① § 11.6 Absolute Convergence and the Ratio and Root Tests

Given a series $\sum a_n$, we consider

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

Motivation: let $s = a_1 + a_2 + a_3 + a_4 + \cdots$
$\tilde{s} = a_2 + a_1 + a_4 + a_3 + \cdots$

Question: When does $s = \tilde{s}$?

Answer: if $s = \sum a_n$ is absolutely convergent, then $s = \tilde{s}$.

Def 1. A series $\sum a_n$ is called absolutely convergent if ~~the~~ ~~series of~~ $\sum |a_n|$ is convergent.
(the series of absolute values)

Ex 1. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is convergent (p-series with $p = 2$).

Ex 2. The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent, but it is not abosolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent (p-series with $p = 1$).

Def 2. A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

Ex: the alternating harmonic series is conditionally convergent.

Theorem 3. If $\sum a_n$ is absolutely convergent, then it is convergent.

Pf: Observation: $0 \le a_n + |a_n| \le 2|a_n|$ for $n \ge 1$.

Since $\sum a_n$ is absolutely convergent, $\sum |a_n|$ is convergent.

Thus, $\sum 2|a_n|$ is convergent.

② By the Comparison Test, $\sum(a_n + |a_n|)$ is convergent.

Then
$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$
is also convergent.

Ex 3. Determine the convergence of
$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$
(This series is not alternating, but has positive and negative terms)

Taking the series of absolute values, we have
$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$
Since $|\cos n| \le 1$ for $n \ge 1$, we get
$$\frac{|\cos n|}{n^2} \le \frac{1}{n^2}$$
By the Comparison test, we know $\sum \frac{|\cos n|}{n^2}$ is convergent.

By Theorem 3, $\sum \frac{\cos n}{n^2}$ is convergent. Nov 8

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, the $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is ~~conclusive~~ inconclusive.

Outline of Proof: 1° ε-N definition of limit

2° the Comparison test with geometric series.

Note: 1. part (iii) of the above test means if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the test gives no information.

③ Ex: $\sum \frac{1}{n^2}$ is convergent

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2} \to 1 \text{ as } n \to \infty$$

$\sum \frac{1}{n}$ is ~~convergent~~ divergent

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1 \text{ as } n \to \infty$$

~~Ex 4. Test the convergence of $\sum\limits_{n=1}$~~

2. The Ratio Test is usually conclusive if a_n contains an exponential or a factorial.

Ex 4. Test the absolute convergence of $\sum\limits_{n=1}^{\infty} (-1)^n \cdot \frac{n^3}{3^n}$

Let $a_n = (-1)^n n^3 / 3^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

By the Ratio Test, the given series is absolutely convergent.

Ex 5. Test the convergence of $\sum\limits_{n=1}^{\infty} \frac{n^n}{n!}$

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdots n}{1 \cdot 2 \cdot 3 \cdots n} \geq n$$

Thus, $a_n \to \infty$ as $n \to \infty$

By the divergence test, $\sum a_n$ is divergent.

If nth powers occur in $\sum a_n$, then the following test is convienient to apply.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, the $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Ex 6. Test the convergence of $\sum \left(\frac{2n+3}{3n+2}\right)^n$

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

Thus, the given series is absolutely convergent by the Root Test.