

# Mahler Discrete Residues and Summability for Rational Functions

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# Linear Mahler equations

Let  $\mathbb{K}$  be an algebraically closed field of char 0,  $x$  be an indeterminate, and  $p \in \mathbb{Z}_{\geq 2}$ .

Consider

$$\ell_r(x)y(x^{p^r}) + \ell_{r-1}(x)y(x^{p^{r-1}}) + \cdots + \ell_0(x)y(x) = f(x), \quad (1)$$

where  $\ell_i, f \in \mathbb{K}[x]$  are given,  $y(x)$  is unknown. A solution of (1) is called a **Mahler function**.

(Mahler 1929): study Mahler equations to prove the transcendence of values of some functions.

**Fact:** the generating series of any  $p$ -automatic sequence (such as the Baum–Sweet and the Rudin–Shapiro sequences) is a Mahler function.

# Differential Galois Theory

Using differential Galois theory, we can determine the differential-algebraic relations between solutions of Mahler equations.

**Example (Roques 2018):** A Galoisian proof that the generating series of the Baum-Sweet and Rudin-Shapiro sequences are algebraic independent over  $\bar{\mathbb{Q}}(x)$ .

**Goal:** Design effective algorithms for computing differential Galois groups of a given linear Mahler equations.

# Discrete residues, telescopers, and Galois theory

Endow  $\mathbb{K}(x)$  with one of the  $\sigma\delta$ -field structures:

(S)  $\sigma : f(x) \mapsto f(x+1)$  and  $\delta = \frac{d}{dx}$ ;

(Q)  $\sigma : f(x) \mapsto f(qx)$  with  $q \in \mathbb{K}^\times$  not root of unity and  $\delta = x \frac{d}{dx}$ .

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Let  $z_1, \dots, z_n \in F$ , a  $\sigma\delta$ -extension of  $\mathbb{K}(x)$  with  $F^\sigma = \mathbb{K}$ , satisfying

$$\sigma(z_i) = a_i z_i \quad \text{for some } a_1, \dots, a_n \in \mathbb{K}(x)^\times.$$

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**Proposition (Hardouin-Singer 2008)**  $z_1, \dots, z_n$  are  $\delta$ -dependent over  $\mathbb{K}(x)$  iff  $\exists \mathcal{L}_1, \dots, \mathcal{L}_n \in \mathbb{K}[\delta]$ , linear  $\delta$ -operators with coefficients in  $\mathbb{K}$ , not all 0, and  $g \in \mathbb{K}(x)$ :

$$\mathcal{L}_1 \left( \frac{\delta(a_1)}{a_1} \right) + \dots + \mathcal{L}_n \left( \frac{\delta(a_n)}{a_n} \right) = \sigma(g) - g.$$

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(Arreche 2017, Arreche-Z. 2022): Using  $(q)$ -discrete residues, there exist constants  $m_1, \dots, m_n \in \mathbb{K}$ , not all 0, such that

$$m_1 \frac{\delta(a_1)}{a_1} + \dots + m_n \frac{\delta(a_n)}{a_n} = \sigma(g) - g + c$$

for some  $g \in \mathbb{K}(x)$  and  $c \in \mathbb{K}$  (with  $c = 0$  in case (S)).

# Motivation

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It also holds for the Mahler case. **Question:** How to derive the explicit formulae for  $\mathcal{L}_1, \dots, \mathcal{L}_n$ ?



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It also holds for the Mahler case. **Question:** How to derive the explicit formulae for  $\mathcal{L}_1, \dots, \mathcal{L}_n$ ?

**Idea:** Develop the notion of Mahler discrete residues and derive an effective version of Hardouin-Singer's Proposition in the Mahler case.

## Continuous residues

Let  $\mathbb{K}$  be an algebraically closed field of char 0, and let  $f(x) \in \mathbb{K}(x)$ . Make the partial fraction decomposition

$$f(x) = r(x) + \sum_{\alpha \in \mathbb{K}} \sum_{k \geq 1} \frac{c_{\alpha}(k)}{(x - \alpha)^k},$$

where  $r(x) \in \mathbb{K}[x]$  and  $c_{\alpha}(k) \in \mathbb{K}$  (almost all 0).

