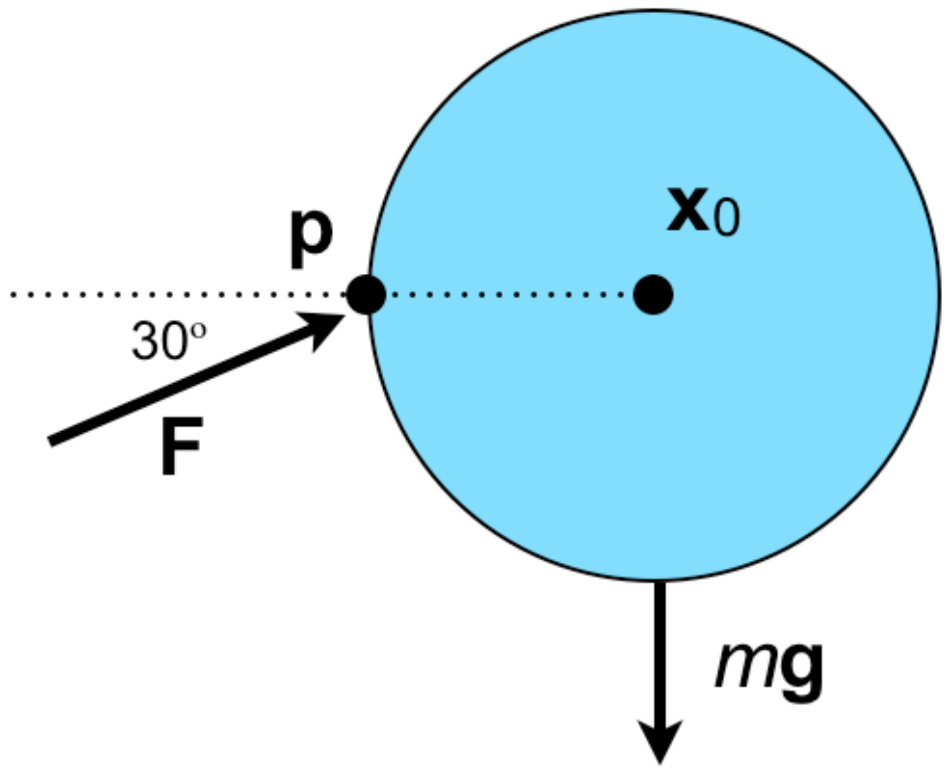
Previously in CS248B…

Quiz

- **Consider a 3D sphere with radius 1m, mass 1kg, and inertia I_{body}. The initial linear and angular velocity are both zero. The initial position and the initial orientation are x_0 and R_0. The forces applied on the sphere include gravity (g) and an initial push F applied at point p. Note that F is only applied for one time step at t_0. If we use Explicit Euler method with time step h to integrate , what are the position and the orientation of the sphere at t_2? Use the actual numbers defined as below to compute your solution (except for g and h).**



$\mathbf{x}_0 = (0, 0, 0)$

$$\mathbf{R}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{p} = (-1, 0, 0)$

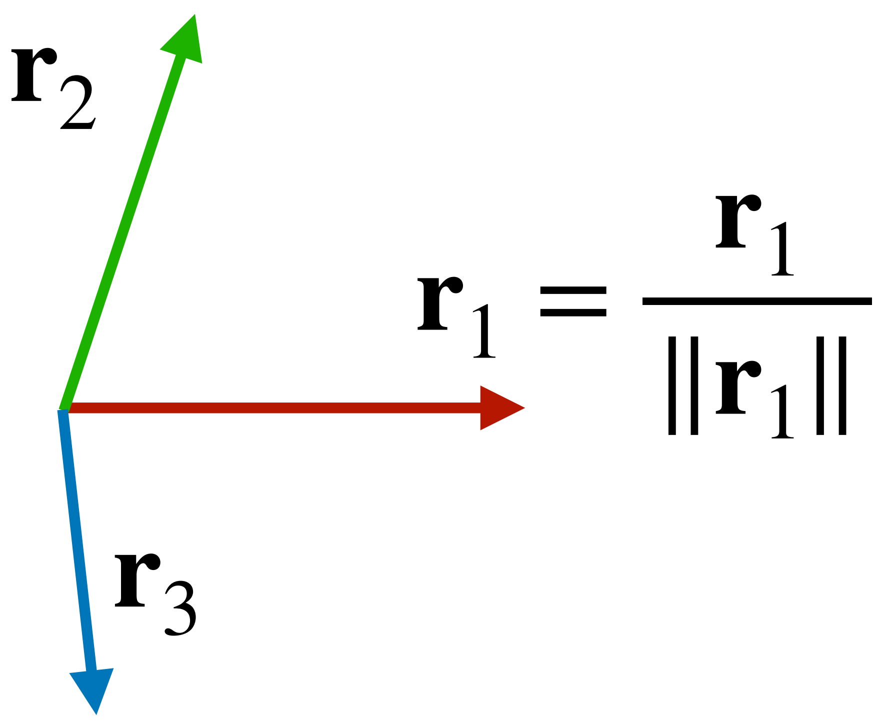
$\mathbf{F} = (4\cos(30^\circ), 4\sin(30^\circ), 0)$

$m = 1$

$$\mathbf{I}_{body} = \begin{pmatrix} 2/5 & 0 & 0 \\ 0 & 2/5 & 0 \\ 0 & 0 & 2/5 \end{pmatrix}$$

Issues with rotation matrix

- The rotational matrix might no longer be orthonormal due to accumulated numerical errors.
- Rectifying a rotational matrix is not trivial.
  - Could use Gram-Schmidt process to make $\mathbf{R}$ orthonormal.



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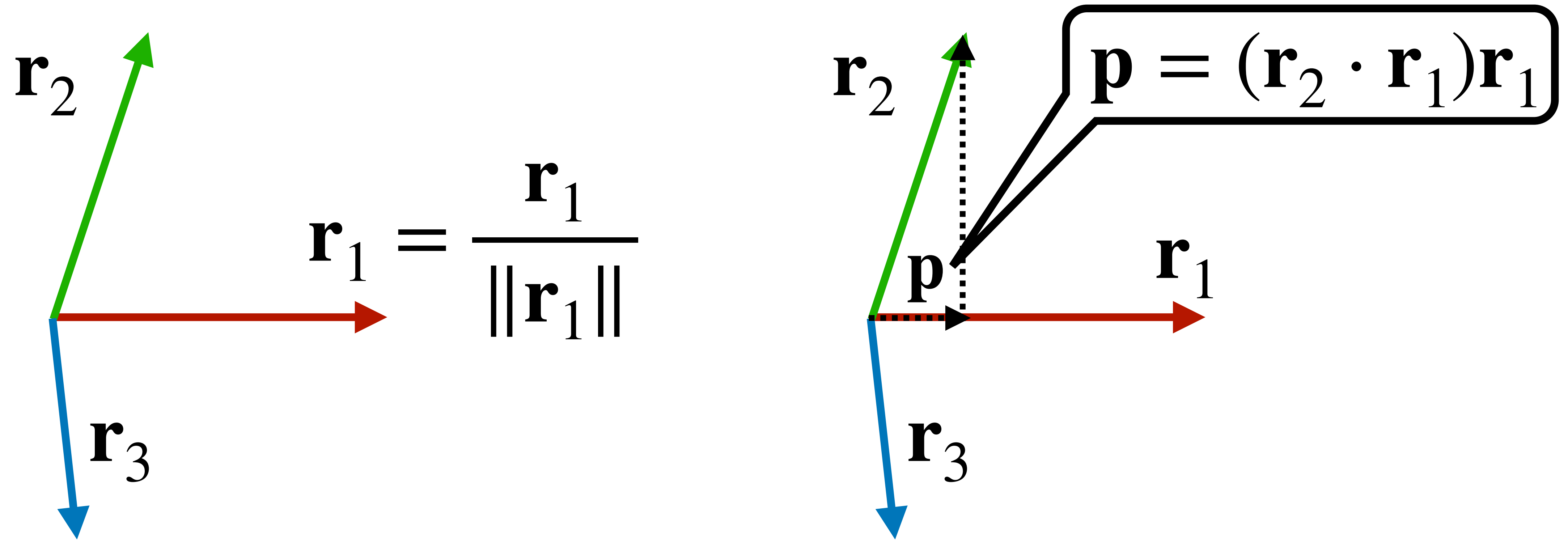

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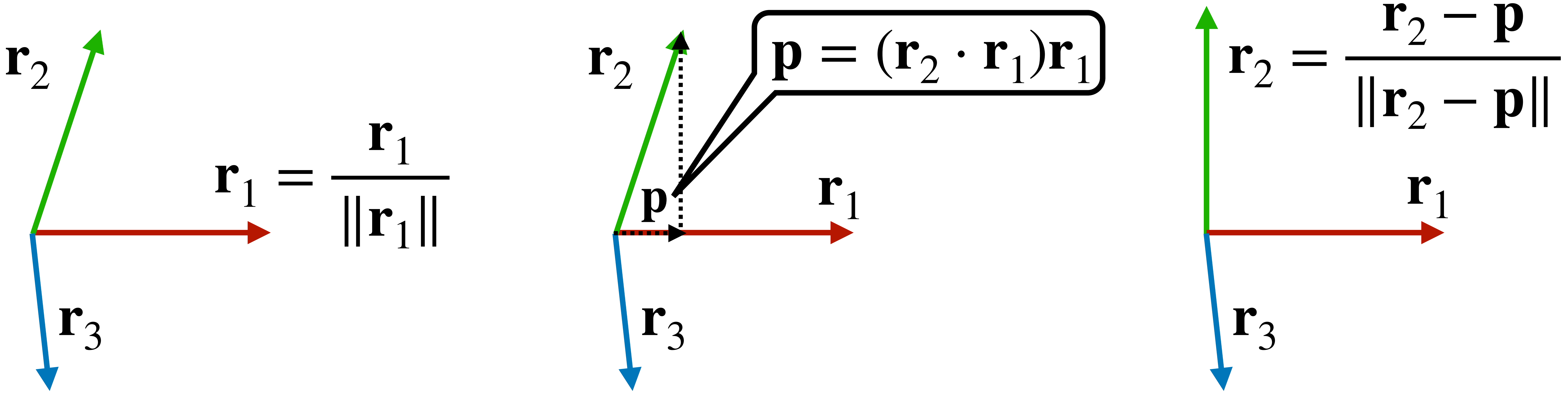

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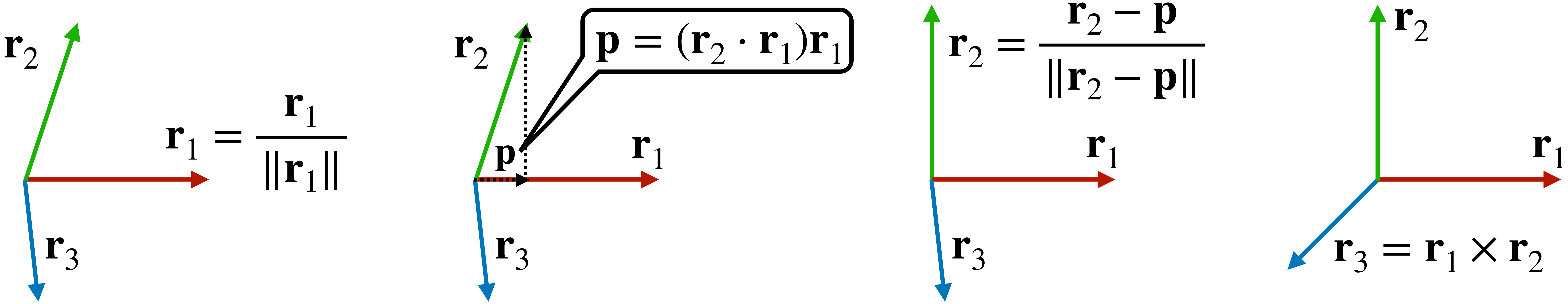


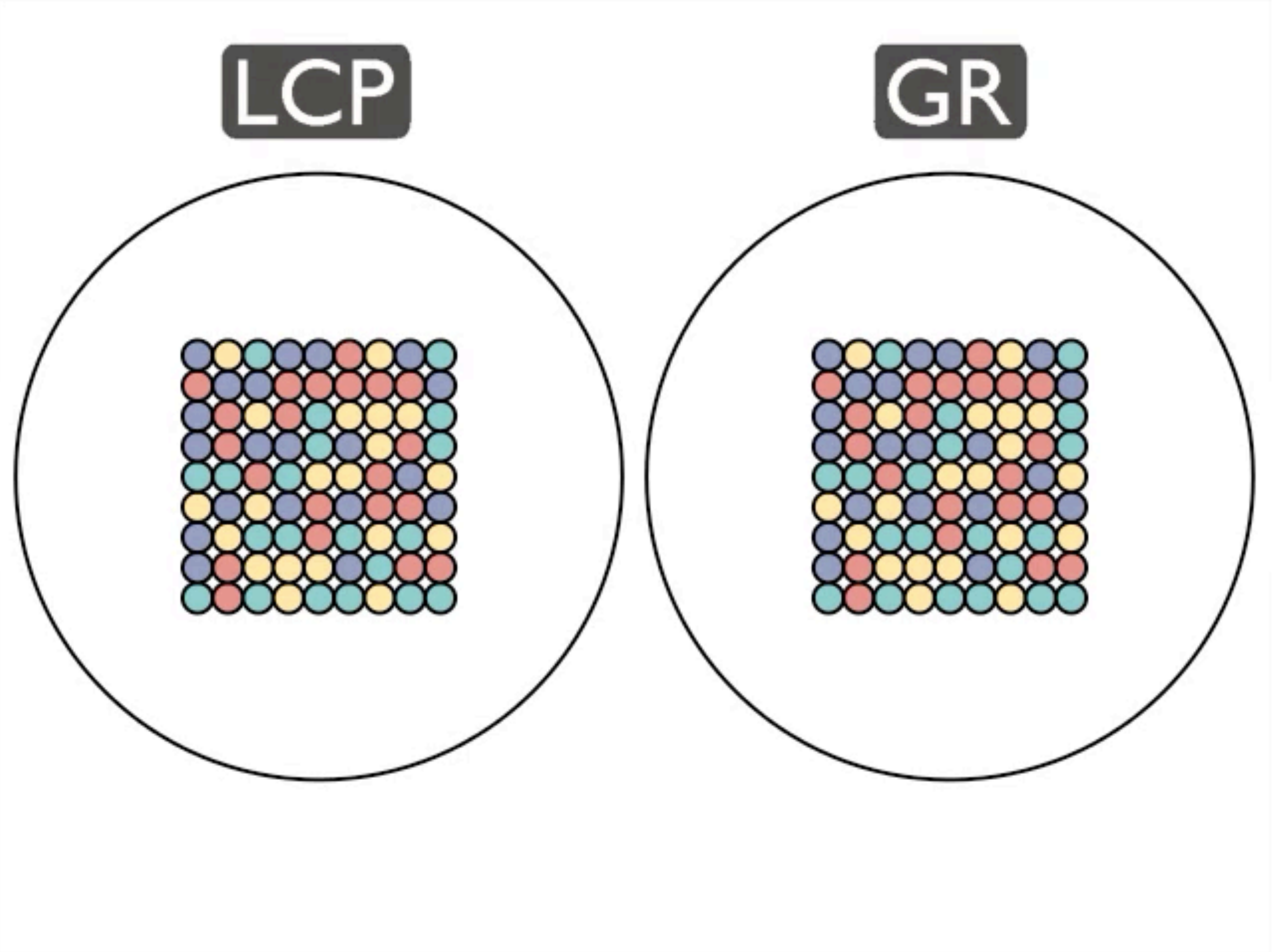
- We will use Quaternion representation for 3D orientation instead.

Momentum vs velocity

- Why do we use momentum in the state space instead of velocity?
  - Because the relation of angular momentum and torque is simpler.
  - Because the angular momentum is constant when there is no torques acting on the object.
- Use linear momentum $\mathbf{p}(t)$ to be consistent with angular velocity.

Constrained rigid body simulation

- Handling contacts and collisions is a very important topic that will be partially covered in later lectures.
- Idealized contact models can produce visually plausible results for graphics applications, but they are often a major source of error when predicting the motion of real-world objects.



