A New Semantic Model, Part I

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Abstract

In this paper, a feature space is proposed as a new mathematical model for the semantics of linguistics. First, meaning is defined as a set based on a feature space. Then linguistic objects such as nouns, determiners, descriptions of nouns (noun phrases), and verbs, are investigated upon feature spaces. Finally, solutions to Frege's two problems are presented.

Key Word: Feature space; Meaning; Noun; Determiner; Description; Relation; Action; Frege's problems.

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1 Introduction

In this paper, we will present a new mathematical model for the semantics of linguistics. Before starting the new model, we'd like to review existing theories of semantics.

1.1 Current Theories of Semantics

Meaning is a key concept of cognition and communication of human thoughts. A main goal of semantic theories is to understand the nature of meaning. Formal semantics began with the work of Gottlob Frege on the philosophy of language in the late nineteenth century. Since then, it has been developed into what is known as the model-theoretic semantics by a group of eminent philosophers and logicians like Russell, Tarski, Montague and others. In his seminal paper "Über Sinn und Bedeutung" [4], Frege divided the semantic content of every expression, including sentences, into two parts — sense (sinn) and reference (bedeutung). The sense of a proper name is the thought that it expresses. It is abstract, universal and objective. The reference (also known as denotation) of a proper name is the object it means or indicates. The sense of a sentence is thought, while its reference is a truth value (true or false). Furthermore, both the sense and denotation of a sentence are functions of the senses and denotations of its parts.

Fregean semantics is truth-conditional in which the meaning of each sentence is reflected by its truth value. This is problematic because there are declarative sentences that can not be valued by truth values. The most notable example of them is the liar paradox such as "The only sentence on the board is not true", which clearly does not have a truth value because either being true or false would lead to contradiction. More importantly, we claim that the meaning of a sentence generally can not be reflected by its truth value alone (if it has one), a key fact that is illustrated in the following discussion.

Truth-conditional semantics [6] assigns a truth value to a linguistic expression with domains of entities and a function to interpret the meaning. For instance, suppose the denotation of verb *eat* is as follows.

$$[\![eat]\!] = f\colon D_1 \times D_2 \to \{0,1\},\ \forall x \in D_1, \forall y \in D_2,\ f(x,y) = 1 \text{ iff } x \text{ eat } y$$

Where f is a function from $D_1 \times D_2$ to a truth value, D_1 is a domain of individuals and D_2 a domain of food.

Despite of its correctness, this model nonetheless fails to reflect the rich content in the meaning of *eat*. The eating process[1] can be modeled as processes of change in the ingestion of food involving the mouth, teeth, tongue, esophagus, stomach and intestines of a person. So "John ate eggs" means that John had processes of change involving eggs in his mouth, teeth, tongue, esophagus, stomach, intestines and so on. He also tasted flavors of eggs and had other feelings. However, "John ate eggs" being true only reflects the fact that John had finished an action of eating eggs, but fails to capture other meaningful contents of *eat*. Let's look at another example.

(1-1) (a) "John is in an elevator in South College" means that John's position is in an elevator in South College, or a set inclusion (5-6), $A \subset B$.

(b) "John is in a room in South College" means that John's position is in a room in South College, or $A \subset C$.

Clearly, even if (a) and (b) are both true, they have different meanings because being in a room is (very) different from in an elevator ($B \neq C$). This fact again suggests that the meaning of a sentence can not be reflected by its truth value alone, but with other factors. As the result, Fregean semantics (as well as the model-theoretic semantics) is not the genuine model for semantics. There are also other theories of semantics such as conceptual, cognitive, computational semantics, and so on [7]. But none of them are good models for semantics as well. Therefore, a new scheme for theory of meaning is called upon.

1.2 A New Theory of Semantics

Since semantics is the field of linguistics to study meaning of words, phrases, sentences and so on, a real theory of semantics must answer the fundamental question — exactly what is *meaning*

[1] *Eat* is a dynamic verb. See section 5.3.

for the linguistic expressions. Language is the most common way for human beings to express their thoughts. Thus words, phrases and sentences are representation of objects, both in the physical and mental worlds. Our minds use those expressions in the form of words and sentences to reflect what we perceive and think. For instances, when we conceive a thing like a person, we think of many attributes possessed by a human being such as his name, age, gender, occupations, and so on; When we think of a country, we reckon its people, location, cities, economy, government, and so forth. This characteristics of the mind suggests that a semantic object (noun or noun phrase) can be described by a feature space consisting of all of its properties in the form of the Cartesian product. Let's exemplify this idea further.

Since snow is a white and solid substance comprised of individual ice crystals, its feature space takes on the following form. (Assume that the total number of properties for *snow* is finite.)

$$X_{snow} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_m$$

Where $\mathfrak{A}_1$ ($= \{$"H_2O"$\}$) is "chemical component", $\mathfrak{A}_2$ ($= \{0°\text{C}\}$) is "melting temperature", $\mathfrak{A}_3$ ($= \{$"white"$\}$) is "color", $\mathfrak{A}_4$ ($= \{$"solid"$\}$) is "state", and so on.[2]

Since linguistic expressions are used for recognition and communication of human thoughts, X_{snow} is certainly qualified for the meaning of *snow*. Note that a semantic object itself does not have a truth value. So a statement like "Snow is true" is considered nonsense. A semantic object has a truth value only if it is put in a sentence (proposition). For examples, "Snow is white" is to check whether a property in the meaning of *snow* is true, i.e. if the color of *snow* is white; "Snow is solid" is to verify if the state of *snow* is solid. So "Snow is white" is true for $\mathfrak{A}_3 =$ "white", and "Snow is solid" is true for $\mathfrak{A}_4 =$ "solid". "Snow is black" is false because of $\mathfrak{A}_3 \neq$ "black".

Next, let's consider a countable noun *human*. Obviously, a human can be described by his names, age, gender, nationalities, appearance, occupations, education, and many others. Since each person has these features different from others, his feature space can be generated from an abstract feature space of *human* known as an abstract form of the noun. So suppose an abstract form of *human* is follows.

$$X_{mankind} = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \cdots \times \mathfrak{R}_n$$

Where $\mathfrak{R}_1$ is "name", $\mathfrak{R}_2$ is "age", $\mathfrak{R}_3$ is "nationality", $\mathfrak{R}_4$ is "gender", $\mathfrak{R}_5$ is "occupation", $\mathfrak{R}_6$ is "appearance", and so on. Since all humans can be generated from $X_{mankind}$, each $\mathfrak{R}_k$ is a range of feature k of the humans. For examples, $\mathfrak{R}_1$ is a list of all the names of humans; $\mathfrak{R}_2$ is a maximum life span of the humans; $\mathfrak{R}_3$ is a list of all the countries in the world; $\mathfrak{R}_4 = \{$"M","F"$\}$, and so on. Each person generated from $X_{mankind}$ is a realization (or instance) of it (definition 2.7). For example, suppose a feature space for *Socrates* is

$$X_{Socrates} = \{\text{"Socrates"}\} \times \{70\} \times \{\text{"Greece"}\} \times \{\text{"M"}\} \times \{\text{"Philosopher"}\} \times \cdots \times \mathfrak{R}_n^{So}$$

For any k, $\mathfrak{R}_k^{So} \subset \mathfrak{R}_k$. Obviously, $X_{Socrates}$ can be considered as the meaning of *Socrates*. Likewise, "Socrates is mortal" is true for $\mathfrak{R}_2^{So} = 70$, and "Socrates is a Roman philosopher" is false for $\mathfrak{R}_3^{So} \neq$ "Roman".

Consequently, we have reached a fundamental assumption in (the new theory of) semantics — the meaning of each noun[3] can be described by sufficient number of distinct features and properties

[2] *Snow* is a primitive uncountable noun (definition 2.18).

[3] This does not include the second class of uncountable nouns, each meaning of which is also a set (axiom 2.21).

in the form of its feature space (2.2). Next, we will demonstrate that the meaning of each linguistic expression can be represented by a set based on feature spaces as well.

Noun phrases are very important not only in linguistics, but in the theory of knowledge. A noun phrase contains a noun plus the modifiers of the noun that can come before or after the noun. Modifiers that come before a noun might include determiners and adjectives. Examples are "John's watch", "his car", "Greek people", and so on. Modifiers that come after a noun might include prepositional phrases and relative clauses. Examples are "people of Greece", "the present king of France", "the person who wrote *The Adventures of Tom Sawyer*", and so on.

We can see that the meaning of a noun phrase can be represented by a set of reduced feature spaces. For example, every Greek can be generated from $X_{mankind}$ by limiting its "nationality" feature to "Greece", i.e.

$$X_{TheGreeks} = X_{mankind}\,(\mathfrak{R}_3 = \text{“Greece”}) = \mathfrak{R}_1^G \times \mathfrak{R}_2^G \times \{\text{“Greece”}\} \times \cdots \times \mathfrak{R}_n^G$$

For any k, $\mathfrak{R}_k^G$ is a reduced range and $\mathfrak{R}_k^G \subset \mathfrak{R}_k$. So the meaning of "people of Greece" can be represented by a collection of instances from $X_{TheGreeks}$ that is denoted as $X_{TheGreeks}^P$ (definition 2.11).

Notice that different noun phrases can refer to the same objects and have the same meaning, e.g. "people of Greece" and "Greek people" have the same meaning. "The person who wrote *The Adventures of Tom Sawyer*" refers to the same person as "Mark Twain" or "Samuel Clemens". So they have the same meaning in the following feature space.

$$X_{MarkTwain} = \{\text{“Mark Twain/Samuel Clemens”}\} \times \{74\} \times \{\text{“US”}\} \times \{\text{“M”}\} \times \{\text{“Author”}\} \times \cdots$$

The fact that a proper noun could have multiple names is crucial in understanding the first Frege's problem (section 6.1). On the other hand, meanings of some noun phrases could be the empty set. We call these phrases no meaning or meaningless. For example, by setting $\mathfrak{R}_3 =$ "France" and $\mathfrak{R}_5 =$ "king" in $X_{mankind}$, no such instance would be generated for the present time. Thus "the present king of France" corresponds to the empty set and so has no meaning.

Above discussion shows that the meaning of a noun phrase can be described by a set based on reduced feature spaces (of the noun). In the latter section, we will show that meanings of stative and dynamic verbs can also be described by sets based on (dynamic) feature spaces. Since each sentence contains at least one main verb that is either stative or dynamic, the meaning of a sentence can be described by a set as well as a truth value reflected by a formula on components of the set. For example, the meaning of (1-1)(a) can be decided by $A \times B$ and (b) by $A \times C$. Both (a) and (b) are true because $A \subset B$ and $A \subset C$, but they have different meanings because $A \times B \neq A \times C$.

Consequently, we have reached another fundamental assumption in semantics — the meaning of every linguistic expression can be represented by a set (upon feature spaces). This exact definition of meaning in linguistics can base semantics upon a solid and rigorous footing so that the investigation of natural language will be greatly facilitated.

1.3 Mathematical Representation for Feature Spaces and Meanings

From previous discussion, we realize that meanings of nouns and phrases can be represented by (reduced) feature spaces. Next, we need to find out what mathematical model is needed for feature spaces. Since many mathematical tools have been developed for number, numerical representation is most desirable for feature spaces. For examples, a noun for a physical quantity such as *weight*, *length*, *size*, *volume*, *time*, and so on, can be described by a single number. Such noun has only one dimension in its feature space and can no longer be divided. Others like *position*, *picture*, *movie*, and so on, are $2D$ or $3D$ models in the form of a tuple of numbers such as a (x, y, z) coordinate for a position, or an array of tuples of numbers for a picture, or an array of pictures for a movie. We call a noun with numerical representation quantifiable. So these nouns are also quantifiable with feature spaces of dimension higher than one.

Another important class of nouns involves strings of alphabets in languages. Examples of this class are *name*, *word*, *book*, *address*, *gender*, *nationality*, *country*, *city*, etc, and the names of the terminologies in all fields, and so forth. This class of nouns can also have numerical representation if each string is assigned with a number by either ASCII codes or the Godel numbering.

The structure of a feature space could be complicated and its features can be other nouns. Generally, there is no strict order in defining features. For example, a feature space of *city* contains a feature "country" that consists of a country where the city situates, while that of *country* has a feature "city" that includes the cities within the country. A feature space can also be created from other feature spaces through processes such as abstraction or inheritance (definition 4.7).

Human is one of the most complex nouns. Basic features that can describe a human include his names, age, weight, height, gender, nationalities, ethnicities, languages, occupations, education, kinship, and so on. These features are clearly quantifiable. More sophisticated features in biological and psychological sections of a human include a body, genetics, health, a life cycle, diet, skills, hobbies, and many others. Furthermore, each of these categories can be decomposed into other features. For example, the human body has two arms and legs, a torso, a neck, a head that has two eyes, a nose, a mouth, and so on. The human life cycle includes a number of stages: infancy, childhood, adolescence, adulthood and old age. Diet consists of carbohydrates, fats, fiber, proteins, vitamins, minerals, and so on. Again, it is reasonably to assume these features are quantifiable and created from other features.

Unfortunately, little is known about emotion which involves happiness, pain, anger, anxiety, sadness, fear, surprise and so on. Generally, we know little about the so-called abstract nouns such as *love*, *happiness*, *hate*, *anger*, *bravery*, and so forth, which are related to human mind and emotion that can not be experienced with our five senses. In other words, we know nothing about exact (quantitative) models of them. However, this fact of insufficient knowledge does not present us from believing in feature spaces as the mathematical representation of the nouns. As fields in biology, psychology, neuroscience and cognitive science progress, it is pretty certain that there will be definitive answers in quest of such (quantitative) models in the future.

Therefore, all feature spaces involved in this paper are limited to be quantifiable. Since we study mathematical description for thought and language is the most common way to articulate the thought, we desire a form that can reflect (almost) every piece of thought in the mind expressible as language. Since the meaning of a linguistic expression can also be described as a set upon feature spaces, a representation for hierarchical sets based on quantifiable feature spaces is preferred. Since a Cartesian product of numbers (quantifiable feature space) can always be converted into a hierarchical set of numbers by the so-called Wiener-Kuratowski ordered pair, we conclude that the universe of the sets based on numbers (known as the total universe $\mathcal{T}$) is the mathematical model for meaning in linguistics. Please refer to appendix A for more detail.

2 Feature Spaces and Nouns

In this section, we give basic definitions in our new theory of semantics first. Then we will have an in-depth analysis on the semantics of nouns based on their feature spaces.

2.1 Definitions

Generally, a linguistic theory divides human languages into two parts — a lexicon that is essentially a collection of a language's words, and a grammar consisting of a system of rules which allow for the combination of those words into meaningful phrases and sentences. Since we only study the semantics of natural language in this paper, we will omit discussion of syntactic rules of a language.

Definition 2.1 *The **lexicon** of a natural language is a collection of words from the alphabet of the language.*[4] ***Linguistic objects** are words from the lexicon of a language which include nouns, verbs, adjectives, determiners, and so on.*

Definition 2.2 *A **linguistic expression** (or **name**) is a phrase or sentence that is a collection of linguistic objects allowed by grammars of the language. It can be a linguistic object, a noun phrase, a verb, a determiner, adverbial, or preposition phrase, a relative clause, and so on.*

Definition 2.3 *Suppose $\mathcal{T}(\mathbb{R})$ (or $\mathcal{T}$) is the universe of the sets on $\mathbb{R}$ (appendix A). Then the **total feature space** of $\mathcal{T}$ is*

$$\mathcal{F}(\mathbb{R}) = \bigcup_{n \geqslant 1} \mathcal{T}^n, \quad \mathcal{T}^n = \underbrace{\mathcal{T} \times \mathcal{T} \times \cdots \times \mathcal{T}}_{n} \tag{2.1}$$

*$\mathcal{F}(\mathbb{R})$ is often denoted as $\mathcal{F}$. A **feature space** in $\mathcal{T}$ is*

$$X = \mathfrak{R}_1 \times \mathfrak{R}_2 \times \cdots \times \mathfrak{R}_n, \quad \mathfrak{R}_k \subset \mathcal{T}, \ 1 \leqslant k \leqslant n \tag{2.2}$$

*Where n is the **dimension** of X, and each $\mathfrak{R}_k$ is a **range** of feature k. In addition, $N(\mathfrak{R}_k)$ is known as a **name** of feature k, and $\mathfrak{I} = \{N(\mathfrak{R}_1), \cdots, N(\mathfrak{R}_n)\}$ is known as a **header** of X.*

From discussion in previous sections, we have the following fundamental assumption in (the new theory of) semantics.

Axiom 2.4 *The **meaning** of a linguistic expression is a set in $\mathcal{T}$ (based on feature spaces). If the meaning of a linguistic expression is the empty set, it is called **no meaning** or **meaningless**. Different linguistic expressions can have the same meaning. The meaning of a meaningful set is also a set in $\mathcal{T}$.*

Axiom 2.5 *The truth value of a sentence can be defined by a relation of components (sets) in the meaning of the sentence (if it has one). Thus the meaning of a sentence can be measured by both a set and a truth value.*

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[4] In this paper, the language and vocabularies used are English. However, (most) conclusions can be applied to other languages (with some modifications specific to each language).

Most sets in $\mathcal{T}$ do not have names in a natural language because a language of finite length is countable which is a less infinity than $\mathcal{T}$. For a meaningful set, however, we can always give it a name through neologisms. Next we will analyze and classify nouns based on feature spaces. We begin with countable nouns.

2.2 Countable Nouns

A countable noun is a noun that can be counted and modified by a numeral. Our claim is that any countable noun can be created from an abstract feature space (abstract form) of the noun (definition 2.7) in a process called realization. The objects formed in realizations are called instances. For a countable noun, all instances are separable and of a similar sort, and thus are countable.

The advantage of the feature space model for (countable) nouns is that it provides a complete description of the instances of a noun (in terms of their features) as well as the difference among distinct nouns (in terms of their abstract forms), compared to a truth-conditional based semantics which only uses domains of entities and interpretation functions to distinguish various nouns.

2.2.1 Abstract Forms and Realizations

First we introduce a feature of particular importance known as an identity index.

Definition 2.6 *An **identity index** is a feature that takes on number. By adding an identity index to a feature space, any collection of instances of a countable noun can be represented by a collection of identity indexes known as a (**identity) domain**. Often a domain is countable and a subset of* $\mathbb{N}$.

Definition 2.7 *An **abstract form of a countable noun** is a feature space (2.2) with an identity index feature. A **realization** of the noun is a subspace of the abstract form of the noun with a unique identity index assigned which is also known as a **realization index**.*[5] *Each object formed in a realization is known as an **instance**.*

Definition 2.8 *If a realization of a countable noun has real existence, it is a **proper noun** and the realization is the meaning of its proper name. Otherwise, it is a **common noun**. A **singular noun** contains one instance of a noun and has a singleton domain. A **plural noun** contains multiple instances and has a domain of multiple singletons.*

Definition 2.9 *If a realization of a feature is unavailable, then it is set to* $\{\varnothing\}$ *and the feature is known as a **nonessential feature**. If a realization of a feature must be available, it is known as an **essential feature**.*

(2-1) (a) Examples of essential features for a human include "head", "body", "name", "age", "gender", and so on.

