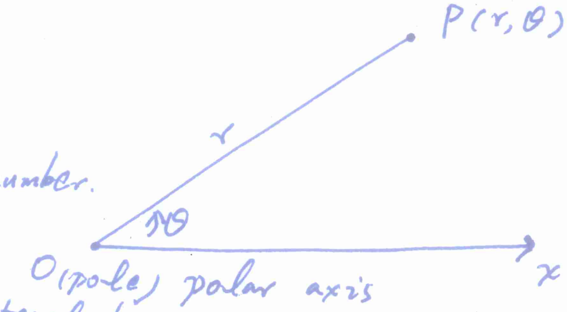
① § 10.3 Polar Coordinates

polar coordinate system:

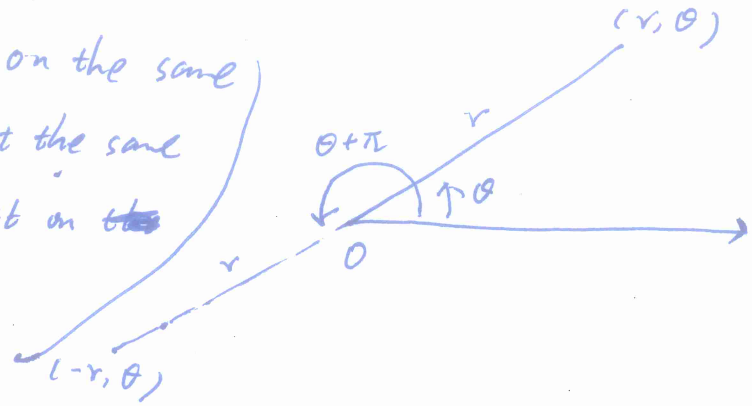


If $P = O$, then $r = 0$, θ is any number.

θ is positive if measured counterclockwise

negative ——— clockwise

($(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on the opposite sides of O.



~~$(-r, \theta)$ represents the same point as $(r, \theta+\pi)$~~

If $r > 0$, (r, θ) lies in the same quadrant as θ,

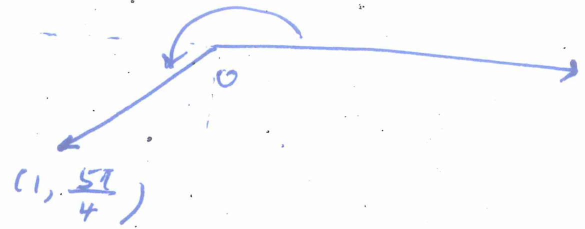
$r < 0$, lies in the quadrant on the opposite side of the pole

Note: $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$

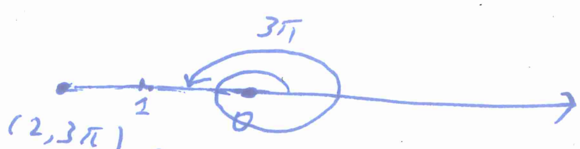
Ex 1. Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

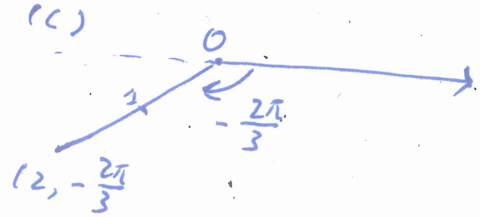
(a)



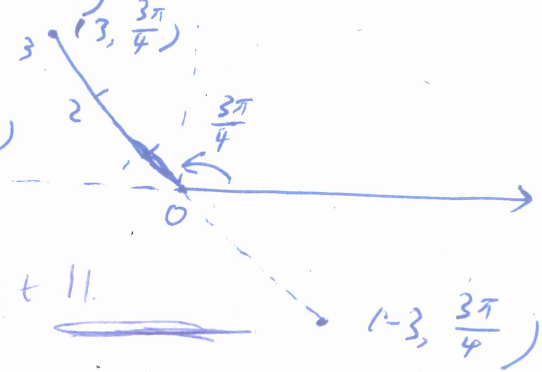
(b)



(c)



(d)

