Mathematical Tripos, Part II
Computational project

Boundary Layer Flow

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Contents

1 Analysis

1.1 Question 1

From the material we have:

$$m(f')^2 - \frac{1}{2}(m+1)ff'' = m + f''' \tag{1.1}$$

With boundary conditions:

$$f' = f = 0 \ on \ \eta = 0, \quad f' \to sgnA \ as \ \eta \to \infty. \tag{1.2}$$

where $f = f(\eta),\ \eta = y/\delta(x),\ A > 0.$
When $\eta \to \infty$, we need to check if $f' \sim 1$ in each case.

(i) **algebraic convergence :**

$$f' \sim 1 + B\eta^{-k} \to 1, \quad as\ \eta \to \infty$$

(ii) **exponential convergence :**

$$f' \sim 1 - (\xi + \eta_0)\sigma'(\xi)e^{-\sigma(\xi)} \to 1, \quad as\ \xi \to \infty$$

where

$$\sigma'(\xi) = k\xi + k' + k''\xi^{-1} + O(\xi^{-2})$$
$$\sigma(\xi) = (k/2)\xi^2 + k'\xi + k''\log\xi + O(\xi^{-1})$$

So

$$f' \sim 1 - \xi^{k''}(k\xi^2 + (k' + k\eta_0)\xi + k'' + k'\eta_0)e^{-(k/2)\xi^2 + k'\xi}$$

(iii) **algebraic divergence :**

$$f' \sim B(1+k)\eta^k \to \infty \quad as\ \eta \to \infty$$

So this case is not possible.

(iv) **exponential divergence :**

$$f' \sim Bke^{k\eta} \to \infty \quad as\ \eta \to \infty$$

, so again it is not possible.

(v) **a finite-distant singularity :**

$$f' \sim B(\eta - \eta_0)^{-2} \to 0 \quad as\ \eta \to \infty$$

, and thus can be incorporated in the asymptotic behaviour alongside (i) or (ii), but cannot be an asymptotic behaviour of f alone.

(a) (i) By means of an asymptotic series solution:

$$f = \sum_{n=-1}^{\infty} a_n \eta^{-n}$$

and substitute into 1.1. By solving for coefficients we can determine B and k without use of conditions at $\eta = 0$.

(ii) f has an asymptotic expansion:

$$f = \sum_{n=0}^{\infty} a_n \xi^{-n} e^{-(k/2)\xi^2 + k'\xi}$$

and substitute into 1.1. By solving for coefficients we can determine k, k' and k" without use of conditions at $\eta = 0$. However, we need them for η_0.

(b) For a simplification of 1.1, we scale the boundary layer thickness

$$\delta = \left(\frac{2\nu x}{(1+m)\left|U_e(x)\right|} \right)^{\frac{1}{2}}$$

and it becomes

$$f''' + ff'' + \beta(1 - f'^2) = 0, \quad \beta = 2m/(m+1) \tag{1.3}$$

with 1.2. Consider f_0 a 'standard solution' of the equation 1.3 which satisfies 1.2. Then the perturbed solution is of the form

$$f = f_0 + \epsilon f_1 + O(\epsilon^2).$$

Substitute the perturbed solution into 1.3, we have

$$f_0''' + \beta(1 - f_0'^2) + f_0 f_0'' + \epsilon(f_1''' + f_1 f_0'' + f_0 f_1'' - 2\beta\epsilon f_0' f_1') + O(\epsilon^2) = 0$$

and thus by solving the coefficient of $O(\epsilon)$, we have

$$f_1''' + f_1 f_0'' + f_0 f_1'' - 2\beta f_0' f_1' = 0$$

assume f_0'' decreases faster than f_1 increases, and as a result:

$$\frac{D^2 f_1'}{D\zeta^2} + f_0 \frac{D f_1'}{D\zeta} - 2\beta f_1' = 0 \tag{1.4}$$

