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Article

Twisted Mahler Discrete Residues

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Recently we constructed Mahler discrete residues for rational functions and showed they comprise a complete obstruction to the Mahler summability problem of deciding whether a given rational function $f(x)$ is of the form $g(x^p) - g(x)$ for some rational function $g(x)$ and an integer $p > 1$. Here we develop a notion of λ-twisted Mahler discrete residues for $\lambda \in \mathbb{Z}$, and show that they similarly comprise a complete obstruction to the twisted Mahler summability problem of deciding whether a given rational function $f(x)$ is of the form $p^\lambda g(x^p) - g(x)$ for some rational function $g(x)$ and an integer $p > 1$. We provide some initial applications of twisted Mahler discrete residues to differential creative telescoping problems for Mahler functions and to the differential Galois theory of linear Mahler equations.

1 Introduction

Continuous residues are fundamental and crucial tools in complex analysis, and have extensive and compelling applications in combinatorics [17]. In the past decade, a theory of discrete and q-discrete residues was proposed in [13] for the study of telescoping problems for bivariate rational functions, and subsequently found applications in the computation of differential Galois groups of second-order linear difference [5] and q-difference equations [8] and other closely-related problems [12, 20]. More recently, the authors of [10, 11] developed a theory of residues for skew rational functions, which has important applications in duals of linearized Reed–Solomon codes [11]. In [19] the authors introduce a notion of elliptic orbit residues, which, in analogy with [13], similarly serves as an obstruction to summability in the context of elliptic shift difference operators. Most recently, we initiated in [9] a theory of Mahler discrete residues aimed at helping bring to the Mahler case the successes of these earlier notions of residues.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $\mathbb{K}(x)$ be the field of rational functions in an indeterminate x over $\mathbb{K}$. Fix an integer $p \geq 2$. For a given $f(x) \in \mathbb{K}(x)$, we considered in [9] the *Mahler summability problem* of deciding effectively whether $f(x) = g(x^p) - g(x)$ for some $g(x) \in \mathbb{K}(x)$; if so, we say $f(x)$ is *Mahler summable*. We defined in [9] a collection of $\mathbb{K}$-vectors, called the *Mahler discrete residues* of $f(x)$ and defined purely in terms of its partial fraction decomposition, having the property that they are all zero if and only if $f(x)$ is Mahler summable. More generally, a (linear) *Mahler equation* is any equation of the form

$$y(x^{p^n}) + a_{n-1}(x)y(x^{p^{n-1}}) + \cdots + a_1(x)y(x^p) + +a_0(x)y(x) = 0, \tag{1.1}$$

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where the $a_i(x) \in \mathbb{K}(x)$ and $y(x)$ is an unknown "function" (or possibly some more general entity, for example, the generating series of a combinatorial object, a Puiseux series, etc.). The motivation to study Mahler equations comes from several directions. They first arose in [22] in connection with transcendence results on values of special functions at algebraic numbers, and have since found other applications, for example to automata theory and automatic sequences since the work of [15]. We refer to [3, 4, 14, 16] and the references therein for more details.

A particularly fruitful approach over the past few decades to study difference equations in general, and Mahler equations such as (1.1) in particular, is through the Galois theory for linear difference equations developed in [25], and the differential (also sometimes called parameterized) Galois theory for difference equations developed in [18]. Both theories associate a geometric object to a given difference equation such as (1.1), called the *Galois group*, that encodes the sought (differential-)algebraic properties of the solutions to the equation. There are now several algorithms and theoretical results (see in particular [6, 7, 16, 24]) addressing qualitative questions about solutions of Mahler equations (1.1), in particular whether they must be (differentially) transcendental, which rely on procedures to compute enough information about the corresponding Galois group (i.e., whether it is "sufficiently large"). These Galois-theoretic arguments very often involve, as a sub-problem, deciding whether a certain auxiliary object—often but not always a rational solution to some Riccati-type equation—is Mahler summable, or more generally whether it becomes Mahler summable after applying some linear differential operator to it, that is, a telescoper. Rather than being able to answer the Mahler summability question for any one individual rational function, the systematic obstructions to the Mahler summability problems developed here serve as essential building blocks for other results and algorithms that rely on determining Mahler summability as an intermediate step. An immediate application of the technology developed here is Theorem 6.3: if $y_1(x), \ldots, y_t(x) \in \mathbb{K}((x))$ are solutions to Mahler equations of the form $y_i(x^p) = a_i(x)y_i(x)$ for some non-zero $a_i(x) \in \mathbb{K}(x)$, then either the $y_1(x), \ldots, y_t(x)$ are differentially independent over $\mathbb{K}(x)$ or else they are multiplicatively dependent over $\mathbb{K}(x)^\times$, that is, there exist integers $k_1, \ldots, k_t \in \mathbb{Z}$, not all zero, such that $\prod_{i=1}^t y_i(x)^{k_i} \in \mathbb{K}(x)$. Let us explain in more detail the technology that we develop here.

For arbitrary $\lambda \in \mathbb{Z}$ and $f(x) \in \mathbb{K}(x)$, we say that $f(x)$ is λ-*Mahler summable* if there exists $g(x) \in \mathbb{K}(x)$ such that $f(x) = p^\lambda g(x^p) - g(x)$. We shall construct certain $\mathbb{K}$-vectors from the partial fraction decomposition of $f(x)$, which we call the *(twisted)* λ-*Mahler discrete residues* of $f(x)$, and prove our main result in Section 5.4:

Theorem 1.1. For $\lambda \in \mathbb{Z}$, f is λ-Mahler summable if and only if all λ-Mahler discrete residues of f are zero.

Our desire to develop an obstruction theory for such a "twisted" λ-Mahler summability problem, beyond the "un-twisted" 0-Mahler summability problem considered in [9], is motivated by our desire to apply this obstruction theory to the following kind of *Mahler creative telescoping problem*. Given $f_1, \ldots, f_n \in \mathbb{K}(x)$ decide whether there exist linear differential operators $\mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathbb{K}[\delta]$, for δ some suitable derivation, such that $\mathcal{L}_1(f_1) + \cdots + \mathcal{L}_n(f_n)$ is suitably Mahler summable. The double-usage of "suitable" above is due to the fact that there are in the Mahler case two traditional and respectable ways to adjoin a Mahler-compatible derivation in order to study differential-algebraic properties of solutions of Mahler equations, as we next explain and recall.

A $\sigma\delta$-field is a field equipped with an endomorphism σ and a derivation δ such that $\sigma \circ \delta = \delta \circ \sigma$. Such are the base fields considered in the δ-Galois theory for linear σ-equations developed in [18]. Denoting by $\sigma : \mathbb{K}(x) \to \mathbb{K}(x) : f(x) \mapsto f(x^p)$ the *Mahler endomorphism*, one can show there is no non-trivial derivation δ on $\mathbb{K}(x)$ that commutes with this σ. In the literature one finds the following two approaches (often used in combination; see e.g., [4, 16]): (1) take $\delta = x\frac{d}{dx}$, and find a systematic way to deal with the fact that σ and δ do not quite commute (but almost do), $\sigma \circ \delta = p \cdot \delta \circ \sigma$; or (2) work over the larger field $\mathbb{K}(x, \log x)$, where $\sigma(\log x) = p \log x$, set $\delta = x \log x \frac{d}{dx}$, and find a systematic way to deal with this new element $\log x$ as the cost of having $\sigma \circ \delta = \delta \circ \sigma$ on the nose. There is, to be sure, a dictionary of sorts between these two approaches.

Let us consider the $\sigma\delta$-field $L := \mathbb{K}(x, \log x)$, and given $F \in L$, let us write the $\log$-Laurent series expansion

$$F = \sum_{\lambda \geq N} f_\lambda(x) \log^\lambda x \in \mathbb{K}(x)((\log x)),$$

where $f_\lambda(x) \in \mathbb{K}(x)$ for each $\lambda \in \mathbb{Z}$, and $\log^\lambda x := [\log x]^\lambda$. Suppose there exists $G \in \hat{L} := \mathbb{K}(x)((\log x))$ such that $F = \sigma(G) - G$ (where σ is applied term-by-term). Writing such a putative $G = \sum_{\lambda\geq N} g_\lambda(x)\log^\lambda x \in \hat{L}$, for some $g_\lambda(x) \in \mathbb{K}(x)$ for $\lambda \in \mathbb{Z}$, we find that F is Mahler summable within $\hat{L}$ if and only if $f_\lambda(x) = p^\lambda g_\lambda(x^p) - g_\lambda(x)$ for each $\lambda \in \mathbb{Z}$. This was our initial motivation for introducing the twisted Mahler discrete residues developed here.

Our strategy expands upon that of [9], which in turn was inspired by that of [13]: for $\lambda \in \mathbb{Z}$, we construct in Section 5.5 a λ-*Mahler reduction* $\bar{f}_\lambda(x) \in \mathbb{K}(x)$ such that

$$\bar{f}_\lambda(x) = f(x) + \big(p^\lambda g_\lambda(x^p) - g_\lambda(x)\big) \tag{1.2}$$

for some $g_\lambda(x) \in \mathbb{K}(x)$ (whose explicit computation it is our purpose to avoid!), with the structure of this $\bar{f}_\lambda(x)$ being such that it cannot possibly be λ-Mahler summable unless $\bar{f}_\lambda(x) = 0$. The λ-Mahler discrete residues of $f(x)$ are (vectors whose components are) the coefficients occurring in the partial fraction decomposition of $\bar{f}_\lambda(x)$. This $\bar{f}_\lambda(x)$ plays the role of a "λ-Mahler remainder" of $f(x)$, analogous to the remainder of Hermite reduction in the context of integration.

The contents of this work are as follows. In §2 we recall some notation and ancillary results from [9]—we also present a new closed formula in §2.5 for the *Mahler coefficients*, used everywhere here and in [9] to track the effect of the Mahler operator on partial fractions, as sums over certain integer partitions. In §3 we develop a linear-algebraic framework for controlling the pre-periodic behavior of roots of unity under the Mahler operator. In §4 we prove that λ-Mahler summable rational functions have Mahler dispersion 0 almost everywhere, which is an essential tool in our proofs in spite of the exceptions that arise for the first time in the Mahler context with positive twists λ. In §5 we define the eponymous twisted Mahler discrete residues and prove our Main Theorem 1.1—we also show the non-obvious agreement of the 0-twisted residues defined here with those of [9], which suggests that the Mahler coefficients of §2.5 enjoy some "umbral" properties that probably deserve further study. In §6 we apply our new technology to study the differential properties of the solutions of any finite collection of first-order homogeneous linear Mahler equations. We conclude in §7 with several concrete examples.

2 Preliminaries

In this section we recall and expand upon some conventions, notions, and ancillary results from [9] that we shall use systematically throughout this work.

2.1 Notation and conventions

We fix once and for all an algebraically closed field $\mathbb{K}$ of characteristic zero and an integer $p \geq 2$ (not necessarily prime). We denote by $\mathbb{K}(x)$ the field of rational functions in the indeterminate x with coefficients in $\mathbb{K}$. We denote by $\sigma : \mathbb{K}(x) \to \mathbb{K}(x)$ the $\mathbb{K}$-linear endomorphism defined by $\sigma(x) = x^p$, called the *Mahler operator*, so that $\sigma(f(x)) = f(x^p)$ for $f(x) \in \mathbb{K}(x)$. For $\lambda \in \mathbb{Z}$, we write $\Delta_\lambda := p^\lambda\sigma - \mathrm{id}$, so that $\Delta_\lambda(f(x)) = p^\lambda f(x^p) - f(x)$ for $f(x) \in \mathbb{K}(x)$. We often suppress the functional notation and write $f \in \mathbb{K}(x)$ instead of $f(x)$. We say that $f \in \mathbb{K}(x)$ is λ-*Mahler summable* if there exists $g \in \mathbb{K}(x)$ such that $f = \Delta_\lambda(g)$.

Let $\mathbb{K}^\times = \mathbb{K}\backslash\{0\}$ denote the multiplicative group of $\mathbb{K}$. Let $\mathbb{K}^\times_t$ denote the torsion subgroup of $\mathbb{K}^\times$, that is, the group of roots of unity in $\mathbb{K}^\times$. For $\zeta \in \mathbb{K}^\times_t$, the *order* of ζ is the smallest $r \in \mathbb{Z}_{>0}$ such that $\zeta^r = 1$. We fix once and for all a compatible system of p-power roots of unity $(\zeta_{p^n})_{n\geq 0} \subset \mathbb{K}^\times_t$, that is, each ζ_{p^n} has order p^n and $\zeta_{p^n}^{p^\ell} = \zeta_{p^{n-\ell}}$ for $0 \leq \ell \leq n$. Each $f \in \mathbb{K}(x)$ decomposes uniquely as

$$f = f_\infty + f_{\mathcal{T}}, \tag{2.1}$$

where $f_\infty \in \mathbb{K}[x, x^{-1}]$ is a Laurent polynomial and $f_{\mathcal{T}} = \frac{a}{b}$ for polynomials $a, b \in \mathbb{K}[x]$ such that either $a = 0$ or else $\deg(a) < \deg(b)$ and $\gcd(a, b) = 1 = \gcd(x, b)$. The reasoning behind our choice of subscripts ∞ and $\mathcal{T}$ for the Laurent polynomial component of f and its complement will become apparent in the sequel.

We obtain similarly as in [9, Lem. 2.2] the following result.

Lemma 2.1. The $\mathbb{K}$-linear decomposition $\mathbb{K}(x) \simeq \mathbb{K}[x, x^{-1}] \oplus \mathbb{K}(x)_{\mathcal{T}}$ given by $f \leftrightarrow f_\infty \oplus f_{\mathcal{T}}$ as in (2.1) is σ-stable. For $f, g \in \mathbb{K}(x)$ and for $\lambda \in \mathbb{Z}$, $f = \Delta_\lambda(g)$ if and only if $f_\infty = \Delta_\lambda(g_\infty)$ and $f_{\mathcal{T}} = \Delta_\lambda(g_{\mathcal{T}})$.

2.2 Mahler trajectories, Mahler trees, and Mahler cycles

We let $\mathcal{P} := \{p^n \mid n \in \mathbb{Z}_{\geq 0}\}$ denote the multiplicative monoid of non-negative powers of p. Then $\mathcal{P}$ acts on $\mathbb{Z}$ by multiplication, and the set of *maximal trajectories* for this action is

$$\mathbb{Z}/\mathcal{P} := \big\{\{0\}\big\} \cup \big\{\{ip^n \mid n \in \mathbb{Z}_{\geq 0}\} \bigm| i \in \mathbb{Z} \text{ such that } p \nmid i\big\}.$$

Definition 2.2. For a maximal trajectory $\theta \in \mathbb{Z}/\mathcal{P}$, we let

$$\mathbb{K}[x, x^{-1}]_\theta := \left\{\textstyle\sum_j c_j x^j \in \mathbb{K}[x, x^{-1}] \,\middle|\, c_j = 0 \text{ for all } j \notin \theta\right\}, \tag{2.2}$$

and call it the *θ-subspace*. The *θ-component* f_θ of $f \in \mathbb{K}(x)$ is the projection of f_∞ as in (2.1) to $\mathbb{K}[x, x^{-1}]_\theta$.

We obtain similarly as in [9, Lem. 2.3] the following result.

Lemma 2.3. For $f, g \in \mathbb{K}(x)$ and for $\lambda \in \mathbb{Z}$, $f_\infty = \Delta_\lambda(g_\infty)$ if and only if $f_\theta = \Delta_\lambda(g_\theta)$ for every $\theta \in \mathbb{Z}/\mathcal{P}$.

Definition 2.4. We denote by $\mathcal{T}$ the set of equivalence classes in $\mathbb{K}^\times$ for the equivalence relation $\alpha \sim \gamma \Leftrightarrow \alpha^{p^r} = \gamma^{p^s}$ for some $r, s \in \mathbb{Z}_{\geq 0}$. For $\alpha \in \mathbb{K}^\times$, we denote by $\tau(\alpha) \in \mathcal{T}$ the equivalence class of α under $\sim$. The elements $\tau \in \mathcal{T}$ are called *Mahler trees*.

We refer to [9, Remark 2.7] for a brief discussion on our choice of nomenclature in Definition 2.4.

Definition 2.5. For a Mahler tree $\tau \in \mathcal{T}$, the *τ-subspace* is

$$\mathbb{K}(x)_\tau := \{f_\mathcal{T} \in \mathbb{K}(x)_\mathcal{T} \mid \text{every pole of } f_\mathcal{T} \text{ is contained in } \tau\}. \tag{2.3}$$

For $f \in \mathbb{K}(x)$, the *τ-component* f_τ of f is the projection of $f_\mathcal{T}$ as in (2.1) to $\mathbb{K}(x)_\tau$.

The following result is proved similarly as in [9, Lem. 2.12].

Lemma 2.6. For $f, g \in \mathbb{K}(x)$ and for $\lambda \in \mathbb{Z}$, $f_\mathcal{T} = \Delta_\lambda(g_\mathcal{T})$ if and only if $f_\tau = \Delta_\lambda(g_\tau)$ for every $\tau \in \mathcal{T}$.

Definition 2.7. For a Mahler tree $\tau \in \mathcal{T}$, the (possibly empty) *Mahler cycle* of τ is

$$\mathcal{C}(\tau) := \{\gamma \in \tau \mid \gamma \text{ is a root of unity of order coprime to } p\}.$$

The (possibly zero) *cycle length* of τ is defined to be $\varepsilon(\tau) := |\mathcal{C}(\tau)|$. For $e \in \mathbb{Z}_{\geq 0}$, let $\mathcal{T}_e := \{\tau \in \mathcal{T} \mid \varepsilon(\tau) = e\}$. We refer to $\mathcal{T}_0$ as the set of *non-torsion Mahler trees*, and to $\mathcal{T}_+ := \mathcal{T} - \mathcal{T}_0$ as the set of *torsion Mahler trees*.

Remark 2.8. Let us collect as in [9, Rem. 2.10] some immediate observations about Mahler cycles that we shall use, and refer to, throughout the sequel.

For $\tau \in \mathcal{T}$ it follows from the Definition 2.4 that either $\tau \subset \mathbb{K}_t^\times$ or else $\tau \cap \mathbb{K}_t^\times = \emptyset$ (that is, either τ consists entirely of roots of unity or else τ contains no roots of unity at all). In particular, $\tau \cap \mathbb{K}_t^\times = \emptyset \Leftrightarrow \mathcal{C}(\tau) = \emptyset \Leftrightarrow \varepsilon(\tau) = 0 \Leftrightarrow \tau \in \mathcal{T}_0$ (the *non-torsion case*). On the other hand, $\mathbb{K}_t^\times$ consists of the pre-periodic points for the action of the monoid $\mathcal{P}$ on $\mathbb{K}^\times$ given by $\alpha \mapsto \alpha^{p^n}$ for $n \in \mathbb{Z}_{\geq 0}$. For $\tau \subset \mathbb{K}_t^\times$ (the *torsion case*), the Mahler cycle $\mathcal{C}(\tau)$ is a non-empty set endowed with a simply transitive action of the quotient monoid $\mathcal{P}/\mathcal{P}^e \simeq \mathbb{Z}/e\mathbb{Z}$, where $\mathcal{P}^e := \{p^{ne} \mid n \in \mathbb{Z}\}$, and $e := \varepsilon(\tau)$. We emphasize that in general $\mathcal{C}(\tau)$ is only a set, and not a group. The Mahler tree $\tau(1)$ consists precisely of the roots of unity $\zeta \in \mathbb{K}_t^\times$ whose order r is such that $\gcd(r, p^n) = r$ for some $p^n \in \mathcal{P}$, or equivalently such that every prime factor of r divides p. When $\tau \subset \mathbb{K}_t^\times$ but $\tau \neq \tau(1)$, the cycle length $\varepsilon(\tau) = e$ is the order of p in the group of units $(\mathbb{Z}/r\mathbb{Z})^\times$, where $r > 1$ is the common order of the roots of unity $\gamma \in \mathcal{C}(\tau)$, and $\mathcal{C}(\tau) = \{\gamma^{p^\ell} \mid 0 \leq \ell \leq e - 1\}$ for any given $\gamma \in \mathcal{C}(\tau)$.

2.3 Mahler supports and singular supports in Mahler trees

As in [9] we utilize Mahler trees to define the following useful variants of the singular support $\mathrm{sing}(f)$ of a rational function f (i.e., its set of poles) and the order $\mathrm{ord}_\alpha(f)$ of a pole of f at $\alpha \in \mathbb{K}$.

Definition 2.9. For $f \in \mathbb{K}(x)$, we define $\mathrm{supp}(f) \subset \mathcal{T} \cup \{\infty\}$, called the *Mahler support* of f, as follows:

- $\infty \in \mathrm{supp}(f)$ if and only if $f_\infty \neq 0$; and
- for $\tau \in \mathcal{T}$, $\tau \in \mathrm{supp}(f)$ if and only if τ contains a pole of f.

For $\tau \in \mathcal{T}$, the *singular support* of f in τ, denoted by $\mathrm{sing}(f,\tau)$, is the (possibly empty) set of poles of f contained in τ, and the *order* of f at τ is $\mathrm{ord}(f,\tau) := \max\big(\{0\} \cup \big\{\mathrm{ord}_\alpha(f) \mid \alpha \in \mathrm{sing}(f,\tau)\big\}\big)$.

For the sake of completeness, we include the straightforward proof of the following lemma, which was omitted from [9, Section 2.2] for lack of space.

Lemma 2.10. For $f, g \in \mathbb{K}(x)$, $\tau \in \mathcal{T}$, $\lambda \in \mathbb{Z}$, and $0 \neq c \in \mathbb{K}$, we have the following:

1) $\mathrm{supp}(f) = \emptyset \Longleftrightarrow f = 0$;
2) $\mathrm{supp}(\sigma(f)) = \mathrm{supp}(f) = \mathrm{supp}(c \cdot f)$; and
3) $\mathrm{supp}(f+g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$.
4) $\tau \in \mathrm{supp}(\Delta_\lambda(g)) \Longleftrightarrow \tau \in \mathrm{supp}(g)$;
5) $\mathrm{ord}(\sigma(f),\tau) = \mathrm{ord}(f,\tau) = \mathrm{ord}(c \cdot f,\tau)$;
6) $\mathrm{ord}(f+g,\tau) \leq \max\big(\mathrm{ord}(f,\tau), \mathrm{ord}(g,\tau)\big)$; and
7) $\mathrm{ord}(\Delta_\lambda(g),\tau) = \mathrm{ord}(g,\tau)$.

Proof. (1). $f = 0 \Longleftrightarrow f_\infty = 0$ and $f_{\mathcal{T}} = 0$, and $f_{\mathcal{T}} = 0 \Longleftrightarrow f$ has no poles in $\mathbb{K}^\times$.

(2) and (5). For $0 \neq c \in \mathbb{K}$, $cf_\infty \neq 0$ if and only if $f_\infty \neq 0$, and f and cf have the same poles and the orders of these poles are the same, and therefore $\mathrm{supp}(f) = \mathrm{supp}(cf)$ and $\mathrm{ord}(f,\tau) = \mathrm{ord}(cf,\tau)$ for every $\tau \in \mathcal{T}$. Moreover, $\sigma(f_\infty) \neq 0$ if and only if $f_\infty \neq 0$, since σ is an injective endomorphism of $\mathbb{K}(x)$, and $\alpha \in \mathbb{K}^\times$ is a pole of $\sigma(f)$ if and only if α^p is a pole of f, whence τ contains a pole of f if and only if τ contains a pole of $\sigma(f)$. In this case, it is clear that $\mathrm{ord}(\sigma(f),\tau) \leq \mathrm{ord}(f,\tau)$. Moreover, since f has only finitely many poles in τ of maximal order $m := \mathrm{ord}(f,\tau)$, there exists $\alpha \in \mathrm{sing}(\sigma(f),\tau)$ such that $\mathrm{ord}_{\alpha^p}(f) = m > \mathrm{ord}_\alpha(f)$, and it follows that $\mathrm{ord}_\alpha(\sigma(f)) = m = \mathrm{ord}(\sigma(f),\tau)$.