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Then  $f(x)$  is **rationally integrable**, i.e., there exists  $g(x) \in \mathbb{K}(x)$  such that  $g'(x) = f(x)$ , if and only if the (continuous first-order) *residues*

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Chen and Singer (2012) created a notion of *discrete residues* that plays an analogous role (where *integrability*  $\mapsto$  *summability*) for the *shift* ( $x \mapsto x + 1$ ) and *q-dilation* ( $x \mapsto qx$ ) difference operators.

## Discrete residues: shift case

Rewrite the partial fraction decomposition of  $f(x) \in \mathbb{K}(x)$ :

$$f(x) = r(x) + \sum_{k \geq 1} \sum_{[\alpha] \in \mathbb{K}/\mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{c_{\alpha}(k, n)}{(x - \alpha + n)^k}$$

where  $r(x) \in \mathbb{K}[x]$ ,  $\alpha \in \mathbb{K}$  is a coset representative for  $[\alpha] := \alpha + \mathbb{Z} \in \mathbb{K}/\mathbb{Z}$ , and  $c_{\alpha}(k, n) \in \mathbb{K}$ .

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**Proposition (Chen-Singer 2012)**  $f(x)$  is **rationally summable**, i.e., there exists  $g(x) \in \mathbb{K}(x)$  such that  $f(x) = g(x+1) - g(x)$ , if and only if  $\text{dres}(f, [\alpha], k) = 0$  for each  $[\alpha] \in \mathbb{K}/\mathbb{Z}$  and  $k \in \mathbb{N}$ .

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## Why use residues?

An advantage of using residues is to answer *whether* (yes/no)  $f(x) \in \mathbb{K}(x)$  is

- ▶ rationally integrable:  $f(x) = g'(x)$ ; or
- ▶ rationally summable:  $f(x) = g(x+1) - g(x)$ ; or
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In the differential case, there is a better way: if  $f = \frac{a}{b}$  with  $a, b \in \mathbb{K}[x]$ ,  $\gcd(a, b) = 1$ ,  $\deg(a) < \deg(b)$ , and  $b$  squarefree, then the roots of the **Rothstein-Trager resultant**

$$RT(f) := \operatorname{Res}_x(a - z \cdot b', b) \in \mathbb{K}[z]$$

are precisely the first-order continuous residues of  $f(x)$ , which implies  $f(x)$  is rationally integrable iff  $RT(f)$  is a monomial in  $z$ .

# Mahler summability for rational functions

Fix  $p \in \mathbb{Z}_{\geq 2}$  and let the Mahler difference operator  $\sigma : g(x) \mapsto g(x^p)$  for  $g(x) \in \mathbb{K}(x)$ .

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**Mahler Summability Problem:** given  $f(x) \in \mathbb{K}(x)$ , decide effectively whether  $f(x)$  is Mahler summable.

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**Our Goal:** Construct a ( $\mathbb{K}$ -linear) complete obstruction to the Mahler summability of  $f(x) \in \mathbb{K}(x)$ .

# Mahler summability for rational functions

More precisely, for the  $\mathbb{K}$ -linear map  $\Delta : g(x) \mapsto g(x^p) - g(x)$ , we wish to construct explicitly a  $\mathbb{K}$ -linear map  $\nabla$  on  $\mathbb{K}(x)$  such that  $\text{im}(\Delta) = \ker(\nabla)$ , bypassing computation of certificates.

We call  $\nabla$  the **Mahler reduction** operator. Given  $f \in \mathbb{K}(x)$ , set  $\bar{f} = \nabla(f)$ . Then  $f$  is Mahler summable if and only if  $\bar{f} = 0$ . The numerators in the partial fraction decomposition of  $\bar{f}$  are **Mahler discrete residues** of  $f$ .

# Mahler trajectories and Mahler trees

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We denote by  $\mathbb{Z}/\mathcal{P}$  the set of **maximal trajectories** for the action of  $\mathcal{P}$  on  $\mathbb{Z}$  by multiplication:

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We denote by  $\mathcal{T}_M$  the set of equivalence classes for the equivalence relation on  $\mathbb{K}^\times$  defined by  $\alpha \sim \gamma$  if and only if  $\alpha^{p^s} = \gamma^{p^r}$  for some  $r, s \in \mathbb{Z}_{\geq 0}$ .

