Chapter 3

Some Models used in Quantitative Finance

3.1 Introduction

The Black-Scholes model (see e.g. [209], [26], [54], [99], [102], and [145]) is one of the most important models in the evolution of the quantitative finance. The model is used extensively for pricing derivatives in financial markets. As we shall see, using a backward parabolic differential equation or a probabilistic argument it produces an analytical formula for European type options.

In this chapter, we start by deriving the Black-Scholes differential equation for the price of an option. The boundary conditions for different types of options and their analytical solutions will also be discussed. We present the assumptions for the derivation of the Black-Scholes equation. We then present a series of other models used in Finance. Finally, we introduce the concept of volatility modeling.

3.2 Assumptions for the Black-Scholes-Merton derivation

1. We assume that the risk free interest rate r, is constant and the same for all maturities. In practice, r is actually stochastic and this will lead to complex formulas for fixed rate instruments. It is also assumed that the lending and borrowing rate are the same.

2. The next assumption is that trading and consequently Delta hedging is done continuously. Obviously, we cannot hedge continuously, portfolio re-balancing must be done at discrete time intervals.

3. We assume that there are no transaction costs on the underlying asset in the market. This is done for simplicity, much like friction due to air

resistance is neglected in simple physics equations. In reality, there are transaction costs, and in fact the hedging strategy often depends on these costs for the underlying asset in the market. We will obviously re-hedge more often when transaction costs for the underlying are inexpensive.

4. The asset price follows a log-normal random walk, as described in the next section. There are other models, but most of these models lack formulae.

5. Another assumption is that there are no arbitrage opportunities. This is an interesting assumption, since arbitrage opportunities actually do exist, and traders typically search for these opportunities. However, the market dynamics today is such that if such opportunities exist they will be exploited and they will disappear very fast.

6. Finally, we assume that short selling is permitted and that assets are infinitely divisible.

3.3 The Black-Scholes model

In Table 3.1, we present an example of a continuous hedging. We will apply Itô's lemma for this example. To begin, we let V be the value of an option at

Table 3.1: Continuous Hedging Example

Holding	Value Today(time t)	Value Tomorrow(t + dt)
Option	V	V + dV
-Δ Stock	-ΔS	-Δ(S + dS)
Cash	$-V + \Delta S$	$(-V + \Delta S)(1 + rdt)$
Total	0	$dV - \Delta dS - (V - \Delta S)rdt$

time t. Suppose we want to go short some amount of stock, say ΔS. Then when we look at the total cash amount, we have $-V + \Delta S$, since we bought the option V and sold some Δ quantity of stock. This is all done at time t, so we can sum these values to get $V + (-\Delta S) + (-V + \Delta S) = 0$. An interesting time to consider in the future to make profit is time T. Then the value of the stock at time T is given as $V + dV$, and the amount that we went short is now worth $-\Delta(S + dS)$. The cash we had after borrowing and going short is now worth $-(V + \Delta S)(1 + rdt)$, where r is the risk free interest rate. Then if we sum both columns we get,

$$dV - \Delta dS - (V - \Delta S)rdt = 0. \tag{3.1}$$

Applying Itô's lemma to (3.1) we obtain the equation :

$$(\frac{\partial V}{\partial S} - \Delta)dS + (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\Delta - rV)dt = 0.$$

Finally, we hedge with $\Delta = \frac{\partial V}{\partial S}$ to eliminate fluctuations in the stock and we obtain:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{3.2}$$

This equation (3.2) is called the Black-Scholes differential equation. Any European type option written on the stock which follows the geometric Brownian motion dynamics will solve this equation.

In practice one hedges against risky market movements. This is done using derivatives of the option price with respect to various parameters in the model. These derivatives are denoted with greek letters and are thus known as the greeks

$$\Theta = \frac{\partial V}{\partial t}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2}, \quad \Delta = \frac{\partial V}{\partial s}, \quad \rho = \frac{\partial V}{\partial r}, \quad \text{Vega} = \frac{\partial V}{\partial \sigma}.$$

Note that "Vega" is not a greek letter. In the Black Scholes model the volatility is constant therefore a derivative with respect to a constant is nonsense (it is always zero). Nevertheless, the volatility moves and this was observed long time ago. To cope with this movement produced the greek which is not a greek.

Substituting these values into the Black-Scholes equation we obtain the following equation:

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rs\Delta - rv = 0. \tag{3.3}$$

Remarks 3.3.1. *1.* $\Theta = \frac{\partial V}{\partial t}$ *measures the variation of* V *with time, keeping the asset price constant.*

2. $\Gamma = \frac{\partial^2 V}{\partial S^2}$ *measures the delta variation with respect to the asset price.*

3. The delta, $\Delta = \frac{\partial V}{\partial S}$ *for a whole portfolio is the rate of change of the value of the portfolio with respect to changes in the underlying asset.*

4. $\rho = \frac{\partial V}{\partial r}$ *measures the variation of the asset price with respect to the risk-free interest rate.*

5. Vega $= \frac{\partial V}{\partial \sigma}$ *represents the variation of the price with the volatility.*

Next consider the problem of pricing an European Call option in the Black-Scholes equation. Denote with $C(S,t)$ the value of this option.

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0. \tag{3.4}$$

The payoff for the call at the final time T is

$$C(S,T) = \max(S-K, 0),$$

where K is the option's strike. If the asset price S reaches 0 at any time the geometric Brownian motion becomes 0, and thus the boundary conditions are:

$$C(0,t) = 0, \quad C(S,t) \sim S \quad \text{as } S \to \infty.$$

We will now look at a more rigorous version of this idea, that involves continuously re-balancing our portfolio in order to eliminate risk. This will allow us to value European Puts and Calls.

Theorem 3.3.1. *(Black-Scholes formula for European Call options) The solution to the Black-Scholes equation for European Call options is given by*

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{3.5}$$

where

$$d_1 = \frac{\log(\frac{S}{E}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$
$$d_2 = \frac{\log(\frac{S}{E}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}},$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} ds.$$

Proof. Equation (3.4) is a backward equation, and we want to transform it into a forward equation. Using the following change of variables,

$$S = Ee^x, \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \quad C = Ec(x,\tau),$$

equation (3.4) becomes:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \tag{3.6}$$

where we set $k = \frac{r}{\frac{1}{2}\sigma^2}$. Hence the initial condition is now

$$v(x,0) = \max(e^x - 1, 0).$$

To transform a backward equation into a forward equation we need to use the diffusion equation. This can be done by making a change of variable

$$v = e^{\alpha x + \beta\tau}u(x,\tau)$$

for some α and β that are to be determined. After the change of variable we differentiate the equation to obtain:

$$\beta u + \frac{\partial u}{\partial r} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)(\alpha u + \frac{\partial u}{\partial x}) - ku.$$

In order to eliminate the u terms we choose

$$\beta = \alpha^2 + (k-1)\alpha - k$$

and to eliminate the $\frac{\partial u}{\partial x}$ terms, we note that

$$0 = 2\alpha + (k-1).$$

Then solving for α and β we get

$$\alpha = -\frac{1}{2}(k-1), \quad \beta = -\frac{1}{4}(k+1)^2.$$

Plugging the values of α and β into our change of variable we obtain,

$$v = e^{\frac{-1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x,\tau),$$

where

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty, \tau > 0,$$

with

$$u(x,0) = u_0(x) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0).$$

This is now the diffusion equation, and the solution to this equation is given by

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{\frac{-(x-s)^2}{4\tau}}\, ds.$$

We can then evaluate this integral making a change of variable $x' = (s-x)\sqrt{2\tau}$:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(x'\sqrt{2\tau} + x) e^{-\frac{1}{2}x'^2} dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x+x'\sqrt{2\tau})} e^{-\frac{1}{2}x'^2} dx'$$

$$-\frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k-1)(x+x'\sqrt{2\tau})} e^{-\frac{1}{2}x'^2} dx' = I_1 - I_2.$$

Next, by completing the square in the exponent, we can solve for I_1 i.e.

