

ON D-FINITENESS OF A SYMMETRIC FUNCTION

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ABSTRACT. We show that a symmetric function induced by a Weyl character of a given representation of a simple complex Lie algebra is D-finite. Moreover, we prove that the fake degree sequence associated to the representation and the Lie algebra is q -holonomic.

In this note, we consider symmetric functions (series) as formal power series in infinitely many variables [13, Section 7.1] $X = x_1, x_2, \dots$, which is equivalent to the definition in [12], over a field \mathbb{K} of characteristic zero. Let $Y = y_1, y_2, \dots$ be another set of variables. Then the *product* of X and Y is defined by $X.Y = \sum_{i,j} x_i y_j$. Let $\mathbb{K}[[X]]$ be the ring of formal power series in X and Λ be the ring of symmetric functions in X . Let $f \in \Lambda$ and $g \in \mathbb{K}[[X]]$ with $g = \sum_{i=1}^{\infty} t_i$, where t_i is a monomial in x_i 's. We define the *plethysm* (or *composition*) [7] of f by g to be $f(t_1, t_2, \dots)$, and denote it by $f[g]$. For each $k \in \mathbb{N}$, let $h_k(X)$ be the k -th complete homogenous symmetric function in X . As a matter of notation, we set $h_k(X) = 0$ for $k < 0$. When it is clear from context, we may abbreviate $h_k(X)$ to h_k . Then the corresponding generating function [3] is denoted by

$$H(t) = \sum_{k=0}^{\infty} h_k(X) t^k = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}.$$

We say that $F \in \mathbb{K}[[X]]$ is *D-finite* [6, section 5] in the infinitely many variables x_i if, for any choice of a finite set S of X , the specification to 0 of each $x_i \in X \setminus S$ gives rise to a power series that is *D-finite*, in the classical sense, in each variable $x_i \in S$. Clearly, $H(t)$ is *D-finite* in the x_i 's and t . Next, let us recall the closure properties of *D-finite* series in finitely many variables.

Theorem 0.1. (1) *The set of D-finite power series forms a \mathbb{K} -algebra of $\mathbb{K}[[x_1, \dots, x_n]]$ for the usual product of series;*

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- (2) If F is D -finite in x_1, \dots, x_n then for any finite subset of variables x_{i_1}, \dots, x_{i_k} the specialization of F at $x_{i_1} = \dots = x_{i_k} = 0$ is D -finite in the remaining variables;
- (3) If P is a polynomial in x_1, \dots, x_n , then $\exp P(x)$ is D -finite in the x_1, \dots, x_n ;
- (4) If F and G are D -finite in the variables x_1, \dots, x_{m+n} 's, then the Hadamard product $F \odot G$ with respect to x_1, \dots, x_n is D -finite in the x_1, \dots, x_{m+n} ;
- (5) Assume that F is a D -finite series in x_1, \dots, x_n . Let f_1, \dots, f_d be algebraic functions in $\mathbf{z} = z_1, \dots, z_e$ which means there exist nonzero polynomials $q_1, \dots, q_d \in \mathbb{K}[h, z_1, \dots, z_e]$ such that $q_i(f_i(\mathbf{z}), z_1, \dots, z_e) = 0$. Then

$$G(\mathbf{z}, x_{d+1}, \dots) = F(f_1(\mathbf{z}), \dots, f_d(\mathbf{z}), x_{d+1}, \dots, x_n)$$

is D -finite in the $\mathbf{z}, x_{d+1}, \dots, x_n$.

The proofs of the first three properties are available in [13], and the fourth one is due to Lipshitz [11]. The last one is given in [9, Theorem 2.18, page 35]. It is straightforward to see that the above properties also hold for D -finite series in an infinitely number of variables.

For each $k \geq 1$, let $p_k(X)$ be the k -th power sum symmetric function in X . We say that a symmetric function is D -finite [6, page 272] if it is D -finite in the power sum symmetric functions p_k 's. By item (5) of Theorem 0.1, a D -finite symmetric function is also D -finite in the x_i 's. Since $H(t) = \exp(\sum_{k=1}^{\infty} p_k t^k / k)$, it is straightforward to see from item (3) of Theorem 0.1 that $H(1)$ is a D -finite symmetric function and $H(t)$ is D -finite in the p_k 's and t .

Let $f \in \mathbb{K}[[X]]$ be a quasisymmetric function. We say that f is D -finite if it is D -finite in the fundamental (or monomial) quasisymmetric functions. The next proposition gives the relation between D -finite symmetric functions and D -finite quasisymmetric functions.

Proposition 0.2. *Let $f \in \mathbb{K}[[X]]$ be a symmetric function. Then f is a D -finite symmetric function if and only if it is a D -finite quasisymmetric functions.*

Proof. \implies : Assume that f is a D -finite symmetric function. Since each power sum symmetric function is a \mathbb{Q} -linear combination of Schur functions, it follows from [13, Theorem 7.19.7] that each power sum symmetric function is a \mathbb{Q} -linear combination of fundamental quasisymmetric functions. By item (5) of Theorem 0.1, we see that f is D -finite in the quasisymmetric functions.

