ON D-FINITENESS OF A SYMMETRIC FUNCTION

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Abstract. We show that a symmetric function induced by a Weyl character of a given representation of a simple complex Lie algebra is D-finite. Moreover, we prove that the fake degree sequence associated to the representation and the Lie algebra is q-holonomic.

In this note, we consider symmetric functions (series) as formal power series in infinitely many variables [13, Section 7.1] $X = x_1, x_2, \ldots$, which is equivalent to the definition in [12], over a field $\mathbb{K}$ of characteristic zero. Let $Y = y_1, y_2, \ldots$ be another set of variables. Then the *product* of X and Y is defined by $X.Y = \sum_{i,j} x_i y_j$. Let $\mathbb{K}[[X]]$ be the ring of formal power series in X and Λ be the ring of symmetric functions in X. Let $f \in \Lambda$ and $g \in \mathbb{K}[[X]]$ with $g = \sum_{i=1}^{\infty} t_i$, where t_i is a monomial in x_i's. We define the *plethysm* (or *composition*) [7] of f by g to be $f(t_1, t_2, \ldots)$, and denote it by $f[g]$. For each $k \in \mathbb{N}$, let $h_k(X)$ be the k-th complete homogenous symmetric function in X. As a matter of notation, we set $h_k(X) = 0$ for $k < 0$. When it is clear from context, we may abbreviate $h_k(X)$ to h_k. Then the corresponding generating function [3] is denoted by

$$H(t) = \sum_{k=0}^{\infty} h_k(X) t^k = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}.$$

We say that $F \in \mathbb{K}[[X]]$ is *D-finite* [6, section 5] in the infinitely many variables x_i if, for any choice of a finite set S of X, the specification to 0 of each $x_i \in X \setminus S$ gives rise to a power series that is D-finite, in the classical sense, in each variable $x_i \in S$. Clearly, $H(t)$ is D-finite in the x_i's and t. Next, let us recall the closure properties of D-finite series in finitely many variables.

Theorem 0.1. (1) *The set of D-finite power series forms a $\mathbb{K}$-algebra of $\mathbb{K}[[x_1, \ldots, x_n]]$ for the usual product of series;*

(2) *If F is D-finite in $x_1, \ldots, x_n$ then for any finite subset of variables $x_{i_1}, \ldots, x_{i_k}$ the specialization of F at $x_{i_1} = \cdots = x_{i_k} = 0$ is D-finite in the remaining variables;*

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(3) *If P is a polynomial in $x_1, \ldots, x_n$, then* $\exp P(x)$ *is D-finite in the* $x_1, \ldots, x_n$*;*
(4) *If F and G are D-finite in the variables $x_1, \ldots, x_{m+n}$'s, then the Hadamard product $F \odot G$ with respect to $x_1, \ldots, x_n$ is D-finite in the $x_1, \ldots, x_{m+n}$;*
(5) *Assume that F is a D-finite series in $x_1, \ldots, x_n$. Let $f_1, \ldots, f_d$ be algebraic functions in $\mathbf{z} = z_1, \ldots, z_e$ which means there exist nonzero polynomials $q_1, \ldots, q_d \in \mathbb{K}[h, z_1, \ldots, z_e]$ such that $q_i(f_i(\mathbf{z}), z_1, \ldots, z_e) = 0$. Then*

$$G(\mathbf{z}, x_{d+1}, \ldots) = F(f_1(\mathbf{z}), \ldots, f_d(\mathbf{z}), x_{d+1}, \ldots, x_n)$$

is D-finite in the $\mathbf{z}, x_{d+1}, \ldots, x_n$.

The proofs of the first three properties are available in [13], and the fourth one is due to Lipshitz [11]. The last one is given in [9, Theorem 2.18, page 35]. It is straightforward to see that the above properties also hold for D-finite series in an infinitely number of variables.

For each $k \geq 1$, let $p_k(X)$ be the k-th power sum symmetric function in X. We say that a symmetric function is *D-finite* [6, page 272] if it is D-finite in the power sum symmetric functions p_k's. By item (5) of Theorem 0.1, a D-finite symmetric function is also D-finite in the x_i's. Since $H(t) = \exp\left(\sum_{k=1}^{\infty} p_k t^k / k\right)$, it is straightforward to see from item (3) of Theorem 0.1 that $H(1)$ is a D-finite symmetric function and $H(t)$ is D-finite in the p_k's and t.

Let $f \in \mathbb{K}[[X]]$ be a quasisymmetric function. We say that f is D-finite if it is D-finite in the fundamental (or monomial) quasisymmetric functions. The next proposition gives the relation between D-finite symmetric functions and D-finite quasisymmetric functions.

Proposition 0.2. *Let $f \in \mathbb{K}[[X]]$ be a symmetric function. Then f is a D-finite symmetric function if and only if it is a D-finite quasisymmetric functions.*

Proof. $\Longrightarrow$: Assume that f is a D-finite symmetric function. Since each power sum symmetric function is a $\mathbb{Q}$-linear combination of Schur functions, it follows from [13, Theorem 7.19.7] that each power sum symmetric function is a $\mathbb{Q}$-linear combination of fundamental quasisymmetric functions. By item (5) of Theorem 0.1, we see that f is D-finite in the quasisymmetric functions.