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x+x'\sqrt{2\tau})} e^{-\frac{1}{2}x'^2} dx'$$

$$= \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{4}(k+1)^2\tau} e^{-\frac{1}{2}(x' - \frac{1}{2}(k+1)\sqrt{2\tau})^2} dx'$$

$$= \frac{e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau} - \frac{1}{2}(k+1)\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}\rho^2 d\rho}$$

$$I_1 = e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2 r} N(d_1). \tag{3.7}$$

Wherein the above equation,

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau},$$

and

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{1}{2}s^2} ds.$$

The last equation is the cumulative distribution function(CDF) for a normal distribution. Now that we have calculated I_1, we can similarly calculate I_2 by replacing the $(k+1)$ in (3.7) by $(k-1)$. Thus we obtain,

$$v(x, \tau) = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x, \tau).$$

Then setting $x = \log(\frac{S}{E})$, $\tau = \frac{1}{2}\sigma^2(T-t)$, and $C = Ev(x, \tau)$, we get back to

$$C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

and with the change of variables, we obtain:

$$d_1 = \frac{\log(\frac{S}{E}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2 = \frac{\log(\frac{S}{E}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

Hence the proof. □

We state a similar theorem for European Put options.

Theorem 3.3.2. *(Black-Scholes formula for European Put options) The solution to the Black-Scholes equation for European Put options is given by:*

$$P(S, t) = e^{-r(T-t)}KN(-d_2) - S_0N(-d_1),$$

where d_1, d_2 and $N(x)$ are given in Theorem 3.3.1.

Proof. The Black Scholes formula can also be obtained by solving equation (3.4) with the specific boundary conditions for the put. However, we shall use [85] results to get to the formula using a probabilistic approach. Suppose we wish to obtain the formula for the Put option denoted $p(t)$. We know that at time T (maturity) the value is $p(T) = (K - S_T)_+$, where K is the strike price of the option. Using Harrison and Pliska [85], the discounted price is a martingale, therefore:

$$e^{-rt}p(t) = \mathbf{E}\left[e^{-rT}P(T)|\mathcal{F}_t\right].$$

Furthermore, under this martingale measure the equation of the stock has drift r and thus the formula (2.14) becomes:

$$S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t} \tag{3.8}$$

For simplicity of notation, we set $t = 0$. We use the fact that the conditional expectation becomes the regular expectation if the variable is independent of the conditional filtration. This is our case since S_T only depends on the increment

$B_T - B_0$ and that is independent of $\mathcal{F}_0$. Taking this observation into account the price of the Put is:

$$\begin{aligned}
p(0) &= \mathbf{E}\left[e^{-rT}(K - S_T)_+ | \mathcal{F}_0\right] \\
&= \mathbf{E}\left[e^{-rT}(K - S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma B_T})_+ | \mathcal{F}_0\right] \\
&= \mathbf{E}\left[e^{-rT}\left(K - S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\frac{B_T}{\sqrt{T}}}\right)_+ \Bigg| \mathcal{F}_0\right].
\end{aligned}$$

Now we recognize that $B_T/\sqrt{T}$ is distributed as a normal distribution with mean zero and variance 1 (i.e. $N(0,1)$) and thus we obtain:

$$p(0) = e^{-rT}\int_{-\infty}^{\infty}\left(K - S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z}\right)_+ \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz.$$

The question that arises is, when exactly is the expression inside positive? We note that,

$$\begin{aligned}
K - S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} \geq 0 \quad &\iff e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} \leq \frac{K}{S_0} \\
&\iff \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z \leq \log\frac{K}{S_0}
\end{aligned}$$

$$\iff z \leq \frac{\log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -\frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -d_2$$

using the notation given in equation (3.5). We also use the same notation $N(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz$ for the CDF of the normal distribution with mean 0 and variance 1. Thus we obtain:

$$\begin{aligned}
p(0) &= e^{-rT}\int_{-\infty}^{-d_2}\left(K - S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= e^{-rT}K\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz - e^{-rT}S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T}\int_{-\infty}^{-d_2}e^{\sigma\sqrt{T}z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= e^{-rT}KN(-d_2) - S_0 e^{-\frac{\sigma^2 T}{2}}\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}+\sigma\sqrt{T}z-\frac{\sigma^2 T}{2}+\frac{\sigma^2 T}{2}}dz \\
&= e^{-rT}KN(-d_2) - S_0 e^{-\frac{\sigma^2 T}{2}+\frac{\sigma^2 T}{2}}\int_{-\infty}^{-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\sigma\sqrt{T})^2}{2}}dz \\
&= e^{-rT}KN(-d_2) - S_0\int_{-\infty}^{-d_2-\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du,
\end{aligned}$$

where we have completed the square in the second integral and made the change of variables $z - \sigma\sqrt{T} = u$. Now recognizing that $d_2 + \sigma\sqrt{T} = d_1$ we finally obtain:

$$p(0) = e^{-rT}KN(-d_2) - S_0 N(-d_1),$$

which is exactly the formula for the Put option in the statement of the theorem.

□

3.4 Some remarks on the Black-Scholes model

3.4.1 Remark 1

Many assets, such as equities, pay dividends. As mentioned in chapter 2 section 2.3, dividends are payments to shareholders out of the profits made by the respective company. The future dividend stream of a company is reflected in today's share price. Suppose that we have an European Call option on an asset that does not pay dividends. Then the Black-Scholes (B-S) model to evaluate is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial V^2} + rS\frac{\partial V}{\partial S} = rV$$

In this case,

$$V\left(S,T\right) = \left(S-K\right)^+ = \max\{S-K, 0\}$$

and we can assume that V is a function of S, t, K and T

$$V = V\left(S, t, K, T\right)$$

In general we assume that the market "knows" the price of the option. More precisely, the "fair price" of an option. The market keeps the prices at equilibrium. Assuming that the present values given in the market are " true values" we can obtain a function $V^*\left(K,T\right)$. The "true market values" are the highest estimated price that a buyer would pay and a seller would accept for an item in an open and competitive market. If the B-S model is correct, we obtain:

$$V\left(S,t,K,T\right) = V^*\left(K,T\right)$$

We can assume that we know the risk free interest rate r (rate corresponding to a risk free bond). If we know r and σ we can solve the previous equation. In order to estimate σ so that the B-S model holds, we need to compute the implied volatility. That is, using market option values, estimate the value of σ implied by these market prices. The estimated parameter $\sigma = \sigma(S,t)$ will be a function of time.

3.4.2 Remark 2

We recall from chapter 2 that an American option is the type of option that may be exercised at any time prior to expiry. Suppose we construct the portfolio,

$$\pi = \begin{cases} 1 & \text{option in long} \\ 1 & \text{unit of share in short} \\ ke^{-r(T-t)} & \text{bonds in long} \end{cases}$$

If we exercise at time t, for a Call option $S > K$ (otherwise we do not exercise) we have that:

$$S - K + Ke^{-r(T-t)} - S < 0,$$

that is, the portfolio exercised is negative. If we keep the option without exercising it until $t = T$, we have that:

$$\max\{S_T - K, 0\} + K - S = \begin{cases} 0 & \text{if } S \geq K \\ (K - S) > 0 & \text{if } S < K \end{cases}$$

So it is more convenient to exercise at the final time T. We remark that this result is valid for a Call option on an asset without dividends (see [208]). However, an American Put option has a higher price than an European Put option. In this case the payoff is:

$$(K - S)^+ = \max\{K - S; 0\}.$$

In order to proceed, we assume a claim. Without this claim we obtain a contradiction.
Claim: We need the inequality:

$$V\,(S, t) \geq \max\{K - S; 0\}$$

This is related to the free boundary - obstacle problem, and we use the non-arbitrage condition if the condition does not hold. Suppose that there exists t so that $0 \leq V\,(S, t) < \max\{K - S; 0\}$. In this case $\max\{K - S; 0\} = K - S$. Then we consider the following strategy where we buy a share and an option that we will be exercise immediately (i.e. we buy the option to exercise it). In this case we are paying $S + V$ and we receive K. Then the net profit is $K - (S + V) > 0$ (assumption). Then there is arbitrage and the proof is done! The obstacle problem ([90]) assumes that:

1. V has a lower constraint,

$$V\,(S, t) \geq \max\{K - S; 0\}$$

2. V and $\frac{\partial V}{\partial S}$ are continuous (i.e. the contact between V and the obstacle is of first order). In this case the Black-Scholes model (PDE) is replaced by a differential.

