CS389: Foundations of Data Science Homework III

Zihao Ye

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EXERCISE 5.9
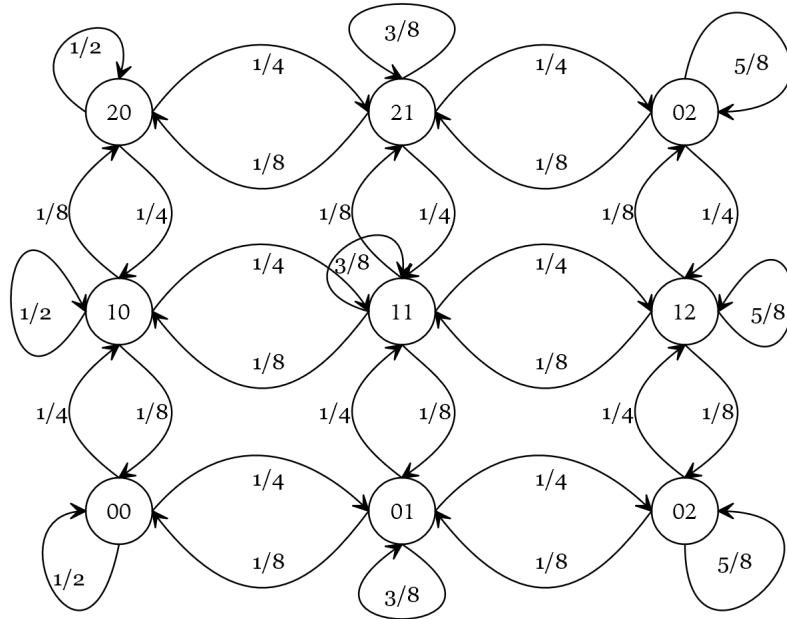

Figure 0.1: Metropolis-Hasting.

EXERCISE 5.13
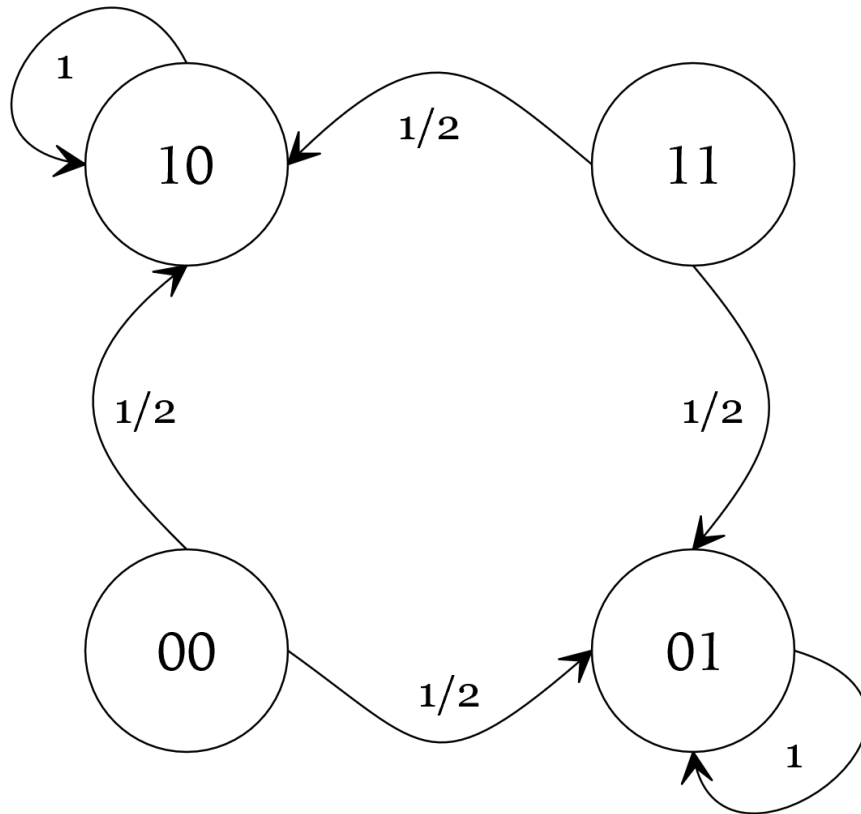
Figure 0.2: Gibbs-Sampling.
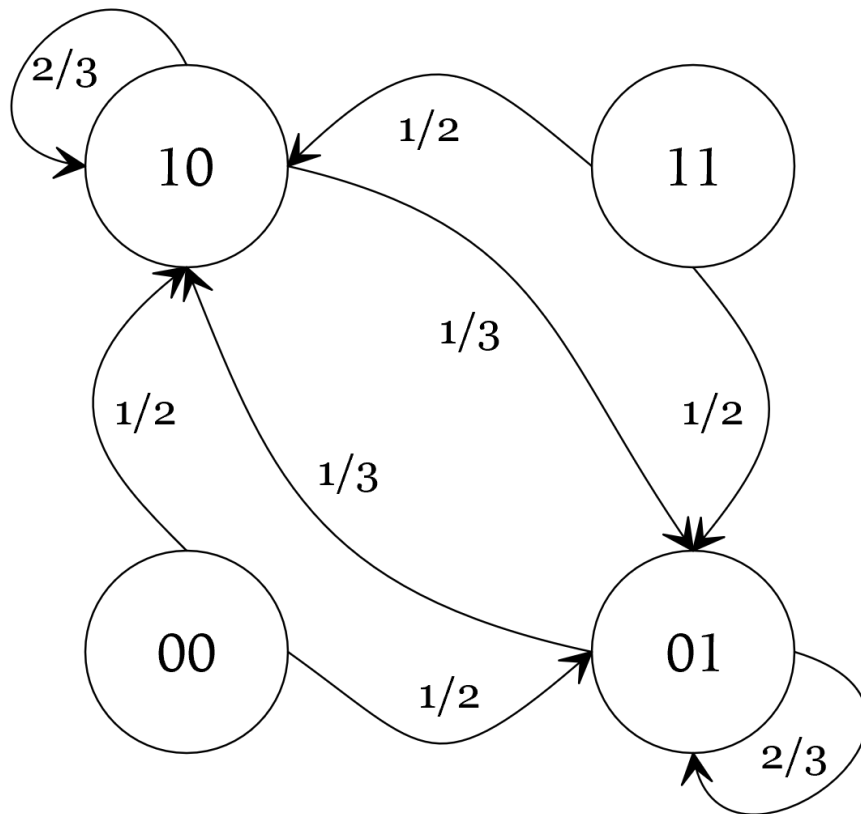
Figure 0.3: Metropolis-Hasting.

