

## ① § 11.8 Power Series

### Motivation

Represent some of the most important functions that arise in mathematics, physics, and chemistry.

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \quad (1)$$

where  $x$  is a variable and  $c_n$ 's are constants called coefficients of the series.

For each fixed  $x$ , (1) is a series of constants. (that we can test for convergence or divergence).

The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

whose domain is the set of all  $x$  for which the series converges. ( $f$  is similar to a polynomial, but has infinitely many terms.)

Ex. Taking  $c_n = 1$  for all  $n$ , we have

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$$

which converges when  $-1 < x < 1$  and diverges when  $|x| \geq 1$ .

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots \quad (3)$$

is called a power series in  $(x-a)$  or a power series centered at  $a$ .

Note: 1.  $(x-a)^0 = 1$

2. (3) always converges when  $x=a$ .

② Ex 1 For what values of  $x$  is  $\sum_{n=0}^{\infty} n! x^n$  convergent?

~~Idea:~~ use the Ratio Test.

Let  $a_n = n! x^n$ . If  $x \neq 0$ , we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| \\ = (n+1)|x| \rightarrow \infty \text{ as } n \rightarrow \infty$$

By the Ratio Test, the series diverges when  $x \neq 0$ .

The given series converges only when  $x=0$ .

Ex 2. For what values of  $x$  does the series  $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$  converge?

Let  $a_n = (x-3)^n/n$ . Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| \\ = \frac{|x-3|}{1 + \frac{1}{n}} \rightarrow |x-3| \text{ as } n \rightarrow \infty$$

By the Ratio Test, the given series is absolutely convergent when  $|x-3| < 1$  and divergent when  $|x-3| > 1$ .

If  $|x-3|=1$ , then  $x=2$  or  $x=4$ .

Set  $x=4$  in the series. It becomes  $\sum \frac{1}{n}$ , which is divergent.

Set  $x=2$  in the series. It becomes  $\sum \frac{(-1)^n}{n}$ , which is convergent by the Alternating Series Test.

Thus, the given power series converges for  $2 \leq x < 4$ .

Ex 3. Find the domain of the Bessel function of order 0, which arose when Bessel solved Kepler's equation for describing planetary motion, defined by

③

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Let  $a_n = (-1)^n x^{2n} / [2^{2n}(n!)^2]$ . Then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)} [(n+1)!]^2} \cdot \frac{2^{2n} (n!)^2}{(-1)^n x^{2n}} \right| \\ &= \frac{x^2}{4(n+1)^2} \rightarrow 0 < 1 \text{ for all } x \end{aligned}$$

By the Ratio Test, the given series converges for all values of  $x$ .

(From previous examples, we know that the domain of a power series is an interval. The following theorem tells us that this is true in general.)

Theorem 4 For a given power series  $\sum_{n=0}^{\infty} c_n (x-a)^n$ , there are only three possibilities:

(i) The series converges only when  $x=a$ .

(ii) The series converges for all  $x$ .

(iii) There is a positive number  $R$  such that the series converges if  $|x-a| < R$  and diverges if  $|x-a| > R$ .

The number  $R$  in (iii) is called the radius of convergence of the power series. By convention, ~~R=0~~  $R=0$  in case (i) and  $R=\infty$  in case (ii).

The interval of convergence of a power series is the interval that consists of all values of  $x$  for which the series converges.

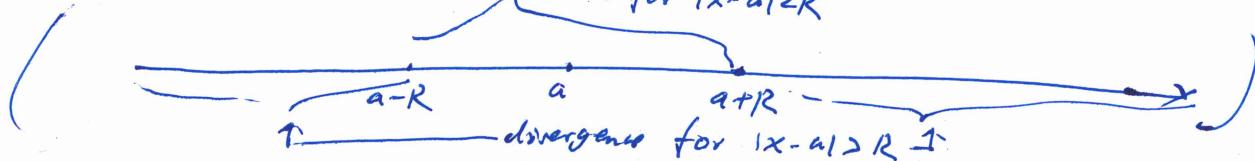
In case (i), the ~~interval~~ is a

(ii)  $\underline{\hspace{1cm}}$   $\overline{\hspace{1cm}}$   $\rightarrow (-\infty, \infty)$

(iii),  $\underline{\hspace{1cm}}$   $\overline{\hspace{1cm}}$  has four possibilities,

$(a-R, a+R)$ ,  $(a-R, a+R]$ ,  $[a-R, a+R)$ ,  $[a-R, a+R]$ .

convergence for  $|x-a| < R$



divergence for  $|x-a| \geq R$

(4) Note: 1. the Ratio Test (sometimes the Root Test) should be used to determine the radius of convergence  $R$ .

2. the Ratio or Root Test always fail when  $x$  is an endpoint of the interval of convergence, so the endpoints must be checked with some other tests.

Ex 4. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Let  $a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$ . Then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\ &= \left| -3x \sqrt{\frac{n+1}{n+2}} \right| \\ &= 3 \sqrt{\frac{1 + 1/n}{1 + 2/n}} \cdot |x| \rightarrow 3|x| \text{ as } n \rightarrow \infty \end{aligned}$$

By the Ratio Test, the given series converges if  $3|x| < 1$  and diverges if  $3|x| > 1$ . Thus, it converges if  $|x| < \frac{1}{3}$  and diverges if  $|x| > \frac{1}{3}$ . So, the radius of convergence is  $R = \frac{1}{3}$ .

The interval of convergence contains  $(-\frac{1}{3}, \frac{1}{3})$ . If  $x = -\frac{1}{3}$ , the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n (-\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \cancel{-}, \text{ which diverges}$$

If  $x = \frac{1}{3}$ , the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n (\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which converges by the Alternating Series Test.

Thus, the interval of convergence is  $[-\frac{1}{3}, \frac{1}{3}]$ .

Ex 5. Find the Radius of convergence and interval of convergence of the series  $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

Let  $a_n = \frac{n(x+2)^n}{3^{n+1}}$ , Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} \right|$$

$$= \left(1 + \frac{1}{n}\right) \frac{|x+2|}{3} \rightarrow \frac{|x+2|}{3} \text{ as } n \rightarrow \infty$$

Using the Ratio Test, we see that the series converges if  $|x+2| < 1$  and it diverges if  $\frac{|x+2|}{3} > 1$ .

So, it converges if  $|x+2| < 3$  and diverges if  $|x+2| > 3$ .

Thus, the radius of convergence is  $R=3$ .

The interval of convergence contains  $(-5, 1)$ .

When  $x = -5$ , the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence ( $(-1)^n n$  does not converge to 0). When  $x=1$ , the series is

$$\sum_{n=0}^{\infty} \frac{n3^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence.

Thus, the interval of convergence is  $(-5, 1)$ .