

CS 615 – Deep Learning

Convolutional Neural Networks

Work adapted from that of Cameron Graybill

Objectives

- Convolutional Neural Networks

Issues with Generic Deep Networks

- There are some problems with an arbitrarily deep ANN:
 - Late layers (ones near the output) learn quickly/well and therefore early layers (near the input stage) have little error and therefore become relatively useless.
- Therefore, in practice generic deep ANNs beyond depth two aren't successful.
- Which leads us to variations of these.....
- One that has seen success in the area of image recognition is *convolutional neural networks (CNNs)*.

Convolution

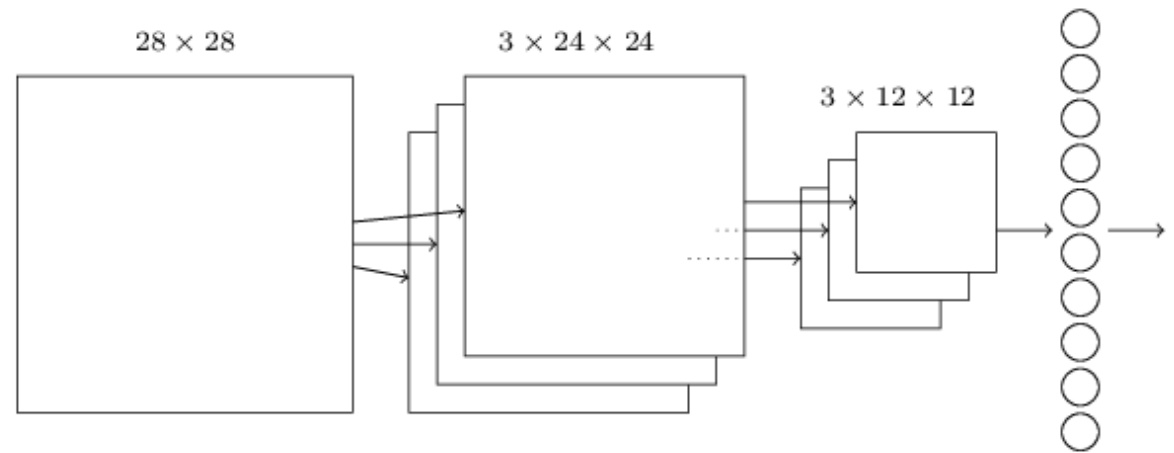
- *Convolution* is an operation used often in image processing and computer vision.
- Given a filter/kernel K and a sub-image, convolution gives a *filter response*.
 - Basically saying how well the sub-image matches the filter/kernel.
- If we know what we're looking for, we can *design* filters and apply them to an image to get filter responses to be used for machine learning.
- This is what is often done in traditional machine learning.
- But what if we don't know the kernels?
- Is there a way to discover/learn useful ones?

Convolution Neural Networks

- Convolutional Neural Networks (CNNs) attempt to do exactly that!
- They have become popular and successful in images and audio classification where traditionally a kernel/filter was applied to the raw data to extract features.
- This is because kernels take spatial relationships into account.
 - What is important in image and audio classification.

CNNs

- CNNs typically have
 - One or more convolution layer
 - A final fully connected shallow ANN.
- The convolution layer contains several parts
 - Feature Map Extraction
 - Pooling
- Let's look at each of these



Convolution

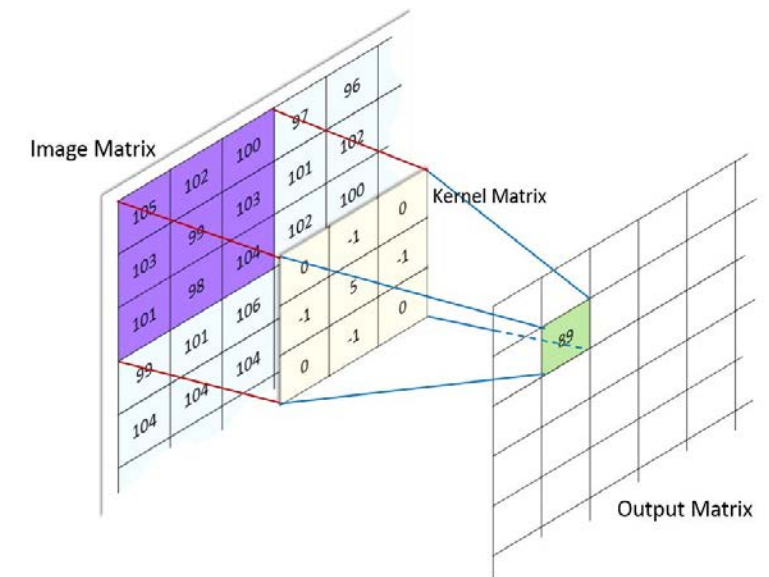
- Again, convolution is a mathematical operation that takes a filter/function, and applies it to some area to produce a single value at the center of that area.
- It's almost identical to cross-correlation
 - Flipping the filter and then doing cross-correlation is equivalent to doing convolution.
 - The pro of using convolution instead of cross-correlation is that it exhibits the commutative property.

Convolution

- We use the $*$ operator to denote convolution.
- To formalize the convolution operator, given an $M \times M$ filter, K , and an input matrix X , we can compute the convolution at location (a, b) as:

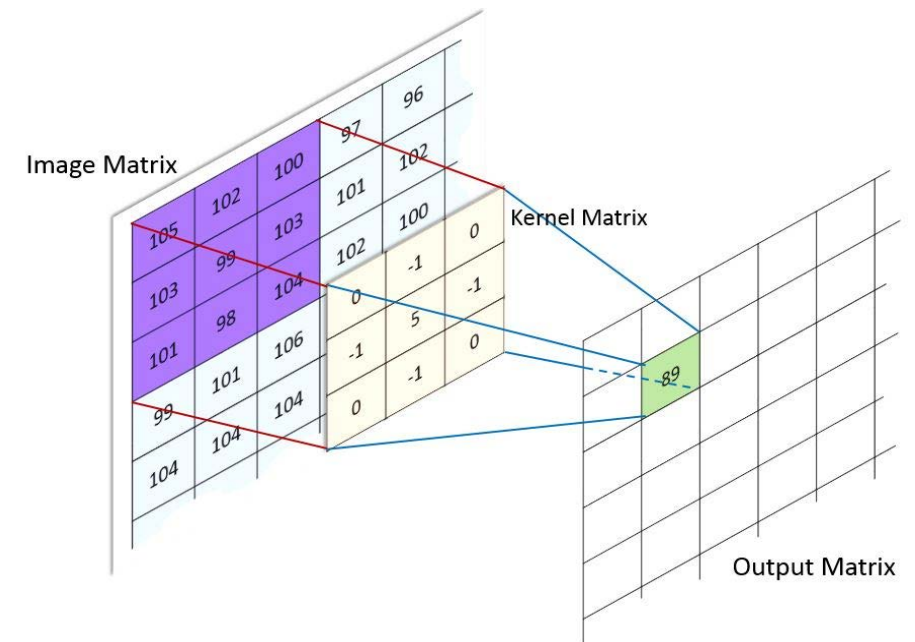
$$F_{ab} = (X_{ab} * K) = \sum_{i=-\frac{M}{2}}^{\frac{M}{2}} \sum_{j=-\frac{M}{2}}^{\frac{M}{2}} X_{(a-i, b-j)} K_{ij}$$

