1 § 11.4 The Comparison Tests

General idea: compare a given series with a series that is known to be convergent or divergent.

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$

is similar to

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

Observation ~~Fact~~:

$$\frac{1}{2^n+1} < \frac{1}{2^n} \quad \text{for } n \geq 1$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} < \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

By Monotonic Sequence Theorem, we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} \text{ is convergent.}$$

The Comparison Test Suppose $\sum a_n$ and $\sum b_n$ are series with ~~positive~~ positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for ~~all n~~ $n \geq N_0$, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for $n \geq N_0$, then $\sum a_n$ is also divergent.

(Pf: Use Monotonic Sequence Theorem).

For the application of the comparison test, most of the time we use one of these series:

1. A p-series ($\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$)
2. A geometric series ($\sum a r^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$)

Ex 1. Determine the convergence of $\sum \frac{5}{2n^2+4n+3}$

(2) (The dominant term in the denominator is $2n^2$)

~~Obs~~ Observation:

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2} \quad (1)$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} \text{ is convergent } (2)$$

By (1) and (2), we conclude that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} \text{ is convergent.}$$

Ex 2. Test the convergence of $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

Observation: $\ln k > 1$ for $k \geq 3$

Thus, $\frac{\ln k}{k} > \frac{1}{k}$ for $k \geq 3$

Since $\sum \frac{1}{k}$ is divergent, it follows from the comparison test that $\sum \frac{\ln k}{k}$ is also divergent.

Consider the series (the case when the Comparison Test fails)

$$\sum \frac{1}{2^n-1}$$

We have

$$\frac{1}{2^n-1} > \frac{1}{2^n},$$

~~But~~ but the Comparison Test fails because $\sum \frac{1}{2^n}$ converges.

~~Obs~~ Observation:

$$\frac{1}{2^n-1} \approx \frac{1}{2^n} \text{ as } n \text{ ~~is large~~ increases}$$

The Limit Comparison Test Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a positive number, then either both series converge or both diverge.

(3)

Ex 3. Test the ~~o~~convergence of $\sum \frac{1}{2^n-1}$

let $a_n = \frac{1}{2^n-1}$, $b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n-1}$$

$$= \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}}$$

$$= 1 > 0$$

Since $\sum \frac{1}{2^n}$ is convergent, the given series converges by the Limit Comparison Test.

Ex 4. Determine the convergence of $\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$

~~Obers~~ Observation: $2n^2+3n \approx 2n^2$ as $n\to\infty$

$\sqrt{5+n^5} \approx \sqrt{n^5}$ as $n\to\infty$

let $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}}$$

$$= \frac{2+0}{2\sqrt{0+1}}$$

$$= 1$$

Since $\sum b_n = 2\sum \frac{1}{n^{\frac{1}{2}}}$ is divergent, the given series diverges by the Limit Comparison Test.

Note: We find a ~~suitable~~ suitable comparison series $\sum b_n$ by taking the highest powers in the numerator and denominator.

- Estimating Sums

④ Assume $\sum a_n$ converges by comparison with $\sum b_n$.

Estimate $\sum a_n$ by comparing remainders.

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

$$T_n = t - t_n = b_{n+1} + b_{n+1} + \cdots$$

Since $a_n \le b_n$ for all n, we have $R_n \le T_n$.

If we can ~~bound~~ estimate T_n, then we can estimate R_n.

Ex 5. Approximate $\sum \frac{1}{n^3+1}$ by taking $n = 100$.

Since

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

Let T_n be the remainder of $\sum \frac{1}{n^3}$. Then

$$T_n \le \int_n^\infty \frac{dx}{x^3} = \frac{1}{2n^2}$$

Thus, the remainder R_n for the given series satisfies

$$R_n \le T_n \le \frac{1}{2n^2}$$

Let $n = 100$. We have

$$R_{100} \le \frac{1}{2(100)^2} = 0.00005$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538$$

with error less than 0.00005.