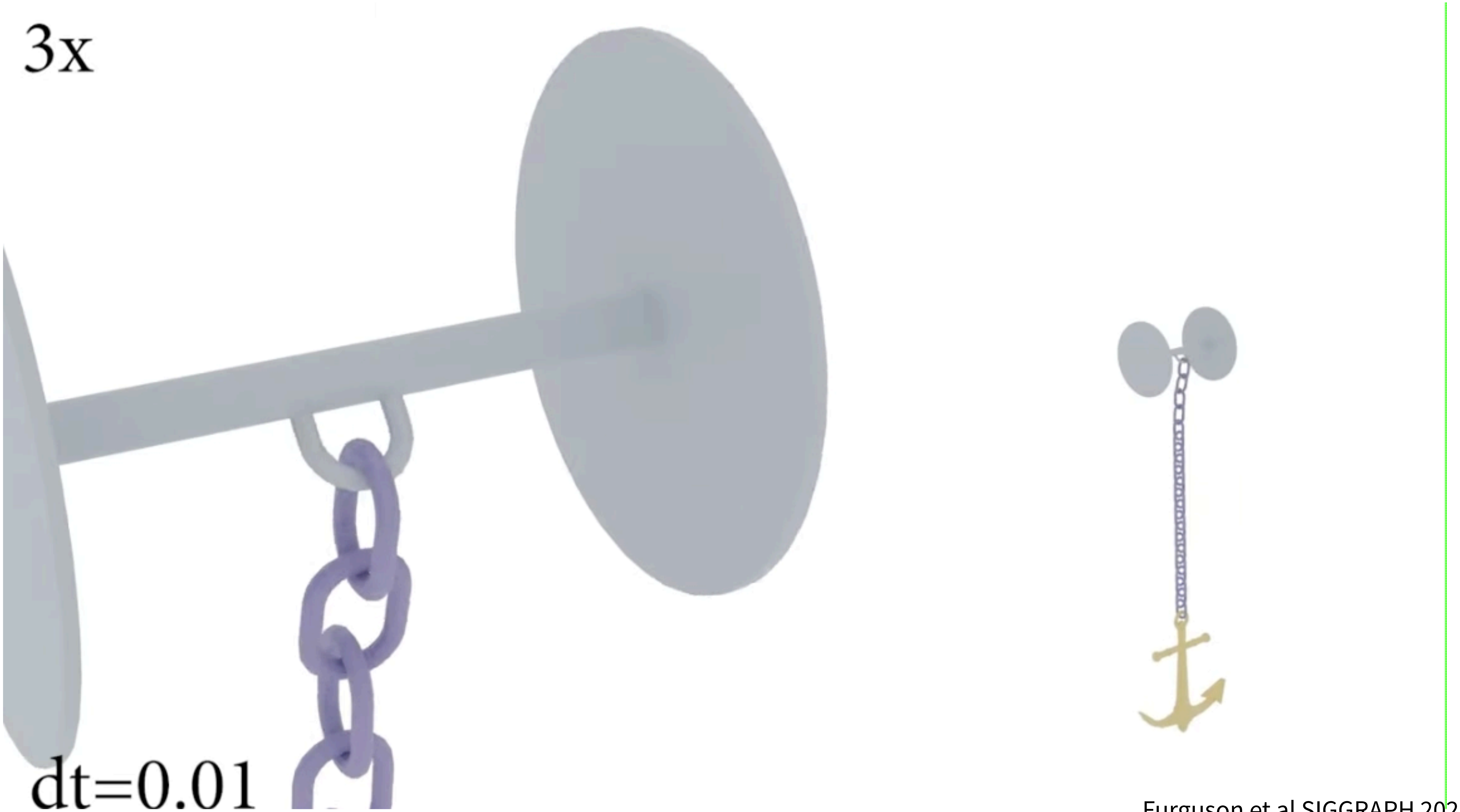
Lecture 13:

Constrained Rigid Body Systems

FUNDAMENTALS OF COMPUTER GRAPHICS

Animation & Simulation

Stanford CS248B, Fall 2022



Furguson et al SIGGRAPH 2021

Collision is detected! What now?

- Collision detector is responsible for returning a list of collisions at every time step.
- If the list is not empty, collision handler will take over and resolve the collisions.
- For each collision on the list, it should contains
  - IDs of a pair of rigid bodies in collision
  - Coordinate of the contact point
  - Normal vector at the contact point

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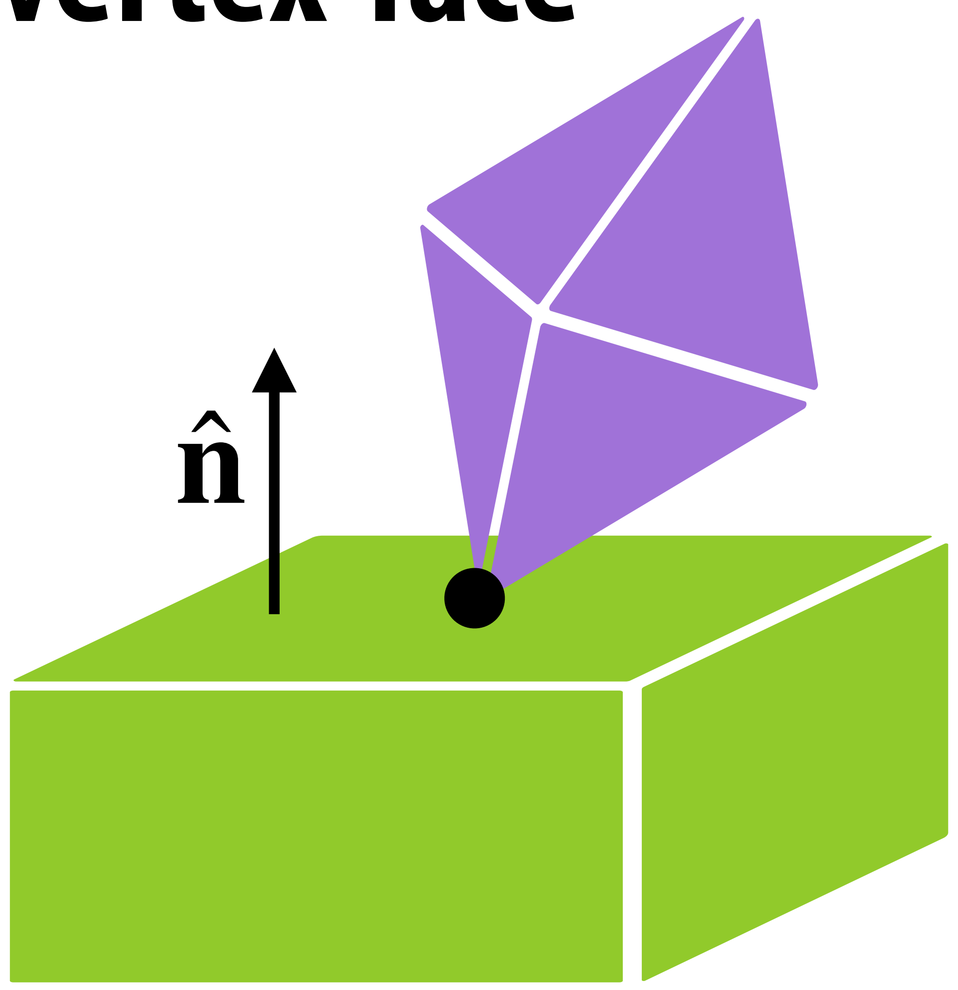
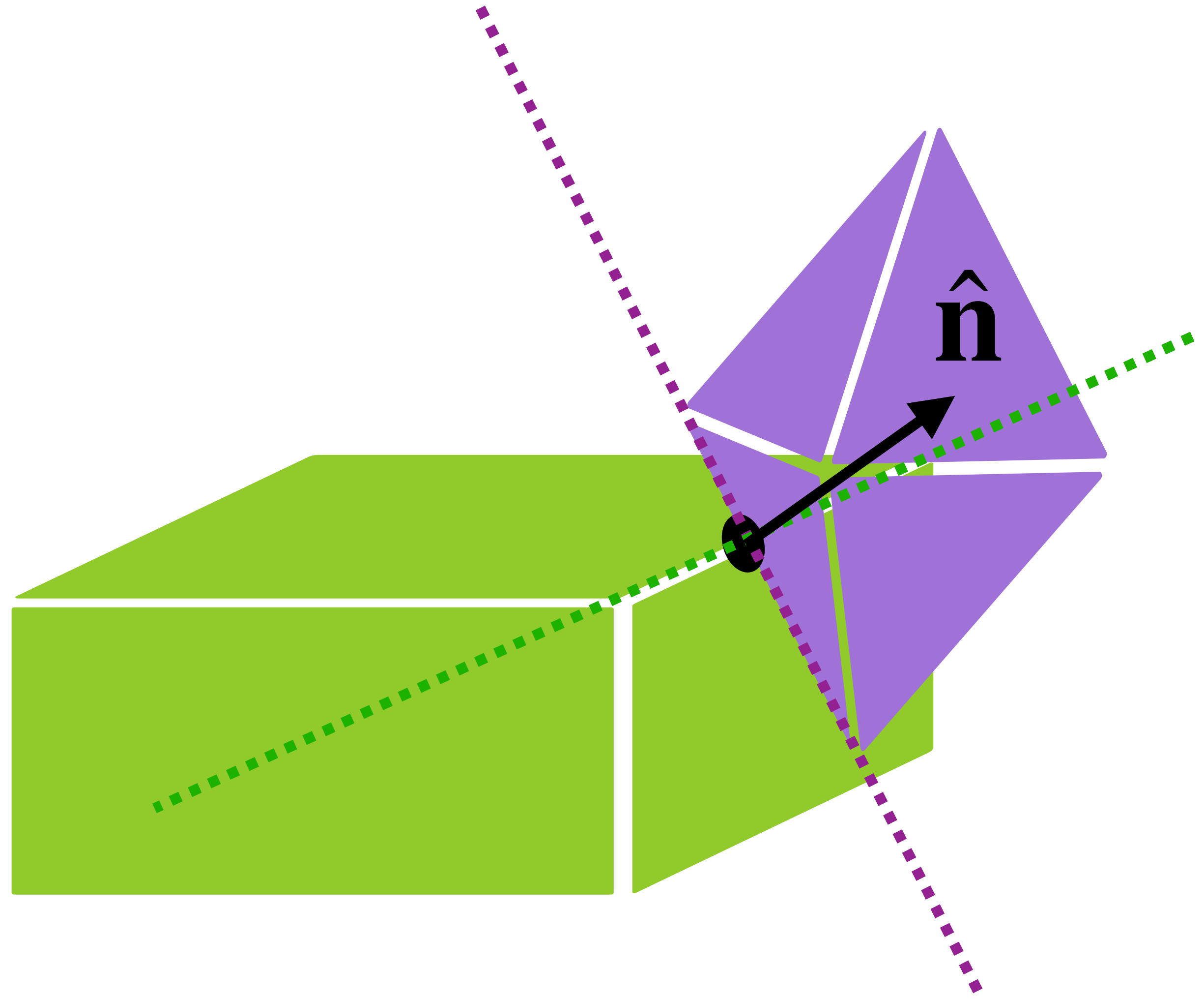
vertex-face

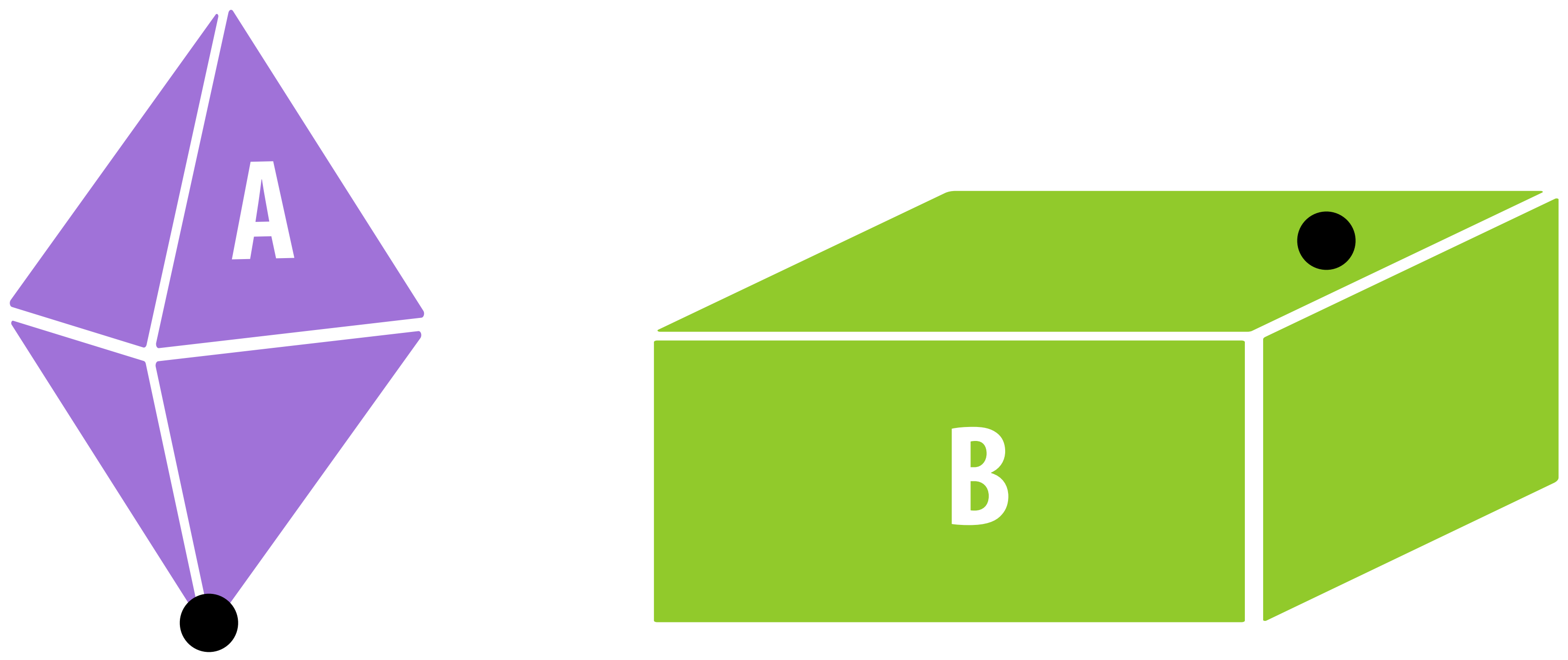


Edge-edge



Contact Points

Collision handler tells us that a point on A and a point on B are in collision



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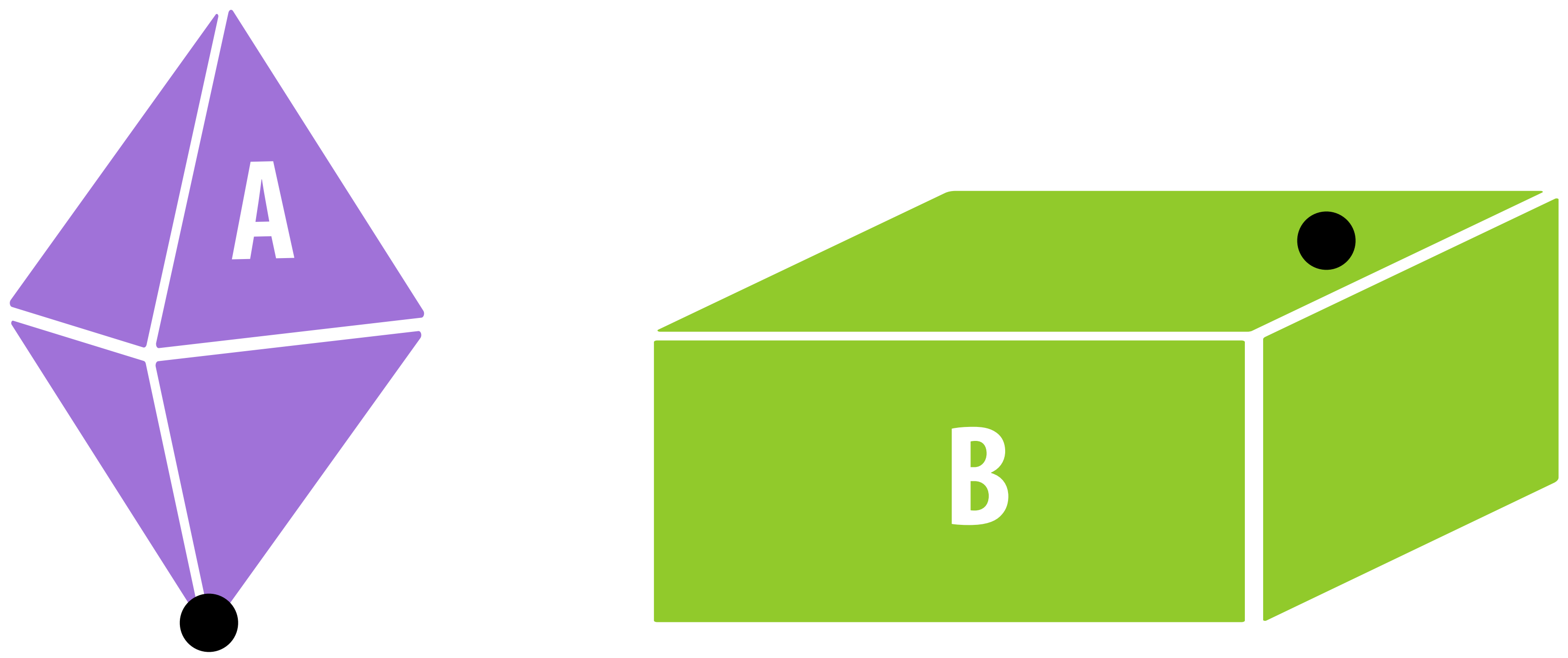
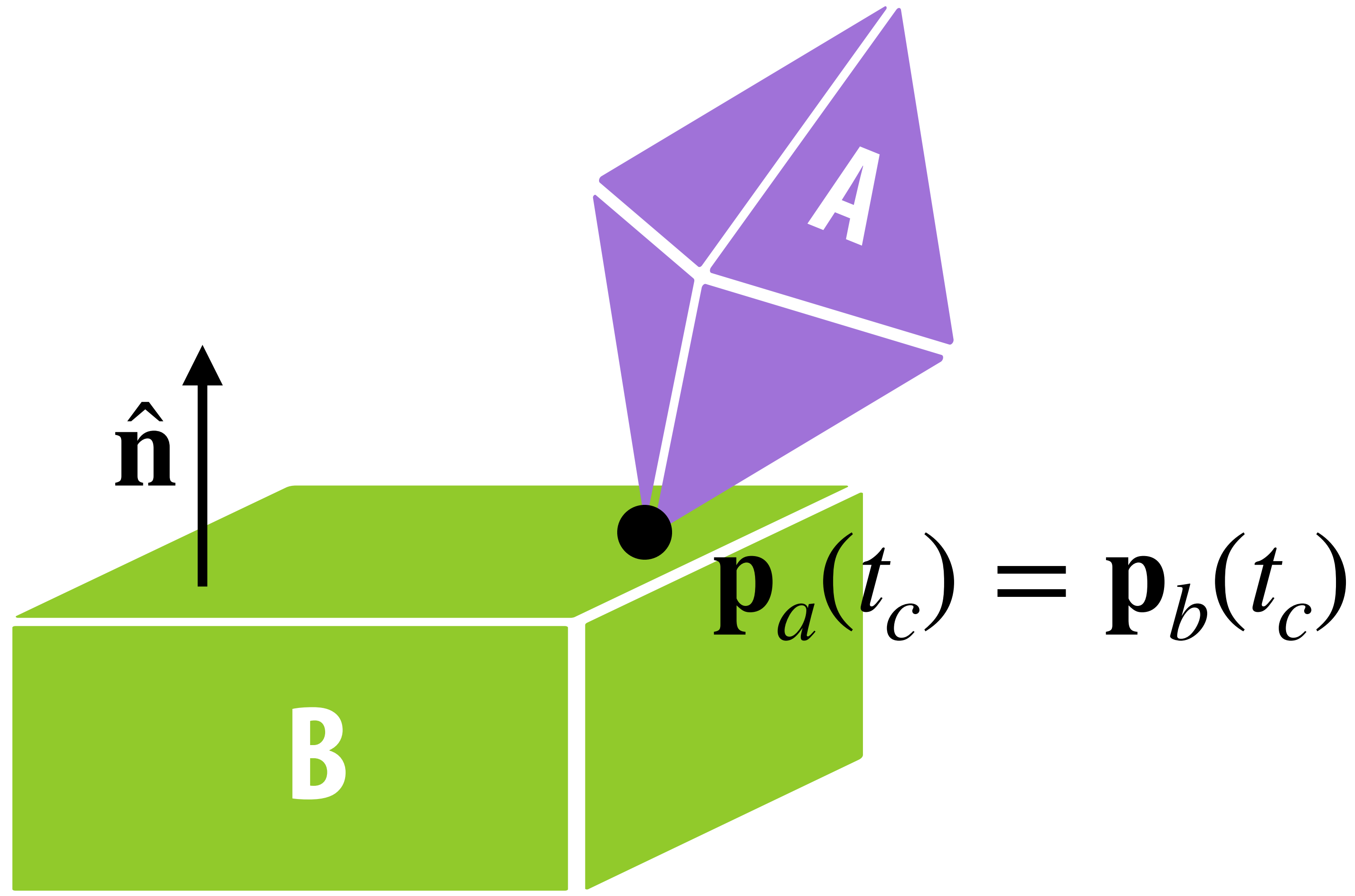


