On Formal Power Series Solutions of Algebraic Ordinary Differential Equations*

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Abstract

We propose a computational method to determine when a solution modulo a certain power of the independent variable of a given algebraic differential equation (AODE) can be extended to a formal power series solution. The existence and the uniqueness conditions for the initial value problems for AODEs at singular points are included. Moreover, when the existence is confirmed, we present the algebraic structure of the set of all formal power series solutions satisfying the initial value conditions.

1 Introduction

The problem of finding formal power series solutions of algebraic ordinary differential equations (AODEs) has a long history and it has been extensively studied in literature. The Newton polygon method is a well-known method developed for studying this problem. In (BB56), Briot and Bouquet use the Newton polygon method to study the singularities of first-order and first degree ODEs. Fine gave a generalization of the method for arbitrary order AODEs in (Fin89). By using

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Newton polygon method, one can obtain interesting results on a larger class of series solutions which is called generalized formal power series solutions, *i.e.*, power series with real exponents. In (GS91), Grigoriev and Singer proposed a parametric version of the Newton polygon method and use it to study generalized formal power series solutions of AODEs. A worth interpretation of the parametric Newton polygon method can be found in (CF09) and (Can05). However, it has been shown in (DDRJ97) that Newton polygon method for AODEs has its own limits: In general, there is no bound on the number of terms which have to be computed in order to guarantee the existence nor the uniqueness of a solution when we try to extend a given truncation of a potential solution. In the case of AODEs of order one with constant coefficients, this problem can be avoided and all formal power series solutions can be found, see (FS19), and its generalization to formal power series solutions with fractional exponents in (CFS19). For a more recent exposition about the history of the Newton polygon method, we refer to (DG19).

We present an ansatz-based general method for determining formal power series solutions of AODEs. Its general strategy is to make an ansatz of unknown coefficients, plug the corresponding power series formally into the differential equation and then make coefficient comparison. If a particular initial value is given, this approach might give rise to a unique solution. However, it might be the case that after some steps there is no solution for the next coefficient. In other words, the computed truncation can not be continued to a formal power solution of the given differential equation. In order to overcome this difficulty, we follow the method which is inherited from the work by Hurwitz (Hur89), Limonov in (Lim15) and Denef and Lipshitz in (DL84). There the authors give an expression of the derivatives of a differential polynomial with respect to the independent variable in terms of lower order differential polynomials (see (Hur89, page 328–329), (Lim15, Corollary 1) and (DL84, Lemma 2.2)). Our first contribution is to enlarge the class of differential equations where all formal power series solutions with a given initial value can be computed algorithmically. This class is given by a sufficient condition on the given differential equation and initial value which is described by the local vanishing order. Moreover, we give a necessary and sufficient condition on the given differential equation such that for every initial value all formal power series solutions can be computed in this way. For differential equations satisfying this condition, we give an algorithm to compute all formal power series solutions up to an arbitrary order and illustrate it by some examples.

The rest of the paper is organized as follows. Section 2 is devoted to present a conjecture (Conjecture 2.4) which may occur by simply performing coefficient comparison to compute formal power series solutions of AODEs and we show how the well-known formula of Ritt (Lemma 2.5) can be used in partially solving this problem, see Proposition 2.6. Since not all formal power series solutions can be found in this way (For instance, see Example 2.3), one may use a refinement of Ritt's formula presented in (Hur89; Lim15). We summarize it by Theorem 3.2 in Section 3. In order to simplify some of the subsequent reasonings, we also use a slightly different notation and define *separant matrices*. Moreover, we give some sufficient conditions on the given differential equation, which is called the *vanishing order*, such that the refined formula can be used in an algorithmic way for computing all formal power series solutions and present new results in this direction, see Theorems 4.4, 5.8 and Algorithm 1. In Section 4 we focus on solutions with given initial values and study the vanishing order locally, whereas in Section 5 we generalize the vanishing order to arbitrary initial values. For the global situation, Proposition 5.3 and 5.4 show that a large class of AODEs indeed satisfy our sufficient conditions.

An interesting application of our method is to give an explicit statement of a result by Hurwitz in (Hur89). Hurwitz proved that for every formal power series solution of an AODE there exists a large enough positive integer N such that coefficients of order greater than N are determined by a recursion formula. Our result can be used to determine a sharp upper bound for such an N and the corresponding recursion formula (compare with (Hoe14)).

2 Implicit Function Theorem for AODEs

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We consider an algebraic ordinary differential equation (AODE) of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{1}$$

where $n \in \mathbb{N}$ and F is a differential polynomial in $\mathbb{K}[x]\{y\}$ of order n. For simplicity we may also write (1) as $F(y) = 0$ and call n the *order* of (1).

Let $\mathbb{K}[[x]]$ be the ring of formal power series with respect to x around the origin. For each formal power series $f \in \mathbb{K}[[x]]$ and $k \in \mathbb{N}$, we use the notation $[x^k]f$ to refer to the coefficient of x^k in f. The coefficient of x^k in a formal power series can be expressed by means of the constant coefficient of its k-th formal derivative, as stated in the following lemma (see (KP10, Theorem 2.3, page 20)).

Lemma 2.1. *Let $f \in \mathbb{K}[[x]]$ and $k \in \mathbb{N}$. Then $[x^k]f = [x^0]\left(\frac{1}{k!} f^{(k)}\right)$.*

Let $n \in \mathbb{N}$. We define the projection map π_n as

$$\begin{array}{cccc} \pi_n : & \mathbb{K}^{\mathbb{N}} & \longrightarrow & \mathbb{K}^{n+1} \\ & (c_0, c_1, \ldots) & \longmapsto & (c_0, \ldots, c_n). \end{array}$$

Assume that $y(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$ is a formal power series solution of $F(y) = 0$ around the origin, where $c_i \in \mathbb{K}$ are unknowns and F is of order n. Set

$\mathbf{c} = (c_0, c_1, \dots) \in \mathbb{K}^{\mathbb{N}}$. By Lemma 2.1, we know that $F(y(x)) = 0$ if and only if

$$[x^0](F^{(k)}(y(x))) = F^{(k)}(0, \pi_{n+k}(\mathbf{c})) = 0 \text{ for each } k \geq 0.$$

The above fact motivates the following definition.