Heuristically, we can assume that when $V > \max\{K - S; 0\}$ the Black-Scholes model holds, and we have an European Call option. But when we have that $V\,(S, t) = \max\{K - S; 0\}$ in an interval $[A, B]$, it is better to exercise. In this case, the B-S model does not hold because $V\,(S, t) = \max\{K - S; 0\}$. At what time will this change take effect? There exists a free boundary condition $S_f\,(t)$ such that if $S < S_f\,(t)$ we have to exercise. We can conclude that: if P is the price of an American Put option, $P \in C^1$ (i.e. P is continuous and the first derivative is also continuous) and also

$$P\,(S, t) \geq \max\{K - S; 0\}$$

then if $S < S_f\,(t)$ it is better to exercise. If the boundary satisfies that

$$0 \leq P\,(S, t) = \max\{K - S; 0\}$$

and $S < S_f$ then the following equation must hold:

$$P\left(S_f\left(t\right), t\right) = K - S_f\left(t\right)$$

Intuitively, taking the derivative with respect to S, we obtain:

$$\frac{\partial P}{\partial S}\left(S_f\left(t\right), t\right) = -1$$

This is a classical first-order contact free-boundary conditions for the obstacle problem. The equality holds only when V "does not have contact" with the obstacle.

In the next section, we will discuss a mathematical model used to describe the evolution of the volatility of an underlying asset.

3.5 Heston Model

One of the main assumptions in the Black Scholes Merton model is that the volatility is constant. However, judging from observed market prices the volatility is not constant but rather changing in time. The Heston model is a stochastic volatility model that assumes that the volatility of the asset is non constant and follows a random process.

In this model the underlying stock price S_t follows:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^1 \tag{3.9}$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2$$

where the two Brownian motions are correlated with coefficients ρ i.e.,

$$\mathbb{E}^p[dW_t^1, dW_t^2] = \rho dt,$$

with parameters defined as follows: μ is the stock drift, $\kappa > 0$ is the mean reversion speed for variance, $\theta > 0$ is the mean reversion level for variance, $\sigma > 0$ is the volatility of the variance, V_0 is the initial (time zero) variance level, $\rho \in [-1, 1]$ is the correlation between the two Brownian motions. The initial variance level, V_0 is often treated as a constant.

If we define

$$\widetilde{W}_t^1 = W_t^1 + \frac{\mu - r}{\sqrt{V_t}} t,$$

Girsanov's theorem ([152]) states that there exists a measure Q under which $\widetilde{W}_t^1$ is a Brownian motion. $\frac{\mu - r}{\sqrt{V_t}}$ is known as the market price of risk. then equation (3.9) becomes

$$dS_t = rS_t dt + \sqrt{V_t} S_t \widetilde{W}_t^1.$$

We can also write an equation for $\log(S_t)$ and using Itô's lemma we obtain:

$$d \ln S_t = (\mu - \frac{V_t}{2})dt + dW_t^1$$

or under the risk neutral measure Q or simply measure Q (a probability measure such that each share price is exactly equal to the discounted expectation of the share price under this measure) we obtain:

$$d\ln S_t = (r - \frac{V_t}{2})dt + dW_t^1.$$

For the variance, we take

$$\widetilde{W}_t^2 = W_t^2 + \frac{\lambda(S_t, V_t, t)}{\sigma\sqrt{V_t}}t$$

and we obtain

$$dV_t = [\kappa(\theta - V_t) - \lambda(S_t, V_t, t)]\, dt + \sigma\sqrt{V_t}\widetilde{W}_t^2,$$

which is the dynamics under measure Q and also $\lambda(S_t, V_t, t)$ is the volatility risk premium (a measure of the extra amount investors demand in order to hold a volatile security, above what can be computed based on expected returns). If we take $\lambda(S_t, V_t, t) = \lambda V_t$, we obtain $dV_t = (\kappa\theta - (\kappa + \lambda)V_t)dt + \sigma\sqrt{V_t}\widetilde{W}_t^2$ (see [91] for more details). If we take $\kappa^* = \kappa + \lambda$ and $\theta^* = \frac{\kappa\theta}{\kappa+\theta}$, we can rewrite the Heston model under risk neutral measure Q as:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t}S_t d\widetilde{W}_t^1 \\ dV_t &= \kappa^*(\theta^* - V_t)dt + \sigma\sqrt{V_t}d\widetilde{W}_t^2, \end{aligned}$$

with

$$\mathbb{E}^Q[d\widetilde{W}_t^1, d\widetilde{W}_t^2] = \rho dt.$$

When $\lambda = 0$, $\kappa^* = \kappa$ and $\theta^* = \theta$. Henceforth, we will assume $\lambda = 0$, but this is not necessarily needed. Thus, we will always assume that the Heston model is specified under the risk neutral problem measure Q. We shall not use the asterisk or the tilde notation from now on.

3.5.1 Heston PDE Derivation

This derivation of the Heston differential equation is similar to the Black-Scholes PDE derivation. Before we derive the Heston PDE, we need another derivation to hedge volatility. So we form a portfolio made of: $V(S, v, t)-$ option, $\Delta-$ units of stock, $\mu-$ units of another option and $U(s, v, t)-$ used for the volatility hedge. Therefore the portfolio value is $\Pi = V + \Delta S + \mu U$. Assuming the self-financing condition we have: $d\Pi = dV + \Delta dS + \mu dU$. The idea of the derivation is to apply Itô's lemma for V and U and then find Δ and μ that make the portfolio riskless.

Using the $2-$dimensional Itô's lemma [183],

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}dt + \rho\sigma v S\frac{\partial^2 V}{\partial S\partial v}dt \tag{3.10}$$

because $(dS_t)^2 = v_t S_t^2 (dW_t^1)^2 = v_t S_t dt$, $(dv)^2 = \sigma_t^2 v_t dt$, $dW_t^1 dt = dW_t^2 dt = 0$. Substituting (3.10) in $d\Pi$, we get:

$$\begin{aligned} d\Pi &= \left[\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma v S \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 V}{\partial S^2}\right] dt \\ &+ \mu \left[\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U}{\partial S^2}\right] dt \\ &+ \left[\frac{\partial V}{\partial S} + \mu \frac{\partial U}{\partial S} + \Delta\right] dS + \left[\frac{\partial V}{\partial v} + \mu \frac{\partial U}{\partial v}\right] dv \end{aligned}$$

So in order to make the portfolio risk free, we need to set $\frac{\partial V}{\partial S} + \mu \frac{\partial U}{\partial S} + \Delta$ and $\frac{\partial V}{\partial v} + \mu \frac{\partial U}{\partial v}$ to zero. Therefore the hedge parameters are: $\mu = -(\frac{\partial V}{\partial v})/(\frac{\partial U}{\partial v})$ and $\Delta = -\mu \frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}$. If we substitute these values back into the portfolio dynamics, we get:

$$\begin{aligned} d\Pi &= \left[\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 V}{\partial v^2}\right] dt \\ &+ \left[\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U}{\partial v^2}\right] dt \end{aligned}$$

since the portfolio earns the riskfree rate:

$$d\Pi = r\Pi dt = r(V + \Delta S + \mu U)dt.$$

Now we set the 2 terms equal and substitute μ and Δ and rearrange the terms as follows:

$$\begin{aligned} &\frac{\left[\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma v S \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 V}{\partial v^2}\right] - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}} = \\ &\frac{\left[\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U}{\partial v^2}\right] - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}. \end{aligned}$$

However since the left side of the above equation is a function of V while the right side is a function of U only, it implies we can write both sides as function of S, v and t only i.e. $f(S, v, t)$. Now let $f(S, v, t) = -\kappa(\theta - v) + \lambda(S, v, t)$ and take $\lambda(S, v, t) = \lambda v$, then the Heston PDE is given as:

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U}{\partial v^2} \\ &- rU + rS\frac{\partial U}{\partial S} + (\kappa(\theta - v) - \lambda(S, v, t)) \frac{\partial U}{\partial v} = 0 \end{aligned} \quad (3.11)$$

with boundary conditions:

$$U(S,v,T) = \max(S-k,0) \quad \text{for the Call option}$$
$$U(0,v,t) = 0 \quad \text{(as stock price gets to 0, Call is worthless)}$$
$$\frac{\partial U}{\partial S}(\infty,v,t) = 1 \quad \text{(when stock gets to } \infty\text{, change in stock implies change in option value)}$$
$$U(S,\infty,t) = S \quad \text{(when volatility is } \infty\text{, option price is } S\text{)}$$

Taking $x = \ln S$ in (3.11) and simplifying, we obtain:

$$\begin{aligned}&\frac{\partial U}{\partial t} + \frac{1}{2}v\frac{\partial^2 U}{\partial x^2} + (v-\frac{1}{2})\frac{\partial U}{\partial x} + \rho\sigma v\frac{\partial^2 U}{\partial v\partial x} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}\\ &- rU + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} = 0.\end{aligned} \tag{3.12}$$

Example 3.5.1. *Consider the volatility process in the Heston model*

$$U_t = \sqrt{V_t}.$$

Use Itô to calculate the dynamic for the volatility process U_t. Under what conditions on the parameters κ, θ, σ the process becomes an Ornstein-Uhlenbeck (O-U) process (i.e. $dX_t = X_t dt + \sigma dW_t$)?