Put in the world space…

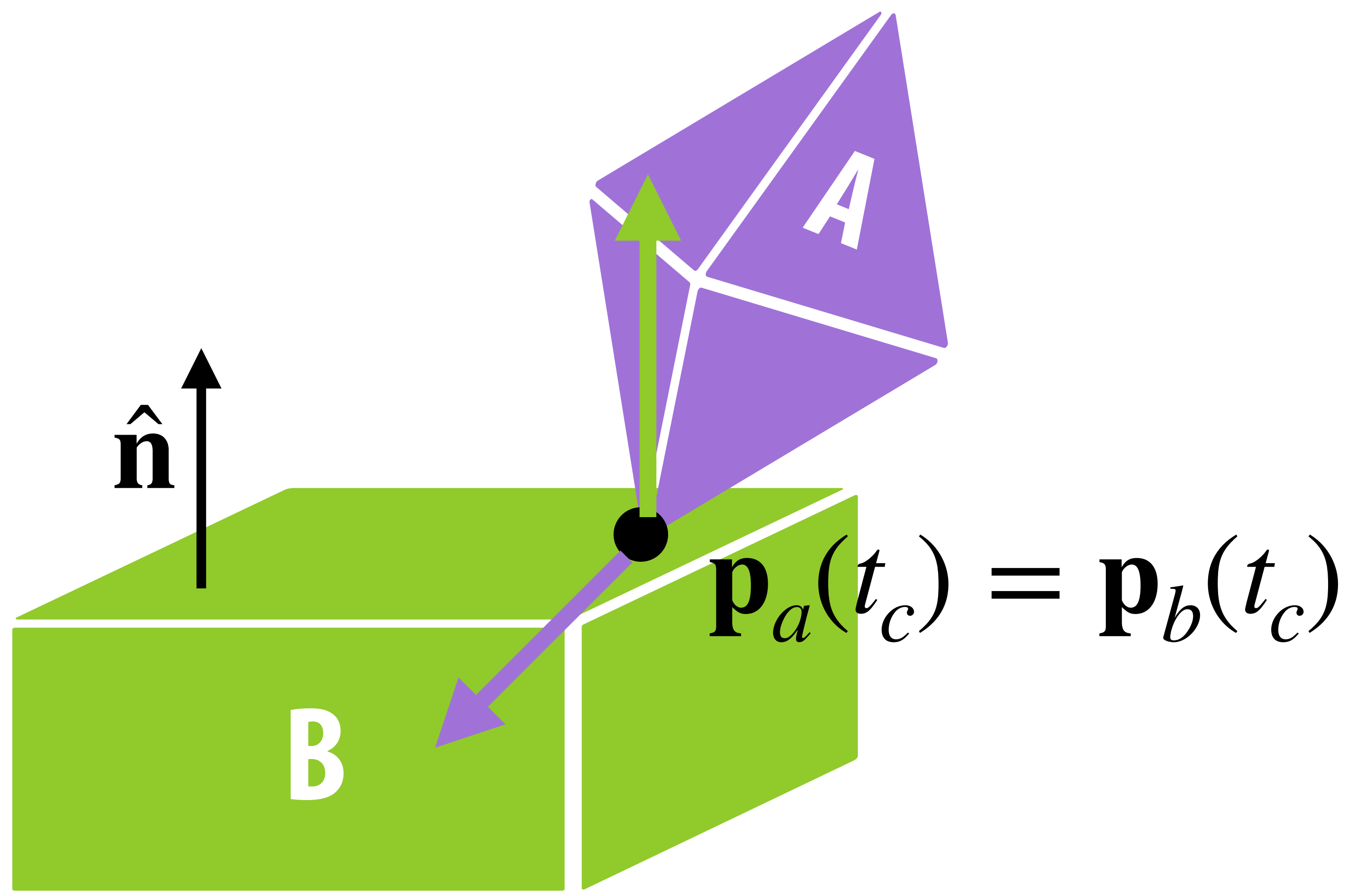


Although $\mathbf{p}_a$ and $\mathbf{p}_b$ are coincident at time t_c, the velocity of the two points may be different!

Velocity of a Contact Point

$$\dot{\mathbf{p}}_a(t_c) = \mathbf{v}_a(t_c) + \boldsymbol{\omega}_a(t_c) \times \left(\mathbf{p}_a(t_c) - \mathbf{x}_a(t_c)\right)$$

$$\dot{\mathbf{p}}_b(t_c) = \mathbf{v}_b(t_c) + \boldsymbol{\omega}_b(t_c) \times \left(\mathbf{p}_b(t_c) - \mathbf{x}_b(t_c)\right)$$



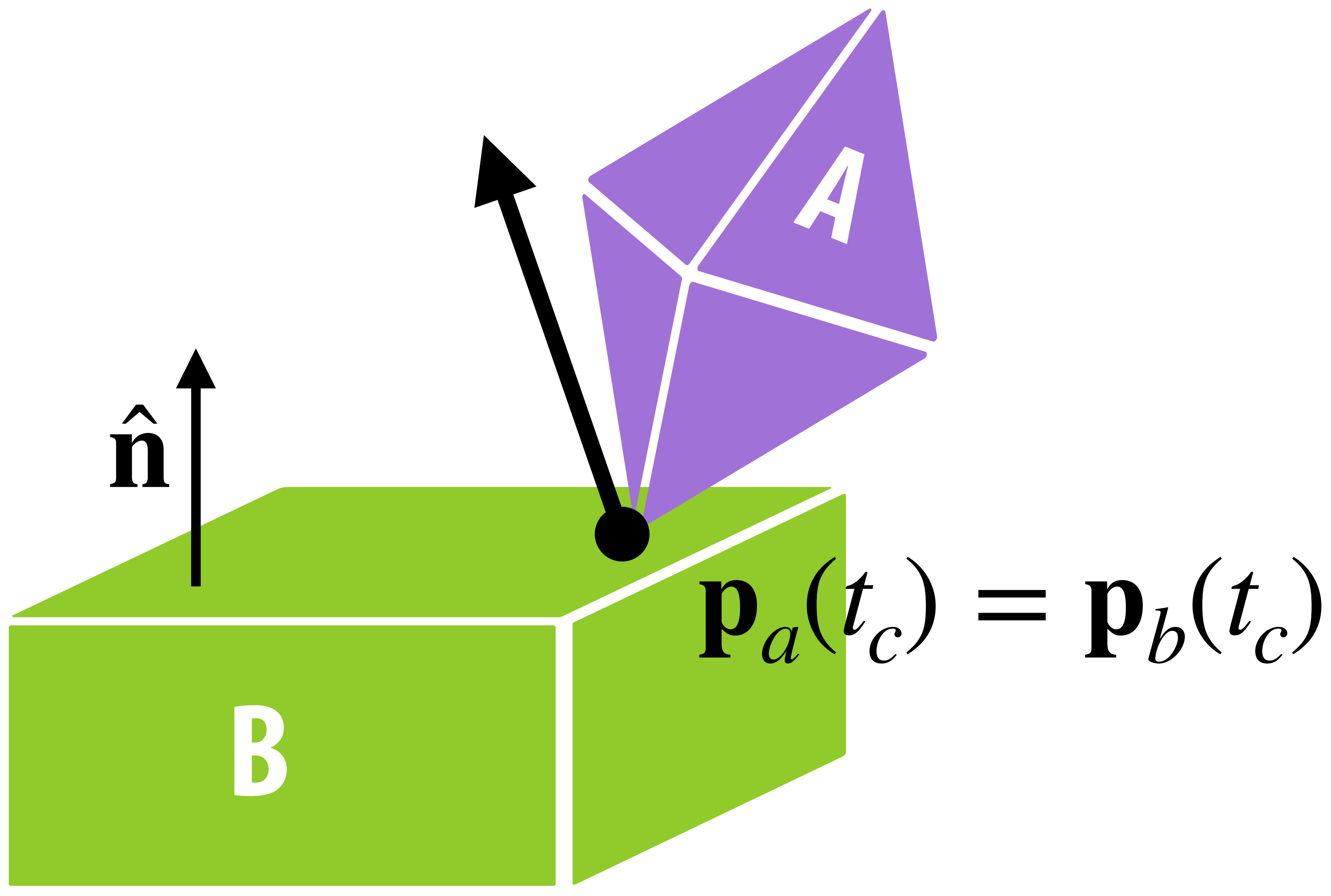
$$v_r = \hat{\mathbf{n}} \cdot \left(\dot{\mathbf{p}}_a(t_c) - \dot{\mathbf{p}}_b(t_c)\right)$$

v_r is the magnitude of the relative velocity in the normal direction

Relative Normal Velocity

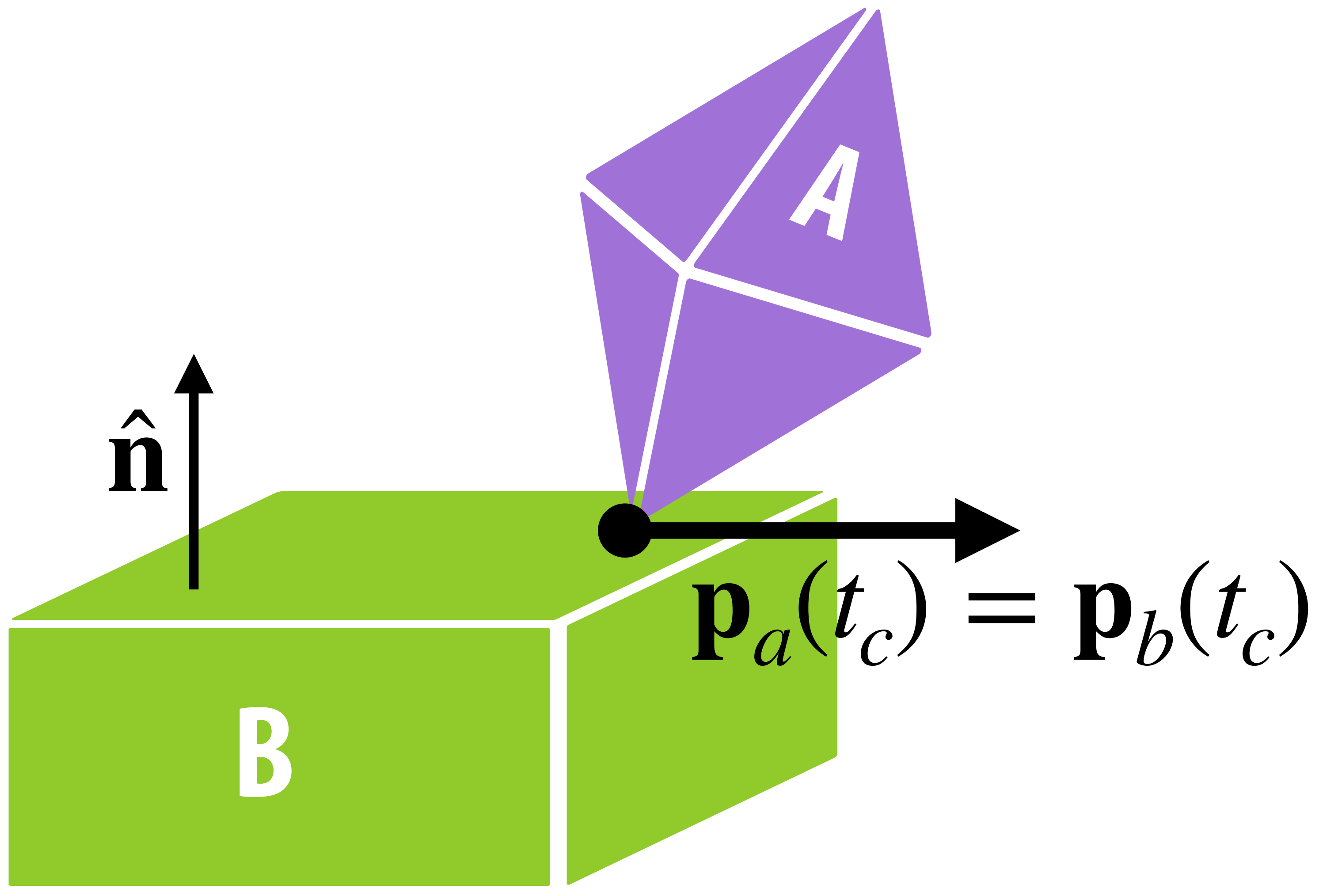
$v_r > 0$

separation



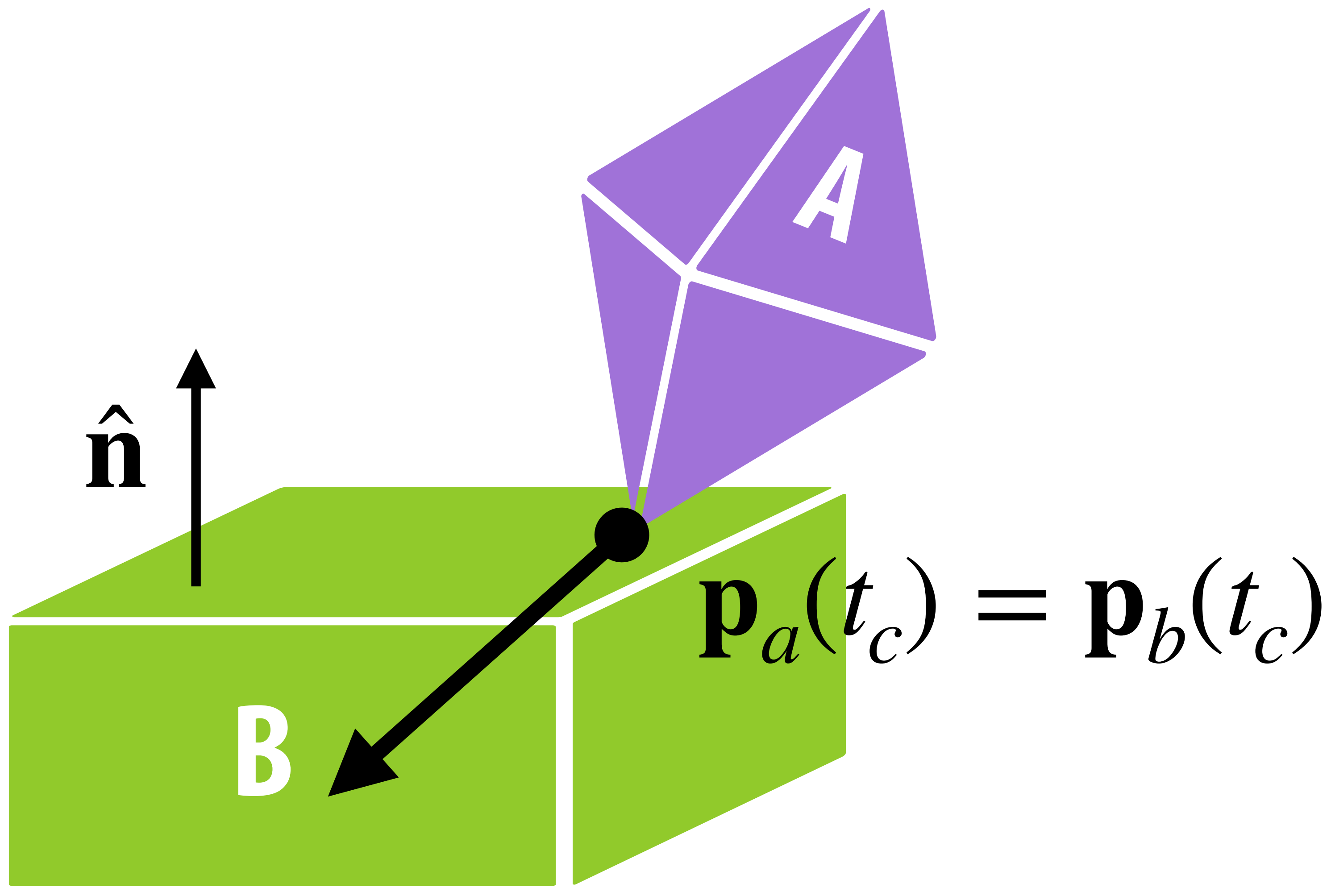
$v_r = 0$

resting contact



$v_r < 0$

colliding contact



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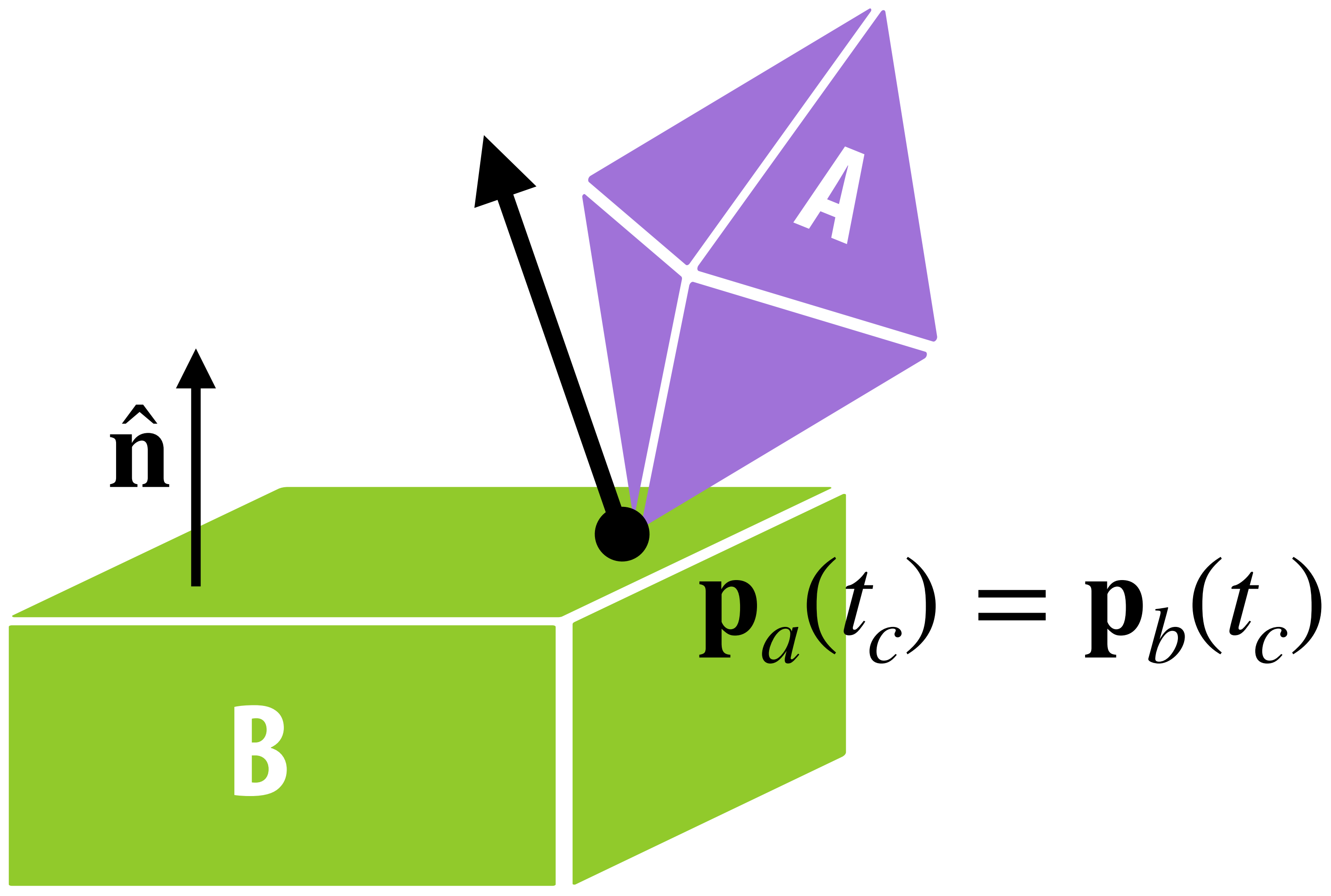
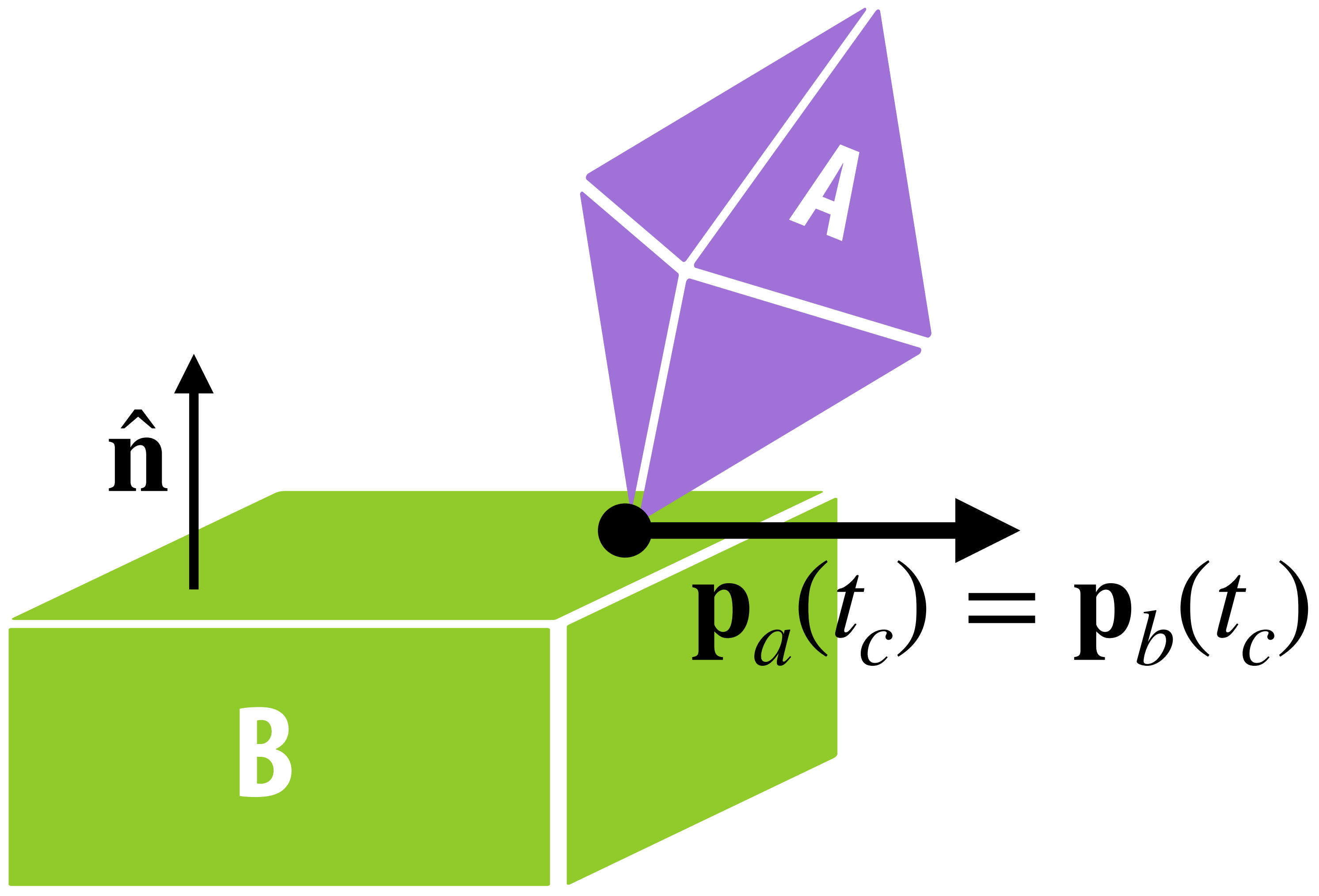


