

On Sequences Associated to the Invariant Theory of Rank Two Lie Algebras

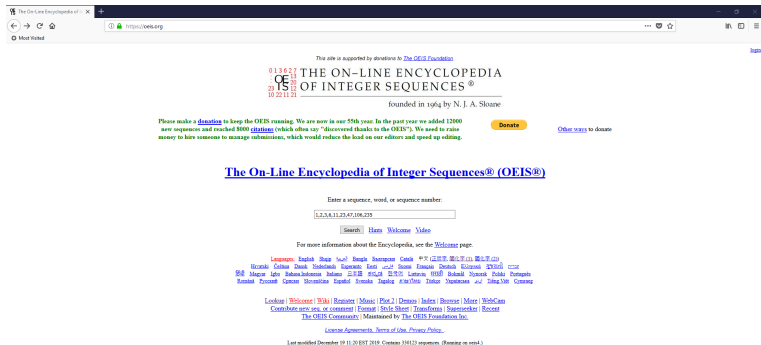
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Joint work with Alin Bostan, Jordan Tirrell and Bruce Westbury



The On-Line Encyclopedia of Integer Sequences (OEIS)



The screenshot shows the OEIS website homepage in a web browser. The browser's address bar displays "https://oeis.org". The page features the OEIS logo, which consists of a grid of numbers (0, 1, 3, 6, 2, 1, 1, 2, 2, 1, 1, 2, 1) and the text "THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®". Below the logo, it states "founded in 1964 by N. J. A. Sloane". A paragraph of text encourages donations to keep the site running, mentioning that they have added 12,000 new sequences and reached 1000 citations. A "Donate" button is visible. Below this, the title "The On-Line Encyclopedia of Integer Sequences® (OEIS®)" is displayed. A search bar with the placeholder text "Enter a sequence, word, or sequence number:" and a "Search" button is present. Below the search bar, there are links for "Home", "Welcome", and "Video". A paragraph of text states "For more information about the Encyclopedia, see the Welcome page." Below this, there is a list of languages: English, Spanish, Arabic, Hindi, Vietnamese, Catalan, Chinese (Simplified), Chinese (Traditional), Korean, Japanese, French, German, Italian, Russian, Polish, Portuguese, Romanian, Dutch, Czech, Slovak, Hungarian, Greek, Hebrew, Persian, Urdu, Bengali, Nepali, Thai, Vietnamese, and others. At the bottom, there are links for "License", "Welcome", "FAQ", "Register", "Login", "Index", "Browse", "More", "WebCams", "Contribute", "Help", "Contact", "Feedback", "Style Sheet", "Translations", "Superchecker", "Recent", "The OEIS Foundation", "Maintained by The OEIS Foundation Inc.", "License Agreements", "Terms of Use", "Privacy Policy", and "Last modified December 19 11:20 EST 2019. Contains 358123 sequences. (Running on oeis4.)"

OEIS is an online database of integer sequences, such as Fibonacci numbers ([A000045](#)), Catalan numbers ([A000108](#)).

A family of sequences in OEIS

a (OEIS tag)	0	1	2	3	4	5
A059710	1	0	1	1	4	10
A108307	1	1	2	5	15	51
A108304	1	2	5	15	52	202

Octant Sequences

- ▶ Those sequences are associated to the invariant theory of the exceptional simple Lie algebra G_2 of rank 2.
- ▶ They can be interpreted as lattice walks restricted to the octant. We call them **octant sequences**.

Octant sequences

- ▶ [A059710](#): enumerates the multiplicities of the trivial representation in the tensor powers of V , which is the 7-D fundamental representation of G_2 .
- ▶ [A108307](#): enumerates **enhanced** 3-noncrossing set partitions.
- ▶ [A108304](#): enumerates 3-noncrossing set partitions.

(Lin, 2018; Gil and Tirrell, 2019): [A108307](#) and [A108304](#) are related by the binomial transform.

Motivation and Contribution

(Bostan, Tirrell, Westbury and Z., 2019): [A059710](#) and [A108307](#) are also related by the binomial transform.

Mihailovs' conjecture: Let $T_3(n)$ be the n -th term of [A059710](#). Then T_3 is determined by $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and

$$14(n+1)(n+2)T_3(n) + (n+2)(19n+75)T_3(n+1) \\ + 2(n+2)(2n+11)T_3(n+2) - (n+8)(n+9)T_3(n+3) = 0.$$

(Bostan, Tirrell, Westbury and Z., 2019): Three independent proofs of Mihailovs' conjecture.

