

This report overviews our methods, and focus on implementation details. We will skip the proofs and derivation.

1. NOTATION AND SET-UP

We consider a historical pricing policy $\pi_0(\cdot)$ which maps a feature vector \mathbf{x} to a deterministic price $\pi_0(\mathbf{x}) \in \mathbb{R}_+$, and a new candidate policy $\pi_1(\cdot)$ of similar form. We have data $\{(\mathbf{x}_1, D_1), \dots, (\mathbf{x}_n, D_n)\}$ consisting of feature and binary demand pairs, where

$$D_i \sim \text{Ber}(d(\mathbf{x}_i, \pi_0(\mathbf{x}_i))) \quad i = 1, \dots, n.$$

It will be convenient to define $R_i = \pi_0(\mathbf{x}_i)D_i$, which is the observed revenue in the data, and the expected revenue function $r(\mathbf{x}, p) = pd(\mathbf{x}, p)$.

Our goal is to estimate the expected revenue earned under the new pricing policy $\pi_1(\cdot)$, applied to a new set of covariates $\{\mathbf{x}_{n+1}, \dots, \mathbf{x}_{2n}\}$, i.e.,

$$\mathcal{R} \equiv \frac{1}{n} \sum_{i=n+1}^{2n} \pi_1(\mathbf{x}_i) d(\mathbf{x}_i, \pi_1(\mathbf{x}_i)) = \frac{1}{n} \sum_{i=n+1}^{2n} r(\mathbf{x}_i, \pi_1(\mathbf{x}_i)). \quad (\text{Target})$$

Of particular interest is when the new covariates are equal to the old sample, i.e., $\mathbf{x}_i = \mathbf{x}_{n+i}$ for $1 \leq i \leq n$.

We focus on weighted revenue estimators of the form

$$\hat{\mathcal{R}}(\mathbf{w}) \equiv \sum_{j=1}^n w_j R_j, \quad (\text{Estimator})$$

We assume

Assumption 1 (Revenue is in RKHS). *There exists an RKHS \mathcal{H} , a known reference revenue $r_0(\cdot, \cdot)$, and a perturbation function $\Delta(\cdot, \cdot) \in \mathcal{H}$ such that $r(\cdot, \cdot) = r_0(\cdot, \cdot) + \Delta(\cdot, \cdot)$ and $\|\Delta(\cdot, \cdot)\|_{\mathcal{H}} = \Gamma < \infty$.*

Importantly, under Assumption 1 there exists an $\boldsymbol{\alpha} \in \mathbb{R}^{2n}$ such that

$$\boldsymbol{\Delta} = \mathbf{G}\boldsymbol{\alpha}, \quad \text{and} \quad \boldsymbol{\alpha}^\top \mathbf{G}\boldsymbol{\alpha} = \Gamma^2. \quad (\text{RKHS norm})$$

When \mathbf{G} is invertible, this is equivalent to the condition

$$\boldsymbol{\Delta}^\top \mathbf{G}^{-1} \boldsymbol{\Delta} = \Gamma^2.$$

2. MSE

Define

$$\mathbf{b}(\mathbf{w}) = (w_1, \dots, w_n, -1/n, \dots, -1/n)^\top \in \mathbb{R}^{2n},$$

$$\mathbf{v}(\mathbf{w}) = (w_1^2 \pi_1, \dots, w_n^2 \pi_n, 0, \dots, 0)^\top \in \mathbb{R}^{2n}$$

$$\text{Bias}(\mathbf{w}; \mathbf{r}) \equiv \mathbb{E} \left[\hat{\mathcal{R}}(\mathbf{w}) - \mathcal{R} \right] = \mathbf{b}(\mathbf{w})^\top \mathbf{r}$$

$$\text{Var}(\mathbf{w}; \mathbf{r}) \equiv \mathbb{E} \left[\left(\hat{\mathcal{R}}(\mathbf{w}) - \mathbb{E} \left[\hat{\mathcal{R}}(\mathbf{w}) \right] \right)^2 \right] = \mathbf{v}(\mathbf{w})^\top \mathbf{r} - \mathbf{r}^\top \begin{pmatrix} \text{diag}(w_1^2, \dots, w_n^2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{r},$$

$$\text{MSE}(\mathbf{w}; \mathbf{r}) \equiv \mathbf{v}(\mathbf{w})^\top \mathbf{r} + \mathbf{r}^\top \left(\mathbf{b}(\mathbf{w}) \mathbf{b}(\mathbf{w})^\top - \begin{pmatrix} \text{diag}(w_1^2, \dots, w_n^2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right) \mathbf{r}.$$

We seek weights that minimize some worst-case metric of our estimator, where the worst-case is taken over the convex set

$$\mathcal{U}_\Gamma \equiv \{\mathbf{r}_0 + \mathbf{\Delta} \in \mathbb{R}^{2n} : \mathbf{0} \leq \mathbf{r}_0 + \mathbf{\Delta} \leq \mathbf{\pi}, \mathbf{\Delta}^\top \mathbf{G}^{-1} \mathbf{\Delta} \leq \Gamma^2\}.$$

The optimization problem is

$$\min_{\mathbf{w}} \max_{\mathbf{r} \in \mathcal{U}_\Gamma} MSE(\mathbf{w}; \mathbf{r}).$$

