MAHLER DISCRETE RESIDUES AND CREATIVE TELESCOPING

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Abstract. We construct Mahler discrete residues for rational functions and show that they comprise a complete obstruction to the Mahler summability problem of deciding whether a given rational function $f(x)$ is of the form $g(x^p) - g(x)$ for some rational function $g(x)$ and an integer $p > 1$. This extends to the Mahler case the analogous notions, properties, and applications of discrete residues (in the shift case) and q-discrete residues (in the q-difference case) developed by Chen and Singer. Along the way we define several additional notions that promise to be useful for addressing related questions involving Mahler difference fields of rational functions, including in particular telescoping problems and problems in the (differential) Galois theory of Mahler difference equations.

1. Introduction

Continuous residues are fundamental and crucial tools in complex analysis, and have extensive and compelling applications in combinatorics [FS09]. In the last decade, a theory of discrete and q-discrete residues was proposed in [CS12] for the study of telescoping problems for bivariate rational functions, and subsequently found applications in the computation of differential Galois groups of second-order linear difference [Arr17] and q-difference equations [AZ22]. More recently, the authors of [Car21, CD75] developed a theory of residues for skew rational functions, which has important applications in duals of linearized Reed-Solomon codes [CD75]. We propose here a theory of Mahler discrete residues aimed at bringing to the Mahler case the successes of these earlier notions of residues.

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{K}(x)$ be the field of rational functions in an indeterminate x over $\mathbb{K}$. Fix an integer $p \geq 2$. We study *Mahler summability problem for rational functions*: given $f(x) \in \mathbb{K}(x)$, decide effectively whether $f(x) = g(x^p) - g(x)$ for some $g(x) \in \mathbb{K}(x)$; if so, we say $f(x)$ is *Mahler summable*.

The motivation to study Mahler equations comes from several directions: they have applications to automata theory (automatic sequences), transcendence, and number theory, to name a few. We refer to [CDDM18] for more details, and for a different approach to the Mahler summability problem: the authors of [CDDM18] develop efficient algorithms to find, in particular, all the rational solutions to a linear Mahler equation. Thus [CDDM18] decides *efficiently* whether any *given* $f(x) \in \mathbb{K}(x)$ is Mahler summable.

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Our goal here is different: we wish to construct a complete obstruction to Mahler summability. Let us elaborate. The map $\Delta : g(x) \mapsto g(x^p) - g(x)$ is $\mathbb{K}$-linear, so its image is a $\mathbb{K}$-linear subspace, and therefore the kernel of some $\mathbb{K}$-linear map — but what is it? We shall construct explicitly such a $\mathbb{K}$-linear map on $\mathbb{K}(x)$, whose kernel is precisely the image of Δ (see Section 3.6). In essence, this amounts to solving the Mahler summability problem for all $f(x) \in \mathbb{K}(x)$ at once. We accomplish this by introducing the notion of *Mahler discrete residues* for rational functions, and prove:

Main Theorem. *$f(x) \in \mathbb{K}(x)$ is Mahler summable if and only if every Mahler discrete residue of f is zero.*

The discrete and q-discrete residues developed in [CS12] comprise complete obstructions to the summability problem of deciding whether $f(x) = g(x+1) - g(x)$ for some $g(x) \in \mathbb{K}(x)$ and the q-summability problem of deciding whether $f(x) = g(qx) - g(x)$ for some $g(x) \in \mathbb{K}(x)$ and $q \in \mathbb{K}$ neither zero nor a root of unity, respectively. It is precisely this theoretical property of (q-)discrete residues what enables their applications to the telescoping problems considered in [CS12] and their indispensable role in the development of the algorithms in [Arr17, AZ22]. We envision analogous applications of Mahler discrete residues to telescoping problems and in the development of algorithms to compute differential Galois groups for Mahler difference equations.

Our strategy is inspired by that of [CS12]: we utilize the coefficients occurring in the partial fraction decomposition of $f(x)$ to construct an aspiring certificate $g(x) \in \mathbb{K}(x)$ such that

$$\bar{f}(x) := f(x) + \big(g(x^p) - g(x)\big) \tag{1.1}$$

is Mahler summable if and only if $\bar{f}(x) = 0$. The Mahler discrete residues of $f(x)$ are (vectors whose components are) the coefficients occurring in the partial fraction decomposition of $\bar{f}(x)$. This $\bar{f}(x)$ plays the role of a "Mahler remainder" of $f(x)$, analogous to the remainder of Hermite reduction in the context of integration.

2. Preliminaries

In this section we define the notation and conventions that we shall use throughout this work, and prove some ancillary results.

We fix once and for all an algebraically closed field $\mathbb{K}$ of characteristic zero and an integer $p \geq 2$ (not necessarily prime). We denote by $\mathbb{K}(x)$ the field of rational functions in the indeterminate x with coefficients in $\mathbb{K}$. We denote by $\sigma : \mathbb{K}(x) \to \mathbb{K}(x)$ the $\mathbb{K}$-linear endomorphism defined by $\sigma(x) = x^p$, called the *Mahler operator*, so that $\sigma(f(x)) = f(x^p)$ for $f(x) \in \mathbb{K}(x)$. We write $\Delta := \sigma - \mathrm{id}$, so that $\Delta(f(x)) = f(x^p) - f(x)$ for $f(x) \in \mathbb{K}(x)$. We often suppress the functional notation and write simply $f \in \mathbb{K}(x)$ instead of $f(x)$ whenever no confusion is likely to arise. We say that $f \in \mathbb{K}(x)$ is *Mahler summable* if there exists $g \in \mathbb{K}(x)$ such that $f = \Delta(g)$.

Let $\mathbb{K}^\times = \mathbb{K}\backslash\{0\}$ denote the multiplicative group of $\mathbb{K}$. Let $\mathbb{K}_t^\times$ denote the torsion subgroup of $\mathbb{K}^\times$, i.e., the group of roots of unity in $\mathbb{K}^\times$. For $\zeta \in \mathbb{K}_t^\times$, the *order* of ζ is the smallest $r \in \mathbb{Z}_{\geq 0}$ such that $\zeta^r = 1$. We fix once and for all a compatible system of p-power roots of unity $(\zeta_{p^n})_{n\geq 0} \subset \mathbb{K}_t^\times$, that is, each ζ_{p^n} has order p^n and $\zeta_{p^n}^{p^\ell} = \zeta_{p^{n-\ell}}$ for $0 \leq \ell \leq n$, and we denote by $\pi_\ell^n : \mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^\ell\mathbb{Z}$ and by $\pi_n : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$ the canonical projections.

Each $f \in \mathbb{K}(x)$ decomposes uniquely as

$$f = f_L + f_T, \tag{2.1}$$

where $f_L \in \mathbb{K}[x, x^{-1}]$ is a Laurent polynomial and $f_T = \frac{a}{b}$ for polynomials $a, b \in \mathbb{K}[x]$ such that either $a = 0$ or else $\deg(a) < \deg(b)$ and $\gcd(a, b) = 1 = \gcd(x, b)$. We obtain a $\mathbb{K}$-linear decomposition

$$\mathbb{K}(x) \simeq \mathbb{K}[x, x^{-1}] \oplus \mathbb{K}(x)_T. \tag{2.2}$$

We see that $\sigma(f_L) \in \mathbb{K}[x, x^{-1}]$ for any $f_L \in \mathbb{K}[x, x^{-1}]$. Moreover, for any $a, b \in \mathbb{K}[x]$ we have that $\gcd(\sigma(a), \sigma(b)) = \sigma(\gcd(a, b))$. It follows that the $\mathbb{K}$-subspace $\mathbb{K}(x)_T$ in (2.2) is also stabilized by σ, and therefore for any $f, g \in \mathbb{K}(x)$ we have that $f = \Delta(g)$ if and only if $f_L = \Delta(g_L)$ and $f_T = \Delta(g_T)$.

2.1. **Mahler trajectories, Mahler trees, Mahler cycles, and Mahler forests.**

Definition 2.1. We let $\mathcal{P} = \{p^n \mid n \in \mathbb{Z}_{\geq 0}\}$ denote the multiplicative monoid of non-negative powers of p. Then $\mathcal{P}$ acts on $\mathbb{Z}$ by multiplication, and the set of *maximal trajectories* for this action is

$$\mathbb{Z}/\mathcal{P} := \big\{\{0\}\big\} \cup \big\{\{ip^n \mid n \in \mathbb{Z}_{\geq 0}\} \bigm| i \in \mathbb{Z} \text{ such that } p \nmid i\big\}.$$

The elements of $\mathbb{Z}/\mathcal{P}$ are pairwise disjoint sets whose union is all of $\mathbb{Z}$. For any $j \in \mathbb{Z}$ (regardless of whether it is divisible by p), we denote by $[j]_\mathcal{P}$ the unique maximal trajectory in $\mathbb{Z}/\mathcal{P}$ containing j.

For $[i]_\mathcal{P} \in \mathbb{Z}/\mathcal{P}$, let us define the σ-stable subspace of $\mathbb{K}[x, x^{-1}]$:

$$\mathbb{K}[x, x^{-1}]_{[i]_\mathcal{P}} = \left\{ \sum_j c_j x^j \in \mathbb{K}[x, x^{-1}] \;\middle|\; c_j = 0 \ \forall\ j \notin [i]_\mathcal{P} \right\}. \tag{2.3}$$

We obtain a decomposition of $\mathbb{K}[x, x^{-1}]$ into σ-stable subspaces:

$$\mathbb{K}[x, x^{-1}] \simeq \bigoplus_{[i]_\mathcal{P} \in \mathbb{Z}/\mathcal{P}} \mathbb{K}[x, x^{-1}]_{[i]_\mathcal{P}}. \tag{2.4}$$

Definition 2.2. For $f \in \mathbb{K}(x)$ and $[i]_\mathcal{P} \in \mathbb{Z}/\mathcal{P}$, the $[i]_\mathcal{P}$*-component* of f, denoted by $f_{[i]_\mathcal{P}}$, is the projection of the Laurent polynomial component f_L of f in (2.1) to the subspace $\mathbb{K}[x, x^{-1}]_{[i]_\mathcal{P}}$ in (2.3).

Remark 2.3. Since the decomposition (2.4) is σ-stable, it follows that for any $f_L, g_L \in \mathbb{K}[x, x^{-1}]$ we have $f_L = \Delta(g_L)$ if and only if $f_{[i]_\mathcal{P}} = \Delta(g_{[i]_\mathcal{P}})$ for every $[i]_\mathcal{P} \in \mathbb{Z}/\mathcal{P}$.

Definition 2.4. For $\alpha \in \mathbb{K}^\times$, the *Mahler tree* containing α is

$$[\alpha]_M := \left\{ \beta \in \mathbb{K}^\times \;\middle|\; \exists\ s, t \in \mathbb{Z}_{\geq 0} \text{ such that } \beta^{p^s} = \alpha^{p^t} \right\}.$$

The set of Mahler trees consists of pairwise disjoint subsets of $\mathbb{K}^\times$ whose union is all of $\mathbb{K}^\times$. We denote by $\mathcal{T}_M$ the set of Mahler trees.

