CS 615 – Deep Learning

Convolutional Neural Networks

Work adapted from that of Cameron Graybill

Objectives

- Convolutional Neural Networks

Issues with Generic Deep Networks

- There are some problems with an arbitrarily deep ANN:
  - Late layers (ones near the output) learn quickly/well and therefore early layers (near the input stage) have little error and therefor become relatively useless.
- Therefore, in practice generic deep ANNs beyond depth two aren't successful.
- Which leads us to variations of these…..
- One that has seen success in the area of image recognition is *convolutional neural networks (CNNs)*.
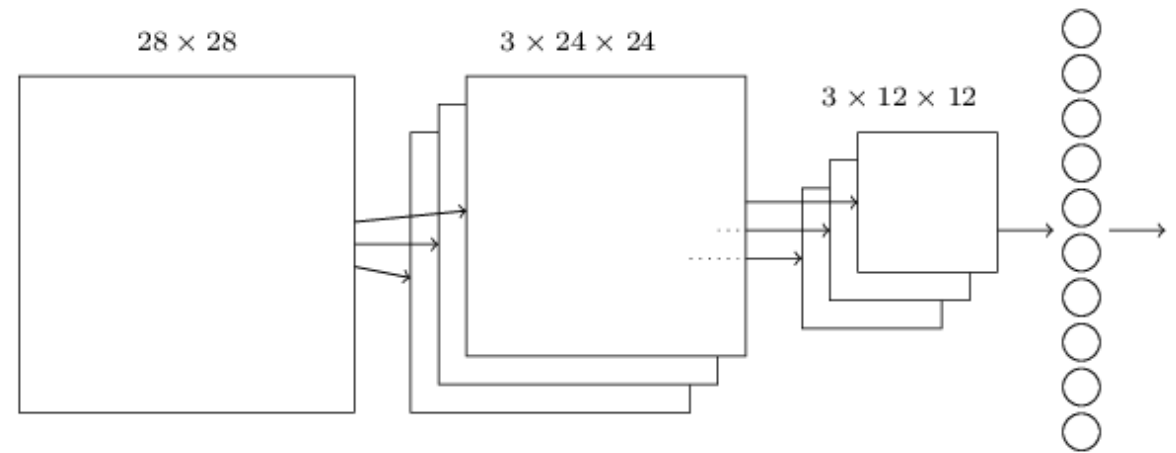
Convolution

- *Convolution* is an operation used often in image processing and computer vision.

- Given a filter/kernel *K* and a sub-image, convolution gives a *filter response*.
    - Basically saying how well the sub-image matches the filter/kernel.

- If we know what we're looking for, we can *design* filters and apply them to an image to get filter responses to be used for machine learning.

- This is what is often done in traditional machine learning.

- But what if we don't know the kernels?

- Is there a way to discover/learn useful ones?

Convolution Neural Networks

- Convolutional Neural Networks (CNNs) attempt to do exactly that!
- They have become popular and successful in images and audio classification where traditionally a kernel/filter was applied to the raw data to extract features.
- This is because kernels take spatial relationships into account.
  - What is important in image and audio classification.

CNNs

- CNNs typically have
  - One or more convolution layer
  - A final fully connected shallow ANN.

- The convolution layer contains several parts
  - Feature Map Extraction
  - Pooling

- Let's look at each of these

Convolution

- Again, convolution is a mathematical operation that takes a filter/function, and applies it to some area to produce a single value at the center of that area.

- It's almost identical to cross-correlation
  - Flipping the filter and then doing cross-correlation is equivalent to doing convolution.
  - The pro of using convolution instead of cross-correlation is that it exhibits the commutative property.

Convolution

- We use the $*$ operator to denote convolution.

- To formalize the convolution operator, given an $M \times M$ filter, K, and an input matrix X, we can compute the convolution at location (a, b) as:

$$F_{ab} = (X_{ab} * K) = \sum_{i=-\frac{M}{2}}^{\frac{M}{2}} \sum_{j=-\frac{M}{2}}^{\frac{M}{2}} X_{(a-i, b-j)} K_{ij}$$

- If we apply convolution to several locations in the image, what we get back is a new matrix, that is the filter response at each location, and which we call the **Feature Map**
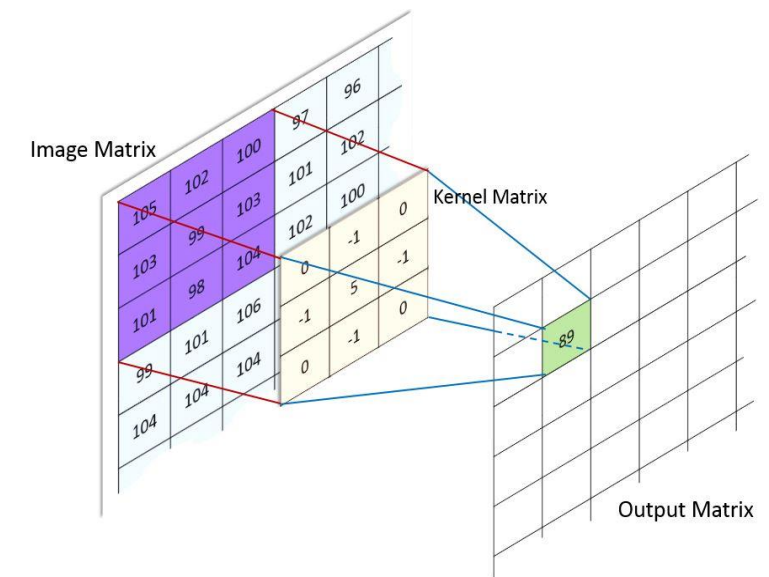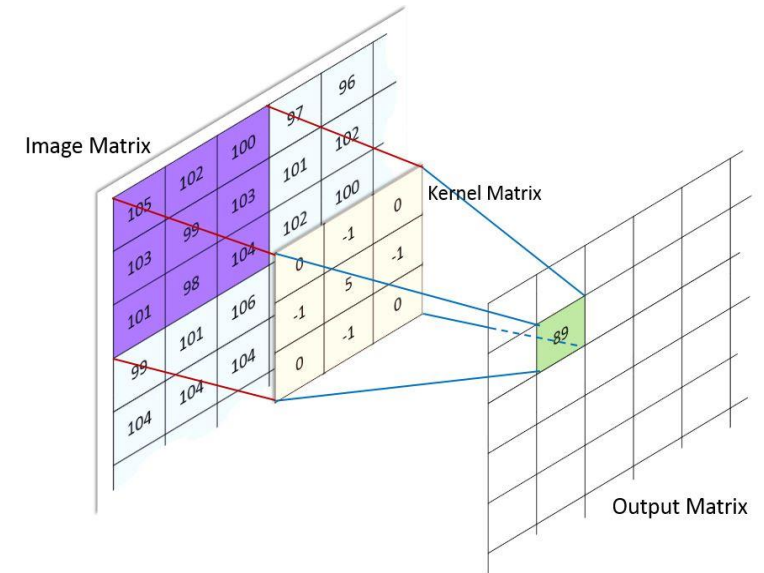
Feature Map Size

- Valid convolution can only be computed where there are $M \times M$ pixels to process.

