

Lecture 3

Particle Systems & ODEs

FUNDAMENTALS OF COMPUTER GRAPHICS
Animation & Simulation

Stanford CS248B, Fall 2022

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Announcements

- HW1 out last Thursday
 - Due Thursday
 - Last question is related to material from this class (ODE integrators)
- Grade breakdown for course:

<u>Material</u>	<u>Weight</u>
HW1	6%
HW2	6%
HW3	6%
HW4	6%
HW5	6%
P1	12%
P2	12%
P3	13%
P4	13%
Final exam	20%

Overview

- **Particle Systems**
- **Ordinary Differential Equations (ODEs)**
 - Reduction to first-order form. Autonomous vs nonautonomous. Linearization. Model equation. Stability.
 - Time-stepping schemes (forward Euler, backward Euler, symplectic Euler, midpoint, ...).
 - Stiffness and stability.

Particle Systems

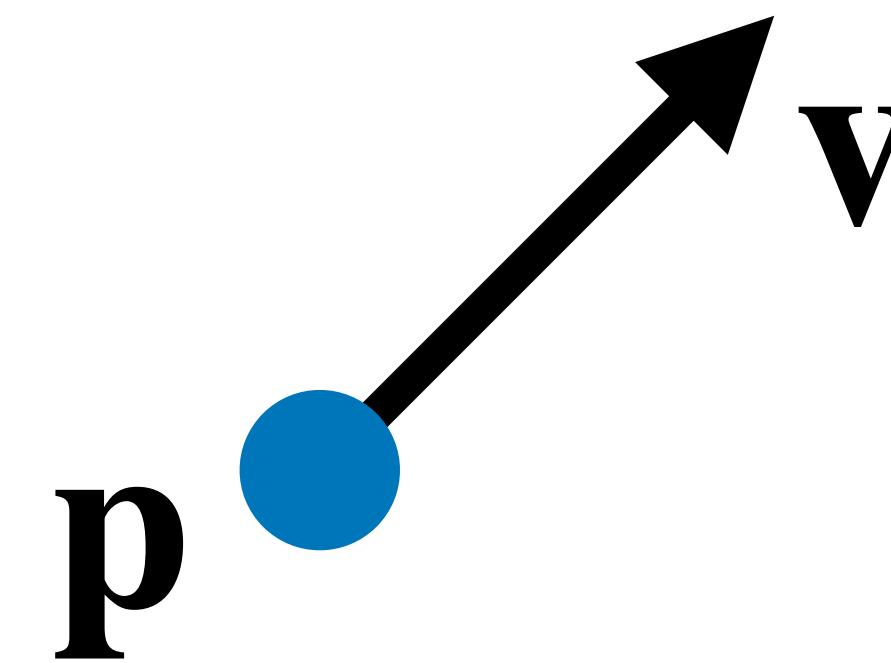
One lousy particle

- Consider a single particle in n dimensions
- Kinematic model of particle motion (*vs dynamics which involves forces*)
- Position, $\mathbf{p} = \mathbf{p}(t) \in \mathbb{R}^n$
- Velocity, $\mathbf{v} = \mathbf{v}(t) \in \mathbb{R}^n$
- Differential relationship

$$\mathbf{v} = \frac{d\mathbf{p}}{dt} = \dot{\mathbf{p}}$$

- Example: 2D ball

$$\mathbf{p}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \text{ and } \mathbf{v}(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$$



Particle systems

- Collection of N particles in \mathbb{R}^n
 - Created or destroyed by various means.

- Indexed particle values:

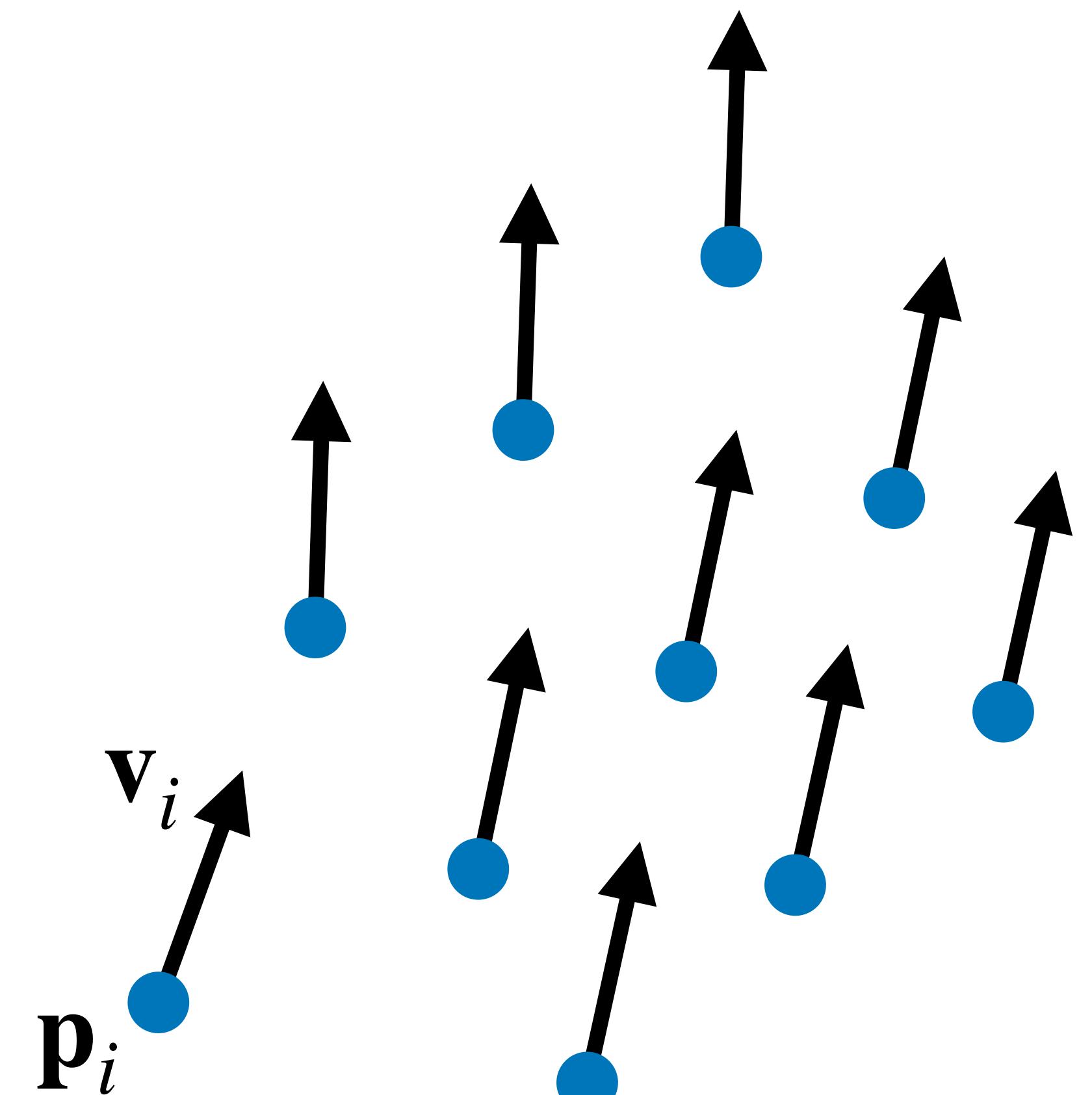
$$\mathbf{p}_i = \mathbf{p}_i(t) \in \mathbb{R}^n, \quad \text{for } i = 1, \dots, N.$$

$$\mathbf{v}_i = \mathbf{v}_i(t) \in \mathbb{R}^n, \quad \text{for } i = 1, \dots, N.$$

- Vector form: $\mathbf{v} = \dot{\mathbf{p}} \in \mathbb{R}^{nN}$ where

$$\mathbf{p} = \begin{pmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_i \\ \vdots \\ \mathbf{p}_N \end{pmatrix}$$

$$\text{and } \mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_N \end{pmatrix}$$



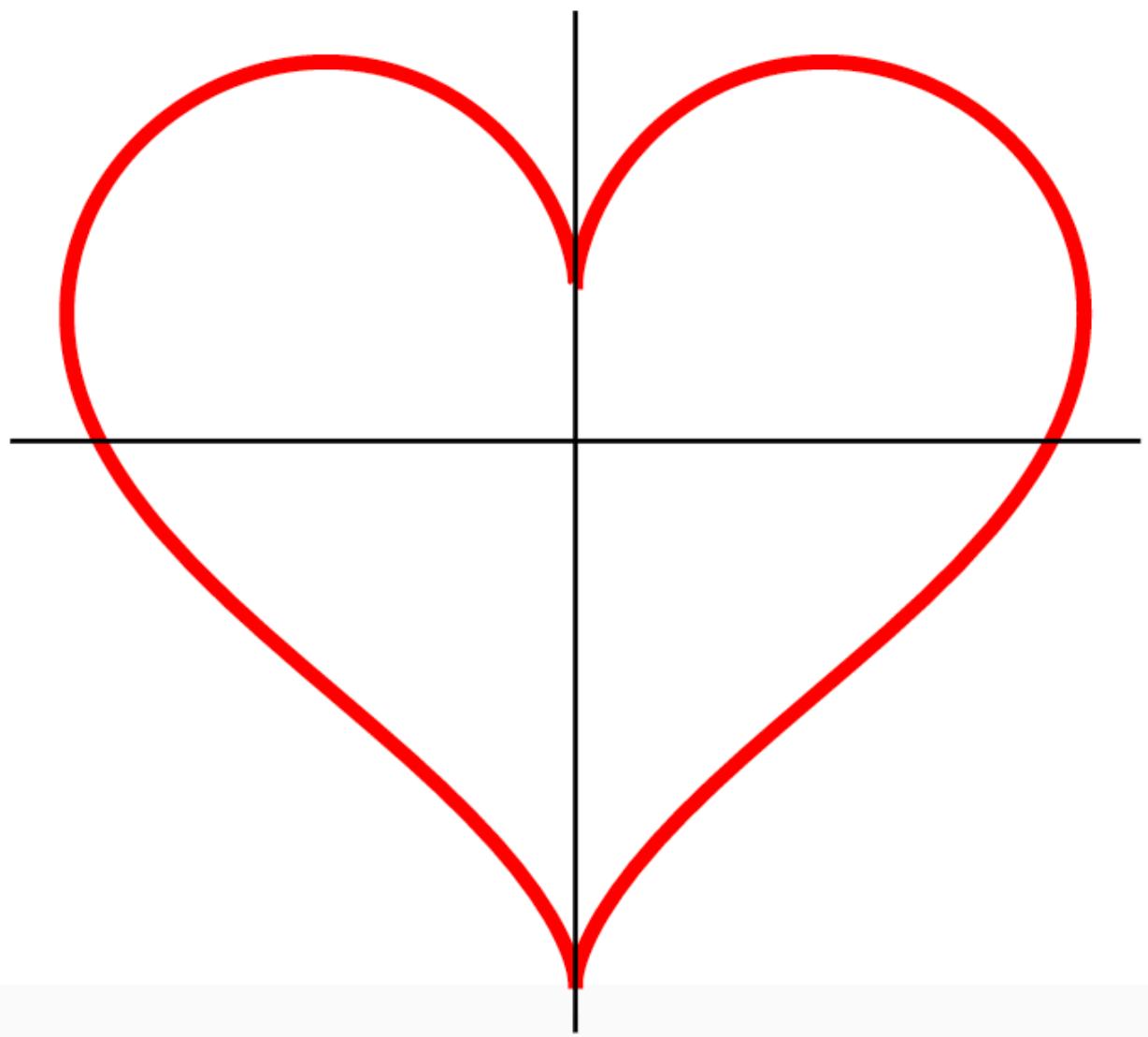
Zeroth-order dynamics

- Directly specify position, $p(t)$, function
 - Zeroth position derivative
- Examples:
 - Parametric equations, e.g., heart curve
 - Keyframe animation (more later)
- Pros:
 - Full control
 - No simulation or dynamics
- Cons:
 - Tedious & hard to do complex motions
 - No physics, collisions or other interactions