(3) and (6). If $f_\infty + g_\infty \neq 0$ then at least one of $f_\infty \neq 0$ or $g_\infty \neq 0$. The set of poles of $f+g$ is contained in the union of the set of poles of f and the set of poles of g, and therefore if τ contains a pole of $f+g$ then τ must contain a pole of f or a pole of g. This shows that $\mathrm{supp}(f+g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$. For m the maximal order of a pole of $f+g$ in τ we see that at least one of f or g must contain a pole of order $\geq m$ in τ. This shows that $\mathrm{ord}(f+g,\tau) \leq \mathbf{max}(\mathrm{ord}(f,\tau), \mathrm{ord}(g,\tau))$.

(4) and (7). By (2) and (3), $\mathrm{supp}(\Delta_\lambda(g)) \subseteq \mathrm{supp}(g)$, and by (5) and (6), $\mathrm{ord}(\Delta_\lambda(g),\tau) \leq \mathrm{ord}(g,\tau)$. Suppose $\tau \in \mathrm{supp}(g)$, and let $\alpha_1, \dots, \alpha_s \in \mathrm{sing}(g,\tau)$ be all the elements, pairwise distinct, with $\mathrm{ord}_{\alpha_j}(g) = \mathrm{ord}(g,\tau) =: m \geq 1$, and choose $\gamma_j \in \tau$ such that $\gamma_j^p = \alpha_j$, we find as in the proof of (5) that $\mathrm{ord}_{\zeta_p^i \gamma_j}(\sigma(g)) = m$ and the elements $\zeta_p^i \gamma_j$ are pairwise distinct for $0 \leq i \leq p-1$ and $1 \leq j \leq s$, whence at least one of the $\zeta_p^i \gamma_j$ is different from every $\alpha_{j'}$ for $1 \leq j' \leq s$, and therefore $\mathrm{ord}(\Delta_\lambda(g),\tau) = m$, which implies $\tau \in \mathrm{supp}(\Delta_\lambda(g))$. ∎

2.4 Mahler dispersion

We now recall from [9] the following Mahler variant of the notion of (polar) dispersion used in [13], following the original definitions in [1, 2].

Definition 2.11. For $f \in \mathbb{K}(x)$ and $\tau \in \mathrm{supp}(f)$, the *Mahler dispersion* of f at τ, denoted by $\mathrm{disp}(f,\tau)$, is defined as follows.

1) For $\tau \in \mathcal{T}$, $\mathrm{disp}(f,\tau)$ is the largest $d \in \mathbb{Z}_{\geq 0}$ (if it exists) for which there exists $\alpha \in \mathrm{sing}(f,\tau)$ such that $\alpha^{p^d} \in \mathrm{sing}(f,\tau)$. If there is no such $d \in \mathbb{Z}_{\geq 0}$, then we set $\mathrm{disp}(f,\tau) = \infty$.
2) For $\tau = \infty$, let us write $f_\infty = \sum_{i=n}^N c_i x^i \in \mathbb{K}[x, x^{-1}]$ with $c_n c_N \neq 0$.
   - If $f_\infty = c_0 \neq 0$ then we set $\mathrm{disp}(f,\infty) = 0$; otherwise

- $\mathrm{disp}(f,\infty)$ is the largest $d \in \mathbb{Z}_{\geq 0}$ for which there exists an index $i \neq 0$ such that $c_i \neq 0$ and $c_{ip^d} \neq 0$.

For $f \in \mathbb{K}(x)$ and $\tau \in \mathcal{T} \cup \{\infty\}$ such that $\tau \notin \mathrm{supp}(f)$, we do not define $\mathrm{disp}(f,\tau)$ at all (cf. [1, 2, 13]).

Similarly as in the shift and q-difference cases (cf. [18, Lemma 6.3] and [13, Lemma 2.4 and Lemma 2.9]), Mahler dispersions will play a crucial role in what follows. As we prove in Theorem 4.2, they already provide a partial obstruction to summability: if $f \in \mathbb{K}(x)$ is λ-Mahler summable then almost every Mahler dispersion of f is non-zero. Moreover, Mahler dispersions also detect whether f has any "bad" poles (i.e., at roots of unity of order coprime to p) according to the following result proved in [9, Lem. 2.16].

Lemma 2.12. Let $f \in \mathbb{K}(x)$ and $\tau \in \mathrm{supp}(f)$. Then $\mathrm{disp}(f,\tau) = \infty$ if and only if $\mathrm{sing}(f,\tau) \cap \mathcal{C}(\tau) \neq \emptyset$.

2.5 Mahler coefficients

Here we extend the study of the effect of the Mahler operator σ on partial fractions initiated in [9, §2.4]. For $\alpha \in \mathbb{K}^\times$ and $m, k, n \in \mathbb{Z}$ with $n \geq 0$ and $1 \leq k \leq m$, we define the *Mahler coefficients* $V^m_{k,n}(\alpha) \in \mathbb{K}$ implicitly by

$$\sigma^n\left(\frac{1}{(x-\alpha^{p^n})^m}\right) = \frac{1}{(x^{p^n}-\alpha^{p^n})^m} = \sum_{k=1}^{m}\sum_{i=0}^{p^n-1}\frac{V^m_{k,n}(\zeta^i_{p^n}\alpha)}{(x-\zeta^i_{p^n}\alpha)^k}. \tag{2.4}$$

These Mahler coefficients are computed explicitly with the following result, proved analogously to the similar [9, Lem. 2.17] in case $n = 1$.

Lemma 2.13. The universal Mahler coefficients $\mathbb{V}^m_{k,n} := V^m_{k,n}(1)$ are the first m Taylor coefficients at $x = 1$ of

$$(x^{p^n-1}+\cdots+x+1)^{-m} = \sum_{k=1}^{m}\mathbb{V}^m_{k,n}\cdot(x-1)^{m-k} + O((x-1)^m). \tag{2.5}$$

For arbitrary $\alpha \in \mathbb{K}^\times$, the Mahler coefficients $V^m_{k,n}(\alpha) = \mathbb{V}^m_{k,n}\cdot\alpha^{k-mp^n}$.

Although Lemma 2.13 serves to compute the $V^m_{k,n}(\alpha)$ for $\alpha \in \mathbb{K}^\times$, $n \in \mathbb{Z}_{\geq 0}$, and $1 \leq k \leq m$ efficiently in practice, the following result provides an explicit symbolic expression for these Mahler coefficients.

Definition 2.14. For $k, n \in \mathbb{Z}_{\geq 0}$, let $\Pi_n(k)$ be the set of integer partitions $\mu = (\mu_1, \ldots, \mu_\ell)$ of k with greatest part $\mu_1 < p^n$, and denote by $\ell(\mu) := \ell$ the length of μ and by $\ell_i(\mu)$ the multiplicity of i in μ for $1 \leq i \leq p^n - 1$. We adopt the conventions that $\Pi_n(0) = \{\emptyset\}$ for every $n \geq 0$ and $\Pi_0(k) = \emptyset$ for every $k \geq 1$. The empty partition $\mu = \emptyset$ has length $\ell(\emptyset) = 0$ and multiplicity $\ell_i(\emptyset) = 0$ for every $1 \leq i \leq p^n - 1$ (vacuously so when $n = 0$).

Proposition 2.15. For $n \geq 0$ and $1 \leq k \leq m$,

$$\mathbb{V}^m_{k,n} = p^{-nm}\cdot\sum_{\mu\in\Pi_n(m-k)}(-p^n)^{-\ell(\mu)}\binom{m-1+\ell(\mu)}{m-1,\ell_1(\mu),\ldots,\ell_{p^n-1}(\mu)}\prod_{i=1}^{p^n-1}\binom{p^n}{i+1}^{\ell_i(\mu)}.$$

Proof. By Lemma 2.13, $V^m_{k,n}(\alpha) = \mathbb{V}^m_{k,n}\cdot\alpha^{k-mp^n}$, where the $\mathbb{V}^m_{k,n} \in \mathbb{Q}$ are given by (2.5). Writing $f_m(x) = x^{-m}$ and $g_n(x) = x^{p^n-1}+\cdots+x+1$, and letting $W^m_{k,n} \in \mathbb{Q}$ be the coefficient of $(x-1)^k$ in the Taylor expansion of $(f_m \circ g_n)(x)$ at $x = 1$ as in Lemma 2.13, $\mathbb{V}^m_{k,n} = W^m_{m-k,n}$ for $1 \leq k \leq m$. By Faà di Bruno's formula [21],

$$W^m_{k,n} = \frac{(f_m\circ g_n)^{(k)}(1)}{k!} = \frac{1}{k!}\cdot\sum_{\mu\in\Pi(k)}\frac{k!}{\ell_1(\mu)!\cdots\ell_k(\mu)!}f_m^{(\ell(\mu))}(g_n(1))\prod_{i=1}^{k}\left(\frac{g_n^{(i)}(1)}{i!}\right)^{\ell_i(\mu)}$$

for every $k \geq 0$, where $\Pi(k)$ denotes the set of *all* partitions of k, and $\ell(\mu)$ and $\ell_i(\mu)$ are as in Definition 2.14. For every $\ell, i \in \mathbb{Z}_{\geq 0}$, we compute

$$f_m^{(\ell)}(g_n(1)) = (-1)^\ell p^{-n(m+\ell)}\frac{(m-1+\ell)!}{(m-1)!} \qquad \text{and} \qquad g_n^{(i)}(1) = i!\binom{p^n}{i+1},$$

where we adopt the usual convention that $\binom{p^n}{i+1} = 0$ whenever $i \geq p^n$. Therefore the partitions $\mu \in \Pi(k)\backslash\Pi_n(k)$ with greatest part $\mu_1 \geq p^n$ do not contribute to the sum. ■

We isolate the following special case for ease of reference (cf. [9, Cor. 2.18]), since it arises often.

Corollary 2.16. Let $\alpha \in \mathbb{K}^\times$, $m \in \mathbb{N}$, and $n \in \mathbb{Z}_{\geq 0}$. Then $V_{m,n}^m(\alpha) = p^{-nm}\alpha^{m-p^n m}$.

Proof. In the special case where $k = m$ in Proposition 2.15, the sum is over $\mu \in \Pi(0) = \{\emptyset\}$, and $\ell(\emptyset) = 0 = \ell_i(\emptyset)$ for every $i \in \mathbb{N}$, whence $V_{m,n}^m(\alpha) = p^{-nm}\alpha^{m-p^n m}$ by Lemma 2.13. ■

The Mahler coefficients $V_{k,n}^m(\alpha)$ defined above are the main ingredients in our definition of twisted Mahler discrete residues. Our proofs that these residues comprise a complete obstruction to λ-Mahler summability will rely on the following elementary computations, which we record here once and for all for future reference.

Lemma 2.17. Let $n \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{K}^\times$, and $d_1, \dots, d_m \in \mathbb{K}$ for some $m \in \mathbb{N}$. Then

$$\sigma^n\left(\sum_{k=1}^m \frac{d_k}{(x-\alpha^{p^n})^k}\right) = \sum_{k=1}^m \sum_{i=0}^{p^n-1} \frac{\sum_{s=k}^m V_{k,n}^s(\zeta_{p^n}^i\alpha)d_s}{(x-\zeta_{p^n}^i\alpha)^k}.$$

For $\lambda \in \mathbb{Z}$ and $g \in \mathbb{K}(x)$, the element $\Delta_\lambda^{(n)}(g) := p^{\lambda n}\sigma^n(g) - g$ is λ-Mahler summable.

Proof. The claims are trivial if $n = 0$: $\zeta_1 = 1$, $V_{k,0}^s(\alpha) = \delta_{s,k}$ (Kronecker's δ) for $k \leq s \leq m$, and $\Delta_\lambda^{(0)}(g) = 0$ is λ-Mahler summable. Suppose that $n \geq 1$. For $1 \leq s \leq m$ we have

$$\sigma^n\left(\frac{d_s}{(x-\alpha^{p^n})^s}\right) = \sum_{k=1}^s \sum_{i=0}^{p^n-1} \frac{V_{k,n}^s(\zeta_{p^n}^i\alpha)d_s}{(x-\zeta_{p^n}^i\alpha)^k}$$

by definition (cf. (2.4)), and it follows that

$$\sigma^n\left(\sum_{s=1}^m \frac{d_s}{(x-\alpha^{p^n})^s}\right) = \sum_{s=1}^m\sum_{k=1}^s \sum_{i=0}^{p^n-1} \frac{V_{k,n}^s(\zeta_{p^n}^i\alpha)d_s}{(x-\zeta_{p^n}^i\alpha)^k} = \sum_{k=1}^m \sum_{i=0}^{p^n-1} \frac{\sum_{s=k}^m V_{k,n}^s(\zeta_{p^n}^i\alpha)d_s}{(x-\zeta_{p^n}^i\alpha)^k}.$$

Finally, the λ-Mahler summability of $\Delta_\lambda^{(n)}(g)$ follows from the computation

$$\Delta_\lambda^{(n)}(g) = p^{\lambda n}\sigma^n(g) - g = p^\lambda\sigma\left(\sum_{j=0}^{n-1} p^{\lambda j}\sigma^j(g)\right) - \left(\sum_{j=0}^{n-1} p^{\lambda j}\sigma^j(g)\right) = \Delta_\lambda\left(\sum_{j=0}^{n-1} p^{\lambda j}\sigma^j(g)\right).$$

■

3 Cycle Maps and Their ω-Sections

The goal of this section is to define and study the properties of two auxiliary maps $\mathcal{D}_{\lambda,\tau}$ and $\mathcal{I}_{\lambda,\tau}^{(\omega)}$ that will help us retain some control over the perverse periodic behavior of the roots of unity $\gamma \in \mathcal{C}(\tau)$ under the p-power map $\gamma \mapsto \gamma^p$. The following definitions and results are relevant only for torsion Mahler trees $\tau \in \mathcal{T}_+$.

Definition 3.1. Let $\tau \in \mathcal{T}_+$ be a torsion Mahler tree, let $g \in \mathbb{K}(x)$, and let us write $g_\tau = \sum_{k\in\mathbb{N}}\sum_{\alpha\in\tau}\frac{d_k(\alpha)}{(x-\alpha)^k}$ as in Definition 2.5. We define the *cyclic component* $\mathcal{C}(g_\tau) := \sum_{k\in\mathbb{N}}\sum_{\gamma\in\mathcal{C}(\tau)}\frac{d_k(\gamma)}{(x-\gamma)^k}$.

Definition 3.2. Let $\mathcal{S} := \bigoplus_{k\in\mathbb{N}}\mathbb{K}$ denote the $\mathbb{K}$-vector space of finitely supported sequences in $\mathbb{K}$. For $\tau \in \mathcal{T}_+$, we let $\mathcal{S}^{\mathcal{C}(\tau)} := \bigoplus_{\gamma\in\mathcal{C}(\tau)}\mathcal{S}$. For $\lambda \in \mathbb{Z}$, we define *cycle map* $\mathcal{D}_{\lambda,\tau}$ to be the $\mathbb{K}$-linear endomorphism

$$\mathcal{D}_{\lambda,\tau} : \mathcal{S}^{\mathcal{C}(\tau)} \to \mathcal{S}^{\mathcal{C}(\tau)} : \big(d_k(\gamma)\big)_{\substack{k\in\mathbb{N}\\ \gamma\in\mathcal{C}(\tau)}} \mapsto \left(-d_k(\gamma) + p^\lambda \sum_{s\geq k} V^s_{k,1}(\gamma)\cdot d_s(\gamma^p)\right)_{\substack{k\in\mathbb{N}\\ \gamma\in\mathcal{C}(\tau)}}, \tag{3.1}$$

where the Mahler coefficients $V^s_{k,1}(\gamma)$ are defined as in (2.4).

We treat the $\mathbb{K}$-vector space $\mathcal{S}^{\mathcal{C}(\tau)}$ introduced in the preceding Definition 3.2 as an abstract receptacle for the coefficients occurring in the partial fraction decomposition of $\mathcal{C}(g_\tau)$ for $\tau \in \mathcal{T}_+$ and arbitrary elements $g \in \mathbb{K}(x)$. Note that the infinite summation in (3.1) is harmless, since $d_s(\gamma^p) = 0$ for every $\gamma \in \mathcal{C}(\gamma)$ for large enough $s \in \mathbb{N}$. The cycle map $\mathcal{D}_{\lambda,\tau}$ for $\lambda = 0$ is the negative of the (truncated) linear map introduced in [9, Lemma 4.14]. The relevance of $\mathcal{D}_{\lambda,\tau}$ to our study of λ-Mahler summability is captured by the following immediate computation.

Lemma 3.3. Let $\lambda \in \mathbb{Z}$, $g \in \mathbb{K}(x)$, and $\tau \in \mathcal{T}_+$. Let us write $\mathcal{C}(g_\tau) = \sum_{k,\gamma}\frac{d_k(\gamma)}{(x-\gamma)^k}$ and $\mathcal{C}\left(\Delta_\lambda(g_\tau)\right) = \sum_{k,\gamma}\frac{c_k(\gamma)}{(x-\gamma)^k}$ as in Definition 3.1. Writing $\mathbf{d} := (d_k(\gamma))_{k,\gamma}$ and $\mathbf{c} := (c_k(\gamma))_{k,\gamma}$ as in Definition 3.2, we have $\mathbf{c} = \mathcal{D}_{\lambda,\tau}(\mathbf{d})$.

Proof. It follows from Lemma 2.17 that $\mathcal{C}(\sigma(g_\tau)) = \sum_{k\in\mathbb{N}}\sum_{\gamma\in\mathcal{C}(\tau)}\frac{\sum_{s\geq k}V^s_{k,1}(\gamma)d_s(\gamma^p)}{(x-\gamma)^k}$, and therefore, for every $k \in \mathbb{N}$ and $\gamma \in \mathcal{C}(\tau)$, $c_k(\gamma) = -d_k(\gamma) + p^\lambda\sum_{s\geq k}V^s_{k,1}(\gamma)d_s(\gamma^p)$. ■

The following Lemma is essential to our study of λ-Mahler summability at torsion Mahler trees $\tau \in \mathcal{T}_+$.

Lemma 3.4. Let $\lambda \in \mathbb{Z}$, $\tau \in \mathcal{T}_+$, and set $e := |\mathcal{C}(\tau)|$ as in Definition 2.7. Let $\mathcal{D}_{\lambda,\tau}$ be as in Definition 3.2.

1) If $\lambda \leq 0$ then $\mathcal{D}_{\lambda,\tau}$ is an isomorphism.
2) If $\lambda \geq 1$ then $\mathrm{im}(\mathcal{D}_{\lambda,\tau})$ has codimension 1 in $\mathcal{S}^{\mathcal{C}(\tau)}$ and $\ker(\mathcal{D}_{\lambda,\tau}) = \mathbb{K}\cdot\mathbf{w}^{(\lambda)}$, where $(w^{(\lambda)}_k(\gamma)) = \mathbf{w}^{(\lambda)}$ is recursively determined by the conditions

$$w^{(\lambda)}_k(\gamma) := \begin{cases} 0 & \text{for } k > \lambda; \\ \gamma^\lambda & \text{for } k = \lambda; \\ \dfrac{p^\lambda\gamma^k}{1-p^{(\lambda-k)e}}\displaystyle\sum_{j=0}^{e-1}\sum_{s=k+1}^{\lambda} p^{(\lambda-k)j}\mathbb{V}^s_{k,1}\gamma^{-sp^{j+1}}w^{(\lambda)}_s\left(\gamma^{p^{j+1}}\right) & \text{for any remaining } k < \lambda; \end{cases} \tag{3.2}$$

for each $\gamma \in \mathcal{C}(\tau)$, where the universal Mahler coefficients $\mathbb{V}^s_{k,1} \in \mathbb{Q}$ are as in Proposition 2.15.

Proof. Let $(d_k(\gamma)) = \mathbf{d} \in \mathcal{S}^{\mathcal{C}(\tau)} - \{\mathbf{0}\}$, let $m \in \mathbb{N}$ be as large as possible such that $d_m(\gamma) \neq 0$ for some $\gamma \in \mathcal{C}(\tau)$, and let us write $(c_k(\gamma)) = \mathbf{c} := \mathcal{D}_{\lambda,\tau}(\mathbf{d})$.

Let us first assume that $\mathbf{d} \in \ker(\mathcal{D}_{\lambda,\tau})$. Then by the Definition 3.2 and our choice of m, for each $\gamma \in \mathcal{C}(\tau)$,

$$0 = c_m(\gamma) = p^\lambda V^m_{m,1}(\gamma)d_m(\gamma^p) - d_m(\gamma) = p^{\lambda-m}\gamma^{m-pm}d_m(\gamma^p) - d_m(\gamma), \tag{3.3}$$

where the second equality results from Corollary 2.16. Since (3.3) holds for every $\gamma \in \mathcal{C}(\tau)$ simultaneously, it follows that $d_m(\gamma^{p^{j+1}}) = p^{m-\lambda}\gamma^{(p^{j+1}-p^j)m}d_m(\gamma^{p^j})$ for every $j \geq 0$ and for each $\gamma \in \mathcal{C}(\tau)$, whence none of the $d_m(\gamma^{p^j})$ can be zero. Since $\gamma^{p^e} = \gamma$, we find that

$$1 = \frac{d_m(\gamma^{p^e})}{d_m(\gamma)} = \prod_{j=0}^{e-1}\frac{d_m(\gamma^{p^{j+1}})}{d_m(\gamma^{p^j})} = \prod_{j=0}^{e-1}p^{m-\lambda}\gamma^{(p^{j+1}-p^j)m} = p^{(m-\lambda)e}\gamma^{(p^e-1)m} = p^{(m-\lambda)e}, \tag{3.4}$$

which is only possible if $m = \lambda$. Therefore $d_k(\gamma) = 0$ for every $k > \lambda$, whence $\mathcal{D}_{\lambda,\tau}$ is injective in case $\lambda \leq 0$. In case $\lambda \geq 1$, it also follows from (3.3) with $m = \lambda$ that $\gamma^{-p\lambda}d_\lambda(\gamma^p) = \gamma^{-\lambda}d_\lambda(\gamma) = \omega$ must be a constant that does not depend on $\gamma \in \mathcal{C}(\tau)$ (recall $\mathcal{C}(\tau)$ is stable by taking pth powers). We claim that if we further impose that this $\omega = 1$, then the remaining components of our $\mathbf{d}$ are uniquely determined by the recursion (3.2). Indeed, if $\lambda = 1$ then there are no more components to determine, whereas if $\lambda \geq 2$ then we must have, for $1 \leq k \leq \lambda - 1$,

$$0 = -d_k(\gamma) + p^\lambda \sum_{s=k}^{\lambda} \mathbb{V}_{k,1}^s(\gamma) d_s(\gamma^p) \qquad \Longleftrightarrow \qquad d_k(\gamma) - p^{\lambda-k}\gamma^{k-pk} d_k(\gamma^p) = p^\lambda \sum_{s=k+1}^{\lambda} \mathbb{V}_{k,1}^s(\gamma) d_s(\gamma^p)$$

by Corollary 2.16. Replacing the arbitrary γ above with γ^{p^j} for $j = 0, \ldots, e-1$, we find the telescoping sum

$$\gamma^{-k}(1 - p^{(\lambda-k)e}) d_k(\gamma) = \sum_{j=0}^{e-1} p^{(\lambda-k)j} \gamma^{-kp^j} \cdot \left(d_k(\gamma^{p^j}) - p^{\lambda-k}\gamma^{kp^j - kp^{j+1}} d_k(\gamma^{p^{j+1}}) \right)$$

$$= \sum_{j=0}^{e-1} p^{(\lambda-k)j}\gamma^{-kp^j} \cdot p^\lambda \sum_{s=k+1}^{\lambda} \mathbb{V}_{k,1}^s(\gamma^{p^j}) d_s(\gamma^{p^{j+1}}) = p^\lambda \sum_{j=0}^{e-1}\sum_{s=k+1}^{\lambda} p^{(\lambda-k)j} \mathbb{V}_{k,1}^s \gamma^{-sp^{j+1}} d_s(\gamma^{p^{j+1}}),$$

which is clearly equivalent to the expression defining the components $w_k^{(\lambda)}(\gamma)$ for $k < \lambda$ in (3.2), and where we have once again used Lemma 2.13 to obtain the last equality, since $\mathbb{V}_{k,1}^s(\gamma^{p^j}) = \mathbb{V}_{k,1}^s \gamma^{kp^j - sp^{j+1}}$. This concludes the proof of the statements concerning $\ker(\mathcal{D}_{\lambda,\tau})$.

