① § 10.2 Calculus with Parametric Curves

Combined with parametric equations, we use calculus to compute tangents, areas, arc length, and surface area ~~of~~ related to parametric curves.

- Tangents.

A parametric curve is given by:

$$x = f(t), \quad y = g(t), \quad \alpha \le t \le \beta$$

By the Chain Rule: $y = y(x) = y(x(t))$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad (1)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Ex 1: A curve C is described by:

$$x = t^2, \ y = t^3 - 3t$$

(a) Show that C has two tangents at $(3, 0)$, and find their equations

(a) $y = t^3 - 3t = t(t^2 - 3) = 0 \Rightarrow t = 0$ or $t = \pm\sqrt{3}$

Thus, $(3, 0)$ on C arise from two parameters $t = \sqrt{3}$ and $t = -\sqrt{3}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

Thus, the slope of the tangent when $t = \pm\sqrt{3}$ is $\pm\sqrt{3}$. Thus, the equations of tangents at $(3, 0)$ are

(2)

$y=\sqrt{3}(x-3)$ and $y=-\sqrt{3}(x-3)$

(b) Find the points on C where the tangent is horizontal or vertical

C has a horizontal tangent when $dy/dx=0 \Leftrightarrow \frac{dy}{dt}=0,\ \frac{dx}{dt}\neq 0$

$$\frac{dy}{dt}=3t^2-3=0 \Rightarrow t=\pm 1$$

The corresponding points are $(1,-2)$ and $(1,2)$

C has a vertical tangent when $\frac{dx}{dt}=0,\ \frac{dy}{dt}\neq 0$

$$\frac{dx}{dt}=2t=0 \Rightarrow t=0$$

The ________ is $(0,0)$

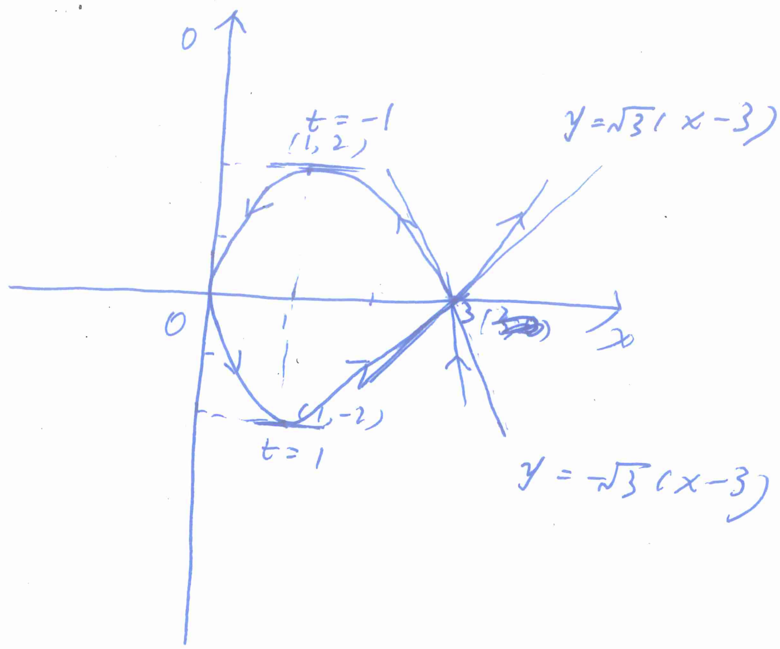
(c) Determine where the curve is concave upward or downward

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{3(t^2+1)}{4t^3}$$

Thus, the ~~a~~curve is concave upward when $t>0$

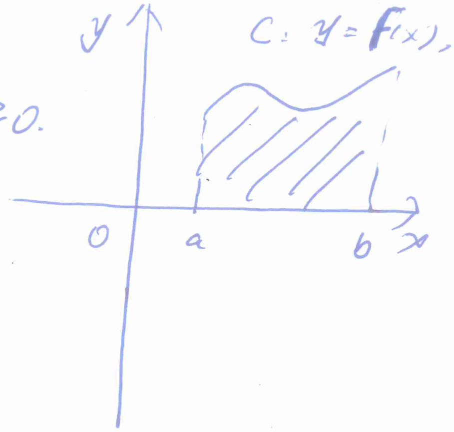
downward when $t<0$.

(d) Sketch the curve



③ Areas

$$A=\int_a^b F(x)\,dx, \text{ where } F(x)\geq 0.$$



Assume C is traced out by

$x=f(t),\ y=g(t),\ \alpha\leq t\leq\beta.$

Then

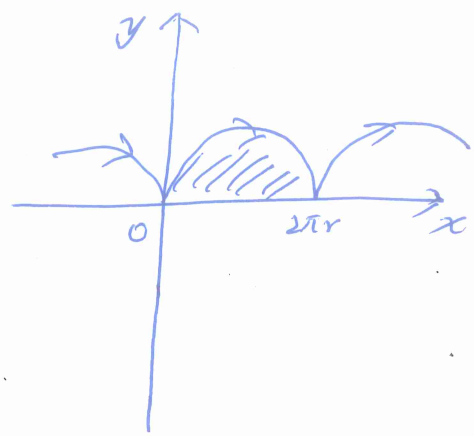
$$A=\int_a^b y\,dx$$

$$=\int_a^b y(t)\,dx(t)$$

$$=\int_\alpha^\beta g(t)f'(t)\,dt$$

Ex 3. Find the area under one arch of the cycloid

$$x=r(\theta-\sin\theta),\quad y=r(1-\cos\theta)$$



One arch of the cycloid is given by $0\leq\theta\leq 2\pi$.