The solution of the example is as follows:
The volatility process $\sqrt{V_t}$ follows an O-U process. Let $U_t = \sqrt{V_t}$. Applying Itô's lemma to $f(x,t) = \sqrt{x}$, we obtain

$$\begin{aligned}dU_t = d\sqrt{V_t} &= \frac{1}{2\sqrt{V_t}}dV_t - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{\sqrt{V_t^3}}d\langle V,V\rangle_t\\ &= \frac{1}{2\sqrt{V_t}}\left(\kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^2\right) - \frac{1}{8}\cdot\frac{1}{\sqrt{V_t^3}}\sigma^2 V_t dt\\ &= \frac{\kappa\theta}{2\sqrt{V_t}}dt - \frac{\kappa\sqrt{V_t}}{2}dt - \frac{\sigma^2}{8\sqrt{V_t}}dt + \frac{\sigma}{2}dW_t^2\end{aligned}$$

If we take $\frac{\kappa\theta}{2} = \frac{\sigma^2}{8}$, we obtain the O-U process.

3.6 The Cox-Ingersoll-Ross (CIR) model

The CIR model is typically used to describe the evolution of interest rates. It describes the interest rate movements as driven by only one source of market risk. The model is also relevant providing the dynamics of the variance process in the Heston model. The model follows the SDE:

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t \tag{3.13}$$

If (3.13) is satisfied then $V_t|V_s = 0$ for $s < t$ has a scaled non-central χ^2 distribution ([47]). Specifically:

$$2C_tV_t|V_\Delta \sim \chi_d^2 = \frac{4\kappa\theta}{\sigma^2}\left(2C_tV_\Delta \exp(-\kappa(t-\Delta))\right)$$

where $C_t = \frac{2\kappa}{\sigma^2(1-e^{\kappa(t-\Delta)})}$ and $\frac{4\kappa\theta}{\sigma^2}$ denotes the degrees of freedom.

We can also calculate the conditional mean and variance as follows:

$$\mathbb{E}[V_t|V_\Delta] = \theta + (V_\Delta - \theta)e^{-\kappa(t-\Delta)}$$

and

$$Var[V_t|V_\Delta] = V_\Delta \frac{\sigma^2 e^{\kappa(t-\Delta)}}{\kappa}\left(1 - e^{-\kappa(t-\Delta)}\right) + \frac{\theta\sigma^2}{2\kappa}\left(1 - e^{-\kappa(t-\Delta)}\right)^2.$$

Therefore as $k \to \infty$, the conditional expectation goes to θ and the variance goes to 0. Also as $k \to 0$ the conditional expectation goes to V_Δ and the variance goes to $\sigma^2 V_t(t-\Delta)$. If $2\kappa\theta > \sigma^2$ (Feller condition) then the variance process will not go negative.

3.7 Stochastic α, β, ρ (SABR) model

The SABR model is a stochastic volatility model, which is used to capture the volatility smile in derivatives markets and also used in the forward rate modeling of fixed income instruments. SABR is a dynamic model in which both S and α are represented by stochastic state variables whose time evolution is given by the following system of stochastic differential equations:

$$dS_t = \alpha_t S_t^\beta dW_t^1$$
$$d\alpha_t = v\alpha_t dW_t^2$$

with $\mathbb{E}[dW_t^1, dW_t^2] = \rho dt$, where S_t is a typical forward rate, α_0- is the initial variance, v is the volatility of variance, β is the exponent for the rate, and ρ is the correlation coefficient. The constant parameters β, v satisfy the conditions $0 \leq \beta \leq 1, v \geq 0$.

Remarks 3.7.1. *If $\beta = 0$ the SABR model reduces to the stochastic Normal model, if $\beta = 1$, we have the stochastic log normal model and if $\beta = 1/2$, we obtain the stochastic CIR model.*

3.7.1 SABR implied volatility

In order to obtain the option price one uses the usual B-S model. For example an European Call with forward rate S_0, strike price K , maturity T and implied volatility σ_B is:

$$C_B(S_0, K, \sigma_B, T) = e^{-rT(S_0N(d_1) - KN(d_2))} \tag{3.14}$$

where

$$d_{12} = \frac{\ln \frac{S_0}{K} \pm \frac{1}{2}\sigma_B^2 T}{\sigma_B \sqrt{T}}.$$

Equation (3.14) is the same as the B-S model. However, σ_B is the implied volatility from the SABR model.

In the next section, we elaborate on the concept of implied volatility introduced in section 3.4.1 and then we discuss some methods used to compute implied volatility.

3.8 Methods for finding roots of functions. Implied Volatility.

3.8.1 Introduction.

This section addresses the following problem. Assume that we have a function $f : \mathbb{R} \to \mathbb{R}$ which perhaps has a very complicated form. We wish to look at the equation $f(x) = 0$. If f would be invertible it would be easy to find the roots as the set $\{x : x = f^{-1}(0)\}$. We also assume that we can calculate $f(x)$ for all x. More specifically, this is needed in the context when f is the value given by the Black Scholes formula. Suppose that we look at a Call option whose value is denoted by $C_{BS} = C_{BS}(S_0, K, T, r, \sigma)$. Note that S_0 and r are values that are known at the current time (stock value and the short term interest rate). K and T are strike price and maturity (in years) which characterize each option. Therefore the unknown in this context is the volatility σ. What one does is: observe the bid-ask spread, (denoted B and A) for the specific option value (characterized by K, T above) from the market. Then one considers the function

$$f(x) = C_{BS}(S_0, K, T, r, x) - (B + A)/2.$$

The value x for which the above function is zero is called the volatility implied by the marked or simply the *implied volatility*[1]. Note that this value is the root of the function f. Thus finding the implied volatility is the same as finding the roots of a function. Below we present several methods of finding roots of a function. For more details of the root finding methods, see [116] and [12].

3.8.2 The Bisection method.

The bisection method is a very simple method. It is based on the fact that if the function f has a single root at x_0 then the function changes the sign from negative to positive or vice versa. Suppose there exists a neighborhood of x_0 which contains only that one root x_0. Take any 2 points a and b in the neighborhood. Then since the function changes sign at x_0, if x_0 is between a

[1]Note that the value $(B + A)/2$ is just a convention, one could also use them separately and obtain two implied volatility values and infer that the real implied volatility value is somewhere inside the interval.

and b the values $f(a)$ and $f(b)$ will have opposite signs. Therefore, the product $f(a)f(b)$ will be negative. However, if x_0 is not between a and b then $f(a)$ and $f(b)$ will have the same sign (be it negative or positive) and the product $f(a)f(b)$ will be positive. This is the main idea of the bisection method. The algorithm starts by finding two points a and b such that the product $f(a)f(b)$ is negative. In the case of implied volatility the suggested starting points are 0 and 1. *Pseudo-Algorithm:*

Step 1: Check if the distance between a and b (i.e., $b-a$) is less than some tolerance level ε. If Yes STOP report either point you want (a or b). If No step further.

Step 2: Calculate $f(\frac{a+b}{2})$. Evaluate $f(a)f(\frac{a+b}{2})$ and $f(\frac{a+b}{2})f(b)$

Step 3: If $f(a)f(\frac{a+b}{2}) < 0$ make $b \leftarrow \frac{a+b}{2}$. Go to Step 1

Step 4: If $f(\frac{a+b}{2})f(b) < 0$ make $a \leftarrow \frac{a+b}{2}$. Go to Step 1

If there are more than one root in the starting interval $[a, b]$ or the root is a multiple root, the algorithm fails. There are many improvements that can be found in [116], [12] and [11]. Examples of these improvements are presented in the following subsequent subsections.

