

Mahler Discrete Residues and Summability for Rational Functions

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Joint work with Carlos E. Arreche



Linear Mahler equations

Let \mathbb{K} be an algebraically closed field of char 0, x be an indeterminate, and $p \in \mathbb{Z}_{\geq 2}$.

Consider

$$\ell_r(x)y(x^{p^r}) + \ell_{r-1}(x)y(x^{p^{r-1}}) + \cdots + \ell_0(x)y(x) = f(x), \quad (1)$$

where $\ell_i, f \in \mathbb{K}[x]$ are given, $y(x)$ is unknown. A solution of (1) is called a **Mahler function**.

(Mahler 1929): study Mahler equations to prove the transcendence of values of some functions.

Fact: the generating series of any p -automatic sequence (such as the Baum–Sweet and the Rudin–Shapiro sequences) is a Mahler function.

Differential Galois theory

Using differential Galois theory, we can determine the differential-algebraic relations between solutions of Mahler equations.

Example (Roques 2018): A Galoisian proof that the generating series of the Baum-Sweet and Rudin-Shapiro sequences are algebraic independent over $\bar{\mathbb{Q}}(x)$.

Goal: Design effective algorithms for computing differential Galois groups of a given linear Mahler equations.

Discrete residues, telescopers, and Galois theory

Endow $\mathbb{K}(x)$ with one of the $\sigma\delta$ -field structures:

(S) $\sigma : f(x) \mapsto f(x+1)$ and $\delta = \frac{d}{dx}$;

(Q) $\sigma : f(x) \mapsto f(qx)$ with $q \in \mathbb{K}^\times$ not root of unity and $\delta = x \frac{d}{dx}$.

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Let $z_1, \dots, z_n \in F$, a $\sigma\delta$ -extension of $\mathbb{K}(x)$ with $F^\sigma = \mathbb{K}$, satisfying

$$\sigma(z_i) = a_i z_i \quad \text{for some } a_1, \dots, a_n \in \mathbb{K}(x)^\times.$$

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Proposition (Hardouin-Singer 2008) z_1, \dots, z_n are δ -dependent over $\mathbb{K}(x)$ iff $\exists \mathcal{L}_1, \dots, \mathcal{L}_n \in \mathbb{K}[\delta]$, linear δ -operators with coefficients in \mathbb{K} , not all 0, and $g \in \mathbb{K}(x)$:

$$\mathcal{L}_1 \left(\frac{\delta(a_1)}{a_1} \right) + \dots + \mathcal{L}_n \left(\frac{\delta(a_n)}{a_n} \right) = \sigma(g) - g.$$

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(Arreche 2017, Arreche-Z. 2022): Using (q) -discrete residues, there exist constants $m_1, \dots, m_n \in \mathbb{K}$, not all 0, such that

$$m_1 \frac{\delta(a_1)}{a_1} + \dots + m_n \frac{\delta(a_n)}{a_n} = \sigma(g) - g + c$$

for some $g \in \mathbb{K}(x)$ and $c \in \mathbb{K}$ (with $c = 0$ in case (S)).

Motivation

Proposition (Hardouin-Singer 2008) z_1, \dots, z_n are δ -dependent over $\mathbb{K}(x)$ iff $\exists \mathcal{L}_1, \dots, \mathcal{L}_n \in \mathbb{K}[\delta]$, linear δ -operators with coefficients in \mathbb{K} , not all 0, and $g \in \mathbb{K}(x)$:

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Idea: Develop the notion of Mahler discrete residues and derive an effective version of Hardouin-Singer's Proposition in the Mahler case.

Continuous residues

Let \mathbb{K} be an algebraically closed field of char 0, and let $f(x) \in \mathbb{K}(x)$. Make the partial fraction decomposition

$$f(x) = r(x) + \sum_{\alpha \in \mathbb{K}} \sum_{k \geq 1} \frac{c_{\alpha}(k)}{(x - \alpha)^k},$$

where $r(x) \in \mathbb{K}[x]$ and $c_{\alpha}(k) \in \mathbb{K}$ (almost all 0).

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Then $f(x)$ is **rationally integrable**, i.e., there exists $g(x) \in \mathbb{K}(x)$ such that $g'(x) = f(x)$, if and only if the (continuous first-order) *residues*

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Chen and Singer (2012) created a notion of *discrete residues* that plays an analogous role (where *integrability* \mapsto *summability*) for the *shift* ($x \mapsto x + 1$) and *q-dilation* ($x \mapsto qx$) difference operators.

