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Chapter 9. Differential Equation

§ 9.1 Modeling with Differential Equations.

Models for Population Growth

1. Natural growth population model

Set

t = time (the independent variable)

P = the number of individuals in the population (the dependent variable)

The rate of growth of the population is dP/dt.

Assumption: (the rate of growth of the population is proportional to the population size;) i.e.

$$\frac{dP}{dt} = kP, \quad (1)$$

where $k>0$

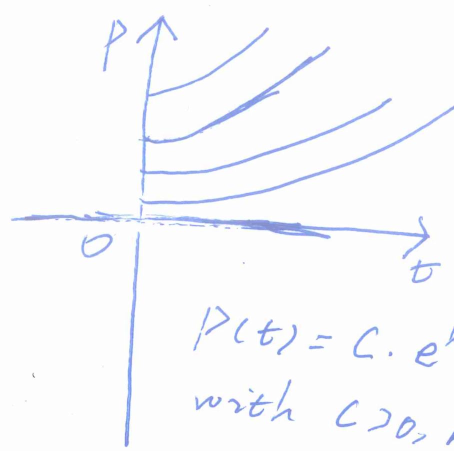
(This happens under ideal conditions: unlimited resources, immunity from disease, and so on)

Equation (1) is a differention equation.

Set $P = C \cdot e^{kt}$, then P is a solution of (1)

Since $P>0$, we have $C>0$.

Set $t=0$, $P(0) = C \cdot e^0 = C$ - initial population.



$P(t) = C \cdot e^{kt}$, with $C>0$, $t \geq 0$.

2. Logistic population model

Let M be the carring capacity of the environment.

Assumptions:

- $\frac{dP}{dt} \approx kP$ if P is small
- $\frac{dP}{dt} < 0$ if $P > M$ (P decreases if it exceeds M)

② Consider

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \quad (2)$$

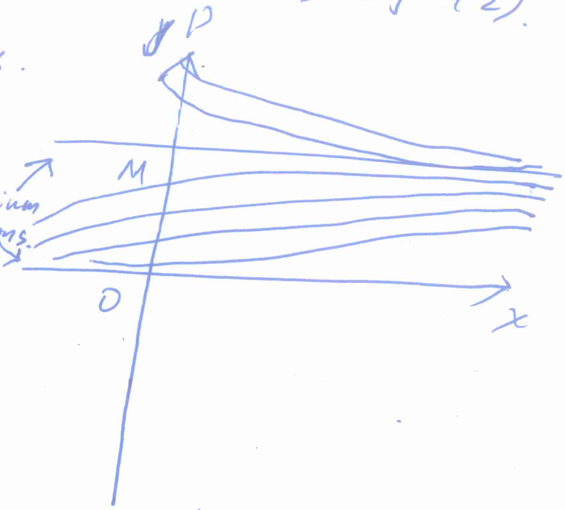
Note: If $P << M$, $\frac{dP}{dt} \approx kP$

If $P > M$, $1 - \frac{P}{M} < 0 \Rightarrow \frac{dP}{dt} < 0$

Equation (2) is called the logistic differential equation

Observation: $P(t) = 0$ and $P(t) = M$ are solutions of (2).

They are <u>equilibrium solutions</u>.



If $0 < P(0) < M$, then $\frac{dP}{dt} > 0$

If $P > M$, then $\frac{dP}{dt} < 0$

In either case, $\frac{dP}{dt} \to 0$ ($P \to M$)

General Differential Equations.

A differential equation is an equation that contains an unkown function and one or more of its derivatives.

Ex: $\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad (3)$

The order of a differential equation is the order of the highest derivative that occurs in the equation.

Ex: The order of (3) is 2.

A function f is called a solution of a differential equation if f satisfies it.

Ex: $y' = x^3$

has a general solution $y = \frac{x^4}{4} + C$

③ Ex 1. Show ~~the~~ that the family of functions

$$y = \frac{1+Ce^t}{1-Ce^t}$$

is a solution of $y' = \frac{1}{2}(y^2-1)$

$$y' = \frac{(1-Ce^t)(C\cdot e^t) - (1+Ce^t)(-Ce^t)}{(1-Ce^t)^2}$$

$$= \frac{2Ce^t}{(1-Ce^t)^2}$$

On the right side,

$$\frac{1}{2}(y^2-1) = \frac{1}{2}\left[\left(\frac{1+Ce^t}{1-Ce^t}\right)^2 - 1\right]$$

$$= \cancel{2\,2Ce^t}\ \frac{1}{2}\left[\frac{(1+Ce^t)^2-(1-Ce^t)^2}{(1-Ce^t)^2}\right]$$

$$= \frac{2Ce^t}{(1-Ce^t)^2}$$

Initial Condition: $y(t_0) = y_0$

Inital-value problem: ~~differential equatial + initial condid~~

find a solution of a differential equation under the initial condition.

Ex 2: Find a solution of $y' = \frac{1}{2}(y^2-1)$ with $y(0)=2$.

By Ex 1, $y = \frac{1+Ce^t}{1-Ce^t}$ is a general solution.

Set $t=0$, $y=2$, we get

$$2 = \frac{1+Ce^0}{1-Ce^0} = \frac{1+C}{1-C}$$

$$\Rightarrow C = \frac{1}{3}$$

Thus, the solution of the ~~int~~ initial-value problem is

$$y = \frac{1+\frac{1}{3}e^t}{1-\frac{1}{3}e^t} = \frac{3+e^t}{3-e^t}$$