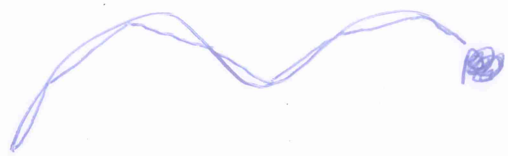
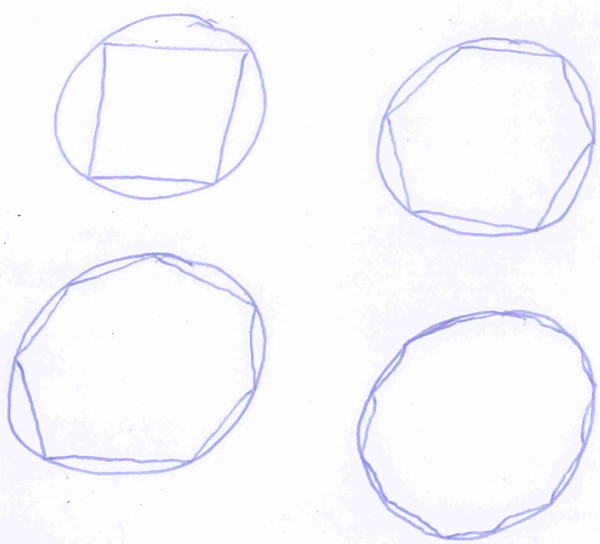
~~8.1①~~ §8.1 Arc Length

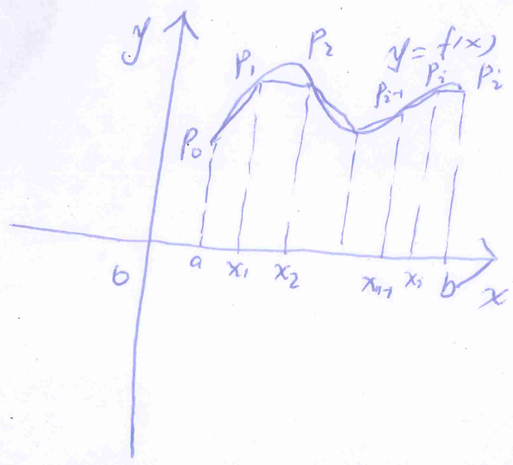
Question: Given a curve, how to compute its length?



Idea: Approximate it by a polygon and then take a limit as the number of segments of the polygon is increased.

Ex:





Suppose: C is defined by $y=f(x)$, $x\in[a,b]$,

Let $P_i=(x_i, f(x_i))$, $i=0,1,\cdots,n$.

The Length L of C is defined by

$$L=\lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i|$$

②

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

By the Mean Value Theorem, there exists $x_i^* \in (x_{i-1}, x_i)$, s.t.

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

Thus, $|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + [f'(x_i^*)]^2(\Delta x)^2}$

$$= \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x$$

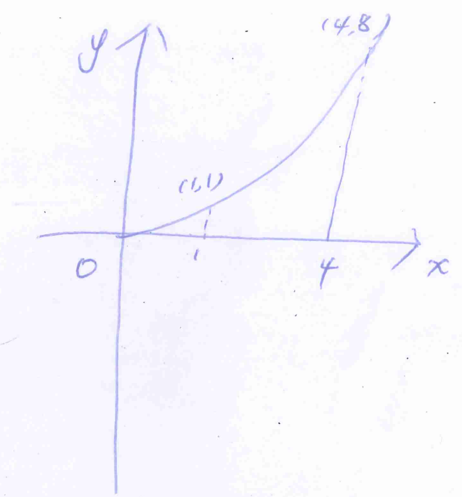
$$\Rightarrow L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+[f'(x_i^*)]^2}\ \Delta x$$

$$= \int_a^b \sqrt{1+[f'(x)]^2}\, dx \Leftarrow \text{The Arc Length Formula.}$$

provided f' is continuous on $[a,b]$ (C is smooth)

Sep 16

Ex 1. Find the Length of $y^2 = x^3$ betwemeen $(1,1)$ and $(4,8)$.



$$y = x^{3/2}$$

$$y' = \frac{3}{2}x^{1/2}$$

$$L = \int_1^4 \sqrt{1+(y')^2}\, dx$$

$$= \int_1^4 \sqrt{1+\frac{9}{4}x}\, dx$$

$$\underset{u=1+\frac{9}{4}x}{=} \frac{4}{9}\int_{13/4}^{10} \sqrt{u}\, du$$

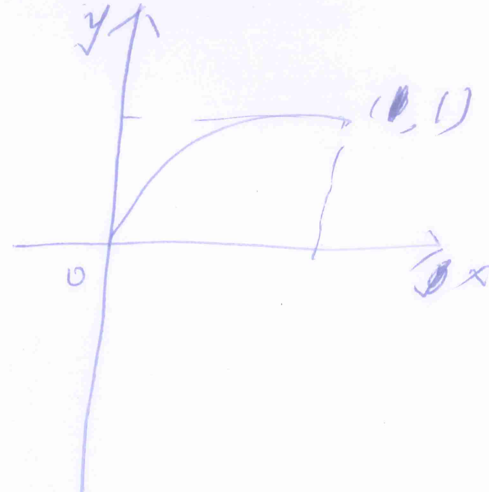
$$= \frac{4}{9}\cdot\frac{2}{3}u^{3/2}\Big|_{13/4}^{10}$$

$$= \frac{1}{27}(80\sqrt{10} - 13\sqrt{13}).$$

If a curve has equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then

$$L = \int_c^d \sqrt{1+(g'(y))^2}\, dy$$

③ Ex 2. Find the Length of the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$.