Additional reading

- Skew symmetric matrix: https://en.wikipedia.org/wiki/Skew-symmetric_matrix
- Rigid body lecture notes from David Baraff:
  - https://www.cs.cmu.edu/~baraff/sigcourse/notesd1.pdf
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- Brian Mirtich's thesis
  - https://people.eecs.berkeley.edu/~jfc/mirtich/thesis/mirtichThesis.pdf

Lecture 12:

3D Orientation

FUNDAMENTALS OF COMPUTER GRAPHICS

Animation & Simulation

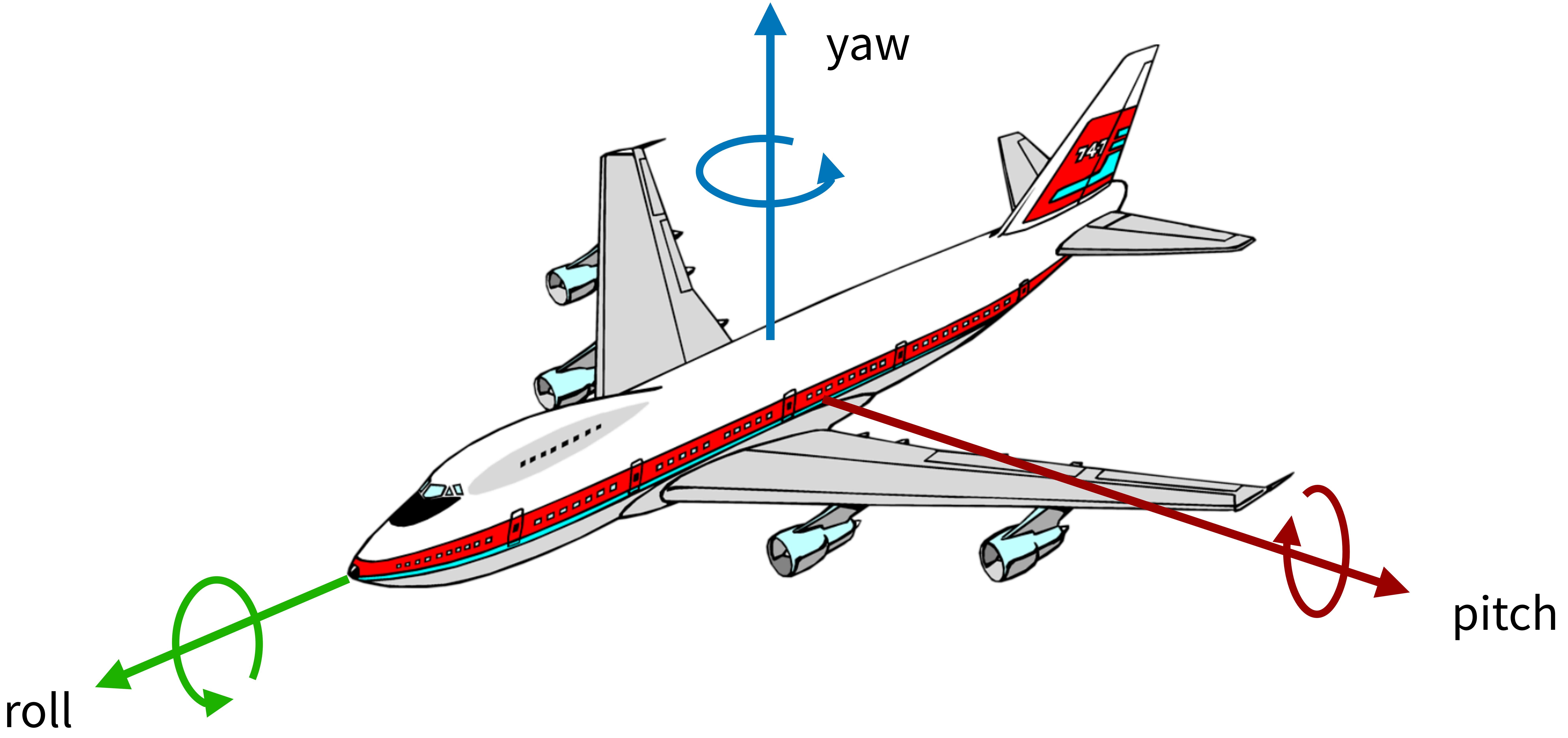
Stanford CS248B, Fall 2022

Learning Objectives

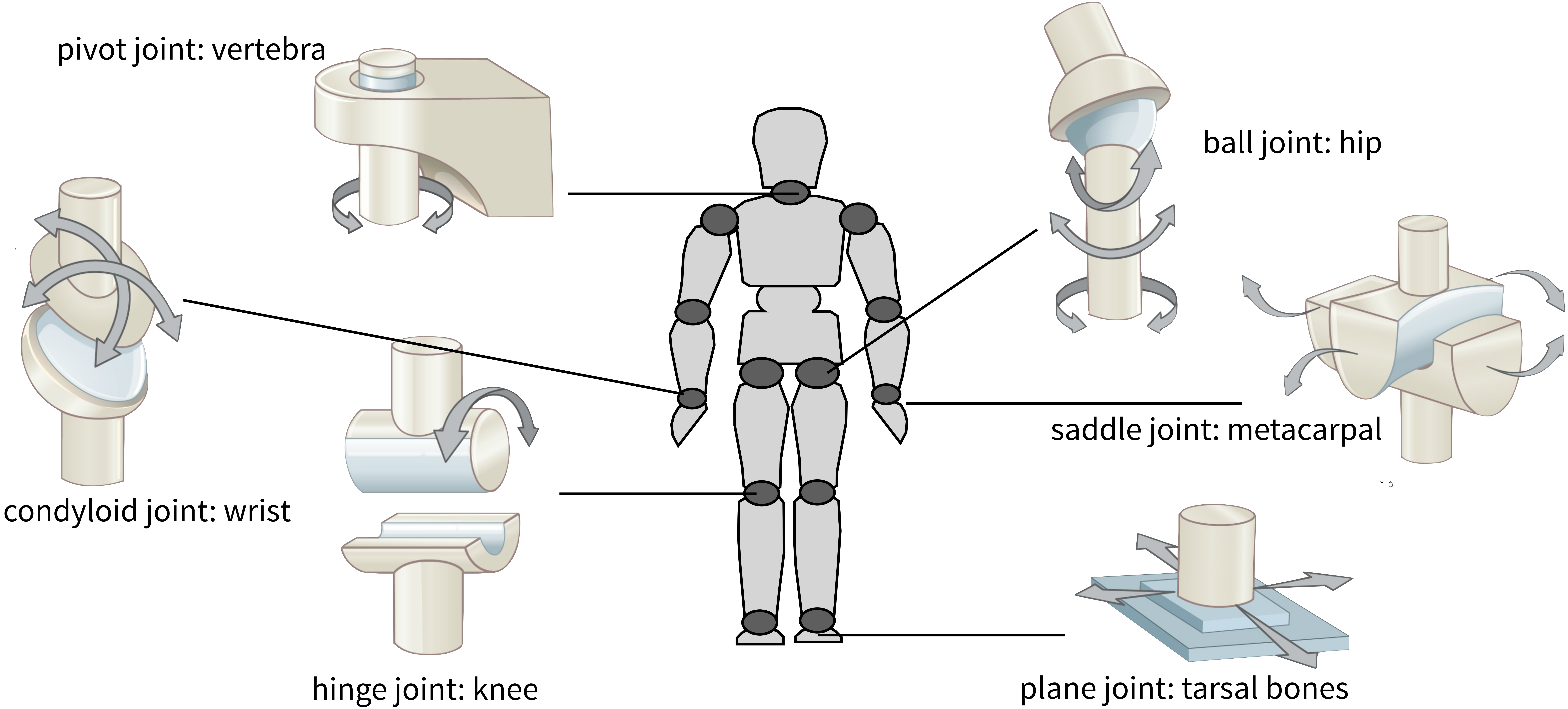
- Learn different representations for 3D orientation
- Understand the properties of each representation
- Learn the concept of quaternion and its applications

3D orientation

Orientation describes the angular position of a 3D object in the space it occupies.



Parameterizing human model



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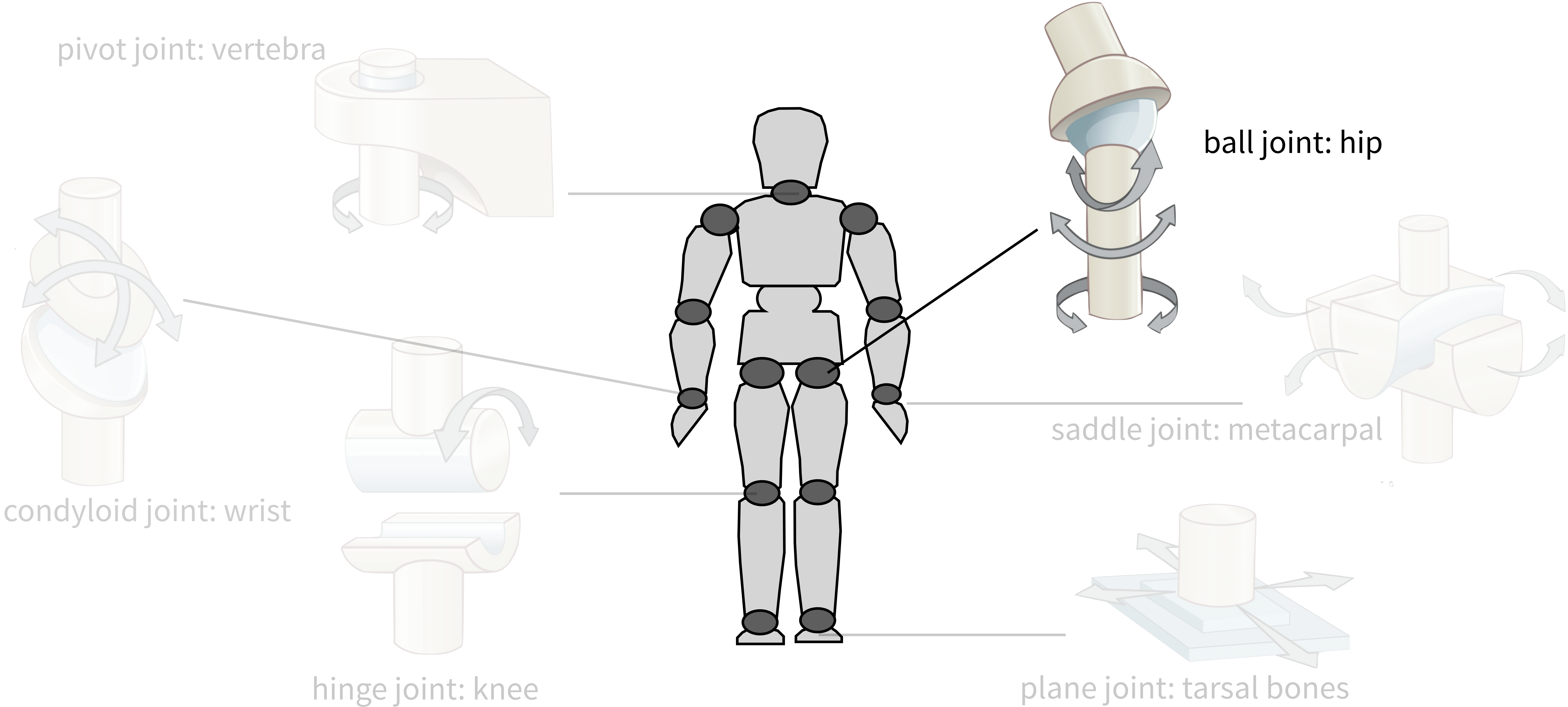

Matrix representation

- 3D orientation can be represented by a matrix $\mathbf{R}$.
- $\mathbf{R}$ is an orthonormal matrix.
- $\mathbf{R}\mathbf{R}^T = \mathbf{I}$
- Multiple rotations can be compose to one matrix
  - $\mathbf{R} = \mathbf{R}_x(30)\mathbf{R}_y(60)\mathbf{R}_z(90)$

about x axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

about y axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

about z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Interpolation

- Interpolating translation is easy, but what about rotations? How about interpolating each entry of the rotation matrix?
- The interpolated matrix might no longer be orthonormal, leading to nonsense for the in-between rotations.
  - Example: interpolate linearly from a $+90^\circ$ rotation about y axis to -90°rotation about y-axis:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Linearly interpolate each component and halfway between, you get this...

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

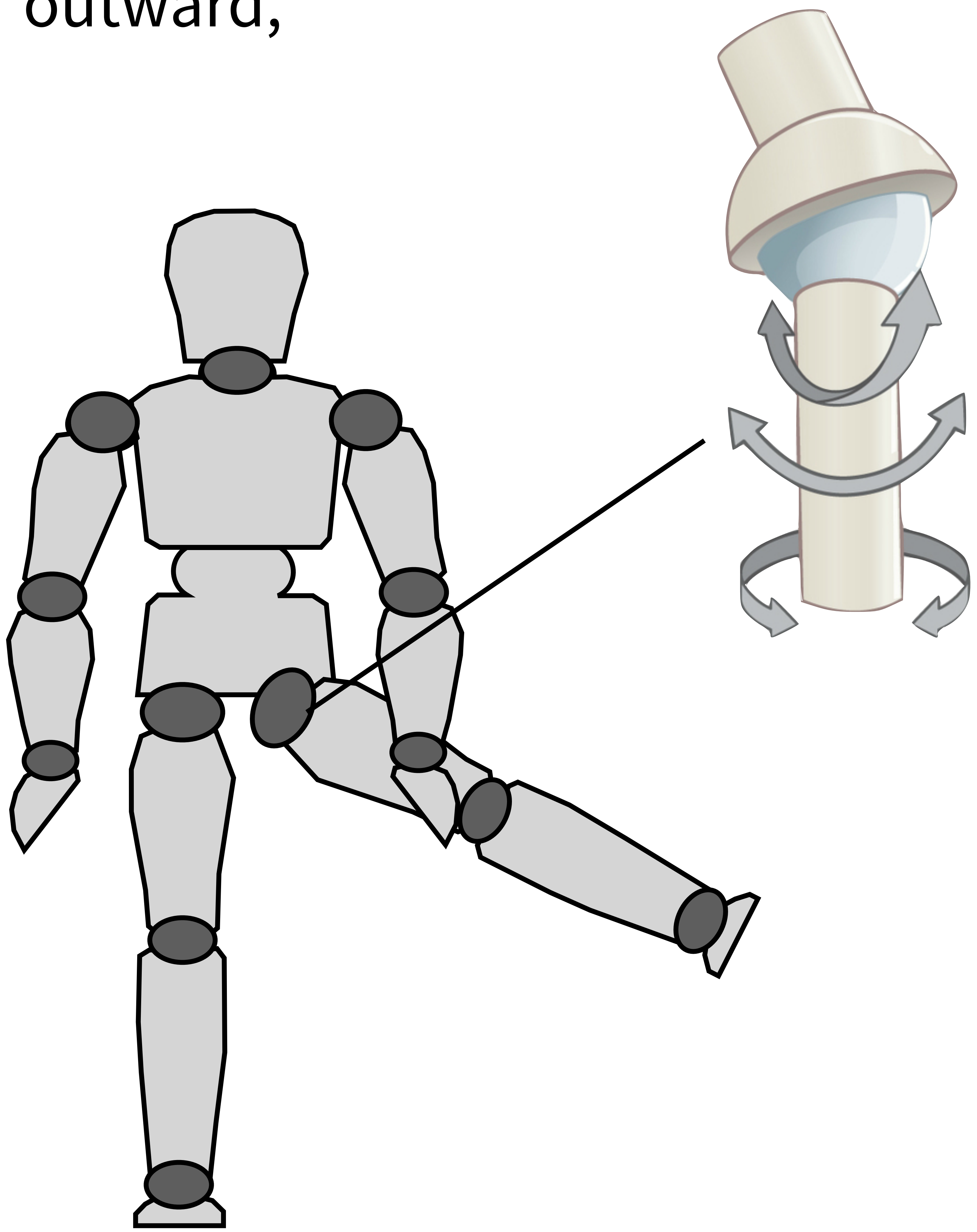
Enforce joint limits

Suppose the hip joint can not abduct more than $80°$ outward,

Does $\mathbf{R}_{l-hip} = \begin{bmatrix} -0.069 & 0.012 & 0.998 \\ 0.601 & 0.799 & 0.032 \\ -0.796 & 0.601 & -0.062 \end{bmatrix}$

violate the joint limits?

Hard to tell!



Properties of rotation matrix

- Easy to compose? ✓
- Easy to interpolate? ✗
- Easy to enforce joint limits? ✗

Fixed angle and Euler angle

- Both fixed angle and Euler angle representations specify 3D orientation by 3 parameters that represent angles used to rotate about 3 ordered axes.
  - Each angle is a rotation about a single Cartesian axis.
  - Concatenating 3 rotations to create a multi-DOF joint.
  - For example, $\mathbf{R}_x(30)\mathbf{R}_y(60)\mathbf{R}_z(90)$ indicates rotating about x-axis by $30°$, about y-axis by $60°$, and finally about z-axis by $90°$.
- Different rotating orders result in different orientations
  - For example, $\mathbf{R}_x(30)\mathbf{R}_y(60)\mathbf{R}_z(90)$ is different from $\mathbf{R}_z(90)\mathbf{R}_y(60)\mathbf{R}_x(30)$

Fixed angle and Euler angle

Two different conventions to rotate $\mathbf{R}_x(30)\mathbf{R}_y(60)\mathbf{R}_z(90)$

Fixed angle: axes do not move with object

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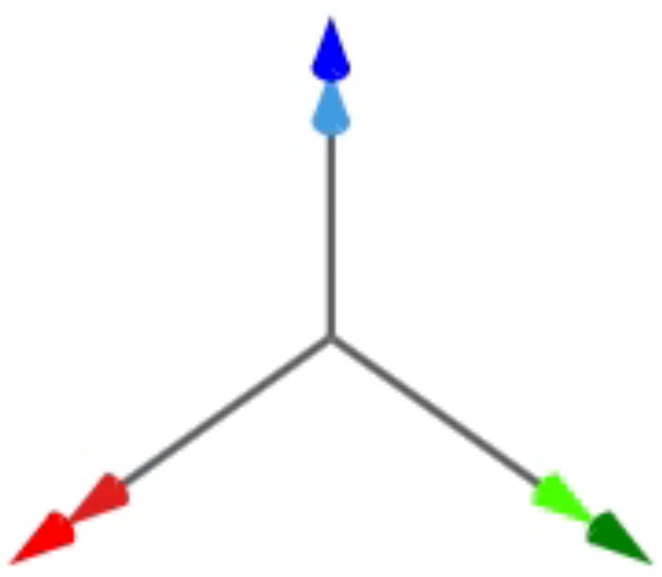
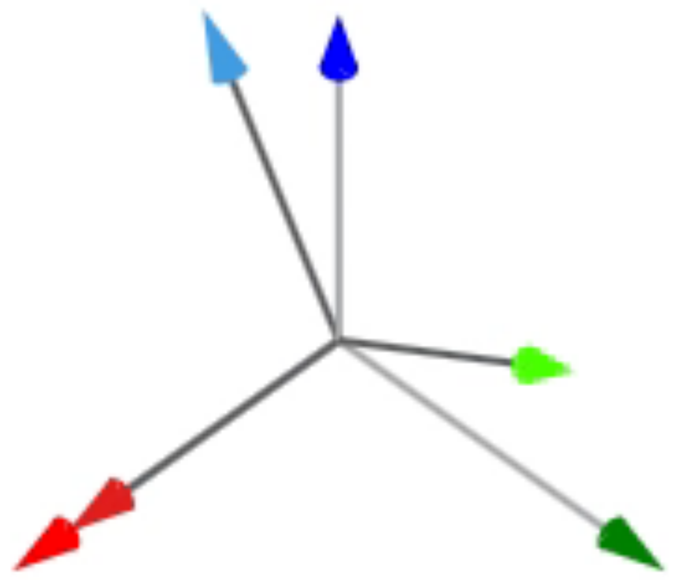
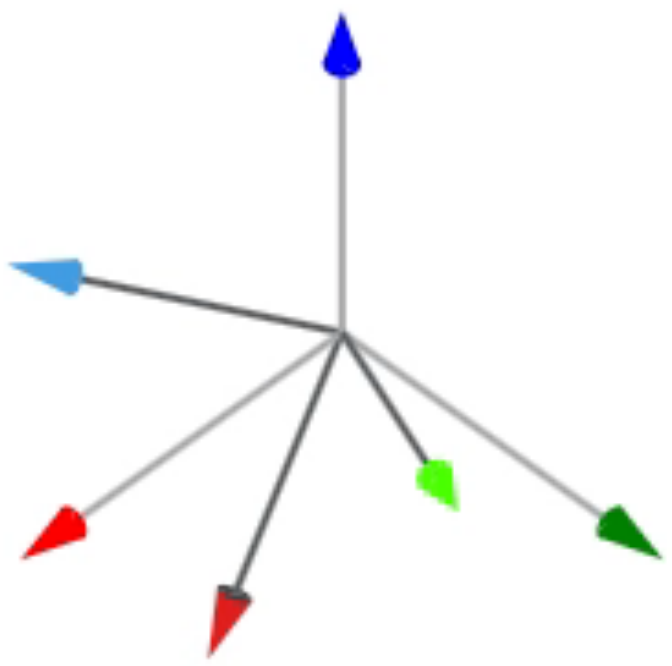
$\mathbf{R}_x^F(30)$



