§9.4 Models for Population Growth

• The Law of Natural Growth

Under ideal conditions,

$$\frac{dP}{dt} = kP \quad (1) \text{ where } k > 0$$

$$\frac{dP}{P} = k\,dt$$

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C$$

$$|P| = e^C \cdot e^{kt}$$

$$P = A \cdot e^{kt}, \text{ where } A = \pm e^C$$

Set $t=0$, then $P(0) = A \cdot e^0 = A$ ← initial population

The solution of the initial-value problem is

$$\frac{dP}{dt} = kP, \quad P(0) = P_0$$

is $P(t) = P_0 e^{kt}$ (2)

By (1),

$$\frac{1}{P}\frac{dP}{dt} = k$$

It means the relative growth rate is a constant.

((2) means a population with constant relative growth must grow exponentially.)

If we account for emigration (or "harvesting"), ~~the natural growth model~~ the corresponding model is,

$$\frac{dP}{dt} = kP - m \quad (3).$$

② The Logistic Model

Let M be the carrying capacity of the ~~o~~population.

Assumptions:

- $\frac{dP}{dt} \approx kP$ if P is small
- $\frac{dP}{dt} < 0$ if $P > M$

The logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

By Ex 0, $P(t) = \frac{M}{1 + Ae^{-kt}}$, $A = \frac{M - P_0}{P_0}$ (4)

Thus, $\lim_{t \to \infty} P(t) = M$

(The population levels off ~~to~~ towards the equilibrium solution $y = M$.

Ex 2. Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right), \quad P(0) = 100$$

and use it ~~p~~ to find $P(40)$ and $P(80)$. At what time does the population reach 900?

$k = 0.08$

$M = 1000$

$P_0 = 100$

By (4), $P(t) = \frac{1000}{1 + Ae^{-0.08t}}$, where $A = \frac{M - P_0}{P_0} = 9$

Thus, $P(t) = \frac{1000}{1 + 9e^{-0.08t}}$

③ $P(40) = \frac{1000}{1+9e^{-3.2}} \approx 731.6$, $P(80) = \frac{1000}{1+9e^{-6.4}} \approx 985.3$

The population reaches 900 when

$$\frac{1000}{1+9e^{-0.08t}} = 900$$

$$1+9e^{-0.08t} = \frac{10}{9}$$

$$e^{-0.08t} = \frac{1}{81}$$

$$-0.08t = \ln\frac{1}{81} = -\ln 81$$

$$t = \frac{\ln 81}{0.08} \approx 54.9$$

(If we account for emigration, the logistic model can be modified as:

$$\frac{dP}{dt} = kP\left(1-\frac{P}{M}\right) - c$$
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