every orthonormal set is linearly independent

Theorem : An orthonormal set of vectors in an inner product space is linearly independent.

Proof. We denote by $\langle\cdot,\cdot\rangle$ the inner product of L. Let S be an orthonormal set of vectors. Let us first consider the case when S is finite, i.e., $S=\{e_1,\dots,e_n\}$ for some n. Suppose

$$\lambda_1 e_1+\cdots+\lambda_n e_n=0$$

for some scalars λ_i (belonging to the field on the underlying vector space of L). For a fixed k in $1,\dots,n$, we then have

$$0=\langle e_k,0\rangle=\langle e_k,\lambda_1 e_1+\cdots+\lambda_n e_n\rangle=\lambda_1\langle e_k,e_1\rangle+\cdots+\lambda_n\langle e_k,e_n\rangle=\lambda_k,$$

so $\lambda_k=0$, and S is linearly independent. Next, suppose S is infinite (countable or uncountable). To prove that S is linearly independent, we need to show that all finite subsets of S are linearly independent. Since any subset of an orthonormal set is also orthonormal, the infinite case follows from the finite case. □

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