CS389: Foundations of Data Science Homework IV

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EXERCISE 5.9

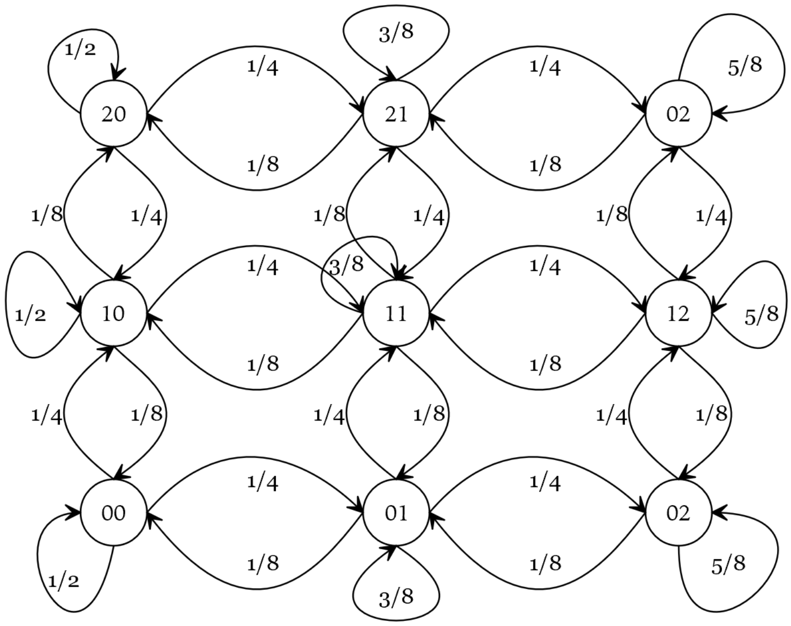

Figure 0.1: Metropolis-Hasting.

EXERCISE 5.13

(A)

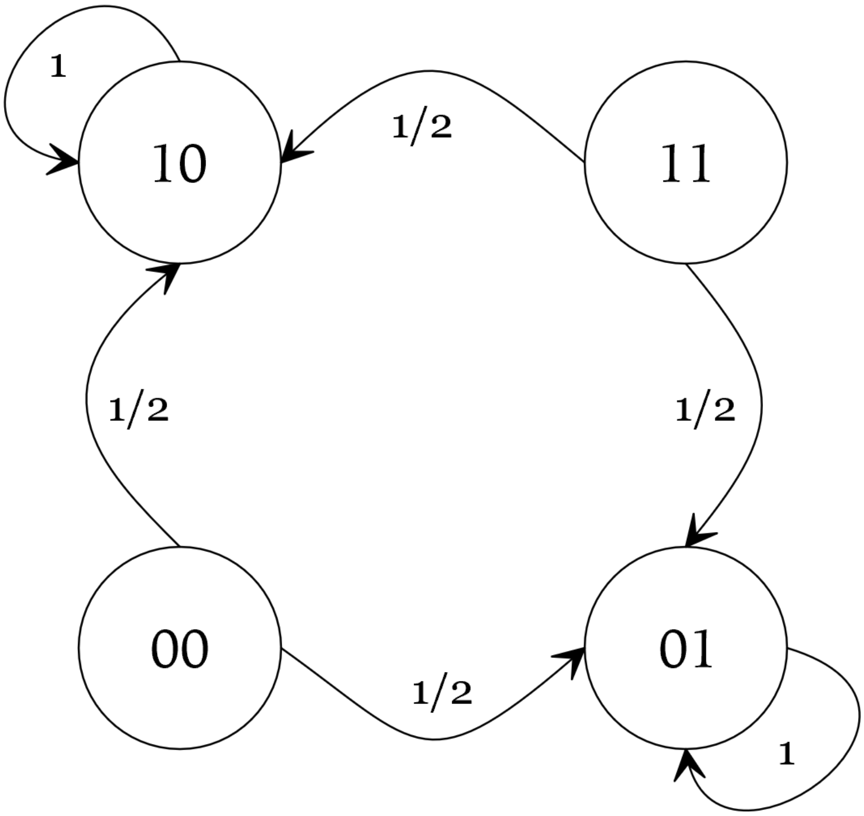

Figure 0.2: Gibbs-Sampling.

