Log-concavity and log-convexity of series containing multiple Pochhammer symbols

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Abstract

1 Introduction and preliminaries

Define the formal power series

$$f(\mu;x)=\sum_{k=0}^{\infty}f_k(\mu)x^k \tag{1}$$

with non-negative coefficients $f_k(\mu)$ which depend continuously on a non-negative parameter μ. Our main focus will be on logarithmic concavity (convexity) of the function $\mu\to f(\mu;x)$, i.e. concavity (convexity) of $\mu\to\log(f(\mu;x))$ for a fixed $x>0$ in the convergence domain of the series (1). To this end, we define the so-called "generalized Turánian" for any $\alpha,\beta\geq 0$ by the expression

$$\Delta_f(\alpha,\beta;x)=f(\mu+\alpha,x)f(\mu+\beta,x)-f(\mu,x)f(\mu+\alpha+\beta,x)=\sum_{k=0}^{\infty}\delta_k x^k. \tag{2}$$

It is well-known and easy to see that the condition $\Delta_f(\alpha,\beta;x)\geq 0(\leq 0)$ implies log-concavity (log-convexity) of $\mu\to f(\mu;x)$. It is less trivial that for continuous functions the reverse implication also holds [11]. In this paper we will mostly deal with the stronger property: if the coefficients δ_k at all powers of x in (2) are non-negative (non-positive) for all $\alpha,\beta\geq 0$ we will say that $\mu\to f(\mu;x)$ is *coefficient-wise log-concave (log-convex)*. If this property holds for $(\alpha,\beta)\in A$

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for some subset A of $\mathbf{R}_+ \times \mathbf{R}_+$, we say that $\mu \to f(\mu;x)$ is coefficient-wise log-concave (log-convex) for shifts α, β in A.

Furthermore, both in our previous work [6, 7, 9] and in this paper we will assume that the coefficients $f_k(\mu)$ permit factorization of the form $f_k(\mu) = f_k\phi_k(\mu)$, where the functions $\phi_k(\mu)$ will be specified explicitly (in terms of gamma functions and/or rising factorials), while $f_k \geq 0$ will remain a generic numerical sequence. In many cases this sequence will be required to satisfy the following property.

Definition 1.1. *Let $\{f_k\}_{k=0}^{\infty}$ be a non-negative real sequence. We call $\{f_k\}_{k=0}^{\infty}$ PF_2* (Pólya frequency sub two) *or doubly positive if $f_k^2 \geq f_{k-1}f_{k+1}$ for each $k \in \mathbb{N}$, the sequence $\{f_k\}_{k=0}^{\infty}$ is nontrivial and has no internal zeros,i.e., $f_N = 0$ implies either $f_{N+i} = 0$ for each $i \in \mathbb{N}_0$ or $f_{N-i} = 0$ for $i = 0, \ldots, N$. If $f_k^2 \leq f_{k-1}f_{k+1}$ for all $k \in \mathbb{N}$, the sequence is called log-convex (this inequality implies that the sequence is strictly positive).*

In [9], S.M.Sitnik and the first author proved the following result. The function

$$\mu \to f(\mu;x) = \sum_{n=0}^{\infty} f_n \frac{(\mu)_n}{n!} x^n. \tag{3}$$

is coefficient-wise log-concave if the sequence $\{f_n\}$ is doubly positive and coefficient-wise log-convex if $\{f_n\}$ is log-convex. This kind of duality is made possible by log-neutrality of the function $f(\mu, x)$ for $f_n \equiv C > 0$ which is both doubly positive and log-convex (and it is the only sequence satisfying both these properties). Indeed, in this case $f(\mu, x) = (1-x)^{-\mu}$, so that $\Delta_f(\alpha, \beta; x) = 0$ for all α, β. The first goal of this paper is to investigate the generalization of this result to the case when $(\mu)_n/n!$ in (3) is replaced by $(\mu)_{nr}/(nr)!$ for $r = 2, 3, \ldots$. Setting $f_n \equiv C > 0$ in this case does not lead to a log-neutral function, so the above kind of duality is not possible here. Hence, instead of one series like in (3) we consider two different series depending on whether $\{f_n\}$ is doubly positive or log-convex. The second, modified series was suggested by Ahn Ninh (private communication, 2017) by replacing $(\mu)_n/(n)!$ in (3) with $(\mu)_{nr}/(nr-1)!$ and starting summation from $n = 1$. These two types of series are discussed in Section 2.

Nevertheless, there exist other cases when setting $f_n \equiv C > 0$ does lead to a log-neutral function. In particular, writing $(\mu)_{2n}/(\mu+1)_n$ in place of $(\mu)_n$ in (3) results in such series. However, we only managed to prove coefficient-wise log-concavity/log-convexity of $\mu \to f(\mu, x)$ for natural shifts in this case. The corresponding results and conjectures are discussed in Section 3.

In Section 4, we deal with the series of the form (1) with $f_n(\mu) = f_n(\mu)_n/(2\mu)_n$. We demonstrate that it is coefficient-wise log-convex for each positive sequence $\{f_n\}_{n\geq 0}$.

In Section 5 we illustrate our results with several applications. In particular, we show that all our claims can be immediately generalized to the series containing the so-called k-shifted factorials and k-Gamma functions. We give examples involving generalized hypergeometric, k-hypergeometric, extended hypergeometric (????), incomplete hypergeometric (????) and parameter derivatives of hypergeometric functions...

We conclude this introduction with several lemmas which will serve as our main tools in the subsequent investigation.

Lemma 1.2. *[7, Lemma 5] Suppose $u, v, r, s > 0$, $u = \max(u, v, r, s)$ and $uv > rs$. Then $u + v > r + s$.*

In the next lemma we say that a sequence has no more than one change of sign if it has the pattern $(- - \cdots - -00 \cdots 00 + + \cdots + +)$, where any of the three parts may be missing. The following lemma is a slight generalization of [7, Lemma 6].

Lemma 1.3. *[7, Lemma 6] Suppose that $\{f_k\}_{k=0}^{\infty}$ is doubly positive (log-convex). If for certain $n \in \mathbb{N}$ the real sequence*

$$A_0, A_1, \ldots, A_{[n/2]}$$

has no more than one change of sign and $\sum_{0 \le k \le [n/2]} A_k \ge 0$ (≤ 0), then

$$\sum_{0 \le k \le n/2} f_k f_{n-k} A_k \ge 0 \ (\le 0).$$

Proof. It is literally the same as that of [6, Lemma 2.1]. □

2 Multiple Pochhammer series

Define a generalization of (3) as follows:

$$g(\mu; x) = \sum_{n=0}^{\infty} g_n \frac{(\mu)_{nr}}{(nr)!} x^n, \quad r = 2, 3, \ldots. \tag{4}$$

Note that

$$[\log((\mu)_{nr})]'' = \psi'(\mu + rn) - \psi'(\mu) \le 0, \tag{5}$$

where the inequality is strict for all $n \ge 1$ in view of the inequalities

$$\psi'(x) = \int_0^{\infty} \frac{te^{-tx}}{1 - e^{-t}} dt > 0, \quad \psi''(x) = -\int_0^{\infty} \frac{t^2 e^{-tx}}{1 - e^{-t}} dt < 0.$$

Hence, we are dealing the the infinite sum of log-concave functions which maybe log-concave, log-convex or neither. Setting $g_n = C > 0$ here does not lead to a log-neutral function of μ (as will be explicitly seen below), so that there is no hope to get a duality between log-concave/log-convex sequences $\{g_n\}$ and log-concave/log-convex functions as outlined in the Introduction. Ahn Ninh (private communication, 2017) suggested to modify (4) as follows

$$\mu \to f(\mu; x) = \sum_{n=1}^{\infty} f_n \frac{(\mu)_{nr}}{(nr-1)!} x^n \tag{6}$$

and conjectured that this functions is log-concave if f_n is doubly positive. In this section we will prove this conjecture for $r=2$ and disprove numerically for $r=4$ (strongly suggesting that is it also wrong for $r>4$). The case $r=3$ remains open and is formulated in the form of Conjecture ?. We will further proof that $\mu \to g(\mu;x)$ defined in (4) is coefficient-wise log-convex for $r=2$ and is neither log-convex nor log-concave for $r=3$ (strongly suggesting that it remains so for $r>3$).