[5] We do not use a name to identify a realization as it is commonly used in linguistics because it is not a unique measure. For example, a person (realization) could have many names and different persons could have the same name.

(b) Examples of nonessential features include “education”, “nationality”, “profession”, some body parts, and so on.

It is important to take every possible realization of a countable noun into account because it is the meaning of the quantifier *all* (section 3.1.1).

Definition 2.10 *Suppose $X = D_X \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n$ is an abstract form of a countable noun. Then the* ***all-realization*** *of X is denoted by X^* and determined by a domain D_X consisting of all identity indexes of X^*.*

A collection of realizations of a countable noun is needed in definite and indefinite quantifiers such as *the*, *any*, *some*, and so on.

Definition 2.11 *A* ***partial-realization*** *of X is a collection of realizations of X that is determined by a domain $D \subsetneq D_X$ and denoted as X^P.*

Definition 2.12 *Suppose $X = D_X \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n$ is an abstract form of a countable noun. Then the feature space after the identity index being removed is denoted as $\overline{X} = \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n$.*

Definition 2.13 *For $X = D_X \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n$, $\mathfrak{R}_k$ is known as the kth$-$projection of X and denoted as $\pi_k(X)$. $\pi(X)$ is a projection to any k ($0 \leqslant k \leqslant n$, $k = 0$ for the identity index). The projection of multiple dimension of X is denoted as $\pi_S(X) = \mathfrak{R}_{k_1} \times \cdots \times \mathfrak{R}_{k_l}$, where $S = \{k_1, \cdots, k_l\}$.*

2.2.2 Two Types of Countable Nouns

Often an abstract form of a noun does not have a name. For example, we do not use $X_{Country}$ for the abstract form of *country* because $X_{Country}$ is for a realization of *country* or common noun like *a country.* In this case, we use $X_{TheCountry}$ for it.[6] Suppose an abstract form for *country* is

$$X_{TheCountry} = D_C \times \mathfrak{C}_1 \times \mathfrak{C}_2 \times \cdots \times \mathfrak{C}_m \tag{2.3}$$

Where $D_C \subset \mathbb{N}$ is an identity domain for all the countries (past, present, future and fictional countries) in the world, $\mathfrak{C}_1$ is “name of country” (a list of all names of countries), $\mathfrak{C}_2$ is “alias of country” (a list of all names), $\mathfrak{C}_3\,(=\mathbb{N})$ is “population”, $\mathfrak{C}_4$ is “capital city” (a list of all names of cities), $\mathfrak{C}_5$ is “official language” (a list of all the languages in the world), $\mathfrak{C}_6$ is “continent” (a list of seven continents in the world), and so on. Next suppose feature spaces of (real) nations of US and Canada are

$$\begin{aligned} Y_{US} = \{I_1\} \times \mathfrak{C}_1^U \times \cdots \times \mathfrak{C}_m^U &= \{I_1\} \times \{\text{“USA”}\} \times \{\text{“US”}, \text{ “America”}, \cdots\}\times \\ &\{330\text{m}\} \times \{\text{“Washington”}\} \times \{\text{“En”}\} \times \{\text{“North America”}\} \times \cdots \\ Y_{Canada} = \{I_2\} \times \mathfrak{C}_1^C \times \cdots \times \mathfrak{C}_m^C &= \{I_2\} \times \{\text{“Canada”}\} \times \{\text{“Great White North”}, \cdots\} \\ &\times \{36\text{m}\} \times \{\text{“Ottawa”}\} \times \{\text{“En”},\text{“Fr”}\} \times \{\text{“North America”}\} \times \cdots \end{aligned}$$

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[6]We will do this to other nouns in the rest of this paper, i.e. add ”The” ahead of the noun if there is no name for the abstract form of it.

Where I_1 and I_2 are identity indexes of *US* and *Canada*, $\mathfrak{C}_k^U \subset \mathfrak{C}_k$ and $\mathfrak{C}_k^C \subset \mathfrak{C}_k$ for $1 \leqslant k \leqslant m$. If any $\mathfrak{C}_k^U$ or $\mathfrak{C}_k^C$ is unavailable, then it is set to $\{\varnothing\}$. We can see that Y_{US} is the meaning of proper name *US* or *USA*, and Y_{Canada} is that of *Canada*. Any realization like $\{i\} \times \mathfrak{C}_1^i \times \cdots \times \mathfrak{C}_m^i$ is the meaning of "a country". Notice that each realization generates only one distinct country because no two countries are identical. In other words, only one instance of *country* is generated in each realization.

Now let's consider the noun *human* again. We believe that an abstract form of *human* does correspond to a noun like *mankind*. In addition, we assume that the total number of humans ever been born (and to be born) in the world is at most countable. So suppose an abstract form for *human* is

$$X_{mankind} = D_H \times \mathfrak{R}_1 \times \mathfrak{R}_2 \times \cdots \times \mathfrak{R}_n \tag{2.4}$$

Where $D_H \subset \mathbb{N}$ is a domain for all the humans, $\mathfrak{R}_1$ is "name" (a list of all the names of humans), $\mathfrak{R}_2\,(=\{(0,150)\})$[7] is "age", $\mathfrak{R}_3$ is "nationality" (a list of all the countries in the world), $\mathfrak{R}_4\,(=\{\text{“M”},\text{“F”}\})$ is "gender", $\mathfrak{R}_5$ is "occupation", $\mathfrak{R}_6\,\left(=\mathbb{R}^3\right)$[8] is "appearance", $\mathfrak{R}_7$ is "position" (GPS coordinates), $\mathfrak{R}_8$ is "address", $\mathfrak{R}_9$ is "alias" (a list of all the names), and many others.

The meaning of "all humans" is the all-realization of $X_{mankind}$, i.e.

$$X_{AllHumans} = X^*_{mankind} = \{\{i\} \times \mathfrak{R}_1^i \times \cdots \times \mathfrak{R}_n^i \colon i \in D_H\} \tag{2.5}$$

Where i is the identity index for each human,[9] and for any k, $1 \leqslant k \leqslant n$ and any $i \in D_H$, $\mathfrak{R}_k^i \subset \mathfrak{R}_k$. The meaning of *Socrates* is the realization as follows.

$$X_{Socrates} = \{I_{So}\} \times \{\text{“Socrates”}\} \times \{70\} \times \{\text{“Greece”}\} \times \{\text{“M”}\} \times \{\text{“Philosopher”}\} \times \cdots \times \mathfrak{R}_n^{So}$$

Where I_{So} is the identity index of Socrates and $I_{So} \in D_H$. Clearly, $X_{Socrates} \in X_{AllHumans}$, which is also the meaning of sentence "Socrates is a human". Furthermore, the indefinite quantifier *some* specifies a partial-realization of $X_{mankind}$, i.e.

$$X_{SomeHumans} = X^P_{mankind} = \{\{i\} \times \mathfrak{R}_1^i \times \cdots \times \mathfrak{R}_n^i \colon i \in D_0\}$$

Where $D_0 \subsetneq D_H$. So $X_{SomeHumans}$ is the meaning of "Some humans". Normally we only consider a relativized domain $E_H \subset D_H$ for "all humans". (Include some past, all present and fictional humans, but not future ones.). So we have

Definition 2.14 *Suppose $X = D_X \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n$ is an abstract form of a countable noun. Then a **relativized domain** or **relativization** is a partial-realization of X that is determined by a domain $E \subsetneq D_X$.*

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[7]This is based on the fact that the longest life span for human beings is no more than 150 years (from official records and not including unconfirmed cases as well as legend).

[8]"Appearance" is one of the most important features of human beings because the most common way to recognize a creature as a human is through its face and body. Unfortunately, there is no exact mathematical representation for human faces and bodies. But we do know that they are subsets of $\mathbb{R}^3$.

[9]This can be a Social Security number (SSN) in US or a Social Insurance number (SIN) in Canada.

Next let's consider another example of countable nouns. Suppose an abstract form of *book* is

$$X_{TheBook} = D_B \times \mathcal{K}_1 \times \cdots \times \mathcal{K}_l \tag{2.6}$$

Where $D_B \subset \mathbb{N}$ is "ISBN number", $\mathcal{K}_1$ is "book name", $\mathcal{K}_2$ is "author", $\mathcal{K}_3$ is "owner", $\mathcal{K}_4$ is "position" (GPS coordinates), $\mathcal{K}_5$ is "language" (a list of all the languages in the world), $\mathcal{K}_6$ is "content" (lexicons of language), and so on. Notice that in this case, each realization (ISBN number) produces many identical copies of a book (all copies with the same ISBN number are identical). So we have to add another index known as an instance index to identify each copy of an edition. So (2.6) becomes

$$X_{TheBook} = D_B \times \mathcal{K}_0 \times \mathcal{K}_1 \times \cdots \times \mathcal{K}_l \tag{2.7}$$

Where $\mathcal{K}_0 \subset \mathbb{N}$ is a domain of the instances of *book*. Obviously, this type of noun can generate multiple (identical) instances in each realization, and is very different from *country* or *human* in which each realization only generates one instance.

This type of noun actually applies to most physical or intangible (countable) products, in which an index generally known as a serial number is used to uniquely identify each individual product. For different products, serial numbers may have different names. For examples, a serial number is known as an IMEI number for smartphones; it remains the same for electronic devices such as computers, TVs, etc; it is known as a VIN number for vehicles.

Likewise, a realization index generally corresponds to the model number of a product. For examples, each edition of a book is assigned with an ISBN number; a model number of a specific vehicle consists of a vehicle model plus a model year, and so on.

So we reach a fundamental assumption about countable nouns.

Axiom 2.15 *Any instance of a* ***countable noun*** *can be generated from an abstract form of the noun through realization. All instances of a countable noun are separable and of a similar sort, and so can be counted. A single or multiple instances of the noun can be generated in a realization depending on the type of the noun. If multiple instances are created in one realization, a new identity index known as an* ***instance index*** *is needed to identify each instance.*

Definition 2.16 *There are two types of countable nouns. A countable noun that has only one instance for each realization is known as a* ***countable noun of the first class****. A countable noun that can generate many instances in a realization is known as a* ***countable noun of the second class****.*

Conclusion 2.17 *After an instance index is added to an abstract form of a noun, each realization generates only one instance. So a countable noun of the second class can be converted to a first class noun after an instance index is added.*[10]

Here are more examples of countable nouns.

(2-1) (a) Countable nouns of the first class include *human*, *student*, *teacher*, *boy*, *girl*, *university*, *city*, *country*, many members of *animals* and *plants*, and so on.

[10] In the rest of this paper, we will only consider countable nouns of the first class unless further specified.

(b) Countable nouns of the second class include most physical or intangible (countable) products such as *computer*, *car*, *chair*, *plate*, *book*, *periodical*, *movie*, *song*, and so on.

In above example, any feature space of *country* also includes the feature "city", which is linked to a feature space of *city*. This fact, together with identity indexes, suggest an analogy between feature spaces and relational databases [3]. Despite some similarities, however, feature spaces are generally far more complex than relational databases because of the following reasons.

First, feature spaces contain general sets that are far more complicated than the data types provided in relational databases. Second, in order to describe linguistic variables, feature spaces contain relations much more sophisticated than those of relational databases. So for feature spaces, it is in general not possible to list all the joins at once, and often it needs multiple keys to reference all the features as well as functions like dynamically creating and merging tables from the others.

2.3 Uncountable Nouns

An uncountable noun is a noun that can not be counted.[11] In other words, it refers to an extended substance rather than to a set of isolable objects. It also refers to a set of many (countable) objects which are of different sorts, and thus can not be counted. Our claim is that uncountable nouns can be divided into two classes — a primitive uncountable noun which can be generated from an abstract form of the noun like a countable noun, but with a unit of measurement instead of an identity index; a composite uncountable noun which consists of many sorts of objects such as primitive uncountable nouns, countable nouns, other composite uncountable nouns, and everything in its transitive closure.

2.3.1 Primitive Uncountable Nouns

Since a primitive uncountable noun is inseparable and continuous, each of its realizations can only be distinguished by a quantity, not by an identity index as for a countable noun. In other words, each realization is the measurement of an uncountable set (or a countable set of large size). For examples, any realization of liquid is an uncountable set that is measured by volume; any realization of metal is measured by weight. First we have

Definition 2.18 *An* ***abstract form of a primitive uncountable noun*** *is a feature space (2.2) with a measurement feature. A* ***realization*** *of the noun is a subspace of the abstract form of the noun with a unit of measurement.*

First let's look at a specific example of primitive uncountable nouns. Suppose an abstract form of *water* is as follows.

$$X_{TheWater} = M \times \mathfrak{R}_1 \times \mathfrak{R}_2 \times \cdots \times \mathfrak{R}_n \tag{2.8}$$

Where M (= {"volume"}) is "measurement", $\mathfrak{R}_1$ (= {"H_2O"}) is "chemical component", $\mathfrak{R}_2$ (= {0°C}) is "freezing temperature", $\mathfrak{R}_3$ (= {100°C}) is "boiling temperature", $\mathfrak{R}_4$ (= {"incompressible"}) is "mechanical property", $\mathfrak{R}_5$ (= {"Yes"}) is "flowability", $\mathfrak{R}_6$ (= {"No"}) is "fixed shape", $\mathfrak{R}_7$ (= {"Yes"}) is "fixed volume", and so on.

[11] Because different languages have distinct sets of uncountable nouns, only those in English are studied in this paper.

Since all features (except "measurement") are singletons, for any k, $1 \leqslant k \leqslant n$ and all i, $\mathfrak{R}_k^i \subset \mathfrak{R}_k$ implies $\mathfrak{R}_k^i = \mathfrak{R}_k$. So any realization of *water* is

$$X_{water} = M \times \mathfrak{R}_1^i \times \cdots \times \mathfrak{R}_n^i = M \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n$$

This means that each realization of *water* depends only on the volume of continuous measure. In other words, there is only one realization of *water* minus the unit of measurement. Moreover, since volume is measurable and incompressible, the sum of two volumes of water is still water, i.e.

$$M_1 \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n \cup M_2 \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n = (M_1 \cup M_2) \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n \tag{2.9}$$

Thus *water* has cumulative reference. Other examples of primitive uncountable nouns are *mercury*, *hydrogen*, *oxygen*, *iron*, *gold*, *silver*, *wood*, *glass*, *snow*, *rice*, and so forth. So we have

Axiom 2.19 *There are two classes of uncountable nouns. The first class is known as* ***primitive uncountable nouns****, each of which can be generated from an abstract form whose features (except the measurement) are exclusively singletons. Each realization of a primitive uncountable noun generates an inseparable (uncountable) set or a countable set of large size, which can not be counted but only be measured by a unit of measurement. There is only one realization of the noun minus the unit of measurement which is also the meaning of the noun.*

Since (2.9) applies to any primitive uncountable noun, we have

Corollary 2.20 *Any primitive uncountable noun has cumulative reference.*

2.3.2 Composite Uncountable Nouns

Compared to a primitive uncountable noun, however, a composite uncountable noun is not generated from a feature space, but consists of many objects, some of which may be countable. Since objects are of different sorts, a composite uncountable noun can not be counted either and only has one distinct instance as a whole.

Let's take a look at an example of composite uncountable nouns. Furniture includes chairs, sofas, tables, beds, and so on, with each of its members being a countable noun. Suppose *chair*, *sofa*, *table*, and whatever can be considered as a member of furniture, have abstract forms as follows.

$$\begin{aligned} X_{TheChair} &= D_1 \times \mathfrak{R}_1^1 \times \cdots \times \mathfrak{R}_{n_1}^1 \\ X_{TheSofa} &= D_2 \times \mathfrak{R}_1^2 \times \cdots \times \mathfrak{R}_{n_2}^2 \\ X_{TheTable} &= D_3 \times \mathfrak{R}_1^3 \times \cdots \times \mathfrak{R}_{n_3}^3 \\ &\vdots \end{aligned}$$

Where all D_k are disjoint. So their all-realizations are

$$\begin{aligned} \text{“all chairs”} = X^*_{TheChair} &= \{\text{ chair1, chair2, } \cdots\} \\ \text{“all sofas”} = X^*_{TheSofa} &= \{\text{ sofa1, sofa2, } \cdots\} \\ \text{“all tables”} = X^*_{TheTable} &= \{\text{ table1, table2, } \cdots\} \\ &\vdots \end{aligned} \tag{2.10}$$

We form a concept *furniture* by wrapping all instances of $X_{TheChair}$, $X_{TheSofa}$, $X_{TheTable}$, etc, in it. More precisely, a feature space of *furniture* is[12]

$$\begin{aligned} X_{furniture} &= \{X_{TheChair}, X_{TheSofa}, X_{TheTable}, \cdots, *X^*_{TheChair}, *X^*_{TheSofa}, *X^*_{TheTable}, \cdots\} \\ &= \{\text{ chair1, chair2, } \cdots, \text{ sofa1, sofa2, } \cdots, \text{ table1, table2, } \cdots\} \end{aligned} \tag{2.11}$$

$X_{furniture}$ is also the meaning of *furniture*. Obviously, the addition of any collection of chairs, sofas, tables, etc, to *furniture* will not change itself. More precisely,

$$X_{furniture} \cup \{\text{chair}i_1, \cdots, \text{sofa}j_1, \cdots, \text{table}k_1, \cdots\} = X_{furniture}$$

The above holds because Chairi_l is a member of $X^*_{TheChair}$ and so belongs to $X_{furniture}$ as well. The same holds for sofaj_l and tablek_l. Thus *furniture* also has cumulative reference.

Now let's look at a more complicated example of composite uncountable nouns. Food is any substance consumed that can provide nutritional support for the body. It is usually of plant or animal origin, and contains essential nutrients, such as carbohydrates, fats, proteins, vitamins, and minerals.

Food can be divided into categories like staple food, non-staple food, raw food, cooked food, cuisine, and so on. Staple food includes raw food like wheat, rice, potatoes, corn, as well as cooked food like bread, cake, cookie, and so on. Non-staple food involves meat, eggs, fruits, vegetable,[13] seafood, and so forth, in which meat includes chicken, beef, ham, and so on; fruits involve apples, bananas, grapes, oranges, and so forth; vegetable includes cabbages, tomatoes, cucumbers, spinach, tofu, and so on. Cuisine includes Chinese food, Italian food, and so forth.