Remark 2.5. The usage of *trajectory* in Definition 2.1 is perhaps unfamiliar to some readers: it is standard in the context of monoid (and more generally semigroup) actions, and replaces the more familiar notion of *orbit* for group actions.

The usage of *tree* in Definition 2.4 is motivated by the fact that one can define a digraph structure $D([\alpha]_M)$ on the vertex set $[\alpha]_M$ with an edge from β to γ whenever $\beta^p = \gamma$, whose underlying (undirected) graph is connected and acyclic provided that $\alpha \notin \mathbb{K}_t^\times$. We find the terminology useful and suggestive even when $\alpha \in \mathbb{K}_t^\times$, because even in this exceptional case we do obtain a tree after collapsing the unique cycle in $D([\alpha]_M)$ defined below.

Definition 2.6. For $\alpha \in \mathbb{K}^\times$, the *Mahler cycle* of $[\alpha]_M$ is

$$\mathcal{C}([\alpha]_M) := \{\gamma \in [\alpha]_M \mid \gamma \text{ is a root of unity of order coprime to } p\}.$$

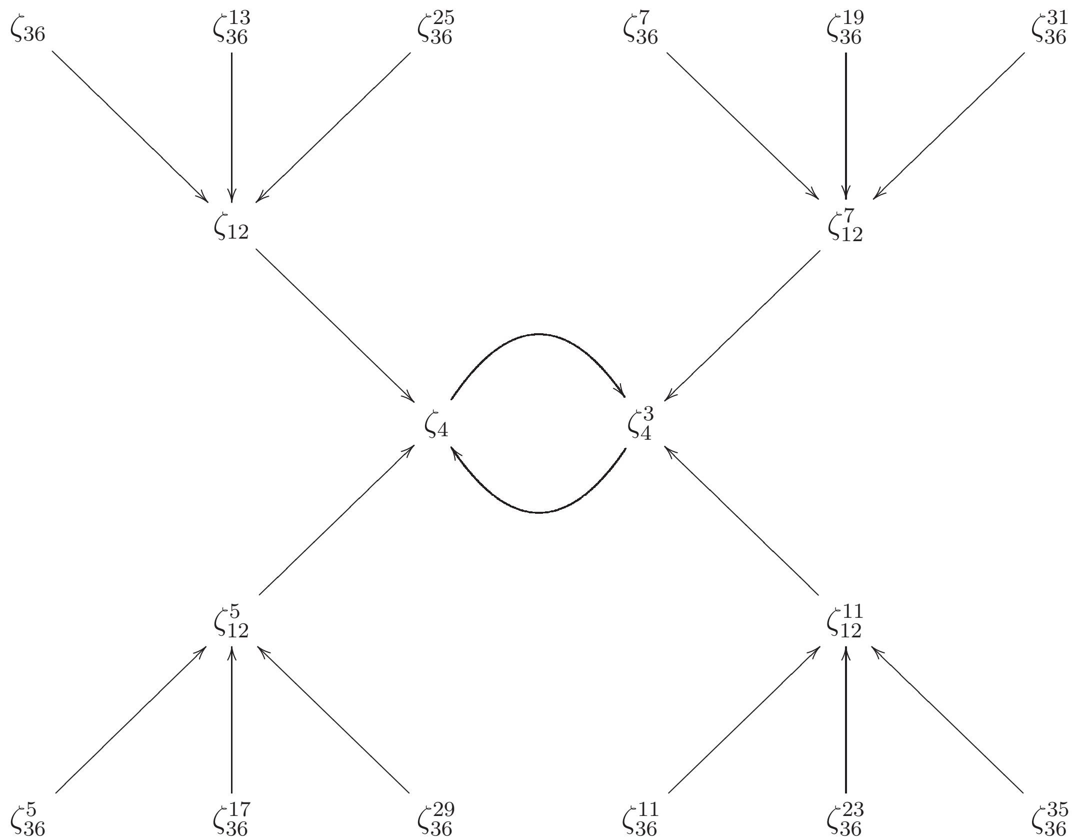
The *Mahler forest* of $[\alpha]_M$ is $\mathcal{F}([\alpha]_M) := \{\beta \in [\alpha]_M \mid \beta \notin \mathcal{C}([\alpha]_M)\}$. The *cycle length* of $[\alpha]_M$ is defined to be $e([\alpha]_M) := |\mathcal{C}([\alpha]_M)|$.

Example 2.7. Let us illustrate the definitions of Mahler trees and Mahler cycles with $\mathbb{K} = \mathbb{C}$ and $p = 3$. In this example we write $\zeta_n := e^{\frac{2\pi\sqrt{-1}}{n}} \in \mathbb{C}^\times$, for concreteness.

The vertices in the digraph $D([2]_M)$ near $\alpha = 2$ are:

$$\begin{tikzcd}
 & \zeta_9\sqrt[9]{2} \arrow[dr] & \zeta_9^4\sqrt[9]{2} \arrow[d] & \zeta_9^7\sqrt[9]{2} \arrow[dl] & \\
\sqrt[9]{2} \arrow[dr] & & \sqrt[3]{2} \arrow[d] & & \zeta_9^2\sqrt[9]{2} \arrow[dl] \\
\zeta_3\sqrt[9]{2} \arrow[r] & \sqrt[3]{2} \arrow[r] & 2 \arrow[d] & \zeta_3^2\sqrt[3]{2} \arrow[l] & \zeta_9^5\sqrt[9]{2} \arrow[l] \\
\zeta_3^2\sqrt[9]{2} \arrow[ur] & & 8 & & \zeta_9^8\sqrt[9]{2} \arrow[ul]
\end{tikzcd}$$

For $\alpha = \zeta_4 = \sqrt{-1}$, we have that $\mathcal{C}([\zeta_4]_M) = \{\zeta_4, \zeta_4^3\}$, and the vertices in the digraph $D([\zeta_4]_M)$ near these two vertices are:



Remark 2.8. Let us collect some observations about Mahler cycles that we shall use repeatedly. For $\alpha \in \mathbb{K}^\times$ it follows from the Definition 2.4 that either $[\alpha]_M \subset \mathbb{K}_t^\times$ or else $[\alpha]_M \cap \mathbb{K}_t^\times = \emptyset$. In particular, $\mathcal{C}([\alpha]_M) = \emptyset \Leftrightarrow e([\alpha]_M) = 0$ whenever $\alpha \notin \mathbb{K}_t^\times$. On the other hand, $\mathbb{K}_t^\times$ consists precisely of the pre-periodic points for the action $\alpha \mapsto \alpha^{p^n}$ for $n \in \mathbb{Z}_{\geq 0}$ of the monoid $\mathcal{P}$ on $\mathbb{K}^\times$. In general, for $\alpha \in \mathbb{K}_t^\times$ the Mahler cycle $\mathcal{C}([\alpha]_M)$ is a non-empty set (not a group!) endowed with a simply transitive action of the quotient monoid $\mathcal{P}/\mathcal{P}^e \simeq \mathbb{Z}/e\mathbb{Z}$, where $\mathcal{P}^e := \{p^{ne} \mid n \in \mathbb{Z}\}$, and $e := e([\alpha]_M)$. The Mahler forest $\mathcal{F}([\alpha]_M)$ coincides with the Mahler tree $[\alpha]_M$ when $\alpha \notin \mathbb{K}_t^\times$, and when $\alpha \in \mathbb{K}_t^\times$ its induced digraph structure (cf. Remark 2.5) makes $\mathcal{F}([\alpha]_M)$ into a finite union of rooted trees.

The Mahler tree $[1]_M$ consists precisely of the roots of unity $\zeta \in \mathbb{K}_t^\times$ whose order r is such that $\gcd(r, p^n) = r$ for some $p^n \in \mathcal{P}$, or equivalently such that every prime factor of r divides p. When $[\alpha]_M \neq [1]_M$, the cycle length $e([\alpha]_M)$ coincides with the order of p in the group of units $(\mathbb{Z}/r\mathbb{Z})^\times$, where $r > 1$ is the common order of the roots of unity $\gamma \in \mathcal{C}([\alpha]_M)$. For any $\alpha \in \mathbb{K}_t^\times$ and for any given $\gamma \in \mathcal{C}([\alpha]_M)$ we have that $\mathcal{C}([\alpha]_M) = \{\gamma^{p^\ell} \mid 0 \leq \ell \leq e-1\}$, and every other element of $[\alpha]_M$ admits a unique representation in terms of γ and the ζ_{p^n} for $n \geq 1$ as in Lemma 3.11.

2.2. **Mahler supports and singular supports in Mahler trees.** Mahler trees allow us to define the following bespoke variants of the singular support $\mathrm{sing}(f)$ of a rational function f (i.e., its set of poles), which are particularly well-suited to the Mahler context.

Definition 2.9. For $f \in \mathbb{K}(x)$, we define $\mathrm{supp}(f) \subset \mathcal{T}_M \cup \{\infty\}$, called the *Mahler support* of f, as follows:

- $\infty \in \mathrm{supp}(f)$ if and only if $f_L \neq 0$; and
- for $\alpha \in \mathbb{K}^\times$, $[\alpha]_M \in \mathrm{supp}(f)$ if and only if $[\alpha]_M$ contains a pole of f.

For $[\alpha]_M \in \mathcal{T}_M$, the *singular support* of f in $[\alpha]_M$, denoted by $\mathrm{sing}(f, [\alpha]_M)$, is the (possibly empty) set of poles of f in $[\alpha]_M$.

We omit the straightforward proof of the following lemma.

Lemma 2.10. *For $f, g \in \mathbb{K}(x)$ and $0 \neq c \in \mathbb{K}$ we have the following:*
(1) $\mathrm{supp}(f) = \emptyset \Leftrightarrow f = 0$*;*
(2) $\mathrm{supp}(\sigma(f)) = \mathrm{supp}(f) = \mathrm{supp}(c \cdot f)$*; and*
(3) $\mathrm{supp}(f + g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$*.*

For $\alpha \in \mathbb{K}^\times$, let us define

$$\mathbb{K}(x)_{[\alpha]_M} := \left\{ f_T \in \mathbb{K}(x)_T \;\middle|\; \mathrm{supp}(f_T) \subseteq \{[\alpha]_M\} \right\}. \tag{2.5}$$

We obtain the following further decomposition of the $\mathbb{K}$-subspace $\mathbb{K}(x)_T$ in (2.2) into the σ-stable $\mathbb{K}$-subspaces $\mathbb{K}(x)_{[\alpha]_M}$ in (2.5) as a direct consequence of Lemma 2.10.

Corollary 2.11. *The following $\mathbb{K}$-linear decomposition is σ-stable:*

$$\mathbb{K}(x)_T \simeq \bigoplus_{[\alpha]_M \in \mathcal{T}_M} \mathbb{K}(x)_{[\alpha]_M}.$$

Definition 2.12. For $f \in \mathbb{K}(x)$ and $[\alpha]_M$, the $[\alpha]_M$*-component* of f, denoted by $f_{[\alpha]_M}$, is the projection of the component $f_T \in \mathbb{K}(x)_T$ of f in (2.1) to the subspace $\mathbb{K}(x)_{[\alpha]_M}$ in (2.5).

