Coherent Interactions in a Three-Level System

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1 Hamiltonian

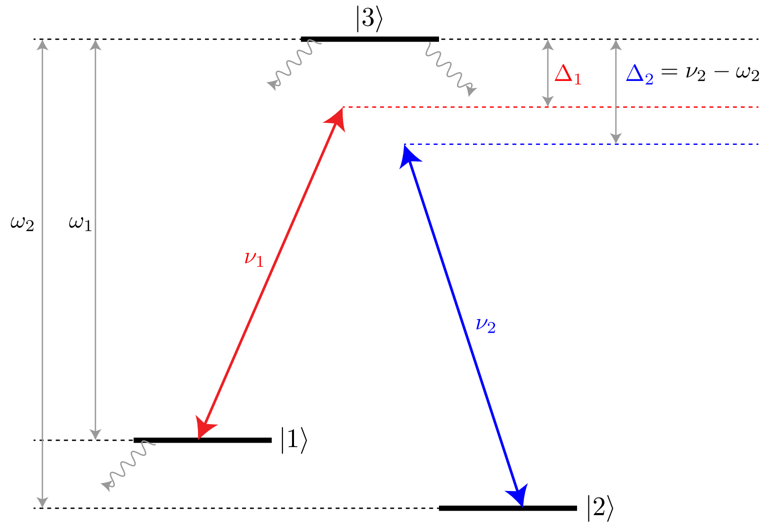

Figure 1: Three-level system. We define $|3\rangle$ to be the zero-energy point and assume $|2\rangle$ is the ground state.

The Hamiltonian for (fig. 1) is,

$$H = -\hbar\omega_1 |1\rangle\langle 1| - \hbar\omega_2 |2\rangle\langle 2| - \boldsymbol{d}\cdot\mathbf{E} \quad \text{where} \quad \mathbf{E}(t) = (\boldsymbol{\varepsilon_1} e^{-i\nu_1 t} + \boldsymbol{\varepsilon_2} e^{-i\nu_2 t} + \text{h.c.}) \tag{1.1}$$

For $\mu_{ij} = \langle i|\hat{d}|j\rangle$ real, we can then write the Hamiltonian as,

$$H = -\hbar\omega_1 |1\rangle\langle 1| - \hbar\omega_2 |2\rangle\langle 2| - \mu_{13}(|1\rangle\langle 3| + |3\rangle\langle 1|)(\varepsilon_1 e^{-i\nu_1 t} + \varepsilon_1^* e^{i\nu_1 t}) - \mu_{23}(|2\rangle\langle 3| + |3\rangle\langle 2|)(\varepsilon_2 e^{-i\nu_2 t} + \varepsilon_2^* e^{i\nu_2 t}) \tag{1.2}$$

2 Moving to Rotating Frame

Now we move into the rotating frame via the unitary,

$$U = e^{-i\nu_1 t} |1\rangle\langle 1| + e^{-i\nu_1 t} |2\rangle\langle 2| + |3\rangle\langle 3| \tag{2.1}$$

In other words, in the rotating frame, the state and Hamiltonian become,

$$|\tilde{\psi}\rangle = U|\psi\rangle \tag{2.2}$$

$$\tilde{H} = UHU^\dagger + i\hbar\dot{U}U^\dagger \tag{2.3}$$

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$$\tilde{H} = \begin{pmatrix} -\hbar\omega_1 & 0 & \mu_{13}(\varepsilon_1^* + \varepsilon_1 e^{-2i\nu_1 t}) \\ 0 & -\hbar\omega_2 & \mu_{23}(\varepsilon_2^* + \varepsilon_2 e^{-2i\nu_2 t}) \\ \mu_{13}(\varepsilon_1 + \varepsilon_1^* e^{2i\nu_1 t}) & \mu_{23}(\varepsilon_2 + \varepsilon_2^* e^{2i\nu_2 t}) & 0 \end{pmatrix} + \begin{pmatrix} \hbar\nu_1 & 0 & 0 \\ 0 & \hbar\nu_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

$$= \begin{pmatrix} -\hbar(\omega_1 - \nu_1) & 0 & \mu_{13}(\varepsilon_1^* + \varepsilon_1 e^{-2i\nu_1 t}) \\ 0 & -\hbar(\omega_2 - \nu_2) & \mu_{23}(\varepsilon_2^* + \varepsilon_2 e^{-2i\nu_2 t}) \\ \mu_{13}(\varepsilon_1 + \varepsilon_1^* e^{2i\nu_1 t}) & \mu_{23}(\varepsilon_2 + \varepsilon_2^* e^{2i\nu_2 t}) & 0 \end{pmatrix} \tag{2.5}$$

