① # §11.3 The Integral Test and Estimates of Sums

In general, it is impossible to find the exact sum of a series.

Goal: develop tests to determine whether a series is convergent or divergent without explicitly finding its sum.

First test: Improper Integrals.

Consider:

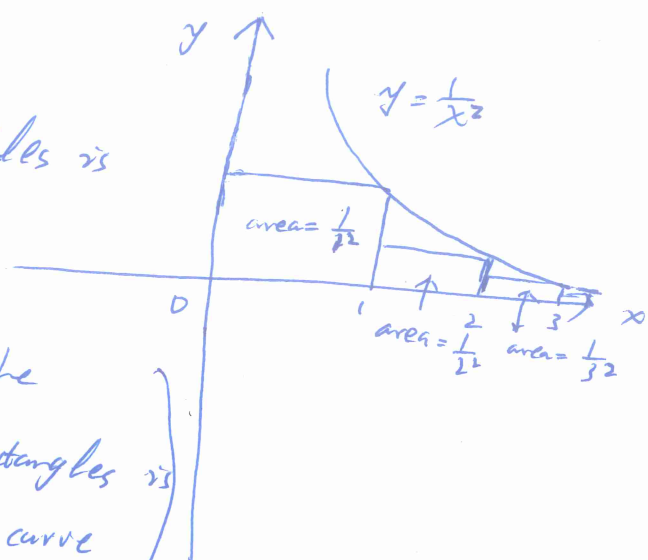
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

(numerical computation for the first few partial sums suggests the series is convergent)

~~Geometric point of view~~

n	$S_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439

Geometric point of view:



The sum of areas of the rectangles is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

(Except the first rectangle, the total area of the remaining rectangles is smaler the the area under the curve $y = \frac{1}{x^2}$ for $x \geq 1$, which is $\int_1^{+\infty} \frac{dx}{x^2}$)

We have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{1}{1^2} + \int_1^{\infty} \frac{1}{x^2} dx = 2$$

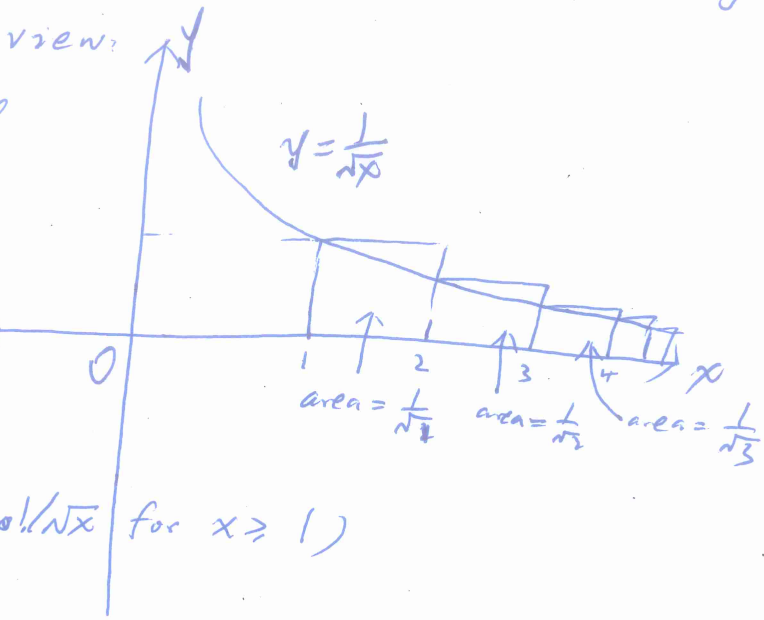
② Since $\left\{ S_n = \sum_{i=1}^{n} \frac{1}{n^2} \right\}_{n=1}^{\infty}$ is increasing, it follows from Monotonic Sequence Theorem that $\sum \frac{1}{n^2}$ is convergent.

Consider

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

Numerical computation for S_n's suggests $\sum \frac{1}{\sqrt{n}}$ diverges.

Geometric point of view:



The sum of areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

(The total area is $\geq$ that under the curve $y = 1/\sqrt{x}$ for $x \geq 1$)

We have

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}} dx = \infty$$

Thus, $\sum \frac{1}{\sqrt{n}}$ is divergent.

