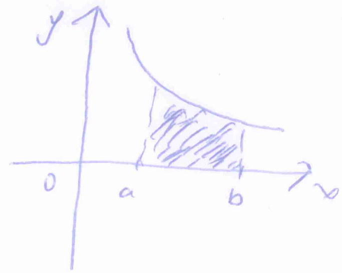
§7.8 Improper Integrals

$$\int_a^b f(x)\,dx$$



Improper Integrals : 1. Infinite Intervals

2. Discontinous Integrands (at a finite point) ~~at a fi~~

Applications: probability distributions.

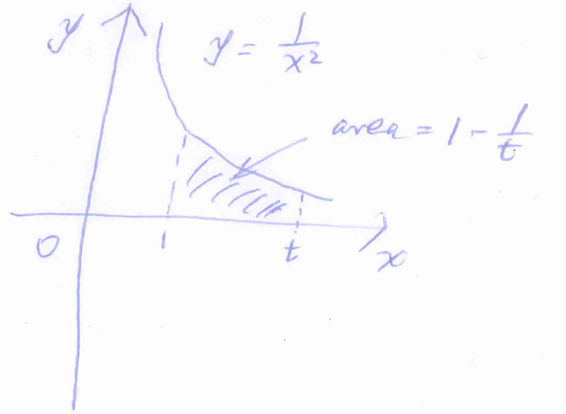
1. Infinite Intervals

Consider

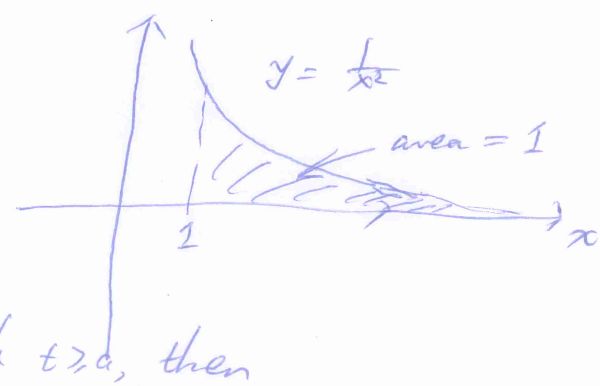
$$A(t) = \int_1^t \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\Big|_1^t$$

$$= 1 - \frac{1}{t}$$



$$\lim_{t\to\infty} A(t) = 1$$



Def 1.

(a) If $\int_a^t f(x)dx$ exists for each $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided the limit exists.

(b) If $\int_t^b f(x)dx$ exists for each $t \leq b$, then

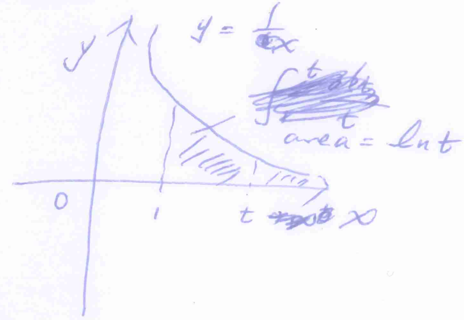
$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$$

$\int_a^\infty f(x)dx$ is called convergent if the limit exists.

② (c) If $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

Ex 1. Determine the convergence of $\int_1^\infty \frac{1}{x}dx$.