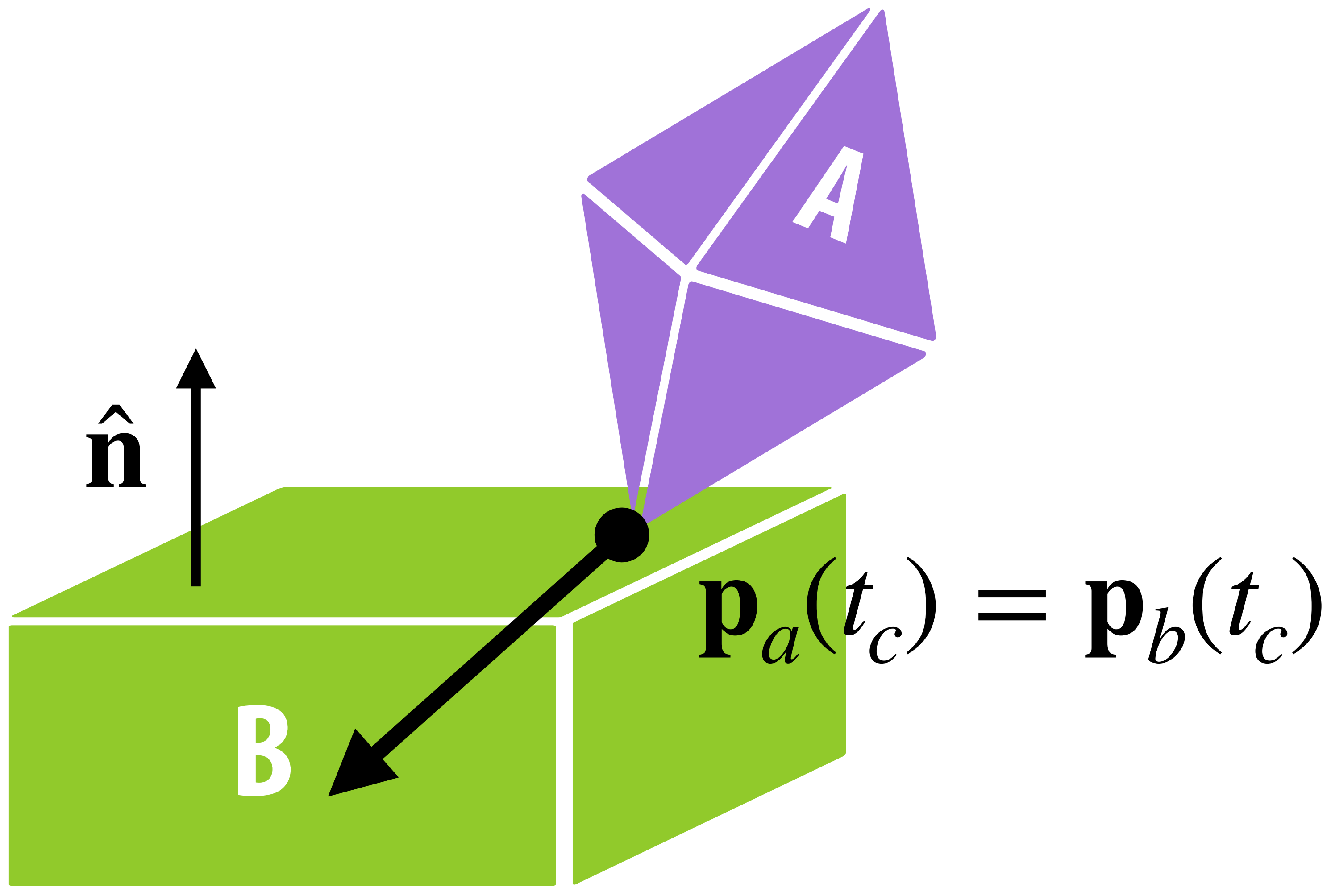
$v_r = 0$

resting contact

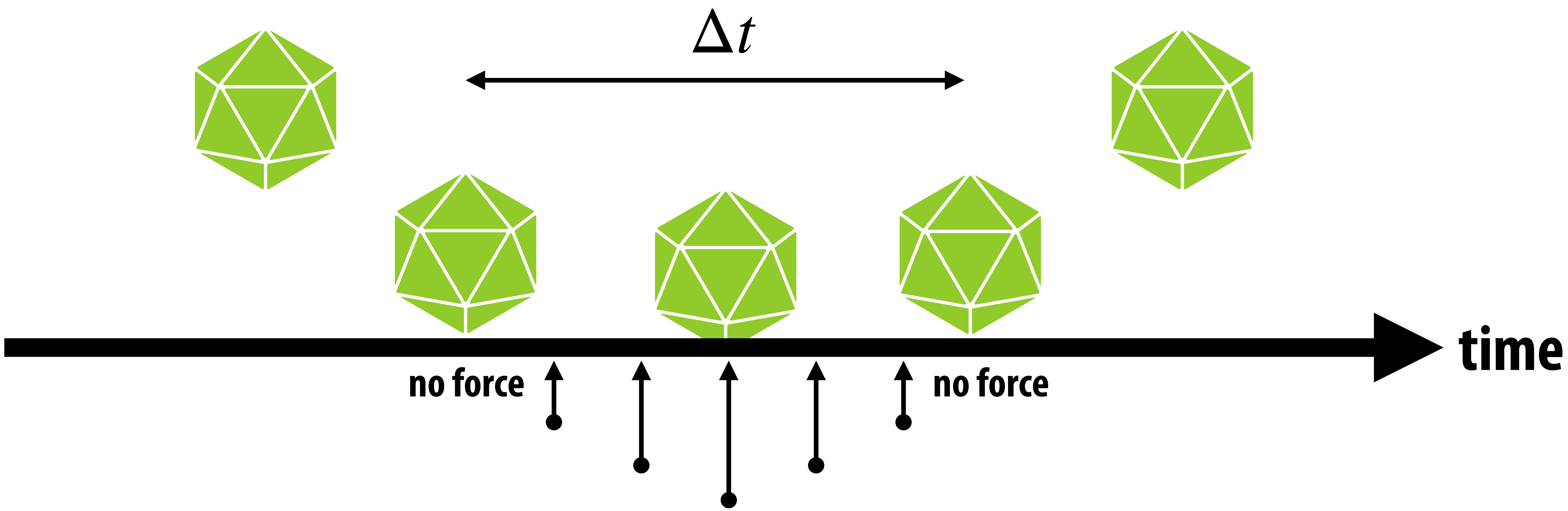


$v_r < 0$

colliding contact

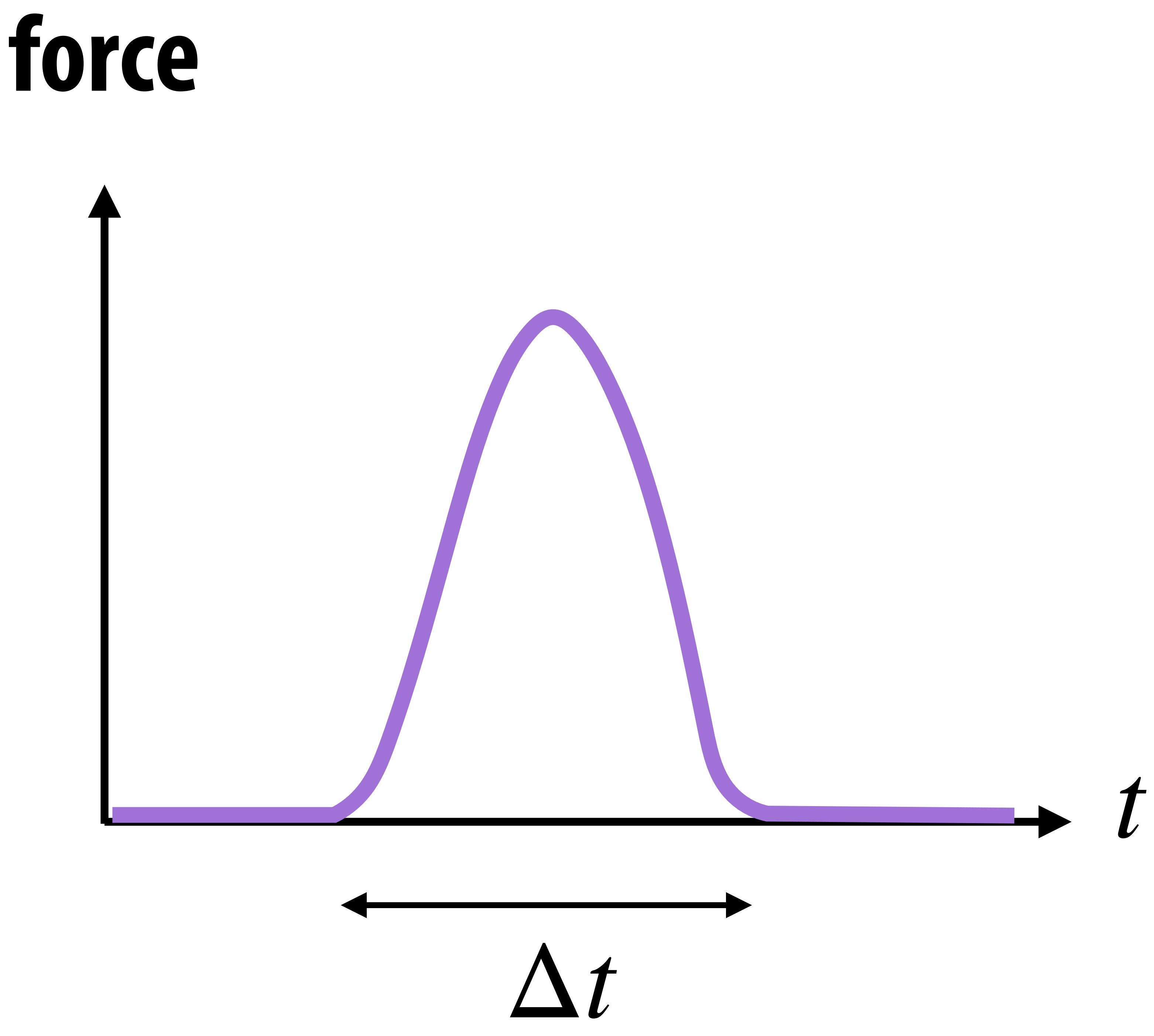


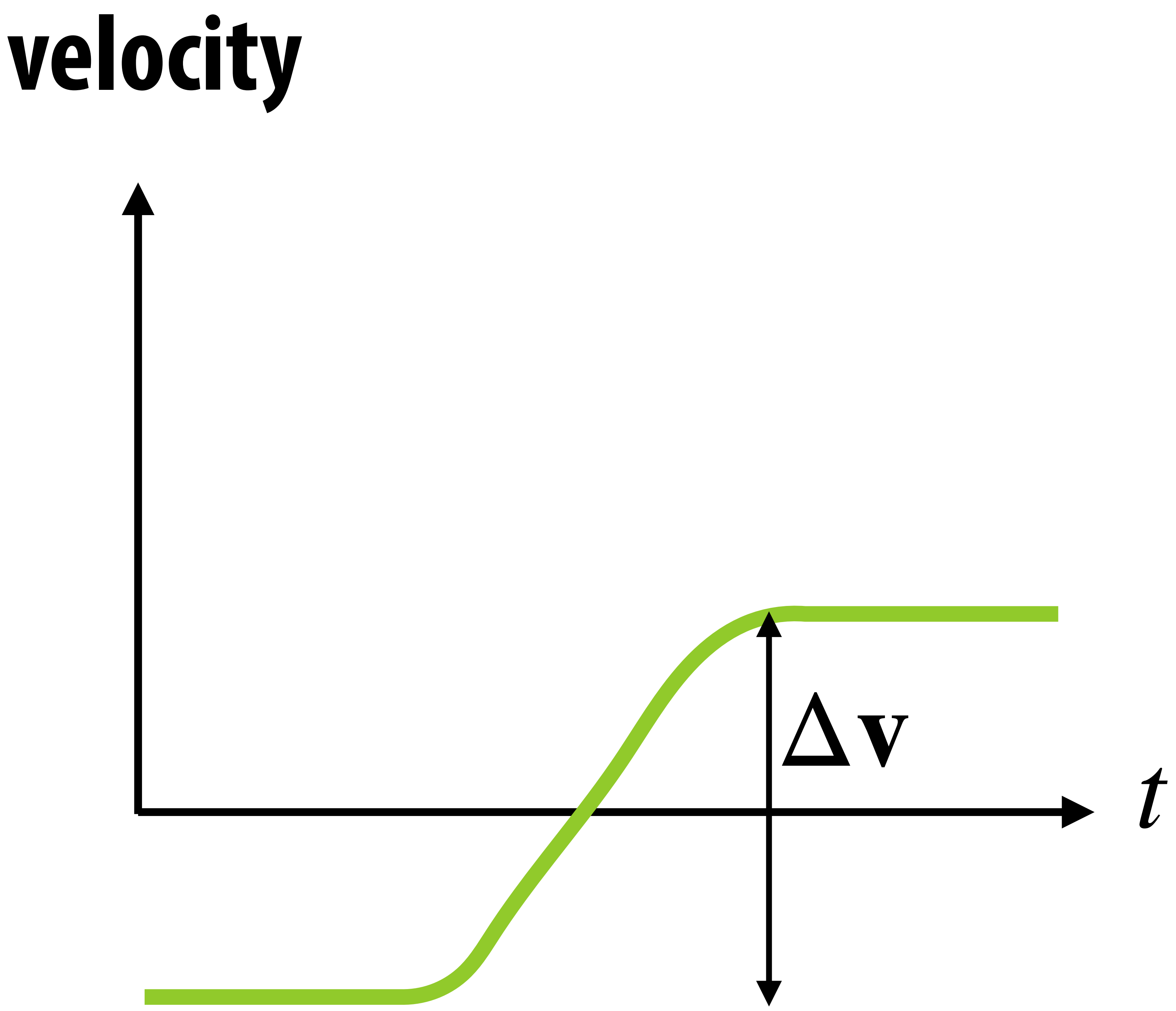
Collision Process



$$\mathbf{J} \equiv \int_0^{\Delta t} \mathbf{f}_t \, dt = m\Delta\mathbf{v}$$

A Soft Collision

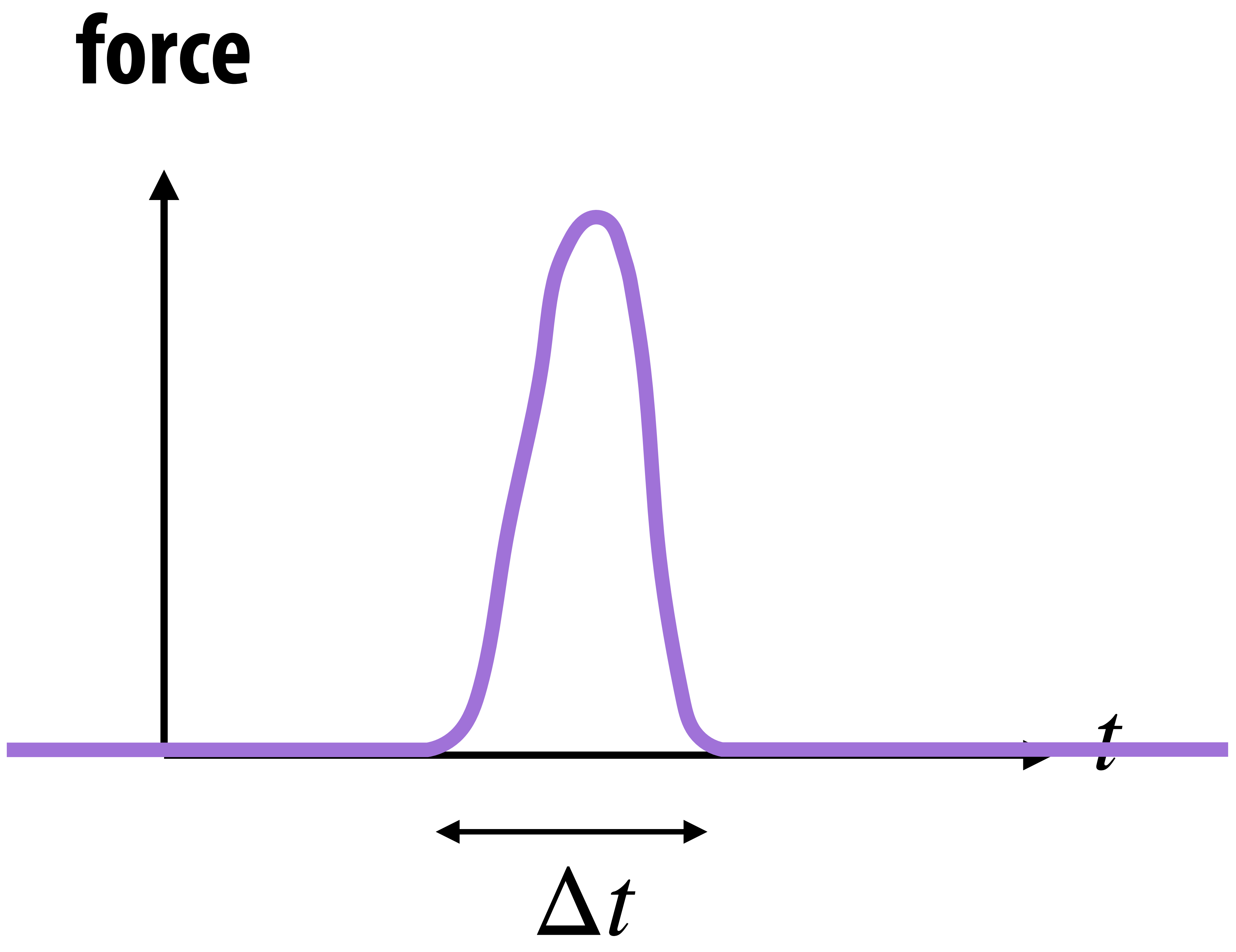


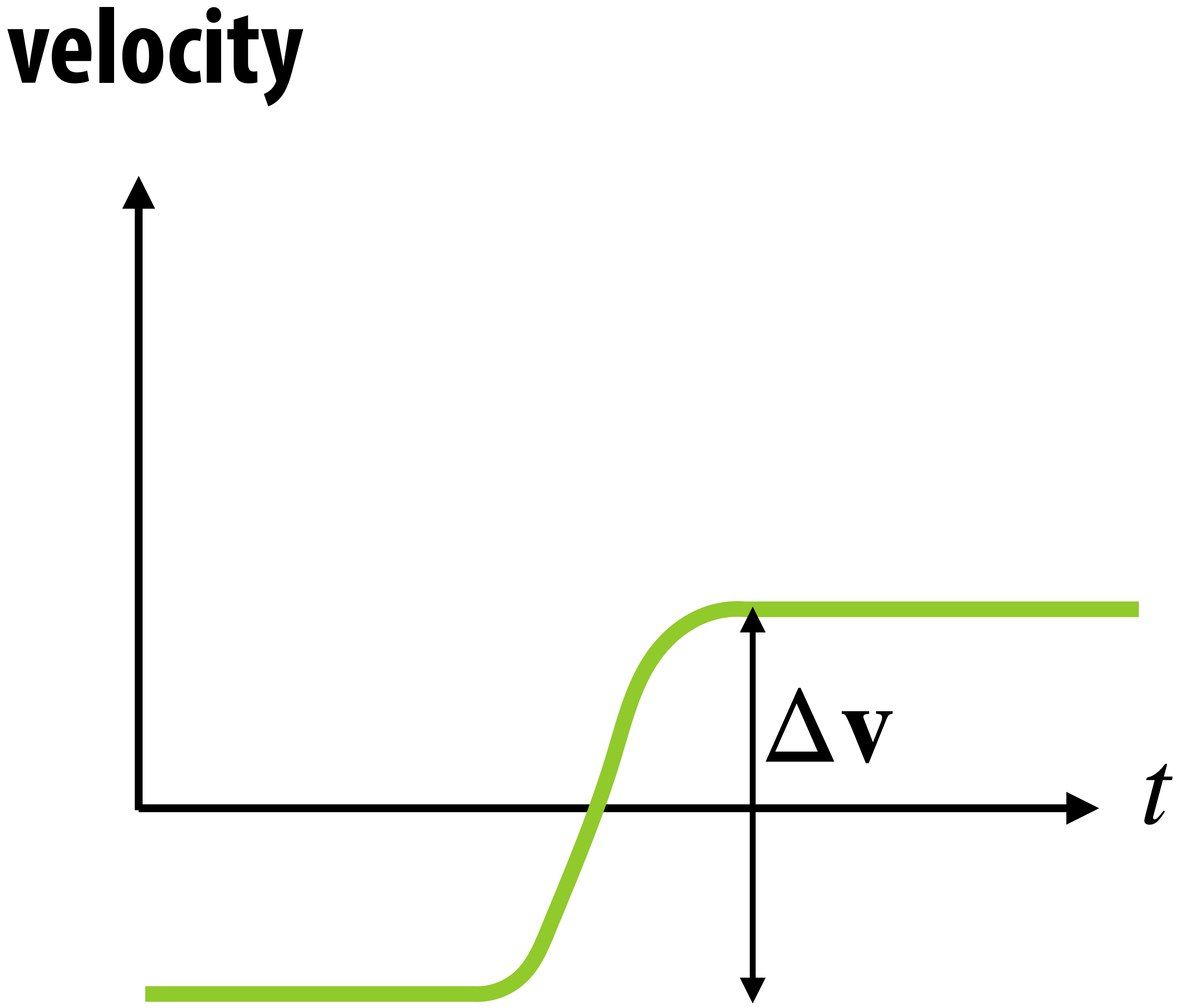


$$\mathbf{J} = \int_0^{\Delta t} \mathbf{f}_t \, dt$$

$$\mathbf{J} = m\Delta\mathbf{v}$$

A Hard Collision

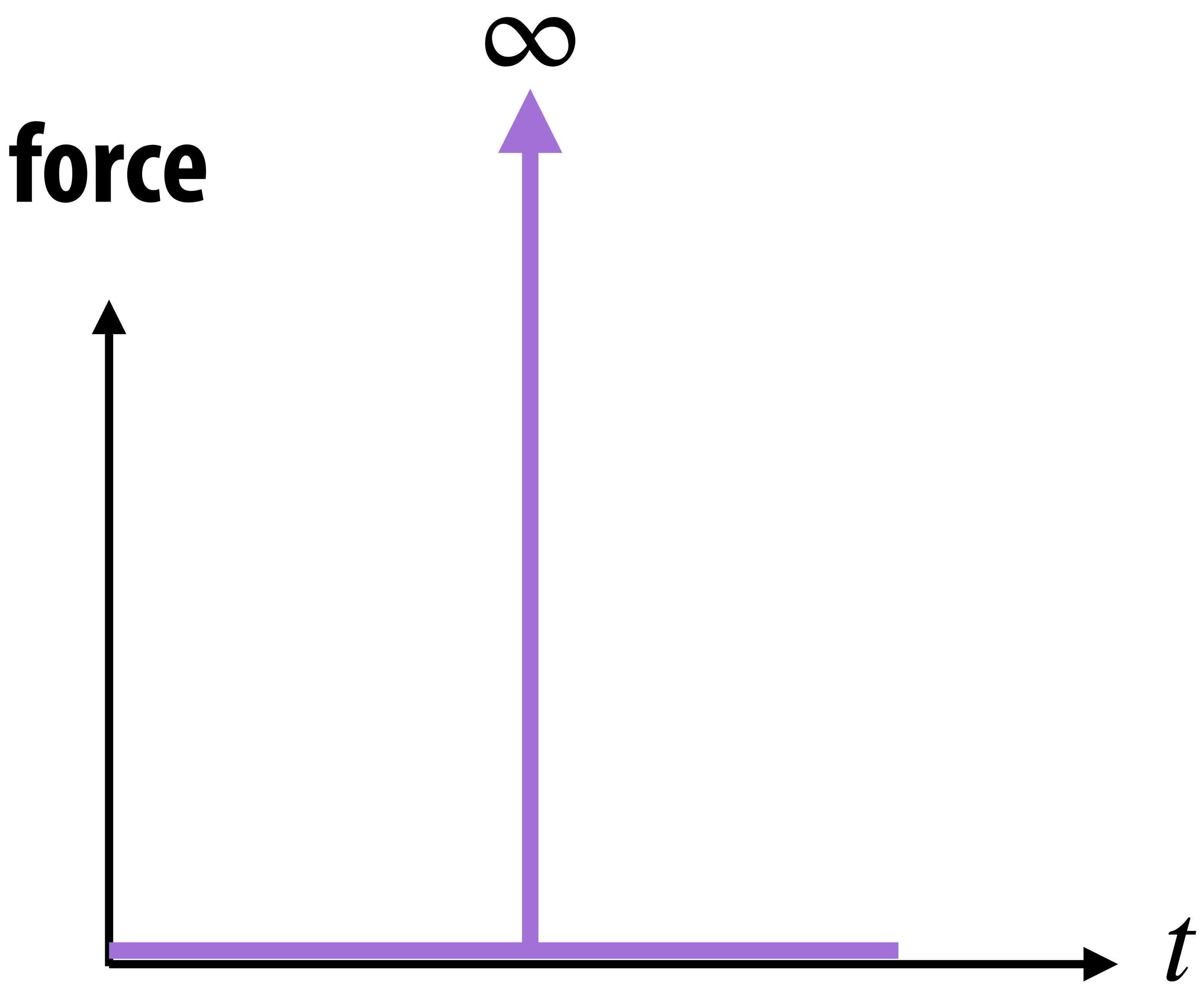




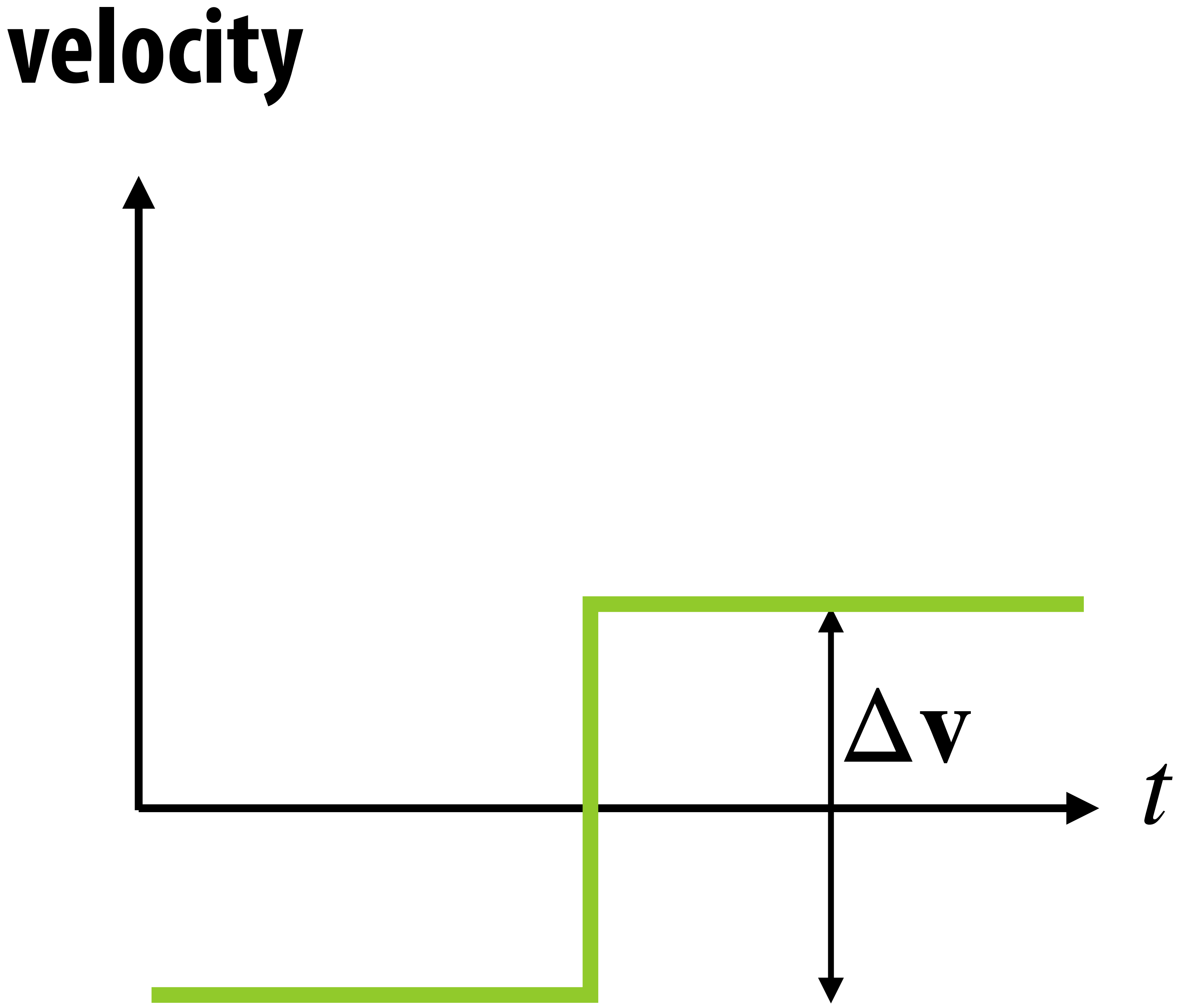
$$\mathbf{J} = \int_0^{\Delta t} \mathbf{f}_t \, dt$$

$$\mathbf{J} = m\Delta\mathbf{v}$$

An Infinitely Hard Collision



$\Delta t = 0$



$\mathbf{J} = ?$