Lemma 2.1. *Let $i,n \in \mathbb{N}$ with $i \le n$, $\mu \ge 0$ and $\alpha, \beta > 0$. Set*

$$T_{i,n} := (\mu+\alpha)_i(\mu+\beta)_{n-i} + (\mu+\alpha)_{n-i}(\mu+\beta)_i - (\mu)_i(\mu+\alpha+\beta)_{n-i} - (\mu)_{n-i}(\mu+\alpha+\beta)_{n-i}. \quad (7)$$

If $T_{i,n} \le 0$, then $T_{i-1,n} < 0$.

Proof. See [9, Theorem 1]. □

Theorem 2.2. *Let*

$$g(\mu;x) = \sum_{n=0}^{\infty} g_n \frac{(\mu)_{2n}}{(2n)!} x^n = \sum_{n=0}^{\infty} g_n \frac{(\mu/2)_n((\mu+1)/2)_n}{(1/2)_n n!} x^n, \quad (8)$$

where $\{g_n\}_{n=0}^{\infty}$ is log-convex and independent of μ. Then the formal power series $g(\mu;x)$ is coefficient-wise log-convex.

Proof. For each $\beta \ge \alpha > 0$, ($\alpha, \beta > 0$ - sufficient!) we have

$$g(\mu+\alpha;x)g(\mu+\beta;x) - g(\mu;x)g(\mu+\alpha+\beta;x) := \sum_{m=1}^{\infty} \phi_m x^m,$$

where $\phi_m = \sum_{k=0}^{m} g_k g_{m-k} M_k$ and

$$M_k = \frac{1}{(2k)!\,(2(m-k))!}\left[(\mu+\alpha)_{2k}(\mu+\beta)_{2(m-k)} - (\mu)_{2k}(\mu+\alpha+\beta)_{2(m-k)}\right].$$

Furthermore, we may write ϕ_m in the following form:

$$\phi_m = \sum_{k=0}^{[m/2]} g_k g_{m-k} A_k,$$

where $A_k = M_k + M_{m-k}$ if $k < m/2$, and $A_k = M_k$ if $k = m/2$. Set

$$\tilde{A}_k = (2k)!(2(m-k))!A_k, \quad (9)$$

which has the same sign as A_k. Then

$$\tilde{A}_k = \begin{cases} \underbrace{(\mu+\alpha)_{2k}(\mu+\beta)_{2(m-k)}}_{u_k} + \underbrace{(\mu+\alpha)_{2(m-k)}(\mu+\beta)_{2k}}_{v_k} \\ - \underbrace{(\mu)_{2k}(\mu+\alpha+\beta)_{2(m-k)}}_{r_k} - \underbrace{(\mu)_{2(m-k)}(\mu+\alpha+\beta)_{2k}}_{s_k} & \text{if} \quad k < m/2, \\ (\mu+\alpha)_{2k}(\mu+\beta)_{2k} - (\mu)_{2k}(\mu+\alpha+\beta)_{2k} & \text{if} \quad k = m/2. \end{cases}$$

First, we show that $\sum_{k=0}^{[m/2]} A_k < 0$ for $m \geq 1$. To this end, we set

$$\psi(\mu;x) := \sum_{n=0}^{\infty} \frac{(\mu)_{2n}}{(2n)!} x^n = \frac{1}{2}\left[\frac{1}{(1+\sqrt{x})^{\mu}} + \frac{1}{(1-\sqrt{x})^{\mu}}\right], \tag{10}$$

where the second equality is immediate by expanding each of the two summands by the binomial theorem. Then

$$\begin{aligned}\xi_\mu(\alpha,\beta;x) &:= \psi(\mu+\alpha;x)\psi(\mu+\beta;x) - \psi(\mu;x)\psi(\mu+\alpha+\beta;x)\\ &= \sum_{m=1}^{\infty}\left(\sum_{k=0}^{[m/2]} A_k\right) x^m.\end{aligned}$$

Using (10), we find that

$$\xi_\mu(\alpha,\beta;x) = -\frac{1}{4}(1-x)^{-\mu}\left[(1+\sqrt{x})^{-\alpha} - (1-\sqrt{x})^{-\alpha}\right]\left[(1+\sqrt{x})^{-\beta} - (1-\sqrt{x})^{-\beta}\right]. \tag{11}$$

Since

$$\begin{aligned}&(1+\sqrt{x})^{-\alpha} - (1-\sqrt{x})^{-\alpha}\\ =&\sum_{n=0}^{\infty}\binom{-\alpha}{n}(\sqrt{x})^n - \sum_{n=0}^{\infty}\binom{-\alpha}{n}(-\sqrt{x})^n\\ =&2\sqrt{x}\sum_{n=0}^{\infty}\binom{-\alpha}{2n+1}x^n\end{aligned}$$

has only negative coefficients, it follows from (11) that $\psi(\mu;x)$ is coefficient-wise log-convex, and thus $\sum_{k=0}^{[m/2]} A_k < 0$ for $m \geq 1$.

Next, we prove that for $m \geq 2$ the sequence $\tilde{A}_0, \dots, \tilde{A}_{[m/2]}$ has exactly one change of sign, *i.e.*, (i) $\tilde{A}_{[m/2]} > 0$; (ii) If $\tilde{A}_k \leq 0$ for some $k < m/2$, then $\tilde{A}_{k-1} \leq 0$.

(i) If $k = m/2 \geq 1$, then $\tilde{A}_k = (\mu+\alpha)_{2k}(\mu+\beta)_{2k} - (\mu)_{2k}(\mu+\alpha+\beta)_{2k} > 0$ because the function $x \mapsto (x+\tau)/x$ is strictly decreasing for any positive x and τ.

If $k = (m-1)/2 \geq 1$, *i.e.*, $m = 2k+1$, then

$$\begin{aligned}\tilde{A}_k = &\underbrace{(\mu+\alpha)_{2k}(\mu+\beta)_{2(k+1)}}_{a_k} + \underbrace{(\mu+\alpha)_{2(k+1)}(\mu+\beta)_{2k}}_{b_k}\\ &- \underbrace{(\mu)_{2k}(\mu+\alpha+\beta)_{2(k+1)}}_{c_k} - \underbrace{(\mu)_{2(k+1)}(\mu+\alpha+\beta)_{2k}}_{d_k}.\end{aligned}$$

We prove that $\tilde{A}_k > 0$ for $k \geq 1$ by induction. For $k = 1$, we have

$$\begin{aligned}\tilde{A}_1 = \alpha\beta \big(&\alpha^3\beta + 2\alpha^3\mu + \alpha^3 + 4\alpha^2\beta\mu + 6\alpha^2\beta + 4\alpha^2\mu^2 + 12\alpha^2\mu + 6\alpha^2 + \alpha\beta^3 \\ &+ 4\alpha\beta^2\mu + 6\alpha\beta^2 + 6\alpha\beta\mu^2 + 30\alpha\beta\mu + 22\alpha\beta + 4\alpha\mu^3 + 30\alpha\mu^2 + 44\alpha\mu + 17\alpha \\ &+ 2\beta^3\mu + \beta^3 + 4\beta^2\mu^2 + 12\beta^2\mu + 6\beta^2 + 4\beta\mu^3 + 30\beta\mu^2 + 44\beta\mu + 17\beta + 2\mu^4 \\ &+20\mu^3 + 44\mu^2 + 34\mu + 12\big) > 0.\end{aligned}$$

