

# 量子场论

## 第7章 Feynman图

### 7.1节和7.2节

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第 7 章 Feynman 图

上一章告诉我们，为了预言像 **散射截面**  $\sigma = \frac{1}{\mathcal{S}} \frac{1}{4E_{\mathcal{A}} E_{\mathcal{B}} v_{\text{Mol}}} \int d\Pi_n |\mathcal{M}|^2$  和 **衰变宽度**  $\Gamma = \frac{1}{\mathcal{S}} \frac{1}{2m_{\mathcal{A}}} \int d\Pi_n |\mathcal{M}|^2$  这样的实验观测量，需要计算 **不变振幅**  $i\mathcal{M}$ ，它来自  **$S$  矩阵元的相互作用部分**  $iT_{fi} = \langle f | iT | i \rangle = (2\pi)^4 \delta^{(4)}(p_i - p_f) i\mathcal{M}$

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对于  $S$  矩阵元所涉及的场算符的时序乘积, [Wick 定理](#)提供了处理方法

对微扰论某一阶应用 Wick 定理, 能够得到散射振幅的表达式, 而且相应的相互作用过程可以用 Feynman 图 (diagram) 表示出来

 Feynman 图上的元素具有对应的表达式，这种对应就是 **Feynman 规则 (rule)**



Richard Feynman  
(1918–1988)

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宽度  $\Gamma = \frac{1}{S} \frac{1}{2m_A} \int d\Pi_n |\mathcal{M}|^2$  这样的实验观测量，需要计算**不变振幅**  $i\mathcal{M}$ ，它来自

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使用 **Feynman** 图方法可以大大地简化计算过程。

## 将一个相互作用理论的 Feynman 规则归纳出来

绕开应用 Wick 定理的繁琐计算，对于特定相互作用过程直接画出 Feynman 图

依照 Feynman 规则写出散射振幅表达式



Richard Feynman  
(1918–1988)

## 7.1 节 Yukawa 理论

实标量场  $\phi(x)$  和 Dirac 旋量场  $\psi(x)$  的 Yukawa 理论拉氏量  $\mathcal{L} = \mathcal{L}_S + \mathcal{L}_D + \mathcal{L}_Y$

$$\mathcal{L}_S = \frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi - \frac{1}{2}m_\phi^2\phi^2, \quad \mathcal{L}_D = i\bar{\psi}\gamma^\mu\partial_\mu \psi - m_\psi\bar{\psi}\psi, \quad \mathcal{L}_Y = -\kappa\phi\bar{\psi}\psi$$

由于  $\mathcal{L}_Y$  不包含场的时空导数，场的共轭动量密度不受  $\mathcal{L}_Y$  影响

👉 相互作用哈密顿量密度  $\mathcal{H}_1(x) = -\mathcal{L}_Y(x) = \kappa \phi(x) \bar{\psi}(x) \psi(x)$

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🍉  $S$  算符表达为  $S = \mathbb{I} + iT = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \mathsf{T}[\mathcal{H}_1(x_1) \cdots \mathcal{H}_1(x_n)]$

$$\text{👉 } iT = \sum_{n=1}^{\infty} iT^{(n)}, \quad iT^{(n)} \equiv \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \mathsf{T}[\mathcal{H}_1(x_1) \cdots \mathcal{H}_1(x_n)]$$

🎀 这是  $iT$  在微扰论中的级数展开式,  $n$  是展开式的阶 (order)

在 Yukawa 理论中, 第  $n$  阶通项  $iT^{(n)}$  的表达式为

$$iT^{(n)} = \frac{(-i\kappa)^n}{n!} \int d^4x_1 \cdots d^4x_n \mathsf{T}[\phi(x_1)\bar{\psi}(x_1)\psi(x_1) \cdots \phi(x_n)\bar{\psi}(x_n)\psi(x_n)]$$

当耦合常数  $\kappa$  比较小时, 计算最低阶就能够得到比较精确的结果



## 应用 Wick 定理



在  $iT$  展开式的第 1 阶, 即  $\kappa^1$  阶, 根据 Wick 定理, 有

$$iT^{(1)} = -i\kappa \int d^4x \mathsf{T}[\phi(x)\bar{\psi}(x)\psi(x)] = -i\kappa \int d^4x \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x) + \phi(x)\overline{\bar{\psi}(x)\psi(x)}]$$



实标量场  $\phi(x)$  和 Dirac 旋量场  $\psi(x)$  具有不相关的产生湮灭算符



$$\phi(x)\bar{\psi}(x) = \phi(x)\psi(x) = 0$$



非平庸的场缩并只有  $\phi(x)\bar{\psi}(x)\psi(x)$



注意  $\langle 0 | N \text{ (产生湮灭算符的乘积) } | 0 \rangle = 0$



为了得到非零的  $T$  矩阵元  $\langle f | iT | i \rangle$ ，初态  $|i\rangle$  和末态  $|f\rangle$  应当包含适当类型和数的产生湮灭算符，使它们刚好能够与场算符——发生缩并（定义见下文）



## 粒子态

引入三种具有确定动量和螺旋度的单粒子态：

Dirac 旋量场  $\psi(x)$  的正费米子态  $|p^+, \lambda\rangle = \sqrt{2E_p} a_{p^+, \lambda}^\dagger |0\rangle$

Dirac 旋量场  $\psi(x)$  的反费米子态  $|p^-, \lambda\rangle = \sqrt{2E_p} b_{p, \lambda}^\dagger |0\rangle$

实标量场  $\phi(x)$  的玻色子态  $|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} c_{\mathbf{p}}^\dagger |0\rangle$