Clearly, *food* contains countable nouns like *egg*, *apple*, etc, primitive uncountable nouns like *wheat*, *rice*, etc, and composite uncountable nouns like *meat*, *cuisine*, etc. It is much more complicated than *furniture* which only contains countable nouns. We can define a composite uncountable noun based on types of its members. For examples, a countable noun contains its abstract form and all the instances as in (2.10); a primitive uncountable noun consists of its abstract form; a composite uncountable noun is the transitive closure (definition B.3) of its members. So we have

$$\begin{aligned} \text{“food”} &= \{\text{ “staple food”, “non-staple food”, “raw food”, “cuisine”, } \cdots\} \\ \text{“staple food”} &= \{\text{ “wheat”, “rice”, “potatoes”, “corn”, “bread”, } \cdots\} \\ \text{“non-staple food”} &= \{\text{ “meat”, “eggs”, “fruits”, “vegetable”, “seafood”, } \cdots\} \\ \text{“cuisine”} &= \{\text{ “Chinese food”, “Italian food”, } \cdots\} \\ \text{“meat”} &= \{\text{ “chicken”, “pork”, “beef”, } \cdots\} \\ \text{“eggs”} &= \{\text{ egg1, egg2, } \cdots\} \\ \text{“vegetable”} &= \{\text{ “cabbages”, “tomatoes”, “spinach”, “tofu”, } \cdots\} \\ &\vdots \end{aligned}$$

[12] $*S$ is the unpacking operator of S (definition B.1).

[13] In this paper, we consider *vegetable* as a composite uncountable noun because it contains uncountable nouns such as *tofu* and *spinach*.

Therefore, *food* is the transitive closure of "food", i.e. its feature space is

$$X_{food} = TC\{ \text{"staple food"}, \text{"non-staple food"}, \text{"raw food"}, \text{"cuisine"}, \cdots \} \tag{2.12}$$

Above discussion leads to the following assumption.

Axiom 2.21 *The second class of uncountable nouns is known as* ***composite uncountable nouns****, each of which is not generated from an abstract form, but a transitive closure involving primitive uncountable nouns, countable nouns and other composite uncountable nouns, which is also the meaning of the noun. A composite uncountable noun can not be counted and only has one instance as a whole.*

Here are more examples of composite uncountable nouns.

(2-2) (a) *Equipment*, *clothing*, *money*, *data*, *information*, *knowledge*, and so on.

(b) All field names such as *literature*, *music*, *art*, *mathematics*, *physics*, *psychology*, *economics*, *engineering*, *science*, and so on.

(c) All languages and sports such as *English*, *French*, *Chinese*, *basketball*, *football*, *chess*, and so on.

A composite uncountable noun also has cumulative reference because the transitive closure of a set includes everything in the set.

Corollary 2.22 *Any composite uncountable noun has cumulative reference.*

3 Determiners

In this section, we will analyze determiners based on feature spaces. A determiner is a word or phrase that occurs together with a noun or noun phrase to express the reference of that noun or noun phrase in the context. Most determiners have been traditionally classified as either adjectives, pronouns or articles. We begin with determiners for countable nouns.

3.1 Determiners for Countable Nouns

According to the new theory of semantics, meanings of a linguistic expression can be represented by sets. Thus our claim is that meanings of determiners for a countable noun are decided by subdomains (sets) on the all-realization of the noun (definition 2.10). Let's start with quantifiers.

3.1.1 Definite and Indefinite Quantifiers

The study of definiteness and indefiniteness begins with the hypothesis that the definite and indefinite articles in English correspond to two primitive building blocks of linguistic structure with fixed and distinct meanings [9]. Russell proposed that the definite and indefinite can be modeled by propositions [14] which are implemented in truth-conditional semantics as follows.

(3-1) (a) $[\![the]\!] = \lambda P.\lambda Q.\exists x \, [\forall y \, [P(y) \longleftrightarrow x = y] \wedge Q(x)]$

(b) $[\![a]\!] = \lambda P.\lambda Q.\exists x\,[P(x) \wedge Q(x)]$

On the other hand, Frege [4] argued that what the definite expresses over the indefinite is not an additional assertion but a presupposition. As the result, *the* is modeled as

(3-2) $[\![the]\!] = \lambda P\colon \exists x \forall y\,[P(y) \longleftrightarrow x = y]\,.\lambda Q.\exists x\,[P(x) \wedge Q(x)]$

Notice that the definite and indefinite articles denote functions of type <*et*, <*et*, *t*>>. Now let's look at the following example.

(3-3) (a) The book arrived.

(b) A book arrived.

(c) The book I purchased from the Amazon website arrived.

For Russell, the difference between *the* and *a* is in their truth conditions, i.e. substituting *the* for *a* leads to a stronger assertion. So in (3-3), (a) entails (b), but not vice versa.

In the new theory of semantics, *the* and *a* are considered simply as an instance of a countable noun among its all-realization. In (3-3), "the book" just specifies an instance of *book* with certain property, while "a book" means any instance of *book* without a specific criterion. In (a), the property associated with *the* is implicit, whereas in (c), the property is stated in a relative clause.

The type of functions is (completely) abandoned in the new theory of semantics because an action exerted by a verb is automatically applied to an instance (feature space) of the noun. So (3-3)(a) indicates that the action *arrive* ((5-14)(b)) exists on a particular instance of *book*, and (b) indicates that the action exists on an instance of *book*.

(3-4) (a) Books arrived.

(b) The books arrived.

A plural (countable) noun means that there is more than one instance of the noun, whose meaning is a non-singleton subdomain of the all-realization of the noun. A plural noun preceded by *the* indicates all instances in the all-realization. So (3-4)(a) means that there is a domain of *book*, each (instance) of which has the action of *arrive*, while (b) means that all instances of *book* (in a range) has the action.

Often we do not need to consider all the realizations of a noun. Rather we only work on a relativized domain of the all-realization (definition 2.14). For instance, we do not need (or is not possible) to consider all the countries in the past and in the future, and so "all countries" generally means all the countries in the world up to today.

Axiom 3.1 *Suppose E is a relativized domain of a countable noun X and $E = \{i_1, i_2, \cdots\}$, $i_k \in \mathbb{N}$ for all $k \in \mathbb{N}$. Then*

(a) *The meaning of "the X" is $\{i_j\colon \varphi(i_j)\}$, where φ is a function defining the property and uniqueness of i_j.*

(b) *The meaning of "a X" is $\{i_k\}$.*

(c) *The meaning of "Xs" is a $D \subset E$ that* $|D| \geqslant 2$.

(d) *The meaning of "the Xs" is E.*

Above idea can be extended to more quantifiers. Let's look at the following examples.

(3-5) Suppose E_C is a relativized domain of *country* with an abstract form $X_{TheCountry}$ (2.3).

(a) "*All* countries" contains all instances in E_C and can be defined by E_C.
(b) "*Every/each* country" means all singletons in E_C, and can be defined by $D = \{\{i_k\} \colon i_k \in E_C \wedge i_k \in \mathbb{N}\}$.
(c) "*Some* countries" means a collection of more than one but not all instances of *country*, and can be described by a $D \subsetneq E_C$ that $|D| \geqslant 2$.
(d) "*Any* countries" means any collection of instances of *country*, and can be described by a $D \subset E_C$.
(e) "*No* country" means no instance of *country* and can be defined by $D = \varnothing$ or $|D| = 0$.
(f) "10 countries" contains 10 instances of *country*, and can be decided by a $D \subset E_C$ that $|D| = 10$.
(g) "*At least* 5 countries" can be decided by a $D \subset E_C$ that $|D| \geqslant 5$.
(h) "*Less than* 10 countries" can be decided by a $D \subset E_C$ that $|D| < 10$.

So we have

Axiom 3.2 *Suppose E is defined in axiom 3.1. Then*

(a) *The meaning of "all Xs" is E.*

(b) *The meaning of "every/each X" is* $\{\{i_k\} \colon i_k \in E\}$.

(c) *The meaning of "some Xs" is a $D \subsetneq E$ that* $|D| \geqslant 2$.

(d) *The meaning of "any Xs" is a $D \subset E$.*

(e) *The meaning of "no/none X" is $\varnothing$.*

(f) *The meaning of "n Xs" is a $D \subset E$ that $|D| = n$.*

(g) *The meaning of "half Xs" is a $D \subset D_0$ that $|D| = |D_0|/2$ (D_0 is the meaning of "Xs").*

(h) *The meaning of "at least/at most n Xs" is a $D \subset E$ that $|D| \geqslant n$ or $|D| \leqslant n$ respectively.*

(i) *The meaning of "less than/more than n Xs" is a $D \subset E$ that $|D| < n$ or $|D| > n$ respectively.*

Unfortunately, determiners such as *many*, *most*, *few*, *more*, *several*, and so on, can not be modeled by classic set theory because each of them can not be represented by a (classic) set. Consequently, we will not pursue this type of determiner further in this paper.

3.1.2 Ranges of Meaning for Quantifiers

In this section, we will introduce an importance notion known as the range of meaning for a quantifier.

Definition 3.3 *The* ***range of meaning*** $\mathcal{C}$ *for a quantifier* $\mathcal{Q}$ *with respect to a relativization* E *is defined as the collection of all (possible) meanings of* $\mathcal{Q}$ *with respect to* E*, i.e.*

$$\mathcal{C}(\mathcal{Q}) \;=\; \{D\colon D \subset E \,\wedge\, D \textit{ is a meaning of } \mathcal{Q}\}$$

Clearly, the range of meaning for a quantifier is a subset of the power set (definition A.1) of a relativization for the quantifier.

Lemma 3.4 $\mathcal{C}(\mathcal{Q}) \;\subset\; \mathcal{P}(E)$

Theorem 3.5 *Suppose* E *is a relativization and* $E = \{i_1, i_2, \cdots\}$, $i_k \in \mathbb{N}$ *for all* $k \in \mathbb{N}$*. Then*

(a) $\mathcal{C}(\textit{all}) \;=\; \{E\}$

(b) $\mathcal{C}(\textit{every}) \;=\; \mathcal{C}(\textit{each}) \;=\; \{\{i_1\}, \{i_2\}, \cdots\}$

(c) $\mathcal{C}(\textit{some}) \;=\; \mathcal{P}(E) - \{\varnothing, E\} - \mathcal{C}(\textit{every})$

(d) $\mathcal{C}(\textit{any}) \;=\; \mathcal{P}(E)$

(e) $\mathcal{C}(\textit{no}) \;=\; \mathcal{C}(\textit{none}) \;=\; \{\varnothing\}$

Proof. (a) and (e) clearly follow from (a) and (e) of axiom 3.2 since E is the only meaning for *all* and $\varnothing$ for *no/none*.

(b) By axiom 3.2(b).

(c) By axiom 3.2(c), meanings of *some* include any sets in $\mathcal{P}(E)$ minus $\varnothing, E$ and all singletons of E.

(d) By axiom 3.2(d), any subset of E can be a meaning of *any*. ■

Corollary 3.6 *The ranges of meaning for "some", "all", "every" and "none" form a partition of that of "any", i.e.*

$$\mathcal{C}(\textit{any}) \;=\; \mathcal{C}(\textit{all}) \,\cup\, \mathcal{C}(\textit{some}) \,\cup\, \mathcal{C}(\textit{every}) \,\cup\, \mathcal{C}(\textit{none})$$

Proof. It follows by theorem 3.5. Clearly all four on the right are disjoint. ■

Corollary 3.6 shows that *any* has the most comprehensive range of meaning which equals the combination of those of *all*, *some*, *every* and *none*, a fact that allows us to solve what is described in [13] as a serious problem for semantic analysis of *any*. In [13, p385], it states that the determiner *any* has been a notorious problem for semantic analysis, since it is sometimes equivalent to the universal *every*, which is called the free choice *any* as in "Any book is readable". But there are contexts where it is not equivalent to universal quantification and in other contexts it is simply unacceptable. First let's look at the following examples.[14]

[14] Many examples come from [13, p385 - 386].

(3-6) (a) John did not read any books.
(b) *John read any books.
(c) John read some books.
(d) John can read any books.
(e) John left without any books.
(f) *John left with any books.
(g) May I borrow some books from you?
(h) Can I borrow any books from you?
(i) Any book is readable.

In (3-6), (a) is right because it means the domain of books read by John is empty, and *any* includes $\varnothing$. (b) is wrong because the sentence is positive which means the domain of books read by John is not empty, but *any* includes $\varnothing$, a possible conflict. (c) is right because *some* does not include $\varnothing$. (d) is correct because the auxiliary verb *can* changes the meaning so that *any* can be used in this situation since "can read" does not mean actually reading books. (e) is correct and (f) is wrong for the same reasons as in (a) and (b). (g) and (h) are both correct because, (g) means that I will borrow some books from you positively and so the domain of books I borrow from you is not empty, while (h) means that it is uncertain weather I will get your books and so the domain of books I borrow from you may be empty. (i) is right because the empty domain of books is also readable. So in this case, *any* means both *every* and *none*.

(3-7) (a) At most ten students who read anything passed.
(b) *At least ten students who read anything passed.
(c) Before anyone enters, he must bow.
(d) *After anyone enters, he must bow.
(e) After someone enters, he must bow.
(f) Never may anyone touch it!
(g) *Always may anyone touch it!

In (3-7), (a) is right because meaning of *at most* can be $\varnothing$ (axiom 3.2(h)), and so *any* can be used. (b) is wrong because meaning of *at least* can not be $\varnothing$, and so *any* can not be used. (c) is correct because it means that the domain of persons who enter is empty, and so *any* can be used. (d) is wrong because it means that the domain of persons who enter is not empty, and so *any* can not be used. (e) is correct because the domain of persons is not empty, and so *some* can be used. (f) is right and (g) is wrong because, (f) means that the domain of persons who touch it could be empty, while (g) means that the domain of persons who touch it is not empty.

From above discussion, we reach the following conclusion.

Conclusion 3.7 *If the meaning of a sentence involves empty domain, then "any" can be used and "some" can not be used. Otherwise, "any" can not be used. This can explain why "some" is mostly used in positive sentences, whereas "any" is often used in negative sentences and questions.*

3.2 Determiners for Uncountable Nouns

The article *a/an* can not be used before an uncountable noun, but *the* can be used. Quantifiers like *some*, *any*, *all*, etc, are available for an uncountable noun, but *every* and *each* are not. For a primitive uncountable noun, a numeral must be followed by a unit of measurement (e.g. 10 liters of water is OK but 10 water is not), and quantifiers for it are based on a relativization of measurement.

First let's take a look at the following examples. Suppose M is the total amount of water (in the number of units) in a place.

(3-8) (a) "*The* water" means a specific amount of water M_0 that $0 < M_0 \leqslant M$.

(b) "*All* water" or "*all* of the water" can be described by M.

(c) "*No* water" or "*none* of the water" means zero amount of water or $M_0 = 0$.

(d) "*Some* water" or "*some* of the water" can be described by a M_0 that $0 < M_0 < M$.

(e) "*Any* water" or "*any* of the water" can be described by a M_0 that $0 \leqslant M_0 \leqslant M$.

(f) "3 liters of water" means $M_0 = 3 < M$.

If we let a set E correspond to M and $\varnothing$ to 0,[15] above can be translated to what are similar to quantifiers for a countable noun (axiom 3.2) as follows.

(3-9) (a) "*The* water" can be described by a specific D that $\varnothing \neq D \subsetneq E$.

(b) "*All* water" can be described by E.

(c) "*No* water" can be described by $\varnothing$.

(d) "*Some* water" can be described by a D that $\varnothing \neq D \subsetneq E$.

(e) "*Any* water" can be described by a $D \subset E$.

(f) "3 liters of water" means a $D \subset E$ that $m(D) = 3$.

For a composite uncountable noun, a quantifier before it specifies a subset of a feature space of the noun (transitive closure), and a numeral must be followed by a unit (noun).

(3-10) (a) "*The* food" can be described by a specific $S \subsetneq X_{food}$ (2.12) that $S \neq \varnothing$.

(b) "*All* meat" refers the transitive closure of *meat* or $TC\{$"chicken", "pork", "beef", $\cdots\}$.

(c) "*No* food" means nothing from X_{food} or $\varnothing$.

(d) "*Some* food" can be described by a $S \subsetneq X_{food}$ and $S \neq \varnothing$.

(e) "*Any* food" can be described by a $S \subset X_{food}$.

(f) "5 pieces of furniture" means 5 instances in $X_{furniture}$ (2.11) or a $S \subset X_{furniture}$ that $|S| = 5$.

From above examples, we can see that certain quantifiers for an uncountable noun can be determined by subsets of either a relativization or the transitive closure of the noun.

[15] This can be done by a mapping m from a set to a measure that $m(E) = M$ and $m(\varnothing) = 0$.

Axiom 3.8 *Suppose E is a relativization (corresponding to the measure M) for a primitive uncountable noun X or the transitive closure for a composite uncountable noun X. Then the following **quantifiers** for X can be decided by subsets of E.*

(a) *"All X" is defined by E.*

(b) *"The X" is defined by a specific $D \subsetneq E$ and $D \neq \varnothing$.*

(c) *"No/none X" is defined by $D = \varnothing$ or $m(D) = 0$.*

(d) *"Some X" is defined by a $D \subsetneq E$ and $D \neq \varnothing$.*

(e) *"Any X" is defined by a $D \subset E$.*

(f) *"n units of X" is defined by a $D \subset E$ that $|D| = n$ (D is countable) or $m(D) = n$ (D is uncountable).*

Furthermore, theorem 3.5 and corollary 3.6 except *every* and *each* hold for uncountable nouns as well.

Corollary 3.9 *Suppose E is defined as in axiom 3.8. Then*

(a) $\mathcal{C}(all) = \{E\}$

(b) $\mathcal{C}(some) = \mathcal{P}(E) - \{\varnothing, E\}$

(c) $\mathcal{C}(any) = \mathcal{P}(E)$

(d) $\mathcal{C}(no/none) = \{\varnothing\}$

(e) $\mathcal{C}(any) = \mathcal{C}(all) \cup \mathcal{C}(some) \cup \mathcal{C}(none)$

Conclusion 3.7 also holds for uncountable nouns.

(3-11) (a) Do you want some more coffee?
(b) Do you like any water?
(c) *I eat any food.
(d) Any food is needed.

In (3-11), (a) is right because it means the amount of coffee you want is not zero, and so *some* can be used. (b) is right because it is uncertain whether you want water or not so that the amount of water may be zero, and thus *any* is used. (c) is wrong because it means the food I eat is not empty, and so *any* can not be used. (d) is correct because the amount of food needed could be zero, and so *any* can be used.

4 Descriptions of Nouns

In this section, we will investigate descriptions of a noun in the form of a description phrase of the noun, which is essentially the noun with modifiers that come before and after it. Modifiers that come before a noun might include articles, nouns, pronouns, adjectives, gerunds and participles. Examples are "a man", "John's book", "set theory", "his car", "Greek people", "red apple", "a green silk dress", "boiling water", "hidden secret", and so on. Modifiers that come after a noun might include preposition phrases, relative clauses, participle phrases and infinitives. Examples are "people of Greece", "a dress of silk in green color", "the present king of France", "the person who wrote *The Adventures of Tom Sawyer*", "an important job to be done", and so forth.

Therefore, a description phrase of a noun can include a noun phrase, a determiner (article, pronoun or adjective) phrase, a preposition or participle phrase, a relative clause, or a combination of them. In other words, description phrases[16] are essentially linguistic expressions (2.2) except sentences. Description phrases are very important not only in linguistics, but in theory of knowledge.

The notion of a feature space provides an ideal understanding on descriptions of a noun because, the effect of the descriptions is simply to reduce a feature space of the noun to a subspace of it. For instances, "people of Greece" contains one description that reduces the nationality of people from any country to Greece; "a green silk dress" has two descriptions that reduce the color of a dress to green and material to silk. We start with descriptions on countable nouns.

