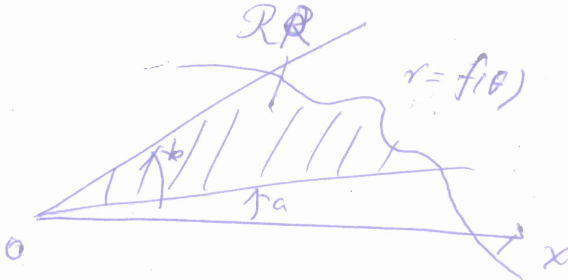
① §10.4 Areas ~~and~~ and Lengths in Polar Coordinates.

Consider a polar curve C describeb by the polar equation

$$r = f(\theta), \quad a \le \theta \le b \quad (*)$$

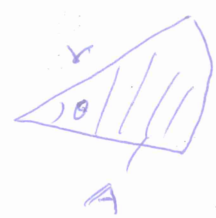
where $f(\theta) \ge 0$, $a \le \theta \le b$

Question: how to compute the area of a region whose boundary is given by (*)?



Recall: the formula for the area of a sector of a circle.

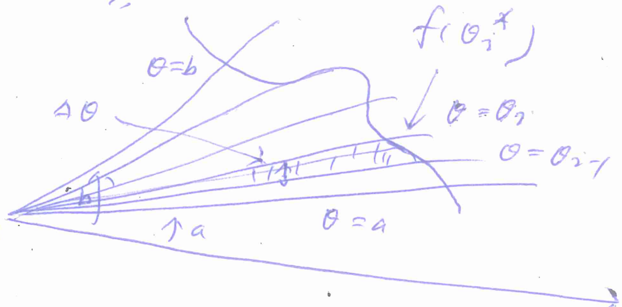
$$A = \frac{1}{2} r^2 \theta$$



Idea: 1. Divide $[a,b]$ into n intervals, with endpoints $\theta_0, \cdots, \theta_n$,

$$\Delta\theta = \theta_i - \theta_{i-1} = \frac{b-a}{n}$$

2. Choose $\theta_i^* \in [\theta_{i-1}, \theta_i]$. ~~to be~~



Then ~~ΔAi of the ith~~ the area of the ith region.

$$\Delta A_i \approx \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta$$

~~to~~ the total area A of R

$$A \approx \sum_{i=1}^{n} \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta$$

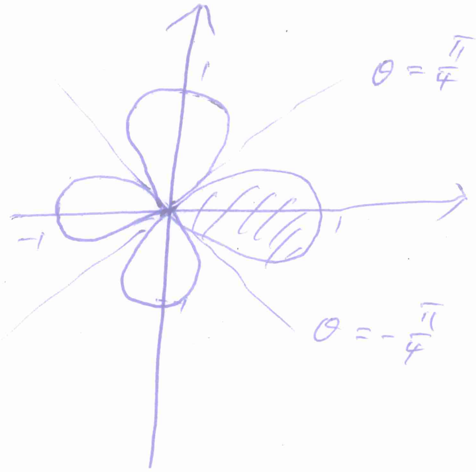
Thus,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta$$

$$= \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta \quad (3)$$

$$= \int_a^b \frac{1}{2} r^2 d\theta \quad (4)$$

② Ex 1. Find the area enclosed by one loop of the four-leaves rose $r = \cos 2\theta$



$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \cos^2 2\theta \, d\theta$$

$$= \int_{0}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

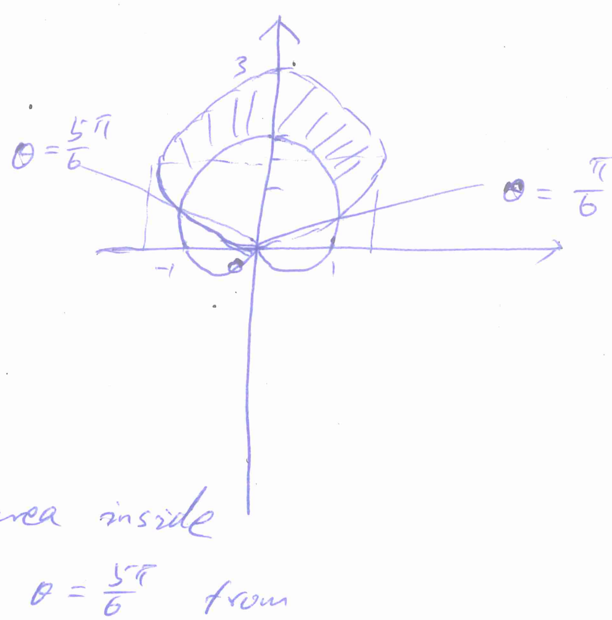
$$= \int_{0}^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 4\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8}$$