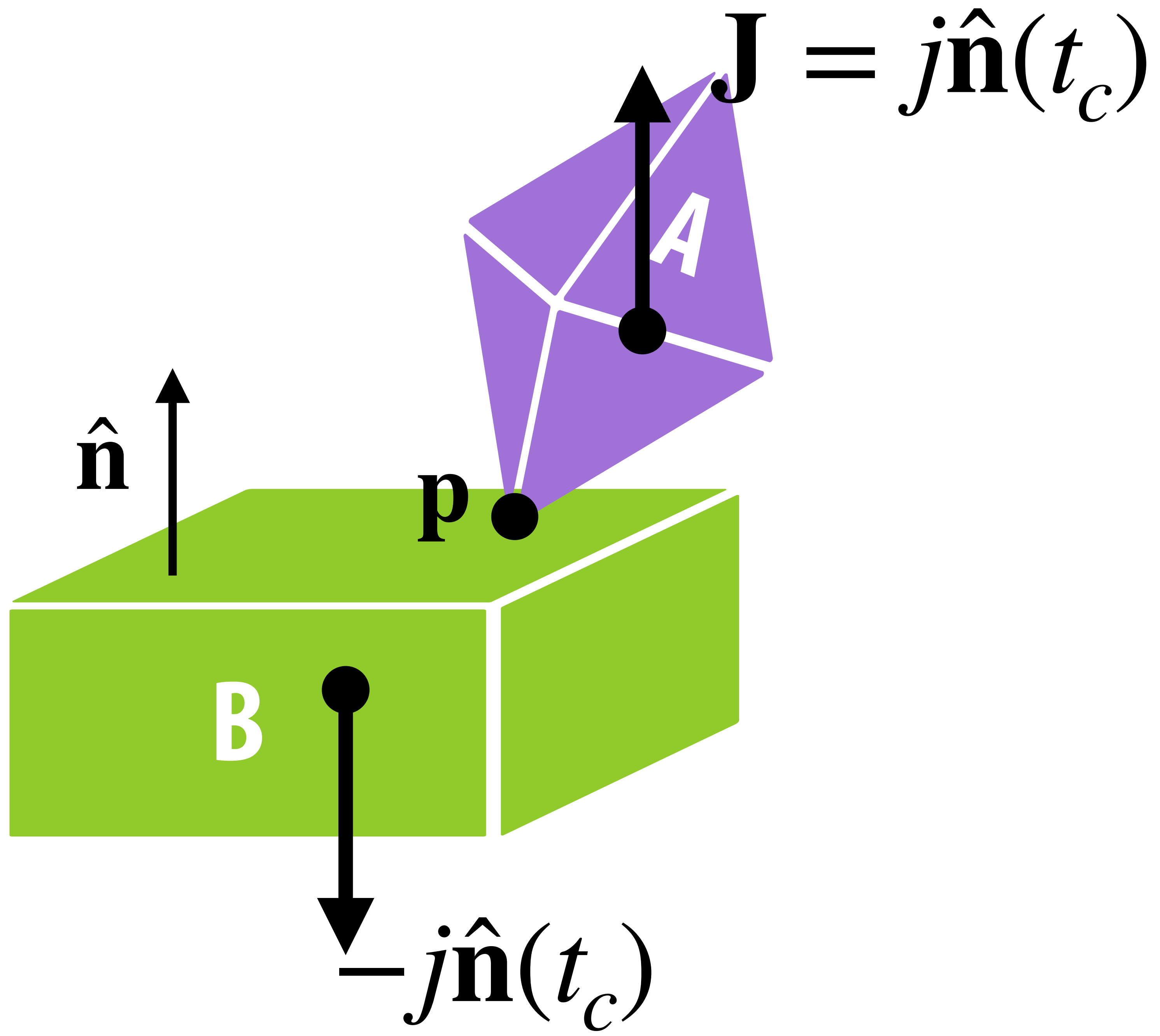
$\mathbf{J} = m\Delta\mathbf{v}$

Impulse

- In the rigid body world, we want the velocity to change instantaneously if there is a collision contact.
- Use finite impulse to change velocity instead of infinite force: $\mathbf{J} = \Delta\mathbf{P} = m\Delta\mathbf{v}$
- If the impulse acts on a point $\mathbf{p}$, the impulse produces an impulsive torque
  - $\boldsymbol{\tau}_{imp} = \big(\mathbf{p} - \mathbf{x}(t)\big) \times \mathbf{J}$
  - Impulsive torque results in a change in angular momentum: $\boldsymbol{\tau}_{imp} = \Delta\mathbf{L}$

Colliding Contact

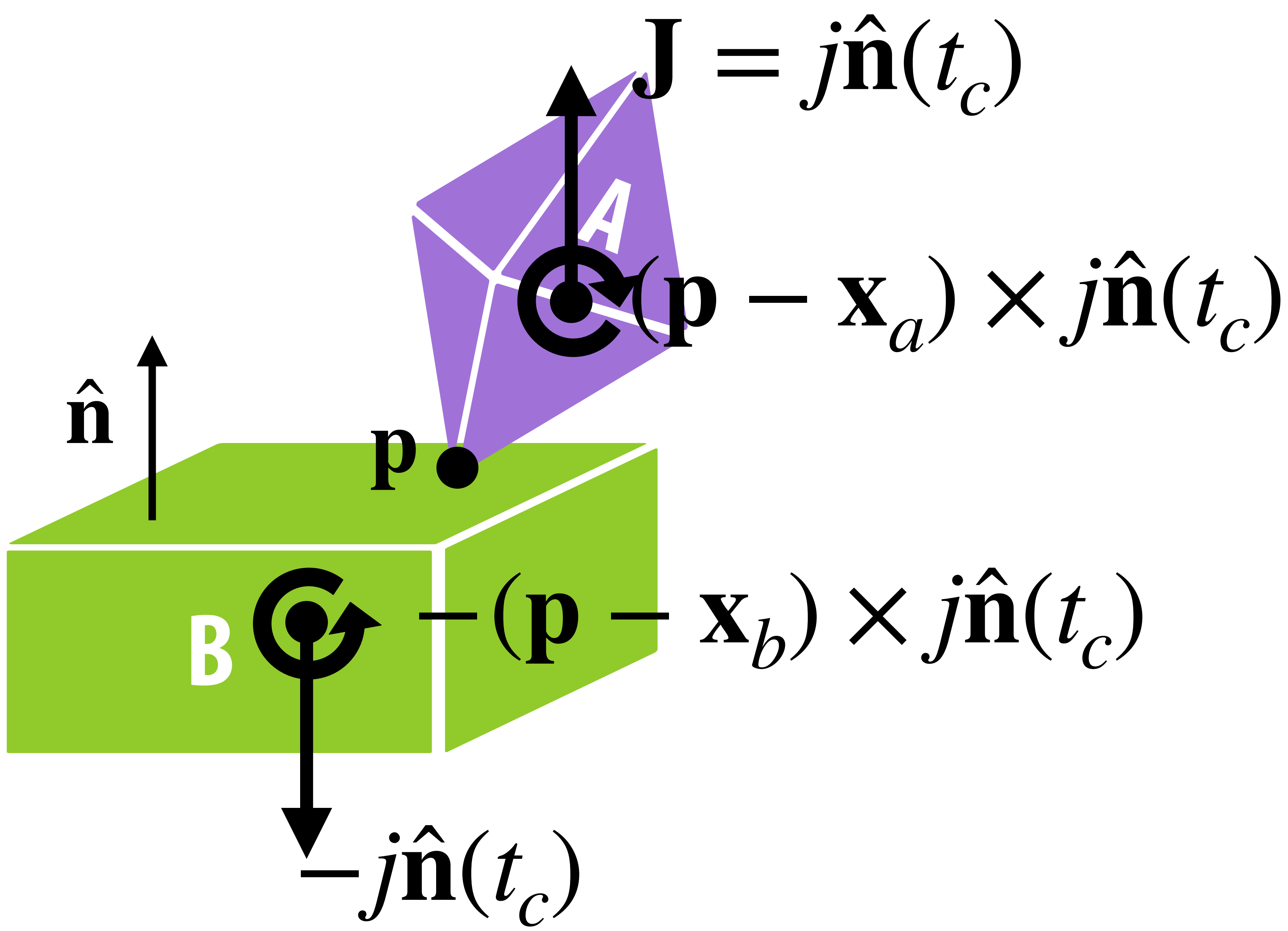
- For frictionless bodies, the direction of the impulse will be in the normal direction $\hat{\mathbf{n}}(t_c)$.