$$x = 16 \sin^3(t)$$

$$y = 13 \cos(t)$$

$$5 \cos(2t) \quad 2 \cos(3t) \quad \cos(4t)$$

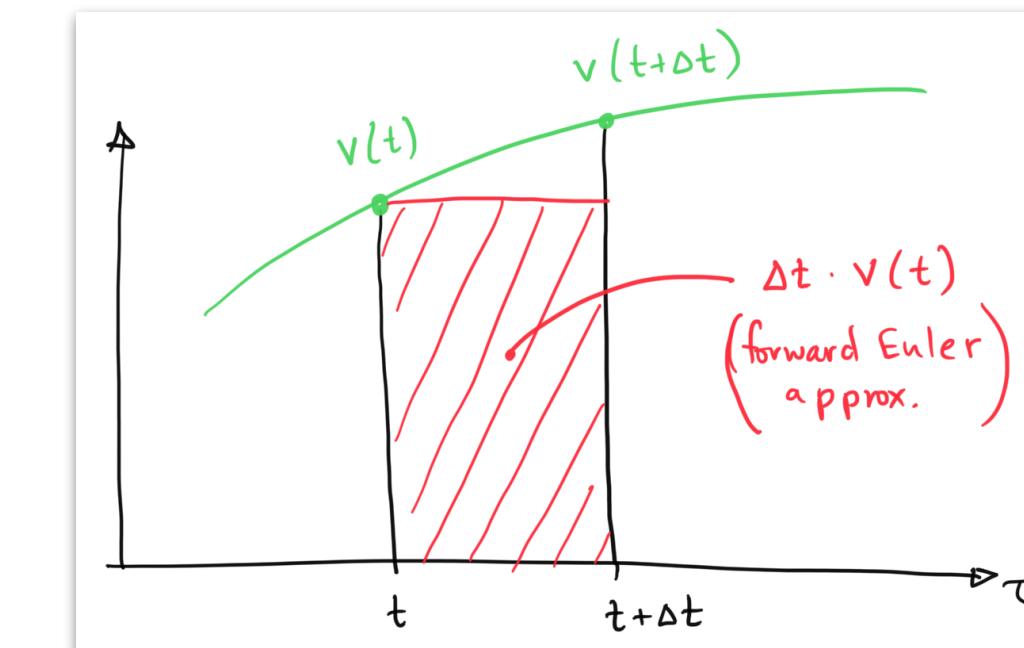


<https://mathworld.wolfram.com/HeartCurve.html>

<https://www.openprocessing.org/sketch/966495>

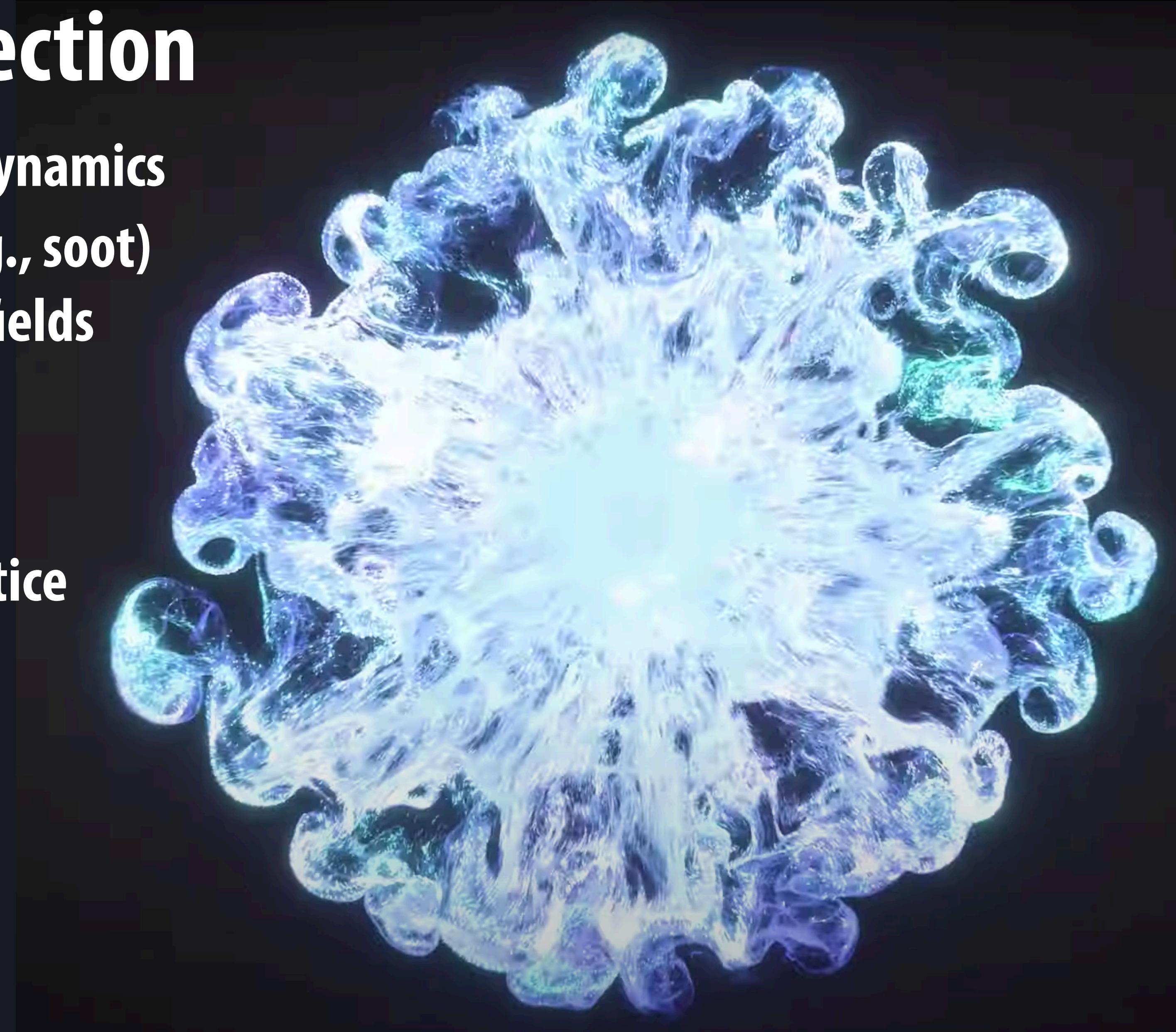
First-order dynamics

- Specify velocity, $v(t)$
 - Integrate $v(t)$ to get $p(t)$ of particles
 - Solve an ordinary differential equation (ODE)
- Solve an *initial value problem* (IVP):
 - Solve $\dot{p} = v$ for $p(t)$ such that $p(0) = \bar{p}$.
 - In general, input velocity is given by some gradient function $v(t) = V(p(t), t)$
- Solution: $p(t + \Delta t) = p(t) + \int_t^{t+\Delta t} v(\tau) d\tau$
- Numerical approximation: Forward Euler scheme
 - $p(t + \Delta t) = p(t) + v(t) \Delta t$
 - $O(\Delta t^2)$ truncation error



Application: Particle advection

- Important application of first-order dynamics
 - Advection of massless particles (e.g., soot) in known spatiotemporal velocity fields
- Integrate $\dot{\mathbf{p}} = \mathbf{V}(\mathbf{p}, t)$
 - subject to initial condition, $\mathbf{p}(0) = \bar{\mathbf{p}}$.
- Millions to billions of particles in practice
- Pleasantly parallel integration



https://www.youtube.com/watch?v=f51ScQ162wA&ab_channel=HAGI

Application: Energy minimization

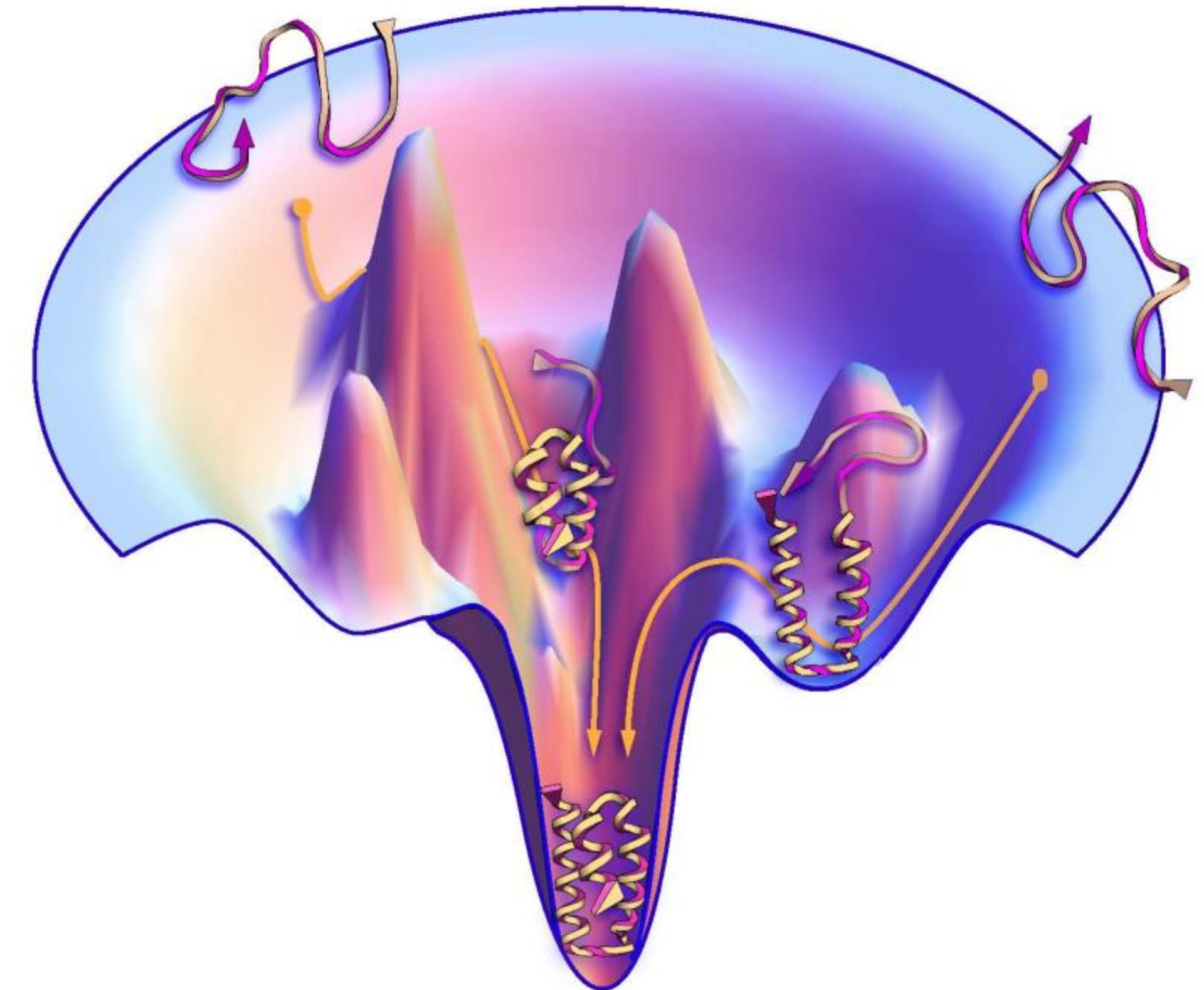
- Consider particular motion under gradient descent

$$\min_{\mathbf{p}} E(\mathbf{p})$$

- Numerical approximation

$$\mathbf{p}_{i+1} = \mathbf{p}_i - h \nabla E(\mathbf{p}_i)$$

$$\xrightarrow{h \rightarrow 0} \frac{d\mathbf{p}}{dt} = - \nabla E(\mathbf{p})$$



<https://www.sciencedaily.com/releases/2012/11/121122152910.htm>

Application: Sampling Implicit Surfaces

- Witkin and Heckbert, "Using Particles to Sample and Control Implicit Surfaces," SIGGRAPH 94.

- Particles

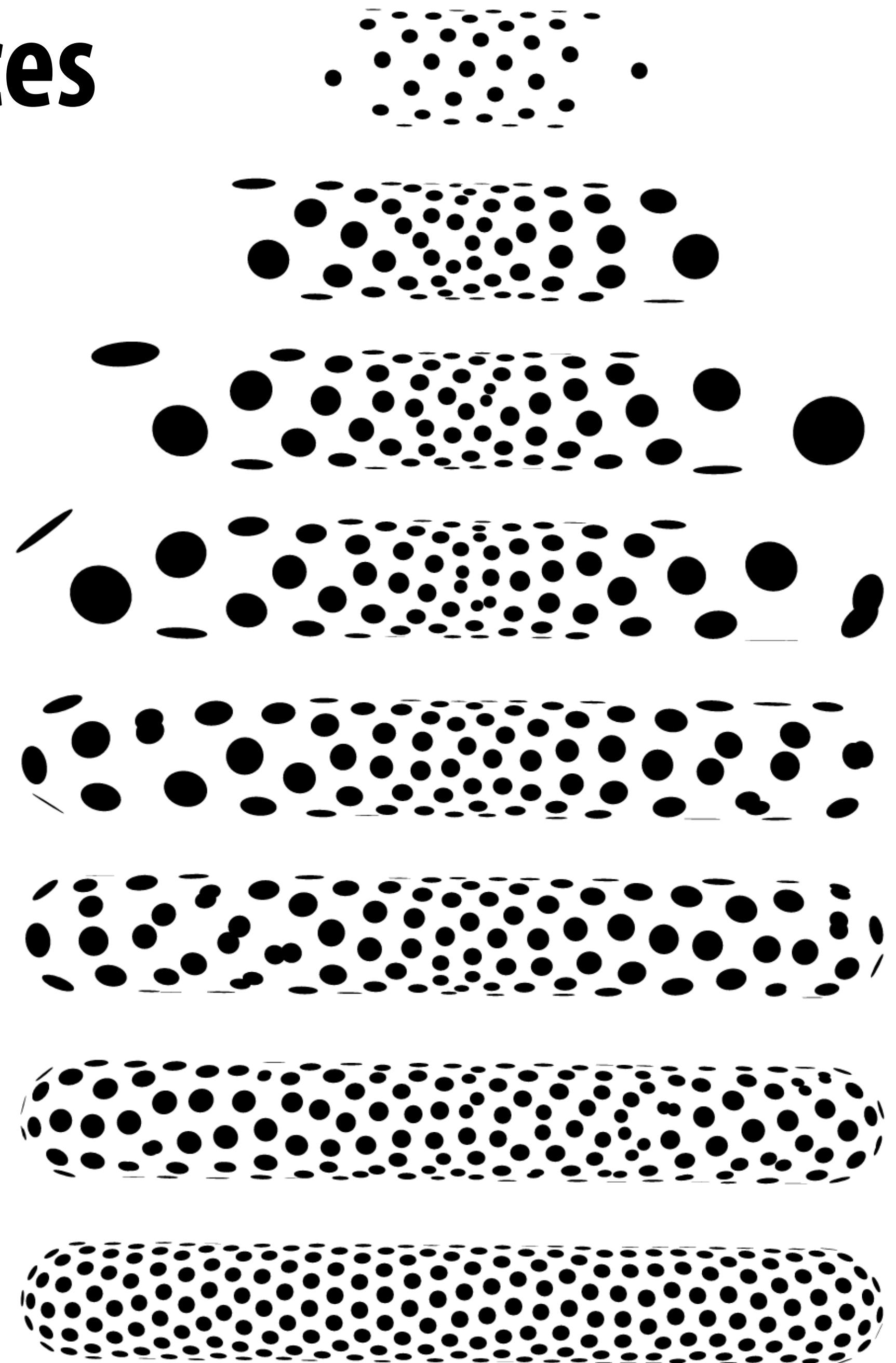
- Constrained to lie on an iso-surface, $d(\mathbf{p}) = 0$
- Repel each other using the energy:

