

On Sequences Associated to the Invariant Theory of Rank Two Lie Algebras

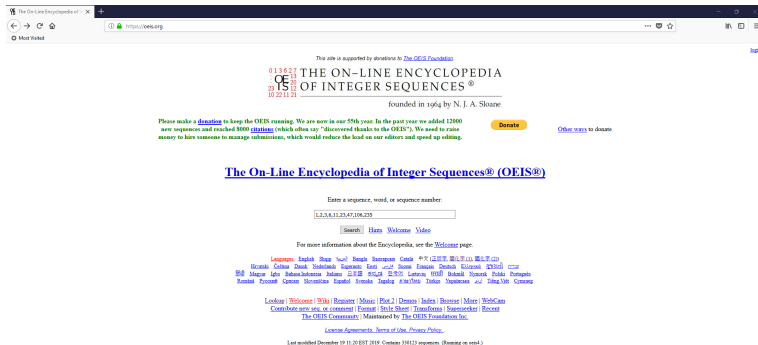
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Joint work with Alin Bostan, Jordan Tirrell and Bruce Westbury



The On-Line Encyclopedia of Integer Sequences (OEIS)



The screenshot shows the OEIS website homepage in a web browser. The browser's address bar displays "https://oeis.org". The page features the OEIS logo, which consists of a grid of numbers (0, 1, 3, 6, 2, 1, 1, 2, 2, 1, 1, 2, 1) and the text "THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®". Below the logo, it states "founded in 1964 by N. J. A. Sloane". A green banner encourages donations to keep the site running, mentioning that 12,000 new sequences were added in the past year. A "Donate" button is visible. The main heading "The On-Line Encyclopedia of Integer Sequences® (OEIS®)" is followed by a search bar with the placeholder text "Enter a sequence, word, or sequence number:". Below the search bar are links for "Search", "Home", "Welcome", and "Video". A section titled "For more information about the Encyclopedia, see the Welcome page:" provides links to various resources, including "Languages" (English, Spanish, etc.), "Help" (Frequently Asked Questions, etc.), "Contact" (Email, etc.), and "License" (Creative Commons, etc.). The footer indicates the last modification date as December 19, 2018, and mentions that the site contains 354,125 sequences.

OEIS is an online database of integer sequences, such as Fibonacci numbers ([A000045](#)), Catalan numbers ([A000108](#)).

Two families of sequences in OEIS

a (OEIS tag)	0	1	2	3	4	5
A059710	1	0	1	1	4	10
A108307	1	1	2	5	15	51
A108304	1	2	5	15	52	202

The first family of sequences (**octant sequences**)

a (OEIS tag)	0	1	2	3	4	5
A151366	1	0	2	2	12	30
A236408	1	1	3	9	33	131
A001181	1	2	6	22	92	422
A216947	1	3	11	49	221	1113

The second family of sequences (**quadrant sequences**)

- ▶ Those sequences are associated to the invariant theory of the exceptional simple Lie algebra G_2 of rank 2.
- ▶ The quadrant sequences are related to the octant sequences by the branching rules for $SL(3)$ of G_2 .

Octant sequences

The first family of sequences can be interpreted as lattice walks restricted to the octant. We call them **octant sequences**.

- ▶ **A059710**: enumerates the multiplicities of the trivial representation in the tensor powers of V , which is the 7-D fundamental representation of G_2 .
- ▶ **A108307**: enumerates **enhanced** 3-noncrossing set partitions.
- ▶ **A108304**: enumerates 3-noncrossing set partitions.

(Gil and Tirrell, 2019): **A108307** and **A108304** are related by the binomial transform.

Motivation and Contribution

(Bostan, Tirrell, Westbury and Z., 2019): [A059710](#) and [A108307](#) are also related by the binomial transform.

Mihailovs' conjecture: Let $T_3(n)$ be the n -th term of [A059710](#). Then T_3 is determined by $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and

$$14(n+1)(n+2)T_3(n) + (n+2)(19n+75)T_3(n+1) + 2(n+2)(2n+11)T_3(n+2) - (n+8)(n+9)T_3(n+3) = 0.$$

(Bostan, Tirrell, Westbury and Z., 2019): Three independent proofs of Mihailovs' conjecture.

- ▶ Two proofs are based on binomial relation between [A059710](#) and [A108307](#), together with a result by Bousquet-Mélou and Xin.
- ▶ The third one is a direct proof by the method of algebraic residues, which leads to closed formulae for the generating function of T_3 in terms of hypergeometric functions.

Quadrant sequences

The second family of sequences can be interpreted as lattice walks restricted to the quadrant. We call them **quadrant sequences**.