为避免混淆, 此处将  $\phi(x)$  的产生算符改记为  $c_p^\dagger$

正反粒子不同时，动量的右上角用正号代表正粒子态，负号代表反粒子态

这些右矢可以单独作为初态，相应的左矢可以单独作为末态

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这些右矢可以单独作为初态，相应的左矢可以单独作为末态

 对真空态作用多个产生算符，就得到包含多个粒子的初态

比如,  $|\mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k}\rangle = \sqrt{8E_p E_q E_k} a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'}^\dagger c_{\mathbf{k}}^\dagger |0\rangle$  描述的初态包含

1 个动量为  $p$ 、螺旋度为  $\lambda$  的 Dirac 正费米子  $\psi$

1 个动量为  $q$ 、螺旋度为  $\lambda'$  的 Dirac 反费米子  $\psi$

1 个动量为  $k$  的实标量玻色子  $\phi$

这里用  $\psi$ 、 $\bar{\psi}$  和  $\phi$  分别作为正费米子、反费米子和实标量玻色子的粒子名称

## 粒子态

另一方面, 初态  $|p^+, \lambda; q^-, \lambda'; k\rangle = \sqrt{8E_p E_q E_k} a_{p, \lambda}^\dagger b_{q, \lambda'}^\dagger c_k^\dagger |0\rangle$  对应的末态是

$$\langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | = \sqrt{8E_{\mathbf{p}}E_{\mathbf{q}}E_{\mathbf{k}}} \langle 0 | a_{\mathbf{p},\lambda} b_{\mathbf{q},\lambda'} c_{\mathbf{k}}$$

这个末态也包含 1 个动量为  $p$ 、螺旋度为  $\lambda$  的 Dirac 正费米子  $\psi$

1 个动量为  $q$ 、螺旋度为  $\lambda'$  的 Dirac 反费米子  $\psi$

1 个动量为  $k$  的实标量玻色子  $\phi$

 注意，在这两个初末态的表达式中，特意让态矢符号中的动量排列次序与相应产生湮灭算符的排列次序相同，使得下文在表示场算符与初末态缩并方面比较方便

这种约定使末态记法与第二章和第四章中关于双粒子态的记法有所不同，对双费米子态实际上相差一个负号，但不会引起物理本质上的差异

## 场算符与初态的缩并

现在，利用旋量场和实标量场的正负能解展开式讨论场算符与初末态的非零缩并

场算符的正能解部分包含湮灭算符，它对具有相应产生算符的初态作用非零

我们将这种作用定义为场算符与初态的缩并

$\psi(x)$  与正费米子初态的缩并定义为

$$\begin{aligned}
& \psi_a(x) \left| \mathbf{p}^+, \lambda \right\rangle \equiv \psi_a^{(+)}(x) \left| \mathbf{p}^+, \lambda \right\rangle \\
&= \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} u_a(\mathbf{q}, \lambda') \color{red} a_{\mathbf{q}, \lambda'} e^{-i\mathbf{q} \cdot \mathbf{x}} \sqrt{2E_{\mathbf{p}}} \color{red} a_{\mathbf{p}, \lambda}^\dagger |0\rangle \\
&= \int \frac{d^3 q}{(2\pi)^3} \frac{\sqrt{E_{\mathbf{p}}}}{\sqrt{E_{\mathbf{q}}}} \sum_{\lambda'=\pm} u_a(\mathbf{q}, \lambda') e^{-i\mathbf{q} \cdot \mathbf{x}} \{ a_{\mathbf{q}, \lambda'}, a_{\mathbf{p}, \lambda}^\dagger \} |0\rangle \\
&= \int d^3 q \frac{\sqrt{E_{\mathbf{p}}}}{\sqrt{E_{\mathbf{q}}}} \sum_{\lambda'=\pm} u_a(\mathbf{q}, \lambda') e^{-i\mathbf{q} \cdot \mathbf{x}} \color{brown} \delta_{\lambda' \lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) \color{black} |0\rangle = \color{blue} u_a(\mathbf{p}, \lambda) e^{-i\mathbf{p} \cdot \mathbf{x}} |0\rangle
\end{aligned}$$

## 第四步用到产生湮灭算符的反对易关系

## 场算符与初态的缩并



$\bar{\psi}(x)$  与反费米子初态的缩并定义为

$$\begin{aligned} & \bar{\psi}_a(x) | \mathbf{p}^-, \lambda \rangle \equiv \bar{\psi}_a^{(+)}(x) | \mathbf{p}^-, \lambda \rangle \\ & \equiv \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \bar{v}_a(\mathbf{q}, \lambda') \color{red} b_{\mathbf{q}, \lambda'} e^{-iq \cdot x} \sqrt{2E_{\mathbf{p}}} b_{\mathbf{p}, \lambda}^\dagger | 0 \rangle \equiv \bar{v}_a(\mathbf{p}, \lambda) e^{-ip \cdot x} | 0 \rangle \end{aligned}$$



$\phi(x)$  与实标量玻色子初态的缩并定义为

$$\begin{aligned} \overline{\phi(x)|\mathbf{p}\rangle} &\equiv \phi^{(+)}(x)|\mathbf{p}\rangle = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{q}}}} c_{\mathbf{q}} e^{-iq \cdot x} \sqrt{2E_{\mathbf{p}}} c_{\mathbf{p}}^\dagger |0\rangle \\ &= \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{E_{\mathbf{p}}}}{\sqrt{E_{\mathbf{q}}}} e^{-iq \cdot x} [c_{\mathbf{q}}, c_{\mathbf{p}}^\dagger] |0\rangle = \int d^3q \frac{\sqrt{E_{\mathbf{p}}}}{\sqrt{E_{\mathbf{q}}}} e^{-iq \cdot x} \delta^{(3)}(\mathbf{q} - \mathbf{p}) |0\rangle = e^{-ip \cdot x} |0\rangle \end{aligned}$$



