Chapter 2

Basics of finance

2.1 Introduction

The value of money over time is a very important concept in financial mathematics. A dollar that is earned today is valued higher than a dollar that will be earned in a years time. This is because a dollar earned today can be invested today and accrue interest, making this dollar worth more in a year. We can then calculate how much money we will need to invest now to have k dollars in a certain time frame. For example, if we invest \$100 at 5% interest for a year, we will have 100(5%) + 100 = \$105 at the end of the year. A question that often arises is, how can one model interest mathematically? This can be modeled by an amount $M(t)$ in the bank at time t. The amount of interest gained during an elapsed time of dt is represented as $M(t + dt) - M(t) = \frac{dM}{dt}dt + \cdots$. Since interest is proportional to the amount of money in the account, given the interest rate r and time step dt, we can then write $\frac{dM}{dt}dt = rM(t)dt$. This is now a differential equation that can be solved to obtain $M(t) = M(0)e^{rt}$. In regards to pricing options, which we will study in section **??**, we are concerned with the present valuation of an option. We will use the time value of money to answer the question, how much we would be willing to pay now in order to receive an exercise amount E at some time T in the future?

In the subsequent sections, we introduce some concepts that will be used throughout this book.

2.2 Arbitrage

Arbitrage is one of the fundamental concepts of mathematical finance. The easiest way to understand arbitrage is through the phrase, "free lunch". Arbitrage is basically investing nothing and being able to get a positive return without undergoing any risk at all, like getting a free meal without having to pay for it. This is a very crucial principle for the mathematical modeling of option pricing. We will present an example of an arbitrage opportunity, but first we will define a forward contract.

Definition 2.2.1. A forward contract is an agreement between two parties to buy and sell specified asset from the other for a specified price, known as the forward price, on a specified date in the future (the delivery date or maturity date).

The agreed upon price of the underlying asset at the delivery date is called the delivery price, which we will denote by F. In a forward contract, there is no choice to buy or sell the underlying asset, the underlying asset must be paid for and delivered, unlike "options" which we will define in the next section. In a forward contract there is no exchange of money until the delivery date. The objective is to find an appropriate value of the delivery price F. If $S(T)$ represents the stock price at time T, then the buyer of the forward is entitled to an amount $S(T) - F$ at expiry, and we do not know this value. In Table 2.1, we present an example of pricing F using an arbitrage argument.

Table 2.1: Arbitrage Example

Holding	Value Today(time t)	Value upon exercise(T)
Forward	0	$S(T) - F$
-Stock	$-S(t)$	$-S(T)$
Cash	$S(t)$	$S(t)e^{r(T-t)}$
Cashflow	0	$S(t)e^{r(T-t)} - F$

For this particular example of arbitrage opportunity, we will enter into a forward contract. At the same time we will sell the asset, or"go short". It costs nothing to enter into a forward contract and we will sell our stock that is worth $S(t)$. In order to make the market arbitrage free, we must have $S(t)e^{r(T-t)} - F = 0$, that is $F = S(t)e^{r(T-t)}$. This is called the arbitrage free payoff for a forward contract.

Consider the case where $S(t)e^{r(T-t)} - F > 0$, and we enter into the situation explained above. At the end of the contract we will have $S(t)e^{r(T-t)}$ in a bank, a shorted asset, and a long forward. The position you take on the asset then washes out when you give back F, so we have obtained a profit of $S(t)e^{r(T-t)} - F$ without spending any money or taking any risk! This is an arbitrage opportunity. Considering the case where $S(t)e^{r(T-t)} - F < 0$, we may construct a similar arbitrage opportunity as well.

Therefore, according to this argument the price of the forward at time t is exactly:

$$F = S(t)e^{r(T-t)}.$$

However, note that the rate $r(T - t)$ is typically unknown and estimated. It is supposed to be the expected value of the future rate and much of quantitative finance is about finding this value.

In practice, traders profit from finding and exploiting more complex opportunities than the simple one described above. They will act quickly on these opportunities, and the prices will adjust.

