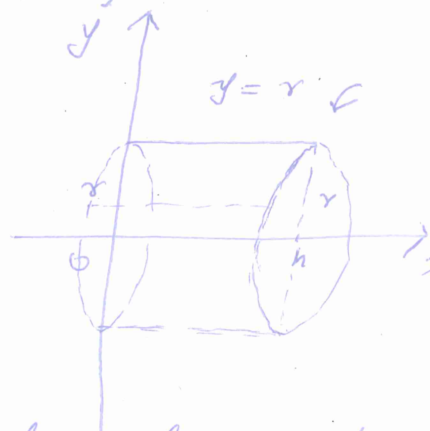
① # § 8.2 Area of a Surface of Revolution.

A surface of revolution is formed when a curve is rotated about a line. (water bottles)

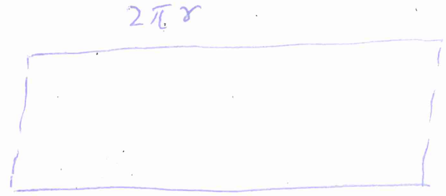
Ex: The ~~lateral~~ surface of a circular cylinder



Question: how to compute its lateral area?

~~Cutting~~ Cut the cylinder along the dash line and unroll it.

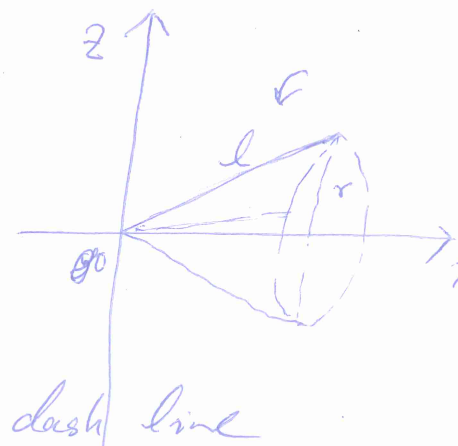
circumference [sər'kʌmfərəns]



First Exam: September 27th, 7pm - 8:15 pm.

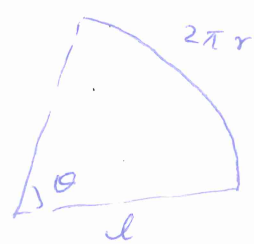
lateral area $A = 2\pi r \cdot h$

Ex: The circular cone with radius r and slant height l. What's its lateral area?



Cut the ~~cylin~~ cone along the dash line and unroll it.

$\theta = \frac{2\pi r}{l}$



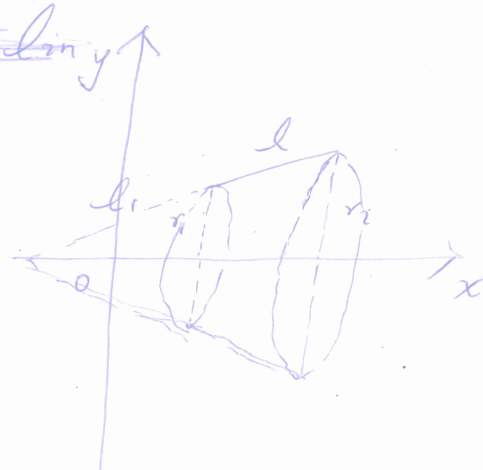
lateral area ~~A~~

$A = \frac{1}{2} l^2 \theta$

$= \frac{1}{2} l^2 \left(\frac{2\pi r}{l}\right)$

$= \pi r l$

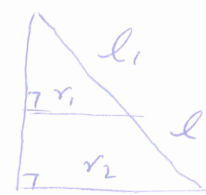
② Ex: The band ~~which formed by rotating a line~~ is a portion of a circular cone.



Its latera area is

$$A = \pi r_2(l_1 + l) - \pi r_1 l_1$$
$$= \pi[(r_2 - r_1)l_1 + r_2 l]$$

By similar triangles, we have



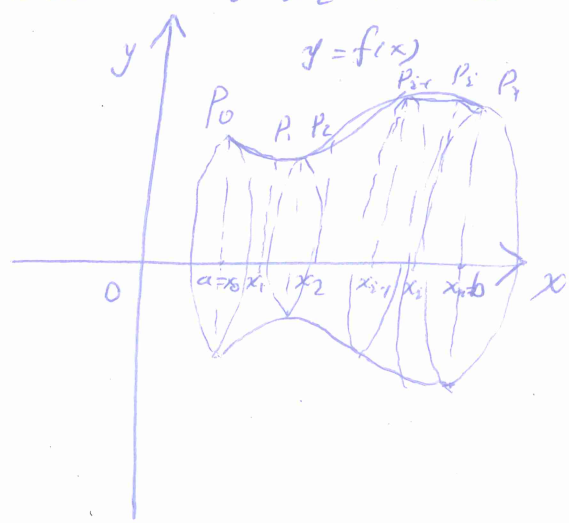
$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

$$\Rightarrow (r_2 - r_1)l_1 = r_1 l$$

Thus, $A = \pi(r_1 + r_2)l$

$= 2\pi r l$, where $r = \frac{1}{2}(r_1 + r_2)$.

