CS-521-900, Assignment 1

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1.
(a).
When $n = 0$, Left side:

$$\sum_{i=0}^{0} i(i+1) = 0 \cdot (0+1) = 0$$

Right side:

$$\frac{1}{6}n(n+1)(2n+4) = \frac{1}{6} \cdot 0 \cdot (0+1)(2 \cdot 0 + 4) = 0$$

Both sides are equal to 0, therefore the base case $n = 0$ is true.

Suppose the equation is true for $n = k$:

$$\sum_{i=0}^{k} i(i+1) = \frac{1}{6}k(k+1)(2k+4)$$

then when $n = k+1$, Left side:

$$\sum_{i=0}^{k+1} i(i+1) = \sum_{i=0}^{k} i(i+1) + (k+1)(k+1+1)$$

$$= \sum_{i=0}^{k} i(i+1) + (k+1)(k+2)$$

By induction hypothesis:

$$= \frac{1}{6}k(k+1)(2k+4) + (k+1)(k+2)$$

$$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$

$$= (k+1)(k+2)(\frac{1}{3}k+1)$$

Right side:

$$\begin{aligned}&\frac{1}{6}(k+1)(k+1+1)(2\cdot(k+1)+4)\\&=\frac{1}{6}(k+1)(k+1+1)(2k+6)\\&=(k+1)(k+2)(\frac{1}{3}k+1)\end{aligned}$$

Left = Right, therefore it is true for all $n \geq 0$.

(b).
When $n = 0$, Left side:

$$\sum_{i=0}^{0} i2^i = 0 \cdot 2^0 = 0$$

Right side:

$$(n-1)2^{n+1} + 2 = (0-1) \cdot 2^{0+1} + 2 = -1 \cdot 2 + 2 = 0$$

Both sides are equal to 0, therefore the base case $n = 0$ is true.

Suppose the equation is true for $n = k$:

$$\sum_{i=0}^{k} i2^i = (k-1)2^{k+1} + 2$$

then when $n = k+1$, Left side:

$$\sum_{i=0}^{k+1} i2^i = \sum_{i=0}^{k} i2^i + (k+1)2^{k+1}$$

By induction hypothesis:

$$\begin{aligned}&= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}\\&= 2^{k+1}(k-1+k+1) + 2\\&= 2^{k+1}2k + 2\\&= 2^{k+2}k + 2\end{aligned}$$

Right side:

$$\begin{aligned}&(k+1-1)2^{k+1+1} + 2\\&= k2^{k+2} + 2\\&= 2^{k+2}k + 2\end{aligned}$$

Left = Right, therefore it is true for all $n \geq 0$.

2.
(a).

$$f(n) = O(r(n)) \iff \exists constant\, c_1, n_1\, s.t.\, \forall n \geq n_1 : 0 \leq f(n) \leq c_1 r(n)$$

$$g(n) = O(s(n)) \iff \exists constant\, c_2, n_2\, s.t.\, \forall n \geq n_2 : 0 \leq g(n) \leq c_2 s(n)$$

Let $n_0 = max(n_1, n_2)$:

$$f(n) \cdot g(n) \iff 0 \leq f(n) \cdot g(n) \leq (c_1 c_2) \cdot r(n) \cdot s(n), \forall n \geq n_0$$

Since c_1 and c_2 are constants, therefore $c_1 c_2$ is a constant.
By definition of Big-O, which means:

$$f(n) \cdot g(n) = O(r(n) \cdot s(n))$$

Therefore the claim is true.

(b).
If

$$r(n) = n^2,\, s(n) = n$$

Then

$$f(n) = O(r(n)) = n^2,\, g(n) = O(s(n)) = n^2$$

$$\frac{f(n)}{g(n)} = \frac{n^2}{n^2} = 1$$

$$O(\frac{r(n)}{s(n)}) = O(\frac{n^2}{n}) = O(n)$$

Therefore the claim is not true.

(c).
Since a positive-valued, monotonically-increasing function doesn't have to be continuous, therefore let

$$f(n) = n$$

$$g(n) = \begin{cases} n - 1 & \text{if n is odd} \\ n & \text{if n is even} \end{cases}$$

In this case, neither $f(n) = O(g(n))$ nor $g(n) = O(f(n))$. Therefore the claim is not true.

3.
(a).
$g(n) = O(f(n))$.
Proof: Consider $c = 1$ and $n_0 = 10$. Then it is true that $\forall n \geq n_0$, $n^3 \geq 5n$, $\log_2 n \geq \log_{10} n$. Since $n \geq 0, \log_{10} 5n \geq 0$. Therefore, $\forall n \geq n_0, 0 \leq \log_{10} 5n \leq 1 \cdot log_2 n^3 = c \cdot log_2 n^3$. By definition of Big-O, we have that

$g(n) = O(f(n))$.

