FE 621 Computational Methods in Finance: Lecture 1

We will start the class learning about basic methods applied to Finance. The context is for the classical model of the stock price, Black-Scholes-Merton model. First I am going to review basic facts.

1 Itô integrals, Itô formula, Stochastic differential equations

Let $(\Omega, \mathscr{F}, \mathbf{P})$ a probability space, $\{\mathcal{F}_t\}$ a filtration on this space. Define for some fixed $S \le T$ a class of functions $\nu = \nu(S, T)$:

$$f(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}$$

such that:

1. $(t, \omega) \mapsto f(t, \omega)$ is a $\mathscr{B} \times \mathscr{F}$ measurable function, where $\mathscr{B} = \mathscr{B}[S, T]$ is the Borel sigma algebra on that interval.
2. $\omega \mapsto f(t, \omega)$ is $\mathcal{F}_t$-adapted for all t.
3. $\mathbf{E}[\int_S^T f^2(t, \omega) dt] < \infty$

Then for every such $f \in \nu$ we can construct:

$$\int_S^T f_t dB_t = \int_S^T f(t, \omega) dB_t(\omega),$$

where B_t is a standard Brownian motion with respect to the same filtration $\{\mathcal{F}_t\}_t$. This quantity is called a stochastic integral with respect to the Brownian motion B_t. Note that the stochastic integral is a random quantity.

1.1 Properties of the stochastic integral

- Linearity:

$$\int_S^T (af_t + bg_t)dB_t = a\int_S^T f_t dB_t + b\int_S^T g_t dB_t, \quad a.s.$$

- $$\int_S^T f_t dB_t = \int_S^U f_t dB_t + \int_U^T f_t dB_t, \quad a.s., \forall S < U < T$$

- $$\mathbf{E}\left[\int_S^T f_t dB_t\right] = 0$$

- Itô Isometry:

$$\mathbf{E}\left[\left(\int_S^T f_t dB_t\right)^2\right] = \mathbf{E}\left[\int_S^T f_t^2 dt\right]$$

- If $f \in \nu(0,T)$ for all T then $M_t(\omega) = \int_0^t f(s,\omega)dB_s(\omega)$ is a martingale with respect to $\mathcal{F}_t$ and:

$$\mathbf{P}\left(\sup_{0\le t\le T} |M_t| \ge \lambda\right) \le \frac{1}{\lambda^2}\mathbf{E}\left[\int_0^T f_t^2 dt\right], \lambda, T > 0 \text{ (Doob's inequality)}$$

1.2 Itô process:

Let B_t be a standard Brownian motion. Then the process:

$$X_t = X_0 + \int_0^t \mu(s,\omega)ds + \int_0^t \sigma(s,\omega)dB_s, \tag{1}$$

is called a Itô process. Note that the functions $\mu, \sigma \in \nu(0,\infty)$ are random in general. Sometimes for brevity of notation the equation above is written as:

$$dX_t = \mu dt + \sigma dB_t.$$

This later equation has no meaning whatsoever other than providing a symbolic notation for the equation (1). We shall use this later notation at all times meaning an equation of the type (1).

1.3 Itô formula

Let X be an Itô process as in equation (1). Suppose that $g(t,x)$ is a function defined on $[0,\infty)\times\mathbb{R}$ twice differentiable in x and one time differentiable in t. Symbolically $g \in \mathcal{C}^{1,2}\left([0,\infty)\times\mathbb{R}\right)$. Then the process $Y_t = g(t, X_t)$ is again an Itô process and:

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2,$$

where $(dX_t)^2$ is calculated using the rules:

$$\begin{gathered} dtdt = dB_t dt = dt dB_t = 0 \\ dB_t dB_t = dt, \end{gathered}$$

and we have used the symbolic notation.

Examples:

Let $X_t = B_t$ (i.e. $dX_t = dB_t$). Let $g(t,x) = \frac{1}{2}x^2$. Then $Y_t = \frac{1}{2}B_t^2$. We can also calculate easily $\frac{\partial g}{\partial t} = 0$, $\frac{\partial g}{\partial x} = x$, $\frac{\partial^2 g}{\partial x^2} = 1$. Thus Itô formula gives:

$$d\left(\frac{1}{2}B_t^2\right) = B_t dB_t + \frac{1}{2}(dB_t)^2 = B_t dB_t + \frac{1}{2}dt,$$

or in the proper notation:

$$\frac{1}{2}B_t^2 - \frac{1}{2}B_0^2 = \int_0^t B_s dB_s + \frac{1}{2}t$$

Rewriting this we obtain the integral of the Brownian motion with respect to itself:

$$\int_0^t B_s dB_s = \frac{1}{2}\left(B_t^2 - t\right)$$

(recall that $B_0 = 0$).