(B)

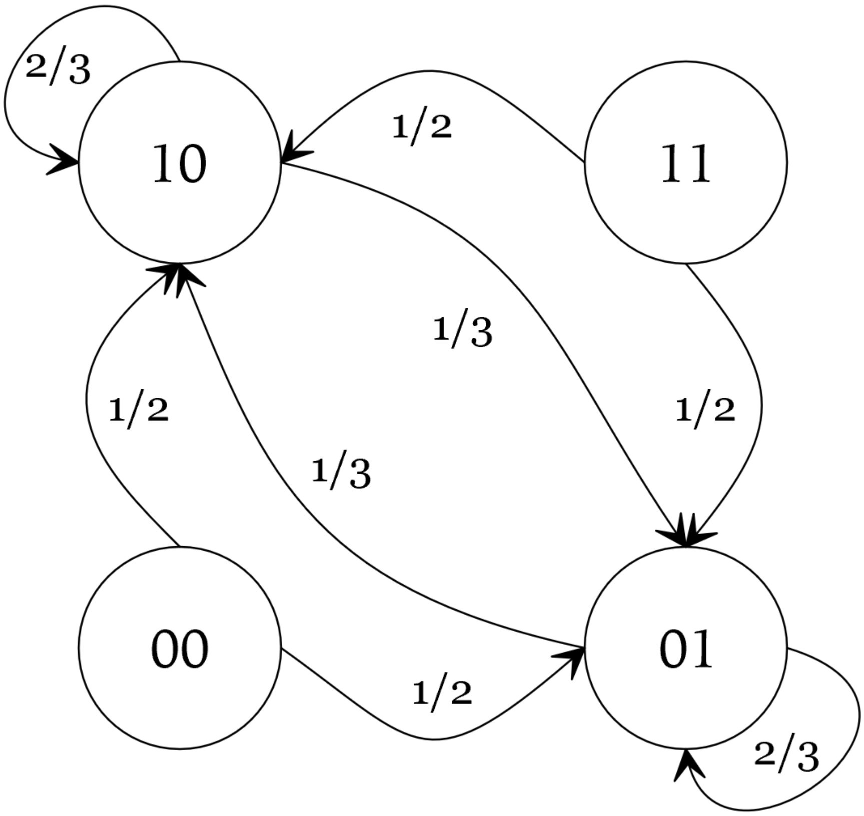

Figure 0.3: Metropolis-Hasting.

EXERCISE 5.16

Construct a *Markov Chain* in the following way(*Metropolis-Hasting Algorithm*):

$$\begin{aligned} V &= \binom{G}{k} \\ d &= k(n-k) \\ p(S_1, S_2) &= \begin{cases} \frac{1}{d}\max\left\{1, \frac{|E_{S_2}|}{|E_{S_1}|}\right\} & |S_1 \cap S_2| = k-1 \\ 1 - \sum\limits_{S' \neq S_2} p(S_1, S') & S_1 = S_2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

In which $|E_S|$ is the number of edges in the subgraph induced on S.

EXERCISE 5.18

(A)

Suppose the sizes of two cliques are n and m respectively($n \geq m$). Then the maximal degree $d = n + m$, use *Metropolis-Hasting Algorithm* to determine the transition probability

$$p_{xy} = \frac{1}{d}\min\left\{1, \frac{\pi_y}{\pi_x}\right\} = \frac{1}{d}$$

Use V to denote the vertex set of the graph, and V_1, V_2 to denote the vertex set of the two cliques, respectively. Let $S = S_1 \cup S_2, S_1 \subseteq V_1, S_2 \subseteq V_2, S_1 \cap S_2 = \emptyset$ be a subset of V with $|S| \leq \frac{n+m}{2}$.

Suppose $v_1 \in V_1$, $v_2 \in V_2$ are connected. Compute Φ under different conditions:

$v_1 \in S_1, v_2 \in S_2$:

$$\frac{1}{|S|}\left(|S_1|(n-|S_1|) + |S_2|(m-|S_2|)\right)\frac{1}{d}$$

$v_1 \notin S_1, v_2 \in S_2$:

$$\frac{1}{|S|}\left(|S_1|(n-|S_1|) + |S_2|(m-|S_2|) + 1\right)\frac{1}{d}$$

$v_1 \in S_1, v_2 \notin S_2$:

$$\frac{1}{|S|}\left(|S_1|(n-|S_1|) + |S_2|(m-|S_2|) + 1\right)\frac{1}{d}$$

$v_1 \notin S_1, v_2 \notin S_2$:

$$\frac{1}{|S|}\left(|S_1|(n-|S_1|) + |S_2|(m-|S_2|)\right)\frac{1}{d}$$

Φ will reach its minima at $S = V_2$, where $d\Phi = \frac{1}{m}$. If $S \cap V_1 \neq$, the numerator of $d\Phi$ will be greater than $n-1$. To make $d\Phi \leq \frac{1}{m}$, the denominator $|S|$ has to be greater than $m(n-1)$ while it's impossible because $\frac{n+m}{2} < m(n-1)$. Thus $S \cap V_1 =$, under this condition, $d\Phi = \frac{|S|(m-|S|)}{|S|} = m - |S|$ will reach its minima at $|S| = m$.

