量 子 场 论

第 5 章　量子旋量场

5.4 节和 5.5 节

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5.4 节 Dirac 旋量场的平面波展开
5.4.1 小节 平面波解的一般形式

- 本小节讨论与表象选取无关的平面波解一般形式
- 自由 Dirac 旋量场 $\psi_a(x)$ 满足 Klein-Gordon 方程 $(\partial^2 + m^2)\psi(x) = 0$
- 因而在无界空间中具有平面波解
- 对于确定的动量 $\mathbf{k}$，假设 Dirac 方程具有如下形式的平面波解：

$$\varphi_a(x, \mathbf{k}) = w_a(k^0, \mathbf{k})\mathrm{e}^{-\mathrm{i}k\cdot x}$$

- 系数 $w_a(k^0, \mathbf{k})$ 是 Dirac 旋量，带着一个旋量指标 a

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- 系数 $w_a(k^0, \mathbf{k})$ 是 Dirac 旋量，带着一个旋量指标 a
- 隐去旋量指标，将这个平面波解代入到 Dirac 方程 $(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) = 0$ 中，得

$$0 = (\mathrm{i}\gamma^\mu\partial_\mu - m)\varphi(x, \mathbf{k}) = (\gamma^\mu k_\mu - m)w(k^0, \mathbf{k})\mathrm{e}^{-\mathrm{i}k\cdot x} = (k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)w(k^0, \mathbf{k})\mathrm{e}^{-\mathrm{i}k\cdot x}$$

- 因此

$$(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)w(k^0, \mathbf{k}) = 0$$

本征值方程

- $(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)w(k^0,\mathbf{k}) = 0$ 左乘 γ^0 得 $[k^0 - \gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) - m\gamma^0]w(k^0,\mathbf{k}) = 0$
- 移项，推出

$$[\gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) + m\gamma^0]w(k^0,\mathbf{k}) = k^0 w(k^0,\mathbf{k})$$

- 这是**矩阵** $\gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) + m\gamma^0$ 的**本征方程**
- 它具有**非平庸解**的条件是**特征多项式** $\det[k^0 - \gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) - m\gamma^0]$ 为零，即

$$\det[k^0 - \gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) - m\gamma^0] = 0$$

- 这个方程的根给出 k^0 的**本征值**，相应的非平庸解是**本征矢量**

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$$\det[k^0 - \gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) - m\gamma^0] = 0$$

- 这个方程的根给出 k^0 的**本征值**，相应的非平庸解是**本征矢量**
- 利用 $(\gamma^0)^2 = \mathbf{1}$，将这个方程化为

$$\begin{aligned} 0 &= \det[k^0\mathbf{1} - \gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) - m\gamma^0] = \det[\gamma^0(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)] \\ &= \det(\gamma^0)\det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m) \end{aligned}$$

- 因而它等价于

$$\det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m) = 0$$

$[\det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)]^2$

利用 $(\gamma^5)^2 = \mathbf{1}$，将方程 $\det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m) = 0$ 左边化为

$$\begin{aligned}\det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m) &= \det[(\gamma^5)^2(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)] = \det[\gamma^5(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)\gamma^5] \\ &= \det[(\gamma^5)^2(-k^0\gamma^0 + \mathbf{k}\cdot\boldsymbol{\gamma} - m)] = \det[-(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma}) - m]\end{aligned}$$

这里第二步用到**行列式性质** $\det(AB) = \det(BA)$，第三步用到 $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$

$[\det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)]^2$

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这里第二步用到**行列式性质** $\det(AB) = \det(BA)$，第三步用到 $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$

利用
$$\begin{aligned}(k_\mu\gamma^\mu)^2 &= k_\mu k_\nu \gamma^\mu\gamma^\nu = \frac{1}{2}k_\mu k_\nu(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \\ &= k_\mu k_\nu g^{\mu\nu}\,\mathbf{1} = k^2\,\mathbf{1} = [(k^0)^2 - |\mathbf{k}|^2]\,\mathbf{1}\end{aligned}$$

推出 $[\det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)]^2$

$$\begin{aligned}&= \det[(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma}) - m]\,\det[-(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma}) - m] \\ &= \det(k_\mu\gamma^\mu - m)\det(-k_\nu\gamma^\nu - m) = \det[(k_\mu\gamma^\mu - m)(-k_\nu\gamma^\nu - m)] \\ &= \det[-(k_\mu\gamma^\mu)^2 + m^2] = \det\{[-(k^0)^2 + |\mathbf{k}|^2 + m^2]\,\mathbf{1}\} \\ &= [-(k^0)^2 + |\mathbf{k}|^2 + m^2]^4 = [E_{\mathbf{k}}^2 - (k^0)^2]^4\end{aligned}$$

其中 $E_{\mathbf{k}} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$

本征矢量

- $[\det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)]^2 = [E_{\mathbf{k}}^2 - (k^0)^2]^4$ 表明

$$\det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m) = [E_{\mathbf{k}}^2 - (k^0)^2]^2 = (E_{\mathbf{k}} + k^0)^2(E_{\mathbf{k}} - k^0)^2$$

- 因此方程 $\det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m) = 0$ 有 **2 个根** $k^0 = \pm E_{\mathbf{k}}$
- 这 2 个根都是 **2 重根**，各自对应于 **2 个线性独立**的**本征矢量**
- 它们是本征方程 $[\gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) + m\gamma^0]w(k^0,\mathbf{k}) = k^0 w(k^0,\mathbf{k})$ 的**解**

本征矢量

- $[\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)]^2=[E_{\mathbf{k}}^2-(k^0)^2]^4$ 表明

$$\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)=[E_{\mathbf{k}}^2-(k^0)^2]^2=(E_{\mathbf{k}}+k^0)^2(E_{\mathbf{k}}-k^0)^2$$

- 因此方程 $\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)=0$ 有 **2 个根** $k^0=\pm E_{\mathbf{k}}$
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1. $k^0=E_{\mathbf{k}}$ 对应于 **2 个**本征矢量 $w^{(+)}(E_{\mathbf{k}},\mathbf{k},\sigma),\quad \sigma=1,2$
   - 因而 $\mathrm{e}^{-ik\cdot x}=\exp[-\mathrm{i}(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})]$，平面波解中有 2 个**正能解**，形式为

$$w^{(+)}(E_{\mathbf{k}},\mathbf{k},\sigma)\exp[-\mathrm{i}(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})],\quad \sigma=1,2$$

2. $k^0=-E_{\mathbf{k}}$ 对应于 **2 个**本征矢量 $w^{(-)}(-E_{\mathbf{k}},\mathbf{k},\sigma),\quad \sigma=1,2$
   - 因而 $\mathrm{e}^{-ik\cdot x}=\exp[-\mathrm{i}(-E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})]$，平面波解中有 2 个**负能解**，形式为

$$w^{(-)}(-E_{\mathbf{k}},\mathbf{k},\sigma)\exp[\mathrm{i}(E_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})],\quad \sigma=1,2$$

正能解和负能解

将这 **4 个本征矢量**的**正交归一关系**取为

$$
\begin{aligned}
w^{(+)\dagger}(E_{\mathbf{k}},\mathbf{k},\sigma)\,w^{(+)}(E_{\mathbf{k}},\mathbf{k},\sigma') &= 2E_{\mathbf{k}}\delta_{\sigma\sigma'} \\
w^{(-)\dagger}(-E_{\mathbf{k}},\mathbf{k},\sigma)\,w^{(-)}(-E_{\mathbf{k}},\mathbf{k},\sigma') &= 2E_{\mathbf{k}}\delta_{\sigma\sigma'} \\
w^{(+)\dagger}(E_{\mathbf{k}},\mathbf{k},\sigma)\,w^{(-)}(-E_{\mathbf{k}},\mathbf{k},\sigma') &= w^{(-)\dagger}(-E_{\mathbf{k}},\mathbf{k},\sigma)\,w^{(+)}(E_{\mathbf{k}},\mathbf{k},\sigma') = 0
\end{aligned}
$$

正能解和负能解

将这 4 个本征矢量的正交归一关系取为

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w^{(+)\dagger}(E_{\mathbf{k}},\mathbf{k},\sigma)\,w^{(-)}(-E_{\mathbf{k}},\mathbf{k},\sigma') &= w^{(-)\dagger}(-E_{\mathbf{k}},\mathbf{k},\sigma)\,w^{(+)}(E_{\mathbf{k}},\mathbf{k},\sigma') = 0
\end{aligned}
$$

引入 Dirac 旋量 $u(\mathbf{k},\sigma)$ 和 $v(\mathbf{k},\sigma)$，定义为

$$
u(\mathbf{k},\sigma) \equiv w^{(+)}(E_{\mathbf{k}},\mathbf{k},\sigma), \quad v(\mathbf{k},\sigma) \equiv w^{(-)}(-E_{\mathbf{k}},-\mathbf{k},\sigma), \quad \sigma = 1,2
$$

于是，Dirac 方程的正能解和负能解可以分别写作

$$
\begin{aligned}
\varphi^{(+)}(x,\mathbf{k},\sigma) &\equiv w^{(+)}(E_{\mathbf{k}},\mathbf{k},\sigma)\exp[-\mathrm{i}(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})] = u(\mathbf{k},\sigma)\exp[-\mathrm{i}(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})] \\
\varphi^{(-)}(x,\mathbf{k},\sigma) &\equiv w^{(-)}(-E_{\mathbf{k}},-\mathbf{k},\sigma)\exp[\mathrm{i}(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})] = v(\mathbf{k},\sigma)\exp[\mathrm{i}(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})]
\end{aligned}
$$

替换动量记号，得到 $\varphi^{(+)}(x,\mathbf{p},\sigma) = u(\mathbf{p},\sigma)\mathrm{e}^{-\mathrm{i}p\cdot x}$ 和 $\varphi^{(-)}(x,\mathbf{p},\sigma) = v(\mathbf{p},\sigma)\mathrm{e}^{\mathrm{i}p\cdot x}$

其中 $p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2+m^2} > 0$

平面波展开

从而，**Dirac 旋量场算符** $\psi(\mathbf{x},t)$ 的**平面波展开式**可写作

$$\begin{aligned}\psi(\mathbf{x},t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=1}^{2}\left[\varphi^{(+)}(x,\mathbf{p},\sigma)\,c_{\mathbf{p},\sigma}+\varphi^{(-)}(x,\mathbf{p},\sigma)\,d^\dagger_{\mathbf{p},\sigma}\right]\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=1}^{2}\left[u(\mathbf{p},\sigma)c_{\mathbf{p},\sigma}e^{-ip\cdot x}+v(\mathbf{p},\sigma)d^\dagger_{\mathbf{p},\sigma}e^{ip\cdot x}\right]\end{aligned}$$