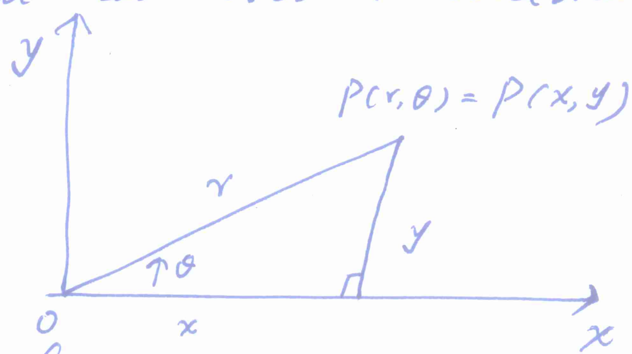


Oct 11

② Note: (r, θ) is also represented by

$(r, \theta + 2n\pi)$ and $(-r, \theta + (2n+1)\pi)$

Connection between polar and ~~Cat~~ Cartesian coordinates.



$\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$

$\Rightarrow$

$x = r\cos\theta$, $y = r\sin\theta$ (1)

Note: (1) is valid for all values of r and θ.

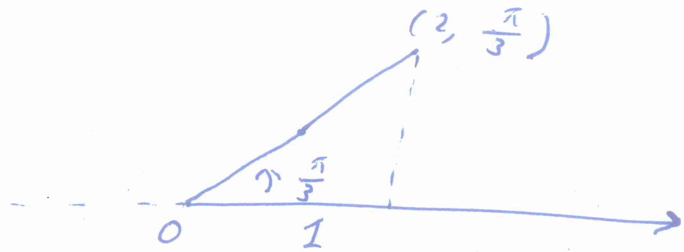
$r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$ (2)

Note: determine θ by ~~xxxx~~ the quadrant of (x, y).

Ex 2: Convert $(2, \pi/3)$ ~~fo~~ from polar to Cartesian coordinates.

$x = r\cos\theta = 2\cos\frac{\pi}{3} = 1$

$y = r\sin\theta = 2\sin\frac{\pi}{3} = \sqrt{3}$



Ex 3. Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

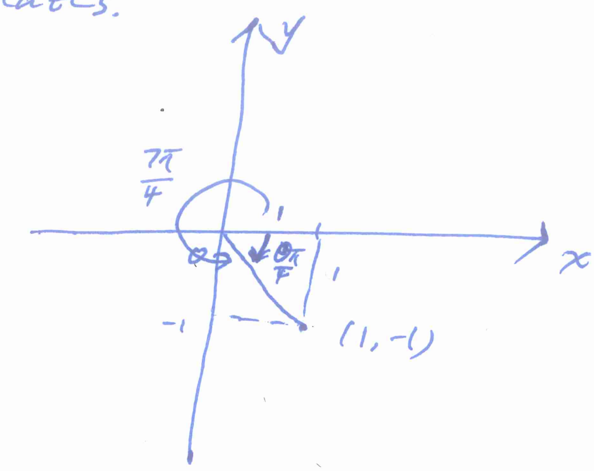
$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\tan\theta = \frac{y}{x} = -1$

$\Rightarrow \theta = -\frac{\pi}{4}$ or $\frac{7\pi}{4}$

polar coordinates:

$(\sqrt{2}, -\frac{\pi}{4})$ or $(\sqrt{2}, \frac{7\pi}{4})$.



③

Polar Curves

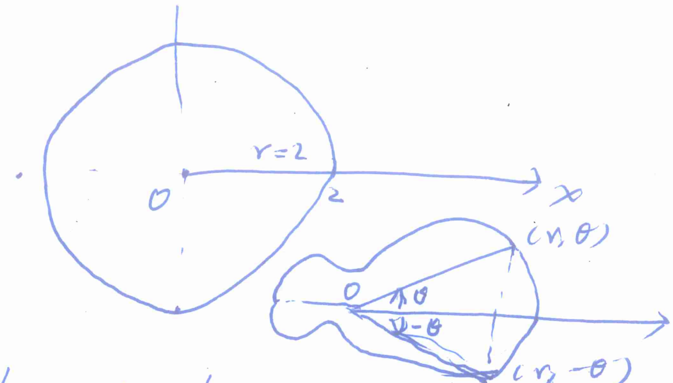
The graph of a polar equation $r = f(\theta)$ consits of all points P whose polar representation (r, θ) satisfy the equation.