Definition 2.2. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$.*

1. *Assume that $\boldsymbol{c} = (c_0, c_1, \dots) \in \mathbb{K}^{\mathbb{N}}$ is an infinite tuple of indeterminates and $m \in \mathbb{N}$. We call the ideal*

$$\mathcal{J}_m(F) = \langle F(0, \pi_n(\boldsymbol{c})), \dots, F^{(m)}(0, \pi_{n+m}(\boldsymbol{c}))\rangle \subseteq \mathbb{K}[c_0, \dots, c_{n+m}]$$

   the m-th jet ideal of F. Moreover, we denote the zero set of $\mathcal{J}_m(F)$ by $\mathbb{V}(\mathcal{J}_m(F)) \subseteq \mathbb{K}^{n+m+1}$.

2. *Let $k \in \mathbb{N}$. Assume that $\tilde{\boldsymbol{c}} = (c_0, c_1, \dots, c_k) \in \mathbb{K}^{k+1}$ and $\tilde{y}(x) = \sum_{i=0}^{k} \frac{c_i}{i!} x^i$. We say that $\tilde{\boldsymbol{c}}$, or $\tilde{y}(x)$, can be extended to a formal power series solutions of $F(y) = 0$ if there exists $y(x) \in \mathbb{K}[[x]]$ such that $F(y(x)) = 0$ and*

$$y(x) \equiv \tilde{y}(x) \mod x^{k+1}.$$

By the above definition, we know that $y(x) \in \mathbb{K}[[x]]$ is a solution of $F(y) = 0$ if and only if $\pi_{n+k}(\mathbf{c}) \in \mathbb{V}(\mathcal{J}_k(F))$ for each $k \geq 0$, where $\mathbf{c} \in \mathbb{K}^{\mathbb{N}}$ denotes the infinite tuple of coefficients of $y(x)$. Since it is impossible to check the latter relation for each $k \in \mathbb{N}$, we need to find an upper bound k_0 such that $\pi_{n+k}(\mathbf{c}) \in \mathbb{V}(\mathcal{J}_k(F))$ for each $k \geq k_0$ or $\pi_{n+k_0}(\mathbf{c}) \notin \mathbb{V}(\mathcal{J}_{k_0}(F))$. The following example shows that in general this bound, if it exists, can be arbitrarily large.

Example 2.3. For each $m \in \mathbb{N}^*$, consider the AODE

$$F = \frac{(y' + y)^2}{2} + x^{2m} = 0$$

with the initial tuple $(c_0, c_1) = (0, 0) \in \mathbb{V}(\mathcal{J}_0(F))$. It is straightforward to see that $((y' + y)^2)^{(k)}$ is a $\mathbb{K}$-linear combination of terms of the form

$$(y^{(i)} + y^{(i+1)})(y^{(j)} + y^{(j+1)}) \text{ with } i + j = k \text{ and } i, j \geq 0.$$

Therefore, we have that $(c_0, \dots, c_{k+1}) \in \mathbb{V}(\mathcal{J}_k(F))$ for all $0 \leq k < 2m$ if and only if $c_1 = \dots = c_{k+1} = 0$. However,

$$F^{(2m)}(0, \dots, 0, c_{2m+1}) = (2m)! \neq 0$$

for every c_{2m+1} and $(c_0, c_1) = (0, 0)$ cannot be extended to a formal power series solution of $F(y) = 0$.

Note that Example 2.3 does not exclude the existence of such an upper bound in terms of the coefficients and exponents of F. The following Conjecture would allow us to determine the existence of a formal power series by extending a given initial tuple.

Conjecture 2.4. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. The descending chain*

$$\mathbb{V}(\mathcal{J}_0(F)) \supseteq \pi_n(\mathbb{V}(\mathcal{J}_1(F))) \supseteq \pi_n(\mathbb{V}(\mathcal{J}_2(F))) \supseteq \cdots \tag{2}$$

stabilizes, i.e., there exists an $N \in \mathbb{N}$ such that $\pi_n(\mathbb{V}(\mathcal{J}_N(F))) = \pi_n(\mathbb{V}(\mathcal{J}_{N+k}(F)))$ for each $k \geq 0$.

Let us assume that the chain (2) indeed stabilizes for some $N \in \mathbb{N}$. Then

$$\pi_n(\mathbb{V}(\mathcal{J}_N(F))) = \pi_n(\mathbf{PSol}(F)), \tag{3}$$

where $\mathbf{PSol}(F)$ denotes the set of tuples of coefficients for formal power series solutions of $F(y) = 0$, *i.e.*,

$$\mathbf{PSol}(F) = \left\{ (c_0, c_1, \ldots) \in \mathbb{K}^{\mathbb{N}} \;\middle|\; F\left(\sum_{i \geq 0} \frac{c_i}{i!}\, x^i\right) = 0 \right\}.$$

Equation (3) implies that every $(c_0, \ldots, c_n) \in \mathbb{K}^{n+1}$ can be extended to a formal power series solution of $F(y) = 0$ if and only if there exist some $c_{n+1}, \ldots, c_{n+N} \in \mathbb{K}$ such that $F^{(k)}(0, c_0, \ldots, c_{n+k}) = 0$ for every $0 \leq k \leq N$, which can be checked algorithmically.

However, up to our knowledge, there is neither proof nor counter examples for Conjecture 2.4. We have to come up with other ideas to calculate formal power series solutions of AODEs.

Let us first recall a lemma which shows that for $k \geq 1$ the highest occurring derivative appears linearly in the k-th derivative of F with respect to x (see (Rit50, page 30)).

Lemma 2.5. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \geq 0$. Then for each $k \geq 1$, there exists a differential polynomial $R_k \in \mathbb{K}[x]\{y\}$ of order at most $n + k - 1$ such that*

$$F^{(k)} = S_F \cdot y^{(n+k)} + R_k, \tag{4}$$

where $S_F = \frac{\partial F}{\partial y^{(n)}}$ is the separant of F.

Based on Lemma 2.5 and the reasonings in the beginning of the section, we have the following proposition.

Proposition 2.6. [1] *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Assume that $\tilde{\mathbf{c}} = (c_0, c_1, \ldots, c_n) \in \mathbb{V}(\mathcal{J}_0(F))$ and $S_F(\tilde{\mathbf{c}}) \neq 0$. For $k > 0$, set*

$$c_{n+k} = -\frac{R_k(0, c_0, \ldots, c_{n+k-1})}{S_F(\tilde{\mathbf{c}})},$$

where R_k is specified in Lemma 2.5. Then $y(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i$ is a solution of $F(y) = 0$.