$$x = y^2, \quad \frac{dx}{dy} = 2y$$

$$L = \int_0^1 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^1 \sqrt{1+4y^2}\, dy$$

$$\underset{y=\frac{1}{2}\tan\theta}{=} \frac{1}{2}\int_0^{\alpha} \sec^3\theta\, d\theta, \quad \text{where } \alpha = \tan^{-1} 2$$

$$\underset{\text{Ex 7.2.8.}}{=} \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]\Big|_0^{\alpha}$$

$$= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)$$

$$= \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4}$$

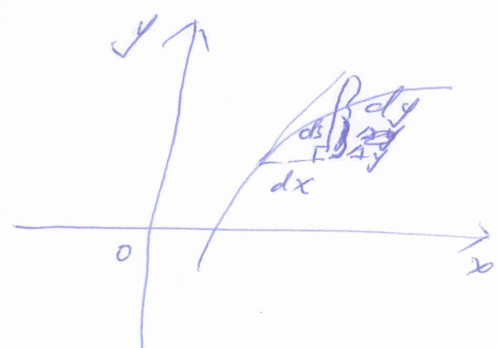
The Arc Length Function.

If a smooth curve C has the equation $y = f(x)$, $a \le x \le b$, let $s(x)$ be the distance along C for $(a, f(a))$ to $Q(x, f(x))$. Then $s(x)$ is the arc length function of C.

$$s(x) = \int_a^x \sqrt{1+[f'(t)]^2}\, dt$$

Then
$$\frac{ds}{dx} = \sqrt{1+[f'(x)]^2}$$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$



$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

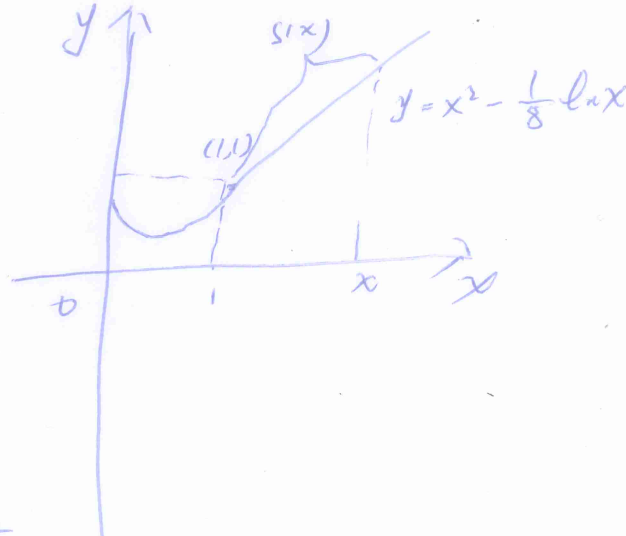
$$\Rightarrow (ds)^2 = (dx)^2 + (dy)^2 \quad (1)$$

$$L = \int ds \qquad (2)$$

By (1), (2), we get

$$L = \int_a^x \sqrt{1+[f'(t)]^2}\, dt$$

Ex 4. Find the arc length function for $y = x^2 - \frac{1}{8}\ln x$ by take the starting point $(1,1)$



$$f'(x) = 2x - \frac{1}{8x}$$

$$\sqrt{1+[f'(x)]^2} = 2x + \frac{1}{8x}$$

Thus,
$$s(x) = \int_1^x \sqrt{1+[f'(t)]^2}\, dt$$

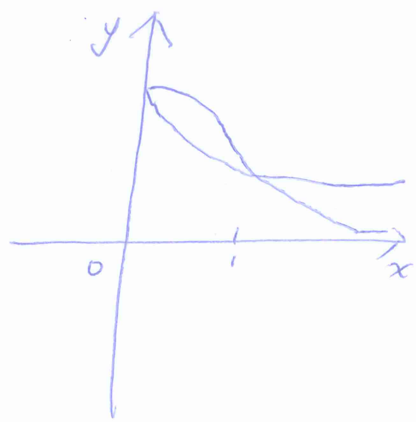
$$= \int_1^x \left(2t + \frac{1}{8t}\right) dt$$

$$= \left(t^2 + \frac{1}{8}\ln t\right)\Big|_1^x$$

$$= x^2 + \frac{1}{8}\ln x - 1. \ \square$$

Additional example.

Ex 9. Show that $\int_0^\infty e^{-x^2}dx$



$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$$

$e^{-x^2} \le e^{-x}$ for $x \ge 1$

$$\int_1^\infty e^{-x}dx = \lim_{t\to\infty}\int_1^t e^{-x}dx$$

$$= \lim_{t\to\infty}(e^{-1} - e^{-t}) = e^{-1}$$