Discrete residues: shift case

Rewrite the partial fraction decomposition of $f(x) \in \mathbb{K}(x)$:

$$f(x) = r(x) + \sum_{k \geq 1} \sum_{[\alpha] \in \mathbb{K}/\mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{c_{\alpha}(k, n)}{(x - \alpha + n)^k}$$

where $r(x) \in \mathbb{K}[x]$, $\alpha \in \mathbb{K}$ is a coset representative for $[\alpha] := \alpha + \mathbb{Z} \in \mathbb{K}/\mathbb{Z}$, and $c_{\alpha}(k, n) \in \mathbb{K}$.

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$$q\text{-dres}(f, [\alpha]_q, k) := \sum_{n \in \mathbb{Z}} q^{-kn} c_\alpha(k, n) \quad (\text{resp., } q\text{-dres}(f, \infty) := r_0).$$

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Why use residues?

An advantage of using residues is to answer *whether* (yes/no) $f(x) \in \mathbb{K}(x)$ is

- ▶ rationally integrable: $f(x) = g'(x)$; or
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In the differential case, there is a better way: if $f = \frac{a}{b}$ with $a, b \in \mathbb{K}[x]$, $\gcd(a, b) = 1$, $\deg(a) < \deg(b)$, and b squarefree, then the roots of the **Rothstein-Trager resultant**

$$RT(f) := \text{Res}_x(a - z \cdot b', b) \in \mathbb{K}[z]$$

are precisely the first-order continuous residues of $f(x)$, which implies $f(x)$ is rationally integrable iff $RT(f)$ is a monomial in z .

Mahler summability for rational functions

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Mahler Summability Problem: given $f(x) \in \mathbb{K}(x)$, decide effectively whether $f(x)$ is Mahler summable.

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Our Goal: Construct a (\mathbb{K} -linear) complete obstruction to the Mahler summability of $f(x) \in \mathbb{K}(x)$.

Mahler summability for rational functions

More precisely, for the \mathbb{K} -linear map $\Delta : g(x) \mapsto g(x^p) - g(x)$, we wish to construct explicitly a \mathbb{K} -linear map ∇ on $\mathbb{K}(x)$ such that $\text{im}(\Delta) = \ker(\nabla)$, bypassing computation of certificates.

We call ∇ the **Mahler reduction** operator. Given $f \in \mathbb{K}(x)$, set $\bar{f} = \nabla(f)$. Then f is Mahler summable if and only if $\bar{f} = 0$. The numerators in the partial fraction decomposition of \bar{f} are **Mahler discrete residues** of f .

Mahler trajectories and Mahler trees

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We denote by \mathbb{Z}/\mathcal{P} the set of **maximal trajectories** for the action of \mathcal{P} on \mathbb{Z} by multiplication:

$$\mathbb{Z}/\mathcal{P} = \{\{0\}\} \cup \{\{ip^n \mid n \in \mathbb{Z}_{\geq 0}\} \mid i \in \mathbb{Z} \text{ such that } p \nmid i\}.$$

The elements $\theta \in \mathbb{Z}/\mathcal{P}$ are pairwise disjoint subsets of \mathbb{Z} whose union is all of \mathbb{Z} .

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We denote by \mathcal{T}_M the set of equivalence classes for the equivalence relation on \mathbb{K}^\times defined by $\alpha \sim \gamma$ if and only if $\alpha^{p^s} = \gamma^{p^r}$ for some $r, s \in \mathbb{Z}_{\geq 0}$.

The elements $\tau \in \mathcal{T}_M$, called **Mahler trees**, are pairwise disjoint subsets of \mathbb{K}^\times whose union is all of \mathbb{K}^\times . We write $\tau(\alpha)$ for the unique Mahler tree containing $\alpha \in \mathbb{K}^\times$.

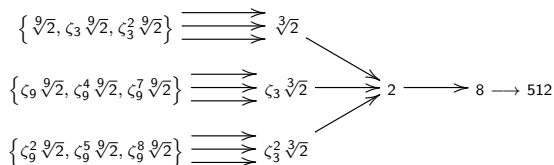
Examples of Mahler trees

We define a digraph on the vertex set τ for each Mahler tree $\tau \in \mathcal{T}_M$ with one directed edge $\alpha \rightarrow \gamma$ whenever $\alpha^p = \gamma$.