- This results in a **smaller** image than the original one.

- If our image is $H \times W$, then our feature map will be $(H - M + 1) \times (W - M + 1)$
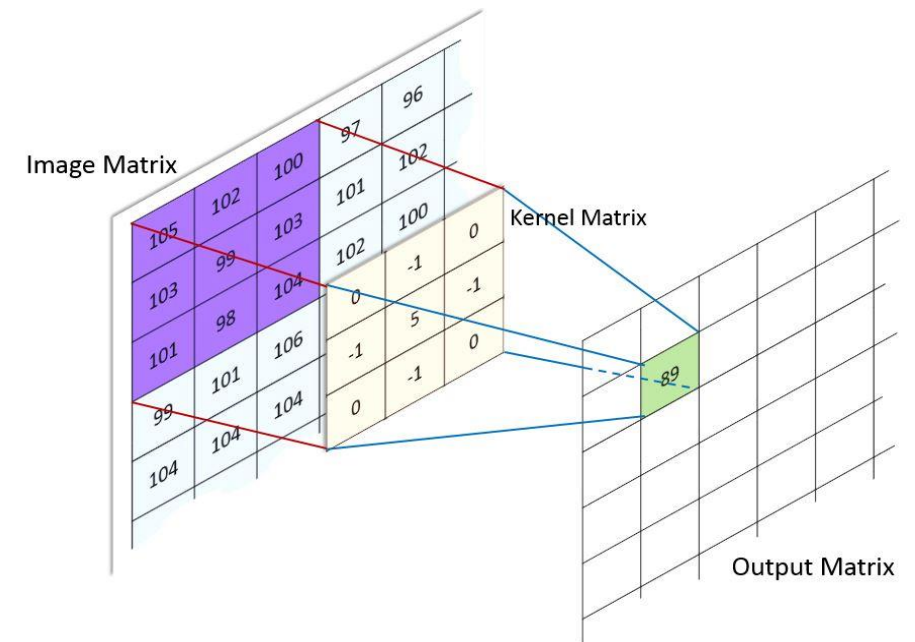
Convolution Example

- Let $X = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}, K = \begin{bmatrix} 0.11 & 0.12 & 0.13 \\ 0.21 & 0.22 & 0.23 \\ 0.31 & 0.32 & 0.33 \end{bmatrix}$

- $X * K = \begin{bmatrix} 49.62 & 51.6 \\ 68.42 & 71.4 \end{bmatrix}$

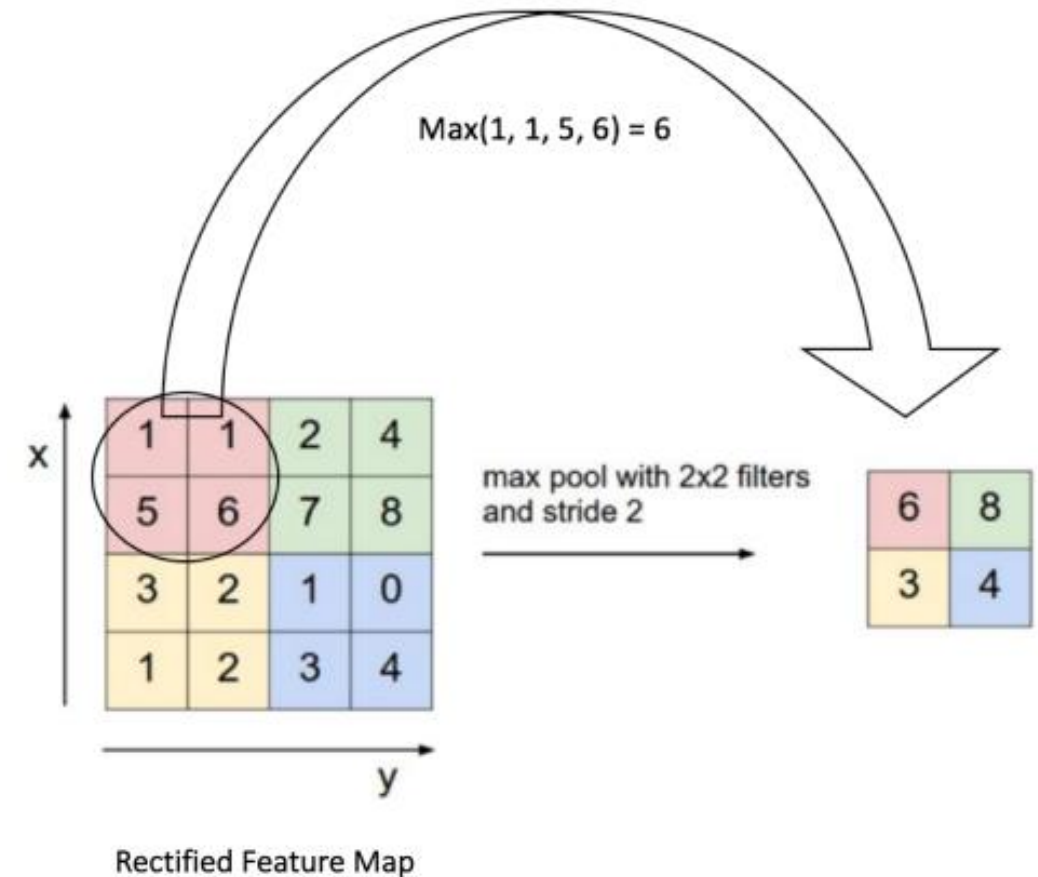Image Matrix

Kernel Matrix

Output Matrix

Pooling

- The next part of a CNN layer is pooling.

- Pooling is essentially down-sampling: we're taking our feature map and making a new, smaller map by "summarizing" the original one.
  - This provides some invariance to translation

- Pooling is also done by moving around a $Q \times Q$ window and extracting a value from each locations

- However, it is common not to move/slide the window by just one pixel, but instead to allow for this to be a user-defined hyperparameter, called the **stride**.

Pooling

- Common pooling techniques include:
  - Max Pooling – Select the maximum value in the square
  - L2 Pooling – Compute the square root of the sum of the values in the square.