$$E_{ij} = e^{-\frac{\|\mathbf{p}_i - \mathbf{p}_j\|^2}{2\sigma_i^2}}$$

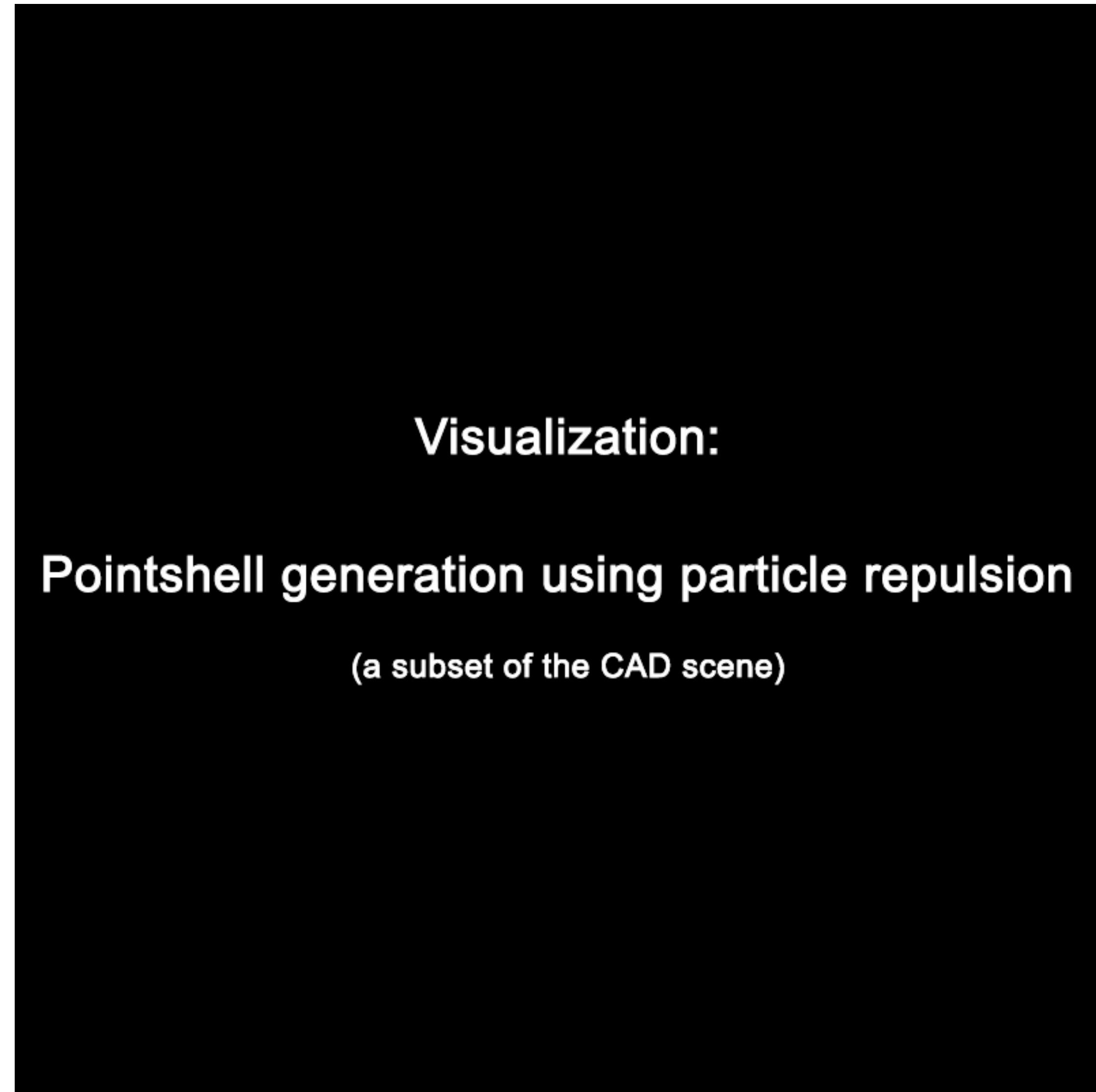
- Particle i energy:

$$E_i(\mathbf{p}) = \sum_{j=1}^N (E_{ij} + E_{ji})$$

- Move particles around using gradient descent



Application: Sampling Implicit Surfaces



From [Barbic and James 2007]

Second-order dynamics: "f=ma"

- Specify particle acceleration, $\mathbf{a} = \frac{d\mathbf{p}^2}{dt^2}$, and integrate twice.

- Newton's equations of motion for a particle system:

$$m_i \frac{d\mathbf{p}_i^2}{dt^2} = \mathbf{f}_i \quad \text{for } i = 1, \dots, N.$$

- m_i is the mass of particle i
- $\mathbf{f}_i = \mathbf{f}_i(\mathbf{p}, \mathbf{v}, t)$ are external forces
- Second-order ODE for $\mathbf{p}(t)$

- Matrix form

- $\mathbf{M} \dot{\mathbf{p}} = \mathbf{f}$ where $\mathbf{M} = \text{diag}(m_1 \mathbf{I}_n, \dots, m_N \mathbf{I}_n)$

Simple particle forces

- Constant gravity:

$$\mathbf{f}_i = m \mathbf{g}$$

- Aerodynamic drag:

$$\mathbf{f}_i = -c \mathbf{v}_i, \quad c > 0$$

- Zero-rest-length springs:

- Pulling particle to anchor point, \mathbf{c} : $\mathbf{f}_i = -k(\mathbf{p}_i - \mathbf{c})$

- Pulling two particles together: $\mathbf{f}_{i/j} = \pm k(\mathbf{p}_i - \mathbf{p}_j)$

- Newton's law of gravitation (c.f. electrostatic forces)

$$\mathbf{f}_{i/j} = \pm G m_i m_j \frac{(\mathbf{p}_i - \mathbf{p}_j)}{\|\mathbf{p}_i - \mathbf{p}_j\|^3}$$

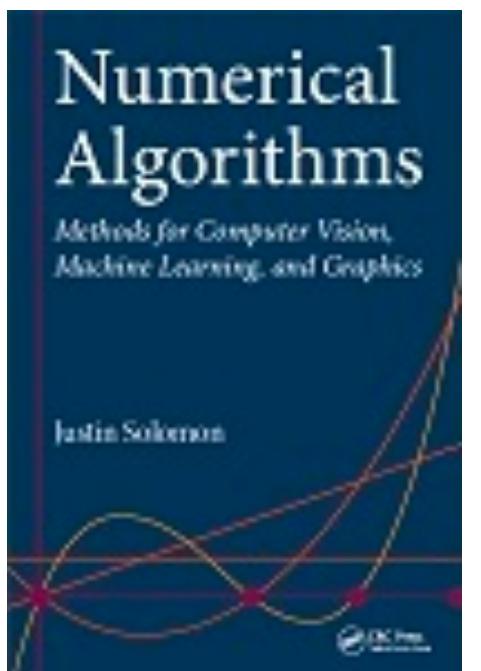
Ordinary Differential Equations (ODEs)

Basic Theory and Numerical Integration

Ordinary Differential Equations (ODEs)

- Initial value problems (IVPs)
 - Common in animation

- Textbook reference:
 - Solomon, Justin. Numerical Algorithms.
Textbook published by AK Peters/CRC Press, 2015



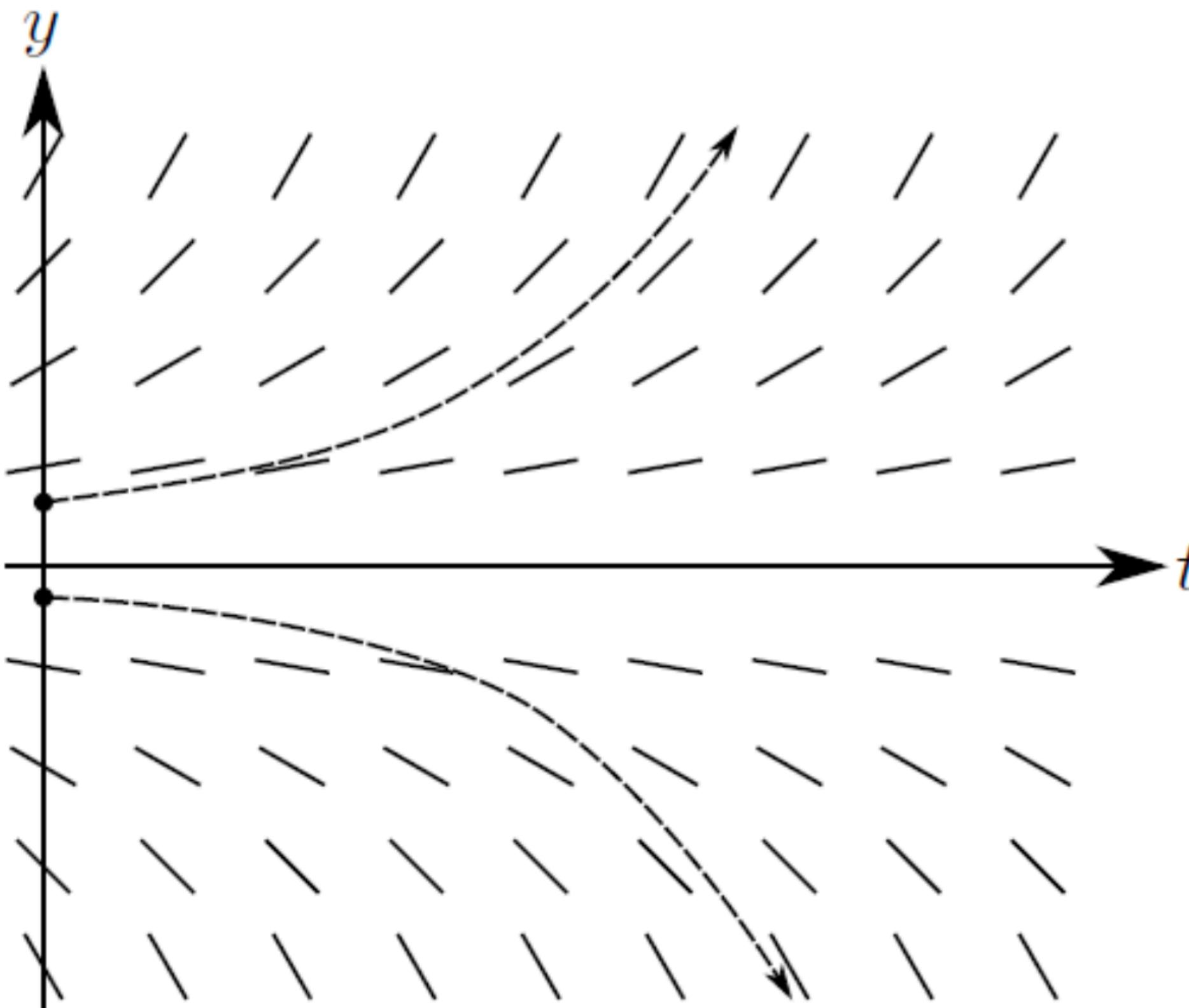
Conversion to first-order ODE form

- Can always convert a system of ODEs to first-order form
 - Useful step for applying standard ODE solvers
 - Explicit first-order form looks like $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t)$
- Convert $\mathbf{M} \ddot{\mathbf{p}} = \mathbf{f}$ by writing

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= \mathbf{v} \\ \frac{d\mathbf{v}}{dt} &= \mathbf{M}^{-1} \mathbf{f}\end{aligned}\quad \iff \quad \begin{aligned}\dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{M}^{-1} \mathbf{f}\end{aligned}$$

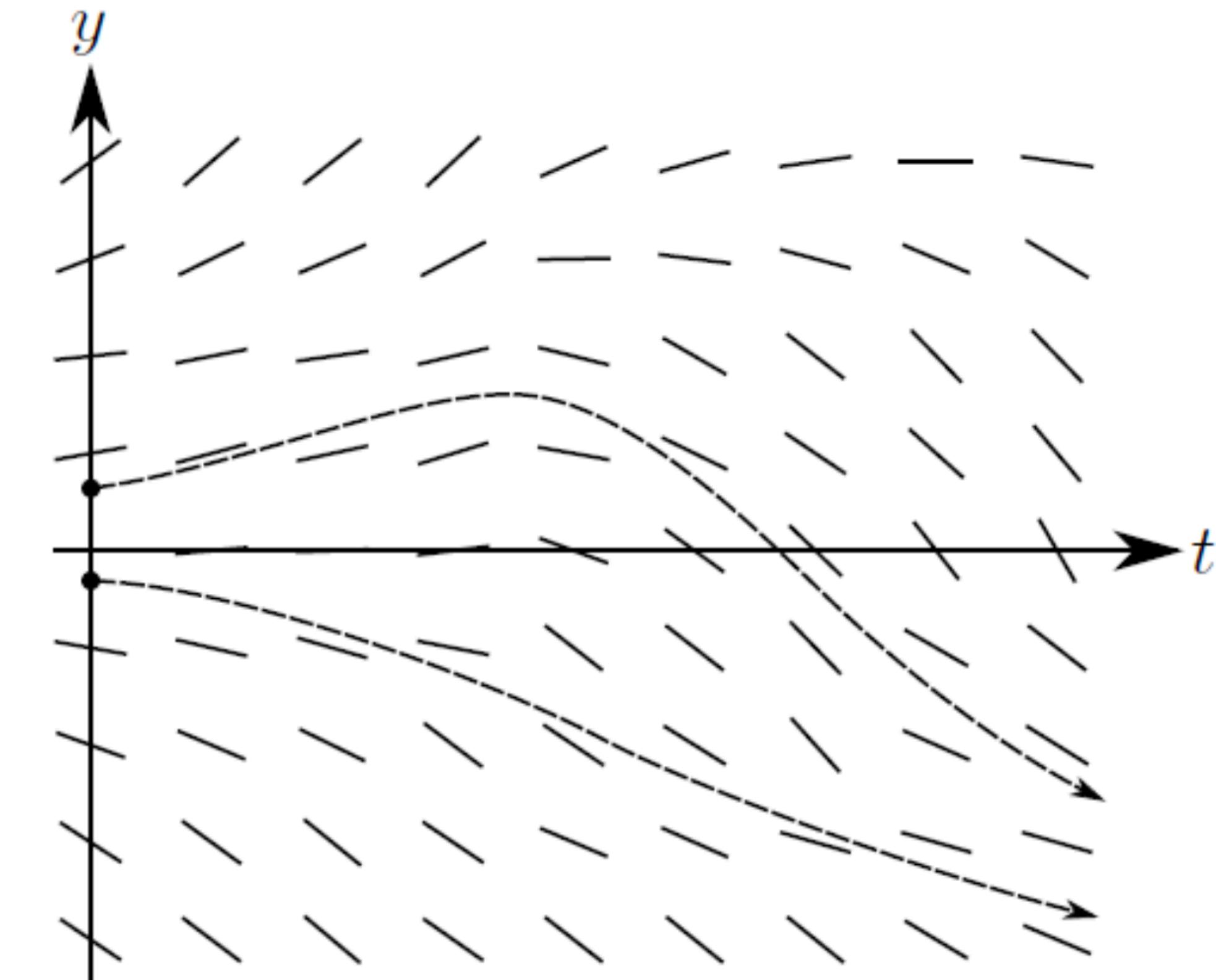
- Let $\mathbf{y} = \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} \in \mathbb{R}^{2nN}$ then $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t) = \begin{pmatrix} \mathbf{v}(t) \\ \mathbf{M}^{-1} \mathbf{f}(\mathbf{y}, t) \end{pmatrix}$ s.t. $\mathbf{y}(0) = \begin{pmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{v}} \end{pmatrix}$

Time dependence of gradient function, F



Time-independent

$$\dot{y} = F(y)$$



Time-dependent

$$\dot{y} = F(y, t)$$

Autonomous & non-autonomous ODEs

- Minor issue: We have a non-autonomous ODE (explicit time dependence)

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t)$$

but some solvers assume autonomous ODE form (no explicit time dependence)

$$\dot{\mathbf{z}} = \mathbf{G}(\mathbf{z})$$

- There's an easy fix to convert a non-autonomous ODE, $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t)$, to autonomous form
- Introduce a time-like variable $\tau = \tau(t)$ with $\dot{\tau} = 1$, and $\tau(0) = 0$ (solution: $\tau = t$).
- Form an augmented system, and replace all RHS references to t with τ :

$$\frac{d}{dt} \begin{pmatrix} \mathbf{y} \\ \tau \end{pmatrix} = \begin{pmatrix} \mathbf{F}(\mathbf{y}, \tau) \\ 1 \end{pmatrix} \iff \dot{\mathbf{z}} = \mathbf{G}(\mathbf{z})$$

where $\mathbf{z}(t) = \begin{pmatrix} \mathbf{y}(t) \\ \tau(t) \end{pmatrix}$ and $\mathbf{z}(0) = \begin{pmatrix} \bar{\mathbf{y}} \\ 0 \end{pmatrix}$.