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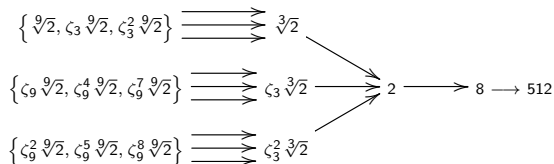
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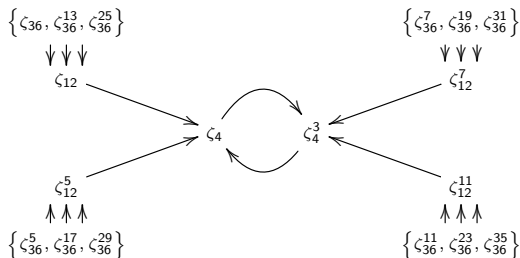
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With $p = 3$, the vertices of $\tau(\zeta_4)$ near $\zeta_4 \in \mathbb{K}^\times$ are



Mahler decomposition of partial fractions

For $f(x) \in \mathbb{K}(x)$, we can decompose it uniquely as

$$f(x) = f_L(x) + f_T(x):$$

$$f_L(x) := \sum_{j \in \mathbb{Z}} r_j x^j \quad \text{and} \quad f_T(x) := \sum_{k \geq 1} \sum_{\alpha \in \mathbb{K}^\times} \frac{c_\alpha(k)}{(x - \alpha)^k},$$

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Moreover, the decompositions $f_L = \sum_{\theta \in \mathbb{Z}/\mathcal{P}} f_\theta$ and $f_T = \sum_{\tau \in \mathcal{T}_M} f_\tau$:

$$f_\theta := \sum_{j \in \theta} r_j x^j \quad \text{and} \quad f_\tau := \sum_{k \geq 1} \sum_{\alpha \in \tau} \frac{c_\alpha(k)}{(x - \alpha)^k}$$

are also σ -stable. Can decide summability of f by deciding for each f_θ ($\theta \in \mathbb{Z}/\mathcal{P}$) and each f_τ ($\tau \in \mathcal{T}_M$) individually.

Mahler residues at infinity

Definition (Arreche-Z. 2022) Let $f(x) \in \mathbb{K}(x)$ and write $f_L(x) = \sum_{j \in \mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$. The **Mahler residue** of $f(x)$ at infinity is the vector

$$\text{dres}(f, \infty) := \left(\sum_{j \in \theta} r_j \right)_{\theta \in \mathbb{Z}/\mathcal{P}} \in \bigoplus_{\theta \in \mathbb{Z}/\mathcal{P}} \mathbb{K}.$$

Mahler residues at infinity

Definition (Arreche-Z. 2022) Let $f(x) \in \mathbb{K}(x)$ and write $f_L(x) = \sum_{j \in \mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$. The **Mahler residue** of $f(x)$ at infinity is the vector

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Proof sketch: For $\theta = \{ip^n\}$ with $p \nmid i$, let $\bar{f}_\theta^{(n)} = f_\theta + \Delta(g_\theta^{(n)})$ with $g_\theta^{(0)} := 0$ and $g_\theta^{(n+1)} := g_\theta^{(n)} + (\sum_{\ell=0}^n r_{ip^\ell}) x^{ip^n}$. Then, for h largest s.t. $r_{ip^h} \neq 0$, $\bar{f}_\theta^{(h)} = \text{dres}(f, \infty)_\theta \cdot x^{ip^h}$. A *dispersion* argument shows $\bar{f}_\theta^{(h)} = 0$ iff f_θ is Mahler summable.

Mahler residues at Mahler trees (1 of 3): coefficients

For $\alpha \in \mathbb{K}^\times$, ζ_p a primitive p -th root of unity, let $V_k^m(\zeta_p^i \alpha) \in \mathbb{K}$:

$$\sigma \left(\frac{1}{(x - \alpha^p)^m} \right) = \frac{1}{(x^p - \alpha^p)^m} = \sum_{k=1}^m \sum_{i=0}^{p-1} \frac{V_k^m(\zeta_p^i \alpha)}{(x - \zeta_p^i \alpha)^k}.$$

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The definition of Mahler residues is more complicated because applying $\sigma : f(x) \mapsto f(x^p)$ at poles of order m “leaks” into the poles of order $k \leq m$.

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Lemma (Arreche-Z. 2022)

$$V_k^m(\zeta_p^i \alpha) = \mathbb{V}_k^m \cdot \frac{(\zeta_p^i \alpha)^k}{\alpha^{pm}},$$

where $\mathbb{V}_k^m \in \mathbb{Q}$ are obtained from the Taylor coefficients at $x = 1$:

$$(x^{p-1} + \cdots + x + 1)^{-m} = \sum_{k=1}^m \mathbb{V}_k^m \cdot (x - 1)^{m-k} + O((x - 1)^m).$$

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The “universal coefficients” \mathbb{V}_k^m can be computed directly (as a sum over partitions) using the Faà di Bruno’s formula.