$\Longleftarrow$: Assume that f is a D-finite quasisymmetric function. Since each monomial symmetric function is a $\mathbb{Q}$-linear combination of power sum symmetric functions, it follows from item (5) of Theorem 0.1 that we just need to prove that f is D-finite in the monomial symmetric functions. Let t be a positive integer and λ be a partition of t. Set $f(m_\lambda) = f|_{m_\mu = 0, \mu \neq \lambda}$. Without loss of generality, it suffices to prove that $f(m_\lambda)$ satisfies a nontrivial linear ODE with polynomial coefficients in

m_λ. Let A be the set of all distince permutations of parts of λ. Set

$$M = \{M_\alpha \mid \alpha \in A\}.$$

Then f is D-finite in the monomial quasisymmetric functions M. Take $\alpha \in A$. Then there exists a non-negative integer r and polynomial $p_j \in \mathbb{K}[M]$ for $j = 0, \ldots, r$ with $p_r(M) \neq 0$ such that

$$p_r(M)\frac{\partial^r f(m_\lambda)}{\partial M_\alpha^r} + p_{r-1}(M)\frac{\partial^{r-1} f(m_\lambda)}{\partial M_\alpha^{r-1}} + \cdots + p_0(M)f(m_\lambda) = 0. \tag{1}$$

Since $m_\lambda = M_\alpha + \sum_{\beta\in A,\beta\neq\alpha} M_\beta$, we have that

$$\frac{\partial^k f(m_\lambda)}{\partial M_\alpha^k} = \frac{\partial^k f(m_\lambda)}{\partial m_\lambda^k} \quad \text{for each} \quad k \in \mathbb{N}.$$

Thus, (1) becomes

$$p_r(M)\frac{\partial^r f(m_\lambda)}{\partial m_\lambda^r} + p_{r-1}(M)\frac{\partial^{r-1} f(m_\lambda)}{\partial m_\lambda^{r-1}} + \cdots + p_0(M)f(m_\lambda) = 0. \tag{2}$$

Let $n = |M|$ and $\sigma \in \mathfrak{S}_n$. For each $p \in \mathbb{K}[[M]]$, we define

$$\sigma(p(M)) := p(M_{\alpha_{\sigma(1)}}, \ldots, M_{\alpha_{\sigma(n)}}).$$

Set $\tilde{p}_j = \sum_{\sigma\in\mathfrak{S}_n} p_j(\sigma(M))$ for $j = 0, \ldots, r$. Since $p_r \neq 0$, so is $\tilde{p}_r$. Moreover, it is straightforward to see that $\tilde{p}_j$ is a symmetric function of finite degree. Thus, we may assume that $\tilde{p}_j \in \mathbb{K}[m_\lambda, m_{\mu_1}, \ldots, m_{\mu_\ell}]$. For each $\sigma \in \mathfrak{S}_n$, applying σ to (2) and then taking the sum, we get

(3)

$$\tilde{p}_r(m_\lambda, m_{\mu_1}, \ldots, m_{\mu_\ell})\frac{\partial^r f(m_\lambda)}{\partial m_\lambda^r} + \tilde{p}_{r-1}(m_\lambda, m_{\mu_1}, \ldots, m_{\mu_\ell})\frac{\partial^{r-1} f(m_\lambda)}{\partial m_\lambda^{r-1}} + \cdots + \tilde{p}_0(m_\lambda, m_{\mu_1}, \ldots, m_{\mu_\ell})f(m_\lambda) = 0.$$

By taking the content over $\mathbb{K}[m_\lambda, m_{\mu_1}, \ldots, m_{\mu_\ell}]$, we may assume that $\tilde{p}_r, \ldots, \tilde{p}_0$ are relatively prime. By specifying $m_{\mu_i} = 0$ in (3) and taking the content over $\mathbb{K}[m_\lambda, m_{\mu_1}, \ldots, m_{\mu_\ell}]$ iteratively for $i \in \{1, \ldots, \ell\}$, we see that there exists $\bar{p}_j \in \mathbb{K}[m_\lambda]$ for $j = 0, \ldots, \bar{r}$ with $\bar{r} \leq r$ and $\bar{p}_{\bar{r}} \neq 0$ such that

$$\bar{p}_{\bar{r}}(m_\lambda)\frac{\partial^r f(m_\lambda)}{\partial m_\lambda^r} + \bar{p}_{\bar{r}-1}(m_\lambda)\frac{\partial^{r-1} f(m_\lambda)}{\partial m_\lambda^{r-1}} + \cdots + \bar{p}_0(m_\lambda)f(m_\lambda) = 0.$$

□

Lemma 0.3. *Let $a \in \mathbb{Z}, v_1, \ldots, v_d \in \mathbb{Z}$ and $t_1, \ldots, t_d$ be variables. Then the series*

$$F = \prod_{i=1}^{d} \sum_{\ell_i=-\infty}^{\infty} h_{a+v_1\ell_1+\cdots+v_d\ell_d} t_1^{\ell_1} \cdots t_d^{\ell_d}$$

is D-finite in the p_i's and t_j's.

Proof. Without loss of generality, we assume that $v_1, \dots, v_d \in \mathbb{Z} \setminus \{0\}$. Otherwise, by reordering the indexes, we assume $v_1, \dots, v_e \in \mathbb{Z} \setminus \{0\}$ and $v_{e+1} = \cdots = v_d = 0$. Then

$$F = \tilde{F} \cdot \prod_{i=e+1}^{d} \left(\frac{t_i^{-1}}{1 - t_i^{-1}} + \frac{1}{1 - t_i} \right),$$

where

$$\tilde{F} = \prod_{i=1}^{e} \sum_{\ell_i=-\infty}^{\infty} h_{a+v_1\ell_1+\cdots+v_e\ell_e} t_1^{\ell_1} \cdots t_e^{\ell_e}.$$

According to item (1) of Theorem 0.1, we only need to show that $\tilde{F}$ is D-finite.