- If we apply convolution to several locations in the image, what we get back is a new matrix, that is the filter response at each location, and which we call the **Feature Map**



Feature Map Size

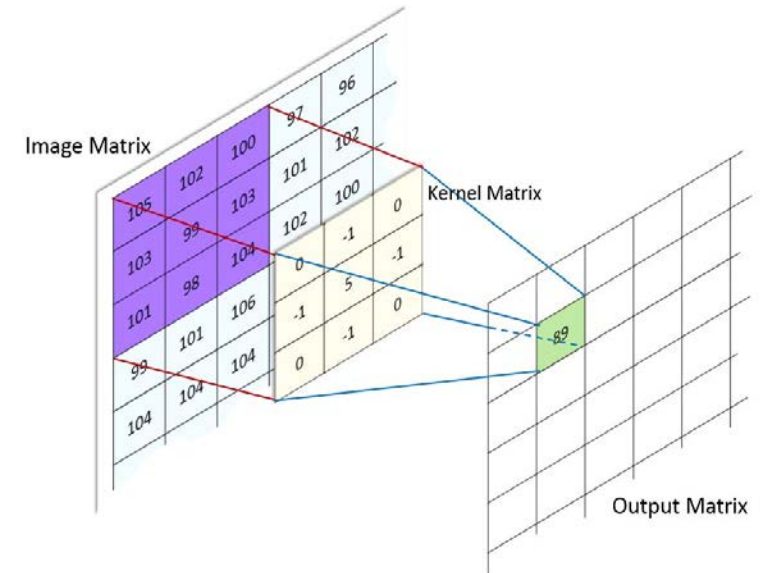
- Valid convolution can only be computed where there are $M \times M$ pixels to process.
- This results in a **smaller** image than the original one.
- If our image is $H \times W$, then our feature map will be $(H - M + 1) \times (W - M + 1)$



Convolution Example

- Let $X = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$, $K = \begin{bmatrix} 0.11 & 0.12 & 0.13 \\ 0.21 & 0.22 & 0.23 \\ 0.31 & 0.32 & 0.33 \end{bmatrix}$

- $X * K = \begin{bmatrix} 37.5 & 39.48 \\ 57.3 & 59.28 \end{bmatrix}$

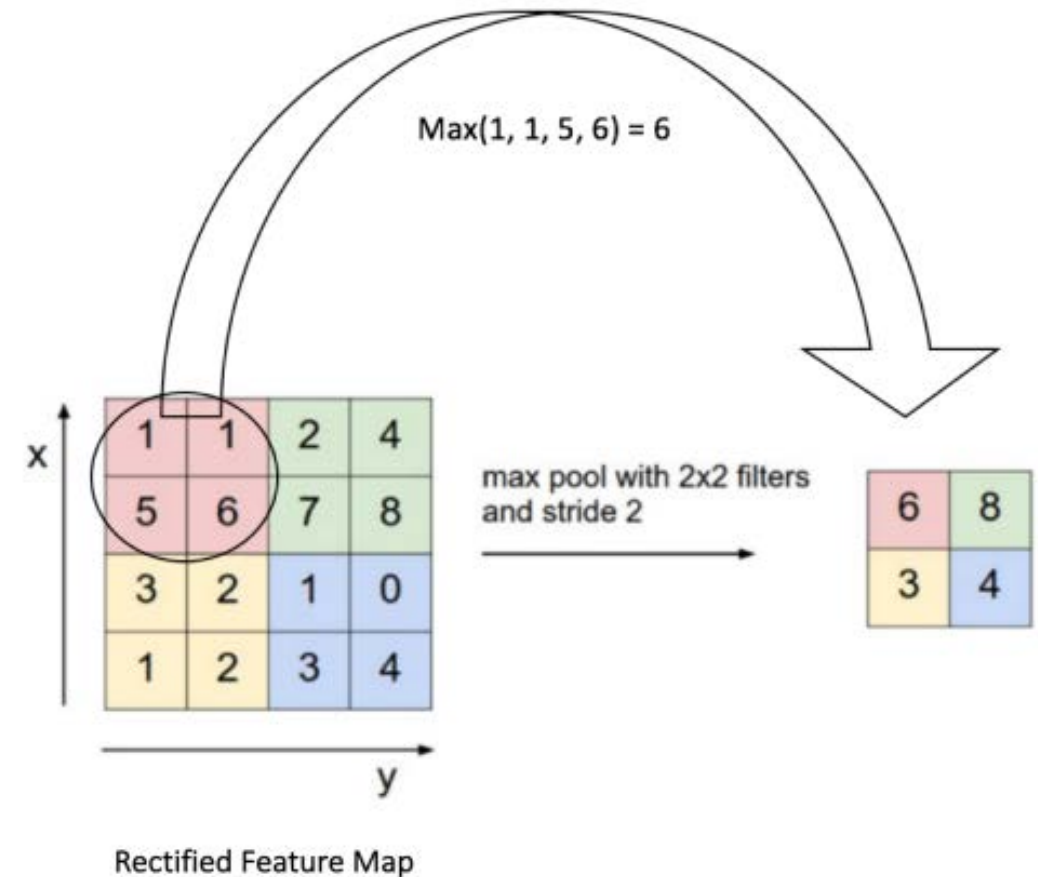


Pooling

- The next part of a CNN layer is pooling.
- Pooling is essentially down-sampling: we're taking our feature map and making a new, smaller map by “summarizing” the original one.
 - This provides some invariance to translation
- Pooling is also done by moving around a $Q \times Q$ window and extracting a value from each locations
- However, it is common not to move/slide the window by just one pixel, but instead to allow for this to be a user-defined hyperparameter, called the ***stride***.