2.3 Options

Definition 2.3.1. An option is a financial contract that gives the holder the right to trade in the future a specified underlying asset at an agreed and specified price. The details of the underlying asset (S), the calculation of the settlement price (strike K), and the time at which the transaction is to take place (maturity or exercise time) are typically written in the contract. The sum of money received at maturity of the contract is called the Payoff. The payoff is typically positive.

The party that gives the right to trade is called the option underwriter or short the option writer. The writer can either own the particular underlying asset in which case the option is called "covered" or provide he has means to purchase the asset in which case the option is called "naked".

The party that buys the option becomes the holder of the contract. The holder has the option to exercise the trade - this gives the name of the contract. In return for this option right the holder must pay a certain amount called *the option premium*. Most of quantitative finance is dedicated to calculate this option premium.

There are two fundamental option types. A *Call* type option gives the holder the right to **buy** the underlying asset. A *Put* type option gives the holder the right to **sell** the underlying.

In the following subsection, we briefly describe the most commonly encountered option types the so called vanilla options. In English language (owing we assume to England and US) apparently the most commonly encountered flavor of ice cream is vanilla thus the most basic types of anything are typically called vanilla. This moniker is most certainly not true in other languages.

2.3.1 Vanilla Options

The most basic type of option is an *European option contract.* At a time specified in the option contract called the maturity time or the expiration date and which we will denote with T, the holder of the option may exercise the right to purchase or sell the underlying asset, for a specified amount, called the strike price and which will be denoted with K. The option to buy is an European Call and the option to sell is an European Put.

The writer is of course obligated by the contract to provide the trade counter party should the option holder decide to exercise it.

Consider a Call option. If at time T the price of the underlying asset $S > K$, then the Call option is woth exercising. The holder would buy the asset from the writer for the specified amount K and then sell it immediately for the market price S. Thus the option's payoff in this case is $S - K$. On the other hand if

$S < K$ at maturity, the option is not worth exercising as the holder can purchase the asset cheaper by paying its market price. Thus the value of the Call option at maturity can be written as:

$$V(S,T) = \max(S - K, 0). \tag{2.1}$$

Using a similar argument the Put option payoff is:

$$V(S,T) = \max(K - S, 0).$$

An American option contract is similar with a European option except the option to purchase or sell the asset may be exercised at any time prior to the contract maturity date T. For this reason, the underlying asset path is very important for American options, as opposed to European where only the final value at time T is important.

As we shall see next if the underlying asset does not pay dividends for the lifetime of the option then an American Call has the same value as an European Call option. Dividend is defined as payments made to shareholders out of the profits made by the company. This is very counterintuitive but logically correct. The same is not true for the Put.

2.3.2 Put-Call parity

This parity refers to the relation between the price of European Put and Call Contracts. The equation derived is valid regardless of the dynamics of the stock which may be following any complicated model.

Suppose first that the asset pays no dividend. We denote the price of the asset at time t with S_t. Construct a portfolio by buying a call and selling a put with the same exact maturity and strike price. At time T the payoff of the portfolio is:

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K$$

regardless of whether S_T is above or below K. But this payoff may be replicated by buying a unit of stock and buying a bond that pays exactly K at time T.

A zero coupon bond is the price at time t of an instrument that pays 1 dollar at some specified time in the future T. We denote this number with $P(t,T)$. A zero coupon bond assumes no default (i.e., we always get the dollar at time T), and in the simple case where the interest rate stays constant at r between t and T its price is simply:

$$P(t,T) = e^{-r(T-t)}$$

Coming back to our portfolio story, since the two portfolios have the same exact payoff, using standard no arbitrage arguments their value must be the same at all times. Thus, at any time t we must have the following relationship between the price of a European Call and Put:

$$C(t) - P(t) = S_t - KP(t,T),$$

where $C(t)$ is the European Call price, $P(t)$ is the European put both at t. As mentioned, $P(t,T)$ is the price of a zero coupon bond that pays \$1 at time T. This is known as the Put-Call parity.