Remark 2.13. By Corollary 2.11, for any $f_T, g_T \in \mathbb{K}(x)_T$ we have $f_T = \Delta(g_T)$ if and only if $f_{[\alpha]_M} = \Delta(g_{[\alpha]_M})$ for every $[\alpha]_M \in \mathcal{T}_M$.

2.3. Mahler dispersion.

Definition 2.14. For $f \in \mathbb{K}(x)$ and $[\alpha]_M \in \mathcal{T}_M$, we define the *Mahler dispersion* of f at $[\alpha]_M$, denoted by $\mathrm{disp}(f, [\alpha]_M)$, to be the largest $d \in \mathbb{Z}_{\geq 0}$ (if it exists) for which there exists $\beta \in \mathrm{sing}(f, [\alpha]_M)$ such that $\beta^{p^d} \in \mathrm{sing}(f, [\alpha]_M)$. If there is no such $d \in \mathbb{Z}_{\geq 0}$, then we set $\mathrm{disp}(f, [\alpha]_M) = \infty$.

If $\infty \in \mathrm{supp}(f)$, we may write $f_L = \sum_{i=n}^{N} c_i x^i \in \mathbb{K}[x, x^{-1}]$ with $c_n c_N \neq 0$, and we define the *Mahler dispersion* of f at ∞, denoted by $\mathrm{disp}(f, \infty)$, as follows:

- if $f_L = c_0 \neq 0$ then we set $\mathrm{disp}(f, \infty) = 0$; otherwise
- $\mathrm{disp}(f, \infty)$ is the largest $d \in \mathbb{Z}_{\geq 0}$ for which there exists an index $i \neq 0$ such that $c_i \neq 0$ and $c_{ip^d} \neq 0$.

If $\infty \notin \mathrm{supp}(f)$, then we set $\mathrm{disp}(f, \infty) = \infty$.

Similarly as in the shift and q-difference cases [HS08, Lemma 6.3], Mahler dispersions will play a crucial role in what follows. As we prove in Corollary 3.6, they already provide a partial obstruction to summability: if $f \in \mathbb{K}(x)$ is Mahler summable then every Mahler dispersion of f is non-zero. Moreover, Mahler dispersions also detect whether f has any "bad" poles (i.e., at roots of unity of order coprime to p) according to the following result.

Lemma 2.15. *Let* $f(x) \in \mathbb{K}(x)$ *and* $[\alpha]_M \in \mathrm{supp}(f)$*. We have that* $\mathrm{disp}(f, [\alpha]_M) = \infty$ *if and only if* $\mathrm{sing}(f, [\alpha]_M) \cap \mathcal{C}([\alpha]_M) \neq \emptyset$*.*

Proof. ($\Rightarrow$). Assuming that $\mathrm{disp}(f, [\alpha]_M) = \infty$, let us write $\mathrm{sing}(f, [\alpha]_M) = \{\beta_1, \ldots, \beta_n\}$. Since for arbitrarily large d there exist $1 \leq i, j \leq n$ such that $\beta_i^{p^d} = \beta_j$, it follows that at least one β_i is a root of unity, and therefore the whole Mahler tree $[\alpha]_M \subset \mathbb{K}_t^\times$ (cf. Remark 2.8). There exists a root of unity $\zeta \in \mathbb{K}_t^\times$, say of order r (which we can take to be the least common multiple of the orders of the roots of unity β_i), such that each $\beta_i = \zeta^{e_i}$ for some $e_1, \ldots, e_n \in \mathbb{Z}_{\geq 0}$. For any $\ell \in \mathbb{Z}_{\geq 0}$, the order of ζ^{p^ℓ} is $r_\ell := r/\gcd(p^\ell, r)$, and we see that for every sufficiently large d we have that r_d is coprime to p, and therefore at least one β_i is a root of unity of order coprime to p.

($\Leftarrow$). For $\gamma \in \mathrm{sing}(f, [\alpha]_M) \cap \mathcal{C}([\alpha]_M)$ we have $\gamma^{p^{e([\alpha]_M)\cdot n}} = \gamma$ for every $n \in \mathbb{Z}_{\geq 0}$ (cf. Remark 2.8), whence $\mathrm{disp}(f, [\alpha]_M) = \infty$. □

3. Mahler discrete residues

In this section we define the Mahler discrete residues of $f(x) \in \mathbb{K}(x)$ and prove the Corollary 3.6: if f is Mahler summable then it has non-zero dispersion everywhere. This Corollary 3.6 is an essential ingredient in our proofs of our Main Theorem in Section 3.5, which states that Mahler discrete residues comprise a complete obstruction to Mahler summability. We begin by studying the effect of the Mahler operator σ on partial fraction decompositions, which we express explicitly in terms of *Mahler coefficients*. We then construct Mahler residues at infinity and at Mahler trees $[\alpha]$, first for $\alpha \notin \mathbb{K}_t^\times$, and then finally for $\alpha \in \mathbb{K}_t^\times$, in increasing order of complexity.

3.1. Mahler coefficients for partial fractions.

For $\alpha \in \mathbb{K}^\times$, $m \in \mathbb{N}$, and $1 \leq k \leq m$, we define the *Mahler coefficients* $V_k^m(\alpha) \in \mathbb{K}$ implicitly by

$$\sigma\left(\frac{1}{(x - \alpha^p)^m}\right) = \frac{1}{(x^p - \alpha^p)^m} = \sum_{k=1}^{m} \sum_{i=0}^{p-1} \frac{V_k^m(\zeta_p^i \alpha)}{(x - \zeta_p^i \alpha)^k}. \tag{3.1}$$

These coefficients are computed explicitly with the following result.

Lemma 3.1. *For every* $\alpha \in \mathbb{K}^\times$*, the Mahler coefficients*

$$V_k^m(\alpha) = \mathbb{V}_k^m \cdot \alpha^{k-pm},$$

where the universal coefficients $\mathbb{V}_k^m \in \mathbb{Q}$ *are independent of* α*. Moreover, the* $\mathbb{V}_k^m$ *are the first* m *Taylor coefficients at* $x = 1$ *of*

$$(x^{p-1} + \cdots + x + 1)^{-m} = \sum_{k=1}^{m} \mathbb{V}_k^m \cdot (x-1)^{m-k} + O((x-1)^m). \tag{3.2}$$

Proof. We claim that $V_k^m(\alpha) = V_k^m(1) \cdot \alpha^{k-pm}$ for every $\alpha \in \mathbb{K}^\times$. To see this, set $x = \alpha y$ for a new indeterminate y, and note that

$$\sum_{k=1}^{m} \sum_{i=0}^{p-1} \frac{V_k^m(\zeta_p^i \alpha)}{(x - \zeta_p^i \alpha)^k} = \frac{1}{(x^p - \alpha^p)^m} = \alpha^{-pm} \cdot \frac{1}{(y^p - 1)^m} =$$
$$= \alpha^{-pm} \sum_{k=1}^{m} \sum_{i=0}^{p-1} \frac{V_k^m(\zeta_p^i)}{(y - \zeta_p^i)^k} = \alpha^{-pm} \sum_{k=1}^{m} \sum_{i=0}^{p-1} \frac{V_k^m(\zeta_p^i)\alpha^k}{(x - \zeta_p^i \alpha)^k}.$$

It follows that $V_k^m(\zeta_p^i \alpha) = V_k^m(\zeta_p^i)\alpha^{k-pm}$ for $i = 0, \ldots, p-1$. In particular for $i = 0$ we obtain $V_k^m(\alpha) = V_k^m(1)\alpha^{k-pm}$, as claimed. Let us set $\mathbb{V}_k^m := V_k^m(1)$, and note that $V_k^m(\zeta_p^i) = \mathbb{V}_k^m \cdot \zeta_p^{ik}$. Then

$$\frac{1}{(x^p - 1)^m} = \sum_{k=1}^{m} \sum_{i=0}^{p-1} \frac{V_k^m(\zeta_p^i)}{(x - \zeta_p^i)^k} = \sum_{k=1}^{m} \frac{\mathbb{V}_k^m}{(x-1)^k} + \sum_{k=1}^{m} \sum_{i=1}^{p-1} \frac{\mathbb{V}_k^m \cdot \zeta_p^{ik}}{(x - \zeta_p^i)^k},$$

and after multiplying through by $(x-1)^m$ we obtain

$$\left(\sum_{i=0}^{p-1} x^i \right)^{-m} = \sum_{k=1}^{m} \mathbb{V}_k^m \cdot (x-1)^{m-k} + (x-1)^m \cdot \sum_{k=1}^{m} \sum_{i=1}^{p-1} \frac{\mathbb{V}_k^m \cdot \zeta_p^{ik}}{(x - \zeta_p^i)^k},$$

where the double sum on the right-hand side has no poles at $x = 1$. Finally, since $(x^{p-1} + \cdots + x + 1)^{-m} \in \mathbb{Q}[[x-1]]$, the $\mathbb{V}_k^m \in \mathbb{Q}$. $\square$

The following immediate consequence of Lemma 3.1 is obtained by evaluating (3.2) at $x = 1$.

Corollary 3.2. *For $\alpha \in \mathbb{K}^\times$, $V_m^m(\alpha) = p^{-m}\alpha^{m-pm}$.*

Remark 3.3. We see in (3.1) two phenomena that arise for the first time in the Mahler context and which do not occur in the shift nor in the q-dilation settings considered in [CS12], which is the main inspiration for the present work. For $0 \neq f_T \in \mathbb{K}(x)_T$, we have that: (1) the number of poles of $\sigma(f_T)$ (counted without multiplicity!) is strictly larger than that of f_T; and (2) the (classical/continuous) higher-order residues of f_T "leak" into the lower-order residues of $\sigma(f_T)$. These two phenomena are mainly responsible for our need to develop new and somewhat intricate bookkeeping devices in the Mahler setting that were not necessary in the shift and q-dilation settings of [CS12] to develop a working theory of discrete residues.