其中，$c_{\mathbf{p},\sigma}$ 是**湮灭算符**，$d^\dagger_{\mathbf{p},\sigma}$ 是**产生算符**，而且 $c_{\mathbf{p},\sigma}\neq d_{\mathbf{p},\sigma}$

平面波旋量系数 $u(\mathbf{p},\sigma)$ 和 $v(\mathbf{p},\sigma)$ 的**正交归一关系**为

$$\begin{aligned}u^\dagger(\mathbf{p},\sigma)u(\mathbf{p},\sigma') &= w^{(+)\dagger}(E_\mathbf{p},\mathbf{p},\sigma)\,w^{(+)}(E_\mathbf{p},\mathbf{p},\sigma')=2E_\mathbf{p}\delta_{\sigma\sigma'}\\ v^\dagger(\mathbf{p},\sigma)v(\mathbf{p},\sigma') &= w^{(-)\dagger}(-E_\mathbf{p},-\mathbf{p},\sigma)\,w^{(-)}(-E_\mathbf{p},-\mathbf{p},\sigma')=2E_\mathbf{p}\delta_{\sigma\sigma'}\\ u^\dagger(\mathbf{p},\sigma)v(-\mathbf{p},\sigma') &= w^{(+)\dagger}(E_\mathbf{p},\mathbf{p},\sigma)\,w^{(-)}(-E_\mathbf{p},\mathbf{p},\sigma')=0\end{aligned}$$

5.4.2 小节 Weyl 表象中的平面波解

- 本小节在 **Weyl 表象**中讨论 **Dirac 方程**的**平面波解**
- **Dirac 旋量场**描述**自旋为 1/2** 的**有质量粒子**，根据 3.3.1 小节讨论，这样的粒子具有 **2 种独立的自旋极化态**，对应于**螺旋度**的 2 种本征值 $+1/2$ 和 $-1/2$
- 为便于表述，这里采用**归一化**的**螺旋度本征值** $\lambda = \pm$
- 类似于矢量场情况，$\lambda = -$ 是**左旋极化**，$\lambda = +$ 是**右旋极化**

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- 类似于矢量场情况，$\lambda = -$ 是**左旋极化**，$\lambda = +$ 是**右旋极化**
- 因此，无论是平面波正能解还是负能解，都能够以 **2 种螺旋度本征态**作为 **2 个线性独立**的**本征矢量**
- 按照这个思路，把 2 个**正能解**表达为

$$\varphi^{(+)}(x, \mathbf{p}, \lambda) = u(\mathbf{p}, \lambda) e^{-ip\cdot x}, \quad \lambda = \pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$$

- 根据 **Dirac 方程** $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$，有

$$0 = (i\gamma^\mu \partial_\mu - m)\varphi^{(+)}(x, \mathbf{p}, \lambda) = (p_\mu \gamma^\mu - m) u(\mathbf{p}, \lambda) e^{-ip\cdot x}$$

$u(\mathbf{p},\lambda)$ 的运动方程

🚗 $(p_\mu\gamma^\mu - m)u(\mathbf{p},\lambda)\mathrm{e}^{-\mathrm{i}p\cdot x} = 0$ 表明 $u(\mathbf{p},\lambda)$ 满足**运动方程**

$$\boxed{(\not{p} - m)u(\mathbf{p},\lambda) = 0}$$

🪁 其中 $\not{p}$ 的定义为 $\not{p} \equiv p_\mu\gamma^\mu$，这种记号称为 **Dirac 斜线** (slash)，是 Richard Feynman 引进的

Richard Feynman
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🎏 将 $u(\mathbf{p},\lambda)$ 分解为两个**二分量旋量** $f_\lambda(\mathbf{p})$ 和 $g_\lambda(\mathbf{p})$，

$$u(\mathbf{p},\lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ g_\lambda(\mathbf{p}) \end{pmatrix}$$

Richard Feynman (1918–1988)

🎱 根据 **Weyl 表象**中的 Dirac 矩阵表达式 $\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$，**运动方程**化为

$$0 = (\not{p} - m)u(\mathbf{p},\lambda) = \begin{pmatrix} -m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & -m \end{pmatrix} \begin{pmatrix} f_\lambda(\mathbf{p}) \\ g_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} p_\mu\sigma^\mu g_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) \\ p_\mu\bar{\sigma}^\mu f_\lambda(\mathbf{p}) - mg_\lambda(\mathbf{p}) \end{pmatrix}$$

$f_\lambda(\mathbf{p})$ 与 $g_\lambda(\mathbf{p})$ 的关系

🚙 从而得到两条方程

$$(p\cdot\sigma)g_\lambda(\mathbf{p})-mf_\lambda(\mathbf{p})=0,\quad (p\cdot\bar{\sigma})f_\lambda(\mathbf{p})-mg_\lambda(\mathbf{p})=0$$

♟ 由**第二条方程**得

$$g_\lambda(\mathbf{p})=\frac{p\cdot\bar{\sigma}}{m}\,f_\lambda(\mathbf{p})$$

🀄 将上式代入到**第一条方程**左边，得

$$(p\cdot\sigma)g_\lambda(\mathbf{p})-mf_\lambda(\mathbf{p})=\frac{(p\cdot\sigma)(p\cdot\bar{\sigma})}{m}\,f_\lambda(\mathbf{p})-mf_\lambda(\mathbf{p})$$

$f_\lambda(\mathbf{p})$ 与 $g_\lambda(\mathbf{p})$ 的关系

从而得到两条方程

$$(p\cdot\sigma)g_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) = 0, \quad (p\cdot\bar{\sigma})f_\lambda(\mathbf{p}) - mg_\lambda(\mathbf{p}) = 0$$

由第二条方程得

$$g_\lambda(\mathbf{p}) = \frac{p\cdot\bar{\sigma}}{m} f_\lambda(\mathbf{p})$$

将上式代入到第一条方程左边，得

$$(p\cdot\sigma)g_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) = \frac{(p\cdot\sigma)(p\cdot\bar{\sigma})}{m} f_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p})$$

为化简 $(p\cdot\sigma)(p\cdot\bar{\sigma})$，由 $\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$ 得反对易关系

$$2g^{\mu\nu}\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = \{\gamma^\mu, \gamma^\nu\} = \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}$$

因此 $\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2g^{\mu\nu}$, $\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2g^{\mu\nu}$

$u(\mathbf{p},\lambda)$ 的形式

🚕 从而

$$
\begin{aligned}
(p\cdot\sigma)(p\cdot\bar{\sigma}) &= p_\mu p_\nu \sigma^\mu \bar{\sigma}^\nu = \frac{1}{2} p_\mu p_\nu (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu) \\
&= \frac{1}{2} p_\mu p_\nu 2 g^{\mu\nu} = p^2 = m^2
\end{aligned}
$$

❤️ 故

$$
\begin{aligned}
(p\cdot\sigma) g_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p}) &= \frac{(p\cdot\sigma)(p\cdot\bar{\sigma})}{m} f_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p}) \\
&= \frac{m^2}{m} f_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p}) = 0
\end{aligned}
$$

$u(\mathbf{p},\lambda)$ 的形式

🚕 从而

$$\begin{aligned}(p\cdot\sigma)(p\cdot\bar{\sigma}) &= p_\mu p_\nu \sigma^\mu \bar{\sigma}^\nu = \frac{1}{2}p_\mu p_\nu(\sigma^\mu\bar{\sigma}^\nu+\sigma^\nu\bar{\sigma}^\mu)\\ &= \frac{1}{2}p_\mu p_\nu 2g^{\mu\nu} = p^2 = m^2\end{aligned}$$

♥ 故

$$\begin{aligned}(p\cdot\sigma)g_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) &= \frac{(p\cdot\sigma)(p\cdot\bar{\sigma})}{m}f_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p})\\ &= \frac{m^2}{m}f_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) = 0\end{aligned}$$

♣ 可见关系式 $g_\lambda(\mathbf{p}) = \dfrac{p\cdot\bar{\sigma}}{m}f_\lambda(\mathbf{p})$ 也符合**第一条方程**

♦ 于是，任取**非零** $f_\lambda(\mathbf{p})$ 都能使

$$u(\mathbf{p},\lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \dfrac{p\cdot\bar{\sigma}}{m}f_\lambda(\mathbf{p})\end{pmatrix}$$

满足**运动方程** $(\not{p}-m)u(\mathbf{p},\lambda)=0$

螺旋度矩阵

旋量表示中螺旋度矩阵是自旋角动量矩阵 $\boldsymbol{\mathcal{S}}$ 在动量 $\mathbf{p}$ 方向上的投影，即 $\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{S}}$

对于 **Weyl 表象**，由 $\mathcal{S}^i = \dfrac{1}{2}\begin{pmatrix}\sigma^i & \\ & \sigma^i\end{pmatrix}$ 得 $\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{S}} = \dfrac{1}{2}\begin{pmatrix}\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & \\ & \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\end{pmatrix}$

因而归一化螺旋度矩阵为 $2\,\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{S}} = \begin{pmatrix}\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & \\ & \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\end{pmatrix}$

两个对角分块相同，左手和右手 **Weyl** 旋量对应的归一化螺旋度矩阵都是 $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$

代入 **Pauli** 矩阵 $\sigma^1 = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$、$\sigma^2 = \begin{pmatrix} & -\mathrm{i}\\ \mathrm{i} & \end{pmatrix}$ 和 $\sigma^3 = \begin{pmatrix}1 & \\ & -1\end{pmatrix}$，推出

$$\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} = \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{|\mathbf{p}|} = \frac{1}{|\mathbf{p}|}\begin{pmatrix} p^3 & p^1 - \mathrm{i}p^2 \\ p^1 + \mathrm{i}p^2 & -p^3 \end{pmatrix}$$

螺旋态

🚑 引入**归一化螺旋度矩阵** $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$ 的**本征矢量** $\xi_\lambda(\mathbf{p})$，称为**螺旋态**，满足**本征方程**

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = \lambda\,\xi_\lambda(\mathbf{p}), \quad \lambda = \pm$$

🧸 求解这个方程，得到**归一化**本征矢量

$$\xi_+(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} |\mathbf{p}|+p^3 \\ p^1+\mathrm{i}p^2 \end{pmatrix}, \quad \xi_-(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} -p^1+\mathrm{i}p^2 \\ |\mathbf{p}|+p^3 \end{pmatrix}$$

🔫 它们满足**正交归一关系** $\xi^\dagger_\lambda(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'}$ 和**完备性关系** $\sum_{\lambda=\pm}\xi_\lambda(\mathbf{p})\xi^\dagger_\lambda(\mathbf{p}) = \mathbf{1}$

螺旋态

🚑 引入**归一化螺旋度矩阵** $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}$ 的**本征矢量** $\xi_\lambda(\mathbf{p})$，称为**螺旋态**，满足**本征方程**