- If our feature map is $D \times E$ and we have a stride of S, then output from the pooling process will be a $\left\lfloor \dfrac{D}{S} \right\rfloor \times \left\lfloor \dfrac{E}{S} \right\rfloor$ matrix

$$Max(1, 1, 5, 6) = 6$$

max pool with 2x2 filters and stride 2

Rectified Feature Map

CNNs

- Now that our Convolution Layer is made, the output of the pooling process is flattened and becomes the input of a traditional ML system.
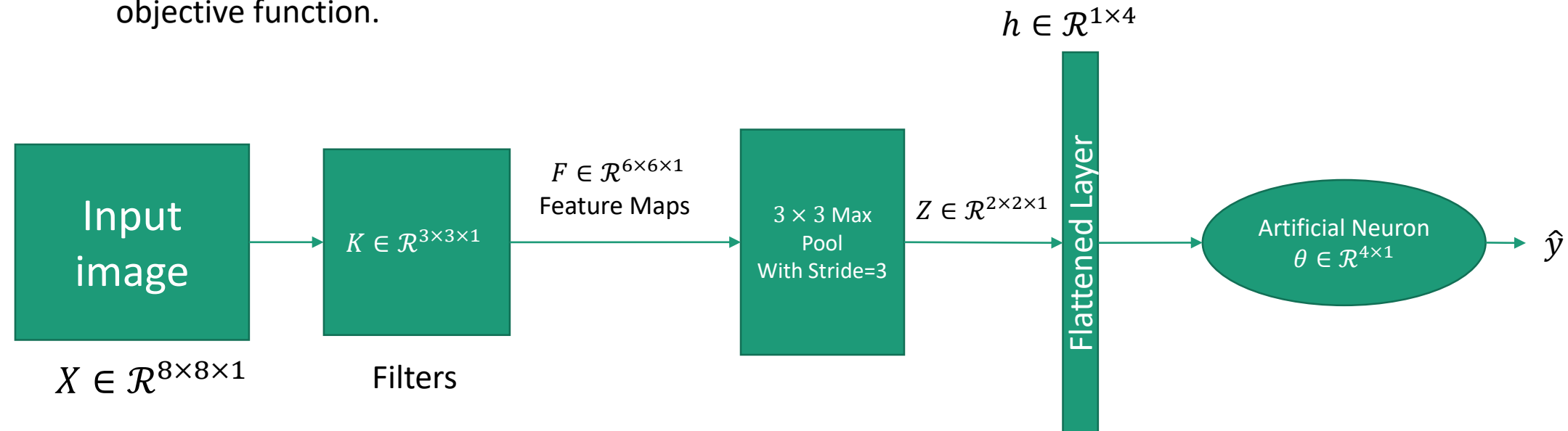 - Potentially adding in bias features as desired.
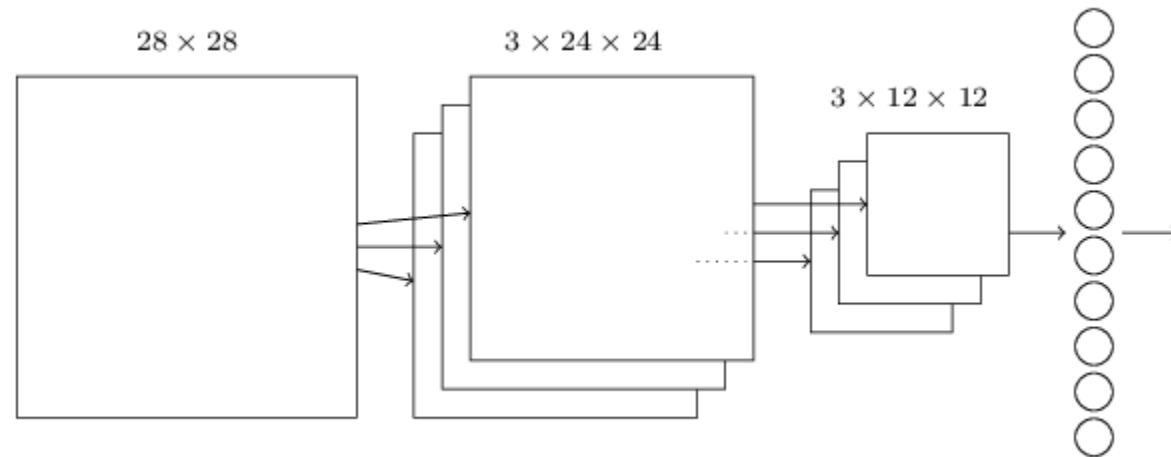
Training

- As with our ANNs, training a CNN requires *forward-backwards propagation.*

- But now our set of parameters to learn are:
  - Any parameters of our "traditional" ML part of our system.
  - The weights of each kernel in the convolution layer(s).

- And once again, given a set of parameters, the forward propagation is relatively simple:
  - Create the feature maps using the current kernels.
  - Create the pooled maps from the feature maps.
  - Feed these into our traditional ML algorithm.

- Of course the tough part is learning the weights.

Example CNN Architecture

- Let's start off with a relatively simple system/example:
  - X is input image of size 8×8
  - K is a single 3×3 kernel
  - Our pooling process using max-pooling on a 3×3 region with stride of 3
  - Our traditional ML layer is an artificial neuron with a logistic activation function and a log likelihood objective function.

$$h \in \mathcal{R}^{1 \times 4}$$

Input image

$$X \in \mathcal{R}^{8 \times 8 \times 1}$$

$$K \in \mathcal{R}^{3 \times 3 \times 1}$$

Filters

$$F \in \mathcal{R}^{6 \times 6 \times 1}$$
Feature Maps

3×3 Max Pool With Stride=3

$$Z \in \mathcal{R}^{2 \times 2 \times 1}$$

Flattened Layer

Artificial Neuron
$$\theta \in \mathcal{R}^{4 \times 1}$$

$$\hat{y}$$

Full Example

- In an attempt to wrap our heads around all of this, let's do a numerical example.
- Let our grayscale image X, whose target class (binary classification) is $y = 0$ be:

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \\ 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 \\ 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 \end{bmatrix}$$
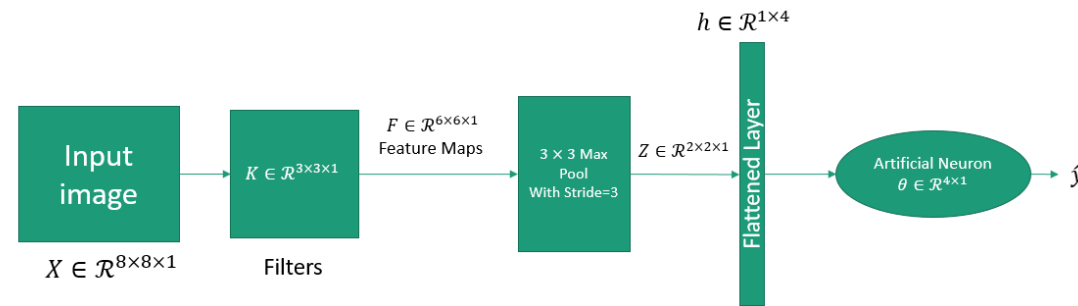
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Full Example

- In addition our kernel will be initialized as $K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

