983}$$

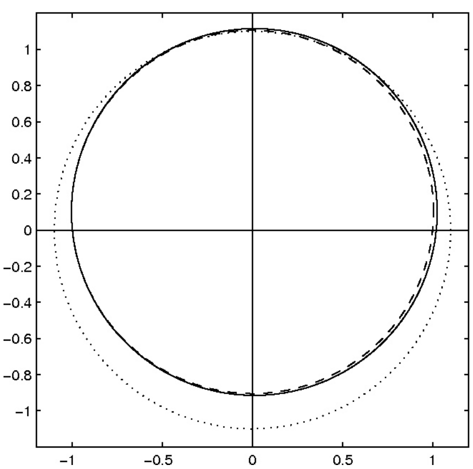
and therefore

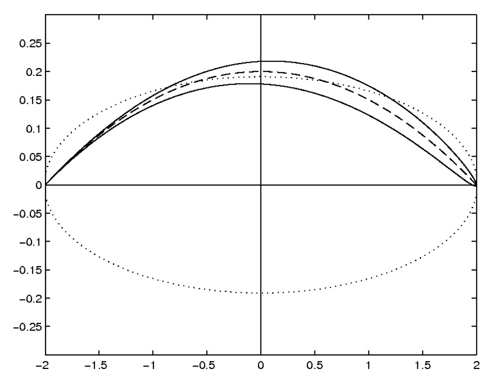
$$u = a\cos\vartheta,\ v = b\sin\vartheta, \quad a := r + \frac{1}{r},\ b := r - \frac{1}{r}. \tag{984}$$

For $r = \text{const.}$, $r \neq 1$, we obtain

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = \cos^2\vartheta + \sin^2\vartheta = 1, \tag{985}$$

and therefore circles $|z| = \text{const.}$, $|z| \neq 1$, in the z-plane are mapped to ellipses in the w-plane. Circles in the z-plane with center slightly away from the origin, however, may be mapped to very different shapes in the w-plane, such as a Joukowski airfoil.





The derivative of f is given by

$$f'(z) = 1 - \frac{1}{z^2} = \frac{z^2 - 1}{z^2} = \frac{(z+1)(z-1)}{z^2}, \tag{986}$$

and therefore the mapping is not conformal at $z = \pm 1$.

3. The mapping $w = f(z) = \exp z$. We find $w = \exp(x+iy) = \exp x \exp(iy)$, so that $|w| = e^x$ and $\arg w = y \pmod{2\pi}$. The derivative of f vanishes nowhere, and so the fundamental region $y \in (-\pi, \pi]$ in the z-plane is mapped bijectively and conformally onto the w-plane without the origin, $\mathbb{C} \setminus \{0\}$ (Section 13.5).

4. The mapping $w = f(z) = \mathrm{Log} z$. To obtain the mapping by the inverse $z = f^{-1}(w)$ of $w = f(z)$, we interchange the roles of the z and the w variables. For this example we obtain $z = \exp w$. We know from the previous example that exp maps the fundamental region $v \in (-\pi, \pi]$ in the w-plane onto the z-plane except for the origin. Hence $w = f(z) = \mathrm{Log} z$ maps the z-plane without the origin and cut along the negative real axis (where $\mathrm{Im}(\mathrm{Log} z)$ jumps by 2π and thus is not differentiable, cf. Thm. 5) conformally onto the horizontal strip $v \in (-\pi, \pi]$ in the w-plane.

By definition of the derivative we have

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)|. \tag{987}$$

Therefore, the mapping $w = f(z)$ magnifies (or shortens) the lengths of short lines in the z-plane by approximately the factor $|f'(z_0)|$. The image (in the w-plane) of a small figure in the z-plane conforms to the original figure in the sense that it has approximately the same shape. Since $f'(z_0)$ varies from point to point, however, a large figure in the z-plane may have an image in the w-plane whose shape is quite different from that of the original figure.

More on the condition $f'(z_0) \neq 0$: From Section 13.4 and from Thm. 2 in particular, we know that

$$|f'(z_0)|^2 = |u_x + iv_x|^2 = (u_x)^2 + (v_x)^2 = u_x v_y - u_y v_x. \tag{988}$$

If we write the mapping in $\mathbb{R}^2$, we have $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $\boldsymbol{F} := (u, v)^\top$. The Jacobian of $\boldsymbol{F}$ is given by

$$\underline{\boldsymbol{DF}} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \quad \Rightarrow \quad \det \underline{\boldsymbol{DF}} = u_x v_y - u_y v_x \tag{989}$$

(we have omitted the arguments), so that we have $|f'(z_0)|^2 = \det(\underline{\boldsymbol{DF}}(x_0, y_0))$. Therefore $f'(z_0) \neq 0$ implies that the mapping $w = f(z)$ is invertible in a small neighborhood of z_0 (Inverse Function Theorem).

17.2 Linear Fractional Transformations

An important class of mappings are linear fraction transformations (or Möbius transformations), which are of the form

$$w = f(z) = \frac{az+b}{cz+d}, \tag{990}$$

$a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$. Then we have

$$f'(z) = \frac{a(cz+d)-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0, \quad z \in \mathbb{C}, \tag{991}$$

and by Thm. 34, the mapping (990) is conformal everywhere in $\mathbb{C}$. Special cases of (990) include

- $a = d = 1$, $c = 0$: $w = z + b$ (translations)
- $|a| = d = 1$, $b = c = 0$: $w = az$, $|a| = 1$ (rotations)
- $c = 0$, $d = 1$: $w = az + b$, $a \neq 0$ (linear transformations)
- $a = d = 0$, $b = c = 1$: $w = 1/z$ (inversion in the unit circle)

Example: The inversion $w = 1/z$. We use polar forms $z = re^{i\vartheta}$, $w = Re^{i\varphi}$ to obtain

$$Re^{i\varphi} = \frac{1}{re^{i\vartheta}} = \frac{1}{r}e^{-i\vartheta} \quad \Rightarrow \quad R = \frac{1}{r},\ \varphi = -\vartheta \pmod{2\pi}. \tag{992}$$

Hence the unit circle $|z| = r = 1$ in the z-plane is mapped onto the unit circle $|w| = R = 1$ in the w-plane (oriented in the opposite direction). We look at points (x, y) in the z-plane which satisfy the equation

$$A(x^2+y^2) + Bx + Cy + D = 0, \quad A, B, C, D \in \mathbb{R}. \tag{993}$$

For $A = 0$, the points lie on a straight line ($Bx + Cy = -D$), whereas for $A \neq 0$, the points lie on a circle:

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \frac{B^2 + C^2 - 4AD}{4A^2} \tag{994}$$

With $z = x + iy$ (993) becomes

$$Az\overline{z} + \frac{B}{2}(z+\overline{z}) + \frac{C}{2i}(z-\overline{z}) + D = 0. \tag{995}$$

Replacing $z = 1/w$ and multiplying by $w\overline{w}$ we have

$$A + \frac{B}{2}(w + \overline{w}) - \frac{C}{2i}(w - \overline{w}) + Dw\overline{w} = 0. \tag{996}$$

With $w = u + iv$, we obtain

$$D(u^2 + v^2) + Bu - Cv + A = 0, \tag{997}$$

which is of the same form as (993). We conclude that circles and straight lines in the z-plane are mapped to circles and straight lines in the w-plane by the mapping $w = 1/z$.

This is not only true for the inversion, but for every linear fractional transformation. Before we prove that, it is useful to notice that every linear fractional transformation can be written as a composition of inversions and linear transformations:

- If $c = 0$, we must have $d \neq 0$. Then we have $w = \frac{a}{d}z + \frac{b}{d}$.
- If $c \neq 0$, we write

$$w = \frac{az+b}{cz+d} = \frac{acz+bc}{c(cz+d)} = \frac{a(cz+d) - ad + bc}{c(cz+d)} \tag{998}$$

$$= \frac{a}{c} - \frac{ad-bc}{c(cz+d)} = -\frac{ad-bc}{c}\frac{1}{cz+d} + \frac{a}{c}. \tag{999}$$

  This is a composition $w = (f_1 \circ f_2 \circ f_3)(z) = f_1(f_2(f_3(z)))$, where

$$f_1(z) = -\frac{ad-bc}{c}z + \frac{a}{c}, \quad f_2(z) = \frac{1}{z}, \quad f_3(z) = cz + d. \tag{1000}$$

  f_2 is the inversion, and f_1, f_3 are linear transformations.

Therefore, if we can prove a theorem for both the inversion and for linear transformation, then this theorem is valid for every linear fractional transformation.

Theorem 35 *Every linear fractional transformation (990) maps circles and lines in the z-plane to circles and lines in the w-plane.*

Proof: We already know that his is true for the inversion $w = 1/z$. We now prove the same property for linear transformations $w = az + b$, $a \neq 0$. We replace z by $(w-b)/a$ in (995) and use $w = u + iv$ to obtain

$$\begin{aligned}
0 &= A\frac{w-b}{a}\overline{\frac{w-b}{a}} + \frac{B}{2}\left(\frac{w-b}{a} + \overline{\frac{w-b}{a}}\right) + \frac{C}{2i}\left(\frac{w-b}{a} - \overline{\frac{w-b}{a}}\right) + D \\
&= \frac{A}{|a|^2}w\overline{w} + \frac{1}{|a|^2}\mathrm{Re}\left(\left((B-iC)\overline{a} - 2A\overline{b}\right)w\right) + A\left|\frac{b}{a}\right|^2 - B\mathrm{Re}\left(\frac{b}{a}\right) - C\mathrm{Im}\left(\frac{b}{a}\right) + D \\
&= \frac{A}{|a|^2}(u^2+v^2) + \frac{B\mathrm{Re}a - C\mathrm{Im}a - 2A\mathrm{Re}b}{|a|^2}u + \frac{B\mathrm{Im}a + C\mathrm{Re}a - 2A\mathrm{Im}b}{|a|^2}v + \\
&\quad + A\left|\frac{b}{a}\right|^2 - B\mathrm{Re}\left(\frac{b}{a}\right) - C\mathrm{Im}\left(\frac{b}{a}\right) + D, \qquad (1001)
\end{aligned}$$

which is again of the save form as (993). Now we use that every linear fractional transformation (990) is a composition of two linear transformations and an inversion. □

Extended Complex Plane Linear fractional transformations may be inverted, which yields another linear fractional transformation:

$$w = f(z) = \frac{az+b}{cz+d}, \quad z = f^{-1}(w) = \frac{dw-b}{-cw+a}. \qquad (1002)$$

For $c \neq 0$, the point $z = -d/c$ in the z-plane is mapped to ∞ in the w-plane. In the same way, $w = a/c$ is the image of $z = \infty$ in the w-plane. Therefore, every linear fractional transformation is defined on the extended complex plane $\mathbb{C} \cup \{\infty\}$, and is a bijective conformal map onto itself.

