① §11.5 Alternating Series

(Previously, we only consider series with positive terms.)

(This time, we learn how to deal with series whose terms are not necessarily positive.)

An alternating series is a series whose terms are ~~altern~~ alternately positive and negative.

Ex.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots$$

The nth term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \quad \text{or} \quad a_n = (-1)^n b_n$$

where $b_n = |a_n|$ is a positive number.

~~If $\{b_n\}$ is decreasing and $b_n \to 0$~~

Alternating Series Test. If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad b_n > 0$$

satisfies

(i) $b_{n+1} \le b_n$ for all n

(ii) $b_n \to 0$ as $n \to \infty$

then the series is convergent

Let $S_n = \sum_{i=1}^{n} (-1)^{i-1} b_i$.

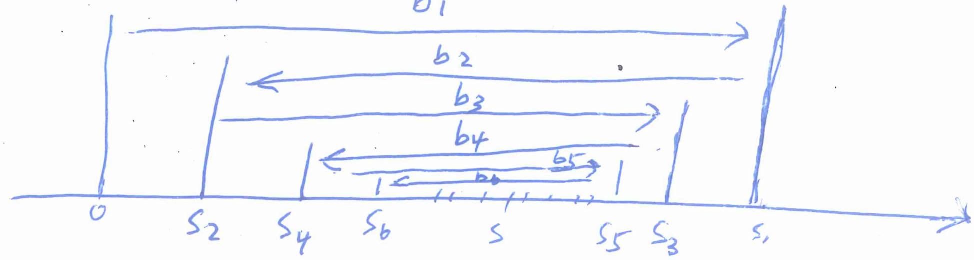


Figure 1.

② Pf of the alternating test:

Since $\{b_n\}$ is decreasing, one can show that

$\{S_{2n}\}$ is ~~also decreasing~~ increasing.

Moreover,

$$S_{2n} = b_1 - \overbrace{(b_2 - b_3)}^{\geq 0} - \overbrace{(b_4 - b_5)}^{\geq 0} - \cdots - \overbrace{(b_{2n-2} - b_{2n-1})}^{\geq 0} - b_{2n}$$

$$\leq b_1 \quad \text{for all } n$$

By the Monotonic Sequence Theorem, $\{S_{2n}\}$ is convergent.

Set $\lim\limits_{n\to\infty} S_{2n} = s$

Then
$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + b_{2n+1})$$
$$= \lim_{n\to\infty} S_n + \lim_{n\to\infty} b_{2n+1}$$
$$= s + 0$$
$$= s$$

Thus, $S_n \to s$ as $n \to \infty$.

Ex 1. The alternating harmonic ~~serie~~ series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies:

(i) $\dfrac{1}{n+1} < \dfrac{1}{n}$

(ii) $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$

So the series is convergent by the Alternating Series Test.

Ex 2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating, but

$$\lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4 - \frac{1}{n}} = \frac{3}{4} \neq 0$$

(3) Furthermore, $\lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$ does not exist.

So, the series is divergent.

Ex 3. Determine the convergence of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

Set $b_n = \frac{n^2}{n^3+1}$. Then ~~lim~~

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}}$$

$$= 0. \quad (1)$$

Set $f(x) = \frac{x^2}{x^3+1}$. Then

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$$

$f'(x) < 0$ if $2 - x^3 < 0$, i.e., $x > \sqrt[3]{2}$

Thus, $f(n+1) < f(n)$ for $n \geq 2$

$\Leftrightarrow b_{n+1} < b_n$ for $n \geq 2$. (2)

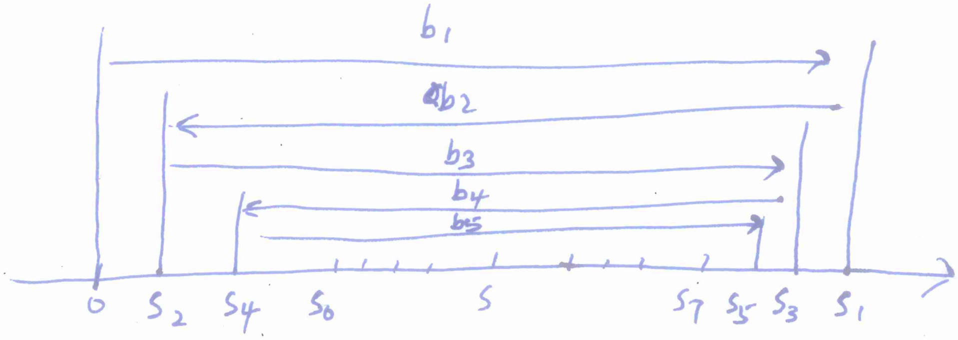
By (1) and (2), the given series is convergent.

• Estimate Sums

Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1} b_n$, where $b_n > 0$, satisfies

$$\text{(i) } b_{n+1} \leq b_n \quad \text{and} \quad \text{(ii) } \lim_{n\to\infty} b_n = 0$$

then $\quad |R_n| = |s - s_n| \leq b_{n+1}$



Pf: $|R_n| = |s - s_n| \le |s_{n+1} - s_n| = b_{n+1}$. ∎

Ex 4. Find the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to 3 decimal places.

(i) $\frac{1}{(n+1)!} < \frac{1}{n!} = b_n$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!} = 0$

(let us write down the first few terms of the series)

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \cdots$$

$$b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$$

and $s_6 \approx 0.368056$.

By the alternating series estimation theorem, we have

$$|s - s_6| \le b_7 < 0.0002.$$

Thus, $s \approx 0.368$ correct to 3 decimal places.