$\mathbf{R}_y^F(60)$



$\mathbf{R}_z^F(90)$



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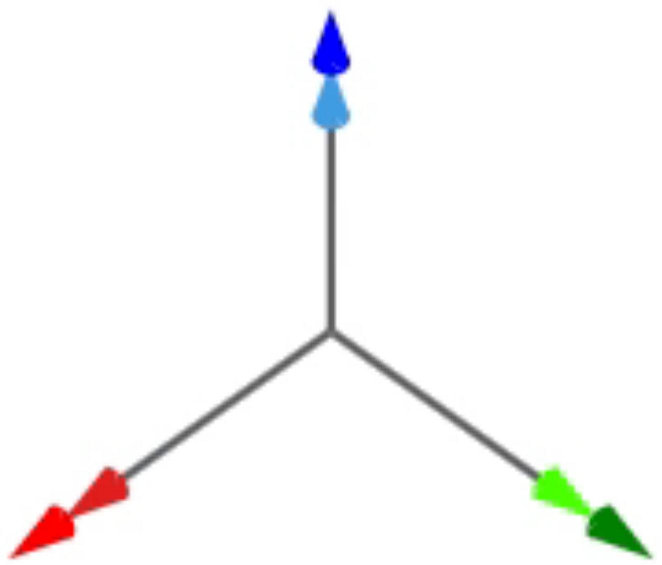
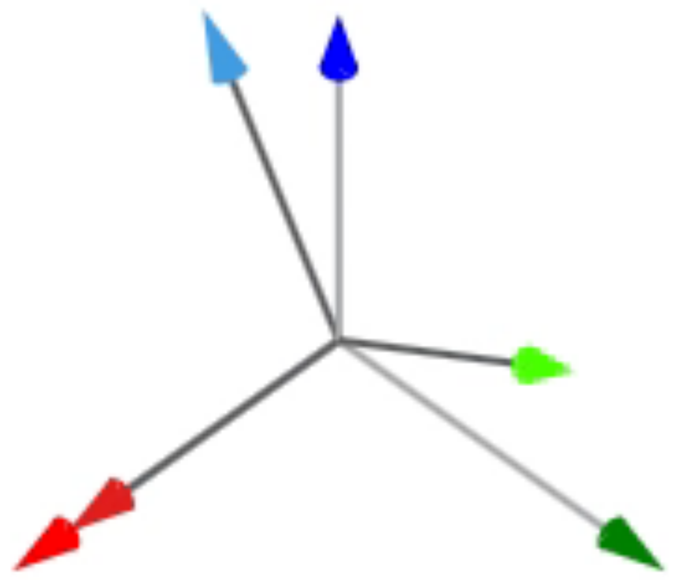
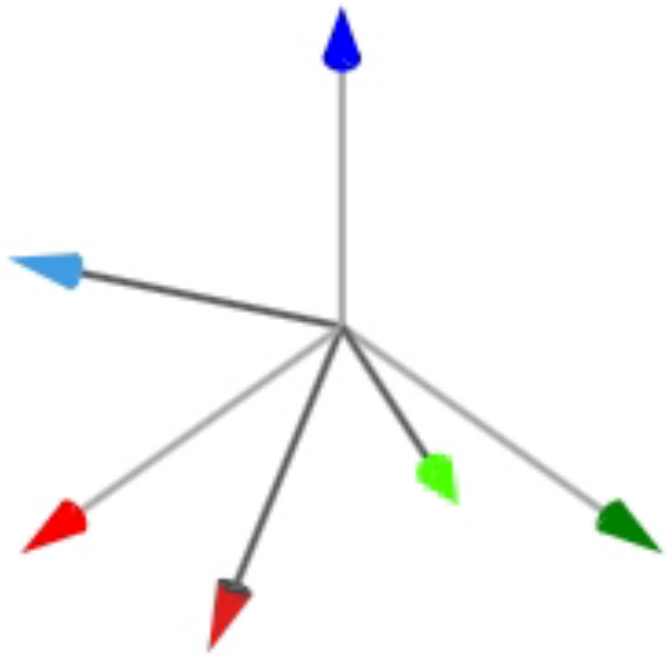


$\mathbf{R}_y^F(60)$

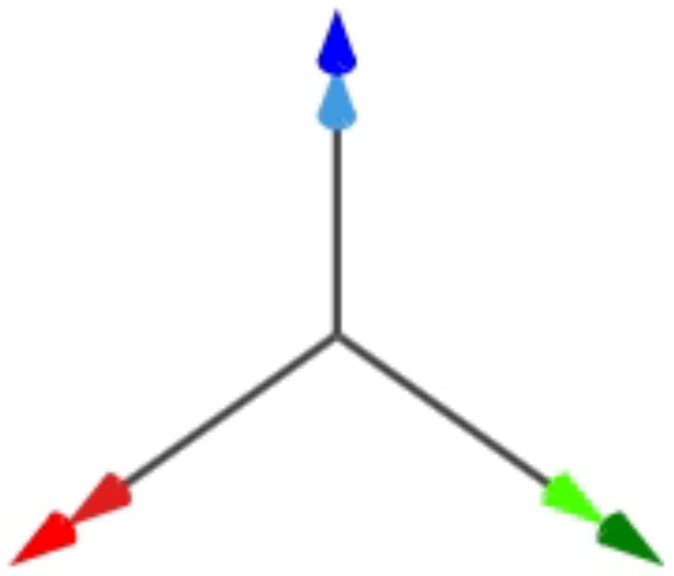


$\mathbf{R}_z^F(90)$

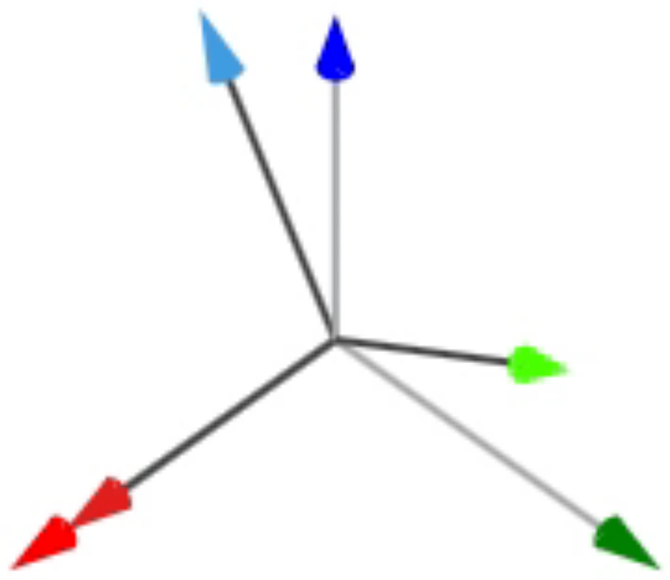


Euler angle: axes move with object

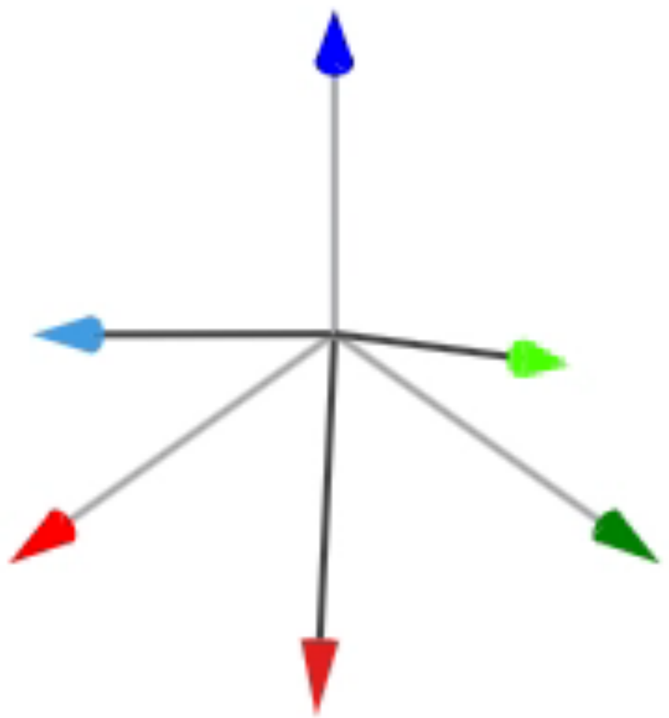
$\mathbf{R}_x^E(30)$



$\mathbf{R}_y^E(60)$



$\mathbf{R}_z^E(90)$



Fixed angle and Euler angle

- Euler angle is the same as fixed angle, except now the axes move with the object.
- Euler angle rotations about moving axes written in reverse order are the same as the fixed axis rotations.

$$\mathbf{R}_x^F(30)\mathbf{R}_y^F(60)\mathbf{R}_z^F(90) = \mathbf{R}_z^E(90)\mathbf{R}_y^E(60)\mathbf{R}_x^E(30)$$

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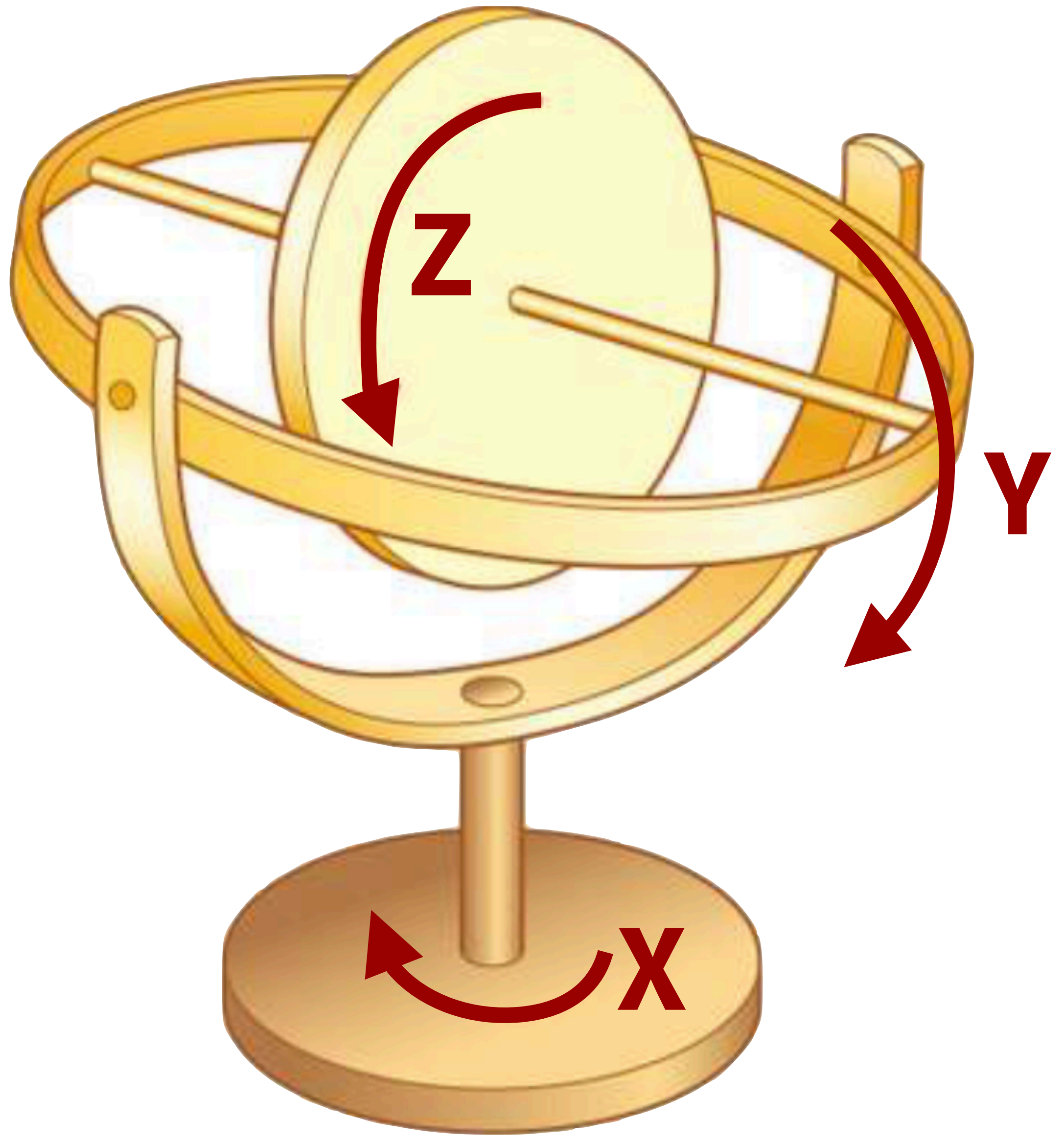
If we rotate the gyroscope in XYZ order, are we using Fixed angle or Euler angle convention?



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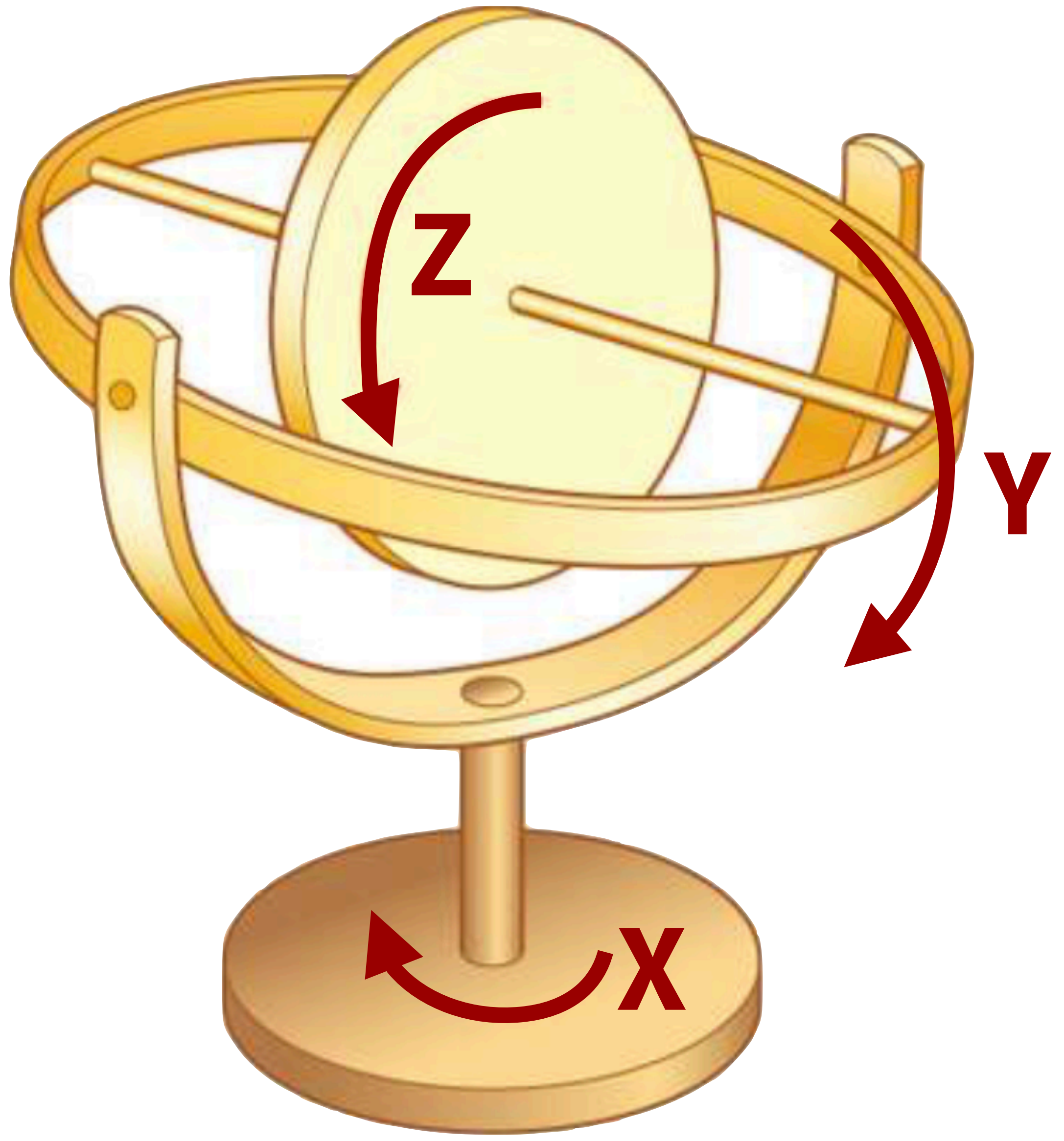
Euler angle.