In the special case when the risk-free interest rate r is assumed constant we obtain the more familiar form of the Put-Call parity:

$$C(t) - P(t) = S_t - Ke^{-r(T-t)},$$

If the asset pays dividends then we simply add all dividend values paid over the option lifetime into some quantity $D(T)$. The options have nothing to do with this dividend, holding an option will not receive any dividend only the asset holder does. Thus, a time t if we held the stock minus the present value of K we will produce at time T: S_T PLUS all the dividends received in the meantime (recall that owning the stock receives dividends). Therefore, for a dividend paying asset if $D(t)$ is the present value of dividends received between time t and maturity T, then the Put-Call parity in this situation is:

$$C(t) - P(t) = S_t - KP(t,T) - D(t), \quad \forall t \leq T$$

For American Options. For clarity let us denote $C_A(t)$ and $P_A(t)$ American call and put prices and with $C_E(t)$ and $P_E(t)$ European option prices. For simplicity, we will also going to use $P(t,T) = e^{-r(T-t)}$. If one prefers just replace the more general term for all the following formulas.

Clearly, since American options are European with added bonus of being able to exercise anytime we must have:

$$C_A(t) \geq C_E(t) \text{ and } P_A(t) \geq P_E(t)$$

Furthermore, we must have:

$$C_A(t) \geq (S_t - K)_+ \text{ and } P_A(t) \geq (K - S_t)_+$$

If one of these relations is not true for example $C_A(t) < (S_t - K)_+$ then buy an American call for $C_A(t)$ and immediately exercise to receive $(S_t - K)_+$. Thus, we can obtain a profit $(S_t - K)_+ - C(t) > 0$ and thus immediate arbitrage opportunity. Therefore, the inequalities as stated must be true at all times.

A slightly more complex argument shows that:

$$C_A(t) \geq S_t - Ke^{-r(T-t)}.$$

Once again assume that this isn't true. Thus, there must be a t at which we have $C_A(t) < S_t - Ke^{-r(T-t)}$. At that exact time form a portfolio by short selling 1 share of the stock for S_t, buying one call at $C_A(t)$ and putting $Ke^{-r(T-t)}$ into a bank account at interest rate r. Total balance at t is thus:

$$S_t - Ke^{-r(T-t)} - C_A(t)$$

which by assumption is strictly positive so at t we make a profit. Then at time T:

- We need to buy back the stock for S_T
- Exercise the option and receive $(S_T - K)_+$
- get the money back from the bank K.

Balance at time T:

$$(S_T - K)_+ + K - S_T \geq 0$$

which is positive for all situations. Thus again, we made money with no risk and therefore it must be that our assumption is false and that:

$$C_A(t) \geq S_t - Ke^{-r(T-t)} \text{ at all times } t.$$

Now, note that for any $t < T$ the term $e^{-r(T-t)} < 1$ therefore:

$$C_A(t) \geq S_t - Ke^{-r(T-t)} > S_t - K, \quad \forall t$$

But on the right hand side is the value received if the call option is exercised at time t. Therefore, IT IS NEVER OPTIMAL TO EXERCISE AN AMERICAN OPTION EARLY in the case whe THE UNDERLYING PAYS NO DIVIDENDS.

This is not true in the case when the underlying pays dividends.

Using similar no arbitrage arguments we may show that for the case of no dividends the following Put-Call inequalities hold:

$$S_t - K \leq C_A(t) - P_A(t) \leq S_t - Ke^{-r(T-t)}$$

for American type options.

For dividend paying stock let $D(t)$ denote the total value of dividends paid between t and T and discounted back at time t. The following Put-Call inequality holds in this case:

$$S_t - K - D(t) \leq C_A(t) - P_A(t) \leq S_t - Ke^{-r(T-t)}$$

2.4 Hedging

When investing in the stock market and developing a portfolio, there is always a chance that the stock market or the assets invested in will drop in value. Hedging is the term used when protecting the investment from such unfavorable events.

Definition 2.4.1. Hedging is the process of reducing the exposure of a portfolio to the movement of an underlying asset.

Mathematically, sensitivity is defined as the derivative of the portfolio value with respect to the particular asset value we want to hedge. Reducing the exposure means setting the derivative equal to zero. This is a process known as *Delta hedging*.

To better understand the concept of delta hedging we will use a simple binomial model. In this example, we will start with a call option on an asset worth \$100 at time t. The contract maturity time is $T = t = \Delta t$. If we decide to exercise the option, in the case when the value of the asset increases, we will receive \$1, and if the option decreases we will not exercise the option, and we will make no profit.

