Linear Algebra: A Brief Overview

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Chapter 1

Vector Spaces

1.1 Fields

Fields are an important structure that come up in the study of linear algebra. We shall define them below.

Definition 1.1.1 (Field)**.** A field is a set of elements $\mathbb{F}$ with two binary operations, addition $+$ and multiplication $\cdot$, that satisfies the field axioms for all $x, y, z \in \mathbb{F}$,

C1 **(Closure under addition)** $x + y \in \mathbb{F}$

C2 **(Closure under multiplication)** $x \cdot y \in \mathbb{F}$

A1 **(Commutativity of addition)** $x + y = y + x$

A2 **(Associativity of addition)** $(x + y) + z = x + (y + z)$

A3 **(Additive identity)** There exists $0 \in \mathbb{F}$ such that $x + 0 = x$

A4 **(Additive inverse)** There exists $-x \in \mathbb{F}$ such that $x + (-x) = 0$

A5 **(Commutativity of multiplication)** $x \cdot y = y \cdot x$

A6 **(Associativity of multiplication)** $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

A7 **(Multiplicative identity)** There exists $1 \in \mathbb{F}$ such that $x \cdot 1 = x$

A8 **(Multiplicative inverse)** There exists $x^{-1} \in \mathbb{F}$ such that $x \cdot x^{-1} = 1$

A9 **(Distributivity of multiplication over addition)** $x \cdot (y + z) = x \cdot y + x \cdot z$

A10 **(Distinct additive and multiplicative identities)** $1 \neq 0$

Note that the final field axiom excludes one-element sets from being fields.

Many of the properties of a field are intuitive to our understanding of introductory mathematics. Here are a few properties of interest.

Theorem 1.1.2 (Unique Identity)**.** The additive identity 0 and the multiplicative identity 1 are both unique.

Proof. Additive identity: suppose 0 and $0'$ are additive identities,

$$0 = 0 + 0' = 0' + 0 = 0' \tag{1.1.1}$$

Multiplicative identity: similarly, suppose 1 and $1'$ are multiplicative identities,

$$1 = 1 \cdot 1' = 1' \cdot 1 = 1' \tag{1.1.2}$$

□

Theorem 1.1.3 (Unique Inverse)**.** For $x \in \mathbb{F}$, the additive inverse $-x$ and multiplicative inverse x^{-1} are unique.

Proof. Additive inverse: suppose α and β are both additive inverses of $x \in \mathbb{F}$,

$$\alpha = \alpha + 0 = \alpha + (x + \beta) = (\alpha + x) + \beta = \beta \tag{1.1.3}$$

Multiplicative inverse: follows the same process as the proof above. □

Theorem 1.1.4. For any $x, y, z \in \mathbb{F}$,

1. $x \cdot 0 = 0$
2. $(-1) \cdot x = -x$
3. $(-x) \cdot y = x \cdot (-y) = -(xy)$
4. $(-x) \cdot (-y) = xy$
5. If $x + z = y + z$, then $x = y$
6. If $xz = yz$ and $z \neq 0$, then $x = y$

Where xy is shorthand for $x \cdot y$.

Proof. (1) For any $x \in \mathbb{F}$,

$$x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0 \tag{1.1.4}$$

Thus,

$$x \cdot 0 = 0 \tag{1.1.5}$$

(2) For any $x \in \mathbb{F}$, we want to show that $x + (-1) \cdot x = 0$,

$$x + (-1) \cdot x = 1 \cdot x + (-1) \cdot x = (1 + (-1)) \cdot x = 0 \cdot x = 0 \tag{1.1.6}$$

(3-6) is left as an exercise. □

1.2 Lists

Definition 1.2.1 (List)**.** A list of length n (or n-tuple) is an ordered collection of n elements,

$$(x_1, x_2, \ldots, x_n) \tag{1.2.1}$$

Two lists are equal if an only if they have the same length and same elements in the same order. Unlike for a set, repeated elements in a list have meaning.

Definition 1.2.2. $\mathbb{F}^n$ is the set of all lists of length n containing elements of $\mathbb{F}$,

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\} \tag{1.2.2}$$

For example, we can think of $\mathbb{F}^2$ as a plane in $\mathbb{F}$.