In the above proposition, if the initial value $\tilde{\mathbf{c}}$ vanishes at the separant of F, we may expand R_k in Lemma 2.5 further in order to find formal power series solutions, as the following example illustrated.

Example 2.7. Consider the AODE

$$F = x\, y' + y^2 - y - x^2 = 0.$$

Since $S_F = x$, we cannot apply Proposition 2.6 to get a formal power series solution around the origin. Instead, we observe that for $k \geq 1$,

$$F^{(k)} = x\, y^{(k+1)} + (2y + k - 1)\, y^{(k)} + \tilde{R}_{k-1}, \tag{5}$$

where $\tilde{R}_{k-1} \in \mathbb{K}[x]\{y\}$ is of order $k - 1$.

Assume that $y(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$ is a formal power series solution of $F(y) = 0$, where $c_i \in \mathbb{K}$ are to be determined. >From $[x^0]F(y(x)) = 0$, we have that $c_0^2 - c_0 = 0$.

If we take $c_0 = 1$, then we can deduce from (5) that for each $k \geq 1$,

$$[x^0]F^{(k)}(y(x)) = (k+1)\, c_k + \tilde{R}_{k-1}(0, 1, c_1, \ldots, c_{k-1}) = 0.$$

Thus,

$$c_k = -\frac{\tilde{R}_{k-1}(0, 1, c_1, \ldots, c_{k-1})}{k+1},$$

where $k \geq 1$. Therefore, we derive uniquely a formal power series solution of $F(y) = 0$ with $c_0 = 1$.

If we take $c_0 = 0$, then we observe that

$$[x^0]F'(y(x)) = 2c_0\, c_1 = 0.$$

It implies that there is no constraint for c_1 in the equation $[x^0]F'(y(x)) = 0$. For $k \geq 2$, it follows from (5) that

$$[x^0]F^{(k)}(y(x)) = (k-1)\, c_k + \tilde{R}_{k-1}(0, 0, c_1, \ldots, c_{k-1}) = 0.$$

[1] This proposition is sometimes called Implicit Function Theorem for AODEs as a folklore.

Thus,

$$c_k = -\frac{\tilde{R}_{k-1}(0, 0, c_1, \ldots, c_{k-1})}{k-1},$$

where $k \geq 2$. Therefore, we derive uniquely a formal power series solutions of $F(y) = 0$ with $c_0 = 0$ for every $c_1 \in \mathbb{K}$. In other words, we can compute a general solution $y(x) \in \mathbb{K}(c_1)[[x]] \setminus \mathbb{K}[[x]]$, where c_1 is a new indeterminate.

In the above example, we expand $F^{(k)}$ up to its second highest derivative $y^{(k)}$. By evaluating at the given initial tuple, the coefficients of this term are non-zero so that all the other coefficients of formal power series solutions of $F(y) = 0$ can be constructed recursively. In the forthcoming sections, we will develop this idea in a systematical way.

3 Generalized separants

In (Hur89; Lim15) the author presents an expansion formula for derivatives of F with respect to x showing that not only the highest occurring derivative appears linearly, also the second-highest one, third-highest one, and so on. This is a refinement of Lemma 2.5 and (DL84)[Lemma 2.2].

Throughout the section we will use the notation $F(\mathbf{c}) = F(0, c_0, c_1, \ldots)$, where $F \in \mathbb{K}[x]\{y\}$, and $\mathbf{c} = (c_0, c_1, \ldots)$ is a infinite tuple of indeterminates or elements in $\mathbb{K}$.

Definition 3.1. *For a differential polynomial $F \in \mathbb{K}[x]\{y\}$ of order $n \in \mathbb{N}$ and $m, k, i \in \mathbb{N}$, we define*

$$f_{n-i} = \begin{cases} \frac{\partial F}{\partial y^{(n-i)}}, & i = 0, \ldots, m; \\ 0, & \textit{otherwise}; \end{cases}$$

and

$$S_{F,k,i} = \sum_{j=0}^{i} \binom{k}{j} f_{n-i+j}^{(j)}.$$

We call $S_{F,k,i}$ the generalized separants of F.

Note that $S_{F,k,0}$ coincides with the usual separant $\frac{\partial F}{\partial y^{(n)}}$ of F and the order of $S_{F,k,i}$ is less than or equal to $n + i$.

Theorem 3.2. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Then for each $m \in \mathbb{N}$ and $k > 2m$ there exists a differential polynomial $r_{n+k-m-1}$ with order less than or equal to $n + k - m - 1$ such that*

$$F^{(k)} = \sum_{i=0}^{m} S_{F,k,i}\, y^{(n+k-i)} + r_{n+k-m-1}. \tag{6}$$

Proof. See (Lim15)[Corollary 1]. $\square$

Let $F \in \mathbb{K}[x]\{y\}$ of order n and $m, k \in \mathbb{N}$. We define

$$\mathcal{B}_m(k) = \begin{bmatrix} \binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{m} \end{bmatrix},$$

and

$$\mathcal{S}_{F,m} = \begin{bmatrix} f_n & f_{n-1} & f_{n-2} & \cdots & f_{n-m} \\ 0 & f_n^{(1)} & f_{n-1}^{(1)} & \cdots & f_{n-m+1}^{(1)} \\ 0 & 0 & f_n^{(2)} & \cdots & f_{n-m+2}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & f_n^{(m)} \end{bmatrix},$$

and

$$Y_m = \begin{bmatrix} y^{(m)} \\ y^{(m-1)} \\ \vdots \\ y \end{bmatrix}.$$

We call $\mathcal{S}_{F,m}$ the *m-th separant matrix of F.*

Then we can also represent formula (6) of Theorem 3.2 as

$$F^{(k)} = \mathcal{B}_m(k) \cdot \mathcal{S}_m(F) \cdot Y_m^{(n+k-m)} + r_{n+k-m-1}. \tag{7}$$

It is straightforward to see that for $m \in \mathbb{N}$ and $\mathbf{c} \in \mathbb{K}^{\mathbb{N}}$ the separant matrix $\mathcal{S}_{F,m}(\mathbf{c}) = 0$ if and only if $S_{F,k,i}(\mathbf{c}) = 0$ holds for all $0 \leq i \leq m$ and $k \in \mathbb{N}$.