Assume that $\tilde{A}_k = a_k + b_k - c_k - d_k > 0$. Consider

$$\begin{aligned}F(\alpha) &:= \tilde{A}_{k+1} \\ &= a_k(\mu+\alpha+2k)_2(\mu+\beta+2k+2)_2 + b_k(\mu+\alpha+2k+2)_2(\mu+\beta+2k)_2 \\ &\quad - c_k(\mu+2k)_2(\mu+\alpha+\beta+2k+2)_2 - d_k(\mu+2k+2)_2(\mu+\alpha+\beta+2k)_2.\end{aligned}$$

Then

$$\begin{aligned}F(0) &= a_k(\mu+2k)_2(\mu+\beta+2k+2)_2 + b_k(\mu+2k+2)_2(\mu+\beta+2k)_2 \\ &\quad - c_k(\mu+2k)_2(\mu+\beta+2k+2)_2 - d_k(\mu+2k+2)_2(\mu+\beta+2k)_2 \\ &= (a_k+b_k-c_k-d_k)(\mu+2k)_2(\mu+\beta+2k+2)_2 \\ &\quad + (b_k-d_k)\left[(\mu+2k+2)_2(\mu+\beta+2k)_2 - (\mu+2k)_2(\mu+\beta+2k+2)_2\right] \\ &= (a_k+b_k-c_k-d_k)(\mu+2k)_2(\mu+\beta+2k+2)_2 \\ &\quad + 2(b_k-d_k)\beta\left(2\beta\mu+3\beta+8k^2+4\beta k+8k\mu+12k+2\mu^2+6\mu+3\right) > 0\end{aligned}$$

because $b_k > d_k$. Regarding a_k, b_k, c_k, d_k as constants and differentiating with respect to α, we have

$$\begin{aligned}F'(\alpha) &= a_k(2\mu+2\alpha+4k+1)(\mu+\beta+2k+2)_2 \\ &\qquad + b_k(2\mu+2\alpha+4k+5)(\mu+\beta+2k)_2 \\ &\qquad - c_k(2\mu+2\alpha+2\beta+4k+5)(\mu+2k)_2 \\ &\qquad - d_k(2\mu+2\alpha+2\beta+4k+1)(\mu+2k+2)_2 \\ &= (a_k+b_k-c_k-d_k)(2\mu+2\alpha+2\beta+4k+5)(\mu+2k)_2 \\ &+ a_k\left[(2\mu+2\alpha+4k+1)(\mu+\beta+2k+2)_2 - (2\mu+2\alpha+2\beta+4k+5)(\mu+2k)_2\right] \\ &+ b_k\left[(2\mu+2\alpha+4k+5)(\mu+\beta+2k)_2 - (2\mu+2\alpha+2\beta+4k+5)(\mu+2k)_2\right] \\ &- d_k\left[(2\mu+2\alpha+2\beta+4k+1)(\mu+2k+2)_2 - (2\mu+2\alpha+2\beta+4k+5)(\mu+2k)_2\right] \\ &> (a_k+b_k-c_k-d_k)(2\mu+2\alpha+2\beta+4k+5)(\mu+2k)_2 \\ &d_k\left[(2\mu+2\alpha+4k+1)(\mu+\beta+2k+2)_2 + (2\mu+2\alpha+4k+5)(\mu+\beta+2k)_2\right. \\ &\left.-(2\mu+2\alpha+2\beta+4k+1)(\mu+2k+2)_2 - (2\mu+2\alpha+2\beta+4k+5)(\mu+2k)_2\right] \\ &= (a_k+b_k-c_k-d_k)(2\mu+2\alpha+2\beta+4k+5)(\mu+2k)_2 \\ &+ 2d_k\beta\big(2\alpha\beta+4\alpha\mu+6\alpha+2\beta\mu+3\beta+8k^2+8\alpha k+4\beta k+8k\mu+12k \\ &+2\mu^2+6\mu-1\big) > 0.\end{aligned}$$

The above first inequality holds since $\min(a_k, b_k) > d_k$, and the second one is true because $k \geq 1$. Thus, we see that $\tilde{A}_{k+1} = F(\alpha) \geq F(0) > 0$.

(ii) Assume that $\tilde{A}_k \leq 0$ for some $k < m/2$. Then $\tilde{A}_k = T_{2k,2m}$, which is given in (7). Thus, it follows from Lemma 2.1 that $\tilde{A}_{k-1} = T_{2k-2,2m} < 0$.

Above all, since $\sum_{k=0}^{[m/2]} A_k < 0$, and the sequence $A_0, \ldots, A_{[m/2]}$ has exactly one change of sign, we conclude from Lemma 1.3 that $g(\mu; x)$ is coefficient-wise log-convex. $\square$

Remark 2.3. *Note that the function $\psi(\mu; x)$ defined in* (10) *is a special case of the so-called hypergeometric superhyperbolic cosine [14, page 74, Definition 2.10]. Its second logarithmic derivative is easily seen to be positive:*

$$\frac{\partial^2}{\partial \mu^2}(\log \psi(\mu; x)) = \frac{4(1-x)^\mu (\operatorname{arctanh}\sqrt{x})^2}{((1-\sqrt{x})^\mu + (1+\sqrt{x})^\mu)^2} > 0,$$

which proves its log-convexity. On the other hand, the function $\mu \to (\mu)_{2n}$ in (8) *is log-concave for every $n \in \mathbb{N}_0$ according to* (5)*. This shows that the condition that $\{g_n\}$ is log-convex cannot be dropped, since taking $g_k = 1$ with $g_j = 0$ for $j \neq k$ yields a log-concave function. Moreover, the sequence $A_0, A_1, \ldots, A_{[m/2]}$ has exactly one change of sign, which shows that restricting all g_n to be strictly positive (which rules out the previous example) is also insufficient, emphasizing the importance of the condition that $\{g_n\}$ is log-convex. A proof of exactly one change of sign is given in the Appendix.*

Remark 2.4. *Theorem 2.2 is a natural generalization of [9, Theorem 1]. For each integer $r \geq 1$ and $\mu > 0$, we set*

$$y(\mu; x) := \sum_{n=0}^{\infty} y_n \Gamma(\mu + rn) x^n,$$

$$h(\mu; x) := \sum_{n=0}^{\infty} \frac{h_n}{(\mu)_{rn}} x^n,$$

where $\{y_n\}_{n=0}^{\infty}$ and $\{h_n\}_{n=0}^{\infty}$ are positive sequences independent of μ, $\Gamma(\cdot)$ is Euler's gamma function. Similar to the proofs of [9, Theorem 2 and 3], we can show that $g(\mu; x)$ and $h(\mu; x)$ are both coefficient-wise log-convex.

Remark 2.5. *In [3], the authors introduced the k-Pochhammer symbol*

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k) \tag{12}$$

for any $k > 0$ and $n = 0, 1, \ldots$ and the k-Gamma function

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{x/k-1}}{(x)_{n,k}} = \int_0^\infty e^{-t^k/k} t^{x-1} dt.$$

It is straightforward to verify that

$$\Gamma_k(x) = k^{x/k-1}\Gamma(x/k), \quad (x)_{n,k} = k^n (x/k)_n = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}.$$

For each integer $r \geq 1$, and $k, \mu > 0$, we set

$$
\begin{aligned}
g_k(\mu; x) &= \sum_{n=0}^{\infty} g_n \frac{(\mu)_{2n,k}}{(2n)!} x^n \\
y_k(\mu; x) &:= \sum_{n=0}^{\infty} y_n \Gamma_k(\mu + rn) x^n, \\
h_k(\mu; x) &:= \sum_{n=0}^{\infty} \frac{h_n}{(\mu)_{rn,k}} x^n,
\end{aligned}
$$

where $\{g_n\}_{n=0}^{\infty}$ is log-convex and independent of μ, the sequences $\{y_n\}_{n=0}^{\infty}$ and $\{h_n\}_{n=0}^{\infty}$ are positive sequences and independent of μ. Using the similar approaches for the case of the usual Pochhammer symbol and Gamma function, we can prove that $g_k(\mu; x), y_k(\mu; x)$ and $h_k(\mu; x)$ are all coefficient-wise log-convex.

