This report overviews our methods, and focus on implementation details. We will skip the proofs and derivation.

1. Notation and Set-up

We consider a historical pricing policy $\pi_0(\cdot)$ which maps a feature vector $\boldsymbol{x}$ to a deterministic price $\pi_0(\boldsymbol{x}) \in \mathbb{R}_+$, and a new candidate policy $\pi_1(\cdot)$ of similar form. We have data $\{(\boldsymbol{x}_1, D_1), \ldots, (\boldsymbol{x}_n, D_n)\}$ consisting of feature and binary demand pairs, where

$$D_i \sim \mathrm{Ber}(d(\boldsymbol{x}_i, \pi_0(\boldsymbol{x}_i))) \quad i = 1, \ldots, n.$$

It will be convenient to define $R_i = \pi_0(\boldsymbol{x}_i)D_i$, which is the observed revenue in the data, and the expected revenue function $r(\boldsymbol{x}, p) = pd(\boldsymbol{x}, p)$.

Our goal is to estimate the expected revenue earned under the new pricing policy $\pi_1(\cdot)$, applied to a new set of covariates $\{\boldsymbol{x}_{n+1}, \ldots, \boldsymbol{x}_{2n}\}$, i.e.,

$$\mathcal{R} \;\equiv\; \frac{1}{n}\sum_{i=n+1}^{2n} \pi_1(\boldsymbol{x}_i) d(\boldsymbol{x}_i, \pi_1(\boldsymbol{x}_i)) \;=\; \frac{1}{n}\sum_{i=n+1}^{2n} r(\boldsymbol{x}_i, \pi_1(\boldsymbol{x}_i)). \tag{Target}$$

Of particular interest is when the new covariates are equal to the old sample, i.e., $\boldsymbol{x}_i = \boldsymbol{x}_{n+i}$ for $1 \le i \le n$.

We focus on weighted revenue estimatrors of the form

$$\hat{\mathcal{R}}(\boldsymbol{w}) \equiv \sum_{j=1}^{n} w_j R_j, \tag{Estimator}$$

We assume

Assumption 1 (Revenue is in RKHS)**.** *There exists an RKHS $\mathcal{H}$, a known reference revenue $r_0(\cdot,\cdot)$, and a perturbation function $\Delta(\cdot,\cdot) \in \mathcal{H}$ such that $r(\cdot,\cdot) = r_0(\cdot,\cdot) + \Delta(\cdot,\cdot)$ and $\|\Delta(\cdot,\cdot)\|_{\mathcal{H}} = \Gamma < \infty$.*

Importantly, under Assumption 1 there exists an $\boldsymbol{\alpha} \in \mathbb{R}^{2n}$ such that

$$\boldsymbol{\Delta} = \boldsymbol{G}\boldsymbol{\alpha}, \qquad \text{and} \qquad \boldsymbol{\alpha}^\top \boldsymbol{G}\boldsymbol{\alpha} = \Gamma^2. \tag{RKHS norm}$$

When G is invertible, this is equivalent to the condition

$$\boldsymbol{\Delta}^\top \boldsymbol{G}^{-1} \boldsymbol{\Delta} = \Gamma^2.$$

2. MSE

Define

$$\boldsymbol{b}(\boldsymbol{w}) = (w_1, \ldots, w_n, -1/n, \ldots, -1/n)^\top \in \mathbb{R}^{2n},$$
$$\boldsymbol{v}(\boldsymbol{w}) = \left(w_1^2\pi_1, \ldots, w_n^2\pi_n, 0, \ldots, 0\right)^\top \in \mathbb{R}^{2n}$$