4 Local vanishing order

In this section, we consider the problem of deciding when a solution modulo a certain power of x of a given AODE can be extended to a full formal power series solution. By using Theorem 3.2, we present a partial answer for this problem. In particular, given a certain number of coefficients satisfying some additional assumptions, we propose an algorithm to check whether there is a formal power series solution whose first coefficients are the given ones, and in the affirmative case, compute all of them (see Theorem 4.4 and Algorithm 1).

Let us start with a technical lemma which we will frequently use later.

Lemma 4.1. *Let $m, n \in \mathbb{N}$ and $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n and $\boldsymbol{c} = (c_0, c_1, \ldots)$ be a infinite tuple of indeterminates. Assume that the generalized separants $S_{F,k,i}(\boldsymbol{c}) = 0$ for all $0 \leq i < m$. Then the differential polynomial $F^{(k)}(\boldsymbol{c})$ involves only*

1. $c_0, \ldots, c_{n+\lfloor k/2 \rfloor}$ *for* $0 \leq k \leq 2m$*;*

2. $c_0, \ldots, c_{n+k-m}$ *for* $k > 2m$*.*

Proof. Assume that $0 \leq k \leq 2m$. Set $\tilde{m} = k - \lfloor k/2 \rfloor$. Then $k > 2\tilde{m} - 1$. By assumption and Theorem 3.2, we have

$$F^{(k)}(\mathbf{c}) = r_{n+k-\tilde{m}}(\mathbf{c}),$$

where $r_{n+k-\tilde{m}}$ is a differential polynomial of order at most $n + k - \tilde{m} = n + \lfloor k/2 \rfloor$.Thus, item 1 follows.

Let $k > 2m$. By (7) and the assumption, we get

$$F^{(k)}(\mathbf{c}) = S_{F,k,m}(\mathbf{c})\, c_{n+k-m} + r_{n+k-m-1}(\mathbf{c}),$$

and thus item 2 holds. $\square$

Definition 4.2. *Let* $m, n \in \mathbb{N}$ *and* $F \in \mathbb{K}[x]\{y\}$ *be a differential polynomial of order* n*. Let* $\tilde{\mathbf{c}} = (c_0, \ldots, c_{n+m}) \in \mathbb{K}^{n+m+1}$*. We say that* F *has vanishing order* m *at* $\mathbf{c}$ *if the following conditions hold:*

1. $\mathcal{S}_{F,i}(\tilde{\mathbf{c}}) = 0$ *for all* $0 \leq i < m$*, and* $\mathcal{S}_{F,m}(\tilde{\mathbf{c}}) \neq 0$*;*

2. $\tilde{\mathbf{c}} \in \mathbb{V}(\mathcal{J}_{2m}(F))$*.*

As a consequence of Lemma 4.1 and item 1 of the above definition, $\mathbb{V}(\mathcal{J}_{2m}(F))$ can be seen as a subset of $\mathbb{K}^{n+m+1}$ and therefore item 2 is well-defined.

Remark 4.3. *Recall that a solution of an AODE* $F(y) = 0$ *of order* n *is called non-singular if it does not vanish at the separant* $S_F = \frac{\partial F}{\partial y^{(n)}}$*. If a formal power series* $y(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$ *is a non-singular solution of* $F(y) = 0$*, then there exists an* $m \in \mathbb{N}$ *such that* $S_F^{(m)}(0, c_0, \ldots, c_{n+m}) \neq 0$*. Therefore,* $\mathcal{S}_{F,m}(0, c_0, \ldots, c_{n+m}) \neq 0$ *and* F *has a vanishing order of at most* m *at* $(c_0, \ldots, c_{n+m})$*.*

Let $m, n \in \mathbb{N}$ and $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Assume that $\mathbf{c} = (c_0, \ldots, c_{n+m}) \in \mathbb{K}^{n+m+1}$ and F has vanishing order m at $\mathbf{c}$. We regard $S_{F,t,m}(\mathbf{c})$ as a polynomial in t and denote

$$\mathbf{r}_{F,\mathbf{c}} = \text{the number of integer roots of } S_{F,t,m}(\mathbf{c}) \text{ which are greater than } 2m,$$

$$\mathbf{q}_{F,\mathbf{c}} = \begin{cases} \text{the largest integer root of } S_{F,t,m}(\mathbf{c}), & \text{if } \mathbf{r}_{F,\mathbf{c}} \geq 1, \\ 2m, & \text{otherwise.} \end{cases}$$

Theorem 4.4. *Let* $F \in \mathbb{K}[x]\{y\}$ *be of vanishing order* $m \in \mathbb{N}$ *at* $\mathbf{c} \in \mathbb{K}^{n+m+1}$*.*

1. *Then* $\mathbf{c}$ *can be extended to a formal power series solution of* $F(y) = 0$ *if and only if it can be extended to a zero of* $\mathbb{V}(\mathcal{J}_{\mathbf{q}_{F,\mathbf{c}}}(F))$.

2. *Let*

$$\mathcal{V}_{\mathbf{c}}(F) = \pi_{n+\mathbf{q}_{F,\mathbf{c}}-m}(\{\tilde{\mathbf{c}} \in \mathbb{V}(\mathcal{J}_{\mathbf{q}_{F,\mathbf{c}}}(F)) \mid \pi_{n+m}(\tilde{\mathbf{c}}) = \mathbf{c}\}).$$

Then $\mathcal{V}_{\tilde{\mathbf{c}}}(F)$ *is an affine variety of dimension at most* $\mathbf{r}_{F,c}$. *Moreover, each point of it can be uniquely extended to a formal power series solution of* $F(y) = 0$.