Consider the surface obtained by rotating the smooth curve $y = f(x)$, $a \le x \le b$ about the x-axis



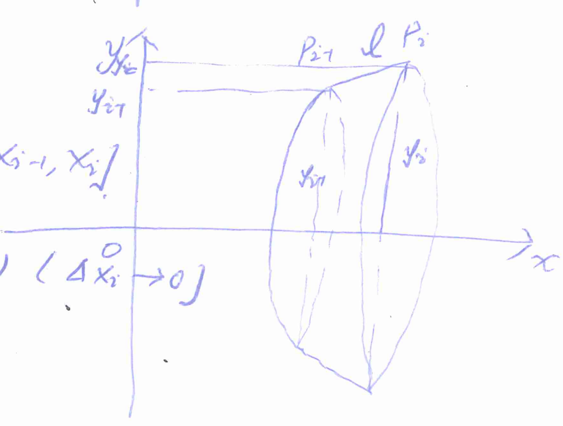
Let $P_i = (x_i, f(x_i))$, $i = 0, 1, \cdots, n$.

$$x_i - x_{i-1} = \frac{b-a}{n}$$

Assume $l_i = |P_{i-1}P_i|$,

The band formed by $P_{i-1}P_i$ has lateral area

$$A_i = 2\pi r_i l_i$$



where $l_i = \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$, $x_i^* \in [x_{i-1}, x_i]$

$$r_i = \frac{y_{i-1} + y_i}{2} \approx \frac{2f(x_i^*)}{2} = f(x_i^*) \quad (\Delta x_i \to 0)$$

since f is continuous.

③ Thus, $A_i \approx 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\,\Delta x$

The area of surface of revolution is

$$S = \lim_{n\to\infty}\sum_{i=1}^{n} A_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\,\Delta x$$

$$= \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$= \int_a^b 2\pi y\, ds$$

If the curve is described by $x=g(y)$, $c\le y\le d$, then

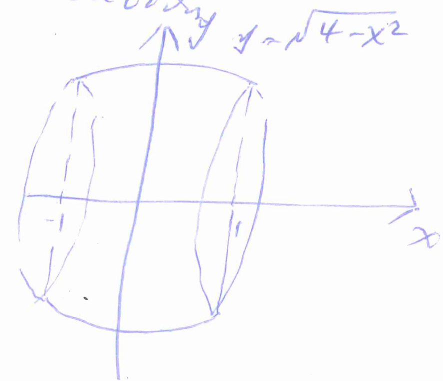
$$S = \int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int 2\pi y\, ds$$

For rotation about the y-axis, the surface area formula is

$$S = \int 2\pi x\, ds$$

Ex 1. The curve $y=\sqrt{4-x^2}$, $-1\le x\le 1$, is an arc of $x^2+y^2=4$. Find the area of the surface of revolution about x-axis.



$$\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$$

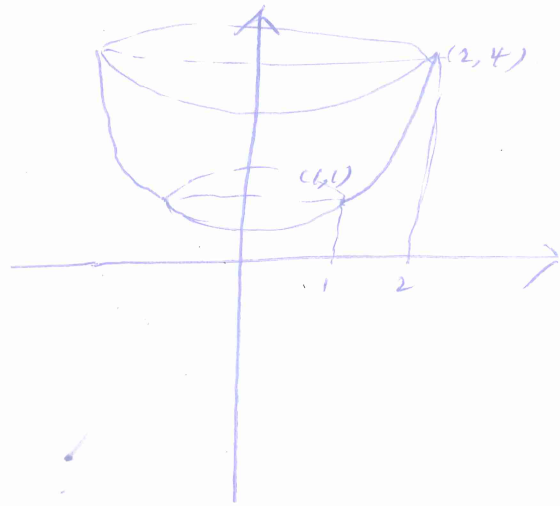
$$S = \int_{-1}^{1} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_{-1}^{1} 2\pi\sqrt{4-x^2}\cdot\frac{2}{\sqrt{4-x^2}}\,dx$$

$$= 4\pi \int_{-1}^{1} dx$$

$$= 8\pi$$

Ex 2. The arc of the parabola $y=x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis. Find the area.



$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$S = \int 2\pi x \, ds$$

$$= \int_1^2 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

$$= 2\pi \int_1^2 x\sqrt{1+4x^2} \, dx$$

$$\overset{u=1+4x^2}{=\!=\!=} 2\pi \int_5^{17} \sqrt{u} \cdot \frac{1}{8} \, du$$

$$= \frac{\pi}{4} \int_5^{17} u^{1/2} \, du$$

$$= \frac{\pi}{4} \left[\frac{2}{3} u^{3/2} \right] \Big|_5^{17}$$

$$= \frac{\pi}{6} \left(17\sqrt{17} - 5\sqrt{5}\right)$$