\impliedby : Assume that f is a D -finite quasisymmetric function. Since each monomial symmetric function is a \mathbb{Q} -linear combination of power sum symmetric functions, it follows from item (5) of Theorem 0.1 that we just need to prove that f is D -finite in the monomial symmetric functions. Let t be a positive integer and λ be a partition of t . Set

$f(m_\lambda) = f|_{m_\mu=0, \mu \neq \lambda}$. Without loss of generality, it suffices to prove that $f(m_\lambda)$ satisfies a nontrivial linear ODE with polynomial coefficients in m_λ . Let A be the set of all distance permutations of parts of λ . Set

$$M = \{M_\alpha \mid \alpha \in A\}.$$

Then f is D -finite in the monomial quasisymmetric functions M . Take $\alpha \in A$. Then there exists a non-negative integer r and polynomial $p_j \in \mathbb{K}[M]$ for $j = 0, \dots, r$ with $p_r(M) \neq 0$ such that

$$(1) \quad p_r(M) \frac{\partial^r f(m_\lambda)}{\partial M_\alpha^r} + p_{r-1}(M) \frac{\partial^{r-1} f(m_\lambda)}{\partial M_\alpha^{r-1}} + \dots + p_0(M) f(m_\lambda) = 0.$$

Since $m_\lambda = M_\alpha + \sum_{\beta \in A, \beta \neq \alpha} M_\beta$, we have that

$$\frac{\partial^k f(m_\lambda)}{\partial M_\alpha^k} = \frac{\partial^k f(m_\lambda)}{\partial m_\lambda^k} \quad \text{for each } k \in \mathbb{N}.$$

Thus, (1) becomes

$$(2) \quad p_r(M) \frac{\partial^r f(m_\lambda)}{\partial m_\lambda^r} + p_{r-1}(M) \frac{\partial^{r-1} f(m_\lambda)}{\partial m_\lambda^{r-1}} + \dots + p_0(M) f(m_\lambda) = 0.$$

Let $n = |M|$ and $\sigma \in \mathfrak{S}_n$. For each $p \in \mathbb{K}[[M]]$, we define

$$\sigma(p(M)) := p(M_{\alpha_{\sigma(1)}}, \dots, M_{\alpha_{\sigma(n)}}).$$

Set $\tilde{p}_j = \sum_{\sigma \in \mathfrak{S}_n} p_j(\sigma(M))$ for $j = 0, \dots, r$. Since $p_r \neq 0$, so is \tilde{p}_r . Moreover, it is straightforward to see that \tilde{p}_j is a symmetric function of finite degree. Thus, we may assume that $\tilde{p}_j \in \mathbb{K}[m_\lambda, m_{\mu_1}, \dots, m_{\mu_\ell}]$. For each $\sigma \in \mathfrak{S}_n$, applying σ to (2) and then taking the sum, we get

$$(3) \quad \tilde{p}_r(m_\lambda, m_{\mu_1}, \dots, m_{\mu_\ell}) \frac{\partial^r f(m_\lambda)}{\partial m_\lambda^r} + \tilde{p}_{r-1}(m_\lambda, m_{\mu_1}, \dots, m_{\mu_\ell}) \frac{\partial^{r-1} f(m_\lambda)}{\partial m_\lambda^{r-1}} + \dots + \tilde{p}_0(m_\lambda, m_{\mu_1}, \dots, m_{\mu_\ell}) f(m_\lambda) = 0.$$

By taking the content over $\mathbb{K}[m_\lambda, m_{\mu_1}, \dots, m_{\mu_\ell}]$, we may assume that $\tilde{p}_r, \dots, \tilde{p}_0$ are relatively prime. By specifying $m_{\mu_i} = 0$ in (3) and taking the content over $\mathbb{K}[m_\lambda, m_{\mu_1}, \dots, m_{\mu_\ell}]$ iteratively for $i \in \{1, \dots, \ell\}$, we see that there exists $\bar{p}_j \in \mathbb{K}[m_\lambda]$ for $j = 0, \dots, \bar{r}$ with $\bar{r} \leq r$ and $\bar{p}_{\bar{r}} \neq 0$ such that

$$\bar{p}_{\bar{r}}(m_\lambda) \frac{\partial^{\bar{r}} f(m_\lambda)}{\partial m_\lambda^{\bar{r}}} + \bar{p}_{\bar{r}-1}(m_\lambda) \frac{\partial^{\bar{r}-1} f(m_\lambda)}{\partial m_\lambda^{\bar{r}-1}} + \dots + \bar{p}_0(m_\lambda) f(m_\lambda) = 0.$$

□

Lemma 0.3. *Let $a \in \mathbb{Z}$, $v_1, \dots, v_d \in \mathbb{Z}$ and t_1, \dots, t_d be variables. Then the series*

$$F = \prod_{i=1}^d \sum_{\ell_i=-\infty}^{\infty} h_{a+v_1\ell_1+\dots+v_d\ell_d} t_1^{\ell_1} \dots t_d^{\ell_d}$$

is D -finite in the p_i 's and t_j 's.

Proof. Without loss of generality, we assume that $v_1, \dots, v_d \in \mathbb{Z} \setminus \{0\}$. Otherwise, by reordering the indexes, we assume $v_1, \dots, v_e \in \mathbb{Z} \setminus \{0\}$ and $v_{e+1} = \dots = v_d = 0$. Then

$$F = \tilde{F} \cdot \prod_{i=e+1}^d \left(\frac{t_i^{-1}}{1 - t_i^{-1}} + \frac{1}{1 - t_i} \right),$$

where

$$\tilde{F} = \prod_{i=1}^e \sum_{\ell_i=-\infty}^{\infty} h_{a+v_1\ell_1+\dots+v_e\ell_e} t_1^{\ell_1} \dots t_e^{\ell_e}.$$

According to item (1) of Theorem 0.1, we only need to show that \tilde{F} is D-finite.