Fixed Points *Fixed points* of a mapping $w = f(z)$ are points in the z-plane that are mapped to the same points in the w-plane: They are the solutions of the nonlinear equation $f(z) = z$.
Examples:

- The *identity mapping* $w = z$ has every point as a fixed point.
- The mapping $w = \overline{z}$ has infinitely many fixed points (the points on the real axis $\mathrm{Im}z = 0$).
- The mapping $w = 1/z$ has two fixed points, $z = \pm 1$

- A rotation $w = az$, $|a| = 1$, has one fixed point, $z = 0$
- A translation $w = z + b$, $b \neq 0$, has no fixed point

For a linear fractional transformation, we obtain the fixed points

$$\frac{az+b}{cz+d} = z \quad \Leftrightarrow \quad cz^2 - (a-d)z - b = 0. \tag{1003}$$

From this we can prove

Theorem 36 *A linear fractional transformation, not the identity, has at most two fixed points. If a linear fractional transformation is known to have three or more fixed points, it must be the identity mapping, $w = z$.*

Proof: For $c \neq 0$, we have a quadratic equation for z, which has two solutions in $\mathbb{C}$. For $c = 0$, $a \neq d$, we have a linear equation with one solution in $\mathbb{C}$. We have infinitely many solutions if and only if $a = d \neq 0$, $b = c = 0$, in which case we have the identity mapping. $\square$

17.4 Conformal Mapping by Other Functions

So far, we have discussed the mapping by z^n, $z + 1/z$, $\exp z$, and $\mathrm{Log} z$, and linear fractional transformations. In this section we shall turn to the mapping by trigonometric and hyperbolic holomorphic functions. Formulas derived for these functions in Section 13.6 will be helpful.

1. Sine Function: The mapping $w = \sin z$ satisfies (648)

$$u = \sin x \cosh y, \quad v = \cos x \sinh y. \tag{1004}$$

   Because sin is 2π-periodic, it will not be one-to-one if we consider it in the full z-plane. Therefore, we restrict z to the vertical strip $S := \{z \in \mathbb{C} \,|\, |\mathrm{Re} z| \leq \pi/2\}$. The derivative $f'(z) = \cos z$ vanishes at $z = \pm\pi/2$, so the mapping is not conformal on the boundary of S. This maps vertical lines in the z-plane ($x =$ const.) onto hyperbolas in the w-plane, and horizontal lines in the z-plane ($y =$ const.) onto ellipses in the w-plane:

$$\begin{aligned} x = \text{const.}: \quad & \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = \cosh^2 y - \sinh^2 y &= 1 \quad & \text{(hyperbolas)}, \\ y = \text{const.}: \quad & \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = \sin^2 x + \cos^2 x &= 1 \quad & \text{(ellipses)}. \end{aligned}$$

Exceptions are the lines $x = \pm\pi/2$, which are mapped onto $[1, \infty)$ and $(-\infty, -1]$, respectively, the line $y = 0$, which is mapped onto $[-1, 1]$, and the line $x = 0$, which is mapped onto the line $u = 0$.

2. Cosine Function: we write

$$w = \cos z = \sin\left(z + \frac{\pi}{2}\right), \tag{1005}$$

so that this mapping is a composition of a translation to the right by $\pi/2$ and of the mapping by sin, which was discussed before.

3. Hyperbolic Sine: Here we use the relation

$$w = \sinh z = -i \sin(iz) \tag{1006}$$

to see that this mapping is a composition of a counterclockwise rotation by $90°$, followed by the sine mapping, and followed by a clockwise rotation by $90°$.

4. Hyperbolic Cosine: We use the relation

$$w = \cosh z = \cos(iz), \tag{1007}$$

which means that this mapping is a composition of a counterclockwise rotation by $90°$, followed by the cosine mapping. The semi-infinite strip $\{z \in \mathbb{C} \,|\, x \geq 0,\ y \in [0, \pi]\}$ in the z-plane is mapped onto the upper half of the w-plane ($v \geq 0$) by the cosh mapping. We have

$$u = \cosh x \cos y, \quad v = \sinh x \sin y. \tag{1008}$$

$y \in [0, \pi]$ implies that $\sin y$ takes on every non-negative value. From $x \geq 0$ we have $\sinh x \geq 0$, and therefore $v \geq 0$. Furthermore, $\cos y$ takes on all values in $[-1, 1]$ and $\cosh x \geq 1$. Therefore u takes on every real number.

5. Tangent Function: We write

$$w = \tan z = \frac{\sin z}{\cos z} = \frac{\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)}{\frac{1}{2}\left(e^{iz} + e^{-iz}\right)} = -i\frac{e^{2iz} - 1}{e^{2iz} + 1}. \tag{1009}$$

With $Z := e^{2iz}$ this becomes

$$w = -i\frac{Z - 1}{Z + 1} = -iW, \quad W := \frac{Z - 1}{Z + 1}. \tag{1010}$$

Therefore, the tangent mapping is a linear fraction transformation, preceded by an exponential mapping and followed by a clockwise rotation by 90°.

With $Z = e^{-2y}e^{2ix}$, we see that the strip $S := \{z \in \mathbb{C} \mid x \in (-\pi/4, \pi/4)\}$ in the z-plane is mapped onto the right half of the Z-plane, $X > 0$. The linear fractional transformation maps the right half of the Z-plane onto the interior of the unit disk in the W-plane. A clockwise rotation by $\pi/2$ follows, so that $w = \tan z$ maps the strip S in the z-plane onto the interior of the unit disk in the w-plane.

18 Complex Analysis and Potential Theory

The theory of solutions of Laplace's equation (cf. Sec. 12.10) is called *potential theory.* Laplace's equations occurs in many fields such as gravitation, electrostatics, heat conduction, fluid flow etc. In two space dimensions, and in cartesian coordinates, we are interested in solutions of the second-order PDE

$$\Delta\Phi = \Phi_{xx} + \Phi_{yy} = 0, \quad \text{in } \Omega \subseteq \mathbb{R}^2, \tag{1011}$$

for the (real-valued) *potential* $\Phi : \Omega \to \mathbb{R}$. We know from Section 13.4 that solutions of (1011) with continuous second derivatives (harmonic functions) are closely related to holomorphic functions $\Phi + i\Psi$ where Ψ is a harmonic conjugate of Φ.

The restriction to two space dimensions is often justified when the potential Φ is independent of one of the space coordinates. In this last chapter of the lecture, we shall consider this connection and its consequences in detail and illustrate it by typical boundary value problems from electrostatics, heat conduction, and hydrodynamics.

18.1 Electrostatic Fields

The electric force $\boldsymbol{F}$ $[N]$ between charged particles is given by Coulomb's law. This force is the gradient of a function Φ: $\boldsymbol{F} = q\boldsymbol{E}$, $\boldsymbol{E} = -\nabla\Phi$, with the electric field $\boldsymbol{E}$ $[\mathrm{Vm}^{-1}] = [\mathrm{NC}^{-1}]$ and the *electrostatic potential* Φ [V] (cf. Sec. 12.10; there we looked at Newton's law of universal gravitation, but Coulomb's law has a similar form). *Equipotential surfaces* are level sets of Φ ($\Phi \equiv$ const.; lines in 2D). The electric field is perpendicular to these surfaces.

<u>Examples:</u>

1. Two parallel plates kept at potentials Φ_1, Φ_2, respectively, extending to infinity. Choose a cartesian coordinate system (x, y, z) such that the plates are parallel to the yz-plane, and located at $x = -1$ and $x = 1$. Due to translation invariance of the problem in both y- and z-direction, we may restrict our consideration to the x-direction, where we have a second-order ODE with Dirichlet boundary conditions:

$$\Phi'' = 0, \quad \text{in } (-1, 1), \qquad \Phi(-1) = \Phi_1,\ \Phi(1) = \Phi_2. \tag{1012}$$

The general solution is given by $\Phi(x) = ax + b$, and the constants a, b

are determined from the boundary conditions:

$$\begin{array}{rcl} \Phi(-1) & = & -a+b=\Phi_1 \\ \Phi(1) & = & a+b=\Phi_2 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} a & = & \frac{1}{2}(\Phi_2-\Phi_1) \\ b & = & \frac{1}{2}(\Phi_1+\Phi_2) \end{array} \tag{1013}$$

Therefore, the potential between two parallel plates is given by

$$\Phi(x) = \frac{1}{2}(\Phi_2 - \Phi_1)\,x + \frac{1}{2}(\Phi_1 + \Phi_2)\,. \tag{1014}$$

The equipotential surfaces are parallel planes, $x = \text{const}$.