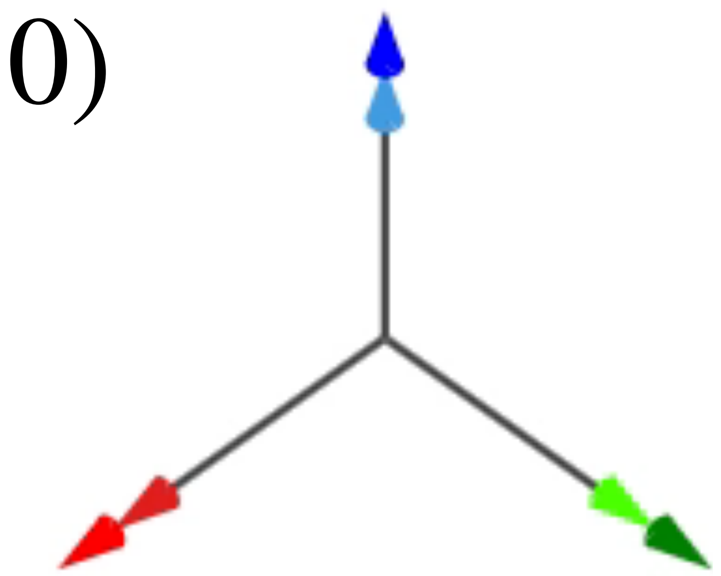
Properties of Fixed/Euler angle

- **Easy to compose?** ✓, but can only consolidate consecutive rotations about the same axis.
- **Easy to interpolate?** ✓, but interpolating each axis separately might not lead to a smooth interpolated path.
- **Easy to enforce joint limits?** ✓
- **What seems to be the problem?** Gimbal lock!

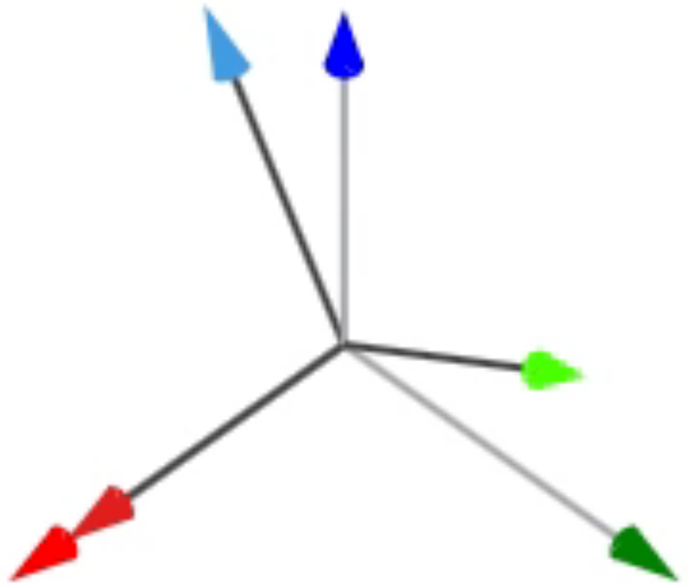
Gimbal lock

When two rotational axis of an object pointing in the same direction, the rotation ends up losing one degree of freedom.

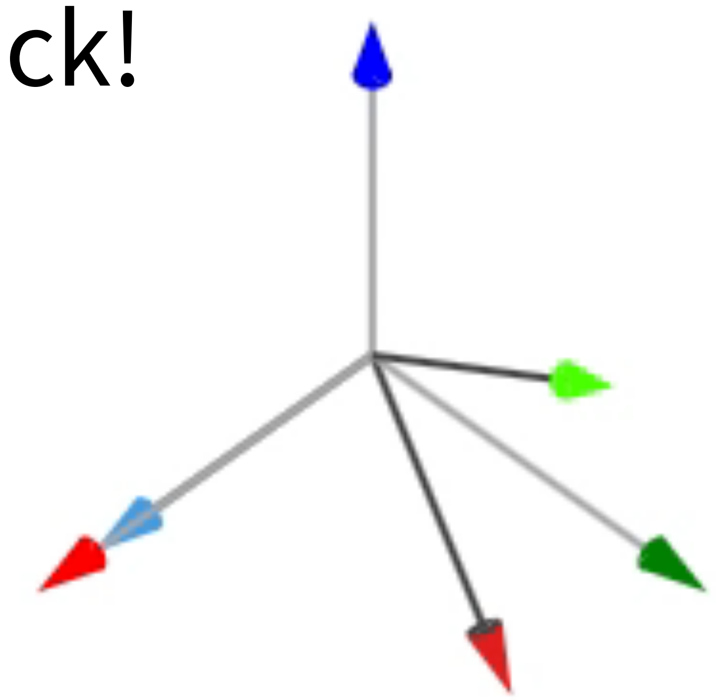
$\mathbf{R}_x^E(30)$



$\mathbf{R}_y^E(90)$



3rd axis aligned with 1st axis
Gimbal lock!



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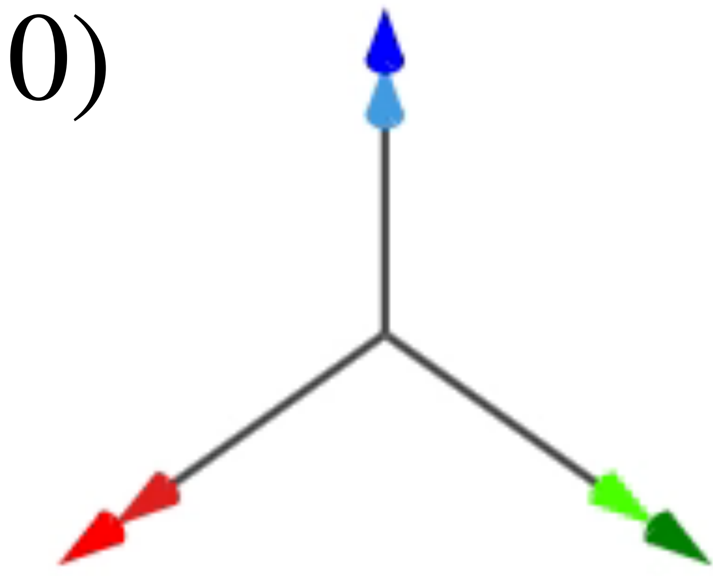
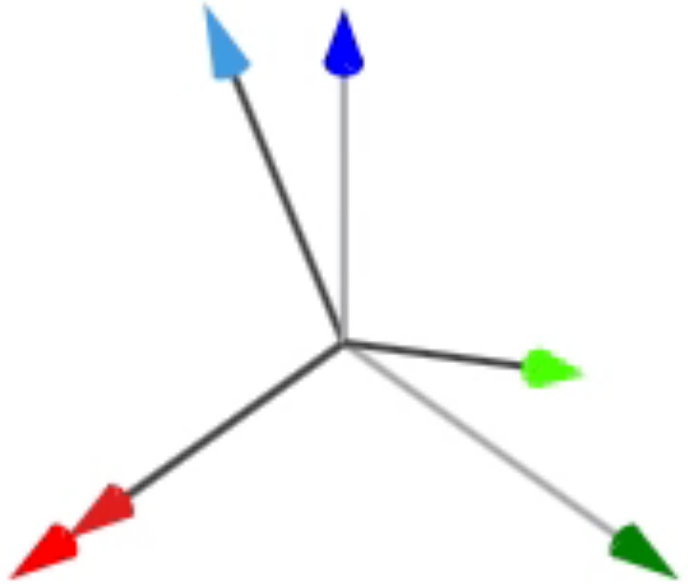


$\mathbf{R}_y^E(90)$



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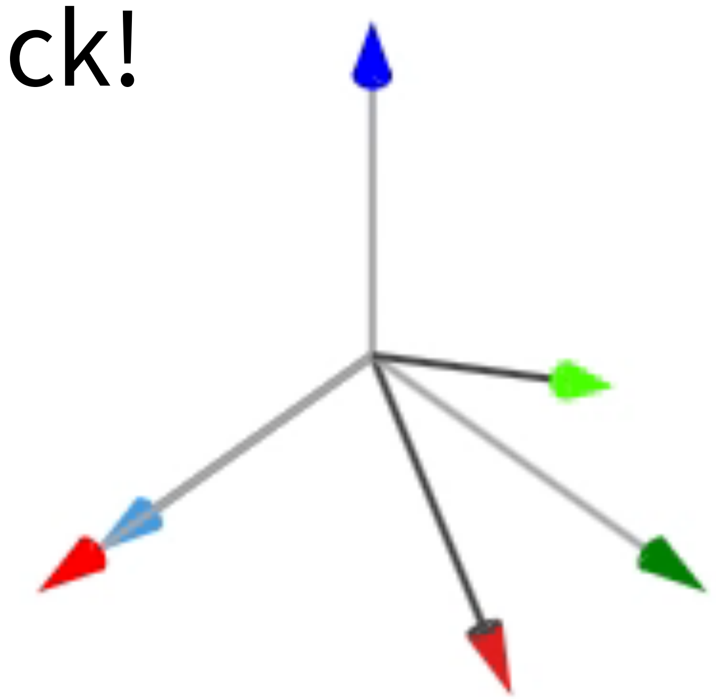
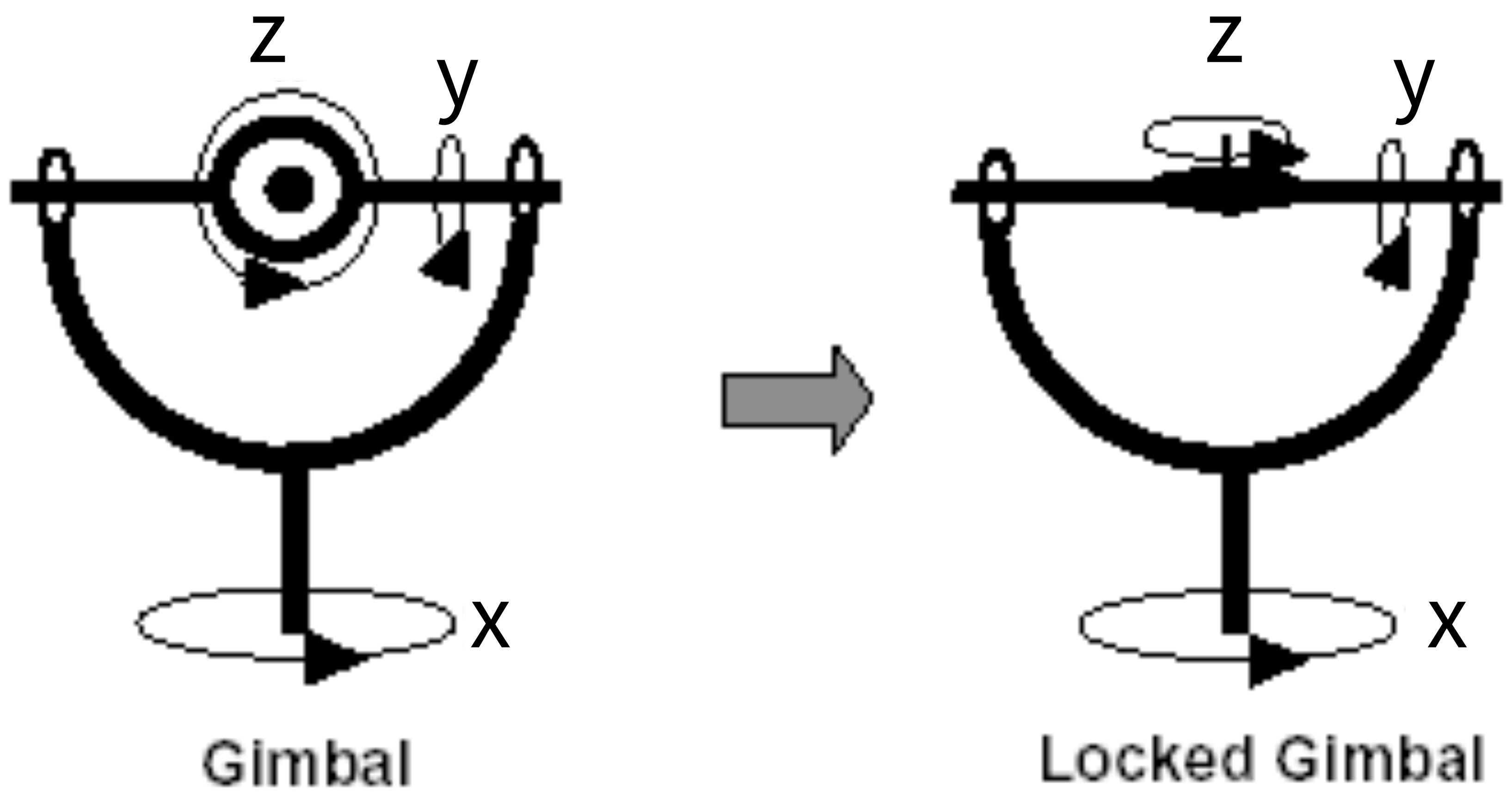
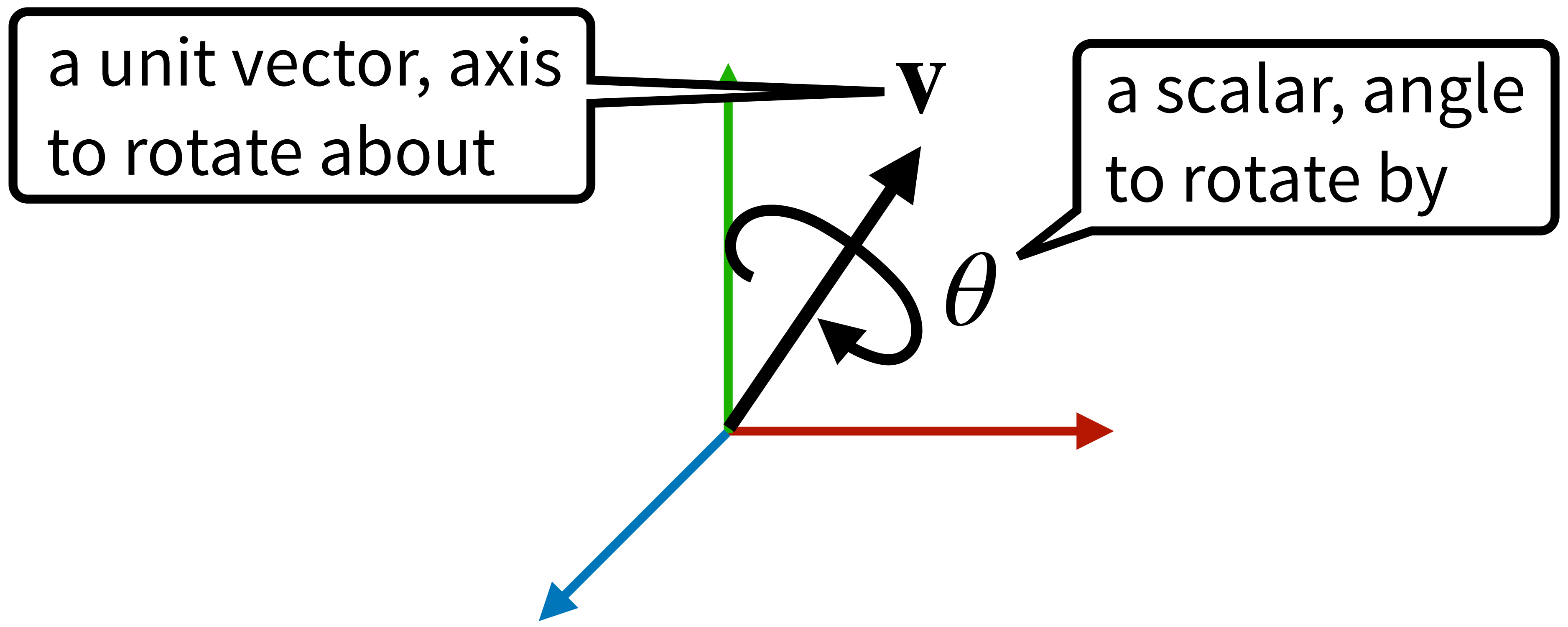




Axis angle

Represent orientation as a scalar and a vector $(\theta, \mathbf{v})$



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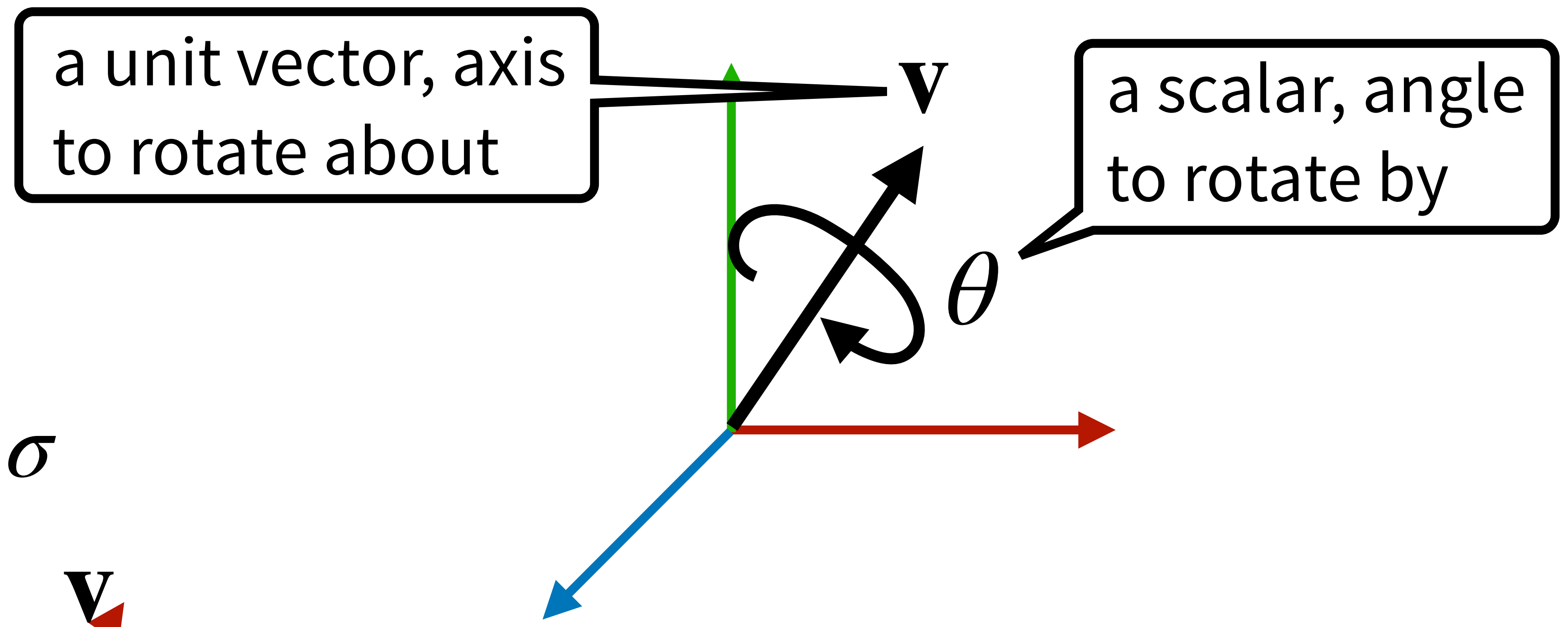
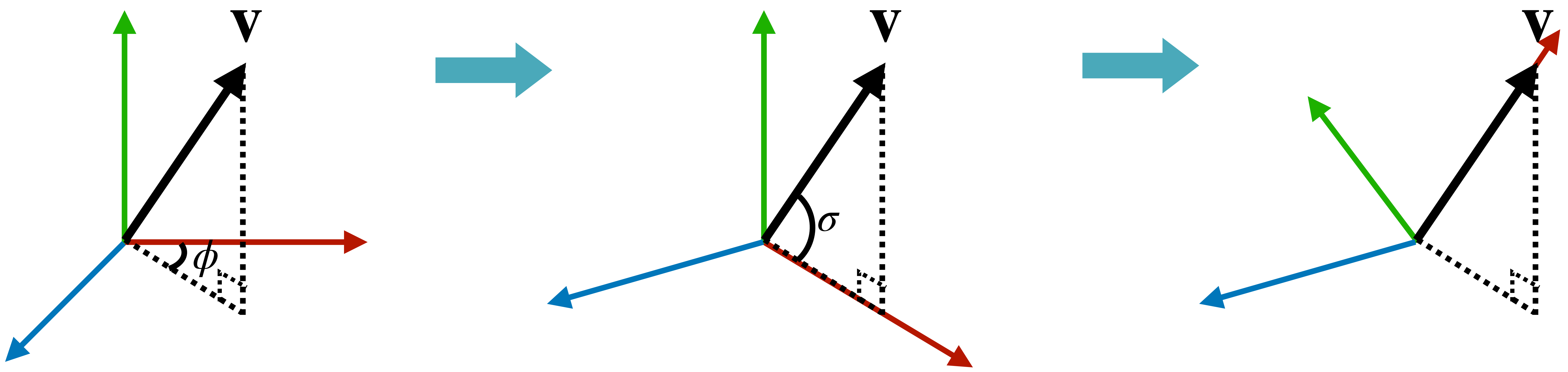