3. BERNSTEIN BOUND

Let

$$q_{\max}(\mathbf{w}) \equiv \max_{1 \leq j \leq n} |w_j| \pi_j$$

The Bernstein Bound says with probability at least $1 - \epsilon$,

$$\hat{\mathcal{R}} - \mathcal{R} \leq \mathbf{b}(\mathbf{w})^\top \mathbf{r} + \sqrt{2Var(\mathbf{w}; \mathbf{r}) \log(1/\epsilon)} + \frac{q_{\max}(\mathbf{w}) \log(1/\epsilon)}{3}.$$

The resulting optimization problem is

$$\min_{\mathbf{w}} \max_{\mathbf{r} \in \mathcal{U}_\Gamma} \mathbf{b}(\mathbf{w})^\top \mathbf{r} + \sqrt{2Var(\mathbf{w}; \mathbf{r}) \log(1/\epsilon)} + \frac{q_{\max}(\mathbf{w}) \log(1/\epsilon)}{3}.$$

Rewriting the inner optimization in terms of $\mathbf{\Delta}$ and dropping constants yields

$$\begin{aligned} \max_{\mathbf{\Delta}, t} \quad & \mathbf{b}(\mathbf{w})^\top \mathbf{\Delta} + t\sqrt{2\log(1/\epsilon)} \\ \text{s.t.} \quad & t^2 \leq Var(\mathbf{w}, \mathbf{r}_0 + \mathbf{\Delta}) \\ & \mathbf{0} \leq \mathbf{r}_0 + \mathbf{\Delta} \leq \mathbf{\pi} \\ & \mathbf{\Delta}^\top \mathbf{G}^{-1} \mathbf{\Delta} \leq \Gamma^2. \end{aligned}$$

We rewrite this problem as a convex QCQP, i.e.,

$$\begin{aligned} \max_{\mathbf{\Delta}, t} \quad & \mathbf{b}(\mathbf{w})^\top \mathbf{\Delta} + t\sqrt{2\log(1/\epsilon)} \\ \text{s.t.} \quad & t^2 + \mathbf{\Delta}^\top \mathbf{Q}(\mathbf{w}) \mathbf{\Delta} + 2\mathbf{r}_0^\top \mathbf{Q}(\mathbf{w}) \mathbf{\Delta} - \mathbf{v}(\mathbf{w})^\top \mathbf{\Delta} \leq \mathbf{v}(\mathbf{w})^\top \mathbf{r}_0 - \mathbf{r}_0^\top \mathbf{Q}(\mathbf{w}) \mathbf{r}_0, \\ & \mathbf{0} \leq \mathbf{r}_0 + \mathbf{\Delta} \leq \mathbf{\pi} \\ & \mathbf{\Delta}^\top \mathbf{G}^{-1} \mathbf{\Delta} \leq \Gamma^2. \end{aligned}$$

where

$$\mathbf{Q}(\mathbf{w}) \equiv \begin{pmatrix} \text{diag}(w_1^2, \dots, w_n^2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Under Assumption 1, above problem is equivalent to

$$\begin{aligned} \max_{\mathbf{\alpha}, t} \quad & (\mathbf{b}(\mathbf{w})^\top \mathbf{G}) \mathbf{\alpha} + t\sqrt{2\log(1/\epsilon)} \\ \text{s.t.} \quad & t^2 + \mathbf{\alpha}^\top (\mathbf{G} \mathbf{Q}(\mathbf{w}) \mathbf{G}) \mathbf{\alpha} + (2\mathbf{r}_0^\top \mathbf{Q}(\mathbf{w}) - \mathbf{v}(\mathbf{w})^\top) \mathbf{G} \mathbf{\alpha} \leq \mathbf{v}(\mathbf{w})^\top \mathbf{r}_0 - \mathbf{r}_0^\top \mathbf{Q}(\mathbf{w}) \mathbf{r}_0, \\ & -\mathbf{G}^{-1} \mathbf{r}_0 \leq \mathbf{\alpha} \leq \mathbf{G}^{-1} (\mathbf{\pi} - \mathbf{r}_0) \\ & \mathbf{\alpha}^\top \mathbf{G} \mathbf{\alpha} \leq \Gamma^2. \end{aligned}$$

4. NATHAN'S METHOD

$$Bias(\mathbf{w}; \mathbf{r})^2 \equiv \mathbb{E} \left[\hat{\mathcal{R}}(\mathbf{w}) - \mathcal{R} \right]^2 = (\mathbf{b}(\mathbf{w})^\top \mathbf{r})^2 = \mathbf{r}^\top (\mathbf{b}(\mathbf{w}) \mathbf{b}(\mathbf{w})^\top) \mathbf{r}$$

Nathan computes weights that minimize the worst-case bias square, plus some constant regularization.

The optimization problem is

$$\min_{\mathbf{w}} \max_{\mathbf{r}: \|\mathbf{r}\|_{\mathcal{H}} \leq \Gamma} \mathbf{r}^\top (\mathbf{b}(\mathbf{w}) \mathbf{b}(\mathbf{w})^\top) \mathbf{r} + \lambda \|\mathbf{w}\|_2^2$$

Under Assumption 1, above problem is equivalent to

$$\min_{\mathbf{w}} \mathbf{b}(\mathbf{w})^\top \left(\mathbf{G} + \begin{pmatrix} \lambda \mathbb{I}^n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right) \mathbf{b}(\mathbf{w})$$