Exercise Please calculate yourself $\int_0^t s dB_s$. To this end once again take $X_t = B_t$ but $g(t,x) = tx$.

1.4 Product rule:

Finally, if X_t, Y_t are two Itô processes of the form (1), then:

$$d\left(X_tY_t\right) = X_tdY_t + Y_tdX_t + dX_tdY_t,$$

where in the last term the same rules as in the Itô formula apply.

1.5 Stochastic Differential equations. SDE's

If the functions μ and σ in (1) depend on ω through the process X_t itself then (1) defines a Stochastic differential equation. There are techniical conditions that guarantee the existence and uniqueness of the solution of the SDE but we will not need them. Specifically, assume that $X_t \in \nu(0,\infty)$. Assume that the functions $\mu = \mu(t,x)$ and $\sigma = \sigma(t,x)$ are twice differentiable with continuous second derivative in both variables. Then the following:

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \tag{2}$$

defines a Stochastic differential equation.

2 The Black-Scholes-Merton model

I am going to state basic facts about this model. For more details consult chapter 1 in Clewlow textbook [1] or chapter 13 in Hull's [3].

In the Black-Scholes-Merton model the stock price follows the equation:

$$dS_t = \mu S_tdt + \sigma S_tdB_t, \tag{3}$$

where μ and σ are constants and B_t is a regular one dimensional Brownian motion. This is the simplest model one can imagine. The process in (3) is also called a geometric Brownian motion.

As a parenthesis a bit more general we can have a model of the form:

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dB_t, \tag{4}$$

where now μ and σ are general (but still deterministic) functions of class $\mathscr{C}^2$. Most of the analysis still proceeds in a fundamentally similar way. The explicit formulas do not exist anymore in this case.

2.1 The return equation. An application of the Itô formula.

Suppose that the stock follows a geometric Brownian motion (equation (3)). Consider the process $R_t = \log S_t$. This is called the return process. Notice that $R_t - R_{t-\Delta t} = \log S_t - \log S_{t-\Delta t} = \log(S_t/S_{t-\Delta t})$ which is the continuously compounded return over the interval $[t-\Delta t, t]$. We can obtain an equation for the return by applying Itô to the function $g(t,x) = \log x$. We obtain:

$$
\begin{aligned}
dR_t &= \frac{\partial g}{\partial t}(t,S_t)dt + \frac{\partial g}{\partial x}(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,S_t)(dS_t)^2 \\
&= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dB_t) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(\mu S_t dt + \sigma S_t dB_t)^2 \\
&= \mu dt + \sigma dB_t - \frac{1}{2S_t^2}\left(\mu^2 S_t^2 dt^2 + 2\mu\sigma S_t^2 dt dB_t + \sigma^2 S_t^2 dB_t^2\right) \\
&= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dB_t,
\end{aligned}
$$

or:

$$R_t - R_0 = \int_0^t \left(\mu - \frac{\sigma^2}{2}\right)ds + \int_0^t \sigma dB_s = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t.$$

Recall that $R_t = \log S_t$. Substituting back and solving for S_t we obtain an explicit formula:

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t} \tag{5}$$

Final examples

- *Ornstein-Uhlenbeck process*:

$$dX_t = \mu X_t dt + \sigma dB_t$$

- *Mean reverting Ornstein-Uhlenbeck process*:

$$dX_t = \alpha(m - X_t)dt + \sigma dB_t$$

Both of these SDE's have explicit solutions which are obtained in either case by applying Itô to the function $g(t,x) = xe^{-\mu t}$ (first process) or $g(t,x) = xe^{\alpha t}$ (second process).