Set $s_i = t_i^{1/v_i}$ for $i = 1, \dots, d$. Then $F = s_1^{-a} \cdot G$, where

$$G = \prod_{i=1}^{d} \sum_{\ell_i=-\infty}^{\infty} h_{a+v_1\ell_1+\cdots+v_d\ell_d} s_1^{a+v_1\ell_1} s_2^{v_2\ell_2} \cdots s_d^{v_d\ell_d}.$$

In light of item (1) and (5) of Theorem 0.1, it suffices to show that G is D-finite in the p_i's and s_j's. Let $z_1, \dots, z_d$ be several auxiliary variables. We may write

$$G = \left(L \odot \prod_{i=1}^{d} \sum_{\ell_i=-\infty}^{\infty} z_1^{a+v_1\ell_1} z_2^{v_2\ell_2} \cdots z_d^{v_d\ell_d} \right)_{z_1=\cdots=z_d=1}, \tag{4}$$

where

$$L = \prod_{i=1}^{d} \sum_{\ell_i=-\infty}^{\infty} h_{\ell_1+\ell_2+\cdots+\ell_d} (s_1z_1)^{\ell_1} (s_2z_2)^{\ell_2} \cdots (s_dz_d)^{\ell_d}.$$

Clearly, we see that

$$\prod_{i=1}^{d} \sum_{\ell_i=-\infty}^{\infty} z_1^{a+v_1\ell_1} z_2^{v_2\ell_2} \cdots z_d^{v_d\ell_d} = z_1^a \prod_{i=1}^{d} \left(\frac{z_i^{-v_i}}{1 - z_i^{-v_i}} + \frac{1}{1 - z_i^{v_i}} \right)$$

is D-finite in the p_i's, s_j's and z_k's. Therefore, it follows from item (2), (4) of Theorem 0.1 and (4) that we only need to prove that L is also D-finite in those variables. We may write

$$\begin{aligned} L &= \prod_{i=1}^{d} \sum_{\ell_i=-\infty}^{\infty} (\frac{s_1z_1}{s_dz_d})^{\ell_1} \cdots (\frac{s_{d-1}z_{d-1}}{s_dz_d})^{\ell_{d-1}} h_{\ell_1+\ell_2+\cdots+\ell_d} (s_dz_d)^{\ell_1+\ell_2+\cdots+\ell_d} \\ &= \left(\prod_{i=1}^{d-1} \sum_{\ell_i=-\infty}^{\infty} (\frac{s_iz_i}{s_dz_d})^{\ell_i} \right) \cdot \left(\sum_{\ell_d=-\infty}^{\infty} h_{\ell_d} (s_dz_d)^{\ell_d} \right) \\ &= \left(\prod_{i=1}^{d-1} \left(\frac{(s_iz_i)^{-1}}{1 - (s_iz_i)^{-1}} + \frac{s_iz_i}{1 - s_iz_i} \right) \right) \cdot H(s_dz_d). \end{aligned}$$

From the last identity and item (1) of Theorem 0.1, we conclude that L is D-finite in the p_i's, s_j's and z_k's. $\square$

Let V be a representation of a simple complex Lie algebra $\mathfrak{g}$ and $\mathrm{ch}_V(Y)$ be the Weyl character [8] of V in variables Y. Take the set of monomials whose sum is $\mathrm{ch}_V(Y)$ and denote it by $m_V(Y)$. Then $H[X.\,\mathrm{m}_V(Y)]$ is an element of the tensor product of symmetric functions in X and Laurent polynomials in Y. Denote by Δ the Laurent polynomial in Y which appears in the Weyl integration formula [1]. Then the product $\Delta\cdot H[X.\,\mathrm{m}_V(Y)]$ is also in the tensor product of symmetric functions in X and Laurent polynomials in Y. Since $\mathrm{ch}_V(Y)$ and Δ are Laurent polynomials in Y, we may assume that $Y = y_1,\dots,y_m$ are variables appeared in $\mathrm{ch}_V(Y)$ and Δ. A monomial in Y is then denoted by $Y^{\mathbf{e}} := y_1^{e_1}\cdots y_m^{e_m}$ for some $\mathbf{e} = (e_1,\dots,e_m)\in\mathbb{Z}^m$. Let $F\in\mathbb{K}[[X]][y_1^{\pm1},\dots,y_m^{\pm1}]$ and $\mathbf{e}\in\mathbb{Z}^m$, we denote the coefficient of $Y^{\mathbf{e}}$ in F by $[Y^{\mathbf{e}}](F)$.

Theorem 0.4. *The symmetric function $S = [Y^{\mathbf{0}}](\Delta\cdot H[X.\,\mathrm{m}_V(Y)])$ is D-finite.*

Proof. We first prove the claim in the case that $\Delta = Y^{\gamma}$ is a monomial in Y. Set $\mathrm{m}_V(Y) = \{Y^{\alpha_1},\dots,Y^{\alpha_s}\}$. Then

$$\begin{aligned}\Delta\cdot H[X.\,\mathrm{m}_V(Y)] &= Y^{\gamma}\cdot\prod_{i=1}^{s}\left(\prod_{j=1}^{\infty}\frac{1}{1-x_j\cdot Y^{\alpha_i}}\right)\\ &= Y^{\gamma}\cdot\prod_{i=1}^{s}\sum_{k_i=0}^{\infty}h_{k_i}(X)Y^{\alpha_i k_i}.\end{aligned}$$

Then we have

$$\begin{aligned}S &= [Y^{\mathbf{0}}](\Delta\cdot H[X.\,\mathrm{m}_V(Y)])\\ &= \sum_{\alpha_1k_1+\alpha_2k_2\cdots+\alpha_sk_s=-\gamma} h_{k_1}(X)\cdot h_{k_2}(X)\cdots h_{k_s}(X).\end{aligned}\tag{5}$$

