Assignment

Problem 1 Consider a gambler who initially has \$0 in hand. In each round, he tosses a fair coin and wins \$1 if it is HEAD and loses \$1 if it is TAIL. Let $a, b > 0$ be two positive integers. The game proceeds in rounds and stops once the gambler gets \$$b$ or $-$\$$a$, and he wins or loses the game respectively.

- What is the probability that the gambler wins when the game stops. Please use the optional stopping time theorem to derive the exact formula of the probability.
- Let T be the number of rounds when the game stops. What is the expectation of T? (Hint: construct a martingale Z_t such that $E[Z_T] = E[Z_0] = 0$ and $E[Z_T]$ is a function of $E[T]$.)

Problem 2 The 2-SAT problem is a special case of k-SAT problem that can be solved in polynomial time. Given as input a 2-CNF ϕ in which each clause contains two literals, the problem is to decide whether ϕ is satisfiable. Let $t > 0$ be a positive integer, consider the following randomized algorithm to solve 2-SAT:

1. Arbitrarily choose an assignment σ.
2. For $i = 1$ to t:
   2.1 If ϕ is satisfied by σ, return YES;
   2.2 Otherwise, choose an arbitrary unsatisfied clause, pick one of the (two) literals in it uniformly at random, and flip the assignment of σ on the literal.
3. return NO.

Assume ϕ is a 2-CNF with n variables. If ϕ is not satisfiable, then the above algorithm always return NO. If ϕ is satisfiable, to analyse the algorithm, we fix a satisfying assignment π and let X_i denote the number of variables on which σ and π disagree in the i-th iteration.

What is the expectation of T at which $X_T = 0$? Suppose you want the above algorithm to return correctly with probability at least 99%, what is the suitable value of t?

Problem 3 Consider a random walk on a graph whose edges have positive real costs: the interpretation of these costs is that every time the random walk traverses an edge (ij), it incurs a given cost $c_{ij} > 0$; $c_{ij} = c_{ji}$, and $c_{ii} = 0$. Consider the random walk on a graph G with m edges that have such costs associated with them, with transition probabilities

$$P_{ij} = \frac{1/c_{ij}}{\sum_k 1/c_{ik}}.$$

Let S_{uv} denote the expected total cost incurred by a walk that begins at vertex u and terminates upon returning to u after having visited v at least once. Show that

$$S_{uv} = 2mR_{uv},$$

where R_{uv} is the effective resistance between node u and node v in an electrical network whose underlying graph is G, and where the branch resistance between i and j is c_{ij}.

Problem 4 Let $G = (V, E)$ be an undirected graph with maximum degree Δ, which is not necessarily regular.

1. Design a lazy, reversible, irreducible and ergodic random walk on G whose stationary distribution is the uniform distribution on V.
2. Design a lazy, reversible, irreducible and ergodic random walk on G whose stationary distribution is an arbitrarily fixed distribution π on V satisfying $\pi(v) > 0$ for every $v \in V$.

Problem 5 An n-dimensional cube is a graph with 2^n vertices, each of which can be encoded as an n-bit string $b_1 b_2 \dots b_n$. There is an edge between two vertices if and only if their encodings have Hamming distance exact one. Consider the following two random walks on the n-dimensional cube:

Random Walk 1: start from an arbitrary vertex.

1. with probability $\frac{1}{n+1}$, do nothing;
2. otherwise, for each of the neighbors, with probability $\frac{1}{n+1}$ go to it.

Random Walk 2: start from an arbitrary vertex.

1. with probability p ($\frac{1}{n+1} \leq p \leq \frac{1}{2}$), do nothing;
2. otherwise, uniformly choose a neighbor and go to it.

Give an upper bound for the mixing time of above two random walks respectively.

Problem 6 Let $n, k \in \mathbb{N}$ satisfying $k \leq \frac{n}{2}$ and $\Omega = \left\{ S \mid S \in \binom{[n]}{k} \right\}$ the family of subsets of $\{1, 2, \dots, n\}$ of cardinality k. Consider the following random walk on Ω: start from a subset $S \in \Omega$, choose $a \in S$ and $b \in [n] \setminus S$ uniformly at random, move to $S + b - a$.

1. Show that the random walk is irreducible and ergodic (you can make it lazy if necessary) with uniform stationary distribution.
2. Apply coupling argument to show that its mixing time is at most $O(n \log k)$.

Final

Problem 1 Let H be a graph and $n > |V(H)|$ be an integer. Assume there exists a graph with n vertices and m edges of which H is not a subgraph. Let $k > \frac{n^2 \ln n}{m}$ be an integer. Prove that there is an edge coloring of the complete graph on n vertices (not necessarily a proper coloring) with k colors such that no monochromatic H exists.

Problem 2 Let $X_{n,p}$ denote the number of triangles in the random graph $G(n, p)$ where p is a function to be specified in your analysis.

- Compute the expectation of $X_{n,p}$ and use at least two techniques introduced in the lectures (first/second-moment method, Chernoff bound, martingale...) to derive concentration bounds of $X_{n,p}$ as tight as possible.
- (Optional) Can you give better concentration bounds using more advanced probabilistic tools (not necessarily introduced in the lecture).

Problem 3 Consider the Jerrum-Sinclair chain introduced in the lecture to sample matchings in a graph. There is an operation of "edge exchange" in the chain. Now consider another chain which does not allow such operation, i.e., we only use the operations of "edge addition" and "edge deletion" in the new chain. Is this modified Markov chain still rapid mixing?

If your answer is YES, please give an analysis of its mixing time.

If your answer is NO, please give an example on which the chain is not rapid mixing and explain why.