Existence and Uniqueness of Solutions to $y' = F(y)$

Theorem: Local existence and uniqueness

Suppose F is continuous and Lipschitz, that is,

$$\|F[\vec{y}] - F[\vec{x}]\|_2 \leq L\|\vec{y} - \vec{x}\|_2 \text{ for some fixed } L \geq 0.$$

Then, the ODE $f'(t) = F[f(t)]$ admits exactly one solution for all $t \geq 0$ regardless of initial conditions.

- **Issue:** Many problems in animation may not be sufficiently continuous due to non-smooth contact, e.g., instantaneous velocity changes.

Linear Model Equations

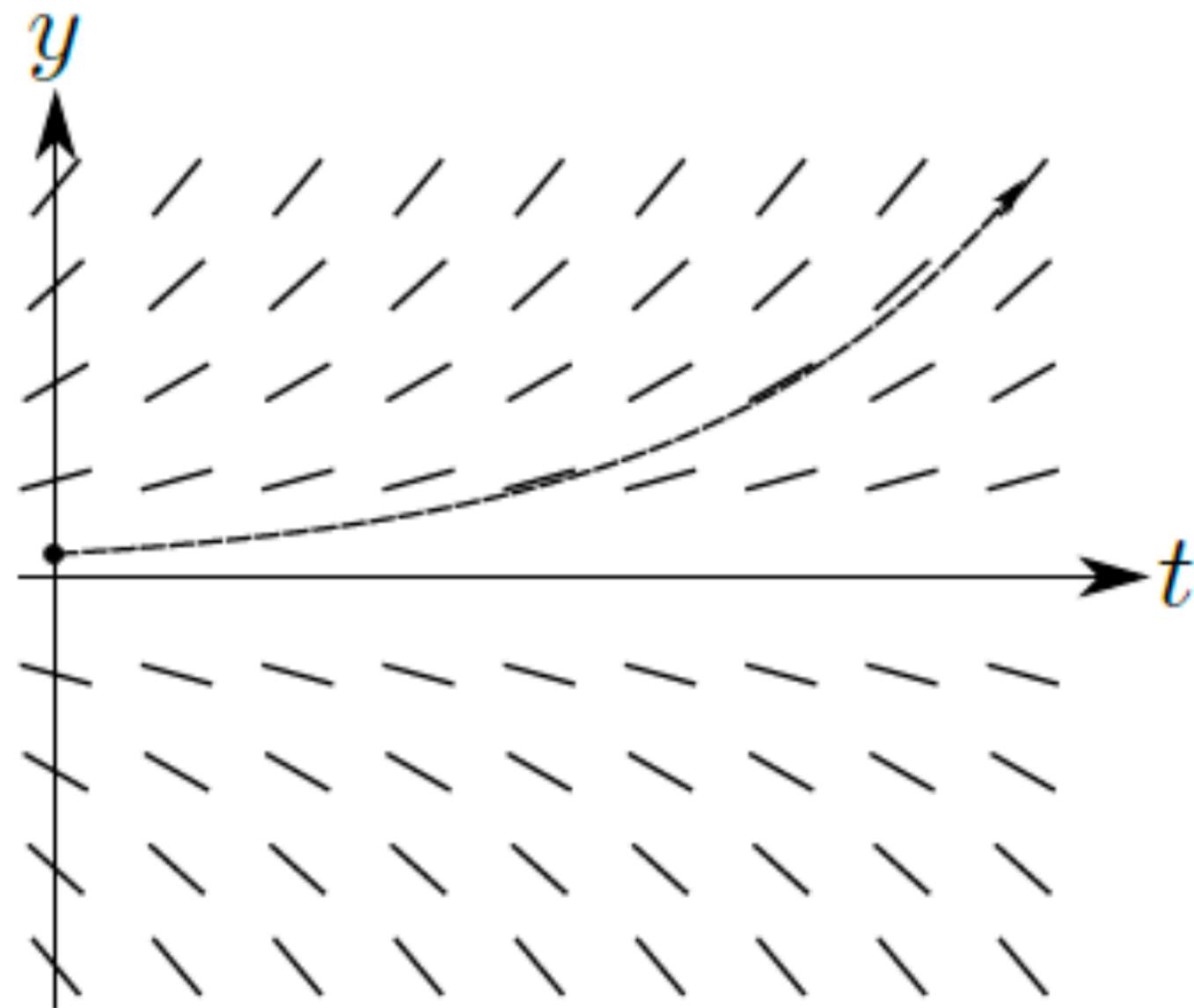
- Common to study stability and behavior of time-stepping schemes on model equations
- Consider the 1D ODE: $\dot{y} = F(y)$
 - Linearizing problem:

- $F(y) = F(0) + F'(0)y \approx a y + b = a \left(y + \frac{b}{a} \right) = a \bar{y}$
- Since $\frac{d}{dt}y' = \frac{d}{dt}\bar{y}'$ it is sufficient to consider...

- Model Problem:

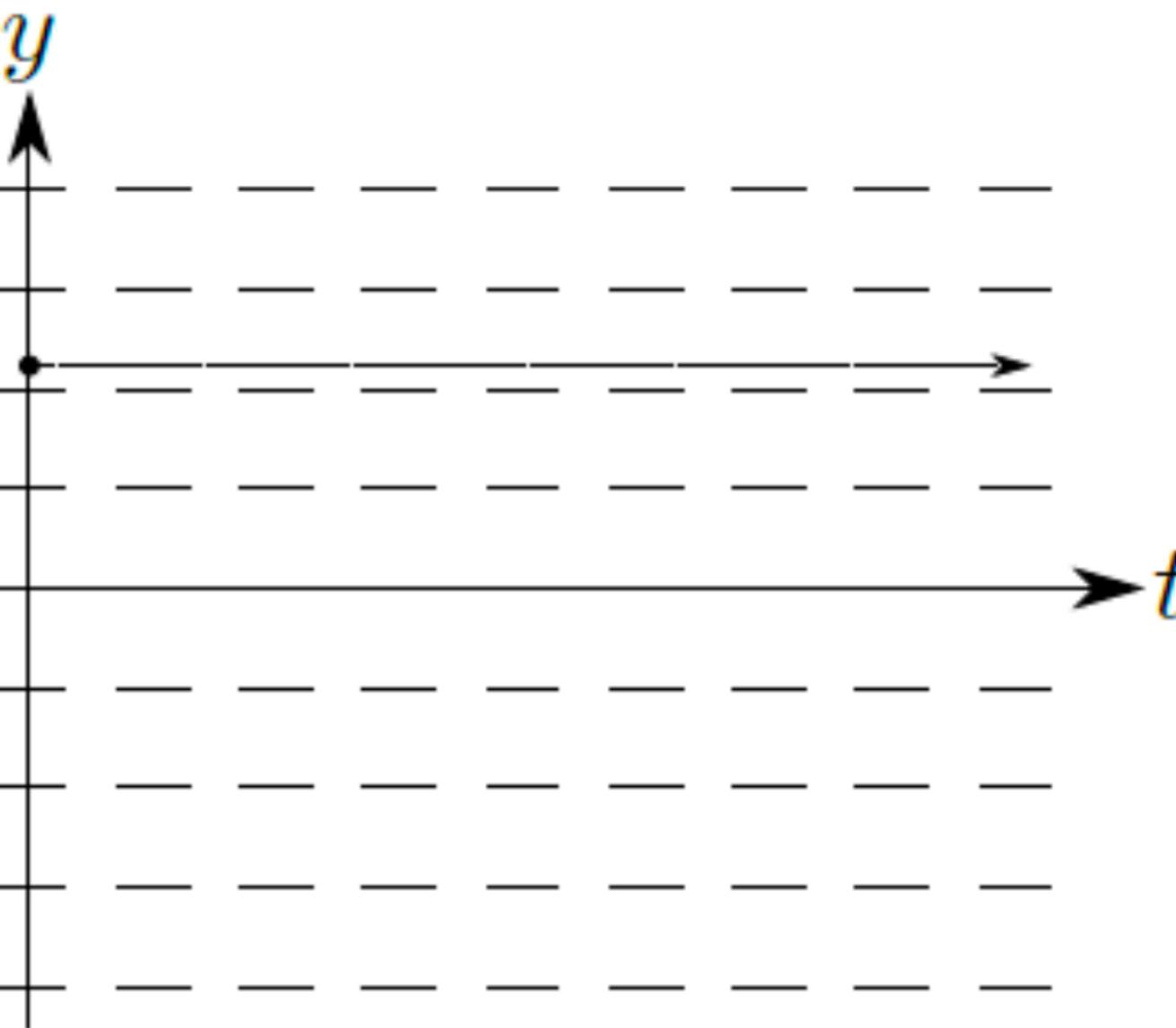
$$\dot{y} = a y \implies y(t) = C e^{at}$$