2. Two coaxial cylinders extending to infinity on both ends, and kept at potentials Φ_1, Φ_2, respectively. We choose a cylindrical coordinate system (r, ϑ, z) where the z-axis coincides with the axis of the cylinders. Because the problem is translation invariant in z-direction, we consider a cross-section of the domain between the cylinders $r \in [R_1, R_2]$, $\vartheta \in [0, 2\pi)$. With the Laplacian in polar coordinates (Section 12.9), we obtain the PDE $r^2\Phi_{rr} + r\Phi_r + \Phi_{\vartheta\vartheta} = 0$. Because the problem is also invariant with respect to rotations around the origin, it turns out that Φ actually depends only on r, which yields an ODE with boundary conditions:

$$r\Phi'' + \Phi' = 0, \quad \text{in } (R_1, R_2), \qquad \Phi(R_1) = \Phi_1,\ \Phi(R_2) = \Phi_2. \tag{1015}$$

The ODE can be solved by separation of variables and integrating:

$$(\log \Phi')' = \frac{\Phi''}{\Phi'} = -\frac{1}{r} \quad \Rightarrow \quad \log \Phi' = -\log r + \tilde{a} \quad \Rightarrow \quad \Phi' = \frac{a}{r}, \tag{1016}$$

and a second integration yields

$$\Phi(r) = a \log r + b. \tag{1017}$$

The constants a, b are again determined from the boundary conditions (cf. Problem Set 6): $a = (\Phi_2 - \Phi_1)/(\log R_2 - \log R_1)$, $b = (\Phi_1 \log R_2 - \Phi_2 \log R_1)/(\log R_2 - \log R_1)$. The equipotential surfaces are concentric cylinders $r = \text{const}$.

3. Two non-parallel plates extending to infinity and intersecting at an angle $\alpha \in (0, \pi]$, which are kept at potentials Φ_1 and Φ_2. We choose a

cartesian coordinate system such that the z-axis is coincides with the line of intersection between the plates. Due to translation invariance in z-direction of the problem, we may restrict ourselves to a cross-section, which we now interpret as the complex plane, $z = x + iy$. The domain between the two plates can be characterized by $\vartheta := \mathrm{Im}\,(\mathrm{Log}z) \in (-\alpha/2, \alpha/2)$, which is a circular sector in the right half of the complex plane.
Remark: The imaginary part of the principal value of the logarithm of a complex number z is sometimes called the principal value of the argument of z, $\mathrm{Im}\,(\mathrm{Log}z) \equiv \mathrm{Arg}z \in (-\pi, \pi]$.
We exclude $z = 0$, so that $r > 0$. The function Log is holomorphic in this domain (Thm. 5), and so its real and imaginary parts are both harmonic functions (Thm. 4). Only the imaginary part is constant along the rays $\vartheta = \text{const.}$ ($\mathrm{Re}\,(\mathrm{Log}z) = \log r$ is not), and therefore the function

$$\Phi(r, \vartheta) = a + b\vartheta \tag{1018}$$

is the general solution of the Laplace equation in this domain. From the boundary conditions, we determine $a = (\Phi_1 + \Phi_2)/2$, $b = (\Phi_2 - \Phi_1)/\alpha$. The equipotential surfaces are half-planes $\vartheta = \text{const.} \in (-\alpha/2, \alpha/2)$, $r > 0$.

Complex Potential Let Φ be harmonic in some domain $\Omega \subseteq \mathbb{R}^2$ and Ψ a harmonic conjugate of Φ in Ω (Section 13.4). Then

$$F(z) = \Phi(x, y) + i\Psi(x, y), \quad z = x + iy, \tag{1019}$$

is a holomorphic function in Ω (by definition of the harmonic conjugate, Section 13.4). This complex function F is called the *complex potential* corresponding to the real potential Φ. The harmonic conjugate and thus the complex potential for a given harmonic function Φ are uniquely determined up to a real additive constant.

Technically, F is easier to handle than Φ and Ψ. Physically, the harmonic conjugate Ψ of Φ is also meaningful: the complex potential F defines a conformal mapping, except where $F'(z) = 0$. Therefore, the curves $\Psi = \text{const.}$ intersect the equipotential lines $\Phi = \text{const.}$ at right angles. Hence they have the direction of the electric force and, therefore, are called *lines of force.* They are the field lines of the electric field $\boldsymbol{E}$. Charged particles (such

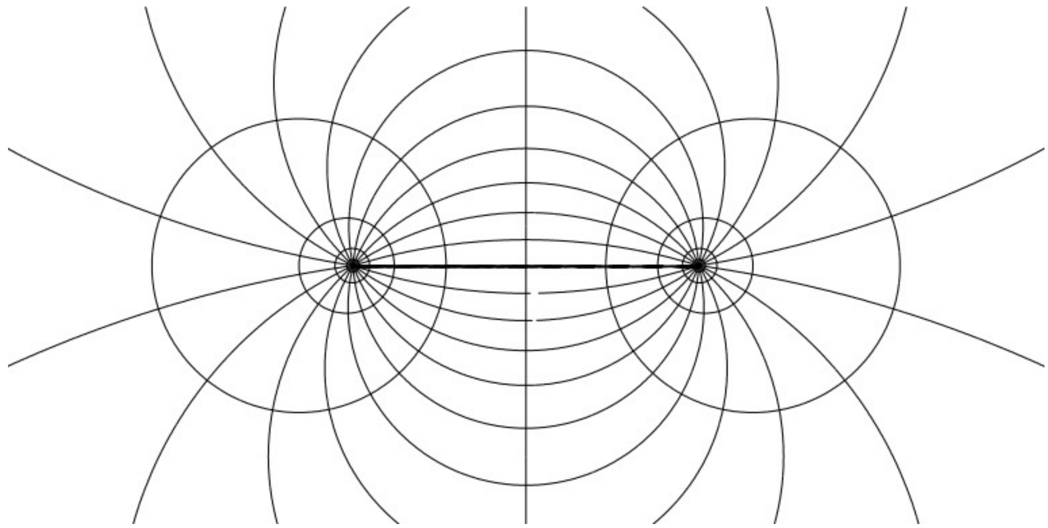
as electrons in an electron microscope, etc.) will travel along these curves.
Examples:

1. For the potential $\Phi(x,y) = ax+b$ from Example 1 before, $\Psi(x,y) := ay$ is a harmonic conjugate, so that the complex potential is given by $F(z) = ax + b + iay = a(x+iy) + b = az + b$. The lines of force (level sets of Ψ) are given by horizontal straight lines $y = \text{const.}$ parallel to the x-axis.

2. For the potential $\Phi(r,\vartheta) = a\log r + b = a\mathrm{Re}\,(\mathrm{Log} z) + b$ from Example 2 before, a conjugate is given by $\Psi(r,\vartheta) = a\mathrm{Im}\,(\mathrm{Log} z)$, so that the complex potential is given by $F(z) = a\mathrm{Log} z + b$. The lines of force are straight lines through the origin.
Remark: With $R_1 \to 0$ and $R_2 \to \infty$, we obtain the solution of an exterior problem, namely for the electric potential of a line source, perpendicular to the complex plane and through the origin. This could be a charged wire, for example. In that case, $F(z) = K\mathrm{Log} z$.

3. For the potential $\Phi(r,\vartheta) = a + b\vartheta = a + b\mathrm{Im}\,(\mathrm{Log} z)$ from Example 3 before, we can get the complex potential
$$F(z) = a - ib\mathrm{Log} z = a + b\mathrm{Im}\,(\mathrm{Log} z) - ib\mathrm{Re}\,(\mathrm{Log} z)\,, \tag{1020}$$
that is, $\Psi(r,\vartheta) = -b\mathrm{Re}\,(\mathrm{Log} z) = -b\log r$. The lines of force are concentric circles, $r = \text{const.}$

Superposition More complicated potentials can often be obtained by superposition (cf. Thm. 1). As an example, we consider the potential of a pair of oppositely charged wires (line sources). Consider charged wires perpendicular to the z-plane and cutting the plane at $z = \pm c$. From the previous examples, we know that the complex potentials are given by
$$F^{+}(z) = K\mathrm{Log}\,(z-c)\,, \quad F^{-}(z) = -K\mathrm{Log}\,(z+c)\,. \tag{1021}$$
The complex potential of the combination of the two line sources is the superposition
$$F(z) = F^{+}(z) + F^{-}(z) = K\left(\mathrm{Log}\,(z-c) - \mathrm{Log}\,(z+c)\right) = K\mathrm{Log}\left(\frac{z-c}{z+c}\right). \tag{1022}$$
Equipotential lines are circles with variable centers and radii, and the lines of force cross these at a right angle:



18.2 Use of Conformal Mapping. Modeling

When we solve boundary value problems in complicated domains, the idea is to use a conformal mapping to map the given domain onto one for which the solution is known or can be found more easily. This solution is then mapped back to the given domain. This works because harmonic functions remain harmonic under conformal mapping:

Theorem 37 *Let Φ^* be harmonic in a domain Ω^* in the w-plane. Suppose that $w = u + iv = f(z)$ is holomorphic in a domain Ω in the z-plane and maps Ω conformally onto Ω^*. Then the function*

$$\Phi(x, y) := \Phi^* \left(u(x, y), v(x, y)\right), \tag{1023}$$

defined in the z-plane, is harmonic in Ω.

Remarks:

1. (Alternative proof) With a harmonic conjugate Ψ^* of Φ^*, we obtain the complex potential $F^*(w) := \Phi^*(u, v) + i\Psi^*(u, v)$ in the w-plane, corresponding to Φ^*. We define the complex function $F(z) := F^*(f(z))$ in the z-plane, which is holomorphic (chain rule). By Thm. 4, the function $\Phi(x, y) := \mathrm{Re}(F(z)) = \mathrm{Re}(F^*(f(z))) = \Phi^*(u(x, y), v(x, y))$ is harmonic.

2. $F = F^* \circ f$ is called the *pullback* of F^* by f. The theorem states that the pullback of a complex potential by a conformal mapping is also a complex potential.