The elements  $\tau \in \mathcal{T}_M$ , called **Mahler trees**, are pairwise disjoint subsets of  $\mathbb{K}^\times$  whose union is all of  $\mathbb{K}^\times$ . We write  $\tau(\alpha)$  for the unique Mahler tree containing  $\alpha \in \mathbb{K}^\times$ .

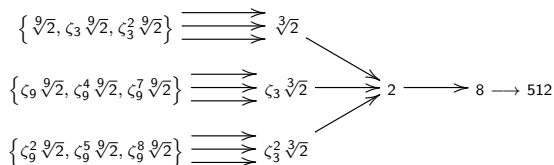
## Examples of Mahler trees

We define a digraph on the vertex set  $\tau$  for each Mahler tree  $\tau \in \mathcal{T}_M$  with one directed edge  $\alpha \rightarrow \gamma$  whenever  $\alpha^p = \gamma$ .

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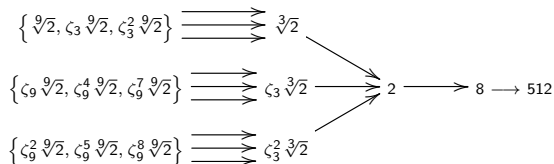
With  $p = 3$ , the vertices of  $\tau(2)$  near  $2 \in \mathbb{K}^\times$  are



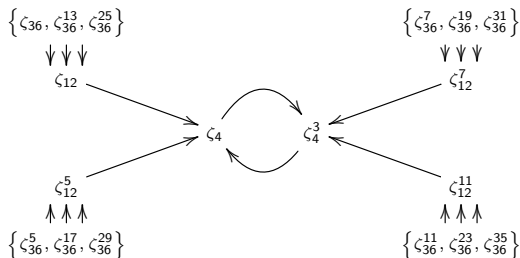
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# Mahler decomposition of partial fractions

For  $f(x) \in \mathbb{K}(x)$ , we can decompose it uniquely as

$$f(x) = f_L(x) + f_T(x):$$

$$f_L(x) := \sum_{j \in \mathbb{Z}} r_j x^j \quad \text{and} \quad f_T(x) := \sum_{k \geq 1} \sum_{\alpha \in \mathbb{K}^\times} \frac{c_\alpha(k)}{(x - \alpha)^k},$$

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Then  $f \in \mathbb{K}(x)$  is Mahler summable if and only if  $f_L$  and  $f_T$  are both Mahler summable. We address each component separately.

# Mahler decomposition of partial fractions

For  $f(x) \in \mathbb{K}(x)$ , we can decompose it uniquely as

$$f(x) = f_L(x) + f_T(x):$$

$$f_L(x) := \sum_{j \in \mathbb{Z}} r_j x^j \quad \text{and} \quad f_T(x) := \sum_{k \geq 1} \sum_{\alpha \in \mathbb{K}^\times} \frac{c_\alpha(k)}{(x - \alpha)^k},$$

where  $r_j, c_\alpha(k) \in \mathbb{K}$ .

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Moreover, the decompositions  $f_L = \sum_{\theta \in \mathbb{Z}/\mathcal{P}} f_\theta$  and  $f_T = \sum_{\tau \in \mathcal{T}_M} f_\tau$ :

$$f_\theta := \sum_{j \in \theta} r_j x^j \quad \text{and} \quad f_\tau := \sum_{k \geq 1} \sum_{\alpha \in \tau} \frac{c_\alpha(k)}{(x - \alpha)^k}$$

are also  $\sigma$ -stable. Can decide summability of  $f$  by deciding for each  $f_\theta$  ( $\theta \in \mathbb{Z}/\mathcal{P}$ ) and each  $f_\tau$  ( $\tau \in \mathcal{T}_M$ ) individually.