#### 第四步用到产生湮灭算符的对易关系



这三种初态缩并均包含正能解因子  $e^{-ip \cdot x}$

## 场算符与末态的缩并

场算符的负能解部分包含产生算符，它对具有相应湮灭算符的末态作用非零

 我们将这种作用定义为**场算符与末态的缩并**

$\bar{\psi}(x)$  与正费米子末态的缩并定义为

$$\begin{aligned}
& \langle \overline{\mathbf{p}^+}, \lambda | \bar{\psi}_a(x) \equiv \langle \mathbf{p}^+, \lambda | \bar{\psi}_a^{(-)}(x) \\
&= \int \frac{d^3q}{(2\pi)^3} \langle 0 | \sqrt{2E_{\mathbf{p}}} \, a_{\mathbf{p},\lambda} \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \bar{u}_a(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}} \\
&= \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{E_{\mathbf{p}}}}{\sqrt{E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \bar{u}_a(\mathbf{q}, \lambda') \langle 0 | \{ a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger \} e^{i\mathbf{q}\cdot\mathbf{x}} = \langle 0 | \bar{u}_a(\mathbf{p}, \lambda) e^{i\mathbf{p}\cdot\mathbf{x}}
\end{aligned}$$

$\psi(x)$  与反费米子末态的缩并定义为

$$\begin{aligned} & \langle \overline{\mathbf{p}^-}, \lambda | \psi_a(x) \equiv \langle \mathbf{p}^- , \lambda | \psi_a^{(-)}(x) \\ &= \int \frac{d^3 q}{(2\pi)^3} \langle 0 | \sqrt{2E_{\mathbf{p}}} b_{\mathbf{p}, \lambda} \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} v_a(\mathbf{q}, \lambda') b_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} = \langle 0 | v_a(\mathbf{p}, \lambda) e^{ip \cdot x} \end{aligned}$$

## 场算符与末态的缩并

$\phi(x)$  与实标量玻色子末态的缩并定义为

$$\langle \mathbf{p} | \phi(x) \equiv \langle \mathbf{p} | \phi^{(-)}(x) = \int \frac{d^3q}{(2\pi)^3} \langle 0 | \sqrt{2E_{\mathbf{p}}} \, c_{\mathbf{p}} \frac{1}{\sqrt{2E_{\mathbf{q}}}} c_{\mathbf{q}}^\dagger e^{iq \cdot x}$$

$$= \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{E_{\mathbf{p}}}}{\sqrt{E_{\mathbf{q}}}} e^{iq \cdot x} \langle 0 | [c_{\mathbf{p}}, c_{\mathbf{q}}^\dagger] = \int d^3q \frac{\sqrt{E_{\mathbf{p}}}}{\sqrt{E_{\mathbf{q}}}} e^{iq \cdot x} \langle 0 | \delta^{(3)}(\mathbf{q} - \mathbf{p}) = \langle 0 | e^{ip \cdot x}$$

这三种末态缩并均包含负能解因子  $e^{ip \cdot x}$



### 7.1.1 小节 $\kappa^1$ 阶 Feynman 图

将上述  $iT^{(1)}$  分为两项,  $iT^{(1)} = iT_1^{(1)} + iT_2^{(1)}$ , 其中

$$iT_1^{(1)} \equiv -i\kappa \int d^4x \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)]$$

$$i T_2^{(1)} \equiv -i\kappa \int d^4x \, N[\phi(x) \bar{\psi}(x) \psi(x)]$$

我们先来讨论  $iT_1^{(1)}$ ，要得到非平庸的  $T$  矩阵元  $\langle f | iT_1^{(1)} | i \rangle$ ，初态和末态中需要包含 3 个粒子，可列出 8 种情况

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第 1 种情况下, 考虑初态  $|i\rangle = |\mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k}\rangle$  包含 1 对正反费米子和 1 个实标量玻色子, 末态  $\langle f| = \langle 0|$  是真空态, 相应的  $T$  矩阵元为

$$\begin{aligned} \langle 0 | iT_1^{(1)} | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle &= -i\kappa \int d^4x \langle 0 | \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \\ &= -i\kappa \int d^4x \langle 0 | \phi^{(+)}(x)\bar{\psi}_a^{(+)}(x)\psi_a^{(+)}(x) | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \end{aligned}$$

 第二步将场算符分解为正负能解部分，本来应该有 8 项，但只有 1 项贡献非零

## iT<sub>1</sub><sup>(1)</sup> 第 1 种情况

$$\begin{aligned}
\langle 0 | iT_1^{(1)} | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle &= -i\kappa \int d^4x \langle 0 | \phi^{(+)}(x) \bar{\psi}_a^{(+)}(x) \psi_a^{(+)}(x) | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \\
&= -i\kappa \int d^4x \langle 0 | \mathbf{N}[\phi(x) \bar{\psi}_a(x) \psi_a(x)] | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \\
&= -i\kappa \int d^4x \langle 0 | e^{-ik \cdot x} \bar{v}_a(\mathbf{q}, \lambda') e^{-iq \cdot x} u_a(\mathbf{p}, \lambda) e^{-ip \cdot x} | 0 \rangle \\
&= -i\kappa \int d^4x \bar{v}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda) e^{-i(p+q+k) \cdot x} \langle 0 | 0 \rangle \\
&= -i\kappa \bar{v}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p+q+k)
\end{aligned}$$

## 第二步用到场算符与初态缩并的定义

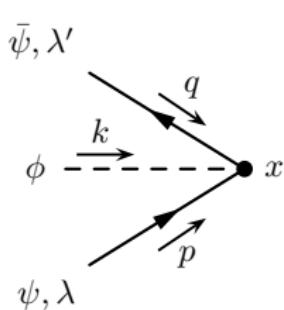
火山 最后一步用到  $\langle 0|0 \rangle = 1$  和 Fourier 变换公式, 得出体现能动量守恒的四维  $\delta$  函数