Ex 2. Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1+\sin\theta$.

points of intersection of the two curves.



$$3\sin\theta = 1 + \sin\theta$$

$$\Rightarrow \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

(~~[illegible]~~ subtracting the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ from that inside the circle - from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$).

By symmetry, we have

③

$$A = 2(A_1 - A_2)$$

$$A_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(3\sin\theta)^2 d\theta, \quad A_2 = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1+\sin\theta)^2 d\theta$$

$$\Rightarrow A = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (8\sin^2\theta - 1 - 2\sin\theta)\, d\theta \quad \left(\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\right)$$

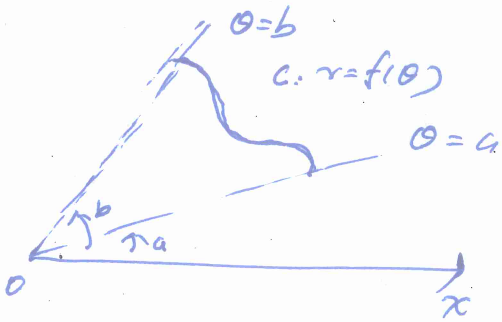
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta$$

$$= \Big[3\theta - 2\sin 2\theta + 2\cos\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \pi$$

Arc Length

Motivation: Find the length of a polar curve $r = f(\theta)$, $a \le \theta \le b$



Then

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

Recall:

$$L = \int ds$$

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$\Downarrow$$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

By the Product Rule,

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Thus,

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$
$$+ \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

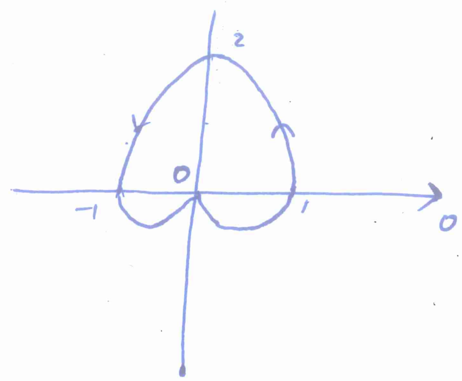
Assume $f' = \frac{dr}{d\theta}$ is continuous, we have

$$L = \int ds$$

$$= \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Ex 4. Find the length of the cardioid $r = 1 + \sin\theta$.



$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2 + 2\sin\theta}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{(2+2\sin\theta)(2-2\sin\theta)}}{\sqrt{2-2\sin\theta}}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\sqrt{\cos^2\theta}}{\sqrt{2-2\sin\theta}}\, d\theta$$

$$= \frac{4}{\sqrt{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta\, d\theta}{\sqrt{1-\sin\theta}}$$

$$= \frac{4}{\sqrt{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\sin\theta}{\sqrt{1-\sin\theta}}$$

$$= \frac{4}{\sqrt{2}} \int_{-1}^{1} \frac{du}{\sqrt{1-u}}$$

$$= \frac{4}{\sqrt{2}} \lim_{t\to 1^-} \int_{-1}^{t} \frac{du}{\sqrt{1-u}}$$

⑤

$$= \frac{4}{\sqrt{2}} \lim_{t \to 1^-} \left[-2\sqrt{1-u} \right]_{-1}^{t}$$

$$= \frac{4}{\sqrt{2}} \lim_{t \to 1^-} \left[-2\sqrt{1-t} + 2\sqrt{2} \right]$$

$$= \frac{4}{\sqrt{2}} \cdot 2\sqrt{2}$$

$$= 8 \quad)$$