Stability: Visualization



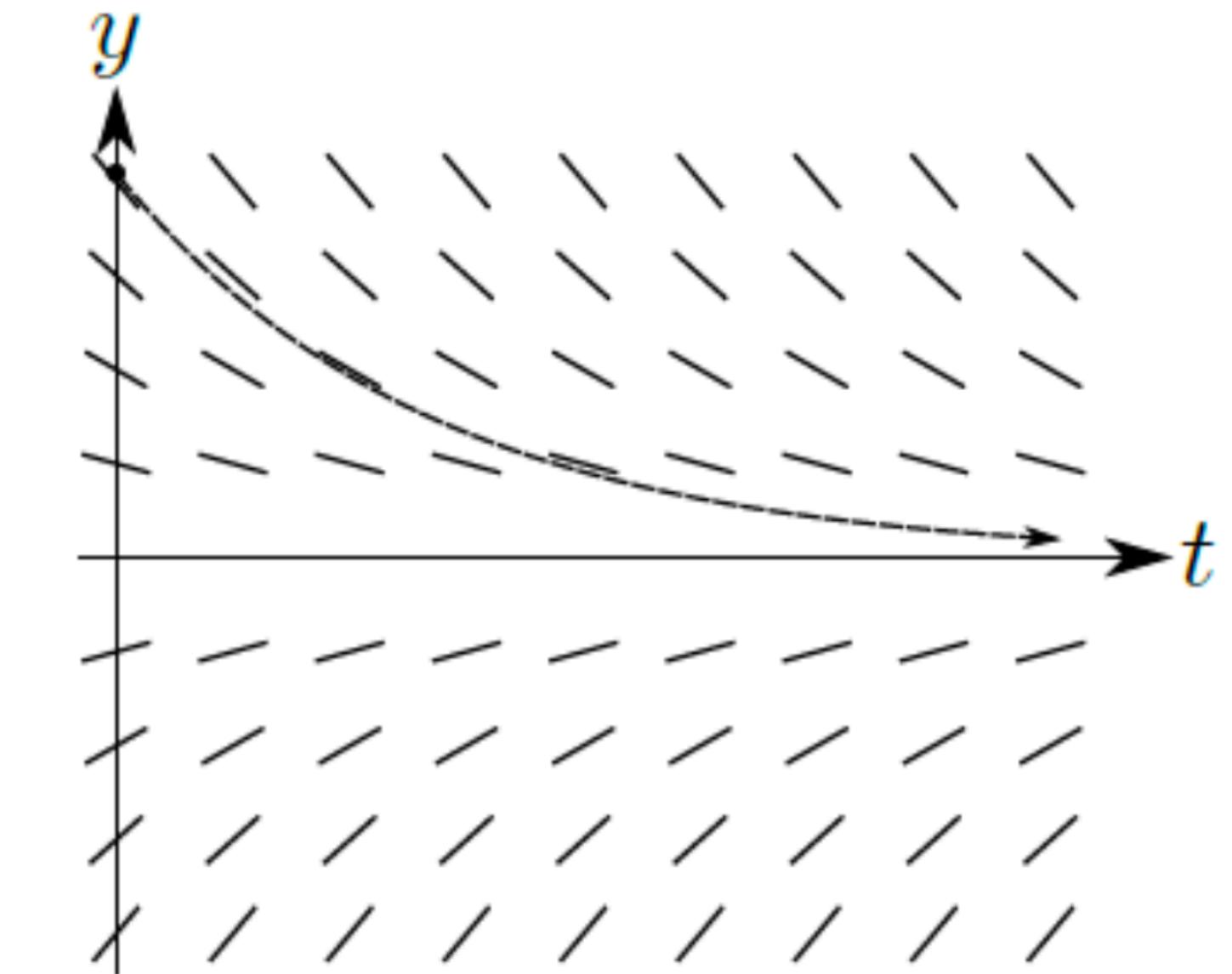
$$a > 0$$

Unstable
Solutions separate



$$a = 0$$

Stable



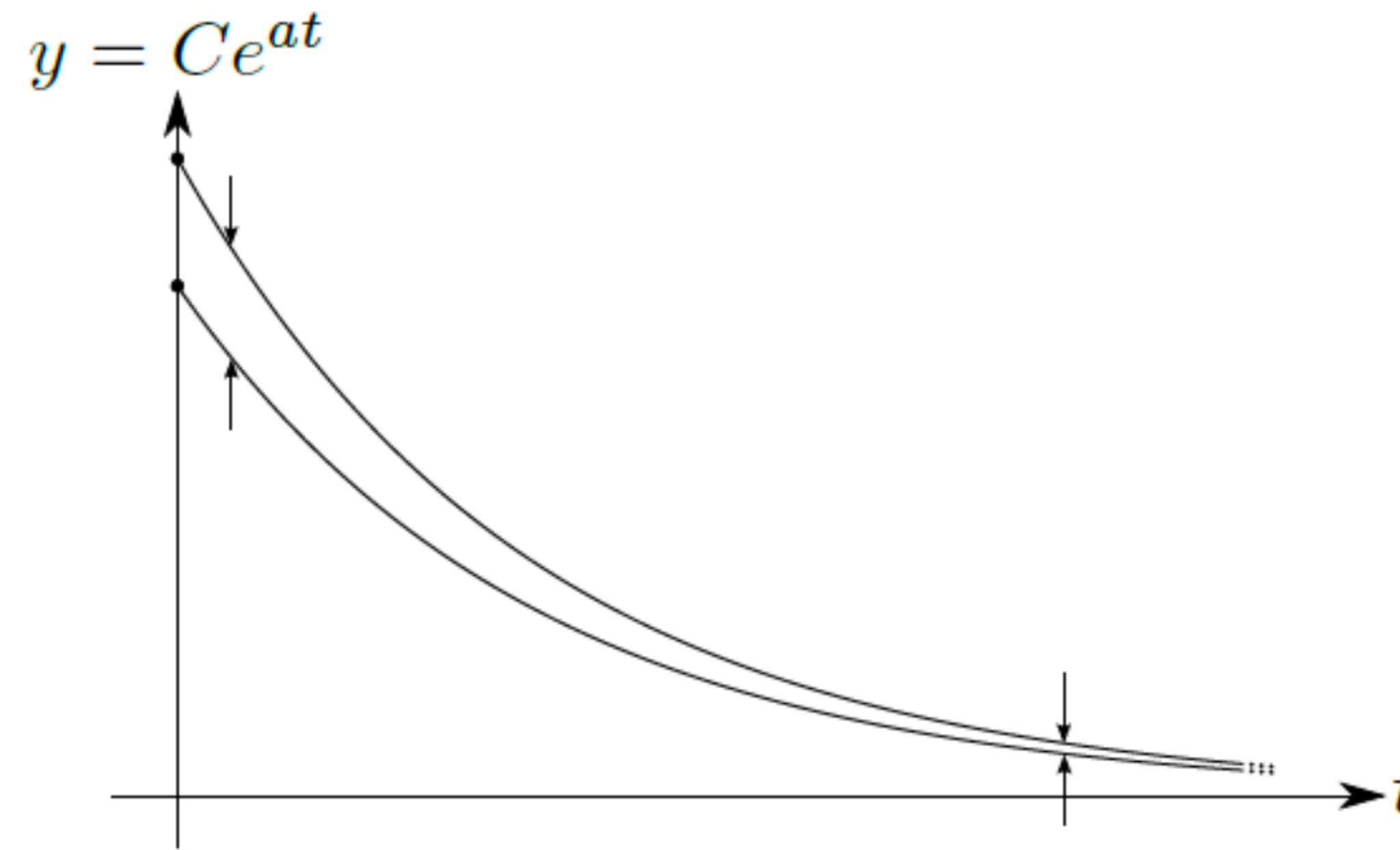
$$a < 0$$

Stable
Solutions get closer

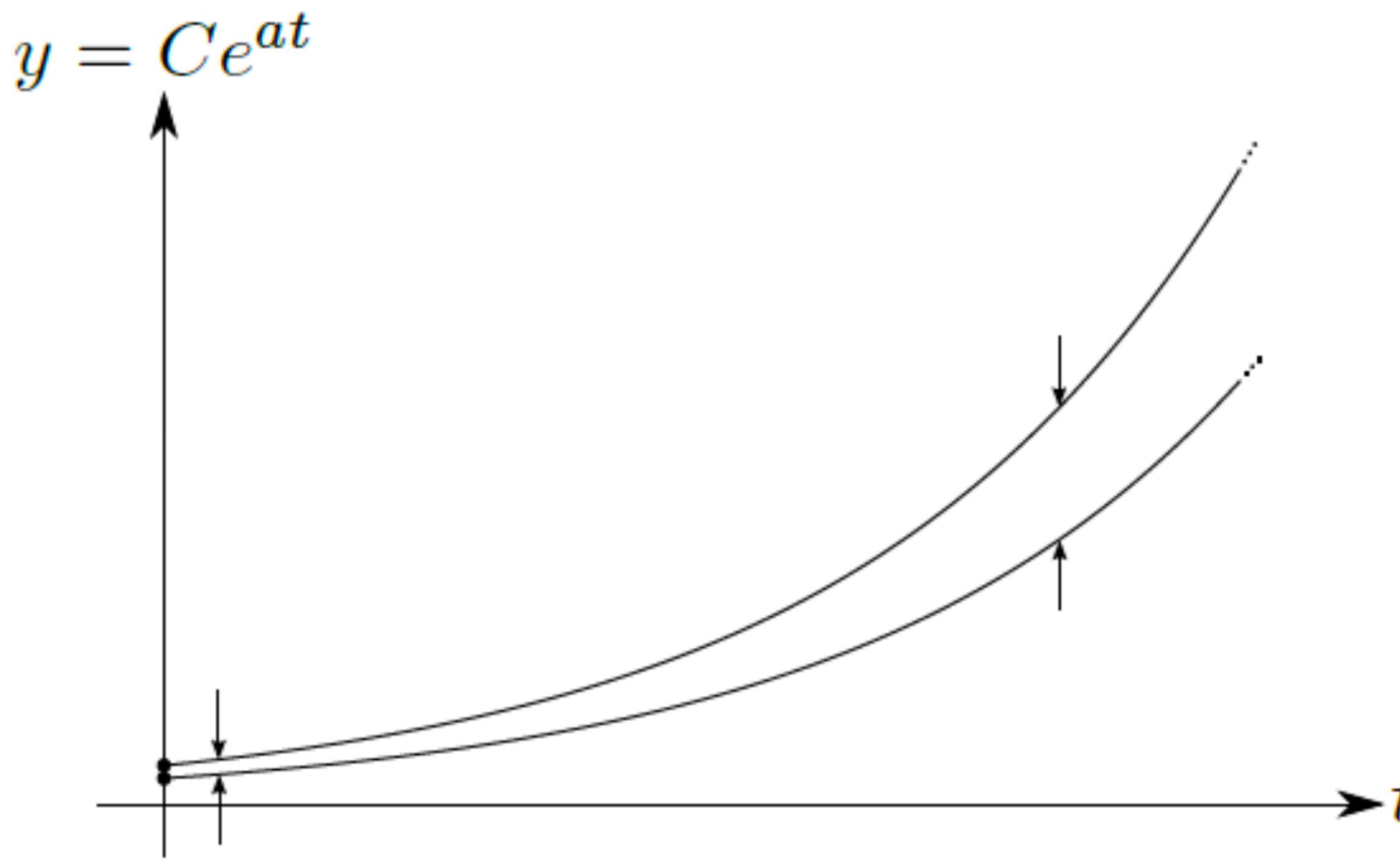
$$\dot{y} = ay$$

Stability and error in initial conditions

An *unstable* ODE magnifies mistakes in the initial conditions $y(0)$.



Stable ($a < 0$)



Unstable ($a > 0$)

Time-stepping schemes

- Consider $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y})$
- Given \mathbf{y}_k at time t_k generate \mathbf{y}_{k+1} at time $t_{k+1} = t_k + h$
- Things to consider
 - Accuracy
 - Local truncation error
 - Global truncation error
 - Stability
 - Analyze model problem $y' = ay$ ($a < 0$)
 - Stable when $|y_{k+1}| \leq |y_k|$
 - Unconditional vs conditional stability
 - Time-step restrictions

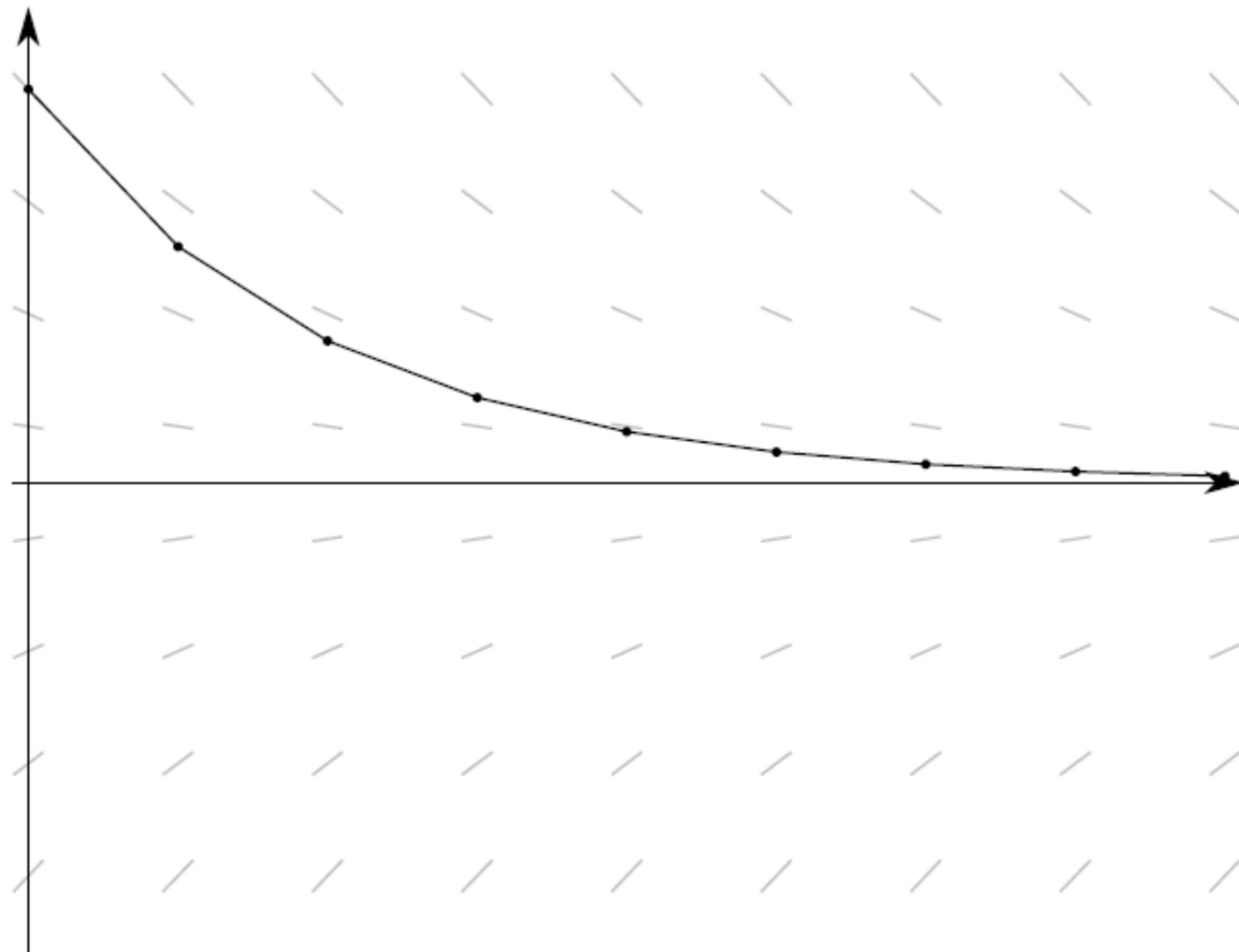
Forward Euler (a.k.a. Explicit Euler)

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h \mathbf{F}(\mathbf{y}_k)$$

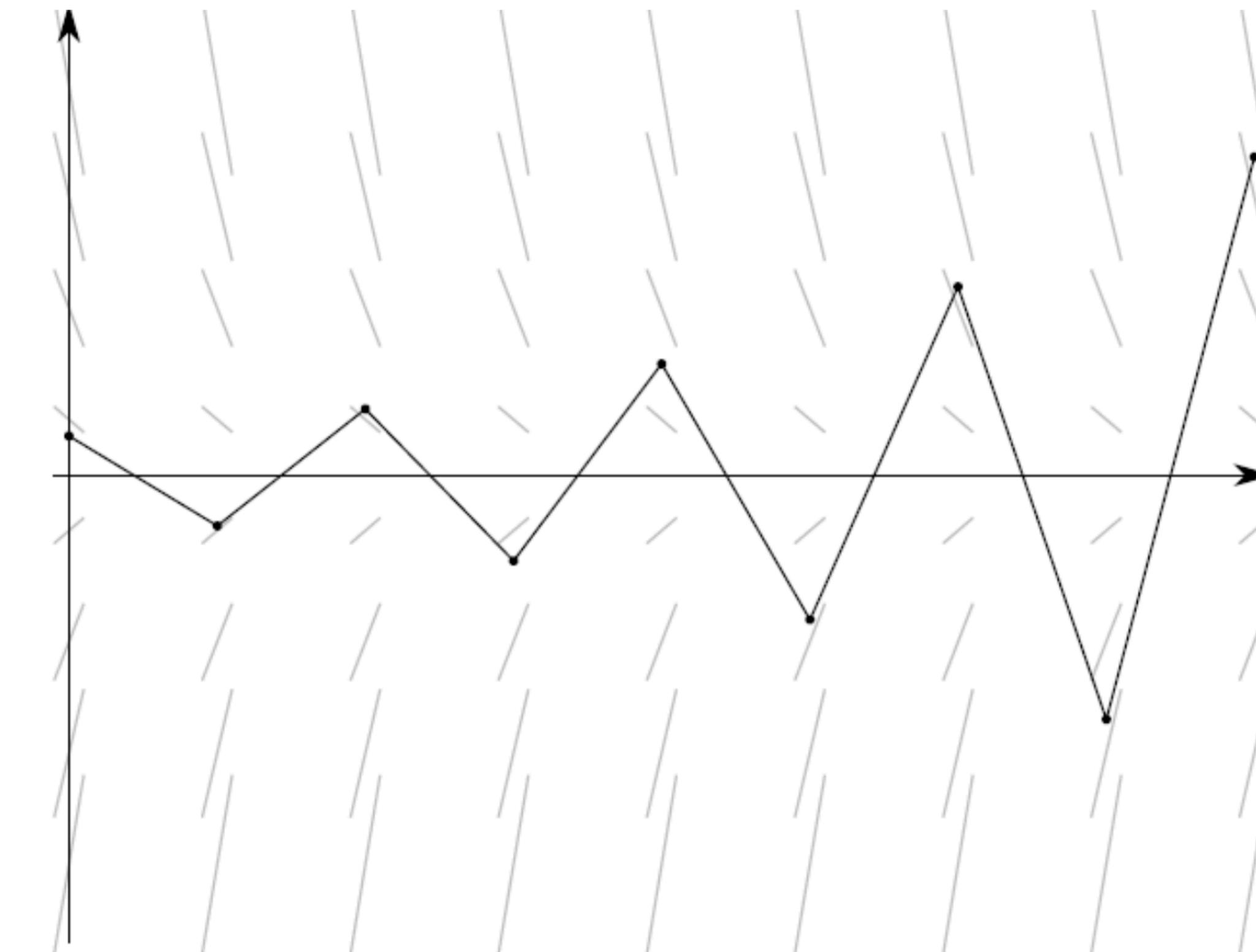
- ▶ Explicit method
- ▶ $O(h^2)$ localized truncation error
- ▶ $O(h)$ global truncation error;
“first order accurate”

Note: Take $O\left(\frac{1}{h}\right)$ steps to advance $O(1)$ time.