此处，对时空坐标  $x^\mu$  积分意味着将所有时空点的贡献叠加起来

这个结果符合  $\langle f | iT | i \rangle = (2\pi)^4 \delta^{(4)}(p_i - p_f) i\mathcal{M}$  的形式，相应的**不变振幅**为

$$i\mathcal{M} = -i\kappa \bar{v}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda)$$

Feynman 10



左边用图形表示  $\langle 0 | iT_1^{(1)} | p^+, \lambda; q^-, \lambda'; k \rangle$  对应的过程

这种图形化表示称为 **Feynman 图**

时间方向自左向右

用虚线表示实标量玻色子的运动

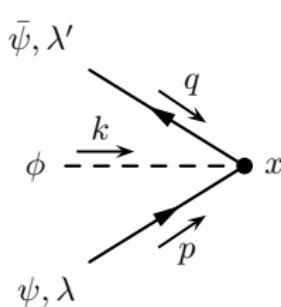
用带箭头的实线表示 Dirac 费米子的运动

图上用箭头标明三个粒子的四维动量  $p^\mu$ 、 $q^\mu$  和  $k^\mu$  的方向

这只是示意性的，不用精确对应于三维空间中三维动量的实际方向

可以认为这些四维动量的相反数  $-p^\mu$ 、 $-q^\mu$  和  $-k^\mu$  的方向与图上方向相反

Feynman 



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用带箭头的实线表示 Dirac 费米子的运动

图上用箭头标明三个粒子的四维动量  $p^\mu$ 、 $q^\mu$  和  $k^\mu$  的方向

这只是示意性的，不用精确对应于三维空间中三维动量的实际方向

可以认为这些四维动量的相反数  $-p^\mu$ 、 $-q^\mu$  和  $-k^\mu$  的方向与图上方向相反

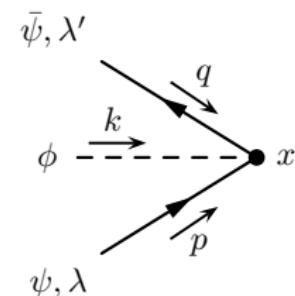
费米子线上的箭头代表费米子线的方向，可以认为这是某种 **U(1)** 荷（比如电荷）流动的方向，或者说 **费米子数** (fermion number) 流动的方向

一条连续费米子线只有一个方向，它与正(反)费米子的运动方向相同(相反)

👉 正费米子的动量方向与费米子线上的箭头方向相同，反费米子则相反

实标量场  $\phi(x)$  描述的玻色子是自身的反粒子，不具有任何  $U(1)$  荷，因而不需要在线上标注箭头，即纯中性粒子的线上没有箭头

正反粒子不同（即复场）时，应当在粒子线上标注箭头



## Feynman 图和 Feynman 规则

实标量场  $\phi(x)$  描述的玻色子是自身的反粒子，不具有任何  $U(1)$  荷，因而不需要在线上标注箭头，即纯中性粒子的线上没有箭头

正反粒子不同（即复场）时，应当在粒子线上标注箭头

三条粒子线相交代表相互作用的发生，称为**顶点** (vertex)

在顶点附近注明相应的时空坐标, 右上图标注了  $x^k$

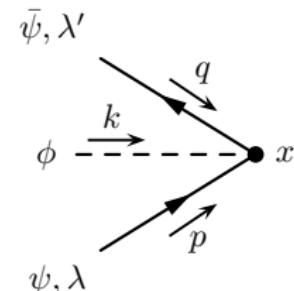
从顶点到初末态粒子的连线称为外线 (external line)，也称为外腿 (external leg)

右上图包含 1 个顶点和 3 条外线

● Feynman 图清晰地体现了运动情况和相互作用过程

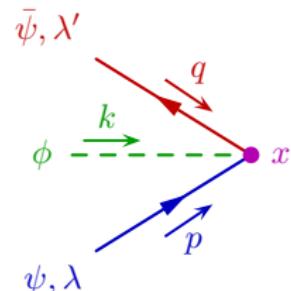
可以让 Feynman 图的每个部分对应于一个代数表达式，将这些表达式拼接起来，就得到  $T$  矩阵元  $\langle f | iT | i \rangle$  的表达式

这样的对应关系形成一套 Feynman 规则



## 入射外线 Feynman 规则

$$\begin{aligned}
& \langle 0 | iT_1^{(1)} | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \\
&= -i\kappa \int d^4x \langle 0 | \mathsf{N}[\phi(x) \bar{\psi}_a(x) \psi_a(x)] | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \\
&= -i\kappa \int d^4x e^{-ik \cdot x} \bar{v}(\mathbf{q}, \lambda') e^{-iq \cdot x} u(\mathbf{p}, \lambda) e^{-ip \cdot x}
\end{aligned}$$



三条外线分别对应于场算符  $\phi(x)$ 、 $\bar{\psi}(x)$ 、 $\psi(x)$  与初态的缩并，从而归纳出如下**位置空间中的入射外线 Feynman 规则**

$$\begin{aligned}
 \psi, \lambda \xrightarrow[p]{\bullet} x &= \langle 0 | \overline{\psi(x)} | \mathbf{p}^+, \lambda \rangle = \langle 0 | \psi^{(+)}(x) | \mathbf{p}^+, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i\mathbf{p} \cdot x} \\
 \bar{\psi}, \lambda \xrightarrow[p]{\bullet} x &= \langle 0 | \overline{\bar{\psi}(x)} | \mathbf{p}^-, \lambda \rangle = \langle 0 | \bar{\psi}^{(+)}(x) | \mathbf{p}^-, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i\mathbf{p} \cdot x} \\
 \phi \xrightarrow[p]{\bullet} x &= \langle 0 | \overline{\phi(x)} | \mathbf{p} \rangle = \langle 0 | \phi^{(+)}(x) | \mathbf{p} \rangle = e^{-i\mathbf{p} \cdot x}
 \end{aligned}$$

