① §11.7 Strategy for Testing Series

(We have learned various tests for the convergence of series)

The main strategy is to classify the series according to its form.

1. If the series ~~if~~ is of the form $\sum 1/n^p$, i.e. p-series, then it is convergent when $p>1$ and divergent when $p\le 1$.
2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, i.e. geometric series, then it is convergent when $|r|<1$ and divergent when $|r|\ge 1$.
3. If the series is similar to a p-series or a geometric series, then comparison tests shall be considered. If a_n is a rational function or an algebraic function of n, then the series ~~sh~~ should be compared with a p-series

~~Ex 1. $\sum \frac{n-1}{2n+1}$~~

Ex 2. $\sum \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

$$\frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \approx \frac{\sqrt{n^3}}{3n^3} = \frac{1}{3n^{3/2}} \quad \text{as } n\to\infty$$

The comparison series is $\sum \frac{1}{3n^{3/2}}$

Ex 6. $\sum \frac{1}{2+3^n}$

$$\frac{1}{2+3^n} \approx \frac{1}{3^n} \quad \text{as } n\to\infty$$

The comparison series is $\sum \frac{1}{3^n}$

4. If $\lim_{n\to\infty} a_n \ne 0$, then the Test for Divergence should be used.

Ex 1. $\sum \frac{n-1}{2n+1}$

$$\frac{n-1}{2n+1} = \frac{1-\frac{1}{n}}{2+\frac{1}{n}} \to \frac{1}{2} \ne 0 \quad \text{as } n\to\infty$$

Thus, the series is divergent by the Test for Divergence.

②

5. If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test shall be considered.

Ex 4. $\sum (-1)^n \frac{n^3}{n^4+1}$

(i) $b_n = \frac{n^3}{n^4+1} = \frac{\frac{1}{n}}{1+\frac{1}{n^4}} \to 0$ as $n \to \infty$

(ii) let $f(x) = \frac{x^3}{x^4+1}$

$f'(x) = -\frac{x^2(x^4-3)}{(1+x^4)^2} < 0, \quad x^4 > 3$ < $f(x)$ is decreasing on $[3, \infty)$

Thus, $f(n+1) < f(n)$ for $n \geq 3$

i.e. $b_{n+1} < b_n$ for $n \geq 3$

The series is convergent by the Alternating Series Test.

6. Series that involves factorials or other products are often tested using the Ratio Test.

Ex 5. $\sum \frac{2^k}{k!}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \frac{2}{n+1} \to 0 \quad \text{as } n \to \infty$$

The series is absolutely convergent by the Ratio Test

Note: For all rational or algebraic functions of n, $\left| \frac{a_{n+1}}{a_n} \right| \to 1$ as $n \to \infty$. The Ratio Test should not be used for such series.

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

Ex. $\sum \left(\frac{n}{n+1} \right)^{n^2}$

③ $$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}$$
$$= \left(\frac{n}{n+1}\right)^{\frac{n^2}{n}}$$
$$= \left(\frac{n}{n+1}\right)^{n}$$
$$= \frac{1}{\left(\frac{n+1}{n}\right)^{n}}$$
$$= \frac{1}{\left(1+\frac{1}{n}\right)^{n}} \to \frac{1}{e} \text{ as } n \to \infty$$

Since $\frac{1}{e} < 1$, the series is convergent by the Root Test.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)dx$ is easily evaluated, then the Integral Test is effective. (assme the hypothesis of this test are satisfied).

Ex 3. $\sum_1 n e^{-n^2}$

$f(x) = x e^{-x^2}$,

$f'(x) = -e^{-x^2}(2x^2-1) < 0$, if $2x^2 > 1$

$f(x)$ is ~~increasing~~ decreasing on $[1, \infty)$

Since $\int_1^\infty f(x)\,dx = \frac{1}{2}e^{-1}$, the series is convergent by the Integral Test.