Remark 2.6. *Numerical experiments suggest that $\mu \to \sum_{n=0}^{\infty} \frac{(\mu)_{nr}}{(nr)!} x^n$ is neither coefficient-wise log-convex nor coefficient-wise log-concave for $r \geq 3$. For instance, we consider*

$$
\psi(\mu, x) = \sum_{n=0}^{\infty} \frac{(\mu)_{3n}}{(3n)!} x^n = \frac{1}{3} \sum_{k=0}^{2} \frac{1}{(1 - \omega_k x^{1/3})^{\mu}},
$$

where $\omega_k = e^{2k\pi i/3}$ are primitive roots of unity and the second equality follows from the binomial theorem combined with the elementary identity

$$
\sum_{k=0}^{r-1} \omega_k^n = \begin{cases} r, & \text{if } r \text{ divides } n \\ 0, & \text{otherwise} \end{cases}
$$

for $\omega_k = e^{2k\pi i/r}$ being r-th primitive roots of unity. Note first that the function $\mu \to \psi(\mu, x)$ is convex for $x > 0$ as a sum of convex functions. Next, by direct calculation for $\beta \geq \alpha > 0$, we have

$$
\begin{aligned}
\xi_\mu(\alpha, \beta; x) &:= \psi(\mu + \alpha; x)\psi(\mu + \beta; x) - \psi(\mu; x)\psi(\mu + \alpha + \beta; x) \\
&= -\frac{1}{2}\alpha\beta(\alpha + \beta + 2\mu + 2)x + c(\mu, \alpha, \beta)x^2 + O(x^3),
\end{aligned}
$$

where

$$
\begin{aligned}
c(\mu, \alpha, \beta) = -\frac{1}{240}\alpha\beta \big(&2\alpha^4 + 5\alpha^3\beta + 10\alpha^3\mu + 25\alpha^3 + 30\alpha^2\beta + 60\alpha^2\mu + 100\alpha^2 \\
&+5\alpha\beta^3 + 30\alpha\beta^2 + 30\alpha\beta\mu + 110\alpha\beta + 30\alpha\mu^2 + 220\alpha\mu + 185\alpha \\
&+2\beta^4 + 10\beta^3\mu + 25\beta^3 + 60\beta^2\mu + 100\beta^2 + 30\beta\mu^2 + 220\beta\mu + 185\beta \\
&+20\mu^3 + 220\mu^2 + 370\mu + 156 - 30\alpha\beta\mu^2 - 20\mu^3(\alpha + \beta) - 10\mu^4\big).
\end{aligned}
$$

It is straightforward to verify that

$$
\begin{aligned}
c(5.76788, 1, 1) &= -0.0998924, \\
c(6.13463, 1, 1) &= 9.99983.
\end{aligned}
$$

Thus, we see that $\psi(\mu, x)$ is neither coefficient-wise log-convex nor coefficient-wise log-concave.

Theorem 2.7. *Let*

$$f(\mu; x) = \sum_{n=1}^{\infty} f_n \frac{(\mu)_{2n}}{(2n-1)!} x^n = x(\mu)_2 \sum_{n=0}^{\infty} f_{n+1} \frac{(\mu/2+1)_n(\mu/2+3/2)_n}{(3/2)_n n!} x^n,$$

where $\{f_n\}_{n=1}^{\infty}$ is doubly positive and independent of μ. Then the formal power series $f(\mu; x)$ is coefficient-wise log-concave.

Proof. For each $\alpha, beta > 0$ $\beta \geq \alpha > 0$, we get

$$f(\mu+\alpha; x)f(\mu+\beta; x) - f(\mu; x)f(\mu+\alpha+\beta; x) := \sum_{m=2}^{\infty} \tilde{\phi}_m x^m,$$

where $\tilde{\phi}_m = \sum_{k=1}^{m} f_k f_{m-k} \tilde{M}_k$ and

$$\tilde{M}_k = \frac{1}{(2k-1)!\,(2(m-k)-1)!} \left[(\mu+\alpha)_{2k}(\mu+\beta)_{2(m-k)} - (\mu)_{2k}(\mu+\alpha+\beta)_{2(m-k)}\right].$$

Moreover, we can write $\tilde{\phi}_m$ in the following form:

$$\tilde{\phi}_m = \sum_{k=1}^{[m/2]} f_k f_{m-k} B_k,$$

where $B_k = \tilde{M}_k + \tilde{M}_{m-k}$ if $k < m/2$, and $B_k = \tilde{M}$ if $k = m/2$. Set

$$\tilde{B}_k = (2k-1)!\,(2(m-k)-1)! B_k,$$

which has the same sign as B_k. It is straightforward to see that $\tilde{B}_k = \tilde{A}_k$, which is defined by (9). Therefore, by the proof of Theorem 2.2, we see that the sequence $B_1, B_2, \ldots, B_{[m/2]}$ has no more than one change of sign and $B_{[m/2]} > 0$.

Next, we prove that $\sum_{k=1}^{[m/2]} B_k > 0$ for $m \geq 2$. Set

$$\tilde{\psi}(\mu; x) := \sum_{n=1}^{\infty} \frac{(\mu)_{2n}}{(2n-1)!} x^n,$$
$$\eta(\mu; x) := \frac{1}{(1-\sqrt{x})^{\mu+1}} - \frac{1}{(1+\sqrt{x})^{\mu+1}}.$$

Using the binomial theorem it is easy to see that

$$\tilde{\psi}(\mu; x) = \frac{\mu\sqrt{x}}{2} \cdot \eta(\mu; x).$$

Thus, we have

$$\tilde{\psi}(\mu+\alpha;x)\tilde{\psi}(\mu+\beta;x)-\tilde{\psi}(\mu;x)\tilde{\psi}(\mu+\alpha+\beta;x)=\sum_{m=2}^{\infty}\left(\sum_{k=0}^{[m/2]}B_k\right)x^m$$
$$=\frac{(\mu+\alpha)(\mu+\beta)x}{4}\eta(\mu+\alpha;x)\eta(\mu+\beta;x)-\frac{\mu(\mu+\alpha+\beta)x}{4}\eta(\mu;x)\eta(\mu+\alpha+\beta;x). \tag{13}$$

Since $(\mu+\alpha)(\mu+\beta)>\mu(\mu+\alpha+\beta)$, we see that the coefficients of (13) are larger than that of

$$\frac{\mu(\mu+\alpha+\beta)x}{4}\left[\eta(\mu+\alpha;x)\eta(\mu+\beta;x)-\eta(\mu;x)\eta(\mu+\alpha+\beta;x)\right].$$

Therefore, it suffices to prove the coefficient-wise log-concavity of $\eta(\mu;x)$. Indeed, we have

$$\begin{aligned}&\eta(\mu+\alpha;x)\eta(\mu+\beta;x)-\eta(\mu;x)\eta(\mu+\alpha+\beta;x)\\&\quad=(1-x)^{-1-\mu}\left[(1+\sqrt{x})^{-\alpha}-(1-\sqrt{x})^{-\alpha}\right]\cdot\left[(1+\sqrt{x})^{-\beta}-(1-\sqrt{x})^{-\beta}\right]\end{aligned} \tag{14}$$

Since the coefficients of $[(1+\sqrt{x})^{-\alpha}-(1-\sqrt{x})^{-\alpha}]$ are negative as explained in the proof of Theorem 2.2, it follows from (14) that $\eta(\mu;x)$ is coefficient-wise log-concave, and therefore $\sum_{k=0}^{[m/2]}B_k>0$ for $m\geq 2$ by (13).