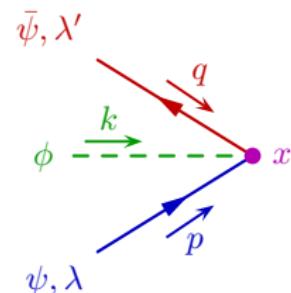
由于正费米子动量方向与线上方向相同，我们可以省略标明动量方向的箭头

反费米子动量方向与线上方向相反，因而将两个箭头都标示出来

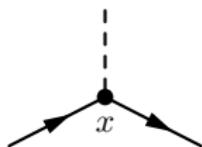
也就是说，如果没有标明动量的方向，则它与粒子线上的方向相同

## 顶点 Feynman 规则

$$\begin{aligned}
& \langle 0 | i T_1^{(1)} | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \\
&= -i\kappa \int d^4x \langle 0 | \mathsf{N}[\phi(x) \bar{\psi}_a(x) \psi_a(x)] | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \\
&= -i\kappa \int d^4x e^{-ik \cdot x} \bar{v}(\mathbf{q}, \lambda') e^{-iq \cdot x} u(\mathbf{p}, \lambda) e^{-ip \cdot x}
\end{aligned}$$



位置空间中 Yukawa 相互作用的顶点 Feynman 规则为

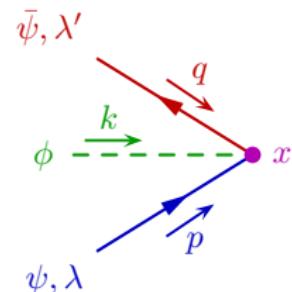


$$= -i\kappa \int d^4x$$

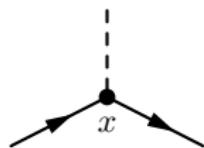
注意，左图从顶点引出来的三条线不是外线，而是三个“线头”，用于连接相应类型的外线或下文引入的内线

## 顶点 Feynman 规则

$$\begin{aligned}
& \langle 0 | iT_1^{(1)} | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \\
&= -i\kappa \int d^4x \langle 0 | \mathsf{N}[\phi(x) \bar{\psi}_a(x) \psi_a(x)] | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \\
&= -i\kappa \int d^4x e^{-ik \cdot x} \bar{v}(\mathbf{q}, \lambda') e^{-iq \cdot x} u(\mathbf{p}, \lambda) e^{-ip \cdot x}
\end{aligned}$$



位置空间中 Yukawa 相互作用的顶点 Feynman 规则为



$$= -i\kappa \int d^4x$$

注意，左图从顶点引出来的三条线不是外线，而是三个“线头”，用于连接相应类型的外线或下文引入的内线



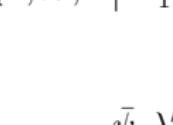
现在可以绕过 Wick 定理，从 Feynman 图出发，根据 Feynman 规则直接写出

$$\langle 0 | iT_1^{(1)} | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle = -i\kappa \int d^4x \bar{v}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda) e^{-i(p+q+k)\cdot x}$$

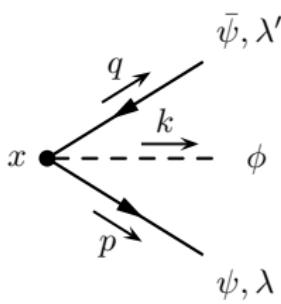
!! 在写下费米子的贡献时, 应当注意次序, 要逆着费米子线上的方向逐项写出来, 这样得到的是数  $\bar{v}(\mathbf{q}, \lambda')u(\mathbf{p}, \lambda)$ , 而非矩阵  $u(\mathbf{p}, \lambda)\bar{v}(\mathbf{q}, \lambda')$

## iT<sub>1</sub><sup>(1)</sup> 第 2 种情况

第 2 种情况下, 考虑初态  $|i\rangle = |0\rangle$  是真空态, 末态  $\langle f| = \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k}|$  包含 1 对正反费米子和 1 个实标量玻色子, 相应的  $T$  矩阵元为



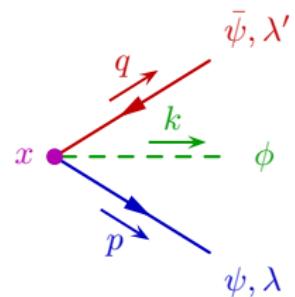
$$\begin{aligned}
 \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | i T_1^{(1)} | 0 \rangle &= -i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | \mathcal{N}[\phi(x) \bar{\psi}(x) \psi(x)] | 0 \rangle \\
 &= +i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | \phi^{(-)}(x) \bar{\psi}_a^{(-)}(x) \psi_a^{(-)}(x) | 0 \rangle \\
 &= +i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | \mathcal{N}[\phi(x) \psi_a(x) \bar{\psi}_a(x)] | 0 \rangle \\
 &= +i\kappa \int d^4x \langle 0 | e^{ik \cdot x} v_a(\mathbf{q}, \lambda') e^{iq \cdot x} \bar{u}_a(\mathbf{p}, \lambda) e^{ip \cdot x} | 0 \rangle \\
 &= +i\kappa \int d^4x \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') e^{i(p+q+k) \cdot x} \\
 &= +i\kappa \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(p+q+k)
 \end{aligned}$$