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = \lambda\,\xi_\lambda(\mathbf{p}), \quad \lambda = \pm$$

🧸 求解这个方程，得到**归一化**本征矢量

$$\xi_+(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} |\mathbf{p}|+p^3 \\ p^1+\mathrm{i}p^2 \end{pmatrix}, \quad \xi_-(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} -p^1+\mathrm{i}p^2 \\ |\mathbf{p}|+p^3 \end{pmatrix}$$

🔫 它们满足**正交归一关系** $\xi_\lambda^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'}$ 和**完备性关系** $\sum_{\lambda=\pm}\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) = \mathbf{1}$

🎨 由 $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ 得 $(\mathbf{p}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = \lambda|\mathbf{p}|\xi_\lambda(\mathbf{p})$

🖼 根据 $\sigma^\mu = (\mathbf{1},\boldsymbol{\sigma})$ 和 $\bar{\sigma}^\mu = (\mathbf{1},-\boldsymbol{\sigma})$，有

$$\begin{aligned}(p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p}) &= (E_\mathbf{p}\mathbf{1}+\mathbf{p}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_\mathbf{p}+\lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ (p\cdot\sigma)\xi_\lambda(\mathbf{p}) &= (E_\mathbf{p}\mathbf{1}-\mathbf{p}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_\mathbf{p}-\lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p})\end{aligned}$$

🪆 其中函数 $\omega_\lambda(\mathbf{p})$ 定义为 $\omega_\lambda(\mathbf{p}) \equiv \sqrt{E_\mathbf{p}+\lambda|\mathbf{p}|}$

$u(\mathbf{p},\lambda)$ 作为螺旋度本征态

🚒 为了让 $u(\mathbf{p},\lambda)$ 作为**螺旋度本征态**，设 $f_\lambda(\mathbf{p})$ **正比**于 $\xi_\lambda(\mathbf{p})$，$f_\lambda(\mathbf{p}) = C_{\mathbf{p},\lambda}\,\xi_\lambda(\mathbf{p})$

🎮 利用 $(p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})$，推出

$$u(\mathbf{p},\lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \dfrac{p\cdot\bar{\sigma}}{m}\,f_\lambda(\mathbf{p}) \end{pmatrix} = C_{\mathbf{p},\lambda}\begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \dfrac{p\cdot\bar{\sigma}}{m}\,\xi_\lambda(\mathbf{p}) \end{pmatrix} = C_{\mathbf{p},\lambda}\begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \dfrac{\omega_\lambda^2(\mathbf{p})}{m}\,\xi_\lambda(\mathbf{p}) \end{pmatrix}$$

$u(\mathbf{p},\lambda)$ 作为螺旋度本征态

🚒 为了让 $u(\mathbf{p},\lambda)$ 作为**螺旋度本征态**，设 $f_\lambda(\mathbf{p})$ **正比**于 $\xi_\lambda(\mathbf{p})$，$f_\lambda(\mathbf{p}) = C_{\mathbf{p},\lambda}\,\xi_\lambda(\mathbf{p})$

🎮 利用 $(p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})$，推出

$$u(\mathbf{p},\lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \dfrac{p\cdot\bar{\sigma}}{m} f_\lambda(\mathbf{p}) \end{pmatrix} = C_{\mathbf{p},\lambda}\begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \dfrac{p\cdot\bar{\sigma}}{m} \xi_\lambda(\mathbf{p}) \end{pmatrix} = C_{\mathbf{p},\lambda}\begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \dfrac{\omega_\lambda^2(\mathbf{p})}{m} \xi_\lambda(\mathbf{p}) \end{pmatrix}$$

🕹️ 为了使 $u(\mathbf{p},\lambda)$ 满足**归一关系** $u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda) = 2E_\mathbf{p}$，取

$$C_{\mathbf{p},\lambda} = \omega_{-\lambda}(\mathbf{p})$$

🎰 注意到

$$\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) = \sqrt{(E_\mathbf{p}+\lambda|\mathbf{p}|)(E_\mathbf{p}-\lambda|\mathbf{p}|)} = \sqrt{E_\mathbf{p}^2-\lambda^2|\mathbf{p}|^2} = \sqrt{E_\mathbf{p}^2-|\mathbf{p}|^2} = m$$

🔮 有 $C_{\mathbf{p},\lambda}\dfrac{\omega_\lambda^2(\mathbf{p})}{m} = \dfrac{\omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})}{m}\,\omega_\lambda(\mathbf{p}) = \omega_\lambda(\mathbf{p})$

$u(\mathbf{p},\lambda)$ 的螺旋态表达式

于是得到 $u(\mathbf{p},\lambda)$ 的螺旋态表达式

$$u(\mathbf{p},\lambda)=\begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p})\\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p})\end{pmatrix}$$

根据 $2\,\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{S}}=\begin{pmatrix}\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & \\ & \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\end{pmatrix}$，$u(\mathbf{p},\lambda)$ 是螺旋度本征态，本征值为 λ：

$$\begin{aligned}(2\,\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{S}})u(\mathbf{p},\lambda) &= \begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p})\\ \omega_\lambda(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p})\end{pmatrix}\\ &= \lambda\begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p})\\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p})\end{pmatrix}=\lambda\,u(\mathbf{p},\lambda)\end{aligned}$$

$v(\mathbf{p},\lambda)$ 的运动方程

另一方面，将 2 个**负能解**表达为

$$\varphi^{(-)}(x,\mathbf{p},\lambda)=v(\mathbf{p},\lambda)\mathrm{e}^{\mathrm{i}p\cdot x},\quad \lambda=\pm,\quad p^0=E_{\mathbf{p}}=\sqrt{|\mathbf{p}|^2+m^2}$$

根据 **Dirac 方程** $(\mathrm{i}\gamma^\mu\partial_\mu-m)\psi(x)=0$，有

$$0=(\mathrm{i}\gamma^\mu\partial_\mu-m)\varphi^{(-)}(x,\mathbf{p},\lambda)=(-p_\mu\gamma^\mu-m)v(\mathbf{p},\lambda)\mathrm{e}^{\mathrm{i}p\cdot x}$$

即 $v(\mathbf{p},\lambda)$ 满足**运动方程**

$$\boxed{(\not{p}+m)v(\mathbf{p},\lambda)=0}$$

$v(\mathbf{p},\lambda)$ 的运动方程

另一方面，将 2 个**负能解**表达为

$$\varphi^{(-)}(x,\mathbf{p},\lambda) = v(\mathbf{p},\lambda)\mathrm{e}^{\mathrm{i}p\cdot x}, \quad \lambda=\pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2+m^2}$$

根据 **Dirac 方程** $(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) = 0$，有

$$0 = (\mathrm{i}\gamma^\mu\partial_\mu - m)\varphi^{(-)}(x,\mathbf{p},\lambda) = (-p_\mu\gamma^\mu - m)v(\mathbf{p},\lambda)\mathrm{e}^{\mathrm{i}p\cdot x}$$

即 $v(\mathbf{p},\lambda)$ 满足**运动方程**

$$\boxed{(\not{p}+m)v(\mathbf{p},\lambda) = 0}$$

同样将 $v(\mathbf{p},\lambda)$ 分解为两个**二分量旋量** $\tilde{f}_\lambda(\mathbf{p})$ 和 $\tilde{g}_\lambda(\mathbf{p})$，$v(\mathbf{p},\lambda) = \begin{pmatrix}\tilde{f}_\lambda(\mathbf{p})\\ \tilde{g}_\lambda(\mathbf{p})\end{pmatrix}$，则

$$0 = (\not{p}+m)v(\mathbf{p},\lambda) = \begin{pmatrix} m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & m\end{pmatrix}\begin{pmatrix}\tilde{f}_\lambda(\mathbf{p})\\ \tilde{g}_\lambda(\mathbf{p})\end{pmatrix} = \begin{pmatrix} p_\mu\sigma^\mu\tilde{g}_\lambda(\mathbf{p}) + m\tilde{f}_\lambda(\mathbf{p}) \\ p_\mu\bar{\sigma}^\mu\tilde{f}_\lambda(\mathbf{p}) + m\tilde{g}_\lambda(\mathbf{p})\end{pmatrix}$$

$v(\mathbf{p},\lambda)$ 的形式

从而得到两个条方程

$$(p\cdot\sigma)\tilde{g}_\lambda(\mathbf{p})+m\tilde{f}_\lambda(\mathbf{p})=0,\quad (p\cdot\bar{\sigma})\tilde{f}_\lambda(\mathbf{p})+m\tilde{g}_\lambda(\mathbf{p})=0$$

由**第二条方程**得

$$\tilde{g}_\lambda(\mathbf{p})=-\frac{p\cdot\bar{\sigma}}{m}\,\tilde{f}_\lambda(\mathbf{p})$$

代入到**第一条方程**左边，由 $(p\cdot\sigma)(p\cdot\bar{\sigma})=m^2$ 式推出

$$(p\cdot\sigma)\tilde{g}_\lambda(\mathbf{p})+m\tilde{f}_\lambda(\mathbf{p})=-\frac{(p\cdot\sigma)(p\cdot\bar{\sigma})}{m}\,\tilde{f}_\lambda(\mathbf{p})+m\tilde{f}_\lambda(\mathbf{p})=-\frac{m^2}{m}\,\tilde{f}_\lambda(\mathbf{p})+m\tilde{f}_\lambda(\mathbf{p})=0$$

可见，关系式 $\tilde{g}_\lambda(\mathbf{p})=-\frac{p\cdot\bar{\sigma}}{m}\,\tilde{f}_\lambda(\mathbf{p})$ 符合**第一条方程**

于是，任取**非零** $\tilde{f}_\lambda(\mathbf{p})$ 都能使

$$v(\mathbf{p},\lambda)=\begin{pmatrix}\tilde{f}_\lambda(\mathbf{p})\\ -\dfrac{p\cdot\bar{\sigma}}{m}\,\tilde{f}_\lambda(\mathbf{p})\end{pmatrix}$$

满足**运动方程** $(\not{p}+m)v(\mathbf{p},\lambda)=0$

$v(\mathbf{p},\lambda)$ 作为螺旋度本征态

🚛 为了让 $v(\mathbf{p},\lambda)$ 作为**螺旋度本征态**，设 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$，$\tilde{f}_\lambda(\mathbf{p}) = \tilde{C}_{\mathbf{p},\lambda}\,\xi_{-\lambda}(\mathbf{p})$

🧧 这里**没有**选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_\lambda(\mathbf{p})$，**原因**将在 **5.5.4 小节**中说明