3 The Black-Scholes Partial Differential Equation for a derivative (option) on the stock process S_t.

Harrison and Pliska in a series of fundamental papers showed the following: Assume that we have *any derivative* whose value depends only on the stock price S at time t, denoted with $V(S,t)$. As long as the price process S solves an equation of the type (3), the derivative process at t will solve the following partial differential equation:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{6}$$

This is remarkable because the equation is constructed without any assumption on the derivative V other than at any time t to be dependent only on the value of the process at that time S_t. This means that the above equation (6) holds for the price of any contract that depends only on S_t: European, American, lookback, barrier and many more options.

However, in order to solve this PDE one has to use the particular boundary conditions specified by the particular type of option analyzed. It turns out that there are very few boundary conditions that will give rise to an explicit solution. One of these particular cases is the European Option. The boundary condition for the European Call option is $C(S,T) = \max(0, S_T - K) := (S_T - K)_+$, and for the European Put option is $P(S,T) = \max(0, K - S_T) := (K - S_T)_+$. The European option case can be solved explicitly and the solution is the well known Black-Scholes formula:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{7}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

For the put option the formula is:

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{8}$$

NOTE: Also recall the put-call parity relation you have learned in FE620 (note that for any $z \in \mathbb{R}$ we have $N(z) + N(-z) = 1$).

These formulae remain even today the most widely used instruments from the Mathematics of Finance, however they are useful only for the European type options. Most other derivative prices have to be estimated rather than calculated. We shall see how to do this in the later courses.

Next we give a derivation for the Black-Scholes PDE as well as for the call and put formulae.

3.1 The derivation for the Black Scholes PDE

Assume that the world contains only two assets:

- A risky asset modeled by a geometric BM as in (3):

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t}$$

- A risk-free (bank account) asset earning r continuous interest, i.e.:

$$W_t = W_0 e^{rt}$$

Let us assume that we have a derivative whose value only depends on the value of the stock at some future time T, i.e., $V_T = V(T, S_T)$. Under these conditions Harrison and Pliska show that there exist an equivalent measure under which the discounted price of the derivative V_t is a martingale (many details are omitted for more see the original article [2]). In any case for us we shall use their result and deduce that this price is only a deterministic function of S_t, i.e., $V_t = V(t, S_t)$. Thus we can apply Itô's rule to obtain the dynamics of V_t:

$$\begin{aligned}
dV_t &= \frac{\partial V}{\partial t}(t, S_t)dt + \frac{\partial V}{\partial S}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)(dS_t)^2 \\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(\mu S_t dt + \sigma S_t dB_t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 dt \\
&= \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S_t \frac{\partial V}{\partial S} dB_t
\end{aligned}$$

Now consider a portfolio Π where we own a portion $h(t)$ of the stock and we sell the derivative V itself. Therefore, the value of this portfolio at any moment in time t is:

$$\Pi(t) = h(t)S_t - V_t.$$

Thus, its change in value is governed by:

$$d\Pi(t) = d\left(h(t)S_t\right) - dV_t = h(t)dS_t - dV_t.$$

This last derivation is actually quite tricky since normally we should have another term $h'(t)S_t dt$ by the product rule given above. I will not enter into details here, the formula presented is indeed correct. Substituting dV_t from above we obtain:

$$d\Pi(t) = \mu h(t)S_t dt + \sigma h(t)S_t dB_t - (\dots)\, dt - \sigma S_t \frac{\partial V}{\partial S} dB_t$$

To make this portfolio non-random we chose $h(t)$ so that the terms in dB_t cancel out. Specifically, we take:

$$h(t) = \frac{\partial V}{\partial S}(t, S_t), \forall t$$

(i.e. we construct what is called the delta neutral portfolio).

If we do this to our portfolio the value equation becomes:

$$\begin{aligned} d\Pi(t) &= \mu \frac{\partial V}{\partial S} S_t dt - \frac{\partial V}{\partial t} dt - \mu S_t \frac{\partial V}{\partial S} dt - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} dt \\ &= -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt \end{aligned}$$

But this portfolio does not contain any randomness (there is no dB_t term). Therefore it should behave like the risk-free asset, that is its equation should be:

$$d\Pi(t) = r\Pi(t)dt,$$

where $\Pi(t)$ is given by $\Pi(t) = h(t)S_t - V_t = \frac{\partial V}{\partial S} S_t - V_t$. Setting the two terms equal we obtain:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r \frac{\partial V}{\partial S} S_t - rV_t = 0$$

or, the Black Scholes PDE. We also note that the drift μ is irrelevant (it does not appear in the equation) and therefore we consider the stock equation under the equivalent measure constructed by Harrison and Pliska[1] as:

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

3.2 The derivation of the Black Scholes formula

The Black Scholes formula can be obtained by solving the equation (6) with specific boundary conditions. I am sure that you have seen this derivation before. Here however, we shall use Harrison and Pliska's result to get to the formula differently.