Define the detunings $\Delta_i = \omega_i - \nu_i$ and Rabi frequencies $\Omega_i = \varepsilon_i \mu_{i3}/\hbar$. To simplify the notation, we'll drop the tilde moving forward, but we will remain in the rotating frame.

$$H = -\hbar\Delta_1 |1\rangle\langle 1| - \hbar\Delta_2 |2\rangle\langle 2| - \hbar \left[(\Omega_1^* + \Omega_1 e^{-2i\nu_1 t}) |1\rangle\langle 3| + (\Omega_2^* + \Omega_2 e^{-2i\nu_2 t}) |2\rangle\langle 3| + \text{h.c.} \right] \tag{2.6}$$

3 Rotating Wave Approximation

We use the rotating wave approximation (RWA) to eliminate the time-dependent terms in (eq. 2.6) under appropriate conditions. Before discussing the RWA in detail, note that in the original definition of the Hamiltonian we have assumed a classical electromagnetic field. If instead we quantized the electromagnetic field (i.e. in the case of Jaynes-Cummings Hamiltonian), then a laser with frequency ν would correspond to,

$$\boldsymbol{E}(t) = \hat{a}\, \boldsymbol{\varepsilon} e^{-i\nu t} + \hat{a}^\dagger\, \boldsymbol{\varepsilon}^* e^{i\nu t} \tag{3.1}$$

Therefore, the time-independent terms in the Hamiltonian (eq. 2.6) correspond to operators like,

$$|1\rangle\langle 3|\, \hat{a}^\dagger \quad \text{and} \quad |3\rangle\langle 1|\, \hat{a} \tag{3.2}$$

which are energy conserving; while time-dependent terms correspond to operators like,

$$|1\rangle\langle 3|\, \hat{a} \quad \text{and} \quad |3\rangle\langle 1|\, \hat{a}^\dagger \tag{3.3}$$

which are *not* energy conserving. Such processes are "virtual" — consistent with the quickly oscillating phase of these operators (in the analogy of virtual particles, these processes must be quick to satisfy energy-time uncertainty). We can eliminate these terms if the following two conditions are satisfied:

1. The driving fields are weak: $\Omega_i \ll \nu_i$ (3.4)
2. The driving fields are near resonant: $\nu_i \approx \Omega_i$ so $\Delta_i \ll \nu_i, \Omega_i$ (3.5)

[TODO: JUSTIFY THE CONDITIONS]

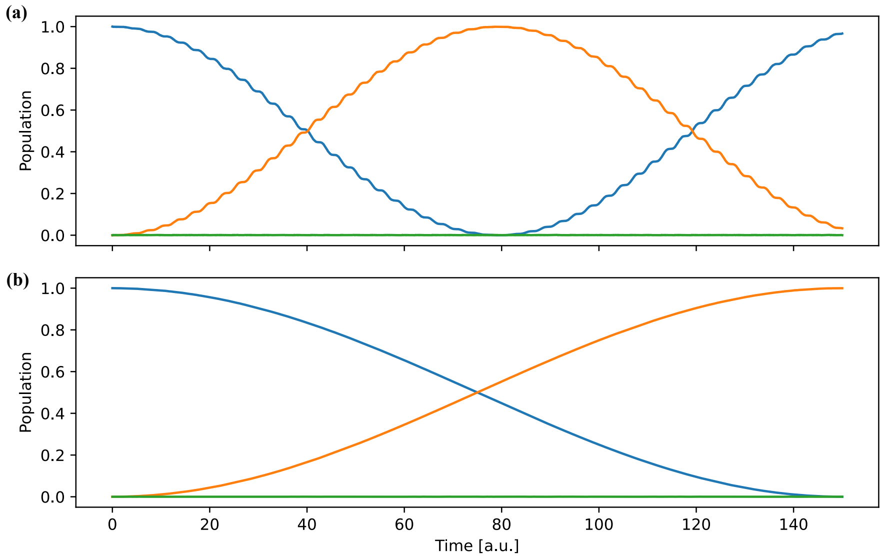

Figure 2: Exact numerical evolution of (eq. 2.6) for $\Omega_i = 1$, $\Delta_i = 100$, and various ν_i. Blue: population in $|1\rangle$; orange: in $|2\rangle$; and green in $|3\rangle$. (a) with $\nu_i = 1$. (b) with $\nu_i = 1000$. Despite constant Ω_i, Δ_i, the effective Rabi frequency increases by a factor of 2 when moving out of the RWA regime (when ν_i becomes similar/smaller than other timescales). This is because in the limit $\nu_i = 0$, Ω_i is effectively doubled in (eq. 2.6).