Example 3.4. Let us illustrate the definition of Mahler coefficients with $p = 3$, $m = 2$, and $\alpha^3 = 1$. Then (3.1) becomes

$$\sigma\left(\frac{1}{(x-1)^2}\right) = \frac{1}{(x^3-1)^2} = \sum_{k=1}^{2} \sum_{i=0}^{2} \frac{\mathbb{V}_k^2 \cdot \zeta_3^{ki}}{(x - \zeta_3^i)^k},$$

since, according to Lemma 3.1, $V_k^2(\zeta_3^i) = \mathbb{V}_k^2 \cdot (\zeta_3^i)^{k-6} = \mathbb{V}_k^2 \cdot \zeta_3^{ki}$ for $k = 1, 2$. We find in this case, using (3.2) in Lemma 3.1, that

$$\mathbb{V}_2^2 = \left(x^2 + x + 1\right)^{-2}\Big|_{x=1} = \frac{1}{9}; \quad \text{and} \quad \mathbb{V}_1^2 = \left((x^2 + x + 1)^{-2}\right)'\Big|_{x=1} = -\frac{2}{9}.$$

One can verify using a computer algebra system (or by hand!) that the partial fraction decomposition of $9\cdot(x^3-1)^{-2}$ is indeed

$$\frac{1}{(x-1)^2}+\frac{\zeta_3^2}{(x-\zeta_3)^2}+\frac{\zeta_3}{(x-\zeta_3^2)^2}+\frac{-2}{x-1}+\frac{-2\zeta_3}{x-\zeta_3}+\frac{-2\zeta_3^2}{x-\zeta_3^2}.$$

3.2. **Mahler dispersion and Mahler summability.**

Proposition 3.5. *For $f,g\in\mathbb{K}(x)$ such that $f=\Delta(g)$ we have that* $\mathrm{disp}(f[\alpha]_M)=\mathrm{disp}(g,[\alpha]_M)+1$ *for every Mahler tree $[\alpha]_M\in\mathcal{T}_M$, and if $\infty\in\mathrm{supp}(f)$ then* $\mathrm{disp}(f,\infty)=\mathrm{disp}(g,\infty)+1$ *also, with the convention that $\infty+1=\infty$.*

Proof. By Lemma 2.10, if $\infty\in\mathrm{supp}(f)$ then $\infty\in\mathrm{supp}(g)$. For $f_L,g_L\in\mathbb{K}[x,x^{-1}]$ as in (2.1), we have $0\neq f_L=\Delta(g_L)$, since $\infty\in\mathrm{supp}(f)$, and in particular g_L is not constant. By Remark 2.2, $f_{[i]_\mathcal{P}}=\Delta(g_{[i]_\mathcal{P}})$ for each $[i]_\mathcal{P}\in\mathbb{Z}/\mathcal{P}$. Since $f_{[0]_\mathcal{P}}=\Delta(g_{[0]_\mathcal{P}})=0$, it follows from Definition 2.14 that

$$\mathrm{disp}(f,\infty)=\max\left\{\mathrm{disp}\left(f_{[i]_\mathcal{P}},\infty\right)\;\middle|\;[i]_\mathcal{P}\in\mathbb{Z}/\mathcal{P},\;f_{[i]_\mathcal{P}}\neq 0\neq i\right\}.$$

We claim that $\mathrm{disp}\left(\Delta\left(g_{[i]_\mathcal{P}}\right),\infty\right)=\mathrm{disp}\left(g_{[i]_\mathcal{P}},\infty\right)+1$ for every $0\neq g_{[i]_\mathcal{P}}\in\mathbb{K}[x,x^{-1}]_{[i]_\mathcal{P}}$ with $i\neq 0$, which will conclude the proof. Let us write $g_{[i]_\mathcal{P}}=\sum_{j=0}^d c_{ip^j}x^{ip^j}$, where we assume $c_i\neq 0$ and $c_{ip^d}\neq 0$, i.e., $\mathrm{disp}(g_{[i]_\mathcal{P}},\infty)=d$. Then

$$\Delta(g_{[i]_\mathcal{P}})=c_{ip^d}x^{ip^{d+1}}-c_ix^i+\sum_{j=1}^d(c_{ip^{j-1}}-c_{ip^j})x^{ip^j},$$

from which it follows that $\mathrm{disp}(\Delta(g_{[i]_\mathcal{P}}),\infty)=d+1$, as desired.

For $[\alpha]_M\in\mathcal{T}_M$ we have that $f_{[\alpha]_M}=\Delta(g_{[\alpha]_M})$, by Remark 2.13. For $[\alpha]_M\notin\mathrm{supp}(f)$ we have that $0=f_{[\alpha]_M}=\Delta(g_{[\alpha]_M})$, and therefore $\mathrm{disp}(f,[\alpha]_M)=\infty$ and $\sigma(g_{[\alpha]_M})=g_{[\alpha]_M}$, which implies that $g_{[\alpha]_M}\in\mathbb{K}\cap\mathbb{K}(x)_{[\alpha]_M}=\{0\}$, and therefore $\mathrm{disp}(g,[\alpha]_M)=\infty$.

We assume from now on that $[\alpha]_M\in\mathrm{supp}(f,[\alpha]_M)$, and therefore $[\alpha]_M\in\mathrm{supp}(g,[\alpha]_M)$ by Lemma 2.10. We consider two cases, depending on whether $\mathrm{disp}(g,[\alpha]_M)$ is finite or not.

If $\mathrm{disp}(g,[\alpha]_M)=:d<\infty$, let $\beta\in[\alpha]_M$ be such that β and β^{p^d} are poles of g. Choose $\gamma\in[\alpha]_M$ such that $\gamma^p=\beta$. Then γ is a pole of $\sigma(g)$ but not of g (by the maximality of d), and therefore γ is a pole of f. On the other hand, $\gamma^{p^{d+1}}=\beta^{p^d}$ is a pole of g but not of $\sigma(g)$, for if β^{p^d} were a pole of $\sigma(g)$ then $\beta^{p^{d+1}}$ would be a pole of g, contradicting the maximality of d. Therefore $\gamma^{p^{d+1}}$ is a pole of f. It follows that $\mathrm{disp}(f,[\alpha]_M)\geq d+1$. One can show equality by contradiction: if $\beta\in[\alpha]_M$ is a pole of f such that β^{p^s} is also a pole of f for some $s>d+1$, then each of β and β^{p^s} is either a pole of g or a pole of $\sigma(g)$. This implies (after some tedious but straightforward casework) that there exist $0\leq i,j\leq 1$ such that $\beta^{p^{s+i}}$ and β^{p^j} are both poles of g, which contradicts the maximality of d since in any case $s+i-j\geq s-1>d$.

If $\mathrm{disp}(g,[\alpha]_M)=\infty$, there exists $\gamma\in\mathrm{sing}(g,[\alpha]_M)\cap\mathcal{C}([\alpha]_M)$ by Lemma 2.15. We claim $\gamma^{p^\ell}\in\mathrm{sing}(f,[\alpha]_M)\cap\mathcal{C}([\alpha]_M)$ for some $\ell\in\mathbb{Z}/e\mathbb{Z}$, where $e:=e([\alpha]_M)\geq 1$ (cf. Remark 2.8). This will imply that $\mathrm{disp}(f,[\alpha]_M)=\infty$ by another application of Lemma 2.15.

Let us prove the claim. It is easy to see[1] that the $\mathbb{K}$-subspace S of $\mathbb{K}(x)_{[\alpha]_M}$, consisting of rational functions none of whose poles is contained in $\mathcal{C}([\alpha]_M)$, or equivalently (by Lemma 2.15), the $\mathbb{K}$-span of the elements of $\mathbb{K}(x)_{[\alpha]_M}$ having finite dispersion, is σ-stable. Although the complementary $\mathbb{K}$-subspace to S, consisting of proper rational functions whose poles are contained in $\mathcal{C}([\alpha]_M)$, is not σ-stable (see Remark 3.18), the projection of $g_{[\alpha]_M} \in \mathbb{K}(x)_{[\alpha]_M}$ onto S is irrelevant for the purpose of witnessing that f has a pole at γ^{p^ℓ} for some $\ell \in \mathbb{Z}/e\mathbb{Z}$, as we contend. So we may further assume without loss of generality that $g_{[\alpha]_M} = \sum_{k=1}^{m} \sum_{\ell \in \mathbb{Z}/e\mathbb{Z}} d(k,\ell) \cdot \left(x - \gamma^{p^\ell}\right)^{-k}$, where $d(k,\ell) \in \mathbb{K}$ such that $d(m,s) \neq 0$ for some $s \in \mathbb{Z}/e\mathbb{Z}$. Then

$$\sigma(g_{[\alpha]_M}) = \sum_{k=1}^{m} \sum_{\ell \in \mathbb{Z}/e\mathbb{Z}} \frac{d(k,\ell)}{\left(x^p - \gamma^{p^\ell}\right)^k} = \sum_{i=0}^{p-1} \left(\sum_{\ell \in \mathbb{Z}/e\mathbb{Z}} \frac{V_m^m\left(\zeta_p^i \gamma^{p^{\ell-1}}\right) \cdot d(m,\ell)}{\left(x - \zeta_p^i \gamma^{p^{\ell-1}}\right)^m} \right) + \\ + (\text{lower-order terms})$$

(see again Remark 3.18), where the V_m^m are as in (3.1), and therefore

$$f_{[\alpha]_M} = \Delta(g_{[\alpha]_M}) = \sum_{\ell \in \mathbb{Z}/e\mathbb{Z}} \frac{V_m^m\left(\gamma^{p^\ell}\right) \cdot d(m,\ell+1) - d(m,\ell)}{\left(x - \gamma^{p^\ell}\right)^m} + \\ + (\text{lower-order terms}) + (\text{elements of } S).$$

But the coefficients $V_m^m\left(\gamma^{p^\ell}\right) \cdot d(m,\ell+1) - d(m,\ell)$ cannot be zero for every $\ell \in \mathbb{Z}/e\mathbb{Z}$, for otherwise we would have that

$$d(m,\ell) = d(m,\ell) \prod_{j=0}^{e-1} V_m^m\left(\gamma^{p^j}\right) = d(m,\ell) \prod_{j=0}^{e-1} \frac{\gamma^{mp^j}}{p^m \gamma^{mp^{j+1}}} = \frac{d(m,\ell)}{p^{em}}$$

for every $\ell \in \mathbb{Z}/e\mathbb{Z}$, where the middle equality is obtained from Corollary 3.2. But this is impossible since $d(m,s) \neq 0$, concluding the proof of the claim that f has a pole at γ^{p^ℓ} for some $\ell \in \mathbb{Z}/e\mathbb{Z}$. □

Corollary 3.6. *Suppose that $f \in \mathbb{K}(x)$ is Mahler summable. Then* $\operatorname{disp}(f,\infty) > 0$ *and* $\operatorname{disp}(f,[\alpha]_M) > 0$ *for every $[\alpha]_M \in \mathcal{T}_M$.*

3.3. **Mahler discrete residues at infinity.** We now define the Mahler discrete residue of $f \in \mathbb{K}(x)$ at ∞ in terms of the component $f_L \in \mathbb{K}[x,x^{-1}]$ of f in (2.1) and show that it forms a complete obstruction to the Mahler summability of f_L.

Definition 3.7. For $f \in \mathbb{K}(x)$, let $f_L = \sum_{j \in \mathbb{Z}} c_j x^j$ with $c_j = 0$ for all but finitely many $j \in \mathbb{Z}$. The *Mahler discrete residue* of f at ∞ is

$$\operatorname{dres}(f,\infty) := \left(\sum_{j \in [i]_{\mathcal{P}}} c_j \right)_{[i]_{\mathcal{P}} \in \mathbb{Z}/\mathcal{P}} \in \bigoplus_{[i]_{\mathcal{P}} \in \mathbb{Z}/\mathcal{P}} \mathbb{K}.$$

[1] If a denominator $b \in \mathbb{K}[x]$ has no roots in $\mathcal{C}([\alpha]_M)$ then neither does $\sigma(b)$, for if $\gamma \in \mathcal{C}([\alpha]_M)$ were a root of $\sigma(b)$ then $\gamma^p \in \mathcal{C}([\alpha]_M)$ would be a root of b.