$$Bias(\boldsymbol{w}; \boldsymbol{r}) \equiv \mathbb{E}\left[\hat{\mathcal{R}}(\boldsymbol{w}) - \mathcal{R}\right] \;=\; \boldsymbol{b}(\boldsymbol{w})^\top \boldsymbol{r}$$
$$Var(\boldsymbol{w}; \boldsymbol{r}) \equiv \mathbb{E}\left[\left(\hat{\mathcal{R}}(\boldsymbol{w}) - \mathbb{E}\left[\hat{\mathcal{R}}(\boldsymbol{w})\right]\right)^2\right] \;=\; \boldsymbol{v}(\boldsymbol{w})^\top \boldsymbol{r} - \boldsymbol{r}^\top \begin{pmatrix} \mathrm{diag}(w_1^2, \ldots, w_n^2) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \boldsymbol{r},$$
$$MSE(\boldsymbol{w}; \boldsymbol{r}) \equiv \boldsymbol{v}(\boldsymbol{w})^\top \boldsymbol{r} + \boldsymbol{r}^\top \left(\boldsymbol{b}(\boldsymbol{w})\boldsymbol{b}(\boldsymbol{w})^\top - \begin{pmatrix} \mathrm{diag}(w_1^2, \ldots, w_n^2) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\right) \boldsymbol{r}.$$

We seek weights that minimize some worst-case metric of our estimator, where the worst-case is taken over the convex set

$$\mathcal{U}_\Gamma \equiv \left\{ \boldsymbol{r}_0 + \boldsymbol{\Delta} \in \mathbb{R}^{2n} \;:\; \boldsymbol{0} \leq \boldsymbol{r}_0 + \boldsymbol{\Delta} \leq \;\boldsymbol{\pi}, \quad \boldsymbol{\Delta}^\top \boldsymbol{G}^{-1} \boldsymbol{\Delta} \leq \Gamma^2 \right\}.$$

The optimization problem is

$$\min_{\boldsymbol{w}} \max_{\boldsymbol{r} \in \mathcal{U}_\Gamma} MSE(\boldsymbol{w}; \boldsymbol{r}).$$

3. Bernstein Bound

Let

$$q_{\max}(\boldsymbol{w}) \equiv \max_{1 \leq j \leq n} |w_j| \, \pi_j$$

The Bernstein Bound says with probability at least $1 - \epsilon$,

$$\hat{\mathcal{R}} - \mathcal{R} \;\leq\; \boldsymbol{b}(\boldsymbol{w})^\top \boldsymbol{r} + \sqrt{2Var(\boldsymbol{w}; \boldsymbol{r}) \log(1/\epsilon)} + \frac{q_{\max}(\boldsymbol{w}) \log(1/\epsilon)}{3}.$$

The resulting optimization problem is

$$\min_{\boldsymbol{w}} \max_{\boldsymbol{r} \in \mathcal{U}_\Gamma} \boldsymbol{b}(\boldsymbol{w})^\top \boldsymbol{r} + \sqrt{2Var(\boldsymbol{w}; \boldsymbol{r}) \log(1/\epsilon)} + \frac{q_{\max}(\boldsymbol{w}) \log(1/\epsilon)}{3}.$$

Rewriting the inner optimization in terms of $\boldsymbol{\Delta}$ and dropping constants yields

$$\begin{aligned}
\max_{\boldsymbol{\Delta}, t} \quad & \boldsymbol{b}(\boldsymbol{w})^\top \boldsymbol{\Delta} + t\sqrt{2\log(1/\epsilon)} \\
\text{s.t.} \quad & t^2 \leq Var(\boldsymbol{w}, \boldsymbol{r}_0 + \boldsymbol{\Delta}) \\
& \boldsymbol{0} \leq \boldsymbol{r}_0 + \boldsymbol{\Delta} \leq \boldsymbol{\pi} \\
& \boldsymbol{\Delta}^\top \boldsymbol{G}^{-1} \boldsymbol{\Delta} \leq \Gamma^2.
\end{aligned}$$