🎁 现在姑且接受这种选择，从而由 $(p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})$ 推出

$$v(\mathbf{p},\lambda) = \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ -\dfrac{p\cdot\bar{\sigma}}{m}\,\tilde{f}_\lambda(\mathbf{p}) \end{pmatrix} = \tilde{C}_{\mathbf{p},\lambda}\begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\dfrac{(p\cdot\bar{\sigma})}{m}\,\xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \tilde{C}_{\mathbf{p},\lambda}\begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\dfrac{\omega_{-\lambda}^2(\mathbf{p})}{m}\,\xi_{-\lambda}(\mathbf{p}) \end{pmatrix}$$

$v(\mathbf{p},\lambda)$ 作为螺旋度本征态

🚛 为了让 $v(\mathbf{p},\lambda)$ 作为**螺旋度本征态**，设 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$，$\tilde{f}_\lambda(\mathbf{p}) = \tilde{C}_{\mathbf{p},\lambda}\,\xi_{-\lambda}(\mathbf{p})$

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$$v(\mathbf{p},\lambda) = \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ -\dfrac{p\cdot\bar{\sigma}}{m}\,\tilde{f}_\lambda(\mathbf{p}) \end{pmatrix} = \tilde{C}_{\mathbf{p},\lambda}\begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\dfrac{(p\cdot\bar{\sigma})}{m}\,\xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \tilde{C}_{\mathbf{p},\lambda}\begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\dfrac{\omega_{-\lambda}^2(\mathbf{p})}{m}\,\xi_{-\lambda}(\mathbf{p}) \end{pmatrix}$$

🧨 为了使 $v(\mathbf{p},\lambda)$ 满足**归一关系** $v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda) = 2E_\mathbf{p}$，取

$$\tilde{C}_{\mathbf{p},\lambda} = \lambda\,\omega_\lambda(\mathbf{p})$$

🪄 由 $\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) = m$ 得

$$-\tilde{C}_{\mathbf{p},\lambda}\frac{\omega_{-\lambda}^2(\mathbf{p})}{m} = -\lambda\,\frac{\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})}{m}\,\omega_{-\lambda}(\mathbf{p}) = -\lambda\,\omega_{-\lambda}(\mathbf{p})$$

$v(\mathbf{p},\lambda)$ 的螺旋态表达式

🚐 于是得到 $v(\mathbf{p},\lambda)$ **的螺旋态表达式**

$$\boxed{v(\mathbf{p},\lambda)=\begin{pmatrix}\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\\ -\lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\end{pmatrix}}$$

🔮 这样一来，$v(\mathbf{p},\lambda)$ 是**螺旋度本征态**，**本征值为** $-\lambda$：

$$(2\,\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{S}})v(\mathbf{p},\lambda)=\begin{pmatrix}\lambda\,\omega_\lambda(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p})\\ -\lambda\,\omega_{-\lambda}(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p})\end{pmatrix}$$

$$=-\lambda\begin{pmatrix}\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\\ -\lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\end{pmatrix}=-\lambda\,v(\mathbf{p},\lambda)$$

平面波旋量系数的关系

可以验证，以上平面波旋量系数 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 满足正交归一关系

$$
\begin{aligned}
u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda') &= v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda') = 2E_{\mathbf{p}}\delta_{\lambda\lambda'} \\
u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda') &= v^\dagger(-\mathbf{p},\lambda)u(\mathbf{p},\lambda') = 0
\end{aligned}
$$

记 $\bar{u}(\mathbf{p},\lambda) = u^\dagger(\mathbf{p},\lambda)\gamma^0$，$\bar{v}(\mathbf{p},\lambda) = v^\dagger(\mathbf{p},\lambda)\gamma^0$，可以推出 Lorentz 不变的关系式

$$
\begin{aligned}
\bar{u}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') &= 2m\delta_{\lambda\lambda'}, \quad \bar{v}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') = -2m\delta_{\lambda\lambda'} \\
\bar{u}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') &= \bar{v}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') = 0
\end{aligned}
$$

另一方面，考虑螺旋度求和式

$$
\begin{aligned}
\sum_{\lambda=\pm} u(\mathbf{p},\lambda)\bar{u}(\mathbf{p},\lambda) &= \sum_{\lambda=\pm} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \\
&= \sum_{\lambda=\pm} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \\ \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix}
\end{aligned}
$$

螺旋度求和

🚎 利用

$$\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) = m, \quad (p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p}), \quad (p\cdot\sigma)\xi_\lambda(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p})$$

💄 以及 $\xi_\lambda(\mathbf{p})$ 的**完备性关系** $\sum_{\lambda=\pm}\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) = \mathbf{1}$，推出

$$\begin{aligned}
\sum_{\lambda=\pm} u(\mathbf{p},\lambda)\bar{u}(\mathbf{p},\lambda) &= \sum_{\lambda=\pm}\begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \\ \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \\
&= \sum_{\lambda=\pm}\begin{pmatrix} m\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & (p\cdot\sigma)\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \\ (p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & m\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \\
&= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = p_\mu\gamma^\mu + m
\end{aligned}$$

自旋求和关系

将 $(p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p})=\omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})$ 和 $(p\cdot\sigma)\xi_\lambda(\mathbf{p})=\omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p})$ 中的 λ 换成 $-\lambda$，得

$$(p\cdot\bar{\sigma})\xi_{-\lambda}(\mathbf{p})=\omega_{-\lambda}^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p}),\quad (p\cdot\sigma)\xi_{-\lambda}(\mathbf{p})=\omega_\lambda^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p})$$

有 $\sum_{\lambda=\pm} v(\mathbf{p},\lambda)\bar{v}(\mathbf{p},\lambda)$

$$= \sum_{\lambda=\pm}\begin{pmatrix}\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\\ -\lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\end{pmatrix}\begin{pmatrix}-\lambda\,\omega_{-\lambda}(\mathbf{p})\,\xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}$$

$$= \sum_{\lambda=\pm}\begin{pmatrix}-\lambda^2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda^2\omega_\lambda^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\\ \lambda^2\omega_{-\lambda}^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda^2\omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}$$

$$= \sum_{\lambda=\pm}\begin{pmatrix}-m\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & (p\cdot\sigma)\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\\ (p\cdot\bar{\sigma})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -m\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix} = \begin{pmatrix}-m & p\cdot\sigma\\ p\cdot\bar{\sigma} & -m\end{pmatrix} = p_\mu\gamma^\mu - m$$

整理一下，有如下螺旋度求和关系，或者说，**自旋求和关系**：

$$\sum_{\lambda=\pm} u(\mathbf{p},\lambda)\bar{u}(\mathbf{p},\lambda)=\not{p}+m,\quad \sum_{\lambda=\pm} v(\mathbf{p},\lambda)\bar{v}(\mathbf{p},\lambda)=\not{p}-m$$

平面波展开

🚃 用 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 把 **Dirac 旋量场算符** $\psi(\mathbf{x},t)$ 的**平面波展开式**写作

$$\psi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda=\pm} \left[\varphi^{(+)}(x,\mathbf{p},\lambda)a_{\mathbf{p},\lambda} + \varphi^{(-)}(x,\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda=\pm} \left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]$$

💠 其中 $a_{\mathbf{p},\lambda}$ 是**湮灭算符**，$b^\dagger_{\mathbf{p},\lambda}$ 是**产生算符**，而且 $a_{\mathbf{p},\lambda} \neq b_{\mathbf{p},\lambda}$，于是

$$\psi^\dagger(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda=\pm} \left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right]$$

$$\bar{\psi}(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda=\pm} \left[\bar{u}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + \bar{v}(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right]$$

5.4.3 小节　哈密顿量和产生湮灭算符

根据**拉氏量** $\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi$，$\psi(x)$ 对应的**共轭动量密度**是

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \mathrm{i}\bar{\psi}\gamma^0 = \mathrm{i}\psi^\dagger$$

它的**平面波展开式**为

$$\pi(\mathbf{x},t) = \mathrm{i}\psi^\dagger(\mathbf{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{\mathrm{i}}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{\mathrm{i}p\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-\mathrm{i}p\cdot x}\right]$$

代入**自由旋量场** $\psi(x)$ 满足的 **Dirac 方程** $(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) = 0$，**拉氏量**化为

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0$$

因此，自由 Dirac 旋量场的**哈密顿量密度**为

$$\mathcal{H} = \pi\partial_0\psi - \mathcal{L} = \pi\partial_0\psi = \mathrm{i}\psi^\dagger\partial_0\psi$$

哈密顿量算符

从而，**哈密顿量算符**为

$$
\begin{aligned}
H &= \int d^3x\, \mathcal{H} = \int d^3x\, \psi^\dagger i\partial_0\psi \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right] \\
&\qquad \times q_0\left[u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x} - v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x\, d^3p\, d^3q\, E_{\mathbf{q}}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x} \\
&\qquad - v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x} \\
&\qquad - u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x} \\
&\qquad + v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\Big]
\end{aligned}
$$

化简哈密顿量

积分，得

$$
\begin{aligned}
H = &\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q\,E_{\mathbf{q}}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t} \\
&\qquad -v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\Big] \\
&\qquad +\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[-u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} \\
&\qquad + v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\Big]\Big\} \\
= &\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}-v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'} \\
&\qquad -\underbrace{u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')}_{=0}a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}+\underbrace{v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')}_{=0}b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}\Big] \\
= &\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left(2E_{\mathbf{p}}\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}-2E_{\mathbf{p}}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\right) \\
= &\sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}-b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\right)
\end{aligned}
$$

正交归一关系

$a_{\mathbf{p},\lambda}$ 和 $a^\dagger_{\mathbf{p},\lambda}$ 的表达式

另一方面，有

$$\begin{aligned}
&\int d^3x\, e^{ip\cdot x} u^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t) \\
&= \int \frac{d^3x\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x} + u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\right] \\
&= \int \frac{d^3q}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}-\mathbf{q})\right. \\
&\qquad\left. + u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a_{\mathbf{p},\lambda'} + u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} (2E_{\mathbf{p}}\delta_{\lambda\lambda'}a_{\mathbf{p},\lambda'}) = \sqrt{2E_{\mathbf{p}}}\, a_{\mathbf{p},\lambda}
\end{aligned}$$

从而将湮灭算符 $a_{\mathbf{p},\lambda}$ 和产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 表示为

$$a_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{ip\cdot x} u^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t), \quad a^\dagger_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{-ip\cdot x}\psi^\dagger(\mathbf{x},t)u(\mathbf{p},\lambda)$$