Definition 1.2.3 (0 in $\mathbb{F}^n$)**.** We denote 0 as the list of length n with coordinates,

$$0 = (0, \ldots, 0) \tag{1.2.3}$$

Definition 1.2.4 (Addition in $\mathbb{F}^n$)**.** We define coordinate-wise addition in $\mathbb{F}^n$ as,

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \tag{1.2.4}$$

Definition 1.2.5 (Scalar Multiplication in $\mathbb{F}^n$)**.** The product of a scalar $\lambda \in \mathbb{F}$ and a list in $\mathbb{F}^n$ is computed by multiplying each coordinate of the vector by λ,

$$\lambda \cdot (x_1, \ldots, x_n) = (\lambda \cdot x_1, \ldots, \lambda \cdot x_n) \tag{1.2.5}$$

1.3 Vector Space

Definition 1.3.1 (Vector Space)**.** A vector space V over a field F is a set with the following operations and satisfying the following closure and vector-space axioms, for all $x, y, z \in V$ and $\lambda, \alpha, \beta \in \mathbb{F}$,

- O1 **(Vector addition operation)** $+ : V \times V \to V$
- O2 **(Scalar multiplication operation)** $\cdot : F \times V \to V$
- C1 **(Closed under vector addition)** $x + y \in V$
- C2 **(Closed under scalar multiplication)** $\lambda \cdot x \in V$
- A1 **(Commutativity of vector addition)** $x + y = y + x$
- A2 **(Associativity of vector addition)** $(x + y) + z = x + (y + z)$
- A3 **(Vector additive identity)** There exists $0 \in V$ such that $x + 0 = x$

A4 **(Vector additive inverse)** There exists $-x \in V$ such that $x + (-x) = 0$

A5 **(Distributivity of vector addition over scalar multiplication)** $(\alpha + \beta) \cdot x = \alpha x + \beta x$

A6 **(Associativity of scalar multiplication)** $(\alpha\beta) \cdot x = \alpha \cdot (\beta x)$

A7 **(Scalar multiplicative identity)** For $1 \in F$, $1 \cdot x = x$

A8 **(Distributivity of scalar multiplication over vector addition)** $\lambda(x + y) = \lambda x + \lambda y$

Notice that the field over which a vector space is defined only serves to specify the set of scalars. Also note that the simplest vector spaces is the zero vector space, $\{0\}$, which contains only the zero vector over an arbitrary field.

Definition 1.3.2 (Vector)**.** A vector is an element of a vector space.

Proposition 1.3.3. $\mathbb{F}^n$ is a vector space over $\mathbb{F}$

Proof. We can show that $\mathbb{F}^n$ satisfies the conditions for a vector space (definition 1.3.1) with definitions of addition and scalar multiplication over $\mathbb{F}^n$ (1.2.4 and 1.2.5) and the field axioms (definition 1.1.1). The full exercise is left for the reader. □

Note that while all $\mathbb{F}^n$ are vector spaces, the converse is not true: a vector space does not have to be $\mathbb{F}^n$. The zero vector space is a simple counterexample—though we can devise all manners of interesting and exotic vector spaces.[1]

The properties of a vector space follow very closely from the properties of a field. Thus, we shall omit the proofs and state some of the highlights,

Theorem 1.3.4. For a vector space V over $\mathbb{F}$,

1. V has a unique additive identity, 0
2. Each element in V has a unique additive inverse.
3. $0 \cdot \boldsymbol{v} = 0$ for all $v \in V$
4. $\lambda \cdot 0 = 0$ for all $\lambda \in \mathbb{F}$
5. $(-1) \cdot \boldsymbol{v} = -\boldsymbol{v}$ for all $v \in V$

1.4 Subspaces

Definition 1.4.1 (Subspace)**.** A subset U of V is a subspace of V if U is also a vector space using the same addition and scalar multiplication as defined on V.

[1]Another counterexample that comes to mind: the vector space of 2×2 real matrices over $\mathbb{R}$ with element-wise vector addition and scalar multiplication. Somewhat confusingly, here we say that a matrix is a "vector" because it is an element of the vector space.

Proposition 1.4.2 (Conditions for a subspace)**.** A subset U of V is a subspace if and only if U satisfies the conditions,

1. U contains the additive identity, 0
2. U is closed under addition and scalar multiplication

Example 1.4.3. The subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, all lines in $\mathbb{R}^3$ that pass through the origin, and all planes in $\mathbb{R}^3$ that intersect the origin.