Proposition 3.8. *For $f \in \mathbb{K}(x)$, the component $f_L \in \mathbb{K}[x, x^{-1}]$ in (2.1) is Mahler summable if and only if* $\mathrm{dres}(f, \infty) = \mathbf{0}$.

Proof. By Remark 2.3, f_L is Mahler summable if and only if the $[i]_\mathcal{P}$-component $f_{[i]_\mathcal{P}}$ is Mahler summable for all $[i]_\mathcal{P} \in \mathbb{Z}/\mathcal{P}$. We shall show that $f_{[i]_\mathcal{P}}$ is Mahler summable if and only if the $[i]_\mathcal{P}$-component of $\mathrm{dres}(f, \infty)$ is 0. First note that $f_{[0]_\mathcal{P}} = \mathrm{dres}(f, \infty)_{[0]_\mathcal{P}}$, and $\mathrm{disp}(f_{[0]_\mathcal{P}}, \infty) = 0$ if and only if $f_{[0]} \neq 0$. We conclude in this case by Corollary 3.6.

Now let us write $f_{[i]_\mathcal{P}} = \sum_{j \in [i]_\mathcal{P}} c_j x^j \in \mathbb{K}[x, x^{-1}]_{[i]_\mathcal{P}}$, where $p \nmid i$, that is, where $i \neq 0$ is initial in its $\mathcal{P}$-trajectory $[i]_\mathcal{P}$, and $c_j = 0$ for all but finitely many $j \in [i]_\mathcal{P}$. The statement is trivial if $f_{[i]_\mathcal{P}} = 0$. Let $h_i \in \mathbb{Z}_{\geq 0}$ be as large as possible such that $c_{ip^{h_i}} \neq 0$.

Let us define recursively: $g^{(0)}_{[i]_\mathcal{P}} := 0$; and, if $h_i \geq 1$, then set

$$g^{(n+1)}_{[i]_\mathcal{P}} := \sum_{k=0}^{n} \left(\sum_{\ell=0}^{k} c_{ip^\ell} \right) x^{ip^k} = g^{(n)}_{[i]_\mathcal{P}} + \left(\sum_{\ell=0}^{n-1} c_{ip^\ell} \right) x^{ip^n}.$$

for $0 \leq n \leq h_i - 1$. A straightforward induction argument shows:

$$\bar{f}^{(n)}_{[i]_\mathcal{P}} := f_{[i]_\mathcal{P}} + \Delta \left(g^{(n)}_{[i]_\mathcal{P}} \right) = \sum_{k=n+1}^{h_i+1} c_{ip^k} x^{ip^k} + \left(\sum_{\ell=0}^{n} c_{ip^\ell} \right) x^{ip^n}$$

for each $0 \leq n \leq h_i$, whence $\bar{f}^{(h_i)}_{[i]_\mathcal{P}} = (\mathrm{dres}(f, \infty)_{[i]_\mathcal{P}}) x^{ip^{h_i}}$. Hence $\mathrm{disp}(\bar{f}^{(h_i)}_{[i]_\mathcal{P}}, \infty) = 0$ if and only if $\mathrm{dres}(f_{[i]_\mathcal{P}}, \infty) \neq 0$, and we conclude by Corollary 3.6. □

Remark 3.9. For $0 \neq i \in \mathbb{Z}$, the $\bar{f}^{(h_i)}_{[i]_\mathcal{P}}, g^{(h_i)}_{[i]_\mathcal{P}} \in \mathbb{K}[x, x^{-1}]$ constructed in the proof of Proposition 3.8 are the $[i]_\mathcal{P}$-components of the $\bar{f}, g \in \mathbb{K}(x)$ in (1.1). We set $\bar{f}_{[0]_\mathcal{P}} := f_{[0]_\mathcal{P}}$ and $g_{[0]_\mathcal{P}} := 0$.

3.4. **Mahler discrete residues at Mahler trees.** We now define the Mahler discrete residues of $f \in \mathbb{K}(x)$ at $[\alpha]_M$ in terms of the partial fraction decomposition of the component $f_{[\alpha]_M} \in \mathbb{K}(x)_{[\alpha]_M}$ in Definition 2.12 and show that it forms a complete obstruction to the Mahler summability of $f_{[\alpha]_M}$. We proceed separately in the cases where $\alpha \notin \mathbb{K}_t^\times$ and where $\alpha \in \mathbb{K}_t^\times$, depending on which we represent the poles of $f_{[\alpha]_M}$ in a particular manner.

Lemma 3.10. *For $f \in \mathbb{K}(x)$ and $[\alpha]_M \in \mathrm{supp}(f)$ such that $\alpha \notin \mathbb{K}_t^\times$, there exists $\beta \in \mathrm{sing}(f, [\alpha]_M)$ and $h \in \mathbb{Z}_{\geq 0}$ such that*

$$\mathrm{sing}(f, [\alpha]_M) \subseteq \tau_h(\beta) := \left\{ \zeta^i_{p^n} \beta^{p^{h-n}} \;\middle|\; 0 \leq n \leq h;\ i \in \mathbb{Z}/p^n\mathbb{Z} \right\}.$$

Moreover, the elements $\zeta^i_{p^n} \beta^{p^{h-n}} \in \tau_h(\beta)$ are uniquely determined by $0 \leq n \leq h$ and $i \in \mathbb{Z}/p^n\mathbb{Z}$, relative to the choice of $\beta \in \mathrm{sing}(f, [\alpha]_M)$.

Proof. Note that the set $\tau_h(\beta)$ is precisely the union of the sets of roots of $y^{p^n} = \beta^{p^h}$ for all $0 \leq n \leq h$. It is clear that the elements of $\tau_h(\beta)$ are uniquely determined by n and i (relative to the choice of β), because if we had $\zeta^j_{p^m} \beta^{p^{h-m}} = \zeta^i_{p^n} \beta^{p^{h-n}}$, then this would force $m = n$, for otherwise $\beta \in \mathbb{K}_t^\times$ contradicting our assumptions, and then $\zeta^j_{p^n} = \zeta^i_{p^n}$ implies that $j = i$.

Let us now show that for any finite set $S \subset [\alpha]_M$ there exist $\beta \in S$ and $h \in \mathbb{Z}_{\geq 0}$ such that $S \subseteq \tau_h(\beta)$. For $\gamma \in S$, let $h(\gamma) \in \mathbb{Z}_{\geq 0}$ be minimal such that

$$\gamma^{h(\gamma)} \in \xi^{\mathcal{P}} := \left\{ \xi^{p^t} \;\middle|\; t \in \mathbb{Z}_{\geq 0} \right\}$$

for every $\xi \in S$. Choose $\beta \in S$ such that $h(\beta) =: h$ is maximal among all elements of S. We claim that for every $\gamma \in S$ we have that $\gamma^{p^{h(\gamma)}} = \beta^{p^h}$, which will conclude the proof, since $h(\gamma) \leq h$ for every $\gamma \in S$. To prove the claim, note that in any case there exist $t, r \in \mathbb{Z}_{\geq 0}$ such that $\gamma^{p^t} = \beta^{p^h}$ and $\gamma^{p^{h(\gamma)}} = \beta^{p^r}$, and the definitions of $h(\gamma)$ and h then imply $t \geq h(\gamma)$ and $r \geq h$. But then

$$\beta^{p^h} = \gamma^{p^t} = \left(\gamma^{p^{h(\gamma)}}\right)^{p^{t-h(\gamma)}} = \left(\beta^{p^r}\right)^{p^{t-h(\gamma)}} = \beta^{p^{r+t-h(\gamma)}},$$

and since $\beta \notin \mathbb{K}_t^\times$ we obtain that $h + h(\gamma) = r + t$, from which it follows that $r = h$ and $t = h(\gamma)$, as claimed. □

Lemma 3.11. *Let $\alpha \in \mathbb{K}_t^\times$, $e := e([\alpha]_M)$, and choose $\gamma \in \mathcal{C}([\alpha]_M)$. Then for any $\beta \in [\alpha]_M$ there exist unique $n \in \mathbb{Z}_{\geq 0}$, $i \in \mathbb{Z}/p^n\mathbb{Z}$, and $\ell \in \mathbb{Z}/e\mathbb{Z}$, with either $n = i = 0$ or else $p \nmid i$, such that*

$$\beta = \zeta_{p^n}^i \gamma^{p^{\ell - \pi_e(n)}}. \tag{3.3}$$

Proof. Since $\beta \in [\alpha]_M = [\gamma]_M$ there exist integers $n, t \in \mathbb{Z}_{\geq 0}$ such that $\beta^{p^n} = \gamma^{p^t}$, and we may take this n to be as small as possible and replace t with $\ell := \pi_e(t)$. The p^n distinct solutions to $y^{p^n} = \gamma^{p^\ell}$, one of which is $y = \beta$, are all of the form (3.3) for $i \in \mathbb{Z}/p^n\mathbb{Z}$. The minimality of n implies that either $n = i = 0$ or else $1 \leq i \leq p^n - 1$ is such that $p \nmid i$. □

Definition 3.12. For $f \in \mathbb{K}(x)$ and $[\alpha]_M \in \mathrm{supp}(f)$, the *height* of f at $[\alpha]_M$, denoted by $h(f, [\alpha]_M)$, is defined as follows.

- If $\alpha \notin \mathbb{K}_t^\times$, $h(f, [\alpha]_M)$ is the smallest $h \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{sing}(f, [\alpha]_M) \subseteq \tau_h(\beta)$ for some $\beta \in \mathrm{sing}(f, [\alpha]_M)$ as in Lemma 3.10.
- If $\alpha \in \mathbb{K}_t^\times$, $h(f, [\alpha]_M)$ is the smallest $h \in \mathbb{Z}_{\geq 0}$ such that $\beta^{p^h} \in \mathcal{C}([\alpha]_M)$ for every $\beta \in \mathrm{sing}(f, [\alpha]_M)$.

Note that we always have $h(f, [\alpha]_M) < \infty$. One can show that $h(f, [\alpha]_M) \geq \mathrm{disp}(f, [\alpha]_M)$ provided that $\mathrm{disp}(f, [\alpha]_M) < \infty$, but the inequality may be strict.

3.4.1. *Mahler discrete residues at Mahler trees: the non-torsion case.*

Lemma 3.13. *Let $f \in \mathbb{K}(x)$ and suppose that $[\alpha]_M \in \mathrm{supp}(f)$ such that $\alpha \notin \mathbb{K}_t^\times$. Then there exists $\beta \in \mathrm{sing}(f, [\alpha]_M)$ such that the partial fraction decomposition of*

$$f_{[\alpha]_M} = \sum_{k=1}^{m} \sum_{n=0}^{h} \left(\sum_{i \in \mathbb{Z}/p^n\mathbb{Z}} \frac{c_\beta(k, n, i)}{\left(x - \zeta_{p^n}^i \beta^{p^{h-n}}\right)^k} \right), \tag{3.4}$$

where $m \geq 1$ is the highest order of a pole of f in $\mathrm{sing}(f, [\alpha]_M)$ and $h := h(f, [\alpha]_M)$ as in Definition 3.12.