- And the output layer weights are initialized as $\theta = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
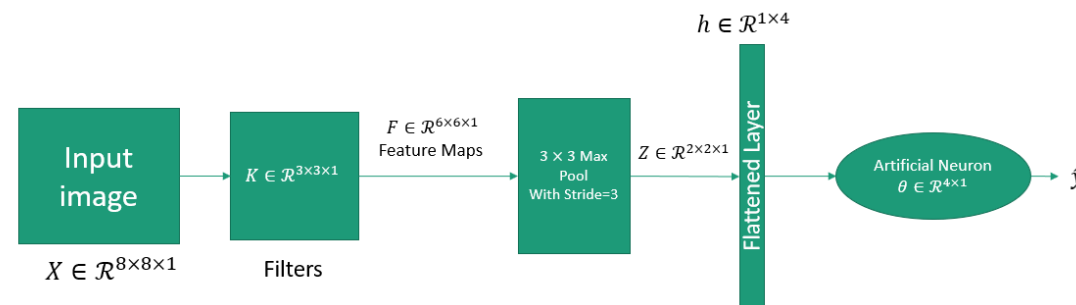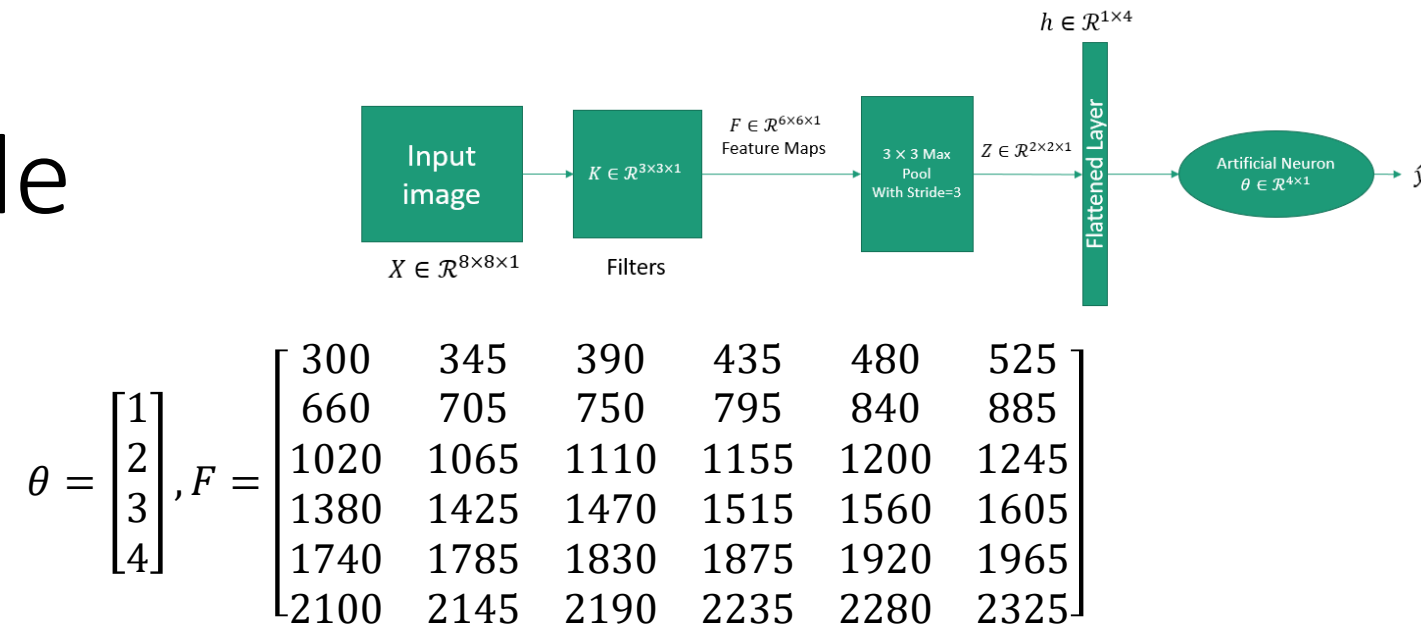
Full Example

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \\ 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 \\ 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 \end{bmatrix}, y = 0, K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \theta = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- First let's forward propagate!

- Here's the convolution

$$F = X * K = \begin{bmatrix} 300 & 345 & 390 & 435 & 480 & 525 \\ 660 & 705 & 750 & 795 & 840 & 885 \\ 1020 & 1065 & 1110 & 1155 & 1200 & 1245 \\ 1380 & 1425 & 1470 & 1515 & 1560 & 1605 \\ 1740 & 1785 & 1830 & 1875 & 1920 & 1965 \\ 2100 & 2145 & 2190 & 2235 & 2280 & 2325 \end{bmatrix}$$
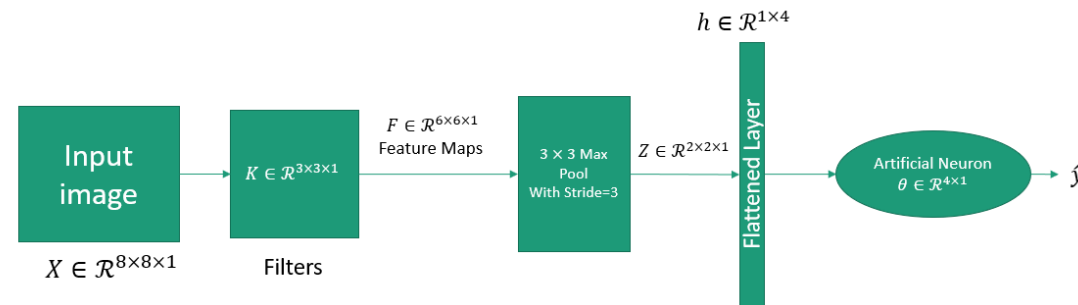
Full Example

$X \in \mathcal{R}^{8\times8\times1}$ — Input image

$K \in \mathcal{R}^{3\times3\times1}$ — Filters

$F \in \mathcal{R}^{6\times6\times1}$ — Feature Maps

3×3 Max Pool With Stride=3

$Z \in \mathcal{R}^{2\times2\times1}$

Flattened Layer

$h \in \mathcal{R}^{1\times4}$

Artificial Neuron $\theta \in \mathcal{R}^{4\times1}$ → $\hat{y}$

$$\theta = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, F = \begin{bmatrix} 300 & 345 & 390 & 435 & 480 & 525 \\ 660 & 705 & 750 & 795 & 840 & 885 \\ 1020 & 1065 & 1110 & 1155 & 1200 & 1245 \\ 1380 & 1425 & 1470 & 1515 & 1560 & 1605 \\ 1740 & 1785 & 1830 & 1875 & 1920 & 1965 \\ 2100 & 2145 & 2190 & 2235 & 2280 & 2325 \end{bmatrix}$$