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then $\sum a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ is convergent.

Note: It is not necessary to start the series or the integral at $n=1$. Ex: in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \quad \text{we use} \quad \int_4^{\infty} \frac{dx}{(x-3)^2}$$

Ex 2. For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

If $p<0$, then $\lim_{n\to\infty} \left(\frac{1}{n^p}\right) = \infty$. If $p=0$, $\lim_{n\to\infty} \frac{1}{n^0} = \lim_{n\to\infty} 1 = 1$

So, $\sum \frac{1}{n^p}$ diverges if $p \leq 0$

③ If $p>0$, $f(x)=\frac{1}{x^p}$ is continuous, positive, and decreasing on $[1,\infty)$. Recall:

$$\int_1^\infty \frac{dx}{x^p} \text{ converges if } p>1 \text{ and diverges if } p\le 1$$

By the Integral Test, $\sum 1/n^p$ converges if $p>1$ and diverges if $0<p\le 1$.

In summary,

The p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p\le 1$.

• Estimating the Sum of a Series.

Assume $\sum a_n$ is convergent and we want find an approximation to the sum s? We can use partial sum s_n to approximate s.

Question: how good is such an approximation?

Idea: estimate the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

< from geometric point of view

Assume f is positive, continuous, decreasing on $[1,\infty)$, and $a_n = f(n)$, $n\ge 1$.

By Figure 3,

$$R_n \le \int_n^\infty f(x)dx$$

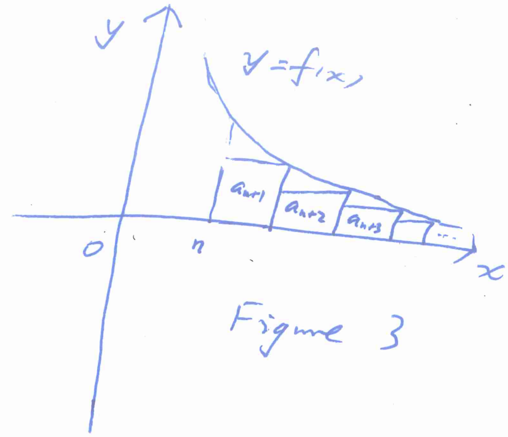


Figure 3

④ Similarly, by Figure 4, we have

$$R_n \geq \int_{n+1}^{\infty} f(x)\,dx$$

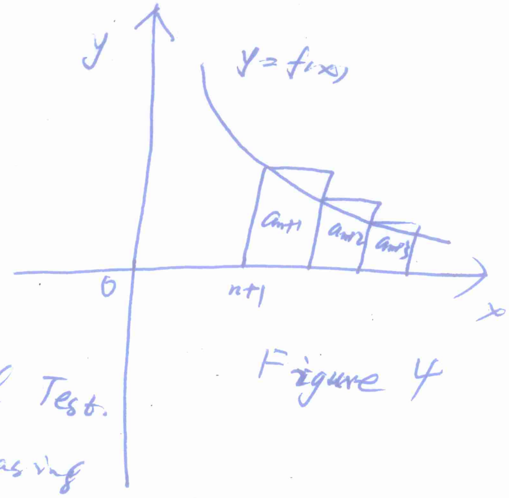


Figure 4

Remainder Estimate for the Integral Test.
Let f be a continous, posive, decreasing function on $[1, \infty)$, $a_n = f(n)$, $\sum a_n$ is convergent.
If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx \quad (2)$$

By (2), we have

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\,dx \quad (3)$$

Since $s_n + R_n = s$.

Ex 6. Use (3) with $n=10$ to estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$$s_{10} + \int_{11}^{\infty} \frac{dx}{x^3} \leq s \leq s_{10} + \int_{10}^{\infty} \frac{dx}{x^3}$$

Since $\int_n^{\infty} \frac{dx}{x^3} = \frac{1}{2n^2}$, we have

$$s_{10} + \frac{1}{2(11)^2} \leq s \leq s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \leq s \leq 1.202532$$

Taking s to be the midpoint of this interval, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \quad \text{with error} < 0.0005.$$