$\mathbf{R}_y(\phi)$: rotate about y by ϕ

$\mathbf{R}_z(\sigma)$: rotate about z by σ



Now that x-axis is aligned with $\mathbf{v}$, we can rotate θ about x axis, $\mathbf{R}_x(\theta)$.

$$\mathbf{R}_y(\phi)\mathbf{R}_z(\sigma)\mathbf{R}_x(\theta) = \mathbf{R}_\mathbf{v}(\theta) = \begin{bmatrix} \cos\theta + v_x^2(1-\cos\theta) & v_xv_y(1-\cos\theta) - v_z\sin\theta & v_xv_z(1-\cos\theta) + v_y\sin\theta \\ v_yv_x(1-\cos\theta) + v_z\sin\theta & \cos\theta + v_y^2(1-\cos\theta) & v_yv_z(1-\cos\theta) - v_x\sin\theta \\ v_zv_x(1-\cos\theta) - v_y\sin\theta & v_zv_y(1-\cos\theta) + v_x\sin\theta & \cos\theta + v_z^2(1-\cos\theta) \end{bmatrix}$$

Axis angle

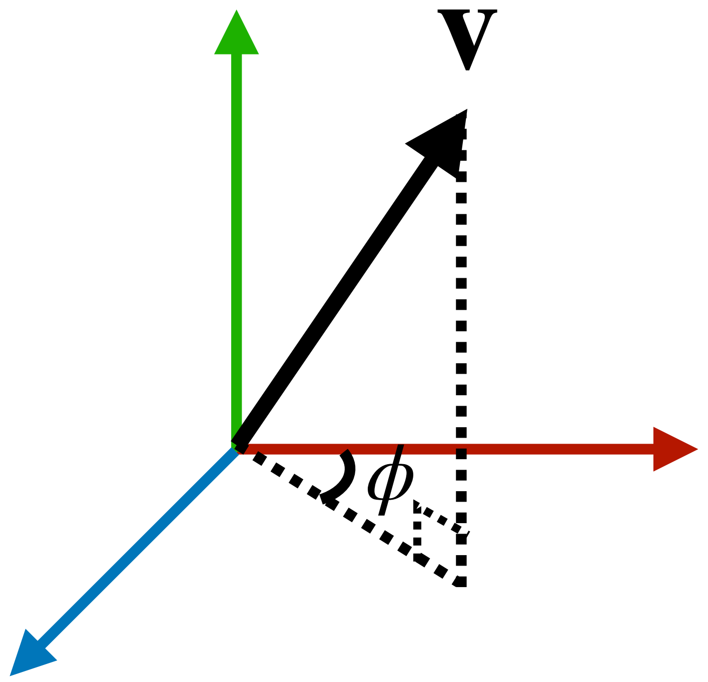
Represent orientation as a scalar and a vector $(\theta, \mathbf{v})$

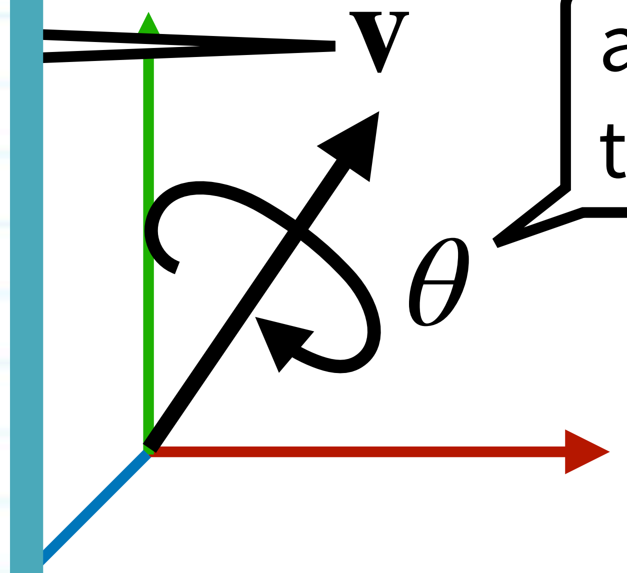
What is the 3D rotation matrix $\mathbf{R}$ for a rotation by θ about axis $\mathbf{k}$?

$$
\begin{aligned}
\mathbf{v}_{\text{rot}} &= \mathbf{v}_{\|} + \mathbf{v}_{\perp\text{rot}} \\
&= \left(\mathbf{v} - \mathbf{v}_{\perp}\right) + \mathbf{v}_{\perp\text{rot}} \\
&= \left(\mathbf{v} + \mathbf{k} \times (\mathbf{k} \times \mathbf{v})\right) + \left(-\cos\theta\, \mathbf{k} \times (\mathbf{k} \times \mathbf{v}) + \sin\theta\, \mathbf{k} \times \mathbf{v}\right) \\
&= \mathbf{v} + (1 - \cos\theta)\, \mathbf{k} \times (\mathbf{k} \times \mathbf{v}) + \sin\theta\, \mathbf{k} \times \mathbf{v} \\
&= \left[\mathbf{I} + (1 - \cos\theta)\, \mathbf{K}^2 + \sin\theta\, \mathbf{K}\right] \mathbf{v} \quad \text{where} \quad \mathbf{K} \equiv \text{"}\mathbf{k}\times\text{"} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \\
&= \mathbf{R}\,\mathbf{v}
\end{aligned}
$$

We just derived the famous Rodrigues' Formula!!!

$\mathbf{R}_y(\phi)$: rot





a scalar, angle to rotate by

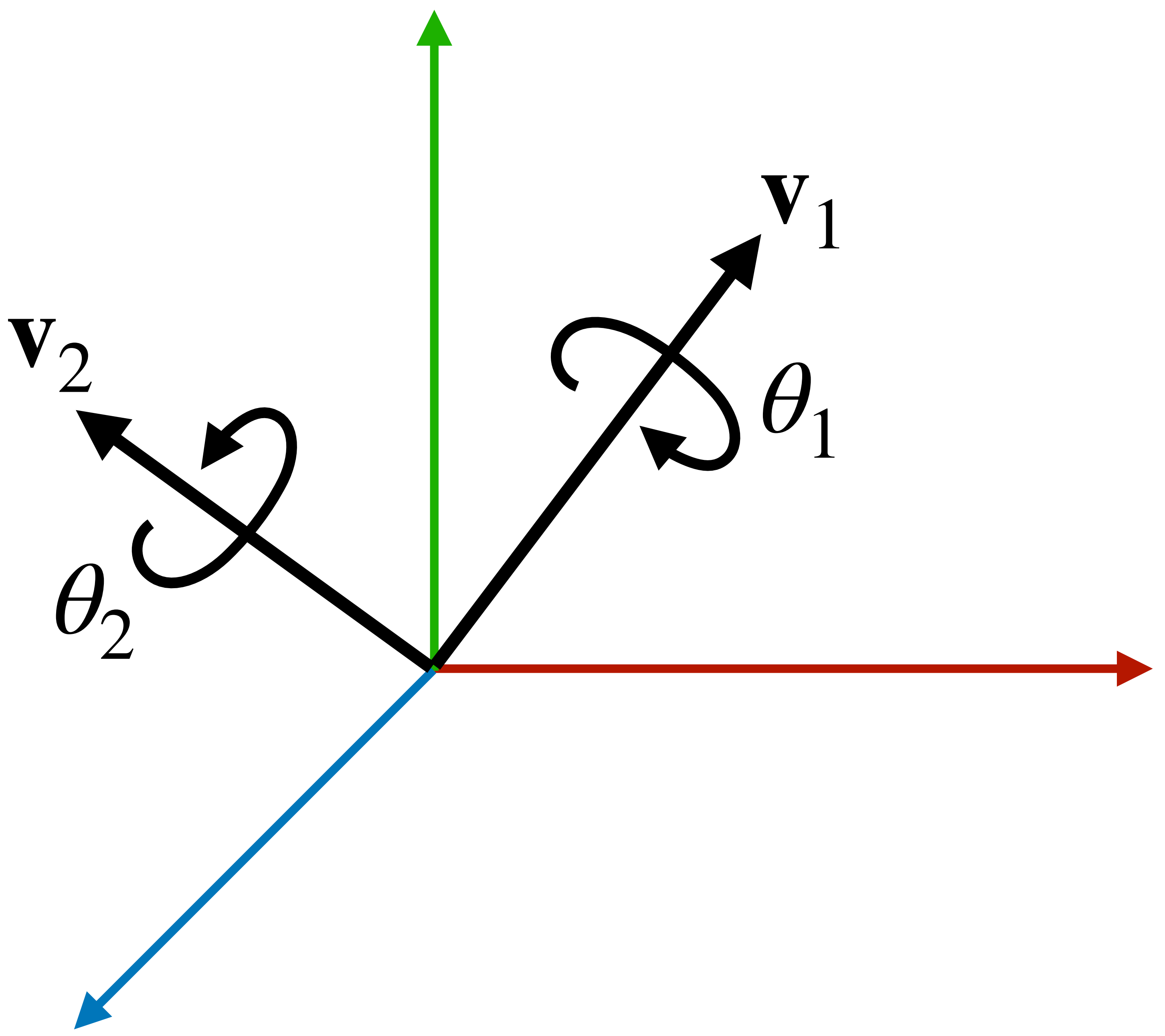
axis is aligned

an rotate θ

, $\mathbf{R}_x(\theta)$.

$$
\mathbf{R}_y(\phi)\mathbf{R}_z(\sigma)\mathbf{R}_x(\theta) = \mathbf{R}_{\mathbf{v}}(\theta) = \begin{bmatrix} \cos\theta + v_x^2(1-\cos\theta) & v_x v_y(1-\cos\theta) - v_z\sin\theta & v_x v_z(1-\cos\theta) + v_y\sin\theta \\ v_y v_x(1-\cos\theta) + v_z\sin\theta & \cos\theta + v_y^2(1-\cos\theta) & v_y v_z(1-\cos\theta) - v_x\sin\theta \\ v_z v_x(1-\cos\theta) - v_y\sin\theta & v_z v_y(1-\cos\theta) + v_x\sin\theta & \cos\theta + v_z^2(1-\cos\theta) \end{bmatrix}
$$

Axis angle interpolation



- **Scalar interpolation:** $\theta_k = (1-k)\theta_1 + k\theta_2$**,** **where** $0 \leq k \leq 1$
- **Vector interpolation is more involved**

Axis angle interpolation

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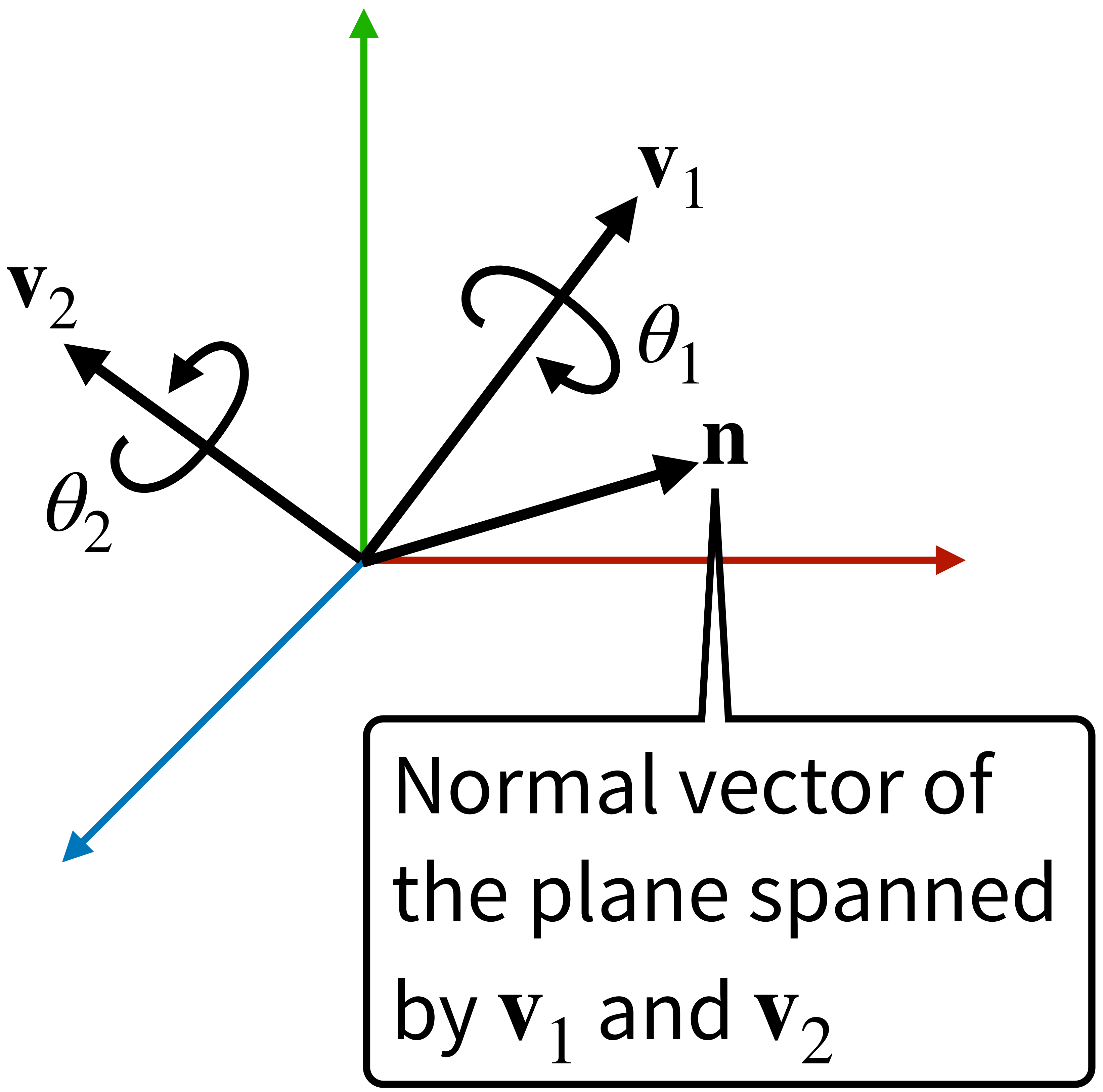