Forward Euler: Stability on model problem, $\dot{y} = a y$



Stable ($a = -0.4$)



Unstable ($a = -2.3$)

Forward Euler: Stability on model problem, $\dot{y} = a y$

$$y' = ay \longrightarrow y_{k+1} = (1 + ah)y_k$$

For $a < 0$, stable when $h < \frac{2}{|a|}$.

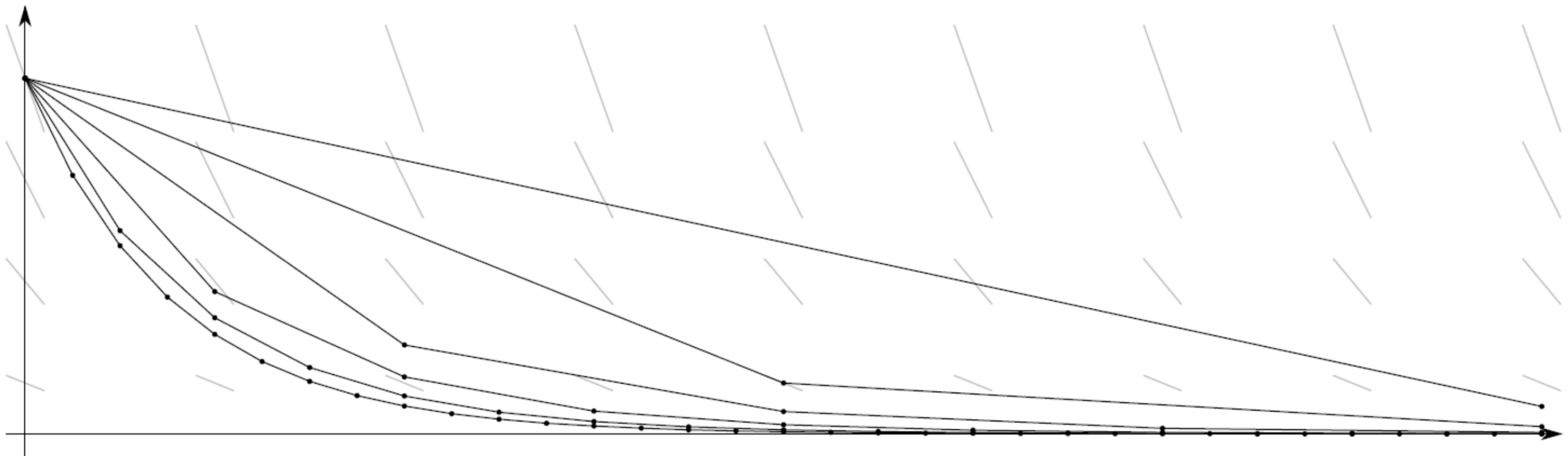
Forward Euler has a well-known time-step restriction for stability.

Backward Euler (a.k.a. Implicit Euler)

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h \mathbf{F}(\mathbf{y}_{k+1})$$

- ▶ Implicit method
- ▶ $O(h^2)$ localized truncation error
- ▶ $O(h)$ global truncation error;
“first order accurate”

Backward Euler: Stability on model problem, $\dot{y} = a y$



Backward Euler: Stability on model problem, $\dot{y} = a y$

$$y' = ay \longrightarrow y_{k+1} = \frac{1}{1 - ah} y_k$$

Unconditionally stable!

But this has nothing to do with accuracy.

Good for *stiff* equations.

Numerical Stiffness for IVP ODEs

An IVP is said to be numerically “stiff” if stability requirements dictate a much smaller time step size than is needed to satisfy the approximation requirements alone.

[Ascher & Petzold 1998]

Midpoint Method

$$\mathbf{y}_{k+\frac{1}{2}} = \mathbf{y}_k + \frac{h}{2} \mathbf{F}(\mathbf{y}_k)$$

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h \mathbf{F}(\mathbf{y}_{k+\frac{1}{2}})$$

Forward Euler half-step

Full-step w/ mid-point gradient

- Explicit method
- Two function evaluations, but ...
- $O(h^3)$ localized truncation error
- $O(h^2)$ global truncation error
 - "second-order accurate"

Midpoint Method: Stability on model problem, $\dot{y} = a y$

$$y_{k+1} = y_k + h \left(a y_{k+\frac{1}{2}} \right) = y_k + h a \left(y_k + \frac{h}{2} a y_k \right) = \left(1 + h a + \frac{h^2 a^2}{2} \right) y_k$$

- Stable when $|y_{k+1}| \leq |y_k|$ or $\left| 1 + h a + \frac{h^2 a^2}{2} \right| \leq 1$
- Requires $h < \frac{2}{|a|}$ time-step restriction (same as forward Euler).

Symplectic Euler Method (a.k.a. semi-implicit Euler)

- So far time-stepping schemes work with $\mathbf{y} = \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix}$, and treat \mathbf{p} and \mathbf{v} similarly.
- Downside: Big forces update \mathbf{v} immediately, but \mathbf{p} only sees them on the next timestep.
 - Bad for collisions.
- Idea: Update \mathbf{v} first, then use it to update \mathbf{p}
- Aside: Could also update \mathbf{p} first, then use it to update \mathbf{v}
 - So-called adjoint version
 - Valid approach, but it delays force integration

Symplectic Euler Method (a.k.a. semi-implicit Euler)

$$\mathbf{v}_{k+1} = \mathbf{v}_k + h \mathbf{a}(\mathbf{p}_k, \mathbf{v}_k)$$

Update velocity

$$\mathbf{p}_{k+1} = \mathbf{p}_k + h \mathbf{v}_{k+1}$$

Update position

- Semi-implicit method
- One function evaluation
- $O(h^2)$ localized truncation error
- $O(h)$ global truncation error
 - "first-order accurate"
- Symplectic structure
 - preserves area in position-momentum phase space

Symplectic Euler Method: Stability on model problem, $\dot{y} = a y$

???

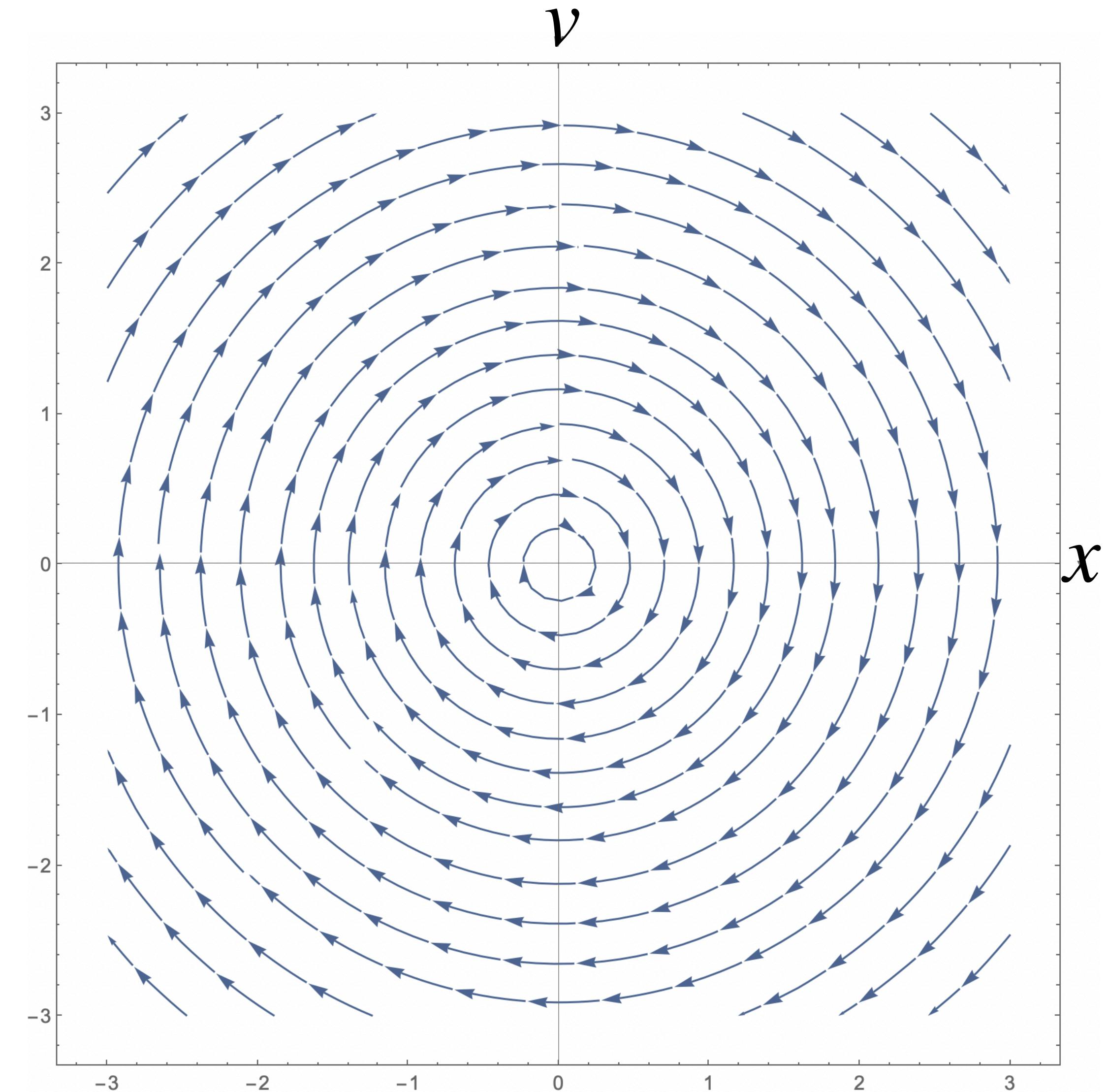
Harmonic oscillator model problem

$$\frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -x \end{pmatrix} \iff \dot{\mathbf{r}} = \mathbf{r}_\perp$$

■ Harmonic oscillator equation

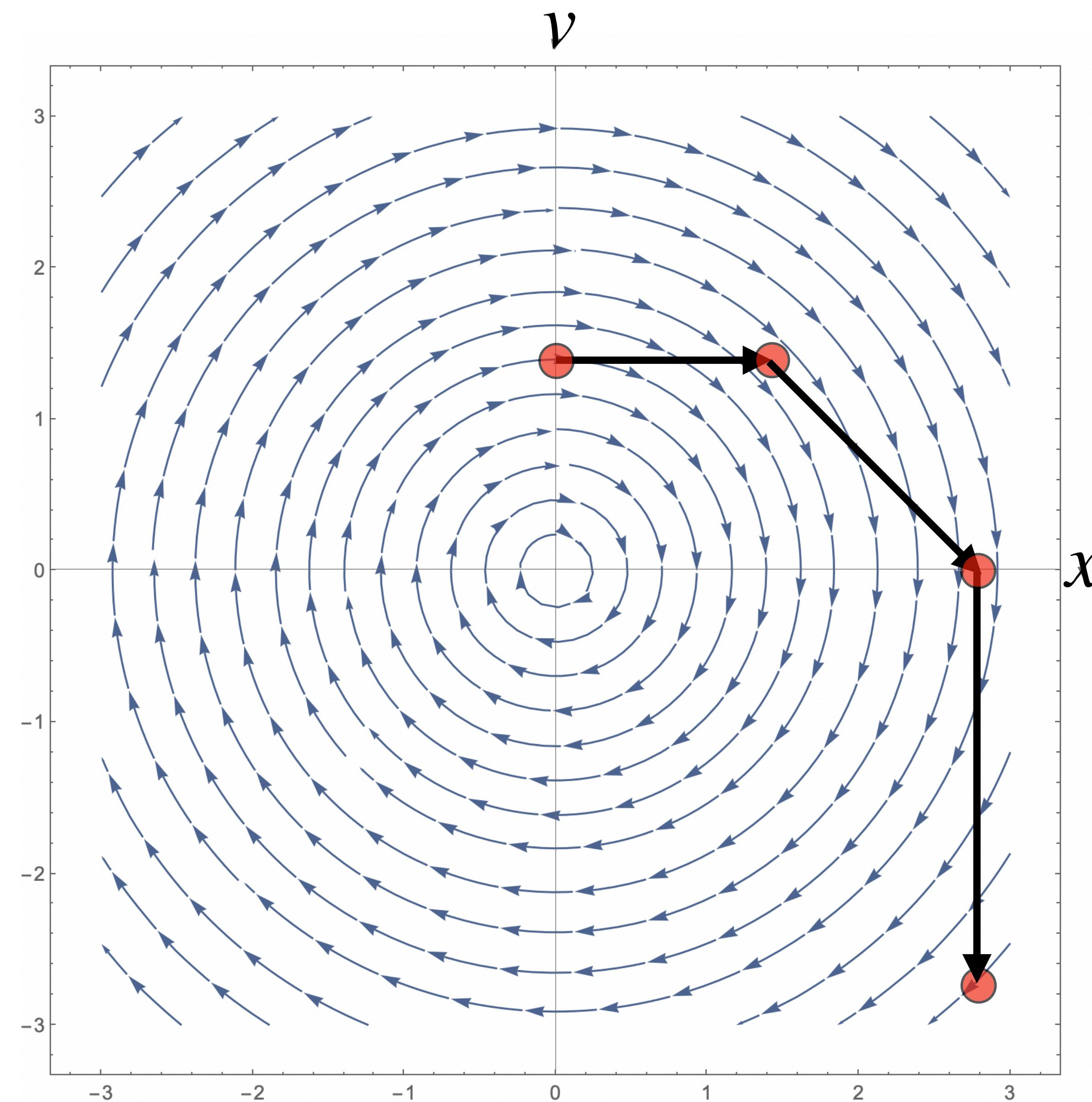
$$\ddot{x} + \omega^2 x = 0$$

with unit natural frequency, $\omega = 1$.