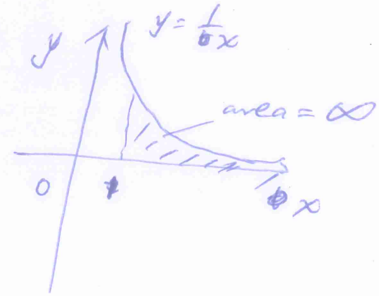


$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{dt}{x}$$

$$= \lim_{t\to\infty} \ln|x| \Big|_1^t$$

$$= \lim_{t\to\infty} \ln t$$

$$= \infty$$



Ex 2. Evaluate $\int_{-\infty}^0 xe^x dx$

$$\int_{-\infty}^0 xe^x dx = \lim_{t\to-\infty}\int_t^0 xe^x dx$$

$$\int_t^0 xe^x dx = \int_t^0 x\,de^x$$

$$= xe^x\Big|_t^0 - \int_t^0 e^x dx$$

$$= -te^t - 1 + e^t$$

$$\lim_{t\to-\infty} e^t = 0$$

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}}$$

$$= \lim_{t\to-\infty}\frac{1}{-e^{-t}}$$

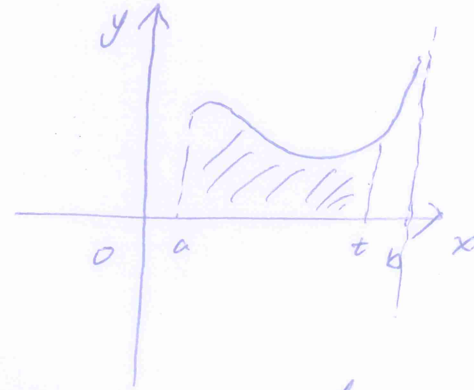
$$= 0$$

Thus, $\int^0 \quad = \lim (-te^t - 1 + e^t)$

③ Type 2: Discontinuous Integrands.

Assume f is a continous, positive function defined on $[a, b)$, ~~but $\lim_{t\to b-} f(t) = \infty$~~ but has a vertial asympstote at b

$$A(t) = \int_a^t f(x)\,dx$$



Define: $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to b-} A(t)$

Def 2. (a) If f is continous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists

(b) If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a+} \int_t^b f(x)\,dx$$

$\int_a^b f(x)\,dx$ is called convergent if the limit exists

Otherwise, it is divergent.

(c) If f has a discontinuity at c, where $a < c < b$, and $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, the we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex 5. Find $\displaystyle\int^{5} \frac{dx}{\ }$

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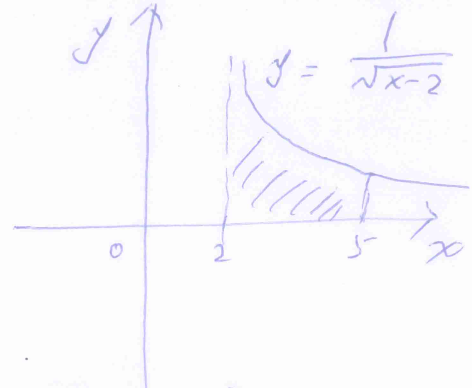
$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}}$$

$$= \lim_{t\to 2^+} 2\sqrt{x-2}\Big|_t^5$$

$$= \lim_{t\to 2^+} 2(\sqrt{3} - \sqrt{t-2})$$

$$= 2\sqrt{3}$$



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Ex 7. Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

$$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1}$$

$$= \lim_{t\to 1^-} \ln|x-1|\Big|_0^t$$

$$= \lim_{t\to 1^-} \ln(1-t)$$

$$= -\infty$$

Thus, $\int_0^3 \frac{dx}{x-1}$ is divergent.

Note: $\int_0^3 \frac{dx}{x-1} \neq \ln|x-1|\Big|_0^3 = \ln 2$

because $x=1$ is a vertical asymptote of $\frac{1}{x-1}$.

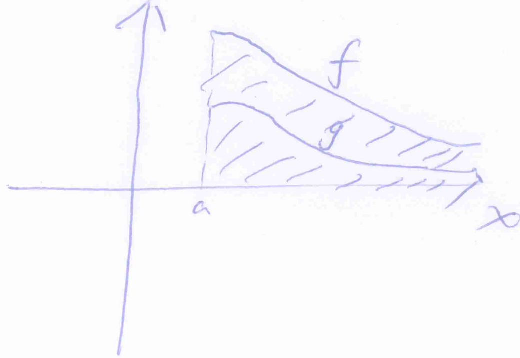
Question: how to test whether an improper integral is convergent or not?

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int^\infty g(x)\,dx$ is divergent, then $\int^\infty$ ...

Ex 10. $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent



$$\frac{1+e^{-x}}{x} > \frac{1}{x} \quad \text{for } x \geq 1$$

Since $\int_1^\infty \frac{dx}{x}$ is divergent, so is $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$.