- And now let's 3x3 max-pool with stride of 3.
- This results in:

$$Z = \begin{bmatrix} 1110 & 1245 \\ 2190 & 2325 \end{bmatrix}$$

Full Example

$h \in \mathcal{R}^{1\times4}$

$F \in \mathcal{R}^{6\times6\times1}$
Feature Maps

$Z \in \mathcal{R}^{2\times2\times1}$

Input image

$K \in \mathcal{R}^{3\times3\times1}$

3×3 Max Pool With Stride=3

Flattened Layer

Artificial Neuron $\theta \in \mathcal{R}^{4\times1}$

$\hat{y}$

$X \in \mathcal{R}^{8\times8\times1}$

Filters

$$\theta = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, F = \begin{bmatrix} 300 & 345 & 390 & 435 & 480 & 525 \\ 660 & 705 & 750 & 795 & 840 & 885 \\ 1020 & 1065 & 1110 & 1155 & 1200 & 1245 \\ 1380 & 1425 & 1470 & 1515 & 1560 & 1605 \\ 1740 & 1785 & 1830 & 1875 & 1920 & 1965 \\ 2100 & 2145 & 2190 & 2235 & 2280 & 2325 \end{bmatrix}, Z = \begin{bmatrix} 1110 & 1245 \\ 2190 & 2325 \end{bmatrix}$$

- Then we'll flatted this to be a single row vector. Since Matlab uses column major indexing, this becomes:

$$h = \begin{bmatrix} 1110 & 2190 & 1245 & 2325 \end{bmatrix}$$

- To which we apply θ

$$net = h \cdot \theta = 18525$$

- Then apply activation function $g(z)$ (for which we'll use the logistic activation function):

$$\hat{y} = \frac{1}{1 + e^{-18525}} \approx 1$$

Gradient of Artificial Neuron Layer

- Ok, now it's time to update our parameters based on the forward propagation.

- As a reminder, at the end of our CNN we have an artificial neuron with a logistic activation function and log likelihood objective function for binary classification.

- Given this configuration, from our previous work, the gradient with respect to our last set of parameters is:

$$\frac{\partial J}{\partial \theta_s} = \frac{\partial J}{\partial g(h|\theta)} \cdot \frac{\partial g(h|\theta)}{\partial \theta_s} = \left(\frac{y - \hat{y}}{\hat{y}(1 - \hat{y})} \right) \cdot \left(h_s \hat{y}(1 - \hat{y}) \right) = (y - \hat{y})h_s$$

$$\frac{\partial J}{\partial \theta} = h^T(y - \hat{y})$$

- For our example this is:

$$\frac{\partial J}{\partial \theta} = \begin{bmatrix} -1110 \\ -2190 \\ -1245 \\ -2325 \end{bmatrix}$$
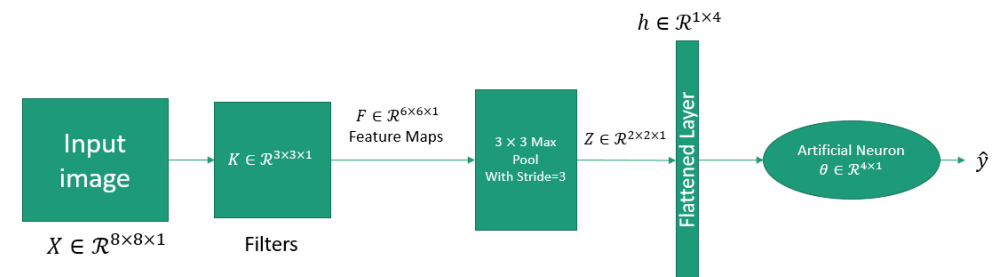
Gradient of Kernel
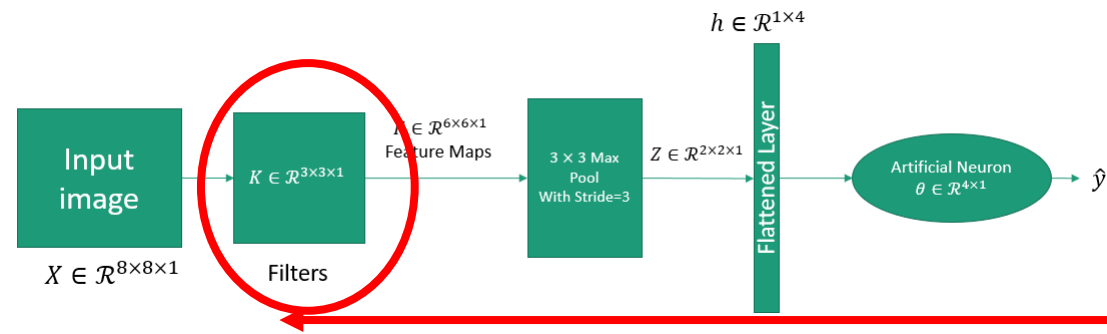
- The next set of parameters we need to update are those of the convolution kernel, K
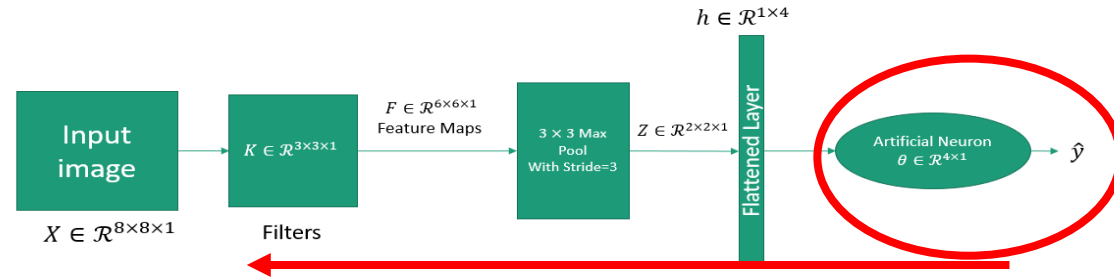
- So the question is, what is:

$$\frac{\partial J}{\partial K_{ij}}?$$

- Well let's try to chain rule this guy!

$$\frac{\partial J}{\partial K_{ij}} = \frac{\partial J}{\partial g(h|\theta)} \cdot \frac{\partial g(h|\theta)}{\partial h} \cdot \frac{\partial h}{\partial Z} \cdot \frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}$$

- Ugh ☹!

Gradient of Kernel

$$\frac{\partial J}{\partial K_{ij}} = \frac{\partial J}{\partial g(h|\theta)} \cdot \frac{\partial g(h|\theta)}{\partial h} \cdot \frac{\partial h}{\partial Z} \cdot \frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}$$

- We already have $\frac{\partial J}{\partial g(h|\theta)}$ from our previous layer:

$$\frac{\partial J}{\partial g(h|\theta)} = \left(\frac{y - \hat{y}}{\hat{y}(1 - \hat{y})} \right)$$

- What is $\frac{\partial g(h|\theta)}{\partial h}$?

$$\frac{\partial g(h|\theta)}{\partial h} = \theta^T \hat{y}(1 - \hat{y})$$

- So:

$$\frac{\partial J}{\partial K_{ij}} = (y - \hat{y})\theta^T \cdot \frac{\partial h}{\partial Z} \cdot \frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}$$

- For the example:

$$\frac{\partial J}{\partial K_{ij}} = (-1)\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T \cdot \frac{\partial h}{\partial Z} \cdot \frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}$$