Let us now prove the statements concerning $\mathrm{im}(\mathcal{D}_{\lambda,\tau})$. We see from Definition 3.2 that $\mathcal{D}_{\lambda,\tau}$ preserves the increasing filtration of $\mathcal{S}^{\mathcal{C}(\tau)}$ by the finite-dimensional subspaces

$$\mathcal{S}^{\mathcal{C}(\tau)}_{<m} := \left\{ (d_k(\gamma)) \in \mathcal{S}^{\mathcal{C}(\tau)} \;\middle|\; d_k(\gamma) = 0 \text{ for } k \geq m \text{ and every } \gamma \in \mathcal{C}(\tau) \right\}. \tag{3.5}$$

In case $\lambda \leq 0$, since $\mathcal{D}_{\lambda,\tau}$ is injective, it must restrict to an automorphism of $\mathcal{S}^{\mathcal{C}(\tau)}_{<m}$ for each $m \in \mathbb{N}$, concluding the proof of (1). In case $\lambda \geq 1$, so long as $m \geq \lambda + 1$, the one-dimensional $\ker(\mathcal{D}_{\lambda,\tau}) \subseteq \mathcal{S}^{\mathcal{C}(\tau)}_{<m}$, whence $\mathcal{D}_{\lambda,\tau}(\mathcal{S}^{\mathcal{C}(\tau)}_{<m})$ has codimension 1 in $\mathcal{S}^{\mathcal{C}(\tau)}_{<m}$. Also for $m \geq \lambda+1$, the computations (3.3) and (3.4) imply $\mathbf{d} \in \mathcal{S}^{\mathcal{C}(\tau)}_{<m} \Leftrightarrow \mathcal{D}_{\lambda,\tau}(\mathbf{d}) \in \mathcal{S}^{\mathcal{C}(\tau)}_{<m}$, and therefore $\mathcal{D}_{\lambda,\tau}(\mathcal{S}^{\mathcal{C}(\tau)}_{<m}) = \mathrm{im}(\mathcal{D}_{\lambda,\tau}) \cap \mathcal{S}^{\mathcal{C}(\tau)}_{<m}$. Thus $\mathrm{im}(\mathcal{D}_{\lambda,\tau})$ has codimension 1 in all of $\mathcal{S}^{\mathcal{C}(\tau)}$. ■

We refer to Remark 4.3 for a small example of the relevance of Lemma 3.4 to λ-Mahler summability. The following maps will mediate our Definition 5.13 of λ-Mahler discrete residues at torsion Mahler trees $\tau \in \mathcal{T}_+$.

Definition 3.5. Let $\lambda \in \mathbb{Z}$, $\tau \in \mathcal{T}_+$, and set $e := |\mathcal{C}(\tau)|$ as in Definition 2.7. We define the 0-*section* $\mathcal{I}^{(0)}_{\lambda,\tau}$ (of the map $\mathcal{D}_{\lambda,\tau}$ of Definition 3.2) as follows. For $(c_k(\gamma)) = \mathbf{c} \in \mathcal{S}^{\mathcal{C}(\tau)}$, let us write $(d_k(\gamma)) = \mathbf{d} = \mathcal{I}^{(0)}_{\lambda,\tau}(\mathbf{c}) \in \mathcal{S}^{\mathcal{C}(\tau)}$. We set $d_k(\gamma) = 0$ for every $\gamma \in \mathcal{C}(\tau)$ whenever $k \in \mathbb{N}$ is such that $c_{\tilde{k}}(\tilde{\gamma}) = 0$ for every $\tilde{\gamma} \in \mathcal{C}(\tau)$ and every $\tilde{k} \geq k$. For any remaining $k \in \mathbb{N}$, we define recursively

$$d_k(\gamma) := \frac{\gamma^k}{p^{(\lambda-k)e} - 1} \sum_{j=0}^{e-1} p^{(\lambda-k)j}\gamma^{-kp^j} \left[c_k(\gamma^{p^j}) - p^\lambda \sum_{s \geq k+1} \mathbb{V}_{k,1}^s(\gamma^{p^j}) d_s(\gamma^{p^{j+1}}) \right] \qquad \text{for } k \neq \lambda; \tag{3.6}$$

and, if $\lambda \geq 1$, we set

$$d_\lambda(\gamma) := \frac{\gamma^\lambda}{e} \sum_{j=0}^{e-1} (j+1-e)\gamma^{-\lambda p^j} \left[c_\lambda(\gamma^{p^j}) - p^\lambda \sum_{s \geq \lambda+1} \mathbb{V}_{\lambda,1}^s(\gamma^{p^j}) d_s(\gamma^{p^{j+1}}) \right]. \tag{3.7}$$

More generally, for any $\omega \in \mathbb{K}$, the ω-*section* $\mathcal{I}^{(\omega)}_{\lambda,\tau}$ (of $\mathcal{D}_{\lambda,\tau}$) is defined by setting

$$\mathcal{I}^{(\omega)}_{\lambda,\tau}(\mathbf{c}) := \begin{cases} \mathcal{I}^{(0)}_{\lambda,\tau}(\mathbf{c}) & \text{if } \lambda \leq 0; \\ \mathcal{I}^{(0)}_{\lambda,\tau}(\mathbf{c}) + \omega \mathbf{w}^{(\lambda)} & \text{if } \lambda \geq 1; \end{cases} \tag{3.8}$$

for every $\mathbf{c} \in \mathcal{S}^{\mathcal{C}(\tau)}$, where $\mathbf{w}^{(\lambda)}$ is the vector defined in (3.2) for $\lambda \geq 1$.

Although, by Lemma 3.4, the map $\mathcal{D}_{\lambda,\tau}$ is not always surjective, in which case it cannot have an honest inverse, we show in the following result that their ω-sections $\mathcal{I}^{(\omega)}_{\lambda,\tau}$ above come as close as possible to inverting $\mathcal{D}_{\lambda,\tau}$.

Proposition 3.6. Let $\lambda \in \mathbb{Z}$, $\tau \in \mathcal{T}_+$, and set $e := |\mathcal{C}(\tau)|$ as in Definition 2.7. Let $\omega \in \mathbb{K}$ and let $\mathcal{I}_{\lambda,\tau}^{(\omega)}$ be as in Definition 3.5. Let $\mathbf{c} \in \mathcal{S}^{\mathcal{C}(\tau)}$, and let us write $\mathbf{d} := \mathcal{I}_{\lambda,\tau}^{(\omega)}(\mathbf{c})$ and $\tilde{\mathbf{c}} := \mathcal{D}_{\lambda,\tau}(\mathbf{d})$ as in Definition 3.2. Then

$$c_k(\gamma) = \tilde{c}_k(\gamma) \qquad \text{whenever } k \neq \lambda, \text{ for every } \gamma \in \mathcal{C}(\gamma); \quad \text{and,} \tag{3.9}$$

$$\text{in case } \lambda \geq 1, \qquad c_\lambda(\gamma) - \tilde{c}_\lambda(\gamma) = \frac{\gamma^\lambda}{e}\sum_{j=1}^{e}\gamma^{-\lambda p^j}\left(c_\lambda\left(\gamma^{p^j}\right) - p^\lambda \sum_{s\geq\lambda+1} V_{\lambda,1}^s\left(\gamma^{p^j}\right)d_s\left(\gamma^{p^{j+1}}\right)\right). \tag{3.10}$$

Moreover, $\mathbf{c} \in \mathrm{im}(\mathcal{D}_{\lambda,\tau})$ if and only if $\mathbf{c} = \tilde{\mathbf{c}}$.

Proof. The expression (3.6) arises from a similar computation as in the proof of Lemma 3.4. Let $\mathbf{c} \in \mathcal{S}^{\mathcal{C}(\tau)}$ be arbitrary, and let us try (and maybe fail), to construct $\mathbf{d} \in \mathcal{S}^{\mathcal{C}(\tau)}$ such that $\mathcal{D}_{\lambda,\tau}(\mathbf{d}) = \mathbf{c}$, that is, with

$$c_k(\gamma) = -d_k(\gamma) + p^\lambda\sum_{s\geq k} V_{k,1}^s(\gamma)d_s(\gamma) \quad \Longleftrightarrow \quad p^{\lambda-k}\gamma^{k-pk}d_k(\gamma^p) - d_k(\gamma) = c_k(\gamma) - p^\lambda\sum_{s\geq k+1}V^s(\gamma)d_s(\gamma^p). \tag{3.11}$$

Then we again have the telescoping sum

$$\begin{aligned}\left(p^{(\lambda-k)e}-1\right)\gamma^{-k}d_k(\gamma) &= \sum_{j=0}^{e-1}p^{(\lambda-k)j}\gamma^{-kp^j}\cdot\left(p^{\lambda-k}\gamma^{kp^j-kp^{j+1}}d_k\left(\gamma^{p^{j+1}}\right) - d_k\left(\gamma^{p^j}\right)\right)\\ &= \sum_{j=0}^{e-1}p^{(\lambda-k)j}\gamma^{-kp^j}\cdot\left(c_k\left(\gamma^{p^j}\right) - p^\lambda\sum_{s\geq k+1}V_{k,1}^s\left(\gamma^{p^j}\right)d_s\left(\gamma^p\right)\right),\end{aligned}$$

which is equivalent to (3.6) provided precisely that $k \neq \lambda$. Thus we see that (3.6) is a *necessary* condition on the $d_k(\gamma)$ in order to satisfy (3.9). In case $\lambda \leq 0$, we know that $\mathcal{D}_{\lambda,\tau}$ is an isomorphism by Lemma 3.4(1), in which case this condition must also be sufficient and we have nothing more to show. Let us assume from now on that $\lambda \geq 1$. Since by Lemma 3.4(2) the restriction of $\mathcal{D}_{\lambda,\tau}$ to

$$\mathcal{S}_{>\lambda}^{\mathcal{C}(\tau)} := \left\{\mathbf{d} \in \mathcal{S}^{\mathcal{C}(\tau)} \;\middle|\; d_k(\gamma) = 0 \text{ for every } k \leq \lambda \text{ and } \gamma \in \mathcal{C}(\gamma)\right\}$$

is injective, and since it preserves the induced filtration (3.5), it follows that $\mathrm{pr}_\lambda \circ \mathcal{D}_{\lambda,\tau}$ restricts to an automorphism of $\mathcal{S}_{>\lambda}^{\mathcal{C}(\tau)}$, where $\mathrm{pr}_\lambda : \mathcal{S}^{\mathcal{C}(\tau)} \twoheadrightarrow \mathcal{S}_{>\lambda}^{\mathcal{C}(\tau)}$ denotes the obvious projection map. Therefore the necessary condition (3.6) must also be sufficient in order to satisfy (3.9) for $k > \lambda$. Since $\mathcal{D}_{\lambda,\tau}$ also restricts to an automorphism of $\mathcal{S}_{<\lambda}^{\mathcal{C}(\tau)}$ (trivially so in case $\lambda = 1$, since $\mathcal{S}_{<1}^{\mathcal{C}(\tau)} = \{\mathbf{0}\}$), it similarly follows that the necessary condition (3.8) must also be sufficient in order to satisfy (3.9) for any $k < \lambda$ also, regardless of how the $d_\lambda(\gamma)$ are chosen. Now for the prescribed choice of $d_\lambda(\gamma)$ in (3.7), we compute

$$\tilde{c}_\lambda(\gamma) - p^\lambda\sum_{s\geq\lambda+1}V_{\lambda,1}^s(\gamma)d_s(\gamma^p) = p^\lambda V_{\lambda,1}^\lambda(\gamma)d_\lambda(\gamma^p) - d_\lambda(\gamma) = \gamma^{\lambda-p\lambda}d_\lambda(\gamma^p) - d_\lambda(\gamma), \tag{3.12}$$

where the first equality follows from the definition of $\tilde{\mathbf{c}} = \mathcal{D}_{\lambda,\tau}(\mathbf{d})$, and the second equality from Corollary 2.16. On the other hand, after re-indexing the sum in (3.7), evaluated at γ^p instead of γ, we find that

$$\gamma^{\lambda-p\lambda}d_\lambda(\gamma^p) = \frac{\gamma^\lambda}{e}\sum_{j=1}^{e}(j-e)\gamma^{-\lambda p^j}\left[c_\lambda\left(\gamma^{p^j}\right) - p^\lambda\sum_{s\geq\lambda+1}V_{\lambda,1}^s\left(\gamma^{p^j}\right)d_s\left(\gamma^{p^{j+1}}\right)\right],$$

and after subtracting $d_\lambda(\gamma)$ exactly as given in (3.7) we find that

$$\begin{aligned}\gamma^{\lambda-p\lambda}d_\lambda(\gamma^p) - d_\lambda(\gamma) &= -\frac{\gamma^\lambda}{e}\sum_{j=1}^{e-1}\gamma^{-\lambda p^j}\left[c_\lambda\left(\gamma^{p^j}\right) - p^\lambda\sum_{s\geq\lambda+1}V_{\lambda,1}^s\left(\gamma^{p^j}\right)d_s\left(\gamma^{p^{j+1}}\right)\right]\\ &\quad - \frac{\gamma^\lambda}{e}(1-e)\gamma^{-\lambda}\left[c_\lambda(\gamma) - p^\lambda\sum_{s\geq\lambda+1}V_{\lambda,1}^s(\gamma)d_s\left(\gamma^p\right)\right]\\ &= -\frac{\gamma^\lambda}{e}\sum_{j=0}^{e-1}\gamma^{-\lambda p^j}\left[c_\lambda\left(\gamma^{p^j}\right) - p^\lambda\sum_{s\geq\lambda+1}V_{\lambda,1}^s\left(\gamma^{p^j}\right)d_s\left(\gamma^{p^{j+1}}\right)\right] + c_\lambda(\gamma) - p^\lambda\sum_{s\geq\lambda+1}V_{\lambda,1}^s(\gamma)d_s(\gamma^p),\end{aligned} \tag{3.13}$$

with the convention that the sum $\sum_{j=1}^{e-1}$ is empty in case $e = 1$. Putting (3.12) and (13) together establishes (3.10). Since $\mathbf{c} = \tilde{\mathbf{c}}$ is a non-trivial sufficient linear condition to have $\mathbf{c} \in \mathrm{im}(\mathcal{D}_{\lambda,\tau})$, by Lemma 3.4(2) it must also be necessary, since $\mathrm{im}(\mathcal{D}_{\lambda,\tau})$ has codimension 1 in $\mathcal{S}^{\mathcal{C}(\tau)}$. This concludes the proof. ■

4 Mahler Dispersion and λ-Mahler Summability

Our goal in this section is to prove Theorem 4.2: if $f \in \mathbb{K}(x)$ is λ-Mahler summable for some $\lambda \in \mathbb{Z}$, then it has non-zero dispersion almost everywhere, generalizing to arbitrary $\lambda \in \mathbb{Z}$ the analogous result for $\lambda = 0$ obtained in [9, Corollary 3.2]. In spite of the exceptions that occur for $\lambda \geq 1$, this will be an essential tool in our proofs that twisted Mahler discrete residues comprise a complete obstruction to λ-Mahler summability.

In the following preliminary result, which generalizes [9, Proposition 3.1] from the special case $\lambda = 0$ to arbitrary $\lambda \in \mathbb{Z}$, we relate the Mahler dispersions of a λ-Mahler summable $f \in \mathbb{K}(x)$ to those of a certificate $g \in \mathbb{K}(x)$ such that $f = \Delta_\lambda(g)$.

Proposition 4.1. Let $f, g \in \mathbb{K}(x)$ and $\lambda \in \mathbb{Z}$ such that $f = \Delta_\lambda(g)$.

1) If $\infty \in \mathrm{supp}(f)$, then $\mathrm{disp}(f, \infty) = \mathrm{disp}(g, \infty) + 1$, except in case $\lambda \neq 0$ and the Laurent polynomial component $f_\infty = c_0 \in \mathbb{K}^\times$, in which case we must have $g_\infty = c_0/(p^\lambda - 1)$.
2) If $\infty \neq \tau \in \mathrm{supp}(f)$, then $\mathrm{disp}(f, \tau) = \mathrm{disp}(g, \tau) + 1$, with the convention that $\infty + 1 = \infty$, except possibly in case that: $\tau \in \mathcal{T}_+$ is torsion ; and $\lambda \geq 1$; and g has a pole of order exactly λ at every $\gamma \in \mathcal{C}(\tau)$.

Proof. (1). First suppose that $\{0\} \neq \theta \in \mathbb{Z}/\mathcal{P}$ is such that $g_\theta \neq 0$, and let us write $g_\theta = \sum_{j=0}^{d} c_{ip^j} x^{ip^j}$, where we assume that $c_i c_{ip^d} \neq 0$, that is, that $\mathrm{disp}(g_\theta, \infty) = d$. Then

$$\Delta_\lambda(g_\theta) = p^\lambda c_{ip^d} x^{ip^{d+1}} - c_i x^i + \sum_{j=1}^{d} (p^\lambda c_{ip^{j-1}} - c_{ip^j}) x^{ip^j},$$

from which it follows that $0 \neq f_\theta = \Delta_\lambda(g_\theta)$ and $\mathrm{disp}(f_\theta, \infty) = \mathrm{disp}(\Delta_\lambda(g_\theta), \infty) = d + 1$. Since in this case

$$\mathrm{disp}(f, \infty) = \max\left\{\mathrm{disp}\left(f_\theta, \infty\right) \mid \{0\} \neq \theta \in \mathbb{Z}/\mathcal{P}, f_\theta \neq 0\right\}$$

by Definition 2.11(2), and similarly for $\mathrm{disp}(g, \infty)$, we find that $\mathrm{disp}(f, \infty) = \mathrm{disp}(g, \infty) + 1$ provided that the Laurent component $g_\infty \in \mathbb{K}[x, x^{-1}]$ is not constant.

In any case, by Lemma 2.10(2,3), if $\infty \in \mathrm{supp}(f)$ then $\infty \in \mathrm{supp}(g)$. In this case, we have $0 \neq f_\infty = \Delta_\lambda(g_\infty)$, since $\infty \in \mathrm{supp}(f)$, and if $\lambda = 0$ it follows in particular $g_\infty \notin \mathbb{K}$. In case $\lambda \neq 0$ and $f_\infty = c_0 \in \mathbb{K}^\times$, the computation above shows that $g_\theta = 0$ for every $\{0\} \neq \theta \in \mathbb{Z}/\mathcal{P}$, and we see that $g_\infty = g_{\{0\}} = c_0/(p^\lambda - 1)$.

(2). Suppose $\tau \in \mathrm{supp}(f)$, and therefore $\tau \in \mathrm{supp}(g)$ by Lemma 2.10(4). We consider two cases, depending on whether $\mathrm{disp}(g, \tau)$ is finite or not.

If $\mathrm{disp}(g, \tau) =: d < \infty$, let $\alpha \in \tau$ be such that α and α^{p^d} are poles of g. Choose $\gamma \in \tau$ such that $\gamma^p = \alpha$. Then γ is a pole of $\sigma(g)$ but not of g (by the maximality of d), and therefore γ is a pole of f. On the other hand, $\gamma^{p^{d+1}} = \alpha^{p^d}$ is a pole of g but not of $\sigma(g)$, for if α^{p^d} were a pole of $\sigma(g)$ then $\alpha^{p^{d+1}}$ would be a pole of g, contradicting the maximality of d. Therefore $\gamma^{p^{d+1}}$ is a pole of f. It follows that $\mathrm{disp}(f, \tau) \geq d + 1$. One can show equality by contradiction: if $\alpha \in \tau$ is a pole of f such that α^{p^s} is also a pole of f for some $s > d + 1$, then each of α and α^{p^s} is either a pole of g or a pole of $\sigma(g)$. If α^{p^s} is a pole of g, then α cannot also be a pole of g, for this would contradict the maximality of d, whence α must be a pole of $\sigma(g)$, but then α^p would have to be a pole of g, still contradicting the maximality of d. Hence α^{p^s} must be a pole of $\sigma(g)$. But then $\alpha^{p^{s+1}}$ is a pole of g, which again contradicts the maximality of d whether α is a pole of $\sigma(g)$ or of g. This concludes the proof that $\mathrm{disp}(f, \tau) = \mathrm{disp}(g, \tau) + 1$ in this case where $\mathrm{disp}(g, \tau) < \infty$.

If $\mathrm{disp}(g, \tau) = \infty$ then g has a pole in $\mathcal{C}(\tau)$ by Lemma 2.12, and therefore $\tau \in \mathcal{T}_+$ (cf. Remark 2.8). If f also has a pole in $\mathcal{C}(\tau)$ then $\mathrm{disp}(f, \tau) = \infty = \mathrm{disp}(g, \tau) + 1$ and we are done. So let us suppose $\mathrm{disp}(f, \tau) < \infty$ and conclude that g has a pole of order exactly λ at every $\gamma \in \mathcal{C}(\tau)$. In this case, writing

$$0 \neq \mathcal{C}(g_\tau) = \sum_{k \in \mathbb{N}} \sum_{\gamma \in \mathcal{C}(\tau)} \frac{d_k(\gamma)}{(x - \gamma)^k} \quad \text{and} \quad 0 = \mathcal{C}(f_\tau) = \sum_{k \in \mathbb{N}} \sum_{\gamma \in \mathcal{C}(\tau)} \frac{c_k(\gamma)}{(x - \gamma)^k}$$

as in Definition 3.1, it follows from Lemma 3.3 that $\mathcal{D}_{\lambda,\tau}(\mathbf{d}) = \mathbf{c}$, where $\mathbf{d} := (d_k(\gamma))$ and $\mathbf{c} := (c_k(\gamma)) = \mathbf{0}$. By Lemma 3.4, $\lambda \geq 1$ and $\mathbf{d} = \omega \mathbf{w}^{(\lambda)}$ for some $0 \neq \omega \in \mathbb{K}$, where $\mathbf{w}^{(\lambda)} = (w_k^{(\lambda)}(\gamma))$ is the unique vector specified in Lemma 3.4(2), whose components satisfy $w_k^{(\lambda)}(\gamma) = 0$ for $k > \lambda$ and $w_\lambda^{(\lambda)}(\gamma) = \gamma^\lambda \neq 0$ for every $\gamma \in \mathcal{C}(\tau)$. ■

In the next result we deduce from Proposition 4.1 that if $f \in \mathbb{K}(x)$ is λ-Mahler summable then f has non-zero dispersion almost everywhere.

Theorem 4.2. Let $\lambda \in \mathbb{Z}$ and and suppose that $f \in \mathbb{K}(x)$ is λ-Mahler summable.