- If we solve for j, we then can update the linear momentum of the rigid body after the collision.
- Body A is subject to impulse $\mathbf{J}$, while B is subject to an equal but opposite impulse $-\mathbf{J}$

Colliding Contact

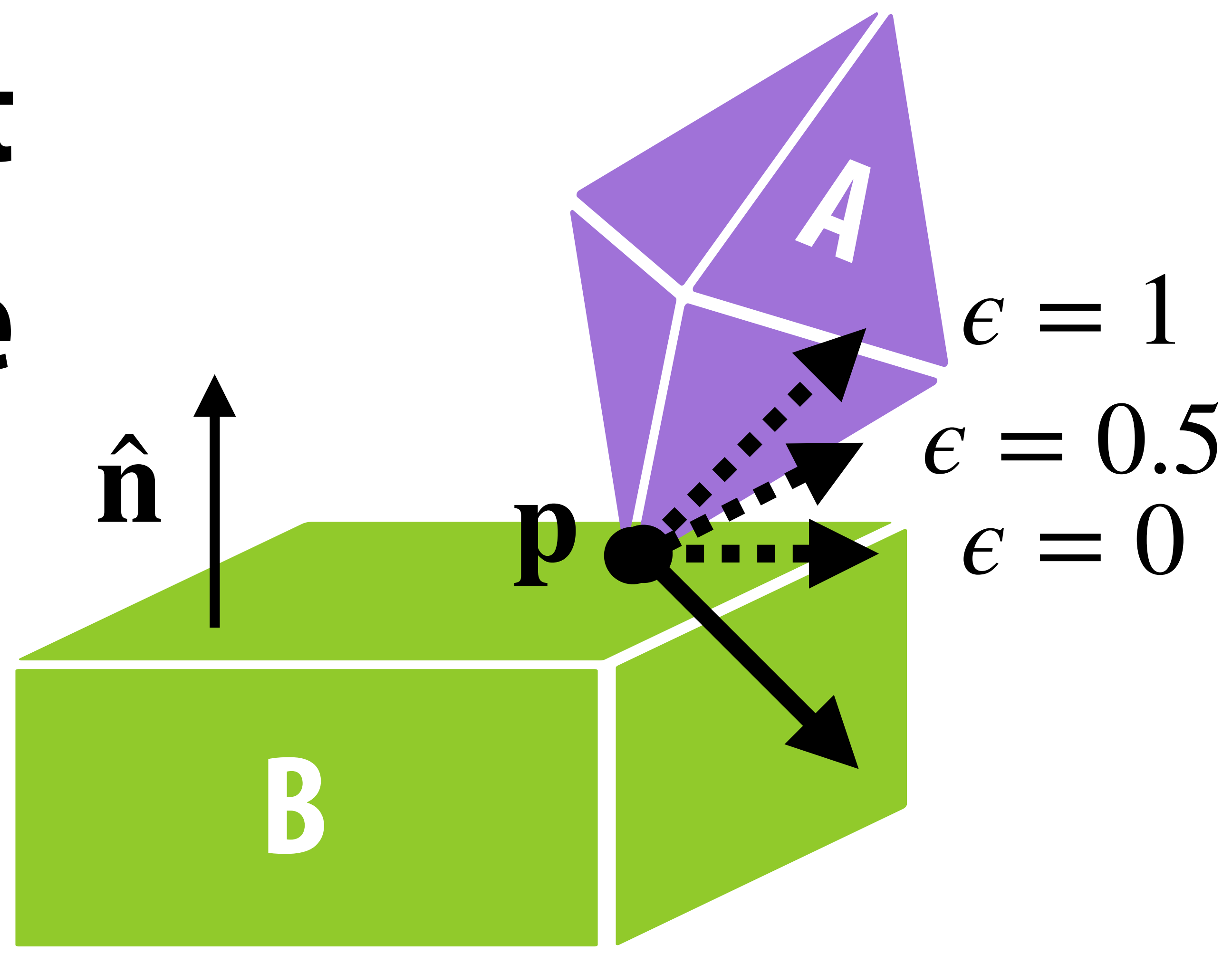
- Similarly, we use impulsive torque to update the angular moment of the rigid bodies



How to solve j?

Colliding Contact

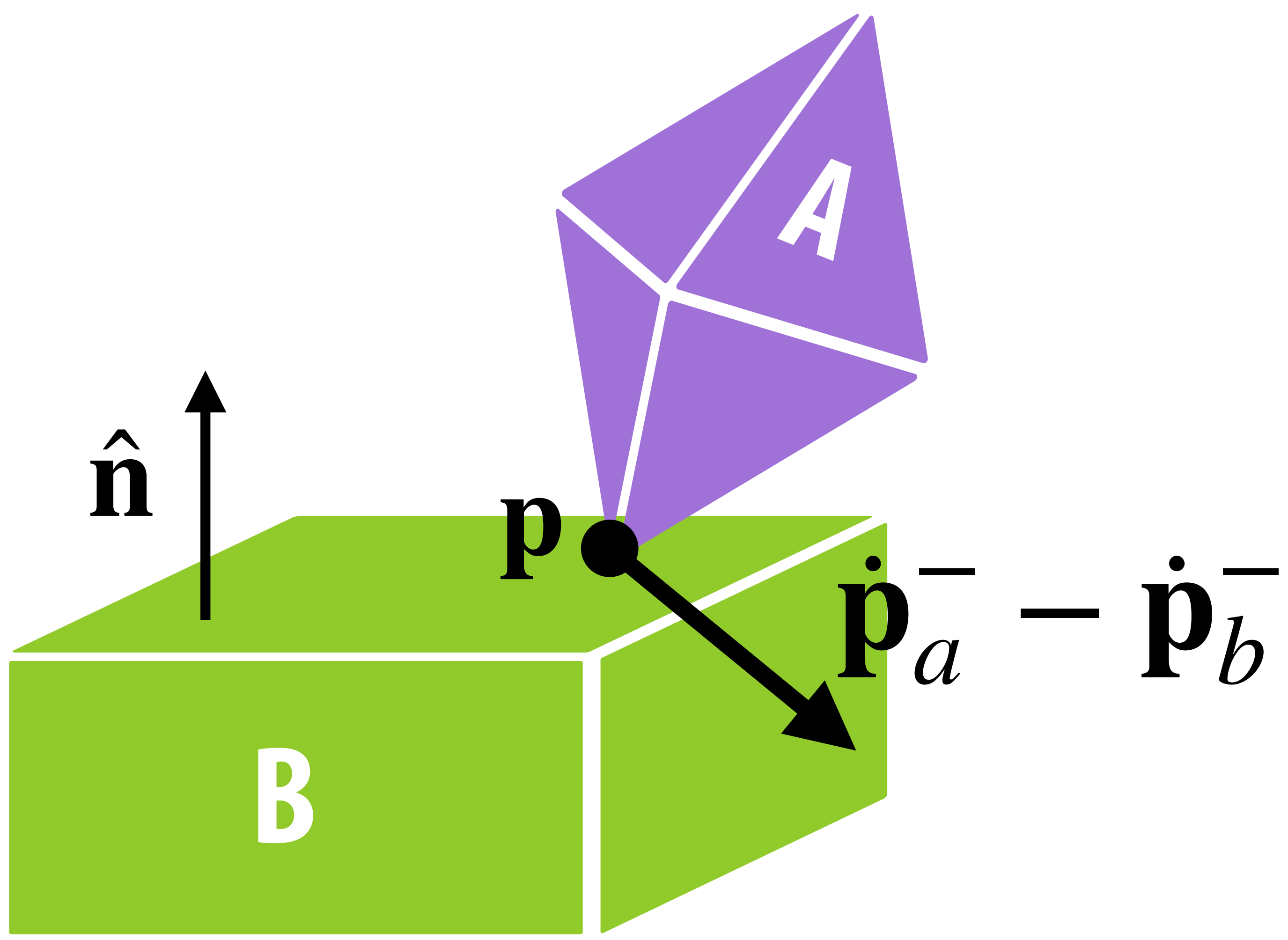
- The change of velocity at the contact point follows the empirical law: $\boxed{v_r^+ = -\epsilon v_r^-}$
- Coefficient of restitution
  - $\epsilon = 0$, resting contact
  - $\epsilon = 1$, perfect bounce



We need to solve for j such that $v_r^+ = -\epsilon v_r^-$

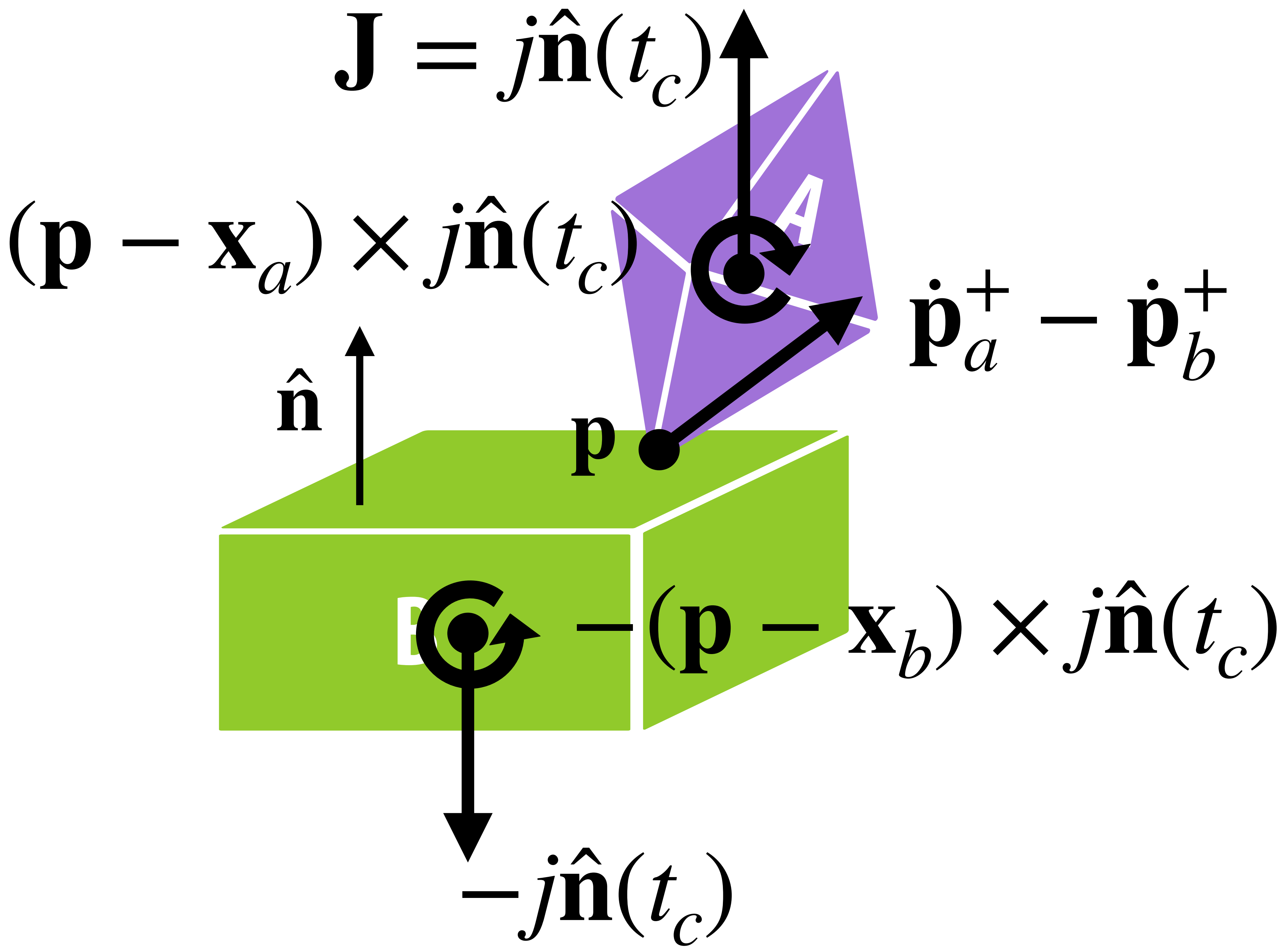
Colliding Contact

before collision



$$v_r^- = \hat{\mathbf{n}}(t_c) \cdot (\dot{\mathbf{p}}_a^- - \dot{\mathbf{p}}_b^-)$$

after collision



$$v_r^+ = \hat{\mathbf{n}}(t_c) \cdot (\dot{\mathbf{p}}_a^+ - \dot{\mathbf{p}}_b^+)$$

Compute the Impulse

- **Define the displacement from center of mass**
  - $\mathbf{r}_a = \mathbf{p}_a - \mathbf{x}_a$
  - $\mathbf{r}_b = \mathbf{p}_b - \mathbf{x}_b$
- **Express contact point velocity in rigid body velocity**
  - $\dot{\mathbf{p}}_a^- = \mathbf{v}_a^- + \boldsymbol{\omega}_a^- \times \mathbf{r}_a$, **similar for** $\dot{\mathbf{p}}_b^-$
  - $\dot{\mathbf{p}}_a^+ = \mathbf{v}_a^+ + \boldsymbol{\omega}_a^+ \times \mathbf{r}_a$, **similar for** $\dot{\mathbf{p}}_b^+$
- **Express post-collision velocity in unknown impulse**
  - $\mathbf{v}_a^+ = \mathbf{v}_a^- + \dfrac{j\hat{\mathbf{n}}}{m_a}$, **similar for** $\mathbf{v}_b^+$
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Substitute post-collision rigid body velocity

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Substitute post-collision rigid body velocity

$$\dot{\mathbf{p}}_a^+ = \mathbf{v}_a^- + \frac{j\hat{\mathbf{n}}}{m_a} + \left(\boldsymbol{\omega}_a^- + \mathbf{I}_a^{-1}(\mathbf{r}_a \times j\hat{\mathbf{n}})\right) \times \mathbf{r}_a$$

Compute the Impulse

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Substitute post-collision rigid body velocity

$$\dot{\mathbf{p}}_a^+ = \boxed{\mathbf{v}_a^-} + \frac{j\hat{\mathbf{n}}}{m_a} + \left(\boxed{\boldsymbol{\omega}_a^-} + \mathbf{I}_a^{-1}(\mathbf{r}_a \times j\hat{\mathbf{n}})\right)\boxed{\times\, \mathbf{r}_a}$$

Recover pre-collision contact velocity, $\dot{\mathbf{p}}_a^-$

$$\dot{\mathbf{p}}_a^+ = \dot{\mathbf{p}}_a^- + j\left(\frac{j\hat{\mathbf{n}}}{m_a} + \left(\mathbf{I}_a^{-1}(\mathbf{r}_a \times j\hat{\mathbf{n}})\right) \times \mathbf{r}_a\right)$$

Compute the Impulse

- **Express the empirical law in contact velocity**

$$v_r^+ = -\epsilon v_r^-$$

$$\boxed{\dot{\mathbf{p}}_a^+ = \dot{\mathbf{p}}_a^- + j\left(\frac{j\hat{\mathbf{n}}}{m_a} + \left(\mathbf{I}_a^{-1}(\mathbf{r}_a \times j\hat{\mathbf{n}})\right) \times \mathbf{r}_a\right)}$$

Compute the Impulse

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$$v_r^+ = \hat{\mathbf{n}} \cdot (\dot{\mathbf{p}}_a^+ - \dot{\mathbf{p}}_b^+)$$

$$= \hat{\mathbf{n}} \cdot (\dot{\mathbf{p}}_a^- - \dot{\mathbf{p}}_b^-) + j(\frac{1}{m_a} + \frac{1}{m_b} + \hat{\mathbf{n}} \cdot (\mathbf{I}_a^{-1}(\mathbf{r}_a \times \hat{\mathbf{n}})) \times \mathbf{r}_a + \hat{\mathbf{n}} \cdot (\mathbf{I}_b^{-1}(\mathbf{r}_b \times \hat{\mathbf{n}})) \times \mathbf{r}_b)$$

Compute the Impulse

- **Express the empirical law in contact velocity**

$$v_r^+ = -\epsilon v_r^-$$

$$\boxed{\dot{\mathbf{p}}_a^+ = \dot{\mathbf{p}}_a^- + j\left(\frac{j\hat{\mathbf{n}}}{m_a} + \left(\mathbf{I}_a^{-1}(\mathbf{r}_a \times j\hat{\mathbf{n}})\right) \times \mathbf{r}_a\right)}$$

$$v_r^+ = \hat{\mathbf{n}} \cdot (\dot{\mathbf{p}}_a^+ - \dot{\mathbf{p}}_b^+)$$

$$= \boxed{\hat{\mathbf{n}} \cdot (\dot{\mathbf{p}}_a^- - \dot{\mathbf{p}}_b^-)} + j\left(\frac{1}{m_a} + \frac{1}{m_b} + \hat{\mathbf{n}} \cdot (\mathbf{I}_a^{-1}(\mathbf{r}_a \times \hat{\mathbf{n}})) \times \mathbf{r}_a + \hat{\mathbf{n}} \cdot (\mathbf{I}_b^{-1}(\mathbf{r}_b \times \hat{\mathbf{n}})) \times \mathbf{r}_b\right)$$

$$= v_r^- + j\left(\frac{1}{m_a} + \frac{1}{m_b} + \hat{\mathbf{n}} \cdot (\mathbf{I}_a^{-1}(\mathbf{r}_a \times \hat{\mathbf{n}})) \times \mathbf{r}_a + \hat{\mathbf{n}} \cdot (\mathbf{I}_b^{-1}(\mathbf{r}_b \times \hat{\mathbf{n}})) \times \mathbf{r}_b\right)$$

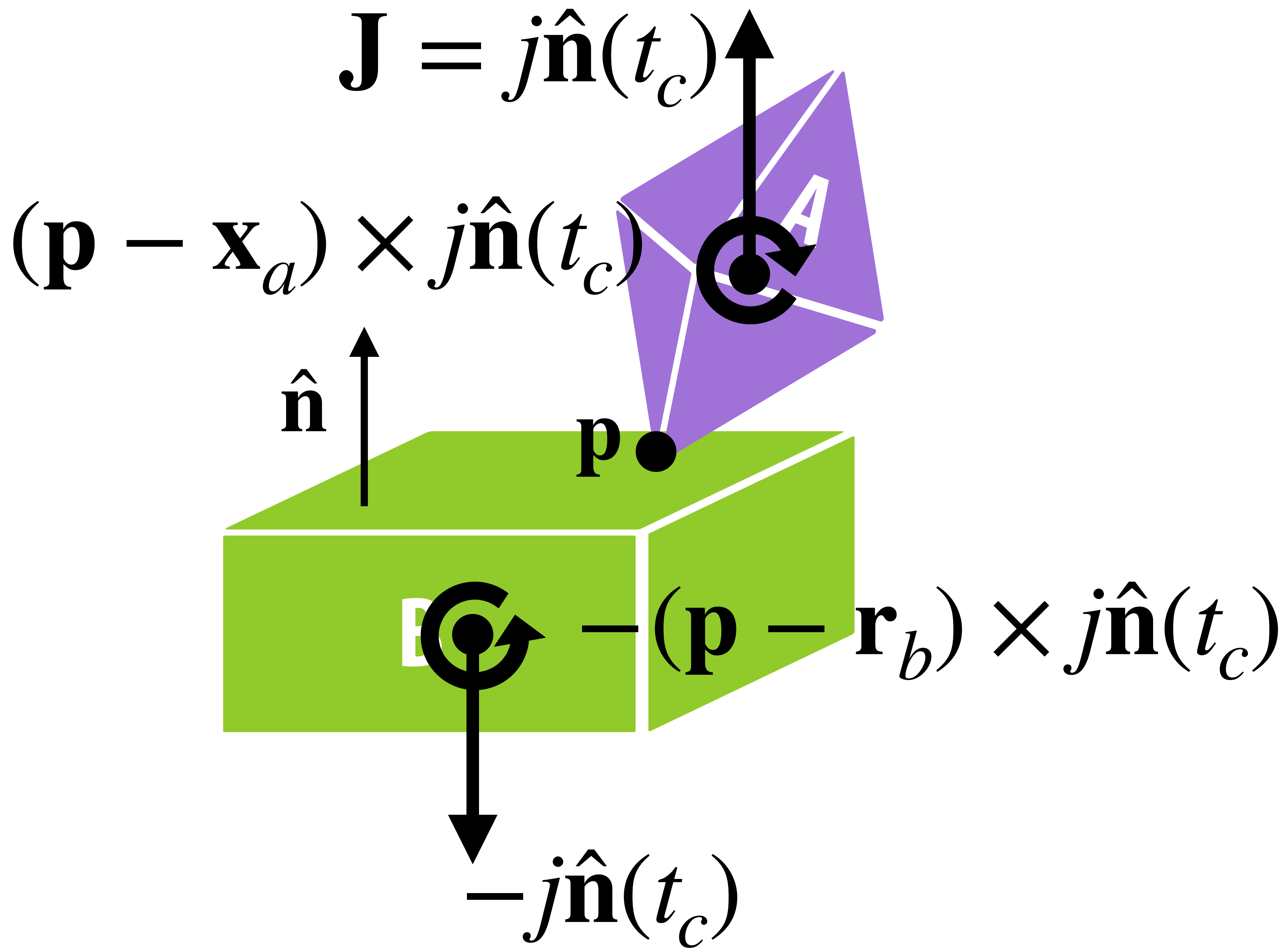
$$-\epsilon v_r^- = v_r^- + j\left(\frac{1}{m_a} + \frac{1}{m_b} + \hat{\mathbf{n}} \cdot (\mathbf{I}_a^{-1}(\mathbf{r}_a \times \hat{\mathbf{n}})) \times \mathbf{r}_a + \hat{\mathbf{n}} \cdot (\mathbf{I}_b^{-1}(\mathbf{r}_b \times \hat{\mathbf{n}})) \times \mathbf{r}_b\right)$$

$$\boxed{j = \frac{-(1+\epsilon)v_r^-}{\frac{1}{m_a} + \frac{1}{m_b} + \hat{\mathbf{n}} \cdot (\mathbf{I}_a^{-1}(\mathbf{r}_a \times \hat{\mathbf{n}})) \times \mathbf{r}_a + \hat{\mathbf{n}} \cdot (\mathbf{I}_b^{-1}(\mathbf{r}_b \times \hat{\mathbf{n}})) \times \mathbf{r}_b}}$$