(b).
$f(n) = O(g(n))$.
Proof: $\log_2 n^5 = 5\log_2 n = \dfrac{5\log_{10} n}{\log_{10} 2} = \dfrac{5}{\log_{10} 2} \cdot \log_{10} n$, consider $c = 1$ and $n_0 = 10^{\dfrac{5}{\log_{10} 2}}$. Then it is true that $\forall n \geq n_0$, $\log_{10} n \cdot \log_{10} n \geq \dfrac{5}{\log_{10} 2} \cdot \log_{10} n$. Since $n \geq 0, \log_2 n^5 \geq 0$, therefore $\forall n \geq n_0, 0 \leq \dfrac{5}{\log_{10} 2} \cdot \log_{10} n \leq 1 \cdot \log_{10} n \cdot \log_{10} n \iff 0 \leq \log_2 n^5 \leq c \cdot (\log_{10} n)^2$. By definition of Big-O, we have that $f(n) = O(g(n))$.

(c).
$g(n) = O(f(n))$.
Proof: Consider $c = 1$ and $n_0 = 1$. Then it is true that $\forall n \geq n_0$, $(\log_2 n)^2 \geq n^{\frac{1}{2}}$. Since $n \geq 0, n^{\frac{3}{2}} \geq 0$. Therefore, $\forall n \geq n_0$, $0 \leq n^{\frac{3}{2}} \leq 1 \cdot n(\log_2 n)^2 = c \cdot n(\log_2 n)^2$. By definition of Big-O, we have that $g(n) = O(f(n))$.

(d).
$f(n) = O(g(n))$.
Proof: $n^2 = 2^{\log_2 n^2} = 2^{2\log_2 n}$. Consider $c = 1$ and $n_0 = 3$. Then it is true that $\forall n \geq n_0$, $2\log_2 n \geq \sqrt{n}$. Since $n \geq 0$, $2^{\sqrt{n}} \geq 0$. Therefore, $\forall n \geq n_0, 0 \leq 2^{\sqrt{n}} \leq 1 \cdot 2^{2\log_2 n} = c \cdot n^2$. By definition of Big-O, we have that $f(n) = O(g(n))$.

4.

$$n^{log_2 n} = 2^{log_2 n^{log_2 n}} = 2^{log_2 n^2}$$

Consider $c = 1$ and $n_0 = 17$. Then it is true that $\forall n \geq n_0$, $n \geq \log_2 n^2$. Since $n \geq 0$, $2^{\log_2 n^2} \geq 0$. Therefore, $\forall n \geq n_0$, $0 \leq 2^{\log_2 n^2} \leq 1 \cdot 2^n \iff 0 \leq n^{\log_2 n} \leq 1 \cdot 2^n = c \cdot 2^n$.
By definition of Big-O, we have that $n^{\log_2 n} = O(2^n)$.

5.

$$\sqrt{n} = \log 10^{\sqrt{n}}$$

Consider $c = 1$ and $n_0 = 1$. Then it is true that $\forall n \geq n_0$, $10^{\sqrt{n}} \geq n$. Since $n \geq 0$, $\log n \geq 0$. Therefore, $\forall n \geq n_0, 0 \leq \log n \leq 1 \cdot \log 10^{\sqrt{n}} \iff 0 \leq \log n \leq 1 \cdot \sqrt{n} = c \cdot \sqrt{n}$.
By definition of Big-O, we have that $\log n = O(\sqrt{n})$.

6.
(a).

$$a = 4, b = 2, f(n) = n, n^{\log_b a} = n^{\log_2 4} = \Theta(n^2)$$

Since $f(n) = n = O(n^{\log_2 4-\epsilon})$, where $\epsilon = 1$, apply case 1 of the master theorem and conclude that the solution is $T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 4}) = \Theta(n^2)$.

(b).

$$a = 4, b = 2, f(n) = n^2, n^{\log_b a} = n^{\log_2 4} = \Theta(n^2)$$

Since $f(n) = n^2 = \Theta(n^{\log_2 4})$, apply case 2 of the master theorem and conclude that the solution is $T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(n^{\log_2 4} \lg n) = \Theta(n^2 \lg n)$.

(c).

$$a = 3, b = 2, f(n) = n^2, n^{\log_b a} = n^{\log_2 3} = \Theta(n^{1.5849625007})$$

Since $f(n) = n^2 = \Omega(n^{\log_2 3+\epsilon})$, where $\epsilon = 0.4150374993$, apply case 3 of the master theorem and conclude that the solution is $T(n) = \Theta(f(n)) = \Theta(n^2)$.

(d).

$$a = 81, b = 3, f(n) = \frac{1}{3}n^4 + 81n^3, n^{\log_b a} = n^{\log_3 8} = \Theta(n^4)$$

Since $f(n) = O(n^4) = \Theta(n^{\log_3 81})$, apply case 2 of the master theorem and conclude that the solution is $T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(n^{\log_3 81} \lg n) = \Theta(n^4 \lg n)$.