The coefficients $c_\beta(k,n,i)\in\mathbb{K}$ *are uniquely determined by* f *and* β*, and moreover for any* $\beta,\tilde{\beta}\in[\alpha]_M$ *as above we have* $\tilde{\beta}=\zeta_{p^h}^j\beta$ *for some* $j\in\mathbb{Z}/p^h\mathbb{Z}$*, and*

$$c_{\tilde{\beta}}(k,n,i)=c_\beta\left(k,n,i+\pi_n^h(j)\right). \tag{3.5}$$

Proof. We obtain the existence of $\beta\in\mathrm{sing}(f,[\alpha]_M)$ such that $\mathrm{sing}(f,[\alpha]_M)\subseteq\tau_h(\beta)$ by Lemma 3.10 and Definition 3.12. The existence and uniqueness of the coefficients $c_\beta(k,n,i)\in\mathbb{K}$ satisfying (3.4) follows directly from the existence and uniqueness of partial fraction decompositions, since in this case the elements $\zeta_{p^n}^i\beta^{p^{h-n}}\in\tau_h(\beta)$ are uniquely determined by n and i (relative to the choice of β). For any possibly different $\tilde{\beta}\in\mathrm{sing}(f,[\alpha]_M)$ such that $\mathrm{sing}(f,[\alpha]_M)\subseteq\tau_h(\tilde{\beta})$ we would have that $\tilde{\beta}^{p^{\tilde{n}}}=\beta^{p^h}$ and $\beta^{p^n}=\tilde{\beta}^{p^h}$ such that $0\le n,\tilde{n}\le h$, which forces $\tilde{n}=h=n$ since $\beta\notin\mathbb{K}_t^\times$. Hence $\tilde{\beta}=\zeta_{p^h}^s\beta$ for some $s\in\mathbb{Z}/p^h\mathbb{Z}$, and the computation

$$\frac{c_{\tilde{\beta}}(k,n,i)}{\left(x-\zeta_{p^n}^i\tilde{\beta}^{p^{h-n}}\right)^k}=\frac{c_{\tilde{\beta}}(k,n,i)}{\left(x-\zeta_{p^n}^{i+s}\beta^{p^{h-n}}\right)^k}=\frac{c_\beta(k,n,i+\pi_n^h(s))}{\left(x-\zeta_{p^n}^{i+s}\beta^{p^{h-n}}\right)^k}$$

implies the transformation formula (3.5). $\square$

Remark 3.14. Writing $f_{[\alpha]_M}$ as in (3.4), let us perform the following computation exactly once:

$$\sigma\left(\sum_{k=1}^m\left(\sum_{i\in\mathbb{Z}/p^n\mathbb{Z}}\frac{c_\beta(k,n,i)}{\left(x-\zeta_{p^n}^i\beta^{p^{h-n}}\right)^k}\right)\right)=$$

$$=\sum_{k=1}^m\left(\sum_{i\in\mathbb{Z}/p^{n+1}\mathbb{Z}}\frac{\sum_{s=k}^m V_k^s\left(\zeta_{p^{n+1}}^i\beta^{p^{h-n-1}}\right)\cdot c_\beta\left(s,n,\pi_n^{n+1}(i)\right)}{\left(x-\zeta_{p^{n+1}}^i\beta^{p^{h-n-1}}\right)^k}\right) \tag{3.6}$$

for each $0\le n\le h-1$, where the V_k^s are as in (3.1) for $k\le s\le m$.

Definition 3.15. For $f\in\mathbb{K}(x)$ and $[\alpha]_M\in\mathcal{T}_M$ with $\alpha\notin\mathbb{K}_t^\times$, the *Mahler discrete residue* of f at $[\alpha]_M$ of degree $k\in\mathbb{N}$ is the vector $\mathrm{dres}(f,[\alpha]_M,k)\in\bigoplus_{\gamma\in[\alpha]_M}\mathbb{K}$ defined in terms of the coefficients $c_\beta(k,n,i)$ in the partial fraction decomposition of $f_{[\alpha]_M}$ in Lemma 3.13 as follows.

We set $\mathrm{dres}(f,[\alpha]_M,k)=\mathbf{0}$ whenever $[\alpha]_M\notin\mathrm{supp}(f)$ or $k>m$, and if $[\alpha]_M\in\mathrm{supp}(f)$ we set the component $\mathrm{dres}(f,[\alpha]_M,k)_\gamma=0$ whenever $\gamma^{p^h}\ne\beta^{p^h}$.

For $1\le k\le m$ and $\gamma=\zeta_{p^h}^i\beta$ with $i\in\mathbb{Z}/p^h\mathbb{Z}$, the component

$$\mathrm{dres}(f,[\alpha]_M,k)_\gamma:=\hat{c}_\beta(k,h,i);$$

where for $0\le n\le h$ and $i\in\mathbb{Z}/p^n\mathbb{Z}$ we define recursively (in n):

$$\hat{c}_\beta(k,0,0):=c_\beta(k,0,0);\quad\text{and, if } h\ge 1,\text{ then set} \tag{3.7}$$

$$\hat{c}_\beta(k,n,i):=c_\beta(k,n,i)+\sum_{s=k}^m V_k^s\left(\zeta_{p^n}^i\beta^{p^{h-n}}\right)\cdot\hat{c}_\beta\left(j,n-1,\pi_{n-1}^n(i)\right)$$

for $1\le n\le h$, where the V_k^s are as in (3.1) for $k\le s\le m$.

Remark 3.16. Note that the definition of $\mathrm{dres}(f,[\alpha]_M,k)$ for $\alpha\notin\mathbb{K}_t^\times$ given above is independent of the choice of $\beta\in\mathrm{supp}(f,[\alpha]_M)$, because for any possibly different $\tilde\beta=\zeta_{p^h}^j\beta$ with $j\in\mathbb{Z}/p^h\mathbb{Z}$ we have that $\zeta_{p^h}^i\tilde\beta=\zeta_{p^h}^{i+j}\beta=:\gamma$, and the equality of expressions

$$\hat c_{\tilde\beta}(k,h,i)=\mathrm{dres}(f,[\alpha]_M,k)_\gamma=\hat c_\beta(k,h,i+j)$$

follows from (3.5), since $\zeta_{p^n}^i\tilde\beta^{p^{h-n}}=\zeta_{p^n}^{i+j}\beta^{p^{h-n}}$ for all $i\in\mathbb{Z}/p^n\mathbb{Z}$, and therefore $\hat c_{\tilde\beta}(k,n,i)=\hat c_\beta(k,n,i+\pi_n^h(j))$ for every $0\leq n\leq h$.

3.4.2. *Mahler discrete residues at Mahler trees: the torsion case.*

Lemma 3.17. *Let $f\in\mathbb{K}(x)$ and suppose that $[\alpha]_M\in\mathrm{supp}(f)$ such that $\alpha\in\mathbb{K}_t^\times$. Then for any $\gamma\in\mathcal{C}([\alpha]_M)$ the partial fraction decomposition of*

$$f_{[\alpha]_M}=\sum_{k=1}^m\sum_{n=0}^h\left(\sideset{}{'}\sum_{i\in\mathbb{Z}/p^n\mathbb{Z}}\left(\sum_{\ell\in\mathbb{Z}/e\mathbb{Z}}\frac{d_\gamma(k,n,i,\ell)}{\left(x-\zeta_{p^n}^i\gamma^{p^{\ell-\pi_e(n)}}\right)^k}\right)\right),\tag{3.8}$$

where: $m\geq1$ is the highest order of a pole of f in $\mathrm{sing}(f,[\alpha]_M)$; $h:=h(f,[\alpha]_M)$ as in Definition 3.12; the restricted sum $\sum'$ is taken over $i\in\mathbb{Z}/p^n\mathbb{Z}$ such that $p\nmid i$ whenever $n\neq0$; and $e:=e([\alpha]_M)\geq1$.

The coefficients $d_\gamma(k,n,i,\ell)\in\mathbb{K}$ are uniquely determined by f and γ, and moreover for any $\gamma,\tilde\gamma\in\mathcal{C}([\alpha]_M)$ we have that $\tilde\gamma=\gamma^{p^j}$ for some $j\in\mathbb{Z}/e\mathbb{Z}$, and

$$d_{\tilde\gamma}(k,n,i,\ell)=d_\gamma(k,n,i,\ell+j).\tag{3.9}$$

Proof. If $\alpha\in\mathbb{K}_t^\times$ then $e\geq1$ (cf. Remark 2.8). We then have by Lemma 3.11 that for any given choice of $\gamma\in\mathcal{C}([\alpha]_M)$ the elements of $\beta\in[\alpha]_M$ can be written uniquely as $\beta=\zeta_{p^n}^i\gamma^{p^{\ell-\pi_e(n)}}$ for some $n\in\mathbb{Z}_{\geq0}$ and $i\in\mathbb{Z}/p^n\mathbb{Z}$ such that either $n=i=0$ or else $p\nmid i$. It follows that the apparent poles in (3.8) are all distinct, and therefore the coefficients $d_\gamma(k,n,i,\ell)\in\mathbb{K}$ are uniquely determined by f and γ. The set of elements $\beta\in[\alpha]_M$ such that $\beta^{p^h}\in\mathcal{C}([\alpha]_M)$ are precisely the $\beta=\zeta_{p^n}^i\gamma^{p^{\ell-\pi_e(n)}}$ with $n\leq h$. It also follows from Lemma 3.11 that for any other $\tilde\gamma\in\mathcal{C}([\alpha]_M)$ there exists $j\in\mathbb{Z}/e\mathbb{Z}$ such that $\tilde\gamma=\gamma^{p^j}$, and therefore the computation

$$\frac{d_{\tilde\gamma}(k,n,i,\ell)}{\left(x-\zeta_{p^n}^i\tilde\gamma^{p^{\ell-\pi_e(n)}}\right)^k}=\frac{d_{\tilde\gamma}(k,n,i,\ell)}{\left(x-\zeta_{p^n}^i\gamma^{p^{\ell+j-\pi_e(n)}}\right)^k}=\frac{d_\gamma(k,n,i,\ell+j)}{\left(x-\zeta_{p^n}^i\gamma^{p^{\ell+j-\pi_e(n)}}\right)^k}$$

implies the transformation formula (3.9). ☐

Remark 3.18. Writing $f_{[\alpha]_M}$ as in (3.8), let us perform each of the following computations exactly once:

$$\sigma\left(\sum_{k=1}^m\sum_{\ell\in\mathbb{Z}/e\mathbb{Z}}\frac{d_\gamma(k,0,0,\ell)}{\left(x-\gamma^{p^\ell}\right)^k}\right)=\sum_{k=1}^m\left(\sum_{\ell\in\mathbb{Z}/e\mathbb{Z}}\frac{\sum_{s=k}^mV_k^s\left(\gamma^{p^\ell}\right)\cdot d_\gamma\left(s,0,0,\ell+1\right)}{\left(x-\gamma^{p^\ell}\right)^k}\right)+$$
$$+\sum_{k=1}^m\sum_{i=1}^{p-1}\left(\sum_{\ell\in\mathbb{Z}/e\mathbb{Z}}\frac{\sum_{s=k}^mV_k^s\left(\zeta_p^i\gamma^{p^{\ell-1}}\right)\cdot d_\gamma\left(s,0,0,\ell\right)}{\left(x-\zeta_p^i\gamma^{p^{\ell-1}}\right)^k}\right);\tag{3.10}$$

and for $n \geq 1$ and each $\ell \in \mathbb{Z}/e\mathbb{Z}$ we have

$$\sigma\left(\sum_{k=1}^{m}\sideset{}{'}\sum_{i\in\mathbb{Z}/p^n\mathbb{Z}}\frac{d_\gamma(k,n,i,\ell)}{\left(x-\zeta_{p^n}^i\gamma^{p^{\ell-\pi_e(n)}}\right)^k}\right)=$$
$$=\sum_{k=1}^{m}\left(\sideset{}{'}\sum_{i\in\mathbb{Z}/p^{n+1}\mathbb{Z}}\frac{\sum_{s=k}^{m}V_k^s\left(\zeta_{p^{n+1}}^i\gamma^{p^{\ell-\pi_e(n+1)}}\right)\cdot d_\gamma\left(s,n,\pi_n^{n+1}(i),\ell\right)}{\left(x-\zeta_{p^{n+1}}^i\gamma^{p^{\ell-\pi_e(n+1)}}\right)^k}\right), \quad (3.11)$$

where the V_k^s are as in (3.1) for $k \leq s \leq m$.

The following technical lemma is essential for the definition of Mahler discrete residues in the torsion case.

Lemma 3.19. *Let $\gamma \in \mathcal{C}([\alpha]_M)$ for $\alpha \in \mathbb{K}_t^\times$ and $e := e([\alpha]_M)$. For $m \in \mathbb{N}$, let $\mathcal{D}_\gamma^{(m)}$: $\mathbb{K}^{m\times e} \to \mathbb{K}^{m\times e}$ be the $\mathbb{K}$-linear map*

$$\mathcal{D}_\gamma^{(m)}:(c_{k,\ell})_{\substack{1\leq k\leq m\\ \ell\in\mathbb{Z}/e\mathbb{Z}}}\mapsto(d_{k,\ell})_{\substack{1\leq k\leq m\\ \ell\in\mathbb{Z}/e\mathbb{Z}}}, \quad (3.12)$$

defined by $d_{k,\ell} := c_{k,\ell} - \sum_{s=k}^{m} V_k^s\left(\gamma^{p^\ell}\right)\cdot c_{s,\ell+1}$, where the V_k^s are as in (3.1). Then $\mathcal{D}_\gamma^{(m)}$ is invertible and its 1-eigenspace is $\{\mathbf{0}\}$.

Proof. Let $\mathbf{0} \neq (c_{k,\ell}) \in \mathbb{K}^{m\times e}$, and let $1 \leq r \leq m$ be as large as possible such that $c_{r,\ell} \neq 0$ for some $\ell \in \mathbb{Z}/e\mathbb{Z}$. Then we have that $\mathcal{D}_\gamma^{(m)}(c_{k,\ell}) =: (d_{k,\ell}) \neq \mathbf{0}$, because for each $\ell \in \mathbb{Z}/e\mathbb{Z}$ we have that

$$d_{r,\ell}=c_{r,\ell}-\sum_{s=r}^{m}V_r^s\left(\gamma^{p^\ell}\right)\cdot c_{s,\ell+1}=c_{r,\ell}-V_r^r\left(\gamma^{p^\ell}\right)\cdot c_{r,\ell+1}$$

because $c_{s,\ell+1} = 0$ whenever $s > r$, and we see just as at the and of proof of Proposition 3.5 that the $d_{r,\ell}$ cannot be zero for every $\ell \in \mathbb{Z}/e\mathbb{Z}$ because this would imply that every $c_{r,\ell} = 0$, contradicting our choice of r. Moreover, we also cannot have $d_{k,\ell} = c_{k,\ell}$ for every $1 \leq k \leq m$ and $\ell \in \mathbb{Z}/e\mathbb{Z}$, for this would also imply that $c_{r,\ell} = 0$ for every $\ell \in \mathbb{Z}/e\mathbb{Z}$, again contradicting our choice of r. □

Definition 3.20. With notation as in Lemma 3.19, the inverse of $\mathcal{D}_\gamma^{(m)}$ is denoted by $\mathcal{L}_\gamma^{(m)}$.

Remark 3.21. One can show from the proof of Lemma 3.19 that the matrix associated to $\mathcal{D}_\gamma^{(m)}$ is block upper triangular, and even that the k-th diagonal block is almost a Jordan block, with 1 along the main diagonal, $-V_k^k(\gamma^{p^\ell})$ above the main diagonal (ranging over $\ell = 0, \ldots, e-1$, in that order), but with lower left-hand corner $-V_k^k(\gamma^{p^{e-1}})$. We omit the explicit computation of the inverse map $\mathcal{L}_\gamma^{(m)}$ in Definition 3.20 due to spacetime constraints.

Definition 3.22. For $f \in \mathbb{K}(x)$ and $[\alpha]_M \in \mathcal{T}_M$ with $\alpha \in \mathbb{K}_t^\times$, the *Mahler discrete residue* of f at $[\alpha]_M$ of degree $k \in \mathbb{N}$ is the vector $\operatorname{dres}(f,[\alpha]_M,k) \in \bigoplus_{\beta\in[\alpha]_M}\mathbb{K}$ defined in terms of the coefficients $d_\gamma(k,n,i,\ell)$ in the partial fraction decomposition of $f_{[\alpha]_M}$ in Lemma 3.17 as follows.

We set $\mathrm{dres}(f,[\alpha]_M,k)=\mathbf{0}$ whenever $[\alpha]_M\notin\mathrm{supp}(f)$ or $k>m$, and if $[\alpha]_M\in\mathrm{supp}(f)$ we set the component $\mathrm{dres}(f,[\alpha]_M,k)_\beta=0$ whenever the smallest integer $r\in\mathbb{Z}_{\geq 0}$ such that $\beta^{p^r}\in\mathbb{C}([\alpha]_M)$ is different from h.

If $h=0$, then for $1\leq k\leq m$ and $\gamma^{p^\ell}\in\mathcal{C}([\alpha]_M)$ with $\ell\in\mathbb{Z}/\ell\mathbb{Z}$, the component

$$\mathrm{dres}(f,[\alpha]_M,k)_{\gamma^{p^\ell}}:=d_\gamma(k,0,0,\ell).$$

If $h\neq 0$, then for $1\leq k\leq m$ and $\beta=\zeta_{p^h}^i\gamma^{p^{\ell-\pi_e(h)}}$ with $i\in\mathbb{Z}/p^h\mathbb{Z}$ such that $p\nmid i$ and $\ell\in\mathbb{Z}/e\mathbb{Z}$, the component

$$\mathrm{dres}(f,[\alpha]_M,k)_\beta:=\hat{d}_\gamma(k,h,i,\ell);\qquad\text{where we set}$$

$$\hat{d}_\gamma(k,0,0,\ell):=c_\gamma(k,\ell),\qquad\text{with}\tag{3.13}$$

$$\big(c_\gamma(k,\ell)\big)_{\substack{1\leq k\leq m\\ \ell\in\mathbb{Z}/e\mathbb{Z}}}:=\mathcal{L}_\gamma^{(m)}\left(\big(d_\gamma(k,0,0,\ell)\big)_{\substack{1\leq k\leq m\\ \ell\in\mathbb{Z}/e\mathbb{Z}}}\right)\tag{3.14}$$

for the linear map $\mathcal{L}_\gamma^{(m)}$ in Definition 3.20; and for $1\leq n\leq h$ and $i\in\mathbb{Z}/p^n\mathbb{Z}$ with $p\nmid i$ we define recursively (in n):

$$\hat{d}_\gamma(k,n,i,\ell):=d_\gamma(k,n,i,\ell)+\sum_{s=k}^{m}V_k^s\left(\zeta_{p^n}^i\gamma^{p^{\ell-\pi_e(n)}}\right)\cdot\hat{d}_\gamma\left(s,n-1,\pi_{n-1}^n(i),\ell\right),\tag{3.15}$$

where the V_k^s are as in (3.1).

Remark 3.23. Note that the definition of $\mathrm{dres}(f,[\alpha]_M,k)$ for $\alpha\in\mathbb{K}_t^\times$ given above is independent of the choice of $\gamma\in\mathcal{C}([\alpha]_M)$, because for any possibly different $\tilde{\gamma}=\gamma^{p^j}$ with $j\in\mathbb{Z}/e\mathbb{Z}$ we have that $\zeta_{p^h}^i\tilde{\gamma}^{p^{\ell-\pi_e(h)}}=\zeta_{p^h}^i\gamma^{p^{\ell+j-\pi_e(h)}}=:\beta$, and the desired equalities

$$\hat{d}_{\tilde{\gamma}}(k,h,i,\ell)=\mathrm{dres}(f,[\alpha]_M,k)_\beta=\hat{d}_\gamma(k,h,i,\ell+j)$$

follow from (3.9), after observing that $\mathcal{D}_{\tilde{\gamma}}^{(m)}\circ\mathrm{cyc}_j=\mathcal{D}_\gamma^{(m)}$, where $\mathrm{cyc}_j:\mathbb{K}^{m\times e}\to\mathbb{K}^{m\times e}:(c_{k,\ell})\mapsto(c_{k,\ell+j})$, which shows that $\mathrm{cyc}_j\circ\mathcal{L}_{\tilde{\gamma}}^{(m)}=\mathcal{L}_\gamma^{(m)}$ and hence $\hat{d}_{\tilde{\gamma}}(k,n,i,\ell)=\hat{d}_\gamma(k,n,i,\ell+j)$ for every $0\leq n\leq h$.