In summary, since $\sum_{k=1}^{[m/2]}B_k>0$, the sequence $B_1,B_2,\ldots,B_{[m/2]}$ has no more than one change of sign and $B_{[m/2]}>0$, it follows from Lemma 1.3 that $f(\mu;x)$ is coefficient-wise log-concave. □

Remark 2.8. *For any $k,\mu>0$, we set*

$$f_k(\mu;x)=\sum_{n=0}^{\infty}f_n\frac{(\mu)_{2n,k}}{(2n-1)!}x^n,$$

where $\{f_n\}_{n=0}^{\infty}$ is doubly positive and independent of μ, and $(\mu)_{2n,k}$ is defined by (12). Similar to the proof of Theorem 2.7, we can show that $f_k(\mu;x)$ is also coefficient-wise log-concave.

Remark 2.9. *Similar to Remark 2.6, we find that the function*

$$\mu\to f(\mu;x):=\sum_{n=1}^{\infty}\frac{(\mu)_{nr}}{(nr-1)!}x^n=\frac{\mu}{r}x^{1/r}\sum_{k=0}^{r-1}\frac{\omega_k}{(1-\omega_k x^{1/r})^{\mu+1}},$$

$\omega_k=e^{2\pi ik/r}$, is neither coefficient-wise log-convex nor coefficient-wise log-concave for $r\geq 4$ by numerical computation. However, the numerical evidence confirms the following conjecture

Conjecture 2.10. *Assume that $\{f_n\}_{n=0}^{\infty}$ is doubly positive and independent of μ. Then the function*

$$\mu \to f(\mu;x) = \sum_{n=1}^{\infty} f_n \frac{(\mu)_{3n}}{(3n-1)!} x^n$$

is coefficient-wise log-concave.

3 Double over single Pochhammer series

We consider the series

$$f(\mu;x) := \sum_{k=0}^{\infty} f_k \frac{(\mu)_{2k}}{(\mu+1)_k k!} x^k = \sum_{k=0}^{\infty} f_k \frac{(\mu/2)_k((\mu+1)/2)_k}{(\mu+1)_k k!}(4x)^k. \tag{15}$$

Along with the series (3), this provides another example of the log-neutral function when $f_k = 1$ for all k. Indeed if we set

$$\psi(\mu;x) := \sum_{k=0}^{\infty} \frac{(\mu)_{2k}}{(\mu+1)_k k!} x^k \quad (|4x| < 1),$$

then $\psi(\mu;x)$ is a hypergeometric function with the following properties.

Lemma 3.1. *[12, Lemma 7] The hypergeometric function $\psi(\mu;x)$ has the following properties.*

1. *Closed form*
$$\psi(\mu;x) = \left(\frac{1-\sqrt{1-4x}}{2x}\right)^{\mu}.$$

2. *Index law*
$$\psi(\alpha;x)\psi(\beta;x) = \psi(\alpha+\beta;x).$$

This makes the following conjecture very natural.

Conjecture 3.2. *Assume that $\{f_k\}_{k=0}^{\infty}$ is doubly positive (log-convex) and independent of μ. Then the function $\mu \to f(\mu;x)$ defined in* (15) *is coefficient-wise log-concave (log-convex).*

Forming the generalized Turánian

$$\begin{aligned}\Delta_\mu(\alpha,\beta;x) &= f(\mu+\alpha;x)f(\mu+\beta;x) - f(\mu;x)f(\mu+\alpha+\beta;x)\\ &= \sum_{m=2}^{\infty} \delta_m x^m,\end{aligned}$$

we will have

$$\delta_m = \sum_{k=0}^{m} f_k f_{m-k} M_k,$$

$$M_k = \frac{1}{k!(m-k)!}\left[\frac{(\mu+\alpha)_{2k}(\mu+\beta)_{2(m-k)}}{(\mu+\alpha+1)_k(\mu+\beta+1)_{m-k}} - \frac{(\mu)_{2k}(\mu+\alpha+\beta)_{2(m-k)}}{(\mu+1)_k(\mu+\alpha+\beta+1)_{m-k}}\right].$$

Furthermore, we can write

$$\delta_m = \sum_{k=0}^{[m/2]} f_k f_{m-k} A_k \quad \text{for each} \quad m \geq 2,$$

where $A_k = M_k + M_{m-k}$ for $k < m/2$ and $A_k = M_k$ for $k = m/2$. As

$$\sum_{m=0}^{\infty}\left(\sum_{k=0}^{[m/2]} A_k\right) x^m = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} M_k\right) x^m$$
$$= \psi(\mu+\alpha;x)\psi(\mu+\beta;x) - \psi(\mu;x)\psi(\mu+\alpha+\beta;x),$$

Lemma 3.1 implies that that $\sum_{k=0}^{[m/2]} A_k = 0$ for $m \geq 0$. Hence, to apply Lemma 1.3 we only need to prove that the sequence $\{A_k\}_{k=0}^{[m/2]}$ has no more than one change of sign. So far, we failed to prove this fact.

Given a formal power series $f(\mu;x)$ defined by (1). We say that $\mu \to f(\mu;x)$ is *discrete coefficient-wise log-concave (log-convex)* if the coefficients of the series

$$\Delta_f(1,1;x) = f(\mu+1,x)^2 - f(\mu,x)f(\mu+2,x)$$

at all powers of x are non-negative (non-positive). It is a special case of coefficient-wise log-concavity (log-convexity) of $f(\mu;x)$ by specifying $\alpha = \beta = 1$.

Theorem 3.3. *Let $f(\mu;x)$ be the formal power series defined by (15). If $\{f_k\}_{k=0}^{\infty}$ is doubly positive (log-convex) and independent of μ, then $\mu \to f(\mu;x)$ is discrete coefficient-wise log-concave (log-convex).*

Proof. Consider the following generalized Turánian

$$\Delta_\mu(1,1;x) = f(\mu+1;x)^2 - f(\mu;x)f(\mu+2;x)$$
$$= \sum_{m=2}^{\infty} \delta_m x^m,$$

where

$$\delta_m = \sum_{k=0}^{m} f_k f_{m-k} M_k,$$

$$M_k = \frac{1}{k!(m-k)!}\left[\frac{(\mu+1)_{2k}(\mu+1)_{2(m-k)}}{(\mu+2)_k(\mu+2)_{m-k}} - \frac{(\mu)_{2k}(\mu+2)_{2(m-k)}}{(\mu+1)_k(\mu+3)_{m-k}}\right].$$

Moreover, we may write

$$\delta_m = \sum_{k=0}^{[m/2]} f_k f_{m-k} A_k \quad \text{for each} \quad m \geq 2,$$

where $A_k = M_k + M_{m-k}$ for $k < m/2$ and $A_k = M_k$ for $k = m/2$. For each $k \geq 0$, we set $\tilde{A}_k = k!(m-k)!A_k$, which has the same sign as A_k. Then

$$\tilde{A}_k = \begin{cases} 2 \cdot \underbrace{\dfrac{(\mu+1)_{2k}(\mu+1)_{2(m-k)}}{(\mu+2)_k(\mu+2)_{m-k}}}_{u_k} - \underbrace{\dfrac{(\mu)_{2k}(\mu+2)_{2(m-k)}}{(\mu+1)_k(\mu+3)_{m-k}}}_{r_k} \\ - \underbrace{\dfrac{(\mu)_{2(m-k)}(\mu+2)_{2k}}{(\mu+1)_{m-k}(\mu+3)_k}}_{s_k}, & \text{if} \quad k < m/2, \\ \frac{(\mu+1)_{2k}^2}{(\mu+2)_k^2} - \frac{(\mu)_{2k}(\mu+2)_{2k}}{(\mu+1)_k(\mu+3)_k}, & \text{if} \quad k = m/2. \end{cases}$$

Next, we show that the sequence $\{A_k\}_{k=0}^{[m/2]}$ has no more than one change of sign, *i.e.*,

$$\tilde{A}_k \leq 0 \quad \Rightarrow \quad \tilde{A}_{k-1} < 0 \quad \text{for} \quad k \geq 1. \tag{16}$$