Proof: (without using the harmonic conjugate) We have to prove that $\Delta\Phi = 0$ in Ω. By the chain rule

$$\Phi_x = \Phi^*_u u_x + \Phi^*_v v_x, \quad \Phi_y = \Phi^*_u u_y + \Phi^*_v v_y \tag{1024}$$

and

$$\begin{aligned} \Phi_{xx} &= \Phi^*_{uu} u_x^2 + \Phi^*_{uv} u_x v_x + \Phi^*_u u_{xx} + \Phi^*_{vu} u_x v_x + \Phi^*_{vv} v_x^2 + \Phi^*_v v_{xx}, \quad (1025) \\ \Phi_{yy} &= \Phi^*_{uu} u_y^2 + \Phi^*_{uv} u_y v_y + \Phi^*_u u_{yy} + \Phi^*_{vu} u_y v_y + \Phi^*_{vv} v_y^2 + \Phi^*_v v_{yy}. \quad (1026) \end{aligned}$$

We take the sum and collect terms:

$$\begin{aligned} \Delta\Phi &= \Phi_{xx} + \Phi_{yy} = \Phi^*_{uu}\left(u_x^2 + u_y^2\right) + \left(\Phi^*_{uv} + \Phi^*_{vu}\right)\left(u_x v_x + u_y v_y\right) + \\ &\quad + \Phi^*_u\left(u_{xx} + u_{yy}\right) + \Phi^*_{vv}\left(v_x^2 + v_y^2\right) + \Phi^*_v\left(v_{xx} + v_{yy}\right) \end{aligned} \tag{1027}$$

Because f is holomorphic, we have $u_x v_x + u_y v_y = 0$ by the Cauchy-Riemann equations (Thm. 2). With the same theorem, we may also eliminate the derivatives with respect to y: $u_y^2 = (-v_x)^2 = v_x^2$ and $v_y^2 = u_x^2$. Because $f(z) = u(x, y) + iv(x, y)$ is holomorphic, we also have $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$ (Thm. 4). The remaining terms in the sum are

$$\Delta\Phi = \Phi^*_{uu}\left(u_x^2 + u_y^2\right) + \Phi^*_{vv}\left(v_x^2 + v_y^2\right) = \left(\Phi^*_{uu} + \Phi^*_{vv}\right)\left(u_x^2 + v_x^2\right) \tag{1028}$$

which is 0 because Φ^* was assumed to be harmonic. $\Box$

Example: Potential between noncoaxial cylinders. In the z-plane, we consider the domain between the circles $C_1 := \{z \in \mathbb{C} \,|\, |z| = 1\}$ and $C_2 := \{z \in \mathbb{C} \,|\, |z - 2/5| = 2/5\}$. We look for the electrostatic potential in this domain, with prescribed values on the circles. This leads to a boundary value problem involving Laplace's equation and Dirichlet boundary conditions. We first show that the linear fractional transformation

$$w = \frac{z - b}{\bar{b}z - 1}, \quad |b| < 1, \tag{1029}$$

maps the unit disk onto the unit disk. Indeed, for $|z| = 1$ we have

$$|z - b| = |\bar{z} - \bar{b}| = |z||\bar{z} - \bar{b}| = |z\bar{z} - \bar{b}z| = |1 - \bar{b}z| = |\bar{b}z - 1|, \tag{1030}$$

and therefore $|w| = 1$. Furthermore, $z = 0$ maps to $w = -b$ with $|w| = |b| < 1$, and so any point on the unit disk in the z-plane is mapped to the unit disk in the w-plane. For $|z - 2/5| = 2/5$, we proceed in a similar way:

$$
\begin{aligned}
|z-b| &= |\overline{z}-\overline{b}| = \frac{5}{2}\left|z-\frac{2}{5}\right| |\overline{z}-\overline{b}| = \left|\frac{5}{2}\left(z-\frac{2}{5}\right)\left(\overline{z}-\overline{b}\right)\right| && (1031)\\
&= \left|\frac{5}{2}\left(z-\frac{2}{5}\right)\left(\overline{z}-\frac{2}{5}+\frac{2}{5}-\overline{b}\right)\right| && (1032)\\
&= \left|\frac{5}{2}\left(z-\frac{2}{5}\right)\left(\overline{z}-\frac{2}{5}\right)+\frac{5}{2}\left(z-\frac{2}{5}\right)\left(\frac{2}{5}-\overline{b}\right)\right| && (1033)\\
&= \left|\frac{2}{5}+z-\frac{5}{2}\overline{b}z-\frac{2}{5}+\overline{b}\right| = \left|\left(\frac{5}{2}\overline{b}-1\right)z-\overline{b}\right| && (1034)\\
&= |b|\left|\left(\frac{5}{2}-\frac{1}{\overline{b}}\right)z-1\right| \overset{!}{=} \alpha\left|\overline{b}z-1\right|. && (1035)
\end{aligned}
$$

This is true if

$$
1 > |b| = \alpha \quad \wedge \quad \overline{b}^2 - \frac{5}{2}\overline{b} + 1 = 0. \tag{1036}
$$

The quadratic equation is satisfied if $\overline{b} \in \{1/2, 2\}$, and $b = \alpha = 1/2$ satisfies $|b| < 1$. With this choice of b, we have $|w| = 1/2$, i. e. the circle C_2 in the z-plane is mapped onto $\{w \in \mathbb{C} \,|\, |w| = 1/2\}$ in the w-plane by the mapping

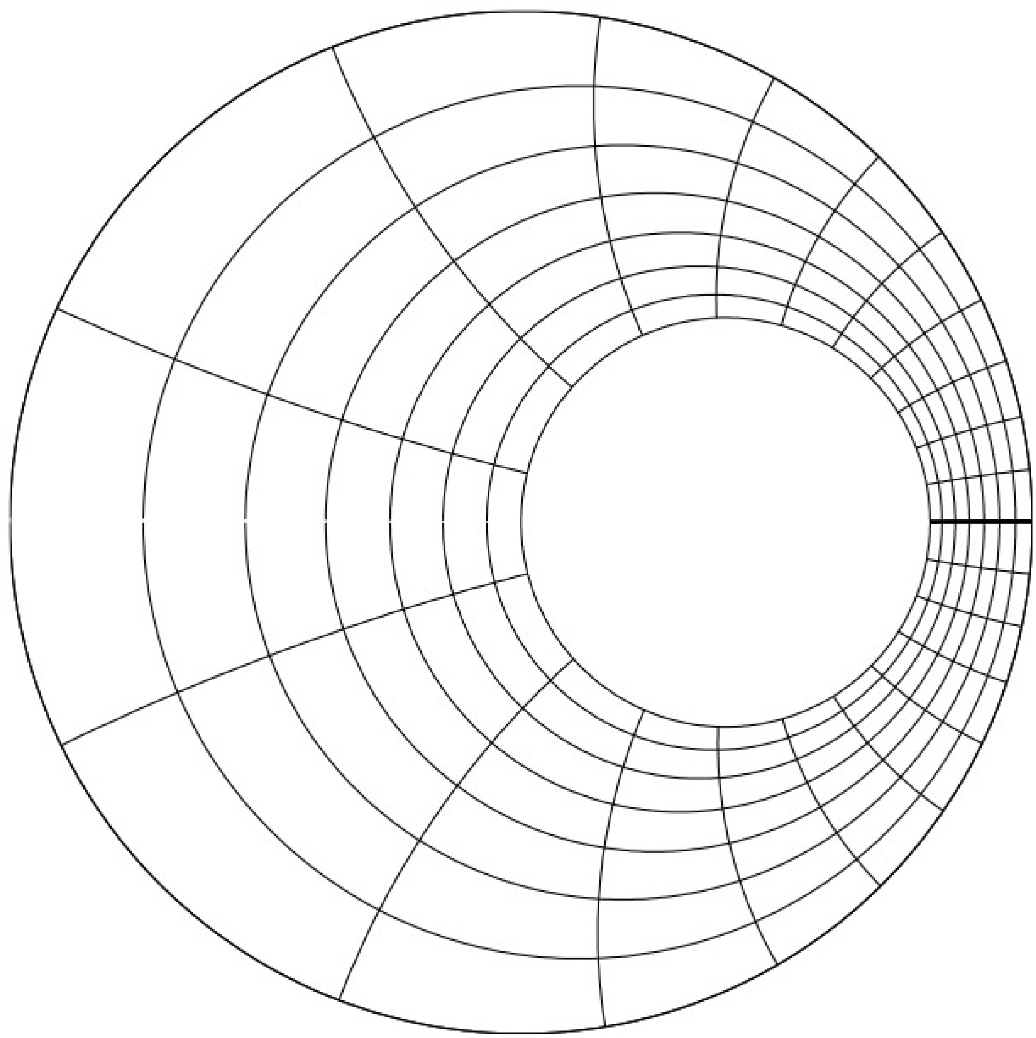
$$
w = f(z) = \frac{2z-1}{z-2}, \tag{1037}
$$

so that we have two *concentric* circles in the w-plane. From Example 2 before, we know that the general complex potential in the w-plane is given by $F^*(w) = a\mathrm{Log}w + b$, where the coefficients a, b are determined from the boundary conditions on Φ^*. The pullback of F^* under the mapping $w = f(z)$ is given by

$$
F(z) = F^*(f(z)) = a\mathrm{Log}\left(\frac{2z-1}{z-2}\right) + b, \tag{1038}
$$

and it is also a complex potential, by Thm. 37, corresponding to the harmonic function

$$
\Phi(x, y) = \mathrm{Re}\left(a\mathrm{Log}\left(\frac{2z-1}{z-2}\right) + b\right) = a\log\left|\frac{2z-1}{z-2}\right| + b. \tag{1039}
$$



18.3 Heat Problems

Laplace's equation also governs steady heat problems (cf. Sections 12.5, 12.6): the heat equation is given by $T_t = c^2 \Delta T$, where T [K] denotes the temperature and where c [$\mathrm{m}^2\mathrm{s}^{-1}$] denotes the thermal diffusivity. We have $T_t \equiv 0$ in steady state, i. e. Laplace's equation. Therefore in two space dimensions, we may again use methods from complex analysis. In this application, T is also called the *heat potential,* and it is the real part of the *complex heat potential*

$$F(z) = T(x, y) + i\Psi(x, y). \tag{1040}$$

The curves $T \equiv$ const., are called *isotherms* and the curves $\Psi \equiv$ const. are called *heat flow lines.* Heat flows along these lines from higher to lower temperatures.
Examples: The examples considered in the previous section can now be reinterpreted as problems on heat flow: the electrostatic potential lines now become isotherms, and the lines of electrical force become lines of heat flow. So we can immediately write down the heat potential between two plates, two cylinders and so on. We consider two new examples with different boundary conditions:

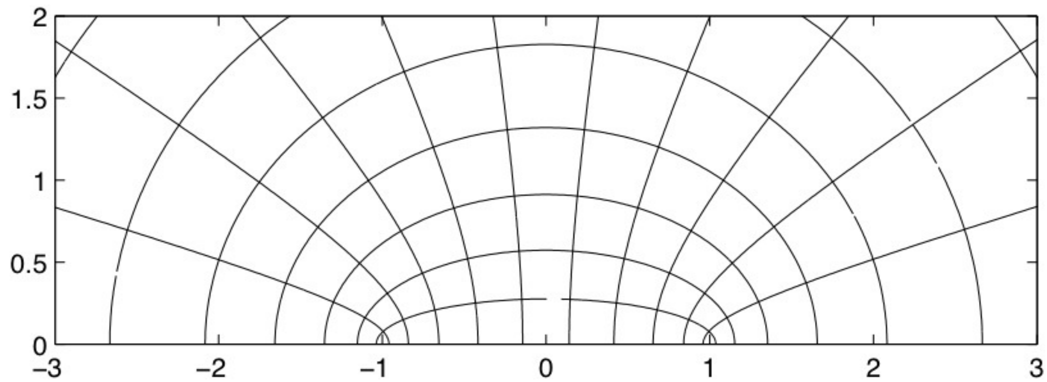
1. We consider the first quadrant of the unit disk. The temperature on the straight line segments is prescribed (Dirichlet boundary conditions), but on the arc, the domain is assumed to be insulated, so that $\partial_n T = 0$ (Neumann boundary condition). We use polar coordinates (r, ϑ), so that $\partial_n \equiv \partial_r$ on the arc. The angular solution from Example 3 before satisfies the Neumann boundary condition on the arc. The complex potential is thus given by $F(z) = a - ib\mathrm{Log} z$ with real part $T(x, y) = a + b\vartheta$, $\vartheta = \mathrm{Im}\,(\mathrm{Log} z)$. Coefficients a, b are determined from the Dirichlet boundary values.

2. We consider the upper half-plane. On the boundary (real axis), the temperature is prescribed for $x < -1$ and $x > 1$ (two different values) and the domain is insulated on $(-1, 1)$. The upper half of the z-plane is mapped onto a semi-infinite vertical strip in the w-plane, $S := \{w \in \mathbb{C} \,|\, u \in (-\pi/2, \pi/2),\ v \geq 0\}$, by the inverse of the sine mapping:

$$w = f(z) = \arcsin z. \tag{1041}$$

   The Dirichlet boundary segments are mapped onto the semi-infinite vertical lines $u = \pm\pi/2$ $(v > 0)$ and the Neumann boundary segment is mapped to $v = (-\pi/2, \pi/2)$, $u = 0$. Thus we have $\partial_n \equiv -\partial_v$. This is related to Example 1 before (the parallel plates), so that we obtain $F^*(w) = aw + b$, where the constants a, b are determined from the Dirichlet boundary values. The pullback by f is given by

$$F(z) = F^*(f(z)) = a \arcsin z + b. \tag{1042}$$



18.4 Fluid Flow

The Navier-Stokes equations for an incompressible, Newtonian fluid are given by the following system of nonlinear PDEs:

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right) = -\nabla p + \mu\boldsymbol{\Delta}\boldsymbol{v} + \boldsymbol{f}, \tag{1043}$$

$$\mathrm{div}\boldsymbol{v} = 0, \tag{1044}$$

where ρ [kgm^{-3}] denotes the mass density of the fluid, $\boldsymbol{v}$ [ms^{-1}] the velocity, p [Pa] the pressure, μ [Pas] the (dynamic) viscosity and $\boldsymbol{f}$ [Nm^{-3}] denotes the volume force density (such as from gravitational forces). The vector Laplacian is given by $\boldsymbol{\Delta} = \nabla\mathrm{div} - \mathrm{curlcurl}$.

These equations are difficult to solve. A simplification can be made when advective inertial forces are small compared to viscous forces. This assumption is valid in the case of small Reynolds numbers, $\mathrm{Re} \ll 1$, i. e. for flow at small length scales, small velocities or high viscosity. In these cases, the fluid exhibits *Stokes flow* or *creeping flow,* which is steady, i. e. $\boldsymbol{v}_t \equiv \boldsymbol{0}$:

$$\mu\boldsymbol{\Delta}\boldsymbol{v} + \boldsymbol{f} = \nabla p, \quad \mathrm{div}\boldsymbol{v} = 0. \tag{1045}$$

This is a system for four linear PDEs. Next we assume irrotational flow, i. e. $\mathrm{curl}\boldsymbol{v} = \boldsymbol{0}$. In this case, we may write $\boldsymbol{v} = \nabla\Phi$ with the *velocity potential* Φ (Helmholtz decomposition). The equations simplify to a system of two decoupled PDEs for the pressure p and for the velocity potential Φ:

$$\Delta p = \mathrm{div}\boldsymbol{f}, \quad \Delta\Phi = 0. \tag{1046}$$

The PDE for the velocity potential Φ is Laplace's equation. We restrict ourselves again to two space dimension, so that we can use complex analysis. In the corresponding complex potential $F(z) = \Phi(x, y) + i\Psi(x, y)$, the function Ψ is called the *stream function.* Lines $\Psi \equiv$ const. are called *streamlines* of the fluid motion. In this context, the velocity vector $\boldsymbol{v} = (v_1, v_2)^\top$ is usually written as a complex number:

$$V = v_1 + iv_2 = \Phi_x + i\Phi_y = \Phi_x - i\Psi_x, \tag{1047}$$

where we have used the second Cauchy-Riemann equation. Therefore we have $V(x, y) = \overline{F'(z)}$.

Similar to what we did for the electrostatic and heat problems before, we may interpret any holomorphic function F as a complex potential, whose real and imaginary part will be the velocity potential and stream function of a certain flow problem.

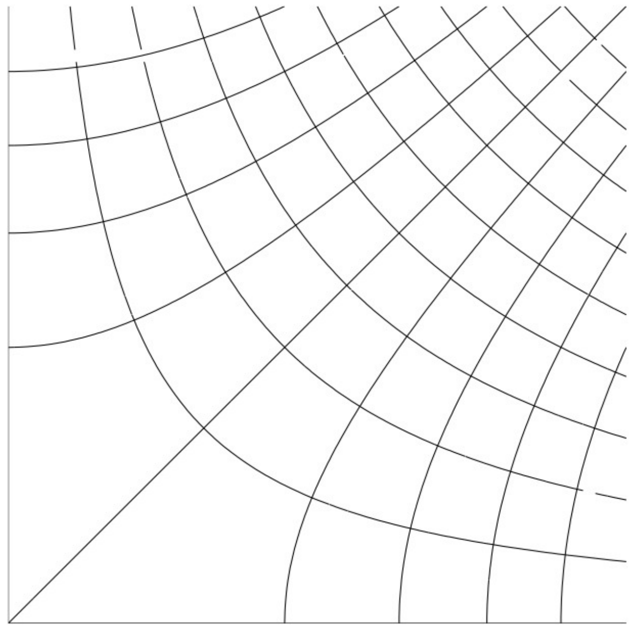
Examples:

1. The complex potential $F(z) = z^2 = x^2 - y^2 + 2ixy$ models a flow with

$$\text{Equipotential lines} \quad \Phi(x, y) = x^2 - y^2 = \text{const.}, \tag{1048}$$

$$\text{Streamlines} \quad \Psi(x, y) = 2xy = \text{const.}, \tag{1049}$$

which are hyperbolas. The velocity vector is given by $V(x, y) = \overline{F'(z)} = 2\overline{z} = 2(x - iy)$, that is $v_1 = 2x$, $v_2 = -2y$, with speed $|V| = 2\sqrt{x^2 + y^2}$. This flow may be interpreted as flow around a corner. The flow speed along the streamline $2xy = c$ is minimal at $x = y = \sqrt{c/2}$, where the cross-section of the channel is large.

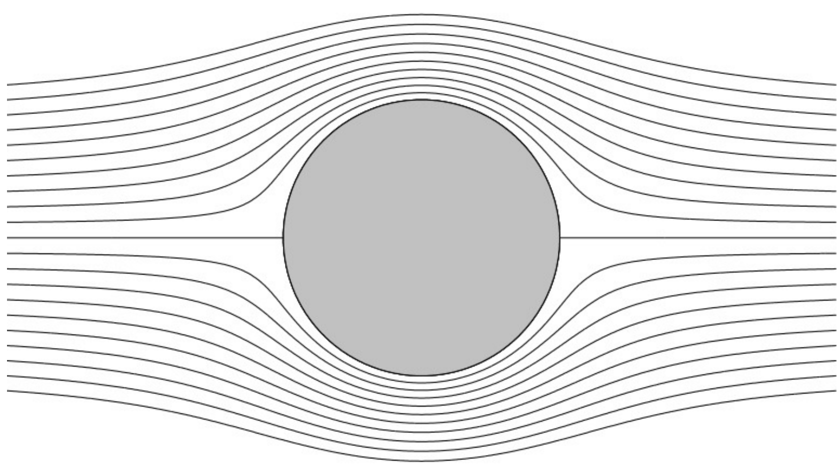


2. The complex potential $F(z) = z + 1/z$ gives, in polar coordinates,

$$F(z) = re^{i\vartheta} + \frac{1}{r}e^{-i\vartheta} = \left(r + \frac{1}{r}\right)\cos\vartheta + i\left(r - \frac{1}{r}\right)\sin\vartheta \qquad (1050)$$

(cf. Section 17.1). Therefore, the streamlines are given by

$$\Psi(x, y) = \left(r - \frac{1}{r}\right)\sin\vartheta = \text{const.} \qquad (1051)$$



18.5 Poisson's Integral Formula for Potentials

We have seen in the previous sections that boundary value problems involving the Laplace equation in complicated domains in two space dimensions can be transformed to problems in standard domains, such as a disk, by a suitable conformal mapping. In this section, we shall investigate the solution of Laplace's equation in a disk in more detail.