1) If $\infty \in \mathrm{supp}(f)$ and either $\lambda = 0$ or $f_\infty \notin \mathbb{K}$ then $\mathrm{disp}(f, \infty) > 0$.
2) If $\lambda \leq 0$ then $\mathrm{disp}(f, \tau) > 0$ for every $\infty \neq \tau \in \mathrm{supp}(f)$.
3) If $\lambda \geq 1$ and $\infty \neq \tau \in \mathrm{supp}(f)$ is such that either $\tau \in \mathcal{T}_0$ or $\mathrm{ord}(f, \tau) \neq \lambda$ then $\mathrm{disp}(f, \tau) > 0$.

Proof. Suppose $f \in \mathbb{K}(x)$ is λ-Mahler summable and let $g \in \mathbb{K}(x)$ such that $f = \Delta_\lambda(g)$.

(1) and (2). If $\infty \in \mathrm{supp}(f)$ then by Proposition 4.1(1) $\mathrm{disp}(f, \infty) = \mathrm{disp}(g, \infty) + 1 > 0$ provided that either $\lambda = 0$ or $f_\infty \notin \mathbb{K}$. If $\lambda \leq 0$ then $\mathrm{disp}(f, \tau) = \mathrm{disp}(g, \tau) + 1 > 0$ for all $\infty \neq \tau \in \mathrm{supp}(f)$ by Proposition 4.1(2).

(3). Assuming that $\lambda \geq 1$, we know by Proposition 4.1(2) that $\mathrm{disp}(f, \tau) = \mathrm{disp}(g, \tau) + 1 > 0$ for every $\infty \neq \tau \in \mathrm{supp}(f)$, except possibly in case $\tau \in \mathcal{T}_+$ and g has a pole of order exactly λ at every $\gamma \in \mathcal{C}(\tau)$. Thus our claim is already proved for $\tau \in \mathcal{T}_0$. So from now on we suppose $\tau \in \mathcal{T}_+$. By Lemma 2.10(7), $\mathrm{ord}(f, \tau) = \mathrm{ord}(g, \tau)$, and therefore if $\mathrm{ord}(f, \tau) < \lambda$, there are no poles of g of order λ anywhere in τ, let alone in $\mathcal{C}(\tau)$, whence $\mathrm{disp}(f, \tau) = \mathrm{disp}(g, \tau) + 1 > 0$ by Proposition 4.1(2) in this case also. Moreover, if f has a pole of any order in $\mathcal{C}(\tau)$, then $\mathrm{disp}(f, \tau) = \infty > 0$ by Lemma 2.12. It remains to show that if $m := \mathrm{ord}(f, \tau) > \lambda$ then $\mathrm{disp}(f, \tau) > 0$. In this case, even though $\mathrm{ord}(g, \tau) = m > \lambda$ by Lemma 2.10(7) it could be the case that g has a pole of order exactly λ at every $\gamma \in \mathcal{C}(\tau)$ and yet the order-m poles of g lie in the complement $\tau - \mathcal{C}(\tau)$, in which case Proposition 4.1 remains silent. So let $\alpha_1, \ldots, \alpha_s \in \mathrm{sing}(g, \tau)$ be all the pairwise-distinct elements at which g has a pole of order $m > \lambda$. Choose $\beta_j \in \tau$ such that $\beta_j^p = \alpha_j$ for $j = 1, \ldots, s$, and let us write $g_\tau = \sum_{j=1}^s \frac{d_j}{(x-\alpha_j)^m} +$ (lower-order terms) so that

$$f_\tau = \sum_{j=1}^{s} \left(\sum_{i=0}^{p-1} \frac{p^\lambda V_{m,1}^m(\zeta_p^i \beta_j) \cdot d_j}{(x - \zeta_p^i \beta_j)^m} - \frac{d_j}{(x - \alpha_j)^m} \right) + \text{(lower-order-terms)} \tag{4.1}$$

by Lemma 2.17. If any $\alpha_j \in \mathcal{C}(\tau)$, then by Proposition 4.1(2) we already have $\mathrm{disp}(f, \tau) = \mathrm{disp}(g, \tau) + 1 > 0$. So let us now further assume that no α_j belongs to $\mathcal{C}(\tau)$. Then there exists $j_0 \in \{1, \ldots, s\}$ such that $\alpha_j^{p^r} \neq \alpha_{j_0}$ for every $j \neq j_0$ and every $r \in \mathbb{Z}_{\geq 0}$, for otherwise we would have at least one $\alpha_j \in \mathcal{C}(\tau)$. Then every $\zeta_p^i \beta_j \neq \alpha_{j_0}$, and it is now clear that the apparent pole of f_τ at this α_{j_0} in (4.1) is a true pole of f_τ (i.e., it does get canceled). Similarly, the p elements $\zeta_p^i \beta_{j_0}$, which are obviously pairwise distinct, are also all different from every α_j, and moreover $\zeta_p^i \beta_{j_0} = \zeta_p^{i'} \beta_{j'}$ if and only if $i' = i$ and $j' = j_0$. Thus, in particular, the apparent pole of f_τ at β_{j_0} in (4.1) is also a true pole of f_τ. Thus $\mathrm{disp}(f, \tau) \geq 1$ also in this last case where $\mathrm{ord}(f, \tau) = m > \lambda$. ■

Remark 4.3. The exceptions in Theorem 4.2 cannot be omitted. If $\lambda \neq 0$ then every $\Delta_\lambda(\frac{c}{p^\lambda - 1}) = c \in \mathbb{K}$ is λ-Mahler summable and has $\mathrm{disp}(c, \infty) = 0$ whenever $c \neq 0$. If $\lambda \geq 1$ then for any $\gamma \in \mathcal{C}(\tau)$ with $\varepsilon(\tau) =: e \geq 1$ one can construct (cf. Section 5.3) $g = \sum_{k=1}^{\lambda} \sum_{\ell=0}^{e-1} c_{k,\ell} \cdot (x - \gamma^{p^\ell})^{-k}$ such that $\mathrm{disp}(\Delta_\lambda(g), \tau) = 0$. The simplest such example is with $\lambda, \gamma, e = 1$ (and $p \in \mathbb{Z}_{\geq 2}$ still arbitrary):

$$f := \Delta_1\left(\frac{1}{x-1}\right) = \frac{p}{x^p - 1} - \frac{1}{x-1} = \frac{pV_{1,1}^1(1) - 1}{x - 1} + \sum_{i=1}^{p-1} \frac{pV_{1,1}^1(\zeta_p^i)}{x - \zeta_p^i} = \sum_{i=1}^{p-1} \frac{\zeta_p^i}{x - \zeta_p^i},$$

which is 1-Mahler summable but has $\mathrm{disp}(f, \tau(1)) = 0$. We provide a slightly more elaborate illustration of this phenomenon in Example 7.3. More generally, all other such examples for arbitrary $\lambda \geq 1$ and $\tau \in \mathcal{T}_+$, of $f \in \mathbb{K}(x)$ such that f_τ is λ-Mahler summable but $\mathrm{disp}(f, \tau) = 0$, arise essentially from the construction $f_\tau := \Delta_\lambda(g_\tau)$ with

$$g_\tau = \sum_{k=1}^{\lambda} \sum_{\gamma \in \mathcal{C}(\tau)} \frac{\omega \cdot w_k^{(\lambda)}(\gamma)}{(x - \gamma)^k}$$

for an arbitrary constant $0 \neq \omega \in \mathbb{K}$ and the vector $\mathbf{w}^{(\lambda)} = (w_k^{(\lambda)}(\gamma))$ defined in Lemma 3.4(2).

5 Twisted Mahler Discrete Residues

Our goal in this section is to define the λ-Mahler discrete residues of $f(x) \in \mathbb{K}(x)$ for $\lambda \in \mathbb{Z}$ and prove our Main Theorem in Section 5.4, that these λ-Mahler discrete residues comprise a complete obstruction to λ-Mahler summability. We begin with the relatively simple construction of λ-Mahler discrete residues at ∞ (for Laurent polynomials), followed by the construction of λ-Mahler discrete residues at Mahler trees $\tau \in \mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_+$ (see Definition 2.7), first for non-torsion $\tau \in \mathcal{T}_0$, and finally for torsion $\tau \in \mathcal{T}_+$, in increasing order of complexity, and prove separately in each case that these λ-Mahler discrete residues comprise a complete obstruction to the λ-Mahler summability of the corresponding components of f.

5.1 Twisted Mahler discrete residues at infinity

We now define the λ-Mahler discrete residue of $f \in \mathbb{K}(x)$ at ∞ in terms of the Laurent polynomial component $f_\infty \in \mathbb{K}[x, x^{-1}]$ of f in (2.1), and show that it forms a complete obstruction to the λ-Mahler summability of f_∞. The definition and proof in this case are both straightforward, but they provide helpful moral guidance for the analogous definitions and proofs in the case of λ-Mahler discrete residues at Mahler trees $\tau \in \mathcal{T}$.

Definition 5.1. For $f \in \mathbb{K}(x)$ and $\lambda \in \mathbb{Z}$, the λ-*Mahler discrete residue* of f at ∞ is the vector

$$\mathrm{dres}_\lambda(f,\infty) = \Big(\mathrm{dres}_\lambda(f,\infty)_\theta\Big)_{\theta\in\mathbb{Z}/\mathcal{P}} \in \bigoplus_{\theta\in\mathbb{Z}/\mathcal{P}} \mathbb{K}$$

defined as follows. Write $f_\infty = \sum_{\theta\in\mathbb{Z}/\mathcal{P}} f_\theta$ as in Definition 2.2, and write each component $f_\theta = \sum_{j=0}^{h_\theta} c_{ip^j} x^{ip^j}$ with $p \nmid i$ whenever $i \neq 0$ (that is, with each i initial in its maximal $\mathcal{P}$-trajectory θ), and where $h_\theta = 0$ if $f_\theta = 0$ and otherwise $h_\theta \in \mathbb{Z}_{\geq 0}$ is as large as possible such that $c_{ip^{h_\theta}} \neq 0$. Then we set

$$\mathrm{dres}_\lambda(f,\infty)_\theta := p^{\lambda h_\theta}\sum_{j=0}^{h_\theta} p^{-\lambda j} c_{ip^j} \quad \text{for } \theta \neq \{0\}; \qquad \text{and} \qquad \mathrm{dres}_\lambda(f,\infty)_{\{0\}} := \begin{cases} c_0 & \text{if } \lambda = 0; \\ 0 & \text{if } \lambda \neq 0. \end{cases}$$

Proposition 5.2. For $f \in \mathbb{K}(x)$ and $\lambda \in \mathbb{Z}$, the component $f_\infty \in \mathbb{K}[x, x^{-1}]$ in (2.1) is λ-Mahler summable if and only if $\mathrm{dres}_\lambda(f,\infty) = \mathbf{0}$.

Proof. By Lemma 2.3, f_∞ is λ-Mahler summable if and only if f_θ is λ-Mahler summable for all $\theta \in \mathbb{Z}/\mathcal{P}$. We shall show that f_θ is λ-Mahler summable if and only if $\mathrm{dres}_\lambda(f,\infty)_\theta = 0$. If $\lambda \neq 0$ then $f_{\{0\}} = \Delta_\lambda(\frac{c_0}{p^\lambda - 1})$ is always λ-Mahler summable, while we have defined $\mathrm{dres}_\lambda(f,\infty)_{\{0\}} = 0$ in this case. On the other hand, for $\lambda = 0$, $f_{\{0\}} = \mathrm{dres}_0(f,\infty)_{\{0\}}$, and $\mathrm{disp}(f_{\{0\}}, \infty) = 0$ if $f_{\{0\}} \neq 0$, while if $f_{\{0\}} = 0$ then it is clearly λ-Mahler summable. By Theorem 4.2(1) in case $\lambda = 0$, and trivially in case $\lambda \neq 0$, we conclude that $f_{\{0\}}$ is λ-Mahler summable if and only if $\mathrm{dres}_\lambda(f,\infty)_{\{0\}} = 0$.

Now let us assume $\{0\} \neq \theta \in \mathbb{Z}/\mathcal{P}$ and let us write $f_\theta = \sum_{j\geq 0} c_{ip^j} x^{ip^j} \in \mathbb{K}[x, x^{-1}]_\theta$, for the unique minimal $i \in \theta$ such that $p \nmid i$. If $f_\theta = 0$ then we have nothing to show, so suppose $f_\theta \neq 0$ and let $h_\theta \in \mathbb{Z}_{\geq 0}$ be maximal such that $c_{ip^{h_\theta}} \neq 0$. Letting $\Delta_\lambda^{(n)} := p^{\lambda n}\sigma^n - \mathrm{id}$ as in Lemma 2.17, we find that

$$\bar{f}_{\lambda,\theta} := f_\theta + \sum_{j=0}^{h_\theta} \Delta_\lambda^{(h_\theta - j)}(c_{ip^j} x^{ip^j}) = \sum_{j=0}^{h_\theta} p^{\lambda(h_\theta - j)} c_{ip^j} x^{ip^{h_\theta}} + 0 = \mathrm{dres}_\lambda(f,\infty)_\theta \cdot x^{ip^{h_\theta}}.$$

By Lemma 2.17, we see that f_θ is λ-Mahler summable if and only if $\bar{f}_{\lambda,\theta}$ is λ-Mahler summable. Clearly, $\bar{f}_{\lambda,\theta} = 0$ if and only if $\mathrm{dres}(f,\infty)_\theta = 0$. We also see that $\mathrm{disp}(\bar{f}_{\lambda,\theta}, \infty) = 0$ if $\mathrm{dres}_\lambda(f,\infty)_\theta \neq 0$, in which case $\bar{f}_{\lambda,\theta}$ cannot be λ-Mahler summable by Theorem 4.2(1), and so f_θ cannot be λ-Mahler summable either. On the other hand, if $\bar{f}_{\lambda,\theta} = 0$ then f_θ is λ-Mahler summable by Lemma 2.17. ■

Remark 5.3. The factor of $p^{\lambda h_\theta}$ in the Definition 5.1 of $\mathrm{dres}_\lambda(f,\infty)_\theta$ for $\{0\} \neq \theta \in \mathbb{Z}/\mathcal{P}$ plays no role in deciding whether f_∞ is λ-Mahler summable, but this normalization allows us to define uniformly the $\bar{f}_{\lambda,\theta} = \mathrm{dres}_\lambda(f,\infty)_\theta \cdot x^{ip^{h_\theta}}$ as the θ-component of the $\bar{f}_\lambda \in \mathbb{K}(x)$ in the λ-*Mahler* *reduction* (1.2). For every $\{0\} \neq \theta \in \mathbb{Z}/\mathcal{P}$, we set $h_\theta(f)$ to be the h_θ defined in the course of the proof of Proposition 5.2 in case $f_\theta \neq 0$, and in all other cases we set $h_\theta(f) := 0$.

5.2 Twisted Mahler discrete residues at Mahler trees: the non-torsion case

We now define the λ-Mahler discrete residues of $f \in \mathbb{K}(x)$ at non-torsion Mahler trees $\tau \in \mathcal{T}_0$ in terms of the partial fraction decomposition of the component $f_\tau \in \mathbb{K}(x)_\tau$ in Definition 2.5, and show that it forms a complete obstruction to the λ-Mahler summability of f_τ. We begin by introducing some auxiliary notions, which already appeared in [9], but with an unfortunately different choice of notation.

Definition 5.4. Let $\tau \in \mathcal{T}_0$, $\gamma \in \tau$, and $h \in \mathbb{Z}_{\geq 0}$. The *bouquet* of height h rooted at γ is

$$\beta_h(\gamma) := \left\{\alpha \in \tau \mid \alpha^{p^n} = \gamma \text{ for some } 0 \leq n \leq h\right\}.$$

Lemma 5.5 (cf. [9, Lem. 4.4]). Let $\tau \in \mathcal{T}_0$ and $S \subset \tau$ be a finite non-empty subset. Then there exists a unique $\gamma \in \tau$ such that $S \subseteq \beta_h(\gamma)$ with h as small as possible.

Proof. This is an immediate consequence of the proof of [9, Lem. 4.4], whose focus and notation was rather different from the one adopted here, so let us complement it here with an alternative and more conceptual argument. As explained in [9, Remark 2.7 and Example 2.9], we can introduce a digraph structure on τ in which we have a directed edge $\alpha \to \xi$ whenever $\alpha^p = \xi$, resulting in an infinite (directed) tree. The "meet" of the elements of S is the unique $\gamma \in \tau$ such that $S \subseteq \beta_h(\gamma)$ with h as small as possible. ■

Definition 5.6 (cf. [9, Def. 4.6]). For $f \in \mathbb{K}(x)$ and $\tau \in \mathrm{supp}(f) \cap \mathcal{T}_0$, the *height* of f at τ, denoted by $\mathrm{ht}(f,\tau)$, is the smallest $h \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{sing}(f,\tau) \subseteq \beta_h(\gamma)$ for the unique $\gamma \in \tau$ identified in Lemma 5.5 with $S = \mathrm{sing}(f,\tau) \subset \tau$. We write $\beta(f,\tau) := \beta_h(\gamma)$, the *bouquet* of f in τ. For $\alpha \in \beta(f,\tau)$, the *height* of α in f, denoted by $\eta(\alpha|f)$, is the unique $0 \leq n \leq h$ such that $\alpha^{p^n} = \gamma$.

Example 5.7. Consider $p = 3$ and $\tau = \tau(2)$ as in [9, Example 2.9], let $\alpha_1 := \zeta_3 \sqrt[9]{2}$ and $\alpha_2 := \zeta_3^2 \sqrt[3]{2}$ and suppose $f \in \mathbb{K}(x)$ is such that $\mathrm{sing}(f,\tau) = \{\alpha_1, \alpha_2\}$. The first common 3-power power of α_1 and α_2 is $\gamma = 2 = \alpha_1^{3^2} = \alpha_2^3$ — this is the "meet" of α_1 and α_2 referred to in the proof of Lemma 5.5, and $h = 2$ is the largest exponent n such that $\alpha^{3^n} = \gamma$ for some $\alpha \in \mathrm{sing}(f,\tau)$. We see that

$$\mathrm{sing}(f,\tau) \subset \beta_2(2) = \{\alpha \in \mathbb{K} \mid \alpha^{3^2} = 2\} \cup \{\alpha \in \mathbb{K} \mid \alpha^{3^1} = 2\} \cup \{\alpha \in \mathbb{K} \mid \alpha^{3^0} = 2\} = \beta(f,\tau),$$

which is the union of the elements α whose $\eta(\alpha|f)$ (height in f) is 2, 1, 0, respectively. So $\eta(\alpha_1|f) = 2$ and $\eta(\alpha_2|f) = 1$ and $\eta(2|f) = 0$ (the latter is defined even though $2 \notin \mathrm{sing}(f,\tau)$, because $2 \in \beta(f,\tau)$).

In [9, Def. 4.10] we gave a recursive definition in the $\lambda = 0$ case of Mahler discrete residues for non-torsion $\tau \in \mathcal{T}_0$. Here we provide a non-recursive definition for $\lambda \in \mathbb{Z}$ arbitrary, which can be shown to agree with the one from [9] in the special case $\lambda = 0$ (see Proposition 5.21).

Definition 5.8. For $f \in \mathbb{K}(x)$, $\lambda \in \mathbb{Z}$, and $\tau \in \mathcal{T}_0$, the λ-*Mahler discrete residue* of f at τ of degree $k \in \mathbb{N}$ is the vector

$$\mathrm{dres}_\lambda(f,\tau,k) = \Big(\mathrm{dres}_\lambda(f,\tau,k)_\alpha\Big)_{\alpha\in\tau} \in \bigoplus_{\alpha\in\tau} \mathbb{K}$$

defined as follows.

We set $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ if either $\tau \notin \mathrm{supp}(f)$ or $k > \mathrm{ord}(f,\tau)$ as in Definition 2.9. For $\tau \in \mathrm{supp}(f)$, let

$$f_\tau = \sum_{k\in\mathbb{N}} \sum_{\alpha\in\tau} \frac{c_k(\alpha)}{(x-\alpha)^k}. \tag{5.1}$$

We set $\mathrm{dres}_\lambda(f,\tau,k)_\alpha = 0$ for every $k \in \mathbb{N}$ whenever $\alpha \in \tau$ is such that either $\alpha \notin \beta(f,\tau)$ or, for $\alpha \in \beta(f,\tau)$, such that $\eta(\alpha|f) \neq h$, where $h := \mathrm{ht}(f,\tau)$ and $\beta(f,\tau)$ are as in Definition 5.6.
Finally, for the remaining $\alpha \in \beta(f,\tau)$ with $\eta(\alpha|f) = h$ and $1 \leq k \leq \mathrm{ord}(f,\tau) =: m$, we define

$$\mathrm{dres}_\lambda(f,\tau,k)_\alpha := \sum_{s=k}^{m}\sum_{n=0}^{h} p^{\lambda n} V^s_{k,n}(\alpha) c_s(\alpha^{p^n}), \tag{5.2}$$

where the Mahler coefficients $V^s_{k,n}(\alpha)$ are as in Proposition 2.15.

We compute explicitly the 1-Mahler discrete residues of a concrete $f \in \mathbb{K}(x)$ in Example 7.1. A computational example of 0-Mahler discrete residues is presented in [9, Example 5.1], but computed differently (see Proposition 5.21).

Proposition 5.9. For $f \in \mathbb{K}(x)$, $\lambda \in \mathbb{Z}$, and $\tau \in \mathcal{T}_0$, the component f_τ is λ-Mahler summable if and only if $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $k \in \mathbb{N}$.

Proof. The statement is trivial for $\tau \notin \mathrm{supp}(f) \Leftrightarrow f_\tau = 0$. So let us suppose $\tau \in \mathrm{supp}(f)$, and let $h := \mathrm{ht}(f,\tau)$, $m := \mathrm{ord}(f,\tau)$, and $\eta(\alpha) := \eta(\alpha|f)$ for each $\alpha \in \beta(f,\tau)$. Writing f_τ as in (5.1), let us also write, for $0 \leq n \leq h$,

$$f_\tau^{(n)} := \sum_{k=1}^{m} \sum_{\substack{\alpha\in\beta(f,\tau)\\ \eta(\alpha)=n}} \frac{c_k(\alpha)}{(x-\alpha)^k} \qquad \text{so that} \qquad f_\tau = \sum_{n=0}^{h} f_\tau^{(n)}.$$

By Lemma 2.17, for each $0 \leq n \leq h$, we have

$$\sigma^n\left(f_\tau^{(h-n)}\right) = \sum_{k=1}^{m} \sum_{\substack{\alpha\in\beta(f,\tau)\\ \eta(\alpha)=h}} \frac{\sum_{s=k}^{m} V^s_{k,n}(\alpha)c_s(\alpha^{p^n})}{(x-\alpha)^k},$$

and therefore

$$\Delta_\lambda^{(n)}\left(f_\tau^{(h-n)}\right) = -f_\tau^{(h-n)} + \sum_{k=1}^{m} \sum_{\substack{\alpha\in\beta(f,\tau)\\ \eta(\alpha)=h}} \frac{p^{\lambda n}\sum_{s=k}^{m} V^s_{k,n}(\alpha)c_s(\alpha^{p^n})}{(x-\alpha)^k}.$$

It follows from the Definition 5.8 that

$$\bar{f}_\tau := f_\tau + \sum_{n=0}^{n} \Delta_\lambda^{(n)}\left(f_\tau^{(h-n)}\right) = \sum_{k=1}^{m}\sum_{\alpha\in\tau} \frac{\mathrm{dres}_\lambda(f,\tau,k)_\alpha}{(x-\alpha)^k}. \tag{5.3}$$

By Lemma 2.17, $\bar{f}_{\lambda,\tau} - f_\tau$ is λ-Mahler summable, and therefore f_τ is λ-Mahler summable if and only if $\bar{f}_{\lambda,\tau}$ is λ-Mahler summable. If $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $1 \leq k \leq m$, then $\bar{f}_{\lambda,\tau} = 0$ and therefore f_τ is λ-Mahler summable. On the other hand, if some $\mathrm{dres}_\lambda(f,\tau,k) \neq \mathbf{0}$, then $0 \neq \bar{f}_{\lambda,\tau}$ has $\mathrm{disp}(\bar{f}_{\lambda,\tau},\tau) = 0$. This is because, in Definition 5.8, the only $\alpha \in \tau$ for which $\mathrm{dres}(f,\tau,k)_\alpha$ could possibly be non-zero in (5.2) are those with $\eta(\alpha) = h$, so it is impossible to have any such α be a p-power power of another (see Definition 5.6 and Definition 2.11(1)). By Theorem 4.2(2,3) $\bar{f}_{\lambda,\tau}$ could not possibly be λ-Mahler summable unless it is 0, and therefore neither could f_τ. This concludes the proof that f_τ is λ-Mahler summable if and only if $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $k \in \mathbb{N}$. ■

Remark 5.10. For $f \in \mathbb{K}(x)$ and $\tau \in \mathrm{supp}(f) \cap \mathcal{T}_0$, the element $\bar{f}_{\lambda,\tau}$ in (5.3) is the τ-component of the $\bar{f}_\lambda \in \mathbb{K}(x)$ in the λ-*Mahler reduction* (1.2).

