§7.4 Integration of Rational Functions by Partial Fractions

Ex 0. $\frac{2}{x-1} - \frac{1}{x+2} = \frac{x+5}{x^2+x-2}$ (partial fraction)

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) dx$$

$$= 2\ln|x-1| - \ln|x+2| + C$$

Consider

$$f(x) = \frac{P(x)}{Q(x)}, \qquad (1)$$

where P and Q are polynomials.

Recall: if

$$P = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where $a_n \neq 0$, then the degree of P is n, denote it by $\deg(P)$

Idea of partial fraction: if $\deg(P) < \deg(Q)$ in (1), then we express f as a sum of fractions. Such a rational function is called <u>proper</u>.

Step 1: If f is improper, i.e., $\deg(P) \geq \deg(Q)$

By Euclidean division, there exists $S(x)$, $R(x)$ s.t.

$$P(x) = S(x)Q(x) + R(x),$$

where $R = 0$ or $\deg(R) < \deg(Q)$.

Then $f = \frac{P}{Q} = SQ + R$

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Ex 1. Find $\int \frac{x^3+x}{x-1}\,dx$

$$x^3+x = (x^2+x+2)(x-1)+2$$

$$\Rightarrow \frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

$$\int \frac{x^3+x}{x-1}\,dx = \int \left(x^2+x+2+\frac{2}{x-1}\right)dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C.$$

Step 2: In (2), Q can be factored as a product of linear factors $(ax+b)$ and irreducible quadratic factors $(ax^2+bx+c,\ b^2-4ac<0)$.

$$Q = x^4-16 = (x-2)(x+2)(x^2+4)$$

Step 3: Express R/Q in (2) as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

Case 1. $Q(x)$ is a product of distinct linear factors.

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

where no factor is repeated. ~~To~~ There exist constant $A_1, \dots, A_k$ s.t.

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k}$$

A_i's can be ~~de~~ determined by solving linear equations.

Ex 2. Compute $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$$2x^3+3x^2-2x = x(2x-1)(x+2)$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2} \quad (2)$$

$$x^2+2x-1=(2A+B+2C)x^2+(3A+2B-C)x-2A$$

$$\Rightarrow \begin{cases} 2A+B+2C=1 \\ 3A+2B-C=2 \\ -2A=-1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=\frac{1}{5} \\ C=-\frac{1}{10} \end{cases}$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int \left(\frac{1}{2}\frac{1}{x}+\frac{1}{5}\frac{1}{2x-1}-\frac{1}{10}\frac{1}{x+2}\right)dx$$

$$= \frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+C.$$

Case 2. $Q(x)$ is a product of linear factors, some of which are repeated.

$$Q(x)=(a_1x+b_1)^{r_1}(a_2x+b_2)^{r_2}\cdots(a_kx+b_k)^{r_k}, \quad r_i\geq 1$$

$$\frac{R(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_{r_1}}{(a_1x+b_1)^{r_1}}+\cdots,$$ where A_i's are unknown constants

Ex. $$\frac{x^3-x+1}{x^2(x-1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}$$

Ex 4. Find $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$

Step 1. $$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}=x+1+\frac{4x}{x^3-x^2-x+1}$$

Step 2. $$x^3-x^2-x+1=(x-1)^2(x+1)$$

Step 3. $$\frac{4x}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1} \quad (3)$$

Multiplying (3) by $(x-1)^2(x+1)$, we get

$$4x=(A+C)x^2+(B-2C)x+(-A+B+C)$$

$$\Rightarrow \begin{cases} A+C=0 \end{cases} \quad \begin{cases} A=1 \end{cases}$$

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$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int \left[x+1+\frac{1}{x-1}+\frac{2}{(x-2)^2}-\frac{1}{x+1}\right]dx$$

$$= \frac{x^2}{2}+x+\ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

Case 3. Q contains irreducible quadratic factors, none of them repeated.

If $Q(x)$ has factor ax^2+bx+C, $b^2-4ac<0$, then

$$\frac{R(x)}{Q(x)} = \frac{Ax+B}{ax^2+bx+C} + \cdots \qquad (4)$$

where A, B are unknown constants.

Ex: $\dfrac{x}{(x-2)(x^2+1)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{x^2+4}$

The term in (4) can be integrated by completing the square and using

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Ex 5. Compute $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx$

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

Multiplying by $x(x^2+4)$, we have

$$2x^2-x+4 = A(x^2+4)+(Bx+C)x$$

$$=(A+B)x^2+Cx+4A$$

$$\Rightarrow \begin{cases} A+B=2 \\ C=-1 \\ 4A=4 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=1 \\ C=-1 \end{cases}$$

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)dx$$

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$$\int \frac{x-1}{x^2+4}\,dx = \int \frac{x\,dx}{x^2+4} - \int \frac{dx}{x^2+4}$$

$$= \frac{1}{2}\int \frac{d(x^2+4)}{x^2+4} - \int \frac{dx}{x^2+4}$$

Thus, $$\int \frac{2x^2-x+4}{x^2(x^2+4)}\,dx = \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}(x/2) + C$$

Case 4. $Q(x)$ contains repeated irreducible quadratic factors.

~~In~~ If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then in ~~a~~ partial fractions decomposition of $\frac{R(x)}{Q(x)}$, it contains

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Ex 8. Evaluate $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we get

$$-x^3+2x^2-x+1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\Rightarrow \begin{cases} A+B=0 \\ C=-1 \\ 2A+B+D=2 \\ C+E=-1 \\ A=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=-1 \\ D=1 \\ E=0 \end{cases}$$

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx = \int\left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx$$

$$= \int \frac{dx}{x} - \int \frac{x}{x^2+1}\,dx - \int \frac{dx}{1+x^2} + \int \frac{x\,dx}{(x^2+1)^2}$$

$$= \ln|x| - \frac{1}{2}\ln(1+x^2) - \tan^{-1}x - \frac{1}{2(x^2+1)} + C$$

Note: 1. in second and foourth, we set u=...

⑥

$$\int \frac{dx}{(1+x^2)^2} \overset{x=\tan\theta}{=\!=} \int \frac{d\theta}{1+\tan^2\theta}$$

$$= \int \cos^2\theta \, d\theta$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{1}{2}\left(\arctan(\cancel{\theta}x) + \frac{x}{1+x^2}\right)$$

3. Some non rational function can be changed into rational functions by substitions.

$$\int \frac{\sqrt{x+4}}{x} dx \overset{u=\sqrt{x+4}}{=\!=} 2\int \frac{u^2}{u^2-4} du$$

Note: 72.

$$\int \frac{dx}{(x^2+a^2)^n} dx \longrightarrow \int \frac{dx}{(x^2+a^2)^n}$$