Consider a circle with radius $R > 0$, $C_R := \{\gamma(\alpha; R) \mid \alpha \in [0, 2\pi]\}$, $\gamma(t) := Re^{i\alpha}$, and a complex function F which is holomorphic in a simply connected domain Ω that contains C_R.

$$\frac{1}{2\pi i}\oint_{C_R} \frac{F(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{F(Re^{i\alpha})Re^{i\alpha}}{Re^{i\alpha} - z}\, d\alpha = \left\{ \begin{array}{ll} F(z), & |z| < R \\ 0, & |z| > R \end{array} \right. , \quad z \in \Omega, \tag{1052}$$

from Thms. 7, 8, and 10. We choose $z \in \mathbb{C}$ with $|z| < R$ and define $Z := R^2/\overline{z}$, $|Z| = R^2/|z| > R$. Therefore

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{F(Re^{i\alpha})Re^{i\alpha}}{Re^{i\alpha} - Z}\, dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{F(Re^{i\alpha})\overline{z}}{\overline{z} - Re^{-i\alpha}}\, d\alpha \tag{1053}$$

Subtracting the integrals, we obtain

$$\begin{aligned} F(z) &= \frac{1}{2\pi}\int_0^{2\pi} F(Re^{i\alpha}) \left(\frac{Re^{i\alpha}}{Re^{i\alpha} - z} - \frac{\overline{z}}{\overline{z} - Re^{-i\alpha}} \right) d\alpha & (1054) \\ &= \frac{1}{2\pi}\int_0^{2\pi} F(Re^{i\alpha}) \frac{R^2 - |z|^2}{|Re^{i\alpha} - z|^2}\, d\alpha. & (1055) \end{aligned}$$

Finally, with $z = re^{i\vartheta}$, we have

$$F(re^{i\vartheta}) = \frac{1}{2\pi}\int_0^{2\pi} F(Re^{i\alpha}) \frac{R^2 - r^2}{R^2 - 2Rr\cos(\vartheta - \alpha) + r^2}\, d\alpha. \tag{1056}$$

This formula is valid for any $z \in \mathbb{C}$, $|z| < R$. Now we write $F(re^{i\vartheta}) =$

$\Phi(r,\vartheta)+i\Psi(r,\vartheta)$ and take the real part on both sides:

$$\Phi(r,\vartheta)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(R,\alpha)\frac{R^2-r^2}{R^2-2Rr\cos(\vartheta-t)+r^2}\,d\alpha,\quad r<R,\ \vartheta\in[0,2\pi). \tag{1057}$$

Poisson's integral formula (1057) expresses the potential Φ inside of the disk with radius $R>0$ centered at the origin using the boundary values, $\Phi(R,\cdot)$.

Series for Potentials in Disks We may also write this formula as a series. For that purpose, we note first that the quotient under the integral is the real part of the complex function

$$\frac{\zeta+z}{\zeta-z}=\frac{(\zeta+z)(\overline{\zeta}-\overline{z})}{(\zeta-z)(\overline{\zeta}-\overline{z})}=\frac{|\zeta|^2+2i\mathrm{Im}(z\overline{\zeta})-|z|^2}{|\zeta-z|^2}, \tag{1058}$$

where $\zeta=Re^{i\alpha}$ and $z=re^{i\vartheta}$. Now we use the geometric series to write

$$\frac{\zeta+z}{\zeta-z} = \frac{1+\frac{z}{\zeta}}{1-\frac{z}{\zeta}}=\left(1+\frac{z}{\zeta}\right)\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n=\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n+\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^{n+1} \tag{1059}$$

$$= 1+2\sum_{n=1}^{\infty}\left(\frac{z}{\zeta}\right)^n=1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n e^{in(\vartheta-\alpha)}. \tag{1060}$$

For the real part, we obtain

$$\begin{aligned}\frac{R^2-r^2}{R^2-2Rr\cos(\vartheta-\alpha)+r^2} &= \mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right)=1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos\left(n(\vartheta-\alpha)\right)\\ &= 1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\left(\cos(n\vartheta)\cos(n\alpha)+\sin(n\vartheta)\sin(n\alpha)\right).\end{aligned}$$

Now Poisson's integral formula (1057) becomes a Fourier series:

$$\Phi(r,\vartheta)=a_0+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\left(a_n\cos(n\vartheta)+b_n\sin(n\vartheta)\right), \tag{1061}$$

which involves the Fourier coefficients of the boundary values $\Phi(R,\cdot)$:

$$a_0 = \frac{1}{2\pi}\int\limits_0^{2\pi} \Phi(R,\alpha)\,d\alpha, \quad a_n = \frac{1}{\pi}\int\limits_0^{2\pi} \Phi(R,\alpha)\cos(n\alpha)\,d\alpha, \quad n = 1,2,\ldots,$$
$$b_n = \frac{1}{\pi}\int\limits_0^{2\pi} \Phi(R,\alpha)\sin(n\alpha)\,d\alpha, \quad n = 1,2,\ldots. \tag{1062}$$

Comparte this with Section 12.10 where we had derived the analogous formula for the interior problem of the 3D Laplace equation. In 3D, this involved spherical harmonics $Y_{nm}(\varphi,\vartheta)$, $0 \le |m| \le n$ ($2n+1$ linearly independent functions for each $n \in \mathbb{N}_0$), whereas in 2D, we have trigonometric functions $\cos(n\vartheta)$, $\sin(n\vartheta)$ (two linearly independent functions for each $n \in \mathbb{N}$, one for $n = 0$).

Example: For $R = 1$, with the boundary values

$$\Phi(1,\alpha) = \begin{cases} -\alpha/\pi, & \alpha \in (-\pi, 0] \\ \alpha/\pi, & \alpha \in (0, \pi] \end{cases}, \tag{1063}$$

we obtain $b_n = 0$, $n \in \mathbb{N}$, because $\Phi(1,\cdot)$ is an even function. We also have

$$a_0 = \frac{1}{2\pi}\left(-\int\limits_{-\pi}^{0} \frac{\alpha}{\pi}\,d\alpha + \int\limits_0^{\pi} \frac{\alpha}{\pi}\,d\alpha\right) = \frac{1}{2}, \tag{1064}$$
$$a_n = \frac{1}{\pi}\left(-\int\limits_{-\pi}^{0} \frac{\alpha}{\pi}\cos(n\alpha)\,d\alpha + \int\limits_0^{\pi} \frac{\alpha}{\pi}\cos(n\alpha)\,d\alpha\right) = \frac{2\left((-1)^n - 1\right)}{n^2\pi^2}, \tag{1065}$$

so that

$$\Phi(r,\vartheta) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{r^{2k+1}}{(2k+1)^2}\cos((2k+1)\vartheta). \tag{1066}$$

18.6 Review of the Second Part of the Lecture

18.6.1 Final Exam

The final exam of MATH 529 will take place on *Tuesday, May 3, 2011, 8:00 – 11:00 AM,* in Phillips Hall, Room 332 (the regular lecture hall). Please read the paragraph on "Final Examinations" in the Academic Procedures section of the 2010–2011 Undergraduate Bulletin (http://www.unc.edu/ugradbulletin/procedures1.html), to be informed about what to do in case you are unable to take the exam at the scheduled time. As stated in the course syllabus, you are allowed to use a summary of the lecture notes on 6 (six) pages (US letter format, double-sided) for the final exam. This is meant as 6 sheets of paper, with text on both sides.

The exam will cover topics from Chapter 12 (up to and including 12.10) and 13–17 (not 16.4). The problems will focus on the *application* of the methods treated in this course; I want to see that students are able to recognize which method should be used for a given problem, and that they apply it correctly. I will not ask for proofs, and I will try to avoid complicated auxiliary calculations (such as partial fraction decompositions, finding roots of high order polynomials, or computing high-order derivatives of complicated functions). There will be ~ 9 problems in the final exam, which means ~ 20 minutes to solve each problem.

18.6.2 Complex Numbers and Functions (13)

Complex Numbers. Complex Plane We had constructed the field of complex numbers $\mathbb{C}$ by considering ordered pairs (x, y) of real numbers. A complex number $z \in \mathbb{C}$ may be represented in various ways in the complex plane $\mathbb{C} \simeq \mathbb{R}^2$, such as

$$z = x + iy = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}, \quad i^2 = -1, \tag{1067}$$

where $x := \mathrm{Re}z$ denotes the *real part* of z, $y := \mathrm{Im}z$ the *imaginary part,* $r = |z| = \sqrt{\bar{z}z} = \sqrt{(\mathrm{Re}z)^2 + (\mathrm{Im}z)^2}$ the *absolute value,* and $\varphi = \arg z$ the *argument* of z. It is defined by the equations

$$\cos\varphi = \frac{x}{r}, \quad \sin\varphi = \frac{y}{r}, \tag{1068}$$

and therefore multi-valued, because of the periodicity of sin and cos. We typically restrict the argument to either $\arg z \in [0, 2\pi)$ or $\mathrm{Arg}z \in (-\pi, \pi]$. The latter is convenient when working with complex logarithms, because $\mathrm{Arg}z = \mathrm{Im}(\mathrm{Log}z)$, and the branch cut of Arg coincides with the branch cut of Log. We have

$$\arg z = \begin{cases} \mathrm{Arg}z, & \mathrm{Im}z \geq 0 \\ 2\pi + \mathrm{Arg}z, & \mathrm{Im}z < 0 \end{cases}, \quad z \neq 0. \tag{1069}$$

The complex conjugate $\overline{z} \in \mathbb{C}$ of a complex number $z \in \mathbb{C}$ is defined by $\overline{z} = \mathrm{Re}z - i\mathrm{Im}z$; geometrically, this corresponds to a mirroring on the real axis in the complex plane, and therefore $\overline{z} = z \Leftrightarrow \mathrm{Im}z = 0$. The absolute value $|z| = \sqrt{\overline{z}z}$ induces a metric on $\mathbb{C}$, $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $d(z_1, z_2) := |z_1 - z_2|$. Thus we can measure distances between complex numbers (Problem Set 7). In particular, we have the triangle inequality $|z_1+z_2| \leq |z_1|+|z_2|$, $z_1, z_2 \in \mathbb{C}$.