为了能让各个场算符依次对末态中相应的粒子进行缩并, 第二步交换了两个费米子场算符的位置, 因而带来一个额外的负号, 使最前面的符号从负号变为正号

## 出射外线 Feynman 规则

$$\begin{aligned}
& \langle 0 | iT_1^{(1)} | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} \rangle \\
&= +i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | \mathbf{N}[\phi(x)\psi_a(x)\bar{\psi}_a(x)] | 0 \rangle \\
&= +i\kappa \int d^4x \bar{u}(\mathbf{p}, \lambda) e^{ip \cdot x} v(\mathbf{q}, \lambda') e^{iq \cdot x} e^{ik \cdot x}
\end{aligned}$$



归纳得到如下位置空间中的出射光线 Feynman 规则

$$\begin{aligned}
 x \bullet \xrightarrow{p} \psi, \lambda &= \langle \overline{\mathbf{p}^+}, \lambda | \bar{\psi}(x) | 0 \rangle = \langle \mathbf{p}^+, \lambda | \bar{\psi}^{(-)}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{i p \cdot x} \\
 x \bullet \xrightarrow{p} \bar{\psi}, \lambda &= \langle \overline{\mathbf{p}^-}, \lambda | \psi(x) | 0 \rangle = \langle \mathbf{p}^-, \lambda | \psi^{(-)}(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{i p \cdot x} \\
 x \bullet \xrightarrow{p} \phi &= \langle \overline{\mathbf{p}}, \phi | \phi(x) | 0 \rangle = \langle \mathbf{p} | \phi^{(-)}(x) | 0 \rangle = e^{i p \cdot x}
 \end{aligned}$$

 初末态粒子满足质壳条件  $p^2 = E_p^2 - |\mathbf{p}|^2 = m^2$ ，而且能量  $E_p > 0$ ，属于在壳 (on-shell) 粒子

入射光线联系着初态粒子，出射光线联系着末态粒子，**光线上的动量是在壳的**



前面提到, 由于计算中交换了两个费米子场算符的位置, **最前面的符号由负变正**



此时顶点 Feynman 规则仍然适用，但需考虑交换费米子场算符带来的额外负号



•  $T$  矩阵元是概率振幅, 它的模方才联系着观测量  整体符号不影响观测量



如果一个过程存在多个概率振幅，则振幅之间的相对符号会影响观测量

## 额外负号

前面提到, 由于计算中交换了两个费米子场算符的位置, 最前面的符号由负变正

此时顶点 Feynman 规则仍然适用，但需考虑交换费米子场算符带来的额外负号

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如果一个过程存在多个概率振幅，则振幅之间的相对符号会影响观测量

$$\begin{aligned}
\langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | i T_1^{(1)} | 0 \rangle &= -i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | N[\phi(x)\bar{\psi}(x)\psi(x)] | 0 \rangle \\
&= +i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | N[\overbrace{\phi(x)\psi_a(x)}^{} \overbrace{\bar{\psi}_a(x)}^{}] | 0 \rangle \\
&= -i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | N[\phi(x)\overbrace{\bar{\psi}(x)\psi(x)}^{}] | 0 \rangle
\end{aligned}$$

上式第三步调换两个费米子场算符的次序，回到相互作用拉氏量中的次序，从而将最前面的符号改回来，但代表场算符缩并的线会纠缠起来



## 额外负号

前面提到, 由于计算中交换了两个费米子场算符的位置, 最前面的符号由负变正

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如果一个过程存在多个概率振幅，则振幅之间的相对符号会影响观测量

$$\begin{aligned}
\langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | i T_1^{(1)} | 0 \rangle &= -i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | N[\phi(x)\bar{\psi}(x)\psi(x)] | 0 \rangle \\
&= +i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | N[\overline{\phi(x)\psi_a(x)}\bar{\psi}_a(x)] | 0 \rangle \\
&= -i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | N[\phi(x)\bar{\psi}(x)\psi(x)] | 0 \rangle
\end{aligned}$$

上式第三步调换两个费米子场算符的次序，回到相互作用拉氏量中的次序，从而将最前面的符号改回来，但代表场算符缩并的线会纠缠起来

计算  $T$  矩阵元时，正规乘积期待值等于将场算符与初末态缩并之后的结果

当算符次序保持相互作用拉氏量中的次序时，**不会**出现额外的负号

## $T$ 矩阵元的缩并规律

计算  $T$  矩阵元时，正规乘积期待值等于将场算符与初末态缩并之后的结果

当场算符次序**保持**相互作用拉氏量中的次序时, **不会**出现额外的负号

 熟悉这个规律后，可在正规乘积期待值上直接添加缩并线，调换场算符解开纠缠的缩并线，确定振幅符号，再往下计算；这样就跳过了用正负能解表达的中间步骤

$$\begin{aligned}
\langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | iT_1^{(1)} | 0 \rangle &= -i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | \mathcal{N}[\phi(x)\bar{\psi}(x)\psi(x)] | 0 \rangle \\
&= -i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | \mathcal{N}[\phi(x)\bar{\psi}(x)\psi(x)] | 0 \rangle \\
&= +i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'; \mathbf{k} | \mathcal{N}[\phi(x)\psi_a(x)\bar{\psi}_a(x)] | 0 \rangle \\
&= +i\kappa \int d^4x \bar{u}(\mathbf{p}, \lambda)v(\mathbf{q}, \lambda')e^{i(p+q+k)\cdot x}
\end{aligned}$$