Set $s_i = t_i^{1/v_i}$ for $i = 1, \dots, d$. Then $F = s_1^{-a} \cdot G$, where

$$G = \prod_{i=1}^d \sum_{\ell_i=-\infty}^{\infty} h_{a+v_1\ell_1+\dots+v_d\ell_d} s_1^{a+v_1\ell_1} s_2^{v_2\ell_2} \dots s_d^{v_d\ell_d}.$$

In light of item (1) and (5) of Theorem 0.1, it suffices to show that G is D -finite in the p_i 's and s_j 's. Let z_1, \dots, z_d be several auxiliary variables. We may write

$$(4) \quad G = \left(L \odot \prod_{i=1}^d \sum_{\ell_i=-\infty}^{\infty} z_1^{a+v_1\ell_1} z_2^{v_2\ell_2} \dots z_d^{v_d\ell_d} \right)_{z_1=\dots=z_d=1},$$

where

$$L = \prod_{i=1}^d \sum_{\ell_i=-\infty}^{\infty} h_{\ell_1+\ell_2+\dots+\ell_d} (s_1 z_1)^{\ell_1} (s_2 z_2)^{\ell_2} \dots (s_d z_d)^{\ell_d}.$$

Clearly, we see that

$$\prod_{i=1}^d \sum_{\ell_i=-\infty}^{\infty} z_1^{a+v_1\ell_1} z_2^{v_2\ell_2} \dots z_d^{v_d\ell_d} = z_1^a \prod_{i=1}^d \left(\frac{z_i^{-v_i}}{1 - z_i^{-v_i}} + \frac{1}{1 - z_i^{v_i}} \right)$$

is D -finite in the p_i 's, s_j 's and z_k 's. Therefore, it follows from item (2), (4) of Theorem 0.1 and (4) that we only need to prove that L is also D -finite in those variables. We may write

$$\begin{aligned} L &= \prod_{i=1}^d \sum_{\ell_i=-\infty}^{\infty} \left(\frac{s_1 z_1}{s_d z_d} \right)^{\ell_1} \dots \left(\frac{s_{d-1} z_{d-1}}{s_d z_d} \right)^{\ell_{d-1}} h_{\ell_1+\ell_2+\dots+\ell_d} (s_d z_d)^{\ell_1+\ell_2+\dots+\ell_d} \\ &= \left(\prod_{i=1}^{d-1} \sum_{\ell_i=-\infty}^{\infty} \left(\frac{s_i z_i}{s_d z_d} \right)^{\ell_i} \right) \cdot \left(\sum_{\ell_d=-\infty}^{\infty} h_{\ell_d} (s_d z_d)^{\ell_d} \right) \\ &= \left(\prod_{i=1}^{d-1} \left(\frac{(s_i z_i)^{-1}}{1 - (s_i z_i)^{-1}} + \frac{s_i z_i}{1 - s_i z_i} \right) \right) \cdot H(s_d z_d). \end{aligned}$$

From the last identity and item (1) of Theorem 0.1, we conclude that L is D -finite in the p_i 's, s_j 's and z_k 's. \square

Let V be a representation of a simple complex Lie algebra \mathfrak{g} and $\text{ch}_V(Y)$ be the Weyl character [8] of V in variables Y . Take the set of monomials whose sum is $\text{ch}_V(Y)$ and denote it by $m_V(Y)$. Then $H[X \cdot m_V(Y)]$ is an element of the tensor product of symmetric functions in X and Laurent polynomials in Y . Denote by Δ the Laurent polynomial in Y which appears in the Weyl integration formula [1]. Then the product $\Delta \cdot H[X \cdot m_V(Y)]$ is also in the tensor product of symmetric functions in X and Laurent polynomials in Y . Since $\text{ch}_V(Y)$ and Δ are Laurent polynomials in Y , we may assume that $Y = y_1, \dots, y_m$ are variables appeared in $\text{ch}_V(Y)$ and Δ . A monomial in Y is then denoted by $Y^{\mathbf{e}} := y_1^{e_1} \dots y_m^{e_m}$ for some $\mathbf{e} = (e_1, \dots, e_m) \in \mathbb{Z}^m$. Let $F \in \mathbb{K}[[X]][y_1^{\pm 1}, \dots, y_m^{\pm 1}]$ and $\mathbf{e} \in \mathbb{Z}^m$, we denote the coefficient of $Y^{\mathbf{e}}$ in F by $[Y^{\mathbf{e}}](F)$.

Theorem 0.4. *The symmetric function $S = [Y^0](\Delta \cdot H[X \cdot m_V(Y)])$ is D -finite.*

Proof. We first prove the claim in the case that $\Delta = Y^\gamma$ is a monomial in Y . Set $m_V(Y) = \{Y^{\alpha_1}, \dots, Y^{\alpha_s}\}$. Then

$$\begin{aligned} \Delta \cdot H[X \cdot m_V(Y)] &= Y^\gamma \cdot \prod_{i=1}^s \left(\prod_{j=1}^{\infty} \frac{1}{1 - x_j \cdot Y^{\alpha_i}} \right) \\ &= Y^\gamma \cdot \prod_{i=1}^s \sum_{k_i=0}^{\infty} h_{k_i}(X) Y^{\alpha_i k_i}. \end{aligned}$$

Then we have

$$\begin{aligned} S &= [Y^0](\Delta \cdot H[X \cdot m_V(Y)]) \\ (5) \quad &= \sum_{\alpha_1 k_1 + \alpha_2 k_2 + \dots + \alpha_s k_s = -\gamma} h_{k_1}(X) \cdot h_{k_2}(X) \cdots h_{k_s}(X). \end{aligned}$$

