Séminaire Lotharingien de Combinatoire XX (2020)
Article #YY, 12 pp.

Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

On sequences associated to the invariant theory of rank two simple Lie algebras

Alin Bostan[*1], Jordan Tirrell[†2] Bruce W. Westbury[‡3], and Yi Zhang[§3,4]

[1]*Inria, Université Paris-Saclay, 1 rue Honoré d'Estienne d'Orves, 91120 Palaiseau, France*
[2]*Department of Mathematics and Computer Science, Washington College, USA*
[3]*Department of Mathematical Sciences, The Unversity of Texas at Dallas, USA*
[4]*Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, Suzhou, China*

Abstract. We study two families of sequences, listed in the On-Line Encyclopedia of Integer Sequences (OEIS), which are associated to invariant theory of Lie algebras. For the first family, we prove combinatorially that the sequences A059710 and A108307 are related by a binomial transform. Based on this, we present two independent proofs of a recurrence equation for A059710, which was conjectured by Mihailovs. Besides, we also give a direct proof of Mihailovs' conjecture by the method of algebraic residues. As a consequence, closed formulae for the generating function of sequence A059710 are obtained in terms of classical Gaussian hypergeometric functions. Moreover, we show that sequences in the second family are also related by binomial transforms.

Keywords: representation theory, Lie algebra, binomial transform, algebraic residues, computer algebra, creative telescoping

1 Introduction

The representation theory of simple Lie algebras is a keystone of algebraic and enumerative combinatorics, giving rise to combinatorial objects such as tableaux, symmetric functions, quantum groups, crystal graphs, and many others. We are interested in two families of sequences in OEIS. Sequences in the first family of sequences are called *octant sequences*. For instance, sequence A059710, which is the first octant sequence, is defined to be a sequence associated to fundamental representations of the exceptional simple Lie algebra G_2, of rank two and dimension fourteen [11]. The second octant sequence is A108307 [13], and it is defined to be the cardinality of the set of set partitions of $[n]$ with no enhanced 3-crossing. Our first contribution is to prove that sequences A059710 and A108307 are tightly related, by a binomial transform.

*alin.bostan@inria.fr
†jtirrell@mtholyoke.edu
‡Bruce.Westbury@utdallas.edu.
§zhangy@amss.ac.cn. Yi Zhang was partially supported by the UTD Program P-1-03246, the Natural Science Foundation of USA Grants CCF-1815108 and CCF-1708884.

Theorem 1.1. *Let $T_3(n)$ and $E_3(n)$ be the n-th terms of A059710 and A108307, respectively. Then E_3 is the binomial transform of T_3, i.e., for $n \geq 0$,*

$$E_3(n) = \sum_{k=0}^{n} \binom{n}{k} T_3(k).$$

Theorem 1.1 provides an unexpected connection between invariant theory of G_2 and combinatorics of set partitions. In the same spirit, [8, Theorem 1] proves a binomial relation between E_3 and A108304, which is the third octant sequence. Together, these two results show that the octant sequences are associated to representations of G_2.

Building on Theorem 1.1, as well as on results by Bousquet-Mélou and Xin [4], our second contribution is to give two independent proofs of a recurrence equation for T_3 conjectured by Mihailovs [17, §3], which was the initial motivation for our study:

Theorem 1.2. *The sequence T_3 is determined by the initial conditions $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and the recurrence relation that for $n \geq 0$,*

$$\begin{aligned} &14\,(n+1)\,(n+2)\,T_3\,(n) + (n+2)\,(19n+75)\,T_3\,(n+1) \\ &\qquad + 2\,(n+2)\,(2n+11)\,T_3\,(n+2) - (n+8)\,(n+9)\,T_3\,(n+3) = 0. \end{aligned} \tag{1.1}$$

Furthermore, we give an alternative proof of Theorem 1.2, using the interpretation of T_3 in terms of G_2 walks and using algorithms for computing Picard-Fuchs differential equations for algebraic residues. As a consequence, closed formulae for the generating function of T_3 are obtained in terms of the classical Gaussian hypergeometric function.