$$A=\int_0^{2\pi r} y\,dx$$

$$=\int_0^{2\pi} r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta$$

$$=r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

$$=r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

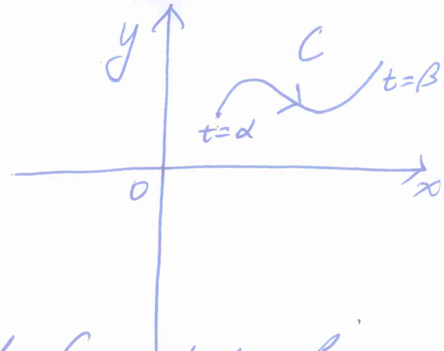
$$=r^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]\Big|_0^{2\pi}$$

$$=3\pi r^2$$

④ Arc Length

Assume a smooth curve C is described by:

$x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$



C is traversed exactly once as t increased from α to β

The Length of C is

$$L = \int ds$$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

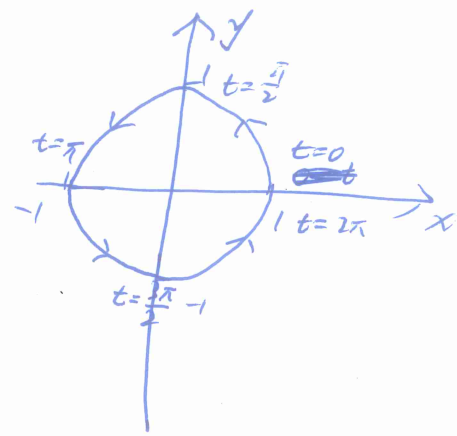
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\Rightarrow L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex 4. Consider the ~~curve~~ unit circle:

$x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

Find its length



$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{2\pi} dt$$

$$= 2\pi.$$

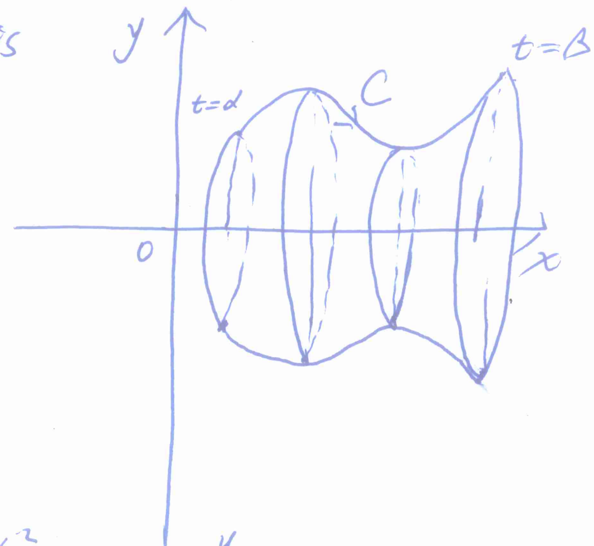
- Surface area

Suppose the smooth curve C given by $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, $g(t) \ge 0$ is rotated about the x-axis. If C is traversed exactly once as t increased from α to β, then

⑤ the area of the resulting surface is

$$S = \int 2\pi y \, ds$$

$$= \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

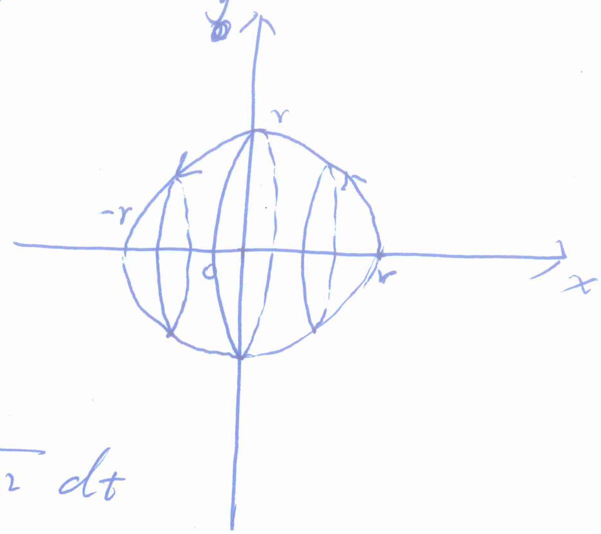


Ex 6. Show that the surface area of a sphere of radius r is $4\pi r^2$

The sphee is obtained by rotating the semicircle

$x = r\cos t$, $y = r\sin t$, $0 \le t \le \pi$

about the x-axis.



Then

$$S = \int_0^{\pi} 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2} \, dt$$

$$= 2\pi r^2 \int_0^{\pi} \sin t \, dt$$

$$= 2\pi r^2 \; (\cos t)\Big|_0^{\pi}$$

$$= 4\pi r^2$$