Gradient of Kernel
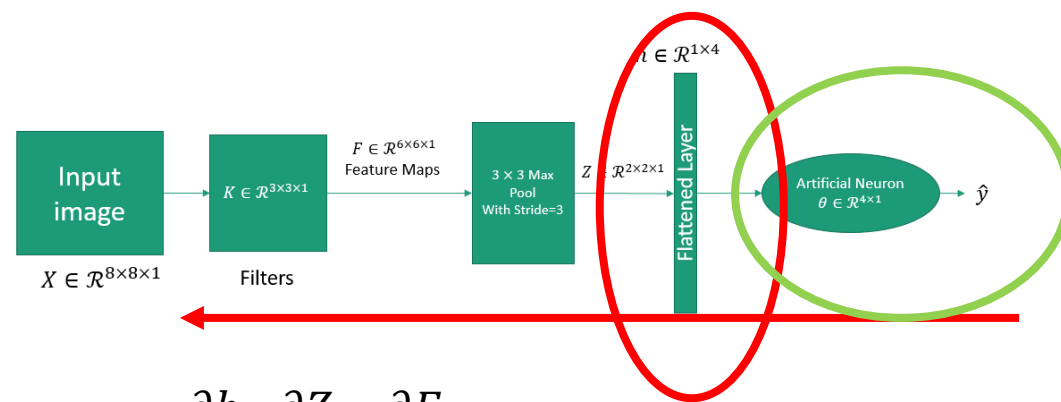
$$\frac{\partial J}{\partial K_{ij}} = (y - \hat{y})\theta^T \cdot \frac{\partial h}{\partial Z} \cdot \frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}$$

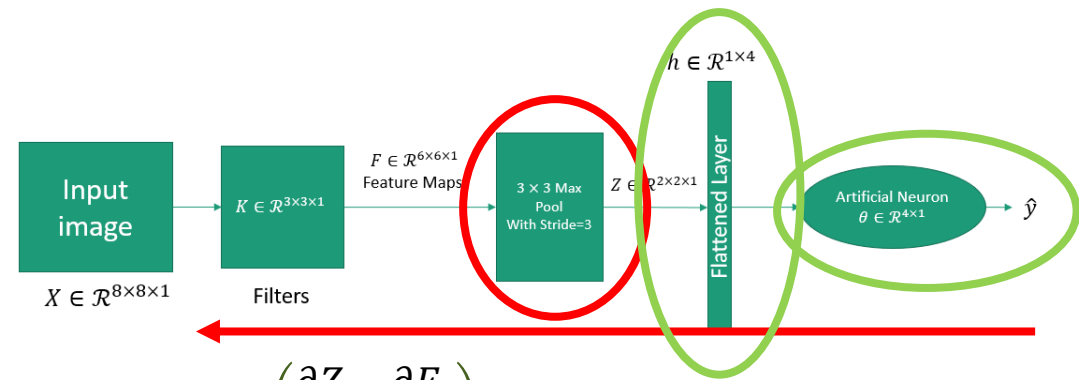- h is just a flattened version of Z, so $\frac{\partial h}{\partial Z}$ is just a reshaping function

$$\frac{\partial J}{\partial K_{ij}} = (y - \hat{y})\theta^T \cdot reshape \left(\frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}} \right)$$

- For our example:

$$\frac{\partial J}{\partial K_{ij}} = (-1) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T \cdot reshape \left(\frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}} \right)$$

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Gradient of Kernel

$$\frac{\partial J}{\partial K_{ij}} = (y - \hat{y})\theta^T \cdot reshape\left(\frac{\partial Z}{\partial F} \cdot \frac{\partial F}{\partial K_{ij}}\right)$$

- Since Z is created by max-pooling, we can think of it as selecting elements from $\frac{\partial F}{\partial K_{ij}}$.

- Let's call this gradient *select*

$$\frac{\partial J}{\partial K_{ij}} = (y - \hat{y}) \cdot \theta^T \cdot reshape\left(select\left(\frac{\partial F}{\partial K_{ij}}\right)\right)$$

- Where were the max locations selected from?

- That leaves us with $\frac{\partial F}{\partial K_{ij}}$

$$F = \begin{bmatrix} 300 & 345 & 390 & 435 & 480 & 525 \\ 660 & 705 & 750 & 795 & 840 & 885 \\ 1020 & 1065 & 1110 & 1155 & 1200 & 1245 \\ 1380 & 1425 & 1470 & 1515 & 1560 & 1605 \\ 1740 & 1785 & 1830 & 1875 & 1920 & 1965 \\ 2100 & 2145 & 2190 & 2235 & 2280 & 2325 \end{bmatrix}$$
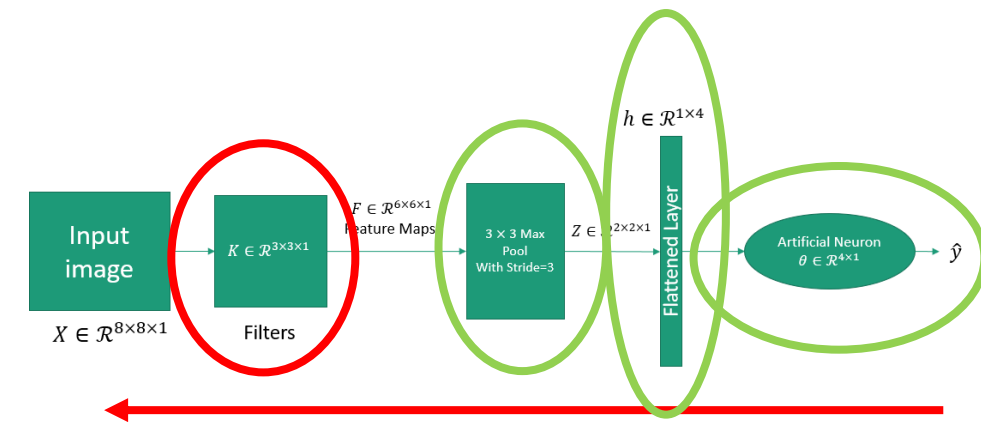
Gradient of Convolution Function

- Recall the convolution function

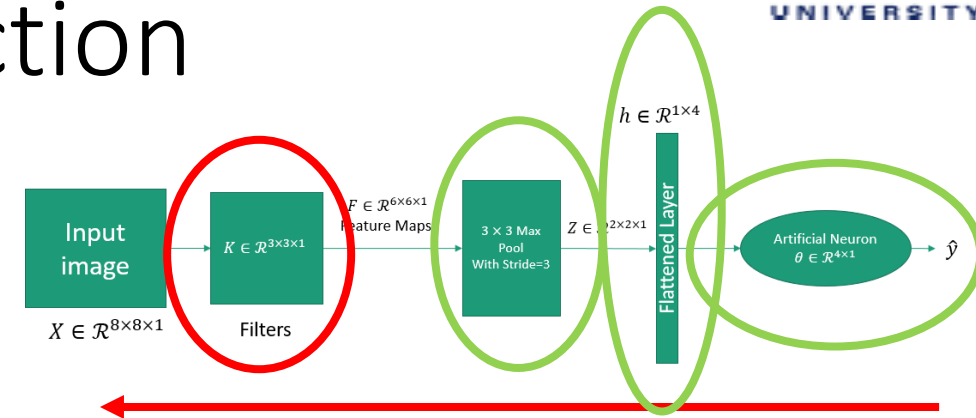$$F_{ab} = \sum_{i=-\frac{M}{2}}^{\frac{M}{2}} \sum_{j=-\frac{M}{2}}^{\frac{M}{2}} X_{a-i,b-j} K_{ij}$$