$b^\dagger_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}$ 的表达式

同理推出

$$\begin{aligned}
&\int d^3x\, e^{-ip\cdot x} v^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t) \\
&= \int \frac{d^3x\, d^3q}{(2\pi)^3\sqrt{2E_\mathbf{q}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'} e^{-i(p+q)\cdot x} + v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'} e^{-i(p-q)\cdot x} \right] \\
&= \int \frac{d^3q}{\sqrt{2E_\mathbf{q}}} \sum_{\lambda'=\pm} \Big[v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'} e^{-i(E_\mathbf{p}+E_\mathbf{q})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \\
&\qquad + v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'} e^{-i(E_\mathbf{p}-E_\mathbf{q})t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \Big] \\
&= \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')a_{-\mathbf{p},\lambda'} e^{-2iE_\mathbf{p}t} + v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b^\dagger_{\mathbf{p},\lambda'} \right] \\
&= \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda'=\pm} \left(2E_\mathbf{p}\delta_{\lambda\lambda'} b^\dagger_{\mathbf{p},\lambda'} \right) = \sqrt{2E_\mathbf{p}}\, b^\dagger_{\mathbf{p},\lambda}
\end{aligned}$$

于是将**产生算符** $b^\dagger_{\mathbf{p},\lambda}$ 和**湮灭算符** $b_{\mathbf{p},\lambda}$ 表示成

$$b^\dagger_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{-ip\cdot x} v^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t), \quad b_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{ip\cdot x} \psi^\dagger(\mathbf{x},t)v(\mathbf{p},\lambda)$$

5.5 节 Dirac 旋量场的正则量子化

5.5.1 小节 用等时对易关系量子化 Dirac 旋量场的困难

- 回顾前面**标量场**和**矢量场**的**正则量子化**程序
- 我们先假设场算符与其共轭动量密度算符满足**等时对易关系**

$$[\Phi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)] = \mathrm{i}\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

$$[\Phi_a(\mathbf{x},t),\Phi_b(\mathbf{y},t)] = [\pi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)] = 0$$

- 然后推导出**产生湮灭算符的对易关系**，再通过计算给出**正定**的**哈密顿量算符**
- 对于**无质量矢量场**，则需要用**弱 Lorenz 规范条件**来得到**正**的**哈密顿量期待值**
- 这些结果说明在量子场论中使用**正则量子化方法**是**合理**的
- 本小节将尝试用类似的**等时对易关系**对 **Dirac 旋量场**进行量子化
- 不过，我们会发现这种方法并**不能**给出正定的哈密顿量算符，因而是**不可行**的

等时对易关系

- 假设 Dirac 旋量场算符 $\psi(x)$ 与其共轭动量密度算符 $\pi(x)$ 满足等时对易关系

$$[\psi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)]=\mathrm{i}\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)]=[\pi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)]=0.$$

- 这里将旋量指标明显地写出来
- 由于 $\pi=\mathrm{i}\psi^\dagger$，这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时对易关系

$$[\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]=\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)]=[\psi_a^\dagger(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]=0$$

等时对易关系

假设 Dirac 旋量场算符 $\psi(x)$ 与其共轭动量密度算符 $\pi(x)$ 满足等时对易关系

$$[\psi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)]=\mathrm{i}\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)]=[\pi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)]=0.$$

这里将旋量指标明显地写出来

由于 $\pi=\mathrm{i}\psi^\dagger$，这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时对易关系

$$[\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]=\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)]=[\psi_a^\dagger(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]=0$$

根据 $a_{\mathbf{p},\lambda}=\dfrac{1}{\sqrt{2E_\mathbf{p}}}\displaystyle\int d^3x\, e^{ip\cdot x}u^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t)$，推出

$$\begin{aligned}[a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}^\dagger] &= \frac{1}{\sqrt{4E_\mathbf{p}E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)[\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]u_b(\mathbf{q},\lambda')\\
&= \frac{1}{\sqrt{4E_\mathbf{p}E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)u_b(\mathbf{q},\lambda')\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})\\
&= \frac{1}{\sqrt{4E_\mathbf{p}E_\mathbf{q}}}\int d^3x\,e^{i(E_\mathbf{p}-E_\mathbf{q})t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\,u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')\\
&= \frac{1}{2E_\mathbf{p}}u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q})\end{aligned}$$

$[b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}]$

根据

$$b^\dagger_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{-ip\cdot x} v^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t), \quad b_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{ip\cdot x} \psi^\dagger(\mathbf{x},t)v(\mathbf{p},\lambda)$$

以及**等时对易关系** $[\psi_a(\mathbf{x},t), \psi_b^\dagger(\mathbf{y},t)] = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})$，得到

$$\begin{aligned}
[b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] &= \frac{1}{\sqrt{4E_\mathbf{p}E_\mathbf{q}}}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} v_b^\dagger(\mathbf{q},\lambda')[\psi_a^\dagger(\mathbf{x},t), \psi_b(\mathbf{y},t)]v_a(\mathbf{p},\lambda) \\
&= \frac{1}{\sqrt{4E_\mathbf{p}E_\mathbf{q}}}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} v_b^\dagger(\mathbf{q},\lambda')v_a(\mathbf{p},\lambda)(-\delta_{ba})\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
&= -\frac{1}{\sqrt{4E_\mathbf{p}E_\mathbf{q}}}\int d^3x\, e^{i(E_\mathbf{p}-E_\mathbf{q})t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\, v^\dagger(\mathbf{q},\lambda')v(\mathbf{p},\lambda) \\
&= -\frac{1}{2E_\mathbf{p}}\, v^\dagger(\mathbf{p},\lambda')v(\mathbf{p},\lambda)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) = -(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q})
\end{aligned}$$

这个结果非同寻常地多了一个**负号**

负能量困难

进一步计算，最终通过等时对易关系得到的产生湮灭算符对易关系为

$$[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = 0$$

$$[b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = -(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [b^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = 0$$

$$[a_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = [b_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = 0$$

利用这样的对易关系，可以把哈密顿量算符化为

$$\begin{aligned} H &= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}) \\ &= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}) + (2\pi)^3\delta^{(3)}(\mathbf{0})\int\frac{d^3p}{(2\pi)^3}\,2E_{\mathbf{p}} \end{aligned}$$

负能量困难

进一步计算，最终通过**等时对易关系**得到的**产生湮灭算符对易关系**为

$$[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = 0$$

$$[b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = -(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [b^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = 0$$

$$[a_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = [b_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = 0$$

利用这样的对易关系，可以把**哈密顿量算符**化为

$$\begin{aligned} H &= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}) \\ &= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}) + (2\pi)^3\delta^{(3)}(\mathbf{0})\int\frac{d^3p}{(2\pi)^3}\,2E_{\mathbf{p}} \end{aligned}$$

第二项是**零点能**，第一项中由 $(a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{p},\lambda})$ 描述的粒子对总能量的贡献为**正**

但第一项中由 $(b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{p},\lambda})$ 描述的粒子对总能量的贡献为**负**

粒子数密度 $b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}$ **越大**，场的**总能量越少**，显然是**非物理**的，出现**负能量困难**

因此，用**等时对易关系**对 **Dirac 旋量场**进行量子化是**行不通**的

5.5.2 小节　用等时反对易关系量子化 Dirac 旋量场

- 从以上哈密顿量算符计算过程看出，如果在交换 $b_{\mathbf{p},\lambda}$ 和 $b^\dagger_{\mathbf{p},\lambda}$ 位置的同时能够改变符号，就可以得到正定的哈密顿量算符
- 因此，需要的不是 $b_{\mathbf{p},\lambda}$ 与 $b^\dagger_{\mathbf{p},\lambda}$ 的对易关系，而是反对易关系
- 为了得到合适的 $b_{\mathbf{p},\lambda}$ 与 $b^\dagger_{\mathbf{p},\lambda}$ 的反对易关系，则需要舍弃等时对易关系

5.5.2 小节　用等时反对易关系量子化 Dirac 旋量场

从以上哈密顿量算符计算过程看出，如果在交换 $b_{\mathbf{p},\lambda}$ 和 $b^\dagger_{\mathbf{p},\lambda}$ 位置的同时能够改变符号，就可以得到正定的哈密顿量算符

因此，需要的不是 $b_{\mathbf{p},\lambda}$ 与 $b^\dagger_{\mathbf{p},\lambda}$ 的对易关系，而是反对易关系

为了得到合适的 $b_{\mathbf{p},\lambda}$ 与 $b^\dagger_{\mathbf{p},\lambda}$ 的反对易关系，则需要舍弃等时对易关系，代之以等时反对易关系

$$\{\psi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)\} = i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})$$
$$\{\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)\} = \{\pi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)\} = 0$$

采用反对易关系进行量子化的方法称为 **Jordan-Wigner 量子化**

由于 $\pi = i\psi^\dagger$，这些关系等价于 ψ 与 $\psi^\dagger$ 的等时反对易关系

$$\{\psi_a(\mathbf{x},t),\psi^\dagger_b(\mathbf{y},t)\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})$$
$$\{\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)\} = \{\psi^\dagger_a(\mathbf{x},t),\psi^\dagger_b(\mathbf{y},t)\} = 0$$

Pascual Jordan
(1902–1980)

Eugene Wigner
(1902–1995)

哈密顿量的正定性

通过**等时反对易关系**得到的**产生湮灭算符反对易关系**为

$$\{a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}\} = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad \{a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}\} = \{a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}\} = 0$$

$$\{b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad \{b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}\} = \{b^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = 0$$

$$\{a_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = \{b_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}\} = \{a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}\} = \{a^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = 0$$

可见，$(a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{p},\lambda})$ 和 $(b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{p},\lambda})$ 互不干扰，各自描述一种粒子

哈密顿量的正定性

通过等时反对易关系得到的产生湮灭算符反对易关系为

$$\{a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}\} = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad \{a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}\} = \{a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}\} = 0$$

$$\{b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad \{b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}\} = \{b^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = 0$$

$$\{a_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = \{b_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}\} = \{a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}\} = \{a^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = 0$$

可见，$(a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{p},\lambda})$ 和 $(b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{p},\lambda})$ 互不干扰，各自描述一种粒子

利用这样的反对易关系，把哈密顿量算符化为

$$\begin{aligned} H &= \sum_{\lambda=\pm}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,E_{\mathbf{p}}(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}) \\ &= \sum_{\lambda=\pm}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,E_{\mathbf{p}}(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} + b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}) - (2\pi)^3\delta^{(3)}(\mathbf{0})\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,2E_{\mathbf{p}} \end{aligned}$$

第二项是零点能

第一项是所有动量模式所有螺旋度所有粒子贡献的能量之和，它是正定的

可见，用等时反对易关系对 Dirac 旋量场进行正则量子化是合适的

哈密顿量与产生湮灭算符的对易

计算哈密顿量 H 与产生湮灭算符的对易子，得到

$$[H, a^\dagger_{\mathbf{p},\lambda}] = E_\mathbf{p} a^\dagger_{\mathbf{p},\lambda}, \qquad [H, a_{\mathbf{p},\lambda}] = -E_\mathbf{p} a_{\mathbf{p},\lambda}$$
$$[H, b^\dagger_{\mathbf{p},\lambda}] = E_\mathbf{p} b^\dagger_{\mathbf{p},\lambda}, \qquad [H, b_{\mathbf{p},\lambda}] = -E_\mathbf{p} b_{\mathbf{p},\lambda}$$