Suppose the price of the stock tomorrow is unknown, but the probability that the stock rises is 0.6, and the probability that the stock falls is 0.4. The expected value of V is $E[V] = 1(0.6) + 0(0.4) = 0.6$. This is actually not the correct value of V, as we will see in the binomial hedging example presented in Table 2.2.

Table 2.2: Binomial Hedging Example

	Value Today(time t)	Value Tomorrow(t + dt)	Value Tomorrow(t + dt)
Option	V	1	0
-Δ Stock	-Δ100	-Δ(101)	-Δ(99)
Cash	$100\Delta - V$	$100\Delta - V$	$100\Delta - V$
Total	0	1 - Δ(101) + $100\Delta - V$	-99Δ + $100\Delta - V$

To hedge we are in some sense "going short" or selling a quantity Δ of our stock to protect against unfavorable outcomes in the market. Our goal is to make our portfolio insensitive to market movement. The question is how much should we sell.

For simplicity, we will ignore the concept of time value of money in this example. To calculate the value of Δ that does not make us dependent on the market in Table 2.2, set the totals in the last two columns equal:

$$1 - \Delta(101) + 100\Delta - V = -99\Delta + 100\Delta - V. \tag{2.2}$$

Solving in terms of Δ in (2.2), we obtain $\Delta = \frac{1}{2}$. Therefore if we short a half share of the stock our portfolio will be insensitive to the market movement. Since we have correctly hedged, we are not concerned whether or not the stock rises or falls, the portfolio is now insensitive to market movement.

More generally, assume that at time t the value of our asset is S and in some elapsed time dt, the price of the asset can either rise or fall. We will denote the value risen as uS and the value fallen as vS, where $0 < v < 1 < u$, and $u = 1 + \sigma\sqrt{dt}, v = 1 - \sigma\sqrt{dt}$. At time of expiry, $t + dt$, the option either takes on the value V^+ or V^- if the asset price rises or falls respectively. Then $\Delta = \frac{V^+ - V^-}{(u-v)S}$. If we let r to be the interest rate, the price of the option is given today by:

$$V = \frac{1}{1 + rdt}(p'V^+ + (1 - p')V^-),$$

where

$$p' = \frac{1}{2} + \frac{r\sqrt{dt}}{2\sigma},$$

is known as the risk-neutral probability. This is the general case for the binomial model that will be studied in chapter 6. Next, we study the modeling of returns in order to proceed to continuous hedging.

2.5 Modeling Return of stocks

When investing in the stock market, it is very important to model the money you make from a stock over a certain time period, or the return.

Definition 2.5.1. A return is a percentage growth in the value of an asset, together with accumulated dividends, over some period of time.

Mathematically the return is given as:

$$\text{Return} = \frac{\text{Change in the value of the asset} + \text{accumulated cashflows}}{\text{Original value of the asset}}.$$

For example, suppose we invest in two different stocks A and B. Stock A is worth \$10, and stock B is worth \$100. Now suppose that both stocks rise by \$10, so that stock A is worth \$20 and stock B is worth \$110. Since both stocks have risen \$10, they have the same absolute growth. Applying our definition of return, we observe that stock A grew 100% and stock B by 10%. When comparing the two stocks, we observe that we had much more return on stock A. Letting S_i denote the asset value on the ith day, the return from day i to day $i+1$ is calculated as,

$$\frac{S_{i+1} - S_i}{S_i} = R_i.$$

If we suppose that the empirical returns are close enough to normal (for a good approximation, then ignoring dividends), we can model the return by

$$R_i = \frac{S_{i+1} - S_i}{S_i} = \mu + \sigma\phi, \tag{2.3}$$

where μ represents the mean, σ represents the standard deviation and ϕ represents the probability density function (pdf) for a standard normal distribution i.e.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