We consider a second family of sequences, called *quadrant sequences*. These are defined to be sequences associated to representations of G_2 restricted to $SL(3)$. By invariant theory, these sequences are also are related by binomial transforms. Based on this, we give a uniform recurrence equation holding *for each quadrant sequence*. Furthermore, we show that sequences in the second family are identical to quadrant sequences because they satisfy the same initial conditions and recurrence equations. They are related to the octant sequences by the branching rules [6] for the maximal subgroup $SL(3)$ of G_2.

2 Preliminaries

In this section, we give a general construction of sequences associated to the invariant theory of algebraic groups and the relations between them.

Definition 2.1. *Let G be a reductive complex algebraic group and let V be a (finite dimensional) representation of G. The sequence associated to (G, V), denoted $\mathbf{a}_V$, is the sequence whose n-th term is the multiplicity of the trivial representation in the tensor power $\otimes^n V$.*

Given the sequence $\mathbf{a}$ with n-th term $a(n)$, the *binomial transform of* $\mathbf{a}$ is the sequence, denoted $\mathcal{B}\mathbf{a}$, whose n-th term is

$$\sum_{i=0}^{n} \binom{n}{i} a(i).$$

Here, $\mathcal{B}$ denotes the *binomial transform operator*. Binomial transforms arise naturally for sequences associated to the representations of reductive complex algebraic groups.

Lemma 2.2. *Assume that $\mathbf{a}_V$ is the sequence associated to (G, V) as specified in Definition 2.1. Then $\mathbf{a}_{V\oplus\mathbb{C}} = \mathcal{B}\mathbf{a}_V$.*

The binomial transform also arises naturally for lattice walks restricted to a domain.

Lemma 2.3. *Assume that a sequence $\mathbf{a}$ enumerates walks in a lattice, confined to a domain D, using a set of steps S. Then $\mathcal{B}\mathbf{a}$ is given by adding a new step corresponding to the zero element of the lattice; that is, S is replaced by the disjoint union $S \coprod \{0\}$ without changing the domain.*

The binomial transform can also be regarded as an operator on the generating function of a sequence. Let $G(t) = \sum_{n=0}^{\infty} a(n)t^n$ be the generating function of the sequence $\mathbf{a}$. We denote the generating function of $\mathcal{B}^k\mathbf{a}$ by $\mathcal{B}^k G$. Then we have

Lemma 2.4. *For $k \in \mathbb{Z}$, the k-th binomial transform of $G(t)$ is*

$$(\mathcal{B}^k G)(t) = \frac{1}{1-k\,t} G\left(\frac{t}{1-k\,t}\right).$$

3 Octant sequences

The first family of sequences we are interested in is illustrated in Figure 1. In Subsection 3.1, we show that A059710 and A108307 are related by the binomial transform, which is Theorem 1.1. Combining with [8, Theorem 1] and Lemma 2.2, sequences in this first family are identical to the sequences $\mathbf{a}_{V\oplus k\mathbb{C}}$ (Definition 2.1), where V is the seven dimensional fundamental representation of the exceptional simple Lie algebra G_2, for $k = 0, 1, 2$. They can also be interpreted in terms of lattice walks restricted to the octant. Thus, we call them *octant sequences*.

Let $T_3(n)$ be the n-th term of the first octant sequence A059710. In Subsection 3.2, based on Theorem 1.1, we present two independent proofs of Theorem 1.2, which gives a linear recurrence equation for $T_3(n)$. Besides, we also give a direct proof of Theorem 1.2 by the method of algebraic residues. As a consequence, closed formulae for the generating function of T_3 are presented in terms of hypergeometric functions in Subsection 3.3.

a (OEIS tag)	0	1	2	3	4	5	6	7	8	9
A059710	1	0	1	1	4	10	35	120	455	1792
A108307	1	1	2	5	15	51	191	772	3320	15032
A108304	1	2	5	15	52	202	859	3930	19095	97566

Figure 1: The first family of sequences **a**, with their OEIS tags, and their first terms.