Suppose we wish to obtain the formula for the put option denoted $p(t)$. We know that at time T maturity the value is $p(T) = (K - S_T)_+$, where K is the strike price of the option. Using Harrison and Pliska, the discounted price is a martingale therefore:

$$e^{-rt} p(t) = \mathbf{E}\left[e^{-rT} P(T) | \mathcal{F}_t\right],$$

which is just the martingale property. Furthermore, under this martingale measure the equation of the stock has drift r not μ and thus the formula (5) becomes:

$$S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t} \tag{9}$$

For simplicity of notation I am going to use $t = 0$. One has to use the fact that the conditional expectation becomes regular expectation if the variable is independent of the conditional filtration. This is our case since S_T only depends on the increment $B_T - B_0$ and that is independent of $\mathcal{F}_0$. With this observation the price of the put is:

$$\begin{aligned} p(0) &= \mathbf{E}\left[e^{-rT}(K - S_T)_+ | \mathcal{F}_0\right] \\ &= \mathbf{E}\left[e^{-rT}(K - S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T})_+ | \mathcal{F}_0\right] \\ &= \mathbf{E}\left[e^{-rT}\left(K - S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\frac{B_T}{\sqrt{T}}}\right)_+ \middle| \mathcal{F}_0\right] \end{aligned}$$

[1] Please consult the article in particular the details of the Girsanov theorem to obtain exactly these dynamics of the stock price.

Now we recognize that $B_T/\sqrt{T}$ is distributed as a normal with mean zero and variance 1 ($N(0,1)$) and we obtain[2]:

$$p(0) = e^{-rT} \int_{-\infty}^{\infty} \left(K - S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z}\right)_+ \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

So, when exactly is the expression inside positive? Note that:

$$\begin{aligned}
K - S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z} \geq 0 \quad &\Leftrightarrow e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z} \leq \frac{K}{S_0} \\
&\Leftrightarrow \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z \leq \log\frac{K}{S_0} \\
\Leftrightarrow z \leq \frac{\log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -\frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} &= -d_2
\end{aligned}$$

using the notation given above in equation (7). We also use the same notation $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$ for the CDF of the normal with mean 0 and variance 1. Thus we obtain:

$$\begin{aligned}
p(0) &= e^{-rT} \int_{-\infty}^{-d_2} \left(K - S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= e^{-rT} K \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - e^{-rT} S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T} \int_{-\infty}^{-d_2} e^{\sigma\sqrt{T}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= e^{-rT} K N(-d_2) - S_0 e^{-\frac{\sigma^2 T}{2}} \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2} + \sigma\sqrt{T}z - \frac{\sigma^2 T}{2} + \frac{\sigma^2 T}{2}} dz \\
&= e^{-rT} K N(-d_2) - S_0 e^{-\frac{\sigma^2 T}{2} + \frac{\sigma^2 T}{2}} \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z - \sigma\sqrt{T})^2}{2}} dz \\
&= e^{-rT} K N(-d_2) - S_0 \int_{-\infty}^{-d_2 - \sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du,
\end{aligned}$$

where we have completed the square in the second integral and then we made the change of variables $z - \sigma\sqrt{T} = u$. Now recognizing that $d_2 + \sigma\sqrt{T} = d_1$ we finally obtain:

$$p(0) = e^{-rT} K N(-d_2) - S_0 N(-d_1),$$

which is exactly the formula for the put given before.

[2]This is an informal derivation that happens to be correct. Normally, one should use the measurability of S_0 with respect to $\mathcal{F}_0$ first.

4 Put-Call parity

This relation refer to the relation between the price of European Put and Call Contracts. It is valid regardless of the dynamics of the stock which may be following any complicated dynamics.