Suppose we let the time step from the asset price S_i to S_{i+1} be denoted δt. Thus, we can write the mean of the return model as as

$$\text{mean} = \mu\delta t,$$

where μ is assumed to be constant. The standard deviation can also be written as

$$\text{standard deviation} = \sigma(\delta t)^{\frac{1}{2}},$$

where σ measures the amount of randomness. Large values of σ suggest a more uncertain return. Therefore, we can now model the return as

$$R_i = \frac{S_{i+1} - S_i}{S_i} = \mu\delta t + \sigma(\delta t)^{\frac{1}{2}}\phi. \tag{2.4}$$

2.6 Continuous time model

From the previous section, we deduced that the return model is given as

$$R_i = \mu\delta t + \sigma(\delta t)^{\frac{1}{2}}\phi. \tag{2.5}$$

If we take $\delta t \to 0$ in (2.5), and define a process $dX = \phi(dt)^{\frac{1}{2}}$, so that $E[dX] = 0$ and $E[(dX)^2] = dt$, we can model the continuous model of an asset price by using the equation:

$$\frac{dS}{S} = \mu dt + \sigma dX. \tag{2.6}$$

This model for the corresponding return on the asset dS/S, decomposes this return into two parts. The first part is a predictable deterministic return, μdt, where μ could be a constant or a function i.e. $\mu = \mu(S,t)$. μ is a measure of the average rate of growth of the asset price also known as the drift. The second part, $\sigma(S,t)dX$ models the random change in the asset price in response to external effects. Here σ is called the volatility. The volatility is a measure of the fluctuation (risk) in the asset prices and it corresponds to the diffusion coefficient, dX. X is known as the Brownian motion, which is a random variable drawn from the normal distribution with mean 0 and variance dt.

Equation (2.6) models what is known as a geometric Brownian motions and has been used to model currency, equities, indices, and commodities.

In classical calculus, given $F = X^2$, the derivative is given as $dF = 2XdX$. Finding the derivatives of stochastic variables do not follow classical calculus. The most important rule for stochastic calculus is known as the Itô lemma which will be explained in section 2.7. In order to present Itô's lemma, we first need to introduce the concept of Itô integral.

Definition 2.6.1 (Itô integral)**.** *Let $f = f(t,X)$ be a differentiable function with $t \in [0,T]$ and X a Brownian motion in $[0,T]$. Consider the partition*

$$\pi = \{t_0, t_1, ..., t_n\}, 0 = t_0 < t_1 < ... < t_n = T,$$

with $\|\pi\| = \max\{|t_{i+1} - t_i| : 0 \le i \le n-1\}$. Associated to each partition π, we define the sum of stochastic variables:

$$S_\pi = \sum_{i=0}^{n-1} f(t_i, X(t_i))\,\Delta X_i = \sum_{i=0}^{n-1} f(t_i, X_i)\,\Delta X_i$$

with

$$\Delta X_i = X(t_{i+1}) - X(t_i) = X_{i+1} - X_i.$$

When the limit,

$$\lim_{\|\pi\|\to 0} \sum_{i=0}^{n-1} f(t_i, X_i)\,\Delta X_i$$

exists, we say that f is an Itô -integrable function *and*

$$\lim_{\|\pi\|\to 0} \sum_{i=0}^{n-1} f(t_i, X_i)\,\Delta X_i = \int_0^T f(t, X(t))\,dX.$$

2.6.1 Itô's lemma

The Itô's lemma (1951) is a Taylor series expansion for stochastic calculus and a very important results. It serves as the stochastic calculus counterpart of the chain rule. In this subsection, we will present the heuristic derivation showing the details that we have to take into account for the proof of the Itô's lemma. For a rigorous proof see [152]. In order to derive Itô's lemma we will first need to introduce various timescales. This derivation can be found in [208]. Suppose we have a stochastic process $X(t)$ and an arbitrary function $F(x)$. Itô lemma is concerned with deriving an expression for the dynamics of $F(X(t))$. We define a timescale,

$$\frac{\delta t}{n} = h.$$

This timescale is so small that $F(X(t+h))$ can be approximated by a Taylor series:

$$F(X(t+h))-F(X(t)) = (X(t+h)-X(t))\frac{\partial F}{\partial X}(X(t))+\frac{1}{2}(X(t+h)-X(t))^2\frac{\partial^2 F}{\partial X^2}(X(t))+\cdots.$$