The equation has been studied by Hartree (1937) [1], and he obtained the asymptotic behaviour:

$$f_1 \sim A e^{-\frac{\zeta^2}{2}} \zeta^{-(2\beta+1)} + B\zeta^{2\beta}$$

Then it is clear that the asymptotic behaviour of f is 1 plus above, as $f_0 \to 1$ when $\zeta \to \infty$. Simultaneously, we have verified that f demonstrates algebraic or exponential convergence as indicated previously. Accordingly, when $\beta > 0$, we must have $B = 0$ to avoid blow-up at infinity. On the other hand, if $\beta < 0$, *i.e.* $-1 < m < 0$, it is possible to have both A and B non-zero, and f_1 tends to 0 for any choice of A and B. When A = 0, the convergence is algebraic. In conclusion, for case (i) to be possible, we must have $-1 < m < 0$ and there is no restriction on (ii).

2 Computational

The programming task is 3.1

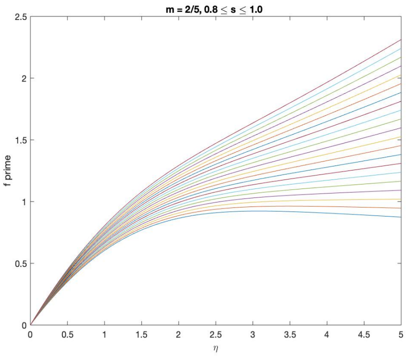
2.1 Question 2

Substitute m = 0 and S = 1 into 3.1, from the data obtained we can see that f' converges to 2.0856... after integrating the system over η. The convergence indicates that as *eta* increases to infinity, i.e. goes close to the top of the boundary layer, the horizontal velocity component tends to a constant which is greater than that of the mainstream flow. There is a sudden change in velocity gradient at the interface, which corresponds to a tremendous tangential stress. Therefore, this solution is not physical.
1a shows the integrated solution with η from 0 to 10, with an interval of 0.25. To deduce a solution f of 1.1 satisfying the asymptotic behaviour at $\eta \to \infty$, we use the substitution $g = af(b\eta)$ suggested in the material and set ab = 2.0856... corrected to 4 significant figures. We then have

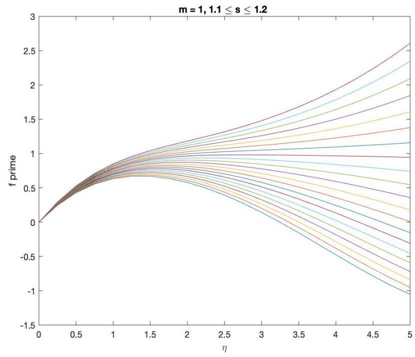
$$\frac{1}{2}ff'' + \frac{b}{a}f''' = 0$$

So let b = a = $\sqrt{2.0856}$ we have $f' \to 1$ when $\eta \to \infty$. Then the solution is $g/\sqrt{2.0856}$, and $f''(0) = (2.0856)^{-\frac{3}{2}}$.

2.2 Question 3



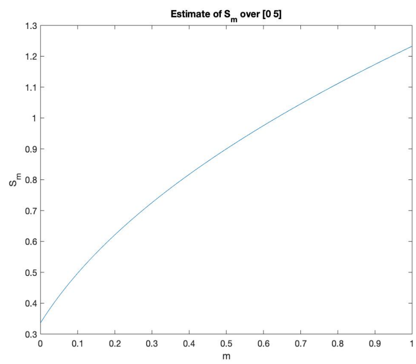
(a) Figure 3.1



(b) Figure 3.2

Shooting method for m =2/5 and 1

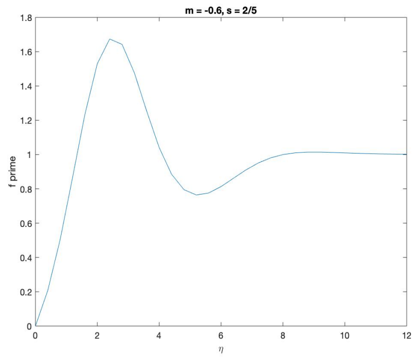
The programming task is at 3.2. From the figures we can see that for in the case where $m = 2/5$, the solution for greater S diverges algebraically, while when $m = 1$ the divergence is exponential. From 3.2 we can find that $S_{2/5} = 0.8173$, and $s_1 = 1.1512$. Over $\eta = [010]$ we have:



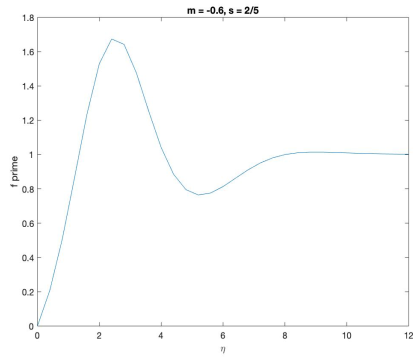
Plot of estimates of S_m against m

Here I choose η up to 5 as this is a value demonstrating a regular asymptotic behaviour of the solution, after integrating further over η. It is noted that when η gets too large the solution diverges considerably. That is because when η gets very large, we come out of the boundary layer and the previous equation do not hold. All values below are accurate to 4 significant figures, by setting the tolerance as 1e-5 in the script. Physically, as m increases, the mainstream flow velocity increases faster as $x \to \infty$. As a result, throughout the boundary layer, the flow increases more rapidly towards the mainstream and thus f”(0) should be larger for a larger m, which also indicates a greater viscous drag due to a steeper velocity gradient at the rigid boundary contact.

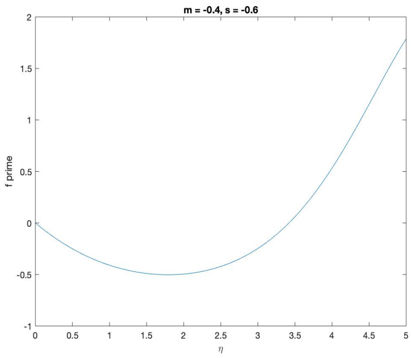
2.3 Question 4



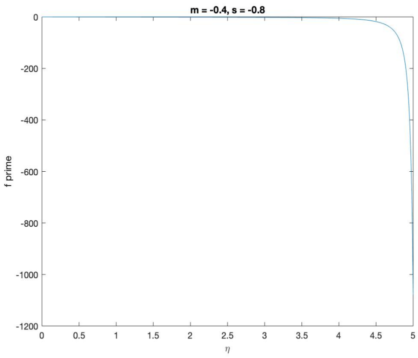
(a) Figure 4.1



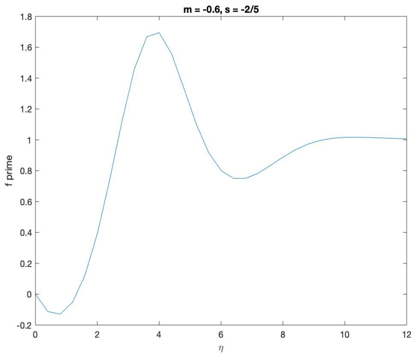