5.3 Twisted Mahler discrete residues at Mahler trees: the torsion case

We now define the λ-Mahler discrete residues of $f \in \mathbb{K}(x)$ at torsion trees $\tau \in \mathcal{T}_+$ (see Definition 2.7) in terms of the partial fraction decomposition of the component $f_\tau \in \mathbb{K}(x)_\tau$ in Definition 2.5, and show that it forms a complete obstruction to the λ-Mahler summability of f_τ. The definitions and proofs in this case are more technical than in the non-torsion case, involving the cycle map $\mathcal{D}_{\lambda,\tau}$ of Definition 3.2 and its ω-section $\mathcal{I}^{(\omega)}_{\lambda,\tau}$ from Definition 3.5, for a particular choice of constant $\omega \in \mathbb{K}$ associated to f, which we construct in Definition 5.12.

We begin by recalling the following definition from [9], which is the torsion analogue of Definition 5.6.

Definition 5.11 (cf. [9, Def. 4.6]). For $\tau \in \mathcal{T}_+$ and $\alpha \in \tau$, the *height* of α, denoted by $\eta(\alpha)$, is the smallest $n \in \mathbb{Z}_{\geq 0}$ such that $\alpha^{p^n} \in \mathcal{C}(\tau)$ (cf. Definition 2.7). For $f \in \mathbb{K}(x)$ and $\tau \in \mathrm{supp}(f) \cap \mathcal{T}_+$, the *height* of f at τ is

$$\mathrm{ht}(f,\tau) := \max\{\eta(\alpha) \mid \alpha \in \mathrm{sing}(f,\tau)\},$$

or equivalently, the smallest $h \in \mathbb{Z}_{\geq 0}$ such that $\alpha^{p^h} \in \mathcal{C}(\tau)$ for every pole α of f in τ.

The following technical definition will allow us to use the correct ω-section $\mathcal{I}^{(\omega)}_{\lambda,\tau}$ from Definition 3.5 in our construction of λ-Mahler discrete residues in the torsion case.

Definition 5.12. For $f \in \mathbb{K}(x)$ and $\tau \in \mathrm{supp}(f) \cap \mathcal{T}_+$, let us write

$$f_\tau = \sum_{k\in\mathbb{N}} \sum_{\alpha\in\tau} \frac{c_k(\alpha)}{(x-\alpha)^k}.$$

For $\lambda \in \mathbb{Z}$, we define the *residual average* $\omega_{\lambda,\tau}(f) \in \mathbb{K}$ of f (relative to λ and τ) as follows. If $\lambda \leq 0$ or if $h := \mathrm{ht}(f,\tau) = 0$ (cf. Definition 5.11), we simply set $\omega_{\lambda,\tau}(f) = 0$. In case both $\lambda, h \geq 1$, let $\tau_h := \{\alpha \in \tau \mid \eta(\alpha) = h\}$ be the set of elements of τ of height h. Let us write $\mathbf{c} = (c_k(\gamma))$, for γ ranging over $\mathcal{C}(\tau)$ only, and let $(d^{(0)}_k(\gamma)) = \mathbf{d}^{(0)} := \mathcal{I}^{(0)}_{\lambda,\tau}(\mathbf{c})$ as in Definition 3.5 and $(\tilde{c}_k(\gamma)) = \tilde{\mathbf{c}} = \mathcal{D}_{\lambda,\tau}(\mathbf{d}^{(0)})$, as in Definition 3.5. Then we define

$$\omega_{\lambda,\tau}(f) := \frac{1}{(p^h - p^{h-1})e} \sum_{\alpha\in\tau_h} \sum_{s\geq\lambda} \sum_{n=0}^{h-1} p^{\lambda n} \mathbb{V}^s_{\lambda,n} \alpha^{-sp^n} c_s(\alpha^{p^n}) - \frac{p^{\lambda(h-1)}}{e} \sum_{\gamma\in\mathcal{C}(\tau)} \sum_{s\geq\lambda} \mathbb{V}^s_{\lambda,h-1} \gamma^{-s} (\tilde{c}_s(\gamma) + d^{(0)}_s(\gamma)), \quad (5.4)$$

where the universal Mahler coefficients $\mathbb{V}^s_{\lambda,n} \in \mathbb{Q}$ are defined as in Section 2.5.

Explicit computations of the residual average $\omega_{\lambda,\tau}(f)$ are presented in Examples 5.15, 7.2, and 7.3. The significance of this definition and our choice of nomenclature is explained in the proof of Proposition 5.19 below (with the aid of Lemma 5.18). We are now ready to define the λ-Mahler discrete residues at torsion Mahler trees. In [9, Def. 4.16] we gave a *recursive* definition of Mahler discrete residues for torsion $\tau \in \mathcal{T}_+$ in the $\lambda = 0$ case. Here we provide a less recursive definition for $\lambda \in \mathbb{Z}$ arbitrary, which can be shown to agree with the one from [9] in the special case $\lambda = 0$ (see Proposition 5.21). This new definition is only *less* recursive than that of [9] because of the intervention of the map $\mathcal{I}^{(\omega)}_{\lambda,\tau}$, for which we have not found a closed form and whose definition is still essentially recursive.

Definition 5.13. For $f \in \mathbb{K}(x)$, $\lambda \in \mathbb{Z}$, and $\tau \in \mathcal{T}$ with $\tau \subset \mathbb{K}^\times_t$, the λ-*Mahler discrete residue* of f at τ of degree $k \in \mathbb{N}$ is the vector

$$\mathrm{dres}_\lambda(f,\tau,k) = \Big(\mathrm{dres}_\lambda(f,\tau,k)_\alpha\Big)_{\alpha\in\tau} \in \bigoplus_{\alpha\in\tau} \mathbb{K}$$

defined as follows.

We set $\mathrm{dres}_\lambda(f,\tau,k)=\mathbf{0}$ if either $\tau\notin\mathrm{supp}(f)$ or $k>\mathrm{ord}(f,\tau)$ as in Definition 2.9. For $\tau\in\mathrm{supp}(f)$, let

$$f_\tau=\sum_{k\in\mathbb{N}}\sum_{\alpha\in\tau}\frac{c_k(\alpha)}{(x-\alpha)^k}. \tag{5.5}$$

We set $\mathrm{dres}_\lambda(f,\tau,k)_\alpha=0$ for every $k\in\mathbb{N}-\{\lambda\}$ whenever $\alpha\in\tau$ is such that $\eta(\alpha)\neq h$, where $h:=\mathrm{ht}(f,\tau)$ and $\eta(\alpha)$ are as in Definition 5.11. In case $\lambda\geq 1$, we set $\mathrm{dres}_\lambda(f,\tau,\lambda)_\alpha=0$ also whenever $\eta(\alpha)\notin\{0,h\}$.

In case $h=0$, so that $\mathrm{sing}(f,\tau)\subseteq\mathcal{C}(\tau)$, we simply set

$$\mathrm{dres}_\lambda(f,\tau,k)_\gamma:=c_k(\gamma) \tag{5.6}$$

for every $1\leq k\leq\mathrm{ord}(f,\tau)$ and $\gamma\in\mathcal{C}(\tau)$.

In case $h\geq 1$, let us write $\mathbf{c}=(c_k(\gamma))$ for γ ranging over $\mathcal{C}(\tau)$ only, and let $(d_k(\gamma))=\mathbf{d}:=\mathcal{I}^{(\omega)}_{\lambda,\tau}(\mathbf{c})$ as in Definition 3.5, where $\omega:=\omega_{\lambda,\tau}(f)$ (cf. Definition 5.12), and $(\tilde{c}_k(\gamma))=\tilde{\mathbf{c}}:=\mathcal{D}_{\lambda,\tau}(\mathbf{d})$ as in Definition 3.2. For $\alpha\in\tau$ such that $\eta(\alpha)=h$ and for $1\leq k\leq\mathrm{ord}(f,\tau)=:m$, we define

$$\mathrm{dres}_\lambda(f,\tau,k)_\alpha:=\sum_{s=k}^{m}\sum_{n=0}^{h-1}p^{\lambda n}V^s_{k,n}(\alpha)c_s(\alpha^{p^n})-p^{\lambda(h-1)}\sum_{s=k}^{m}\mathbb{V}^s_{k,h-1}\alpha^{k-sp^{h+e-1}}\left(\tilde{c}_s\left(\alpha^{p^{h+e-1}}\right)+d_s\left(\alpha^{p^{h+e-1}}\right)\right). \tag{5.7}$$

In case $\lambda\geq 1$, for $\gamma\in\mathcal{C}(\tau)$ we set

$$\mathrm{dres}_\lambda(f,\tau,\lambda)_\gamma:=c_\lambda(\gamma)-\tilde{c}_\lambda(\gamma)=\frac{\gamma^\lambda}{e}\sum_{j=1}^{e}\gamma^{-\lambda p^j}\left(c_\lambda(\gamma^{p^j})-p^\lambda\sum_{s\geq\lambda+1}V^s_{\lambda,1}(\gamma^{p^j})d_s(\gamma^{p^{j+1}})\right). \tag{5.8}$$

As before, the Mahler coefficients $V^s_{k,n}(\alpha)$ and the universal Mahler coefficients $\mathbb{V}^s_{k,h-1}$ are as in Section 2.5.

Remark 5.14. The Definition 5.13 can be expressed equivalently in ways that are easier to compute but more difficult to write. We cannot improve on the definition (5.6) in case $h=0$; so let us address the case $h\geq 1$. The different ingredients used in Definition 5.13 are best computed in the following order. In every case, one should first compute the vector $\mathbf{d}^{(0)}:=\mathcal{I}^{(0)}_{\lambda,\tau}(\mathbf{c})$ of Definition 3.5. Every instance of $\tilde{c}_s$ in (5.4) and in (5.7) can (and should) be replaced with c_s, with the single exception of $\tilde{c}_\lambda$ (if it happens to occur), which should be rewritten in terms of the c_s and $d^{(0)}_s$ using (3.7). There is no need to find $\tilde{\mathbf{c}}$ by applying $\mathcal{D}_{\lambda,\tau}$ to anything. Having made these replacements, and only then, one should then compute the residual average ω from Definition 5.12, if necessary, using (5.4). If this $\omega=0$ then we already have all the required ingredients to compute our discrete residues. Only in case $\omega\neq 0$, we then proceed to compute the vector $\mathbf{w}^{(\lambda)}$ of Lemma 3.4(2), and by Definition 3.5 we can replace the d_s in (5.7) with $d^{(0)}_s+\omega\cdot w^{(\lambda)}_s$, all of which have already been computed, and now we are once again in possession of all the required ingredients.

Example 5.15. In the small example $f=\sum_{i=1}^{p-1}\zeta_p^i/(x-\zeta_p^i)$ considered in Remark 4.3, we have $\lambda=1$, $\tau=\tau(1)$, $\mathcal{C}(\tau)=\{1\}=\tau_0$, $e=1$, $h=1$, and $\tau_1=\{\zeta_p^i\mid i=1,\ldots,p-1\}$. Here we have $\mathbf{c}=(c_k(1))_{k\in\mathbb{N}}=\mathbf{0}$, whence so are $\mathbf{d}^{(0)}=\tilde{\mathbf{c}}=\mathbf{0}$. Moreover, $c_1(\zeta_p^i)=\zeta_p^i$ for $i=1,\ldots,p-1$, and every $c_s(\alpha)=0$ for $s\geq 2$, so we compute from (5.4)

$$\omega_{1,\tau}(f)=\frac{1}{(p^1-p^0)\cdot 1}\sum_{i=1}^{p-1}\sum_{s\geq 1}\sum_{n=0}^{0}p^{1\cdot n}\mathbb{V}^s_{1,n}(\zeta_p^i)^{-1\cdot p^n}c_s\left((\zeta_p^i)^{p^n}\right)-\frac{p^{1\cdot 0}}{1}\sum_{s\geq 1}\mathbb{V}^s_{1,0}1^{-s}(0+0)$$

$$=\frac{1}{p-1}\sum_{i=1}^{p-1}\zeta_p^{-i}\zeta_p^i-1\cdot 0=1.$$

Next we compute the vector $\mathbf{w}^{(1)} = (w_k(1))_{k\in\mathbb{N}}$, which is given by $w_k(1) = \delta_{1,k}$ (Kronecker's δ), whence the vector $\mathbf{d} = \mathcal{I}^{(1)}_{1,\tau} = 1 \cdot \mathbf{w}^{(1)}$. Since $m = 1$, according to Definition 5.13 every $\mathrm{dres}_1(f, \tau(1), k)_\alpha = 0$ except possibly for $k = 1$ and $\alpha \in \{\zeta_p^i \mid i = 0, \dots, p-1\} = \tau_0 \cup \tau_1$. But since $\mathbf{c} = \mathbf{0} = \tilde{\mathbf{c}}$, we find $\mathrm{dres}_1(f, \tau(1), 1)_1 = 0$ immediately from (5.8). The remaining components of the vector $\mathrm{dres}_1(f, \tau(1), 1)$ are computed by (5.7):

$$\mathrm{dres}_1(f, \tau(1), 1)_{\zeta_p^i} = \sum_{s=1}^{1}\sum_{n=0}^{0} p^{1\cdot n}\mathbb{V}^s_{1,n}(\zeta_p^i)c_s\left((\zeta_p^i)^{p^n}\right) - p^{1\cdot 0}\sum_{s=1}^{1}\mathbb{V}^s_{1,0}(\zeta_p^i)^{1-sp^1}\left(0 + d_s\left((\zeta_p^i)^{p^1}\right)\right) = \zeta_p^i - \zeta_p^i = 0.$$

We encourage the reader who would find it helpful at this point to see more instances of the computational strategy of Remark 5.14 in action to consult the more elaborate concrete Examples 7.2 and 7.3.

We next present three preparatory Lemmas that will aid us in streamlining our proof of Proposition 5.19 below that the λ-Mahler discrete residues just defined comprise a complete obstruction to the λ-Mahler summability of f_τ for $\tau \in \mathcal{T}_+$. We hope that the reader who, like us, finds the above Definition 5.13 painfully complicated, especially in comparison with the relatively simpler Definition 5.8 in the non-torsion case, can begin to glimpse in the statements of the following preliminary results the reasons for the emergence of the additional ingredients in Definition 5.13 that are absent from Definition 5.8. This is why we have chosen to present them first, and postpone their proofs until after their usefulness has become apparent in the proof of Proposition 5.19.

Lemma 5.16. If $f \in \mathbb{K}(x)$ and $\tau \in \mathrm{supp}(f, \tau) \cap \mathcal{T}_+$ are such that $\mathrm{ht}(f, \tau) = 0$ then f_τ is not λ-Mahler summable for any $\lambda \in \mathbb{Z}$.

Lemma 5.17. Let $\lambda \in \mathbb{Z}$ and $\tau \in \mathcal{T}_+$, and set $e := |\mathcal{C}(\tau)|$ as in Definition 2.7. Let $f \in \mathbb{K}(x)$, and write $\mathcal{C}(f_\tau) = \sum_{k\in\mathbb{N}}\sum_{\gamma\in\mathcal{C}(\tau)} \frac{c_k(\gamma)}{(x-\gamma)^k}$ as in Definition 3.1. Let us write $\mathbf{c} = (c_k(\gamma)) \in \mathcal{S}^{\mathcal{C}(\tau)}$. Let $\omega \in \mathbb{K}$ be arbitrary, and let $\mathbf{d} = (d_k(\gamma)) = \mathcal{I}^{(\omega)}_{\lambda,\tau}(\mathbf{c})$ as in Definition 3.5 and $\tilde{\mathbf{c}} = \mathcal{D}_{\lambda,\tau}(\mathbf{d})$ as in Definition 3.2. Set

$$g_0 := \sum_{k\in\mathbb{N}}\sum_{\gamma\in\mathcal{C}(\tau)} \frac{d_k(\gamma)}{(x-\gamma)^k} \quad \text{and} \quad g_1 := -\sum_{k\in\mathbb{N}}\sum_{\gamma\in\mathcal{C}(\tau)}\sum_{i=1}^{p-1} \frac{\zeta_p^{ki}(\tilde{c}_k(\gamma) + d_k(\gamma))}{(x - \zeta_p^i\gamma)^k}. \tag{5.9}$$

Then

$$\mathcal{C}(f_\tau) - \Delta_\lambda(g_0) = \begin{cases} g_1 & \text{if } \lambda \le 0; \\ g_1 + \sum_{\gamma\in\mathcal{C}(\tau)} \frac{c_\lambda(\gamma) - \tilde{c}(\gamma)}{(x-\gamma)^\lambda} & \text{if } \lambda \ge 1. \end{cases} \tag{5.10}$$

Moreover, for any $h \ge 1$, writing $\tau_h := \{\alpha \in \tau \mid \eta(\alpha) = h\}$, we have

$$\sigma^{h-1}(g_1) = -\sum_{k\in\mathbb{N}}\sum_{\alpha\in\tau_h} \frac{\sum_{s\ge k}\mathbb{V}^s_{k,h-1}\alpha^{k-sp^{h+e-1}}\left(\tilde{c}_s\left(\alpha^{p^{h+e-1}}\right) + d_s\left(\alpha^{p^{h+e-1}}\right)\right)}{(x-\alpha)^k}. \tag{5.11}$$

Lemma 5.18. Let $\lambda \ge 1$, $h \ge 1$, $\bar{f}_\tau \in \mathbb{K}(x)_\tau$, and $\tau \in \mathrm{supp}(\bar{f}) \cap \mathcal{T}_+$ such that $\mathrm{ord}(\bar{f}, \tau) = \lambda$ and

$$\mathrm{sing}(f, \tau) \subseteq \tau_h = \{\alpha \in \tau \mid \eta(\alpha) = h\}, \qquad \text{so that we can write} \qquad \bar{f}_\tau = \sum_{k=1}^{\lambda}\sum_{\alpha\in\tau_h} \frac{\bar{c}_k(\alpha)}{(x-\alpha)^k}.$$

If $\bar{f}_\tau$ is λ-Mahler summable then all the elements $\alpha^{-\lambda}\bar{c}_\lambda(\alpha)$ are equal to the constant $\bar{\omega} = \frac{1}{|\tau_h|}\sum_{\alpha\in\tau_h}\alpha^{-\lambda}\bar{c}_\lambda(\alpha)$, which is their arithmetic average. Letting $e := |\mathcal{C}(\tau)|$, we have $|\tau_h| = (p^h - p^{h-1})e$.

Proposition 5.19. For $f \in \mathbb{K}(x)$, $\lambda \in \mathbb{Z}$, and $\tau \in \mathcal{T}_+$, the component f_τ is λ-Mahler summable if and only if $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $k \in \mathbb{N}$.

Proof. The statement is trivial for $\tau \notin \mathrm{supp}(f) \Leftrightarrow f_\tau = 0$. If $\mathrm{ht}(f,\tau) = 0$ then $0 \neq f_\tau$ cannot be λ-Mahler summable by Lemma 5.16, whereas in this case we defined $\mathrm{dres}(f,\tau,k)_\gamma = c_k(\gamma)$ in (5.6) of Definition 5.13, and we obtain our conclusion vacuously in this case.

From now on we assume $\tau \in \mathrm{supp}(f)$, and let $h := \mathrm{ht}(f,\tau) \geq 1$, $m := \mathrm{ord}(f,\tau)$, and $\omega := \omega_{\lambda,\tau}(f)$. Writing f_τ as in (5.5), let $\tau_n := \{\alpha \in \tau \mid \eta(\alpha) = n\}$ for $n \in \mathbb{Z}_{\geq 0}$ and let us also write

$$f_\tau^{(n)} := \sum_{k=1}^{m} \sum_{\alpha \in \tau_n} \frac{c_k(\alpha)}{(x-\alpha)^k} \qquad \text{so that} \qquad f_\tau = \sum_{n=0}^{h} f_\tau^{(n)}.$$

Similarly as in the proof of Proposition 5.9, we compute that by Lemma 2.17, for each $0 \leq n \leq h-1$ we have

$$\sigma^n\left(f_\tau^{(h-n)}\right) = \sum_{k=1}^{m} \sum_{\alpha \in \tau_h} \frac{\sum_{s=k}^{m} V_{k,n}^{s}(\alpha) c_s(\alpha^{p^n})}{(x-\alpha)^k},$$

and therefore (notice that the component $f_\tau^{(0)} = \mathcal{C}(f_\tau)$, corresponding to $n = h$, is left untouched!)

$$\tilde{f}_{\lambda,\tau} := f_\tau + \sum_{n=0}^{h-1} \Delta_\lambda^{(n)}(f_\tau^{(h-n)}) = \sum_{k=1}^{m} \sum_{\alpha \in \tau_h} \frac{\sum_{s \geq k} \sum_{n=0}^{h-1} p^{\lambda n} V_{k,n}^{s}(\alpha) c_s(\alpha^{p^n})}{(x-\alpha)^k} + \sum_{k=1}^{m} \sum_{\gamma \in \mathcal{C}(\tau)} \frac{c_k(\gamma)}{(x-\gamma)^k}. \tag{5.12}$$

Let us now write, as in Definition 5.13 (in the present case $h \geq 1$), $\mathbf{c} = (c_k(\gamma))$ for γ ranging over $\mathcal{C}(\tau) = \tau_0$ only, $(d_k(\gamma)) = \mathbf{d} := \mathcal{I}_{\lambda,\tau}^{(\omega)}(\mathbf{c})$, and $(\tilde{c}_k(\gamma)) = \tilde{\mathbf{c}} := \mathcal{D}_{\lambda,\tau}(\mathbf{d})$.