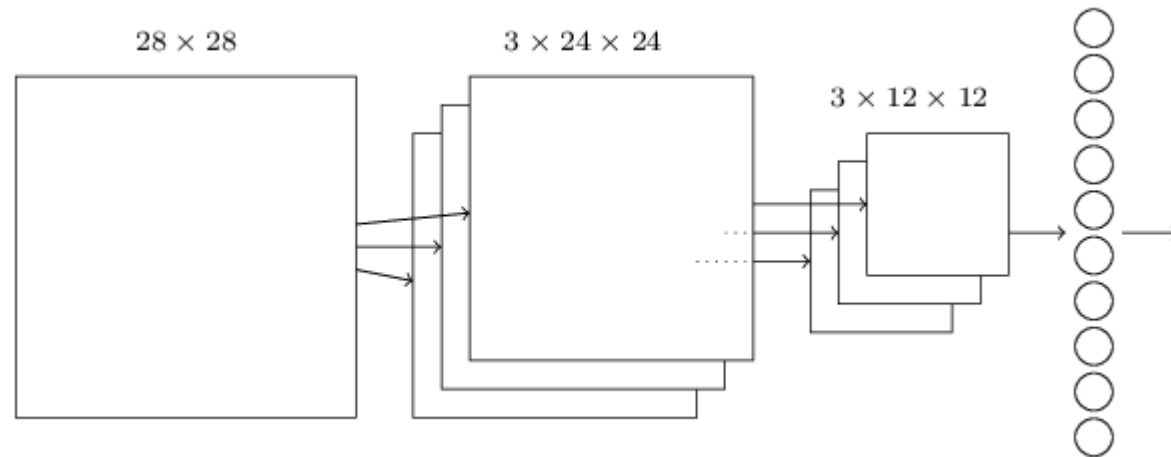
Pooling

- Common pooling techniques include:
 - Max Pooling – Select the maximum value in the square
 - L2 Pooling – Compute the square root of the sum of the values in the square.
- If our feature map is $D \times E$ and we have a stride of S , then output from the pooling process will be a $\left\lfloor \frac{D}{S} \right\rfloor \times \left\lfloor \frac{E}{S} \right\rfloor$ matrix



CNNs

- Now that our Convolution Layer is made, the output of the pooling process is flattened and becomes the input of a traditional ML system.
 - Potentially adding in bias features as desired.

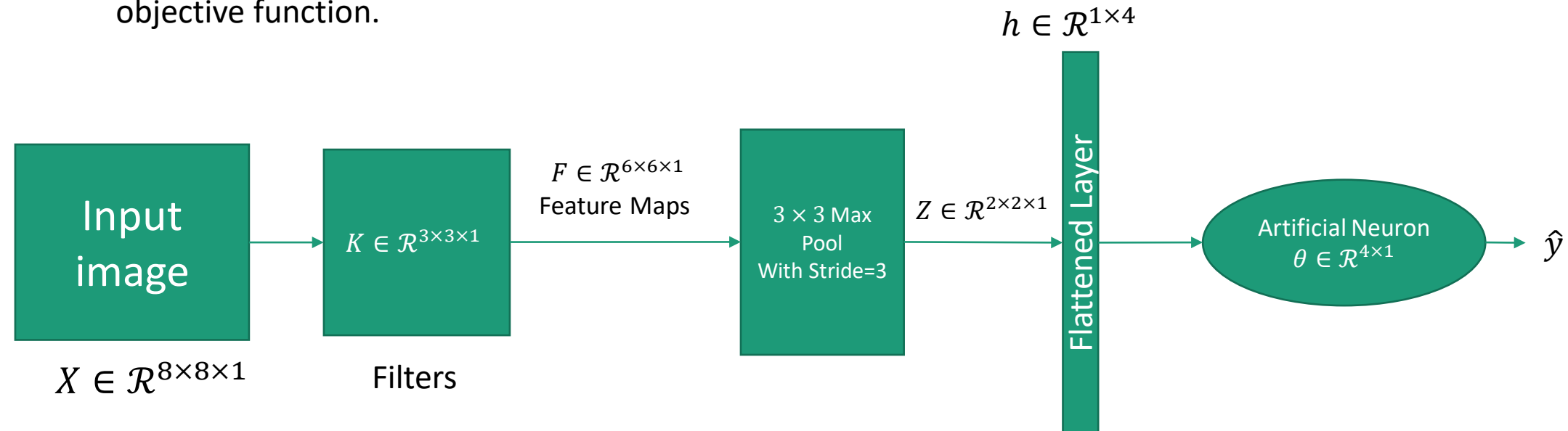


Training

- As with our ANNs, training a CNN requires *forward-backwards propagation*.
- But now our set of parameters to learn are:
 - Any parameters of our “traditional” ML part of our system.
 - The weights of each kernel in the convolution layer(s).
- And once again, given a set of parameters, the forward propagation is relatively simple:
 - Create the feature maps using the current kernels.
 - Create the pooled maps from the feature maps.
 - Feed these into our traditional ML algorithm.
- Of course the tough part is learning the weights.

Example CNN Architecture

- Let's start off with a relatively simple system/example:
 - X is input image of size 8×8
 - K is a single 3×3 kernel
 - Our pooling process using max-pooling on a 3×3 region with stride of 3
 - Our traditional ML layer is an artificial neuron with a logistic activation function and a log likelihood objective function.



Full Example

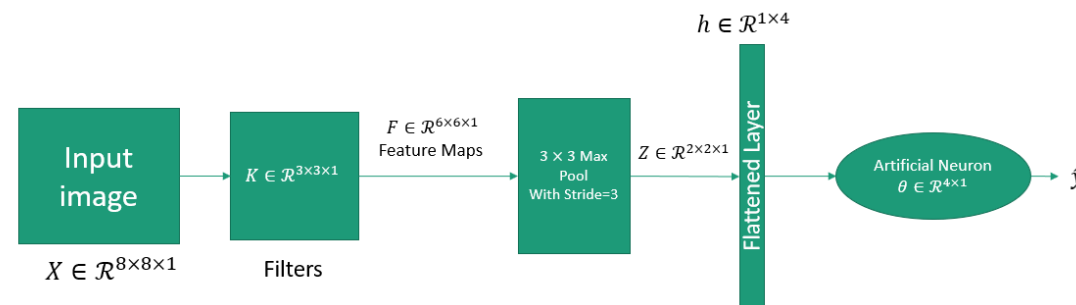
- In an attempt to wrap our heads around all of this, let's do a numerical example.
- Let our grayscale image X , whose target class (binary classification) is $y = 0$ be:

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \\ 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 \\ 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 \end{bmatrix}$$