3.8.3 The Newton's method.

This method assumes the function f has a continuous derivative. It also assumes that the derivative $f'(x)$ can be calculated easily (fast) at any point x. Newton's method may not converge if you start too far away from a root. However, if it does converge, it is faster than the bisection method since its convergence is quadratic rather than linear. Newton's method is also important because it readily generalizes to higher-dimensional problems [2] see [116]. Suppose we have a function f whose zeros are to be determined numerically. Let r be a zero of f and let x be an approximation to r. If $f''(x)$ exists and is continuous, then by Taylor's Theorem,

$$0 = f(r) = f(x + h) = f(x) + hf'(x) + \mathcal{O}(h^2) \tag{3.15}$$

where $h = r - x$. If h is small, then it is reasonable to ignore the $\mathcal{O}(h^2)$ term and solve the remaining equation for h. Ignoring the $\mathcal{O}(h^2)$ term and solving for h in (3.15), we obtain $h = -f(x)/f'(x)$. If x is an approximation to r, then $x - f(x)/f'(x)$ should be a better approximation to r. Newton's method begins with an estimate x_0 of r and then defines inductively

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (n \geq 0). \tag{3.16}$$

[2]The bisection method does not generalize.

In some cases the conditions on the function that are necessary for convergence are satisfied, but the initial point x_0 chosen is not in the interval where the method converges. This can happen, for example, if the function whose root is sought approaches zero asymptotically as x goes to $-\infty$ or ∞. In some cases, Newton's method is combined with other slower methods such as the Bisection method in a hybrid method that is numerically globally convergent (i.e. the iterative method converges for any arbitrary initial approximation).

3.8.4 Secant Method.

In (3.17), we defined the Newton iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (n \geq 0). \tag{3.17}$$

An issue with the Newton's method is that it involves the derivative of the function whose zero is sought. Notice that in (3.17) we need to know the derivative value at x_n to calculate the next point. In most cases we cannot compute this value so instead we need to estimate it. A number of methods have been proposed. For example, Steffensen's iteration

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)} \tag{3.18}$$

gives one approach to this problem. Another method is to replace $f'(x)$ in (3.17) by a difference quotient, such as

$$f'(x) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}. \tag{3.19}$$

The approximation given in (3.19) comes directly from the definition of $f'(x)$ as a limit. When we replace $f'(x)$ in (3.17) with (3.19), the resulting algorithm is called the **secant method**. Its formula is

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] \quad (n \geq 1). \tag{3.20}$$

Since the calculation of x_{n+1} requires x_n and x_{n-1}, two initial points must be prescribed at the beginning. However, each new x_{n+1} requires only one new evaluation of f.

Remark 3.8.1. *Other methods used in finding roots of a function are mentioned below. The respective formulas have been omitted since they are a bit less elegant than the few presented so far.*

- **Muller Method**. *This method is similar to the secant method except that the Muller method uses a second order approximation (using second derivative).*

- **Inverse quadratic interpolation**. *Uses second order approximation for the inverse function f^{-1}.*
- **Brent's method**. *This method is the most powerful deterministic algorithm since it combines three of the methods presented thus far to find the root faster. The three basic algorithms used are bisection, secant and inverse quadratic interpolation methods.*

Besides these methods, new random algorithms exist where perturbations are introduced for a more certain convergence. These algorithms are better known for finding extreme points of functions (the best known one is called simulated annealing), but there are some variations used to find roots of functions as well. Please refer to [23] for details and for details of these methods, please refer to [146] and [11].

We now present an example of calculating the implied volatility using the Newton's method.

3.8.5 Computation of implied volatility using the Newton's method

The inversion of the Black-Scholes formula to get the implied volatility is done by using the root finding methods described in the previous subsections. In this subsection, we consider the Newton's method for computing the implied volatility. The method works very well for a single option.

Let us recall the Black-Scholes formula for the price of an European Call option.

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{3.21}$$

where,

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and N is the distribution function of the standard Normal distribution.

The procedure for the "daily" computation of the implied volatility of an underlying asset is as follows. We compute the volatility $\sigma := \sigma_{K,T-t}$ by solving the equation

$$C(S_*, t_*; K, T-t, r, \sigma) = \nu \tag{3.22}$$

where t_* is time, S_* is the price of the underlying asset, r is the interest rate constant in the future, K is the strike price of an European Call option, T is the maturity, $T-t$ is the term to maturity and $\nu := C_{market}$ is the observed market price of the option. In order to simplify the notation, we set

$$f(\sigma) := C(S_*, t_*; K, T-t, r, \sigma), \quad d_2 := d_2(\sigma, S, K, T-t, r). \tag{3.23}$$

As a result, from (3.22) we have to compute a solution of

$$f(\sigma) - \nu = 0. \tag{3.24}$$

The function $f(\sigma)$ is a smooth function that depends on σ in a highly non-linear manner. Since f is differentiable, we can apply the Newton's method as follows.

1. Choose an initial guess σ_0 for the implied volatility and a stopping-bound $N \in \mathbb{N}$ for the iteration steps.
2. Set $f(\sigma) := C(S_*, t_*; K, T-t, r, \sigma)$ and $f'(\sigma) := \frac{dC}{d\sigma}(S_*, t_*; K, T-t, r, \sigma)$
3. Compute for $n = 0, \ldots, N-1$

$$\sigma_{n+1} = \sigma_n - \frac{f(\sigma_n) - \nu}{f'(\sigma_n)} \tag{3.25}$$

4. Approximate the value σ_n for the implied volatility $\sigma_{implied}(K, T-t)$

3.9 Some remarks of implied volatility (Put-Call Parity)

Put-Call parity which we explained in chapter 2, provides a good starting point for understanding volatility smiles. It is an important relationship between the price C, of a European Call and the price P of a European Put:

$$P + S_0 e^{-qT} = C + K e^{-rT}. \tag{3.26}$$

The Call and the Put have the same strike price, K and time to maturity, T. The variable S_0 is the price of the underlying asset today, r is the risk-free interest rate for the maturity T and q is the yield of the asset.

An essential feature of the Put-Call parity relationship is that it is based on a relatively simple non-arbitrage argument.

Suppose that for a particular value of the volatility P_α and C_α are the values of the European Put and Call options calculated using the Black-Scholes model. Suppose further that P_β and C_β are the market values of these options. Since the Put-Call parity holds for the Black-Scholes model, we must have

$$P_\alpha + S_0 e^{-qT} = C_\alpha + K e^{-rT}. \tag{3.27}$$

Similarly, since the Put-Call parity holds for the market prices, we have

$$P_\beta + S_0 e^{-qT} = C_\beta + K e^{-rT}. \tag{3.28}$$

Subtracting (3.28) from (3.27) gives

$$P_\alpha - P_\beta = C_\alpha - C_\beta. \tag{3.29}$$

This shows that the pricing error when the Black-Scholes model is used to price the European Put option should be exactly the same as the pricing error when it is used to price an European Call option with the same strike price and time to maturity. In the next section, we discuss an important concept known as hedging using volatility.

3.10 Hedging using volatility

Definition 3.10.1. Volatility is a measure for the variation of price of a financial instrument over time. It is usually denoted σ and it corresponds to the squared root of the quadrativ variance of a stochastic process.

There are two types of volatility that we are concerned with, actual volatility and implied volatility. Actual volatility is the unknown parameter or process. Implied volatility is a market suggested value for volatility. We can then take historical values of stock prices and other parameters to figure out the market's opinion on future volatility values. The next example presents a way to use volatility in trading.

Example 3.10.1. *Suppose we calculate the implied volatility to be* 40% *using the price of the stock, but we think that the actual volatility is* 50%. *If our assumption is correct, how can we make money, and which delta should we use?*

It is unclear which delta we should hedge with, so in this section we will use both and analyze the results. For a European Call option in the Black Scholes model we have

$$\Delta = N(d_1),$$

and we saw earlier that

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{1}{2}x^2} dx$$

and

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

We still do not know which volatility value to use for σ. The other variables are easily determined. We will denote actual volatility as σ, and $\tilde{\sigma}$ as implied volatility. In Table 3.2, we present an example of hedging with actual volatility. We will also use Δ^a to represent the delta using the actual volatility, and V^i is the theoretical value of the option found by plugging implied volatility in the formula. So we will buy an option for V^i and hedge with Δ^a.