设 $|E\rangle$ 是 H 的本征态，本征值为 E，则 $H\,|E\rangle = E\,|E\rangle$

从而推出

$$\begin{aligned}
H a^\dagger_{\mathbf{p},\lambda}\,|E\rangle &= (a^\dagger_{\mathbf{p},\lambda} H + E_\mathbf{p} a^\dagger_{\mathbf{p},\lambda})\,|E\rangle = (E + E_\mathbf{p}) a^\dagger_{\mathbf{p},\lambda}\,|E\rangle \\
H a_{\mathbf{p},\lambda}\,|E\rangle &= (a_{\mathbf{p},\lambda} H - E_\mathbf{p} a_{\mathbf{p},\lambda})\,|E\rangle = (E - E_\mathbf{p}) a_{\mathbf{p},\lambda}\,|E\rangle \\
H b^\dagger_{\mathbf{p},\lambda}\,|E\rangle &= (b^\dagger_{\mathbf{p},\lambda} H + E_\mathbf{p} b^\dagger_{\mathbf{p},\lambda})\,|E\rangle = (E + E_\mathbf{p}) b^\dagger_{\mathbf{p},\lambda}\,|E\rangle \\
H b_{\mathbf{p},\lambda}\,|E\rangle &= (b_{\mathbf{p},\lambda} H - E_\mathbf{p} b_{\mathbf{p},\lambda})\,|E\rangle = (E - E_\mathbf{p}) b_{\mathbf{p},\lambda}\,|E\rangle
\end{aligned}$$

当 $a^\dagger_{\mathbf{p},\lambda}\,|E\rangle \neq 0$ 和 $b^\dagger_{\mathbf{p},\lambda}\,|E\rangle \neq 0$ 时，$a^\dagger_{\mathbf{p},\lambda}$ 和 $b^\dagger_{\mathbf{p},\lambda}$ 的作用是使能量本征值增加 $E_\mathbf{p}$

当 $a_{\mathbf{p},\lambda}\,|E\rangle \neq 0$ 和 $b_{\mathbf{p},\lambda}\,|E\rangle \neq 0$ 时，$a_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}$ 的作用是使能量本征值减少 $E_\mathbf{p}$

总动量算符

Dirac 旋量场的总动量算符为

$$\begin{aligned}
\mathbf{P} &= -\int d^3x\,\pi\nabla\psi = \int d^3x\,\psi^\dagger(-\mathrm{i}\nabla)\psi \\
&= \sum_{\lambda\lambda'}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{4E_\mathbf{p}E_\mathbf{q}}}\left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}+v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right] \\
&\qquad\times\left[\mathbf{q}\,u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x}-\mathbf{q}\,v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right] \\
&= \sum_{\lambda\lambda'}\int \frac{d^3p\,d^3q\;\mathbf{q}}{(2\pi)^3\sqrt{4E_\mathbf{p}E_\mathbf{q}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{\mathrm{i}(E_\mathbf{p}-E_\mathbf{q})t} \\
&\qquad -v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-\mathrm{i}(E_\mathbf{p}-E_\mathbf{q})t}\Big] \\
&\qquad +\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[-u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{\mathrm{i}(E_\mathbf{p}+E_\mathbf{q})t} \\
&\qquad +v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-\mathrm{i}(E_\mathbf{p}+E_\mathbf{q})t}\Big]\Big\}
\end{aligned}$$

化简总动量

积分，得

$$\begin{aligned}\mathbf{P} &= \sum_{\lambda\lambda'}\int\frac{\mathrm{d}^3p\ \mathbf{p}}{(2\pi)^3 2E_\mathbf{p}}\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} - v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'} \\ &\qquad + u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_\mathbf{p}t} - v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_\mathbf{p}t}\Big] \\ &= \sum_{\lambda\lambda'}\int\frac{\mathrm{d}^3p\ \mathbf{p}}{(2\pi)^3 2E_\mathbf{p}}\left(2E_\mathbf{p}\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} - 2E_\mathbf{p}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\right) \\ &= \sum_{\lambda=\pm}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathbf{p}\,(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}) \qquad \{b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ &= \sum_{\lambda=\pm}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathbf{p}\,(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} + b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}) - 2\delta^{(3)}(\mathbf{0})\int \mathrm{d}^3p\,\mathbf{p} \\ &= \sum_{\lambda=\pm}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathbf{p}\,(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} + b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda})\end{aligned}$$

总动量是所有动量模式所有螺旋度所有粒子贡献的**动量之和**

5.5.3 小节　U(1) 整体对称性

- 类似于**复标量场**，**Dirac 旋量场**也具有 **U(1) 整体对称性**
- 对 Dirac 旋量场 $\psi(x)$ 作 **U(1) 整体变换** $\psi'(x) = \mathrm{e}^{\mathrm{i}q\theta}\psi(x)$
- 则 $\psi^\dagger(x)$ 和 $\bar{\psi}(x)$ 的相应变换为

$$[\psi^\dagger(x)]' = [\psi'(x)]^\dagger = \psi^\dagger(x)\mathrm{e}^{-\mathrm{i}q\theta}, \quad [\bar{\psi}(x)]' = \bar{\psi}'(x) = [\psi'(x)]^\dagger\gamma^0 = \bar{\psi}(x)\mathrm{e}^{-\mathrm{i}q\theta}$$

- 在此变换下，**拉氏量不变**，

$$\begin{aligned}\mathcal{L}'(x) &= \bar{\psi}'(x)(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi'(x) = \bar{\psi}(x)\mathrm{e}^{-\mathrm{i}q\theta}(\mathrm{i}\gamma^\mu\partial_\mu - m)\mathrm{e}^{\mathrm{i}q\theta}\psi(x)\\ &= \bar{\psi}(x)(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) = \mathcal{L}(x)\end{aligned}$$

- 容易验证，前面列举的**旋量双线性型**都在这种 **U(1) 整体变换**下**不变**
- 因此，用这些旋量双线性型构造的拉氏量都具有 **U(1) 整体对称性**

U(1) 守恒流

U(1) 整体变换的无穷小形式为

$$\psi'(x) = \psi(x) + \mathrm{i}q\theta\psi(x)$$

由于 $\delta x^\mu = 0$，$\bar{\delta}\psi = \delta\psi = \mathrm{i}q\theta\psi$

按照 $j^\mu = \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_a)}\bar{\delta}\Phi_a + \mathcal{L}\delta x^\mu$，相应的 Noether 守恒流为

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\bar{\delta}\psi = \mathrm{i}\bar{\psi}\gamma^\mu(\mathrm{i}q\theta\psi) = -q\theta\bar{\psi}\gamma^\mu\psi$$

扔掉无穷小参数 $-\theta$，定义 U(1) 守恒流算符

$$J^\mu \equiv q\bar{\psi}\gamma^\mu\psi$$

则 Noether 定理给出

$$\partial_\mu J^\mu = 0$$

U(1) 守恒荷

相应的 U(1) 守恒荷算符为

$$Q = \int d^3x\, J^0 = q\int d^3x\,\bar{\psi}\gamma^0\psi = q\int d^3x\,\psi^\dagger\psi$$

$$= q\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{k}}}}\left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}+v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right]$$

$$\times\left[u(\mathbf{k},\lambda')a_{\mathbf{k},\lambda'}e^{-ik\cdot x}+v(\mathbf{k},\lambda')b^\dagger_{\mathbf{k},\lambda'}e^{ik\cdot x}\right]$$

$$= q\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3k}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{k}}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{k})\left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{k},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{i(E_{\mathbf{p}}-E_{\mathbf{k}})t}\right.$$

$$\left.+v^\dagger(\mathbf{p},\lambda)v(\mathbf{k},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{k},\lambda'}e^{-i(E_{\mathbf{p}}-E_{\mathbf{k}})t}\right]$$

$$+\delta^{(3)}(\mathbf{p}+\mathbf{k})\left[u^\dagger(\mathbf{p},\lambda)v(\mathbf{k},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{k},\lambda'}e^{i(E_{\mathbf{p}}+E_{\mathbf{k}})t}\right.$$

$$\left.+v^\dagger(\mathbf{p},\lambda)u(\mathbf{k},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{-i(E_{\mathbf{p}}+E_{\mathbf{k}})t}\right]\Big\}$$

正粒子和反粒子

🚢 积分，得

$$
\begin{aligned}
Q &= q\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} + v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'} \\
&\qquad + u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t} + v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}\Big] \\
&= q\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\left(2E_{\mathbf{p}}\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} + 2E_{\mathbf{p}}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\right) \\
&= q\sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} + b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\right) \quad 👈 \quad \{b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\left(q\,a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - q\,b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}\right) + 2\delta^{(3)}(\mathbf{0})\int d^3p\ q \quad \text{（零点荷）}
\end{aligned}
$$

正粒子和反粒子

积分，得

$$
\begin{aligned}
Q =& \ q\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}+v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\\
&+u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}+v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}\Big]\\
=& \ q\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}(2E_{\mathbf{p}}\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}+2E_{\mathbf{p}}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'})\\
=& \ q\sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}+b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}) \quad 👈 \quad \{b_{\mathbf{p},\lambda},b^\dagger_{\mathbf{q},\lambda'}\}=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q})\\
=& \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}(q\,a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}-q\,b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda})+2\delta^{(3)}(\mathbf{0})\int d^3p\ q \quad \text{(零点荷)}
\end{aligned}
$$

从第一项可以看出，由 $(a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{p},\lambda})$ 描述的粒子是**正粒子**，携带的 **U(1) 荷**为 q

由 $(b_{\mathbf{p},\lambda},b^\dagger_{\mathbf{p},\lambda})$ 描述的粒子是**反粒子**，携带的 **U(1) 荷**为 $-q$

除去**零点荷**，总荷是所有动量模式所有螺旋度所有正反粒子贡献的 **U(1) 荷之和**

5.5.4 小节　粒子态

- 对于**自由 Dirac 旋量场**，**真空态** $|0\rangle$ 定义为被任意 $a_{\mathbf{p},\lambda}$ 和任意 $b_{\mathbf{p},\lambda}$ 湮灭的态，

$$a_{\mathbf{p},\lambda}|0\rangle = b_{\mathbf{p},\lambda}|0\rangle = 0$$

- 满足

$$\langle 0|0\rangle = 1, \quad H|0\rangle = E_{\text{vac}}|0\rangle, \quad E_{\text{vac}} = -2\delta^{(3)}(\mathbf{0})\int d^3p\, E_{\mathbf{p}}$$

