$$Fib(n) = \begin{cases} 0 & \text{if } n=0 \\ 1 & \text{if } n=1 \\ Fib(n-1) + Fib(n-2) & \text{otherwise} \end{cases}$$

Exercise 1.13 pg 1/3

Induction

To prove $\forall n(A(n))$, establish:

1. $A(1)$: base
2. $(\forall n)[A(n) \Longrightarrow A(n+1)]$: Induction

Prove $Fib(n) \approx \varphi^n/\sqrt{5}$ where $\varphi = (1+\sqrt{5})/2$.

Let $\psi = (1-\sqrt{5})/2$. $\qquad \sqrt{5} = 2\varphi - 1$

$= (-\varphi + 1) \qquad = -2\psi + 1$

$$Fib(n+1) = Fib(n) + Fib(n-1)$$

Proving Base cases:

$Fib(0) = (\varphi^0 - \psi^0)/\sqrt{5}$ ✓

$Fib(1) = (\varphi^1 - \psi^1)/\sqrt{5}$ ✓

$$Fib(2) = \frac{(\varphi^1 - \psi^1)}{\sqrt{5}} + \frac{(\varphi^0 - \psi^0)}{\sqrt{5}} \checkmark$$

$$= \frac{(\varphi^2 - \psi^2)}{\sqrt{5}} = \frac{\frac{(1+2\sqrt{5}+5) - (1-2\sqrt{5}+5)}{2^2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 \checkmark$$

$$= \frac{(\varphi^1 - \psi^1)}{\sqrt{5}} = \frac{\frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 \checkmark$$

$$Fib(3) = \frac{(\varphi^3 - \psi^3)}{\sqrt{5}} = \frac{\frac{(1+2\sqrt{5}+5)(1+\sqrt{5}) - (1-2\sqrt{5}+5)(1-\sqrt{5})}{2^3}}{\sqrt{5}} =$$

$$= \frac{\frac{(1+2\sqrt{5}+5+\sqrt{5}+10+5\sqrt{5}) - (1-2\sqrt{5}+5-\sqrt{5}+10-5\sqrt{5})}{2^3}}{\sqrt{5}}$$

$$\frac{\frac{16+8\sqrt{5} - (16-8\sqrt{5})}{8}}{\sqrt{5}} = \frac{\frac{16\sqrt{5}}{8}}{\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{5}} = 2 \checkmark$$

~~Show~~ $\cancel{(\Phi^{n+1} - \Psi^{n+1})/\sqrt{5} = (\Phi^{n} - \Psi^{n})/\sqrt{5} + (\Phi^{n-1} - \Psi^{n-1})/\sqrt{5} = Fib(n+1)}$

$$\cancel{\frac{\Phi\Phi^n - \Psi\Psi^n}{\sqrt{5}} = \frac{\frac{\Phi}{\sqrt{5}} \cdot \frac{\Phi^n}{\sqrt{5}}}{\frac{\Phi}{\sqrt{5}}} - \frac{\left(\frac{\Psi}{\sqrt{5}} \cdot \frac{\Psi^n}{\sqrt{5}}\right)}{\frac{\Phi}{\sqrt{5}}} \quad \Phi^n =}$$

Exercise 1.13 pg. 2/3

$$\Phi^2 = \frac{6+2\sqrt{5}}{4} \qquad \frac{1+\sqrt{5}}{2} \cdot \frac{2}{2} = \frac{2+2\sqrt{5}}{4} + \frac{4}{4} = \frac{6+2\sqrt{5}}{4}$$

$$= (\Phi + 1)$$

$$\Psi^2 = \frac{6-2\sqrt{5}}{4} \qquad \frac{1-\sqrt{5}}{2} \cdot \frac{2}{2} = \frac{2-2\sqrt{5}}{4} + \frac{4}{4} = \frac{6-2\sqrt{5}}{4}$$

$$= (\Psi + 1)$$

Base Case Already Proven!

$Fib(0), (1), (2) = \dfrac{\Phi^0 - \Psi^0}{\sqrt{5}}, \dfrac{\Phi^1 - \Psi^1}{\sqrt{5}} \ldots$

Proving Induction Step using definition of Fib

$$Fib(n+1) = \frac{(\Phi^n - \Psi^n)}{\sqrt{5}} + \frac{(\Phi^{n-1} - \Psi^{n-1})}{\sqrt{5}} =$$

$$\frac{\left[\frac{(1+\sqrt{5})^n}{2^n} - \frac{(1-\sqrt{5})^n}{2^n}\right]}{\sqrt{5}} + \frac{\left[\frac{(1+\sqrt{5})^{n-1}}{2^{n-1}} - \frac{(1-\sqrt{5})^{n-1}}{2^{n-1}}\right] \cdot \frac{2}{2}}{\sqrt{5}} =$$

$$+ \frac{\left[\frac{2((1+\sqrt{5})^{n-1}) - 2((1-\sqrt{5})^{n-1})}{2^n}\right]}{\sqrt{5}} =$$

$$\frac{\left[\frac{(1+\sqrt{5})(1+\sqrt{5})^{n-1} + 2((1+\sqrt{5})^{n-1})}{2^n}\right]}{\sqrt{5}} + \frac{\left[\frac{-(1-\sqrt{5})^n - 2(1-\sqrt{5})^{n-1}}{2^n}\right]}{\sqrt{5}} =$$

$$\frac{\left[\frac{(1+\sqrt{5})^{n-1}((1+\sqrt{5})+2)}{2^n}\right]}{\sqrt{5}} + \frac{\left[\frac{(1-\sqrt{5})^{n-1}(-(1-\sqrt{5})-2)}{2^n}\right]}{\sqrt{5}} =$$

$$\frac{\left[\frac{(1+\sqrt{5})^{n-1}}{2^{n-1}} \cdot \left(\frac{(1+\sqrt{5})}{2} + \frac{2}{2}\right)\right]}{\sqrt{5}} + \frac{\left[\frac{-((1-\sqrt{5})^{n-1})}{2^{n-1}} \cdot \left(\frac{(1-\sqrt{5})}{2} + \frac{2}{2}\right)\right]}{\sqrt{5}} =$$

Next Page...

$$\frac{\varphi^{n-1}(\varphi+1)}{\sqrt{5}} + \frac{(-\psi^{n-1})(\psi+1)}{\sqrt{5}} = \frac{\varphi^{n-1}(\varphi^2)}{\sqrt{5}} - \frac{\psi^{n-1}(\psi^2)}{\sqrt{5}} =$$

$$\frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}} \quad \checkmark$$

By the principle of Mathematical Induction,
$Fib(n) = (\varphi^n - \psi^n)/\sqrt{5}$.

The difference between $Fib(n)$ and $\varphi^n/\sqrt{5}$ is:

$$\left| \frac{\varphi^n}{\sqrt{5}} - \frac{(\varphi^n - \psi^n)}{\sqrt{5}} \right| = \left| \frac{\psi^n}{\sqrt{5}} \right|$$

$\left|\frac{\psi^n}{\sqrt{5}}\right|$ is less than 0.5 for all numbers greater than or equal to 0, hence $Fib(n)$ is the closest integer to $\varphi^n/\sqrt{5}$.