1.4.1 Sum of Subspaces

Definition 1.4.4 (Sum of Subsets)**.** Suppose $U_1, \ldots, U_n$ are subsets of V. The sum of the subsets is the set of all possible sums of an element from each subset,

$$(U_1 + \cdots + U_n) = \{(u_1 + \cdots + u_n) : u_j \in U_j\} \tag{1.4.1}$$

Theorem 1.4.5. Suppose $U_1, \ldots, U_n$ are subspaces of V. Then their sum $(U_1 + \cdots + U_n)$ is a subspace of V. Furthermore, it is the smallest subspace of V containing $U_1, \ldots, U_n$.

Proof. It is simple to show that $0 \in (U_1 + \cdots + U_n)$ and that $(U_1 + \cdots + U_n)$ is closed under addition and scalar multiplication. Thus, $(U_1 + \cdots + U_n)$ is a subspace of V.

$U_1, \ldots, U_n$ are all contained in $(U_1 + \cdots + U_n)$ because the zero vector exists in each subset.[2] Conversely, every subspace of V containing $U_1, \ldots, U_n$ must contain all elements of $(U_1 + \cdots + U_n)$. Thus, $(U_1 + \cdots + U_n)$ is the smallest subspace of V containing $U_1, \ldots, U_n$. $\square$

Definition 1.4.6 (Direct Sum)**.** Suppose $U_1, \ldots, U_n$ are subspaces of V. Then the sum $(U_1 + \cdots + U_n)$ is a direct sum if each element can be written in only one way as a sum $(u_1 + \cdots + u_n)$ where $u_j \in U_j$. A sum that satisfies this condition is denoted $(U_1 \oplus \cdots \oplus U_n)$.

Theorem 1.4.7 (Condition for a Direct Sum)**.** Suppose $U_1, \ldots, U_n$ are subspaces of V. Then $(U_1 + \cdots + U_n)$ is a direct sum if an only if the only way to write,

$$u_1 + \cdots + u_n = 0 \quad \text{for } u_j \in U_j \tag{1.4.2}$$

is by taking each $u_j = 0$.

Proof. First suppose $(U_1 + \cdots + U_n)$ is a direct sum. Clearly, we can write $(0 + \cdots + 0) = 0$. And by the definition of a direct sum, this must be the only way to write 0 as a sum $(u_1 + \cdots + u_n)$.

Now suppose the only way to write 0 as a sum $(u_1 + \cdots + u_n)$ is by taking each $u_j = 0$. Let $v \in (U_1 + \cdots + U_n)$ and suppose that we can write v as,

$$v = u_1 + \cdots + u_n = u_1' + \cdots + u_n' \tag{1.4.3}$$

[2] In the sum $(u_1 + \cdots + u_n)$, we can set all but one of the u's to be the zero vector.

Then,

$$0 = (u_1 - u_1') + \cdots + (u_n - u_n') \tag{1.4.4}$$

But we know that each $(u_j - u_j') = 0$. Thus, $u_j = u_j'$ and each $v \in (U_1 + \cdots + U_n)$ has a unique representation. $\square$

Chapter 2

Finite-Dimensional Vector Spaces

A large portion of linear algebra focuses on a subset of all possible vector spaces that are finite-dimensional—which we shall define more formally later in the chapter.

Notation: note that from now on, it will be implicit that V is a vector space over a field $\mathbb{F}$.

2.1 Span

Definition 2.1.1 (Linear Combination)**.** A linear combination of vectors $v_1, \dots, v_n \in V$ is a vector of the form,

$$v = a_1 v_1 + \cdots + a_n v_n \quad \text{where } a_j \in \mathbb{F} \tag{2.1.1}$$

Definition 2.1.2 (Span)**.** The set of all possible linear combinations of a list of vectors is it's span,

$$\operatorname{span}(v_1, \dots, v_n) = \{(a_1 v_1 + \cdots + a_n v_n) : a_j \in \mathbb{F}\} \tag{2.1.2}$$

The span of the empty list () is defined to be $\{0\}$

Theorem 2.1.3. The span of a list of vectors in V is a subspace. Furthermore, it is the smallest subspace of V containing all the vectors in the list.