3.1 Lattice walks

In this subsection we prove Theorem 1.1, using lattice walks. An *excursion* is a walk which starts and ends at the origin. The main idea of our proof is to exhibit excursions that are enumerated by the two sequences and then to compare these.

Sequence T_3 has an interpretation via enumerating excursions on the weight lattice of G_2. This interpretation is given in [11, 17] and it uses the theory of Kashiwara crystals.

Proposition 3.1. *Let C be the crystal of the representation V. For $n \geqslant 0$, $T_3(n)$ is the number of highest weight words of weight 0 in the tensor power $\otimes^n C$.*

The highest weight words are in bijection with a set of excursions in the weight lattice of G_2. The steps are the seven weights of the fundamental representation V. The walk is constrained to stay in the dominant chamber. These weights and the dominant chamber are shown in Figure 2. There is one further rule, which is the *boundary rule*, stating that if a walk is at a boundary point $(0, y)$, then the step $(0, 0)$ is not permitted.

Let $E_3(n)$ be the n-th term of the second octant sequence A108307. Then the sequence E_3 enumerates hesitating tableaux of height 2, introduced in [5]. The excursion interpretation is given in [4].

A *hesitating tableau* of semilength n is an excursion in the Young lattice with n steps. Each step is one of the following pairs of moves on the Young lattice:

- do nothing, add a cell;
- remove a cell, do nothing;
- add a cell, remove a cell.

A hesitating tableau of *height h* is a hesitating tableau such that every partition in the sequence has height at most h. A hesitating tableau of height 2 can be interpreted as an excursion in $\mathbb{Z}^2$ by identifying partitions with at most two nonzero rows with the set

$$\{(x, y) \in \mathbb{Z}^2 \mid x \geqslant y \geqslant 0.\}$$

There are eight steps since there are two ways to do nothing then add a cell, two ways to remove a cell then do nothing and four ways to remove a cell then add a cell. Two

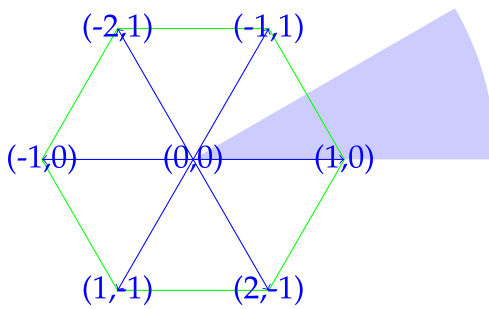

Figure 2: Steps in weight lattice

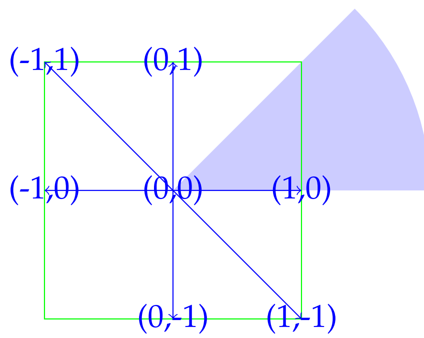

Figure 3: Steps in octant

of the four ways to remove a cell then add a cell give the step $(0,0)$, namely add a cell on the first row then remove this cell and add a cell on the second row then remove this cell. It is always allowed to add and then remove a cell on the first row. The step which adds and removes a cell on the second row is not permitted on the line $x = y$.

Proof of Theorem 1.1. We compare the two descriptions of the walks. The steps of the walks for the sequence T_3 are shown in Figure 2 and the steps of the walks for the sequence E_3 are shown in Figure 3.

In order to compare these, we first make the change of coordinates $(x, y) \to (x, y \pm x)$. This identifies the six non-zero steps, as well as the two domains.