Thus, in order to show that S is D-finite, we need to derive the solutions of the following system of linear Diophantine equations:

$$\alpha_1k_1+\alpha_2k_2\cdots+\alpha_sk_s = -\gamma,\tag{6}$$

where $k_i\in\mathbb{Z}$ is unknown. Set $A = (\alpha_1,\alpha_2,\dots,\alpha_s)\in\mathbb{Z}^{m\times s}$ and $C = -\gamma\in\mathbb{Z}^{m\times1}$. Then (14) can be written into the following matrix form:

$$AX = C.\tag{7}$$

There exist two unimodular matrices $U\in\mathbb{Z}^{m\times m}$ and $V\in\mathbb{Z}^{s\times s}$ such that $B = (b_{i,j}) = UAV$ is the Smith normal form of A with $b_{i,i}\neq 0$ for $i = 1,\dots,s-d$ and the other entries being zeros. Set $Y = V^{-1}X$ and $Q = UC = (q_1,\dots,q_m)^{\intercal}$. Then (7) can be rewritten as

$$BY = Q.\tag{8}$$

If there exists $i \in \{1, \ldots, s-d\}$ such that $b_{i,i} \nmid q_i$, then (8) has no solution. In this case, we have $S = 0$, which is D-finite in the x_i's.

If $b_{i,i} \mid q_i$ for $i = 1, \ldots, s-d$, then the solutions of (7) are

$$V \begin{bmatrix} \frac{q_1}{b_{1,1}} \\ \vdots \\ \frac{q_{s-d}}{b_{s-d,s-d}} \\ \ell_1 \\ \vdots \\ \ell_d \end{bmatrix},$$

where $\ell_1, \ldots, \ell_d$ are arbitrary integers. It implies that the solutions of (14) are

$$k_i = a_i + v_{i,1}\ell_1 + \cdots + v_{i,d}\ell_d,$$

where $a_i, v_{i,j} \in \mathbb{Z}$ for $i = 1, \ldots, s$ and $j = 1, \ldots, d$. By (5), we see that

$$S = \prod_{i=1}^{s} \prod_{j=1}^{d} \sum_{\ell_j=-\infty}^{\infty} h_{a_i+v_{i,1}\ell_1+\cdots+v_{i,d}\ell_d} \tag{9}$$

$$= (F_1 \odot F_2 \odot \cdots \odot F_s)_{t_1=\cdots=t_d=1}, \tag{10}$$

where

$$F_i = \prod_{j=1}^{d} \sum_{\ell_j=-\infty}^{\infty} h_{a_i+v_{i,1}\ell_1+\cdots+v_{i,d}\ell_d} t_1^{\ell_1} \cdots t_d^{\ell_d} \quad \text{for} \quad i = 1, \ldots, s.$$

By Lemma 0.3, we see that F_i is D-finite in the p_j's and t_k's. In light of item (2), (4) of Theorem 0.1 and (10), we conclude that S is D-finite in the p_i's.

It follows from item (1) of Theorem 0.1 that the claim also holds when Δ is a Laurent polynomial in Y. □

By Proposition 0.2, the symmetric function in the above theorem is also a D-finite quasisymmetric function.

Example 0.5. *Let V be an irreducible representation of $\mathfrak{g} := SL(2)$ such that $\mathrm{ch}_V(Y) = y + 1/y$ and $\Delta = 1 - y^2$. Then*

$$\begin{aligned} H[X.\,\mathrm{m}_V(Y)]) &= \left(\prod_{i=1}^{\infty} \frac{1}{1-x_i y}\right) \cdot \left(\prod_{i=1}^{\infty} \frac{1}{1-x_i y^{-1}}\right) \\ &= \left(\sum_{k=0}^{\infty} h_k y^k\right) \cdot \left(\sum_{k=0}^{\infty} h_k y^{-k}\right). \end{aligned}$$

Thus, by Theorem 0.4, the symmetric function

$$\begin{aligned} S &= [Y^{\mathbf{0}}](\Delta \cdot H[X.\mathrm{m}_V(y)]) \\ &= \sum_{k=0}^{\infty} \left(h_k^2 - h_{k-1}h_{k+1}\right) \\ &= \sum_{k=0}^{\infty} \left(h_k^2 - h_k h_{k+2}\right) \end{aligned}$$

is D-finite in the p_i's. For each $n \in \mathbb{N}$, set $S(p_n) = S|_{p_k=0,k\neq n}$.

When $n = 1$, we have

$$\begin{aligned} S(p_1) &= \sum_{i=0}^{\infty} \frac{p_1^{2i}}{i!^2} \left(1 + \frac{p_1^2}{(i+1)(i+2)}\right) \\ &= I_0(2p_1) + I_2(2p_1), \end{aligned}$$

where $I_k(z)$ is the modified Bessel function of the first kind. By item (1) of Theorem 0.1, it is straightforward to see that $S(p_1)$ satisfies the following fourth-order linear ordinary differential equation (ODE):

$$\begin{aligned} p_1^3 \frac{d^4 S(p_1)}{dp_1^4} + 6p_1^2 \frac{d^3 S(p_1)}{dp_1^3} + (3p_1 - 8p_1^3) \frac{d^2 S(p_1)}{dp_1^2} & \\ - (3 + 24p_1^2) \frac{dS(p_1)}{dp_1} + 16p_1^3 S(p_1) &= 0. \end{aligned}$$

When $n = 2$, we have

$$\begin{aligned} S(p_2) &= \sum_{i=0}^{\infty} \frac{p_2^{2i}}{4^i i!^2} \left(1 + \frac{p_2}{2(i+1)}\right) \\ &= I_0(p_2) + I_1(p_2), \end{aligned}$$

which satisfies the following fourth-order linear ODE:

$$\begin{aligned} p_2^2 \frac{d^4 S(p_2)}{dp_2^4} + 6p_2 \frac{d^3 S(p_2)}{dp_2^3} + (6 - 2p_2^2) \frac{d^2 S(p_2)}{dp_2^2} - 6p_2 \frac{dS(p_2)}{dp_2} & \\ - (3 - p_2^2) S(p_2) &= 0. \end{aligned}$$

When $n > 2$, we have $S(p_n) = \sum_{i=0}^{\infty} p_n^{2i}/(n^{2i} i!^2) = I_0(2p_n/n)$, which satisfies the following second-order linear ODE:

$$n^2 p_n \frac{d^2 S(p_n)}{dp_n^2} + n^2 \frac{dS(p_n)}{dp_n} - 4p_n S(p_n) = 0.$$

In Example 0.7, we will show that the fake degree sequence associated to S above is q-holonomic and closely related to q-Catalan numbers [4].