Polar Form of Complex Numbers. Powers and Roots. The polar form of a complex number is useful to represent products, quotients, powers and roots of complex numbers:

$$
\begin{aligned}
z_1 z_2 &= |z_1||z_2| \exp\left(i(\arg z_1 + \arg z_2)\right), && (1070)\\
\frac{z_1}{z_2} &= \frac{|z_1|}{|z_2|} \exp\left(i(\arg z_1 - \arg z_2)\right), && (1071)\\
z^n &= |z|^n \exp\left(in \arg z\right), && (1072)\\
\sqrt[n]{z} &= \sqrt[n]{|z|} \exp\left(i\frac{\arg z}{n}\right) \exp\left(i\frac{2\pi k}{n}\right), \quad k = 0, \ldots, n-1. && (1073)
\end{aligned}
$$

Derivative. Holomorphic Function. With the metric $d(z_1, z_2) := |z_1 - z_2|$ on the complex numbers induced by the absolute value (which defines a norm on $\mathbb{C}$), we may introduce open balls $B_\rho(a) := \{z \in \mathbb{C} \,|\, d(z, a) < \rho\}$, $a \in \mathbb{C}$, $\rho > 0$. The open balls generate a topology on $\mathbb{C}$, making it a topological space. A subset $U \subseteq \mathbb{C}$ is open if every point in U has an open neighborhood contained in U. U is connected if it cannot be written as the union of two disjoint nonempty open sets.

For a complex function $f : z \mapsto w = f(z)$, both the argument and the value may be split into real and imaginary parts: $f(x+iy) = u(x, y)+iv(x, y)$, with real-valued functions $u, v : \mathbb{R}^2 \to \mathbb{R}$. The limit of f as z approaches z_0 is L if

$$\forall\, \varepsilon > 0\; \exists\, \delta > 0 : f(z) \in B_\varepsilon(L) \quad \forall\, z \in B_\delta(z_0),\; z \neq z_0. \tag{1074}$$

in this case we write $\lim_{z\to z_0} f(z) = L$. f is continuous at $z_0 \in \mathbb{C}$ if $f(z_0)$ is defined and $\lim_{z\to z_0} = f(z_0)$. f is differentiable at $z_0 \in \mathbb{C}$, if the limit

$$f'(z_0) := \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1075}$$

exists. In this case, the limit value is the derivative of f at z_0.
Examples:

- $f(z) := z^2$ is differentiable at all $z_0 \in \mathbb{C}$,
- $f(z) := \overline{z}$ is not differentiable at any $z_0 \in \mathbb{C}$.

The rules for differentiation are the same as for real functions (product rule, quotient rule, chain rule).

A complex function f which is differentiable at every point $z \in U \subseteq \mathbb{C}$ (U is typically assumed to be a domain, i. e. open and connected), is *holomorphic in U.* A function which is holomorphic in all of $\mathbb{C}$ is an *entire function.* A holomorphic function is necessarily continuous (Problem Set 7).

Cauchy-Riemann Equations. Laplace's Equation. The following two theorems provide a criterion for differentiability of a complex function:
Thm. 2: $f(x + iy) = u(x, y) + iv(x, y)$ holomorphic in $U \subseteq \mathbb{C} \Rightarrow$ the first partial derivatives of u and v exist, and they satisfy the Cauchy-Riemann equations (system of linear first-order PDEs)

$$u_x = v_y, \quad u_y = -v_x \quad \text{in } U. \tag{1076}$$

Thm. 3: $u, v : \mathbb{R}^2 \to \mathbb{R}$ with *continuous* first partial derivatives satisfy the Cauchy-Riemann equations in some domain $U \subseteq \mathbb{C} \Rightarrow f(x + iy) := u(x, y) + iv(x, y)$ is holomorphic in U.
From the proof of Thm. 2, we also concluded that the derivative of a holomorphic function f can be written as

$$f'(x_0 + iy_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0). \tag{1077}$$

In polar form, we write $f(re^{i\varphi}) = u(r, \varphi) + iv(r, \varphi)$, and the Cauchy-Riemann equations are given by (Problem Set 8)

$$u_r = \frac{1}{r} v_\varphi, \quad v_r = -\frac{1}{r} u_\varphi. \tag{1078}$$

The following theorem was useful in potential theory in 2D (Chapter 18):
Thm. 4: f holomorphic in some domain $U \subseteq \mathbb{C} \Rightarrow$ both the real and imaginary part of f are harmonic functions in U.
A *harmonic function* is a twice continuously differentiable function u which

satisfies Laplace's equation $\Delta u = 0$ in some domain $\Omega \subseteq \mathbb{R}^2$. The conjugate of a harmonic function u is a function v such that $f(x+iy) := u(x,y)+iv(x,y)$ is holomorphic. The harmonic conjugate of a given function u is uniquely determined up to some real additive constant. It can be found by solving the Cauchy-Riemann equations. In potential theory, u is typically some real potential (electric potential, temperature, ...), whereas its harmonic conjugate gives the lines of force, heat flow lines, ..., which are perpendicular to the equipotential lines. We had seen this in Chapter 18.

Exponential Function The function $f(z) = e^z$, $z \in \mathbb{C}$ is defined by

$$\exp(x+iy) := e^x \cos y + ie^x \sin y. \tag{1079}$$

It satisfies $\exp(z_1 + z_2) = \exp z_1 \exp z_2$, from which we infer Euler's formula $e^{iy} = \cos y + i \sin y$, $|e^{iy}| = 1$. We also know that $\exp' z = \exp z$, and that exp never vanishes. The function is periodic with period $2\pi i$: $e^{z+2\pi i} = e^z$, $z \in \mathbb{C}$. It maps the fundamental region $\mathrm{Im} z \in (-\pi, \pi]$ (a horizontal strip) onto the whole complex plane without the origin, $\mathbb{C} \setminus \{0\}$.

Trigonometric and Hyperbolic functions With the complex exponential function, we may define trigonometric and hyperbolic functions for complex numbers in exactly the same way as for real numbers:

$$\cos z := \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \tag{1080}$$

$$\cosh z := \frac{1}{2}(e^{z} + e^{-z}), \quad \sinh z = \frac{1}{2}(e^{z} - e^{-z}). \tag{1081}$$

Addition theorems for these functions remain valid in the complex. In the complex only, trigonometric and hyperbolic functions are related via

$$\cosh(iz) = \cos z, \quad \sinh(iz) = i \sin z, \tag{1082}$$

$$\cos(iz) = \cosh z, \quad \sin(iz) = i \sinh z. \tag{1083}$$

Logarithm. General Power. The natural logarithm is the inverse of the exponential function, and therefore multi-valued (because exp is not one-to-one in all of $\mathbb{C}$). For $z = re^{i\varphi}$, $r > 0$, we obtain

$$w = \log z = \log r + i(\varphi + 2\pi n), \quad n \in \mathbb{Z}. \tag{1084}$$

The *principal value* of log is the one which lies in the fundamental region of exp: $\mathrm{Im}(\mathrm{Log} z) \in (-\pi, \pi]$:

$$\mathrm{Log} z = \begin{cases} \log r + i\varphi, & \varphi \in [0, \pi] \\ \log r + i(\varphi - 2\pi), & \varphi \in (\pi, 2\pi) \end{cases} . \tag{1085}$$

All values of log are then given by $\log z = \mathrm{Log} z + 2\pi i n$, $n \in \mathbb{Z}$.

Thm. 5: The natural logarithm is holomorphic except for $z \leq 0$. Its derivative is given by

$$\log' z = \frac{1}{z}, \quad z \neq 0, \arg z \neq \pi. \tag{1086}$$

The negative real axis $z < 0$ is called the *branch cut* of log.

With the complex exponential and logarithm, we may define the general power of two complex numbers by

$$z^c := \exp\left(c \log z\right), \quad z, c \in \mathbb{C}. \tag{1087}$$

This is multi-valued, because log is multi-valued. The principal value of z^c uses the principal value Log instead of log.