Polar Equation. $r = f(\theta)$

Polar Curves: $\{(r, \theta) \mid r = f(\theta)\}$

Ex 4 What is curve represented by $r = 2$?

A circle with center O and radius 2.



Use Symmetry to sketch polar curves:

(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.

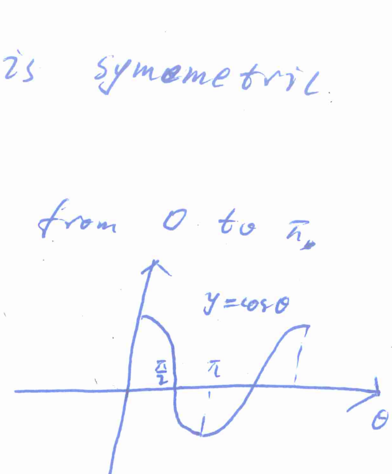
Ex 6. (a) Sketch the curve with polar equation $r = 2\cos\theta$

(b) Find a Cartesian equation for the curve.

(a) (1) Since $\cos\theta = \cos(-\theta)$, the curve is symmetric about the polar axis.

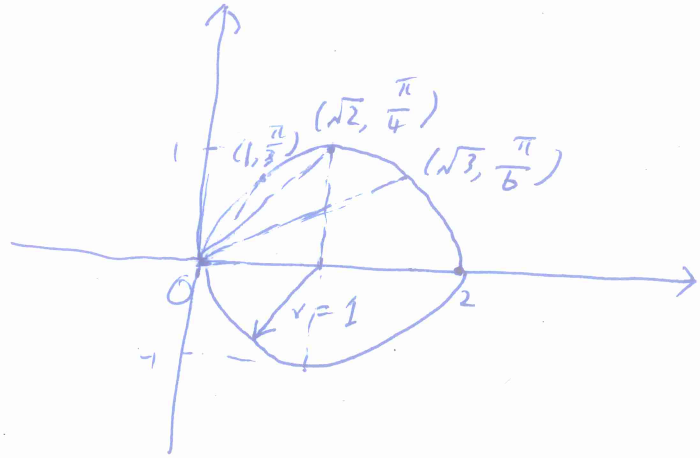
(2) We only need to take values of θ from 0 to π.

By (1), (2), we only need to sketch the curve for $\theta \in [0, \frac{\pi}{2}]$.



(a)

θ	$r=2\cos\theta$
0	2
$\frac{\pi}{6}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{3}$	1
$\frac{\pi}{2}$	0



(b) $x = r\cos\theta$

$\cos\theta = \frac{r}{x}$ (1)

Substitution (1) into $r = 2\cos\theta$, we get

$r = 2x/r$

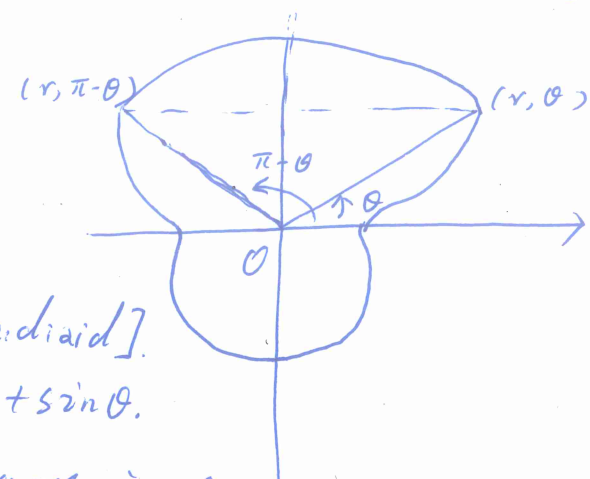
$\Leftrightarrow r^2 = 2x$

$\Leftrightarrow x^2+y^2 = 2x$

$\Leftrightarrow (x-1)^2+y^2 = 1$, which is a circle with center $(1,0)$ and radius 1.

Oct 14th,

(B) If the equation in unchanged when θ is replaced by $\pi-\theta$, the curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$



Cardioid ['kɑːdiɔid]

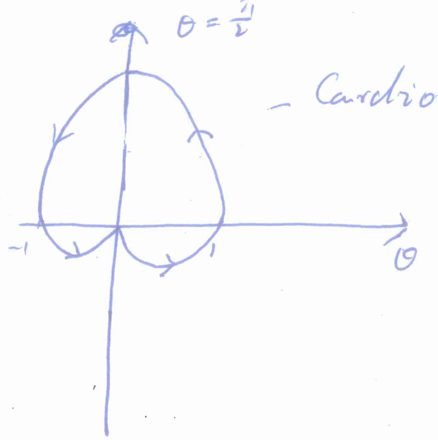
Ex 7 Sketch the curve $r = 1+\sin\theta$.