- 动量为 $\mathbf{p}$、螺旋度为 λ 的**单个正粒子态**和**单个反粒子态**分别定义为

$$|\mathbf{p}^+,\lambda\rangle \equiv \sqrt{2E_{\mathbf{p}}}\, a^\dagger_{\mathbf{p},\lambda}|0\rangle, \quad |\mathbf{p}^-,\lambda\rangle \equiv \sqrt{2E_{\mathbf{p}}}\, b^\dagger_{\mathbf{p},\lambda}|0\rangle$$

5.5.4 小节 粒子态

- 对于**自由 Dirac 旋量场**，**真空态** $|0\rangle$ 定义为被任意 $a_{\mathbf{p},\lambda}$ 和任意 $b_{\mathbf{p},\lambda}$ 湮灭的态，

$$a_{\mathbf{p},\lambda}|0\rangle = b_{\mathbf{p},\lambda}|0\rangle = 0$$

- 满足

$$\langle 0|0\rangle = 1, \quad H|0\rangle = E_{\mathrm{vac}}|0\rangle, \quad E_{\mathrm{vac}} = -2\delta^{(3)}(\mathbf{0})\int \mathrm{d}^3p\, E_{\mathbf{p}}$$

- 动量为 $\mathbf{p}$、螺旋度为 λ 的**单个正粒子态**和**单个反粒子态**分别定义为

$$|\mathbf{p}^+,\lambda\rangle \equiv \sqrt{2E_{\mathbf{p}}}\, a^\dagger_{\mathbf{p},\lambda}|0\rangle, \quad |\mathbf{p}^-,\lambda\rangle \equiv \sqrt{2E_{\mathbf{p}}}\, b^\dagger_{\mathbf{p},\lambda}|0\rangle$$

- 根据**产生湮灭算符的反对易关系**，单粒子态的**内积**是

$$\begin{aligned}
\langle \mathbf{q}^+,\lambda'|\mathbf{p}^+,\lambda\rangle &= \sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}\,\langle 0|a_{\mathbf{q},\lambda'}a^\dagger_{\mathbf{p},\lambda}|0\rangle \\
&= \sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}\,\langle 0|[(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}]|0\rangle \\
&= 2E_{\mathbf{p}}(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
\langle \mathbf{q}^-,\lambda'|\mathbf{p}^-,\lambda\rangle &= \sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}\,\langle 0|b_{\mathbf{q},\lambda'}b^\dagger_{\mathbf{p},\lambda}|0\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
\langle \mathbf{q}^-,\lambda'|\mathbf{p}^+,\lambda\rangle &= \sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}\,\langle 0|b_{\mathbf{q},\lambda'}a^\dagger_{\mathbf{p},\lambda}|0\rangle = -\sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}\,\langle 0|a^\dagger_{\mathbf{p},\lambda}b_{\mathbf{q},\lambda'}|0\rangle = 0
\end{aligned}$$

单粒子态的能量本征值

根据 $Ha^\dagger_{\mathbf{p},\lambda}|E\rangle = (E+E_\mathbf{p})a^\dagger_{\mathbf{p},\lambda}|E\rangle$ 和 $Hb^\dagger_{\mathbf{p},\lambda}|E\rangle = (E+E_\mathbf{p})b^\dagger_{\mathbf{p},\lambda}|E\rangle$，有

$$H\left|\mathbf{p}^+,\lambda\right\rangle = (E_\text{vac}+E_\mathbf{p})\left|\mathbf{p}^+,\lambda\right\rangle, \quad H\left|\mathbf{p}^-,\lambda\right\rangle = (E_\text{vac}+E_\mathbf{p})\left|\mathbf{p}^-,\lambda\right\rangle$$

可见，$\left|\mathbf{p}^+,\lambda\right\rangle$ 和 $\left|\mathbf{p}^-,\lambda\right\rangle$ 都比真空态**多了一份能量** $E_\mathbf{p}=\sqrt{|\mathbf{p}|^2+m^2}$

单粒子态的能量本征值

✈️ 根据 $Ha^\dagger_{\mathbf{p},\lambda}\,|E\rangle = (E+E_{\mathbf{p}})a^\dagger_{\mathbf{p},\lambda}\,|E\rangle$ 和 $Hb^\dagger_{\mathbf{p},\lambda}\,|E\rangle = (E+E_{\mathbf{p}})b^\dagger_{\mathbf{p},\lambda}\,|E\rangle$，有

$$H\left|\mathbf{p}^+,\lambda\right\rangle = (E_{\text{vac}}+E_{\mathbf{p}})\left|\mathbf{p}^+,\lambda\right\rangle, \quad H\left|\mathbf{p}^-,\lambda\right\rangle = (E_{\text{vac}}+E_{\mathbf{p}})\left|\mathbf{p}^-,\lambda\right\rangle$$

🐊 可见，$\left|\mathbf{p}^+,\lambda\right\rangle$ 和 $\left|\mathbf{p}^-,\lambda\right\rangle$ 都比真空态**多了一份能量** $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2+m^2}$

🦖 将 $\psi(x)$ 的平面波解代入 $[\psi(x),\mathbf{J}] = (\hat{\mathbf{L}}+\boldsymbol{\mathcal{S}})\psi(x)$ **左边**，得

$$[\psi(x),\mathbf{J}] = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\left\{u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\mathbf{J}]e^{-ip\cdot x} + v(\mathbf{p},\lambda)[b^\dagger_{\mathbf{p},\lambda},\mathbf{J}]e^{ip\cdot x}\right\}$$

🦕 代入**右边**，得

$$\begin{aligned}
&(\hat{\mathbf{L}}+\boldsymbol{\mathcal{S}})\psi(x)\\
&= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}(-i\,\mathbf{x}\times\nabla+\boldsymbol{\mathcal{S}})\left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]\\
&= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\left[(\mathbf{x}\times\mathbf{p}+\boldsymbol{\mathcal{S}})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + (-\mathbf{x}\times\mathbf{p}+\boldsymbol{\mathcal{S}})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]
\end{aligned}$$

$[a_{\mathbf{p},\lambda},\, 2\,\hat{\mathbf{p}}\cdot\mathbf{J}]$ 和 $[b^\dagger_{\mathbf{p},\lambda},\, 2\,\hat{\mathbf{p}}\cdot\mathbf{J}]$

两相比较，对于动量 $\mathbf{p}$ 和螺旋度 λ，有

$$u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\mathbf{J}] = (\mathbf{x}\times\mathbf{p}+\boldsymbol{\mathcal{S}})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda},\quad v(\mathbf{p},\lambda)[b^\dagger_{\mathbf{p},\lambda},\mathbf{J}] = (-\mathbf{x}\times\mathbf{p}+\boldsymbol{\mathcal{S}})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}$$

按照前面讨论，$u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 分别是本征值为 λ 和 $-\lambda$ 的**螺旋度本征态**，故

$$\begin{aligned} u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\, 2\,\hat{\mathbf{p}}\cdot\mathbf{J}] &= 2\,\hat{\mathbf{p}}\cdot(\mathbf{x}\times\mathbf{p}+\boldsymbol{\mathcal{S}})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda} \\ &= (2\,\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{S}})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda} = \lambda\, u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda} \end{aligned}$$

$[a_{\mathbf{p},\lambda},\, 2\,\hat{\mathbf{p}}\cdot\mathbf{J}]$ 和 $[b^\dagger_{\mathbf{p},\lambda},\, 2\,\hat{\mathbf{p}}\cdot\mathbf{J}]$

两相比较，对于动量 $\mathbf{p}$ 和螺旋度 λ，有

$$u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\mathbf{J}] = (\mathbf{x}\times\mathbf{p}+\boldsymbol{\mathcal{S}})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda},\quad v(\mathbf{p},\lambda)[b^\dagger_{\mathbf{p},\lambda},\mathbf{J}] = (-\mathbf{x}\times\mathbf{p}+\boldsymbol{\mathcal{S}})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}$$

按照前面讨论，$u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 分别是本征值为 λ 和 $-\lambda$ 的**螺旋度本征态**，故

$$\begin{aligned}
u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\, 2\,\hat{\mathbf{p}}\cdot\mathbf{J}] &= 2\,\hat{\mathbf{p}}\cdot(\mathbf{x}\times\mathbf{p}+\boldsymbol{\mathcal{S}})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\\
&= (2\,\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{S}})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda} = \lambda\,u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\\
v(\mathbf{p},\lambda)[b^\dagger_{\mathbf{p},\lambda},\, 2\,\hat{\mathbf{p}}\cdot\mathbf{J}] &= 2\,\hat{\mathbf{p}}\cdot(-\mathbf{x}\times\mathbf{p}+\boldsymbol{\mathcal{S}})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\\
&= (2\,\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{S}})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda} = -\lambda\,v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}
\end{aligned}$$

因而

$$[a_{\mathbf{p},\lambda},\, 2\,\hat{\mathbf{p}}\cdot\mathbf{J}] = \lambda\,a_{\mathbf{p},\lambda},\quad [b^\dagger_{\mathbf{p},\lambda},\, 2\,\hat{\mathbf{p}}\cdot\mathbf{J}] = -\lambda\,b^\dagger_{\mathbf{p},\lambda}$$

由于 $\mathbf{J}$ 是**厄米算符**，对第一式取**厄米共轭**得

$$\lambda\,a^\dagger_{\mathbf{p},\lambda} = [a_{\mathbf{p},\lambda},\, 2\,\hat{\mathbf{p}}\cdot\mathbf{J}]^\dagger = (2\,\hat{\mathbf{p}}\cdot\mathbf{J})a^\dagger_{\mathbf{p},\lambda} - a^\dagger_{\mathbf{p},\lambda}(2\,\hat{\mathbf{p}}\cdot\mathbf{J}) = [2\,\hat{\mathbf{p}}\cdot\mathbf{J},\, a^\dagger_{\mathbf{p},\lambda}]$$

单粒子态的螺旋度本征值

- 于是，$[2\,\hat{\mathbf{p}}\cdot\mathbf{J}, a^\dagger_{\mathbf{p},\lambda}] = \lambda\, a^\dagger_{\mathbf{p},\lambda}$，$[2\,\hat{\mathbf{p}}\cdot\mathbf{J}, b^\dagger_{\mathbf{p},\lambda}] = \lambda\, b^\dagger_{\mathbf{p},\lambda}$

- $\mathbf{J}$ 是**总角动量算符**，真空态 $|0\rangle$ 满足 $\mathbf{J}\,|0\rangle = \mathbf{0}$，由此得到