4.1 Descriptions of Countable Nouns

Definition 4.1 *A* ***description*** *of a countable noun can be represented by a* ***description phrase*** *that is a non-sentence linguistic expression. It reduces one feature of the noun to a proper subset of the original one. More precisely, suppose* $X = D \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n$ *is an abstract form for a noun. Then a description* $\mathfrak{D}_k$ *on feature* k *is*

$$X_k \;=\; X(\mathfrak{R}_k = \mathfrak{D}_k) \;=\; D_k \times \mathfrak{R}_1^k \times \cdots \times \mathfrak{D}_k \times \cdots \times \mathfrak{R}_n^k$$

Where D_k *is a domain for the description,* $\mathfrak{D}_k \subsetneq \mathfrak{R}_k$*, and for any* $j \neq k$*,* $\mathfrak{R}_j^k \subset \mathfrak{R}_j$*,* $\mathfrak{R}_j^k$ *is a reduced range of feature* j *after the description.*

Definition 4.2 *A sequence of descriptions on a noun can be represented by a description phrase that reduces multiple features of the noun to proper subsets of the original ones, and are known as* ***(multiple) descriptions*** *of the noun. More precisely, descriptions on feature* $k_1, \cdots, k_l$*, where* $2 \leqslant l \leqslant n$ *and* $1 \leqslant k_1 < k_2 < \cdots < k_l \leqslant n$*, are denoted as*

$$\begin{aligned} X_{\overrightarrow{k}} &= X(\mathfrak{R}_{k_1} = \mathfrak{D}_{k_1}, \cdots, \mathfrak{R}_{k_l} = \mathfrak{D}_{k_l}) \\ &= D_k \times \mathfrak{R}_1^{\overrightarrow{k}} \times \cdots \times \mathfrak{D}_{k_1} \times \cdots \times \mathfrak{D}_{k_l} \times \cdots \times \mathfrak{R}_n^{\overrightarrow{k}} \end{aligned}$$

Where D_k *is a domain for the description phrase,* $\overrightarrow{k} = (k_1, \cdots, k_l)$*. For any* $1 \leqslant j \leqslant l$*,* $\mathfrak{D}_{k_j} \subsetneq \mathfrak{R}_{k_j}$*, and for any* $i \neq k_j$*,* $\mathfrak{R}_i^{\overrightarrow{k}} \subset \mathfrak{R}_i$*,* $\mathfrak{R}_i^{\overrightarrow{k}}$ *is a reduced range of feature* i *after the descriptions.*

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[16] Russell called it denoting phrases in [14].

For examples, all Asian countries can be realized by setting $\mathfrak{C}_4$ = "Asia" in (2.3); all Greeks can be realized by setting $\mathfrak{R}_3$ = "Greece" in (2.4). More precisely, an abstract form for "all Greeks" is

$$X_{TheGreeks} = X_{mankind}(\mathfrak{R}_3 = \text{"Greece"}) = D_G \times \mathfrak{R}_1^G \times \mathfrak{R}_2^G \times \{\text{"Greece"}\} \times \mathfrak{R}_4^G \times \cdots \times \mathfrak{R}_n^G$$

Where D_G is a domain for "all Greeks", $D_G \subsetneq D_H$, $\mathfrak{R}_k^G$ is a reduced range of feature k after the description $\mathfrak{R}_3$ = "Greece". So all Greeks can be realized from $X_{TheGreeks}$, i.e.

$$X_{AllGreeks} = X^*_{TheGreeks} = \{\{i\} \times \mathfrak{R}_1^{G,i} \times \cdots \times \{\text{"Greece"}\} \times \cdots \times \mathfrak{R}_n^{G,i} : i \in D_G\} \tag{4.1}$$

Where $\mathfrak{R}_k^G = \bigcup\limits_{i \in D_G} \mathfrak{R}_k^{G,i}$, $\mathfrak{R}_k^{G,i}$ is the k^{th} feature of each individual Greek. Often there is $\mathfrak{R}_k^G \subsetneq \mathfrak{R}_k$, e.g. all Greek names are a proper subset of the names of mankind. But for some features, $\mathfrak{R}_k^G = \mathfrak{R}_k$, e.g. $\mathfrak{R}_4^G = \{\text{"M"}, \text{"F"}\} = \mathfrak{R}_4$. So we have

Definition 4.3 *If a feature remains the same after a description, then the feature is* ***independent*** *of the description. If a feature becomes a proper subset of the original one after a description, then it is* ***dependent*** *on the description.*

As the result, $\mathfrak{R}_1^G$ is dependent on $\mathfrak{R}_3$ = "Greece", but $\mathfrak{R}_4^G$ is independent of it for Greeks have both men and women. Next let an abstract form for *man* (male mankind) be

$$X_{man} = D_M \times \mathfrak{R}_1^M \times \mathfrak{R}_2^M \times \mathfrak{R}_3^M \times \{\text{"M"}\} \times \cdots \times \mathfrak{R}_n^M$$

Clearly, all men (males) can be realized from X_{man} as

$$X_{allmen} = X^*_{man} = \{\{i\} \times \mathfrak{R}_1^{M,i} \times \cdots \times \{\text{"M"}\} \times \cdots \times \mathfrak{R}_n^{M,i} : i \in D_M\} \tag{4.2}$$

Where D_M is a domain for "all men", $D_M \subsetneq D_H$, $\mathfrak{R}_k^{M,i}$ is the k^{th} feature of each individual male, and $\mathfrak{R}_k^M = \bigcup\limits_{i \in D_M} \mathfrak{R}_k^{M,i}$. Obviously, an abstract form for "all Greek men" is

$$X_{TheGreekMen} = D_{GM} \times \mathfrak{R}_1^{GM} \times \mathfrak{R}_2^{GM} \times \{\text{"Greece"}\} \times \{\text{"M"}\} \times \cdots \times \mathfrak{R}_n^{GM}$$

Where $D_{GM} = D_G \cap D_M$, $\mathfrak{R}_k^{GM} = \mathfrak{R}_k^G \cap \mathfrak{R}_k^M$, $1 \leqslant k \leqslant n$. And all Greek men can be realized from $X_{TheGreekMen}$ as

$$X_{AllGreekMen} = \{\{i\} \times \mathfrak{R}_1^{GM,i} \times \cdots \times \{\text{"Greece"}\} \times \{\text{"M"}\} \times \cdots \times \mathfrak{R}_n^{GM,i} : i \in D_{GM}\}$$

Where $\mathfrak{R}_k^{GM} = \bigcup\limits_{i \in D_{GM}} \mathfrak{R}_k^{GM,i}$. Consequently, $X_{TheGreekMen} = X_{TheGreeks} \cap X_{man}$ for

$$\mathfrak{R}_3^M \cap \{\text{"Greece"}\} = \{\text{"Greece"}\} \quad \text{and} \quad \mathfrak{R}_4^G \cap \{\text{"M"}\} = \{\text{"M"}\}$$

In general, we have the following theorem on descriptions of a noun.

Theorem 4.4 *Suppose $\mathfrak{D}_k$ is a description on feature k of a countable noun whose abstract form is $X = D \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n$. Then*

(a) $X(\mathfrak{D}_k) \subset X$ *and* $X^*(\mathfrak{D}_k) \subset X^*$.

(b) *If* $\mathfrak{D}_{k_1}, \cdots, \mathfrak{D}_{k_l}$ *are* l *descriptions on* X*, then*

$$X(\mathfrak{D}_{k_1}, \cdots, \mathfrak{D}_{k_l}) = X(\mathfrak{D}_{k_1}) \cap \cdots \cap X(\mathfrak{D}_{k_l})$$

Proof. (a) By definition 4.1, for any k, $1 \leqslant k \leqslant n$, there is

$$X(\mathfrak{D}_k) = D_k \times \mathfrak{R}_1^k \times \cdots \times \mathfrak{D}_k \times \cdots \times \mathfrak{R}_n^k \subset D \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_k \times \cdots \times \mathfrak{R}_n = X$$

Where $D_k \subset D$, $\mathfrak{D}_k \subsetneq \mathfrak{R}_k$, and for any $j \neq k$, $\mathfrak{R}_j^k \subset \mathfrak{R}_j$. Also by definition 2.10, it is obvious $X^*(\mathfrak{D}_k) \subset X^*$.

(b) We only prove $l = 2$. By definition 4.2 ($k_1 < k_2$)

$$\begin{aligned} X(\mathfrak{D}_{k_1}, \mathfrak{D}_{k_2}) &= D_{k_1,k_2} \times \mathfrak{R}_1^{k_1,k_2} \times \cdots \times \mathfrak{D}_{k_1} \times \cdots \times \mathfrak{D}_{k_2} \times \cdots \times \mathfrak{R}_n^{k_1,k_2} \\ &= (D_{k_1} \cap D_{k_2}) \times \left(\mathfrak{R}_1^{k_1} \cap \mathfrak{R}_1^{k_2}\right) \times \cdots \times (D) \times \cdots \\ &\times \left(\mathfrak{R}_{k_2}^{k_1} \cap \mathfrak{D}_{k_2}\right) \times \cdots \times \left(\mathfrak{R}_n^{k_1} \cap \mathfrak{R}_n^{k_2}\right) \\ &= D_{k_1} \times \mathfrak{R}_1^{k_1} \times \cdots \times \mathfrak{D}_{k_1} \times \cdots \times \mathfrak{R}_n^{k_1} \ \cap \ D_{k_2} \times \mathfrak{R}_1^{k_2} \times \cdots \times \mathfrak{D}_{k_2} \times \cdots \times \mathfrak{R}_n^{k_2} \\ &= X(\mathfrak{D}_{k_1}) \cap X(\mathfrak{D}_{k_2}) \end{aligned}$$

The general case can be proved by induction. ∎

Unfortunately, adjectives like *long/short*, *young/old*, *more/less*, *big/small*, and so on, do not have clear boundaries and so can not be modeled by classic set theory. As the result, we will not pursue this type of adjective further in this paper.

4.2 Descriptions of Uncountable Nouns

Descriptions on a primitive uncountable noun is trivial because all of its features are singletons. So we only consider descriptions on composite uncountable nouns. Examples of this type of description are: "Chinese food", "antique furniture", "tender meat", "fresh vegetable", and so on.

Definition 4.5 *A* ***description*** *of a composite uncountable noun is a noun phrase that is either a noun after the noun through a preposition, or an adjective ahead of the noun. It results in a proper subset of the feature space* X *of the noun. A sequence of descriptions (known as* ***multiple descriptions****) on a composite uncountable noun results in a descending proper subsets of* X*. More specifically, a k-descriptor of* $\mathfrak{D}_1, \cdots, \mathfrak{D}_k$ *on* X *is denoted as* $X(\mathfrak{D}_1, \cdots, \mathfrak{D}_k)$*.*

Let's look at more examples. Suppose X_{food} is the transitive closure for *food* (2.12), $X_{chinesefood}$ for "Chinese food" and $X_{seafood}$ for "seafood". Obviously,

$$X_{chinesefood} \subset X_{food} \quad \text{and} \quad X_{seafood} \subset X_{food}$$

Furthermore, suppose $X_{chineseseafood}$ is the transitive closure for "Chinese seafood", then

$$X_{chineseseafood} = X_{chinesefood} \cap X_{seafood}$$

So theorem 4.4 also holds for uncountable nouns.

Corollary 4.6 *Suppose X is the transitive closure of a composite uncountable noun with a k-descriptor of $\mathfrak{D}_1, \cdots, \mathfrak{D}_k$ on it. Then*

(a) For any $1 \leqslant i \leqslant k$, $X(\mathfrak{D}_i) \subset X$.

(b) $X(\mathfrak{D}_1, \cdots, \mathfrak{D}_k) = X(\mathfrak{D}_1) \cap \cdots \cap X(\mathfrak{D}_k)$

4.3 Inheritance

In this section, we introduce an important way to create new feature spaces from existing ones known as inheritance, which allows existing feature spaces to be reused and shared for new ones. First, let's take a look at the following examples.

Houses, cars, electronic devices, etc, are things that share some common features such as names, types, owners, users, appearance, and so on. They also have many distinct features that set themselves apart. For examples, houses have rooms; cars have engines; electronic devices have chips, and so on. This suggests that we can put all common features from above nouns into a base feature space known as "thing", i.e.

$$X_{thing} = \mathfrak{B}_1 \times \mathfrak{B}_2 \times \cdots \times \mathfrak{B}_p \tag{4.3}$$

Where $\mathfrak{B}_1$ is "type", $\mathfrak{B}_2$ is "name", $\mathfrak{B}_3$ is "appearance", $\mathfrak{B}_4$ is "cost", $\mathfrak{B}_5$ is "value", and so on. Since users are (generally) persons, we need also to create another base feature space as "user" related to $X_{AllHumans}$ (2.5), i.e.

$$X_{user} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_q \tag{4.4}$$

Where $\mathcal{U}_1$ is "user's name", $\mathcal{U}_2$ (= {"owner", "renter", "occupant", $\cdots$}) is "user's type", $\mathcal{U}_3$ is "user's identity index", and so on. Since *house* contains all features in X_{thing} and X_{user}, its feature space can be inherited from them, i.e.

$$X_{TheHouse} = D_{Ho} \times X_{thing} \times X_{user} \times \cdots = D_{Ho} \times \mathcal{H}_1 \times \cdots \times \mathcal{H}_s \tag{4.5}$$

Where D_{Ho} is a domain of "all houses", and $\mathcal{H}_j$ are rearranged features of *house*, e.g. $\mathcal{H}_1$ is "type of house", $\mathcal{H}_2$ (= {"owner", "renter", "occupant", $\cdots$}) is "house user's type", $\mathcal{H}_3$ is "house user's identity index", $\mathcal{H}_4$ is "price of house", and so on. Likewise, a feature space of *car* can be inherited as

$$X_{TheCar} = D_{Car} \times X_{thing} \times X_{user} \times \cdots = D_{Car} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_t \tag{4.6}$$

Where D_{Car} is a domain of "all cars", and $\mathcal{C}_j$ are rearranged features of *car*, e.g. $\mathcal{C}_1$ is "make of car", $\mathcal{C}_2$ is "model of car", $\mathcal{C}_3$ is "car user's identity index", $\mathcal{C}_4$ (= {"owner", "renter", "driver", $\cdots$}) is "car user's type", and so on. So we have

Definition 4.7 *The process to obtain a new feature space from other feature spaces is known as* ***inheritance****. Suppose $\mathcal{F}$ is a collection of feature spaces known as the* **base feature spaces***. Then a new feature space X inherited from $\mathcal{F}$ is denoted as $X|\mathcal{F}$.*

The importance of inheritance is that results on base feature spaces can be reused for others without changes, which makes it possible to share features of different semantic objects and also easier to manage feature spaces of various sorts.

4.4 More Types of Determiners

In this section, we investigate other types of determiners such as personal, demonstrative and interrogative determiners. Since any persons[17] are part of $X_{AllHumans}$, personal pronouns such as *I/me*, *you*, *he/him*, *she/her*, *we/us* and *they/them*, can be decided by subdomains of E_H. More specifically, each of singular pronouns such as *I/me*, *you* and *he/him/she/her* can be determined by a singleton domain, whereas each of plural pronouns like *we/us*, *you* and *they/them* can be determined by a domain of multiple singletons.

The composite of a personal pronoun with a quantifier like *some*, *any*, and so on, can be determined by a subdomain with respect to a domain of the pronoun. Let's take a look at the following examples. Suppose $D_{us} = \{i_1, \cdots, i_m\}$ ($D_{us} \subset E_H$) is a domain of *we/us*, where $i_1, \cdots, i_m$ are identity indexes of the persons in *we/us*.

(4-1) (a) "*Every/each* of us" can be decided by any $i_k \in D_{us}$ or $\{\{i_1\}, \cdots, \{i_m\}\}$.
(b) "*Some* of us" can be decided by a non-singleton and nonempty $D_0 \subsetneq D_{us}$ or $\mathcal{P}(D_{us}) - \{\varnothing, D_{us}\} - \{\{i_1\}, \cdots, \{i_m\}\}$.
(c) "*Any* of us" can be determined by any $D_0 \subset D_{us}$ or $\mathcal{P}(D_{us})$.
(d) "*None* of us" can be determined by $\varnothing$ or D_{us}^c.

In addition, personal pronouns such as *everyone* and *someone* can be determined by a singleton domain of E_H, and *anyone* is decided by a singleton as well as $\varnothing$, while *nobody* and *no one* is decided by $\varnothing$.[18]

Furthermore, demonstrative pronouns such as *this*, *that*, *these* and *those*, can be decided by subdomains of a relativization similar to personal pronouns. Suppose $D_A \subset E_C$ is a domain of all Asian countries.

(4-2) (a) "*This* country in Asia" can be determined by a $i \in D_A$.
(b) "*That* country in Asia" can be determined by a $j \in D_A - \{i\}$.
(c) "*These* countries in Asia" can be decided by a subdomain $D_0 \subsetneq D_A$.
(d) "*Those* countries in Asia" can be decided by $D_1 \subset D_A - D_0$.

Interrogative pronouns such as *who*, *whom*, *which* and *what*, are in fact attempts to locate particular persons or things of interest, while *where* and *when* try to find locations or times of interest. These pronouns can be reduced to find particular features in feature spaces of objects.

(4-3) (a) "*What* is the nationality of Socrates?" is to get the "nationality" field under Socrates' identity index in $X_{AllHumans}$.
(b) "*Who* is Aristotle?" is to find the identity and profession of Aristotle.
(c) "*Where* is Canada?" is to find the continent and location of Canada.

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[17] We assume that *human* and *person* are the same in this paper despite of some subtle difference between the two nouns.

[18] Note that *someone* and *anyone* are always singlular.

(d) "*When* will the train leave?" is to find the departure time of the train.

In section 4.4, we have studied personal, demonstrative and interrogative determiners. Now we investigate more types of determiners such as possessive and quantitative adjectives based upon feature spaces.

Possessive determiners modify a noun by attributing possession to someone or something, e.g. *my*, *your*, *his/her*, *our*, *their*, *whose*, and so on. In terms of feature spaces, possessive determiners simply specify user related features if nouns are things.

(4-4) (a) "*My* house" means that an owner of the house (4.5) is "me", i.e. $\{I_{me}\} \subset \mathcal{H}_3$ and $\mathcal{H}_2 = \{$"owner"$\}$ where I_{me} is the identity index of "me".

(b) "*His* car" means that an owner of the car (4.6) is "him", i.e. $\{I_{him}\} \subset \mathcal{C}_3$ and $\mathcal{C}_4 = \{$"owner"$\}$ where I_{him} is the identity index of "him".

(c) "*Our* office" means that occupants of the office are "us", i.e. $\mathcal{U}_3 = D_{us}$ and $\mathcal{U}_2 = \{$"occupant"$\}$ where D_{us} is a domain of "us".[19]

(d) "*Whose* car is this?" is to locate an owner of this car.