Therefore we get:

$$\text{Profit} = dV^i - \Delta^a dS - (V^i - \Delta^a S)rdt.$$

Table 3.2: Example of Hedging with actual volatility

Holding	Value Today(time t)	Value Tomorrow(t + dt)
Option	V^i	$V^i + dV^i$
-Δ^aS	-Δ^aS	-Δ^a(S + dS)
Cash	$-V^i + \Delta^a S$	$(-V^i + \Delta^a S)(1 + rdt)$
Total	0	$dV^i - \Delta^a dS - (V^i - \Delta^a S)rdt$

Then the Black-Scholes equation yields

$$dV^a - \Delta^a dS - (V^a - \Delta^a s)rdt = 0.$$

Hence the profit for a time step is given by

$$\begin{aligned} &= dV^i - dV^a + (V^a - \Delta^a S)rdt - (V^i - \Delta^a S)rdt \\ &= dV^i - dV^a - (V^i - V^a)rdt \\ &= e^{rt} d(e^{-rt(V^i - V^a)}). \end{aligned}$$

Then the profit at time t_0 is

$$e^{-r(t-t_0)} e^{rt} d(e^{-rt}(V^i - V^a)) = e^{rt_0} d(e^{-rt}(V^i - V^a))$$

and the total profit is given by

$$e^{rt_0} \int_{t_0}^{T} d(e^{-rt}(V^i - V^a)) = V^a(t_0) - V^i(t_0).$$

Thus, if you are good at approximating actual volatility, you can hedge in such a way that there will always be arbitrage. The profit in this case is

$$\text{Profit} = V(S, t, \sigma) - V(S, t, \tilde{\sigma}).$$

With this approach, we can determine the profit from time t_0 to time T, but the profit at each step is random. In Table 3.3, an example of hedging with implied volatility is presented. If we hedge with implied volatility $\tilde{\sigma}$ we have

Table 3.3: Example of Hedging with implied volatility

Holding	Value Today(time t)	Value Tomorrow(t + dt)
Option	V^i	$V^i + dV^i$
-Δ^iS	-Δ^iS	-Δ^i(S + dS)
Cash	$-V^i + \Delta^i S$	$(-V^i + \Delta^i S)(1 + rdt)$
Total	0	$dV^i - \Delta^i dS - (V^i - \Delta^i S)rdt$

Then Profit is given from [208] by

$$\text{Profit} = dV^i - \Delta^i dS - r(V^i - \Delta^i S)dt$$

$$\text{Profit} = \Theta^i dt + \frac{1}{2}\sigma^2 S^2 \Gamma^i dt - r(V^i - \Delta^i S)dt$$

$$\text{Profit} = \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2) S^2 \Gamma^i dt.$$

Then the profit present value is

$$\frac{1}{2}(\sigma^2 - \tilde{\sigma}^2) e^{-r(t-t_0)} S^2 \Gamma^i dt.$$

Unlike hedging with actual volatility, we can determine profit at every step. In [155, 232] the authors have shown that if you hedge using σ_h, then the total profit is modeled by

$$V(S,t;\sigma_h) - V(S,t,\tilde{\sigma}) + \frac{1}{2}(\sigma^2 - \sigma_h^2) \int_{t_0}^{T} e^{-r(t-t_0)} S^2 \Gamma^h dt,$$

where the superscript h on the Γ means that it uses the Black-Scholes formula using volatility of σ_h. We have now modeled profit for σ, $\tilde{\sigma}$, and σ_h. We will now examine the advantages and disadvantages of using these different volatilities.

We discuss the pros and cons of hedging with actual volatility, implied volatility, and then finally we discuss hedging using other volatilities.

When using actual volatility, the main advantage is that we know exactly what profit we will have at expiration. This is a very desirable trait, as we can better see the risk and reward before hand. A drawback is that the profit and loss can fluctuate drastically during the life of an option. We are also going to have a lack of confidence when hedging with this volatility, since the number we have chosen is likely to be wrong. On the other hand, we can play around with numbers, and since profit is deterministic in this case, we can see what will happen if we do not have the correct figure for actual volatility. We can then decide if there is a small margin for success not to hedge.

Unlike hedging with actual volatility, hedging with implied volatility gives three distinct advantages. First and foremost, there are not any local fluctuations with profit and loss, instead you are continually making a profit. The second advantage is that we only need to be on the right side of a trade to make a profit. We can simply buy when the actual volatility is higher than implied, and sell when lower. The last advantage is that we are using implied volatility for delta, which is easily found. The disadvantage is that we do not know how much we will make, but we do know that this value will be positive.

We can balance these advantages and disadvantages by choosing another volatility entirely. This is an open problem and is beyond the scope of this book. When using this in the real world, applied and implied volatility have their purposes. If we are not concerned with the day to day profit and loss, then actual volatility is the natural choice. However, if we are expected to report profits or losses on a regular basis, then we will need to hedge with implied volatility, as profit or loss can be measured periodically.

3.11 Functional Approximation Methods

An option pricing model establishes a relationship between several variables. They are the traded derivatives, the underlying asset and the market variables, e.g., volatility of the underlying asset. These models are used in practice to price derivative securities given knowledge of the volatility and other market variables.

The classical Black-Scholes model assumes that the volatility is constant across strikes and maturity dates. However it is well known in the world of option pricing that this assumption is very unrealistic. Option prices for different maturities change significantly, and option prices for different strikes also experience significant variations. In this section we consider the numerical problems arising in the computation of the implied volatilities and the implied volatility surface.

Volatility is one of the most basic market variables in financial practice and theory. However, it is not directly observable in the market. The Black-Scholes model can be used to estimate volatility from observable option prices. By inverting the Black-Scholes formula with option market data, we obtain estimates which are known as implied volatility. These estimates show strong dependence of volatility values on strike price and time to maturity. This dependence, (popularly called volatility smile) cannot be captured by the Black-Scholes model.

The constant implied volatility approach, which uses different volatilities for options with different strikes and maturities, works well for pricing simple European options but it fails to provide adequate solutions for pricing exotic or American options. This approach also produces incorrect hedge factors even for simple options. We already discussed several stochastic models that do not have this problem (e.g., Heston SABR, etc.). However, we will present next a very popular practitioner approach. In this approach, we use a one-factor diffusion process with a volatility function depending on both the asset price and time i.e. $\sigma(S_t, t)$. This is a deterministic approach and the function $\sigma(S_t, t)$ is also known as the "local volatility" or the "forward volatility". We discuss the local volatility model next.

3.11.1 Local Volatility model

We recall from section 2.6 that if the underlying asset follows a continuous one factor diffusion process for some fixed time horizon $t \in [0, T]$, then we can write

$$dS_t = \mu(S_t, t)S_t dt + \sigma(S_t, t)S_t dW_t, \tag{3.30}$$

where W_t is a Brownian notion, $\mu(S_t, t)$ is the drift and $\sigma(S_t, t)$ is a local volatility function. The local volatility function is assumed to be continuous and sufficiently smooth so that (3.30) with corresponding initial and boundary conditions have a unique solution.

The Local volatility models are widely used in the financial industry to hedge barrier options (see [57]). These models try to stay close to the Black-Scholes model by introducing more flexibility into the volatility. In the local volatility model the only stochastic behaviour comes from the underlying asset price dynamics. Thus there is only one source of randomness, ensuring that the completeness of the Black-Scholes model is preserved. Completeness guarantees that the prices are unique. Please refer to [111] for more details about local volatility modeling.

Next we present the derivation of local volatility from Black-Scholes implied volatility.

3.11.2 Dupire's Equation

The Dupire's equation [55] is a forward equation for the Call option price C as a function of the strike price K and the time to maturity T. According to standard financial theory, the price at time t of a Call option with strike price K and maturity time T is the discounted expectation of its payoff, under the risk-neutral measure. Let

$$D_{0,T} = \exp\left(-\int_{t_0}^{T} r_s ds\right) \tag{3.31}$$

denotes the discount rate from the current time t_0 to maturity T and $\phi(T, s)$ denotes the risk neutral probability density of the underlying asset at maturity. More accurately the density should be written as $\phi(T, s; t_0, S_0)$, since it is the transition probability density function of going from state (t_0, S_0) to (T, s). However, since t_0 and S_0 are considered to be constants, for brevity of notation it is written as $\phi(T, s)$. It is assumed that the term structure for the short rate r_t is a deterministic function known at the current time t_0.