Since

$$\begin{aligned} \tilde{A}_0 &= 2\frac{(\mu+1)_{2m}}{(\mu+2)_m} - \frac{(\mu+2)_{2m}}{(\mu+3)_m} - \frac{(\mu)_{2m}}{(\mu+1)_m} \\ &= \frac{(\mu+2)_{2m-2}}{(\mu+3)_{m-2}}\left[2\frac{(\mu+1)(\mu+2m)}{(\mu+2)(\mu+m+1)} - \frac{(\mu+2m)(\mu+2m+1)}{(\mu+m+1)(\mu+m+2)} - \frac{\mu}{\mu+2}\right] \\ &= -\frac{(\mu+2)_{2m-2}}{(\mu+3)_{m-2}} \cdot \frac{(m-1)m(\mu+4)}{(\mu+2)(\mu+m+1)(\mu+m+2)} < 0, \end{aligned}$$

we see that the claim (16) holds for $k = 1$. If $k \geq 2$, then we have

$$\begin{aligned} \tilde{A}_k &= 2u_k - r_k - s_k \\ &= \frac{(\mu+2)_{2k-2}(\mu+2)_{2(m-k)-2}}{(\mu+3)_{k-2}(\mu+3)_{(m-k)-2}}\left(2g_1 - g_2 - g_3\right), \end{aligned}$$

where

$$\begin{aligned} g_1 &= \frac{(\mu+1)^2(\mu+2k)(\mu+2(m-k))}{(\mu+2)^2(\mu+k+1)(\mu+(m-k)+1)}, \\ g_2 &= \frac{\mu(\mu+2(m-k))(\mu+2(m-k)+1)}{(\mu+2)(\mu+(m-k)+1)(\mu+(m-k)+2)}, \\ g_3 &= \frac{\mu(\mu+2k)(\mu+2k+1)}{(\mu+2)(\mu+k+1)(\mu+k+2)}. \end{aligned}$$

Set $k = q+2$ and $m-k = k+t = q+t+2$. Then $q \geq 0$ and $t \geq 0$. A direct calculation implies that

$$2g_1 - g_2 - g_3 = \frac{n_1}{(\mu+2)^2(\mu+q+3)(\mu+q+4)(\mu+q+t+3)(\mu+q+t+4)},$$

where

$$n_1 = 4\mu^4 + 52\mu^3 + 256\mu^2 + 576\mu + 8q^4 + 4\mu^2 q^3 + 32\mu q^3 + 16q^3 t + 96q^3 + 6\mu^3 q^2 + 66\mu^2 q^2 + 264\mu q^2 + 8q^2 t^2 + 6\mu^2 q^2 t + 48\mu q^2 t + 144q^2 t + 416q^2 + 2\mu^4 q + 38\mu^3 q + 244\mu^2 q + 688\mu q + 2\mu^2 q t^2 + 16\mu q t^2 + 48 q t^2 + 6\mu^3 q t + 66\mu^2 q t + 264\mu q t + 416 q t + 768 q - \mu^4 t^2 - 7\mu^3 t^2 - 10\mu^2 t^2 + 24\mu t^2 + 64 t^2 + \mu^4 t + 19\mu^3 t + 122\mu^2 t + 344\mu t + 384 t + 512. \tag{17}$$

Thus, we see that $\tilde{A}_k \le 0$ is equivalent to $n_1 \le 0$ with $q \ge 0, t \ge 0$, and $\mu \ge 0$. Set

$$\begin{aligned}
I_1 &= \frac{u_{k-1}}{u_k} = \frac{(k+\mu+1)(2k-\mu-2m-2)(2k-\mu-2m-1)}{(2k+\mu-1)(2k+\mu)(-k+\mu+m+2)},\\
I_2 &= \frac{r_{k-1}}{r_k} = \frac{(k+\mu)(2k-\mu-2m-3)(2k-\mu-2m-2)}{(2k+\mu-2)(2k+\mu-1)(-k+\mu+m+3)},\\
I_3 &= \frac{s_{k-1}}{s_k} = \frac{(k+\mu+2)(2k-\mu-2m-1)(2k-\mu-2m)}{(2k+\mu)(2k+\mu+1)(-k+\mu+m+1)}.
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{A}_{k-1} &= 2u_{k-1} - r_{k-1} - s_{k-1}\\
&= 2I_1 u_k - I_2 r_k - I_3 s_k\\
&= \frac{(\mu+2)_{2k-2}(\mu+2)_{2(m-k)-2}}{(\mu+3)_{k-2}(\mu+3)_{(m-k)-2}}\left(2I_1 g_1 - I_2 g_2 - I_3 g_3\right),
\end{aligned}$$

where

$$2I_1 g_1 - I_2 g_2 - I_3 g_3 = \frac{(\mu+2q+2t+4)(\mu+2q+2t+5)n_2}{(\mu+2)^2(\mu+q+3)(\mu+2q+2)(\mu+2q+3)} \cdot \frac{1}{(\mu+q+t+3)(\mu+q+t+4)(\mu+q+t+5)}$$

with

$$n_2 = 18\mu^3 + 150\mu^2 + 408\mu + 8q^4 + 4\mu^2 q^3 + 32\mu q^3 + 16q^3 t + 96q^3 + 6\mu^3 q^2 + 66\mu^2 q^2 + 264\mu q^2 + 8q^2 t^2 + 6\mu^2 q^2 t + 48\mu q^2 t + 128 q^2 t + 400 q^2 + 2\mu^4 q + 38\mu^3 q + 240\mu^2 q + 656\mu q + 2\mu^2 q t^2 + 16\mu q t^2 + 32 q t^2 + 6\mu^3 q t + 62\mu^2 q t + 232\mu q t + 304 q t + 672 q - \mu^4 t^2 - 7\mu^3 t^2 - 12\mu^2 t^2 + 8\mu t^2 + 24 t^2 - 3\mu^4 t - 15\mu^3 t + 14\mu^2 t + 160\mu t + 192 t + 360. \tag{18}$$

Therefore, we have that $\tilde{A}_{k-1} < 0$ is equivalent to $n_2 < 0$ with $q \ge 0, t \ge 0$, and $\mu \ge 0$. By (17) and (18), we find that

$$n_2 = n_1 - 4\mu^4 - 34\mu^3 - 106\mu^2 - 168\mu - 16q^2 t - 16q^2 - 4\mu^2 q - 32\mu q - 16 q t^2 - 4\mu^2 q t - 32\mu q t - 112 q t - 96 q - 2\mu^2 t^2 - 16\mu t^2 - 40 t^2 - 4\mu^4 t - 34\mu^3 t - 108\mu^2 t - 184\mu t - 192 t - 152. \tag{19}$$

From the above identity, we see that $n_1 \le 0$ implies $n_2 < 0$, and thus the claim (16) holds for $k \ge 2$.

By Lemma 3.1, we see that $\sum_{k=0}^{[m/2]} A_k = 0$. Together with the sequence $\{A_k\}_{k=0}^{[m/2]}$ has no more than one change of sign, it give rise to the implication that $A_{[m/2]} > 0$. Above all, it follows from Lemma 1.3 that the conclusion holds. $\square$

Remark 3.4. *Using the approach in the proof of Theorem 3.3 , we proved the coefficient-wise log-concavity (log-convexity) of $f(\mu; x)$ at $(\alpha, \beta) = (1, 2), (1, 3)$, respectively. The problem still attributes to prove that $n_1 \le 0$ implies $n_2 < 0$ with $q \ge 0, t \ge 0$, and $\mu \ge 0$, where n_1 and n_2 are polynomials in q, t, and μ. However, in those two cases, there are no nice identities as* (19). *Instead, we utilize Cylindrical Algebraic Decomposition [2, 10], which is implemented in* `Mathematica`*, to prove the desired identities. The detailed calculation involved in the proof can be found in [8].*

4 Pochhammer over twice Pochhammer series

Given $\mu \ge 0$. Assume that$\{g_k\}_{k=0}^{\infty}$ is positive and independent of μ. We show that the formal power series

$$g(\mu; x) := \sum_{k=0}^{\infty} g_k \frac{(\mu)_k}{(2\mu)_k k!} x^k \tag{20}$$

is coefficient-wise log-convex.