Thus, in order to show that S is D -finite, we need to derive the solutions of the following system of linear Diophantine equations:

$$(6) \quad \alpha_1 k_1 + \alpha_2 k_2 + \dots + \alpha_s k_s = -\gamma,$$

where $k_i \in \mathbb{Z}$ is unknown. Set $A = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^{m \times s}$ and $C = -\gamma \in \mathbb{Z}^{m \times 1}$. Then (14) can be written into the following matrix form:

$$(7) \quad AX = C.$$

There exist two unimodular matrices $U \in \mathbb{Z}^{m \times m}$ and $V \in \mathbb{Z}^{s \times s}$ such that $B = (b_{i,j}) = UAV$ is the Smith normal form of A with $b_{i,i} \neq 0$ for $i = 1, \dots, s - d$ and the other entries being zeros. Set $Y = V^{-1}X$ and $Q = UC = (q_1, \dots, q_m)^\top$. Then (7) can be rewritten as

$$(8) \quad BY = Q.$$

If there exists $i \in \{1, \dots, s-d\}$ such that $b_{i,i} \nmid q_i$, then (8) has no solution. In this case, we have $S = 0$, which is D -finite in the x_i 's.

If $b_{i,i} \mid q_i$ for $i = 1, \dots, s-d$, then the solutions of (7) are

$$V \begin{bmatrix} \frac{q_1}{b_{1,1}} \\ \vdots \\ \frac{q_{s-d}}{b_{s-d,s-d}} \\ \ell_1 \\ \vdots \\ \ell_d \end{bmatrix},$$

where ℓ_1, \dots, ℓ_d are arbitrary integers. It implies that the solutions of (14) are

$$k_i = a_i + v_{i,1}\ell_1 + \dots + v_{i,d}\ell_d,$$

where $a_i, v_{i,j} \in \mathbb{Z}$ for $i = 1, \dots, s$ and $j = 1, \dots, d$. By (5), we see that

$$(9) \quad S = \prod_{i=1}^s \prod_{j=1}^d \sum_{\ell_j=-\infty}^{\infty} h_{a_i+v_{i,1}\ell_1+\dots+v_{i,d}\ell_d}$$

$$(10) \quad = (F_1 \odot F_2 \odot \dots \odot F_s)_{t_1=\dots=t_d=1},$$

where

$$F_i = \prod_{j=1}^d \sum_{\ell_j=-\infty}^{\infty} h_{a_i+v_{i,1}\ell_1+\dots+v_{i,d}\ell_d} t_1^{\ell_1} \dots t_d^{\ell_d} \quad \text{for } i = 1, \dots, s.$$

By Lemma 0.3, we see that F_i is D -finite in the p_j 's and t_k 's. In light of item (2), (4) of Theorem 0.1 and (10), we conclude that S is D -finite in the p_i 's.

It follows from item (1) of Theorem 0.1 that the claim also holds when Δ is a Laurent polynomial in Y . \square

By Proposition 0.2, the symmetric function in the above theorem is also a D -finite quasisymmetric function.

Example 0.5. Let V be an irreducible representation of $\mathfrak{g} := SL(2)$ such that $\text{ch}_V(Y) = y + 1/y$ and $\Delta = 1 - y^2$. Then

$$\begin{aligned} H[X \cdot \text{m}_V(Y)] &= \left(\prod_{i=1}^{\infty} \frac{1}{1 - x_i y} \right) \cdot \left(\prod_{i=1}^{\infty} \frac{1}{1 - x_i y^{-1}} \right) \\ &= \left(\sum_{k=0}^{\infty} h_k y^k \right) \cdot \left(\sum_{k=0}^{\infty} h_k y^{-k} \right). \end{aligned}$$

Thus, by Theorem 0.4, the symmetric function

$$\begin{aligned} S &= [Y^0](\Delta \cdot H[X.\mathbf{m}_V(y)]) \\ &= \sum_{k=0}^{\infty} (h_k^2 - h_{k-1}h_{k+1}) \\ &= \sum_{k=0}^{\infty} (h_k^2 - h_k h_{k+2}) \end{aligned}$$

is D -finite in the p_i 's. For each $n \in \mathbb{N}$, set $S(p_n) = S|_{p_k=0, k \neq n}$.

When $n = 1$, we have

$$\begin{aligned} S(p_1) &= \sum_{i=0}^{\infty} \frac{p_1^{2i}}{i!^2} \left(1 + \frac{p_1^2}{(i+1)(i+2)} \right) \\ &= I_0(2p_1) + I_2(2p_1), \end{aligned}$$

where $I_k(z)$ is the modified Bessel function of the first kind. By item (1) of Theorem 0.1, it is straightforward to see that $S(p_1)$ satisfies the following fourth-order linear ordinary differential equation (ODE):

$$\begin{aligned} p_1^3 \frac{d^4 S(p_1)}{dp_1^4} + 6p_1^2 \frac{d^3 S(p_1)}{dp_1^3} + (3p_1 - 8p_1^3) \frac{d^2 S(p_1)}{dp_1^2} \\ - (3 + 24p_1^2) \frac{dS(p_1)}{dp_1} + 16p_1^3 S(p_1) = 0. \end{aligned}$$