In [16], the authors proved that the symmetric function in Theorem 0.4 is equal to

$$S = \sum_{n=0}^{\infty} \mathbf{fr}((\otimes^n V)^{\mathfrak{g}}), \tag{11}$$

where $\mathbf{fr}(V)$ is the Frobenius character [13, § 7.18] of the representation V of $\mathfrak{S}_n$, which is a homogeneous symmetric function of degree n and is an invariant of the representation V.

Let q be a transcendental indeterminate over the field $\mathbb{K}$. For $n \in \mathbb{N}$, the q-integer $[n]_q$ is the polynomial

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1},$$

and the q-factorial $[n]_q! := [n]_q \cdot [n-1]_q \cdots [1]_q$. For $k_1, \ldots, k_r \in \mathbb{Z}$, the q-multinomial coefficient is defined by

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_r \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[k_1]_q![k_2]_q!\cdots[k_r]_q!} & \text{if} \quad k_i \geq 0, \quad \text{and} \quad \sum_{i=1}^r k_i = n, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the fake degree function

$$\mathbf{fd} : \Lambda \longrightarrow \mathbb{K}(q),$$

which a linear map from symmetric functions to polynomials in q. The interpretation in terms of representation theory is that if V has an action of $\mathfrak{S}_n$ then the fake degree of the Frobenius character is a polynomial which describes the action of the long cycle. This can be defined on the basis of complete homogeneous functions and extended by linearity. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be an integer partition. Then the evaluation of $\mathbf{fd}$ at h_λ is defined by:

$$\mathbf{fd}(h_\lambda) = \begin{bmatrix} |\lambda| \\ \lambda_1, \lambda_2, \ldots, \lambda_s \end{bmatrix}_q.$$

The fake degree map is not a ring homomorphism but has the following property: Let f_r and g_s be a homogeneous symmetric functions of degre r and s, respectively. Then

$$\mathbf{fd}(f_r g_s) = \begin{bmatrix} r+s \\ r \end{bmatrix}_q \mathbf{fd}(f_r)\, \mathbf{fd}(g_s).$$

A univariate sequence $(b_n(q))_{n\in\mathbb{N}}$ is called *q-holonomic* [5] if it satisfies a nontrivial linear q-difference equation with coefficients that are polynomials in q and q^n; that is, there exists a non-negative integer r and bivariate polynomial $c_j(x, y) \in \mathbb{K}[x, y]$ for $j = 0, \ldots, r$ with $c_r(x, y) \neq 0$ such that for each $n \in \mathbb{N}$ the following identity holds:

$$c_r(q, q^n)b_{n+r}(q) + c_{r-1}(q, q^n)b_{n+r-1}(q) + \cdots + c_0(q, q^n)b_n(q) = 0.$$

Similar to D-finite functions, the class of q-holonomic sequences satisfy closure properties under certain operations such as addition and multiplication. For instance, see [9, Section 2.3] for details.

Given $\mathfrak{g}$ and $V \in \mathcal{Rep}(\mathfrak{g})$. We define $f_n(q) = \mathbf{fd}(\mathbf{fr}((\otimes^n V)^{\mathfrak{g}}))$ for each $n \in \mathbb{N}$ and call $(f_n(q))_{n\in\mathbb{N}}$ the fake degree sequence associated to $\mathfrak{g}$ and V. A natural question is whether this sequence is q-holonomic or not. In the below theorem, we give an answer for this problem.

Theorem 0.6. *Given a simple complex Lie algebra $\mathfrak{g}$ and $V \in \mathcal{R}ep(\mathfrak{g})$. Then the fake degree sequence associated to $\mathfrak{g}$ and V is q-holonomic.*

Proof. Let $\mathrm{ch}_V(Y)$ be the Weyl character of V in Y and $m_V(Y)$ be the set of monomials whose sum is $\mathrm{ch}_V(Y)$. Set Δ to be the Laurent polynomial in Y which appears in the Weyl integration formula. We first prove the claim in the case that $\Delta = Y^\gamma$ is a monomial in Y. Set $\mathrm{m}_V(Y) = \{Y^{\alpha_1}, \ldots, Y^{\alpha_s}\}$. Then

$$\begin{aligned} S &= [Y^{\mathbf{0}}](\Delta \cdot H[X.\,\mathrm{m}_V(Y)]) \\ &= \sum_{\alpha_1 k_1 + \alpha_2 k_2 \cdots + \alpha_s k_s = -\gamma} h_{k_1}(X) \cdot h_{k_2}(X) \cdots h_{k_s}(X). \end{aligned}$$

By the arguments in the proof of Theorem 0.4, there exist $a_i, v_{i,j} \in \mathbb{Z}$ for $i = 1, \ldots, s$ and $j = 1, \ldots, d$ such that

$$S = \prod_{i=1}^{s} \prod_{j=1}^{d} \sum_{\ell_j=-\infty}^{\infty} h_{a_i + v_{i,1}\ell_1 + \cdots + v_{i,d}\ell_d}.$$

