

# Desingularization in the $q$ -Weyl Algebra

Yi Zhang

Johann Radon Institute for Computational and Applied Mathematics (RICAM)  
Austrian Academy of Sciences, Austria

Joint work with Christoph Koutschan



Der Wissenschaftsfonds.

# Garoufalidis' conjecture

**Conjecture:** Let  $J_K(n) \in \mathbb{Q}(q)$  be the Jones polynomial of a “colored” knot  $K$ . Then  $(J_K(n))_{n \in \mathbb{N}}$  has the following properties:

1.  $(1 - q^n)J_K(n)$  satisfies a bimononic recurrence relation,
2.  $J_K(n)$  does not satisfy a monic recurrence relation.

►  $J_K(n)$  satisfies a nonzero linear  $q$ -difference equation, i.e.,

$$p_r(q, q^n)J_K(n+r) + (\cdots)J_K(n+r-1) + \cdots + p_0(q, q^n)J_K(n) = 0,$$

where  $p_i(n) \in \mathbb{Q}[q, q^n]$ .

► If  $J_K(n) = \sum_{k=0}^n \sum_{j=0}^k f(j, k)$  with  $f(j, k) \in \mathbb{Q}(q)$ , one can use “Guess” to find such an equation.

## Example for Garoufalidis' conjecture

Let  $f(n) = (1 - q^n)J_K(n)$ . Assume that

$$p_r(q, q^n)f(n+r) + (\cdots)f(n+r-1) + \cdots + p_0(q, q^n)f(n) = 0. \quad (1)$$

- ▶ If  $p_r(n) = q^{an+b}$ , then we call (1) **monic**.
- ▶ If  $p_r(n) = q^{an+b}$  and  $p_0(n) = q^{cn+d}$ , then we call (1) **bimonic**.

## Example for Garoufalidis' conjecture

Let  $f(n) = (1 - q^n)J_K(n)$ . Assume that

$$p_r(q, q^n)f(n+r) + (\cdots)f(n+r-1) + \cdots + p_0(q, q^n)f(n) = 0. \quad (1)$$

▶ If  $p_r(n) = q^{an+b}$ , then we call (1) **monic**.

▶ If  $p_r(n) = q^{an+b}$  and  $p_0(n) = q^{cn+d}$ , then we call (1) **bimonic**.

**Example 1** Consider the equation of  $(1 - q^n)J_K(n)$  with  $K = K_{-1}^{\text{twist}}$ :

$$q^{2n+2}(q^{2n+1} - 1)f(n+2) + (\cdots)f(n+1) + q^{2n+2}(q^{2n+3} - 1)f(n) = 0.$$

## Example for Garoufalidis' conjecture

Let  $f(n) = (1 - q^n)J_K(n)$ . Assume that

$$p_r(q, q^n)f(n+r) + (\cdots)f(n+r-1) + \cdots + p_0(q, q^n)f(n) = 0. \quad (1)$$

► If  $p_r(n) = q^{an+b}$ , then we call (1) **monic**.

► If  $p_r(n) = q^{an+b}$  and  $p_0(n) = q^{cn+d}$ , then we call (1) **bimonic**.

**Example 1** Consider the equation of  $(1 - q^n)J_K(n)$  with  $K = K_{-1}^{\text{twist}}$ :

$$q^{2n+2}(q^{2n+1} - 1)f(n+2) + (\cdots)f(n+1) + q^{2n+2}(q^{2n+3} - 1)f(n) = 0.$$

Our algorithm yields:

$$q^{2n+4}f(n+3) + (\cdots)f(n+2) + (\cdots)f(n+1) + q^{3n+7}f(n) = 0.$$

# Rings of $q$ -difference operators

Let  $x = q^n$ .