(b) Figure 4.2



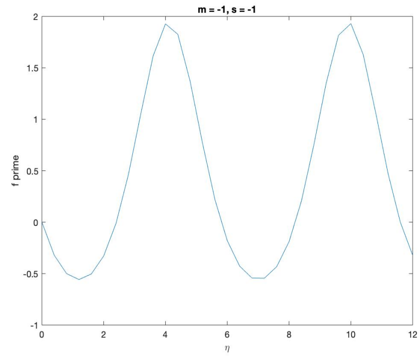
(c) Figure 4.3



(d) Figure 4.4



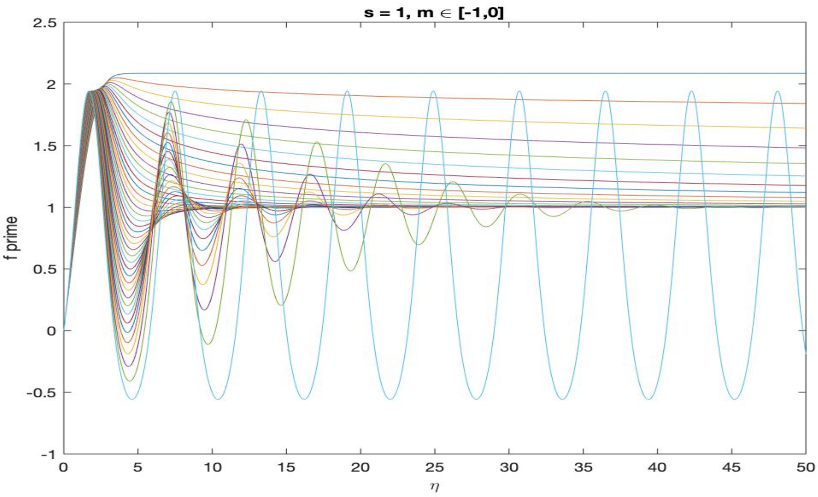
(e) Figure 4.5



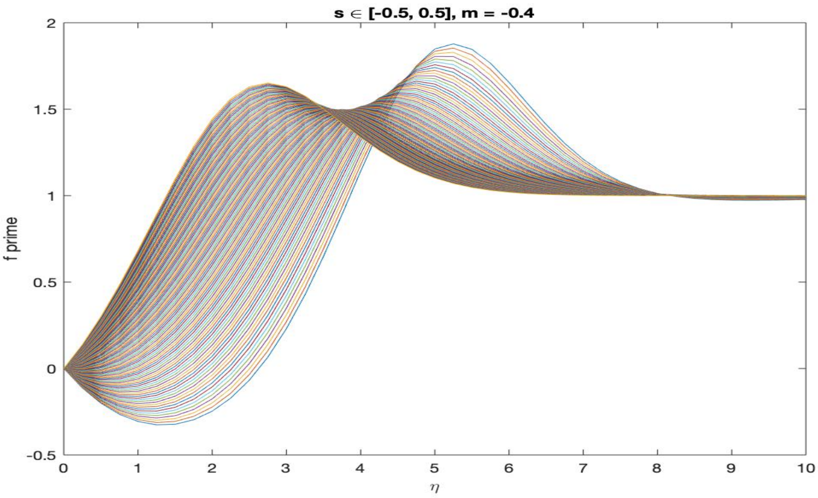
(f) Figure 4.6

In 3f, the curve diverges algebraically as it wobbles periodically without a decrease in amplitude. In 3a, 3b and 3e, the curves converge quickly over

η, and they converge exponentially as η becomes large. In 3c and 3d, they diverge exponentially. The value
Fix s = 1, for a range of values of m $\in [-1, 0]$, the solutions are:



If we take a full spectrum of m we have:



from which we can see one of the branches is a continuation of $m > 0$, the other is of a different kind.

3 Codes

3.1 Programming Task 1

```
function y = Prog_Task(S,tspan,m)
y0 = [0 0 S];
options = odeset('RelTol', 1e-8,'AbsTol',1e-8);
[t,y] = ode45(@(t,y) FalknerSkan(t,y,m), tspan, y0,options);
function dy = FalknerSkan(t,y,m)
dy = zeros(3,1);
dy(1) = y(2);
dy(2) = y(3);
dy(3) = m*y(2)^2 - (1/2)*(1+m)*y(1)*y(3) - m;
end
end
```