这样一个  $T$  矩阵元的缩并规律是普遍的，接下来的计算会不断地验证它

后面我们将利用这个规律确定多个振幅之间的相对符号

## iT<sub>1</sub><sup>(1)</sup> 第 3 种情况

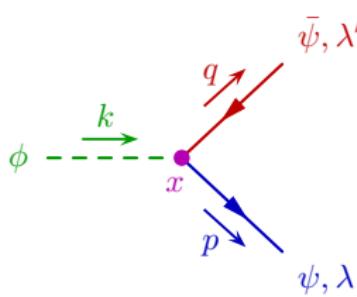
上述 Feynman 规则适用于各种情况，比如与  $iT_1^{(1)}$  相关的余下 6 种情况

👉 顶点规则只有一种形式，适用于各种不同的时间方向和动量方向

第 3 种情况中, 考虑初态  $|i\rangle = |k\rangle$  包含 1 个实标量玻色子, 末态  $\langle f| = \langle p^+, \lambda; q^-, \lambda' |$  包含 1 对正反费米子, 相应的  $T$  矩阵元为

Diagram illustrating the scattering process: an incoming particle with momentum  $k$  and spin  $\psi, \lambda$  (blue dashed line) and momentum  $p$  (blue arrow) scatters into an outgoing particle with momentum  $q$  (red arrow) and spin  $\bar{\psi}, \lambda'$  (red dashed line).

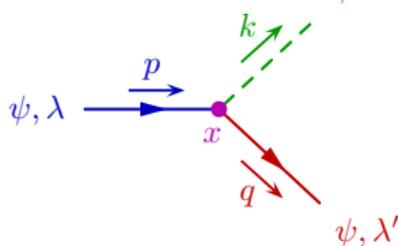
$$\begin{aligned}
 \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' | iT_1^{(1)} | \mathbf{k} \rangle &= -i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' | \mathcal{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{k} \rangle \\
 &= +i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' | \psi_a^{(-)}(x)\bar{\psi}_a^{(-)}(x)\phi^{(+)}(x) | \mathbf{k} \rangle \\
 &= +i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' | \mathcal{N}[\psi_a(x)\bar{\psi}_a(x)\phi(x)] | \mathbf{k} \rangle \\
 &= +i\kappa \int d^4x v_a(\mathbf{q}, \lambda') \bar{u}_a(\mathbf{p}, \lambda) e^{-i(\mathbf{k}-\mathbf{p}-\mathbf{q}) \cdot \mathbf{x}} \\
 &= +i\kappa \int d^4x \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') e^{-i(\mathbf{k}-\mathbf{p}-\mathbf{q}) \cdot \mathbf{x}} \\
 &= +i\kappa \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(\mathbf{k} - \mathbf{p} - \mathbf{q}) \\
 &= -i\kappa \int d^4x \langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' | \mathcal{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{k} \rangle
 \end{aligned}$$



iT<sub>1</sub><sup>(1)</sup> 第 4 种情况

第 4 种情况下, 考虑初态  $|i\rangle = |\mathbf{p}^+, \lambda\rangle$  包含 1 个正费米子, 末态  $\langle f| = \langle \mathbf{q}^+, \lambda'; \mathbf{k}|$  包含 1 个正费米子和 1 个实标量玻色子, 相应的  $T$  矩阵元为

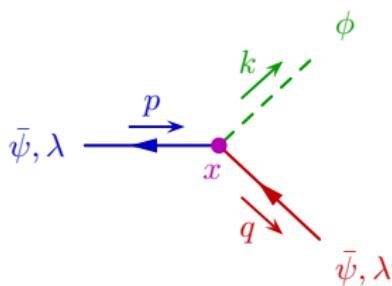
$$\begin{aligned}
 \langle \mathbf{q}^+, \lambda'; \mathbf{k} | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= -i\kappa \int d^4x \langle \mathbf{q}^+, \lambda'; \mathbf{k} | \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{p}^+, \lambda \rangle \\
 &= -i\kappa \int d^4x \langle \mathbf{q}^+, \lambda'; \mathbf{k} | \phi^{(-)}(x)\bar{\psi}^{(-)}(x)\psi^{(+)}(x) | \mathbf{p}^+, \lambda \rangle \\
 &= -i\kappa \int d^4x \langle \mathbf{q}^+, \lambda'; \mathbf{k} | \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{p}^+, \lambda \rangle \\
 &= -i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda')u(\mathbf{p}, \lambda) e^{-i(\mathbf{p}-\mathbf{q}-\mathbf{k}) \cdot \mathbf{x}} \\
 &= -i\kappa \bar{u}(\mathbf{q}, \lambda')u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p - q - k)
 \end{aligned}$$



## iT<sub>1</sub><sup>(1)</sup> 第 5 种情况

熊 第 5 种情况下, 考虑初态  $|i\rangle = |\mathbf{p}^-, \lambda\rangle$  包含 1 个反费米子, 末态  $\langle f| = \langle \mathbf{q}^-, \lambda'; \mathbf{k}|$  包含 1 个反费米子和 1 个实标量玻色子, 相应的  $T$  矩阵元为

$$\begin{aligned}
\langle \mathbf{q}^-, \lambda'; \mathbf{k} | iT_1^{(1)} | \mathbf{p}^-, \lambda \rangle &= -i\kappa \int d^4x \langle \mathbf{q}^-, \lambda'; \mathbf{k} | \mathbf{N}[\phi(x)\bar{\psi}_a(x)\psi_a(x)] | \mathbf{p}^-, \lambda \rangle \\
&= +i\kappa \int d^4x \langle \mathbf{q}^-, \lambda'; \mathbf{k} | \phi^{(-)}(x)\psi_a^{(-)}(x)\bar{\psi}_a^{(+)}(x) | \mathbf{p}^-, \lambda \rangle \\
&= +i\kappa \int d^4x \langle \mathbf{q}^-, \lambda'; \mathbf{k} | \mathbf{N}[\phi(x)\psi_a(x)\bar{\psi}_a(x)] | \mathbf{p}^-, \lambda \rangle \\
&= +i\kappa \int d^4x v_a(\mathbf{q}, \lambda') \bar{v}_a(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
&= +i\kappa \int d^4x \bar{v}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') e^{-i(p-q-k)\cdot x} \\
&= +i\kappa \bar{v}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(p-q-k) \\
&= -i\kappa \int d^4x \langle \mathbf{q}^-, \lambda'; \mathbf{k} | \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{p}^-, \lambda \rangle
\end{aligned}$$