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$$

Axis angle interpolation

- **Scalar interpolation:** $\theta_k = (1-k)\theta_1 + k\theta_2$, **where** $0 \leq k \leq 1$
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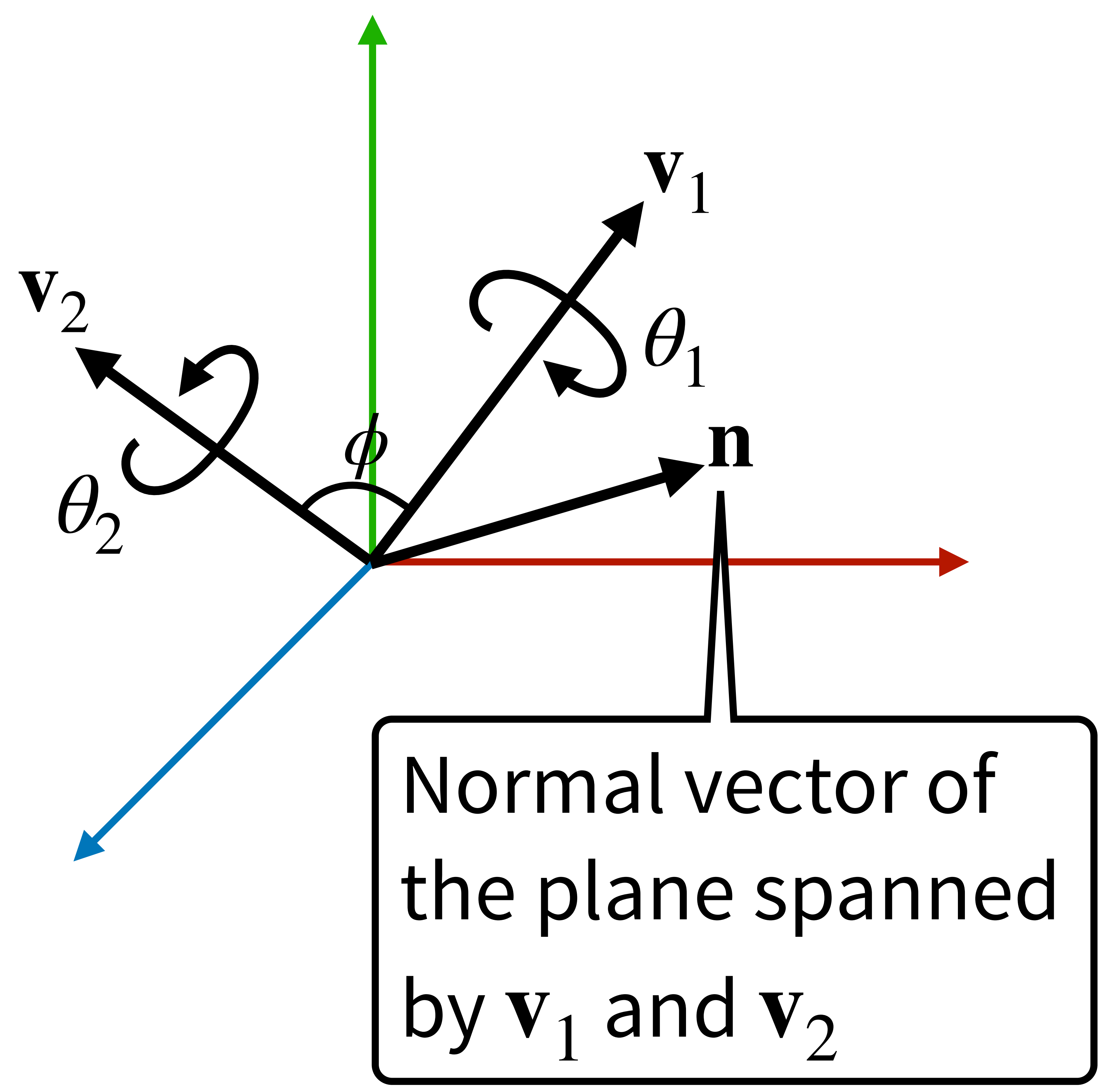
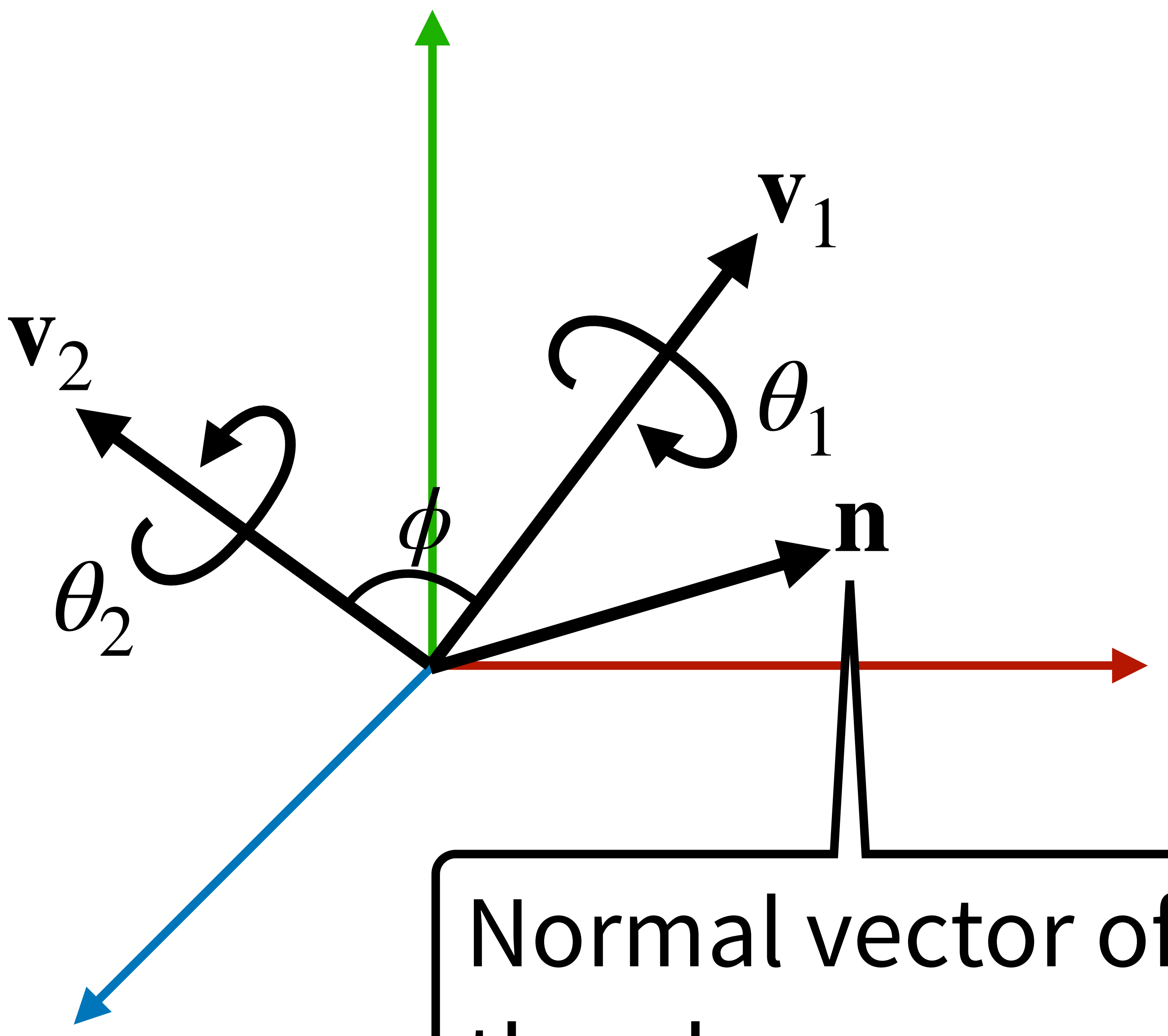


$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$$

$$\phi = \cos^{-1}(\mathbf{v}_1 \cdot \mathbf{v}_2)$$

Axis angle interpolation



Normal vector of the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$

- **Scalar interpolation:** $\theta_k = (1-k)\theta_1 + k\theta_2$, where $0 \le k \le 1$
- **Vector interpolation is more involved**

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$$

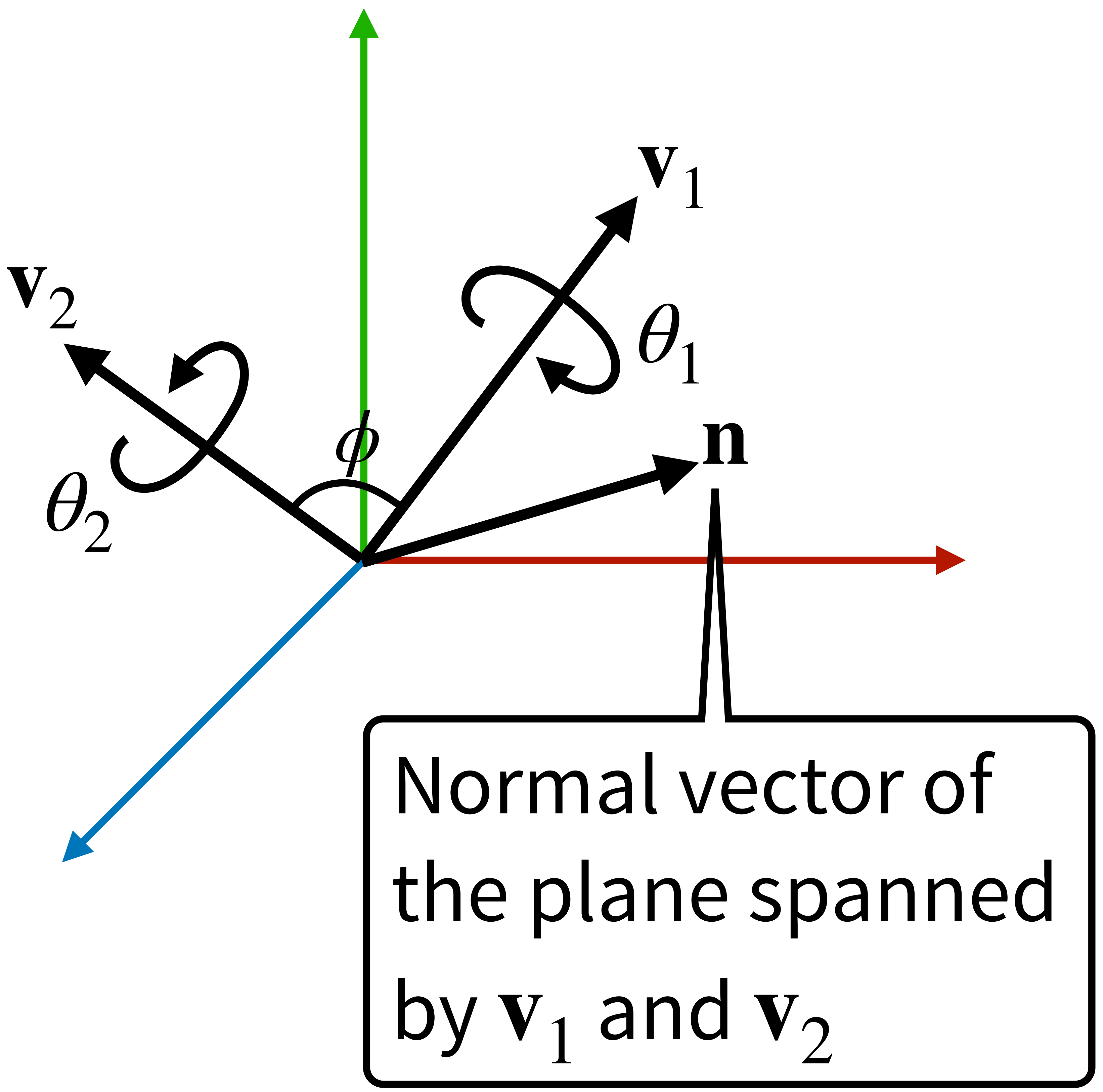
$$\phi = \cos^{-1}(\mathbf{v}_1 \cdot \mathbf{v}_2)$$

$$\mathbf{v}_k = \mathbf{R}_\mathbf{n}(k\phi)\mathbf{v}_1$$

Convert axis angle $(k\phi, \mathbf{n})$ to matrix $\mathbf{R}_\mathbf{n}(k\phi)$

Axis angle interpolation

- **Scalar interpolation:** $\theta_k = (1-k)\theta_1 + k\theta_2$, **where** $0 \le k \le 1$
- **Vector interpolation is more involved**



$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$$

$$\phi = \cos^{-1}(\mathbf{v}_1 \cdot \mathbf{v}_2)$$

$$\mathbf{v}_k = \mathbf{R}_\mathbf{n}(k\phi)\mathbf{v}_1 = \frac{\sin\big((1-k)\phi\big)}{\sin\phi}\mathbf{v}_1 + \frac{\sin(k\phi)}{\sin\phi}\mathbf{v}_2$$

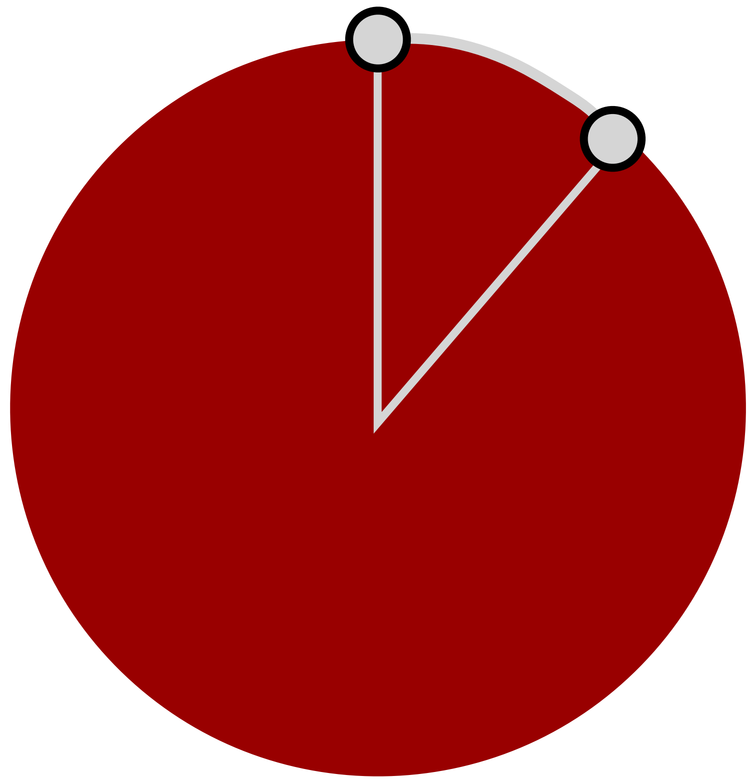
Or, we can use Spherical Linear Interpolation (SLERP) formula.

Convert axis angle $(k\phi, \mathbf{n})$ to matrix $\mathbf{R}_\mathbf{n}(k\phi)$

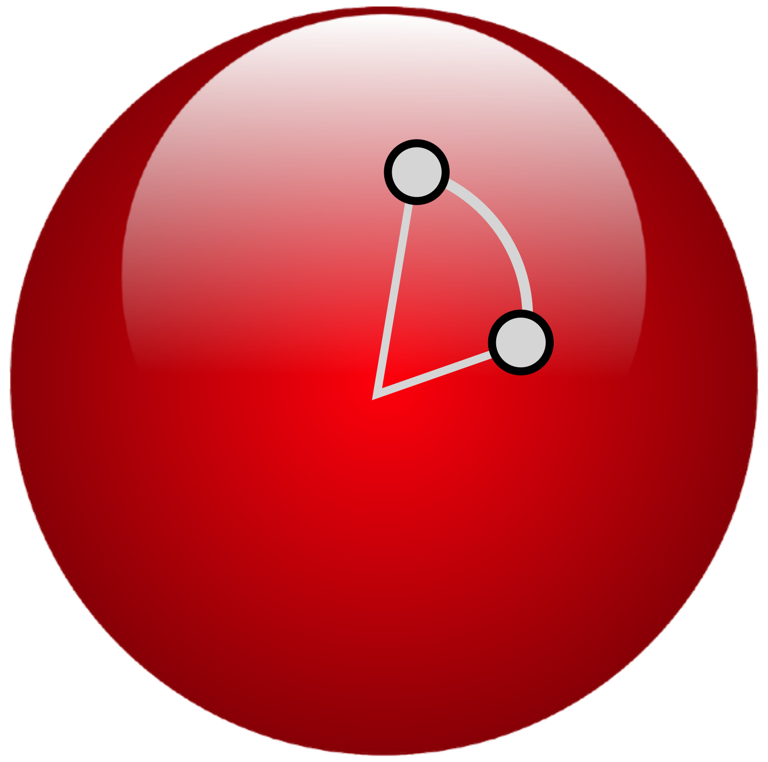
Properties of axis angle

- Easy to compose? ✗ Must convert back to matrix form.
- Easy to interpolate? ✓
- Easy to enforce joint limits? ✓
- Avoid gimbal lock? ✓ It rotates all three axes at once.

Quaternion: geometric view



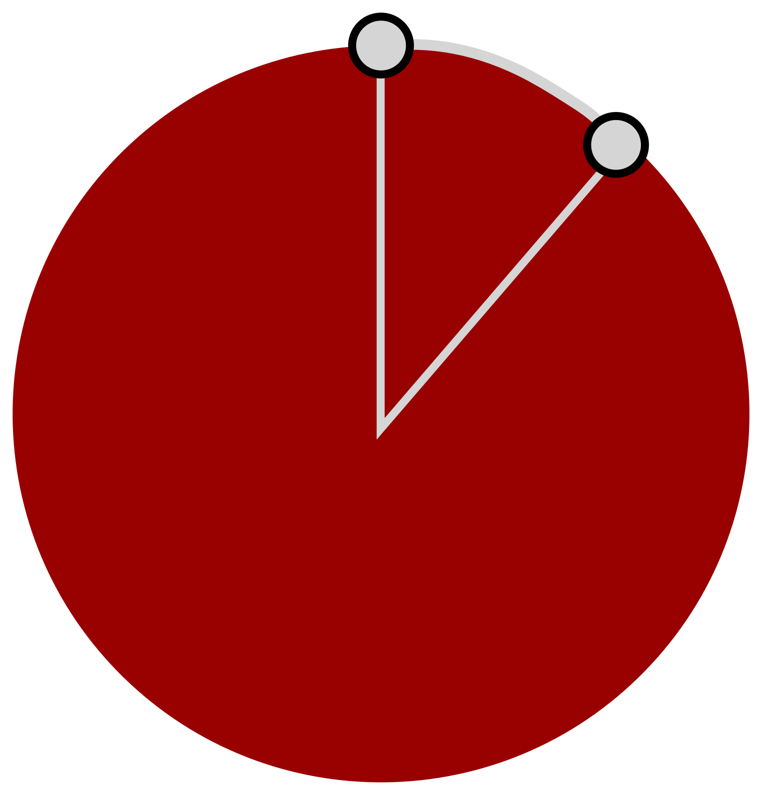
1-angle rotation can be represented by a unit circle



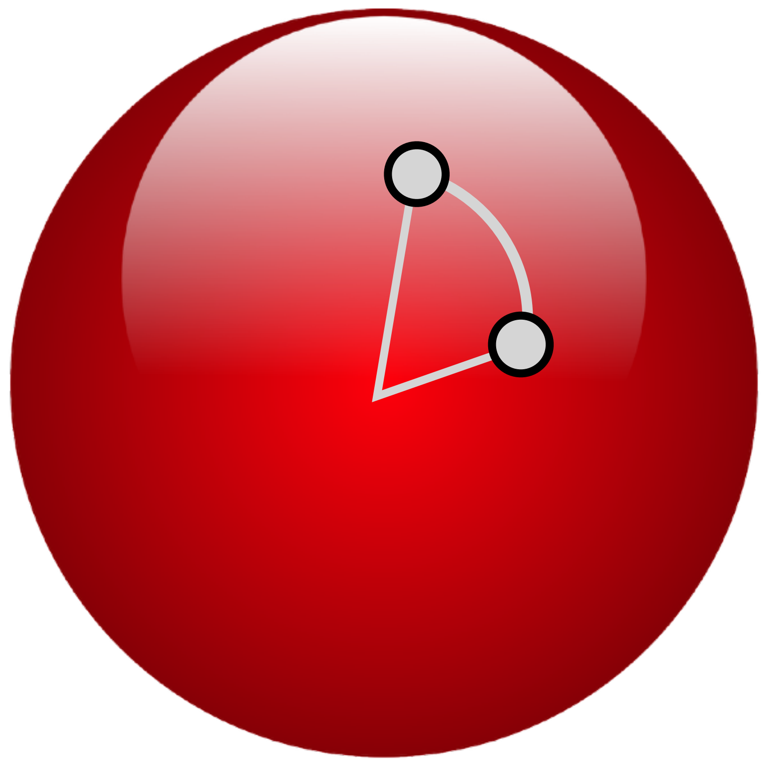
2-angle rotation can be represented by a unit sphere

What about 3-angle rotation?