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{J})a^\dagger_{\mathbf{p},\lambda}\,|0\rangle = [a^\dagger_{\mathbf{p},\lambda}(2\,\hat{\mathbf{p}}\cdot\mathbf{J}) + \lambda\, a^\dagger_{\mathbf{p},\lambda}]\,|0\rangle = \lambda\, a^\dagger_{\mathbf{p},\lambda}\,|0\rangle$$

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{J})b^\dagger_{\mathbf{p},\lambda}\,|0\rangle = [b^\dagger_{\mathbf{p},\lambda}(2\,\hat{\mathbf{p}}\cdot\mathbf{J}) + \lambda\, b^\dagger_{\mathbf{p},\lambda}]\,|0\rangle = \lambda\, b^\dagger_{\mathbf{p},\lambda}\,|0\rangle$$

- 自由的单粒子态**没有轨道角动量**，而 $2\,\hat{\mathbf{p}}\cdot\mathbf{J}$ 相当于**归一化**的**螺旋度算符**

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(2\,\hat{\mathbf{p}}\cdot\mathbf{J})b^\dagger_{\mathbf{p},\lambda}\,|0\rangle &= [b^\dagger_{\mathbf{p},\lambda}(2\,\hat{\mathbf{p}}\cdot\mathbf{J}) + \lambda\, b^\dagger_{\mathbf{p},\lambda}]\,|0\rangle = \lambda\, b^\dagger_{\mathbf{p},\lambda}\,|0\rangle
\end{aligned}
$$

- 自由的单粒子态没有轨道角动量，而 $2\,\hat{\mathbf{p}}\cdot\mathbf{J}$ 相当于归一化的螺旋度算符
- 因此，上面两式说明 $|\mathbf{p}^+,\lambda\rangle$ 和 $|\mathbf{p}^-,\lambda\rangle$ 都是螺旋度本征态，本征值为 λ：

$$
(2\,\hat{\mathbf{p}}\cdot\mathbf{J})\,|\mathbf{p}^\pm,\lambda\rangle = \lambda\,|\mathbf{p}^\pm,\lambda\rangle
$$

- 以上讨论表明，产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 的作用是产生一个动量为 $\mathbf{p}$、螺旋度为 λ 的正粒子
- 另一种产生算符 $b^\dagger_{\mathbf{p},\lambda}$ 的作用是产生一个动量为 $\mathbf{p}$、螺旋度为 λ 的反粒子
- 正粒子和反粒子具有相同的质量 m

湮灭算符的作用

- 在 $\tilde{f}_\lambda(\mathbf{p}) = \tilde{C}_\lambda\, \xi_{-\lambda}(\mathbf{p})$ 中，我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ **正比于** $\xi_{-\lambda}(\mathbf{p})$，使得 $v(\mathbf{p},\lambda)$ 的**螺旋度本征值**为 $-\lambda$，从而得到 $b^\dagger_{\mathbf{p},\lambda}\,|0\rangle$ 的**螺旋度本征值**为 λ 的结果
- 如果我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ **正比于** $\xi_\lambda(\mathbf{p})$，依照上述推导，$b^\dagger_{\mathbf{p},\lambda}\,|0\rangle$ 的**螺旋度本征值**就会变成 $-\lambda$，则 $(b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{p},\lambda})$ 将描述**螺旋度为** $-\lambda$ 的**反粒子**
- 这**不符合**我们的记号，因此，我们将 $\tilde{f}_\lambda(\mathbf{p})$ 取为 $\tilde{f}_\lambda(\mathbf{p}) = \tilde{C}_\lambda\, \xi_{-\lambda}(\mathbf{p})$

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- 由**产生湮灭算符的反对易关系**，有

$$
\begin{aligned}
a_{\mathbf{p},\lambda}\,|\mathbf{q}^+,\lambda'\rangle &= \sqrt{2E_\mathbf{q}}\, a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}\,|0\rangle = \sqrt{2E_\mathbf{q}}\,[(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - a^\dagger_{\mathbf{q},\lambda'}a_{\mathbf{p},\lambda}]\,|0\rangle \\
&= \sqrt{2E_\mathbf{q}}\,(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q})\,|0\rangle \\
b_{\mathbf{p},\lambda}\,|\mathbf{q}^-,\lambda'\rangle &= \sqrt{2E_\mathbf{q}}\, b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}\,|0\rangle = \sqrt{2E_\mathbf{q}}\,[(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - b^\dagger_{\mathbf{q},\lambda'}b_{\mathbf{p},\lambda}]\,|0\rangle \\
&= \sqrt{2E_\mathbf{q}}\,(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q})\,|0\rangle
\end{aligned}
$$

- 可以看出，**湮灭算符** $a_{\mathbf{p},\lambda}$ 的作用是**减少**一个动量为 $\mathbf{p}$、螺旋度为 λ 的**正粒子**
- **湮灭算符** $b_{\mathbf{p},\lambda}$ 的作用是**减少**一个动量为 $\mathbf{p}$、螺旋度为 λ 的**反粒子**

粒子交换

将包含 2 个正粒子和 2 个反粒子的态记为

$$|\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2; \mathbf{p}_3^-, \lambda_3; \mathbf{p}_4^-, \lambda_4\rangle \equiv \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{p}_3}E_{\mathbf{p}_4}}\, a^\dagger_{\mathbf{p}_1,\lambda_1} a^\dagger_{\mathbf{p}_2,\lambda_2} b^\dagger_{\mathbf{p}_3,\lambda_3} b^\dagger_{\mathbf{p}_4,\lambda_4} |0\rangle$$

多次利用反对易关系 $\{a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}\} = \{b^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = \{a^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = 0$

调换产生算符的位置，可得

$$a^\dagger_{\mathbf{p}_1,\lambda_1} a^\dagger_{\mathbf{p}_2,\lambda_2} b^\dagger_{\mathbf{p}_3,\lambda_3} b^\dagger_{\mathbf{p}_4,\lambda_4} |0\rangle = -b^\dagger_{\mathbf{p}_4,\lambda_4} a^\dagger_{\mathbf{p}_2,\lambda_2} b^\dagger_{\mathbf{p}_3,\lambda_3} a^\dagger_{\mathbf{p}_1,\lambda_1} |0\rangle$$

负号源自奇数次反对易，从而

$$|\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2; \mathbf{p}_3^-, \lambda_3; \mathbf{p}_4^-, \lambda_4\rangle = -\,|\mathbf{p}_4^-, \lambda_4; \mathbf{p}_2^+, \lambda_2; \mathbf{p}_3^-, \lambda_3; \mathbf{p}_1^+, \lambda_1\rangle$$

即交换第 1 和第 4 个粒子得到的态与原来的态相差一个负号

同理，交换其中任意两个粒子，也会出现一个负号

费米子与 Pauli 不相容原理

- 一般地，对于多个全同粒子的态，**交换**任意两个**全同粒子**，需要对**产生算符**进行**奇数次反对易**，得到的态与原态相差一个**负号**
- 也就是说，态对**全同粒子交换**是**反对称**的
- 这说明 **Dirac 旋量场**描述的粒子是一种**费米子**，称为 **Dirac 费米子**，服从 **Fermi-Dirac 统计**
- 得到这个结论的关键在于两个**产生算符反对易**

Enrico Fermi
(1901–1954)

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这说明 Dirac 旋量场描述的粒子是一种费米子，称为 Dirac 费米子，服从 Fermi-Dirac 统计

得到这个结论的关键在于两个产生算符反对易

对于两个相同的产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 或 $b^\dagger_{\mathbf{p},\lambda}$，反对易关系导致

$$a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle = -a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle, \quad b^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle = -b^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle$$

故

$$a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle = 0, \quad b^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle = 0$$

没有其它自由度时，不存在动量和螺旋度都相同的两个正费米子或两个反费米子组成的态，这符合 Pauli 不相容原理

Enrico Fermi
(1901–1954)

Wolfgang Ernst Pauli
(1900–1958)

自旋–统计定理

- 在第 2 章和第 4 章中，我们分别讨论了自旋为 0 的标量场和自旋为 1 的矢量场，合适的处理方式是通过对易关系对它们进行量子化，因而它们都描述玻色子
- 在本章中，我们需要采用反对易关系才能对自旋为 1/2 的旋量场进行合适的量子化，因而旋量场描述的粒子是费米子
- 实际上，这样的状况是普遍的，存在下列自旋–统计定理

自旋–统计定理

- 整数自旋的物理场必须用对易关系进行量子化，对应的粒子是玻色子
- 半奇数自旋的物理场必须用反对易关系进行量子化，对应的粒子是费米子

自旋—统计定理

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可从多个角度证明这个定理成立，前面已说明哈密顿量算符的正定性要求它成立

此外，也可以从交换全同粒子的路径依赖性、散射矩阵的 Lorentz 不变性、因果性的角度加以证明（详细讨论见 M. D. Schwartz 的书 *Quantum Field Theory and the Standard Model* 第 12 章）

双粒子态内积

- 记两个正费米子组成的双粒子态为 $|\mathbf{p}_1^+,\lambda_1;\,\mathbf{p}_2^+,\lambda_2\rangle \equiv \sqrt{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}\,a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}|0\rangle$
- 双粒子态的内积关系是

$$
\begin{aligned}
&\langle\mathbf{q}_1^+,\lambda_1';\,\mathbf{q}_2^+,\lambda_2'\,|\,\mathbf{p}_1^+,\lambda_1;\,\mathbf{p}_2^+,\lambda_2\rangle\\
&=\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\langle 0|\,a_{\mathbf{q}_2,\lambda_2'}a_{\mathbf{q}_1,\lambda_1'}a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}\,|0\rangle\\
&=\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\Big[(2\pi)^3\delta_{\lambda_1\lambda_1'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\,\langle 0|\,a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_2,\lambda_2}\,|0\rangle\\
&\qquad\qquad -\langle 0|\,a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_1,\lambda_1}a_{\mathbf{q}_1,\lambda_1'}a^\dagger_{\mathbf{p}_2,\lambda_2}\,|0\rangle\Big]\\
&=\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\Big[(2\pi)^3\delta_{\lambda_1\lambda_1'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\,\langle 0|\,a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_2,\lambda_2}\,|0\rangle\\
&\qquad\qquad -(2\pi)^3\delta_{\lambda_2\lambda_1'}\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\,\langle 0|\,a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_1,\lambda_1}\,|0\rangle\Big]\\
&=4E_{\mathbf{p}_1}E_{\mathbf{p}_2}(2\pi)^6\,\big[\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2)\\
&\qquad\qquad -\delta_{\lambda_1\lambda_2'}\delta_{\lambda_2\lambda_1'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\big]
\end{aligned}
$$

- 最后两行方括号中第二项前面有一个负号，由产生湮灭算符的反对易关系引起
- 这是双费米子态内积关系与双玻色子态内积关系在形式上的不同之处