When $n = 2$, we have

$$\begin{aligned} S(p_2) &= \sum_{i=0}^{\infty} \frac{p_2^{2i}}{4^i i!^2} \left(1 + \frac{p_2}{2(i+1)} \right) \\ &= I_0(p_2) + I_1(p_2), \end{aligned}$$

which satisfies the following fourth-order linear ODE:

$$\begin{aligned} p_2^2 \frac{d^4 S(p_2)}{dp_2^4} + 6p_2 \frac{d^3 S(p_2)}{dp_2^3} + (6 - 2p_2^2) \frac{d^2 S(p_2)}{dp_2^2} - 6p_2 \frac{dS(p_2)}{dp_2} \\ - (3 - p_2^2) S(p_2) = 0. \end{aligned}$$

When $n > 2$, we have $S(p_n) = \sum_{i=0}^{\infty} p_n^{2i} / (n^{2i} i!^2) = I_0(2p_n/n)$, which satisfies the following second-order linear ODE:

$$n^2 p_n \frac{d^2 S(p_n)}{dp_n^2} + n^2 \frac{dS(p_n)}{dp_n} - 4p_n S(p_n) = 0.$$

In Example 0.7, we will show that the fake degree sequence associated to S above is q -holonomic and closely related to q -Catalan numbers [4].

In [16], the authors proved that the symmetric function in Theorem 0.4 is equal to

$$(11) \quad S = \sum_{n=0}^{\infty} \mathbf{fr}((\otimes^n V)^g),$$

where $\mathbf{fr}(V)$ is the Frobenius character [13, § 7.18] of the representation V of \mathfrak{S}_n , which is a homogeneous symmetric function of degree n and is an invariant of the representation V .

Let q be a transcendental indeterminate over the field \mathbb{K} . For $n \in \mathbb{N}$, the q -integer $[n]_q$ is the polynomial

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1},$$

and the q -factorial $[n]_q! := [n]_q \cdot [n-1]_q \cdots [1]_q$. For $k_1, \dots, k_r \in \mathbb{Z}$, the q -multinomial coefficient is defined by

$$\left[\begin{matrix} n \\ k_1, k_2, \dots, k_r \end{matrix} \right]_q = \begin{cases} \frac{[n]_q!}{[k_1]_q! [k_2]_q! \cdots [k_r]_q!} & \text{if } k_i \geq 0, \text{ and } \sum_{i=1}^r k_i = n, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the fake degree function

$$\mathbf{fd} : \Lambda \longrightarrow \mathbb{K}(q),$$

which is a linear map from symmetric functions to polynomials in q . The interpretation in terms of representation theory is that if V has an action of \mathfrak{S}_n then the fake degree of the Frobenius character is a polynomial which describes the action of the long cycle. This can be defined on the basis of complete homogeneous functions and extended by linearity. Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be an integer partition. Then the evaluation of \mathbf{fd} at h_λ is defined by:

$$\mathbf{fd}(h_\lambda) = \left[\begin{matrix} |\lambda| \\ \lambda_1, \lambda_2, \dots, \lambda_s \end{matrix} \right]_q.$$

The fake degree map is not a ring homomorphism but has the following property: Let f_r and g_s be homogeneous symmetric functions of degree r and s , respectively. Then

$$\mathbf{fd}(f_r g_s) = \left[\begin{matrix} r+s \\ r \end{matrix} \right]_q \mathbf{fd}(f_r) \mathbf{fd}(g_s).$$

A univariate sequence $(b_n(q))_{n \in \mathbb{N}}$ is called *q -holonomic* [5] if it satisfies a nontrivial linear q -difference equation with coefficients that are polynomials in q and q^n ; that is, there exists a non-negative integer r and bivariate polynomial $c_j(x, y) \in \mathbb{K}[x, y]$ for $j = 0, \dots, r$ with $c_r(x, y) \neq 0$ such that for each $n \in \mathbb{N}$ the following identity holds:

$$c_r(q, q^n) b_{n+r}(q) + c_{r-1}(q, q^n) b_{n+r-1}(q) + \cdots + c_0(q, q^n) b_n(q) = 0.$$

Similar to D -finite functions, the class of q -holonomic sequences satisfy closure properties under certain operations such as addition and multiplication. For instance, see [9, Section 2.3] for details.

Given g and $V \in \mathcal{Rep}(g)$. We define $f_n(q) = \mathbf{fd}(\mathbf{fr}((\otimes^n V)^g))$ for each $n \in \mathbb{N}$ and call $(f_n(q))_{n \in \mathbb{N}}$ the fake degree sequence associated to g and V . A natural question is whether this sequence is q -holonomic or not. In the below theorem, we give an answer for this problem.

Theorem 0.6. *Given a simple complex Lie algebra \mathfrak{g} and $V \in \mathcal{R}ep(\mathfrak{g})$. Then the fake degree sequence associated to \mathfrak{g} and V is q -holonomic.*

Proof. Let $\text{ch}_V(Y)$ be the Weyl character of V in Y and $m_V(Y)$ be the set of monomials whose sum is $\text{ch}_V(Y)$. Set Δ to be the Laurent polynomial in Y which appears in the Weyl integration formula. We first prove the claim in the case that $\Delta = Y^\gamma$ is a monomial in Y . Set $m_V(Y) = \{Y^{\alpha_1}, \dots, Y^{\alpha_s}\}$. Then

$$\begin{aligned} S &= [Y^0](\Delta \cdot H[X, m_V(Y)]) \\ &= \sum_{\alpha_1 k_1 + \alpha_2 k_2 + \dots + \alpha_s k_s = -\gamma} h_{k_1}(X) \cdot h_{k_2}(X) \cdots h_{k_s}(X). \end{aligned}$$