If nouns are persons, possessive determiners specify features related to the persons. This can be done first by creating a feature space known as "relative" as[20]

$$X_{relative} = D_R \times D_I \times \mathfrak{C}_1$$

Where D_R contains identity indexes of persons (in $X_{AllHumans}$) who are relatives, D_I contains identity indexes of persons whom are relatives to D_R, and

$$\mathfrak{C}_1 \,(= \{\text{“son”, “daughter”, “wife”, “husband”,“friend”, “colleague”, } \cdots\})$$

is "relation" that specifies relations of persons in D_R to one in D_I. Often, we do not put $X_{relative}$ into $X_{mankind}$ (2.4) because it could be large and not fixed. Rather, we set a feature "relative" in $X_{mankind}$ that contains links[21] to $X_{relative}$.

(4-5) (a) "*My* son" refers to an identity index in $X_{relative}$ whose relation is "son" to that of "me".

(b) "*His* wife" refers to an identity index in $X_{relative}$ whose relation is "wife" to that of "him".

(c) "*My* friends" refers to a domain of D_R in $X_{relative}$ whose relation is "friend" to that of "me".

Quantitative adjective *how many* before a noun is to find a number of realizations or relations for the noun, and a numeral is to specify a number of realizations for the noun, whereas *how much* tries to get the value in "cost" field of the noun.

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[19] A feature space of *office* can be inherited from X_{thing} and X_{user} as well.

[20] Here "relative" is more general than noun *relative* because it also includes friends, colleagues and so on.

[21] This is similar to tables and foreign keys in a relational database.

(4-6) (a) "*How many* children do you have?" is to find the number of identity indexes in $X_{relative}$ that have relations "son" and "daughter" to that of "you".

(b) "He has *two* cars" means that there are two cars, each of whose "owner" field $\mathcal{C}_3$ contains the identity index of "him".

(c) "*How much* does the house cost?" is to get the "price of house" field $\mathcal{H}_4$ in the instance of the house.

5 Relations and Actions of Nouns

In this section, we will discuss actions and relations of nouns that are commonly known as verbs in linguistics.[22] There are two main categories of verbs — stative and dynamic verbs. A stative verb is a verb used primarily to describe a relation such as a state of being (I am) or situation (I have). Stative verbs describe situations that are static or unchanging for a long or indefinite period of time. A dynamic verb, however, describes an action which involves processes of change over time. In general, dynamic verbs can be used in the progressive tense whereas stative verbs can not.

Please note that there is not always a clear cut to decide whether a verb is stative or dynamic. Some verbs are regarded as stative if only end states are considered, whereas regarded as dynamic if processes of change are mainly focused. In other words, some verbs may act as both stative and dynamic,[23] e.g. *think*, *become*, *feel*, *look*, etc. In order to investigate verbs upon feature spaces, we need the following notion first.

5.1 Dynamic Feature Spaces

Since an action involves changes of events in progress and a relation may change over a period of time (such as in the past tense),[24] feature spaces for verbs have to be dynamic — feature spaces change with respect to time. So we have

Definition 5.1 *Suppose $X = D_X \times \mathfrak{R}_1 \times \cdots \times \mathfrak{R}_n$ is an abstract form of a countable noun and t be time. Then a **dynamic feature space** of the noun is*

$$X(t) = D_X(t) \times \mathfrak{R}_1(t) \times \cdots \times \mathfrak{R}_n(t) \tag{5.1}$$

Following definition 2.10, 2.11 and 4.1, we have the following definitions.

Definition 5.2 *Suppose X is the same as the above. Then the **all-realization** of $X(t)$ up to a time t is denoted by $X^*(t)$ and determined by a domain $D_X(t)$ consisting of all identity indexes of $X^*(t)$ (up to t).*

Definition 5.3 *A **partial-realization** of $X(t)$ up to a time t is a subset of the all-realization of $X(t)$ that is determined by a domain $D(t) \subsetneq D_X(t)$ and denoted as $X^P(t)$.*

[22]Since we investigate semantics in this paper, we will mainly focus on understanding meanings of verbs based on feature spaces and omit most grammatical features of verbs.

[23]These are also known as inchoative verbs.

[24]Grammatical tense and aspect upon feature spaces will be studied in another paper.

Definition 5.4 *Suppose $X(t)$ is in (5.1). Then a* ***description*** *$\mathfrak{D}_k(t)$ on feature k of $X(t)$ (by a time t) is*

$$X(\mathfrak{R}_k(t) = \mathfrak{D}_k(t)) \;=\; X(\mathfrak{D}_k(t)) \;=\; D_k(t) \times \mathfrak{R}_1^k(t) \times \cdots \times \mathfrak{D}_k(t) \times \cdots \times \mathfrak{R}_n^k(t)$$

Where for any t, $\mathfrak{D}_k(t) \subsetneq \mathfrak{R}_k(t)$, and for any $j \neq k$, $\mathfrak{R}_j^k(t) \subset \mathfrak{R}_j(t)$. The ***multiple descriptions*** *of $X(t)$ (by a time t) is*

$$X(\mathfrak{R}_{k_1}(t) = \mathfrak{D}_{k_1}(t), \cdots, \mathfrak{R}_{k_l}(t) = \mathfrak{D}_{k_l}(t)) \quad or \quad X(\mathfrak{D}_{k_1}(t), \cdots, \mathfrak{D}_{k_l}(t))$$

We use (2.5) for all humans ever been and to be born in the world. However, the dynamic version of (2.5) has to be adopted for all persons who were born by a time or within a period of time. Suppose $X_{mankind}(t) = D_H(t) \times \mathfrak{R}_1(t) \times \cdots \times \mathfrak{R}_n(t)$.

(5-1) (a) "All humans were born before 2000" is $X^*_{mankind}(2000)$.
(b) "All humans were born in the 20th century" is $X^*_{mankind}(2000) - X^*_{mankind}(1900)$.
(c) "All Greeks were born before 1950" is $X_{AllGreeks}(1950) = X^*_{mankind}(\mathfrak{R}_3 = \text{“Greece”})(1950)$.
(d) "All philosophers before the 20th century" is $X^*_{mankind}(\mathfrak{R}_5 = \text{“Philosopher”})(1900)$.

The significance of dynamic feature spaces lies in the following claim — stative verbs can be described by relations on dynamic feature spaces of the subject and object, whereas dynamic verbs can be modeled by processes of change on them. We start with stative verbs first.

5.2 Stative Verbs

We will show by examples that stative verbs can be described by relations on dynamic feature spaces of the subject and object (subject complement). Let's begin with an important type of stative verb known as state of being verbs.

5.2.1 State of Being Verbs

The state of being verb *be* identifies who or what a noun is, was, or will be. There are eight states of *be* (used in different tense and aspect), namely *am*, *are*, *is*, *was*, *were*, *been*, *being* plus *be* itself. Suppose dynamic feature spaces for "he" and "they" are

$$H(t) = \{i\} \times \mathfrak{R}_1^i(t) \times \cdots \times \mathfrak{R}_n^i(t) \tag{5.2}$$
$$T(t) = \{\{j\} \times \mathfrak{R}_1^j(t) \times \cdots \times \mathfrak{R}_n^j(t) : j \in D_T(t)\} \tag{5.3}$$

Where $i \in E_H(t)$, $D_T(t)$ is a domain for "they" and $D_T(t) \subset E_H(t)$.($E_H(t)$ is a relativization of $D_H(t)$ up to t (2.5).)

(5-2) (a) "He *is* a lawyer" means that he is an instance of "all lawyers" and this profession will last indefinitely, i.e. $i \in D_L(t)$ ($\{i\} \subset D_L(t)$) at the current time t (and an indefinite period of time into the future, $D_L(t)$ is a domain of "all lawyers" up to t).
(b) "He *was* a lawyer" implies that he once was an instance of "all lawyers" but is not now, i.e. $i \in D_L(t')$ (t' is in a past period of time, $D_L(t')$ is a domain of "all lawyers" up to t'), but $i \notin D_L(t)$ at the current time t.

(c) "He *will be* a lawyer" implies that he will be an instance of "all lawyers" sometime in the future, i.e. $i \in D_L(t)$ for $t > t'$, t' is a future time.

(d) "They *are* Greeks" means that they are instances of "all Greeks", i.e. $D_T(t) \subset D_G(t)$ at the current time t (and an indefinite period of time into the future, $D_G(t)$ is a domain of "all Greeks" up to t).

(e) "He *was* a Greek" indicates that he once was an instance of "all Greeks" but is not now, i.e. $i \in D_G(t')$ (t' is in a past period of time), but $i \notin D_G(t)$ at the current time t (he either is not alive or has changed nationality).

(f) "He *is not* a Greek" implies that he is not an instance of "all Greeks", i.e. $i \notin D_G(t)$ ($\{i\} \not\subset D_G(t)$ and $\{i\} \subset D_G^c(t)$) at the current time t.

From these examples, we can see that if a subject and object noun are countable, *be* actually specifies a relation (set inclusion) between feature spaces of the subject and object, i.e. whether the domain of the subject is part of that of the all-realization of the object.

Moreover, since the simple present tense of a state of being verb implies that the state of being can last for a long or indefinite period of time, we do not use dynamic feature spaces in the simple present tense. So we have

Axiom 5.5 *Suppose X_A and X_B are abstract forms for countable noun (or NP) A and B. Then*

(a) *"A is/am/are B"*[25] *(A is singular)* $\iff D_{X_A} \subset D_{X_B^*}$

(b) *"A are B" (A is plural)* $\iff D_{X_A} \subset D_{X_B^*}$

(c) *"A is/am/are not B" (A is singular)* $\iff D_{X_A} \not\subset D_{X_B^*} \left(D_{X_A} \subset D^c_{X_B^*}\right)$

(d) *"A are not B" (A is plural)* $\iff D_{X_A} \not\subset D_{X_B^*}$

(e) *"A being B"* $\iff D_{X_A} \subset D_{X_B^*}$

(f) *"A was/were B"* $\iff D_{X_A}(t') \subset D_{X_B^*}(t') \wedge D_{X_A}(t) \not\subset D_{X_B^*}(t)$*, t' is in a past period of time and t is the current time.*

(g) *"A has/have been B"* $\iff D_{X_A}(t) \subset D_{X_B^*}(t)$*, for $t > t'$, t' is a past time and t includes the current time.*

(h) *"A will be B"* $\iff D_{X_A}(t) \subset D_{X_B^*}(t)$*, for $t > t'$, t' is a future time.*

For uncountable nouns, *be* also specifies a relation between feature spaces of the subject and object because, a primitive uncountable noun has only one realization (minus the unit of measurement), and a composite uncountable noun is the transitive closure of a set.

(5-3) (a) "A bottle of water *is* water" is true by axiom 3.8(f).

(b) "Chairs *are* furniture" is true since any collection of instances of *chair* is part of $X_{furniture}$ (2.11).

[25] The only singular subject that is followed by *are* is *you*.

(c) "Rice *is* food" is true for $X_{rice} \subset X_{food}$.

Furthermore, since uncountable nouns generally refer to unchanging states of being, we only use static feature spaces and the simple present tense. So we have

Axiom 5.6 *Suppose X_A and X_B are feature spaces (or transitive closures) for noun (or NP) A and uncountable noun (or NP) B. Then*

(a) *"A is/are/being B"* $\iff X_A \subset X_B$

(b) *"A is/are/being not B"* $\iff X_A \not\subset X_B$

The following examples suggest that object nouns can be replaced by subject complements with the same meaning.

(5-4) (a) "He is *male*" is the same as "he is a *man*" because the complement *male* (adjective) has the same meaning as *man* (noun).

(b) "They are *human*" is the same as "they are *humans*" because the complement *human* (adjective) has the same meaning as *humans* (noun).

Axiom 5.7 *In axiom 5.5 and 5.6, if a subject complement (adjective) corresponds to an object noun with the same meaning, then the noun can be replaced by the complement.*

5.2.2 More Stative Verbs

Now let's look at more stative verbs like *have*, *own*, *possess*, and so on. Suppose H and T are static feature spaces for "he" and "they" as in (5.2) and (5.3), as well as $\mathcal{H}_3^H$ and $\mathcal{H}_3^T = \bigcup_{j \in D_T} \mathcal{H}_3^j$ are feature $\mathcal{H}_3$ (4.5) for "his house" and "their houses".

(5-5) (a) "He *has* a house" means that the identity index of "he" is part of the owners of the house, i.e. $\{i\} \subset \mathcal{H}_3^H$, but H is not an instance of $X^*_{TheHouse}$.

(b) "They *have* houses" implies that identity indexes of "they" are part of all the owners of the houses, i.e. $D_T \subset \mathcal{H}_3^T$, but T are not part of $X^*_{TheHouse}$.

(c) "He *owns/possesses* a house" means that the identity index of "he" is part of the owners of the house, i.e. $\{i\} \subset \mathcal{H}_3^H$, but H is not an instance of $X^*_{TheHouse}$.

From these examples, we can see that unlike *be*, *have/own/possess* can not reduce the subject to be part of the all-realization of the object, but can relate features of the subject to those of the object.

Furthermore, stative verbs (or VP) such as *include*, *be part of*, *be a member of*, *belong to*, *contain*, *consist of*, and so on, can be reduced to set inclusions as well. So we have

Axiom 5.8 *Suppose X_A and X_B are feature spaces for countable noun (or NP) A and B. Then*[26]

[26] π is the projection operator as in definition 2.13.

(a) *"A have B"* $\iff \pi_i(X_A) \subset \pi_j(X_B)$

(b) *"A own/possess B"* $\iff \pi_i(X_A) \subset \pi_j(X_B)$

(c) *"A is a member of/belongs to B"(A is singular)* $\iff D_{X_A} \subset D_{X_B^*}$

(d) *"A are members of/included in/part of B"(A is plural)* $\iff D_{X_A} \subset D_{X_B^*}$

(e) *"A contain/consist of B"* $\iff D_{X_B} \subset D_{X_A^*}$

In addition, *be* can be used together with a preposition such as *in*, *at*, *on*, *to*, *before*, *after*, *for*, *from*, *since*, and so on, to specify a spatial or temporal relationship.

(5-6) (a) "They *are in* Toronto now" implies that $\pi_7(T(t)) \subset Coor$ ("Toronto") at the current time t.[27]

(b) "He *was in* London last week" implies that $\pi_7(H(t')) \subset Coor$ ("London") for t' in a period of last week.

(c) "He *is at* the office *from* 9 *to* 5 every weekday" means that $\pi_7(H(t)) \subset Coor$ ("the office") for $t \in$ [9am, 5pm] every weekday.

(d) "He *has been at* this address *for* 3 years" implies that $\pi_7(H(t)) \subset Coor$ ("this address") or $\pi_8(H(t)) =$ "this address" for some $t \in [t_0 - 3\,(\text{year})\,,\, t_0]$, t_0 is the current time.

So we have

Axiom 5.9 *When "be" is used together with certain prepositions to specify either a spatial or a temporal relationship, it can be modeled by a set relation on features in dynamic feature spaces of the subject and object.*

From above discussion, we reach the following for stative verbs.

Axiom 5.10 *A* ***stative verb*** *is defined as a relation on (dynamic) feature spaces of the subject and object.*

5.3 Dynamic Verbs

In this section, we will investigate dynamic verbs that describe actions over time. We will show by examples that dynamic verbs can be modeled by processes of change on dynamic feature spaces of the subject (and object).[28]

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[27] Coor is an abbreviation of *coordinate*. $\mathcal{R}_7$ is "position" (GPS coordinates) and $\mathcal{R}_8$ is "address" in $X_{mankind}$.

[28] A transitive verb links the subject with an object in a sentence, whereas an intransitive verb does not have a direct object.

5.3.1 Definitions

Since dynamic verbs are based on actions, a fundamental question that we have to solve first of all, is exactly what is action and how to describe it mathematically. Linguistically, an action is defined as the process of doing an activity in order to make something happen or to deal with a situation. This is a sort of circular answer and rather unhelpful because it is unclear on words like *activity*, *do*, *deal*, etc. Therefore, we adopt the method of inductive reasoning by understanding it through examples.

Let's begin with verbs involving position and movement that are among the easiest ones to comprehend.

(5-7) (a) "He has *gone* from A to B" means that his position has changed from A to B.

(b) "They *left* my house at 10pm yesterday" means that their positions were away from my house from 10pm yesterday.

(c) "He is *entering* the room" implies that his position is in the process of moving inside the room.

(d) "I am *walking*" implies that I am moving at certain speeds in the range and shapes of human walk.

(e) "The apple is *falling* to the ground" means that the height of the apple is approaching to the ground level.

We can see that each of these dynamic verbs can be reduced to a process of change in the "position" feature of the subject, while the differences are reflected in other features.

Consequently, *start* marks the beginning of a process of change in dynamic feature spaces of the subject, and *stop* marks the end of the process of change, while *continue* signifies the ongoing of the process. All of these verbs can also be modeled by changes in feature spaces. Furthermore, quantitative verbs such as *increase/decrease*, *shorten/lengthen*, etc, can be modeled by changes of the corresponding features in dynamic feature spaces of nouns involving amount, size, length, height, and so forth.

Some actions need approval in order to be performed. In addition, often there are reasons or purposes for an action, which are in the form of other actions or statements.

(5-8) (a) "He is *allowed* to *leave* the prison because he wants to *go to* hospital" implies that his position is permitted to be out of the prison, and the reason for this action is another action of going to a hospital for medical treatment.

(b) "He must *go to* the hospital in order to *take* the medical exam" implies that in order to fulfill the action of taking the exam, the action for him to go to the hospital is necessary.

(c) "I want to *eat* something because I *am* hungry" means that the action for me to eat is driven by the feeling of hunger. The verb *eat* can be modeled as processes of change in the ingestion of food involving the mouth, teeth, tongue, esophagus, stomach and intestines of a person.

Generally, an action is done by a subject known as a performer or agent. Sometimes there is an object known as a receiver or patient to receive the action.

(5-9) (a) "I *gave* him a book of mine (he *accepted* it)" means that I transferred one book from my possession to his directly, and so there is one book less in my possession and one more in his. Note that "I" is the performer of the action of giving a book and the book is the receiver of the action.

(b) "I *received* a letter he *sent* to me" means that he transferred one letter from his possession to mine indirectly, and "he" is the performer of the action, while the letter is the receiver.

(c) "He *got* a master degree in science" means that he added a master degree to his "education" feature. "He" is the performer of the action, while the master degree is the receiver.

The processes of change in some dynamic verbs are determined by procedures, each of which could have a final outcome in the end.

(5-10) (a) "He *bought* a house" implies that he added a house to his possession in exchange for a payment out of his money. The process of buying a house could involve many steps like acquiring mortgage, signing contract, etc, which can be decided by a procedure. The final outcome is a new house and less money in his possession.

(b) "It is *computed* that the Morning Star is the Evening Star" implies that the orbit of the Morning Star is computed to be the same as that of the Evening Star. The verb *compute* can be modeled as processes of change on states of a Turing machine which are decided by procedures of instructions for operations like addition, subtraction, multiplication, and so on. The final result is in the last (halting) state.

