① §9.3 Separable Equations

Consider the following separable equation:

$$\frac{dy}{dx} = g(x) f(y) \qquad (*)$$

(Since ~~they~~ the right side of $(*)$ is separable, we can try to ~~slove~~ solve $(*)$ explicitly.)

Assume $f(y) \neq 0$. Set $h(y) = 1/f(y)$. Then we can write $(*)$ as

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \qquad (1)$$

$$\Longleftrightarrow \quad h(y)\,dy = g(x)\,dx$$

$$\Rightarrow \quad \int h(y)\,dy = \int g(x)\,dx \qquad (2)$$

Assume $y = y(x)$ satisfies (2), then

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

$$h(y)\cdot\frac{dy}{dx} = g(x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

Thus, (1) is equivalent to (2).

Ex 0. Solve the logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

$$\frac{dP}{P(1-P/M)} = k\,dt$$

$$\Rightarrow \quad \int \frac{dP}{P(1-P/M)} = \int k\,dt \qquad (3)$$

$$\frac{1}{P(1-P/M)} = \frac{1}{P} + \frac{1}{M-P}$$

$$\int \frac{dP}{P} + \int \frac{dP}{M-P} = \int kt$$

$$\ln|P| - \ln|M-P| = kt + C$$

$$\ln\left|\frac{M-P}{P}\right| = -kt - C$$

$$\left|\frac{M-P}{P}\right| = e^{-kt-C}$$

$$\frac{M-P}{P} = A \cdot e^{-kt} \quad (A = \pm e^{-C})$$

Thus, $P = \dfrac{M}{1 + A \cdot e^{-kt}}$, where $A = \dfrac{M - P_0}{P_0}$.

Ex 1. (a) Solve ~~the~~

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

(b) Find the solution of this equation with the ~~initial~~ initial condition $y(0) = 2$.

(a)

$$y^2 dy = x^2 dx$$

$$\int y^2 dy = \int x^2 dx$$

$$\frac{1}{3} y^3 = \frac{1}{3} x^3 + C$$

$$y^3 = x^3 + 3C$$

$$\Rightarrow y = \sqrt[3]{x^3 + K} \quad (4), \text{ where } K = 3C.$$

(b) Set ~~y=0~~ $x = 0$ in (4), we get

$$2 = y(0) = \sqrt[3]{K}$$

$$\Rightarrow K = 8$$

Thus, $y = \sqrt[3]{x + 8}$.

③ Ex 2. Solve

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$$

$$(2y + \cos y)\,dy = 6x^2\,dx$$

$$\int (2y + \cos y)\,dy = \int 6x^2\,dx$$

$$y^2 + \sin y = 2x^3 + C$$