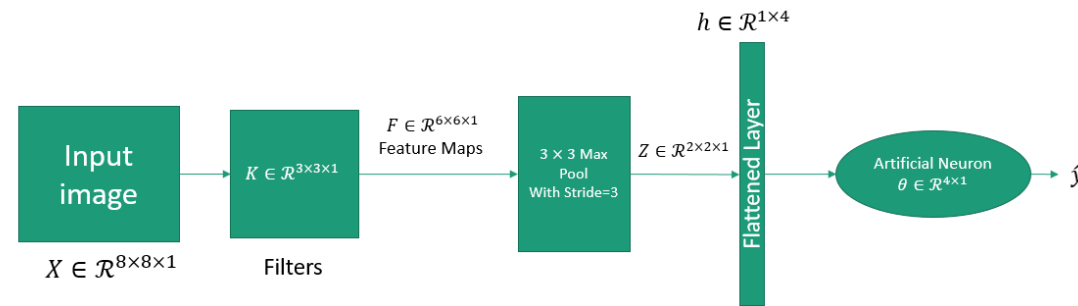
Full Example

- In addition our kernel will be initialized as $K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

- And the output layer weights are initialized as $\theta = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$



Full Example

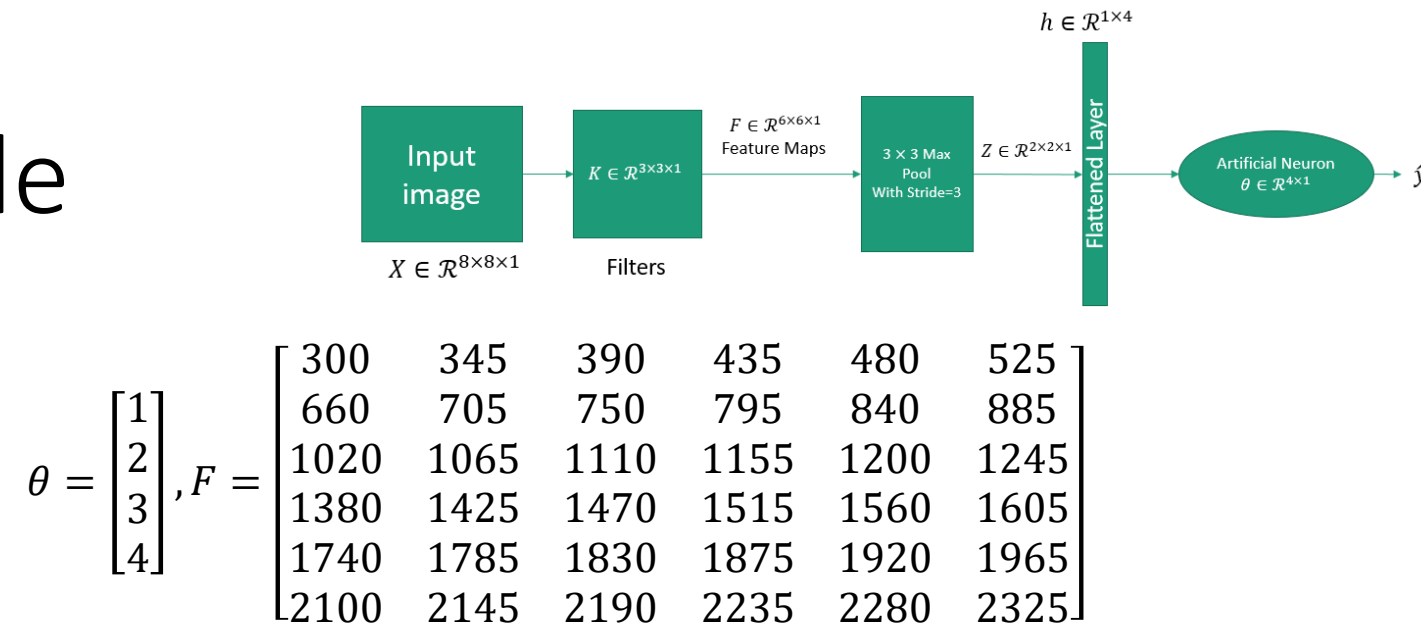


$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \\ 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 \\ 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 \end{bmatrix}, y = 0, K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \theta = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- First let's forward propagate!
- Here's the convolution

$$F = X * K = \begin{bmatrix} 300 & 345 & 390 & 435 & 480 & 525 \\ 660 & 705 & 750 & 795 & 840 & 885 \\ 1020 & 1065 & 1110 & 1155 & 1200 & 1245 \\ 1380 & 1425 & 1470 & 1515 & 1560 & 1605 \\ 1740 & 1785 & 1830 & 1875 & 1920 & 1965 \\ 2100 & 2145 & 2190 & 2235 & 2280 & 2325 \end{bmatrix}$$

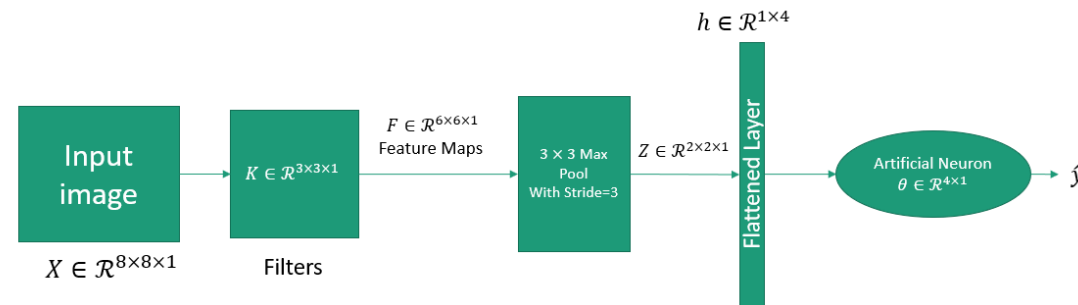
Full Example



- And now let's 3x3 max-pool with stride of 3.
- This results in:

$$Z = \begin{bmatrix} 1110 & 1245 \\ 2190 & 2325 \end{bmatrix}$$

Full Example



$$\theta = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, F = \begin{bmatrix} 300 & 345 & 390 & 435 & 480 & 525 \\ 660 & 705 & 750 & 795 & 840 & 885 \\ 1020 & 1065 & 1110 & 1155 & 1200 & 1245 \\ 1380 & 1425 & 1470 & 1515 & 1560 & 1605 \\ 1740 & 1785 & 1830 & 1875 & 1920 & 1965 \\ 2100 & 2145 & 2190 & 2235 & 2280 & 2325 \end{bmatrix}, Z = \begin{bmatrix} 1110 & 1245 \\ 2190 & 2325 \end{bmatrix}$$