Proof. 1. Let $\bar{\mathbf{c}} = (c_0, c_1, \ldots) \in \mathbb{K}^{\mathbb{N}}$ be such that $\pi_{n+m}(\bar{\mathbf{c}}) = \mathbf{c}$, where c_k is to be determined for $k > n + m$. Recall that $\sum_{i\geq 0} \frac{c_i}{i!} x^i$ is a solution of $F(y) = 0$ if and only if $F^{(k)}(\bar{\mathbf{c}}) = 0$ for each $k > 2m$. Since F has a vanishing order m, and by Theorem 3.2, there is a differential polynomial $r_{n+k-m-1}$ of order at most $n + k - m - 1$ such that

$$F^{(k)}(\bar{\mathbf{c}}) = S_{F,k,m}(\mathbf{c})\, c_{n+k-m} + r_{n+k-m-1}(\bar{\mathbf{c}}) = 0. \tag{8}$$

If $\mathbf{c}$ can be extended to a solution of $F(y) = 0$, then it follows from Definition 2.2 that $\mathbf{c}$ can be extended to a zero of $\mathcal{J}_{\mathbf{q}_{F,\mathbf{c}}}(F)$. Vice versa, if $\mathbf{c}$ can be extended to a zero of $\mathcal{J}_{\mathbf{q}_{F,\mathbf{c}}}(F)$, then there exist $c_{n+m+1}, \ldots, c_{n+\mathbf{q}_{F,\mathbf{c}}-m} \in \mathbb{K}$ such that equation (8) holds for $k = 2m, \ldots, \mathbf{q}_{F,\mathbf{c}}$. For $k > \mathbf{q}_{F,\mathbf{c}}$, we set

$$c_{n+k-m} = -\frac{r_{n+k-m-1}(0, c_0, \ldots, c_{n+k-m-1})}{S_{F,k,m}(\mathbf{c})} \tag{9}$$

and thus$y(x) = \sum_{i\geq 0} \frac{c_i}{i!} x^i$ is a solution of $F(y) = 0$.

2. By item 1, $\mathcal{V}_{\mathbf{c}}(F)$ is the set of points satisfying

(i) $\tilde{c}_k = c_k \quad$ for $0 \leq k \leq n + m$;

(ii) $S_{F,k,m}(\mathbf{c})\, \tilde{c}_{n+k-m} + r_{n+k-m-1}(\tilde{c}_0, \ldots, \tilde{c}_{n+k-m-1}) = 0 \quad$ for $n + m < k \leq \mathbf{q}_{F,\mathbf{c}}$,

and therefore it is an affine variety.

If $\mathbf{r}_{F,\mathbf{c}} = 0$, it is straightforward to verify that $\mathcal{V}_{\mathbf{c}}(F)$ contains at most one point and hence it is the empty set or has dimension zero.

Assume that $\mathbf{r}_{F,\mathbf{c}} \geq 1$. Let $k_1 < \cdots < k_{\mathbf{r}_{F,\mathbf{c}}} = \mathbf{q}_{F,\mathbf{c}}$ be integer roots of $S_{F,t,m}(\mathbf{c})$ which are greater than $2m$. If $k \notin \{k_1, \ldots, k_{\mathbf{r}_{F,\mathbf{c}}}\}$, then it follows from (8) that $\tilde{c}_{n+k-m}$ is uniquely determined from the previous coefficients and

$$\begin{array}{cccc} \phi: & \mathcal{V}_{\mathbf{c}}(F) & \longrightarrow & \mathbb{K}^{\mathbf{r}_{F,\mathbf{c}}} \\ & \tilde{\mathbf{c}} & \longmapsto & (\tilde{c}_{n+k_1-m}, \ldots, \tilde{c}_{n+k_{\mathbf{r}_{F,c}}-m}) \end{array}$$

defines an injective map. Therefore, we conclude that $\mathcal{V}_{\mathbf{c}}(F)$ is of dimension at most $\mathbf{r}_{F,\mathbf{c}}$. Moreover, it follows from item 1 that each point of $\mathcal{V}_{\mathbf{c}}(F)$ can be uniquely extended to a formal power series solution of $F(y) = 0$. □

The proof of the above theorem is constructive. More precisely, if a tuple $\mathbf{c} \in \mathbb{K}^{n+m+1}$ satisfies the condition that F has vanishing order m at $\mathbf{c}$, then the proof gives an algorithm to decide whether $\mathbf{c}$ can be extended to a formal power series solution of $F(y) = 0$ or not, and in the affirmative case determine all of them. We summarize them as in Algorithm 1.

Algorithm 1 DirectMethodLocal

Input: $\ell \in \mathbb{N}$, $\mathbf{c} = (c_0, \dots, c_{n+m}) \in \mathbb{K}^{n+m+1}$, and a differential polynomial F of order n which has vanishing order m at $\mathbf{c}$.

Output: All formal power series solutions of $F(y) = 0$, with $\mathbf{c}$ as initial tuple, described as the truncation of the series up to order ℓ including a finite number of indeterminates and the algebraic conditions on these indeterminates. The truncations are in one-to-one correspondence to the formal power series solutions.

1: Compute $S_{F,k,m}(\mathbf{c}), \mathbf{r}_{F,\mathbf{c}}, \mathbf{q}_{F,\mathbf{c}}$ and the defining equations of $\mathcal{V}_{\mathbf{c}}(F)$:

(i) $\tilde{c}_k = c_k \quad$ for $0 \leq k \leq n+m$;

(ii) $S_{F,k,m}(\mathbf{c})\,\tilde{c}_{n+k-m} + r_{n+k-m-1}(\tilde{c}_0, \dots, \tilde{c}_{n+k-m-1}) = 0 \quad$ for $n+m < k \leq \mathbf{q}_{F,\mathbf{c}}$,

where $\tilde{c}_k$'s are indeterminates.

2: Check whether $\mathcal{V}_{\mathbf{c}}(F)$ is empty or not by using Gröbner bases.

3: **if** $\mathcal{V}_{\mathbf{c}}(F) = \emptyset$ **then**

4: Output the string "$\mathbf{c}$ can not be extended to a formal power series solution of $F(y) = 0$".

5: **else**

6: Compute $\tilde{c}_{\mathbf{q}_{F,\mathbf{c}}}, \dots, \tilde{c}_\ell$ by using (9).

7: **return** $\sum_{i=0}^{\ell} \frac{\tilde{c}_i}{i!} x^i$ and $\mathcal{V}_{\mathbf{c}}(F)$.

8: **end if**

The termination of the above algorithm is evident. The correctness follows from Theorem 4.4.

