Supplementary Material

Heterogenous firing responses leads to diverse coupling to presynaptic activity in a simplified morphological model of layer V pyramidal neurons

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Contents

1 Derivation of the mean membrane potential solution

Equation for $\mu_v(X)$:

$$\begin{cases} \dfrac{\partial^2 \mu_v}{\partial X^2} = \mu_v(X) - v_0^p \quad \forall X \in [0, L_p] \\ \dfrac{\partial^2 \mu_v}{\partial X^2} = \mu_v(X) - v_0^d \quad \forall X \in [L_p, L] \\ \dfrac{\partial \mu_v}{\partial X}|_{X=0} = \gamma^p \Big(\mu_v(0) - V_0 \Big) \\ \mu_v(X \to L_p^-) = \mu_v(X \to L_p^+) \\ \dfrac{\partial \mu_v}{\partial X}_{X \to L_p^-} = \dfrac{\lambda^p}{\lambda^d} \dfrac{\partial \mu_v}{\partial X}_{X \to L_p^+} \\ \dfrac{\partial \mu_v}{\partial X}_{X=L} = 0 \end{cases} \tag{1}$$

We write the solution on the form:

$$\begin{cases} \mu_v(X) = v_0^p + A\, \cosh(X) + C\, \sinh(X) \quad \forall\, X \in [0, L_p] \\ \mu_v(X) = v_0^d + B\, \cosh(X-L) + D\, \sinh(X-L) \quad \forall\, X \in [L_p L] \end{cases} \tag{2}$$

- Sealed-end boundary condition at cable end implies $D = 0$
- Somatic boudary condition imply: $C = \gamma^p\, (v_0^p - V_0 + A)$
- Then v continuity imply : $v_0^p + A\, \cosh(L_p) + \gamma^p\, (v_0^p - V_0 + A)\, \sinh(L_p) = v_0^d + B\, \cosh(L_p - L)$
- Then current conservation imply: $A\, \sinh(L_p) + \gamma^p\, (v_0^p - V_0 + A)\, \cosh(L_p) = \frac{\lambda^p}{\lambda^d}\, B\, \sinh(L_p - L)$

We rewrite those condition on a matrix form:

$$\begin{pmatrix} \cosh(L_p) + \gamma^p \sinh(L_p) & -\cosh(L_p - L) \\ \sinh(L_p) + \gamma^p \cosh(L_p) & -\frac{\lambda^p}{\lambda^d}\, \sinh(L_p - L) \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} v_0^d - v_0^p - \gamma^p\, (v_0^p - V_0)\, \sinh(L_p) \\ -\gamma^p\, (v_0^p - V_0)\, \cosh(L_p) \end{pmatrix} \tag{3}$$

And we solved this equation with the `solve_linear_system_LU` method of `sympy`

The coefficients A and B are given by:

$$A = \frac{\alpha}{\beta} \qquad B = \frac{\gamma}{\delta} \tag{4}$$

where:

$$\begin{aligned}
\alpha &= V_0\gamma^P\lambda^D\cosh(L_p)\cosh(L-L_p) + V_0\gamma^P\lambda^P\sinh(L_p)\sinh(L-L_p) \\
&\quad - \gamma^P\lambda^D v_0^d\cosh(L_p)\cosh(L-L_p) - \gamma^P\lambda^P v_0^d\sinh(L_p)\sinh(L-L_p) \\
&\quad - \lambda^P v_0^d\sinh(L-L_p) + \lambda^P v_0^p\sinh(L-L_p) \\
\beta &= \gamma^P\lambda^D\cosh(L_p)\cosh(L-L_p) + \gamma^P\lambda^P\sinh(L_p)\sinh(L-L_p) + \\
&\quad \lambda^D\sinh(L_p)\cosh(L-L_p) + \lambda^P\sinh(L-L_p)\cosh(L_p) \\
\gamma &= \lambda^D\big(V_0\gamma^P + \gamma^P v_0^d\cosh(L_p) - \gamma^P v_0^d \\
&\quad - \gamma^P v_0^p\cosh(L_p) + v_0^d\sinh(L_p) - v_0^p\sinh(L_p)\big) \\
\delta &= \gamma^P\lambda^D\cosh(L_p)\cosh(L-L_p) + \gamma^P\lambda^P\sinh(L_p)\sinh(L-L_p) \\
&\quad + \lambda^D\sinh(L_p)\cosh(L-L_p) + \lambda^P\sinh(L-L_p)\cosh(L_p)
\end{aligned} \tag{5}$$