Proof. First, we wish to show that $\operatorname{span}(v_1, \dots, v_n)$ is a subspace of V. We know that,

$$0 = 0 \cdot v_1 + \cdots + 0 \cdot v_n \in \operatorname{span}(v_1, \dots, v_n) \tag{2.1.3}$$

Furthermore, $\operatorname{span}(v_1, \dots, v_n)$ is closed under addition and scalar multiplication because,

$$(a_1 v_1 + \cdots + a_n v_n) + (b_1 v_1 + \cdots + b_n v_n) = (a_1 + b_1) v_1 + \cdots + (a_n + b_n) v_n \tag{2.1.4}$$

$$\lambda(a_1 v_1 + \cdots + a_n v_n) = \lambda a_1 v_1 + \cdots + \lambda a_n v_n \tag{2.1.5}$$

Thus, $\operatorname{span}(v_1, \dots, v_n)$ is a subspace of V.

It is clear that $\operatorname{span}(v_1, \dots, v_n)$ contains $v_1, \dots, v_n$. Conversely, because subspaces are closed under addition and scalar multiplication, every subspace containing $v_1, \dots, v_n$ must also contain $\mathrm{span}(v_1, \ldots, v_n)$. Thus, the span is the smallest subspace of V containing all of the vectors $v_1, \ldots, v_n$. □

Definition 2.1.4 (Finite-Dimensional Vector Space)**.** A vector space is finite-dimensional if it spanned by a finite list of vectors. A vector space is infinite-dimensional if it is not finite-dimensional.

2.2 Linear Independence

Definition 2.2.1 (Linearly Independent)**.** A list of vectors $v_1, \ldots, v_n$ is linearly independent if and only if each vector in $\mathrm{span}(v_1, \ldots, v_n)$ has a unique representation as a linear combination of $v_1, \ldots, v_n$. The empty list () is also defined to be linearly independent.

Theorem 2.2.2 (Condition for Linear Independence)**.** A list of vectors $v_1, \ldots, v_n$ is linearly independent if and only if,

$$a_1 v_1 + \cdots + a_n v_n = 0 \quad \text{for } a_j \in \mathbb{F} \tag{2.2.1}$$

implies $a_j = 0$.

Proof. The proof is similar to the proof of theorem 1.4.7. □

Definition 2.2.3 (Linearly Dependent)**.** A list of vectors in V is linearly dependent if it is not linearly independent. In other words, there exists $a_j \in \mathbb{F}$ that are not all zero, such that,

$$a_1 v_1 + \cdots + a_n v_n = 0 \tag{2.2.2}$$

Lemma 2.2.4 (Linear Dependence Lemma)**.** Suppose $v_1, \ldots, v_n$ is a linearly dependent list in V. Then there exists $k \in \{1, 2, \ldots, n\}$ such that,

1. $v_k \in \mathrm{span}(v_1, \ldots, v_n)$
2. The span of the list is unchanged if v_k is removed.

Proof. Left for the reader. □

Theorem 2.2.5. In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof. This follows from the linear dependence lemma 2.2.4. □

2.3 Bases

Definition 2.3.1 (Basis)**.** A basis of V is a list of vectors in V that are linearly independent and spans V.

Proposition 2.3.2 (Criterion for Basis)**.** If $v_1, \ldots, v_n$ is a basis of V, then every vector in V can be uniquely represented as a linearly combination of $v_1, \ldots, v_n$.

Proof. The proof is closely related to the ideas of linear independence (definition 2.2.1) and is left for the reader. □

2.4 Dimension

Theorem 2.4.1. Any two bases of a finite-dimensional vector space have the same length. This theorem serves as the motivation for our definition of "dimension."

Proof. Suppose V is finite-dimensional and B_1 and B_2 are two bases of V. We know B_1 is linearly independent in V and B_2 spans V. Thus, by theorem 2.2.5, the length of B_2. Interchanging B_1 and B_2, we also see that the length of B_2 is less than or equal to the length of B_1. Thus, the length of B_1 is equal to the length of B_2. □

Definition 2.4.2 (Dimension)**.** The dimension of a finite-dimensional vector space (definition 2.1.4) is the length of any basis of the vector space.