It follows that,

$$\begin{aligned}
&(F(X(t+h))-F(X(t)))+(F(X(t+2h))-F(X(t+h)))+\cdots+(F(X(t+n))-F(X(t+(n-1)h)))\\
&\quad=\sum_{j=1}^{n}(X(t+jh)-X(t+(j-1)h))\frac{\partial F}{dX}(X(t+(j-1)h))\\
&\quad+\frac{1}{2}\frac{\partial^2 F}{\partial X^2}(X(t))\sum_{j=1}^{n}(X(t+jh)-X(t+(j-1)h))^2+\cdots
\end{aligned}$$

The above equation uses the approximation,

$$\frac{\partial^2 F}{\partial X^2}(X(t+(j-1)h)) = \frac{\partial^2 F}{\partial X^2}(X(t)).$$

The first line then becomes

$$F(X(t+nh))-F(X(T)) = F(X(t+\delta t))-F(X(T)). \tag{2.7}$$

The second line is the definition of

$$\int_t^{t+\delta t}\frac{\partial F}{\partial X}dX \tag{2.8}$$

and the last line becomes

$$\frac{1}{2}\frac{\partial^2 F}{\partial X^2}(X(t))\delta t, \tag{2.9}$$

in the mean squared sense. Combining (2.7), (2.8) and (2.9) we obtain

$$F(X(t+\delta t))-F(X(t)) = \int_t^{t+\delta t}\frac{\partial F}{\partial X}(X(\tau))dX(\tau)+\frac{1}{2}\int_t^{t+\delta t}\frac{\partial^2 F}{\partial X^2}(X(\tau))d\tau. \tag{2.10}$$

Lemma 2.6.1. *(Itô's Lemma) Given a stochastic process $X(t)$ and a twice differentiable function $F(x)$, the dynamics of the process $F(X(t))$ can be written as:*

$$F(X(t)) = F(X(0)) + \int_0^t \frac{\partial F}{\partial X}(X(\tau))dX(\tau) + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial X^2}(X(\tau))d\tau.$$

This is the integral version of Itô's lemma which is conventionally written as follows:

$$dF = \frac{\partial F}{\partial X}dX + \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right)dt. \tag{2.11}$$

This is the most famous version of the Itô's lemma.

Remarks 2.6.1. 1. *In* (2.11)*, the Itô's lemma is the "stochastic version" of the fundamental theorem of calculus.*

2. *In* (2.11)*, the first term is a stochastic component and the second one is a deterministic component.*

   The stochastic component is due to the presence of dX. We recall that X is a Brownian motion.

Next we present examples and applications of the Itô's lemma.

Example 2.6.1 (The return process.)**.** *We will state basic facts about the return equation. More details about the model can be found in [38] and [99]. In a general setting the stock price follows the equation:*

$$dS = \mu(S,t)dt + \sigma(S,t)dX \tag{2.12}$$

where μ and σ are general functions of class $\mathscr{C}^2$ (i.e. first and second derivatives are continuous) and X is a regular one dimensional Brownian motion.

The Black Scholes Merton model is a particular form of the above where the stock follows a geometric Brownian motion (2.12)*. Specifically in this case $\mu(S,t) = \mu S$ and $\sigma(S,t) = \sigma S$, i.e., the functions are linear. The process $R_t = \log S_t$ increment is $R_t - R_{t-\Delta t} = \log S_t - \log S_{t-\Delta t} = \log(S_t/S_{t-\Delta t})$, the continuously compounded return over the interval $[t-\Delta t, t]$. We can obtain an equation for the continuously compounded return applying Itô's lemma to the function $g(t,x) = \log x$. This is given as:*

$$\begin{aligned}
dR_t &= \frac{\partial g}{\partial t}(t,S_t)dt + \frac{\partial g}{\partial x}(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,S_t)(dS_t)^2 \\
&= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dB_t) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(\mu S_t dt + \sigma S_t dB_t)^2 \\
&= \mu dt + \sigma dB_t - \frac{1}{2S_t^2}\left(\mu^2 S_t^2 dt^2 + 2\mu\sigma S_t^2 dt dB_t + \sigma^2 S_t^2 dB_t^2\right) \\
&= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dB_t,
\end{aligned}$$