Writing g_0 and g_1 as in (5.9), it follows from Lemma 5.17 and Definition 5.13 (where in case $\lambda \leq 0$, we use (5.7) alone; and in case $\lambda \geq 1$, we use (5.7) for $\alpha \in \tau_h$ and (5.8) for $\alpha \in \mathcal{C}(\tau)$) that

$$\bar{f}_{\lambda,\tau} := \tilde{f}_{\lambda,\tau} - \Delta_\lambda(g_0) + \Delta_\lambda^{(h-1)}(g_1) = \sum_{k=1}^{m} \sum_{\alpha \in \tau} \frac{\mathrm{dres}_\lambda(f,\tau,k)_\alpha}{(x-\alpha)^k}. \tag{5.13}$$

By a twofold application of Lemma 2.17, to (5.12) and to (5.13), we find that

$$f_\tau \text{ is } \lambda\text{-Mahler summable} \Longleftrightarrow \tilde{f}_{\lambda,\tau} \text{ is } \lambda\text{-Mahler summable} \Longleftrightarrow \bar{f}_{\lambda,\tau} \text{ is } \lambda\text{-Mahler summable.}$$

On the other hand, we see from (5.13) that $\bar{f}_{\lambda,\tau} = 0$ if and only if $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $k \in \mathbb{N}$. Therefore we immediately conclude that if $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $k \in \mathbb{N}$ then f_τ is λ-Mahler summable. Moreover, in case $\lambda \leq 0$, if f_τ is λ-Mahler summable, so that $\bar{f}_{\lambda,\tau}$ is also λ-Mahler summable, then we must have $\bar{f}_{\lambda,\tau} = 0$. Otherwise we would have, in contradiction with Theorem 4.2(2), that $\mathrm{disp}(\bar{f}_{\lambda,\tau},\tau) = 0$. This is because, by (5.10) in Lemma 5.17 in case $\lambda \leq 0$ (cf. (5.7) in Definition 5.13), we have $\mathrm{dres}_\lambda(f,\tau,k)_\alpha = 0$ for every $k \in \mathbb{N}$ and $\alpha \notin \tau_h$, and it is impossible to have any such α being a p-power power of another when $h \geq 1$ (see Definition 5.11 and Definition 2.11(1)). This concludes the proof of the Proposition in case $\lambda \leq 0$.

It remains to prove the converse in the case where $\lambda \geq 1$: assuming f_τ is λ-Mahler summable, we must have $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $k \in \mathbb{N}$. By Proposition 3.6, we must have $\mathbf{c} = \tilde{\mathbf{c}}$, and therefore in (5.8) $\mathrm{dres}_\lambda(f,\tau,k)_\gamma = c_\lambda(\gamma) - \tilde{c}_\lambda(\gamma) = 0$ for every $\gamma \in \mathcal{C}(\tau)$, whence $\mathrm{sing}(\bar{f}_{\lambda,\tau},\tau) \subseteq \tau_h$ by the Definition 5.13 of $\mathrm{dres}_\lambda(f,\tau,k)$ (since we set $\mathrm{dres}_\lambda(f,\tau,k)_\alpha = 0$ whenever $\alpha \in \tau$ is neither in τ_h nor in $\mathcal{C}(\tau)$). Moreover, if we had $\bar{f}_{\lambda,\tau} \neq 0$, contrary to our contention, then we would have $\mathrm{disp}(\bar{f}_{\lambda,\tau},\tau) = 0$ (for the same reasons as those just discussed above in the case $\lambda \leq 0$), and by Theorem 4.2(3) this can only happen in case $\mathrm{ord}(\bar{f}_{\lambda,\tau},\tau) = \lambda$. So we already conclude that $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $k > \lambda$ if f_τ is λ-Mahler summable. If we can further show that $\mathrm{dres}_\lambda(f,\tau,\lambda) = \mathbf{0}$ also, then this will force $\mathrm{ord}(\bar{f}_{\lambda,\tau},\tau) \neq \lambda$ and we will be able to conclude that actually $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $k \in \mathbb{N}$, as we contend, by another application of Theorem 4.2(3).

Thus it remains to show that if f_τ is λ-Mahler summable then $\mathrm{dres}_\lambda(f,\tau,\lambda)=\mathbf{0}$, which task will occupy us for the rest of the proof. We already know that $\mathrm{dres}_\lambda(f,\tau,k)=\mathbf{0}$ for every $k>\lambda$ and $\mathrm{dres}_\lambda(f,\tau,\lambda)_\gamma=0$ for every $\gamma\in\mathcal{C}(\tau)$, and therefore $\bar{f}_{\lambda,\tau}$ satisfies the hypotheses of Lemma 5.18 by (5.13) and the Definition 5.13 (where we had set $\mathrm{dres}_\lambda(f,\tau,k)_\alpha=0$ whenever $\alpha\in\tau$ is neither in τ_h nor in $\mathcal{C}(\tau)$). So let us write $\bar{c}_k(\alpha):=\mathrm{dres}_\lambda(f,\tau,k)_\alpha$ (given here as in (5.7)) as in Lemma 5.18, so that $\bar{f}_{\lambda,\tau}=\sum_{k=1}^{\lambda}\sum_{\alpha\in\tau_h}\frac{\bar{c}_k(\alpha)}{(x-\alpha)^k}$, and compute the arithmetic average $\bar{\omega}$ of the elements $\alpha^{-\lambda}\bar{c}_\lambda(\alpha)$ for α ranging over τ_h, which must be equal to $\alpha^{-\lambda}\bar{c}_\lambda(\alpha)$ for each $\alpha\in\tau_h$ by Lemma 5.18. Firstly, we see that

$$\frac{1}{|\tau_h|}\sum_{\alpha\in\tau_h}\alpha^{-\lambda}\left(\sum_{s\geq\lambda}\sum_{n=0}^{h-1}p^{\lambda n}V^s_{\lambda,n}(\alpha)c_s(\alpha^{p^n})\right)=\frac{1}{(p^h-p^{h-1})e}\sum_{\alpha\in\tau_h}\sum_{s\geq\lambda}\sum_{n=0}^{h-1}p^{\lambda n}\mathbb{V}^s_{\lambda,n}\alpha^{-sp^n}c_s(\alpha^{p^n}),$$

since $V^s_{\lambda,n}(\alpha)=\mathbb{V}^s_{\lambda,n}\cdot\alpha^{\lambda-sp^n}$ by Lemma 2.13. Secondly, we find that in the remaining portion of the average of $\alpha^{-\lambda}\bar{c}_\lambda(\alpha)=\alpha^{-\lambda}\mathrm{dres}_\lambda(f,\tau,\lambda)_\alpha$ for α ranging over τ_h,

$$\begin{aligned}&\frac{1}{|\tau_h|}\sum_{\alpha\in\tau_h}\alpha^{-\lambda}\left(-p^{\lambda(h-1)}\sum_{s\geq\lambda}\mathbb{V}^s_{\lambda,h-1}\alpha^{\lambda-sp^{h+e-1}}\left(\tilde{c}_s\left(\alpha^{p^{h+e-1}}\right)+d_s\left(\alpha^{p^{h+e-1}}\right)\right)\right)\\&\quad=\frac{-p^{\lambda(h-1)}}{(p^h-p^{h-1})e}\sum_{\alpha\in\tau_h}\sum_{s\geq\lambda}\mathbb{V}^s_{\lambda,h-1}\left(\left(\alpha^{p^h}\right)^{p^{e-1}}\right)^{-s}\left(\tilde{c}_s\left(\left(\alpha^{p^h}\right)^{p^{e-1}}\right)+d_s\left(\left(\alpha^{p^h}\right)^{p^{e-1}}\right)\right),\end{aligned}\tag{5.14}$$

the summands depend only on $\alpha^{p^h}=\gamma\in\mathcal{C}(\tau)$. For each $\gamma\in\mathcal{C}(\tau)$, the set $\{\alpha\in\tau_h\mid\alpha^{p^h}=\gamma\}$ has p^h-p^{h-1} elements: there are $(p-1)$ distinct p^{th}-roots of γ that do not belong to $\mathcal{C}(\tau)$, and then there are p^{h-1} distinct $(p^{h-1})^{\text{th}}$ roots of each of those elements. Therefore the expression in (14) is equal to the simpler

$$-\frac{p^{\lambda(h-1)}}{e}\sum_{\gamma\in\mathcal{C}(\tau)}\sum_{s\geq\lambda}\mathbb{V}^s_{\lambda,h-1}\gamma^{-s}(\tilde{c}_s(\gamma)+d_s(\gamma)),\qquad\text{whence the average}$$

$$\bar{\omega}:=\frac{1}{|\tau_h|}\sum_{\alpha\in\tau_h}\alpha^{-\lambda}\bar{c}_\lambda(\alpha)=\frac{1}{(p^h-p^{h-1})e}\sum_{\alpha\in\tau_h}\sum_{n=0}^{h-1}\sum_{s\geq\lambda}p^{\lambda n}\mathbb{V}^s_{\lambda,n}\alpha^{-sp^n}c_s(\alpha^{p^n})\tag{5.15}$$

Note that this is not necessarily the same as the similar expression for the residual average $\omega_{\lambda,\tau}(f)$ given by (5.4) in Definition 5.12, which was defined with respect to $(d_k^{(0)}(\gamma))=\mathbf{d}^{(0)}:=\mathcal{I}^{(0)}_{\lambda,\tau}(\mathbf{c})$ as

$$\omega_{\lambda,\tau}(f)=\frac{1}{(p^h-p^{h-1})e}\sum_{\alpha\in\tau_h}\sum_{s\geq\lambda}\sum_{n=0}^{h-1}p^{\lambda n}\mathbb{V}^s_{\lambda,n}\alpha^{-sp^n}c_s(\alpha^{p^n})-\frac{p^{\lambda(h-1)}}{e}\sum_{\gamma\in\mathcal{C}(\tau)}\sum_{s\geq\lambda}\mathbb{V}^s_{\lambda,h-1}\gamma^{-s}(\tilde{c}_s(\gamma)+d_s^{(0)}(\gamma)).$$

And yet, by (3.8) in Definition 3.5 and the definition (3.2) of $\mathbf{w}^{(\lambda)}$ in Lemma 3.4, we have $d_s(\gamma)=d_s^{(0)}(\gamma)$ for every $s>\lambda$ and $\gamma\in\mathcal{C}(\tau)$ and $d_\lambda(\gamma)=\omega_{\lambda,\tau}(f)\cdot\gamma^\lambda+d_\lambda^{(0)}(\gamma)$ for each $\gamma\in\mathcal{C}(\tau)$. By Corollary 2.16, $\mathbb{V}^\lambda_{\lambda,h-1}=p^{-\lambda(h-1)}$, and therefore we find from (13), with $\omega:=\omega_{\lambda,\tau}(f)$, that

$$\begin{aligned}\bar{\omega}&=\frac{1}{(p^h-p^{h-1})e}\sum_{\alpha\in\tau_h}\sum_{n=0}^{h-1}\sum_{s\geq\lambda}p^{\lambda n}\mathbb{V}^s_{\lambda,n}\alpha^{-sp^n}c_s(\alpha^{p^n})\\&\quad-\frac{p^{\lambda(h-1)}}{e}\sum_{\gamma\in\mathcal{C}(\tau)}\sum_{s\geq\lambda+1}\mathbb{V}^s_{\lambda,h-1}\gamma^{-s}(\tilde{c}_s(\gamma)+d_s^{(0)}(\gamma)))-\frac{p^{\lambda(h-1)}}{e}\sum_{\gamma\in\mathcal{C}(\tau)}\mathbb{V}^\lambda_{\lambda,h-1}\gamma^{-\lambda}(\tilde{c}_\lambda(\gamma)+\omega\gamma^\lambda+d_\lambda^{(0)}(\gamma))\\&=\omega-\frac{p^{\lambda(h-1)}}{e}\sum_{\gamma\in\mathcal{C}(\tau)}p^{-\lambda(h-1)}\gamma^{-\lambda}\gamma^\lambda\omega=\omega-\omega=0.\end{aligned}$$

Since we must have $\bar{c}_\lambda(\alpha)=\mathrm{dres}_\lambda(f,\tau,\lambda)_\alpha=\alpha^\lambda\bar{\omega}$ in (13) for each $\alpha\in\tau_h$ by Lemma 5.18, it follows that $\mathrm{dres}_\lambda(f,\tau,\lambda)=\mathbf{0}$, concluding the proof of Proposition 5.19. ■

Remark 5.20. For $f\in\mathbb{K}(x)$ and $\tau\in\mathrm{supp}(f)\cap\mathcal{T}_+$, the element $\bar{f}_{\lambda,\tau}$ in (5.13) is the τ-component of the $\bar{f}_\lambda\in\mathbb{K}(x)$ in the *λ-Mahler reduction* (1.2).

Next we provide the proofs of the preliminary Lemmas that we used in the proof of Proposition 5.19.

Proof of Lemma 5.16. It suffices to show that for any $g \in \mathbb{K}(x)$ such that $g_\tau \neq 0$, $\mathrm{ht}(\Delta_\lambda(g), \tau) \geq 1$. So let $m := \mathrm{ord}(g, \tau)$, $h := \mathrm{ht}(g, \tau)$, $\tau_n := \{\alpha \in \tau \mid \eta(\alpha) = n\}$ for $n \in \mathbb{Z}_{\geq 0}$, and $0 \neq g_\tau = \sum_{k=1}^{m} \sum_{n=0}^{h} \sum_{\alpha \in \tau_n} \frac{d_k(\alpha)}{(x-\alpha)^k}$. Then

$$\Delta_\lambda(g) = \sum_{\alpha \in \tau_{h+1}} \frac{p^\lambda V_{m,1}^m(\alpha) d_m(\alpha^p)}{(x-\alpha)^m} + (\text{lower-order or lower-height terms}),$$

and since $p^\lambda V_{m,1}^m(\alpha) = p^{\lambda-m}\alpha^{m-pm}$ by Corollary 2.16 and at least one $d_m(\alpha^p) \neq 0$ for some $\alpha \in \tau_{h+1}$ by assumption, we conclude that $\Delta_\lambda(g)$ has at least one pole in τ_{h+1} and therefore $\mathrm{ht}(\Delta_\lambda(g), \tau) = h + 1 \geq 1$. ■

Proof of Lemma 5.17. It follows from (2.4) and Lemma 3.3 that

$$\Delta_\lambda(g_0) = \sum_{k \in \mathbb{N}} \sum_{\gamma \in \mathcal{C}(\tau)} \frac{\tilde{c}_k(\gamma)}{(x-\gamma)^k} + \sum_{k \in \mathbb{N}} \sum_{\gamma \in \mathcal{C}(\tau)} \sum_{i=1}^{p-1} \frac{p^\lambda \sum_{s \geq k} V_{k,1}^s(\zeta_p^i \gamma) d_s(\gamma^p)}{(x - \zeta_p^i \gamma)^k}.$$

To see that $p^\lambda \sum_{s \geq k} V_k^s(\zeta_p^i \gamma) d_s(\gamma^p) = \zeta_p^{ki}(\tilde{c}_k(\gamma) + d_k(\gamma))$, note that $V_{k,1}^s(\zeta_p^i\gamma) = (\zeta_p^i\gamma)^{k-sp} \cdot \mathbb{V}_{k,1}^s = \zeta_p^{ki} V_{k,1}^s(\gamma)$ for every $s \geq k$ simultaneously by Lemma 2.13, and $p^\lambda \sum_{s \geq k} V_{k,1}^s(\gamma) d_s(\gamma^p) = \tilde{c}_k(\gamma) + d_k(\gamma)$ by the Definition 3.2 of $\tilde{\mathbf{c}} = \mathcal{D}_{\lambda,\tau}(\mathbf{d})$. This concludes the proof of (5.10).

For $\gamma \in \mathcal{C}(\tau)$ and $1 \leq i \leq p-1$, let $S(\gamma, i) := \left\{\alpha \in \tau \;\middle|\; \alpha^{p^{h-1}} = \zeta_p^i \gamma\right\}$. Then τ_h is the disjoint union of the sets $S(\gamma, i)$, and it follows from Lemma 2.17 that, for each $\gamma \in \mathcal{C}(\tau)$ and $1 \leq i \leq p-1$,

$$\sigma^{h-1}\left(\sum_{k \in \mathbb{N}} \frac{\zeta_p^{ik}(\tilde{c}_k(\gamma) + d_k(\gamma))}{(x - \zeta_p^i\gamma)^k}\right) = \sum_{k \in \mathbb{N}} \sum_{\alpha \in S(\gamma,i)} \frac{\sum_{s \geq k} V_{k,h-1}^s(\alpha) \zeta_p^{is}(\tilde{c}_s(\gamma) + d_s(\gamma))}{(x-\alpha)^k}. \tag{5.16}$$

For each $\alpha \in S(\gamma, i) \Leftrightarrow \alpha^{p^{h-1}} = \zeta_p^i \gamma$, we compute $\alpha^{p^{h+e-1}} = \gamma$ and $\zeta_p^{is} = \alpha^{sp^{h-1}(1-p^e)}$, and therefore each

$$V_{k,h-1}^s(\alpha)\zeta_p^{is}(\tilde{c}_s(\gamma) + d_s(\gamma)) = V_{k,h-1}^s(\alpha)\alpha^{sp^{h-1}(1-p^e)}\left(\tilde{c}_s\left(\alpha^{p^{h+e-1}}\right) + d_s\left(\alpha^{p^{h+e-1}}\right)\right).$$

By Lemma 2.13, $V_{k,h-1}^s(\alpha) = \mathbb{V}_{k,h-1}^s \cdot \alpha^{k-sp^{h-1}}$, and therefore $V_{k,h-1}^s(\alpha)\alpha^{sp^{h-1}(1-p^e)} = \mathbb{V}_{k,h-1}^s \alpha^{k-sp^{h+e-1}}$. It follows that (5.16) is equal to

$$\sum_{k \in \mathbb{N}} \sum_{\alpha \in S(\gamma,i)} \frac{\sum_{s \geq k} \mathbb{V}_{k,h-1}^s \alpha^{k-sp^{h+e-1}} \left(\tilde{c}_s\left(\alpha^{p^{h+e-1}}\right) + d_s\left(\alpha^{p^{h+e-1}}\right)\right)}{(x-\alpha)^k},$$

and (5.11) now follows by summing over $\gamma \in \mathcal{C}(\gamma)$ and $1 \leq i \leq p-1$. ■

Proof of Lemma 5.18. First of all, $|\tau_h| = (p^h - p^{h-1})e$ because there are e elements in $\mathcal{C}(\tau)$, each of which has $(p-1)$ distinct p^{th} roots (of height 1) that do not belong to $\mathcal{C}(\tau)$, and each of these latter elements has p^{h-1} distinct $(p^{h-1})^{\text{th}}$ distinct roots—it follows from the Definition 5.11 that $\alpha \in \tau$ has height $\eta(\alpha) = h$ if and only if α is a $(p^{h-1})^{\text{th}}$ root of an element of height 1. Moreover, the $\alpha^{-\lambda}\bar{c}_\lambda(\alpha)$ are all equal to one another if and only if they are all equal to their arithmetic average. It remains to show that $\alpha^{-\lambda}\bar{c}_\lambda(\alpha)$ is independent of α.

Now let $g_\tau \in \mathbb{K}(x)_\tau$ such that $\bar{f}_\tau = \Delta_\lambda(g_\tau)$. By Lemma 2.10(7), $\mathrm{ord}(g, \tau) = \mathrm{ord}(f, \tau) = \lambda$, so we can write

$$g_\tau = \sum_{k=1}^{\lambda} \sum_{n=0}^{h-1} \sum_{\alpha \in \tau_n} \frac{d_k(\alpha)}{(x-\alpha)^k},$$

because if g had a pole in τ_n for some $n \geq h$ then $\Delta_\lambda(g_\tau) = \bar{f}_\tau$ would have a pole in τ_{n+1}, contradicting our assumptions. Let $\mathbf{d} = (d_k(\gamma))$ for γ ranging over $\mathcal{C}(\tau)$ only. Since $\Delta_\lambda(g_\tau) = \bar{f}_\tau$ has no poles in $\mathcal{C}(\tau)$, we must have $\mathbf{d} \in \ker(\mathcal{D}_{\lambda,\tau})$ by Lemma 3.3. In particular, for each $\gamma \in \mathcal{C}(\tau)$ we must have

$$0 = c_\lambda(\gamma) = (\mathcal{D}_{\lambda,\tau}(\mathbf{d}))_{\lambda,\gamma} = -d_\lambda(\gamma) + \sum_{s\geq\lambda} p^\lambda V^s_{\lambda,1}(\gamma) = \gamma^{\lambda-p\lambda} d_\lambda(\gamma^p) - d_\lambda(\gamma),$$

since $d_s(\gamma) = 0$ for every $s > \lambda$ and $\gamma \in \mathcal{C}(\tau)$ and $V^\lambda_{\lambda,1}(\gamma) = p^{-\lambda}\gamma^{\lambda-p\lambda}$ by Corollary 2.16, and therefore $\gamma^{-\lambda} d_\lambda(\gamma) = \bar{\omega}$ is a constant that does not depend on $\gamma \in \mathcal{C}(\tau)$. This is the base case $n = 0$ of an induction argument showing that $\alpha^{-\lambda} d_\lambda(\alpha) = \bar{\omega}$ is independent of $\alpha \in \tau_n$ for $0 \leq n \leq h - 1$. Indeed, it follows from Lemma 2.17 and our assumption that $\mathrm{sing}(f, \tau) \cap \mathcal{C}(\tau) = \emptyset$ that

$$\begin{aligned}\Delta_\lambda\left(\sum_{n=0}^{h-1}\sum_{\alpha\in\tau_n} \frac{d_\lambda(\alpha)}{(x-\alpha)^\lambda}\right) &= \sum_{n=0}^{h-1}\sum_{\alpha\in\tau_{n+1}} \frac{p^\lambda V^\lambda_{\lambda,1}(\alpha) d_\lambda(\alpha^p) - d_\lambda(\alpha)}{(x-\alpha)^\lambda} + (\text{lower-order terms}) \\ &= \sum_{n=0}^{h-1}\sum_{\alpha\in\tau_{n+1}} \frac{\alpha^\lambda\cdot((\alpha^p)^{-\lambda} d_\lambda(\alpha^p)) - d_\lambda(\alpha)}{(x-\alpha)^\lambda} + (\text{lower-order terms}) \\ &= \sum_{\alpha\in\tau_h} \frac{\bar{c}_\lambda(\alpha)}{(x-\alpha)^\lambda} + (\text{lower-order terms}), \end{aligned} \tag{5.17}$$

where the second equality follows from the computation $V^\lambda_{\lambda,1}(\alpha) = p^{-\lambda}\alpha^{\lambda-p\lambda}$ in Corollary 2.16. In case $h = 1$ we have already concluded our induction argument. In case $h \geq 2$, we proceed and find from (17) that

$$\alpha^\lambda \cdot ((\alpha^p)^{-\lambda} d_\lambda(\alpha^p)) - d_\lambda(\alpha) = 0 \qquad \Longleftrightarrow \qquad \alpha^{-\lambda} d_\lambda(\alpha) = (\alpha^p)^{-\lambda} d_\lambda(\alpha^p) = \bar{\omega}$$

for each $\alpha \in \tau_{n+1}$ whenever $n+1 \leq h-1$, since $\alpha^p \in \tau_n$ for such an α, concluding our induction argument. Finally, since $d_\lambda(\alpha) = 0$ for $\alpha \in \tau_h$, we find again that $\bar{c}_\lambda(\alpha) = \alpha^\lambda \cdot ((\alpha^p)^{-\lambda} d_\lambda(\alpha^p)) = \alpha^\lambda\bar{\omega}$ for $\alpha \in \tau_h$, since $d_\lambda(\alpha) = 0$ and $\alpha^p \in \tau_{h-1}$ for such α, whence each $d_\lambda(\alpha^p) = \alpha^{p\lambda}\bar{\omega}$. ∎

5.4 Proof of the Main Theorem

Let us now gather our earlier results to prove the Main Theorem 1.1 stated in the introduction, that the λ-Mahler discrete residue at ∞ constructed in Definition 5.1 for the Laurent polynomial component f_∞, together with the λ-Mahler discrete residues at Mahler trees $\tau \in \mathcal{T}$ constructed in Definition 5.8 for non-torsion $\tau \in \mathcal{T}_0$ and in Definition 5.13 for torsion $\tau \in \mathcal{T}_+$, comprise a complete obstruction to the λ-Mahler summability problem.

Main Theorem (Theorem 1.1). For $\lambda \in \mathbb{Z}$, $f \in \mathbb{K}(x)$ is λ-Mahler summable if and only if the λ-Mahler discrete residues $\mathrm{dres}_\lambda(f,\infty) = \mathbf{0}$ and $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $\tau \in \mathcal{T}$ and every $k \in \mathbb{N}$.