Theorem 2.4.3. The following are a few theorems that follow from the definition of a dimension. We shall state them here without proof.

1. If V is finite-dimensional and U is a subspace of V, then $\dim(U) \leq \dim(V)$
2. If V is finite-dimensional, then every linearly independent list of vectors in V with length $\dim(V)$ is a basis for V
3. If V is finite-dimensional, then every spanning list of vectors in V with length $\dim(V)$ is a basis for V
4. If U_1 and U_2 are subspaces of a finite-dimensional vector space, then,

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2) \tag{2.4.1}$$

Chapter 3

Linear Maps

Definition 3.0.1 (Linear Maps)**.** A linear map, or linear transformation, from vector space V to vector space W[1] is an operation $T : V \to W$ with the properties, for any $v, v_1, v_2 \in V$ and $\lambda \in \mathbb{F}$

P1 **(Additivity)** $T(v_1 + v_2) = T(v_1) + T(v_2)$

P2 **(Homogeneity)** $T(\lambda \cdot v) = \lambda \cdot T(v)$

The set of all possible linear maps from V to W is denoted $\mathcal{L}(V, W)$

Theorem 3.0.2 (Linear Maps take 0 to 0)**.** Suppose $T \in \mathcal{L}(V, W)$. Then, $T(0) = 0$.

Proof. By additivity,

$$T(0) = T(0 + 0) = T(0) + T(0) \tag{3.0.1}$$

Thus,

$$T(0) = 0 \tag{3.0.2}$$

□

Definition 3.0.3 (Algebraic Operations on $\mathcal{L}(V, W)$)**.** Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. For all $v \in V$,

1. The sum is defined as $(S + T)v = Sv + Tv$
2. The scalar product is defined as $(\lambda S)v = \lambda \cdot Sv$

Notice that these two conditions are sufficient to show that $\mathcal{L}(V, W)$ is a vector space. Finally,

3. The product of linear maps is defined as $(ST)u = S(Tu)$ for $T \in \mathcal{L}(U, V)$, $S \in \mathcal{K}(V, W)$, and $u \in U$

Theorem 3.0.4 (Algebraic Properties of Linear Maps)**.** For $T, T_1, T_2 \in \mathcal{L}(U, V)$ an $S, S_1, S_2 \in \mathcal{L}(V, W)$,

[1] It is implicit that both vector spaces are over $\mathbb{F}$.

1. **Associativity**: $(T_1T_2)T_3 = T_1(T_2T_3))$
2. **Identity**: $TI = IT = T$ (notice the first I is the identity map on V while the second I is the identity map on W)
3. **Distributive**: $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$

3.1 Null Spaces and Ranges

Definition 3.1.1. For $T \in \mathcal{L}(V, W)$, the null space of T, $\operatorname{null} T$, is the subset of V consisting of vectors that T maps to 0,

$$\operatorname{null} T = \{v \in V : Tv = 0\} \tag{3.1.1}$$

Theorem 3.1.2. For $T \in \mathcal{L}(V, W)$, $\operatorname{null} T$ is a subspace of V.

Proof. T is a linear map, so $T(0) = 0$ (theorem 3.0.2). Thus, $0 \in \operatorname{null} T$.

Suppose $u, v \in T$. Then, $T(u + v) = T(u) + T(v) = 0 + 0 = 0$. Thus, $\operatorname{null} T$ is closed under addition.

Suppose $u \in \operatorname{null} T$ and $\lambda \in \mathbb{F}$. Then, $T(\lambda u) = \lambda T(u) = \lambda 0 = 0$. This, $\operatorname{null} T$ is closed under scalar multiplication.

The conditions of a subspace are met (1.4.2). □

Definition 3.1.3. For T, a function from V to W, the range of T is,

$$\operatorname{range} T = \{Tv : v \in V\} \tag{3.1.2}$$

Theorem 3.1.4. If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W.

Proof. Since T is a linear map, $T(0) = 0$. Thus, $0 \in \operatorname{range} T$

If $w_1, w_2 \in \operatorname{range} T$, then there must exist $v_1, v_2 \in V$ such that $Tv_i = w_i$. Thus, =

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2 \tag{3.1.3}$$

Hence, $w_1 + w_2 \in \operatorname{range} T$; $\operatorname{range} T$ is closed under addition.

