① §11.9 Representations of Functions as Power Series

Motivation: Represent functions as sums of power series by using geometric series or by differentiating or integrating such a series.

Application: ① Integrating functions that do not have elementary ~~antideri~~ antiderivatives;

② Solving differential equations;

③ Approximating functions by polynomials.

Geometric series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x|<1 \qquad (1)$$

(Regrad (1) as expressing function $f(x) = \frac{1}{1-x}$ in terms of power series)

Ex 1. Express $\frac{1}{1+x^2}$ in terms of power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \quad (\text{set } -x^2 = u, \text{ and apply } (1))$$

$$\overset{(1)}{=} \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

It converge when $|-x^2|<1$, i.e. $x^2<1$ or $|x|<1$

Thus, the interval of convergence is $(-1, 1)$.

Ex 2. Find a power series representation for $1/(x+2)$

$$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})}$$

$$= \frac{1}{2} \cdot \frac{1}{1+\frac{x}{2}}$$

$$= \frac{1}{2} \frac{1}{1-(-\frac{x}{2})}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

② $= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$

The series converges when $|-\frac{x}{2}| < 1$, i.e., $|x| < 2$.

Thus, the interval of convergence is $(-2, 2)$.

Ex 3. Find a power series representation of $\frac{x^3}{x+2}$

$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2}$$
$$\stackrel{\text{Ex 2}}{=} x^3 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}$$
$$= \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

As in Example 2, the interval of convergence is $(-2, 2)$.

- Differentiation and Integration of Power Series

Assume $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ whose domain is the interval of convergence of the series.

Goal: differentiate and integrate $f(x)$.

(The following theorem tells us that we can do so by differentiating or integrating ~~to~~ each individual term in the series, just as we do for polynomials). (term-by-term differentiation and integration)

Theorem 2. If $\sum c_n (x-a)^n$ has radius of convergence $R > 0$, the the function defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable on $(a-R, a+R)$ and

(i) $$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right]$$
$$= \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n (x-a)^n\right]$$
$$= \sum_{n=0}^{\infty} c_n n (x-a)^{n-1}$$
$$= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

(ii) $$\int f(x)dx = \int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx$$

③

$$= \sum_{n=0}^{\infty} \int c_n (x-a)^n dx$$

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$$= \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

$$= C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \cdot \frac{(x-a)^3}{3} + \cdots$$

The radii of convergence of power series in (i) and (ii) are both R.

Note 2: 1. Theorem 2 says that the radius of convergence remain the same after we ~~diff~~ differentiate or integrate a power series, ~~it~~ It is not ~~~~ true that the interval of convergence always remains the same.

2. Theorem 2 is useful in solving differential equations.

Ex ~~3~~ 4. The Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

is defined for all x.

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left[\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right]$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2n \cdot x^{2n-1}}{2^{2n}(n!)^2}$$

Ex 5. Express $\frac{1}{(1-x)^2}$ as a power series by differentiating $\frac{1}{1-x}$. What is the radius of convergence?

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\left[\frac{1}{1-x}\right]' = 1' + x' + (x^2)' + (x^3)' + \cdots = \sum_{n=0}^{\infty} (x^n)'$$

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}$$

By Theorem 2, the radius of convergence is $R = 1$.

Ex 6. Find a power series representation for $\ln(1+x)$ and its radius of convergence.

Observation: $\ln(1+x) = \int \frac{dx}{1+x}$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots, \quad |x|<1 \quad (2)$$

Integrating both sides of (2), we get

$$\ln(1+x) = \int \frac{dx}{1+x} = \int (1 - x + x^2 - x^3 + \cdots)\,dx$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C, \quad |x|<1$$

Set $x=0$, we get

$$0 = \ln 1 = C$$

Thus, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x|<1$$

The radius of convergence is $R=1$.

Ex 7. Find a power series representation for $f(x) = \tan^{-1}x$.

Observation: $\tan^{-1}x = \int \frac{dx}{1+x^2}$

$$= \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Set $x=0$, we get $C=0$.

Thus, $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$$

Since the radius of convergence for $\frac{1}{1+x^2}$ is 1, ~~so is that of $\tan^{-1}x$.~~ ~~the radius~~ $\tan^{-1}x$ is also 1.

⑤ Ex 8. (a) Evaluate $\int \frac{dx}{1+x^7}$ as a power series

(b) use (a) to approximat $\int_0^{0.5} \frac{dx}{1+x^7}$ correct to within 10^{-7}.

(a)
$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)}$$
$$= \sum_{n=0}^{\infty} (-x^7)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{7n}$$
$$= 1 - x^7 + x^{14} - \cdots$$

Now we integrate term by term:

$$\int \frac{dx}{1+x^7} = \int \sum_{n=0}^{\infty} (-1)^n x^{7n} dx$$
$$= C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{7n+1}}{7n+1}$$
$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$$

The series converges for $|-x^7| < 1$, i.e., for $|x| < 1$.

(b) By the Fundamental Theorem of Calculus, $(C=0)$

$$\int_0^{0.5} \frac{dx}{1+x^7} = \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots \right]_0^{1/2}$$
$$= \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \cdots + \frac{(-1)^n}{(7n+1)2^{7n+1}} + \cdots$$

By the Alternating Series Estimation theorem, if ~~one take n=3~~, then the err is smaller than the term with $n=4$:

$$\frac{1}{29 \cdot 2^{29}} \approx 6.4 \times 10^{-11}$$

So we have

$$\int_0^{0.5} \frac{dx}{1+x^7} \approx \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374.$$