- So let's start with, what is $\frac{\partial F_{ab}}{\partial K_{ij}}$?

$$\frac{\partial F_{ab}}{\partial K_{ij}} = X_{a-i,b-j}$$

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Gradient of Convolution Function

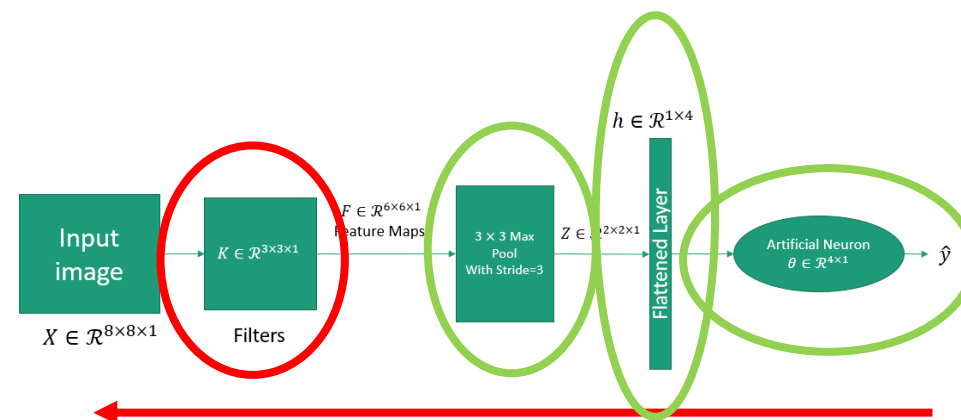$$\frac{\partial F_{ab}}{\partial K_{ij}} = X_{a-i,b-j}$$

- So then, what is $\frac{\partial F}{\partial K_{ij}}$?

- We need to think about what elements of F will K_{ij} affect?

- This is actually (somewhat) simple!

$$\frac{\partial F}{\partial K_{ij}} = X_{H-i+1:-1:M-i+1,\ W-j+1:-1:M-j+1}$$

- Note: $H - i + 1: -1: M - i + 1$ means from $H - i + 1$ ***down to*** $M - i + 1$
  - This achieves the flipping done by convolution.

Full Example

$$\frac{\partial J}{\partial K_{ij}} = (-1) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T \cdot reshape\left(select\left(X_{H-i+1:-1:M-i+1,\ W-j+1:-1:M-j+1}\right)\right)$$

- Let's find $\frac{\partial J}{\partial K_{11}}$ for our example:

$$\frac{\partial J}{\partial K_{11}} = (-1) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T \cdot reshape\left(select\left(\begin{bmatrix} 64 & 63 & 62 & 61 & 60 & 59 \\ 56 & 55 & 54 & 53 & 52 & 51 \\ 48 & 47 & 46 & 45 & 44 & 43 \\ 40 & 39 & 38 & 37 & 36 & 35 \\ 32 & 31 & 30 & 29 & 28 & 27 \\ 24 & 23 & 22 & 21 & 20 & 19 \end{bmatrix}\right)\right) = [-1]\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T reshape\left(\begin{bmatrix} 46 & 43 \\ 22 & 19 \end{bmatrix}\right)$$

$$\frac{\partial J}{\partial K_{11}} = (-1) \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T [46\ 22\ 43\ 19] = -295$$

$$F = \begin{bmatrix} 300 & 345 & 390 & 435 & 480 & 525 \\ 660 & 705 & 750 & 795 & 840 & 885 \\ 1020 & 1065 & 1110 & 1155 & 1200 & 1245 \\ 1380 & 1425 & 1470 & 1515 & 1560 & 1605 \\ 1740 & 1785 & 1830 & 1875 & 1920 & 1965 \\ 2100 & 2145 & 2190 & 2235 & 2280 & 2325 \end{bmatrix}$$

Full Example

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \\ 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 \\ 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 \end{bmatrix}, K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- Doing this for all elements in K

$$\frac{\partial J}{\partial K} = \begin{bmatrix} -295 & -285 & -275 \\ -215 & -205 & -195 \\ -135 & -125 & -115 \end{bmatrix}$$
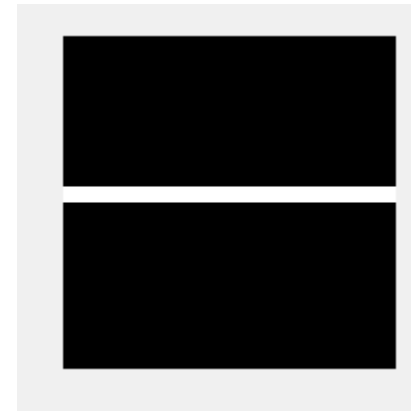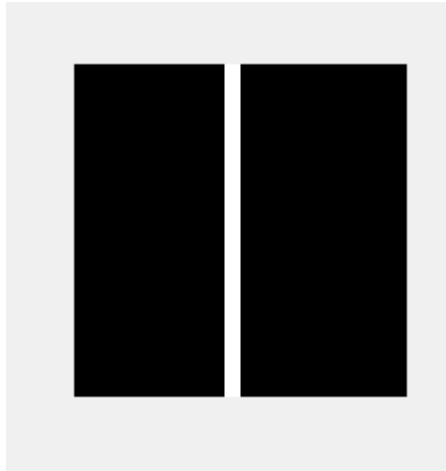
- And since our objective function is log likelihood, we want to maximize it and thus:

$$K = K + \eta \frac{\partial J}{\partial K}$$
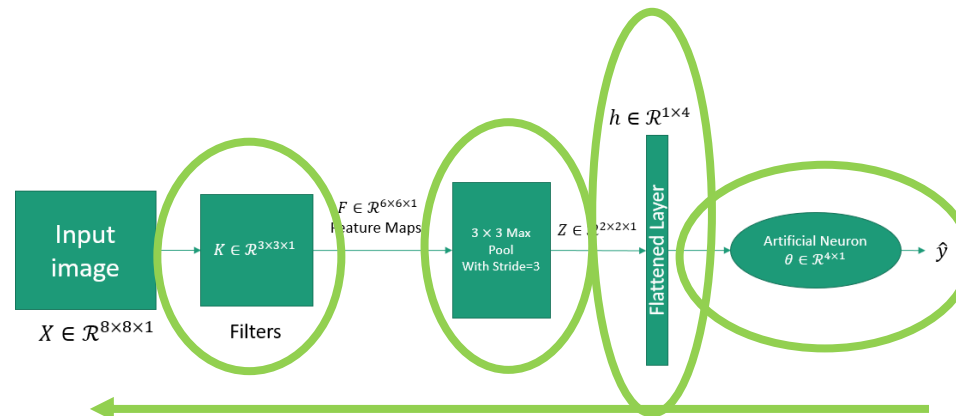
Another Example