By the arguments in the proof of Theorem 0.4, there exist $a_i, v_{i,j} \in \mathbb{Z}$ for $i = 1, \dots, s$ and $j = 1, \dots, d$ such that

$$S = \prod_{i=1}^s \prod_{j=1}^d \sum_{\ell_j=-\infty}^{\infty} h_{a_i + v_{i,1}\ell_1 + \dots + v_{i,d}\ell_d}.$$

Set $a = \sum_{i=1}^s a_i \in \mathbb{Z}$, and $v_j = \sum_{i=1}^s v_{i,j} \in \mathbb{Z}$ for $j = 1, \dots, d$. Let

$$\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{Z}^d, \quad \boldsymbol{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{Z}^d,$$

and $\mathbf{v}_i = (v_{i,1}, \dots, v_{i,d}) \in \mathbb{Z}^d$ for $i = 1, \dots, s$. On the other hand, since $S = \sum_{n=0}^{\infty} \mathbf{fr}((\otimes^n V)^{\mathfrak{g}})$, we see that

$$\mathbf{fr}((\otimes^n V)^{\mathfrak{g}}) = \sum_{a + \mathbf{v} \cdot \boldsymbol{\ell} = n} \prod_{i=1}^s h_{a_i + \mathbf{v}_i \cdot \boldsymbol{\ell}} \quad \text{for } n \geq 0.$$

Then the n -th term of the fake degree sequence associated to \mathfrak{g} and V is

$$(12) \quad f_n(q) = \mathbf{fd}(\mathbf{fr}((\otimes^n V)^{\mathfrak{g}})) = \sum_{a + \mathbf{v} \cdot \boldsymbol{\ell} = n} r(\boldsymbol{\ell}, n),$$

where

$$r(\boldsymbol{\ell}, n) = \left[a_1 + \mathbf{v}_1 \cdot \boldsymbol{\ell}, \quad \dots, \quad a_s + \mathbf{v}_s \cdot \boldsymbol{\ell} \right]_q^n.$$

Set $g = \gcd(v_1, \dots, v_d)$. Without loss of generality, we may assume that $g > 0$. If $g \nmid n - a$, then it follows from (12) that $f_n(q) = 0$. For $k \geq k_0 := \lfloor -a/g \rfloor$, we define $\tilde{f}_k(q) = f_{a+gk}(q)$. Then we have

$$(13) \quad f_n(q) = \begin{cases} \tilde{f}_k(q) & \text{if } n = a + gk, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we prove that the sequence $(\tilde{f}_k(q))_{k \geq k_0}$ is q -holonomic. For each $k \geq k_0$, consider the following linear Diophantine equation:

$$(14) \quad v_1 \ell_1 + v_2 \ell_2 + \dots + v_d \ell_d = gk.$$

By the extended Euclidean algorithm, there exist $u_{i,j} \in \mathbb{Z}$, $1 \leq i \leq d$, $0 \leq j \leq i$, which are independent of k , with $u_{d,d} = 0$ such that the solutions of (14) are

$$(15) \quad \ell_i = u_{i,0}k + u_{i,1}t_1 + \cdots + u_{i,i}t_i, \quad i = 1, \dots, d,$$

where t_i is an arbitrary integer. Set $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{Z}^{d-1}$. Substituting (15) into $r(\ell, a + gk)$, we denote the corresponding term by $\tilde{r}(\mathbf{t}, k)$. Then we can write $f_k(q)$ as

$$(16) \quad \tilde{f}_k(q) = \sum_{\mathbf{t}} \tilde{r}(\mathbf{t}, k).$$

Since $\tilde{r}(\mathbf{t}, k)$ is a proper q -hypergeometric term in \mathbf{t} and k , it follows from [17, section 5.2] that $(\tilde{f}_k(q))_{k \geq k_0}$ is a q -holonomic. Moreover, it follows from the proof of [5, Theorem 1] that there exists a nontrivial q -difference equation for $\tilde{f}_k(q)$ of the following form:

$$(17) \quad p_{\tilde{d}}(q, q^{gk})\tilde{f}_{k+\tilde{d}}(q) + p_{\tilde{d}-1}(q, q^{gk})\tilde{f}_{k+\tilde{d}-1}(q) + \cdots + p_0(q, q^{gk})\tilde{f}_k(q) = 0,$$

where $p_j(x, y) \in \mathbb{K}[x, y]$ for $j = 0, \dots, \tilde{d}$. Then it follows from (13) and (17) that the fake degree sequence satisfies

$$\begin{aligned} p_{\tilde{d}}(q, q^{n-a})f_{n+g\tilde{d}}(q) + p_{\tilde{d}-1}(q, q^{n-a})f_{n+g(\tilde{d}-1)}(q) \\ + \cdots + p_0(q, q^{n-a})f_n(q) = 0. \end{aligned}$$

By clearing denominators of the above equation, we see that $(f_n(q))_{n \in \mathbb{N}}$ is indeed q -holonomic.

Since the class of q -holonomic sequences is closed under the \mathbb{K} -linear combination, the claim also holds when Δ is a Laurent polynomial in Y . \square

Given a simple complex Lie algebra \mathfrak{g} and $V \in \mathcal{R}ep(\mathfrak{g})$. By the integral expression (16) of the fake degree sequence associated to \mathfrak{g} and V , we can utilize the method of creative telescoping [18] to derive the corresponding linear q -difference equations with polynomial coefficients.