Colliding Contact

- Apply change in momentum to current state:
  - Body A:
    - $\mathbf{P}(t_c + h) = \mathbf{P}(t_c) + \mathbf{J}$
    - $\mathbf{L}(t_c + h) = \mathbf{L}(t_c) + (\mathbf{p} - \mathbf{x}_a) \times \mathbf{J}$
  - Body B:
    - $\mathbf{P}(t_c + h) = \mathbf{P}(t_c) - \mathbf{J}$
    - $\mathbf{L}(t_c + h) = \mathbf{L}(t_c) + (\mathbf{p} - \mathbf{x}_b) \times (-\mathbf{J})$

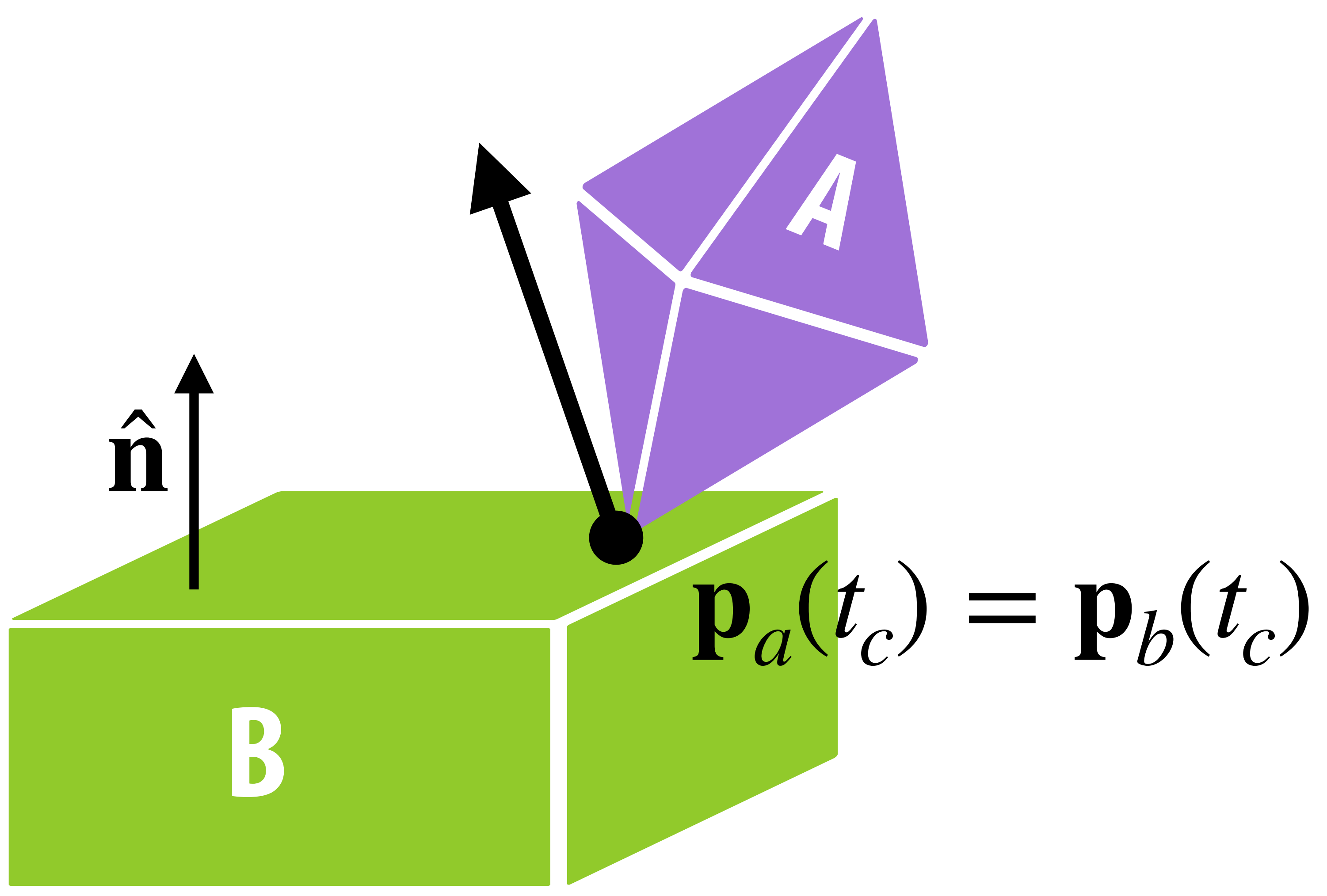
after collision



Relative Normal Velocity

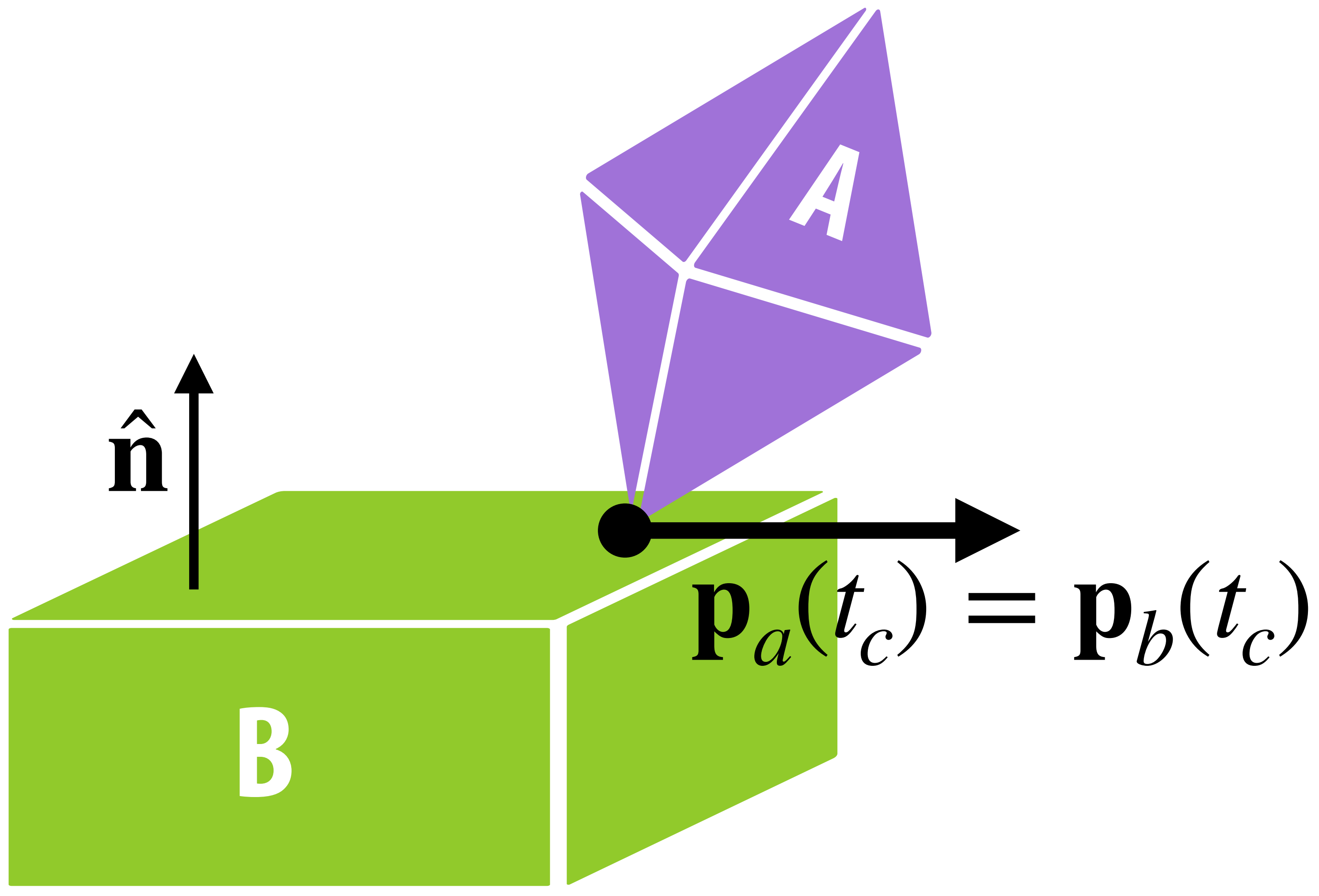
$v_r > 0$

separation



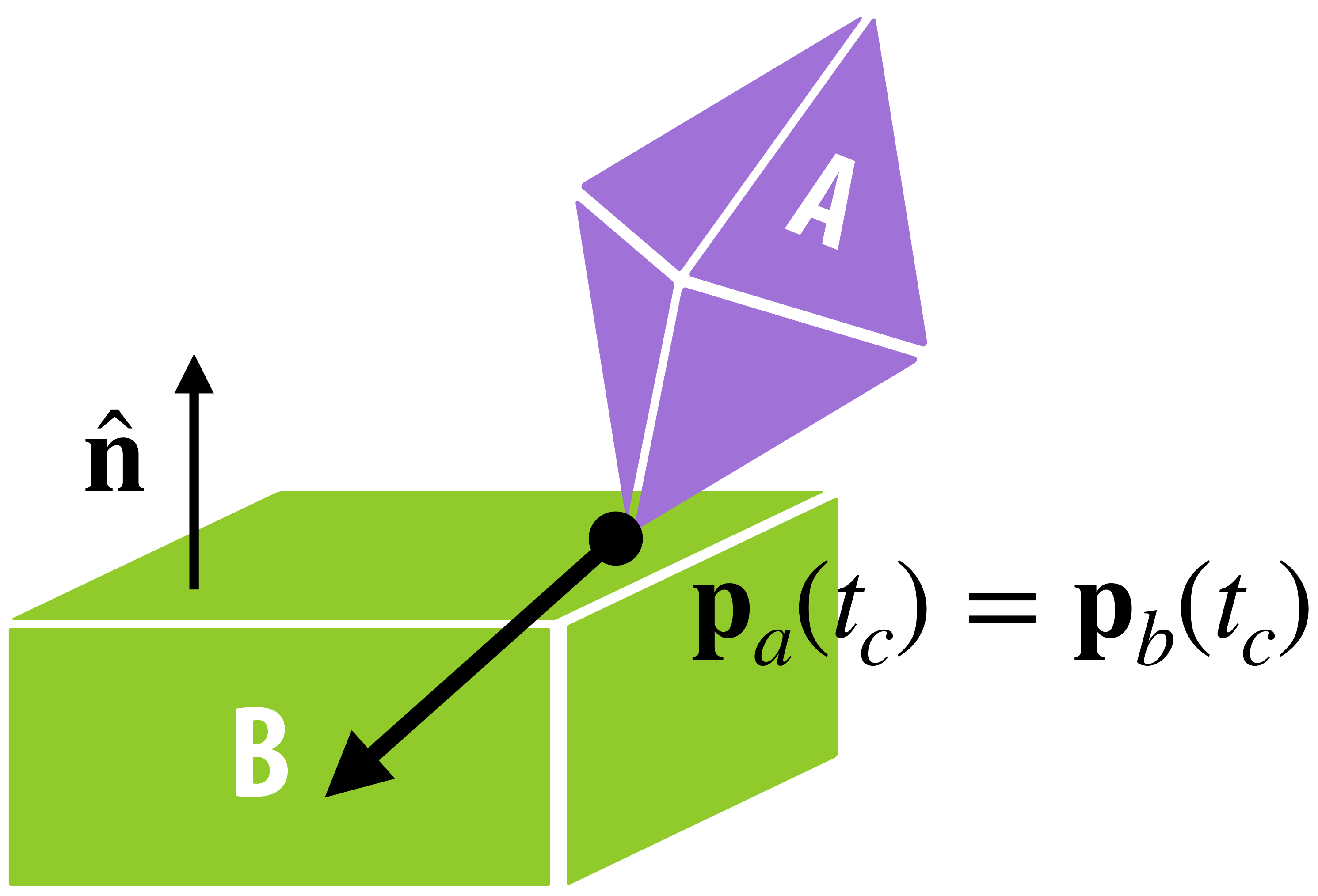
$v_r = 0$

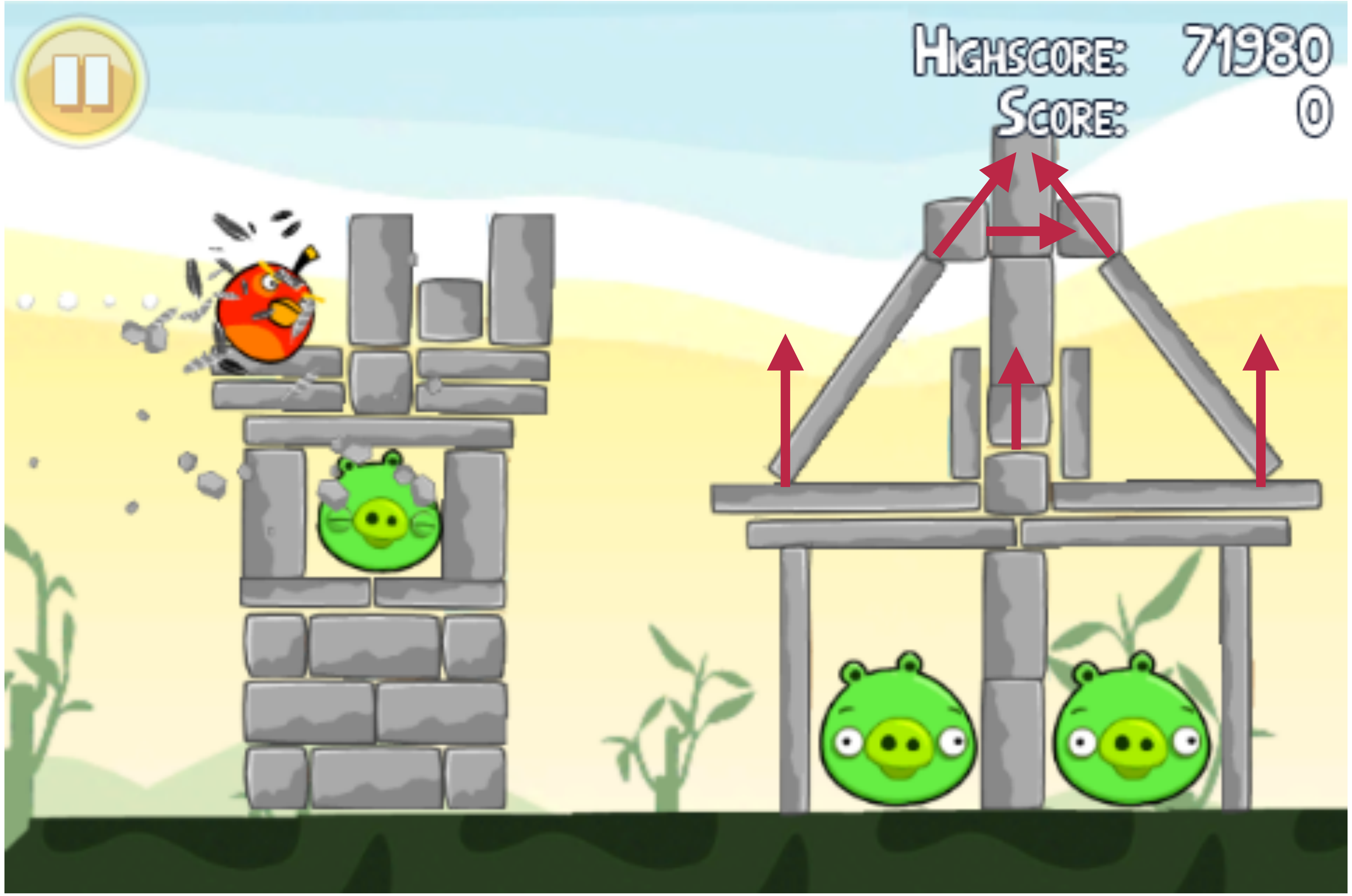
resting contact



$v_r < 0$

colliding contact

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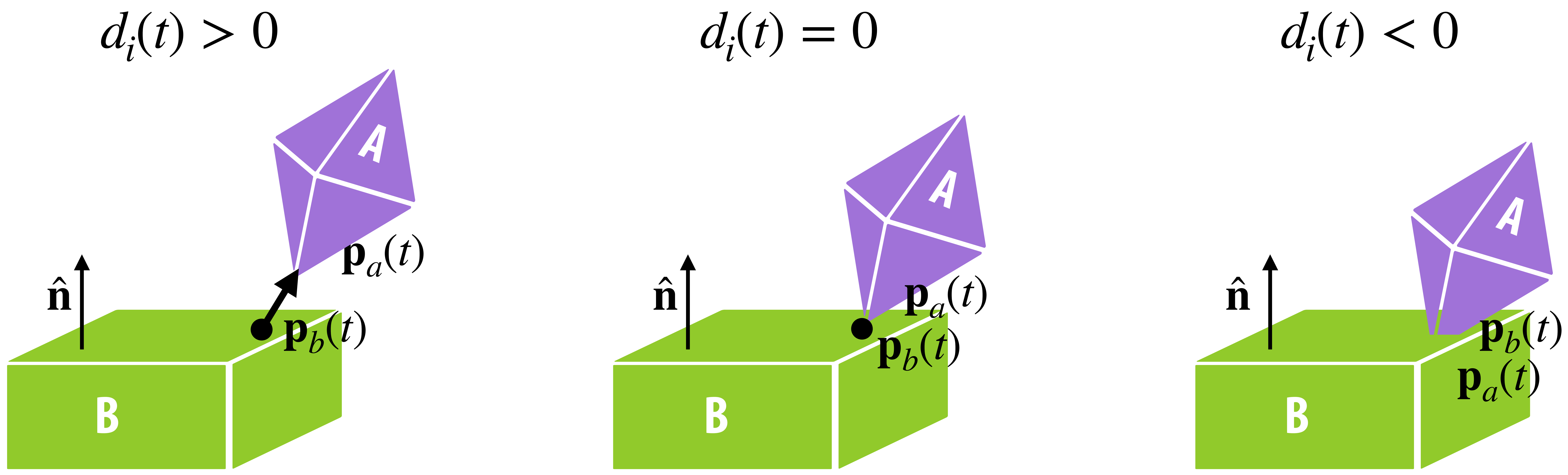
SCORE: 0

Resting Contact

- **In this case, all n contact points have the zero relative velocity**
- **At each contact point there is some force $f_i\hat{\mathbf{n}}_i$, where f_i is an unknown scalar and $\hat{\mathbf{n}}_i$ is a defined normal at that contact point**
- **Our goal is to determine what each f_i is by solving all of them simultaneously**
- **What are the conditions for f_i?**

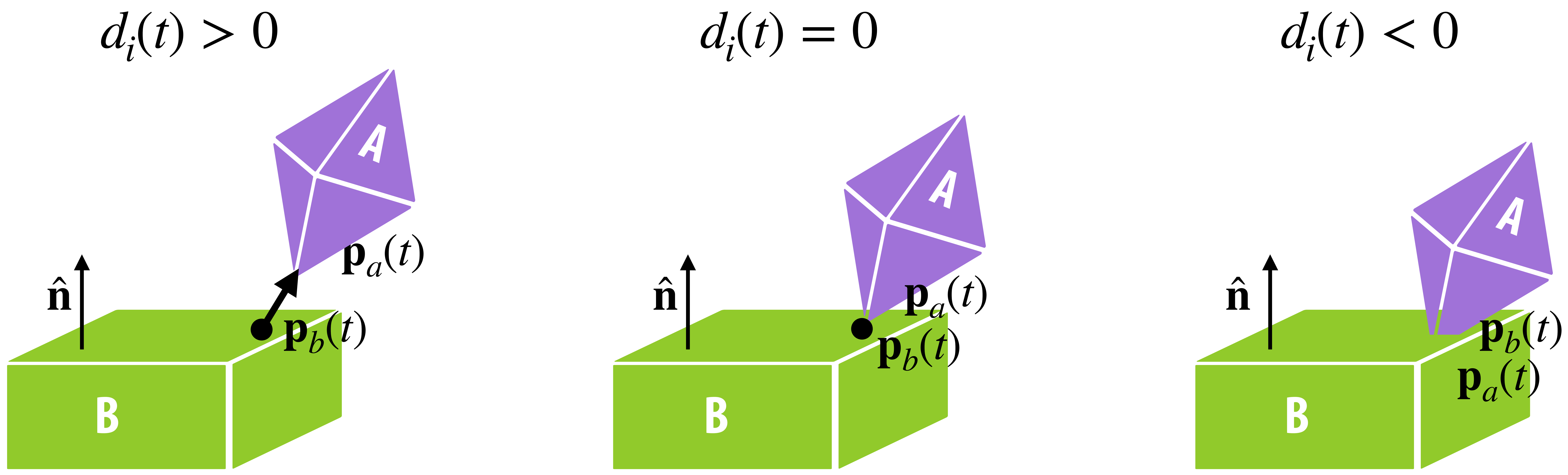
Non-penetration

- **Let's define penetration:**
  - $d_i = \hat{\mathbf{n}} \cdot (\mathbf{p}_a - \mathbf{p}_b)$