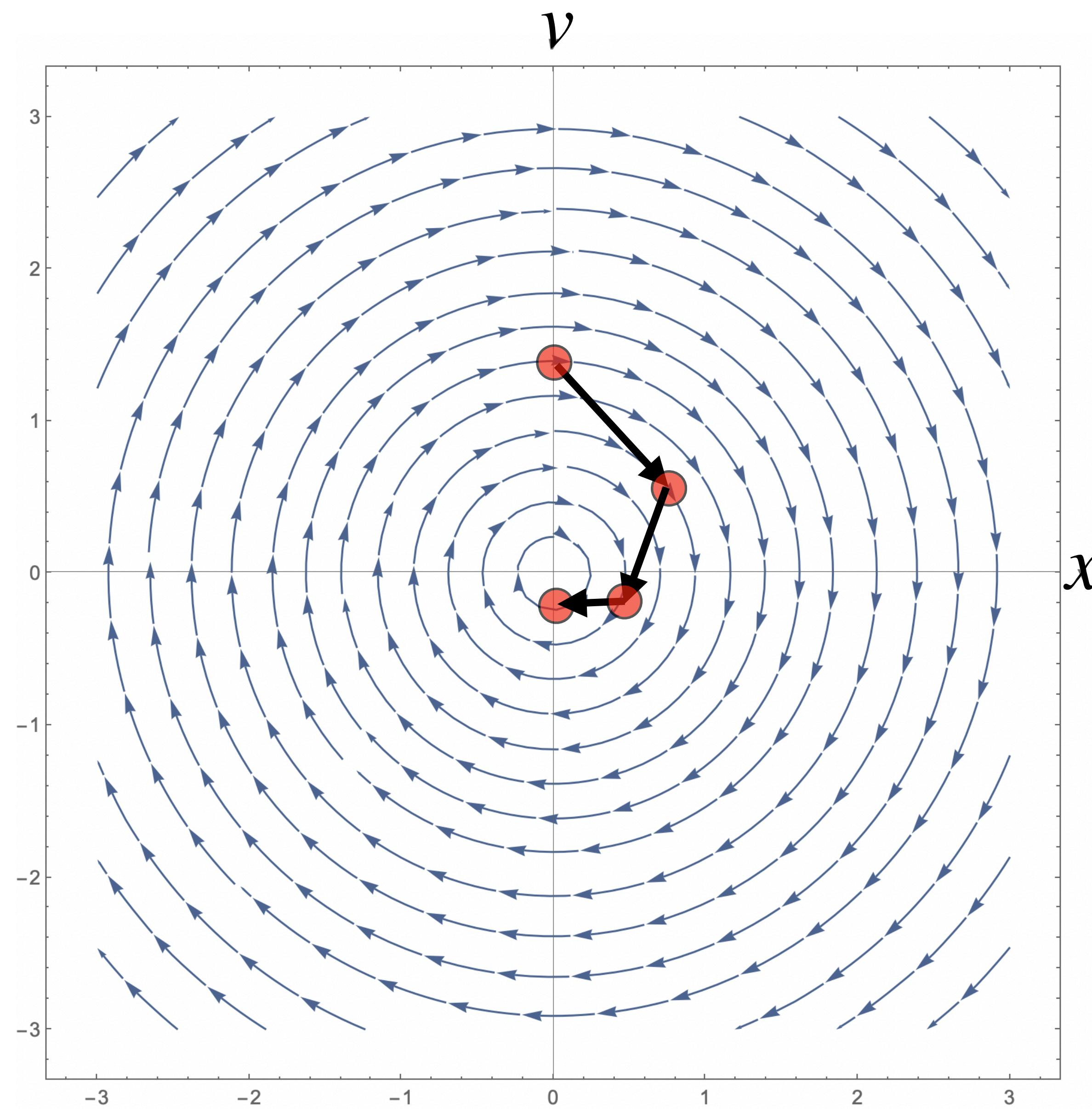
Stability on harmonic oscillator model problem

Forward Euler

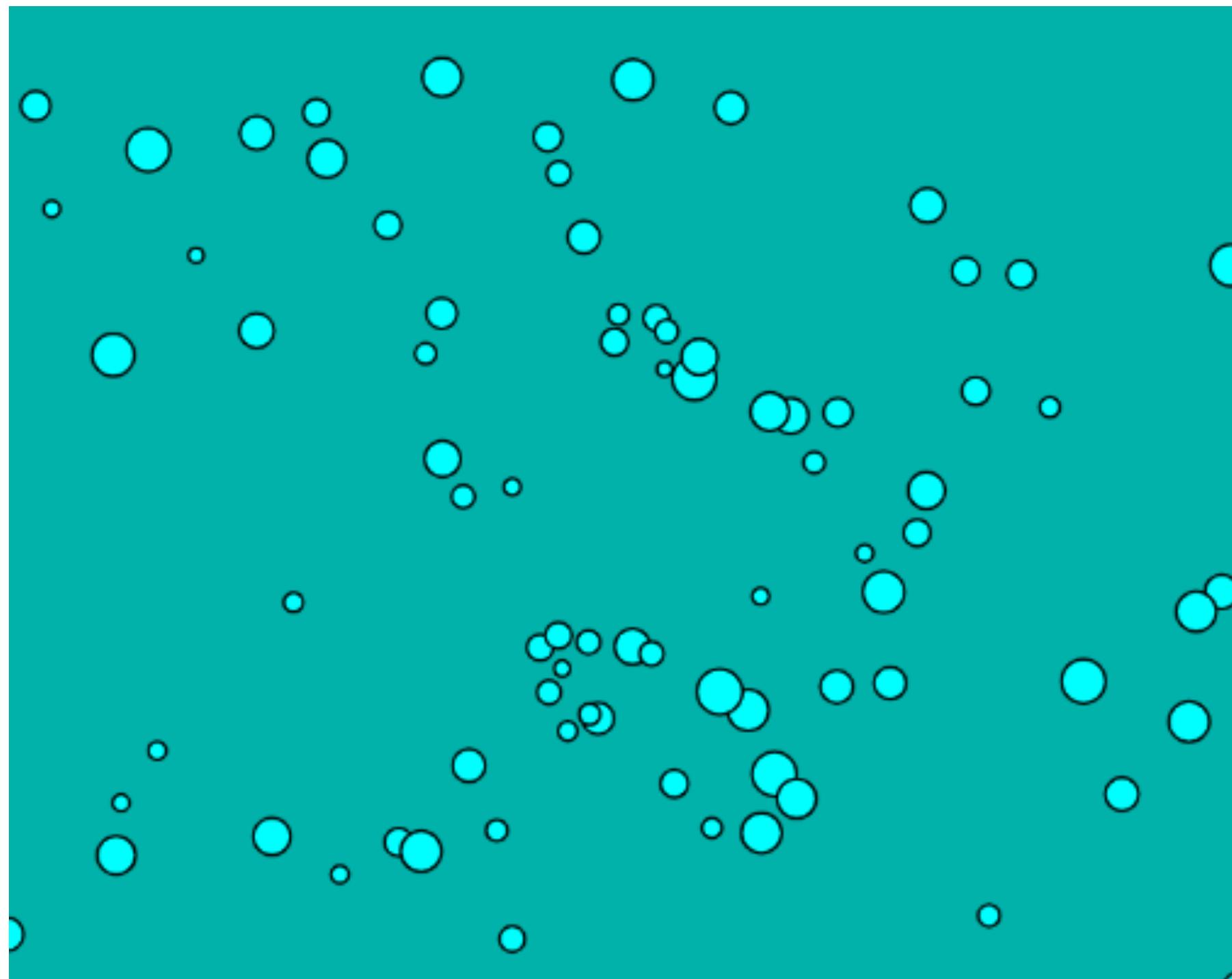


Stability on harmonic oscillator model problem

Backward Euler



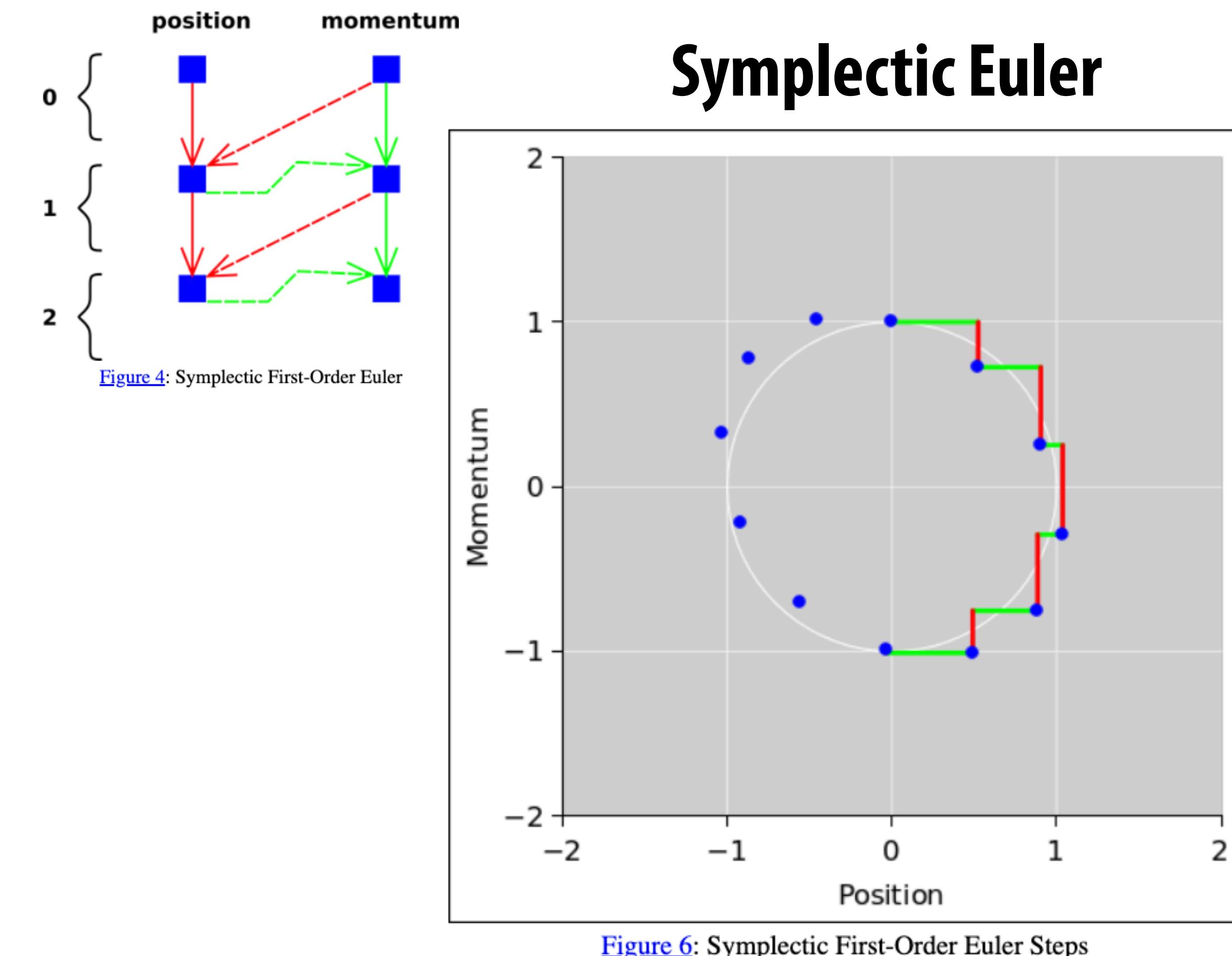
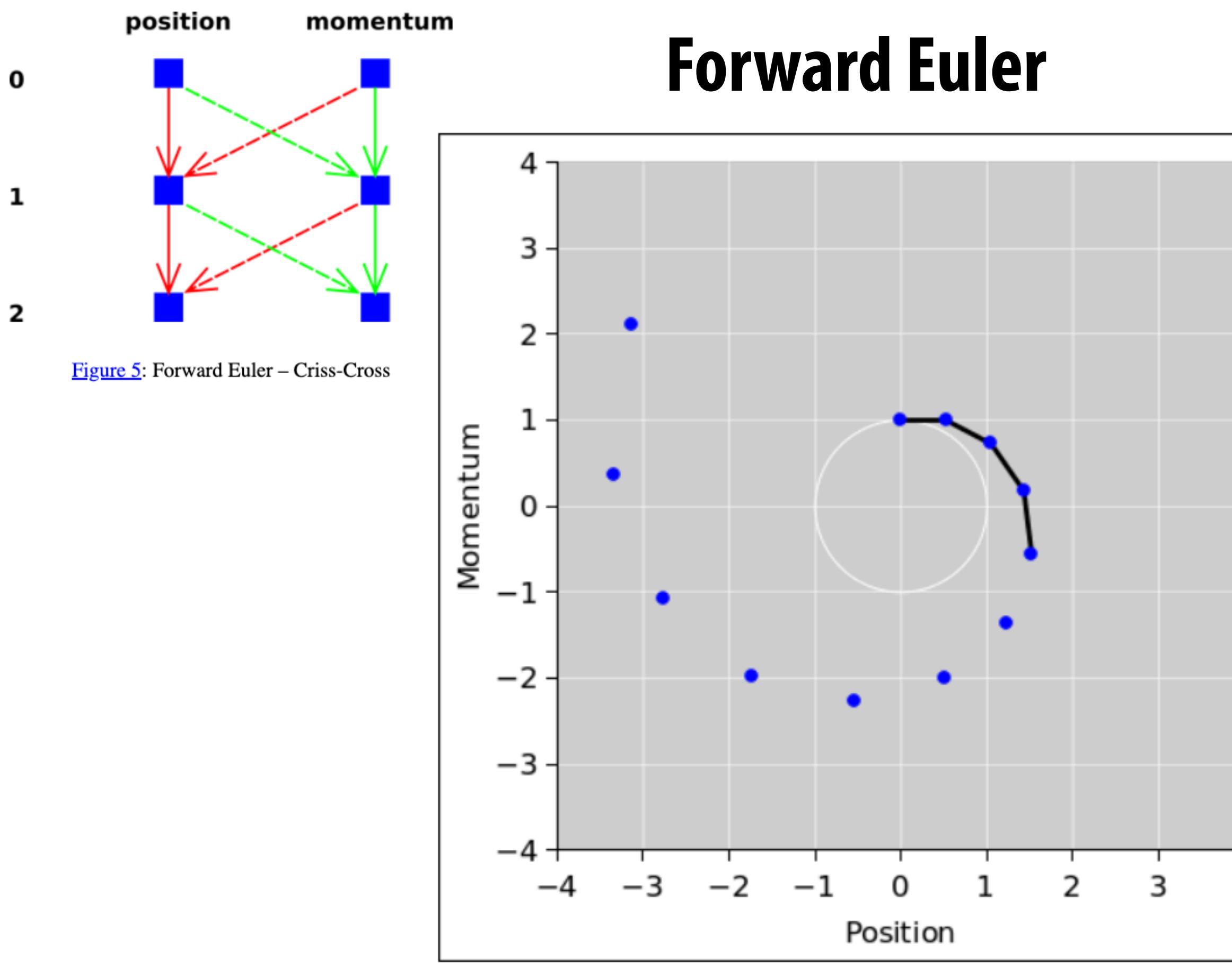
Example: Particle advection on vortex