Example 0.7. Let V be an irreducible representation of $\mathfrak{g} := SL(2)$ such that $\text{ch}_V(Y) = y + 1/y$ and $\Delta = 1 - y^2$. Consider the symmetric function in Example 0.5:

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \mathbf{fr}((\otimes^n V)^{\mathfrak{g}}) \\ &= \sum_{k=0}^{\infty} (h_k^2 - h_{k-1}h_{k+1}). \end{aligned}$$

For each $k \in \mathbb{N}$, set

$$\begin{aligned} g_k(q) &= \begin{bmatrix} 2k \\ k \end{bmatrix}_q - \begin{bmatrix} 2k \\ k+1 \end{bmatrix}_q \\ &= q^k \cdot \text{Cat}(k; q), \end{aligned}$$

where

$$\text{Cat}(k; q) = \frac{1}{[k+1]_q} \begin{bmatrix} 2k \\ k \end{bmatrix}_q$$

is the MacMahon q -analog of the Catalan number. Then for $n \in \mathbb{N}$, the n -th term of the fake degree sequence associated to V and g is

$$f_n(q) = \begin{cases} g_k(q) & \text{if } n = 2k, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 0.6, the fake degree sequence $(f_n)_{n \in \mathbb{N}}$ is q -holonomic and satisfies the following fourth-order linear q -difference equation:

$$\begin{aligned} (q^{n+6} - 1) f_{n+4}(q) \\ - q(q+1) (q^{n+3} - 1) (q^{n+4} + 1) f_{n+2}(q) \\ + q^3 (q^{n+1} - 1) (q^{n+2} - 1) (q^{n+3} - 1) f_n(q) = 0. \end{aligned}$$

Example 0.8. Let V be an irreducible representation of $\mathfrak{g} := SL(2)$ such that $\text{ch}_V(Y) = y^2 + 1 + y^{-2}$ and $\Delta = 1 - y^2$. Consider the symmetric function:

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \text{fr}((\otimes^n V)^{\mathfrak{g}}) \\ &= [Y^0](\Delta \cdot H[X, \text{m}_V(Y)]) \\ &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} (h_{k_1}^2 h_{k_2} - h_{k_1} h_{k_1+1} h_{k_2}). \end{aligned}$$

Then for $n \in \mathbb{N}$, the n -th term of the fake degree sequence associated to V and g is

$$\begin{aligned} f_n(q) &= \sum_{2k_1+k_2=n} \begin{bmatrix} n \\ k_1, k_1, k_2 \end{bmatrix}_q - \sum_{2k_1+k_2+1=n} \begin{bmatrix} n \\ k_1, k_1+1, k_2 \end{bmatrix}_q \\ &= \sum_{t=-\infty}^{\infty} \left(\begin{bmatrix} n \\ n-t, n-t, -n+2t \end{bmatrix}_q - \begin{bmatrix} n \\ n-1-t, n-t, -n+1+2t \end{bmatrix}_q \right) \\ &= \sum_{t=-\infty}^{\infty} \frac{q^{n-t} - q^{-n+1+2t}}{1 - q^{-n+1+2t}} \begin{bmatrix} n \\ n-t, n-t, -n+2t \end{bmatrix}_q. \end{aligned}$$

Using Koutschan's `Mathematica` package `HolonomicFunctions.m` [10] that implements Chyzak's algorithm [2], we derive the following sixth-order linear q -difference equation for the fake degree sequence $(f_n)_{n \in \mathbb{N}}$:

$$\begin{aligned} & (q^{n+7} - 1) f_{n+6}(q) - 2q (q^{n+5} - 1) f_{n+5}(q) \\ & \quad - (q^{n+5} - 1) (2q^{n+6} - q^2 + q + 1) f_{n+4}(q) \\ & \quad - q(q+1) (q^{n+4} - 1) (q^{n+5} - 1) (q^{n+5} + 2) f_{n+3}(q) \\ & \quad - q(q^2 + q - 1) (q^{n+3} - 1) (q^{n+4} - 1) (q^{n+5} - 1) f_{n+2}(q) \\ & \quad + 2q^2 (q^{n+2} - 1) (q^{n+3} - 1) (q^{n+4} - 1) (q^{n+5} - 1) f_{n+1}(q) \\ & + q^3 (q^{n+1} - 1) (q^{n+2} - 1) (q^{n+3} - 1) (q^{n+4} - 1) (q^{n+5} - 1) f_n(q) = 0. \end{aligned}$$

Example 0.9. Let V be the four-dimensional defining representation of $g := Sp(4)$ such that

$$\text{ch}_V(Y) = y_2 + y_2^{-1} + y_1 y_2^{-1} + y_1^{-1} y_2,$$

and

$$\Delta = y_1^{-2} y_2^{-2} - y_2^{-4} - y_1^{-3} + y_1^{-3} y_2^2 + y_1 y_2^{-4} - y_1 y_2^{-2} - y_1^{-2} y_2^2 + 1.$$

The dimension of the invariant subspace of $\otimes^{2n} V$ gives [A005700](#).