Quaternion: geometric view



1-angle rotation can be
represented by a unit circle



2-angle rotation can be
represented by a unit sphere

What about 3-angle rotation?

A unit quaternion is a point on the 4D sphere!

Quaternion: algebraic view

- **A quaternion can be represented by a 4-tuple of real numbers:** w, x, y, z

$$\mathbf{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} \begin{array}{l} \text{scalar} \\ \text{3D vector} \end{array}$$

Quaternion: algebraic view

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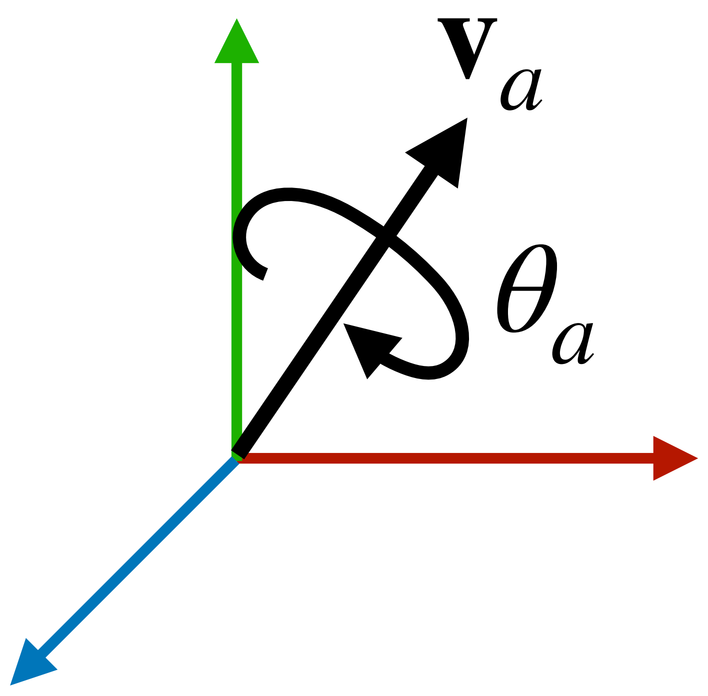
$$\mathbf{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} \begin{matrix} \text{scalar} \\ \text{3D vector} \end{matrix}$$

- **Same information as axis angles but in a different form:**



axis angle $(\theta_a, \mathbf{v}_a)$ ➡

$$\mathbf{q} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta_a}{2} \\ \sin\frac{\theta_a}{2}\mathbf{v}_a \end{bmatrix}$$

Quaternion basic definitions

- **Unit quaternion**

$$\|\mathbf{q}\| = \sqrt{x^2 + y^2 + z^2 + w^2} = 1$$

- **Inverse quaternion**

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\|\mathbf{q}\|} \qquad \mathbf{q}^* = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix}^* = \begin{bmatrix} w \\ -\mathbf{v} \end{bmatrix}$$

- **Identity**

$$\mathbf{q}\mathbf{q}^{-1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Quaternion multiplication

$$\mathbf{q}_1 \otimes \mathbf{q}_2 = \begin{bmatrix} w_1 \\ \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} w_2 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \\ w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \end{bmatrix}$$

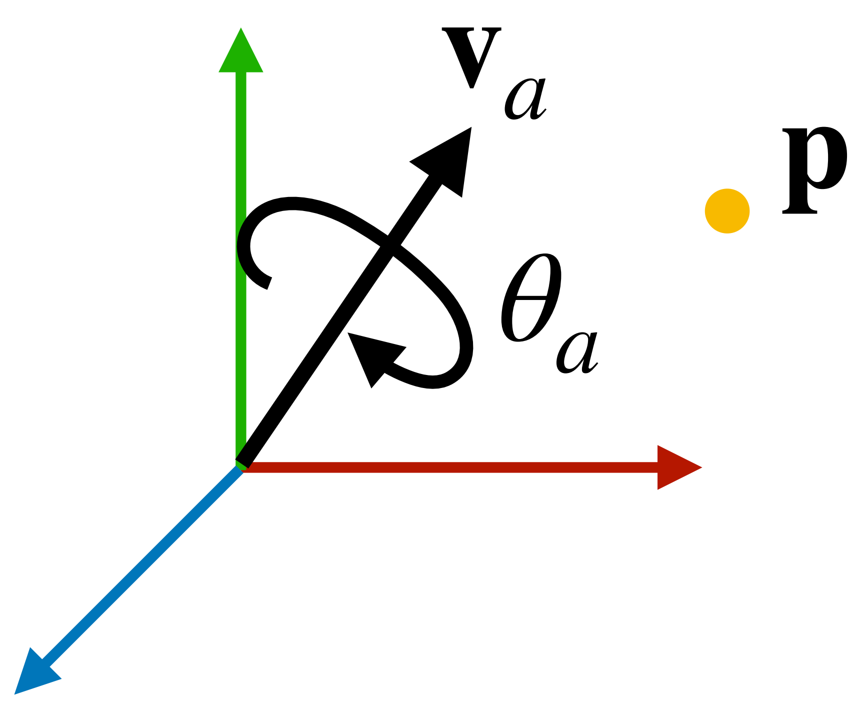
- **Commutativity** ✗

$$\mathbf{q}_1\mathbf{q}_2 \neq \mathbf{q}_2\mathbf{q}_1$$

- **Associativity** ✓

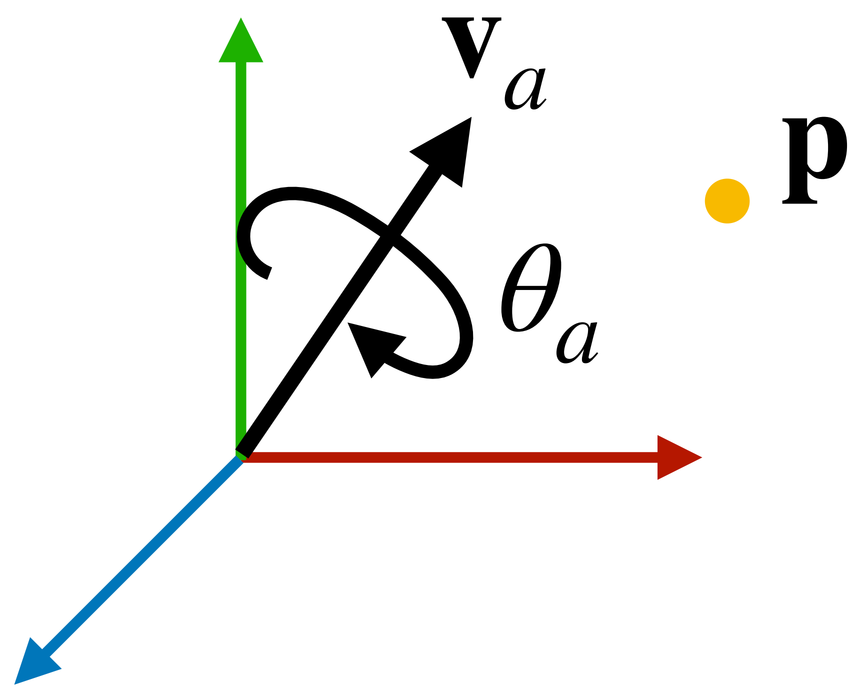
$$\mathbf{q}_1(\mathbf{q}_2\mathbf{q}_3) = (\mathbf{q}_1\mathbf{q}_2)\mathbf{q}_3$$

Quaternion rotation



Let a unit quaternion $\mathbf{q} \equiv \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix}$ be $\begin{bmatrix} \cos\frac{\theta_a}{2} \\ \sin\frac{\theta_a}{2}\mathbf{v}_a \end{bmatrix}$. Let a vector $\mathbf{q}_p \in \mathbb{R}^4$ be $\begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix}$, where $\mathbf{p}$ is a 3D point.

Quaternion rotation



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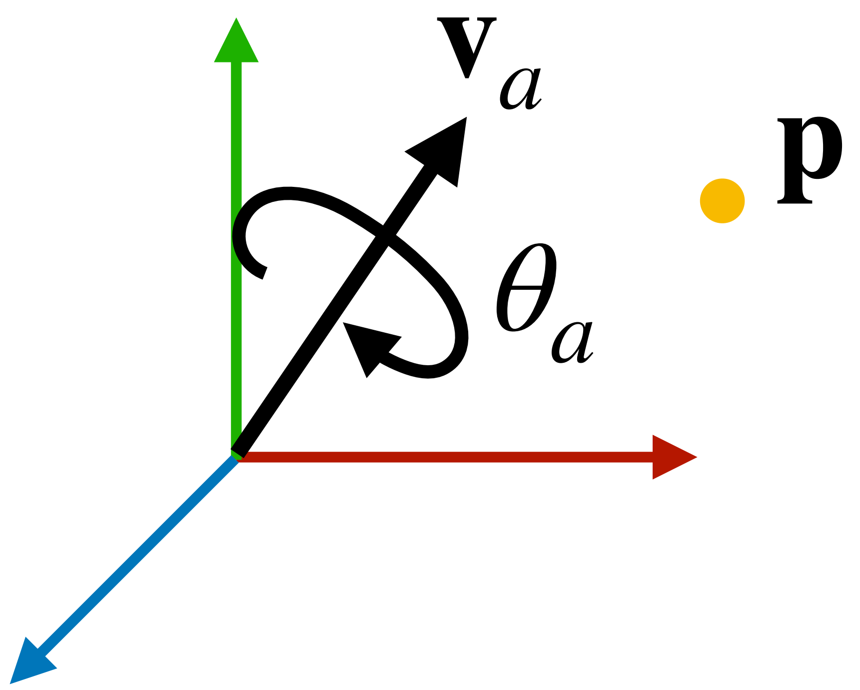
Then $\mathbf{p}'$ is the result of $\mathbf{p}$ rotating about $\mathbf{v}_a$ by θ_a, where

$$\mathbf{q}\mathbf{q}_p\mathbf{q}^{-1} = \begin{bmatrix} 0 \\ \mathbf{p}' \end{bmatrix}.$$

These are "quaternion multiplications", which product is in the form of $[0, \mathbf{p}']$

Proof: see Quaternions by Shoemaker

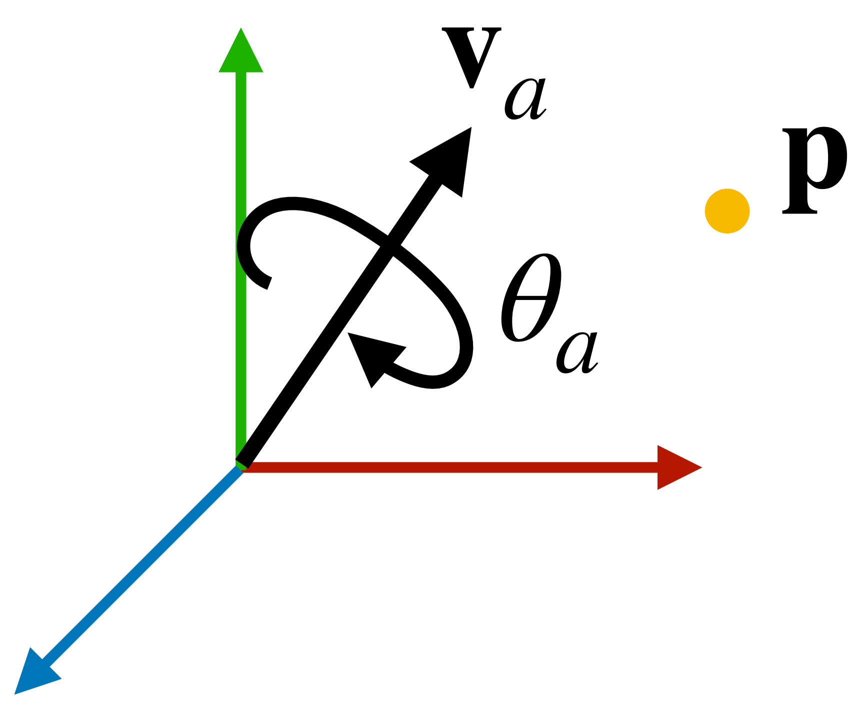
Quaternion rotation



$$\mathbf{q}\mathbf{q}_p\mathbf{q}^{-1} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \begin{bmatrix} w \\ -\mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ w(w\mathbf{p} - \mathbf{p} \times \mathbf{v}) + (\mathbf{p} \cdot \mathbf{v})\mathbf{v} + \mathbf{v} \times (w\mathbf{p} - \mathbf{p} \times \mathbf{v}) \end{bmatrix}$$

where $w = \cos\frac{\theta_a}{2}$ and $\mathbf{v} = \sin\frac{\theta_a}{2}\mathbf{v}_a$

Quaternion rotation



$$\mathbf{q}\mathbf{q}_p\mathbf{q}^{-1} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \begin{bmatrix} w \\ -\mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ w(w\mathbf{p} - \mathbf{p} \times \mathbf{v}) + (\mathbf{p} \cdot \mathbf{v})\mathbf{v} + \mathbf{v} \times (w\mathbf{p} - \mathbf{p} \times \mathbf{v}) \end{bmatrix}$$

where $w = \cos\dfrac{\theta_a}{2}$ and $\mathbf{v} = \sin\dfrac{\theta_a}{2}\mathbf{v}_a$

Express $w(w\mathbf{p} - \mathbf{p} \times \mathbf{v}) + (\mathbf{p} \cdot \mathbf{v})\mathbf{v} + \mathbf{v} \times (w\mathbf{p} - \mathbf{p} \times \mathbf{v})$ in terms of $\mathbf{v}_a$ and θ_a and compare it with

Convert the axis angle to a rotation matrix

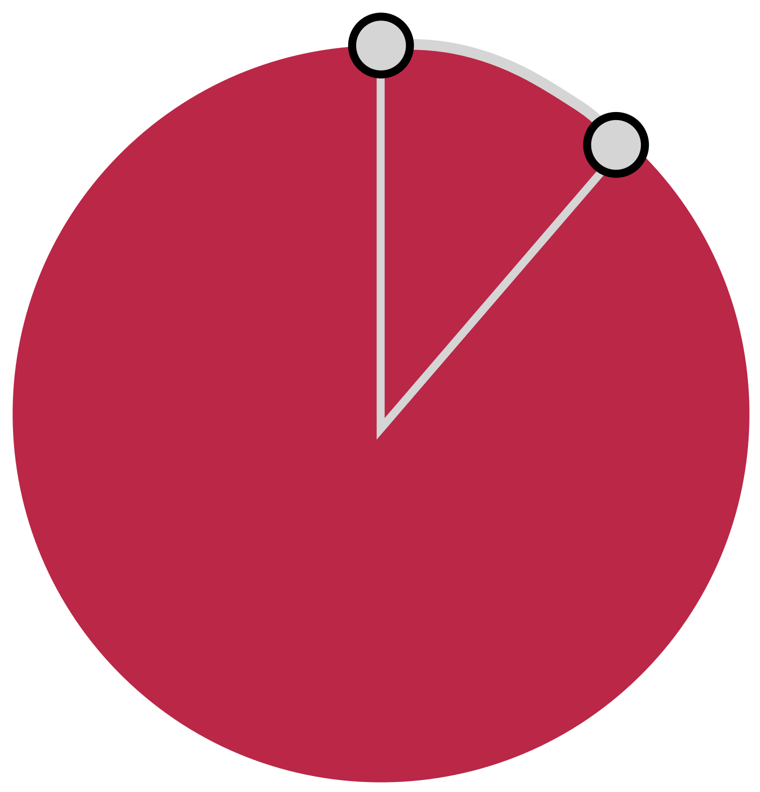
$$\mathbf{R}_{\mathbf{v}_a}(\theta_a)\mathbf{p} = \begin{bmatrix} \cos\theta_a + v_x^2(1-\cos\theta_a) & v_xv_y(1-\cos\theta_a) - v_z\sin\theta_a & v_xv_z(1-\cos\theta_a) + v_y\sin\theta_a \\ v_yv_x(1-\cos\theta_a) + v_z\sin\theta_a & \cos\theta_a + v_y^2(1-\cos\theta_a) & v_yv_z(1-\cos\theta_a) - v_x\sin\theta_a \\ v_zv_x(1-\cos\theta_a) - v_y\sin\theta_a & v_zv_y(1-\cos\theta_a) + v_x\sin\theta_a & \cos\theta_a + v_z^2(1-\cos\theta_a) \end{bmatrix}\mathbf{p}$$

Quaternion composition

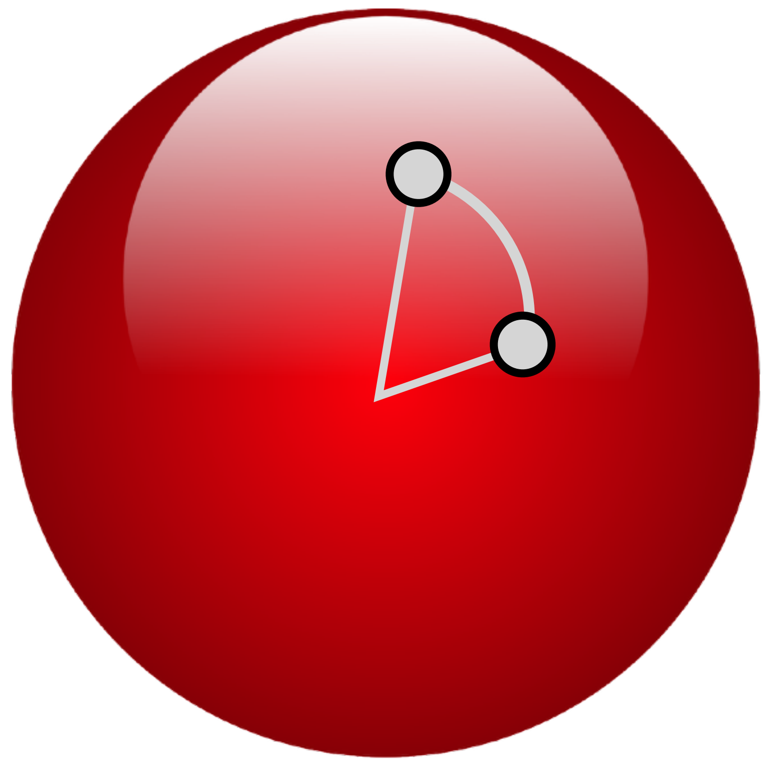
- If $\mathbf{q}_1$ and $\mathbf{q}_2$ are unit quaternions, the combined rotation of first rotating by $\mathbf{q}_1$ and then by $\mathbf{q}_2$ is equivalent to $\mathbf{q}_3 = \mathbf{q}_2\mathbf{q}_1$.
- Quaternion $\mathbf{q} = [w, x, y, z]$ can also be converted to a rotation matrix in homogeneous coordinate.