By Lemma 2.3, it remains to compare the zero steps. There is one zero step in the T_3 case and two in the E_3 case. In the E_3 case we have the zero step which adds and then removes a cell on the second row. The boundary condition is that this step is not allowed on the line $x = y$. After the change of coordinates, this is the same boundary condition as the zero step in the T_3 case. The second zero step in the E_3 case adds and then removes a cell in the first row. This is always allowed. $\square$

3.2 Algebra

In this subsection, we give three proofs of Theorem 1.2, with different flavors. The first two proofs use Theorem 1.1 and a result of Bousquet-Mélou and Xin [4] on partitions that avoid 3-crossings. The last proof relies on the connection with G_2 walks.

3.2.1 First proof of Theorem 1.2

The first proof is based on Theorem 1.1, on Proposition 2 in [4] and on the method of creative telescoping [18, 9] for the summation of (bivariate) holonomic sequences.

Proposition 3.2. *[4, Proposition 2] The number $E_3(n)$ of partitions of $[n]$ having no enhanced 3-crossing is given by $E_3(0) = E_3(1) = 1$, and for $n \geq 0$,*

$$8(n+3)(n+1)E_3(n) + \left(7n^2+53n+88\right)E_3(n+1) \\ -(n+8)(n+7)E_3(n+2) = 0. \quad (3.1)$$

Equivalently, the associated generating function $\mathcal{E}(t) = \sum_{n\geq 0} E_3(n)t^n$ satisfies

$$t^2(1+t)(1-8t)\frac{d^2}{dt^2}\mathcal{E}(t) + 2t\left(6-23t-20t^2\right)\frac{d}{dt}\mathcal{E}(t) \\ +6\left(5-7t-4t^2\right)\mathcal{E}(t) - 30 = 0.$$

First Proof of Theorem 1.2. By Theorem 1.1, we have

$$T_3(n) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}E_3(k).$$

Set $f(n,k) = (-1)^{n-k}\binom{n}{k}E_3(k)$. By Proposition 3.2, and by the closure properties for holonomic functions, it follows that $f(n,k)$ is holonomic. Thus, we can apply Chyzak's algorithm [9] for creative telescoping to derive a recurrence relation for T_3. In particular, using the Koutschan's Mathematica package `HolonomicFunctions.m` [10] that implements Chyzak's algorithm, we find exactly the recurrence equation in Theorem 1.2. □

The detailed calculation involved in the above proof can be found in [3].

3.2.2 Second proof of Theorem 1.2

The second proof is also based on Proposition 2 in [4] and on Theorem 1.1, namely on the relation between the generating functions of $T_3(n)$ and of $E_3(n)$ implied by Theorem 1.1.

Second Proof of Theorem 1.2. Let $\mathcal{T}(t) = \sum_{n\geq 0} T_3(n)t^n$. By Theorem 1.1 and Lemma 2.4,

$$\mathcal{T}(t) = \frac{1}{1+t} \cdot \mathcal{E}\left(\frac{t}{1+t}\right).$$

We know from Proposition 3.2 a differential equation for $\mathcal{E}(t)$. By (univariate) closure properties of D-finite functions, we deduce a differential equation for $\mathcal{T}(t)$, and convert it into a linear recurrence for $T_3(n)$, which is exactly the recurrence in Theorem 1.2. □

3.2.3 Third proof of Theorem 1.2

The third proof of Theorem 1.2 relies on G_2 walks and the method in [2]. First we define two elements K and W in the group ring of the root lattice of G_2. Since the root lattice has rank 2, these become Laurent polynomials in two variables once we choose a basis of the root lattice. The element K is the character of the representation V; W is given by

$$W = \sum_{w\in W} \varepsilon(w)\,[w(\rho) - \rho]$$

Here W is the Weyl group, $\varepsilon\colon W \to \{\pm 1\}$ is the sign character and ρ is half the sum of the positive roots.

