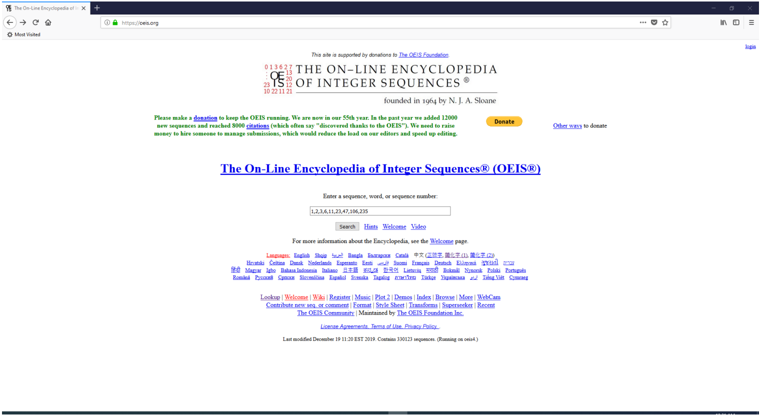
On Sequences Associated to the Invariant Theory of Rank Two Lie Algebras

Yi Zhang

Department of Applied Mathematics
Xi'an Jiaotong-Liverpool University, Suzhou, China

Joint work with Alin Bostan, Jordan Tirrell and Bruce Westbury

The On-Line Encyclopedia of Integer Sequences (OEIS)



OEIS is an online database of integer sequences, such as Fibonacci numbers (A000045), Catalan numbers (A000108).

A family of sequences in OEIS

a (OEIS tag)	0	1	2	3	4	5
A059710	1	0	1	1	4	10
A108307	1	1	2	5	15	51
A108304	1	2	5	15	52	202

Octant Sequences

- Those sequences are associated to the invariant theory of the exceptional simple Lie algebra G_2 of rank 2.
- They can be interpreted as lattice walks restricted to the octant. We call them octant sequences.

Octant sequences

- A059710: enumerates the multiplicities of the trivial representation in the tensor powers of V, which is the 7-D fundamental representation of G_2.

- A108307: enumerates enhanced 3-noncrossing set partitions.

- A108304: enumerates 3-noncrossing set partitions.

(Lin, 2018; Gil and Tirrell, 2019): A108307 and A108304 are related by the binomial transform.

Motivation and Contribution

(Bostan, Tirrell, Westbury and Z., 2019): A059710 and A108307 are also related by the binomial transform.

Mihailovs' conjecture: Let $T_3(n)$ be the n-th term of A059710. Then T_3 is determined by $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and

$$
\begin{aligned}
&14(n+1)(n+2)\,T_3(n) + (n+2)(19n+75)\,T_3(n+1) \\
&+2(n+2)(2n+11)\,T_3(n+2) - (n+8)(n+9)\,T_3(n+3) = 0.
\end{aligned}
$$

(Bostan, Tirrell, Westbury and Z., 2019): Three independent proofs of Mihailovs' conjecture.

- Two proofs are based on binomial relation between A059710 and A108307, together with a result by Bousquet-Mélou and Xin.
- The third one is a direct proof by the method of algebraic residues.

Outline

- binomial relation between the first and second octant sequences
- Three independent proofs of Mihailovs' conjecture

Preliminaries

Definition 1 Let G be a reductive complex algebraic group and let V be a representation of G. The sequence associated to (G, V), denoted $\mathbf{a}_V$, is the sequence whose n-th term is the multiplicity of the trivial representation in the tensor power $\otimes^n V$.

Example 1 Let V be the 7-D fundamental representation of G_2. Then A059710 is the sequence associated with (G_2, V).

Let $\mathbf{a}$ be a sequence with n-th term $a(n)$, the binomial transform of $\mathbf{a}$ is the sequence, denoted $\mathcal{B}\mathbf{a}$, whose n-th term is

$$\sum_{i=0}^{n} \binom{n}{i} a(i).$$

Preliminaries

Lemma 1 Assume $\mathbf{a}_V$ is the sequence associated to (G, V) as specified in Definition 1. Then $\mathbf{a}_{V\oplus\mathbb{C}} = \mathcal{B}\mathbf{a}_V$.

Lemma 2 Assume $\mathbf{a}$ enumerates walks in a lattice, confined to a domain D, using a set of steps S. Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S\coprod\{0\}$.

Lemma 3 Let $G(t)$ be the generating function of $\mathbf{a}$. For $k \in \mathbb{Z}$, denote the generating function of $\mathcal{B}^k\mathbf{a}$ by $\mathcal{B}^k G$. Then

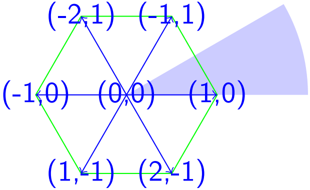
$$(\mathcal{B}^k G)(t) = \frac{1}{1-k\ t} G\left(\frac{t}{1-k\ t}\right).$$

Binomial relation between A059710 and A108307

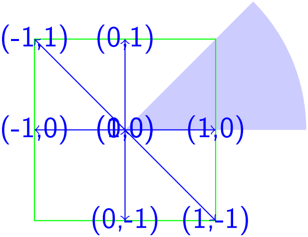
Let V be the 7-D fundamental representation of G_2. Then

- A059710 is the sequence associated to (G_2, V). Let $T_3(n)$ be its n-th term.
- A108307 enumerates enhanced 3-noncrossing set partitions. Let $E_3(n)$ be its n-th term.