# Mahler residues at infinity

**Definition (Arreche-Z. 2022)** Let  $f(x) \in \mathbb{K}(x)$  and write  $f_L(x) = \sum_{j \in \mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$ . The **Mahler residue** of  $f(x)$  at infinity is the vector

$$\text{dres}(f, \infty) := \left( \sum_{j \in \theta} r_j \right)_{\theta \in \mathbb{Z}/\mathcal{P}} \in \bigoplus_{\theta \in \mathbb{Z}/\mathcal{P}} \mathbb{K}.$$

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**Proposition (Arreche-Z. 2022)** For  $f(x) \in \mathbb{K}(x)$  the component  $f_L(x) \in \mathbb{K}[x, x^{-1}]$  is Mahler summable if and only if  $\text{dres}(f, \infty) = 0$  (the zero vector).

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**Proof sketch:** For  $\theta = \{ip^n\}$  with  $p \nmid i$ , let  $\bar{f}_\theta^{(n)} = f_\theta + \Delta(g_\theta^{(n)})$  with  $g_\theta^{(0)} := 0$  and  $g_\theta^{(n+1)} := g_\theta^{(n)} + (\sum_{\ell=0}^n r_{ip^\ell}) x^{ip^n}$ . Then, for  $h$  largest s.t.  $r_{ip^h} \neq 0$ ,  $\bar{f}_\theta^{(h)} = \text{dres}(f, \infty)_\theta \cdot x^{ip^h}$ . A *dispersion* argument shows  $\bar{f}_\theta^{(h)} = 0$  iff  $f_\theta$  is Mahler summable.

## Mahler residues at Mahler trees (1 of 3): coefficients

For  $\alpha \in \mathbb{K}^\times$ ,  $\zeta_p$  a primitive  $p$ -th root of unity, let  $V_k^m(\zeta_p^i \alpha) \in \mathbb{K}$ :

$$\sigma \left( \frac{1}{(x - \alpha^p)^m} \right) = \frac{1}{(x^p - \alpha^p)^m} = \sum_{k=1}^m \sum_{i=0}^{p-1} \frac{V_k^m(\zeta_p^i \alpha)}{(x - \zeta_p^i \alpha)^k}.$$



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The definition of Mahler residues is more complicated because applying  $\sigma : f(x) \mapsto f(x^p)$  at poles of order  $m$  “leaks” into the poles of order  $k \leq m$ .

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Lemma (Arreche-Z. 2022)

$$V_k^m(\zeta_p^i \alpha) = \mathbb{V}_k^m \cdot \frac{(\zeta_p^i \alpha)^k}{\alpha^{pm}},$$

where  $\mathbb{V}_k^m \in \mathbb{Q}$  are obtained from the Taylor coefficients at  $x = 1$ :

$$(x^{p-1} + \cdots + x + 1)^{-m} = \sum_{k=1}^m \mathbb{V}_k^m \cdot (x - 1)^{m-k} + O((x - 1)^m).$$

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The “universal coefficients”  $\mathbb{V}_k^m$  can be computed directly (as a sum over partitions) using the Faà di Bruno’s formula.

## Small example of Mahler coefficients

Let  $p = 3$ ,  $m = 2$ , and  $\alpha^3 = 1$ . Then

$$\sigma \left( \frac{1}{(x-1)^2} \right) = \frac{1}{(x^3-1)^2} = \sum_{k=1}^2 \sum_{i=0}^2 \frac{V_k^2(\zeta_3^i)}{(x-\zeta_3^i)^k},$$

By the previous Lemma,  $V_k^2(\zeta_3^i) = \mathbb{V}_k^2 \cdot (\zeta_3^i)^{k-6} = \mathbb{V}_k^2 \cdot \zeta_3^{ki}$  for  $k = 1, 2$ . We find that

$$\mathbb{V}_2^2 = (x^2+x+1)^{-2} \Big|_{x=1} = \frac{1}{9}; \text{ and } \mathbb{V}_1^2 = ((x^2+x+1)^{-2})' \Big|_{x=1} = -\frac{2}{9}.$$