The following definition is given in [12] in the paragraph following Conjecture 3.3. The general principle is discussed in [7].

Definition 3.3. *The n-th term $T_3(n)$ is the constant term of the Laurent polynomial $W\,K^n$, where*

$$K = (1 + x + y + x\,y + x^{-1} + y^{-1} + (xy)^{-1}),$$

and W is the Laurent polynomial

$$\begin{aligned} W = x^{-2}y^{-3}(&x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} \\ &+ x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2). \end{aligned}$$

Third Proof of Theorem 1.2. Let $\mathcal{T}(t) = \sum_{n\geq 0} T_3(n)t^n$. By Definition 3.3, $\mathcal{T}(t)$ is the constant coefficient $[x^0y^0]$ of $W/(1-tK)$. In other words, $\mathcal{T}(t)$ is equal to the algebraic residue of $W/(xy - txyK)$, and thus it is D-finite [14] and is annihilated by a linear differential operator that cancels the contour integral of $W/(xy - txyK)$ over a cycle. Using the integration algorithm for multivariate rational functions in [1] we compute the following operator of order 6, that cancels $\mathcal{T}(t)$:

$$\begin{aligned}
L_6 = t^5\,(t+1)\,(7\,t-1)\,(2\,t+1)^2\,\partial^6 + \\
3\,t^4\,(2\,t+1)\left(168\,t^3+211\,t^2+40\,t-11\right)\partial^5 + \\
6\,t^3\left(2100\,t^4+3475\,t^3+1616\,t^2+79\,t-61\right)\partial^4 + \\
6\,t^2\left(11200\,t^4+17400\,t^3+7556\,t^2+268\,t-273\right)\partial^3 + \\
36\,t\left(4200\,t^4+6100\,t^3+2442\,t^2+54\,t-77\right)\partial^2 + \\
36\,\left(3360\,t^4+4540\,t^3+1646\,t^2+16\,t-35\right)\partial + \\
20160\,t^3+25200\,t^2+8064\,t,
\end{aligned}$$

where $\partial = \frac{\partial}{\partial t}$ denotes the derivation operator with respect to t.

The operator L_6 factors as $L_6 = QL_3$, where

$$Q = (2\,t+1)\,t^3\partial^3 + (24\,t+13)\,t^2\partial^2 + 6\,(12\,t+7)\,t\partial + 48\,t + 30,$$

and

$$\begin{aligned}
L_3 = t^2\,(2\,t+1)\,(7\,t-1)\,(t+1)\,\partial^3 + 2\,t\,(t+1)\left(63\,t^2+22\,t-7\right)\partial^2 + \\
\left(252\,t^3+338\,t^2+36\,t-42\right)\partial + 28\,t\,(3\,t+4)\,.
\end{aligned}$$

This shows that $f(t) := L_3(\mathcal{T}(t))$ is a solution of the differential operator Q. Hence, by denoting $f(t) = \sum_{n\geq 0} f_n t^n$, one deduces that for all $n \geq 0$ we have

$$2\,(n+2)\,f_n + (n+6)\,f_{n+1} = 0.$$

One the other hand, from $\mathcal{T}(t) = 1 + t^2 + O(t^3)$, it follows that $f_0 = 0$, therefore $f(t) = 0$, in other words $\mathcal{T}(t)$ is also a solution of L_3. From there, deducing the recurrence relation of Theorem 1.2 is immediate. □