Example 4.5. Consider the following AODE of order two:

$$F = x\,y'' - 3y' + x^2 y^2 = 0.$$

Let $\mathbf{c} = (c_0, 0, 0, 2c_0^2) \in \pi_3(\mathbb{V}(\mathcal{J}_2(F)))$, where c_0 is an arbitrary constant in $\mathbb{K}$. One can verify that each point of $\pi_3(\mathbb{V}(\mathcal{J}_2(F)))$ is of the form $\mathbf{c}$. A direct calculation shows that F has vanishing order 1 at $\mathbf{c}$. Moreover, we have that $S_{F,t,m}(\mathbf{c}) = t$ and $\mathbf{q}_{F,\mathbf{c}} = 0$. Thus, it follows that

$$\mathcal{V}_{\mathbf{c}}(F) = \{\tilde{\mathbf{c}} = (c_0, 0, 0, 2c_0^2, c_4) \in \mathbb{K}^5 \mid c_4 \in \mathbb{K}\}.$$

So, the dimension of $\mathcal{V}_{\mathbf{c}}(F)$ is equal to one and the corresponding formal power series solutions are

$$y(x) \equiv c_0 + \frac{c_0^2}{3}\,x^3 + \frac{c_4}{24}\,x^4 - \frac{c_0^3}{18}\,x^6 - \frac{c_0\,c_4}{252}\,x^7 - \frac{c_0^2\,c_4}{3024}\,x^{10} \mod x^{11}. \tag{10}$$

Above all, the set of formal power series solutions of $F(y) = 0$ at the origin is in bijection with $\mathbb{K}^2$ and can be represented as in (10).

5 Global vanishing order

The input specification of Algorithm 1 is that the given initial tuple $\mathbf{c}$ is of length $n+m+1$ and that the differential polynomial F has vanishing order m at $\mathbf{c}$. In general, the natural number m can be arbitrarily large. In this section, we give a necessary and sufficient condition for differential polynomials for which the existence for an upper bound of m is guaranteed. If this condition is satisfied, Algorithm 1 can be applied to every initial tuple of appropriate length.

Definition 5.1. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Assume that $\boldsymbol{c} = (c_0, c_1, \ldots)$ is an infinite tuple of indeterminates and $m \in \mathbb{N}$.*

1. *We define $\mathcal{I}_m(F) \subseteq \mathbb{K}[c_0, \ldots, c_{n+m}]$ to be the ideal generated by the entries of the separant matrix $\mathcal{S}_{F,m}(\boldsymbol{c})$.*

2. *We say that F has vanishing order m if m is the smallest natural number such that $1 \in \mathcal{I}_m(F) + \mathcal{J}_{2m}(F) \subseteq \mathbb{K}[c_0, \ldots, c_{n+2m}]$, where $\mathcal{J}_{2m}(F)$ is the $2m$-th jet ideal of F. If there does not exist such $m \in \mathbb{N}$, then we define the vanishing order of F to be ∞.*

Example 5.2. Consider the AODE from Example 2.7:

$$F = x\,y' + y^2 - y - x^2 = 0.$$

By computation, we find that $1 \notin \langle c_0^2 - c_0 \rangle_{\mathbb{K}[c_0]} = \mathcal{I}_0 + \mathcal{J}_0$. Furthermore, we have $1 \in \mathcal{I}_1 + \mathcal{J}_2$ because $(S_F)' = 1$. Therefore, the differential polynomial F has vanishing order 1.

Proposition 5.3. *Every differential polynomial of the form*

$$A(x)\,y^{(m)} + B(x, y, \ldots, y^{(m-1)}, y^{(m+1)}, \ldots, y^{(n)})$$

has the vanishing order of at most $\deg_x(A) + n - m$.

Proof. The separant matrix $\mathcal{S}_{F,\deg_x(A)+n-m}$ has the non-zero entry

$$f_{n-m}^{(\deg_x(A))} = \deg_x(A)!\ \mathrm{lc}(A).$$

Hence, $1 \in \mathcal{I}_{\deg_x(A)} \subseteq \mathcal{I}_{\deg_x(A)} + \mathcal{J}_{2\deg_x(A)}$. ☐

Proposition 5.3 is a generalization of (Lim15)[Corollary 2] where only the case $A(x) \in \mathbb{K}$ is treated. The following proposition gives a characterization of differential polynomials with finite vanishing order.

Proposition 5.4. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Then F has finite vanishing order if and only if the differential system*

$$F = \frac{\partial F}{\partial y} = \cdots = \frac{\partial F}{\partial y^{(n)}} = 0 \tag{11}$$

has no solution in $\mathbb{K}[[x]]$. In particular, if the differential ideal $\left[F, \frac{\partial F}{\partial y}, \ldots, \frac{\partial F}{\partial y^{(n)}}\right]$ in $\mathbb{K}(x)\{y\}$ contains 1, then F has finite vanishing order.

Proof. Assume that the system (11) has a solution $y(x) = \sum_{i\geq 0} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$. Then for every $m \in \mathbb{N}$ we have $(c_0, \ldots, c_{n+m}) \in \mathbb{V}(\mathcal{I}_m(F) + \mathcal{J}_{2m}(F))$. Therefore, it follows from Hilbert's weak Nullstellensatz that $1 \notin \mathcal{I}_m(F) + \mathcal{J}_{2m}(F)$ and thus F does not have finite vanishing order.

Conversely, assume that F does not have finite vanishing order, *i.e.*, for each $k \in \mathbb{N}$ we have $1 \notin \mathcal{I}_{k+n}(F) + \mathcal{J}_{2(k+n)}(F)$ and the ideals have a common root $\mathbf{c} = (c_0, \ldots, c_{3n+2k}) \in \mathbb{K}^{3n+2k+1}$. In particular, we have for all $i = 0, \ldots, k$

$$F^{(i)}(\mathbf{c}) = \left(\frac{\partial F}{\partial y}\right)^{(i)}(\mathbf{c}) = \ldots = \left(\frac{\partial F}{\partial y^{(n)}}\right)^{(i)}(\mathbf{c}) = 0, \tag{12}$$

and therefore, by Lemma 2.1, for $\tilde{y}(x) = \sum_{i=0}^{3n+2k} \frac{c_i}{i!} x^i$ and every $i = 0, \ldots, k$ also

$$[x^i]F(\tilde{y}(x)) = [x^i]\frac{\partial F}{\partial y}(\tilde{y}(x)) = \cdots = [x^i]\frac{\partial F}{\partial y^{(n)}}(\tilde{y}(x)) = 0.$$

Thus, $\tilde{y}(x)$ is a solution of $F = \frac{\partial F}{\partial y} = \cdots = \frac{\partial F}{\partial y^{(n)}} = 0$ modulo x^k. Due to the Strong Approximation Theorem (DL84, Theorem 2.10), we conclude that the system (11) admits a solution in $\mathbb{K}[[x]]$ and the equivalence is proven.