Non-penetration

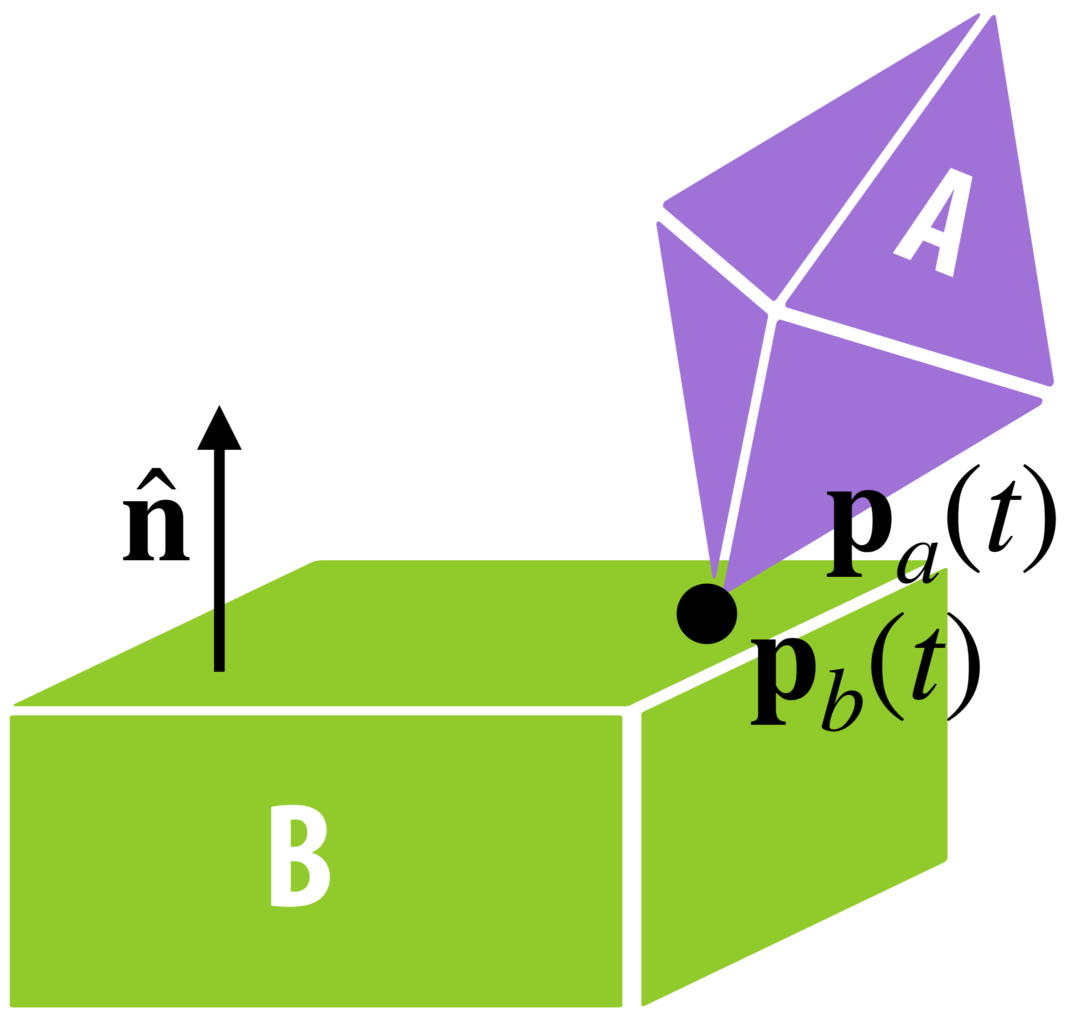
- Let's define penetration:
  - $d_i = \hat{\mathbf{n}} \cdot (\mathbf{p}_a - \mathbf{p}_b)$
- We want to avoid $d_i < 0$



Non-penetration

- Let's define penetration:
  - $d_i = \hat{\mathbf{n}} \cdot (\mathbf{p}_a - \mathbf{p}_b)$
- We want to avoid $d_i < 0$
- Since collision is detected, $d_i(t) = 0$

$$d_i(t) = 0$$



Non-penetration

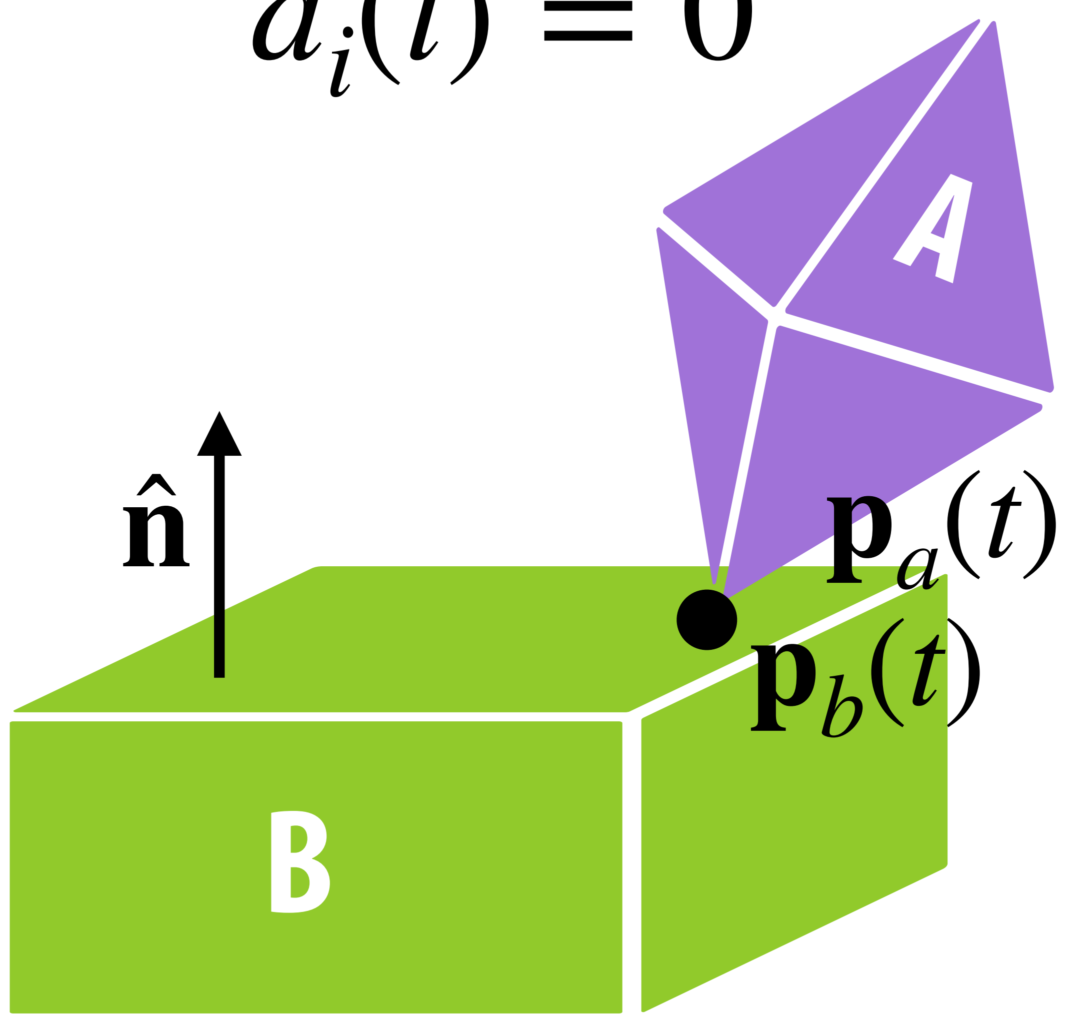
$$\dot{d}_i(t) = \cancel{\dot{\hat{\mathbf{n}}}_i(t) \cdot \left(\mathbf{p}_a(t) - \mathbf{p}_b(t)\right)} + \hat{\mathbf{n}}_i(t) \cdot \left(\dot{\mathbf{p}}_a(t) - \dot{\mathbf{p}}_b(t)\right)$$

$\dot{d}_i(t) = v_r = 0$ **because it is a resting contact**

- **Let's define penetration:**
  - $d_i = \hat{\mathbf{n}} \cdot (\mathbf{p}_a - \mathbf{p}_b)$
- **We want to avoid** $d_i < 0$
- **Since collision is detected,** $d_i(t) = 0$
- **What about** $\dot{d}_i(t)$?

$$d_i(t) = 0$$

$$\dot{d}_i(t) = 0$$



Non-penetration

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  - $d_i = \hat{\mathbf{n}} \cdot (\mathbf{p}_a - \mathbf{p}_b)$
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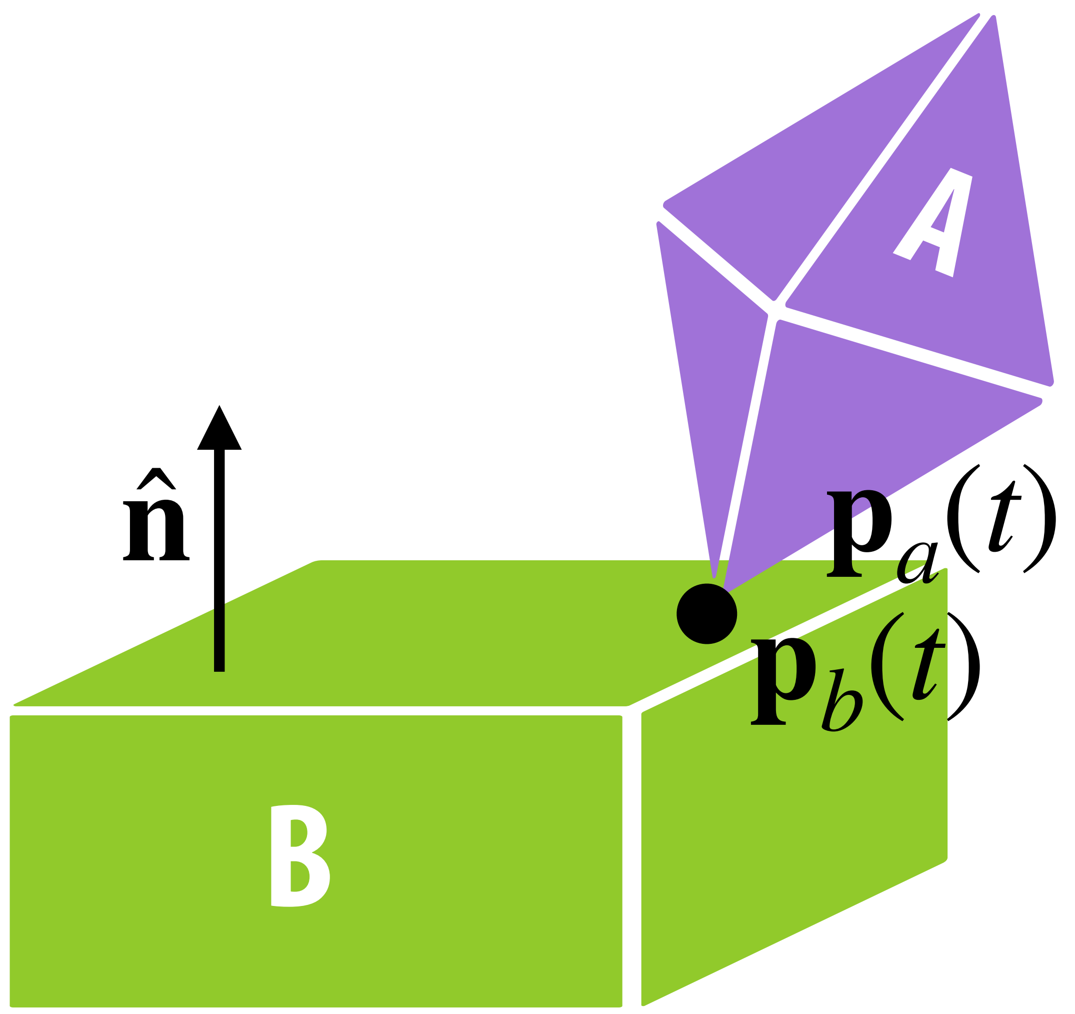
- At rest contact, $d_i(t) = 0$ and $\dot{d}_i(t) = 0$



Non-penetration

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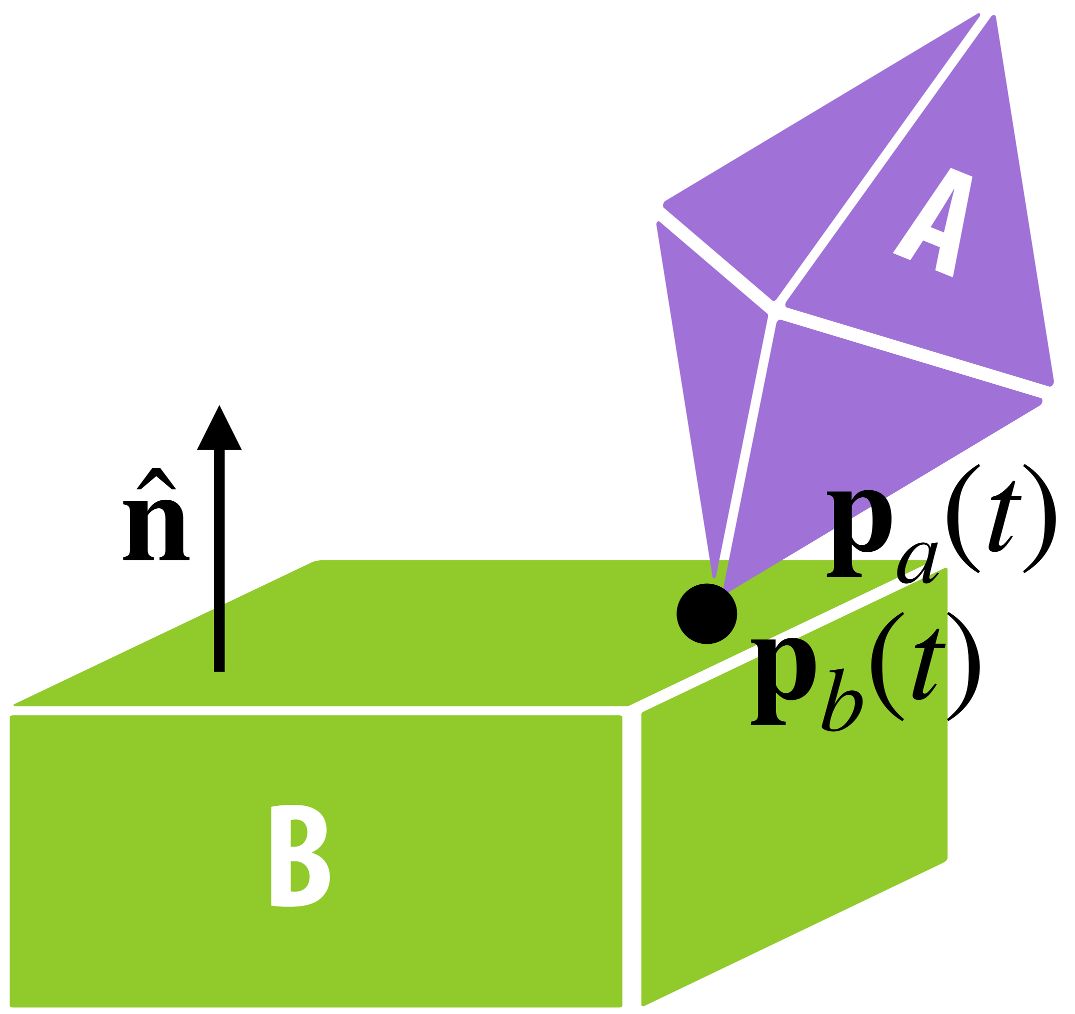
- If $\ddot{d}(t) < 0$, bodies have an acceleration toward each other and the penetration will occur.

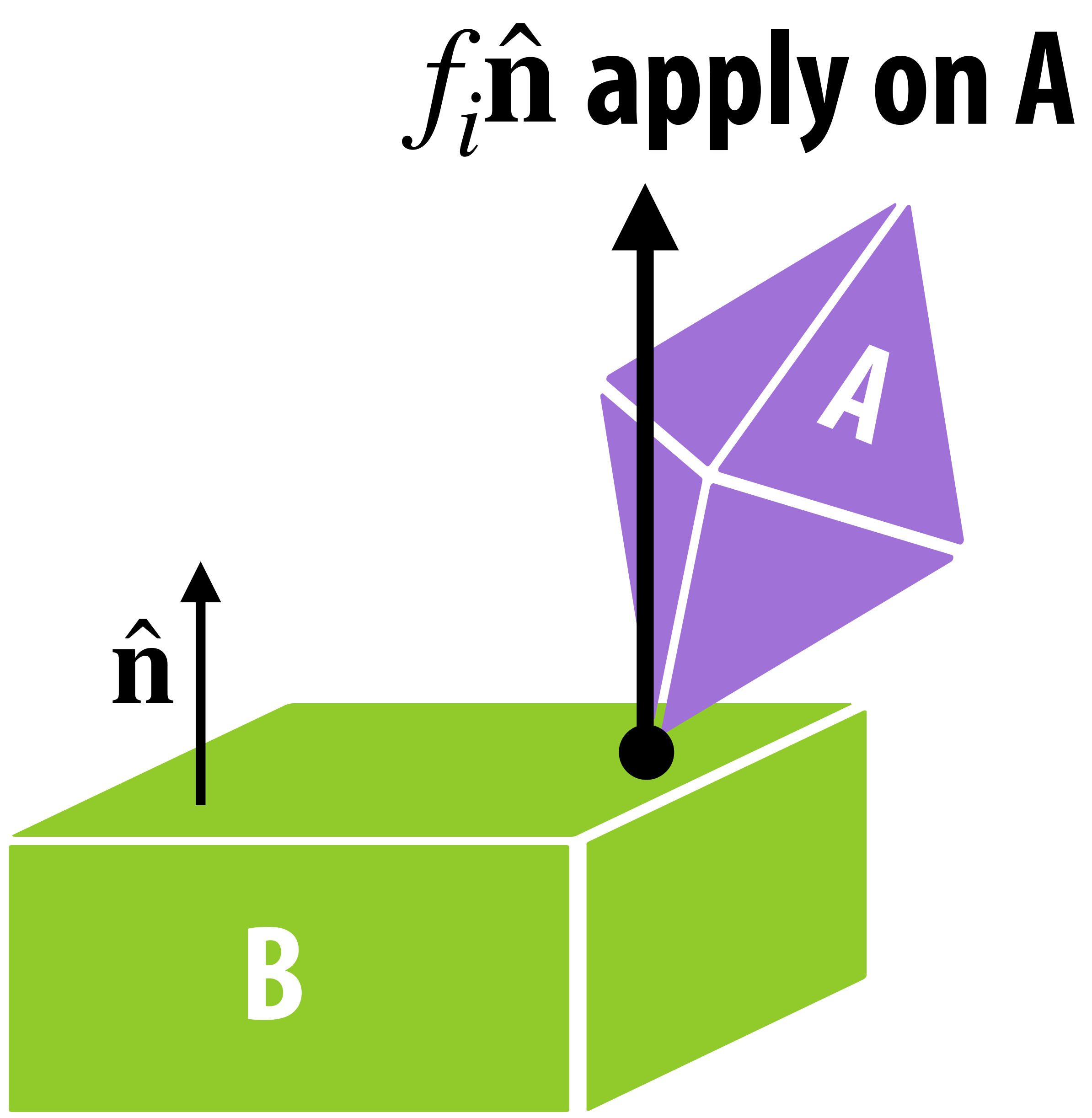


Non-penetration

- Let's define penetration:
  - $d_i = \hat{\mathbf{n}} \cdot (\mathbf{p}_a - \mathbf{p}_b)$
- We want to avoid $d_i < 0$
- At rest contact, $d_i(t) = 0$ and $\dot{d}_i(t) = 0$
- If $\ddot{d}(t) < 0$, bodies have an acceleration toward each other and the penetration will occur.
- Therefore, the first condition is $\boxed{\ddot{d}(t) \geq 0}$

Repulsive force

- The contact forces can push bodies apart, but can never act like "glue" and hold bodies together.
- Therefore, each contact force must act outward: $\boxed{f_i \geq 0}$



Workless force

- The contact force at the a contact point becomes zero if the bodies begin to separate.
- If contact is breaking, that is, $\ddot{d}_i(t) > 0$, then f_i should be zero.
- If f_i is not zero, then the contact is not breaking, that is, $\ddot{d}_i(t) = 0$.
- What is the equation that satisfies these two conditions?

$$\boxed{f_i \ddot{d}_i(t) = 0}$$

Compute contact forces

- **Non- penetration**

$$\ddot{d}_i(t) \geq 0$$

- **Repulsive force**

$$f_i \geq 0$$

- **Workless force**

$$f_i \ddot{d}_i(t) = 0$$

Compute contact forces

- Non- penetration

$$\ddot{d}_i(t) \geq 0$$

- Repulsive force

$$f_i \geq 0$$

- Workless force

$$f_i \ddot{d}_i(t) = 0$$

Express $\ddot{d}$'s in terms of f's:

$$\ddot{d}_i = \hat{\mathbf{n}} \cdot (\ddot{\mathbf{p}}_a - \ddot{\mathbf{p}}_b) + 2\dot{\hat{\mathbf{n}}} \cdot (\dot{\mathbf{p}}_a - \dot{\mathbf{p}}_b)$$
$$= a_{i1}f_1 + a_{i2}f_2 + \cdots + a_{in}f_n + b_i$$