$$\varepsilon\text{-mixing time} = O\left(\frac{\log\frac{1}{\pi_{min}}}{\Phi^2\varepsilon^3}\right) = O\left(\frac{1}{\varepsilon^3}m^2(n+m)^2\log(n+m)\right)$$

(B)

It's the same case as in the first question if we set the smaller clique's size as 1:

$$\varepsilon\text{-mixing time} = O\left(\frac{1}{\varepsilon^3}(n+1)^2\log(n+1)\right)$$

EXERCISE 5.23

Construct a circuit in the following way(Use the property $p_{xy} = \frac{C_{xy}}{C_x}$ to derive R_{xy}):

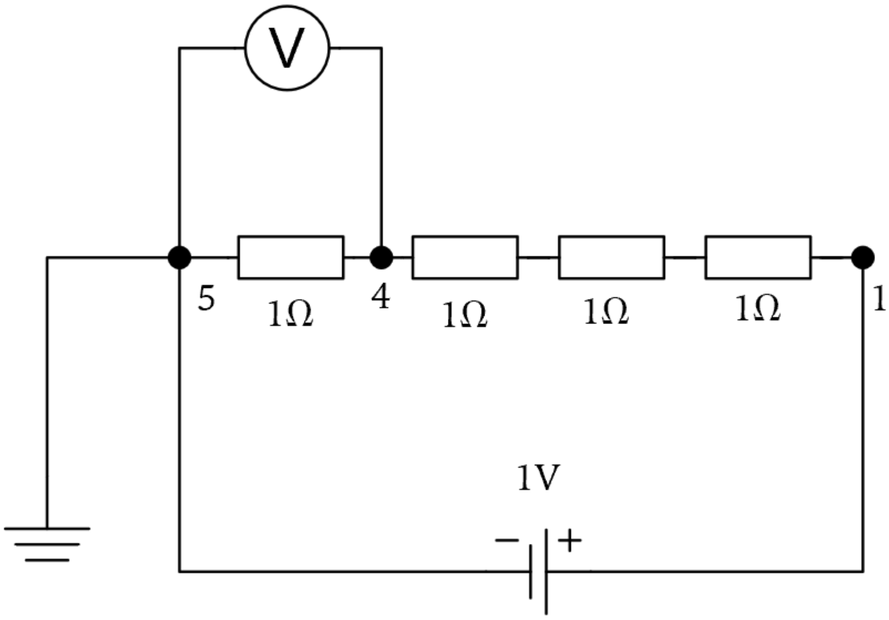

Figure 0.4: Circuit.

$$V_4 = \frac{1}{4}V$$

Thus the probability of a random walk starting at 4 reaching 1 before 5 is $\frac{1}{4}$.

EXERCISE 5.44

DESCRIPTION

Prove that two independent random walks starting at the origin on a two dimensional lattice will eventually meet with probability one.

SOLUTION

Use A, B to denote the two independent random walks. Suppose B's position is fixed, then A's move could be regarded as the graph showed below:

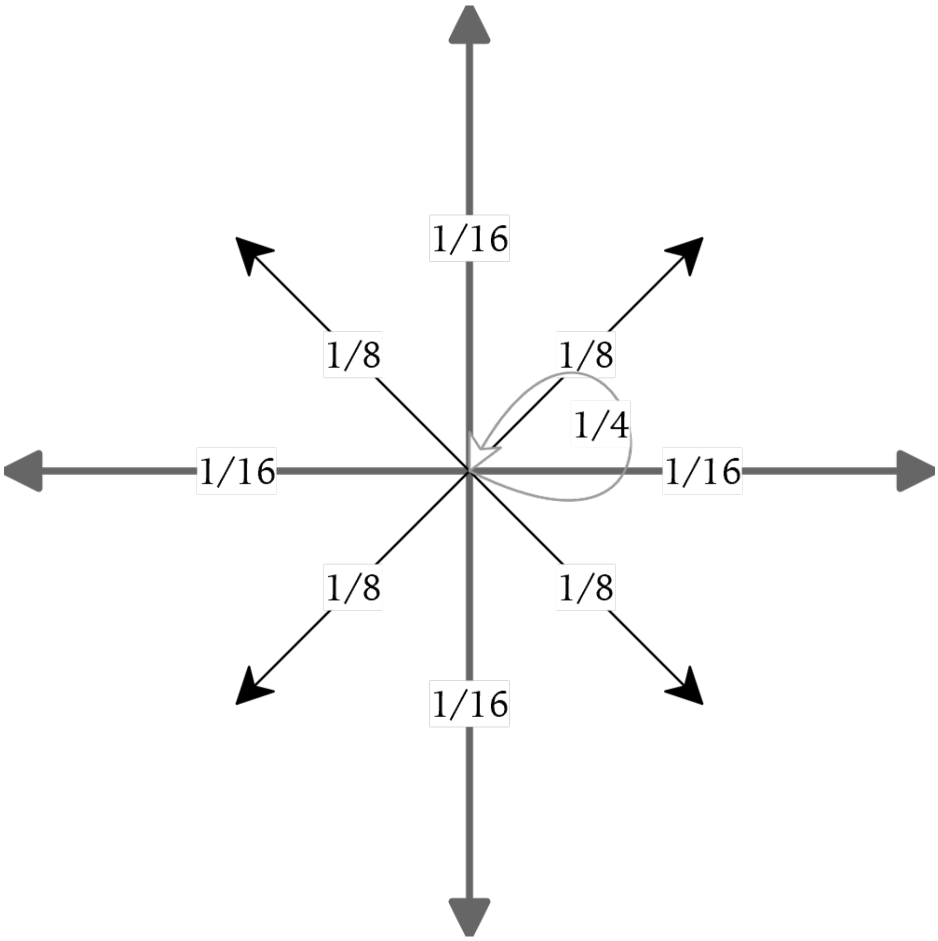

Figure 0.5: Random Walk.

We just need to calculate the escape probability of A. However, the electrical circuit corresponding to this random walk is:

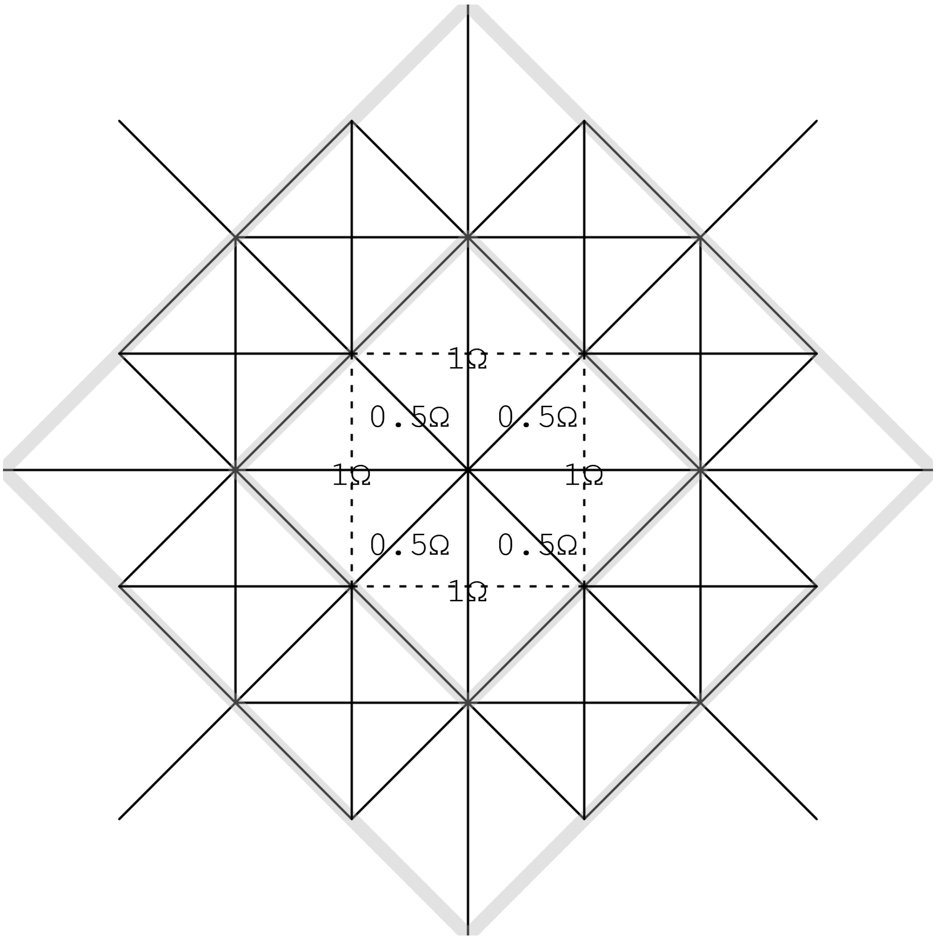

Figure 0.6: Electrical Circuit.

Let r_n be the resistance between $(0,0)$ and the n-th boundary($|x|+|y|=n$), From the circuit we derive:

$$r_n \geq \frac{1}{12\Omega} + \sum_{i=2}^{n} \frac{1}{32i-20}$$

$$p_{escape} = \frac{c_{eff}}{c_a} = \frac{1}{16r_{eff}} = \lim_{n\to\infty} \frac{1}{\frac{4}{3} + \sum_{i=2}^{n} \frac{4}{8i-5}} = 0$$

Thus with probability 1, A will return to the origin(A), which implies A will meet B at some moment.