If $w \in \operatorname{range} T$ and $\lambda \in \mathbb{F}$, then there exists $v \in V$ such that $T(v) = w$. Thus,

$$T(\lambda v) = \lambda T(v) = \lambda w \tag{3.1.4}$$

Thus, T is closed under scalar multiplication. □

Definition 3.1.5 (Injective)**.** A function $T : V \to W$ is injective if $T(u) = T(v)$ implies $u = v$. In other words, distinct inputs are mapped to distinct outputs.

Theorem 3.1.6. Let $T \in \mathcal{L}(V, W)$. T is injective if and only if $\operatorname{null} T = \{0\}$

Definition 3.1.7 (Surjective)**.** A function T from V to W is surjective if it's range equals W,

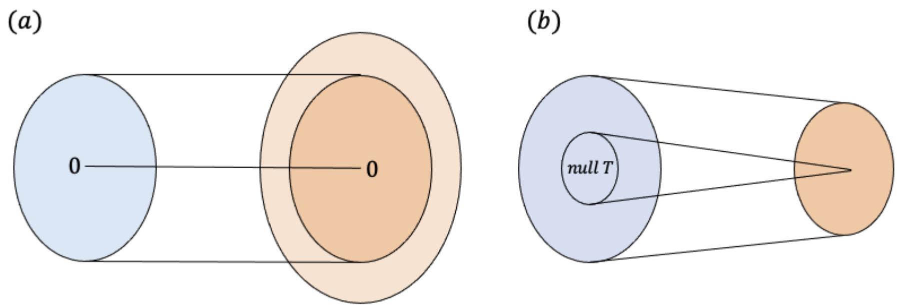

Figure 3.1: Suppose $T \in \mathcal{L}(V, W)$ is a linear map from V (blue) to W (orange) with some range T (darker orange). (a) T is injective. (b) T is surjective. Notice, a map to a smaller dimensional space can not be injective and a map to a larger dimensional space can not be surjective. We can also show this with the fundamental theorem of linear maps.

3.2 Fundamental Theorem of Linear Maps

Theorem 3.2.1. Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and,

$$\dim V = \dim(\operatorname{null} T) + \dim(\operatorname{range} T) \tag{3.2.1}$$

Proof. Let $u_1, \ldots, u_m$ be a basis of null T. This can be extended to a basis for V,

$$u_1, \ldots, u_m, v_1, \ldots, v_n \tag{3.2.2}$$

Thus, $\dim V = m + n$. To finish, we need to show that $\dim(\operatorname{range} T) = n$. We first prove that $Tv_1, \ldots, Tv_n$ is a basis of range T. For $a_i, b_j \in \mathbb{F}$,

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n \tag{3.2.3}$$

$$\begin{align} Tv &= T(a_1 u_1) + \cdots + T(a_m u_m) + T(b_1 v_1) + \cdots + T(b_n v_n) \tag{3.2.4} \\ &= b_1 T(v_1) + \cdots + b_n T(v_n) \tag{3.2.5} \end{align}$$

Thus, $Tv_1, \ldots, Tv_n$ spans range V. Now, to show that they are linearly independent, suppose $c_i \in \mathbb{F}$ and,

$$c_1 T(v_1) + \cdots + c_n T(v_n) = 0 \tag{3.2.6}$$

Then,

$$T(c_1 v_2 + \cdots + c_n v_n) = 0 \tag{3.2.7}$$

$$c_1 v_2 + \cdots + c_n v_n \in \operatorname{null} T \tag{3.2.8}$$

Because $u_1, \ldots, u_m$ spans $\operatorname{null} T$,

$$c_1 v_2 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m \tag{3.2.9}$$

But $u_1, \ldots, u_m, v_1, \ldots, v_n$ are linearly independent, so $c_i = d_j = 0$. Thus, $Tv_1, \ldots, Tv_n$ are linearly independent and span $\operatorname{range} T$. Hence, $\dim(\operatorname{range} T) = n$ $\square$

We have shown that $\dim V = \dim(\operatorname{null} T) + \dim(\operatorname{range} T)$

3.3 Matrices

Bibliography

[1] S. Axler, *Linear algebra done right.* Springer, 2015.