The risk-neutral probability density distribution of the underlying asset price at maturity is known to be $\phi(T, s)$. Therefore since this is a probability density function, its time evolution satisfies the equation:

$$\frac{\partial \phi(t,s)}{\partial t} = -(r_t - q_t)\frac{\partial}{\partial s}[s\phi(t,s)] + \frac{1}{2}\frac{\partial^2}{\partial s^2}[\sigma(t,s)^2 s^2 \phi(t,s)]. \tag{3.32}$$

Next, let the price of an European Call with strike price K be denoted $C = C(S_t, K)$ such that,

$$\begin{aligned} C &= E[D_{0,T}(S_T - K)^+ | F_0] \\ &= D_{0,T}\int_K^{\infty} (s-K)\phi(T,s)ds. \end{aligned} \tag{3.33}$$

Taking the first derivative of (3.33) with respect to strike K , we obtain

$$\begin{aligned} \frac{\partial C}{\partial K} &= D_{0,T}\frac{\partial}{\partial K}\int_K^{\infty}(s-K)\phi(T,s)ds \\ &= D_{0,T}\left[-(K-K)\phi(T,K) - \int_K^{\infty}\phi(T,s)ds\right] \\ &= -D_{0,T}\int_K^{\infty}\phi(T,s)ds, \end{aligned} \tag{3.34}$$

and the second derivative with respect to the strike price yields

$$\begin{aligned}\frac{\partial^2 C}{\partial K^2} &= -D_{0,T}\frac{\partial}{\partial K}\int K^{\infty}\phi(T,s)ds \\ &= D_{0,T}\phi(T,K). \qquad (3.35)\end{aligned}$$

Assuming that $\lim_{s\to\infty}\phi(T,s)=0$.
Taking the first derivative of (3.33) with respect maturity T and using the chain rule, we obtain:

$$\frac{\partial C}{\partial T} + r_T C = D_{0,T}\int_K^{\infty}(s-K)\frac{\partial \phi(T,s)}{\partial T}ds. \qquad (3.36)$$

Substituting (3.32) into (3.36) yields,

$$\begin{aligned}\frac{\partial C}{\partial T} + r_T C &= D_{0,T}\int_K^{\infty}(s-K)\left(\frac{1}{2}\frac{\partial^2}{\partial s^2}[\sigma(T,s)^2 s^2\phi(T,s)] - (r_T-q_T)\frac{\partial}{\partial S_T}[S_T\phi(T,s)]\right)ds \\ &= -\frac{1}{2}D_{0,T}[\sigma(T,s)^2 S_T^2\phi(T,s)]_{s=K}^{\infty} + (r_T-q_T)D_{0,T}\int_K^{\infty}s\phi(T,s)ds \\ &= \frac{1}{2}D_{0,T}\sigma(T,K)^2K^2\phi(T,K) + (r_T-q_T)\left(C + KD_{0,T}\int_K^{\infty}\phi(T,s)ds\right). \qquad (3.37)\end{aligned}$$

Finally, substituting (3.34) and (3.35) into the (3.37), we get:

$$\frac{\partial C}{\partial T} + r_T C = \frac{1}{2}\sigma(T,K)^2K^2\frac{\partial^2 C}{\partial K^2} + (r_T-q_T)\left(C + K\frac{\partial C}{\partial K}\right) \qquad (3.38)$$

In this context, it is assumed that $\phi(T,S_T)$ behaves appropriately at the boundary condition of $S_t=\infty$ (for instance this is the case when ϕ decays exponentially fast for $S_T\to\infty$). Therefore

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma(T,K)^2K^2\frac{\partial^2 C}{\partial K^2} - q_T C - (r_T-q_T)K\frac{\partial C}{\partial K}.$$

Thus

$$\sigma(T,K)^2 = 2\frac{\frac{\partial C}{\partial T} + (r_T-q_T)K\frac{\partial C}{\partial K} + q_T C}{K^2\frac{\partial^2 C}{\partial K^2}}. \qquad (3.39)$$

Equation (3.39) is commonly known as *Dupire formula* ([75]). Since at any point in time the value of Call options with different strikes and times to maturity can be observed in the market, the local volatility is a deterministic function, even when the dynamics of the spot volatility is stochastic.

Given a certain local volatility function, the price of all sorts of contingent claims on the underlying can be priced. By using the Black-Scholes model described in section 3.3 this process can be inverted, by extracting the local volatility surface from option prices given as a function of strike and maturity. The hidden assumption being that the option price is a continuous $C^{2,1}$ function (i.e. twice continuously differentiable function), known over all possible strikes and maturities. Even if this assumption holds, a problem arises in the implementation of equation (3.39). In general the option price function will never be known analytically, neither will its derivatives. Therefore numerical approximations for the derivatives have to be made, which are by their very nature approximations. Problems can arise when the values to be approximated are very small, and small absolute errors in the approximation can lead to big relative errors, perturbing the estimated quantity. When the disturbed quantity is added to other values, the effect will be limited. This is not the case in Dupire's formula where the second derivative with respect to the strike in the denominator stands by itself. This derivative will be very small for options that are far in or out of the money (i.e. the effect is particularly large for options with short maturities). Small errors in the approximation of this derivative will get multiplied by the strike value squared resulting in big errors at these values, sometimes even giving negative values that results in negative variances and complex local volatilities.

Another problem is the derivability assumption for option prices. Using market data option prices are known for only discrete points. Beyond the current month, option maturities correspond to the third Friday of the month, thus the number of different maturities is always limited with large gaps in time especially for long maturities. The same holds to a lesser degree for strikes. As a result, in practice the inversion problem is ill-posed: the solution is not unique and is unstable. This is a known problem when dealing with Dupire's formula in practice. One can smooth the option price data using Tikhonov regularisation (see [84]) or by minimizing the functions entropy (see [14]). Both of these methods try to estimate a stable option price function. These methods need the resulting option price function to be convex in the strike direction at every point to avoid negative local variance. This guarantees the positivity of the second derivative in the strike direction. This seems sensible since the non-convexity of the option prices leads to arbitrage opportunities, however this add a considerable amount of complexity to the model. An easier and inherently more stable method is to use implied volatilities and implied volatility surface to obtain the local volatility surface (see [111]). In the subsections that follow, we discuss methods used to represent the local volatility function.

3.11.3 Spline Approximation

In this method the local volatility function is obtained using a bicubic spline approximation which is computed by solving an inverse box-constrained nonlinear optimization problem.

Let $\{\bar{V}_j\}$, $j = 1, \ldots, m$ denote the m given market option prices. Given $\{K_i, T_i\}$ (where $i = 1, \ldots, p$, $p \leq m$) spline knots with corresponding local volatility values $\sigma_i = \sigma(K_i, T_i)$, an interpolating cubic spline $c(K, T, \sigma)$ with a fixed end condition (e.g., the natural spline end condition) is uniquely defined by setting $c(K_i, T_i) = \sigma_i$, $i = 1, \ldots, p$. The local volatility values σ_i at a set of fixed knots are determined by solving the following minimization problem:

$$\min_{\bar{\sigma}\in\mathbb{R}^p} f(\bar{\sigma}) = \frac{1}{2}\sum_{j=1}^{m} w_j(V_j(\bar{\sigma}) - \bar{V}_j)^2, \tag{3.40}$$

subject to $l_i \leq \sigma_i \leq u_i$, for $i = 1, \ldots, p$. Where $V_j(\bar{\sigma}) = V_j(c(K_j, T_j), K_j, T_j)$, $\bar{V}_j$'s are the given option prices at given strike price and expiration time (K_j, T_j), $j = 1, \ldots, m$ pairs, w_j's are weights, σ_i, $i = 1, \ldots, p$ are the model parameters, l's and u's are lower and upper bounds, respectively. The $\{V_j(\bar{\sigma})\}$'s are the model option values.

To guarantee an accurate and appropriate reconstruction of the local volatility, the number of spline knots p should not be greater than the number of available market option prices m.