3.2 Programming Task 2

```
function f = Asympt(S,tspan,m)
y0 = [0 0 S];
options = odeset('RelTol', 1e-5,'AbsTol',1e-5);
[t,y] = ode45(@(t,y) FalknerSkan(t,y,m), tspan, y0,options);
l = length(tspan);
function dy = FalknerSkan(t,y,m)
dy = zeros(3,1);
dy(1) = y(2);
dy(2) = y(3);
dy(3) = m*y(2)^2 - (1/2)*(1+m)*y(1)*y(3) - m;
end
f = y(l,2)-1;
end

function x0 = S_finder(tspan,m,S0)
% Finds S_m for value m in [0,1]
options = odeset('RelTol', 1e-5,'AbsTol',1e-5);
fun = @(x) Asympt(x,tspan,m);
x0 = fzero(fun,S0,options);
end

function x0 = m_finder(tspan,m0,s)
% Finds m
options = odeset('RelTol', 1e-5,'AbsTol',1e-5);
fun = @(m) Asympt(s,tspan,m);
x0 = fzero(fun,m0,options);
end
```

4 Tables

f	f'	f"
0.0000	0.0000	1.0000
0.0312	0.2499	0.9987
0.1249	0.4987	0.9896
0.2802	0.7435	0.9655
0.4959	0.9797	0.9203
0.7689	1.2016	0.8509
1.0949	1.4033	0.7574
1.4682	1.5790	0.6457
1.8820	1.7252	0.5243
2.3284	1.8410	0.4029
2.8000	1.9278	0.2921
3.2901	1.9889	0.2000
3.7929	2.0293	0.1290
4.3037	2.0547	0.0778
4.8194	2.0696	0.0440
5.3379	2.0778	0.0233
5.8580	2.0820	0.0116
6.3788	2.0841	0.0054
6.8999	2.0850	0.0024
7.4212	2.0854	0.0010
7.9426	2.0855	0.0004
8.4640	2.0856	0.0001
8.9854	2.0856	0.0000
9.5068	2.0856	0.0000
10.0282	2.0856	0.0000
10.5496	2.0856	0.0000
11.0710	2.0856	0.0000
11.5924	2.0856	0.0000
12.1138	2.0856	-0.0000
12.6352	2.0856	-0.0000
13.1566	2.0856	-0.0000
13.6780	2.0856	-0.0000
14.1994	2.0856	0.0000
14.7208	2.0856	0.0000
15.2422	2.0856	0.0000
15.7636	2.0856	0.0000
16.2850	2.0856	-0.0000
16.8064	2.0856	0.0000
17.3279	2.0856	-0.0000
17.8493	2.0856	-0.0000
18.3707	2.0856	0.0000

(a) Table 1

m	s_m
0.0000	0.3362
0.0250	0.3816
0.0500	0.4231
0.0750	0.4614
0.1000	0.4972
0.1250	0.5308
0.1500	0.5625
0.1750	0.5927
0.2000	0.6215
0.2250	0.6490
0.2500	0.6755
0.2750	0.7011
0.3000	0.7258
0.3250	0.7497
0.3500	0.7728
0.3750	0.7954
0.4000	0.8173
0.4250	0.8386
0.4500	0.8595
0.4750	0.8798
0.5000	0.8997
0.5250	0.9192
0.5500	0.9383
0.5750	0.9570
0.6000	0.9753
0.6250	0.9933
0.6500	1.0110
0.6750	1.0284
0.7000	1.0455
0.7250	1.0624
0.7500	1.0789
0.7750	1.0953
0.8000	1.1114
0.8250	1.1272
0.8500	1.1429
0.8750	1.1583
0.9000	1.1735
0.9250	1.1886
0.9500	1.2034
0.9750	1.2181
1.0000	1.2326

(b) Table 2

References

[1] D. R. Hartree. On an equation occurring in falkner and skan's approximate treatment of the equations of the boundary layer. *Mathematical Proceedings of the Cambridge Philosophical Society*, 33(2):223–239, 1937.