2 Derivation of the post-synaptic membrane potential event

For the PSP events we need to solve:

$$\begin{cases}
\dfrac{\partial^2\hat{\delta v}}{\partial X^2} = \big(\alpha_f^p + (\alpha_f^d - \alpha_f^p)\mathcal{H}(X-L_p)\big)^2\,\hat{\delta v} \\
\dfrac{\partial\hat{\delta v}}{\partial X}_{|X=0} = \gamma_f^p\,\hat{\delta v}(0,f) \\
\hat{\delta v}(X_{src}^-, f) = \hat{\delta v}(X_{src}^+, f) \\
\dfrac{\partial\hat{\delta v}}{\partial X}_{X_{src}^-} = \dfrac{\partial\hat{\delta v}}{\partial X}_{X_{src}^+} - \big(\mu_v(X_{src}) - E_{rev}\big)\,\big(r_f^p + (r_f^d - r_f^p)\mathcal{H}(X_{src}-L_p)\big)\,\hat{g(f)} \\
\hat{\delta v}(L_p^-, f) = \hat{\delta v}(L_p^+, f) \\
\dfrac{\partial\hat{\delta v}}{\partial X}_{L_p^-} = \dfrac{\lambda^p}{\lambda^d}\,\dfrac{\partial\hat{\delta v}}{\partial X}_{L_p^+} \\
\dfrac{\partial\hat{\delta v}}{\partial X}_{X=L} = 0
\end{cases} \tag{6}$$

To obtain the solution, we need to split the solution into two cases:

1. $X_{src} \leq L_p$

   Let's write the solution to this equation as the form (already including the boundary conditions at $X = 0$ and $X = L$):

$$\hat{\delta v}(X, X_{src}, f) = \begin{cases} A_f(X_{src}) \left(\cosh(\alpha_f^p\, X) + \gamma^p \sinh(\alpha_f^p\, X) \right) \\ \qquad\qquad \text{if :} 0 \leq X \leq X_{src} \leq L_p \leq L \\ B_f(X_{src}) \cosh(\alpha_f^p\, (X - L_p)) + C_f(X_{src}) \sinh(\alpha_f^p\, (X - L_p)) \\ \qquad\qquad \text{if :} 0 \leq X_{src} \leq X \leq L_p \leq L \\ D_f(X_{src}) \cosh(\alpha_f^d\, (X - L)) \\ \qquad\qquad \text{if :} 0 \leq X_{src} \leq L_p \leq X \leq L \end{cases} \tag{7}$$

   We write the 4 conditions correspondingto the conditions in X_{src} and L_p to get A_f, B_f, C_f, D_f. On a matrix form, this gives:

$$M = \begin{pmatrix} \cosh(\alpha_f^p\, X_{src}) + \gamma_f^p \sinh(\alpha_f^p\, X_{src}) & -\cosh(\alpha_f^p\, (X_{src} - L_p)) & -\sinh(\alpha_f^p\, (X_{src} - L_p)) & 0 \\ \alpha_f^p \big(\sinh(\alpha_f^p\, X_{src}) + \gamma_f^p \cosh(\alpha_f^p\, X_{src}) \big) & -\alpha_f^p \sinh(\alpha_f^p\, (X_{src} - L_p)) & -\alpha_f^p \cosh(\alpha_f^p\, (X_{src} - L_p)) & 0 \\ 0 & 1 & 0 & -\cosh(\alpha_f^d\, (L_p - L)) \\ 0 & 0 & \alpha_f^p & -\alpha_f^d \frac{\lambda^p}{\lambda^d} \sinh(\alpha_f^d\, (L_p - L)) \end{pmatrix} \tag{8}$$

$$M \cdot \begin{pmatrix} A_f \\ B_f \\ C_f \\ D_f \end{pmatrix} = \begin{pmatrix} 0 \\ -r_f^p I_f \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