Set $a = \sum_{i=1}^{s} a_i \in \mathbb{Z}$, and $v_j = \sum_{i=1}^{s} v_{i,j} \in \mathbb{Z}$ for $j = 1, \cdots, d$. Let

$$\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{Z}^d, \qquad \boldsymbol{\ell} = (\ell_1, \ldots, \ell_d) \in \mathbb{Z}^d,$$

and $\mathbf{v}_i = (v_{i,1}, \ldots, v_{i,d}) \in \mathbb{Z}^d$ for $i = 1, \ldots, s$. On the other hand, since $S = \sum_{n=0}^{\infty} \mathbf{fr}((\otimes^n V)^{\mathfrak{g}})$, we see that

$$\mathbf{fr}((\otimes^n V)^{\mathfrak{g}}) = \sum_{a + \mathbf{v}\cdot\boldsymbol{\ell} = n} \prod_{i=1}^{s} h_{a_i + \mathbf{v}_i \cdot \boldsymbol{\ell}} \quad \text{for} \quad n \geq 0.$$

Then the n-th term of the fake degree sequence associated to $\mathfrak{g}$ and V is

$$f_n(q) = \mathbf{fd}(\mathbf{fr}((\otimes^n V)^{\mathfrak{g}})) = \sum_{a + \mathbf{v}\cdot\boldsymbol{\ell} = n} r(\boldsymbol{\ell}, n), \tag{12}$$

where

$$r(\boldsymbol{\ell}, n) = \begin{bmatrix} & n & \\ a_1 + \mathbf{v}_1 \cdot \boldsymbol{\ell}, & \cdots, & a_s + \mathbf{v}_s \cdot \boldsymbol{\ell} \end{bmatrix}_q.$$

Set $g = \gcd(v_1, \ldots, v_d)$. Without loss of generality, we may assume that $g > 0$. If $g \nmid n - a$, then it follows from (12) that $f_n(q) = 0$. For $k \geq k_0 := \lfloor -a/g \rfloor$, we define $\tilde{f}_k(q) = f_{a+gk}(q)$. Then we have

$$f_n(q) = \begin{cases} \tilde{f}_k(q) & \text{if} \quad n = a + gk, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Next, we prove that the sequence $(\tilde{f}_k(q))_{k \geq k_0}$ is q-holonomic. For each $k \geq k_0$, consider the following linear Diophantine equation:

$$v_1\ell_1 + v_2\ell_2 + \cdots + v_d\ell_d = gk. \tag{14}$$

By the extended Euclidean algorithm, there exist $u_{i,j} \in \mathbb{Z}, 1 \le i \le d$, $0 \le j \le i$, which are independent of k, with $u_{d,d} = 0$ such that the solutions of (14) are

$$\ell_i = u_{i,0}k + u_{i,1}t_1 + \cdots + u_{i,i}t_i, \quad i = 1, \ldots, d, \tag{15}$$

where t_i is an arbitrary integer. Set $\mathbf{t} = (t_1, \ldots, t_{d-1}) \in \mathbb{Z}^{d-1}$. Substituting (15) into $r(\boldsymbol{\ell}, a + gk)$, we denote the corresponding term by $\tilde{r}(\mathbf{t}, k)$. Then we can write $\tilde{f}_k(q)$ as

$$\tilde{f}_k(q) = \sum_{\mathbf{t}} \tilde{r}(\mathbf{t}, k). \tag{16}$$

Since $\tilde{r}(\mathbf{t}, k)$ is a proper q-hypergeometric term in $\mathbf{t}$ and k, it follows from [17, section 5.2] that $(\tilde{f}_k(q))_{k \ge k_0}$ is a q-holonomic. Moreover, it follows from the proof of [5, Theorem 1] that there exists a nontrivial q-difference equation for $\tilde{f}_k(q)$ of the following form:

$$p_{\tilde{d}}(q, q^{gk})\tilde{f}_{k+\tilde{d}}(q) + p_{\tilde{d}-1}(q, q^{gk})\tilde{f}_{k+\tilde{d}-1}(q) + \cdots + p_0(q, q^{gk})\tilde{f}_k(q) = 0, \tag{17}$$

where $p_j(x, y) \in \mathbb{K}[x, y]$ for $j = 0, \ldots, \tilde{d}$. Then it follows from (13) and (17) that the fake degree sequence satisfies

$$\begin{aligned} p_{\tilde{d}}(q, q^{n-a}) f_{n+g\tilde{d}}(q) + p_{\tilde{d}-1}(q, q^{n-a}) f_{n+g(\tilde{d}-1)}(q) \\ + \cdots + p_0(q, q^{n-a}) f_n(q) = 0. \end{aligned}$$

By clearing denominators of the above equation, we see that $(f_n(q))_{n \in \mathbb{N}}$ is indeed q-holonomic.

Since the class of q-holonomic sequences is closed under the $\mathbb{K}$-linear combination, the claim also holds when Δ is a Laurent polynomial in Y. □

Given a simple complex Lie algebra $\mathfrak{g}$ and $V \in \mathcal{R}ep(\mathfrak{g})$. By the integral expression (16) of the fake degree sequence associated to $\mathfrak{g}$ and V, we can utilize the method of creative telescoping [18] to derive the corresponding linear q-difference equations with polynomial coefficients.