- Then we'll flattened this to be a single row vector. Since Matlab uses column major indexing, this becomes:

$$h = [1110 \quad 2190 \quad 1245 \quad 2325]$$

- To which we apply θ

$$net = h \cdot \theta = 18525$$

- Then apply activation function $g(z)$ (for which we'll use the logistic activation function):

$$\hat{y} = \frac{1}{1 + e^{-18525}} \approx 1$$

Gradient of Artificial Neuron Layer

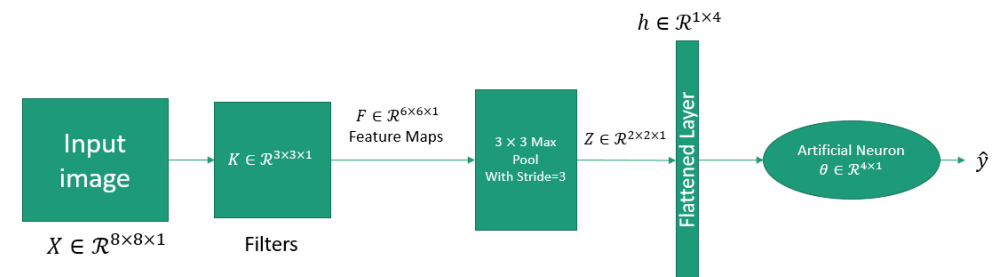
- Ok, now it's time to update our parameters based on the forward propagation.
- As a reminder, at the end of our CNN we have an artificial neuron with a logistic activation function and log likelihood objective function for binary classification.
- Given this configuration, from our previous work, the gradient with respect to our last set of parameters is:

$$\frac{\partial J}{\partial \theta_s} = \frac{\partial J}{\partial g(h|\theta)} \cdot \frac{\partial g(h|\theta)}{\partial \theta_s} = \left(\frac{y - \hat{y}}{\hat{y}(1 - \hat{y})} \right) \cdot (h_s \hat{y}(1 - \hat{y})) = (y - \hat{y})h_s$$

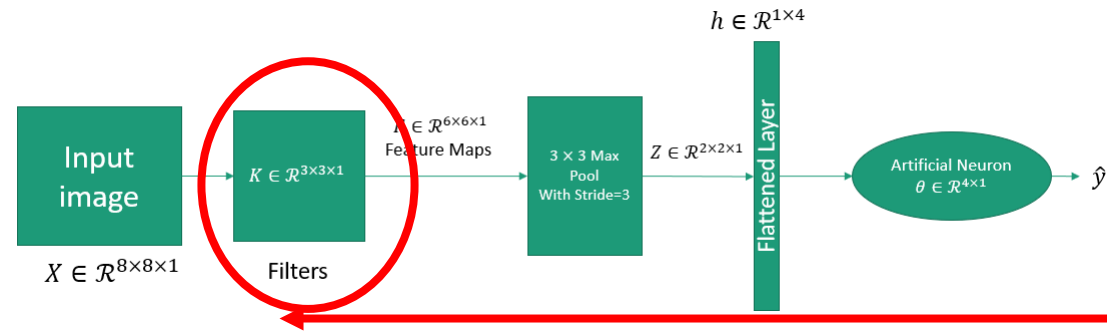
$$\frac{\partial J}{\partial \theta} = h^T (y - \hat{y})$$

- For our example this is:

$$\frac{\partial J}{\partial \theta} = \begin{bmatrix} -1110 \\ -2190 \\ -1245 \\ -2325 \end{bmatrix}$$



Gradient of Kernel



- The next set of parameters we need to update are those of the convolution kernel, K
- So the question is, what is:

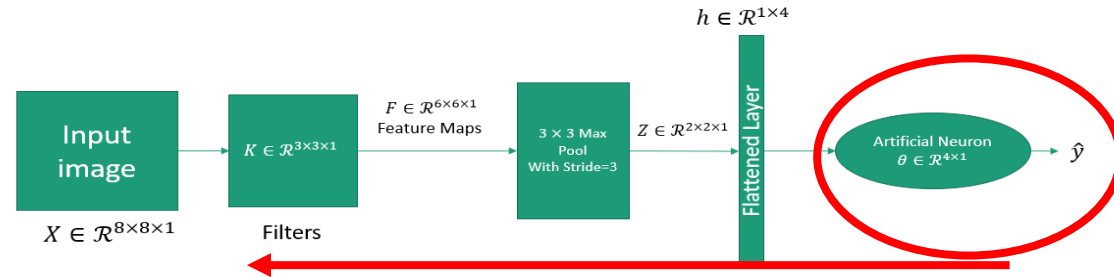
$$\frac{\partial J}{\partial K_{ij}}?$$

- Well let's try to chain rule this guy!

$$\frac{\partial J}{\partial K_{ij}} = \frac{\partial J}{\partial g(h|\theta)} \cdot \frac{\partial g(h|\theta)}{\partial h} \cdot \frac{\partial h}{\partial Z} \cdot \frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}$$

- Ugh 😞!

Gradient of Kernel



$$\frac{\partial J}{\partial K_{ij}} = \frac{\partial J}{\partial g(h|\theta)} \cdot \frac{\partial g(h|\theta)}{\partial h} \cdot \frac{\partial h}{\partial Z} \cdot \frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}$$

- We already have $\frac{\partial J}{\partial g(h|\theta)}$ from our previous layer:

$$\frac{\partial J}{\partial g(h|\theta)} = \left(\frac{y - \hat{y}}{\hat{y}(1 - \hat{y})} \right)$$

- What is $\frac{\partial g(h|\theta)}{\partial h}$?

$$\frac{\partial g(h|\theta)}{\partial h} = \theta^T \hat{y}(1 - \hat{y})$$

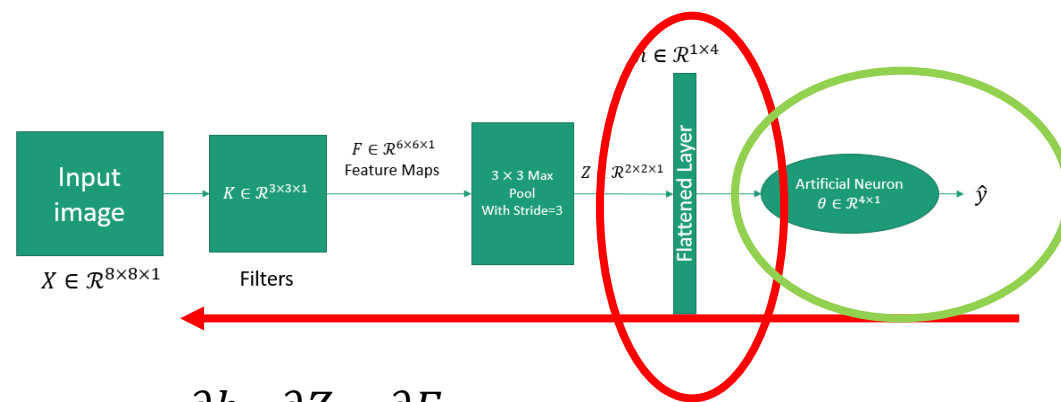
- So:

$$\frac{\partial J}{\partial K_{ij}} = (y - \hat{y})\theta^T \cdot \frac{\partial h}{\partial Z} \cdot \frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}$$