18.6.3 Complex Integration (14, 16)

Line Integral in the Complex Plane A curve $C \subset \mathbb{C}$ is defined by

$$C := \{\gamma(t) \,|\, t \in [a, b]\}, \quad a, b \in \mathbb{R},\ \gamma : [a, b] \to \mathbb{C}. \tag{1088}$$

We usually assume a *regular* curve, i. e. γ is continuous and piecewise continuously differentiable, and that $\dot{\gamma}(t) \neq 0$, $t \in (a, b)$. The line integral of a complex function f along the curve C can then be defined in a similar way as the Riemann integral in $\mathbb{R}$, with a partition $a = t_0 < t_1 \cdots < t_n = b$, $n \in \mathbb{N}$:

$$S_n := \sum_{m=1}^{n} f(\zeta_m)(z_m - z_{m-1}), \quad z_m = \gamma(t_m),\ \zeta_m = \gamma(\tau_m),\ \tau_m \in [t_{m-1}, t_m]. \tag{1089}$$

Then

$$\int\limits_C f(z)\, dz := \lim_{n \to \infty} S_n. \tag{1090}$$

From this definition, we infer properties such as the linearity of the integral. It is not, however, used to evaluate integrals in practice. For that purpose,

we have two other theorems:
Thm. 6: f holomorphic in $U \subseteq \mathbb{C}$, U a simply connected domain (!). Then the antiderivative F of f exists ($F'(z) = f(z)\ \forall\, z \in U$) and is also holomorphic in U. For any path in U joining the points $z_0, z_1 \in U$, we have

$$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0). \tag{1091}$$

Notice that the domain U does not appear explicitly in the integral. $U \subseteq \mathbb{C}$ is typically not prescribed, but it has to be chosen such that it is open and simply connected, contains C and such that f is holomorphic in U. This domain $U \subseteq \mathbb{C}$ occurs in several other integral theorems. In general, it may be impossible to find such a domain, in which case these theorems are not applicable.
Example: $f(z) = z^2$, then we can choose $U = \mathbb{C}$.

$$\int_0^{1+i} z^2\, dz = \left.\frac{z^3}{3}\right|_0^{1+i} = -\frac{2}{3} + \frac{2}{3}i. \tag{1092}$$

When Thm. 6 is not applicable (such as for functions with singularities inside of a closed curve C), we may need to use a parametrization of C to write the integral as a real integral:
Thm. 7: C regular, piecewise C^1 curve, f continuous complex function.

$$\int_C f(z)\, dz = \int_a^b f(\gamma(t))\dot{\gamma}(t)\, dt. \tag{1093}$$

Example: $z_0 \in \mathbb{C}$, $f(z) = (z - z_0)^m$, $m \in \mathbb{Z}$, $C := \{z \in \mathbb{C}\,|\,|z - z_0| = \rho\}$, $\rho > 0$. For $m < 0$, f is not differentiable at z_0 and C encloses z_0: therefore, there is no simply connected domain $U \subseteq \mathbb{C}$ which contains C and in which f is holomorphic, so that Thm. 6 cannot be applied (we could use Thm. 6 for $m \geq 0$, though, in which case f is entire). With Thm. 7, we obtain

$$\oint_C (z - z_0)^m\, dz = \cdots = \begin{cases} 2\pi i, & m = -1 \\ 0, & m \in \mathbb{Z} \setminus \{-1\} \end{cases}. \tag{1094}$$

We have seen several theorems which are usually easier to use than Thm. 7, but require less strong conditions than Thm. 6:

CIT, CIF, CDF, Residue theorem The integral theorems by Cauchy (Thms. 8, 10, 11) are all special cases of the residue theorem (Chapter 16):
Thm. 33: f holomorphic on a closed curve C and inside of C, except for finitely many singular points $z_1, \ldots, z_k$, $k \in \mathbb{N}$ inside of C (!). Then the integral of f taken counterclockwise around C is given by

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, z_j). \tag{1095}$$

In the special cases of CIT, CIF, CDF, the integrand has at most one singular point inside of C ($k = 1$), and it is either holomorphic in a simply connected domain which contains C (CIT), or it has a pole of order $n+1$ at $z_0 \in \mathbb{C}$, which lies inside of C (CDF; for $n = 0$ (simple pole), this is CIF).

Here, the residue of f at the singular point z_0 is defined as the coefficient a_{-1} of the Laurent series of f at z_0:

$$f(z) = \sum_{m=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R, \quad \Rightarrow \quad \operatorname{Res}(f, z_0) = a_{-1}. \tag{1096}$$

If f has an essential singularity at z_0, we typically need to compute the Laurent series of f explicitly in order to determine the value of a_{-1} (by substitution in a Maclaurin series, for example). For poles, however, there is an easier way to compute the residue, which involves differentiation: the residue of f at a pole of order $m \in \mathbb{N}$, $z_0 \in \mathbb{C}$, is given by

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left(\frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z) \right) \right). \tag{1097}$$

For a function of the form $f(z) = p(z)/q(z)$ (p, q not necessarily polynomials!) with $p(z_0) \neq 0$ and where q has a *simple* zero at z_0 ($\Rightarrow f$ has a simple pole at z_0), this simplifies further to

$$\operatorname{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}. \tag{1098}$$

18.6.4 Power Series, Taylor Series, Laurent Series (15, 16)

In order to get to the residue theorem in Chapter 16, we had to consider a few more theoretical topics, concerning power series. We had seen that every power series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad z_0 \in \mathbb{C},\ a_n \in \mathbb{C},\ n \in \mathbb{N}_0 \tag{1099}$$

(no negative powers!) converges at the center z_0. Furthermore, if a power series converges at $z = z_1 \neq z_0$, it converges absolutely for all z which are closer to z_0 than z_1, $|z - z_0| < |z_1 - z_0|$. On the other hand, if a power series diverges at $z = z_2$, it diverges for every z farther away from z_0 than z_2, $|z - z_0| > |z_2 - z_0|$ (Thm. 19). We had defined the convergence radius R of a power series as the radius of the smallest circle with center z_0 containing all points $z \in \mathbb{C}$ for which the power series converges; $R = 0$ and $R = \infty$ are also possible. We have also found that termwise integration and differentiation do not change the radius of convergence (Thms. 23 & 24). The radius of convergence can be computed with the Cauchy-Hadamard formula:
Thm. 22: Suppose that the sequence $|a_{n+1}/a_n|$, $n \in \mathbb{N}$, converges with limit L^*. If $L^* = 0$, then $R = \infty$ (convergence everywhere). If $L^* > 0$, then

$$R = \frac{1}{L^*} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|. \tag{1100}$$

If $|a_{n+1}/a_n| \to \infty$, $n \to \infty$, then $R = 0$ (convergence only at the center).
We may need to bring a given series into the form (1099) before we can apply the formula.
Example: We consider the following function from Problem Set 11:

$$z \cos\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n+1}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{z}\right)^{2n-1}. \tag{1101}$$

We use termwise integration to eliminate the constant in the power of $w := 1/z$: The power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{w^{2n}}{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!2n} (w^2)^n \tag{1102}$$

thus obtained is a power series in w^2 in the form (1099) with the same radius of convergence as the original power series (it is not a power series in z, because it contains negative powers of z; the function has an essential singularity at $z = 0$). We look at the quotient

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(-1)^{n+1}}{(2(n+1))!2(n+1)}}{\frac{(-1)^n}{(2n)!2n}} \right| = \frac{(2n)!2n}{(2n+2)!(2n+2)} = \frac{2n}{(2n+1)(2n+2)^2} \quad (1103)$$

With de l'Hôpital's rule, for example, we verify that the limit satisfies $L^* = 0$. Therefore, the power series converges for

$$|w|^2 = \frac{1}{|z|^2} < \infty \quad \Leftrightarrow \quad |z| > 0. \quad (1104)$$

We call a function f analytic at $z_0 \in \mathbb{C}$ if it can be represented as a power series in z of the form (1099) which converges in $B_R(z_0)$, $R > 0$. Thm. 25 says that such a function is holomorphic in $B_R(z_0)$, and Thm. 26 (Taylor's theorem) states that every holomorphic function f can be locally represented at z_0 as a power series, namely the *Taylor series* of f at z_0, where the coefficients $a_n = f^{(n)}(z_0)/n!$ are given by the values of the derivatives of f at z_0. This representation is valid in the largest open disk with center z_0 in which f is holomorphic. From these two theorems we concluded that holomorphic functions are analytic and vice versa. Practical methods to obtain Taylor series without using the coefficient formulas include substitution into or termwise integration of already known Taylor series. The geometric series is also often used as a tool.

In the *Laurent series* of a complex function f at z_0 we also allow for negative powers of $(z - z_0)$. These are therefore not power series in $(z - z_0)$ (of the form (1099)) anymore, but may be brought into that form with a transformation of variables, as we had just seen in the example for the Laurent series of $z\cos(1/z)$. We are usually interested in the Laurent series which converges for $0 < |z - z_0| < R$, where $R = \infty$ is also possible. This series allows us to analyze an (isolated) singularity of f at z_0, by looking at the principal part

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}. \quad (1105)$$

Depending on the highest order non-vanishing coefficient, we identify the singularity as removable, a pole, or an essential singularity. Poles and zeros of functions are related in the sense that if a function f has a zero of order $n \in \mathbb{N}$ at z_0, then the function $1/f$ has a pole of order n at z_0.

We had defined the complex "infinity" via stereographic projection onto the Riemann sphere. ∞ gets mapped onto the north pole of this sphere under stereographic projection. This motivates the *extended complex plane,* which includes ∞. The behavior of a function f at ∞ is analyzed by looking at the behavior of the function $g(w) := f(1/w)$ at $w = 0$.

18.6.5 Conformal Mapping (17)

Holomorophic functions $f : \mathbb{C} \to \mathbb{C}$ can be interpreted as conformal mappings (from $\mathbb{R}^2$ to $\mathbb{R}^2$), conformality meaning that angles between oriented curves are preserved, in magnitude as well as in sense. Thm. 34 states that the mapping $w = f(z)$ by a holomorphic function is conformal except at critical points, where $f'(z_0) = 0$.

We looked at linear fractional transformations (LFTs) in particular, which are of the form

$$w = f(z) := \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0. \tag{1106}$$

Such a mapping is conformal everywhere in $\mathbb{R}^2$; special cases of LFTs include translations, rotations, linear transformations and the inversion on the unit circle. An LFT maps circles and lines in the z-plane to circles and lines in the w-plane (Thm. 35). It is a bijective conformal mapping of the extended complex plane $\mathbb{C} \cup \infty$, and the inverse is given by

$$z = f^{-1}(w) = \frac{dw - b}{-cw + a}. \tag{1107}$$

For $c \neq 0$, the image of $z = -d/c$ is ∞ (in the w-plane), and the image of ∞ (in the z-plane) is $w = a/c$. An LFT has at most two fixed points, $f(z) = z$, unless it is the identity mapping (Thm. 36). The fixed points of an LFT are the solutions of the polynomial equation

$$cz^2 - (a - d)z - b = 0. \tag{1108}$$

References

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