From above discussion, we conclude that an action in a dynamic verb can be described by (a collection of) processes of change on dynamic feature spaces of the subject (and object). An important issue that needs be cleared, however, is exactly how to describe a process of change on a dynamic feature space $X(t)$ (5.1). For a process of change, we mean a sequence of $X(t)$. However, $X(t_1)$, $X(t_2), \cdots$ are not good enough since it does not contain actual times. In order to reflect on time directly, we need the Cartesian product of time with $X(t)$, i.e. $\{t_1\} \times X(t_1)$, $\{t_2\} \times X(t_2)$, $\cdots$.

A dynamic verb can be regarded as a collection of actions since the verb may involve many actions (different meanings corresponding to distinct actions). For instance, *work* is defined as an activity that a person uses physical or mental effort to achieve a purpose or result, usually for money. So working in distinct fields requires (completely) different actions, e.g. working in an office and in a factory could involve totally different actions. So we have the following definitions.

Definition 5.11 *Suppose $X(t)$ is a dynamic feature space. Then a **process of change** $P(t)$ on $X(t)$ is*

$$P(t) = \{t\} \times X(t) \text{ for } t \in D_P,\ D_P \text{ is a domain of } P(t) \tag{5.4}$$

Definition 5.12 *Suppose $X_s(t)$ is a dynamic feature space for the subject (and $X_o(t)$ for the object).*[29] *Then an **action** A on $X_s(t)$ is defined as a collection of processes of change on $X_s(t)$ and others satisfying certain criterion. More precisely,*

$$A\left(X_s(t), X_o(t), \mathcal{C}, \mathcal{R}, \mathcal{A}, \mathcal{L}, \mathcal{O}\right) = \left\{P(t) : \varphi\left(P(t), X_s(t), X_o(t), \mathcal{C}, \mathcal{R}, \mathcal{A}, \mathcal{L}\right),\ t \in D\right\} \tag{5.5}$$

[29] $X_o(t)$ may be absent because some actions do not have direct objects as in the case of intransitive verbs. In addition, there could be more than one $X_o(t)$.

*Where $X_s(t), X_o(t), \mathcal{C}, \mathcal{R}, \mathcal{A}, \mathcal{L}, \mathcal{O}$ are **arguments** of A,*[30] *and D is a domain of A.*

(a) *φ is a formula for defining A based on $X_s(t)$ and other arguments.*

(b) *$\mathcal{C}$ is "procedure" of A (how A operates) in the form of formulas and relations involving $X_s(t)$ and other arguments.*

(c) *$\mathcal{R}$ is "reason" of A in the form of a statement or actions.*

(d) *$\mathcal{A}$ is "authorization" of A in the form of a statement or actions.*

(e) *$\mathcal{L}$ is "parameter" of A in the form of a statement (or PP).*

(f) *$\mathcal{O}$ is "outcome" of A in the form of (new) outcomes or feature spaces.*

Definition 5.13 *In the active voice, the subject is the **performer** (or **agent**) of the action and the object (if any) is the **receiver** (or **patient**) of the action. In the passive voice, the performer is the object or (often) anonymous, and the receiver is the subject.*

Remark 5.14 *The action A on $X_o(t)$ can be obtained by the passive voice in which the subject and object are switched.*

Axiom 5.15 *A **dynamic verb** is defined as a collection of actions on dynamic feature spaces of the subject (and object), in which different actions define distinct **meanings** of the verb.*

Next, we will study more examples of dynamic verbs by finding their defining formulas for actions as in (5.5).

5.3.2 Movement Verbs

Suppose $H(t)$ and $T(t)$ are dynamic feature spaces for "he" and "they" as in (5.2) and (5.3).

(5-11) The verb *move* generally has the following three meanings (actions) when movement is involved.

(a) Someone changes his temporary location. For example, "He *moved* to another seat" implies that $\mathfrak{R}_7^i(t') \subset Coor$ ("another seat"), t' is in a past period of time. So this action in *move* can be defined as [31]

$$\varphi\left(P(t), X_s(t), L\right) = \left(P(t) = \{t\} \times X_s(t) \wedge \pi_7(X_s(t_e)) \subset Coor(L)\right)$$

Where L is a parameter of location (specified in PP), t_e is an end time of the action with $t < t_e$.

[30] $X_o(t), \mathcal{C}, \mathcal{R}, \mathcal{A}, \mathcal{L}, \mathcal{O}$ may be absent.

[31] In the rest of discussion, $X_s(t)$ is assumed to be singular unless further specified.

(b) Someone changes the location of an object. For example, "He *moved* the sofa to the dining room" implies that $\mathfrak{R}^i_7(t') \subset Coor$ ("the room") and $\pi_l(X_o(t')) \subset Coor$ ("the room") (t' is in a past period of time) by a means of hand or dolly. So this action in *move* can be defined as

$$\varphi\left(P(t), X_s(t), X_o(t), L\right) = \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ \pi_7(X_s(t_e)) \subset Coor(L) \wedge \\ \pi_l(X_o(t_e)) \subset Coor(L) \ \wedge \ R(X_s(t), X_o(t)) \end{array} \right)$$

Where L is a parameter of location, $\pi_l(X_o)$ is the "location" feature of X_o, t_e is an end time of the action with $t < t_e$, R is a relation of $X_s(t)$ and $X_o(t)$ on how $X_s(t)$ moves $X_o(t)$.

(c) Someone changes the place where he lives. For example, "He *moved* to London last month" means that $\mathfrak{R}^i_7(t') \subset Coor$ ("a place in London") and $\mathfrak{R}^i_8(t')$ = "an address in London" (t' is in a period of last month), as well as his possessions were also moved. So this action in *move* can be defined as [32]

$$\varphi\left(P(t), X_s(t), L\right) = \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \ \wedge \ \pi_7(X_s(t_e)) \cup \pi_p(X_s(t_e)) \subset Coor(L) \\ \wedge\, \pi_8(X_s(t_e)) \subset Addr(L) \ \wedge \ R(X_s(t)) \end{array} \right)$$

Where L is a parameter of location, $\pi_p(X_s)$ is the "possession" feature of X_s, t_e is an end time of the action with $t < t_e$, R is a relation of $X_s(t)$ on how $X_s(t)$ moves.

(5-12) The verb *go* specifies an action to move to a place that is away from the speaker. For example, "He *went to* Germany last month" means $\mathfrak{R}^i_7(t') \subset Coor$ ("Germany") (t' is in last month), and moving away from the speaker means $Dist\left(\mathfrak{R}^i_7(t'), Coor\,(\text{"speaker"})\right) \leqslant Dist\left(Coor\,(\text{"speaker"}), Coor\,(\text{"Germany"})\right)$.[33] So it can be defined as

$$\varphi\left(P(t), X_s(t), L\right) = \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \ \wedge \ \pi_7(X_s(t_e)) \subset Coor(L) \wedge \\ Dist\left(\pi_7(X_s(t)), Coor(L_s)\right) \leqslant Dist\left(Coor(L_s), Coor(L)\right) \end{array} \right)$$

Where L is a parameter of location, L_s is the location of the speaker, t_e is an end time of the action with $t < t_e$.

(5-13) The verb *come* specifies an action to move to a place that is towards the speaker. For example, "He *came to* my house yesterday" means $\mathfrak{R}^i_7(t') \subset Coor$ ("my house") (t' is in yesterday), and moving towards the speaker means $Dist\left(\mathfrak{R}^i_7(t'), Coor\,(\text{"speaker"})\right) \geqslant Dist\left(Coor\,(\text{"speaker"}), Coor\,(\text{"my house"})\right)$. So it can be defined as

$$\varphi\left(P(t), X_s(t), L\right) = \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \ \wedge \ \pi_7(X_s(t_e)) \subset Coor(L) \wedge \\ Dist\left(\pi_7(X_s(t)), Coor(L_s)\right) \geqslant Dist\left(Coor(L_s), Coor(L)\right) \end{array} \right)$$

Where L is a parameter of location, L_s is the location of the speaker, t_e is an end time of the action with $t < t_e$.

[32] Addr is an abbreviation of *address*.

[33] Dist is an abbreviation of *distance*.

(5-14) The verb *arrive* specifies an action to get to a place. It has the following forms.

(a) The place is specified in a proposition phrase of *at/in*. For example, "He arrived in London" implies that $\mathfrak{R}_7^i(t') \subset \mathit{Coor}\,(\text{"London"})$ (t' is a past time). So it can be described as

$$\varphi\,(P(t), X_s(t), L) \;=\; (P(t) = \{t\} \times X_s(t) \,\wedge\, \pi_7(X_s(t_e)) \subset \mathit{Coor}(L))$$

Where L is a parameter of location (specified in PP), t_e is an end time of the action with $t < t_e$.

(b) If something arrives (without destination), it comes or is brought to the location of the speaker. For example, "A book arrived" implies that an instance of *book* was delivered to the location of the speaker, i.e. $\pi_4(X_{b_0}(t')) \subset \mathit{Coor}\,(\text{"speaker"})$ where t' is a past time, X_{b_0} is an instance of $X_{TheBook}$ (2.6). So it can be described as

$$\varphi\,(P(t), X_s(t)) \;=\; (P(t) = \{t\} \times X_s(t) \,\wedge\, \pi_p(X_s(t_e)) \subset \mathit{Coor}(L_s) \,\wedge\, R(X_s(t)))$$

Where $\pi_p(X_s)$ is the "position" feature of X_s, L_s is the location of the speaker, t_e is an end time of the action with $t < t_e$, R is a relation of $X_s(t)$ on how $X_s(t)$ comes or is brought.

If subjects are plural, above conclusions still hold.

(5-15) "They *went to* Germany last month" means for each $i \in D_T(t)$, $\mathfrak{R}_7^i(t') \subset \mathit{Coor}\,(\text{"Germany"})$ and $\mathit{Dist}\left(\mathfrak{R}_7^i(t'), \mathit{Coor}\,(\text{"speaker"})\right) \leqslant \mathit{Dist}\,(\mathit{Coor}\,(\text{"speaker"})\,, \mathit{Coor}\,(\text{"Germany"}))$. So it can be defined as

$$\varphi\,(P(t), X_s(t), L) \;=\; \left(P(t) = \{t\} \times \{X_i(t)\colon X_i(t) \in X_s(t)\} \,\wedge\, \bigwedge_{i \in D_{X_s}} \phi\,(X_i(t), L)\right)$$

Where

$$\phi\,(X(t), L) \;=\; \left(\begin{array}{c} \pi_7(X(t_e)) \subset \mathit{Coor}(L) \wedge \\ \mathit{Dist}\,(\pi_7(X(t)), \mathit{Coor}(L_s)) \leqslant \mathit{Dist}\,(\mathit{Coor}(L_s), \mathit{Coor}(L)) \end{array}\right)$$

Remark 5.16 *A defining formula of an action for a plural subject is a generalization of that of a singular subject.*

Verbs for movement such as *leave*, *return*, *visit* and so on, can be modeled similarly and we omit the detail.

Claim 5.17 *Verbs (or VP) related to movement and position such as "move", "go", "come", "leave", "return", "depart", "travel", "visit", "go in/out", "get in/out", "enter", "pass", and so on, can be modeled by the corresponding features in dynamic feature spaces of the subject and object (when their meanings involve movement and position).*

Here are more examples of movement verbs. Suppose $\mathfrak{R}_{s_m}$ is the "shape of movement" feature in $X_{mankind}$ (2.4), $\mathcal{S}_w$ and $\mathcal{S}_r$ be sets of all the shapes of human walks and runs, and a dynamic feature space for "I" be

$$I(t) \;=\; \{j\} \times \mathfrak{R}_1^j(t) \times \cdots \times \mathfrak{R}_n^j(t) \tag{5.6}$$

(5-16) The verb *walk* specifies an action that moves in the range of certain speeds and shapes (by legs and feet). For example, "He *walked* last night" implies that an action of his moving at certain speeds and in shapes of human walks happened last night, i.e. $0 < \frac{d}{dt}\mathfrak{R}_7^i(t') < c$ (t' is in last night and c is a speed limit of human walks), and $\mathfrak{R}_{s_m}^i(t') \in \mathcal{S}_w$. Since each shape of human walks is a subset of $\mathbb{R}^3 \times t$, $\mathcal{S}_w \subset \mathcal{P}(\mathbb{R}^3) \times \{t\}$.[34] So it can be described as

$$\varphi\left(P(t), X_s(t)\right) \;=\; \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \wedge \\ 0 < \frac{d}{dt}\pi_7(X_s(t)) < c \;\wedge\; \pi_{s_m}(X_s(t)) \in \mathcal{S}_w \end{array}\right) \tag{5.7}$$

(5-17) The verb *run* specifies an action that moves in the range of certain speeds and shapes (by legs and feet). For example, "He *run* last night" implies that an action of his moving at certain speeds and in shapes of human runs happened last night, i.e. $c < \frac{d}{dt}\mathfrak{R}_7^i(t') < r$ [35] (t' is in last night and r is a speed limit of human runs), and $\mathfrak{R}_{s_m}^i(t') \in \mathcal{S}_r \subset \mathcal{P}(\mathbb{R}^3) \times \{t\}$. So it can be described as

$$\varphi\left(P(t), X_s(t)\right) \;=\; \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \wedge \\ c < \frac{d}{dt}\pi_7(X_s(t)) < r \;\wedge\; \pi_{s_m}(X_s(t)) \in \mathcal{S}_r \end{array}\right) \tag{5.8}$$

(5-18) The verb *follow* specifies an action that moves behind or after someone or something. For example, "He has *followed* me for some time" implies that he has kept a distance from me for a while, i.e. $Dist\left(\mathfrak{R}_7^i(t), \mathfrak{R}_7^j(t)\right) < r_0$ for some $t \in [t_0 - c, t_0]$, where r_0 is a positive number, t_0 is the current time and c the duration. So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) \;=\; \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ Dist\left(\pi_7(X_s(t)), \pi_7(X_o(t))\right) < r_0 \;\wedge\; R(X_s(t), X_o(t)) \end{array}\right)$$

Where R is a relation that $X_s(t)$ is behind $X_o(t)$.

(5-19) The verb *fall* specifies an action of downward movement. For example, "He is *falling* to the ground" implies that the height in his "position" feature is dropping to zero with certain acceleration a, i.e. $\frac{d}{dt}\pi_z(\mathfrak{R}_7^i(t)) < 0$ and $\frac{d^2}{dt^2}\pi_z(\mathfrak{R}_7^i(t)) = a$ $\left(\mathfrak{R}_7^i(t) = (x, y, z)\right)$ at the current time t. So it can be described as

$$\varphi\left(P(t), X_s(t)\right) \;=\; \left(P(t) = \{t\} \times X_s(t) \;\wedge\; \frac{d}{dt}\pi_z(\pi_7(X_s(t))) < 0 \;\wedge\; \frac{d^2}{dt^2}\pi_z(\pi_7(X_s(t))) = a\right)$$

Movement verbs such as *fly*, *meet*, *approach* and so on, can be modeled similarly and we omit the detail. On the other hand, verbs such as *stand*, *sit* and *lie* represent standstill shapes and belong to stative verbs.

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[34] No exact mathematical representation for human movement and vision is known so far.

[35] Running is always faster than walking.

Claim 5.18 *Verbs (or VP) such as "walk", "run", "fly", "follow", "meet", "approach", "fall/rise", "go up/down", "raise/lower", and so on, can be modeled by changes of the corresponding features in dynamic feature spaces of the subject (and object) (when their meanings involve movement and position).*

5.3.3 Process and Change Verbs

In this section, we will investigate dynamic verbs involving changes in quantity such as amount, degree, size, and so on, as well as describing processes.

(5-20) The verb *change* means something becomes different. For example, "He has *changed* jobs" means that $\mathfrak{R}_5^i(t) \neq \mathfrak{R}_5^i(t')$ for $t > t'$ (t' is a past time). In general, it can be modeled by any difference between feature spaces at different times as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) \;=\; \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ (X_s(t) \neq X_s(t') \vee X_o(t) \neq X_o(t')) \end{array}\right)$$

(5-21) The verb *increase* specifies an action in which a quantity becomes bigger in amount, number or degree. It has the following forms.

(a) No specific amount is given. For example, "The price of the house has *increased* significantly" implies that $\mathcal{H}_4\,(t_0) - \mathcal{H}_4(t') > c$ (4.5), t' is a past time, t_0 is the current time, and c is a (large) positive number. So it can be described as

$$\varphi\left(P(t), X_s(t)\right) \;=\; \big(P(t) = \{t\} \times X_s(t) \,\wedge\, \pi_q(X_s(t_e)) - \pi_q(X_s(t')) > c\big)$$

(b) A specific amount of increase is given by a phrase of *by*. For example, "The price of the house has *increased by* 50 percent" implies that $\mathcal{H}_4\,(t_0) \;=\; 1.5\mathcal{H}_4(t')$. So it can be described as

$$\varphi\left(P(t), X_s(t), C\right) \;=\; \big(P(t) = \{t\} \times X_s(t) \,\wedge\, \pi_q(X_s(t_e)) = (1 + C)\,\pi_q(X_s(t'))\big)$$

(c) A specific amount of increase is given by a phrase of *to*. For example, "The price of the house has *increased to* 1 million" implies that $\mathcal{H}_4\,(t_0) \;=\; 1m$. So it can be described as

$$\varphi\left(P(t), X_s(t), C\right) \;=\; (P(t) = \{t\} \times X_s(t) \,\wedge\, \pi_q(X_s(t_e)) = C)$$

In above three cases, t_e is an end time of the action and $t' < t_e$, $\pi_q(X_s)$ is a quantity related feature of X_s, C is a parameter.

Verbs such as *modify*, *gain*, *reduce*, *expand*, and so on, can be modeled similarly and we omit the detail.

Claim 5.19 *Verbs such as "change", "modify", "adjust", "become", "turn", "grow", "develop", "increase/decrease", "reduce/expand", "gain/lose", "shorten/lengthen", "enlarge/shrink", "narrow/widen", and so on, can be modeled by changes of the corresponding features in dynamic feature spaces of the subject (and object).*

(5-22) One meaning of the verb *start* is to mark the beginning of an action, i.e. there is a first moment after which the action is performed. For example, “He *starts* to run” implies that there are t_0, t_1 and t_2 that for any $t \in [t_1, t_2]$, $c < \frac{d}{dt}\mathfrak{R}^i_7(t) < r$ and $\mathfrak{R}^i_{s_m}(t) \in \mathcal{S}_r$. So $\{t\} \times H(t) \in A_r$ on $[t_1, t_2]$ (5.4, 5.5), and $\{t\} \times H(t) \notin A_r$ on $[t_0, t_1]$, where A_r is the action for *run* involving movement (5.8). So it can be defined as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) \;=\; \exists t_0 \exists t_1 \exists t_2 \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \;\wedge\; \forall t \in [t_1, t_2], \\ P(t) \in A_{X_o} \;\wedge\; \forall t \in [t_0, t_1], P(t) \notin A_{X_o} \end{array} \right)$$

Where A_{X_o} is an action specified in $X_o(t)$.