Proof. Let $f \in \mathbb{K}(x)$. By Lemma 2.1, f is λ-Mahler summable if and only if both f_∞ and $f_{\mathcal{T}}$ are Mahler summable. By Proposition 5.2, f_∞ is λ-Mahler summable if and only if $\mathrm{dres}(f,\infty) = \mathbf{0}$. By Lemma 2.6, $f_{\mathcal{T}}$ is λ-Mahler summable if and only if f_τ is λ-Mahler summable for each $\tau \in \mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_+$. By Proposition 5.9 in the non-torsion case $\tau \in \mathcal{T}_0$, and by Proposition 5.19 in the torsion case $\tau \in \mathcal{T}_+$, f_τ is λ-Mahler summable if and only if $\mathrm{dres}_\lambda(f,\tau,k) = \mathbf{0}$ for every $k \in \mathbb{N}$. ∎

5.5 Mahler reduction

We can now define the λ-Mahler reduction $\bar{f}_\lambda$ of $f \in \mathbb{K}(x)$ in (1.2), in terms of the local reductions constructed in the proofs of Proposition 5.2, Proposition 5.9, and Proposition 5.19:

$$\bar{f}_\lambda := \sum_{\theta\in\mathbb{Z}/\mathcal{P}} \bar{f}_{\lambda,\theta} + \sum_{\tau\in\mathcal{T}} \bar{f}_{\lambda,\tau} = \sum_{\theta\in\mathbb{Z}/\mathcal{P}} \mathrm{dres}_\lambda(f,\infty)_\theta \cdot x^{i_\theta h_\theta(f)} + \sum_{k\in\mathbb{N}}\sum_{\tau\in\mathcal{T}}\sum_{\alpha\in\tau} \frac{\mathrm{dres}_\lambda(f,\tau,k)_\alpha}{(x-\alpha)^k}. \tag{5.18}$$

We refer to Remark 5.3, Remark 5.10, and Remark 5.20 for more details.

In the un-twisted case where $\lambda = 0$, we had already defined 0-Mahler discrete residues in [9], where we proved that they comprise a complete obstruction to what we call here the 0-Mahler summability problem. That the $\mathrm{dres}(f, \infty)$ of [9, Def. 4.1] agrees with the $\mathrm{dres}_0(f, \infty)$ of Definition 5.1 is immediately clear from their respective formulas. In contrast, the Mahler discrete residues $\mathrm{dres}(f, \tau, k)$ at non-torsion Mahler trees $\tau \in \mathcal{T}_0$ in [9, Def. 4.10] were defined recursively, using the Mahler coefficients $V^{s}_{k,1}(\alpha)$ only, whereas here we provide closed formulas using the full set of Mahler coefficients $V^{s}_{k,n}(\alpha)$ with $n \geq 1$ for $\mathrm{dres}_0(f, \tau, k)$ in Definition 5.8. Similarly, the Mahler discrete residues at torsion Mahler trees $\tau \in \mathcal{T}_+$ in [9, Def. 4.16] are defined recursively and in terms of an auxiliary $\mathbb{K}$-linear map (see [9, Def. 4.15]), whereas here we provide closed formulas in terms of a different auxiliary $\mathbb{K}$-linear map $\mathcal{I}^{(0)}_{0,\tau}$ in Definition 5.13 (of which the auxiliary $\mathbb{K}$-linear map introduced in [9, Def. 4.15] is essentially a truncated version). It is not clear at all (to us) from their respective definitions that the $\mathrm{dres}(f, \tau, k)$ of [9] should agree with the $\mathrm{dres}_0(f, \tau, k)$ defined here. And yet, they do.

Proposition 5.21. The Mahler discrete residues $\mathrm{dres}(f, \tau, k)$ of [9] coincide with the 0-Mahler discrete residues $\mathrm{dres}_0(f, \tau, k)$ in Definitions 5.8 and 5.13.

Proof. It is clear from [9, Defs. 4.10 and 4.16] and Definitions 5.8 and 5.13 that the support of both vectors $\mathrm{dres}(f, \tau, k)$ and $\mathrm{dres}_0(f, \tau, k)$ is contained in the set of $\alpha \in \tau$ such that $\eta(\alpha|f) = \mathrm{ht}(f, \tau)$ in the non-torsion case (see Definition 5.6) and such that $\eta(\alpha) = \mathrm{ht}(f, \tau)$ in the torsion case (see Definition 5.11). In case $\tau \in \mathcal{T}_+$ such that $\mathrm{ht}(f, \tau) = 0$, it is immediately clear from the case $h = 0$ in [9, Def. 4.16] *vis-à-vis* (5.6) in Definition 5.13 that $\mathrm{dres}(f, \tau, k) = \mathrm{dres}_0(f, \tau, k)$. So let us assume without loss of generality that either $\tau \in \mathcal{T}_0$ or $\mathrm{ht}(f, \tau) \geq 1$. In [9, Eq. (4.16)] we constructed a Mahler reduction

$$\bar{f}_\tau = \sum_{k\in\mathbb{N}} \sum_{\alpha\in\tau} \frac{\mathrm{dres}(f, \tau, k)_\alpha}{(x-\alpha)^k}$$

such that $\bar{f}_\tau - f$ is Mahler summable (see [9, §4.4]), whereas here we have constructed an analogous $\bar{f}_{0,\tau}$ in (5.18) with the same property that $\bar{f}_{0,\tau} - f_\tau$ is 0-Mahler summable. Therefore

$$(\bar{f}_{0,\tau} - f_\tau) - (\bar{f}_\tau - f_\tau) = \bar{f}_{0,\tau} - \bar{f}_\tau = \sum_{k\in\mathbb{N}} \sum_{\alpha\in\tau} \frac{\mathrm{dres}_0(f, \tau, k)_\alpha - \mathrm{dres}(f, \tau, k)_\alpha}{(x-\alpha)}$$

is 0-Mahler summable. For $\tau \in \mathcal{T}_0 \cap \mathrm{supp}(f)$, we always have $\mathrm{dres}_0(f, \tau, k)_\alpha = 0$ (resp., $\mathrm{dres}(f, \tau, k)_\alpha = 0$), except possibly for $\alpha \in \beta(f, \tau)$ with $\eta(\alpha|f) = h$ in Definition 5.8 (resp., in [9, Def. 4.10]). Similarly, for $\tau \in \mathcal{T}_+ \cap \mathrm{supp}(f)$, we always have $\mathrm{dres}(f, \tau, k)_\alpha = 0$ except possibly for $\alpha \in \tau_h$ in [9, Def. 4.16], and notice that when $\lambda = 0$ and $h \geq 1$ we only use (5.7) in Definition 5.13, so $\mathrm{dres}_0(f, \tau, k)_\alpha = 0$ is also possibly non-zero for $\alpha \in \tau_h$ only. Thus if we had $\bar{f}_{0,\tau} \neq \bar{f}_\tau$, then we would have $\mathrm{disp}(\bar{f}_{0,\tau} - \bar{f}_\tau, \tau) = 0$. But this would contradict Theorem 4.2(2), so we conclude that $\mathrm{dres}_0(f, \tau, k) = \mathrm{dres}(f, \tau, k)$ for every $\tau \in \mathcal{T}$ and $k \in \mathbb{N}$. ■

6 Differential Relations Among Solutions of First-Order Mahler Equations

Let us now consider the differential structures that are relevant for the most immediate applications of our λ-Mahler discrete residues. We denote by $\partial := x\frac{d}{dx}$ the unique $\mathbb{K}$-linear derivation on $\mathbb{K}(x)$ such that $\partial(x) = x$. We immediately compute that $p\sigma \circ \partial = \partial \circ \sigma$ as $\mathbb{K}$-linear endomorphisms of $\mathbb{K}(x)$. In order to remedy this, one can proceed as proposed by Michael Singer (see [16, Introduction]), to work in the overfield $\mathbb{K}(x, \log x)$ and introduce the derivation $\delta = x \log x \frac{d}{dx} = \log x \cdot \partial$. We insist that the notation $\log x$ is meant to be suggestive only: here $\log x$ is a new transcendental element satisfying $\sigma(\log x) = p \cdot \log x$ and $\partial(\log x) = 1$. Using these properties alone, one can verify that $\delta \circ \sigma = \sigma \circ \delta$ as $\mathbb{K}$-linear endomorphisms on all of $\mathbb{K}(x, \log x)$.

The following computational result is a Mahler analogue of [8, Lem. 3.4], and of an analogous and more immediate computation in the shift case, which occurs in the proof of [5, Cor. 2.1].

Lemma 6.1. Let $0 \neq a \in \mathbb{K}(x)$ and $\lambda \geq 1$. Then $\operatorname{dres}_\lambda\left(\partial^{\lambda-1}\left(\frac{\partial a}{a}\right), \infty\right) = \mathbf{0}$ and, for every $\tau \in \mathcal{T}$ and $\alpha \in \tau$,

$$\operatorname{dres}_\lambda\left(\partial^{\lambda-1}\left(\frac{\partial(a)}{a}\right), \tau, \lambda\right)_\alpha = (-1)^{\lambda-1}(\lambda-1)!\,\alpha^{\lambda-1} \cdot \operatorname{dres}_1\left(\frac{\partial(a)}{a}, \tau, 1\right)_\alpha \in \mathbb{Q}\cdot\alpha^\lambda.$$

Proof. Let $a = b\prod_{\alpha\in\mathbb{K}}(x-\alpha)^{m(\alpha)}$, where $0 \neq b \in \mathbb{K}$ and the $m(\alpha) \in \mathbb{Z}$ are almost all zero, and let

$$f := \frac{\partial(a)}{a} = m(0) + \sum_{\alpha\in\mathbb{K}^\times}\frac{m(\alpha)x}{x-\alpha} = \sum_{\alpha\in\mathbb{K}} m(\alpha) + \sum_{\alpha\in\mathbb{K}^\times}\frac{\alpha\cdot m(\alpha)}{x-\alpha}. \tag{6.1}$$

We then see immediately by induction that, for $\lambda \geq 1$, $\partial^{\lambda-1}(f)_\infty \in \mathbb{K}$, and therefore by Definition 5.1 $\operatorname{dres}_\lambda\left(\partial^{\lambda-1}\left(\frac{\partial a}{a}\right), \infty\right) = \mathbf{0}$. We also compute, using a similar induction argument as in [8, Lem. 3.4], that for $\tau \in \mathcal{T}$ and $\lambda \geq 1$:

$$\partial^{\lambda-1}(f)_\tau = \sum_{\alpha\in\tau}\frac{(-1)^{\lambda-1}(\lambda-1)!\,\alpha^\lambda m(\alpha)}{(x-\alpha)^\lambda} + (\text{lower-order terms}) = \sum_{k=1}^{\lambda}\sum_{\alpha\in\tau}\frac{c_k^{[\lambda]}(\alpha)}{(x-\alpha)^k}, \tag{6.2}$$

where the notation $c_k^{[\lambda]}(\alpha)$ is meant to let us directly apply the definitions of λ-Mahler discrete residues of degree λ of $\partial^{\lambda-1}(f)$ and more easily compare them with one another. In fact, as we shall see, it will only be necessary for us to know that $c_1^{[1]}(\alpha) = \alpha\cdot m(\alpha)$, and more generally that

$$c_\lambda^{[\lambda]}(\alpha) = (-1)^{\lambda-1}(\lambda-1)!\,\alpha^\lambda m(\alpha) = (-1)^{\lambda-1}(\lambda-1)!\,\alpha^{\lambda-1}c_1^{[1]}(\alpha). \tag{6.3}$$

We shall also repeatedly use the results from Lemma 2.13 and Corollary 2.16, that $V_{\lambda,n}^\lambda(\alpha) = p^{-\lambda n}\alpha^{\lambda-\lambda p^n}$.
For $\tau \in \operatorname{supp}(f)\cap\mathcal{T}_0$, let $h := \operatorname{ht}(f,\tau)$, and let $\alpha \in \beta(f,\tau)$ such that $\eta(\alpha|f) = h$ (cf. Definition 5.6). Then

$$\begin{aligned}\operatorname{dres}_\lambda(\partial^{\lambda-1}(f),\tau,\lambda)_\alpha &= \sum_{n=0}^{h} p^{\lambda n}V_{\lambda,n}^\lambda(\alpha)c_\lambda^{[\lambda]}(\alpha^{p^n}) = \sum_{n=0}^{h} p^{\lambda n}p^{-n\lambda}\alpha^{\lambda-\lambda p^n}c_\lambda^{[\lambda]}(\alpha^{p^n})\\ &= (-1)^{\lambda-1}(\lambda-1)!\,\alpha^\lambda\sum_{n=0}^{h} m(\alpha^{p^n}) = (-1)^{\lambda-1}(\lambda-1)!\,\alpha^{\lambda-1}\operatorname{dres}_1(f,\tau,1)_\alpha \in \mathbb{Q}\cdot\alpha^\lambda,\end{aligned}$$

by Definition 5.8. For $\tau \in \operatorname{supp}(f)\cap\mathcal{T}_+$, let us first suppose $\operatorname{ht}(f,\tau) = 0$ as in Definition 5.11, and compute for $\gamma \in \mathcal{C}(\tau)$, using (5.6) in Definition 5.13 that

$$\operatorname{dres}_\lambda(\partial^{\lambda-1}(f),\tau,\lambda)_\gamma = c_\lambda^{[\lambda]}(\gamma) = (-1)^{\lambda-1}(\lambda-1)!\,\gamma^\lambda m(\gamma) = (-1)^{\lambda-1}(\lambda-1)!\,\gamma^{\lambda-1}\operatorname{dres}_1(f,\tau,1)_\gamma,$$

which clearly belongs to $\mathbb{Q}\cdot\gamma^\lambda$. On the other hand, if $h := \operatorname{ht}(f,\tau) \geq 1$, we compute for $\gamma \in \mathcal{C}(\tau)$ using (5.8):

$$\begin{aligned}\operatorname{dres}_\lambda(\partial^{\lambda-1}(f),\tau,\lambda)_\gamma &= \frac{\gamma^\lambda}{e}\sum_{j=1}^{e}\gamma^{-\lambda p^j}c_\lambda^{[\lambda]}(\gamma^{p^j}) = \frac{\gamma^\lambda}{e}\sum_{j=1}^{e}\gamma^{-\lambda p^j}(-1)^{\lambda-1}(\lambda-1)!\,\gamma^{\lambda p^j}m(\gamma^{p^j})\\ &= (-1)^{\lambda-1}(\lambda-1)!\,\frac{\gamma^\lambda}{e}\sum_{j=1}^{e} m(\gamma^{p^j}) = (-1)^{\lambda-1}(\lambda-1)!\,\gamma^{\lambda-1}\operatorname{dres}_1(f,\tau,1)_\gamma \in \mathbb{Q}\cdot\gamma^\lambda\end{aligned} \tag{6.4}$$

Before computing the α-component of $\operatorname{dres}_\lambda(\partial^{\lambda-1}(f),\tau,\lambda)$ for $\alpha \in \tau$ such that $\eta(\alpha) = h$, we must first compute a few preliminary objects (cf. Remark 5.14). Consider the vector $\mathbf{d}^{[\lambda]} := \mathcal{I}_{\lambda,\tau}^{(0)}(\mathbf{c}^{[\lambda]})$ as in Definition

3.5, and let us compute in particular as in (3.7):

$$d_\lambda^{[\lambda]}(\gamma) = \frac{\gamma^\lambda}{e}\sum_{j=0}^{e-1}(j+1-e)\gamma^{-\lambda p^j}c_\lambda^{[\lambda]}(\gamma^{p^j}) = \frac{\gamma^\lambda}{e}\sum_{j=0}^{e-1}(j+1-e)\gamma^{-\lambda p^j}\cdot(-1)^{\lambda-1}(\lambda-1)!\,\gamma^{\lambda p^j}m(\gamma^{p^j})$$

$$= (-1)^{\lambda-1}(\lambda-1)!\,\frac{\gamma^\lambda}{e}\sum_{j=0}^{e-1}(j+1-e)m(\gamma^{p^j}) = (-1)^{\lambda-1}(\lambda-1)!\,\gamma^{\lambda-1}d_1^{[1]}(\gamma), \quad (6.5)$$

where the last equality results again from (3.7), since $m(\gamma^{p^j}) = \gamma^{-p^j}c_1^{[1]}(\gamma^{p^j})$ for each j. For $\tilde{\mathbf{c}}^{[\lambda]} := \mathcal{D}_{\lambda,\tau}(\mathbf{d}^{[\lambda]})$, the λ-components $\tilde{c}_\lambda^{[\lambda]}(\gamma) = c_\lambda^{[\lambda]}(\gamma) - \mathrm{dres}_\lambda(\partial^{\lambda-1}(f),\tau,\lambda)_\gamma$, by (5.8) in Definition 5.13. Thus, putting together (6.3), (4), and (5), we obtain

$$\tilde{c}_\lambda^{[\lambda]}(\gamma) + d_\lambda^{[\lambda]}(\gamma) = \frac{(-1)^{\lambda-1}(\lambda-1)!\,\gamma^\lambda}{e}\sum_{j=1}^{e}(j-e)m(\gamma^{p^j}). \quad (6.6)$$

With this, we next compute the residual average $\omega_{\lambda,\tau}(\partial^{\lambda-1}(f))$ of Definition 5.12, for which we compute separately the two long sums appearing in (5.4). First, the sum over elements of positive height, using (6.3) and Corollary 2.16, yields

$$\omega_{\lambda,\tau}^{(+)}(\partial^{\lambda-1}(f)) = \frac{1}{(p^h-p^{h-1})e}\sum_{\alpha\in\tau_h}\sum_{n=0}^{h-1}p^{\lambda n}\mathbb{V}_{\lambda,n}^{\lambda}\alpha^{-\lambda p^n}c_\lambda^{[\lambda]}(\alpha^{p^n})$$

$$= \frac{(-1)^{\lambda-1}(\lambda-1)!}{(p^h-p^{h-1})e}\sum_{\alpha\in\tau_h}\sum_{n=0}^{h-1}m(\alpha^{p^n}) = (-1)^{\lambda-1}(\lambda-1)!\cdot\omega_{1,\tau}^{(+)}(f). \quad (6.7)$$

Second, the sum over the elements of height zero, using (6.6), results in

$$\omega_{\lambda,\tau}^{(0)}(\partial^{\lambda-1}(f)) = \frac{p^{\lambda(e-1)}}{e}\sum_{\gamma\in\mathcal{C}(\tau)}\mathbb{V}_{\lambda,h-1}^{\lambda}\gamma^{-\lambda}(\tilde{c}^{[\lambda]}(\gamma)+d_\lambda^{[\lambda]}(\gamma))$$

$$= \frac{(-1)^{\lambda-1}(\lambda-1)!}{e^2}\sum_{\gamma\in\mathcal{C}(\tau)}\sum_{j=1}^{e}(j-e)m(\gamma^{p^j}) = (-1)^{\lambda-1}(\lambda-1)!\cdot\omega_{1,\tau}^{(0)}(f). \quad (6.8)$$

Now putting together (7) and (8) we obtain

$$\omega_{\lambda,\tau}(\partial^{\lambda-1}(f)) = \omega_{\lambda,\tau}^{(+)}(\partial^{\lambda-1}(f)) - \omega_{\lambda,\tau}^{(0)}(\partial^{\lambda-1}(f)) = (-1)^{\lambda-1}(\lambda-1)!\cdot\omega_{1,\tau}(f), \quad (6.9)$$

where

$$\omega_{1,\tau}(f) = \omega_{1,\tau}^{(+)}(f) - \omega_{1,\tau}^{(0)}(f) = \frac{1}{(p^h-p^{h-1})e}\sum_{\substack{\alpha\in\tau\\ \eta(\alpha)>0}}m(\alpha) - \frac{e-e^2}{2e^2}\sum_{\gamma\in\mathcal{C}(\tau)}m(\gamma)\in\mathbb{Q}. \quad (6.10)$$

Since the vector $\mathbf{w}^{(\lambda)}$ of Lemma 3.4(2) satisfies $w_\lambda^{(\lambda)}(\gamma) = \gamma^\lambda = \gamma^{\lambda-1}w_1^{(1)}(\gamma)$, we finally compute from (5.7):

$$\mathrm{dres}_\lambda(\partial^{\lambda-1}(f),\tau,\lambda)_\alpha = \sum_{n=0}^{h-1}p^{n\lambda}V_{\lambda,n}^{\lambda}(\alpha)c_\lambda^{[\lambda]}(\alpha^{p^n})$$

$$- p^{\lambda(h-1)}\mathbb{V}_{\lambda,h-1}^{\lambda}\alpha^{\lambda-\lambda p^{h+e-1}}(\tilde{c}_\lambda^{[\lambda]}(\alpha^{p^{h+e-1}}) + d_\lambda^{[\lambda]}(\alpha^{p^{h+e-1}}) + \omega_{\lambda,\tau}(\partial^{\lambda-1}(f))w_\lambda^{(\lambda)}(\alpha^{p^{h+e-1}}))$$

$$= (-1)^{\lambda-1}(\lambda-1)!\,\alpha^\lambda\left[\sum_{n=0}^{h-1}m(\alpha^{p^n}) - \frac{1}{e}\sum_{j=1}^{e}(j-e)m(\alpha^{p^{h+j-1}}) + \omega_{1,\tau}(f)\right]$$

$$= (-1)^{\lambda-1}(\lambda-1)!\,\alpha^{\lambda-1}\mathrm{dres}_1(f,\tau,1)_\alpha \in \mathbb{Q}\cdot\alpha^\lambda. \quad (6.11)$$

This concludes the proof of the Lemma. ∎

Proposition 6.2. Let U be a $\sigma\partial$-$\mathbb{K}(x,\log x)$-algebra such that $U^\sigma=\mathbb{K}$. Suppose $y_1,\dots,y_t\in U^\times$ satisfy $\sigma(y_i)=a_iy_i$ for some $a_1,\dots,a_t\in\mathbb{K}(x)^\times$. Then $y_1,\dots,y_t$ are ∂-dependent over $\mathbb{K}(x)$ if and only if there exist $\mathbf{0}\neq(k_1,\dots,k_t)\in\mathbb{Z}^t$ and $g\in\mathbb{K}(x)$ with $\sum_{i=1}^t k_i\frac{\partial a_i}{a_i}=p\sigma(g)-g$.

Proof. First, suppose there exist $k_1,\dots,k_t\in\mathbb{Z}$ and $g\in\mathbb{K}(x)$ satisfying the conclusion of Proposition 6.2. Then

$$\sigma\left(\sum_{i=1}^t k_i\frac{\delta y_i}{y_i}-g\log x\right)-\left(\sum_{i=1}^t k_i\frac{\delta y_i}{y_i}-g\log x\right)=\log x\left(\sum_{i=1}^t k_i\frac{\partial a_i}{a_i}-(p\sigma(g)-g)\right)=0,$$

and therefore $\sum_{i=1}^t k_i\frac{\delta y_i}{y_i}-g\log x\in U^\sigma=\mathbb{K}$, whence $y_1,\dots,y_t$ are δ-dependent over $\mathbb{K}(x,\log x)$, which is equivalent to them being ∂-dependent over $\mathbb{K}(x)$, since $\log x$ is ∂-algebraic over $\mathbb{K}(x)$.