3.3 Closed formulae

By factorization of the operator L_3 and using algorithms for solving second order differential equations, as described in [2], we derive the following closed formula for $\mathcal{T}(t)$:

$$\mathcal{T}(t) = \frac{1}{30\,t^5}\left[R_1\cdot {}_2F_1\left(\begin{matrix}\frac{1}{3}\;\frac{2}{3}\\ 2\end{matrix};\phi\right) + R_2\cdot {}_2F_1\left(\begin{matrix}\frac{2}{3}\;\frac{4}{3}\\ 3\end{matrix};\phi\right) + 5\,P\right],$$

where

$$R_1 = \frac{(t+1)^2\,(214\,t^3+45\,t^2+60\,t+5)}{t-1}, \qquad R_2 = 6\,\frac{t^2\,(t+1)^2\,(101\,t^2+74\,t+5)}{(t-1)^2},$$

and

$$\phi = \frac{27\,(t+1)\,t^2}{(1-t)^3}, \qquad P = 28\,t^4 + 66\,t^3 + 46\,t^2 + 15\,t + 1.$$

Following the approach in [2, §3.3], one can obtain an alternative expression using a more geometric flavor. The key point is that the denominator of $W/(xy - txyK)$ is a family of *elliptic* curves, thus integrating $W/(xy - txyK)$ over a small torus amounts to computing the periods of the two forms (of the first and second kind). Working out the details, this approach yields an expression for $\mathcal{T}(t)$ in terms of the Weierstrass invariant

$$g_2 = (t-1)\left(25\,t^3 + 21\,t^2 + 3\,t - 1\right)$$

and of the j-invariant

$$J = \frac{(t-1)^3\left(25\,t^3 + 21\,t^2 + 3\,t - 1\right)^3}{t^6\,(1-7\,t)\,(2\,t+1)^2\,(t+1)^3}$$

of our family of curves. As in [2], we introduce the expression

$$H(t) = \frac{1}{{g_2}^{1/4}} \cdot {}_2F_1\left(\begin{matrix}\frac{1}{12} & \frac{5}{12} \\ & 1\end{matrix}; \frac{1728}{J}\right).$$

Then $\mathcal{T}(t)$ is proved to be equal to

$$\frac{P}{6\,t^5} + \frac{(7\,t-1)\,(2\,t+1)\,(t+1)}{360\,t^5}\Big(\left(155\,t^2 + 182\,t + 59\right)(11\,t+1)\,H(t) + \left(341\,t^3 + 507\,t^2 + 231\,t + 1\right)(5\,t+1)\,H'(t)\Big).$$

4 Quadrant sequences

The second family of sequences we are interested in is given in Figure 4. We show that sequences in this second family are identical to sequences associated to representations of G_2 restricted to $SL(3)$. Thus, by Lemma 2.2, those sequences are also related by binomial transforms. Moreover, they can also be interpreted as lattice walks restricted to the quadrant. This is why we call them *quadrant sequences*. They are related to the octant sequences by the branching rules [6] for the maximal subgroup $SL(3)$ of G_2.

Definition 4.1. *Let V be the defining representation of $SL(3)$ and denote the dual by V^*. For $k \geqslant 0$, the quadrant sequences $\mathcal{S}_k$ are the sequence associated to $(SL(3), V \oplus V^* \oplus k\,\mathbb{C})$.*

Lemma 4.2. *Let C_k be the generating function of $\mathcal{S}_k$, where $k \geq 0$. Then C_k is the constant coefficient of $[x^0y^0]$ of $W/(1-tK)$, where*

$$K = k + x + y + x^{-1} + y^{-1} + \frac{x}{y} + \frac{y}{x} \tag{4.1}$$

a (OEIS tag)	0	1	2	3	4	5
A151366	1	0	2	2	12	30
A236408	1	1	3	9	33	131
A001181	1	2	6	22	92	422
A216947	1	3	11	49	221	1113

Figure 4: The second family of sequences **a**, their OEIS tags and their first terms.

and

$$W = 1 - \frac{x^2}{y} + x^3 - x^2y^2 + y^3 - \frac{y^2}{x}. \tag{4.2}$$

Let $C_2(n)$ be the n-th term of the sequence A216947 in the second family. Marberg [15, Theorem 1.7] showed $C_2(n)$ is the constant term $[x^0y^0]$ of $W\tilde{K}^n$, where $\tilde{K} = K|_{k=3}$, the Laurent polynomials K and W are specified in (4.1) and (4.2). By Lemma 4.2, the sequence $\mathcal{S}_3$ is identical to the sequence A216947. Therefore, we have:

Proposition 4.3. *[15, Theorem 1.7] The n-th term $C_2(n)$ of the sequence $\mathcal{S}_3$ is given by $C_2(0) = 1, C_2(1) = 3$ and for $n \geq 0$:*

$$(n+5)(n+6)\cdot C_2(n+2) - 2(5n^2+36n+61)\cdot C_2(n+1) \\ + 9(n+1)(n+4)\cdot C_2(n) = 0,$$

Equivalently, the associated generating function $\mathcal{C}(t) = \sum_{n\geq 0} C_2(n)t^n$ satisfies

$$72\mathcal{C}(t) + 4(-61+117t)\frac{d}{dt}\mathcal{C}(t) + 2(15-184t+234t^2)\frac{d^2}{dt^2}\mathcal{C}(t) \\ + 2t(-6+7t)(-1+9t)\frac{d^3}{dt^3}\mathcal{C}(t) + (-1+t)t^2(-1+9t)\frac{d^4}{dt^4}\mathcal{C}(t) = 0.$$

Next, for $k = 0,1,2,3$, we prove a uniform recurrence equation for the sequence $\mathcal{S}_k$. It is given by a single formula with k as a parameter. Moreover, we show that $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are identical to sequences in the second family.

Theorem 4.4. *For $k = 0,1,2,3$, the n-th term $a(n)$ of the sequence $\mathcal{S}_k$ satisfies the following recurrence equation:*

$$(k-9)(k-1)k^2(n+1)(n+2)a(n)+ \\ 2k(n+2)(2k^2n-15kn+8k^2+9n-56k+36)a(n+1)+ \\ (6k^2n^2+54k^2n-30kn^2+114k^2+9n^2-254kn+81n-510k+162)a(n+2) \\ +2(2kn^2+24kn-5n^2+70k-56n-153)a(n+3) \\ +(n+7)(n+8)a(n+4) = 0 \tag{4.3}$$

Proof. By Lemma 2.2, sequences $\mathcal{S}_2, \mathcal{S}_1, \mathcal{S}_0$ are the first, second, and third binomial transforms of S_3, respectively. Thus, by Lemma 2.4, the generating function of S_k is

$$\mathcal{A}(t) := \frac{1}{1-kt} \cdot \mathcal{C}\left(\frac{t}{1-kt}\right) \quad \text{for} \quad k = 0, 1, 2, 3,$$

where $\mathcal{C}(t)$ is the generating function of S_3. Regarding k as a parameter in the above expression, we deduce the differential equation for $\mathcal{A}(t)$ in the claim by using Proposition 4.3 and closure properties of D-finite functions. By converting the differential equation for $\mathcal{A}(t)$, we get the corresponding recurrence equation for the sequence $a(n)$, which is exactly the recurrence equation in the claim. $\square$

In [15], Marberg proved that A151366, A001181, and A216947 are related by binomial transforms by using combinatorial methods. Here, we give another proof by the representation theory of simple Lie algebras.

Corollary 4.5. *The sequences S_0, S_1, S_2, S_3 are identical to sequences in the second family specified in Figure 4. Moreover, sequences in the second family are related by binomial transforms.*

Proof. In (4.3), by setting k to $0, 1, 2, 3$, we find that the corresponding recurrence operators are left multiples [16, page 618] of those of A216947, A001181, A236408, and A151366 specified in OEIS. It implies that S_0, S_1, S_2, S_3 satisfy the same recurrence equations as the sequences in the second family. To verify they are the same sequences, we just need to check finitely many initial terms. The details of the verification can be find in [3]. Since S_k's are related by binomial transforms, so are sequences in the second family. $\square$

References

[1] Alin Bostan, Pierre Lairez, and Bruno Salvy. "Creative telescoping for rational functions using the Griffiths-Dwork method". In: *ISSAC 2013*. ACM, New York, 2013, pp. 93–100. DOI: `10.1145/2465506.2465935`.