(5-23) One meaning of the verb *stop* is to mark the end of an action, i.e. there is a last moment after which the action is terminated. For example, “He *stops* to run” implies that there are t_0, t_1 and t_2 that $\{t\} \times H(t) \in A_r$ on $[t_0, t_1]$, and $\{t\} \times H(t) \notin A_r$ on $[t_1, t_2]$. So it can be defined as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) \;=\; \exists t_0 \exists t_1 \exists t_2 \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \;\wedge\; \forall t \in [t_0, t_1], \\ P(t) \in A_{X_o} \;\wedge\; \forall t \in [t_1, t_2], P(t) \notin A_{X_o} \end{array} \right)$$

Where A_{X_o} is an action specified in $X_o(t)$.

(5-24) One meaning of the verb *continue* means a nonstop of an action after it starts, i.e. there is a moment before and after which the action is performed. For example, “He *continues* to run” implies that there are t_0, t_1 and t_2 that $\{t\} \times H(t) \in A_r$ on both $[t_0, t_1]$ and $[t_1, t_2]$. So it can be defined as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) \;=\; \exists t_0 \exists t_1 \exists t_2 \left(P(t) = \{t\} \times X_s(t) \;\wedge\; \forall t \in [t_0, t_1] \cup [t_1, t_2], P(t) \in A_{X_o}\right)$$

Where A_{X_o} is an action specified in $X_o(t)$.

(5-25) One meaning of the verb *bear(born)* is to have a new life (human) or mark the beginning of a dynamic feature space, while *die* marks the end of it. For example, “He was *born* at t_0 and *died* at t_1” implies that $H(t)$ begins at t_0 and ends at t_1. In other words, the life of “he” can be represented by $H(t)$ between $t_0 \leqslant t \leqslant t_1$. So an action of *bear(born)* can be defined as

$$\varphi\left(P(t), X_s(t)\right) \;=\; \exists t_0 \exists t' \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \;\wedge\; \forall t \in [t_0, t'], \\ R(X_s(t)) \;\wedge\; \forall t < t_0, X_s(t) = \varnothing \end{array} \right)$$

And an action of *die* can be defined as

$$\varphi\left(P(t), X_s(t)\right) \;=\; \exists t_1 \exists t' \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \;\wedge\; \forall t \in [t', t_1], \\ R(X_s(t)) \;\wedge\; \forall t > t_1, \pi_{\mathcal{B}}(X_s(t)) = \varnothing \end{array} \right)$$

Where $R(X_s(t))$ is a relation that specifies $X_s(t)$ as a human (infant), $\pi_{\mathcal{B}}(X_s)$ is the body related features of X_s (definition 2.13).

Verbs such as *begin*, *initiate*, *end*, *go on*, and so on, can be modeled similarly and we omit the detail.

Claim 5.20 *Verbs (or VP) such as “start”, “stop”, “begin”, “end”, “initiate”, “terminate”, “continue”, “go on”, “bear (born)”, “die”, and so on, can be modeled by changes of the corresponding features in dynamic feature spaces of the subject (and object) (when their meanings involve action).*

5.3.4 Possession Verbs

In this section, we will study verbs related to possession. First, we create a feature space known as "possession" or $X_{possession}$ that is inherited from X_{thing} (4.3) and X_{user} (4.4) so that it can include (most) physical items in the world.[36] In this way, all personal and nonpersonal belongings can be stored, added, removed, and processed in $X_{possession}$.

We do not put $X_{possession}$ into $X_{mankind}$ directly because it could be of large and varied size. Rather, we set a feature "possession" as $\mathfrak{R}_{po}$ in $X_{mankind}$ that contains links[37] to it. All the operations in $X_{possession}$ can be regarded as operating on $\mathfrak{R}_{po}$. Also we assume $\mathfrak{R}_w$ is the "wealth" feature in $X_{mankind}$ (representing the total asset of a person). In addition, we assume that everything in $X_{possession}$ has the same owner as the one of the linked identity index. In other words, the owner field of an item is always set to the identity index of the linked person when the item is accepted and stored in $X_{possession}$.

As the result, verbs like *give*, *accept*, *get*, *send*, *receive*, *buy*, *sell*, and so on, can be modeled by changes in $\mathfrak{R}_{po}$ and other related features of the subject (and object). Suppose X_{b_0} is an instance of $X_{TheBook}$ (2.6), $I(t)$ and $H(t)$ be dynamic feature spaces for "I" (5.6) and "He" (5.2).

(5-26) The verb *give* specifies an action that someone offers something to others. It has the following forms.[38]

(a) Someone offers something physical to others directly. For example, "I *gave* him a book of mine" means that I transferred one book from my possession to his directly, i.e. $X_{b_0}(t') \in \mathfrak{R}^j_{po}(t')$, $X_{b_0}(t) \notin \mathfrak{R}^j_{po}(t)$, $X_{b_0}(t) \in \mathfrak{R}^i_{po}(t)$ (t and t' are past times with $t' < t$). If it is accepted, $\pi_4(X_{b_0}(t)) = \{i\}$.[39] So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t), X_{o'}(t)\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \times X_{o'}(t) \wedge \\ X_o(t') \in \pi_{po}(X_s(t')) \wedge X_o(t_e) \notin \pi_{po}(X_s(t_e)) \\ \wedge X_o(t_e) \in \pi_{po}(X_{o'}(t_e)) \\ \wedge R\left(X_s(t), X_o(t), X_{o'}(t)\right) \end{pmatrix}$$

(b) Wealth (in the form of money) is offered to someone. For example, "An organization *gave* him \$10000 to continue his work" means that \$10000 was in his "wealth" feature for a nonstop of his work, i.e. $\mathfrak{R}^i_w(t) = \mathfrak{R}^i_w(t') + \$10000$ (t and t' are past times with $t' < t$). So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t), X_{o'}(t), \mathcal{L}\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ (\pi_w(X_o(t_e)) = \pi_w(X_o(t')) + X_{o'}) \wedge \\ R\left(X_s(t), X_o(t), X_{o'}(t)\right) \wedge \mathcal{L} \end{pmatrix}$$

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[36]Normally, we do not put nonphysical things into $X_{possession}$. For example, "He obtained a master degree in engineering" means that he added a master degree to his "education" feature.

[37]This is similar to tables and foreign keys in a relational database.

[38]There are many meanings for *give* and we only list a few here.

[39]This means the owner of the book is set to the new owner whenever it is accepted and formally stored in $X_{possession}$.

(c) Something nonphysical is offered to someone. This means that some features other than "possession" would change. For example, "They *gave* him the job for his qualification (and he accepted it)" means that his "occupation" feature changed (because he accepted the job). So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t), X_{o'}(t), \mathcal{L}\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \times X_{o'}(t) \\ \wedge\ (\pi_5(X_o(t_e)) = X_{o'}(t_e)) \\ \wedge\, R\left(X_s(t), X_o(t), X_{o'}(t)\right) \wedge \mathcal{L} \end{pmatrix}$$

In above three cases, $X_o(t), X_{o'}(t)$ are objects, t_e is an end time of the action with $t' < t < t_e$, R is a relation of $X_s(t)$, $X_o(t)$ and $X_{o'}(t)$ on how the offer is done, $\mathcal{L}$ is a parameter of reason.

(5-27) The verb *accept* specifies an action that takes something offered by someone, whereas *refuse* specifies an action that does not take it. *Accept* can be modeled by a Boolean function for decision involving the subject and object, and *refuse* is the negation of it. If an item is accepted, its owner is changed to the new one. So an action of *accept* can be described as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ F(X_s(t), X_o(t)) \Longrightarrow \begin{pmatrix} X_o(t_e) \in \pi_{po}(X_s(t_e)) \wedge \\ \pi_d(X_o(t_e)) = \pi_0(X_s(t_e)) \end{pmatrix} \end{pmatrix}$$

And an action of *refuse* can be described as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ \neg F(X_s(t), X_o(t)) \Longrightarrow X_o(t_e) \notin \pi_{po}(X_s(t_e)) \end{pmatrix}$$

Where $\pi_0(X_s(t))$ is the identity field of $X_s(t)$ (definition 2.13), $\pi_d(X_o(t))$ is the owner field of $X_o(t)$, F is a Boolean function of $X_s(t)$ and $X_o(t)$, t_e is an end time of the action.

(5-28) The verb *send* specifies an action that someone offers something to others indirectly (by a means or media). So its defining formula is the same as (5-26) except R specifies a different relation on how the offer is done.

(5-29) The verb *receive* specifies a reversed action of *give* in which the performer and receiver are switched. It has the following forms.

(a) Something physical is offered from someone. For example, "I *received* a book from him" is the same as "He *gave* me a book", i.e. $X_{b_0}(t') \in \mathfrak{R}^i_{po}(t')$, $X_{b_0}(t) \notin \mathfrak{R}^i_{po}(t)$, $X_{b_0}(t) \in \mathfrak{R}^j_{po}(t)$ (t and t' are past times with $t' < t$). So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t), X_{o'}(t)\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \times X_{o'}(t) \wedge \\ X_o(t') \in \pi_{po}(X_{o'}(t')) \wedge X_o(t_e) \notin \pi_{po}(X_{o'}(t_e)) \wedge \\ X_o(t_e) \in \pi_{po}(X_s(t_e)) \wedge R\left(X_s(t), X_o(t), X_{o'}(t)\right) \end{pmatrix}$$

(b) Wealth (in the form of money) is offered to someone. For example, "He *received* \$10000 to continue his work" means that \$10000 was in his "wealth" feature for a nonstop of his work, i.e. $\mathfrak{R}^i_w(t) = \mathfrak{R}^i_w(t') + \$10000$ (t and t' are past times with $t' < t$). So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t), \mathcal{L}\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ (\pi_w(X_s(t_e)) = \pi_w(X_s(t')) + X_o) \wedge \\ R(X_s(t), X_o(t)) \wedge \mathcal{L} \end{pmatrix}$$

(c) Something nonphysical is offered to someone, resulting in changes of some features other than "possession". For example, "He *received* a master degree from the department" means that his "education" feature changed. So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t), X_{o'}(t)\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \times X_{o'}(t) \\ \wedge\ (\pi_h(X_s(t_e)) = \pi_h(X_s(t')) \cup X_o) \\ \wedge\, R\left(X_s(t), X_o(t), X_{o'}(t)\right) \end{pmatrix}$$

In above three cases, $X_o(t), X_{o'}(t)$ are objects, $\pi_h(X_s)$ is an honor related feature of X_s, t_e is an end time of the action with $t' < t < t_e$, R is a relation of $X_s(t)$, $X_o(t)$ (and $X_{o'}(t)$) on how the offer is done, $\mathcal{L}$ is a parameter of reason.

The verb *pay* involves an action to give out money out of someone's pocket. The verb *buy* involves an action to obtain new items by paying money out of the buyer, and *sell* is vice versa. Suppose X_{h_0} is an instance of $X_{TheHouse}$ (4.5).

(5-30) The verb *pay* specifies an action that money is given out of someone's wealth. For example, "He *paid* \$1000 for rent" implies that his "wealth" feature was reduced by \$1000 because of rent, i.e. $\mathfrak{R}^i_w(t) = \mathfrak{R}^i_w(t') - \$1000$ (t and t' are past times with $t' < t$). So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t), \mathcal{L}\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ (\pi_w(X_s(t_e)) = \pi_w(X_s(t')) - X_o) \wedge \mathcal{L} \end{pmatrix}$$

Where t_e is an end time of the action and $t' < t_e$, $\mathcal{L}$ is a parameter of reason.

(5-31) The verb *buy* specifies an action that something is acquired by paying money for it. For example, "He *bought* a house" implies that he added a house to his possession by paying money out of his wealth for the ownership of the house, i.e. $X_{h_0}(t') \notin \mathfrak{R}^i_{po}(t')$, $X_{h_0}(t) \in \mathfrak{R}^i_{po}(t)$ and $\mathfrak{R}^i_w(t) = \mathfrak{R}^i_w(t') - c$ (c is the price of X_{h_0}, t and t' are past times with $t' < t$). It also involves a buying procedure like acquiring mortgage, signing contract, etc. So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge R\left(X_s(t), X_o(t), \mathcal{C}\right) \\ \wedge\, X_o(t') \notin \pi_{po}(X_s(t')) \wedge X_o(t_e) \in \pi_{po}(X_s(t_e)) \\ \wedge\, \pi_w(X_s(t_e)) = \pi_w(X_s(t')) - \pi_c(X_o(t_e)) \end{pmatrix}$$

Where $\pi_c(X_o)$ is the cost field of X_o, t_e is an end time of the action with $t' < t < t_e$, $\mathcal{C}$ is a purchase procedure (optional) involving $X_s(t)$ and $X_o(t)$, R is a relation of $X_s(t)$, $X_o(t)$ and $\mathcal{C}$.

(5-32) The verb *sell* specifies an opposite action of *buy* that gives something to someone in exchange for money. For example, "He *sell* a house" implies that he removed a house from his possession in exchange for more money in his wealth, i.e. $X_{h_0}(t') \in \mathfrak{R}^i_{po}(t')$, $X_{h_0}(t) \notin \mathfrak{R}^i_{po}(t)$ and $\mathfrak{R}^i_w(t) = \mathfrak{R}^i_w(t') + c$. It also involves a selling procedure. So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \,\wedge\, R\left(X_s(t), X_o(t), \mathcal{C}\right) \\ \wedge\, X_o(t') \in \pi_{po}(X_s(t')) \,\wedge\, X_o(t_e) \notin \pi_{po}(X_s(t_e)) \\ \wedge\, \pi_w(X_s(t_e)) \,=\, \pi_w(X_s(t')) + \pi_c(X_o(t_e)) \end{pmatrix}$$

Where $\pi_c(X_o)$ is the cost field of X_o, t_e is an end time of the action with $t' < t < t_e$, $\mathcal{C}$ is a selling procedure (optional) involving $X_s(t)$ and $X_o(t)$, R is a relation of $X_s(t)$, $X_o(t)$ and $\mathcal{C}$.

More possession verbs such as *obtain*, *reject*, *borrow*, and so on, can be modeled likewise and we omit the detail.

Claim 5.21 *Verbs such as "give", "receive", "send", "offer", "obtain", "get", "reject", "lose", "pay", "buy", "sell", "borrow", and so on, can be modeled by changes in the "possession" feature of dynamic feature spaces of the subject and object (when their meanings involve possession). More specifically, "buy" and "sell" also involve money in exchange for ownership.*

Now we have another way of modelling stative verbs like *have*, *own*, *include*, *possess*, etc, through possession.

(5-33) (a) "He *has/owns* a cottage near the lake" implies that a house of a type near the lake is in his possession.

(b) "His properties *include* a house and two cars" means that a house and two cars are in his possession.

Axiom 5.22 *Suppose X_A and X_B are feature spaces for the subject/object A and B. Then*

(a) *"A have/own/possess B"* $\iff X_B \in \pi_{po}(X_A)$

(b) *"A includes B"(B is plural)* $\iff X_B \subset \pi_{po}(X_A)$

5.3.5 Physical Verbs

In this section, we will investigate dynamic verbs involving movement by force, activity by physical effort, and so on. Suppose X_{b_0} is an instance of $X_{TheBook}$ (2.6), $I(t)$ and $H(t)$ are dynamic feature spaces for "I" (5.6) and "He" (5.2).

(5-34) The verb *hold* specifies an action that has something in your hand/hands or arms. For example, "He is *holding* a book" means that he is having a book at his hand, i.e. $\pi_4(X_{b_0}(t)) \subset \mathfrak{R}^i_7(t)$ for all $t \in [t_0 - c_1, t_0 + c]$ (t_0 is the current time, c and c_1 are durations). The way how he is holding the book can be represented by a relation. So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \,\wedge \\ (\pi_p(X_o(t)) \,\subset\, \pi_7(X_s(t))) \,\wedge\, R(X_s(t), X_o(t)) \end{pmatrix}$$

Where $\pi_p(X_o)$ is the "position" feature of X_o, R is a relation on how $X_s(t)$ holds $X_o(t)$.

(5-35) The verb *put* specifies an action that moves something to a particular place (by your hands). For example, "He *put* the book on the desk" means that he moved the book by his hand onto the desk, i.e. $\pi_4(X_{b_0}(t')) \subset \mathit{Coor}(\text{"the desk"})$ (t' is a past time). How he put the book on the desk can be represented by a relation. So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t), L\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ (\pi_p(X_o(t_e)) \subset \mathit{Coor}(L)) \wedge R(X_s(t), X_o(t)) \end{pmatrix}$$

Where L is a parameter of location, t_e is an end time of the action, R is a relation on how $X_s(t)$ moves $X_o(t)$ to a place.

(5-36) The verb *pull* specifies an action that uses your hands to make something or someone move towards you. For example, "He *pulled* the door open" implies that he moved the door towards himself with his hand, i.e. $\mathit{Dist}\left(\mathfrak{R}_7^i(t'), \mathit{Coor}\left(\text{"door"}\right)\right) \geqslant \mathit{Dist}\left(\mathfrak{R}_7^i(t), \mathit{Coor}\left(\text{"door"}\right)\right)$ (t and t' are past times with $t' < t$). How he pulled the door can be represented by a relation. So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ \mathit{Dist}\left(\pi_7(X_s(t')), \pi_p(X_o(t'))\right) \geqslant \\ \mathit{Dist}\left(\pi_7(X_s(t)), \pi_p(X_o(t))\right) \wedge R(X_s(t), X_o(t)) \end{pmatrix}$$

Where $\pi_p(X_o)$ is the "position" feature of X_o, $t' < t$, R is a relation on how $X_s(t)$ acts on $X_o(t)$.

(5-37) The verb *push* specifies an opposite action of *pull* that uses your hands to make something or someone move away from you. For example, "He *pushed* the door open" implies that he moved the door away from himself with his hand, i.e. $\mathit{Dist}\left(\mathfrak{R}_7^i(t'), \mathit{Coor}\left(\text{"door"}\right)\right) \leqslant \mathit{Dist}\left(\mathfrak{R}_7^i(t), \mathit{Coor}\left(\text{"door"}\right)\right)$ (t and t' are past times with $t' < t$). How he pushed the door can be represented by a relation. So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t)\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ \mathit{Dist}\left(\pi_7(X_s(t')), \pi_p(X_o(t'))\right) \leqslant \\ \mathit{Dist}\left(\pi_7(X_s(t)), \pi_p(X_o(t))\right) \wedge R(X_s(t), X_o(t)) \end{pmatrix}$$

Where $\pi_p(X_o)$ is the "position" feature of X_o, $t' < t$, R is a relation on how $X_s(t)$ acts on $X_o(t)$.