The integral form of the last equation is given as:

$$R_t - R_0 = \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) ds + \int_0^t \sigma dB_s = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t. \tag{2.13}$$

We recall that $R_t = \log S_t$. *Therefore substituting* R_t *into* (2.13) *and solving for* S_t *we obtain an explicit formula for the stock dynamics:*

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t}. \tag{2.14}$$

Example 2.6.2. *Many other stock dynamics may be considered. For example:*

- Ornstein-Uhlenbeck process*:*

$$dS_t = \mu S_t dt + \sigma dB_t.$$

- Mean reverting Ornstein-Uhlenbeck process*:*

$$dS_t = \alpha(m - S_t)dt + \sigma dB_t.$$

Both of these Stochastic differential equation's (SDE's) have explicit solutions that may be found using Itô's lemma. See problem 8.

2.7 Problems

1. Prove the following version of the Fundamental Theorem of arbitrage (the fundamental theorem of arbitrage provides the necessary and sufficient conditions for a market to be arbitrage free and for a market to be complete).

   Let $\mathcal{M}$ be a market with N securities, one of them a risk free security and M futures at time T. Then the following statements are equivalent.

   (a) There are no strict arbitrage opportunities.

   (b) There exists a probability vector $(\hat{p}_1, \cdots, \hat{p}_M)$ so that the present value of each security is its expected value at time T discounted with the risk free interest rate factor.

2. Show that the previous theorem does not hold if the market can take infinitely many future states. Consider a market with three securities, where the third one has a bond with return 0, and the other two with present values $S_1 = 0$ and $S_2 = 2$. Suppose that the future states vectors for $(S_1(T), S_2(T))$ are given by the set:

$$\Omega = \{(a_1, a_2) \in \mathbb{R}^2 : a_1, a_2 > 0, a_1 + a_2 > 1\} \cup \{(0, 1)\}$$

   Show that:

(i) There are no strict arbitrage opportunities.

(ii) The analogous to 1(b) does not hold. What can we conclude about the original version of the fundamental theorem?

3. Consider a market with two securities, one of them risk free with rate of return R, and three future states. Suppose that the risky security has a present value S_1, and can take three future values $S_1^1 > S_1^2 > S_1^3$.

   (i) Give an "if and only if" condition for no arbitrage.

   (ii) Show an example of an option O that can not be replicated (i.e.: there is no hedging strategy). Why is it possible to find O?

   (iii) Show that the present value of O is not uniquely determined. More precisely, verify that the set of possible present values for O is an interval.

4. Explain carefully the difference between the European option and the American option. Give an example for each type of option.

5. Suppose a trader buys an European Put on a share for \$4. The stock price is \$53 and the strike price is \$49.

   (a) Under what circumstances will the trader make a profit?

   (b) Under what circumstances will the option be exercised?

   (c) Draw a diagram showing the variation of the trader's profit with the stock price at the maturity of the option.

6. Suppose that an August Call option to buy a share for \$600 cost \$60.5 and is held until August.

   (a) Under what circumstances will the holder of the option make a profit?

   (b) Under what circumstances will the option be exercised?

   (c) Draw a diagram showing how the profit from a long position in the option depends on the stock price at maturity of the option.

7. Use Itô's lemma to express dF given that $F(x) = x^{1/2}$, where the stochastic process $\{S_t, t \geq 0\}$ satisfies the stochastic differential equation:

$$dS_t = \alpha(\beta - S_t)dt + \sigma\sqrt{S_t}dZ_t$$

   where α, β and σ are positive constants and $\{Z_t, t \geq 0\}$ a standard Brownian motion.

8. Find the explicit solutions of the following stochastic differential equations:

   (a) Ornstein-Uhlenbeck process:

$$dX_t = \mu X_t dt + \sigma dB_t.$$

(b) Mean reverting Ornstein-Uhlenbeck process:

$$dX_t = \alpha(m - X_t)dt + \sigma dB_t.$$

Hint: Apply Itô's lemma to the function $g(t,x) = xe^{-\mu t}$ (Ornstein-Uhlenbeck process) and $g(t,x) = xe^{\alpha t}$ (mean reverting Ornstein-Uhlenbeck process).