CS389: Foundations of Data Science Homework I

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EXERCISE 2.9

I

$$\#face(k) = 2^{d-k}\binom{d}{k}$$

II

$$\sum_{k=0}^{d} \#face(k) = 3^d$$

III

$$\#surface\ area = \#face(d-1) = 2d$$

IV

$$\#surface\ area = 2^{d-1} \cdot 2d = 2^d \cdot d$$

V

Let θ be the width of surface.

$$V(surface)/V = 1 - (1-\theta)^d \geq 1 - e^{-\theta d}$$

When d is large enough, no matter how small θ is, this ratio shall be close to 1.

EXERCISE 2.10

I

$$\textit{incremental unit of area} = 4\pi\sin(2\theta)$$

II

$$I(\theta) = \int_0^{\frac{\pi}{2}} 4\pi\sin(2\theta)d\theta = 4\pi\sin^2(x)$$

III

$$I(36°) = \frac{5-\sqrt{5}}{2}$$

EXERCISE 2.11

Since $V(d) = \frac{2}{d}\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$, to find its maximum, we consider the ratio:

$$\frac{V(d)}{V(d-1)} = \pi^{\frac{1}{2}}\left(1-\frac{1}{d}\right)\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}$$

$$\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} = \begin{cases} \pi^{\frac{1}{2}}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{d-2}\right) & even \\ 2\pi^{-\frac{1}{2}}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{d-2}\right) & odd \end{cases}$$

The formulas above shows that no matter whether d is even or odd, the radio decreases as d grows, which implies we need to find the largest d that the corresponding ratio is no less than 1(from both cases: d is even or odd).

After necessary computation, we derive that $d = 4, 5$ are the largest d's that has a ratio $\frac{V(d)}{V(d-1)} \geq 1$ for even d and odd d respectively.

However, $V(5) > V(4)$, thus 5-dimensional unit ball has largest volume, about 5.26.

EXERCISE 2.12

I

$$V_2(d) = \frac{2}{d}\frac{(4\pi)^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

As d grows, the numerator's magnitude is far less than the dominator's magnitude, as a result, its volume would converge to 0.

II

$$V_c(d) = \frac{2}{d}\frac{\left(c^2\pi\right)^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

It will converge to 0 too, for the same reason as stated above.

III

$$V_f(d) = \frac{2}{d}\frac{\left(f(d)^2\pi\right)^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

By using *Stirling's Approximation*, we derive that

$$f(d) = \sqrt{\frac{d}{2\pi e}} \cdot \sqrt[d]{\pi d}$$

is a feasible solution for this task.

Exercise 2.13

This is because the definition of "volume" changes as d differs. A d-dimension ball with volume V, however, must have a width 1 (like a cylinder) in $d+1$-dimension space to maintain its original volume V.

Exercise 2.14

1.
$$\frac{A(4)}{2}$$

2.
$$V(3)$$