We rewrite the inner problem as a convex QCQP, i.e.,

$$\begin{aligned}
\max_{\boldsymbol{\Delta}, t} \quad & \boldsymbol{b}(\boldsymbol{w})^\top \boldsymbol{\Delta} + t\sqrt{2\log(1/\epsilon)} \\
\text{s.t.} \quad & t^2 + \boldsymbol{\Delta}^\top \boldsymbol{Q}(\boldsymbol{w}) \boldsymbol{\Delta} + 2\boldsymbol{r}_0^\top \boldsymbol{Q}(\boldsymbol{w}) \boldsymbol{\Delta} - \boldsymbol{v}(\boldsymbol{w})^\top \boldsymbol{\Delta} \leq \boldsymbol{v}(\boldsymbol{w})^\top \boldsymbol{r}_0 - \boldsymbol{r}_0^\top \boldsymbol{Q}(\boldsymbol{w}) \boldsymbol{r}_0, \\
& \boldsymbol{0} \leq \boldsymbol{r}_0 + \boldsymbol{\Delta} \leq \boldsymbol{\pi} \\
& \boldsymbol{\Delta}^\top \boldsymbol{G}^{-1} \boldsymbol{\Delta} \leq \Gamma^2.
\end{aligned}$$

where

$$\boldsymbol{Q}(\boldsymbol{w}) \equiv \begin{pmatrix} diag(w_1^2, \ldots, w_n^2) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}.$$

Under Assumption 1, the inner problem is equivalent to

$$\begin{aligned}
\max_{\boldsymbol{\alpha}, t} \quad & (\boldsymbol{b}(\boldsymbol{w})^\top \boldsymbol{G}) \boldsymbol{\alpha} + t\sqrt{2\log(1/\epsilon)} \\
\text{s.t.} \quad & t^2 + \boldsymbol{\alpha}^\top (\boldsymbol{G}\boldsymbol{Q}(\boldsymbol{w})\boldsymbol{G}) \boldsymbol{\alpha} + (2\boldsymbol{r}_0^\top \boldsymbol{Q}(\boldsymbol{w}) - \boldsymbol{v}(\boldsymbol{w})^\top) \boldsymbol{G}\boldsymbol{\alpha} \leq \boldsymbol{v}(\boldsymbol{w})^\top \boldsymbol{r}_0 - \boldsymbol{r}_0^\top \boldsymbol{Q}(\boldsymbol{w}) \boldsymbol{r}_0, \\
& -\boldsymbol{r}_0 \leq G\boldsymbol{\alpha} \leq (\boldsymbol{\pi} - \boldsymbol{r}_0) \\
& \boldsymbol{\alpha}^\top \boldsymbol{G} \boldsymbol{\alpha} \leq \Gamma^2.
\end{aligned}$$

By Danskin' Theorem, the gradient of the outer problem is

$$\boldsymbol{r}[0:n] + \frac{\sqrt{2\log(1/\epsilon)}}{2\sqrt{Var(\boldsymbol{w};\boldsymbol{r})}}\frac{\partial Var(\boldsymbol{w};\boldsymbol{r})}{\partial \boldsymbol{w}} + \frac{\log(1/\epsilon)}{3}\frac{\partial q_{\max}(\boldsymbol{w})}{\partial \boldsymbol{w}}$$

where

$$\frac{\partial Var(\boldsymbol{w};\boldsymbol{r})}{\partial \boldsymbol{w}_i} = 2\boldsymbol{w}_i\boldsymbol{r}_i(\boldsymbol{\pi} - \boldsymbol{r}_i),$$

and

$$\frac{\partial q_{\max}(\boldsymbol{w})}{\partial \boldsymbol{w}_i} = \boldsymbol{\pi}_i \text{sign}(\boldsymbol{w}_i), \text{ if } i = \operatorname*{argmax}_j |\boldsymbol{w}_j|\boldsymbol{\pi}_j \quad \text{and equals zero otherwise}$$

4. Nathan's method

$$Bias(\boldsymbol{w};\boldsymbol{r})^2 \equiv \mathbb{E}\left[\hat{\mathcal{R}}(\boldsymbol{w}) - \mathcal{R}\right]^2 \;=\; (\boldsymbol{b}(\boldsymbol{w})^\top \boldsymbol{r})^2 = \boldsymbol{r}^\top(\boldsymbol{b}(\boldsymbol{w})\boldsymbol{b}(\boldsymbol{w})^\top)\boldsymbol{r}$$

Nathan computes weights that minimize the worst-case bias square, plus some constant regularization.