[2] Alin Bostan et al. "Hypergeometric expressions for generating functions of walks with small steps in the quarter plane". In: *European J. Combin.* 61 (2017), pp. 242–275. DOI: `10.1016/j.ejc.2016.10.010`.

[3] Alin Bostan et al. *Supplementary electronic material to the article "On sequences associated to the invariant theory of rank two simple Lie algebras"*. `https://yzhang1616.github.io/ct/ct.html`. 2019.

[4] Mireille Bousquet-Mélou and Guoce Xin. "On partitions avoiding 3-crossings". In: *Seminaire Lotharingien de Combinatoire 54 (2005) Paper B54e (21 pp.)* (June 27, 2005). arXiv: `math/0506551v2 [math.CO]`.

[5] William Y. C. Chen et al. "Crossings and nestings of matchings and partitions". In: *Trans. Amer. Math. Soc.* 359.4 (2007), pp. 1555–1575. DOI: 10.1090/S0002-9947-06-04210-3.

[6] R. Gaskell and R. T. Sharp. "Generating functions for G_2 characters and subgroup branching rules". In: *J. Math. Phys.* 22.12 (1981), pp. 2736–2739. ISSN: 0022-2488. DOI: 10.1063/1.525178.

[7] Ira M. Gessel and Doron Zeilberger. "Random walk in a Weyl chamber". In: *Proc. Amer. Math. Soc.* 115.1 (1992), pp. 27–31. ISSN: 0002-9939. DOI: 10.2307/2159560.

[8] Juan B. Gil and Jordan O. Tirrell. "A simple bijection for classical and enhanced k-noncrossing partitions". In: *Discrete Mathematics* (2019). In press. DOI: 10.1016/j.disc.2019.111705.

[9] Christoph Koutschan. "Advanced applications of the holonomic systems approach". PhD thesis. Johannes Kepler University Linz, 2009.

[10] Christoph Koutschan. "HolonomicFunctions User's Guide". In: *RISC Report Series* (2010).

[11] Greg Kuperberg. "Spiders for rank 2 Lie algebras". In: *Comm. Math. Phys. 180(1):109-151, 1996* (Nov. 29, 1997). DOI: 10.1007/BF02101184.

[12] Greg Kuperberg. "The quantum G_2 link invariant". In: *Internat. J. Math.* 5.1 (1994), pp. 61–85. ISSN: 0129-167X. DOI: 10.1142/S0129167X94000048.

[13] Zhicong Lin. "Restricted inversion sequences and enhanced 3-noncrossing partitions". In: *European J. Combin.* 70 (2018), pp. 202–211. DOI: 10.1016/j.ejc.2018.01.002.

[14] Leonard Lipshitz. "The diagonal of a D-finite power series is D-finite". In: *J. Algebra* 113.2 (1988), pp. 373–378. ISSN: 0021-8693. DOI: 10.1016/0021-8693(88)90166-4.

[15] Eric Marberg. "Crossings and nestings in colored set partitions". In: *Electron. J. Combin. 20 (2013), no. 4, Research Paper 6* (2012). arXiv: 1203.5738v3 [math.CO].

[16] Manuel Kauers Shaoshi Chen and Michael F. Singer. "Desingularization of Ore Operators". In: *Journal of Symbolic Computation* 74 (2016), pp. 617–626.

[17] Bruce W. Westbury. "Enumeration of non-positive planar trivalent graphs". In: *J Algebr Comb (2007) 25:357-373* (July 6, 2005). DOI: 10.1007/s10801-006-0041-4.

[18] Doron Zeilberger. "The method of creative telescoping". In: *J. Symbolic Comput.* 11.3 (1991), pp. 195–204. ISSN: 0747-7171.