(5-38) The verb *take* specifies an action that moves with someone or something from one place to another. For example, "He *took* me to the hospital" implies that $\mathit{Dist}\left(\mathfrak{R}_7^i(t'), \mathfrak{R}_7^j(t')\right) < r_0$, $\mathfrak{R}_7^i(t) \subset \mathit{Coor}\left(\text{"the hospital"}\right)$ and $\mathfrak{R}_7^j(t) \subset \mathit{Coor}\left(\text{"the hospital"}\right)$ (t' and t are past times with $t' < t$, r_0 is a positive number). So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t), L\right) = \begin{pmatrix} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ \pi_7(X_s(t_e)) \subset \mathit{Coor}(L) \wedge \pi_7(X_o(t_e)) \subset \mathit{Coor}(L) \wedge \\ \mathit{Dist}\left(\pi_7(X_s(t)), \pi_7(X_o(t))\right) < r_0 \wedge R(X_s(t), X_o(t)) \end{pmatrix}$$

Where L is a parameter of location, t_e is an end time of the action, r_0 is a positive number, R is a relation on how $X_s(t)$ moves with $X_o(t)$.

(5-39) The verb *build* specifies an action that makes something such as a building. For example, "They *built* a house" implies that an instance of *house* X_{h_0} was created by them, i.e. there are t_0, t_1 that $X_{h_0}(t) = \varnothing$ for $t < t_0$ and $X_{h_0}(t_1)$ is a (completed) house for $t \geqslant t_1$, where t_0 and t_1 are past times with $t_0 < t_1$, and $[t_0, t_1]$ is the period when the house was built. In addition, there is a building procedure, and an outcome of the action is a new house. So it can be described as

$$\varphi\left(P(t), X_s(t), X_o(t), \mathcal{C}\right) = \exists t_0 \exists t_1 \left(\begin{array}{c} P(t) = \{t\} \times X_s(t) \times X_o(t) \wedge \\ \forall t < t_0, X_o(t) = \varnothing \wedge \forall t > t_1, R'(X_o(t)) \\ \wedge \forall t \in [t_0, t_1], R\left(X_s(t), X_o(t), \mathcal{C}\right) \end{array} \right)$$

Where $\mathcal{C}$ is a building procedure involving $X_s(t)$ and $X_o(t)$, R is a relation that specifies the building process involving $X_s(t)$, $X_o(t)$ and $\mathcal{C}$, R' is a relation that $X_o(t)$ meets a specification of a product.

More physical verbs such as *carry*, *throw*, *produce*, and so on, can be modeled likewise and we omit the detail.

Claim 5.23 *Verbs such as "hold", "put", "push", "pull", "take", "build", "bring", "carry", "insert", "throw", "catch", "place", "produce", "make", "beat", "hit", and so on, can be modeled by changes of the corresponding features in dynamic feature spaces of the subject and object (when their meanings involve action of force and physical activity).*

6 Solutions to Frege's Problems

In this section, we will give solutions to Frege's two problems in the semantics of linguistics based on feature spaces.

6.1 First Frege's Problem

For the first Frege's problem, consider the following two sentences:

(6-1) Hesperus is Hesperus. (The Evening Star is the Evening Star.)

(6-2) Hesperus is Phosphorus. (The Evening Star is the Morning Star.)

Since 'Hesperus' refers to the same object as 'Phosphorus' (the planet Venus), both (6-1) and (6-2) are true. However, (6-1) is considered uninformative or has no semantic significance, whereas (6-2) is informative or has semantic significance. In other words, (6-1) is just a truth of logic that always holds, but (6-2) is an empirical truth that was confirmed by astronomical evidence. This is known as the first Frege's problem which has puzzled linguists and philosophers for over a century, and yet no valid solution has been found.

Our solution to this problem is to separate a semantic object (noun) from its name. Since a (countable) noun can always be represented by a feature space, it could have different names that can be enclosed in another feature like "alias". On the other hand, a name is just a string that may not be linked to a semantic object. In (6-1), since string 'Hesperus' is always equal to itself, it may not be linked to a semantic object. Thus (6-1) always holds and has no semantic significance.

In (6-2), nonetheless, since 'Hesperus' is not equal to 'Phosphorus' as a string, 'Hesperus' must be linked to the same semantic object as 'Phosphorus'.(Otherwise they can not be equal.) In other words, 'Hesperus' and 'Phosphorus' must be different names of the same planet — the planet Venus.

The new semantic model in the form of feature spaces provides a simple solution to the first Frege's problem. To illustrate it, first we create an abstract form for *planet* as

$$X_{ThePlanet} = D_p \times \mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_p$$

Where $D_p \subset \mathbb{N}$ is a domain for "all planets", $\mathfrak{A}_1$ is "name", $\mathfrak{A}_2$ is "alias", $\mathfrak{A}_3$ is "mean radius", $\mathfrak{A}_4$ is "mass", $\mathfrak{A}_5$ is "orbit", $\mathfrak{A}_6$ is "orbital period", $\mathfrak{A}_7$ is "atmosphere", and so on. So the planet Venus is a realization of $X_{ThePlanet}$ as

$$\begin{aligned} X_{Venus} = {} & \{i\} \times \{\text{“Venus”}\} \times \{\text{“Hesperus”}, \text{“Phosphorus”}, \text{“Morning Star”}, \text{“Evening Star”}, \cdots\} \\ & \times \{\text{“6051.8}km\text{”}\} \times \{\text{“4.8675}\times 10^{24}kg\text{”}\} \times \cdots \end{aligned}$$

If we assume that a feature space can be referenced by any of its name and aliases, then "Hesperus is Phosphorus" means $X_{Venus} = X_{Venus}$ which is obviously true. Thus it has semantic significance. On the other hand, "Hesperus is Hesperus" is always true for 'Hesperus' is only a string, and so has no semantic significance. So we have

Conclusion 6.1 *The first Frege's problem is solved by separating a (countable) noun in terms of its feature space from its name in terms of a string. Since a feature space can be referenced by either its name or any of its aliases, the equality of distinctive names has semantic significance because they refer to the same feature space.*

6.2 Second Frege's Problem

The second Frege's problem concerns about substitution of coreferential or equivalent expressions in belief reports. For instance,

(6-3) (a) John believes that Hesperus is Hesperus.
(b) Hesperus is Phosphorus.
(c) John believes that Hesperus is Phosphorus.

In (6-3), (a) must be true even if John knows nothing about Hesperus, because "Hesperus is Hesperus" is always true as discussed in the previous section. However, (c) may be false if John does not know that Hesperus and Phosphorus are the same planet. Thus the principle that coreferring names are substitutable salva veritate is violated in the context of belief reports.

A context in which substitution of coreferential expressions does not always preserve truth is known as an opaque context, whereas the laws of predicate logic hold without restrictions is known as a transparent context.

(6-4) (a) Cicero was a great orator.
(b) Cicero was Tully.
(c) Tully was a great orator.

In (6-4), we can substitute 'Cicero' for 'Tully' in (a) to get (c) so that (a) and (c) are equivalent. So (6-4)(a) gives rise to a transparent context. Next, let's look at the following example.

(6-5) (a) John believes that Cicero was a great orator.
(b) Cicero was Tully.
(c) John believes that Tully was a great orator.

In (6-5), even if (b) is true, we can not simply say that (a) and (c) are equivalent because John may not know that 'Cicero' and 'Tully' refer to the same person. Thus (6-5)(a) gives rise to an opaque context. This example suggests that epistemic verbs[40] such as *believe*, *know*, etc, give rise to opacity.

Careful examination of above examples suggests that an opaque context is caused by the scope of knowledge specified in an epistemic verb. For instances, in (6-4), "Cicero was Tully" is a universal fact, and (6-4)(c) holds since no scope is specified; (6-5)(c), however, may not hold if John does not know that Cicero and Tully were the same person, i.e. "Cicero was Tully" is not in the scope (the mind) of John. So a modification of (6-5) as follows would make substitution valid. (The same applies to (6-3) as well.)

(6-6) (a) John believes that Cicero was a great orator.
(b) John knows that Cicero was Tully.
(c) John believes that Tully was a great orator.

Other verbs that give rise to opacity include *discover*, *compute*, and so on.[41]

(6-7) (a) An astronomer discovered that Hesperus and Phosphorus are the same planet as the planet Venus.
(b) An astronomer computed that Hesperus passed the same orbit at the same time as the planet Venus.
(c) An astronomer computed that Phosphorus passed the same orbit at the same time as the planet Venus.

In (6-7), even if (b) and (c) are true, we can still not conclude that (a) is true because in order to draw a conclusion from (b) and (c), an astronomer must know the fact that any two planets with the same position at the same time must be the same. In other words, this universal knowledge must be in the scope (the mind) of the astronomer.[42] Thus a modification of (6-7) as follows would make it transparent.

(6-8) (a) An astronomer discovered that Hesperus and Phosphorus are the same planet as the planet Venus.

[40]We leave the detailed discussion of epistemic verbs in another paper.

[41]The following examples are essentially from [13, p411–412].

[42]The difference between (6-5) and (6-7) is that the universal knowledge is not explicitly stated in (6-7), whereas it is in (6-5)(b).

(b) An astronomer computed that Hesperus passed the same orbit at the same time as the planet Venus.
(c) An astronomer computed that Phosphorus passed the same orbit at the same time as the planet Venus.
(d) An astronomer knew that any two planets with the same position at the same time must be the same.

So we have

Conclusion 6.2 *For epistemic verbs such as "believe", "know", etc, and others like "discover", "compute", etc, an opaque context is caused by the scope of knowledge specified in such a verb (often in the form of human minds). In other words, if the scope specified by a verb and subject does not include all necessary knowledge and facts, it will give rise to opacity in which substitution of coreferential or equivalent expressions in belief reports fails.*

Communication verbs like *tell*, *say*, etc, can also give rise to opacity.

(6-9) (a) Ortcutt told Ralph about his profession.
(b) Ortcutt works as a professional spy.
(c) Ortcutt told Ralph that he is a spy.

In (6-9), (c) may not be true even if (a) and (b) are true, because either Ortcutt could lie and did not tell Ralph about his true profession, or Ortcutt has other professions than a spy and told Ralph about that. This instance differs from the above examples in that the problem is about the range of Ortcutt's profession and whether or not Ortcutt told the truth about his profession. Thus a solution is to specify the precise ranges of meanings in the sentences. For example, if (6-9) is modified as follows, substitution of coreferential expressions will work.

(6-10) (a) Ortcutt told Ralph correctly about his profession.
(b) Ortcutt works only as a professional spy (in his life).
(c) Ortcutt told Ralph that he is a spy.

Another modification for making substitution work is

(6-11) (a) Ortcutt told Ralph correctly about his profession.
(b) Ortcutt works as a professional spy.
(c) Ortcutt told Ralph that one of his profession is a spy.

Conclusion 6.3 *For communication verbs such as "tell", "say", etc, substitution of coreferential or equivalent expressions in belief reports holds (or a context is transparent) if the ranges of meanings specified by the verbs are precise.*

Consequently, we have reached the following conclusion.

Conclusion 6.4 *The second Frege's problem is solved by either having the scope specified by a subject and verb with all necessary knowledge, or specifying the precise ranges of meanings in the sentences.*

Appendix A Universe of the Sets

In this paper, we investigate quantifiable feature spaces which are based on a universe of the sets on (real) numbers. Let $\mathbb{R}$ denote all real numbers, i.e. $\mathbb{R} = (-\infty, \infty)$. We need to have a system to include all subsets of the reals, i.e. $\{1, 2, 5\}$, $\{(-\infty, 0), 1\}, \cdots$, as well as all subsets of the subsets of the reals, i.e. $\{\{1, 2\}, 3, \{5\}\}$ and so on. First we need the power set operation for this end.

Definition A.1 *The* ***power set*** *of a set A is the collection of all subsets of A, i.e.*

$$\mathcal{P}(A) = \{X \colon X \subset A\}$$

A universe including all subsets, all subsets of subsets of the reals, and so on, can be achieved by successively applying power set operations on the reals. The following scheme known as the von Neumann universe ($\mathcal{V}$) is the universe of all well-founded sets of the reals by applying power set operations for all ordinal numbers.

$$\begin{aligned}
&\mathcal{V}_0 = \mathbb{R}; \\
&\mathcal{V}_\alpha = \mathcal{P}(\mathcal{V}_{\alpha-1}) \cup \mathcal{V}_{\alpha-1}, && \alpha \text{ is any successor ordinal;} \\
&\mathcal{V}_\alpha = \bigcup_{\beta<\alpha} \mathcal{V}_\beta\,, && \alpha \text{ is any limit ordinal;} \\
&\mathcal{V} = \bigcup_{\alpha \in Ord} \mathcal{V}_\alpha.
\end{aligned}$$

In this paper, the universe of the sets of the reals is adopted as the total universe ($\mathcal{T}$) as follows. It is an extension of the von Neumann universe that includes the non-well-founded sets as well.

$$\begin{aligned}
&\mathcal{T}_0 = \mathbb{R}; \\
&\mathcal{T}_\alpha = \mathcal{P}(\mathcal{T}_{\alpha-1}) \cup \mathcal{T}_{\alpha-1}, && \alpha \text{ is any successor ordinal;} \\
&\mathcal{T}_\alpha = \bigcup_{\beta<\alpha} \mathcal{T}_\beta \cup \left. \left(\bigcup_{\beta<\alpha} \mathcal{T}_\beta \right) \right|_{\aleph_0}, && \alpha \text{ is any limit ordinal;} \\
&\mathcal{T} = \bigcup_{\alpha \in Ord} \mathcal{T}_\alpha.
\end{aligned} \tag{A.1}$$

Note that the Cartesian product of $\mathcal{T}$ is still within $\mathcal{T}$.

Proposition A.2 *Suppose $\mathcal{F}$ and $\mathcal{T}^n$ are defined in (2.1).*

(a) $\mathcal{T}^n \subset \mathcal{T}$

(b) $\mathcal{F} = \mathcal{T}$

Proof. (a) We only prove $n = 2$ case. The rest follow from induction.

For any $(x,y) \in \mathcal{T} \times \mathcal{T}$, $x \in \mathcal{T}$ and $y \in \mathcal{T}$. So by (A.1), there is a α that $x \in \mathcal{T}_\alpha$, $y \in \mathcal{T}_\alpha$. Thus

$$\{x\} \subset \mathcal{T}_\alpha \text{ and } \{x,y\} \subset \mathcal{T}_\alpha. \text{ So } \{x\} \in \mathcal{P}(\mathcal{T}_\alpha) \text{ and } \{x,y\} \in \mathcal{P}(\mathcal{T}_\alpha).$$

Since $\mathcal{P}(\mathcal{T}_\alpha) \subset \mathcal{T}_{\alpha+1}$, by Wiener-Kuratowski ordered pair and (A.1)

$$(x,y) = \{\{x\},\{x,y\}\} \subset \mathcal{T}_{\alpha+1}. \text{ So } (x,y) \in \mathcal{P}(\mathcal{T}_{\alpha+1}) \subset \mathcal{T}_{\alpha+2} \subset \mathcal{T}.$$

Thus $\mathcal{T} \times \mathcal{T} \subset \mathcal{T}$.

(b) By (2.1), we have

$$\mathcal{T} \subset \bigcup_{n=1}^{\infty} \mathcal{T}^n = \mathcal{F}$$

Then by (a)

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{T}^n \subset \mathcal{T}$$

■

Corollary A.3 *Suppose A_k are n sets in $\mathcal{T}$. Then*

$$A_1 \times \cdots \times A_n \in \mathcal{T}$$

Proof. We only prove $n = 2$ case. The rest follow from induction.

Since $A_1 \in \mathcal{T}$ and $A_2 \in \mathcal{T}$, there is a α that $A_1 \in \mathcal{T}_\alpha$, $A_2 \in \mathcal{T}_\alpha$. For any $(x,y) \in A_1 \times A_2$, $x \in A_1$ and $y \in A_2$. Since $\mathcal{T}_\alpha$ is transitive, $x \in \mathcal{T}_\alpha$, $y \in \mathcal{T}_\alpha$. Thus by the proof in proposition A.2(a)

$$(x,y) = \{\{x\},\{x,y\}\} \in \mathcal{T}_{\alpha+2}$$

Hence

$$A_1 \times A_2 = \{(x,y) : x \in A_1,\ y \in A_2\} \subset \mathcal{T}_{\alpha+2}$$

And

$$A_1 \times A_2 \in \mathcal{P}(\mathcal{T}_{\alpha+2}) \subset \mathcal{T}_{\alpha+3} \subset \mathcal{T}$$

■

Obviously, any collection of Cartesian sets is still in $\mathcal{T}$.

Corollary A.4 *Suppose A_k^i are sets in $\mathcal{T}$ $(1 \leqslant k \leqslant n,\ i \in D)$. Then*

$$\{A_1^i \times \cdots \times A_n^i : i \in D\} \in \mathcal{T}$$

Example A.5 *Suppose $A_1 = \{1,2\}$ and $A_2 = \{3,4,5\}$. Show $A_1 \times A_2 \in \mathcal{T}$.*

Solution. By Cartesian product and Wiener-Kuratowski ordered pair

$$\begin{aligned} A_1 \times A_2 &= \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\} \\ &= \{\{\{1\},\{1,3\}\},\{\{1\},\{1,4\}\},\{\{1\},\{1,5\}\},\{\{2\},\{2,3\}\},\{\{2\},\{2,4\}\},\{\{2\},\{2,5\}\}\} \end{aligned}$$

Since $\{1\} \in \mathcal{T}_1$ and $\{1,3\} \in \mathcal{T}_1$, $\{\{1\},\{1,3\}\} \subset \mathcal{T}_1$. So $\{\{1\},\{1,3\}\} \in \mathcal{T}_2$. This holds for all other objects in $A_1 \times A_2$. Thus $A_1 \times A_2 \subset \mathcal{T}_2$ and $A_1 \times A_2 \in \mathcal{T}_3 \subset \mathcal{T}$. ■

Appendix B Additional Operators in Set Theory

The unpacking operator is the operation to remove the curly brackets of a set.

Definition B.1 *Suppose $G = \{a_1, a_2, \cdots\}$. The **unpacking operator** $*G$ of G is defined as $\{*G\} = G$, i.e. $*G = a_1, a_2, \cdots$.*

Intuitively, $*G$ can be considered as the collection a_i of G without the curly brackets. For example, $S = \{a_1, a_2, \cdots, b_1, b_2, \cdots\}$, $G_1 = \{a_1, a_2, \cdots\}$ and $G_2 = \{b_1, b_2, \cdots\}$. Then $S = \{*G_1, *G_2\}$.

The Union operator on a set is to obtain the collection of objects, each of which belongs to a member of the set.

Definition B.2 *The **union operator** is defined as*

$$\bigcup X = \{y \colon \exists z\, (y \in z \wedge z \in X)\}$$

The transitive closure of a set is the collection of all the objects contained in various levels of sets in the set.

Definition B.3 *Let $\bigcup^n S = \underbrace{\bigcup \cdots \bigcup}_{n} S$ (n^{th} union operator). Then the **transitive closure** of S is*

$$TC(S) = \bigcup_{n=0}^{\infty} \bigcup^n S$$

Example B.4 *Suppose $X_1, X_2, X_3 \in \mathcal{T}$. Then*

$$TC(\{\{X_1\}, \{\{X_1, X_2\}, X_3\}\}) = \{\{X_1\}, X_1, \{\{X_1, X_2\}, X_3\}, \{X_1, X_2\}, X_2, X_3\}$$

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