Small example of Mahler coefficients

Let $p = 3$, $m = 2$, and $\alpha^3 = 1$. Then

$$\sigma\left(\frac{1}{(x-1)^2}\right) = \frac{1}{(x^3-1)^2} = \sum_{k=1}^2 \sum_{i=0}^2 \frac{V_k^2(\zeta_3^i)}{(x-\zeta_3^i)^k},$$

By the previous Lemma, $V_k^2(\zeta_3^i) = \mathbb{V}_k^2 \cdot (\zeta_3^i)^{k-6} = \mathbb{V}_k^2 \cdot \zeta_3^{ki}$ for $k = 1, 2$. We find that

$$\mathbb{V}_2^2 = (x^2+x+1)^{-2} \Big|_{x=1} = \frac{1}{9}; \text{ and } \mathbb{V}_1^2 = ((x^2+x+1)^{-2})' \Big|_{x=1} = -\frac{2}{9}.$$

Using a computer algebra system (or by hand!), one can verify that the partial fraction decomposition of $9 \cdot (x^3-1)^{-2}$ is indeed

$$\frac{1}{(x-1)^2} + \frac{\zeta_3^2}{(x-\zeta_3)^2} + \frac{\zeta_3}{(x-\zeta_3^2)^2} + \frac{-2}{x-1} + \frac{-2\zeta_3}{x-\zeta_3} + \frac{-2\zeta_3^2}{x-\zeta_3^2}.$$

Mahler residues at Mahler trees (2 of 3): definition

Suppose $\gamma \in \mathbb{K}^\times$ is not a root of unity, $f \in \mathbb{K}(x)$, and $h \in \mathbb{Z}_{\geq 0}$ s.t.:

$$\text{sing}(f) \cap \tau(\gamma) \subseteq \{\zeta_{p^n}^i \gamma^{p^{h-n}} \mid 0 \leq n \leq h, i \in \mathbb{Z}/p^n\mathbb{Z}\}.$$

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Then we can write, for $\tau = \tau(\gamma)$,

$$f_\tau = \sum_{k=1}^m \sum_{n=0}^h \sum_{i \in \mathbb{Z}/p^n\mathbb{Z}} \frac{c_\gamma(k, n, i)}{(x - \zeta_{p^n}^i \gamma^{p^{h-n}})^k}.$$

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Set recursively: $\tilde{c}_{k,0,0} := c_\gamma(k, 0, 0)$, and for $1 \leq n \leq h$; $i \in \mathbb{Z}/p^n\mathbb{Z}$:

$$\tilde{c}_{k,n,i} := c_\gamma(k, n, i) + \sum_{j=k}^m \tilde{c}_{j,n-1,\pi_{n-1}(i)} V_k^j(\zeta_{p^n}^i \gamma^{p^{h-n}}),$$

where $\pi_{n-1} : \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^{n-1}\mathbb{Z}$ is the canonical projection.

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Definition (Arreche-Z. 2022) The **Mahler discrete residue** at τ of order k is the vector $\text{dres}(f, \tau, k) \in \bigoplus_{\alpha \in \tau} \mathbb{K}$ with α -component $:= 0$ except possibly at $\alpha = \zeta_{p^h}^i \gamma$ for $i \in \mathbb{Z}/p^h\mathbb{Z}$, given by $\tilde{c}_{k,h,i}$.

Mahler residues at Mahler trees (3 of 3): proof

Proposition (Arreche-Z. 2022) For $f \in \mathbb{K}(x)$ the component $f_{\mathcal{T}}$ is Mahler summable if and only if $\text{dres}(f, \tau, k) = 0$ (the zero vector) for each $\tau \in \mathcal{T}_M$ and $k \in \mathbb{N}$.

Mahler residues at Mahler trees (3 of 3): proof

Proposition (Arreche-Z. 2022) For $f \in \mathbb{K}(x)$ the component f_τ is Mahler summable if and only if $\text{dres}(f, \tau, k) = 0$ (the zero vector) for each $\tau \in \mathcal{T}_M$ and $k \in \mathbb{N}$.

Proof idea. Similar to the Laurent polynomial case, one adds to f_τ a sequence of “small” summable elements until one obtains a “remainder”

$$\bar{f}_\tau = \sum_{k \geq 1} \sum_{\alpha \in \tau} \frac{\text{dres}(f, \tau, k)_\alpha}{(x - \alpha)^k}$$

such that f_τ is Mahler summable iff $\bar{f}_\tau = 0$.

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- ▶ The definition (and proofs) for Mahler discrete residues at $\tau(\zeta)$ for $\zeta \in \mathbb{K}_t^\times$ a root of unity is similar in spirit, but more technical, due to the perverse (pre-)periodic behavior of roots of unity under the p -power map.

Main Result

Theorem (Arreche-Z. 2022) Given $f \in \mathbb{K}(x)$. Then f is Mahler summable if and only if $\text{dres}(f, \infty) = 0$ and $\text{dres}(f, \tau, k) = 0$ for all $k \in \mathbb{N}$ and $\tau \in \mathcal{T}_M$.

Thanks!