$$\mathbb{Q}(q)[x][\partial] \subset \mathbb{Q}(q, x)[\partial]$$

$q$ -Weyl algebra

$q$ -rational algebra

Assume  $L = \ell_r \partial^r + \cdots + \ell_1 \partial + \ell_0 \in \mathbb{Q}(q)[x][\partial]$ . Then

$$L \circ f(n) = \ell_r f(n+r) + \cdots + \ell_1 f(n+1) + \ell_0 f(n)$$

- ▶ Call  $L$  an **annihilator** of  $f$  if  $L \circ f = 0$ .
- ▶ Call  $\deg_{\partial}(L) := r$  the **order** of  $L$ ,  $\text{lc}_{\partial}(L) := \ell_r$  the **leading coeff**
- ▶ Let  $T \in \mathbb{Q}(q)[x][\partial]$ . Call  $T$  a **left multiple** of  $L$  if  $T = PL$ , where  $P \in \mathbb{Q}(q, x)[\partial]$ .

# Rings of $q$ -difference operators

**Example 2** Let  $g(n) = [n]_q := \frac{1-q^n}{1-q}$ . Then

$$(q^n - 1)g(n+1) - (q^{n+1} - 1)g(n) = 0.$$

It is equivalent to

$$[(x-1)\partial - qx + 1] \circ g(n) = 0.$$

Set  $P = (x-1)\partial - qx + 1$  and  $Q = \frac{1}{qx-1}(\partial - q)$ . Then

$$\begin{aligned} T &= QP \\ &= 1\partial^2 - (q+1)\partial + q \end{aligned}$$

is a left multiple of  $P$ .

# Desingularization

Given  $L \in \mathbb{Q}(q)[x][\partial]$ ,  $p \mid \text{lc}_{\partial}(L)$ .

Let  $T \in \mathbb{Q}(q)[x][\partial]$  and  $\sigma(x) = qx$ . Call  $T$  a  **$p$ -removed operator** of  $L$  if

- ▶  $T$  is a left multiple of  $L$
- ▶  $\sigma^{-k}(g) \mid \frac{1}{p} \text{lc}_{\partial}(L)$ , where  $k = \deg_{\partial}(T) - \deg_{\partial}(L)$ .



# Desingularization

Given  $L \in \mathbb{Q}(q)[x][\partial]$ ,  $p \mid \text{lc}_\partial(L)$ .

Let  $T \in \mathbb{Q}(q)[x][\partial]$  and  $\sigma(x) = qx$ . Call  $T$  a  **$p$ -removed operator** of  $L$  if

- ▶  $T$  is a left multiple of  $L$
- ▶  $\sigma^{-k}(g) \mid \frac{1}{p} \text{lc}_\partial(L)$ , where  $k = \deg_\partial(T) - \deg_\partial(L)$ .

Let  $T$  be a  $p$ -removing operator of  $L$ . Call  $T$  a **desingularized operator** of  $L$  if

$$\deg(\text{lc}_\partial(T)) = \min\{\deg(\text{lc}_\partial(Q)) \mid Q \text{ is a } p\text{-removed operator}\}$$

# Desingularization

**Example 2 (continued)** Let  $P = (x - 1)\partial - qx + 1$  and  $Q = \frac{1}{qx-1}(\partial - q)$ . Then

$$\begin{aligned} T &= QP \\ &= 1\partial^2 - (q + 1)\partial + q \end{aligned}$$

is a desingularized operator of  $P$ .

# Desingularization

**Example 2 (continued)** Let  $P = (x - 1)\partial - qx + 1$  and  $Q = \frac{1}{qx-1}(\partial - q)$ . Then

$$\begin{aligned} T &= QP \\ &= 1\partial^2 - (q + 1)\partial + q \end{aligned}$$

is a desingularized operator of  $P$ .

**Goal:** Given  $P \in \mathbb{Q}(q)[x][\partial]$ , how to compute a desingularized operator of  $P$ ?

# Order bound for desingularized operators

Let  $L \in \mathbb{Q}(q)[x][\partial]$ .

Assume  $p \mid \text{lc}_\partial(L)$ ,  $p$  is irreducible.

- ▶ If  $p = x$ , then  $p$  is not removable from  $L$ .
- ▶ If  $p \neq x$  and  $p$  is removable, then one can **compute** an integer  $k$ , s.t. there exists a  $p$ -removing operator of order  $k$ .
- ▶ Using Euclidean algorithm, one can **compute** an order bound for desingularized operators.