The optimization problem is

$$\min_{\boldsymbol{w}} \max_{\boldsymbol{r}:||\boldsymbol{r}||_{\mathcal{H}}\le\Gamma} \boldsymbol{r}^\top(\boldsymbol{b}(\boldsymbol{w})\boldsymbol{b}(\boldsymbol{w})^\top)\boldsymbol{r} + \lambda||\boldsymbol{w}||_2^2$$

Under Assumption 1, above problem is equivalent to

$$\min_{\boldsymbol{w}} \boldsymbol{b}(\boldsymbol{w})^\top \left(\boldsymbol{G} + \begin{pmatrix} \lambda\mathbb{I}^n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{pmatrix}\right)\boldsymbol{b}(\boldsymbol{w})$$

5. KKT conditions

Lemma 1. *Let $\phi(\boldsymbol{w})$ denote the inner problem objective.*

If outputs $\boldsymbol{w}^*$ satisfy $\boldsymbol{w}^* \in \Delta_n$ and $\nabla_w\phi(\boldsymbol{w}^*) = -\lambda\vec{1}$ for some $\lambda \in \mathbb{R}$, then $\boldsymbol{w}^*$ are optimal.

Furthermore, if $\boldsymbol{w}^* \in \Delta_n$ are all positive, and $\nabla_w\phi(\boldsymbol{w}^*) \neq -\lambda\vec{1}$ for any $\lambda \in \mathbb{R}$, then weights $\boldsymbol{w}^*$ are not optimal.

Proof. We verify the KKT conditions are satisfied. Since $\phi(\boldsymbol{w})$ is convex, KKT conditions are sufficient.

The primal is

$$\min_w \phi(\boldsymbol{w}), \quad s.t. \vec{1}^\top\boldsymbol{w} = 1, \quad w_i \ge 0, \forall i$$

The Lagrangian is

$$\mathcal{L}(\boldsymbol{w},\lambda,\mu) = \phi(\boldsymbol{w}) - \left(\sum_i \mu_i w_i\right) + \lambda\left(\vec{1}^\top\boldsymbol{w} - 1\right).$$

The dual problem is

$$\max_{\mu\ge\vec{0},\lambda\in\mathbb{R}} \quad q(\mu,\lambda), \quad \text{where} \quad q(\mu,\lambda) = \min_w \mathcal{L}(\boldsymbol{w},\mu,\lambda)$$

KKT conditions are as follows:

(1) primal and dual feasibility

(2) (Lagrangian optimality in primal variables $\boldsymbol{w}$) $\nabla_w\mathcal{L}(\boldsymbol{w}^*,\mu^*,\lambda^*) = 0$. Expand the expression, we need

$$\nabla_w\phi(\boldsymbol{w}^*) - \sum_i \mu_i^* + \lambda^*\vec{1}^\top = 0$$

(3) (Complementary slackness) $\forall i, \mu_i^*(-\boldsymbol{w}_i^*) = 0$. That is $\mu_i^* = 0$ whenever $w_i^* > 0$.

Given that outputs $\boldsymbol{w}^*$ satisfy $\boldsymbol{w}^* \in \Delta_n$ and $\nabla_w \phi(\boldsymbol{w}^*) = -\lambda\vec{1}$ for some $\lambda^* \in \mathbb{R}$. We verify that KKT conditions hold with choice of $\mu^* = \vec{0}$.

Primal dual feasibility is trivial. Complementary slackness is satisfied since $\mu^* = \vec{0}$. Lagrangian optimality is satisfied since

$$\nabla_w \phi(\boldsymbol{w}^*) - \sum_i \mu_i^* + \lambda^* \vec{1}^\top = \nabla_w \phi(\boldsymbol{w}^*) + \lambda^* \vec{1}^\top = 0,$$

where the last equality is by our assumption.

To check the second statement, we observe $\boldsymbol{w}_i^* > 0$ forces $\mu_i^* = 0$ for all i. Consequently, Lagrangian optimality in primal variables force $\nabla_w \phi(\boldsymbol{w}^*) \neq -\lambda\vec{1}$ for some $\lambda \in \mathbb{R}$.

$\square$