Factor out the terms that depend on f_j and assign them to a_{ij}

Assign the rest of terms to b_i

Collect all the a_{ij} to form matrix $\mathbf{A}$ and all the b_i to form vector $\mathbf{b}$

$\ddot{\mathbf{d}} = \mathbf{A}\mathbf{f} + \mathbf{b}$, where $\ddot{\mathbf{d}} = [\ddot{d}_1, \cdots \ddot{d}_n]$ and $\mathbf{f} = f_1, \cdots, f_n]$

See details in Baraff and Witkin's course notes

Linear complementarity program (LCP)

- **Solve for** $\mathbf{f} = [f_i, f_2, \cdots, f_n]$

- **Subject to**

$$\mathbf{Af} + \mathbf{b} \geq 0$$

$$\mathbf{f} \geq \mathbf{0}$$

$$(\mathbf{Af} + \mathbf{b})^T \mathbf{f} = 0$$

Can solve it as a Quadratic Program

Solve LCP iteratively

A typical LCP:

$$\mathbf{A}\mathbf{x} + \mathbf{b} \geq \mathbf{0}$$
$$\mathbf{x} \geq \mathbf{0}$$
$$\mathbf{x}^T(\mathbf{A}\mathbf{x} + \mathbf{b}) = 0$$

split $\mathbf{A}$ to $\mathbf{M} + \mathbf{N}$

$$(\mathbf{M} + \mathbf{N})\mathbf{x} + \mathbf{b} \geq \mathbf{0}$$
$$\mathbf{x} \geq \mathbf{0}$$
$$\mathbf{x}^T\big((\mathbf{M} + \mathbf{N})\mathbf{x} + \mathbf{b}\big) = 0$$

Fixed point ite[illegible]: update $\mathbf{x}$ iteratively

old $\mathbf{x}$

$$\mathbf{M}\mathbf{x}_{k+1} + \mathbf{N}\mathbf{x}_k + \mathbf{b} \geq \mathbf{0}$$
$$\mathbf{x}_{k+1} \geq [illegible]$$

new $\mathbf{x}$

$$\mathbf{x}_{k+1}^T(\mathbf{M}\mathbf{x}_{k+1} + \mathbf{N}\mathbf{x}_k + \mathbf{b}) = 0$$

Let $\mathbf{c}_k \equiv \mathbf{N}\mathbf{x}_k + \mathbf{b}$

$$\mathbf{M}\mathbf{x}_{k+1} + \mathbf{c}_k \geq \mathbf{0}$$
$$\mathbf{x}_{k+1} \geq \mathbf{0}$$
$$\mathbf{x}_{k+1}^T(\mathbf{M}\mathbf{x}_{k+1} + \mathbf{c}_k) = 0$$

Solve LCP iteratively

A typical LCP:

$$\mathbf{A}\mathbf{x} + \mathbf{b} \geq \mathbf{0}$$
$$\mathbf{x} \geq \mathbf{0}$$
$$\mathbf{x}^T(\mathbf{A}\mathbf{x} + \mathbf{b}) = 0$$

split $\mathbf{A}$ to $\mathbf{M} + \mathbf{N}$ $\longrightarrow$

$$(\mathbf{M} + \mathbf{N})\mathbf{x} + \mathbf{b} \geq \mathbf{0}$$
$$\mathbf{x} \geq \mathbf{0}$$
$$\mathbf{x}^T\big((\mathbf{M} + \mathbf{N})\mathbf{x} + \mathbf{b}\big) = 0$$

Fixed point iteration: update $\mathbf{x}$ iteratively

old $\mathbf{x}$

$$\mathbf{M}\mathbf{x}_{k+1} + \mathbf{N}\mathbf{x}_k + \mathbf{b} \geq \mathbf{0}$$
$$\mathbf{x}_{k+1} \geq \mathbf{0}$$

new $\mathbf{x}$

$$\mathbf{x}_{k+1}^T(\mathbf{M}\mathbf{x}_{k+1} + \mathbf{N}\mathbf{x}_k + \mathbf{b}) = 0$$

Let $\mathbf{c}_k \equiv \mathbf{N}\mathbf{x}_k + \mathbf{b}$

$$\mathbf{M}\mathbf{x}_{k+1} + \mathbf{c}_k \geq \mathbf{0}$$
$$\mathbf{x}_{k+1} \geq \mathbf{0}$$
$$\mathbf{x}_{k+1}^T(\mathbf{M}\mathbf{x}_{k+1} + \mathbf{c}_k) = 0$$

Projected Gauss Seidel (PGS)

$$\mathbf{M} = \begin{bmatrix} a_{11} & & 0 \\ \vdots & \ddots & \\ a_{11} & \dots & a_{nn} \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} & a_{12} & \dots & a_{1n} \\ & & & \\ 0 & & & \end{bmatrix}$$

for k = 0 to max_iter:

$$x_{k+1}(1) \cdot \big(a_{11}x_{k+1}(1) + c_k(1)\big) = 0$$

$$x_{k+1}(1) = \max(0, -\frac{c_k(1)}{a_{11}})$$

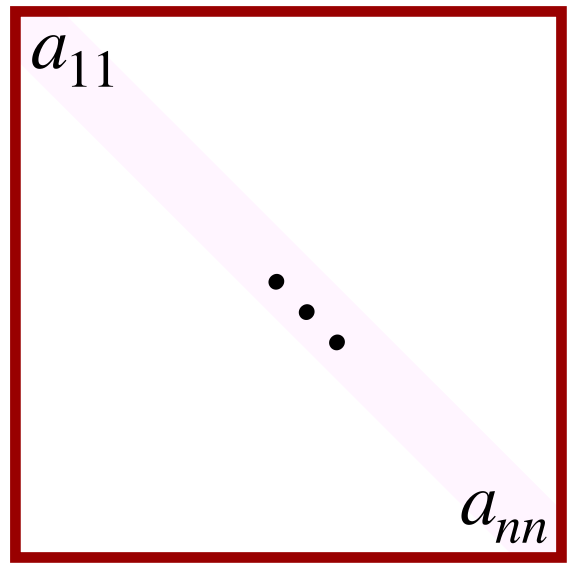
just solved!

$$x_{k+1}(2) \cdot \big(a_{21}x_{k+1}(1) + a_{22}x_{k+1}(2) + c_k(2)\big) = 0$$

$$x_{k+1}(2) = \max(0, -\frac{a_{21}x_{k+1}(1) + c_k(2)}{a_{22}}) \qquad \dots$$

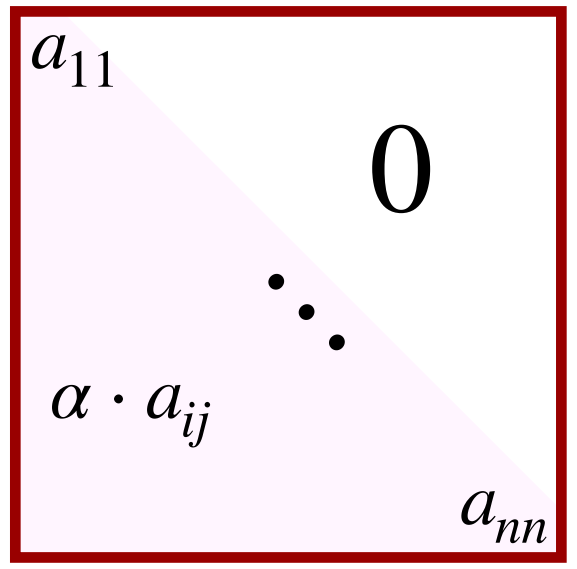
Solve LCP iteratively

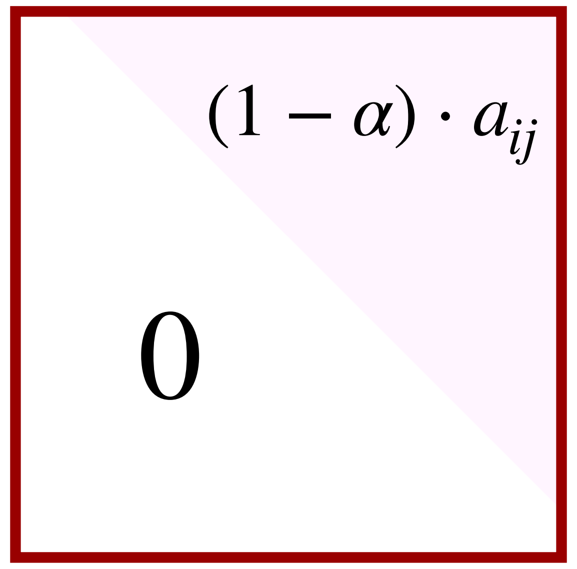
Projected Jacobi:

$\mathbf{M} =$ 

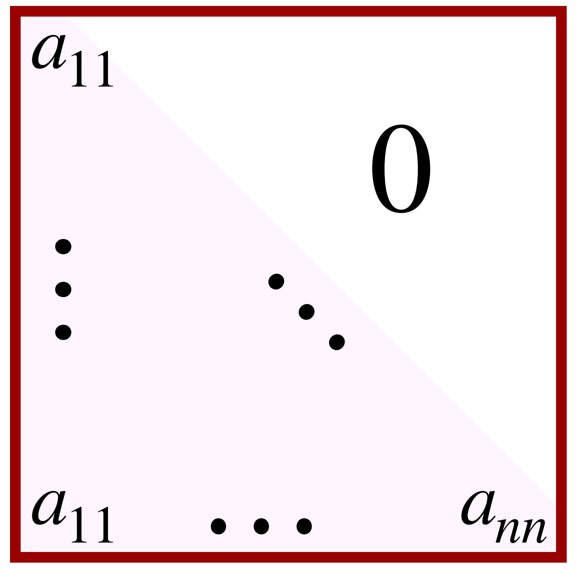
$\mathbf{N} =$ 

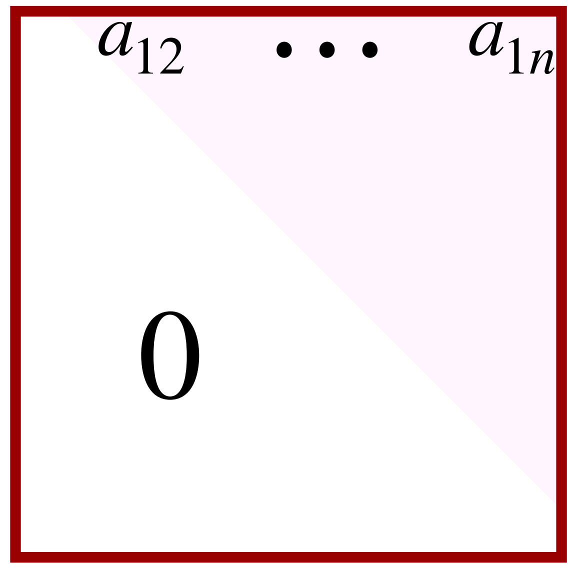
Projected Successive Over Relaxation:

$\mathbf{M} =$ 

$\mathbf{N} =$ 

Projected Gauss Seidel (PGS)

$\mathbf{M} =$ 

$\mathbf{N} =$ 

for k = 0 to max_iter:

$$x_{k+1}(1) \cdot \left(a_{11}x_{k+1}(1) + c_k(1)\right) = 0$$

$$x_{k+1}(1) = \max(0, -\frac{c_k(1)}{a_{11}})$$

$$x_{k+1}(2) \cdot \left(a_{21}x_{k+1}(1) + a_{22}x_{k+1}(2) + c_k(1)\right) = 0$$

$$x_{k+1}(2) = \max(0, -\frac{a_{21}x_{k+1}(1) + c_k(2)}{a_{22}}) \qquad \cdots$$

Velocity-based LCP

- **Non- penetration**

$$\dot{d}_i(t) \geq 0$$

- **Repulsive force**

$$f_i \geq 0$$

- **Workless force**

$$f_i \dot{d}_i(t) = 0$$

Velocity-based LCP

- Non- penetration

$$\frac{\partial d_i}{\partial \mathbf{q}}\dot{\mathbf{q}} \geq 0$$

- Repulsive force

$$f_i \geq 0$$

- Workless force

$$f_i\frac{\partial d}{\partial \mathbf{q}}\dot{\mathbf{q}} = 0$$

General representation of configurations of two rigid bodies

$$\mathbf{q} = [\mathbf{x}_a, \mathbf{R}_a, \mathbf{x}_b, \mathbf{R}_b]$$

Shortest distance between two rigid bodies

$$d(\mathbf{q})$$

Time derivative of $d(\mathbf{q}(t))$

$$\dot{d}_i(\mathbf{q}(t)) = \frac{\partial d_i}{\partial \mathbf{q}}\dot{\mathbf{q}} \geq 0$$

Velocity-based LCP

- Non- penetration

$$\frac{\partial d_i}{\partial \mathbf{q}}\dot{\mathbf{q}} \geq 0$$

- Repulsive force

$$f_i \geq 0$$

- Workless force

$$f_i \frac{\partial d}{\partial \mathbf{q}}\dot{\mathbf{q}} = 0$$

General representation of configurations of two rigid bodies

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$$\dot{d}_i(\mathbf{q}(t)) = \frac{\partial d_i}{\partial \mathbf{q}}\dot{\mathbf{q}} \geq 0$$

Compact expression of LCP: $0 \leq \mathbf{f} \perp \frac{\partial \mathbf{d}}{\partial \mathbf{q}}\dot{\mathbf{q}} \geq 0$

Velocity-based LCP

- Non- penetration

$$\frac{\partial d_i}{\partial \mathbf{q}}\dot{\mathbf{q}} \geq 0$$

- Repulsive force

$$f_i \geq 0$$

- Workless force

$$f_i \frac{\partial d}{\partial \mathbf{q}}\dot{\mathbf{q}} = 0$$

Make it implicit

$$0 \leq \mathbf{f} \perp \frac{\partial \mathbf{d}}{\partial \mathbf{q}}\dot{\mathbf{q}}^+ \geq 0$$

$$\dot{\mathbf{q}}^+ = \dot{\mathbf{q}}^- + M^{-1}\left(\frac{\partial \mathbf{d}}{\partial \mathbf{q}}\right)^T \mathbf{f}$$

Combine with colliding case

$$\boxed{0 \leq \mathbf{f} \perp \frac{\partial \mathbf{d}}{\partial \mathbf{q}}\dot{\mathbf{q}}^+ \geq -\epsilon \frac{\partial \mathbf{d}}{\partial \mathbf{q}}\dot{\mathbf{q}}^-}$$

Compact expression of LCP: $0 \leq \mathbf{f} \perp \frac{\partial \mathbf{d}}{\partial \mathbf{q}}\dot{\mathbf{q}} \geq 0$

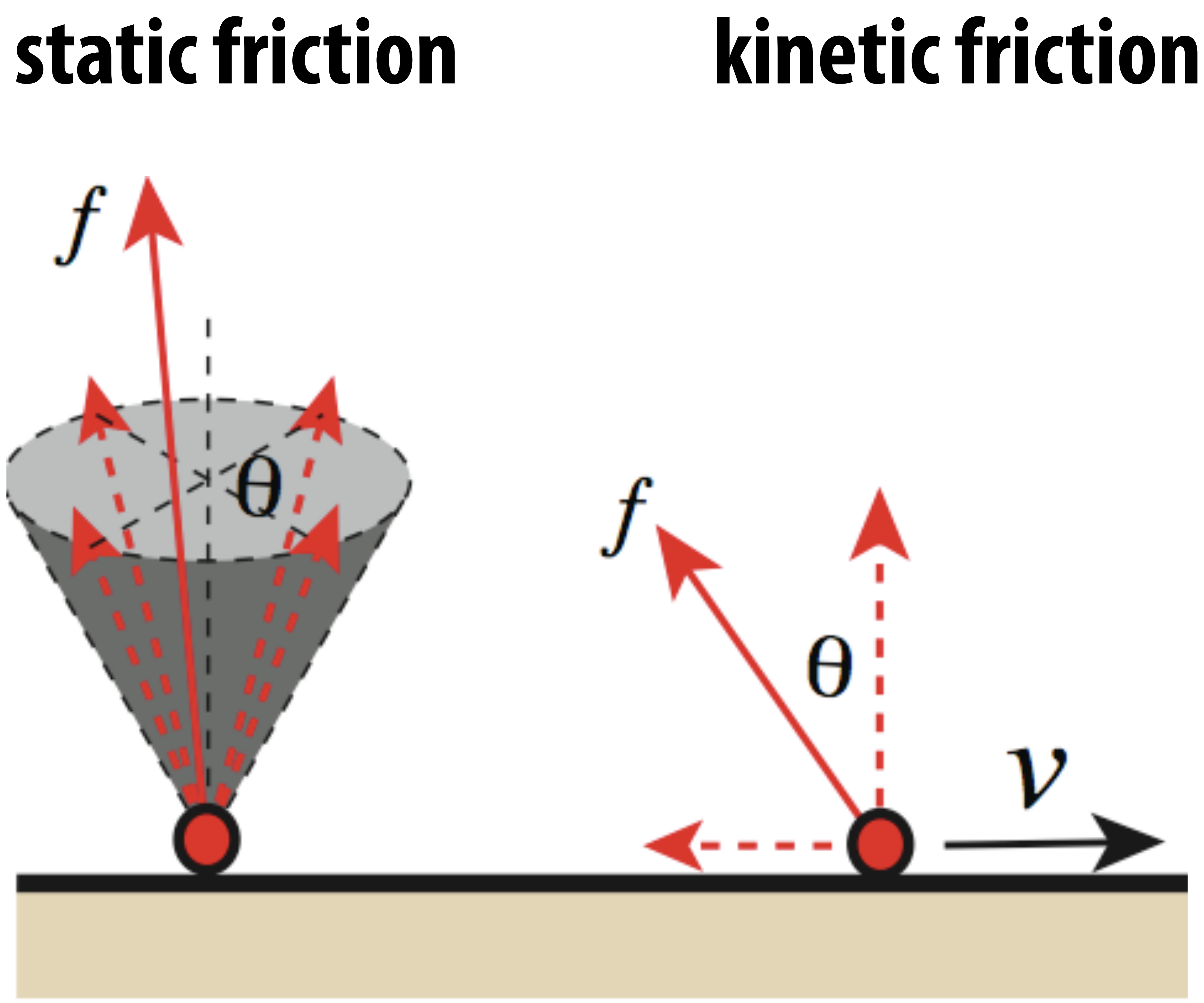
Friction

- **Coulomb's Law of Friction**
  - If sliding, the kinetic friction is

$$\mathbf{f}_{\parallel} = -\mu_k |\mathbf{f}_{\perp}| \frac{\mathbf{v}_{\parallel}}{|\mathbf{v}_{\parallel}|}$$

  - If static, stay static as long as

$$|\mathbf{f}_{\parallel}| \leq \mu_s |\mathbf{f}_{\perp}|$$



$$\theta = \tan^{-1} \mu_s$$

Friction coefficient

Materials		Static Friction, μ_s		Kinetic/Sliding Friction, μ_k	
		Dry and clean	Lubricated	Dry and clean	Lubricated
Aluminium	Steel	0.61[25]		0.47[25]	
Aluminium	Aluminium	1.05–1.35[25]	0.3[25]	1.4[25]–1.5[26]	
Gold	Gold			2.5[26]	
Platinum	Platinum	1.2[25]	0.25[25]	3.0[26]	
Silver	Silver	1.4[25]	0.55[25]	1.5[26]	
Alumina ceramic	Silicon nitride ceramic				0.004 (wet)[27]
BAM (Ceramic alloy $AlMgB_{14}$)	Titanium boride (TiB_2)	0.04–0.05[28]	0.02[29][30]		
Brass	Steel	0.35–0.51[25]	0.19[25]	0.44[25]	
Cast iron	Copper	1.05[25]		0.29[25]	
Cast iron	Zinc	0.85[25]		0.21[25]	
Concrete	Rubber	1.0	0.30 (wet)	0.6–0.85[25]	0.45–0.75 (wet)[25]
Concrete	Wood	0.62[25][31]			
Copper	Glass	0.68[32]		0.53[32]	
Copper	Steel	0.53[32]		0.36[25][32]	0.18[32]
Glass	Glass	0.9–1.0[25][32]	0.005–0.01[32]	0.4[25][32]	0.09–0.116[32]
Human synovial fluid	Human cartilage		0.01[33]		0.003[33]
Ice	Ice	0.02–0.09[34]			
Polyethene	Steel	0.2[25][34]	0.2[25][34]		
PTFE (Teflon)	PTFE (Teflon)	0.04[25][34]	0.04[25][34]		0.04[25]
Steel	Ice	0.03[34]			
Steel	PTFE (Teflon)	0.04[25]–0.2[34]	0.04[25]		0.04[25]
Steel	Steel	0.74[25]–0.80[34]	0.005–0.23[32][34]	0.42–0.62[25][32]	0.029–0.19[32]
Wood	Metal	0.2–0.6[25][31]	0.2 (wet)[25][31]	0.49[32]	0.075[32]
Wood	Wood	0.25–0.62[25][31][32]	0.2 (wet)[25][31]	0.32–0.48[32]	0.067–0.167[32]

Quiz

- **A block is pushed by an increasing horizontal force. The friction force overtime looks like:**