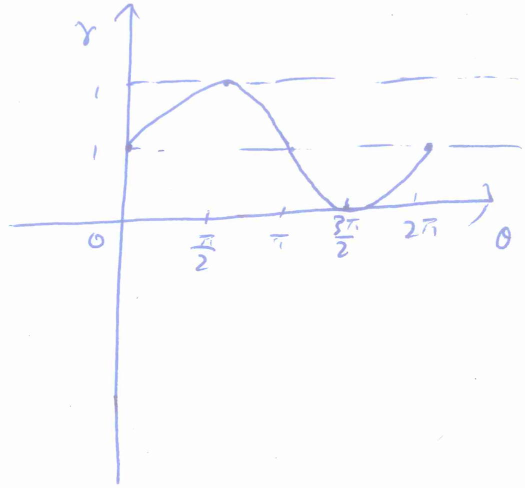
(1) Since $\sin(\pi-\theta) = \sin\theta$, the curve is symmetric about $\theta = \frac{\pi}{2}$.

(2) We only need to take θ from 0 to 2π.

By (1), (2), we only need to sketch the curve for $\theta \in [0, \frac{\pi}{2}]$ and $\theta \in [\frac{3\pi}{2}, 2\pi]$

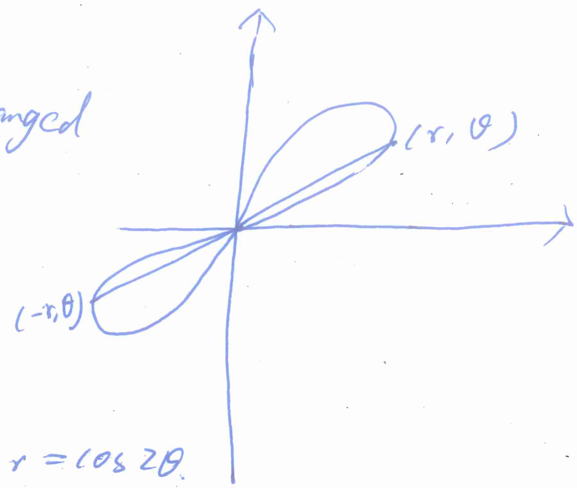
⑤





(c) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta+\pi$, the curve is symmetric about the pole.

(The curve remains unchanged ~~it~~ if we rotate it through 180° about the origin.



Ex 8. Sketch the curve $r = \cos 2\theta$.

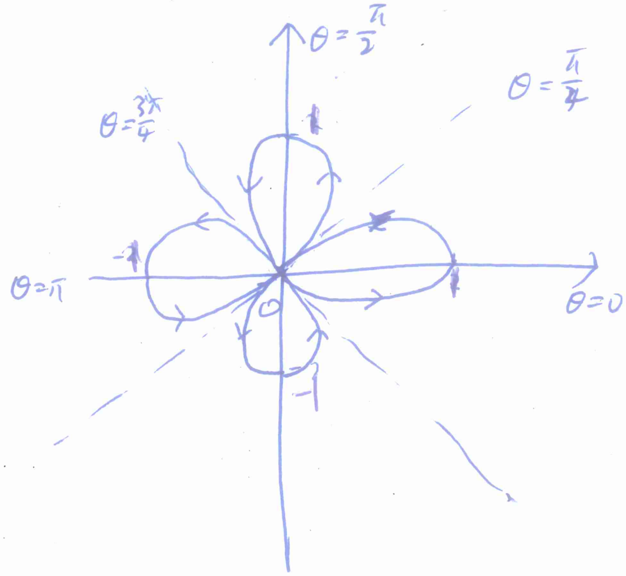
(1) We only need to sketch the curve for $0 \le \theta \le 2\pi$.

