Chapter 11 Infinite Sequences and Series

§11.1 Sequences

A sequence is a list of number in a definite order:

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

a_n is the nth term.

Note: 1. The sequence $\{a_1, a_2, \cdots\}$ is also denoted by

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

2. The first ~~term~~ index ~~is not~~ does not have to be 1:

$a_0, a_1, a_2, \cdots$ is also a sequence.

Ex.

(1) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_n = \dfrac{n}{n+1}$ $\quad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \cdots, \dfrac{n}{n+1}, \cdots\right\}$

(2) The Fibonacci sequence $\{f_n\}$ is defined by:

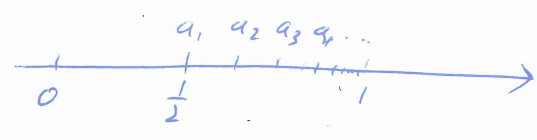
$$f_1 = 1, \ f_2 = 1, \ f_n = f_{n-1} + f_{n-2}, \ n \geq 3$$

The first few terms are

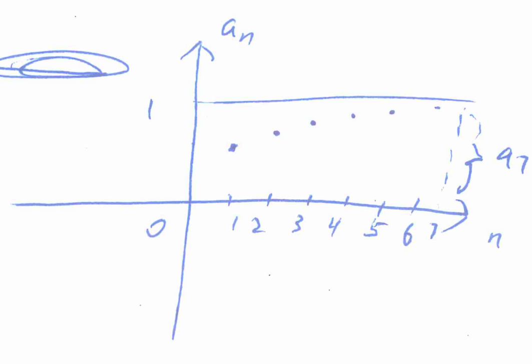
$$\{1, 1, 2, 3, 5, 8, 13, 21, \cdots\}.$$

Plotting a sequence

$a_n = \dfrac{n}{n+1}$



$\{(n, a_n) \mid n = 1, 2, \cdots\}$



②

$$1 - a_n = 1 - \frac{n}{n+1}$$

$$= \frac{1}{n+1} \quad \text{(by taking } n \text{ sufficiently large, } \tfrac{1}{n+1} \text{ approaches } 0\text{)}$$

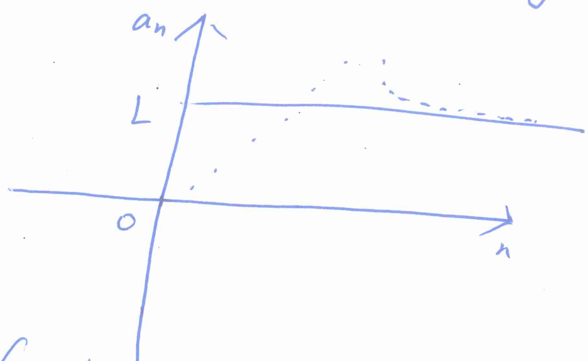
Thus, we write

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

Def 1 A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges. Otherwise, we say the segendence diverges.



Assume f is a continuous function on $[0, \infty)$

The difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer.

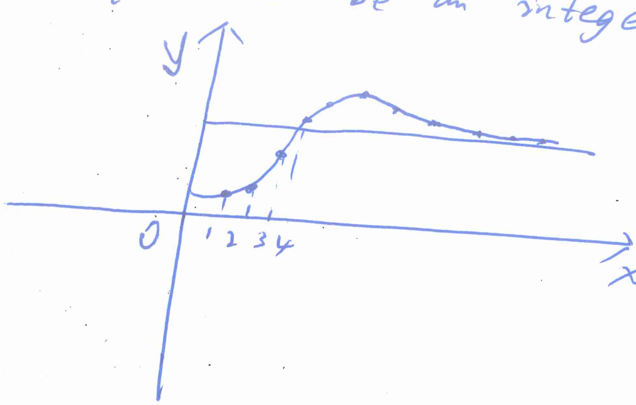


Figure for Theorem 3

Theorem 3 If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, when n is an integer, then $\lim_{n\to\infty} a_n = L$.

③ Ex. Recall: $\lim_{x\to\infty}(1/x^r)=0$ when $r>0$.

Then $\lim_{n\to\infty}\frac{1}{n^r}=0$ if $r>0$.

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n=\infty$. Then we say $\{a_n\}$ diverges to ∞.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent and c is a constant, then

$$\lim_{n\to\infty}(a_n+b_n)=\lim_{n\to\infty}a_n+\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}c\,a_n=c\lim_{n\to\infty}a_n$$

$$\lim_{n\to\infty}(a_n\cdot b_n)=\lim_{n\to\infty}a_n\cdot\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}\frac{1}{b_n}=\frac{1}{\lim_{n\to\infty}b_n}\quad\text{if }\lim_{n\to\infty}b_n\neq 0$$

$$\lim_{n\to\infty}a_n^p=\left(\lim_{n\to\infty}a_n\right)^p\quad\text{if }p>0\text{ and }a_n>0.$$

Ex 4. Find $\lim_{n\to\infty}\frac{n}{n+1}$

$$\frac{n}{n+1}=\frac{1}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty}\frac{n}{n+1}=\frac{1}{\lim_{n\to\infty}(1+\frac{1}{n})}$$

$$=\frac{1}{\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{1}{n}}$$

$$=\frac{1}{1+0}$$

$$=1$$

Ex 5 Is the sequence $a_n = \frac{n}{\sqrt{10+n}}$ convergent or divergent?