EXERCISE 5.16

Construct a *Markov Chain* in the following way(*Metropolis-Hasting Algorithm*):

$$
V = \binom{G}{k}
$$
$$
d = k(n-k)
$$
$$
p(S_1, S_2) = \begin{cases} \frac{1}{d}\max\left\{1, \frac{|E_{S_2}|}{|E_{S_1}|}\right\} & |S_1 \cap S_2| = k-1 \\ 1 - \sum_{S' \neq S_2} p(S_1, S') & S_1 = S_2 \\ 0 & \text{otherwise} \end{cases}
$$

In which $|E_S|$ is the number of edges in the subgraph induced on S.

EXERCISE 5.18

(A)

Suppose the sizes of two cliques are n and m respectively($n > m$). Then the maximal degree $d = n + m$, use *Metropolis-Hasting Algorithm* to determine the transition probability

$$p_{xy} = \frac{1}{d} \min\left\{1, \frac{\pi_y}{\pi_x}\right\} = \frac{1}{d}$$

Use V to denote the vertex set of the graph, and V_1, V_2 to denote the vertex set of the two cliques, respectively. Let $S = S_1 \cup S_2, S_1 \subseteq V_1, S_2 \subseteq V_2, S_1 \cap S_2 = \emptyset$ be a subset of V with $|S| = \lfloor \frac{n+m}{2} \rfloor$.

Suppose $v_1 \in V_1, v_2 \in V_2$ are connected. Compute Φ under different conditions:

$v_1 \in S_1, v_2 \in S_2$:

$$\frac{1}{|S|}(|S_1|(n - |S_1|) + |S_2|(m - |S_2|))\frac{1}{d}$$

$v_1 \notin S_1, v_2 \in S_2$:

$$\frac{1}{|S|}(|S_1|(n - |S_1|) + |S_2|(m - |S_2|) + 1)\frac{1}{d}$$

$v_1 \in S_1, v_2 \notin S_2$:

$$\frac{1}{|S|}(|S_1|(n - |S_1|) + |S_2|(m - |S_2|) + 1)\frac{1}{d}$$

$v_1 \notin S_1, v_2 \notin S_2$:

$$\frac{1}{|S|}(|S_1|(n - |S_1|) + |S_2|(m - |S_2|))\frac{1}{d}$$

In both cases, the minimal Φ is of order $O(\frac{n^2 + m^2}{(n+m)^2})$.

$$\varepsilon\text{-mixing time} = O\left(\frac{\log \frac{1}{\pi_{min}}}{\Phi^2 \varepsilon^3}\right) = O\left(\frac{1}{\varepsilon^3}\frac{(n+m)^4}{(n^2 + m^2)^2}\log(n+m)\right)$$