- ▶ Two proofs are based on binomial relation between [A059710](#) and [A108307](#), together with a result by Bousquet-Mélou and Xin.
- ▶ The third one is a direct proof by the method of algebraic residues.

Outline

- ▶ binomial relation between the first and second octant sequences
- ▶ Three independent proofs of Mihailovs' conjecture

Preliminaries

Definition 1 Let G be a reductive complex algebraic group and let V be a representation of G . The sequence associated to (G, V) , denoted \mathbf{a}_V , is the sequence whose n -th term is the multiplicity of the trivial representation in the tensor power $\otimes^n V$.

Example 1 Let V be the 7-D fundamental representation of G_2 . Then [A059710](#) is the sequence associated with (G_2, V) .

Let \mathbf{a} be a sequence with n -th term $a(n)$, the **binomial transform** of \mathbf{a} is the sequence, denoted $\mathcal{B}\mathbf{a}$, whose n -th term is

$$\sum_{i=0}^n \binom{n}{i} a(i).$$

Preliminaries

Lemma 1 Assume \mathbf{a}_V is the sequence associated to (G, V) as specified in **Definition 1**. Then $\mathbf{a}_{V \oplus \mathbb{C}} = \mathcal{B}\mathbf{a}_V$.

Lemma 2 Assume \mathbf{a} enumerates walks in a lattice, confined to a domain D , using a set of steps S . Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \coprod \{0\}$.

Lemma 3 Let $G(t)$ be the generating function of \mathbf{a} . For $k \in \mathbb{Z}$, denote the generating function of $\mathcal{B}^k \mathbf{a}$ by $\mathcal{B}^k G$. Then

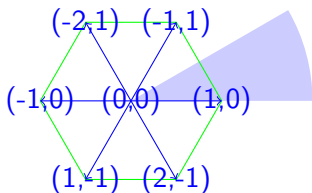
$$(\mathcal{B}^k G)(t) = \frac{1}{1 - k t} G\left(\frac{t}{1 - k t}\right).$$

Binomial relation between A059710 and A108307

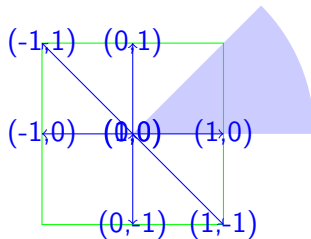
Let V be the 7-D fundamental representation of G_2 . Then

- ▶ A059710 is the sequence associated to (G_2, V) . Let $T_3(n)$ be its n -th term.
- ▶ A108307 enumerates enhanced 3-noncrossing set partitions. Let $E_3(n)$ be its n -th term.

In terms of lattice walks, we can interpret T_3 and E_3 as follows:

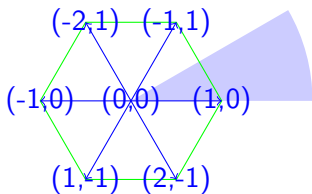


Steps in weight
lattice of G_2

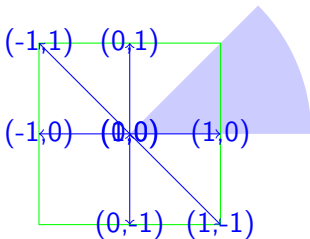


Steps in octant
related to $E_3(n)$

In terms of lattice walks, we can interpret T_3 and E_3 as follows:



Steps in weight
lattice of G_2



Steps in octant
related to $E_3(n)$

If we make a linear transformation $(x, y) \rightarrow (x, y \pm x)$, then it identifies the six non-zero steps, as well as the two domains.

Binomial relation between A059710 and A108307

Recall: **Lemma 2** Assume \mathbf{a} enumerates walks in a lattice, confined to a domain D , using a set of steps S . Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \amalg \{0\}$.

Binomial relation between A059710 and A108307

Recall: [Lemma 2](#) Assume \mathbf{a} enumerates walks in a lattice, confined to a domain D , using a set of steps S . Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \amalg \{0\}$.

By [Lemma 2](#) and the previous figures, we conclude that E_3 is the binomial transform of T_3 .

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By **Lemma 2** and the previous figures, we conclude that E_3 is the binomial transform of T_3 .

(Lin, 2018; Gil and Tirrell, 2019): A108307 and A108304 are related by the binomial transform.

Recall: **Lemma 1** Assume \mathbf{a}_V is the sequence associated to (G, V) as specified in **Definition 1**. Then $\mathbf{a}_{V \oplus \mathbb{C}} = \mathcal{B}\mathbf{a}_V$.