- ▶ [A151366](#): enumerates nonpositive bipartite trivalent graphs.
- ▶ [A236408](#): enumerates pasting diagrams.
- ▶ [A001181](#): enumerates Baxter permutations.
- ▶ [A216947](#): enumerates 2-coloured noncrossing set partitions.

Question: What are relations between quadrant sequences?

Motivation and Contribution

(Marberg, 2013): a combinatorial proof that A151366, A001181, and A216947 are related by binomial transforms.

(Bostan, Tirrell, Westbury and Z., 2019): Derive a uniform recurrence equation for quadrant sequences and show that they are related by binomial transform by the representation theory of simple Lie algebras.

Outline

- ▶ binomial relation between the first and second octant sequences
- ▶ Three independent proofs of Mihailovs' conjecture
- ▶ The quadrant sequences are related by binomial transforms

Preliminaries

Definition 1 Let G be a reductive complex algebraic group and let V be a representation of G . The sequence associated to (G, V) , denoted \mathbf{a}_V , is the sequence whose n -th term is the multiplicity of the trivial representation in the tensor power $\otimes^n V$.

Example 1 Let V be the 7-D fundamental representation of G_2 . Then [A059710](#) is the sequence associated with (G_2, V) .

Let \mathbf{a} be a sequence with n -th term $a(n)$, the **binomial transform** of \mathbf{a} is the sequence, denoted $\mathcal{B}\mathbf{a}$, whose n -th term is

$$\sum_{i=0}^n \binom{n}{i} a(i).$$

Preliminaries

Lemma 1 Assume \mathbf{a}_V is the sequence associated to (G, V) as specified in **Definition 1**. Then $\mathbf{a}_{V \oplus \mathbb{C}} = \mathcal{B}\mathbf{a}_V$.

Lemma 2 Assume \mathbf{a} enumerates walks in a lattice, confined to a domain D , using a set of steps S . Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \coprod \{0\}$.

Lemma 3 Let $G(t)$ be the generating function of \mathbf{a} . For $k \in \mathbb{Z}$, denote the generating function of $\mathcal{B}^k \mathbf{a}$ by $\mathcal{B}^k G$. Then

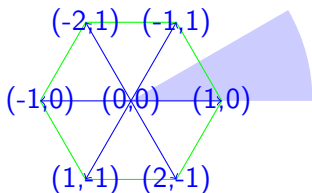
$$(\mathcal{B}^k G)(t) = \frac{1}{1 - k t} G\left(\frac{t}{1 - k t}\right).$$

Binomial relation between A059710 and A108307

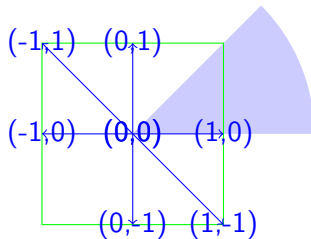
Let V be the 7-D fundamental representation of G_2 . Then

- ▶ A059710 is the sequence associated to (G_2, V) . Let $T_3(n)$ be its n -th term.
- ▶ A108307 enumerates enhanced 3-noncrossing set partitions. Let $E_3(n)$ be its n -th term.

In terms of lattice walks, we can interpret T_3 and E_3 as follows:

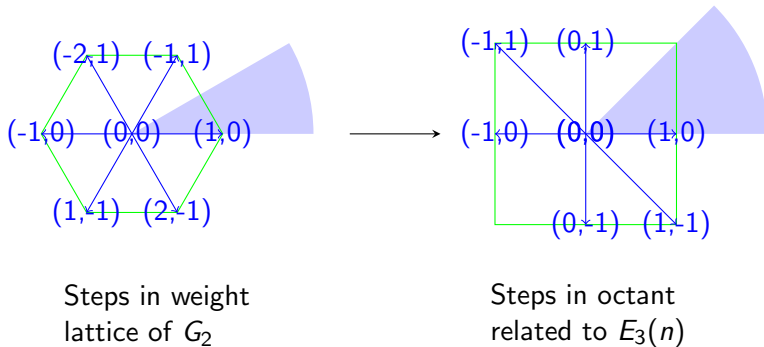


Steps in weight
lattice of G_2



Steps in octant
related to $E_3(n)$

In terms of lattice walks, we can interpret T_3 and E_3 as follows:



If we make a linear transformation $(x, y) \rightarrow (x, y \pm x)$, then it identifies the six non-zero steps, as well as the two domains.

Binomial relation between A059710 and A108307

Recall: **Lemma 2** Assume \mathbf{a} enumerates walks in a lattice, confined to a domain D , using a set of steps S . Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \amalg \{0\}$.

Binomial relation between A059710 and A108307

Recall: [Lemma 2](#) Assume \mathbf{a} enumerates walks in a lattice, confined to a domain D , using a set of steps S . Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \amalg \{0\}$.