(2) Since $\cos(2(\theta+\pi)) = \cos\theta$, the curve is symmetric about the pole

(3) Since $\cos(2\theta) = \cos(-2\theta)$, the curve is symmetric about the polar axis

(4) Since $\cos(2\theta) = \cos(2(\pi-\theta))$, the curve is symmetric about the verticle line $\theta = \frac{\pi}{2}$.

By (1), (2), (3), and (4), we only need to sketch the curve for ~~the first and fourth quadrants.~~ $\theta \in [0, \frac{\pi}{2}]$



Four-leaves rose $r=\cos 2\theta$

Tangents to Polar Curves

~~Given a polar curve r~~

Assume the polar curve is described by: $r=f(\theta)$.

Then

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

By Product Rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Note: 1. Locate horizotal tangents by finding points such that $\frac{dy}{d\theta}=0$ and $\frac{dx}{d\theta}\neq 0$

— vetrical —

— $\frac{dx}{d\theta}=0$ and $\frac{dy}{d\theta}\neq 0$

2. The tangent lines at the pole:

$$r=0, \Rightarrow \frac{dy}{dx}=\tan\theta \quad \text{if } \frac{dr}{d\theta}\neq 0.$$

Ex 9. (a) For the cardioid $r=1+\sin\theta$, find the slope of the tangent line when $\theta=\frac{\pi}{3}$

(7) $\cancel{\frac{dy}{dx}}$ $x = r\cos\theta = (1+\sin\theta)\cos\theta$

$y = r\sin\theta = (1+\sin\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\cos(\frac{\pi}{3})(1+2\sin(\frac{\pi}{3}))}{(1+\sin(\pi/3))(1-2\sin(\pi/3))}$$

$$= \frac{\frac{1}{2}(1+\sqrt{3})}{(1+\frac{\sqrt{3}}{2})(1-\sqrt{3})}$$

$$= \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})}$$

$$= \frac{1+\sqrt{3}}{-1-\sqrt{3}}$$

$$= -1$$

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

$$\frac{dy}{d\theta} = \cos\theta\cdot(1+2\sin\theta) = 0 \quad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{7\pi}{6}$$

$$\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) = 0 \quad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Thus, there are horizontal tangents at $(2, \frac{\pi}{2})$, $(\frac{1}{2}, \frac{7\pi}{6})$, $(\frac{1}{2}, \frac{11\pi}{6})$,

vetrical —— —— $(\frac{3}{2}, \frac{\pi}{6})$ and $(\frac{3}{2}, \frac{5\pi}{6})$.

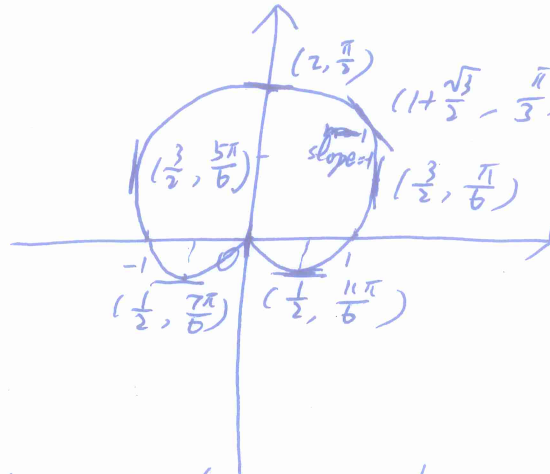
When $\theta = \frac{3\pi}{2}$, $\cancel{dx}\,\frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$

$$\lim_{\theta\to(\frac{3\pi}{2})^-} \frac{dy}{dx} = \lim_{\theta\to(\frac{3\pi}{2})^-} \frac{1+2\sin\theta}{1-2\sin\theta} \cdot \lim_{\theta\to(\frac{3\pi}{2})^-} \frac{\cos\theta}{1+\sin\theta}$$

$$= -\frac{1}{3}\lim_{\theta\to(\frac{3\pi}{2})^-}\frac{\cos\theta}{1+\sin\theta}$$

$$= -\frac{1}{3}\lim_{\theta\to(\frac{3\pi}{2})^-}\frac{-\sin\theta}{\cos\theta}$$

$$= \infty$$



By symmetry,

$$\lim_{\theta\to(\frac{3\pi}{2})^-}\frac{dy}{dx} = -\infty$$

Thus, there is a ~~st~~vertical ~~tagent~~ tangent line at the pole.