① §11.2 Series

Q Consider

$$\pi = 3.14159265\cdots$$

$$= 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

In general, we add terms of $\{a_n\}_{n=1}^{\infty}$ by

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an (infinite) series and is denoted by

$\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

Question: What is the meaning of the sum of infinite term?

It is impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because the partial sums $1 = 1,\ 1+2 = 3,\ 1+2+3 = 6,\ \cdots,\ 1+2+3+\cdots+n = \frac{n(n+1)}{2} \to \infty$ as $n \to \infty$.

~~However,~~ If we add terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots,$$

then the partial sums

$$\frac{1}{2},\quad \frac{1}{2}+\frac{1}{4} = \frac{3}{4},\quad \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8},\quad \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{15}{16},\quad \cdots,\quad \frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = 1 - \frac{1}{2^n} \to 1 \text{ as } n \to \infty$$

Consider partial sums of $\{a_n\}_{n=1}^{\infty}$:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

$$\vdots$$

These partial sums form a new sequence $\{s_n\}$.

Def 2 Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 \cdots$, let s_n be its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the sum of the series. If $\{s_n\}$ is divergent, then the series is called divergent.

Note: ~~the sum of a series~~

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Ex 2. Consider the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \quad a \neq 0$$

If $r = 1$, then $s_n = \underbrace{a + a + \cdots + a}_{n} = na \to \pm\infty$. The series diverges in this case

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} \qquad ①$$

$$r s_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n \qquad ②$$

① − ②, we get

$$s_n - r s_n = a - ar^n$$

Thus,

$$s_n = \frac{a(1-r^n)}{1-r} \qquad ③$$

If $-1 < r < 1$, then $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n\to\infty} r^n = \frac{a}{1-r}$$

③ If $r \le -1$ or $r > 1$, then $\{r^n\}$ is divergent.

By ③, $\lim_{n\to\infty} S_n$ does not exist. Thus, the geometric series diverges in those case.

$$\frac{s}{a} = \frac{a}{a-ar}$$

$$\Rightarrow s = \frac{a}{1-r}$$

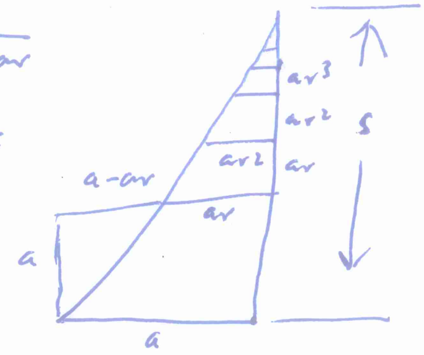


Figure for Ex 2.

In summary,

The geometric series

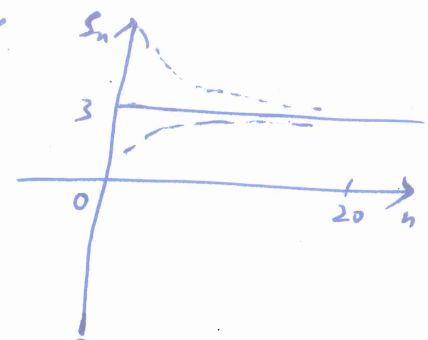
$$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \ge 1$, the geometric series is divergent

Ex 3. Find the sum of the geometric series



$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

intial term $a = 5$, $r = \frac{-\frac{10}{3}}{5} = -\frac{2}{3}$

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1-(-\frac{2}{3})} = 3$$

Ex 8 Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

By partial fraction decomposition,

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Thus, we have

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

telescoping sum

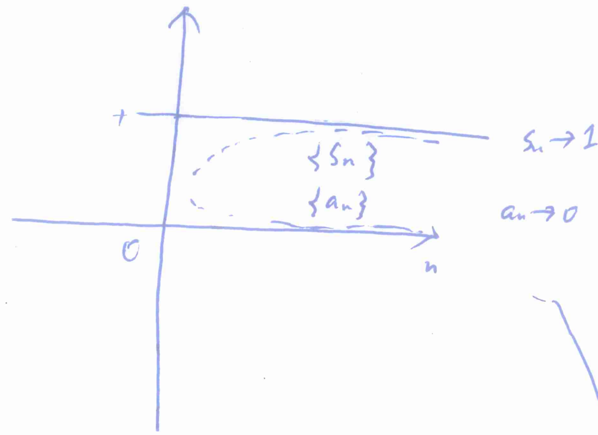
$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

and so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

i.e., $\sum_{i=1}^{\infty} \frac{1}{n(n+1)} = 1$.



~~Sho~~ Ex 9. Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Consider the partial sums $S_2, S_4, S_8, \cdots S_{2^n}, \cdots$.

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$S_8 = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

Similarly, we can show that

$$S_{2^n} > 1 + \frac{n}{2}$$

It implies $S_{2^n} \to \infty$ as $n \to \infty$ and so $\{S_n\}$ is divergent.

Theorem 6 If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$

Pf: Let $S_n = a_1 + a_2 + \cdots + a_n$

Then $a_n = S_n - S_{n-1}$

Since $\sum a_n$ is convergent, $\{S_n\}$ is convergent.

Let $\lim_{n \to \infty} S_n = S$. Then $\lim_{n \to \infty} S_{n-1} = S$

Thus, $\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1})$

$$= \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1}$$

$$= S - S$$

$$= 0.$$

5

Note: the converse of Theorem 6 is not true in general.

Ex: $a_n = \frac{1}{n} \to 0$ as $n \to \infty$

$\sum \frac{1}{n}$ is divergent.

Corollary 7 (Test for Divergence) If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then series $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex 10 Show that $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}}$$

$$= \frac{1}{5+\lim_{n\to\infty}\frac{4}{n^2}}$$

$$= \frac{1}{5+0}$$

$$= \frac{1}{5} \neq 0$$

By Corollary 7, $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Note: if $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ might converge or it might diverge.

Theorem 8 If $\sum a_n$ and $\sum b_n$ are convergent, and c is a constant, then so are the series $\sum c a_n$, $\sum (a_n + b_n)$, and

(i) $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(~~The proof~~ it follows from limit laws for sequences)

Ex 11. Find the sum of $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

By Ex 8, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

⑥ $$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)=\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\frac{1}{2^n}$$
$$=3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\sum_{n=1}^{\infty}\frac{1}{2^n}$$
$$=3\cdot 1+1=4$$

Note: A finite number of terms doesn't affect the convergence or divergence of a series

Assume that $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.