In terms of lattice walks, we can interpret T_3 and E_3 as follows:



Steps in weight lattice of G_2



Steps in octant related to $E_3(n)$

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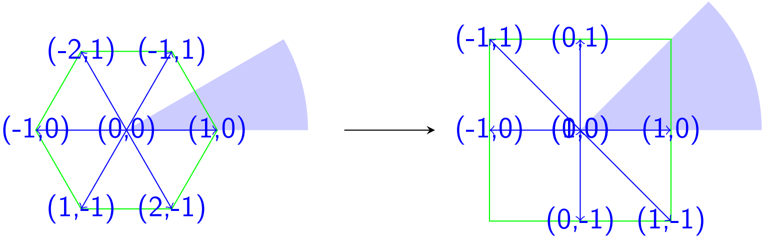

Steps in weight lattice of G_2

Steps in octant related to $E_3(n)$

If we make a linear transformation $(x, y) \to (x, y \pm x)$, then it identifies the six non-zero steps, as well as the two domains.

Binomial relation between A059710 and A108307

Recall: Lemma 2 Assume **a** enumerates walks in a lattice, confined to a domain D, using a set of steps S. Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \coprod \{0\}$.

Binomial relation between A059710 and A108307

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By Lemma 2 and the previous figures, we conclude that E_3 is the binomial transform of T_3.

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Recall: Lemma 1 Assume $\mathbf{a}_V$ is the sequence associated to (G, V) as specified in Definition 1. Then $\mathbf{a}_{V\oplus\mathbb{C}} = \mathcal{B}\mathbf{a}_V$.

Thus, the octant sequences are sequences associated to

$$(G_2, V), \quad (G_2, V \oplus \mathbb{C}), \quad (G_2, V \oplus 2\mathbb{C}).$$

First proof of Mihailovs' conjecture

Mihailovs' conjecture: Let $T_3(n)$ be the n-th term of A059710. Then T_3 is determined by $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and

$$14(n+1)(n+2)\,T_3(n) + (n+2)(19n+75)\,T_3(n+1) \\ +2(n+2)(2n+11)\,T_3(n+2) - (n+8)(n+9)\,T_3(n+3) = 0.$$

(Bousquet-Mélou and Xin, 2005): Let $E_3(n)$ be the n-th term of A108307. Then E_3 is given by $E_3(0) = E_3(1) = 1$, and

$$8(n+3)(n+1)\,E_3(n) + \left(7n^2+53n+88\right)E_3(n+1) \\ - (n+8)(n+7)\,E_3(n+2) = 0.$$

First proof of Mihailovs' conjecture

Recall: We prove that E_3 is the binomial transform of T_3. Thus,

$$T_3(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E_3(k).$$

Set $f(n,k) = (-1)^{n-k}\binom{n}{k} E_3(k)$.

- By Bousquet-Mélou and Xin's result, $f(n,k)$ is holonomic function, which satisfies ordinary difference equations for n and k, respectively.
- Idea: Using creative telescoping method (Zeilberger, 1990), which is an algorithmic approach to compute a differential/difference equation for the integration/sum of holonomic functions, to drive a recurrence equation for T_3.

First proof of Mihailovs' conjecture

- Using the Koutschan's Mathematica package `HolonomicFunctions.m` that implements Chyzak's algorithm for creative telescoping, we find exactly the recurrence equation in Mihailovs' conjecture.

Second proof of Mihailovs' conjecture

Recall: We prove that E_3 is the binomial transform of T_3. Let $\mathcal{T}(t) = \sum_{n\geq 0} T_3(n)t^n$ and $\mathcal{E}(t) = \sum_{n\geq 0} E_3(n)t^n$. Then

$$\mathcal{T}(t) = \frac{1}{1+t} \cdot \mathcal{E}\left(\frac{t}{1+t}\right).$$

- By Bousquet-Mélou and Xin's result, we can derive an ODE for $\mathcal{E}(t)$.
- Using the closure properties of holonomic function (the sum, product and algebraic substitution of holonomic functions is still holonomic), we can derive an ODE for $\mathcal{T}(t)$ and convert it into a linear recurrence for $T_3(n)$, which is exactly the recurrence equation in Mihailovs' conjecture.

Third proof of Mihailovs' conjecture

Idea: In terms of lattice walks, we can interpret $T_3(n)$ to be the constant term of $W\,K^n$, where

$$K = (1 + x + y + x\,y + x^{-1} + y^{-1} + (xy)^{-1}),$$

and

$$W = x^{-2}y^{-3}(x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} + x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2).$$

Let $\mathcal{T}(t) = \sum_{n\geq 0} T_3(n)t^n$. Then $\mathcal{T}(t)$ is the constant coefficient $[x^0y^0]$ of $W/(1-tK)$. In other words, $\mathcal{T}(t)$ is equal to the algebraic residue of $W/(xy - txyK)$, which is proportional to the contour integral of $W/(xy - txyK)$ over a cycle.

Third proof of Mihailovs' conjecture

Using creative telescoping method, we can compute a 6-th order ODE for $\mathcal{T}(t)$. Moreover, by using factorization of differential operators, we can show that $L_3(\mathcal{T}(t)) = 0$, where $\partial = \frac{d}{dt}$ and

$$
\begin{aligned}
L_3 = & t^2 (2t+1)(7t-1)(t+1)\partial^3 + 2t(t+1)\left(63t^2+22t-7\right)\partial^2 + \\
& \left(252t^3+338t^2+36t-42\right)\partial + 28t(3t+4).
\end{aligned}
$$

Converting it into a linear recurrence for $T_3(n)$, we get exactly the recurrence equation in Mihailovs' conjecture.

Summary

- A combinatorial proof of binomial relation between the first and second octant sequences
- Three independent proofs of Mihailovs' conjecture
  - Two proofs are based on binomial relation between the first and second octant sequences
  - A direct proof by the method of algebraic residues

Summary

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Thanks!