Research Statement

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My main research interests are symbolic computation (computer algebra) and its applications in combinatorics, knot theory, statistics and cryptography. It aims to give algorithmic and constructive answers to various problems in mathematics and computer science, such as polynomial factorization, computing solutions of systems of polynomial equations, and quantifier elimination. Systems of algebraic differential equations and difference equations are principal research objects of mathematics, physics and related areas. Algebraic study of those systems gives essential answers to their applications in physics, statistics and so on. Much of my work is devoted to developing algorithms in computing solutions and illustrating algebraic structures of differential equations and difference equations by using constructive tools (such as Gröbner bases, resultant theory) in computer algebra and differential algebra. It has interesting applications such as certifying integer sequences, checking special cases of a conjecture of Krattenthaler and verifying several instrances of the colored Jones polynomial are Laurent polynomial sequences.

Desingularization of linear differential and difference operators

A D-finite function is specified by a linear ordinary differential equation with polynomial coefficients and finitely many initial values. Every singularity of a D-finite function will be a root of the coefficient of the highest order derivative appearing in the corresponding differential equation. For instance, x^{-1} is a solution of the equation $xf'(x) + f(x) = 0$, and the singularity at the origin is also the root of the polynomial x. However, the converse is not true. For example, the solution space of the differential equation $xf'(x) - 4f(x) = 0$ is spanned by x^4 as a vector space, but none of those functions has singularity at the origin.

More specifically, for an ordinary equation $p_0(x)f(x)+\cdots+p_r(x)f^{(r)}(x) = 0$ with polynomial coefficients $p_1, \ldots, p_r$ and $p_r \neq 0$, the roots of p_r are called the singularities of the equation. A root α of p_r is call *apparent* if the differential equation admits r linearly independent formal power series solutions in $x - \alpha$. Deciding whether a singularity is apparent is therefore the same as checking whether the equation admits a fundamental system of formal power series solutions at this point. This can be done by inspecting the so-called *indicial polynomial* of the equation at α and solving a system of finitely many linear equations. If a singularity α of an ordinary differential is apparent, then we can always construct a second ordinary differential equation whose solution space contains all the solutions of the first equation, and which does not have α as a singularity any more. This process is called *desingularization.* There are similar techniques for the difference case. Our contributions are as follows:

- Contraction of Ore ideals with applications. Ore operators form a common algebraic abstraction of linear ordinary differential and recurrence equations. Given an Ore operator L with polynomial coefficients in x, it generates a left ideal I in the Ore algebra over the field $\mathbb{K}(x)$ of rational functions. We present an algorithm for computing a basis of the contraction ideal of I in the Ore algebra over the ring $R[x]$ of polynomials, where R may be either $\mathbb{K}$ or a domain with $\mathbb{K}$ as its fraction field. This algorithm is based on recent work on desingularization for Ore operators by Chen, Jaroschek, Kauers and Singer. Using a basis of the contraction ideal, we compute a completely desingularized operator for L whose leading coefficient not only has minimal degree in x but also has minimal content. Completely desingularized operators have interesting applications such as certifying integer sequences and checking special cases of a conjecture of Krattenthaler.

- Desingularization in the q-Weyl algebra. We give an order bound for desingularized operators, and thus derive an algorithm for computing desingularized operators in the first q-Weyl algebra. Moreover, an algorithm is presented for computing a generating set of the first q-Weyl closure of a given q-difference operator. As an application, we certify that several instances of the colored Jones polynomial are Laurent polynomial sequences by computing the corresponding desingularized operator.

- Apparent singularities of D-finite systems. We generalize the notions of ordinary points and singularities from linear ordinary differential equations to D-finite systems. Ordinary points and apparent singularities of a D-finite system are characterized in terms of its formal power series solutions. We also show that apparent singularities can be removed like in the univariate case by adding suitable additional solutions to the system at hand. Several algorithms are presented for removing and detecting apparent singularities. In addition, an algorithm is given for computing formal power series solutions of a D-finite system at apparent singularities.

Possible future work is as follows:

- Design algorithms for determining a generating set of a contraction ideal in the multivariate Ore algebra.
- Develop the desingularization technique for linear Mahler equations.
- Study the desingularization problem for the multivariate linear difference equations with polynomial coefficients.

Computing symbolic solutions of algebraic differential and difference equations

An algebraic ordinary difference equation (AOΔE) is a difference equation of the form

$$F(x, y(x), y(x+1), \cdots, y(x+m)) = 0,$$

where F is a nonzero polynomial in $y(x), y(x+1), \cdots, y(x+m)$ with coefficients in the field $\mathbb{K}(x)$ of rational functions over an algebraically closed field $\mathbb{K}$ of characteristic zero, and $m \in \mathbb{N}$. AOΔEs naturally appear from various problems, such as symbolic summation, factorization of linear difference operators, analysis of time or space complexity of computer programs with recursive calls. Thus, to determine (closed form) solutions of a given AOΔE is a fundamental problem in difference algebra and is of general interest. We can define algebraic ordinary differential equations (AODEs) analogously. Our contributions are as follows:

- Rational solutions of first-order AOΔEs.We propose an algebraic geometric approach for studying rational solutions of first-order AOΔEs. For an autonomous first-order AOΔE, we give an upper bound for the degrees of its rational solutions, and thus derive a complete algorithm for computing corresponding rational solutions.

- Rational solutions of high-order algebraic ordinary differential equations. We first prove a sufficient condition for the existence of a bound on the degree of the possible polynomial solutions to an AODE. An AODE satisfying this condition is called *noncritical*. Then we prove that some common classes of low-order AODEs are noncritical. For rational solutions, we determine a class of AODEs, which are called *maximally comparable*, such that the possible poles of any rational solutions are recognizable from their coefficients. This generalizes the well-known fact that any rational solutions to a linear ODE are contained in the set of zeros of the leading coefficient. Finally, we develop an algorithm to compute all rational solutions of certain maximally comparable AODEs, which is applicable to 78.54% of the AODEs in Kamke's collection of standard differential equations.

Possible future work is as follows:

- Design algorithms to compute polynomial and rational solutions of high-order AOΔEs.

- Compute rational solutions of non-autonomous first-order AOΔEs.