(B)

It's the same case as in the first question if we set the smaller clique's size as 1:

$$\varepsilon\text{-mixing time} = O\left(\frac{1}{\varepsilon^3}\frac{(n+1)^4}{(n^2 + 1)^2}\log(n+m)\right) = O\left(\frac{1}{\varepsilon^3}\log(n+m)\right)$$

EXERCISE 5.23

Construct a circuit in the following way(Use the property $p_{xy} = \frac{C_{xy}}{C_x}$ to derive R_{xy}):

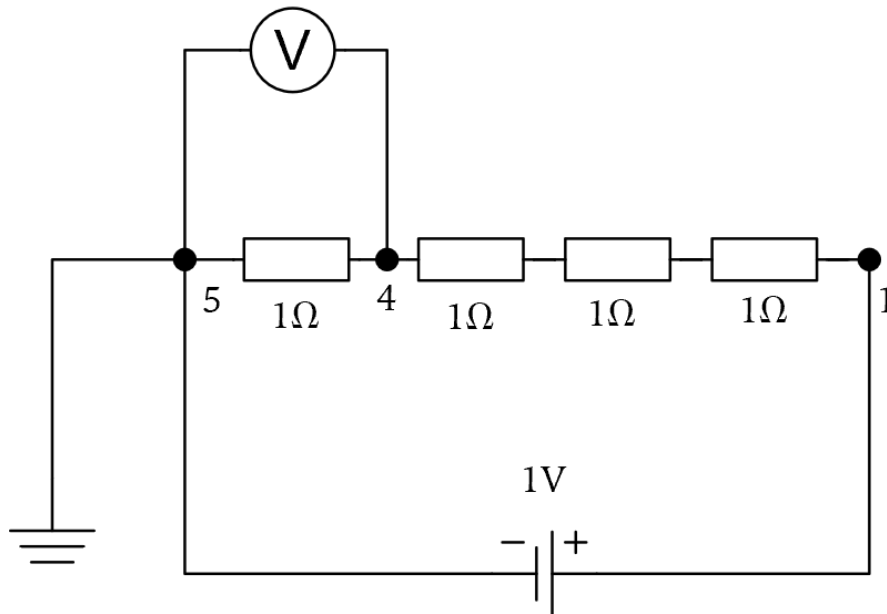

Figure 0.4: Circuit.

$$V_4 = \frac{1}{4}V$$

Thus the probability of a random walk starting at 4 reaching 1 before 5 is $\frac{1}{4}$.

EXERCISE 5.44

DESCRIPTION

Prove that two independent random walks starting at the origin on a two dimensional lattice will eventually meet with probability one.

SOLUTION

Use A, B to denote the two independent random walks. Suppose B's position is fixed, then A's move could be regarded as the graph showed below:
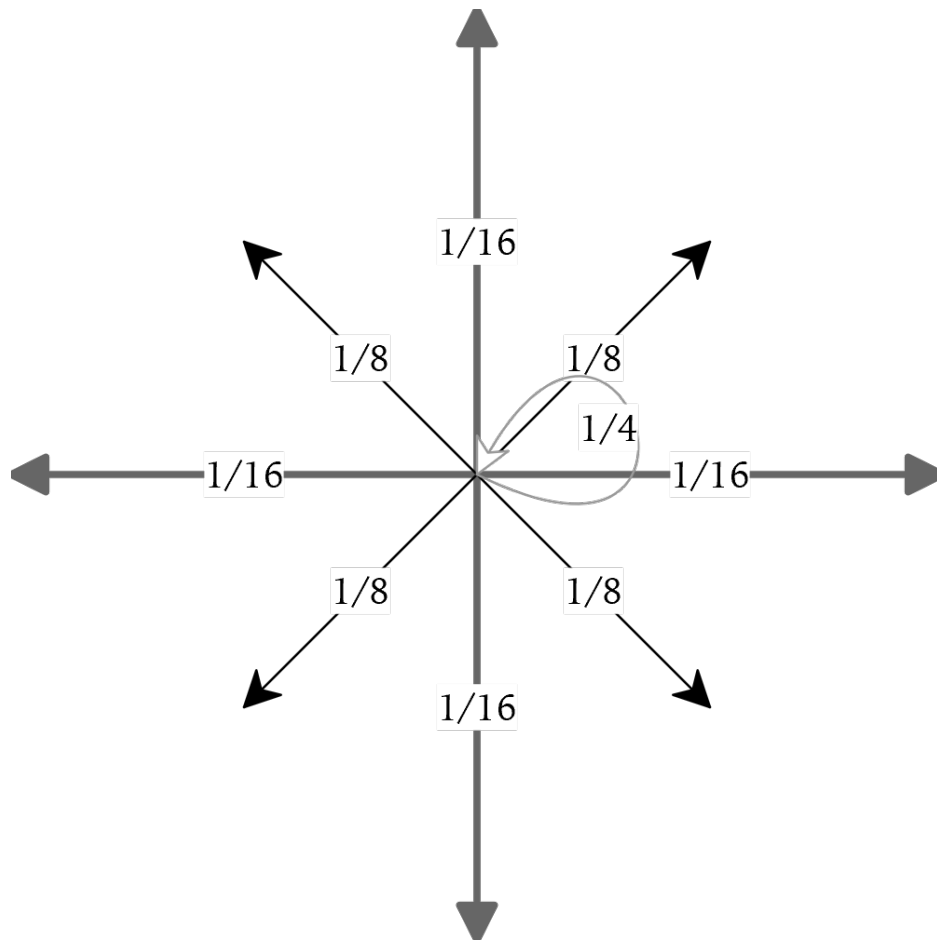
Figure 0.5: Random Walk.