Example 0.7. *Let V be an irreducible representation of $\mathfrak{g} := SL(2)$ such that $\mathrm{ch}_V(Y) = y + 1/y$ and $\Delta = 1 - y^2$. Consider the symmetric function in Example 0.5:*

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \mathbf{fr}((\otimes^n V)^{\mathfrak{g}}) \\ &= \sum_{k=0}^{\infty} \left(h_k^2 - h_{k-1}h_{k+1}\right). \end{aligned}$$

For each $k \in \mathbb{N}$, set

$$\begin{aligned} g_k(q) &= \begin{bmatrix} 2k \\ k \end{bmatrix}_q - \begin{bmatrix} 2k \\ k+1 \end{bmatrix}_q \\ &= q^k \cdot \mathrm{Cat}(k;q), \end{aligned}$$

where

$$\mathrm{Cat}(k;q) = \frac{1}{[k+1]_q} \begin{bmatrix} 2k \\ k \end{bmatrix}_q$$

is the MacMahon q-analog of the Catalan number. Then for $n \in \mathbb{N}$, the n-th term of the fake degree sequence associated to V and $\mathfrak{g}$ is

$$f_n(q) = \begin{cases} g_k(q) & \text{if} \quad n = 2k, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 0.6, the fake degree sequence $(f_n)_{n\in\mathbb{N}}$ is q-holonomic and satisfies the following fourth-order linear q-difference equation:

$$\begin{aligned} &\left(q^{n+6}-1\right) f_{n+4}(q) \\ &\qquad - q(q+1)\left(q^{n+3}-1\right)\left(q^{n+4}+1\right) f_{n+2}(q) \\ &\qquad\qquad + q^3\left(q^{n+1}-1\right)\left(q^{n+2}-1\right)\left(q^{n+3}-1\right) f_n(q) = 0. \end{aligned}$$

Example 0.8. *Let V be an irreducible representation of $\mathfrak{g} := SL(2)$ such that $\mathrm{ch}_V(Y) = y^2+1+y^{-2}$ and $\Delta = 1-y^2$. Consider the symmetric function:*

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \mathbf{fr}((\otimes^n V)^{\mathfrak{g}}) \\ &= [Y^{\mathbf{0}}](\Delta \cdot H[X.\mathrm{m}_V(Y)]) \\ &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \left(h_{k_1}^2 h_{k_2} - h_{k_1}h_{k_1+1}h_{k_2}\right). \end{aligned}$$

Then for $n \in \mathbb{N}$, the n-th term of the fake degree sequence associated to V and $\mathfrak{g}$ is

$$\begin{aligned} f_n(q) &= \sum_{2k_1+k_2=n} \begin{bmatrix} n \\ k_1, k_1, k_2 \end{bmatrix}_q - \sum_{2k_1+k_2+1=n} \begin{bmatrix} n \\ k_1, k_1+1, k_2 \end{bmatrix}_q \\ &= \sum_{t=-\infty}^{\infty} \left(\begin{bmatrix} n \\ n-t, n-t, -n+2t \end{bmatrix}_q - \begin{bmatrix} n \\ n-1-t, n-t, -n+1+2t \end{bmatrix}_q \right) \\ &= \sum_{t=-\infty}^{\infty} \frac{q^{n-t} - q^{-n+1+2t}}{1-q^{-n+1+2t}} \begin{bmatrix} n \\ n-t, n-t, -n+2t \end{bmatrix}_q . \end{aligned}$$

Using Koutschan's `Mathematica` *package* `HolonomicFunctions.m` [10] *that implements Chyzak's algorithm* [2], *we derive the following sixth-order linear q-difference equation for the fake degree sequence* $(f_n)_{n\in\mathbb{N}}$*:*

$$\begin{aligned}\left(q^{n+7}-1\right)f_{n+6}(q) - 2q\left(q^{n+5}-1\right)f_{n+5}(q)\\ -\left(q^{n+5}-1\right)\left(2q^{n+6}-q^2+q+1\right)f_{n+4}(q)\\ -q(q+1)\left(q^{n+4}-1\right)\left(q^{n+5}-1\right)\left(q^{n+5}+2\right)f_{n+3}(q)\\ -q\left(q^2+q-1\right)\left(q^{n+3}-1\right)\left(q^{n+4}-1\right)\left(q^{n+5}-1\right)f_{n+2}(q)\\ +2q^2\left(q^{n+2}-1\right)\left(q^{n+3}-1\right)\left(q^{n+4}-1\right)\left(q^{n+5}-1\right)f_{n+1}(q)\\ +q^3\left(q^{n+1}-1\right)\left(q^{n+2}-1\right)\left(q^{n+3}-1\right)\left(q^{n+4}-1\right)\left(q^{n+5}-1\right)f_n(q)=0.\end{aligned}$$

Example 0.9. *Let V be the four-dimensional defining representation of $\mathfrak{g} := Sp(4)$ such that*

$$\mathrm{ch}_V(Y) = y_2 + y_2^{-1} + y_1 y_2^{-1} + y_1^{-1} y_2,$$

and

$$\Delta = y_1^{-2}y_2^{-2} - y_2^{-4} - y_1^{-3} + y_1^{-3}y_2^2 + y_1 y_2^{-4} - y_1 y_2^{-2} - y_1^{-2}y_2^2 + 1.$$

The dimension of the invariant subspace of $\otimes^{2n}V$ gives A005700.

$$\begin{aligned}S &= \sum_{n=0}^{\infty} \mathbf{fr}((\otimes^n V)^{\mathfrak{g}})\\ &= [Y^{\mathbf{0}}](\Delta \cdot H[X.\,\mathrm{m}_V(Y)])\\ &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left(h_{k_1}h_{k_1+4}h_{k_2}h_{k_2+2} - h_{k_1}h_{k_1+4}h_{k_2}^2 - h_{k_1}h_{k_1+3}h_{k_2}h_{k_2+3}\right.\\ &\qquad + h_{k_1}h_{k_1+1}h_{k_2}h_{k_2+3} + h_{k_1}h_{k_1+3}h_{k_2}h_{k_2+1} - h_{k_1}h_{k_1+1}h_{k_2}h_{k_2+1}\\ &\qquad \left. - h_{k_1}^2 h_{k_2}h_{k_2+2} + h_{k_1}^2 h_{k_2}^2\right). \end{aligned} \tag{18}$$