   And we will solve it with the `solve_linear_system_LU` method of `sympy`. For the $A_f(X_{src})$ coefficient, we obtain:

$$A_f(X_{src}) = \frac{a_f^1(X_{src})}{a_f^2(X_{src})} \tag{10}$$

   with:

$$
\begin{aligned}
a_f^1(X_{src}) = I_f r_f^P \big(&-\alpha_f^D \lambda^P \cosh\left(L\alpha_f^D - L_p\alpha_f^D - L_p\alpha_f^P + X_s\alpha_f^P\right)\\
&+ \alpha_f^D \lambda^P \cosh\left(L\alpha_f^D - L_p\alpha_f^D + L_p\alpha_f^P - X_s\alpha_f^P\right)\\
&+ \alpha_f^P \lambda^D \cosh\left(L\alpha_f^D - L_p\alpha_f^D - L_p\alpha_f^P + X_s\alpha_f^P\right)\\
&+ \alpha_f^P \lambda^D \cosh\left(L\alpha_f^D - L_p\alpha_f^D + L_p\alpha_f^P - X_s\alpha_f^P\right)\\
a_f^2(X_{src}) = \alpha_f^P \big(&-\alpha_f^D \gamma_f^P \lambda^P \cosh\left(-L\alpha_f^D + L_p\alpha_f^D + L_p\alpha_f^P\right)\\
&+ \alpha_f^D \gamma_f^P \lambda^P \cosh\left(L\alpha_f^D - L_p\alpha_f^D + L_p\alpha_f^P\right) -\\
&\alpha_f^D \lambda^P \sinh\left(-L\alpha_f^D + L_p\alpha_f^D + L_p\alpha_f^P\right)\\
&+ \alpha_f^D \lambda^P \sinh\left(L\alpha_f^D - L_p\alpha_f^D + L_p\alpha_f^P\right)\\
&+ \alpha_f^P \gamma_f^P \lambda^D \cosh\left(-L\alpha_f^D + L_p\alpha_f^D + L_p\alpha_f^P\right)\\
&+ \alpha_f^P \gamma_f^P \lambda^D \cosh\left(L\alpha_f^D - L_p\alpha_f^D + L_p\alpha_f^P\right)\\
&+ \alpha_f^P \lambda^D \sinh\left(-L\alpha_f^D + L_p\alpha_f^D + L_p\alpha_f^P\right)\\
&+ \alpha_f^P \lambda^D \sinh\left(L\alpha_f^D - L_p\alpha_f^D + L_p\alpha_f^P\right)
\end{aligned}
\tag{11}
$$

2. \$ $\mathrm{L_p} \leq \mathrm{X_{src}}$\$

   Let's write the solution to this equation as the form (already including the boundary conditions at $X = 0$ and $X = L$:

$$
\begin{aligned}
&\hat{\delta v}(X, X_{src}, f) =\\
&\begin{cases}
E_f(X_{src})\left(\cosh(\alpha_f^p\, X) + \gamma^p\ \sinh(\alpha_f^p\, X)\right)\\
\qquad\qquad \text{if :} 0 \leq X \leq L_p \leq X_{src} \leq L\\
F_f(X_{src})\ \cosh(\alpha_f^d\,(X - L_p)) + G_f(X_{src})\ \sinh(\alpha_f^d\,(X - L_p))\\
\qquad\qquad \text{if :} 0 \leq L_p \leq X \leq X_{src} \leq L\\
H_f(X_{src})\ \cosh(\alpha_f^d\,(X - L))\\
\qquad\qquad \text{if :} 0 \leq L_p \leq X_{src} \leq X \leq L
\end{cases}
\end{aligned}
\tag{12}
$$