Koutschan and Z. Desingularization in the  $q$ -Weyl algebra. *Adv. Appl. Math.* 97, pp. 80–101, 2018

Chen et al. Desingularization explains order-degree curves for Ore operators. *ISSAC 2013*.

## Determining the $k$ -th submodule

Given  $L \in \mathbb{Q}(q)[x][\partial]$ ,  $\deg_{\partial}(L) = r$ .

Set  $k \geq r$ . Call

$$M_k := \{ T \mid T \text{ is a left multiple of } L, \deg_{\partial}(T) \leq k \}$$

the  $k$ -th submodule of  $L$ .

## Determining the $k$ -th submodule

Given  $L \in \mathbb{Q}(q)[x][\partial]$ ,  $\deg_{\partial}(L) = r$ .

Set  $k \geq r$ . Call

$$M_k := \{ T \mid T \text{ is a left multiple of } L, \deg_{\partial}(T) \leq k \}$$

the  $k$ -th submodule of  $L$ .

**Question:** Given  $k \geq r$ , compute a  $\mathbb{Q}(q)[x]$ -spanning set of  $M_k$ ?

## Determining the $k$ -th submodule

Given  $L \in \mathbb{Q}(q)[x][\partial]$ ,  $\deg_{\partial}(L) = r$ .

Set  $k \geq r$ . Call

$$M_k := \{ T \mid T \text{ is a left multiple of } L, \deg_{\partial}(T) \leq k \}$$

the  $k$ -th submodule of  $L$ .

**Question:** Given  $k \geq r$ , compute a  $\mathbb{Q}(q)[x]$ -spanning set of  $M_k$ ?

1. Make an ansatz:  $F = z_k \partial^k + \dots + z_0$ ,  
where  $z_k, \dots, z_0 \in \mathbb{Q}(q)[x]$  are to be determined.
2. Compute  $\text{rrem}(F, L) = 0$ . It gives:

$$(z_k, \dots, z_0)A = \mathbf{0}, \tag{2}$$

where  $A \in \mathbb{Q}(q)[x]^{(k+1) \times r}$ .

3. Using Gröbner bases or linear algebra, solve (2).

# Computing desingularized operators

Let  $L \in \mathbb{Q}(q)[x][\partial]$ ,  $\deg_{\partial}(L) = r$ .

**Question:** Assume  $k$  is an order bound for desingularized operators of  $L$ , compute a desingularized operator?



# Computing desingularized operators

Let  $L \in \mathbb{Q}(q)[x][\partial]$ ,  $\deg_{\partial}(L) = r$ .

**Question:** Assume  $k$  is an order bound for desingularized operators of  $L$ , compute a desingularized operator?

Set  $k \geq r$ . Call

$$I_k := \left\{ [\partial^k]P \mid P \in M_k \right\} \cup \{0\},$$

the  **$k$ -th coefficient ideal** of  $L$ , where  $[\partial^k]P$  is the coefficient of  $\partial^k$  in  $P$ .

# Computing desingularized operators

**Proposition 1** If  $\{B_1, \dots, B_t\}$  is a spanning set of  $M_k$ , then

$$I_k = \langle [\partial^k]B_1, \dots, [\partial^k]B_t \rangle$$

# Computing desingularized operators

**Proposition 1** If  $\{B_1, \dots, B_t\}$  is a spanning set of  $M_k$ , then

$$I_k = \langle [\partial^k]B_1, \dots, [\partial^k]B_t \rangle$$

**Theorem 1** If  $s$  is a nonzero element of  $I_k$  with minimal degree, then  $S$  in  $M_k$  with  $\text{lc}_\partial(S) = s$  is a desingularized operator.

# Computing desingularized operators

**Proposition 1** If  $\{B_1, \dots, B_t\}$  is a spanning set of  $M_k$ , then

$$I_k = \langle [\partial^k]B_1, \dots, [\partial^k]B_t \rangle$$

**Theorem 1** If  $s$  is a nonzero element of  $I_k$  with minimal degree, then  $S$  in  $M_k$  with  $\text{lc}_{\partial}(S) = s$  is a desingularized operator.