By [Lemma 2](#) and the previous figures, we conclude that E_3 is the binomial transform of T_3 .

Binomial relation between A059710 and A108307

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By [Lemma 2](#) and the previous figures, we conclude that E_3 is the binomial transform of T_3 .

([Gil and Tirrell, 2019](#)): [A108307](#) and [A108304](#) are related by the binomial transform.

Recall: [Lemma 1](#) Assume \mathbf{a}_V is the sequence associated to (G, V) as specified in [Definition 1](#). Then $\mathbf{a}_{V \oplus \mathbb{C}} = \mathcal{B}\mathbf{a}_V$.

Thus, the octant sequences are sequences associated to

$$(G_2, V), \quad (G_2, V \oplus \mathbb{C}), \quad (G_2, V \oplus 2\mathbb{C}).$$

First proof of Mihailovs' conjecture

Mihailovs' conjecture: Let $T_3(n)$ be the n -th term of [A059710](#). Then T_3 is determined by $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and

$$14(n+1)(n+2)T_3(n) + (n+2)(19n+75)T_3(n+1) \\ + 2(n+2)(2n+11)T_3(n+2) - (n+8)(n+9)T_3(n+3) = 0.$$

([Bousquet-Mélou and Xin, 2005](#)): Let $E_3(n)$ be the n -th term of [A108307](#). Then E_3 is given by $E_3(0) = E_3(1) = 1$, and

$$8(n+3)(n+1)E_3(n) + (7n^2 + 53n + 88)E_3(n+1) \\ - (n+8)(n+7)E_3(n+2) = 0.$$

First proof of Mihailovs' conjecture

Recall: We prove that E_3 is the binomial transform of T_3 . Thus,

$$T_3(n) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} E_3(k).$$

Set $f(n, k) = (-1)^{n-k} \binom{n}{k} E_3(k)$.

- ▶ By Bousquet-Mélou and Xin's result, $f(n, k)$ is holonomic function, which satisfies ordinary difference equations for n and k , respectively.
- ▶ **Idea:** Using creative telescoping method (Zeilberger, 1990), which is an algorithmic approach to compute a differential/difference equation for the integration/sum of holonomic functions, to drive a recurrence equation for T_3 .

First proof of Mihailovs' conjecture

- ▶ Using the Koutschan's Mathematica package `HolonomicFunctions.m` that implements Chyzak's algorithm for creative telescoping, we find exactly the recurrence equation in Mihailovs' conjecture.

Second proof of Mihailovs' conjecture

Recall: We prove that E_3 is the binomial transform of T_3 . Let $\mathcal{T}(t) = \sum_{n \geq 0} T_3(n)t^n$ and $\mathcal{E}(t) = \sum_{n \geq 0} E_3(n)t^n$. Then

$$\mathcal{T}(t) = \frac{1}{1+t} \cdot \mathcal{E}\left(\frac{t}{1+t}\right).$$

- ▶ By Bousquet-Mélou and Xin's result, we can derive an ODE for $\mathcal{E}(t)$.
- ▶ Using the closure properties of holonomic function (the sum, product and algebraic substitution of holonomic functions is still holonomic), we can derive an ODE for $\mathcal{T}(t)$ and convert it into a linear recurrence for $T_3(n)$, which is exactly the recurrence equation in Mihailovs' conjecture.

Third proof of Mihailovs' conjecture

Idea: In terms of lattice walks, we can interpret $T_3(n)$ to be the constant term of $W K^n$, where

$$K = (1 + x + y + xy + x^{-1} + y^{-1} + (xy)^{-1}),$$

and

$$W = x^{-2}y^{-3}(x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} \\ + x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2).$$

Let $\mathcal{T}(t) = \sum_{n \geq 0} T_3(n)t^n$. Then $\mathcal{T}(t)$ is the constant coefficient $[x^0y^0]$ of $W/(1 - tK)$. In other words, $\mathcal{T}(t)$ is equal to the algebraic residue of $W/(\textcolor{red}{xy} - t\textcolor{red}{xy}K)$, which is proportional to the contour integral of $W/(xy - txyK)$ over a cycle.

Third proof of Mihailovs' conjecture

Using creative telescoping method, we can compute a 6-th order ODE for $\mathcal{T}(t)$. Moreover, by using factorization of differential operators, we can show that $L_3(\mathcal{T}(t)) = 0$, where $\partial = \frac{d}{dt}$ and

$$L_3 = t^2 (2t + 1) (7t - 1) (t + 1) \partial^3 + 2t (t + 1) (63t^2 + 22t - 7) \partial^2 + (252t^3 + 338t^2 + 36t - 42) \partial + 28t (3t + 4).$$

Converting it into a linear recurrence for $T_3(n)$, we get exactly the recurrence equation in Mihailovs' conjecture.