- For the example:

$$\frac{\partial J}{\partial K_{ij}} = (-1) \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T \cdot \frac{\partial h}{\partial Z} \cdot \frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}$$

Gradient of Kernel



$$\frac{\partial J}{\partial K_{ij}} = (y - \hat{y})\theta^T \cdot \frac{\partial h}{\partial Z} \cdot \frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}$$

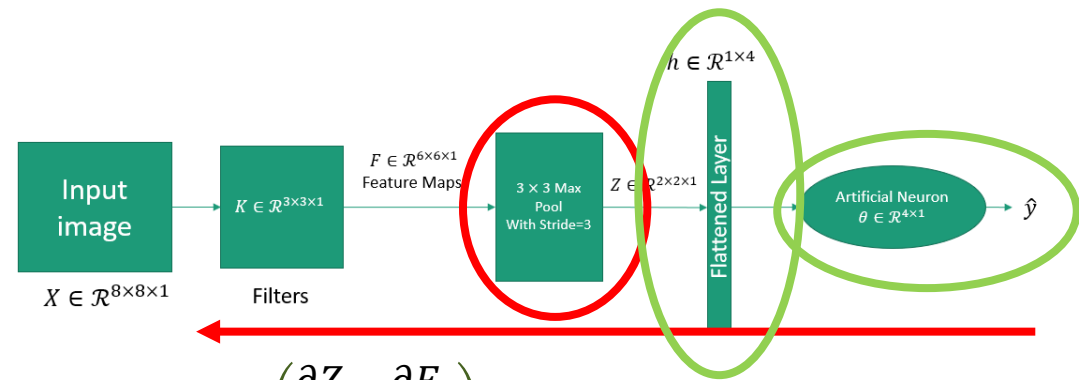
- h is just a flattened version of Z , so $\frac{\partial h}{\partial Z}$ is just a reshaping function

$$\frac{\partial J}{\partial K_{ij}} = (y - \hat{y})\theta^T \cdot \text{reshape} \left(\frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}} \right)$$

- For our example:

$$\frac{\partial J}{\partial K_{ij}} = (-1) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T \cdot \text{reshape} \left(\frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}} \right)$$

Gradient of Kernel



$$\frac{\partial J}{\partial K_{ij}} = (y - \hat{y}) \theta^T \cdot \text{reshape} \left(\frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}} \right)$$

- Since Z is created by max-pooling, we can think of it as selecting elements from $\frac{\partial F}{\partial K_{ij}}$.

- Let's call this gradient *select*

$$\frac{\partial J}{\partial K_{ij}} = (y - \hat{y}) \cdot \theta^T \cdot \text{reshape} \left(\text{select} \left(\frac{\partial F}{\partial K_{ij}} \right) \right)$$

- Where were the max locations selected from?

- That leaves us with $\frac{\partial F}{\partial K_{ij}}$

$$F = \begin{bmatrix} 300 & 345 & 390 & 435 & 480 & 525 \\ 660 & 705 & 750 & 795 & 840 & 885 \\ 1020 & 1065 & 1110 & 1155 & 1200 & 1245 \\ 1380 & 1425 & 1470 & 1515 & 1560 & 1605 \\ 1740 & 1785 & 1830 & 1875 & 1920 & 1965 \\ 2100 & 2145 & 2190 & 2235 & 2280 & 2325 \end{bmatrix}$$

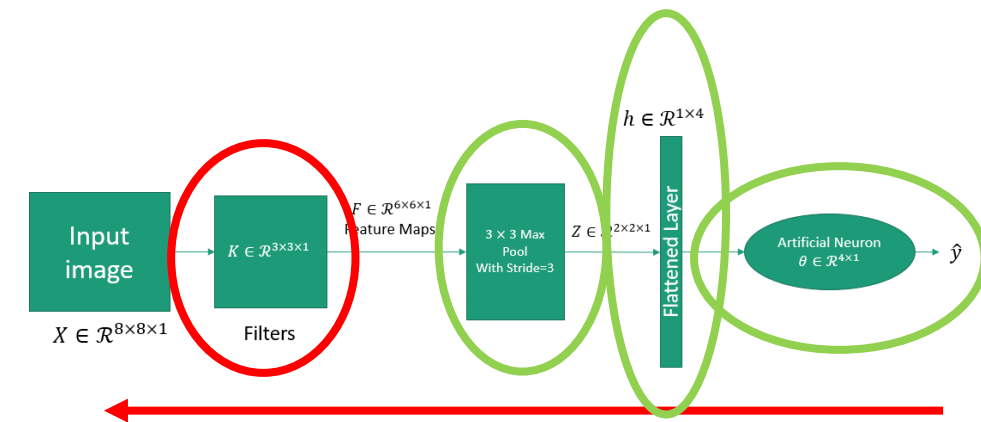
Gradient of Convolution Function

- Recall the convolution function

$$F_{ab} = \sum_{i=-\frac{M}{2}}^{\frac{M}{2}} \sum_{j=-\frac{M}{2}}^{\frac{M}{2}} X_{a-i, b-j} K_{ij}$$

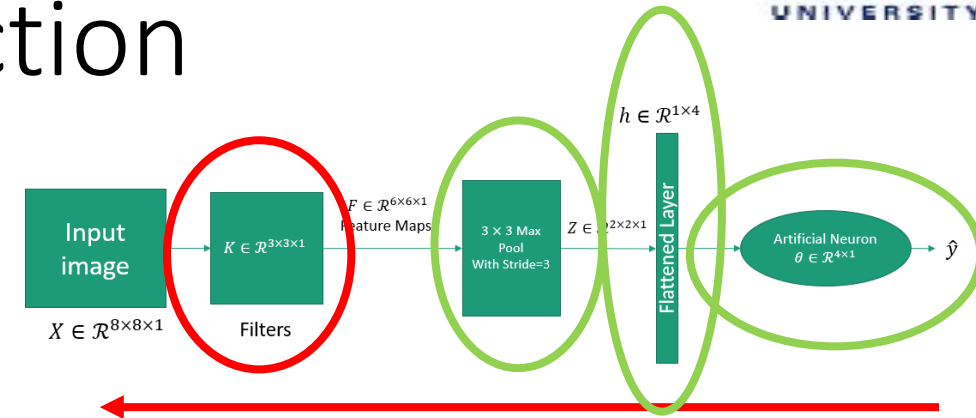
- So let's start with, what is $\frac{\partial F_{ab}}{\partial K_{ij}}$?