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \text{fr}((\otimes^n V)^g) \\ &= [Y^0](\Delta \cdot H[X, \text{m}_V(Y)]) \\ &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} (h_{k_1} h_{k_1+4} h_{k_2} h_{k_2+2} - h_{k_1} h_{k_1+4} h_{k_2}^2 - h_{k_1} h_{k_1+3} h_{k_2} h_{k_2+3} \\ & \quad + h_{k_1} h_{k_1+1} h_{k_2} h_{k_2+3} + h_{k_1} h_{k_1+3} h_{k_2} h_{k_2+1} - h_{k_1} h_{k_1+1} h_{k_2} h_{k_2+1} \\ & \quad - h_{k_1}^2 h_{k_2} h_{k_2+2} + h_{k_1}^2 h_{k_2}^2). \end{aligned} \tag{18}$$

Let $(f_n)_{n \in \mathbb{N}}$ be the fake degree sequence associated to V and g . For $n \geq 0$, it follows from (18) that $f_{2n+1}(q) = 0$. Moreover, the $2n$ -th term of the fake degree sequence is

$$\begin{aligned} f_{2n}(q) &= \sum_{k_1+k_2=n-3} \begin{bmatrix} 2n \\ k_1, k_1+4, k_2, k_2+2 \end{bmatrix}_q - \sum_{k_1+k_2=n-2} \begin{bmatrix} 2n \\ k_1, k_1+4, k_2, k_2 \end{bmatrix}_q \\ & \quad - \sum_{k_1+k_2=n-3} \begin{bmatrix} 2n \\ k_1, k_1+3, k_2, k_2+3 \end{bmatrix}_q + \sum_{k_1+k_2=n-2} \begin{bmatrix} 2n \\ k_1, k_1+1, k_2, k_2+3 \end{bmatrix}_q \\ & \quad + \sum_{k_1+k_2=n-2} \begin{bmatrix} 2n \\ k_1, k_1+3, k_2, k_2+1 \end{bmatrix}_q - \sum_{k_1+k_2=n-1} \begin{bmatrix} 2n \\ k_1, k_1+1, k_2, k_2+1 \end{bmatrix}_q \\ & \quad - \sum_{k_1+k_2=n-1} \begin{bmatrix} 2n \\ k_1, k_1, k_2, k_2+2 \end{bmatrix}_q + \sum_{k_1+k_2=n} \begin{bmatrix} n \\ k_1, k_1, k_2, k_2 \end{bmatrix}_q \end{aligned}$$

$$\begin{aligned}
&= \sum_{t=-\infty}^{\infty} \left(\begin{aligned} &\left[\begin{array}{c} 2n \\ 2n-t-6, 2n-t-2, -n+t+3, -n+t+5 \end{array} \right]_q \\ &- \left[\begin{array}{c} 2n \\ 2n-t-4, 2n-t, -n+t+2, -n+t+2 \end{array} \right]_q \\ &- \left[\begin{array}{c} 2n \\ 2n-t-6, 2n-t-3, -n+t+3, -n+t+6 \end{array} \right]_q \\ &+ \left[\begin{array}{c} 2n \\ 2n-t-4, 2n-t-3, -n+t+2, -n+t+5 \end{array} \right]_q \\ &+ \left[\begin{array}{c} 2n \\ 2n-t-4, 2n-t-1, -n+t+2, -n+t+3 \end{array} \right]_q \\ &- \left[\begin{array}{c} 2n \\ 2n-t-2, 2n-t-1, -n+t+1, -n+t+2 \end{array} \right]_q \\ &- \left[\begin{array}{c} 2n \\ 2n-t-2, 2n-t-2, -n+t+1, -n+t+3 \end{array} \right]_q \\ &+ \left[\begin{array}{c} 2n \\ 2n-t, 2n-t, -n+t, -n+t \end{array} \right]_q \end{aligned} \right).
\end{aligned}$$

Using the method of creative telescoping and the closure properties of the class of q -holonomic sequences, we find a twentieth-order linear q -difference equation for $(f_{2n}(q))_{n \in \mathbb{N}}$, which is given in [15].

Next, let us recall a theorem about scalar products of D -finite symmetric functions.

Theorem 0.10. (Gessel [6, Corollary 8]) *Let f and g be symmetric functions which are D -finite in the p_i 's and in other variables t_j 's. Assume that g involves only finitely many of the p_i 's. Then $\langle f, g \rangle$ is D -finite in the t_j 's as long as it is well defined as a formal power series.*

Note that the above result also holds for $\langle g, f \rangle$ provided that g involves only finitely many of the p_i 's.

Let G be a reductive group and V be a finite dimensional representation. For each $r \geq 0$, we denote the Frobenius character of the G -invariant subspace of $\otimes^r V$ by $I_r(V)$ [14]. Let P be a polynomial functor of degree k ; for instance, the k -th symmetric power functor or the k -th alternating power functor. We denote the character by $\mathbf{ch}(P)$, which is a symmetric function of degree k .

Proposition 0.11. *For each $r, k \geq 0$. Assume that $I_{rk}(V)[Y]$ is a D -finite symmetric function. Then*

$$f(X) := \langle h_r[X, \mathbf{ch}(P[Y])], I_{rk}(V)[Y] \rangle_Y$$

is also a D -finite symmetric function.

Proof. Since $h_r[X.\mathbf{ch}(P[Y])]$ and $I_{rk}(V)[Y]$ are both symmetric functions in X and Y , we see that $f(X)$ is a symmetric function in X . Since $\mathbf{ch}(P[Y])$ is a symmetric function of degree k , it follows that $h_r[X.\mathbf{ch}(P[Y])]$ is a symmetric function of degree $r(k+1)$. Thus, the symmetric function $h_r[X.\mathbf{ch}(P[Y])]$ only involves finitely many $p_i(Y)$'s. Since $h_r = \sum_{\lambda} p_{\lambda}/z_{\lambda}$, where λ runs over all partitions of r and $z_{\lambda} \in \mathbb{Z}_{>0}$, it is straightforward to see that $h_r[X.\mathbf{ch}(P[Y])]$ is D -finite in the $p_i(X)$'s and $p_j(Y)$'s. Taking $t_j = p_j(X)$ in Theorem 0.10, we conclude that the claim holds. \square

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