Closed formulae

By factorization of the operator L_3 and algorithms for solving 2-nd order ODEs, we derive the following closed formula for $\mathcal{T}(t)$:

$$\mathcal{T}(t) = \frac{1}{30 t^5} \left[R_1 \cdot {}_2F_1 \left(\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 2 \end{matrix}; \phi \right) + R_2 \cdot {}_2F_1 \left(\begin{matrix} \frac{2}{3} & \frac{4}{3} \\ 3 \end{matrix}; \phi \right) + 5 P \right],$$

where

$$R_1 = \frac{(t+1)^2 (214 t^3 + 45 t^2 + 60 t + 5)}{t-1},$$

$$R_2 = 6 \frac{t^2 (t+1)^2 (101 t^2 + 74 t + 5)}{(t-1)^2},$$

and

$$\phi = \frac{27 (t+1) t^2}{(1-t)^3}, \quad P = 28 t^4 + 66 t^3 + 46 t^2 + 15 t + 1.$$

Closed formulae

By elliptic curve theory, we derive an alternative formula for $\mathcal{T}(t)$:

$$\frac{P}{6t^5} + \frac{(7t-1)(2t+1)(t+1)}{360t^5} \left((155t^2 + 182t + 59)(11t+1)H(t) \right. \\ \left. + (341t^3 + 507t^2 + 231t + 1)(5t+1)H'(t) \right),$$

where

$$H(t) = \frac{1}{g_2^{1/4}} \cdot {}_2F_1 \left(\begin{matrix} \frac{1}{12} & \frac{5}{12} \\ 1 \end{matrix}; \frac{1728}{J} \right),$$
$$J = \frac{(t-1)^3 (25t^3 + 21t^2 + 3t - 1)^3}{t^6 (1-7t)(2t+1)^2 (t+1)^3},$$

and

$$g_2 = (t-1)(25t^3 + 21t^2 + 3t - 1).$$

Binomial relations between quadrant sequences

Definition 2 Let \tilde{V} be the defining representation of $SL(3)$ and denote the dual by \tilde{V}^* . For $k \geq 0$, we define \mathcal{S}_k to be the sequence associated to $(SL(3), \tilde{V} \oplus \tilde{V}^* \oplus k \mathbb{C})$.

Remark: $SL(3)$ is the maximal subgroup of G_2 . Let V be the 7-D fundamental representation of G_2 . Then \mathcal{S}_k is the the sequence associated to $(SL(3), (V \oplus k\mathbb{C}) \downarrow_{SL(3)})$.

Lemma 4 Let C_k be the generating function of \mathcal{S}_k , where $k \geq 0$. Then C_k is the constant coefficient of $[x^0 y^0]$ of $W/(1 - tK)$, where

$$K = k + x + y + x^{-1} + y^{-1} + \frac{x}{y} + \frac{y}{x}$$

and

$$W = 1 - \frac{x^2}{y} + x^3 - x^2 y^2 + y^3 - \frac{y^2}{x}.$$

Binomial relations between quadrant sequences

By [Lemma 4](#), S_3 is identical to the quadrant sequence [A216947](#).

([Marberg, 2013](#)): The n -th term $C_2(n)$ of S_3 is given by $C_2(0) = 1$, $C_2(1) = 3$ and

$$(n+5)(n+6) \cdot C_2(n+2) - 2(5n^2 + 36n + 61) \cdot C_2(n+1) + 9(n+1)(n+4) \cdot C_2(n) = 0.$$

By [Lemma 1](#), S_k 's are related by binomial transforms. Thus, by [Lemma 3](#), the generating function of S_k is

$$\mathcal{A}(t) := \frac{1}{1-kt} \cdot \mathcal{C}\left(\frac{t}{1-kt}\right)$$

where $\mathcal{C}(t)$ is the generating function of S_3 .

Binomial relations between quadrant sequences

Using closure properties of holonomic functions, we derive a uniform 4-th order recurrence equation for \mathcal{S}_k with k as a parameter.

By comparing the recurrence equations between \mathcal{S}_k 's and quadrant sequences, and then checking initial terms, we show that

Theorem: The sequences $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are identical to quadrant sequences. In particular, quadrant sequences are related by binomial transforms.

Summary

- ▶ A combinatorial proof of binomial relation between the first and second octant sequences
- ▶ Three independent proofs of Mihailovs' conjecture
 - ▶ Two proofs are based on binomial relation between the first and second octant sequences
 - ▶ A direct proof by the method of algebraic residues, which leads to closed formulae for the generating function of the first octant sequence
- ▶ The quadrant sequences are related by binomial transforms

Summary

- ▶ A combinatorial proof of binomial relation between the first and second octant sequences
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Thanks!