Now suppose $y_1,\dots,y_t$ are ∂-dependent over $\mathbb{K}(x)$. Then they are also δ-dependent over $\mathbb{K}(x,\log x)$. Just as in [18, Cor. 3.3], *mutatis mutandis*, there exist $\mathcal{L}_i\in\mathbb{K}[\delta]$, not all zero, such that

$$F:=\sum_{i=1}^t\mathcal{L}_i\left(\frac{\delta(a_i)}{a_i}\right)=\sigma(G)-G \tag{6.12}$$

for some $G\in\mathbb{K}(x,\log x)$. Let $\lambda\geq 1$ be minimal such that $\mathrm{ord}(\mathcal{L}_i)\leq\lambda-1$ for every $1\leq i\leq t$, and let us write $\mathcal{L}_i=\sum_{j=0}^{\lambda-1}k_{i,j}\delta^j$, so that at least one $k_{i,\lambda-1}\neq 0$. Then we see from (6.12) that $F=\sum_{\ell=1}^{\lambda}f_\ell\log^\ell x$ for certain $f_1,\dots,f_\lambda\in\mathbb{K}(x)$, so we must have $G=\sum_{\ell=0}^{\lambda}g_\ell\log^\ell x$ for some $g_1,\dots,g_\lambda\in\mathbb{K}(x)$ and some irrelevant $g_0\in\mathbb{K}$. We then obtain in particular, by comparing $(\log^\lambda x)$-terms on both sides of (6.12), that

$$f_\lambda=\sum_{i=1}^t k_{i,\lambda-1}\partial^{\lambda-1}\left(\frac{\partial a_i}{a_i}\right)=p^\lambda\sigma(g_\lambda)-g_\lambda. \tag{6.13}$$

Let us now conclude the proof in the special case where the following supplementary assumptions hold:

I) for every $\tau\in\bigcup_{i=1}^t\mathrm{supp}(\frac{\partial a_i}{a_i})\cap\mathcal{T}_+$, there exists $h_\tau\in\mathbb{Z}_{\geq 0}$ such that, whenever $\tau\in\mathrm{supp}(\frac{\partial a_i}{a_i})$, we have $\mathrm{ht}(\frac{\partial a_i}{a_i},\tau)=h_\tau$ (cf. Definition 5.11); and

II) for every $\tau\in\bigcup_{i=1}^t\mathrm{supp}(\frac{\partial a_i}{a_i})\cap\mathcal{T}_0$, there exist $h_\tau\in\mathbb{Z}_{\geq 0}$ and $\gamma_\tau\in\tau$ such that, whenever $\tau\in\mathrm{supp}(\frac{\partial a_i}{a_i})$, we have $\alpha^{p^{h_\tau}}=\gamma_\tau$ for each $\alpha\in\mathrm{sing}(\frac{\partial a_i}{a_i},\tau)$ satisfying $\eta(\alpha|\frac{\partial a_i}{a_i})=\mathrm{ht}(\frac{\partial a_i}{a_i},\tau)$ (cf. Definition 5.6).

These supplementary assumptions permit us to naïvely identify

$$\mathrm{dres}_\lambda(f_\lambda,\tau,\lambda)=\sum_{i=1}^t k_{i,\lambda-1}\mathrm{dres}_\lambda\left(\partial^{\lambda-1}\left(\frac{\partial a_i}{a_i}\right),\tau,\lambda\right)\qquad\text{for every}\quad\tau\in\mathcal{T}. \tag{6.14}$$

Indeed, for $\tau\in\mathcal{T}_+$, the α-components of the λ-Mahler discrete residues are set to 0 in all cases of Definition 5.13 for every $\alpha\in\tau$ whose height is different from h_τ or 0, and analogously for $\tau\in\mathcal{T}_0$ the α-components of the λ-Mahler discrete residues are set to 0 in Definition 5.8 for every $\alpha\in\beta_{h_\tau}(\gamma_\tau)$ whose height is non-maximal. Thus, by our Main Theorem 1.1, (6.13) implies that $\sum_{i=1}^t k_{i,\lambda-1}\mathrm{dres}_\lambda\left(\partial^{\lambda-1}\left(\frac{\partial a_i}{a_i}\right),\tau,\lambda\right)=\mathbf{0}$ for every $\tau\in\mathcal{T}$. By Lemma 6.1, this is equivalent to $\sum_{i=1}^t k_{i,\lambda-1}\mathrm{dres}_1\left(\frac{\partial a_i}{a_i},\tau,1\right)=\mathbf{0}$, which in turn is equivalent to

$$\sum_{i=1}^t k_{i,\lambda-1}\mathrm{dres}_1\left(\frac{\partial a_i}{a_i},\tau(\alpha),1\right)_\alpha=0\qquad\Longleftrightarrow\qquad\sum_{i=1}^t k_{i,\lambda-1}\xi_{i,\alpha}=0 \tag{6.15}$$

for every $\alpha\in\mathbb{K}^\times$, where each $\xi_{i,\alpha}:=\alpha^{-1}\mathrm{dres}_1(\frac{\partial a_i}{a_i},\tau(\alpha),1)\in\mathbb{Q}$ (again by Lemma 6.1). Thus we may take our solution $\mathbf{0}\neq(k_{1,\lambda-1},\dots,k_{t,\lambda-1})$ to the $\mathbb{Q}$-linear system (6.15) to belong to $\mathbb{Q}^t$ and, after multiplying by a common denominator, we may further take the $k_i:=k_{i,\lambda-1}\in\mathbb{Z}$. As a consequence of our supplementary assumptions we have so far that

$$\mathrm{dres}_1\left(\sum_{i=1}^{t} k_i \frac{\partial a_i}{a_i}, \tau, 1\right) = \mathbf{0}$$

for every $\tau \in \mathcal{T}$. By another application of Lemma 6.1, we also see that $\mathrm{dres}_1(\sum_{i=1}^t k_i \frac{\partial a_i}{a_i}, \infty) = \mathbf{0}$, and since $\sum_{i=1}^t k_i \frac{\partial a_i}{a_i}$ only has poles of order 1, we conclude by our Main Theorem 1.1 that indeed there exists $g \in \mathbb{K}(x)$ such that $\sum_{i=1}^t k_i \frac{\partial a_i}{a_i} = p\sigma(g) - g$, as claimed.

It remains to show that our supplementary assumptions I and II above indeed incur no loss of generality. For $\tau \in \bigcup_{i=1}^t \mathrm{supp}(\frac{\partial a_i}{a_i}) \cap \mathcal{T}_+$, let h_τ be the largest among the $\mathrm{ht}(\frac{\partial a_i}{a_i}, \tau)$ (see Definition 5.11) such that $\tau \in \mathrm{supp}(\frac{\partial a_i}{a_i})$, and denote $h_{i,\tau} := h_\tau - \mathrm{ht}(\frac{\partial a_i}{a_i}, \tau)$; for the remaining i, we set $h_{i,\tau} := 0$. Analogously for $\tau \in \bigcup_{i=1}^t \mathrm{supp}(\frac{\partial a_i}{a_i}) \cap \mathcal{T}_0$, let h_τ be the smallest non-negative integer such that $\bigcup_{i=1}^t \mathrm{sing}(\frac{\partial a_i}{a_i}, \tau) \subseteq \beta_{h_\tau}(\gamma_\tau)$ (see Definition 5.4) for the (unique) $\gamma_\tau \in \tau$ guaranteed to exist by Lemma 5.5. It follows from the Definition 5.6 that $h_{i,\tau} := h_\tau - \mathrm{ht}(\frac{\partial a_i}{a_i}, \tau) \geq 0$ for each i such that $\tau \in \mathrm{supp}(\frac{\partial a_i}{a_i})$; for the remaining i, we set $h_{i,\tau} := 0$. Writing $a_i = b_i \prod_{\alpha \in \mathbb{K}} (x - \alpha)^{m_i(\alpha)}$, where $0 \neq b_i \in \mathbb{K}$ and the $m_i(\alpha) \in \mathbb{Z}$ are almost all zero, let

$$\tilde{a}_i := a_i \cdot \prod_{\tau \in \mathrm{supp}(f_\lambda)} \prod_{\alpha \in \tau} \left(\frac{x^{p^{h_{i,\tau}}} - \alpha}{x - \alpha}\right)^{m_i(\alpha)}, \qquad \text{so that} \qquad \beta \in \mathrm{sing}\left(\frac{\partial \tilde{a}_i}{\tilde{a}_i}, \tau\right) \iff \beta^{p^{h_{i,\tau}}} \in \mathrm{sing}\left(\frac{\partial a_i}{a_i}, \tau\right)$$

for every $\infty \neq \tau \in \mathrm{supp}(f_\lambda)$. We see that these $\tilde{a}_i$ satisfy the supplementary hypotheses I and II in the special case that we have already established. Moreover, the λ-Mahler summability of f_λ is equivalent to that of

$$\tilde{f}_\lambda := \sum_{i=1}^{t} k_{i,\lambda-1} \partial^{\lambda-1} \left(\frac{\partial \tilde{a}_i}{\tilde{a}_i}\right),$$

as we see from Lemma 2.17 by adding

$$\sum_{i=1}^{t} k_{i,\lambda-1} \partial^{\lambda-1} \left(\frac{\partial \tilde{a}_i}{\tilde{a}_i} - \frac{\partial a_i}{a_i}\right) = \sum_{i=1}^{t} \sum_{\tau \in \mathrm{supp}(f_\lambda)} \sum_{\alpha \in \tau} k_{i,\lambda-1} m_i(\alpha) \Delta_\lambda^{(h_{i,\tau})} \left(\partial^{\lambda-1}\left(\frac{x}{x-\alpha}\right)\right)$$

to both sides of (6.13). ■

Theorem 6.3. Let U be a $\sigma\partial$-$\mathbb{K}(x, \log x)$-algebra such that $U^\sigma = \mathbb{K}$. Suppose $y_1, \dots, y_t \in U^\times$ satisfy $\sigma(y_i) = a_i y_i$ for some $a_1, \dots, a_t \in \mathbb{K}(x)^\times$. The following are equivalent.

1) $y_1, \dots, y_t$ are ∂-dependent over $\mathbb{K}(x)$;
2) there exist $\mathbf{0} \neq (k_1, \dots, k_t) \in \mathbb{Z}^t$ and $g \in \mathbb{K}(x)$ such that $\sum_{i=1}^t k_i \frac{\partial a_i}{a_i} = p\sigma(g) - g$;
3) there exist $\mathbf{0} \neq (k_1, \dots, k_t) \in \mathbb{Z}^t$ such that $\prod_{i=1}^t y_i^{k_i} \in \mathbb{K}(x)$.

Proof. The equivalence of (1) and (2) has already been established in Proposition 6.2. It is obvious that (3) implies (1). It remains to show that (1,2) imply (3). For $\mathbf{0} \neq (k_1, \dots, k_t) \in \mathbb{Z}^t$ and $g \in \mathbb{K}(x)$ as in (2), letting $y := \prod_{i=1}^t y_i^{k_i}$ and $a := \prod_{i=1}^t a_i^{k_i}$ we see immediately that

$$\sigma(y) = ay \qquad \text{and} \qquad \frac{\partial a}{a} = p\sigma(g) - g, \tag{6.16}$$

or what is the same, we can reduce without loss of generality to the case $t = 1$, in which case it is well-known that (1) implies (3) (see for example [23, Thm. 5.1]). ■

7 Examples

In [9, Section 5], we provided two small examples of λ-Mahler discrete residues with $\lambda = 0$. Here we illustrate the definitions and properties of λ-Mahler discrete residues with $\lambda = 1$ in several examples.

Example 7.1 gives a 1-Mahler summable f in the non-torsion case $\tau \subset \mathcal{T}_0$. Example 7.2 gives a 1-Mahler non-summable f in the torsion case $\tau \subset \mathcal{T}_+$. In Example 7.3 we verify the vanishing of 1-Mahler discrete residues for a concrete example of the type described in Remark 4.3, consisting of a 1-Mahler summable rational function with Mahler dispersion 0.

Example 7.1. Let $p = 3$, $\lambda = 1$, and $\tau = \tau(2)$. Consider $f = f_\tau$ with $\mathrm{sing}(f, \tau) = \{2, \sqrt[3]{2}, \zeta_3\sqrt[3]{2}, \zeta_3^2\sqrt[3]{2}\}$ below.

$$f = \frac{-x^6 + 4x^3 + 3x^2 - 12x + 8}{(x-2)^2\left(x^3-2\right)^2}$$

$$= \frac{-1}{(x-2)^2} + \frac{1}{6\sqrt[3]{2}} \cdot \sum_{i=0}^{2} \frac{\zeta_3^{2i}}{(x - \zeta_3^i\sqrt[3]{2})^2} - \frac{1}{3\sqrt[3]{4}} \cdot \sum_{i=0}^{2} \frac{\zeta_3^i}{x - \zeta_3^i\sqrt[3]{2}} = \sum_{k=1}^{2}\sum_{\alpha\in\tau} \frac{c_k(\alpha)}{(x-\alpha)^k},$$

By Definition 5.6, we have $\mathrm{ht}(f, \tau) = 1$. It follows from Definition 5.8 that $\mathrm{dres}_\lambda(f, \tau, k)_\alpha = 0$ except possibly for $k \in \{1, 2\}$ and $\alpha \in \beta_1(2) = \{\zeta_3^i\sqrt[3]{2} \mid i = 0, 1, 2\}$. Now we compute for such $\alpha_i := \zeta_3^i\sqrt[3]{2}$:

$$\mathrm{dres}_1(f, \tau, 2)_{\alpha_i} = \mathbb{V}_{2,0}^2(\alpha_i)c_2(\alpha_i) + 3\mathbb{V}_{2,1}^2(\alpha_i)c_2(\alpha_i^3) = 1 \cdot \frac{\zeta_3^{2i}}{6\sqrt[3]{2}} + 3 \cdot \left(\frac{\zeta_3^{2i}}{3^2\sqrt[3]{2^4}}\right) \cdot (-1) = 0; \qquad \text{and}$$

$$\mathrm{dres}_1(f, \tau, 1)_{\alpha_i} = \mathbb{V}_{1,0}^2(\alpha_i)c_2(\alpha_i) + 3\mathbb{V}_{1,1}^2(\alpha_i)c_2(\alpha_i^3) + \mathbb{V}_{1,0}^1(\alpha_i)c_1(\alpha_i) + 3\mathbb{V}_{1,1}^1(\alpha_i)c_1(\alpha_i^3)$$
$$= 0 \cdot \frac{\zeta_3^{2i}}{6\sqrt[3]{2}} + 3 \cdot \frac{-2\zeta_3^i}{3^2\sqrt[3]{2^5}} \cdot (-1) + 1 \cdot \left(\frac{-\zeta_3^i}{3\sqrt[3]{4}}\right) + 3 \cdot \left(\frac{\zeta_3^i}{3\sqrt[3]{2^2}}\right) \cdot 0 = 0,$$

for each $i = 0, 1, 2$. By Proposition 5.9, our f should be 1-Mahler summable. And indeed, $f = \Delta_1\left(\frac{1}{(x-2)^2}\right)$.

Example 7.2. Let $p = 3$, $\lambda = 1$, and $\tau = \tau(\zeta_4)$. Consider the following $f = f_\tau$ with $\mathrm{sing}(f, \tau) = \{\zeta_4^{\pm1}, \zeta_{12}^{\pm1}, \zeta_{12}^{\pm5}\}$:

$$f = \frac{-2x^4 + 2x^2 + 1}{(x^2+1)\left(x^4 - x^2 + 1\right)} = \frac{1}{2}\left(\frac{\zeta_4}{x - \zeta_4} + \frac{\zeta_4^3}{x - \zeta_4^3} + \frac{\zeta_{12}^7}{x - \zeta_{12}} + \frac{\zeta_{12}^{11}}{x - \zeta_{12}^5} + \frac{\zeta_{12}}{x - \zeta_{12}^7} + \frac{\zeta_{12}^5}{x - \zeta_{12}^{11}}\right) = \sum_{\alpha\in\tau} \frac{c_k(\alpha)}{x - \alpha}.$$

By Definition 2.7, $\mathcal{C}(\tau) = \{\zeta_4^{\pm1}\}$ and $e := e(\tau) = 2$. By Definition 5.11, $\mathrm{ht}(f, \tau) = 1$. We follow the steps outlined in Remark 5.14. First observe that $c_1(\zeta_4^{\pm1}) = \zeta_4^{\pm1}/2$. Now using (3.7) in Definition 3.5, we find for each $\gamma = \zeta_4^{\pm1}$,

$$d_1^{(0)}(\gamma) = \frac{\gamma}{2}\sum_{j=0}^{1}(j-1)\gamma^{-3^j}c_1(\gamma^{3^j}) = -\frac{\gamma}{4},$$

and the remaining components $d_k^{(0)}(\gamma) = 0$ for every $k > 1$. Comparing this with the definition of the vector $\mathbf{w}^{(1)}$ in Lemma 3.4(2), which spans $\ker(\mathcal{D}_{1,\tau})$, we see that $\mathbf{d}^{(0)} = -\mathbf{w}^{(1)}/4$. Therefore $\tilde{\mathbf{c}} = \mathcal{D}_{1,\tau}(\mathbf{d}^{(0)}) = \mathbf{0}$. By Definition 5.12, the residual average in this case is given in (5.4) by

$$\omega := \omega_{1,\tau}(f) = \frac{1}{(3^1 - 3^0)\cdot 2}\sum_{\alpha\in\tau_1} 3^{1\cdot 0}\mathbb{V}_{1,0}^1\alpha^{-1\cdot 3^0}c_1(\alpha^{3^0}) - \frac{p^{1\cdot 0}}{2}\sum_{\gamma\in\mathcal{C}(\tau)} \mathbb{V}_{1,0}^1 \cdot \gamma^{-1}\cdot\left(\tilde{c}_1(\gamma) + d_1^{(0)}(\gamma)\right)$$
$$= \frac{1}{4}\left(\zeta_{12}^{-1}\zeta_{12}^7 + \zeta_{12}^{-5}\zeta_{12}^{11} + \zeta_{12}^{-7}\zeta_{12} + \zeta_{12}^{-11}\zeta_{12}^5\right) - \frac{1}{2}\left(\zeta_4^{-1}\cdot\left(0 - \frac{\zeta_4}{4}\right) + \zeta_4^{-3}\cdot\left(0 - \frac{\zeta_4^3}{4}\right)\right) = -\frac{3}{4}.$$

By Definition 3.5 and Lemma 3.4(2), $\mathbf{d} = \mathcal{I}_{1,\tau}^{(-3/4)}(\mathbf{c}) = \mathbf{d}^{(0)} - \frac{3}{4}\mathbf{w}^{(1)} = -\mathbf{w}^{(1)}$ has coordinates $d_1(\gamma) = -\gamma$ for $\gamma = \zeta_4^{\pm1}$ and all other $d_k(\gamma) = 0$ for $k \geq 2$. Now we observe that $c_1(\alpha) = \alpha^7$ for each $\alpha \in \{\zeta_{12}^{\pm 1}, \zeta_{12}^{\pm 5}\}$, and compute

$$\mathrm{dres}_1(f,\tau,1)_\alpha = 3^{1\cdot 0}\mathbb{V}_{1,0}^{1}(\alpha)c_1(\alpha^{3^0}) - 3^{1\cdot 0}\mathbb{V}_{1,0}^{1}\alpha^{1-1\cdot 3^{1+2-1}}(\tilde{c}_1(\alpha^{3^{1+2-1}}) + d_1(\alpha^{3^{1+2-1}})) = \alpha^7 + \alpha = 0.$$

On the other hand, we compute from (5.8) that $\mathrm{dres}_1(f,\tau,1)_\gamma = c_1(\gamma) - \tilde{c}_1(\gamma) = \gamma/2 \neq 0$ for each $\gamma = \zeta_4^{\pm 1}$, whence according to Proposition 5.19 our f should not be 1-Mahler summable. Let us verify this directly. If we had $f = \Delta_1(g)$ then g could only have poles in $\{\zeta_4^{\pm 1}\}$, so $g = \frac{ax+b}{x^2+1}$ for some impossible $a, b \in \mathbb{K}$ such that

$$\frac{-2x^4 + 2x^2 + 1}{x^6 + 1} = f = \Delta_1(g) = 3\frac{ax^3 + b}{x^6 + 1} - \frac{ax + b}{x^2 + 1} = \frac{-ax^5 - bx^4 + 2ax^3 + bx^2 - ax + 2b}{x^6 + 1}.$$

Example 7.3. Let $p = 7$, $\lambda = 1$, and $\tau = \tau(\zeta_{77})$, where ζ_{77} denotes a primitive 77-th root of unity. Then every ζ_{77}^i for $1 \leq i \leq 76$ such that $11 \nmid i$ belongs to τ, and $\mathcal{C}(\tau) = \{\zeta_{11}^j \mid j = 1, \ldots, 10\}$, where $\zeta_{11} = \zeta_{77}^7$. Now consider

$$f = f_\tau = x \cdot \frac{\Phi_{77}'(x)}{\Phi_{77}(x)} - 60 = \sum_{\substack{1\leq i\leq 76 \\ 7\nmid i;\ 11\nmid i}} \frac{\zeta_{77}^i}{x - \zeta_{77}^i} = \sum_{\alpha\in\tau} \frac{c_1(\alpha)}{x - \alpha},$$

where $\Phi_{77}(x)$ denotes the 77-th cyclotomic polynomial and $\Phi'(x)$ is its usual derivative with respect to x. Then we see from the Definition 2.11(1) that $\mathrm{disp}(f,\tau) = 0$. Since $\mathcal{C}(f_\tau) = 0$, the vectors $\mathbf{c}$, $\mathbf{d}^{(0)} = \mathcal{I}_{1,\tau}^{(0)}(\mathbf{c})$, and $\tilde{\mathbf{c}} = \mathcal{D}_{1,\tau}(\mathbf{d}^{(0)})$ as in Definition 3.5 are all $\mathbf{0}$. Hence we already have $\mathrm{dres}_1(f,\tau,1)_\gamma = c_1(\gamma) - \tilde{c}_1(\gamma) = 0$ for each $\gamma \in \mathcal{C}(\tau)$ by (5.8) in Definition 5.13. The residual average in Definition 5.12 in this case is given in (5.4) by

$$\omega = \omega_{1,\tau}(f) = \frac{1}{(7^1 - 7^0)\cdot 10} \sum_{\alpha\in\tau_1} 7^{1\cdot 0}\mathbb{V}_{1,0}^{1}\alpha^{-1\cdot 7^0}c_1(\alpha) = 1.$$

Therefore the vector $\mathbf{d} = \mathcal{I}^{(1)}(\mathbf{0}) = \mathbf{w}^{(1)}$, the vector spanning the kernel of $\mathcal{D}_{1,\tau}$ in Lemma 3.4(2), and therefore $d_1(\gamma) = \gamma$ for each $\gamma = \zeta_{11}^i$ with $i = 1, \ldots, 10$ and every $d_k(\gamma) = 0$ for $k \geq 2$. Thus, for $\alpha \in \tau_1$,

$$\mathrm{dres}_1(f,\tau,1)_\alpha = 7^{1\cdot 0}\mathbb{V}_{1,0}^{1}(\alpha)c_1(\alpha^{7^0}) - 7^{1\cdot 0}\mathbb{V}_{1,0}^{1}\alpha^{1-1\cdot 7^{1+10-1}}(\tilde{c}_1(\alpha^{7^{1+10-1}}) + d_1(\alpha^{7^{1+10-1}})) = \alpha - \alpha^{1-7^{10}} \cdot \alpha^{7^{10}} = 0.$$

Therefore $\mathrm{dres}_1(f,\tau,1) = \mathbf{0}$, and by Proposition 5.19 f should be 1-Mahler summable. And indeed, letting $\Phi_{11}(x)$ denote the 11-th cyclotomic polynomial, we can verify that $f = \Delta_1(g)$, where

$$g = x \cdot \frac{\Phi_{11}'(x)}{\Phi_{11}(x)} - 10 = -\frac{x^9 + 2x^8 + 3x^7 + 4x^6 + 5x^5 + 6x^4 + 7x^3 + 8x^2 + 9x + 10}{x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1} = \sum_{i=1}^{10} \frac{\zeta_{11}^i}{x - \zeta_{11}^i}.$$

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