Suppose first that the asset pays no dividend. Construct a portfolio by buying a call and selling a put with the same exact maturity and strike price. At time T the payoff of the portfolio is:

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K$$

regardless of whether S_T is above or below K. But this payoff may be replicated by buying a unit of stock and buying a bond that pays exactly K at time T. We shall talk more about these zero coupon bonds later in this class. Therefore since the two instruments have the same exact payoff by standard no arbitrage arguments their value must be the same at all times thus at any time t we must have:

$$C(t) - P(t) = S_t - KP(t,T),$$

where $C(t)$ is the European Call price, $P(t)$ is the European put and $P(t,T)$ is the price of a zero coupon bond that pays \$1 at time T.

In the special case when the risk-free interest rate is assumed constant r then the price of a zero coupon bond at t that pays \$1 at T is:

$$P(t,T) = e^{-r(T-t)},$$

and under this simplifying assumption we obtain the more familiar form

$$C(t) - P(t) = S_t - Ke^{-r(T-t)},$$

If the asset pays dividends then we simply add all dividend value into some quantity $D(T)$. The options have nothing to do with this dividend, owing an option will not receive any dividend. Thus, a time t if we held the stock minus the present value of K we will produce at time T: S_T PLUS all the dividends received in the meantime (recall that owning the stock receives dividends). Therefore, for a dividend paying asset if $D(t)$ is the present value of dividends received between time t and maturity T we have in this situation:

$$C(t) - P(t) = S_t - KP(t,T) - D(t), \quad \forall t \le T$$

For American Options. For clarity let us denote $C_A(t)$ and $P_A(t)$ American call and put prices and with $C_E(t)$ and $P_E(t)$ European option prices. For simplicity I am also going to use $P(t,T) = e^{-r(T-t)}$ in general this term is replaced with more general formulas.

Clearly, since American options are European with added bonus of being able to exercise anytime we must have:

$$C_A(t) \geq C_E(t) \text{ and } P_A(t) \geq P_E(t)$$

Furthermore, we clearly have:

$$C_A(t) \geq (S_t - K)_+ \text{ and } P_A(t) \geq (K - S_t)_+$$

If one of them is not true suppose for example $C_A(t) < (S_t - K)_+$ then buy and American call for $C(t)$ and immediately exercise and receive $(S_t - K)_+$. Thus profit is $(S_t - K)_+ - C(t) > 0$ by assumption and thus immediate arbitrage opportunity.

A slightly more complex argument shows that:

$$C_A(t) \geq S_t - Ke^{-r(T-t)}$$

Why: Once again assume that $C_A(t) < S_t - Ke^{-r(T-t)}$ for some time t. When that happens form a portfolio by selling short 1 share of stock at S_t, buying one call at $C_A(t)$ and putting $Ke^{-r(T-t)}$ into a bank account at interest rate r. Total balance at t is:

$$S_t - Ke^{-r(T-t)} - C_A(t)$$

which by assumption is strictly positive so we make a profit at t. Then at time T:

- We need to buy back the stock for S_T
- Exercise the option and receive $(S_T - K)_+$
- get the money back from the bank K.

Balance at time T:

$$(S_T - K)_+ + K - S_T \geq 0$$

which is positive for all situations. Thus again, we made money with no risk and therefore it must be that:

$$C_A(t) \geq S_t - Ke^{-r(T-t)}.$$

Now, note that for any $t < T$ the term $e^{-r(T-t)} < 1$ therefore:

$$C_A(t) \geq S_t - Ke^{-r(T-t)} > S_t - K, \quad \forall t$$

But on the right hand side is the value received is option is exercised at time t. Therefore, IT IS NEVER OPTIMAL TO EXERCISE AN AMERICAN OPTION EARLY WHEN THE UNDERLYING PAYS NO DIVIDENDS.

The same is not true about the put or about the case when the underlying pays dividends.

Using similar no arbitrage arguments we may show that for the case of no dividends the following put-call inequalities hold:

$$S_t - K \leq C_A(t) - P_A(t) \leq S_t - Ke^{-r(T-t)}$$

For dividend paying stock let $D(t)$ denote the total value of dividends paid between t and T and discounted back at time t. The following inequality holds in this case:

$$S_t - K - D(t) \leq C_A(t) - P_A(t) \leq S_t - Ke^{-r(T-t)}$$

References

[1] L. Clewlow and C. Strickland. *Implementing Derivatives Models*. Wiley, 1998.

[2] M. Harrison and S. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11:215–260, 1981.

[3] John C. Hull. *Options, Futures and Other Derivatives*. Prentice Hall, 7 edition, 2008.