$$\frac{\partial F_{ab}}{\partial K_{ij}} = X_{a-i, b-j}$$



Gradient of Convolution Function

$$\frac{\partial F_{ab}}{\partial K_{ij}} = X_{a-i, b-j}$$

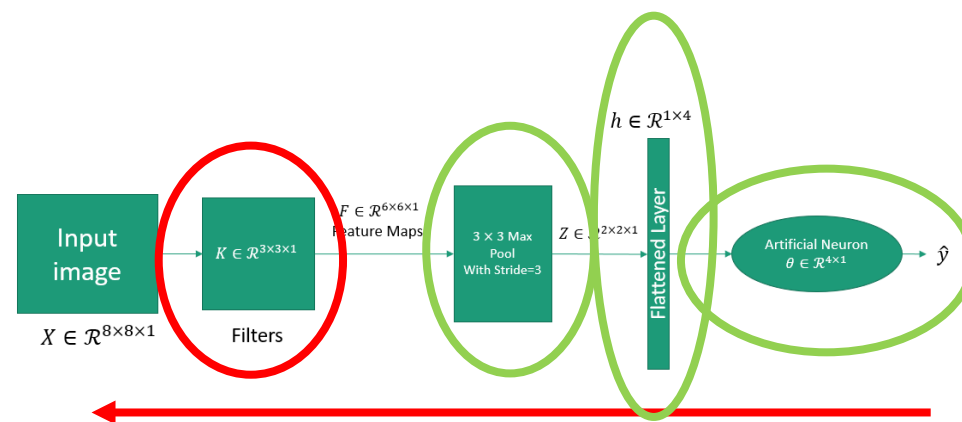


- So then, what is $\frac{\partial F}{\partial K_{ij}}$?
- We need to think about what elements of F will K_{ij} affect?
- This is actually (somewhat) simple!

$$\frac{\partial F}{\partial K_{ij}} = X_{H-i+1:-1:M-i+1, W-j+1:-1:M-j+1}$$

- Note: $H - i + 1 : -1 : M - i + 1$ means from $H - i + 1$ **down to** $M - i + 1$
 - This achieves the flipping done by convolution.

Full Example



$\frac{\partial J}{\partial K_{11}}$
 $\frac{\partial J}{\partial K_{12}}$

- Let's find $\frac{\partial J}{\partial K_{11}}$ for our example:

$$\frac{\partial J}{\partial K_{ij}} = (-1) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T \cdot \text{reshape}(\text{select}(X_{H-i+1:-1:M-i+1, W-j+1:-1:M-j+1}))$$

$$\frac{\partial J}{\partial K_{11}} = (-1) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T \cdot \text{reshape}(\text{select}(\begin{bmatrix} 63 & 62 & 61 & 60 & 59 \\ 55 & 54 & 53 & 52 & 51 \\ 47 & 46 & 45 & 44 & 43 \\ 39 & 38 & 37 & 36 & 35 \\ 31 & 30 & 29 & 28 & 27 \\ 23 & 22 & 21 & 20 & 19 \end{bmatrix})) = [-1] \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T \text{reshape}(\begin{bmatrix} 46 & 43 \\ 22 & 19 \end{bmatrix})$$

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \\ 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 \\ 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 \end{bmatrix}$$

$$\frac{\partial J}{\partial K_{11}} = (-1) \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T \begin{bmatrix} 46 & 22 & 43 & 19 \end{bmatrix} = -295$$

$[45, 21, 42, 18]$

300	345	390	435	480	525
660	705	750	795	840	885
1020	1065	1110	1155	1200	1245
1380	1425	1470	1515	1560	1605
1740	1785	1830	1875	1920	1965
2100	2145	2190	2235	2280	2325

$$K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

3 2 1

Full Example

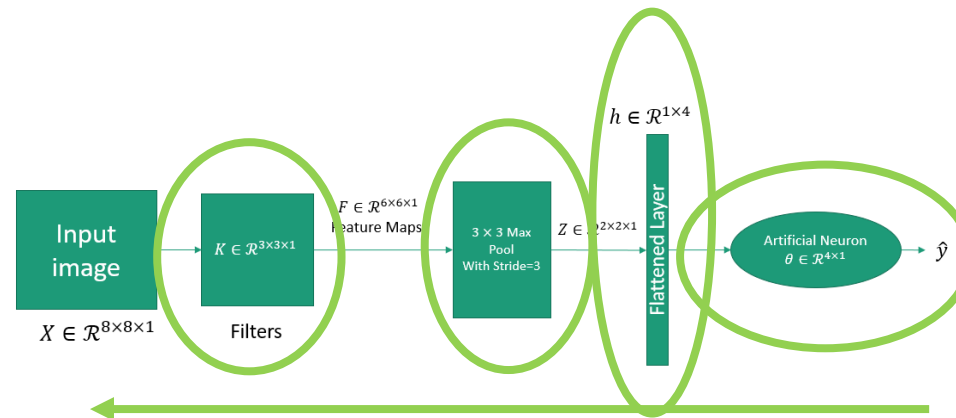
$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \\ 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 \\ 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 \end{bmatrix}, K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- Doing this for all elements in K

$$\frac{\partial J}{\partial K} = \begin{bmatrix} -295 & -285 & -275 \\ -215 & -205 & -195 \\ -135 & -125 & -115 \end{bmatrix}$$

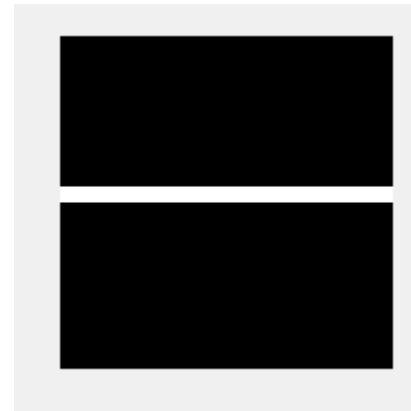
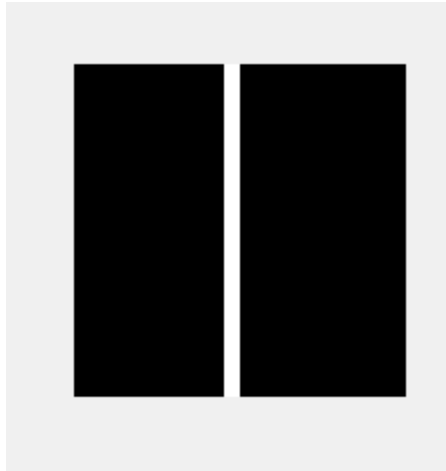
- And since our objective function is log likelihood, we want to maximize it and thus:

$$K = K + \eta \frac{\partial J}{\partial K}$$



Another Example

- Below are two artificially created 20×20 images.
- The left one we'll call "class 1" and the right one "class 0"