$$\frac{n}{\sqrt{10+n}}$$

$$= \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}}$$

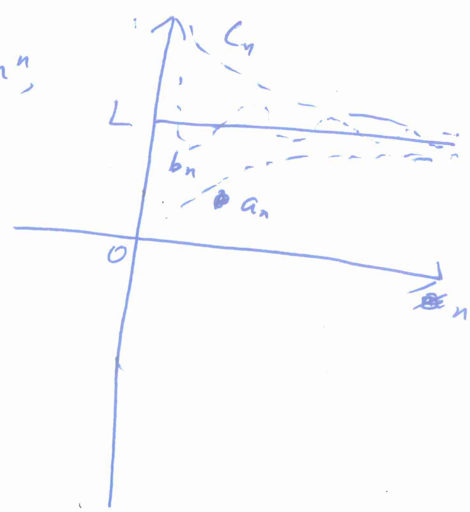
$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \frac{1}{\lim_{n\to\infty}\sqrt{\frac{10}{n^2}+\frac{1}{n}}}$$

$$= \frac{1}{\sqrt{\lim_{n\to\infty}\frac{10}{n^2} + \lim_{n\to\infty}\frac{1}{n}}}$$

$$= \infty$$

Squeeze Theorem for sequences If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Ex 10. Find the convergence of $a_n = n!/n^n$, where $n! = 1\cdot 2\cdot 3 \cdots n$.



$$a_n = \frac{1\cdot 2\cdot 3 \cdots n}{n\cdot n\cdot n \cdots n}$$

$$0 <= \frac{1}{n}\left(\frac{2\cdot 3 \cdots n}{n\cdot n \cdots n}\right) \le \frac{1}{n}$$

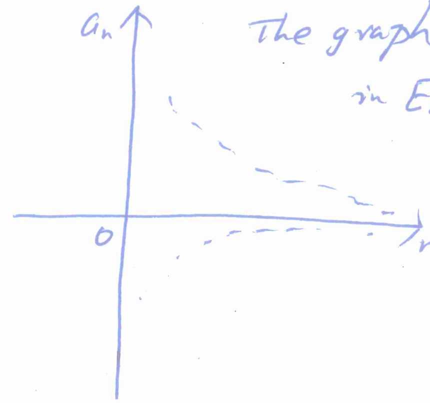
Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we have

$$\lim_{n\to\infty} a_n = 0.$$

Theorem 6 If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Ex 8. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0$$



The graph of a_n in Ex 8

By Theorem 6, we have

$$\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$$

Theorem 7 If $\lim_{n\to\infty} a_n = L$ and the function f is continonous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$ (if we apply a continuous function to terms of a convergent sequence, the result is also convergent)

Ex 9. Find $\lim_{n\to\infty}\sin(\pi/n)$

$$\lim_{n\to\infty}\frac{\pi}{n} = 0$$

Since $\sin x$ is continuous at 0, we have

$$\lim_{n\to\infty}\sin(\pi/n) = \sin\left(\lim_{n\to\infty}\frac{\pi}{n}\right) = \sin(0) = 0.$$

Def 10 A sequence $\{a_n\}$ is called ~~decreasing~~ increasing if $a_n < a_{n+1}$ for $n \geq 1$, i.e., $a_1 < a_2 < a_3 < \cdots$. It is called decreasing if $a_n > a_{n+1}$ for $n \geq 1$. A sequence is monotonic if it is either increasing or decreasing.

Ex 13. Show that $a_n = \frac{n}{n^2+1}$ is decreasing.

Consider the function $f(x) = \frac{x}{x^2+1}$,

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{if } x^2 > 1$$

Thus, f is decreasing on $(1,\infty)$ and so $f(n) > f(n+1)$

~~Hence~~ Hence $\{a_n\}$ is decreasing.

⑥ Def 11. A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \le M \quad \text{for } n \ge 1$$

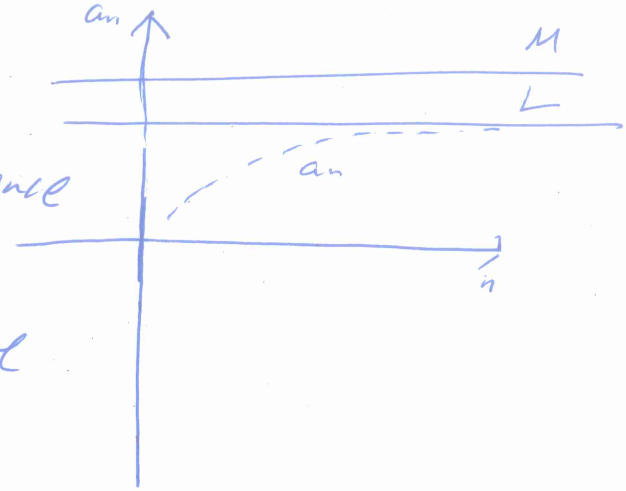
It is bounded below if there is a number m such that

$$m \le a_n \quad \text{for } n \ge 1$$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Ex, $a_n = \frac{n}{n+1}$, $0 < a_n < 1$ for $n \ge 1$.

Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.



Ex: $a_n = \frac{n}{n^2+1}$ is a decreasing sequence

$0 < a_n < 1$

Thus, $\{a_n\}$ is a convergent sequence

$$\frac{n}{n^2+1} = \frac{\frac{1}{n}}{1+\frac{1}{n^2}}$$

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \frac{\lim_{n\to\infty} \frac{1}{n}}{1 + \lim_{n\to\infty} \frac{1}{n^2}} = \frac{0}{1+0} = 0.$$