Using a computer algebra system (or by hand!), one can verify that the partial fraction decomposition of  $9 \cdot (x^3-1)^{-2}$  is indeed

$$\frac{1}{(x-1)^2} + \frac{\zeta_3^2}{(x-\zeta_3)^2} + \frac{\zeta_3}{(x-\zeta_3^2)^2} + \frac{-2}{x-1} + \frac{-2\zeta_3}{x-\zeta_3} + \frac{-2\zeta_3^2}{x-\zeta_3^2}.$$

## Mahler residues at Mahler trees (2 of 3): definition

Suppose  $\gamma \in \mathbb{K}^\times$  is not a root of unity,  $f \in \mathbb{K}(x)$ , and  $h \in \mathbb{Z}_{\geq 0}$  s.t.:

$$\text{sing}(f) \cap \tau(\gamma) \subseteq \{\zeta_{p^n}^i \gamma^{p^{h-n}} \mid 0 \leq n \leq h, i \in \mathbb{Z}/p^n\mathbb{Z}\}.$$

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Then we can write, for  $\tau = \tau(\gamma)$ ,

$$f_\tau = \sum_{k=1}^m \sum_{n=0}^h \sum_{i \in \mathbb{Z}/p^n\mathbb{Z}} \frac{c_\gamma(k, n, i)}{(x - \zeta_{p^n}^i \gamma^{p^{h-n}})^k}.$$

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Set recursively:  $\tilde{c}_{k,0,0} := c_\gamma(k, 0, 0)$ , and for  $1 \leq n \leq h$ ;  $i \in \mathbb{Z}/p^n\mathbb{Z}$ :

$$\tilde{c}_{k,n,i} := c_\gamma(k, n, i) + \sum_{j=k}^m \tilde{c}_{j,n-1,\pi_{n-1}(i)} V_k^j(\zeta_{p^n}^i \gamma^{p^{h-n}}),$$

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**Definition (Arreche-Z. 2022)** The **Mahler discrete residue** at  $\tau$  of order  $k$  is the vector  $\text{dres}(f, \tau, k) \in \bigoplus_{\alpha \in \tau} \mathbb{K}$  with  $\alpha$ -component  $:= 0$  except possibly at  $\alpha = \zeta_{p^h}^i \gamma$  for  $i \in \mathbb{Z}/p^h\mathbb{Z}$ , given by  $\tilde{c}_{k,h,i}$ .

## Mahler residues at Mahler trees (3 of 3): proof

**Proposition (Arreche-Z. 2022)** For  $f \in \mathbb{K}(x)$  the component  $f_{\mathcal{T}}$  is Mahler summable if and only if  $\text{dres}(f, \tau, k) = 0$  (the zero vector) for each  $\tau \in \mathcal{T}_M$  and  $k \in \mathbb{N}$ .

## Mahler residues at Mahler trees (3 of 3): proof

**Proposition (Arreche-Z. 2022)** For  $f \in \mathbb{K}(x)$  the component  $f_\tau$  is Mahler summable if and only if  $\text{dres}(f, \tau, k) = 0$  (the zero vector) for each  $\tau \in \mathcal{T}_M$  and  $k \in \mathbb{N}$ .

**Proof idea.** Similar to the Laurent polynomial case, one adds to  $f_\tau$  a sequence of “small” summable elements until one obtains a “remainder”

$$\bar{f}_\tau = \sum_{k \geq 1} \sum_{\alpha \in \tau} \frac{\text{dres}(f, \tau, k)_\alpha}{(x - \alpha)^k}$$

such that  $f_\tau$  is Mahler summable iff  $\bar{f}_\tau = 0$ .

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- ▶ The definition (and proofs) for Mahler discrete residues at  $\tau(\zeta)$  for  $\zeta \in \mathbb{K}_t^\times$  a root of unity is similar in spirit, but more technical, due to the perverse (pre-)periodic behavior of roots of unity under the  $p$ -power map.

# Main Result

**Theorem (Arreche-Z. 2022)** Given  $f \in \mathbb{K}(x)$ . Then  $f$  is Mahler summable if and only if  $\text{dres}(f, \infty) = 0$  and  $\text{dres}(f, \tau, k) = 0$  for all  $k \in \mathbb{N}$  and  $\tau \in \mathcal{T}_M$ .

Thanks!