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Recall:

$$\int \frac{P(x)}{Q(x)} dx$$

Step 1: if $\deg(P) \geq \deg(Q)$, then

$$\frac{P}{Q} = \frac{SQ + R}{Q} = S + \frac{R}{Q}$$

← proper

Step 2: Q is a product of $(ax+b)$ and $(ax^2+bx+c, b^2-4ac < 0)$ Step 3: Express R/Q as a sum of

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

§7.5 Strategy for Integration

0. Memorize basic formulas

1. Simplify the integrand if possible

$$\begin{aligned} \text{Ex: } \int \frac{\tan \theta}{\sec^2 \theta} d\theta &= \int \frac{\sin \theta}{\cos \theta} \cdot \cos^2 \theta d\theta \\ &= \int \sin \theta \cos \theta d\theta \\ &= \frac{1}{2} \int \sin 2\theta d\theta \end{aligned}$$

2. Look for an obvious substitution.

$$\text{Ex: } \int \frac{x dx}{x^2 - 1}$$

$$\text{Set } u = x^2 - 1, \quad du = 2x dx$$

$$\int \frac{x dx}{x^2 - 1} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 - 1| + C$$

3. Classify the Integrand According to Its Form.

(a) Trigonometric functions.

② (b) Rational functions

$$\int \frac{dx}{x^2(x-1)} \leftarrow \text{partial fraction decomposition.}$$

(c) ~~Int~~ Integration by parts.

$$\int x \sin x dx$$

(d) Radicals

(i) $\sqrt{\pm x^2 \pm a^2} \leftarrow \text{trigonometric substitution}$

(ii) $\sqrt[n]{ax+b} \leftarrow u = \sqrt[n]{ax+b}$

4. Trial and error

Ex 1. $\int \frac{\tan^3 x}{\cos^3 x} dx$

$$= \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos^3 x} dx$$

$$= \int \frac{\sin^3 x}{\cos^4 x} dx$$

$$= \int \frac{(1 - \cos^2 x) \sin x dx}{\cos^4 x}$$

$$\stackrel{u = \cos x}{=} \int \frac{1 - u^2}{u^4} \cdot (-du)$$

$$= \int \frac{u^2 - 1}{u^4} du$$

Ex 2. $\int e^{\sqrt{x}} dx$

$$\stackrel{u = \sqrt{x}}{=} 2 \int u e^u du$$

Ex 3. $\int dx$

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$$\underline{u = \ln x} \quad \int \frac{du}{\sqrt{u}}$$

$$du = \frac{dx}{x}$$

$$\text{Ex 5.} \quad \int \sqrt{\frac{1-x}{1+x}} dx$$

$$= \int \sqrt{\frac{(1-x)(1-x)}{(1+x)(1-x)}} dx$$

$$= \int \frac{1-x}{\sqrt{1-x^2}} dx$$

$$= \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1} x + \sqrt{1-x^2} + C.$$

Note: ~~Not all~~ the integration of elementary functions is not necessarily ^{an} elementary function

$$\text{Ex.} \quad \int e^{t^2} dt$$