Let $(f_n)_{n\in\mathbb{N}}$ be the fake degree sequence associated to V and $\mathfrak{g}$. For $n \geq 0$, it follows from (18) *that $f_{2n+1}(q) = 0$. Moreover, the $2n$-th term of the fake degree sequence is*

$$\begin{aligned}f_{2n}(q) = &\sum_{k_1+k_2=n-3}\begin{bmatrix}2n\\ k_1,k_1+4,k_2,k_2+2\end{bmatrix}_q - \sum_{k_1+k_2=n-2}\begin{bmatrix}2n\\ k_1,k_1+4,k_2,k_2\end{bmatrix}_q\\ &- \sum_{k_1+k_2=n-3}\begin{bmatrix}2n\\ k_1,k_1+3,k_2,k_2+3\end{bmatrix}_q + \sum_{k_1+k_2=n-2}\begin{bmatrix}2n\\ k_1,k_1+1,k_2,k_2+3\end{bmatrix}_q\\ &+ \sum_{k_1+k_2=n-2}\begin{bmatrix}2n\\ k_1,k_1+3,k_2,k_2+1\end{bmatrix}_q - \sum_{k_1+k_2=n-1}\begin{bmatrix}2n\\ k_1,k_1+1,k_2,k_2+1\end{bmatrix}_q\\ &- \sum_{k_1+k_2=n-1}\begin{bmatrix}2n\\ k_1,k_1,k_2,k_2+2\end{bmatrix}_q + \sum_{k_1+k_2=n}\begin{bmatrix}n\\ k_1,k_1,k_2,k_2\end{bmatrix}_q\end{aligned}$$

$$
\begin{aligned}
= \sum_{t=-\infty}^{\infty} \Bigg(& \begin{bmatrix} 2n \\ 2n-t-6, 2n-t-2, -n+t+3, -n+t+5 \end{bmatrix}_q \\
& - \begin{bmatrix} 2n \\ 2n-t-4, 2n-t, -n+t+2, -n+t+2 \end{bmatrix}_q \\
& - \begin{bmatrix} 2n \\ 2n-t-6, 2n-t-3, -n+t+3, -n+t+6 \end{bmatrix}_q \\
& + \begin{bmatrix} 2n \\ 2n-t-4, 2n-t-3, -n+t+2, -n+t+5 \end{bmatrix}_q \\
& + \begin{bmatrix} 2n \\ 2n-t-4, 2n-t-1, -n+t+2, -n+t+3 \end{bmatrix}_q \\
& - \begin{bmatrix} 2n \\ 2n-t-2, 2n-t-1, -n+t+1, -n+t+2 \end{bmatrix}_q \\
& - \begin{bmatrix} 2n \\ 2n-t-2, 2n-t-2, -n+t+1, -n+t+3 \end{bmatrix}_q \\
& + \begin{bmatrix} 2n \\ 2n-t, 2n-t, -n+t, -n+t \end{bmatrix}_q \Bigg).
\end{aligned}
$$

Using the method of creative telescoping and the closure properties of the class of q-holonomic sequences, we find a twentieth-order linear q-difference equation for $(f_{2n}(q))_{n\in\mathbb{N}}$, which is given in [15].

Next, let us recall a theorem about scalar products of D-finite symmetric functions.

Theorem 0.10. *(Gessel* [6, Corollary 8]*) Let f and g be symmetric functions which are D-finite in the p_i's and in other variables t_j's. Assume that g involves only finitely many of the p_i's. Then $\langle f, g\rangle$ is D-finite in the t_j's as along as it is well defined as a formal power series.*

Note that the above result also holds for $\langle g, f\rangle$ provided that g involves only finitely many of the p_i's.

Let G be a reductive group and V be a finite dimensional representation. For each $r \geq 0$, we denote the Frobenius character of the G-invariant subspace of $\otimes^r V$ by $I_r(V)$ [14]. Let P be a polynomial functor of degree k; for instance, the k-th symmetric power functor or the k-th alternating power functor. We denote the character by $\mathbf{ch}(P)$, which is a symmetric function of degree k.

Proposition 0.11. *For each $r, k \geq 0$. Assume that $I_{rk}(V)[Y]$ is a D-finite symmetric function. Then*

$$f(X) := \langle h_r[X.\,\mathbf{ch}(P[Y])], I_{rk}(V)[Y]\rangle_Y$$

is also a D-finite symmetric function.

Proof. Since $h_r[X.\,\mathbf{ch}(P[Y])]$ and $I_{rk}(V)[Y]$ are both symmetric functions in X and Y, we see that $f(X)$ is a symmetric function in X. Since $\mathbf{ch}(P[Y])$ is a symmetric function of degree k, it follows that $h_r[X.\,\mathbf{ch}(P[Y])]$ is a symmetric function of degree $r(k+1)$. Thus, the symmetric function $h_r[X.\,\mathbf{ch}(P[Y])]$ only involves finitely many $p_i(Y)$'s. Since $h_r = \sum_\lambda p_\lambda/z_\lambda$, where λ runs over all partitions of r and $z_\lambda \in \mathbb{Z}_{>0}$, it is straightforward to see that $h_r[X.\,\mathbf{ch}(P[Y])]$ is D-finite in the $p_i(X)$'s and $p_j(Y)$'s. Taking $t_j = p_j(X)$ in Theorem 0.10, we conclude that the claim holds. $\square$

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