1 § 11.10 Taylor and Maclaurin Series

(Previously, we found power series representation for a certain restricted class of functions.)

Question: 1. Which function have power series representations?

2. How can we find such representations?

Assume that f can be represented by a power series.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots, \quad |x-a| < R \quad (1)$$

Goal: determine c_n in terms of f.

Substituting x by a in (1) we get

$$f(a) = c_0$$

Differentiating (1), we get

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots, \quad |x-a| < R \quad (2)$$

Substituting x by a in (2), we get

$$f'(a) = c_1$$

Differentiating (2), we get

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + \cdots, \quad |x-a| < R \quad (3)$$

Substituting x by a in (3), we get

$$f''(a) = 2c_2$$

Differentiating (3), we get

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots, \quad |x-a| < R \quad (4)$$

Substituting x by a in (4), we get

$$f'''(a) = 2\cdot 3c_3 = 3!\,c_3$$

~~Keep d~~ keeping doing the above operations, we obtain

$$f^{(n)}(a) = 2\cdot 3\cdot 4\cdots n\,c_n = n!\,c_n$$

② Thus, $c_n = \frac{f^{(n)}(a)}{n!}$ for $n \geq 0$

By convention $0! = 1$, $f^{(0)}(x) = f(x)$, we have

$$c_0 = \frac{f^{(0)}(a)}{0!}$$

Theorem 5 If f has a power series representation at a, i.e., if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

By Theorem 5, the power series expansion of at a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \quad (6)$$

(6) is called the Taylor series of f at a (about a or centered at a).

If $a = 0$, then the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots \quad (7)$$

(7) is called the Maclaurin series of f.

Ex 1 Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence

$$f^{(n)}(x) = e^x \quad \text{for } n \geq 0$$

$$\Rightarrow f^{(n)}(0) = e^0 = 1$$

Thus, the Maclaurin series of e^x is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Let $a_n = \frac{x^n}{n!}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1 \text{ as } n \to \infty$$

By the Ratio Test, the radius of convergence is $R = \infty$

③ By Theorem 5 and Ex 1, if e^x has a power series expansion at 0, then $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

How can we determine whether e^x has a power series expansion or not?

More general question: if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \ ?$$

$$\Leftrightarrow \quad f(x) = \lim_{n\to\infty} T_n(x), \text{ where}$$

$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ is called the nth-degree Taylor polynomial of f at a.

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

Let $R_n(x) = f(x) - T_n(x)$. Then $f(x) = T_n(x) + R_n(x)$

$R_n(x)$ is called the nth remainder of the Taylor series.

~~Theorem 8 If $f(x)=T_n(x)+R_n(x)$, and~~

$$\cancel{\lim_{n\to\infty} R_n(x) = 0}$$

If we can show that $\lim_{n\to\infty} R_n(x) = 0$, then we have

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}[f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R(x) = f(x)$$

Theorem 8 If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a|<R$, then f is equal to the sum of its Taylor series on the interval $|x-a|<R$.

In order to show that $\lim_{n\to\infty} R_n(x) = 0$ for a given f, we use the following theorem.

④

Taylor's Inequality If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then $R_n(x)$ satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \le d$$

For the application of Theorem 8 and Taylor's Inequality, the following fact is useful:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \text{for every real number } x$$

Ex 2 Prove that e^x is equal to the sum of its Maclaurin Series.

Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$.

If $|x| \le d$, then $|f^{(n+1)}(x)| = e^x \le e^d$.

By Taylor's Inequality, with $a=0$ and $M = e^d$, we have

$$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \quad \text{for } |x| \le d \quad (11)$$

On the other hand, $\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \quad (12)$

By (11), (12), it follows from the Squeeze Theorem that $\lim_{n\to\infty} R_n(x) = 0$

By Theorem 8, we conclude that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

In particular, if we set $x = 1$, then

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

(which is useful for the evaluation of e)

Ex 4. Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

$f(x) = \sin x$, $\quad f(0) = 0$

$f'(x) = \cos x$, $\quad f'(0) = 1$

$f''(x) = -\sin x$, $\quad f''(0) = 0$

$f'''(x) = -\cos x$, $\quad f'''(0) = -1$

$f^{(4)}(x) = \sin x$, $\quad f^{(4)}(0) = 0$

Since the derivatives repeat in a cycle of four, we have:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

⑤ Since $f^{(n+1)}(x)$ is $\pm \sin x$ or $\pm \cos x$, we have $|f^{(n+1)}(x)| \le 1$ for all x.

By Taylor's Inequality:

$$|R_n(x)| \le \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|}{(n+1)!} \quad (M=1)$$

Since $\lim_{n\to\infty} \frac{|x|}{(n+1)!} = 0$, it follows from the Squeeze Theorem that $R_n(x) \to 0$ as $n \to \infty$.

Thus, $\sin x$ is equal to the sum of its Maclaurin ~~Theorem~~ series, i.e.,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x.$$

Ex 5 Find the Maclaurin series for $\cos x$.

$$\cos x = \frac{d}{dx}(\sin x)$$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}\right]$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx}\left[(-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}\right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} \quad \text{~~for all x~~}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x.$$

Ex 8: Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

$f(x) = (1+x)^k, \quad f(0) = 1$

$f'(x) = k(1+x)^{k-1}, \quad f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}, \quad f''(0) = k(k-1)$

$\vdots$

$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n}, \quad f^{(n)}(0) = k(k-1)\cdots(k-n+1)$

Thus, the Maclaurin series of $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

This is called the binomial series.

If k is nonegative integer, then the series is a finite sum.
(For other values of k, none of the term is zero.)

Let $a_n = \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$

⑥ Then $\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n} \right|$

$$= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x| \to |x| \text{ as } n\to\infty$$

By the Ratio Test, the binomial series converges if $|x|<1$ and diverges if $|x|>1$.

Notation: ~~the binomial coefficients~~ the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

and these numbers are called the binomial coefficients.

By Taylor's Inequality, we can show that $(1+x)^k$ is equal to the sum of its Maclaurin series. Thus, we have

(The Binomial Series Theorem) If k is any real number and $|x|<1$, then

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots$$

Ex 11. (a) Evaluate $\int e^{-x^2}dx$ as an infinite series

(b) Evaluate $\int_0^1 e^{-x^2}dx$ correct to within an error of $0.001=10^{-3}$.

(a) For all values of x,

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}$$

$$= 1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots$$

Now we integrate term by term.

$$\int e^{-x^2}dx = \int\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}\right]dx$$

$$= \sum_{n=0}^{\infty}\int(-1)^n\frac{x^{2n}}{n!}dx$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)n!}+C$$

$$= C+x-\frac{x^3}{3\cdot 1!}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}+\cdots$$

7 (b) The fundamental Theorem of Calculus gives

$$\int_0^1 e^{-x^2}dx = \left[x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \cdots\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots$$

$$\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475$$

The Alternating Series Estimation Theorem shows that the error in this ~~ex~~approximation is less than

$$\frac{1}{11\cdot 5!} = \frac{1}{1320} < 0.001.$$