The Inner Product

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January 16, 2022

Definition (Hermitian inner product): Let V be a complex vector space. A Hermitian inner product on V is a function,

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \tag{0.1}$$

which is,

- **conjugate-symmetric**

$$\langle u, v \rangle = \langle v, u \rangle^* \tag{0.2}$$

- **sesquilinear**, linear in the second term (physics convention). For arbitrary scalar λ,

$$\langle u, \lambda v \rangle = \lambda \langle u, v \rangle \tag{0.3}$$
$$\langle \lambda u, v \rangle = \lambda^* \langle u, v \rangle \tag{0.4}$$
$$\langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle \tag{0.5}$$

- **Positive-definite**,

$$\langle \lambda u, u \rangle \geq 0, \quad \text{equality iff } u = 0 \tag{0.6}$$

Corollary: An orthonormal set of vectors $\{e_i\}$ in V is linearly independent. *Note: the converse is not true.*

$$\langle e_i, e_j \rangle = \delta_{ij} \quad \text{(orthonormality)} \tag{0.7}$$

Orthonormal basis: Suppose $v \in V$ and $\{e_i\}$ is an orthonormal basis,

$$v = \sum_i v_i e_i \tag{0.8}$$

We will find that an orthonormal basis has useful consequences, so do all vector spaces have an orthonormal basis? In a finite-dimension space the answer is easy. Yes: all vector spaces have a basis, use the Gram–Schmidt algorithm to determine an orthonormal basis.

Projection: What is the component v_j (linear-expansion coefficient) of v on the orthonormal basis vector e_j? In Einstein notation,

$$\langle e_j, v \rangle = \langle e_j, v_i e_i \rangle = v_i \langle e_j, e_i \rangle = v_j \tag{0.9}$$

Dot product: Suppose $u, v \in V$,

$$\begin{aligned} \langle u, v \rangle &= \langle u_i e_i, v_j e_j \rangle && (0.10) \\ &= u_i^* v_j \langle e_i, e_j \rangle && (0.11) \\ &= u_i^* v_i && (0.12) \end{aligned}$$

Therefore, the inner product of two vectors has particular significance in terms of their expansion coefficients in an *arbitrary* orthonormal basis.

What if $v = u$; i.e. $\langle u, u \rangle$? This forms the notion of a metric.

Definition (norm): Let V be a complex vector space. The norm of a vector $v \in V$ is a function,

$$|| \cdot || : V \to \mathbb{R} \tag{0.13}$$

defined as,

$$||v|| = \sqrt{\langle v, v \rangle} \tag{0.14}$$