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} 1-2y^2-2z^2 & 2xy+2wz & 2xz-2wy & 0 \\ 2xy-2wz & 1-2x^2-2z^2 & 2yz+2wx & 0 \\ 2xz+2wy & 2yz-2wx & 1-2x^2-2y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Quaternion interpolation



1-angle rotation can be represented by a unit circle



2-angle rotation can be represented by a unit sphere

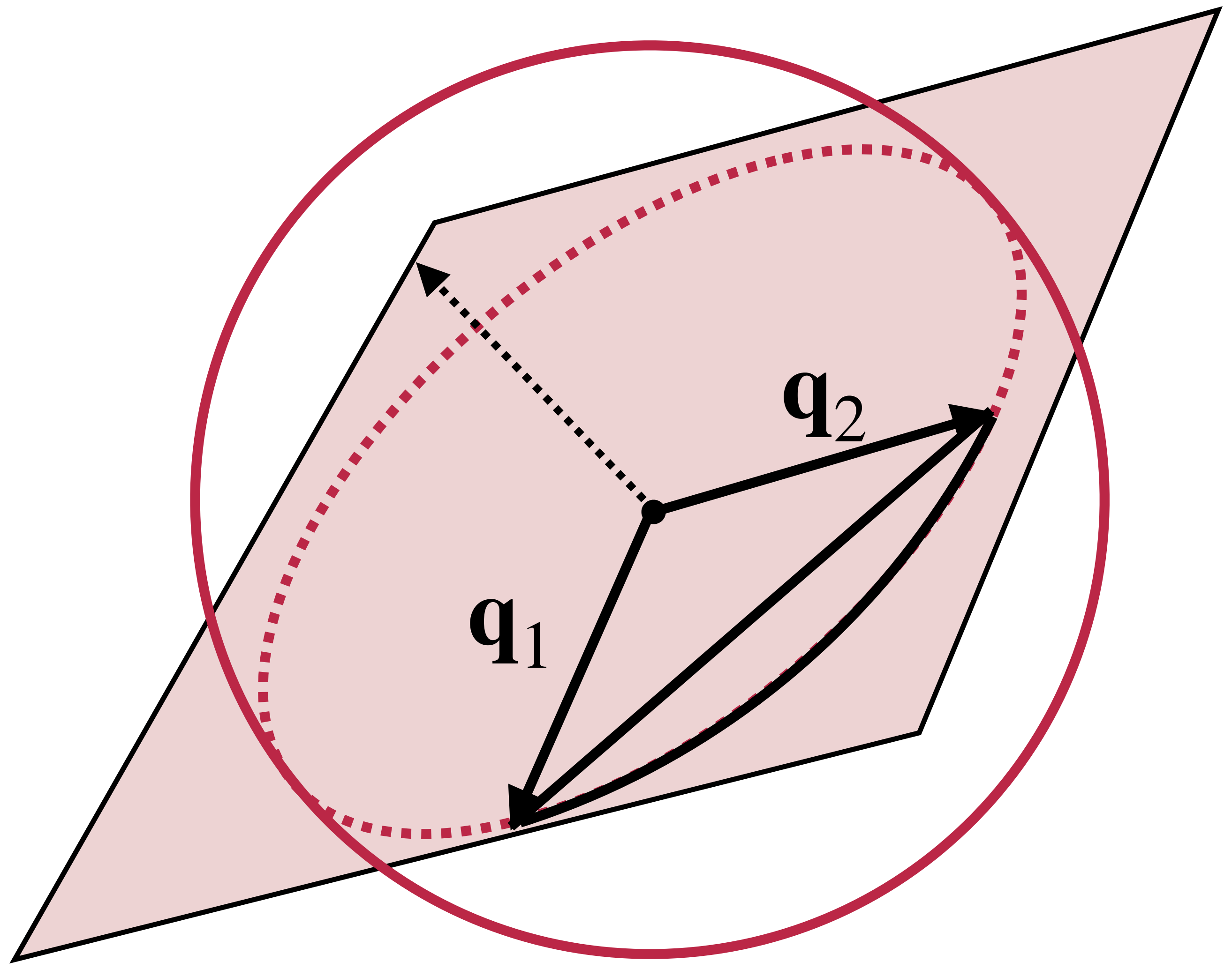
Interpolating quaternion means moving on the surface of 4-D sphere — move with constant angular velocity along the great circle between two points on the 4D unit sphere.

Quaternion interpolation

- Direct linear interpolation does not give motion with constant angular velocity.
- Convert $\mathbf{q}_1$ and $\mathbf{q}_2$ to two axis angles and use Spherical Linear Interpolation (SLERP) to interpolate.

$$\theta_a = 2\cos^{-1}(w), \quad \mathbf{v}_a = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

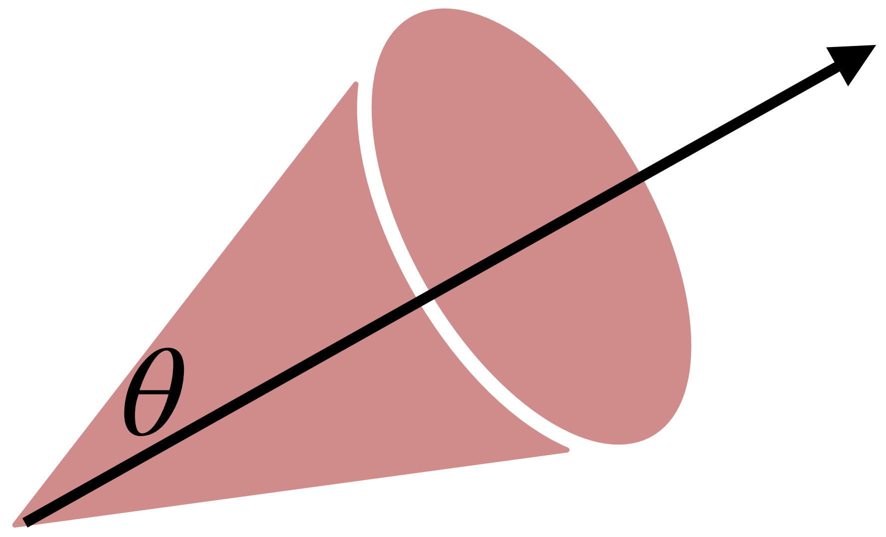
- Use Quaternion SLEPR: $\mathbf{q}_k = \mathbf{q}_1(\mathbf{q}_1^{-1}\mathbf{q}_2)^k$



Interpolating quaternion means moving on the surface of 4-D sphere — move with constant angular velocity along the great circle between two points on the 4D unit sphere.

Quaternion constraints

- **Given a quaternion $\mathbf{q} = [w, x, y, z]$, we can enforce**
  - **Cone constraint, e.g. constrain orientations within a cone aligned with x-axis.**



$$\frac{1 - \cos\theta}{2} = y^2 + z^2$$

Quaternion constraints

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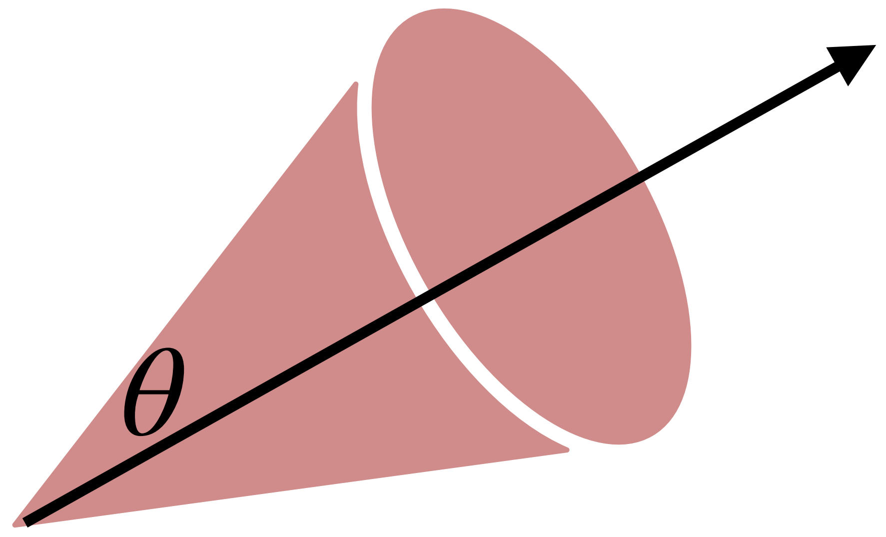
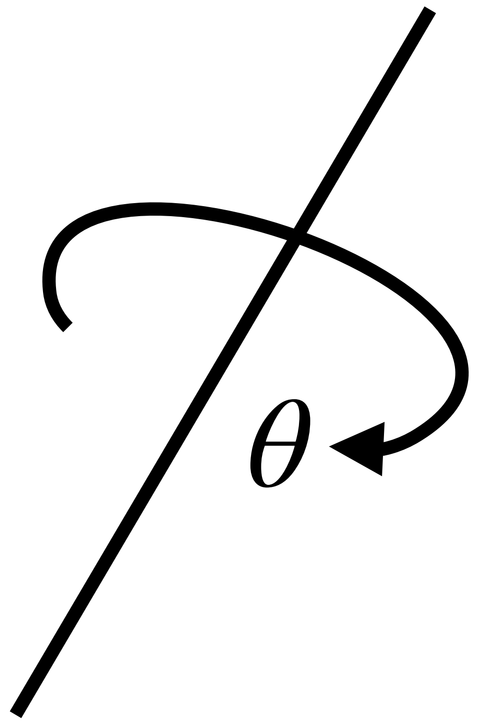


$$\frac{1 - \cos\theta}{2} = y^2 + z^2$$

  - Twist constraint, e.g. constrain the twist about x-axis.



$$\tan\frac{\theta}{2} = \frac{x}{w}$$

Properties of quaternion

- Easy to compose? ✓
- Easy to interpolate? ✓
- Easy to enforce joint limits? ✓
- Avoid Gimbal lock? ✓
- Anything bad about quaternion? Need to be normalized.

Quiz

- **What representation is the best for each of the following requirements?**
  - **Intuitive animation input:** Euler/fixed angles
  - **Enforce joint limits:** Euler/fixed angles, axis angle, quaternion
  - **Interpolation:** axis angle, quaternion
  - **Composition:** rotation matrix, quaternion
  - **Avoid gimbal lock:** rotation matrix, axis angle, quaternion
  - **Rendering:** rotation matrix

Back to rigid bodies…

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{p}(t) \\ \mathbf{L}(t) \end{bmatrix}$$

$$\dot{\mathbf{Y}}(t) = \begin{bmatrix} \mathbf{v}(t) \\ [\boldsymbol{\omega}(t)]\mathbf{R}(t) \\ \mathbf{f}(t) \\ \boldsymbol{\tau}(t) \end{bmatrix}$$

Given the current state $\mathbf{Y}_n$, how to evaluate $\dot{\mathbf{Y}}_n$, assuming the mass, M, and inertia in the body space, $\mathbf{I}_b$ are known?

$$\mathbf{v}(t) = \frac{\mathbf{p}}{M}$$

$$[\boldsymbol{\omega}(t)]\mathbf{R}(t) = [\mathbf{I}(t)^{-1}\mathbf{L}(t)]\mathbf{R}(t) = [\mathbf{R}^T(t)\mathbf{I}_b^{-1}\mathbf{R}(t)\mathbf{L}(t)]\mathbf{R}(t)$$

How to compute $\mathbf{f}(t)$?

Evaluate all the forces, $\mathbf{f}_1, \cdots, \mathbf{f}_n$ currently applied on the rigid body.

$$\boldsymbol{\tau}(t) = \sum_{i=1}^{n} \big(\mathbf{r}_i(t) - \mathbf{x}(t)\big) \times \mathbf{f}_i(t)\big)$$

Back to rigid bodies…

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \cancel{\mathbf{R}(t)} \\ \mathbf{p}(t) \\ \mathbf{L}(t) \end{bmatrix} \quad \mathbf{q}(t)$$

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Given the current state $\mathbf{Y}_n$, how to evaluate $\dot{\mathbf{Y}}_n$, assuming the mass, M, and inertia in the body space, $\mathbf{I}_b$ are known?

$$\mathbf{v}(t) = \frac{\mathbf{p}}{M}$$

$$\cancel{[\boldsymbol{\omega}(t)]\mathbf{R}(t) = [\mathbf{I}(t)^{-1}\mathbf{L}(t)]\mathbf{R}(t) = [\mathbf{R}^{T}}$$

$$\boxed{\begin{aligned} \dot{\mathbf{q}}(t) &= \frac{1}{2}\begin{bmatrix} 0 \\ \boldsymbol{\omega}(t) \end{bmatrix} \otimes \mathbf{q}(t) \\ \boldsymbol{\omega}(t) &= \mathbf{R}^T(t)\mathbf{I}_b^{-1}\mathbf{R}(t)\mathbf{L}(t) \\ \mathbf{R}(t) &= \mathrm{quatToMatrix}(\mathbf{q}(t)) \end{aligned}}$$

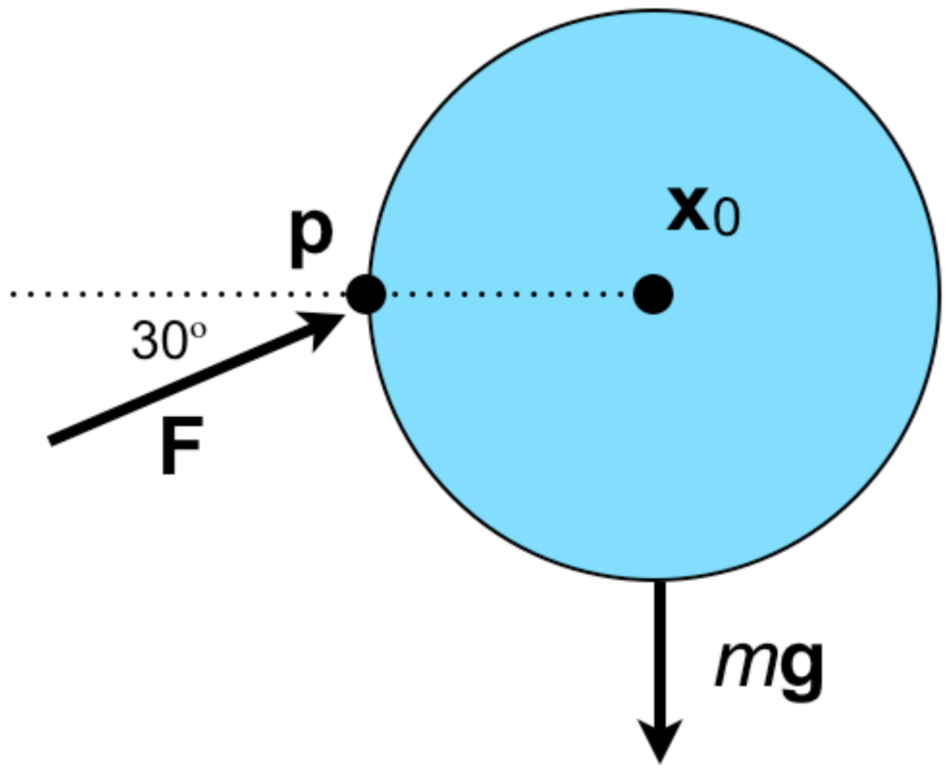
How to compute $\mathbf{f}(t)$?

Evaluate all the forces, $\mathbf{f}_1, \cdots, \mathbf{f}_n$ currently applied on the rigid body.

$$\boldsymbol{\tau}(t) = \sum_{i=1}^{n} \big(\mathbf{r}_i(t) - \mathbf{x}(t)\big) \times \mathbf{f}_i(t)\big)$$

Quiz

- Consider a 3D sphere with radius 1m, mass 1kg, and inertia I_{body}. The initial linear and angular velocity are both zero. The initial position and the initial orientation are x_0 and R_0. The forces applied on the sphere include gravity (g) and an initial push F applied at point p. Note that F is only applied for one time step at t_0. If we use Explicit Euler method with time step h to integrate, what are the position and the orientation of the sphere at t_2? Use Quaternion to represent orientation.



$\mathbf{x}_0 = (0, 0, 0)$

$$\mathbf{R}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{p} = (-1, 0, 0)$

$\mathbf{F} = (4\cos(30^\circ), 4\sin(30^\circ), 0)$

$m = 1$

$$\mathbf{I}_{body} = \begin{pmatrix} 2/5 & 0 & 0 \\ 0 & 2/5 & 0 \\ 0 & 0 & 2/5 \end{pmatrix}$$

Additional reading

- Euler/fixed angles Java script:
  - https://www.mecademic.com/resources/Euler-angles/Euler-angles
- Quaternions by Shoemake:
  - http://www.cs.ucr.edu/~vbz/resources/quatut.pdf
- Quaternions, Interpolation and Animation
  - https://web.mit.edu/2.998/www/QuaternionReport1.pdf
- Spherical Linear Interpolation (SLERP)
  - https://en.wikipedia.org/wiki/Slerp