Theorem 4.1. *Assume that $\beta \ge \alpha > 0$. Then the function*

$$\begin{aligned}\bar{\phi}_\mu(\alpha, \beta; x) &= g(\mu + \alpha; x) g(\mu + \beta; x) - g(\mu; x) g(\mu + \alpha + \beta; x) \\ &= \sum_{m=2}^{\infty} \bar{\phi}_m x^m\end{aligned}$$

has negative power series coefficients $\bar{\phi}_m < 0$ so that the function $\mu \mapsto g(\mu; x)$ is coefficient-wise log-convex.

Proof. By direct computation, we have

$$\bar{\phi}_m = \sum_{k=0}^{m} g_k g_{m-k} \bar{M}_k,$$

where

$$\bar{M}_k = \frac{1}{k!(m-k)!} \left[\frac{(\mu + \alpha)_k (\mu + \beta)_{m-k}}{(2(\mu + \alpha))_k (2(\mu + \beta))_{m-k}} - \frac{(\mu)_k (\mu + \alpha + \beta)_{m-k}}{(2\mu)_k (2(\mu + \alpha + \beta))_{m-k}} \right].$$

Furthermore, we may write $\bar{\phi}_m$ in the form

$$\bar{\phi}_m = \sum_{k=0}^{[m/2]} g_k g_{m-k} \bar{A}_k, \tag{21}$$

where $\bar{A}_k = \bar{M}_k + \bar{M}_{m-k}$ for $k < m/2$ and $\bar{A}_k = \bar{M}_k$ for $k = m/2$. We set $\tilde{A}_k = k!(m-k)!\bar{A}_k$. Then

$$\tilde{A}_k = \begin{cases} \underbrace{\frac{(\mu+\alpha)_k(\mu+\beta)_{m-k}}{(2(\mu+\alpha))_k(2(\mu+\beta))_{m-k}}}_{u_k} + \underbrace{\frac{(\mu+\alpha)_{m-k}(\mu+\beta)_k}{(2(\mu+\alpha))_{m-k}(2(\mu+\beta))_k}}_{v_k} \\ - \underbrace{\frac{(\mu)_k(\mu+\alpha+\beta)_{m-k}}{(2\mu)_k(2(\mu+\alpha+\beta))_{m-k}}}_{r_k} - \underbrace{\frac{(\mu)_{m-k}(\mu+\alpha+\beta)_k}{(2\mu)_{m-k}(2(\mu+\alpha+\beta))_k}}_{s_k} \\ \quad \text{if} \quad k < m/2, \\ \frac{(\mu+\alpha)_k(\mu+\beta)_k}{(2(\mu+\alpha))_k(2(\mu+\beta))_k} - \frac{(\mu)_k(\mu+\alpha+\beta)_k}{(2\mu)_k(2(\mu+\alpha+\beta))_k} \quad \text{if} \quad k = m/2. \end{cases}$$

By (21), it suffices to prove that:

(i) $\tilde{A}_k \le 0$ for $k = [m/2]$, where the equality holds when $k = 0, 1$;

(ii) $\tilde{A}_k < 0$ for $k = 0, 1, \ldots, [m/2] - 1$.

We first prove (i). If $k = 0, 1$, then it is straightforward to verify that $\tilde{A}_k = 0$. If $k \ge 2$, then $\tilde{A}_k < 0$ is equivalent to

$$\frac{(\mu+\alpha)_k(\mu+\beta)_k}{(\mu)_k(\mu+\alpha+\beta)_k} < \frac{(2(\mu+\alpha))_k(2(\mu+\beta))_k}{(2\mu)_k(2(\mu+\alpha+\beta))_k}, \tag{22}$$

which is true because

$$\begin{aligned} &\frac{(2(\mu+\alpha)+i)(2(\mu+\beta)+i)}{(2\mu+i)(2(\mu+\alpha+\beta)+i)} - \frac{(\mu+\alpha+i)(\mu+\beta+i)}{(\mu+i)(\mu+\alpha+\beta+i)} \\ &\quad = \frac{i\alpha\beta(3i+4\mu+2\alpha+4\beta)}{(\mu+i)(2\mu+i)(\mu+\alpha+\beta+i)(2(\mu+\alpha+\beta)+i)} > 0 \quad \text{for} \quad i > 0. \end{aligned}$$

Next, we prove (ii). In this case, we have $u_k < s_k$ is equivalent to

$$\frac{(\mu+\alpha)_k(\mu+\beta)_{m-k}}{(\mu)_{m-k}(\mu+\alpha+\beta)_k} < \frac{(2(\mu+\alpha))_k(2(\mu+\beta))_{m-k}}{(2\mu)_{m-k}(2(\mu+\alpha+\beta))_k}$$

In light of (22), it suffices to prove the above inequality by showing that

$$\frac{(\mu+\beta+k)_{m-2k}}{(\mu+k)_{m-2k}} < \frac{(2(\mu+\beta)+k)_{m-2k}}{(2\mu+k)_{m-2k}},$$

which holds because

$$\frac{2(\mu+\beta)+i}{2\mu+i} - \frac{\mu+\beta+i}{x+i} = \frac{i\beta}{(\mu+i)(2\mu+i)} > 0 \quad \text{for} \quad i > 0.$$

Similarly, we can show that the inequalities $v_k < s_k, r_k < s_k$ and $u_k v_k < r_k s_k$ hold for $k = 0, 1, \ldots, [m/2] - 1$. Thus, it follows from Lemma 1.2 that

$$\tilde{A}_k = u_k + v_k - r_k - s_k < 0.$$

$\square$

Remark 4.2. *By numerical experiments, we observe the following series*

$$h(\mu; x) := \sum_{k=0}^{\infty} h_k \frac{(2\mu)_k}{(\mu)_k k!} x^k,$$

whose coefficients of $x^k/k!$ are reciprocals for that of (20), is coefficient-wise log-concave provided that the sequence $\{h_k\}_{k=0}^{\infty}$ is doubly positive and independent of μ. Currently, we are not able to prove this fact.

5 Applications

Here we give several examples of concrete special functions whose logarithmic concavity/convexity in parameters can be established using the results of Sections 2-4.

5.1 Generalized hypergeometric function

The generalized hypergeometric function is defined by the series

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \;\middle|\; z \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!} z^n. \tag{23}$$

The series (23) converges in the entire complex plane if $p \le q$ and in the unit disk if $p = q + 1$. In the latter case its sum can be extended analytically to the whole complex plane cut along the ray $[1, \infty)$ [1, Chapter 2]. Applications of the previous results to hypergeometric functions is largely based on the following lemma.

Lemma 5.1. *Denote by $e_k(x_1, \ldots, x_q)$ the k-th elementary symmetric polynomial,*

$$e_0(x_1, \ldots, x_q) = 1, \quad e_k(x_1, \ldots, x_q) = \sum_{1 \le j_1 < j_2 \cdots < j_k \le q} x_{j_1} x_{j_2} \cdots x_{j_k}, \quad k \ge 1.$$

Suppose $q \ge 1$ and $0 \le r \le q$ are integers, $a_i > 0$, $i = 1, \ldots, q - r$, $b_i > 0$, $i = 1, \ldots, q$, and

$$\frac{e_q(b_1, \ldots, b_q)}{e_{q-r}(a_1, \ldots, a_{q-r})} \le \frac{e_{q-1}(b_1, \ldots, b_q)}{e_{q-r-1}(a_1, \ldots, a_{q-r})} \le \cdots$$
$$\cdots \le \frac{e_{r+1}(b_1, \ldots, b_q)}{e_1(a_1, \ldots, a_{q-r})} \le e_r(b_1, \ldots, b_q). \tag{24}$$

Then the sequence of hypergeometric terms (if $r = q$ the numerator is 1),

$$f_n = \frac{(a_1)_n \cdots (a_{q-r})_n}{(b_1)_n \cdots (b_q)_n}, \; n = 0, 1, \ldots,$$

is log-concave, i.e. $f_{n-1} f_{n+1} \le f_n^2$, $n = 1, 2, \ldots$ *It is strictly log-concave unless* $r = 0$ *and* $a_i = b_i$, $i = 1, \ldots, q$.