Thus, the octant sequences are sequences associated to

$$(G_2, V), \quad (G_2, V \oplus \mathbb{C}), \quad (G_2, V \oplus 2\mathbb{C}).$$

First proof of Mihailovs' conjecture

Mihailovs' conjecture: Let $T_3(n)$ be the n -th term of [A059710](#). Then T_3 is determined by $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and

$$14(n+1)(n+2)T_3(n) + (n+2)(19n+75)T_3(n+1) \\ + 2(n+2)(2n+11)T_3(n+2) - (n+8)(n+9)T_3(n+3) = 0.$$

([Bousquet-Mélou and Xin, 2005](#)): Let $E_3(n)$ be the n -th term of [A108307](#). Then E_3 is given by $E_3(0) = E_3(1) = 1$, and

$$8(n+3)(n+1)E_3(n) + (7n^2 + 53n + 88)E_3(n+1) \\ - (n+8)(n+7)E_3(n+2) = 0.$$

First proof of Mihailovs' conjecture

Recall: We prove that E_3 is the binomial transform of T_3 . Thus,

$$T_3(n) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} E_3(k).$$

Set $f(n, k) = (-1)^{n-k} \binom{n}{k} E_3(k)$.

- ▶ By Bousquet-Mélou and Xin's result, $f(n, k)$ is holonomic function, which satisfies ordinary difference equations for n and k , respectively.
- ▶ **Idea:** Using creative telescoping method (Zeilberger, 1990), which is an algorithmic approach to compute a differential/difference equation for the integration/sum of holonomic functions, to drive a recurrence equation for T_3 .

First proof of Mihailovs' conjecture

- ▶ Using the Koutschan's Mathematica package `HolonomicFunctions.m` that implements Chyzak's algorithm for creative telescoping, we find exactly the recurrence equation in Mihailovs' conjecture.

Second proof of Mihailovs' conjecture

Recall: We prove that E_3 is the binomial transform of T_3 . Let $\mathcal{T}(t) = \sum_{n \geq 0} T_3(n)t^n$ and $\mathcal{E}(t) = \sum_{n \geq 0} E_3(n)t^n$. Then

$$\mathcal{T}(t) = \frac{1}{1+t} \cdot \mathcal{E}\left(\frac{t}{1+t}\right).$$

- ▶ By Bousquet-Mélou and Xin's result, we can derive an ODE for $\mathcal{E}(t)$.
- ▶ Using the closure properties of holonomic function (the sum, product and algebraic substitution of holonomic functions is still holonomic), we can derive an ODE for $\mathcal{T}(t)$ and convert it into a linear recurrence for $T_3(n)$, which is exactly the recurrence equation in Mihailovs' conjecture.

Third proof of Mihailovs' conjecture

Idea: In terms of lattice walks, we can interpret $T_3(n)$ to be the constant term of $W K^n$, where

$$K = (1 + x + y + xy + x^{-1} + y^{-1} + (xy)^{-1}),$$

and

$$W = x^{-2}y^{-3}(x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} \\ + x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2).$$

Let $\mathcal{T}(t) = \sum_{n \geq 0} T_3(n)t^n$. Then $\mathcal{T}(t)$ is the constant coefficient $[x^0y^0]$ of $W/(1 - tK)$. In other words, $\mathcal{T}(t)$ is equal to the algebraic residue of $W/(\textcolor{red}{xy} - t\textcolor{red}{xy}K)$, which is proportional to the contour integral of $W/(xy - txyK)$ over a cycle.

Third proof of Mihailovs' conjecture

Using creative telescoping method, we can compute a 6-th order ODE for $\mathcal{T}(t)$. Moreover, by using factorization of differential operators, we can show that $L_3(\mathcal{T}(t)) = 0$, where $\partial = \frac{d}{dt}$ and

$$L_3 = t^2 (2t + 1) (7t - 1) (t + 1) \partial^3 + 2t (t + 1) (63t^2 + 22t - 7) \partial^2 + (252t^3 + 338t^2 + 36t - 42) \partial + 28t (3t + 4).$$

Converting it into a linear recurrence for $T_3(n)$, we get exactly the recurrence equation in Mihailovs' conjecture.

Summary

- ▶ A combinatorial proof of binomial relation between the first and second octant sequences
- ▶ Three independent proofs of Mihailovs' conjecture
 - ▶ Two proofs are based on binomial relation between the first and second octant sequences
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Summary

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Thanks!