Assume that $1 \in \left[F, \frac{\partial F}{\partial y}, \ldots, \frac{\partial F}{\partial y^{(n)}}\right]$. Then system (11) does not have a solution in $\mathbb{K}[[x]]$ and thus F has finite vanishing order. ☐

Remark 5.5. *To test whether $1 \in \left[F, \frac{\partial F}{\partial y}, \ldots, \frac{\partial F}{\partial y^{(n)}}\right]$ or not, one can use the* "`RosenfeldGroebner`" *command in the Maple package* `DifferentialAlgebra`.

Below are examples of differential polynomials with vanishing order for each $m \in \mathbb{N} \cup \{\infty\}$.

Example 5.6. Consider the following AODE

$$F = xyy'' - yy' + x(y')^2 = 0.$$

A direct computation implies that for each $m \in \mathbb{N}$, we have

$$\mathcal{I}_m(F) + \mathcal{J}_{2m}(F) \subseteq \langle c_0, c_1, \ldots, c_{2m+1} \rangle.$$

Therefore, it follows from item 2 of Definition 5.1 that F is differential polynomial with infinite vanishing order. Note that in this case $F = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y''} = 0$ has a common solution $y(x) = 0$.

Example 5.7. Assume that $m \in \mathbb{N}$. Consider the AODE

$$F = \frac{(y' + y)^2}{2} + x^{2m} = 0.$$

For $m = 0$, it is straightforward to see that F has vanishing order 0.

Let $m > 0$. By computation, we find that $\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y} = y' + y$. Therefore, we have that

$$\mathcal{I}_m(F) = \langle c_1 + c_0, \ldots, c_{m+1} + c_m \rangle. \tag{13}$$

For each $k \geq 0$, it is straightforward to see that $((y' + y)^2)^{(k)}$ is a $\mathbb{K}$-linear combination of terms of the form $(y^{(i)} + y^{(i+1)})(y^{(j)} + y^{(j+1)})$ with $i + j = k$, and $i, j \geq 0$. Therefore, we conclude that for each $0 \leq k \leq m - 1$, the jet ideal $\mathcal{J}_{2k}(F)$ is contained in $\mathcal{I}_m(F)$. It implies that

$$\mathcal{I}_k(F) + \mathcal{J}_{2k}(F) \subseteq \mathcal{I}_m(F) \quad \text{for } 0 \leq k \leq m - 1.$$

By (13) and the above formula, we have

$$1 \notin \mathcal{J}_{2k}(F) + \mathcal{I}_k(F) \quad \text{for } 0 \leq k \leq m - 1.$$

Furthermore, we have that

$$F^{(2m)}(0, c_1, \ldots, c_{1+2m}) \equiv (2m)! \mod \mathcal{I}_m(F).$$

Thus, it follows that

$$1 \in \mathcal{I}_m(F) + \mathcal{J}_{2m}(F)$$

and by definition, F has vanishing order m.

The following theorem is a generalization of Proposition 2.6.

Theorem 5.8. *Let $F \in \mathbb{K}[x]\{y\}$ be of order n with vanishing order $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathbb{V}(\mathcal{J}_{2m}(F))$.*

1. *There exists $i \in \{0, \dots, m\}$ such that F has vanishing order i at $\pi_{n+i}(\boldsymbol{c})$.*

2. *Let $M = \max\{2m + i, \boldsymbol{q}_{F,\tilde{\mathbf{c}}}\}$. Then $\boldsymbol{c}$ can be extended to a formal power series solution of $F(y) = 0$ if and only if it can be extended to a zero point of $\mathcal{J}_M(F)$.*

3. *Let*
$$\mathcal{V}_{\boldsymbol{c}}(F) = \pi_{n+M-i}(\{\tilde{\mathbf{c}} \in \mathbb{V}(\mathcal{J}_M(F)) \mid \pi_{n+2m}(\tilde{\mathbf{c}}) = \boldsymbol{c}\}).$$
Then $\mathcal{V}_{\boldsymbol{c}}(F)$ is an affine variety of dimension at most $\boldsymbol{r}_{F,\tilde{\mathbf{c}}}$. Moreover, each point of it can be uniquely extended to a formal power series solution of $F(y) = 0$.

Hence, the set of formal power series solutions of $F(y) = 0$ around the origin is in bijection with the set
$$\bigcup_{\boldsymbol{c} \in \mathbb{V}(\mathcal{J}_{2m}(F))} \mathcal{V}_{\boldsymbol{c}}(F).$$

Proof. 1. Since $\mathbf{c} \in \mathbb{V}(\mathcal{J}_{2m}(F))$ and F has vanishing order m, it follows that there exists a minimal $i \in \{0, \dots, m\}$ such that
$$\mathcal{S}_i(F)(\mathbf{c}) \neq 0.$$
By item 2 of Lemma 4.1, we have $F^{(k)}(\mathbf{c}) = 0$ for $k = 0, \dots, 2i$. Taking into account Lemma 4.1, we see that only the first $n+i+1$ coefficients of $\mathbf{c}$ are relevant and therefore F has vanishing order i at $\pi_{n+m}(\mathbf{c})$.

2. and 3. The proofs are literally the same as those in Theorem 4.4. □

As a consequence of Theorem 5.8, for every AODE of finite order, say m, Algorithm 1 can be applied to every given initial tuple $\mathbf{c} \in \mathbb{K}^{n+m+1}$. So we can determine whether there exists a formal power series solution of $F(y) = 0$ extending $\mathbf{c}$ or not and in the affirmative case, all formal power series solutions can be described in finite terms.