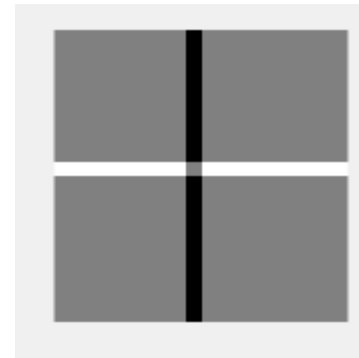
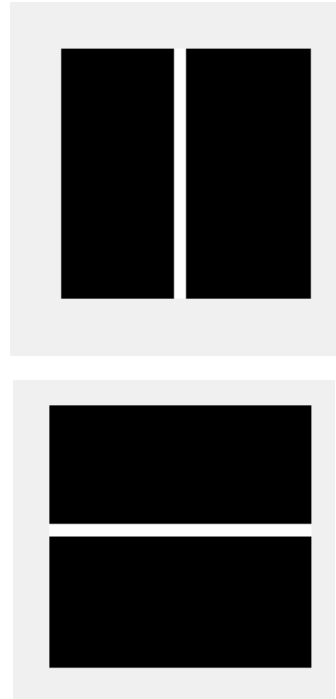


Another Example

- Our CNN will be configured to have a convolutional layer with a single 20×20 kernel filter and a 1×1 max-pool layer with a stride of 1.
 - Note that the kernel size is the same as the image size!
 - Although this is rarely done, I did this for illustrative purposes.
- Therefore, our filter map will be 1×1 and the output of max-pooling will be 1×1 .
- We will then flatten the output of the max-pooling layer to form a 1×1 vector to be fed into an artificial neuron with a logistic activation function and log likelihood objective function.
 - Therefore $\theta \in \mathcal{R}^{1 \times 1}$

Another Example

- After 1,000 iterations, with $\eta = 0.1$ and L^2 regularization of 0.1, I thought it would be interesting to see what filter it learned!
- Thoughts?



Convoluting with Multiple Kernels

- Most CNNs apply more than one kernel to an image.
- This shouldn't change much other than the fact that the output of convolution stage will be a 3D matrix (fyi, a multi-dimensional matrix is called a *tensor*).
- If our input to convolution is a $W \times H$ image and we have P $M \times M$ kernels, we will then obtain multiple feature maps as a tensor.
- The output value for feature map p , at location (a, b) , we now have:

$$F_{(a,b,p)} = \sum_{i=-\frac{M}{2}}^{\frac{M}{2}} \sum_{j=-\frac{M}{2}}^{\frac{M}{2}} X_{(a-i,b-j)} K_{(i,j,p)}$$

- And of course the output of the pooling stage will also be a tensor.
- So we need to keep this in mind when we flatten the output to be the input to our traditional network.

Convoluting with Color Images

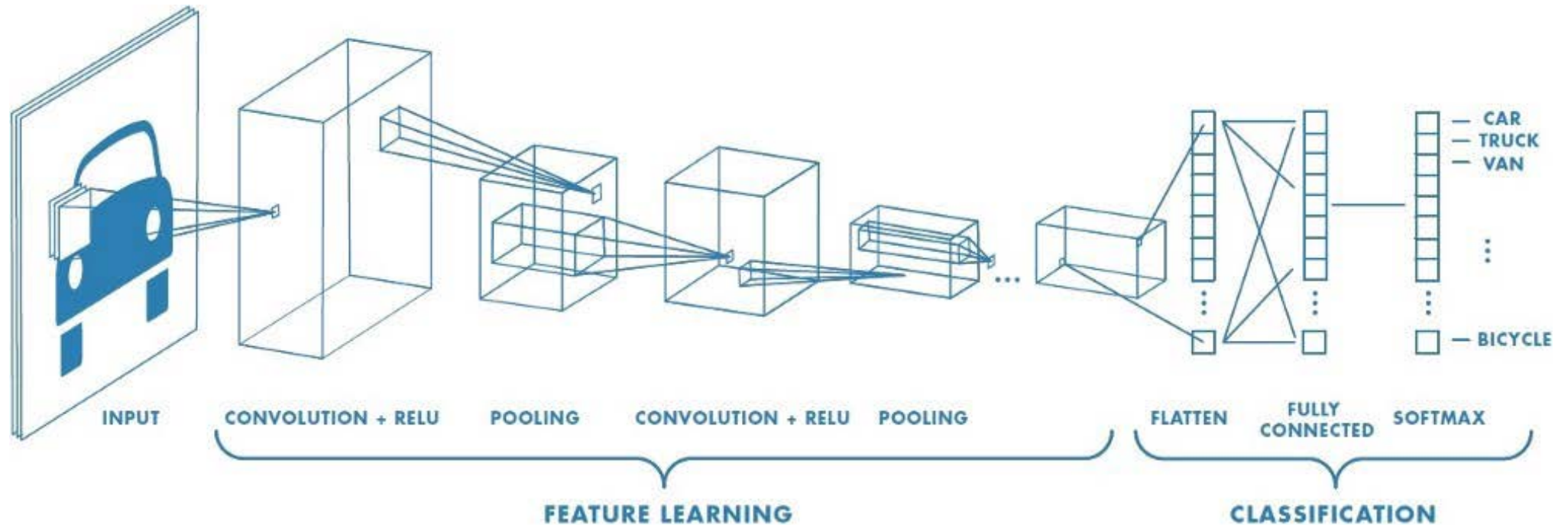
- What if we have an RGB color image?
- Then our image is $W \times H \times 3$
- And therefore our filters/kernels will be $M \times M \times 3$
- And our convolution function will be:

$$F_{a,b} = \sum_{l=1}^3 \sum_{i=-\frac{M}{2}}^{\frac{M}{2}} \sum_{j=-\frac{M}{2}}^{\frac{M}{2}} X_{(a-i,b-j,l)} K_{(i,j,l)}$$

- This will also be important if/when we stack several convolution layers together.
 - Since the output of one layer *could* be a tensor.

Bringing it all together

- Used in series with fully connected layers
- Convolutions find the important patterns in the input
- Fully connected layers abstract the existence of the patterns into a label



Vectorizing/Batching

- Due to CNNs being not fully connected, sharing weights, and the pooling layer, vectorizing some of the partial gradients is a bit tricky, so we'll leave this to figure reading.
- But here's one resource:
 - http://lxu.me/mypapers/vcnn_aaai15.pdf
- Same thing goes for batch processing.
- Of course we can just compute the gradients for each sample then take the average of those gradients over a batch.

Sources

- Articles
 - <http://www.robots.ox.ac.uk/~vgg/practicals/cnn/>
 - <https://becominghuman.ai/back-propagation-in-convolutional-neural-networks-intuition-and-code-714ef1c38199>
- Videos
 - https://www.youtube.com/watch?v=YRhxdVk_sls
 - <https://youtu.be/FTr3n7uBluE>
- Lectures
 - <https://www.youtube.com/watch?v=BvrWiL2fd0M>
 - <http://neuralnetworksanddeeplearning.com/chap6.html>