3.5. **Proof of the Main Theorem.**

Proposition 3.24. *For $f\in\mathbb{K}(x)$ and $[\alpha]_M\in\mathcal{T}_M$, the component $f_{[\alpha]_M}$ is Mahler summable if and only if* $\mathrm{dres}(f,[\alpha]_M,k)=\mathbf{0}$ *for every $k\in\mathbb{N}$.*

Proof. The Proposition is trivial for $[\alpha]_M\notin\mathrm{supp}(f)$. Write $f_{[\alpha]_M}$ as in Lemma 3.13 if $\alpha\notin\mathbb{K}_t^\times$ and as in Lemma 3.17 if $\alpha\in\mathbb{K}_t^\times$. Let us define recursively: $g_{[\alpha]_M}^{(0)}:=0$; and, if $h:=h(f,[\alpha]_M)\geq 1$ as in Definition 3.12, then for $0\leq n\leq h-1$ set

$$g_{[\alpha]_M}^{(n+1)}:=g_{[\alpha]_M}^{(n)}+\sum_{k=1}^{m}\sum_{i\in\mathbb{Z}/p^n\mathbb{Z}}\hat{c}(k,n,i)\cdot\left(x-\zeta_{p^n}^i\beta^{p^{h-n}}\right)^{-k}$$

in case $\alpha \notin \mathbb{K}_t^\times$, with $\hat{c}(k,n,i)$ as in (3.7); and

$$g_{[\alpha]_M}^{(n+1)} := g_{[\alpha]}^{(n)} + \sum_{k=1}^{m} \left({\sum_{i\in\mathbb{Z}/p^n\mathbb{Z}}}' \left(\sum_{\ell\in\mathbb{Z}/e\mathbb{Z}} \frac{\hat{d}_\gamma(k,n,i,\ell)}{\left(x-\zeta_{p^n}^i\gamma^{p^{\ell-\pi_e(n)}}\right)^k} \right)\right)$$

in case $\alpha \in \mathbb{K}_t^\times$, with $\hat{d}(k,n,i,\ell)$ as in (3.13) for $n=0$ and as in (3.15) for any $n>1$. Setting $\bar{f}_{[\alpha]_M}^{(n)} := f_{[\alpha]_M} + \Delta\left(g_{[\alpha]_M}^{(n)}\right)$, an induction argument then shows that, for every $0 \le n \le h$,

$$\bar{f}_{[\alpha]_M}^{(n)} = \sum_{k=1}^{m}\sum_{s=n+1}^{h+1}\sum_{i\in\mathbb{Z}/p^s\mathbb{Z}} \frac{c(k,s,i)}{\left(x-\zeta_{p^s}^i\beta^{p^{h-s}}\right)^k} + \sum_{k=1}^{m}\sum_{i\in\mathbb{Z}/p^n\mathbb{Z}} \frac{\hat{c}(k,n,i)}{\left(x-\zeta_{p^n}^i\beta^{p^{h-n}}\right)^k} \tag{3.16}$$

in case $\alpha \notin \mathbb{K}_t^\times$; and

$$\begin{aligned}\bar{f}_{[\alpha]_M}^{(n)} = \sum_{k=1}^{m}\sum_{s=n+1}^{h+1} \left({\sum_{i\in\mathbb{Z}/p^s\mathbb{Z}}}' \left(\sum_{\ell\in\mathbb{Z}/e\mathbb{Z}} \frac{d_\gamma(k,s,i,\ell)}{\left(x-\zeta_{p^s}^i\gamma^{p^{\ell-\pi_e(s)}}\right)^k} \right)\right) + \\ + \sum_{k=1}^{m} \left({\sum_{i\in\mathbb{Z}/p^n\mathbb{Z}}}' \left(\sum_{\ell\in\mathbb{Z}/e\mathbb{Z}} \frac{\hat{d}_\gamma(k,n,i,\ell)}{\left(x-\zeta_{p^n}^i\gamma^{p^{\ell-\pi_e(n)}}\right)^k} \right)\right)\end{aligned} \tag{3.17}$$

in case $\alpha \in \mathbb{K}_t^\times$. The harmless summand for $s = h+1$ in (3.16) and (3.17) is included so that the sums make sense for $n = h$, but we set every $d(k, h+1, i\ell) := 0$ in (3.17) and every $c_\beta(k, h{+}1, i) := 0$ in (3.16), regardless of the choice of p-th root $\beta^{p^{-1}}$. The induction argument is straightforward, requiring only: the recursive definition of the coefficients $\hat{c}_\beta(k,n,i)$ in (3.7) and the computation (3.6) in case $\alpha \notin \mathbb{K}_t^\times$; the recursive definition of the coefficients $\hat{d}_\gamma(k,n,i,\ell)$ in (3.13) and (3.15) and the computations (3.10) and (3.11) in case $\alpha \in \mathbb{K}_t^\times$; and a moderate amount of space and courage to write it down in both cases. It then follows from (3.16) and (3.17) that

$$\bar{f}_{[\alpha]_M}^{(h)} = \sum_{k=1}^{m}\sum_{\xi\in[\alpha]_M} \frac{\mathrm{dres}(f,[\alpha]_M,k)_\xi}{(x-\xi)^k}.$$

Provided that we do not have both $\alpha \in \mathbb{K}_t^\times$ and $h = 0$, we obtain $\mathrm{disp}\left(\bar{f}_{[\alpha]_M}^{(h)}, [\alpha]_M\right) = 0$ if and only if $\mathrm{dres}(f, [\alpha]_M, k) \neq 0$ for some k, and we conclude by Corollary 3.6. But Corollary 3.6 remains silent in the case where $\alpha \in \mathbb{K}_t^\times$ and $h = 0$, for now we have

$$\bar{f}_{[\alpha]_M}^{(0)} = \sum_{k=1}^{m}\sum_{\ell\in\mathbb{Z}/e\mathbb{Z}} \frac{d_\gamma(k,\ell)}{\left(x-\gamma^{p^\ell}\right)^k} = f_{[\alpha]_M},$$

where we write $d_\gamma(k,\ell) := d_\gamma(k,0,0,\ell)$ to simplify the notation, and we observe that $\mathrm{disp}(f_{[\alpha]_M}, [\alpha]_M) = \infty$ regardless of whether or not $f_{[\alpha]_M} = 0$. A different argument is required in this case: still letting

$$g^{(1)}_{[\alpha]_M} := \sum_{k=1}^{m} \sum_{\ell\in\mathbb{Z}/e\mathbb{Z}} \frac{c(k,\ell)}{\left(x-\gamma^{p^\ell}\right)^k}$$

despite having $h=0$, where the $c_\gamma(k,\ell)$ are as in (3.14), the computation 3.10 shows (after a harmless shift in ℓ) that

$$f_{[\alpha]_M} + \Delta\left(g^{(1)}_{[\alpha]_M}\right) = \sum_{k=1}^{m}\sum_{i=1}^{p-1}\sum_{\ell\in\mathbb{Z}/e\mathbb{Z}} \frac{\sum_{s=k}^m V_k^s\left(\zeta_p^i\gamma^{p^\ell}\right)\cdot c_\gamma(k,\ell+1)}{\left(x-\zeta_p^i\gamma^{p^\ell}\right)^k},$$

which does have dispersion 0 if and only if for some k,i,ℓ the coefficient $\sum_{s=k}^m V_k^s\left(\zeta_p^i\gamma^{p^\ell}\right)\cdot c_\gamma(k,\ell{+}1)$ is non-zero, which we must show is equivalent to some $d_\gamma(k,\ell) =: \mathrm{dres}(f,[\alpha]_M,k)_{\gamma^{p^\ell}}$ being non-zero. To see this, note that

$$\sum_{s=k}^{m} V_k^s\left(\zeta_p^i\gamma^{p^\ell}\right)\cdot c_\gamma(k,\ell+1) = \zeta_p^{ik}\cdot\sum_{s=k}^{m} V_k^s\left(\gamma^{p^\ell}\right)\cdot c_\gamma(k,\ell+1) = \zeta_p^i\cdot\left(c_\gamma(k,\ell)-d_\gamma(k,\ell)\right)$$

where the first equality follows from $V_k^s(\zeta_p^i\gamma^{p^\ell}) = \zeta_p^{ik}V_k^s(\gamma^{p^\ell})$ independently of s by Lemma 3.1, and the second equality follows from the Definition 3.20 of $\mathcal{L}_\gamma^{(m)}$. Finally, we conclude by Lemma 3.19, since the map $\mathcal{D}_\gamma^{(m)}$ has no non-trivial fixed points. □

Remark 3.25. For $f\in\mathbb{K}(x)$, $[\alpha]_M\in\mathrm{supp}(f)$, and $h := h(f,[\alpha]_M)$ as in Definition 3.12, the elements $\bar{f}^{(h)}_{[\alpha]_M}, g^{(h)}_{[\alpha]_M} \in \mathbb{K}(x)_{[\alpha]_M}$ constructed in the proof of Proposition 3.24 are the $[\alpha]_M$-components of the $\bar{f}, g\in\mathbb{K}(x)$ in the *Mahler reduction* (1.1).

Main Theorem. *The rational function $f\in\mathbb{K}(x)$ is Mahler summable if and only if $\mathrm{dres}(f,\infty)=\mathbf{0}$ and $\mathrm{dres}(f,[\alpha]_M,k)=\mathbf{0}$ for each Mahler tree $[\alpha]_M\in\mathcal{T}_M$ and each degree $k\in\mathbb{N}$.*

Proof. For $f\in\mathbb{K}(x)$ we have the decomposition $f=f_L+f_T$ in (2.1), where we observed that f is Mahler summable if and only if each of f_L and f_T is Mahler summable. By Proposition 3.8, the component f_L is Mahler summable if and only if $\mathrm{dres}(f,\infty)=\mathbf{0}$. By Remark 2.13, the component f_T is Mahler summable if and only if the $[\alpha]_M$-component $f_{[\alpha]_M}$ in Definition 2.12 is Mahler summable for each $[\alpha]_M\in\mathcal{T}_M$. By Proposition 3.24, $f_{[\alpha]_M}$ is Mahler summable if and only if $\mathrm{dres}(f,[\alpha]_M,k)=\mathbf{0}$ for every $k\in\mathbb{N}$. □

3.6. **Mahler reduction.** We can now define the Mahler reduction (1.1): $\bar{f} = f+\Delta(g)$ promised in the introduction for any $f\in\mathbb{K}(x)$, in terms of the decompositions $\bar{f}=\bar{f}_L+\bar{f}_T$ and $g=g_L+g_T$ as in (2.1), by setting

$$\bar{f}_L := \sum_{[i]_\mathcal{P}\in\mathbb{Z}/\mathcal{P}} \bar{f}_{[i]_\mathcal{P}} \quad\text{and}\quad g_L := \sum_{[i]_\mathcal{P}\in\mathbb{Z}/\mathcal{P}} g_{[i]_\mathcal{P}}; \quad\text{and}$$

$$\bar{f}_T := \sum_{[\alpha]_M\in\mathrm{supp}(f)} \bar{f}^{(h(f,[\alpha]_M))}_{[\alpha]_M} \quad\text{and}\quad g_T := \sum_{[\alpha]_M\in\mathrm{supp}(f)} g^{(h(f,[\alpha]_M))}_{[\alpha]_M}$$

as in Remark 3.9 and Remark 3.25. It is clear from the definitions that $\overline{c \cdot f} = c \cdot \bar{f}$ for $c \in \mathbb{K}$. Setting $\bar{f}_1 \tilde{+} \bar{f}_2 := \overline{f_1 + f_2}$ defines a $\mathbb{K}$-linear structure on $\{\bar{f} \mid f \in \mathbb{K}(x)\}$ such that M-Rem : $f \mapsto \bar{f}$ is $\mathbb{K}$-linear and has the desired property that $\ker(\text{M-Rem}) = \text{im}(\Delta)$.

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