- Below are two artificially created 20×20 images.
- The left one we'll call "class 1" and the right one "class 0"

Another Example

- Our CNN will be configured to have a convolutional layer with a single 20×20 kernel filter and a 1×1 max-pool layer with a stride of 1.
    - Note that the kernel size is the same as the image size!
    - Although this is rarely done, I did this for illustrative purposes.
- Therefore, our filter map will be 1×1 and the output of max-pooling will be 1×1.
- We will then flatten the output of the max-pooling layer to form a 1×1 vector to be fed into an artificial neuron with a logistic activation function and log likelihood objective function.
    - Therefore $\theta \in \mathcal{R}^{1 \times 1}$

Another Example

- After 1,000 iterations, with $\eta = 0.1$ and L^2 regularization of 0.1, I thought it would be interesting to see what filter it learned!

- Thoughts?

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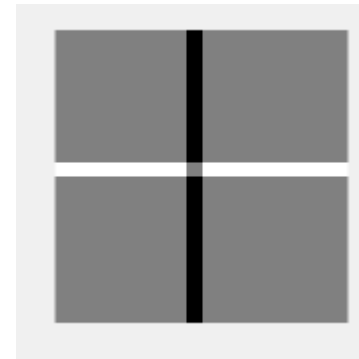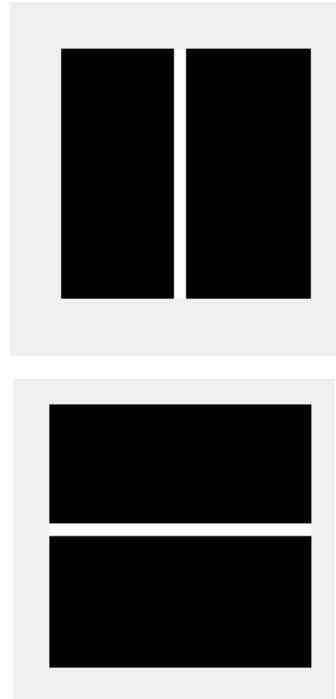
Convolving with Multiple Kernels

- Most CNNs apply more than one kernel to an image.

- This shouldn't change much other than the fact that the output of convolution stage will be a 3D matrix (fyi, a multi-dimensional matrix is called a *tensor)*.

- If our input to convolution is a $W \times H$ image and we have P $M \times M$ kernels, we will then obtain multiple feature maps as a tensor.

- The output value for feature map p, at location (a, b), we now have:

$$F_{(a,b,p)} = \sum_{i=-\frac{M}{2}}^{\frac{M}{2}} \sum_{j=-\frac{M}{2}}^{\frac{M}{2}} X_{(a-i,b-j)} K_{(i,j,p)}$$

- And of course the output of the pooling stage will also be a tensor.

- So we need to keep this in mind when we flatten the output to be the input to our traditional network.

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Convolving with Color Images

- What if we have an RGB color image?

- Then our image is $W \times H \times 3$

- And therefore our filters/kernels will be $M \times M \times 3$

- And our convolution function will be:

$$F_{a,b} = \sum_{l=1}^{3} \sum_{i=-\frac{M}{2}}^{\frac{M}{2}} \sum_{j=-\frac{M}{2}}^{\frac{M}{2}} X_{(a-i,b-j,l)} K_{(i,j,l)}$$

- This will also be important if/when we stack several convolution layers together.
  - Since the output of one layer *could* be a tensor.

Bringing it all together

- Used in series with fully connected layers
- Convolutions find the important patterns in the input
- Fully connected layers abstract the existence of the patterns into a label
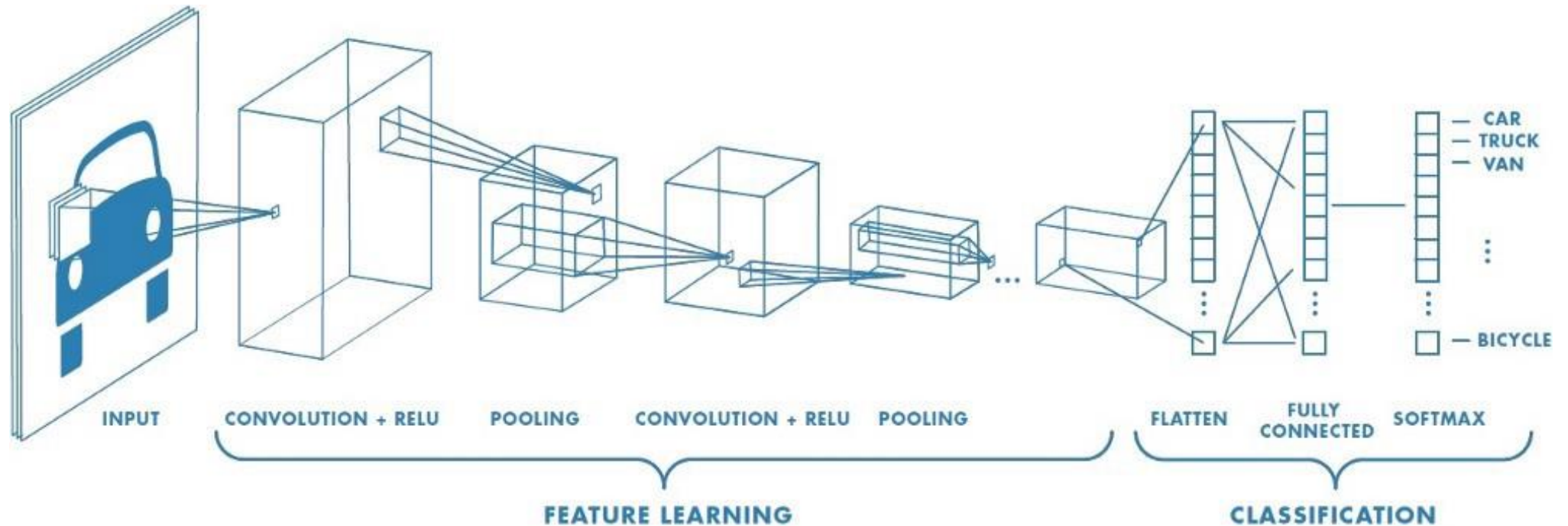
Vectorizing/Batching

- Due to CNNs being not fully connected, sharing weights, and the pooling layer, vectorizing some of the partial gradients is a bit tricky, so we'll leave this to figure reading.

- But here's one resource:
  - http://lxu.me/mypapers/vcnn_aaai15.pdf

- Same thing goes for batch processing.

- Of course we can just compute the gradients for each sample then take the average of those gradients over a batch.

Sources

- Articles
  - http://www.robots.ox.ac.uk/~vgg/practicals/cnn/
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- Videos
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