The proof of this lemma for $r = 0$ can be found in [4, Theorem 4.4] and [9, Lemma 2]. The latter reference also explains how to extend the proof to general r (see the last section of [9]). A simpler sufficient condition for (24) and so for log-concavity of $\{f_n\}$ is given in the following lemma [5, Lemma 4].

Lemma 5.2. *Suppose*

$$\sum\nolimits_{j=1}^{k} b_{n_j} \le \sum\nolimits_{j=1}^{k} a_j \text{ for } k = 1, 2, \ldots, q - r \tag{25}$$

for some $(q-r)$*-dimensional sub-vector* $(b_{n_1}, \ldots, b_{n_{q-r}})$ *of* $(b_1, \ldots, b_q)$. *Then inequalities* (24) *hold true.*

Application of Theorems 2.2 and 2.7 lead immediately to the following statement.

Theorem 5.3. *Let* $0 \le p \le q$ *be integers and suppose that positive parameters* $(a_1, \ldots, a_p)$, $(b_1, \ldots, b_q)$ *satisfy* (24) *or* (25) *with* $r = p - q$. *Then the function*

$$\mu \to {}_{q+2}F_{q+1}(\mu/2, \mu/2 + 1/2, b_1, \ldots, b_q; 1/2, a_1, \ldots, a_q; x)$$

is coefficient-wise log-convex for $\mu > 0$, *and the function*

$$\mu \to (\mu)_2 \cdot {}_{p+2}F_{q+1}(\mu/2 + 1, \mu/2 + 3/2, a_1, \ldots, a_p; 3/2, b_1, \ldots, b_q; x)$$

is coefficient-wise log-concave for $\mu > 0$.

Theorem 3.3 yields the following statement.

Theorem 5.4. *Let* $0 \le p \le q$ *be integers and suppose that positive parameters* $(a_1, \ldots, a_p)$, $(b_1, \ldots, b_q)$ *satisfy* (24) *or* (25) *with* $r = p - q$. *Then the function*

$$f(\mu; x) = {}_{p+2}F_{q+1}(\mu/2, \mu/2 + 1/2, a_1, \ldots, a_p; \mu + 1, b_1, \ldots, b_q; x)$$

satisfies the Turán inequality $[f(\mu+1; x)]^2 - f(\mu; x) f(\mu+2; x) > 0$ *and the coefficients at all powers of* x *in this expression are positive. Then the function*

$$\hat{f}(\mu; x) = {}_{q+2}F_{q+1}(\mu/2, \mu/2 + 1/2, b_1, \ldots, b_q; \mu + 1, a_1, \ldots, a_q; x)$$

satisfies the reverse Turán inequality $[\hat{f}(\mu+1; x)]^2 - \hat{f}(\mu; x) \hat{f}(\mu+2; x) < 0$ *and the coefficients at all powers of* x *in this expression are negative.*

5.2 Bivariate hypergeometric function

The Horn H_4 hypergeometric function is defined by [13, 1.3(12)]

$$H_4(\alpha,\beta,\gamma,\delta;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n x^m y^n}{(\gamma)_m(\delta)_n m!n!}=\sum_{m=0}^{\infty}\frac{(\alpha)_{2m}x^m}{(\gamma)_m m!}{}_2F_1\left(\begin{matrix}\alpha+2m,\beta\\ \delta\end{matrix};y\right). \tag{26}$$

The series converges in certain neighborhood of $x=y=0$, details can be found in [13, Section 2.2]. The series on the right-hand side is of the form (8) if we set

$$g_m=\frac{(2m)!}{(\gamma)_m m!}{}_2F_1\left(\begin{matrix}\alpha+2m,\beta\\ \delta\end{matrix};\,y\right)=\frac{(2m)!}{(\gamma)_m m!}F_m.$$

It is straightforward to verify that the sequence $(2m)!/[(\gamma)_m m!]$ is log-convex if $\gamma>1/2$. Furthermore, it follows from [9, Theorem 7] that the sequence

$$F_m={}_2F_1\left(\begin{matrix}\alpha+2m,\beta\\ \delta\end{matrix};\,y\right)$$

is log-convex if $\delta>\beta>0$ and $y\in(-\infty,1)$. As the product of two log-convex sequences is log-convex, an application of Theorem 2.2 leads to the following statement

Theorem 5.5. *Suppose the point (x,y), $x>0$, is in the domain of convergence of the series* (26) *and $\delta>\beta>0$ and $\gamma>1/2$. Then the function*

$$\alpha\to H_4(\alpha,\beta,\gamma,\delta;x,y)$$

is log-convex on $(0,\infty)$ and, moreover, coefficient-wise log-convex when considered as the single series on the right-hand side of (26).

Further, by taking $\gamma=\alpha+1/2$ and setting

$$f_m={}_2F_1\left(\begin{matrix}\alpha+2m,\beta\\ \delta\end{matrix};\,y\right)$$

we are in the position to apply Theorem 3.3. Indeed, the sequence $\{f_m\}$ is log-concave for all $\alpha>0$ if $y>0$, $\beta>\delta>0$ or $y<0$ and $\delta>0>\beta$ according to [9, Theorem 6]. Hence, we obtain

Theorem 5.6. *Suppose $\alpha>0$ and the point (x,y), $x>0$, lies in the domain of convergence of the series* (26)*. If $y>0$ and $\beta>\delta>0$ or $y<0$ and $\delta>0>\beta$. Then the function*

$$f(\alpha)=H_4(\alpha,\beta,\alpha+1/2,\delta;x,y)$$

satisfies the Turán type inequality $f(\alpha+1)^2\geq f(\alpha)f(\alpha+2)$. If $\delta>\beta>0$ and $y\in(-\infty,1)$, then it satisfies the reverse Turán type inequality $f(\alpha+1)^2\leq f(\alpha)f(\alpha+2)$.

In a similar fashion we can take Appell's functions F_2 and F_3 defined, respectively, by [13, 1.3(2,3)]

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!}x^my^n$$
$$=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m x^m}{(\gamma)_m m!}{}_2F_1\left(\begin{matrix}\alpha+m,\beta'\\ \gamma'\end{matrix};\,y\right),\quad |x|+|y|<1,\quad (27)$$

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!n!}x^my^n$$
$$=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m x^m}{(\gamma)_m m!}{}_2F_1\left(\begin{matrix}\alpha',\beta'\\ \gamma+m\end{matrix};\,y\right),\quad \max|x|,|y|<1.\quad (28)$$

Referring again to [9, Theorems 6,7] we see that the sequence

$$m\to{}_2F_1\left(\begin{matrix}\alpha+m,\beta'\\ \gamma'\end{matrix};\,y\right)$$

is log-concave for all $\alpha>0$ if $y>0$, $\beta'>\gamma'>0$ or $y<0$ and $\gamma'>0>\beta'$ and log-convex if $\gamma'>\beta'>0$ and $y\in(-\infty,1)$. Hence, application of Theorem 3.3 yields.

Theorem 5.7. *Suppose $\alpha>0$ and the point (x,y), $x>0$, lies in the domain of convergence of the series* (27)*. If if $y>0$, $\beta'>\gamma'>0$ or $y<0$ and $\gamma'>0>\beta'$, then the function*

$$f(\alpha)=F_2(\alpha/2,(\alpha+1)/2,\beta';\alpha+1,\gamma';x,y)$$

satisfies the Turán type inequality $f(\alpha+1)^2\ge f(\alpha)f(\alpha+2)$. If $\delta>\beta>0$ and $y\in(-\infty,1)$, then it satisfies the reverse Turán type inequality $f(\alpha+1)^2\le f(\alpha)f(\alpha+2)$.

5.3 Extended and incomplete hypergeometric functions

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