Note that in case that F has infinite vanishing order, there may exist an initial tuple $\mathbf{c}$ with arbitrary size such that the set of formal power series solutions of AODE $F(y) = 0$ extending $\mathbf{c}$ can not be described by an algebraic variety as item 3 of Theorem 5.8. For instance, let us consider the differential polynomial $F = xyy'' + yy' - x(y')^2$ from Example 5.6. For every $k \geq 1$, the set of all formal power series solutions of $F = 0$ extending the zero initial tuple $\mathbf{c} = \mathbf{0} \in \mathbb{K}^k$ is the set $\{0\} \cup \{x^r, \,|\, r \in \mathbb{N}, r \geq k\}$.

Theorem 5.8 gives also a positive answer to Conjecture 2.4 under the additional assumption that F has a finite vanishing order. In this case, the upper bound can be given by M as defined in item 2 of Theorem 5.8.

In (DL84), Lemma 2.3 only concerns non-singular formal power series solutions of a given AODE. Our method also can be used to find singular solutions of AODEs defined by differential polynomials with finite vanishing order, as the following example illustrated.

Example 5.9. Consider the AODE

$$F = y'^2 + y' - 2y - x = 0.$$

>From Proposition 5.3 we know that F has vanishing order of at most 1.

Let $\mathbf{c} = (-\frac{1}{8}, -\frac{1}{2}, 0, c_3)$, where c_3 is an arbitrary constant in $\mathbb{K}$. It is straightforward to verify that $\mathbf{c}$ is a zero point of $\mathcal{J}_2(F)$. Furthermore, we have that F has vanishing order 1 at $\tilde{\mathbf{c}} = \pi_2(\mathbf{c}) = (-\frac{1}{8}, -\frac{1}{2}, 0)$. Therefore, we also know that F has indeed vanishing order equal to 1. We find that $S_{F,k,2}(\tilde{\mathbf{c}}) = -2$ and $M = 3$. >From item 2 of Theorem 5.8, we know that $\mathbf{c}$ can be extended into a formal power series solution of $F(y) = 0$ if and only if it can be extended to a zero point of $\mathcal{J}_3(F)$.

By calculation, we see that $\mathbf{c}$ can be extended to a zero point of $\mathcal{J}_3(F)$ if and only if $c_3 = 0$. In the affirmative case, $\mathcal{V}_{\mathbf{c}}(F) = \{\mathbf{c}\}$ and we can use Theorem 3.2 to extend $\mathbf{c}$ to a unique formal power series solution

$$y_1(x) = -\frac{1}{8} - \frac{1}{2}x.$$

It is straightforward to verify that $y_1(x)$ is a singular solution of $F(y) = 0$.

Similarly, let $\tilde{\mathbf{c}} = (-\frac{1}{8}, -\frac{1}{2}, 1, c_3)$, where c_3 is an arbitrary constant in $\mathbb{K}$. Using item 2 of Theorem 5.8, we deduce that $\tilde{\mathbf{c}}$ can be extended into a formal power solution of $F(y) = 0$ if and only if $c_3 = 0$. In the affirmative case, $\mathcal{V}_{\tilde{\mathbf{c}}}(F) = \{\tilde{\mathbf{c}}\}$ and we find that

$$y_2(x) = -\frac{1}{8} - \frac{1}{2}x + \frac{1}{2}x^2$$

is the corresponding solution.

Actually, one can verify that $y_1(x), y_2(x)$ are all the formal power series solutions of $F(y) = 0$ with $[x^0]S_F(y) = 0$. Therefore, the set of formal power series solutions of $F(y) = 0$ at the origin is equal to

$$\{y_1(x), y_2(x)\} \cup \mathcal{S},$$

where

$$\mathcal{S} = \{y \in \mathbb{K}[[x]] \mid F(y) = 0 \text{ and } [x^0]S_F(y) \neq 0\},$$

which can be determined by Proposition 2.6.

The next example illustrates a differential polynomial with vanishing order two.

Example 5.10. Consider the AODE

$$F = x\,(y'' - 1)^2 + (y - x)\,(y' - 1) = 0,$$

which has vanishing order 2.

Let $\mathbf{c}_1 = (100/9, 1, -1/9, 0, -1/120, 0, c_6) \in \mathbb{V}(\mathcal{J}_4)$, where c_6 is an arbitrary constant in $\mathbb{K}$. Furthermore, the differential polynomial F has vanishing order 1 at $\tilde{\mathbf{c}}_1 = \pi_4(\mathbf{c}_1)$. We find that $S_{F,t,1}(\tilde{\mathbf{c}}_1) = \frac{20(2-t)}{9}$ and $\mathbf{q}_{F,\tilde{\mathbf{c}}_1} = 2$. >From item 2 of Theorem 5.8, we see that $\mathbf{c}_1$ can be extended into a formal power solution if and only if it can be extended to a zero point of $\mathcal{J}_5(F)$. However, a direct calculation shows that this is not possible for any $c_6 \in \mathbb{K}$. Therefore, $\mathcal{V}_{\mathbf{c}_1}(F) = \emptyset$.

Let $\mathbf{c}_2 = (0, 0, 1-i, \frac{3(1+i)}{4}, \frac{-3+4i}{8}, \frac{-2-9i}{64}, c_6) \in \mathbb{V}(\mathcal{J}_4)$, where c_6 is an arbitrary constant in $\mathbb{K}$. Furthermore, the differential polynomial F has vanishing order 1 at $\tilde{\mathbf{c}}_2 = \pi_4(\mathbf{c}_2)$. We find that $S_{F,k,1}(\mathbf{c}_2) = -2(3+t)i$ and $\mathbf{q}_{F,\tilde{\mathbf{c}}_2} = 0$. >From item 2 of Theorem 5.8, the initial tuple $\mathbf{c}_2$ can be extended into a formal power solution if and only if it can be extended to a zero point of $\mathcal{J}_5(F)$. We find that this is only possible for $c_6 = \frac{3(47-11i)}{160}$. In this case, $\mathcal{V}_{\mathbf{c}_2}(F) = \{\mathbf{c}_2\}$ and $\mathbf{c}_2$ can be extended uniquely to the formal power series solution

$$y(x) \equiv \frac{1-i}{2}\,x^2 + \frac{1+i}{8}\,x^3 - \frac{3-4i}{192}\,x^4 - \frac{2+9i}{120\cdot 64}\,x^5 + \frac{47-11i}{240\cdot 160}\,x^6 \mod x^7.$$

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