   We write the 4 conditions correspondingto the conditions in X_{src} and L_p to get A_f, B_f, C_f, D_f. On a matrix form, this gives:

   We rewrite this condition on a matrix form:

$$M_2 = \begin{pmatrix} \cosh(\alpha_f^p\, L_p) + \gamma_f^p \sinh(\alpha_f^p\, L_p) & -1 & 0 & 0 & 0 \\ \alpha_f^p \big(\sinh(\alpha_f^p\, L_p) + \gamma_f^p \cosh(\alpha_f^p\, L_p) \big) & 0 & -\alpha_f^d \frac{\lambda^p}{\lambda^d} & 0 & \\ 0 & \cosh(\alpha_f^d\,(X_{src} - L_p)) & \sinh(\alpha_f^d\,(X_{src} - L_p)) & -\cosh(\alpha_f^d\,(X_{src} - L)) & \\ 0 & \alpha_f^d\, \sinh(\alpha_f^d\,(X_{src} - L_p)) & \alpha_f^d\, \cosh(\alpha_f^d\,(X_{src} - L_p)) & -\alpha_f^d\, \sinh(\alpha_f^d\,(X_{src} - L)) & \end{pmatrix} \tag{13}$$

$$M \cdot \begin{pmatrix} E_f \\ F_f \\ G_f \\ H_f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -r_f^d I_f \end{pmatrix} \tag{14}$$

And we will solve it with the `solve_linear_system_LU` method of `sympy`. For the $E_f(X_{src})$ coefficient, we obtain:

$$E_f(X_{src}) = \frac{e_f^1(X_{src})}{e_f^2(X_{src})} \tag{15}$$

with:

$$\begin{aligned} e_f^1(X_{src}) &= 2 I_f \lambda^P r_f^D \cosh\left(\alpha_f^D\,(L - X_s)\right) \\ e_f^2(X_{src}) &= -\alpha_f^D \gamma_f^P \lambda^P \cosh\left(-L\alpha_f^D + L_p\alpha_f^D + L_p\alpha_f^P\right) \\ &\quad + \alpha_f^D \gamma_f^P \lambda^P \cosh\left(L\alpha_f^D - L_p\alpha_f^D + L_p\alpha_f^P\right) \\ &\quad - \alpha_f^D \lambda^P \sinh\left(-L\alpha_f^D + L_p\alpha_f^D + L_p\alpha_f^P\right) \\ &\quad + \alpha_f^D \lambda^P \sinh\left(L\alpha_f^D - L_p\alpha_f^D + L_p\alpha_f^P\right) \\ &\quad + \alpha_f^P \gamma_f^P \lambda^D \cosh\left(-L\alpha_f^D + L_p\alpha_f^D + L_p\alpha_f^P\right) \\ &\quad + \alpha_f^P \gamma_f^P \lambda^D \cosh\left(L\alpha_f^D - L_p\alpha_f^D + L_p\alpha_f^P\right) \\ &\quad + \alpha_f^P \lambda^D \sinh\left(-L\alpha_f^D + L_p\alpha_f^D + L_p\alpha_f^P\right) \\ &\quad + \alpha_f^P \lambda^D \sinh\left(L\alpha_f^D - L_p\alpha_f^D + L_p\alpha_f^P\right) \end{aligned} \tag{16}$$

3. PSP at the soma

   The main text writes a solution for the PSP at soma of the form:

$$\hat{\delta v}(X=0, X_{src}, f) = K_f(X_{src}) \left(\mu_v(X_{src}) - E_{rev}\right) \hat{g(f)} \tag{17}$$

   The correspondance with the previous calculus is to take a unitary current $I_f = 1$ and $K_f(X_{src})$ given by:

$$K_f(X_{src}) = \begin{cases} A_f(X_{src}) \forall X_{src} \in [0, L_p] \\ E_f(X_{src}) \forall X_{src} \in [L_p, L] \end{cases} \tag{18}$$