The bicubic spline is defined over a rectangle R in the (K, T) plane where the sides of R are parallel to the K and T axes. R is divided into rectangular panels by lines parallel to the axes. Over each panel the bicubic spline is a bicubic polynomial which can be presented in the form:

$$spline(K, T) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}K^iT^j.$$

Each of these polynomials joins the polynomials in adjacent panels with continuity up to the second derivative. The constant K-values of the dividing lines parallel to the T-axis form the set of interior knots for the variable K, corresponding to the set of interior knots of a cubic spline. Similarly, the constant T-values of dividing lines parallel to the K-axis form the set of interior knots for the variable T. Instead of representing the bicubic spline in terms of the above set of bicubic polynomials, in practice it is represented for the sake of computational speed and accuracy in the form:

$$spline(K, T) = \sum_{i=1}^{p}\sum_{j=1}^{q} c_{ij}M_i(K)N_j(T),$$

where $M_i(K)$, $i = 1, \ldots, p$, and $N_j(T)$, $j = 1, \ldots, q$ are normalized B-splines. Please refer to [87] for further details of normalised B-splines and [88] for further details of bicubic splines.

In the next subsection, we briefly discuss a general numerical solution technique for the Dupire's equation.

3.11.4 Numerical Solution Techniques

In the previous subsection, we discussed the bicubic approximation for the volatility surface. In this subsection, both the Black-Scholes equations and the Dupire equations are solved by the finite difference method using the implicit time integration scheme. Please refer to sections 7.2, 7.3 and 7.4 for more details of the finite difference approximation scheme.

A uniform grid with $N \times M$ points in the region $[0, 2S_{init}] \times [0, \tau]$, where S_{init} is the initial money region and τ is the maximum maturity in the market option data used for solving the discretized Black-Scholes and the Dupire equations. The spline knots are also chosen to be on a uniform rectangular mesh covering the region Ω in which volatility values are significant in pricing the market options (the region Ω covers not far out of the money and in the money region of S). This region is not known explicitly, therefore a grid $[\delta_1 S_{init}, \delta_2 S_{init}] \times [0, \tau]$ is used. The parameters δ_1 and δ_2 are suitably chosen.

The general scheme of the volatility calibration algorithm are as follows when we consider the case of the Dupire equation.

1. Set the uniform grid for solving the discretized Dupire equation.

2. Set the spline knots.

3. Set the initial values for the local volatility at chosen spline knots $\{\sigma_i^{(0)}\}$, $i = 1, \ldots, p$.

4. Solve the discretized Dupire equation with the local volatility function approximated by spline with the local volatility values at chosen spline knots $\{\sigma^{(k)}\}$, $i = 1, \ldots, p$. Obtain the set of model based option values $\{V_j(\bar{\sigma})\}$ and the values of the objective function $f(\bar{\sigma})$ for (3.40).

5. Using the objective function $f(\bar{\sigma})$ (3.40) and the chosen minimization algorithm find the undated values of the local volatility at spline knots $\{\sigma_i^{(k+1)}\}$.

6. Repeat steps 4-5 until a local minimum is found. The local minimizer $\{\sigma_i^*\}$, $i = 1, \ldots, p$ defines the local volatility function.

The algorithm remains the same for the case of the Black-Scholes equation.

In the next subsection, we discuss the problem of determining a complete surface of implied volatilities and of option prices.

3.11.5 Pricing surface

Given a set of market prices, we consider the problem of determining a complete surface of the implied volatilities of the option prices. By varying the strike price K and the term to maturity T, we can create a table whose elements represent volatilities for different strikes and maturities. Under this practice, the implied volatility parameters will be different for options with different time to maturity T, and strike price K. This collection of implied volatilities for different strikes and maturities is called the implied volatility surface and the market participants use this table as the option price quotation. For many applications (calibration, pricing of non liquid or non traded options, etc) we are interested in an implied volatility surface which is complete, i.e. which contains an implied volatility for each pair (K, T) in a reasonable large enough set $[0, K_{\max}] \times [0, T_{\max}]$.

However, in a typical option market, one often observes the prices of a few options with the same time to maturity but different strike levels only. Some of these option contracts are not liquid at all, i.e. they are not traded at an adequatable extent. Therefore, we are faced with the problem of how to interpolate or extrapolate the table of implied volatilities. Some well known methods for completing the table of implied volatilities includes: polynomials in two variables, linear, quadratic or cubic splines in one or two variables, parametrization of the surface and fitting of the parameters (See [77]). However, it is appropriate to complete the pricing table of a pricing surface instead of completing the volatilities table. This is due to the fact that the properties of the pricing surface are deeply related to the assumptions concerning the market.

In order to ensure a pricing (using the implied volatility surface) which is arbitrage free, we have to find sufficient conditions for the arbitrage condition to hold. These conditions are related to the pricing surface. In the case of a Call option there exists a mapping $C : (0,\infty) \times (0,\infty) \to [0,\infty)$, such that, $(K,T) \mapsto C(K,T)$. The Call price surface defined by the mapping is called free of (static) arbitrage if and only if there exists a nonnegative martingale, say S, such that $C(K,T) = E((S_T - K)^+)$ for all $(K,T) \in (0,\infty) \times [0,\infty)$. The price surface is called free of (static) arbitrage if there are no arbitrage opportunities. Please refer to [77] and references therein for more details.

3.12 Problems

1. Suppose $V = V(S)$. Find the most general solution of the Black-Scholes equation.

2. Suppose $V = \Lambda_1(t)\Lambda_2(S)$. Find the most general solution of the Black-Scholes equation.

3. Prove that for an European Call option on an asset that pays no dividends the following relations hold:
$$C \le S, \quad C \ge S - E\exp(-r(T-t)).$$

4. Given the formulation of the free boundary problem for the valuation of an American Put option,
$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \le 0$$
$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \quad \text{if} \;\; S < S_f$$
$$P\left(S_f\left(t\right), t\right) = K - S_f\left(t\right)$$
$$\frac{\partial P}{\partial S}\left(S_f\left(t\right), t\right) = -1$$

Using a suitable change of variable, transform the free boundary problem into the modified problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad x > x_f(\tau)$$

$$u(x,\tau) = e^{\frac{1}{2}(k+1)^2\tau}\{e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}\} \quad \text{for} \quad x \leq x_f(\tau)$$

with initial condition

$$u(x,0) = g(x,0) = \{e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}\}$$

where $k = r/\frac{1}{2}\sigma^2$ (Hint: Use the change of variable, $S = Ke^x$ and $t = T - \tau/\frac{1}{2}\sigma^2$.)

This modified problem can be solved numerically using the finite difference method (See chapter 7 for more details).

5. In a market with N securities and M futures, where $(S_1, \cdots, S_N)$ is the present values vector and $(S_1^j(T), \cdots, S_N^j(T))$ the future values vector $(j = 1, \cdots, M)$, we say that a portfolio $\pi = (\pi_1, \cdots, \pi_N)$ produces *strict arbitrage* if one of the following conditions hold:

   (i) $\pi.S < 0$ and $\pi.S^j(T) \geq 0$ for all $j = 1, \cdots, M$.

   (ii) $\pi.S = 0$ and $\pi.S^j(T) > 0$ for all $j = 1, \cdots, M$.

   If R is the risk free interest rate prove that the following statements are equivalent:

   (a) There exists a portfolio that produces strict arbitrage.

   (b) There exists a portfolio satisfying (i).

   (c) There exists a portfolio satisfying (ii).

6. Consider the case of an European Call option whose price is \$1.875 when

$$S = 21, K = 20, r = 0.2, T - t = 0.25.$$

   Compute the implied volatility using the Newton's method by choosing $\sigma_0 = 0.20$.

7. Using the same information in Problem 6, compute the implied volatility using the bisection method in the interval $[0.20, 0.30]$.

8. Use provided data to construct a pricer for Call options using SABR model.

9. Given the general form of Dupire's equation, show that

$$\sigma^2 = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

   asssuming that the risk rate r_T and the dividend yield q_T each equal zero.

10. Explain the reason why it is convenient to represent the bicubic spline in the form

$$spline(K,T) = \sum_{i=1}^{p}\sum_{j=1}^{q} c_{ij} M_i(K) N_j(T),$$

where $M_i(K)$, $i = 1, \ldots, p$, and $N_j(T)$, $j = 1, \ldots, q$ are normalized B-splines instead of

$$spline(K,T) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij} K^i T^j.$$