**Note:** Using Euclidean algorithm over  $\mathbb{Q}(q)[x]$ , one can **compute** an operator  $S$  with  $\text{lc}_{\partial}(S) = s$ .

# Computing desingularized operators

**Algorithm 1:** Given  $L \in \mathbb{Q}(q)[x][\partial]$ , compute a desingularized operator of  $L$ .

1. Compute an order bound  $k$  for desingularized operators of  $L$ .
2. Compute a spanning set of  $M_k$ .
3. Using Euclidean algorithm over  $\mathbb{Q}(q)[x]$ , compute an operator  $S \in M_k$  with  $\text{lc}_\partial(S) = s$  such that  $s$  is a nonzero element of  $I_k$  with minimal degree.
4. Output  $S$ .

# Garoufalidis' conjecture

**Conjecture:** Let  $J_K(n) \in \mathbb{Q}(q)$  be the Jones polynomial of a “colored” knot  $K$ . Then  $(J_K(n))_{n \in \mathbb{N}}$  has:

1.  $(1 - q^n)J_K(n)$  satisfies a bimononic recurrence relation,
2.  $J_K(n)$  does not satisfy a monic recurrence relation.

**Example 1 (Continued)** Consider the equation of  $(1 - q^n)J_K(n)$  with  $K = K_{-1}^{\text{twist}}$ :

$$q^{2n+2}(q^{2n+1} - 1)f(n+2) + (\cdots)f(n+1) + q^{2n+2}(q^{2n+3} - 1)f(n) = 0.$$

It is equivalent to

$$[q^2x^2(qx^2 - 1)\partial^2 + (\cdots)\partial + q^2x^2(q^3x^2 - 1)] \circ f(n) = 0.$$

Set  $L = q^2x^2(qx^2 - 1)\partial^2 + (\cdots)\partial + q^2x^2(q^3x^2 - 1).$

# Garoufalidis' conjecture

Using **Algorithm 1**, we have

1. An order bound for desingularized operators of  $L$  is 3.
2. A spanning set of  $M_k$  over  $\mathbb{Q}(q)[x]$  is  $\{S, L\}$  with

$$S = q^4 x^2 \partial^3 + (\cdots) \partial^2 + (\cdots) \partial + q^7 x^3.$$

3. By **Theorem 1**,  $S$  is a desingularized operator of  $L$ .
4. Output  $S \circ f(n) = 0$ , which is equivalent to

$$q^{2n+4} f(n+3) + (\cdots) f(n+2) + (\cdots) f(n+1) + q^{3n+7} f(n) = 0.$$

# Garoufalidis' conjecture

Using **Algorithm 1**, we have

1. An order bound for desingularized operators of  $L$  is 3.
2. A spanning set of  $M_k$  over  $\mathbb{Q}(q)[x]$  is  $\{S, L\}$  with

$$S = q^4 x^2 \partial^3 + (\cdots) \partial^2 + (\cdots) \partial + q^7 x^3.$$

3. By **Theorem 1**,  $S$  is a desingularized operator of  $L$ .
4. Output  $S \circ f(n) = 0$ , which is equivalent to

$$q^{2n+4} f(n+3) + (\cdots) f(n+2) + (\cdots) f(n+1) + q^{3n+7} f(n) = 0.$$

- ▶ **Algorithm 1** can be used for desingularization of trailing coeff of  $L$ .
- ▶ **Algorithm 1** can be used for verification of item 2 of Garoufalidis' conjecture.



# Conclusion

- ▶ Order bound for desingularized operators
- ▶ An algorithm for computing desingularized operators
- ▶ Certify special cases of Garoufalidis' conjecture

# Conclusion

- ▶ Order bound for desingularized operators
- ▶ An algorithm for computing desingularized operators
- ▶ Certify special cases of Garoufalidis' conjecture

Thanks!