We just need to calculate the escape probability of A. However, the electrical circuit corresponding to this random walk is:
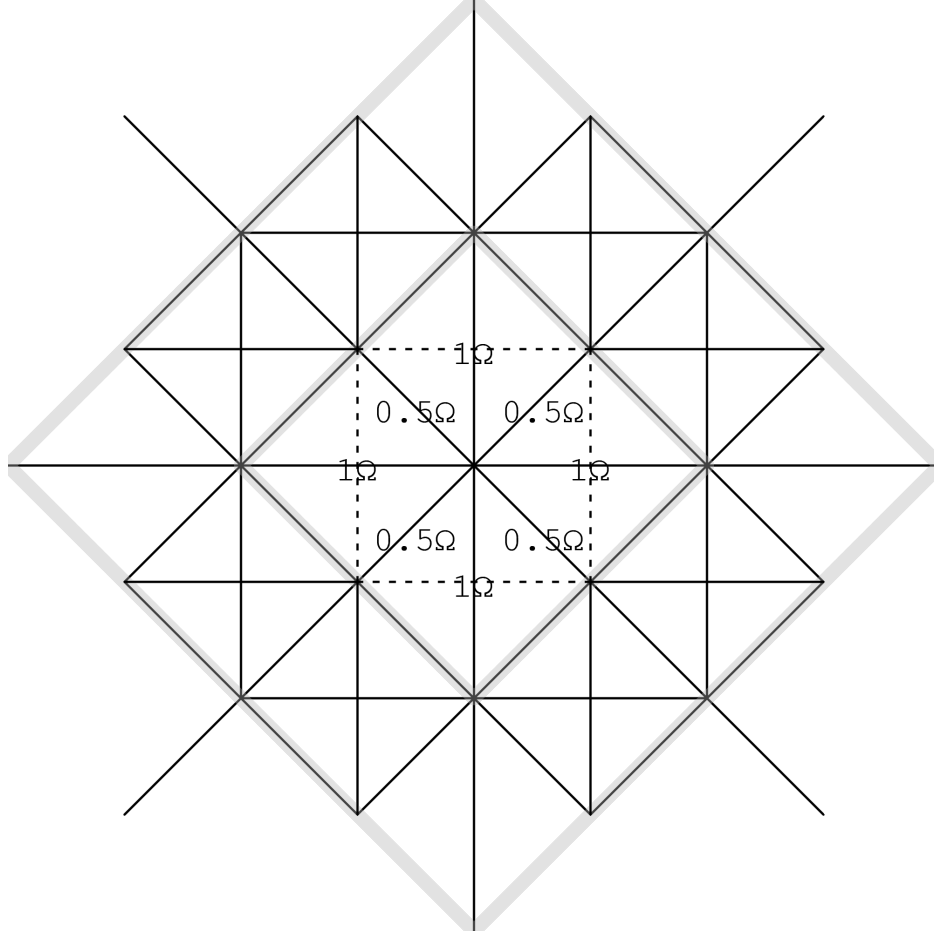
Figure 0.6: Electrical Circuit.

Let r_n be the resistance between $(0,0)$ and the n-th boundary($|x| + |y| = n$), From the circuit we derive:

$$r_n \geq \frac{1}{12\Omega} + \sum_{i=2}^{n} \frac{1}{32i - 20}$$

$$p_{escape} = \frac{c_{eff}}{c_a} = \frac{1}{16r_{eff}} = \lim_{n \to \infty} \frac{1}{\frac{4}{3} + \sum_{i=2}^{n} \frac{4}{8i-5}} = 0$$

Thus with probability 1, A will return to the origin(A), which implies A will meet B at some moment.