<https://www.openprocessing.org/sketch/966498>

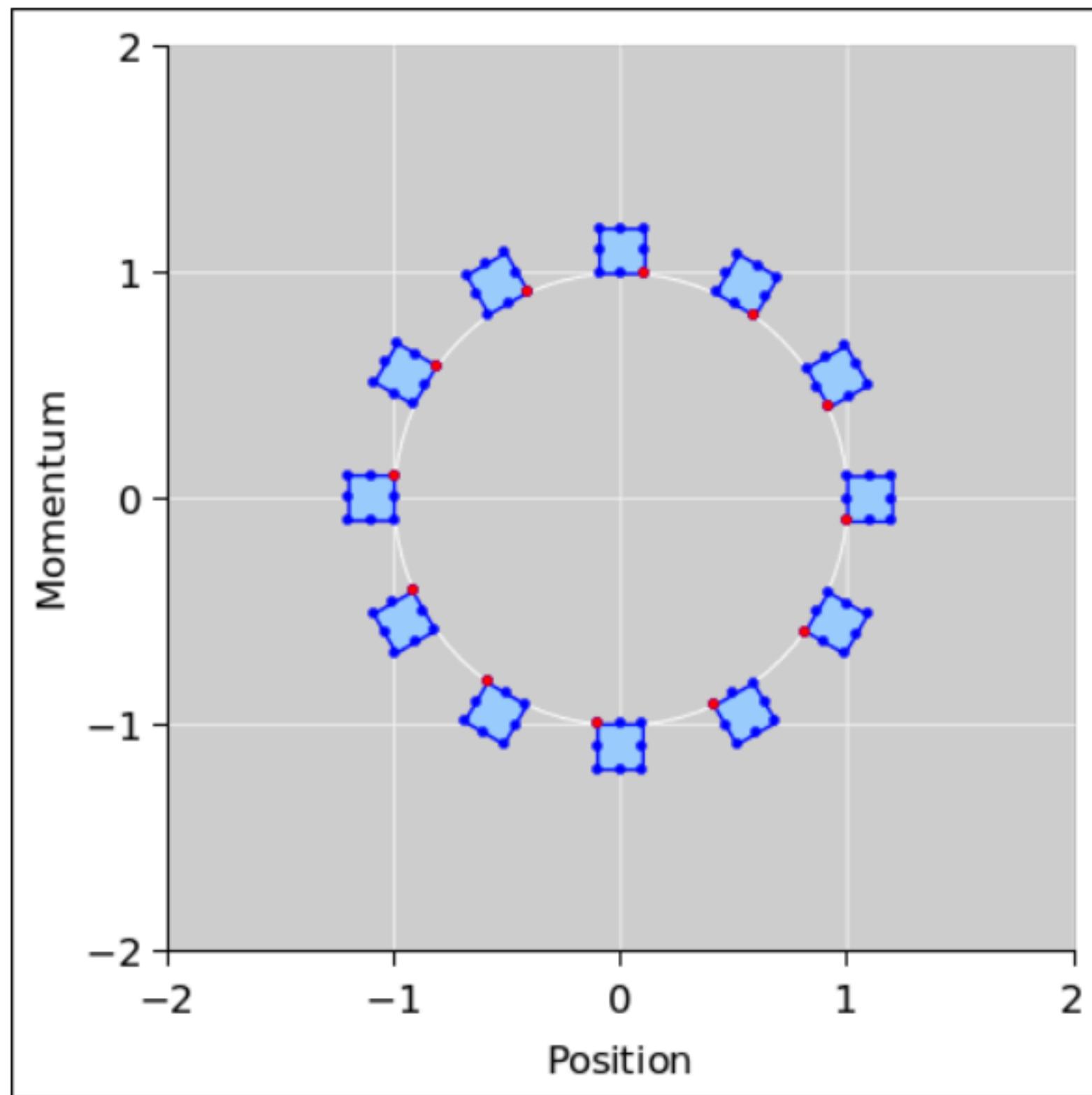
Stability on harmonic oscillator model problem

Forward vs Symplectic Euler

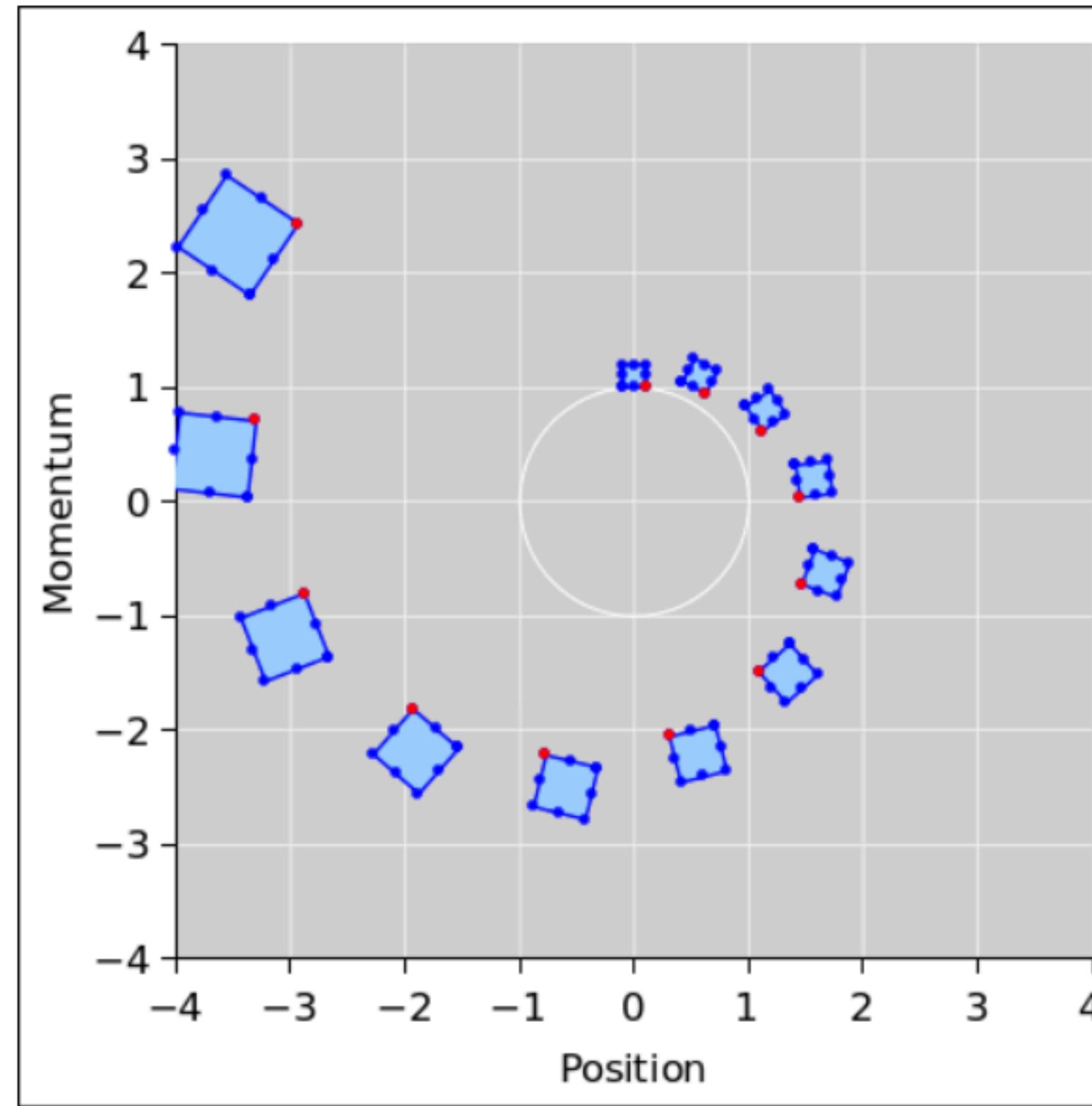


<https://www.av8n.com/physics/symplectic-integrator.htm>

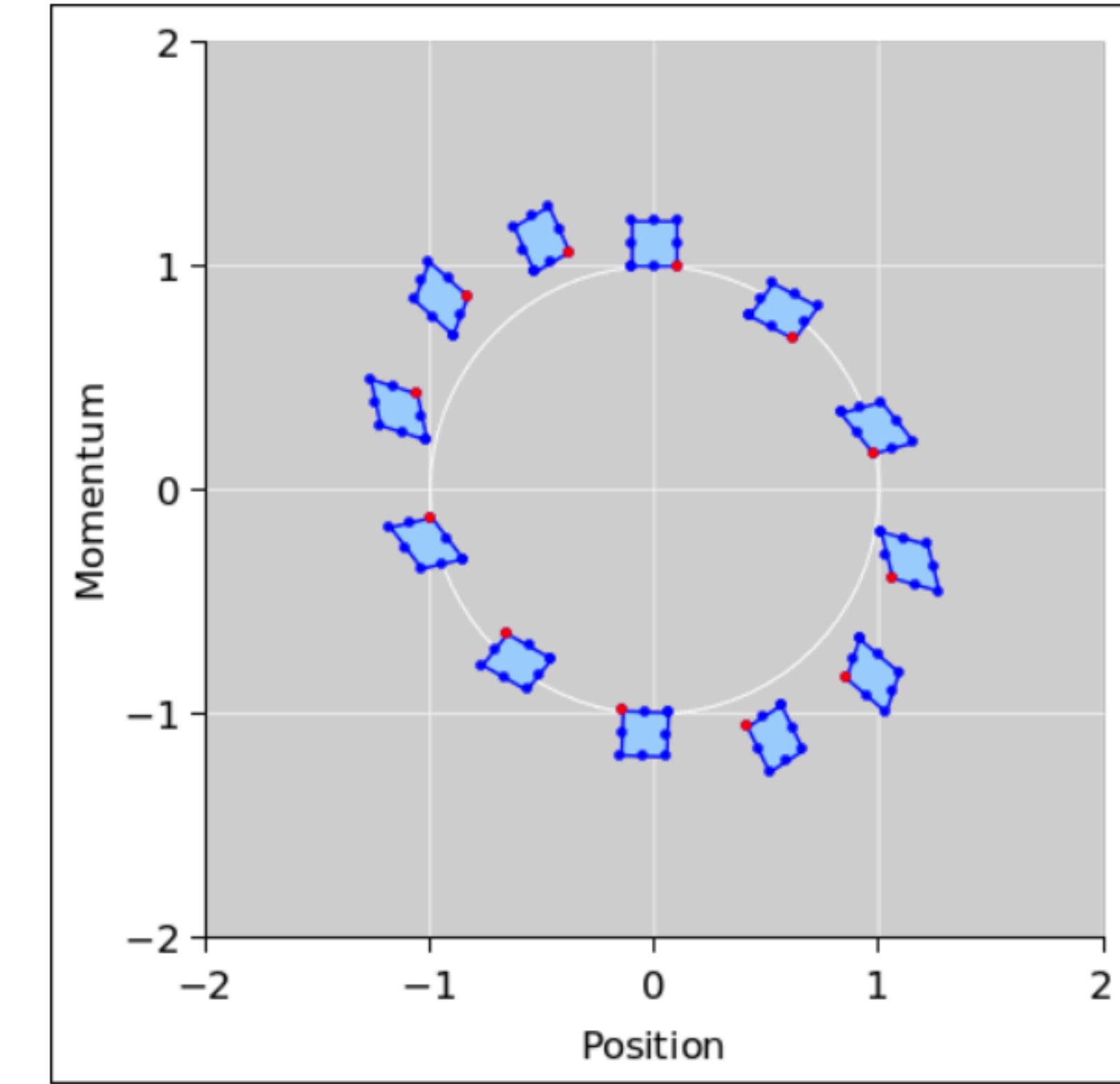
Symplectic Integrators



[Figure 1](#): Phase Space of a Harmonic Oscillator



[Figure 3](#): Non-Symplectic Integration : Harmonic Oscillator



[Figure 2](#): Symplectic Integration : Harmonic Oscillator

<https://www.av8n.com/physics/symplectic-integrator.htm>