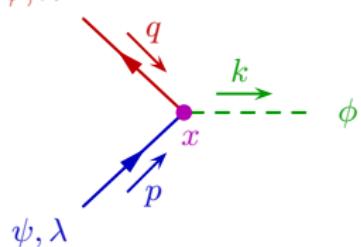


iT<sub>1</sub><sup>(1)</sup> 第 6 种情况

第 6 种情况下, 考虑初态  $|i\rangle = |\mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda'\rangle$  包含 1 对正反费米子, 末态  $\langle f| = \langle \mathbf{k}|$  包含 1 个实标量玻色子, 相应的  $T$  矩阵元为

$$\langle \mathbf{k} | iT_1^{(1)} | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' \rangle = -i\kappa \int d^4x \langle \mathbf{k} | \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' \rangle$$

$$\bar{\psi}, \lambda' \quad = -i\kappa \int d^4x \langle \mathbf{k} | \phi^{(-)}(x) \bar{\psi}^{(+)}(x) \psi^{(+)}(x) | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' \rangle$$



$$= -i\kappa \int d^4x \langle \mathbf{k} | \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' \rangle$$

$$= -i\kappa \int d^4x \bar{v}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda) e^{-i(p+q-k)\cdot x}$$

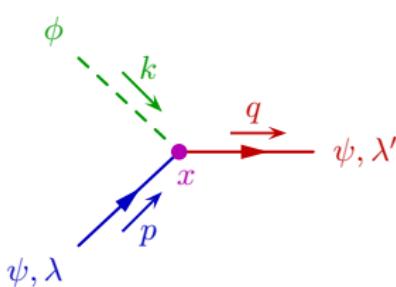
$$= -i\kappa \bar{v}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p + q - k)$$

iT<sub>1</sub><sup>(1)</sup> 第 7 种情况

第 7 种情况下, 考虑初态  $|i\rangle = |\mathbf{p}^+, \lambda; \mathbf{k}\rangle$  包含 1 个正费米子和 1 个实标量玻色子, 末态  $\langle f| = \langle \mathbf{q}^+, \lambda' |$  包含 1 个正费米子, 相应的  $T$  矩阵元为

Diagram illustrating the derivation of the equation. A particle with momentum  $p$  and spin  $\lambda$  (blue arrow) and a virtual particle with momentum  $k$  and spin  $\lambda'$  (green dashed arrow) interact to produce a particle with momentum  $q$  and spin  $\lambda'$  (red arrow). The virtual particle  $k$  is shown as a green dashed arrow pointing to the left.

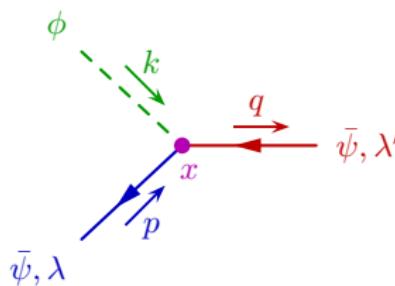
$$\begin{aligned}
 \langle \mathbf{q}^+, \lambda' | i T_1^{(1)} | \mathbf{p}^+, \lambda; \mathbf{k} \rangle &= -i\kappa \int d^4x \langle \mathbf{q}^+, \lambda' | \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{p}^+, \lambda; \mathbf{k} \rangle \\
 &= -i\kappa \int d^4x \langle \mathbf{q}^+, \lambda' | \bar{\psi}_a^{(-)}(x)\phi^{(+)}(x)\psi_a^{(+)}(x) | \mathbf{p}^+, \lambda; \mathbf{k} \rangle \\
 &= -i\kappa \int d^4x \langle \mathbf{q}^+, \lambda' | \mathbf{N}[\bar{\psi}_a(x)\phi(x)\psi_a(x)] | \mathbf{p}^+, \lambda; \mathbf{k} \rangle \\
 &= -i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda) e^{-i(\mathbf{p} + \mathbf{k} - \mathbf{q}) \cdot x} \\
 &= -i\kappa \bar{u}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p + k - q) \\
 &= -i\kappa \int d^4x \langle \mathbf{q}^+, \lambda' | \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{p}^+, \lambda; \mathbf{k} \rangle
 \end{aligned}$$



iT<sub>1</sub><sup>(1)</sup> 第 8 种情况

第 8 种情况下, 考虑初态  $|i\rangle = |\mathbf{p}^-, \lambda; \mathbf{k}\rangle$  包含 1 个反费米子和 1 个实标量玻色子, 末态  $\langle f| = \langle \mathbf{q}^-, \lambda' |$  包含 1 个反费米子, 相应的  $T$  矩阵元为

$$\begin{aligned} \langle \mathbf{q}^-, \lambda' | i T_1^{(1)} | \mathbf{p}^-, \lambda; \mathbf{k} \rangle &= -i\kappa \int d^4x \langle \mathbf{q}^-, \lambda' | \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)] | \mathbf{p}^-, \lambda; \mathbf{k} \rangle \\ &= +i\kappa \int d^4x \langle \mathbf{q}^-, \lambda' | \psi_a^{(-)}(x)\phi(x)\bar{\psi}_a^{(+)}(x) | \mathbf{p}^-, \lambda; \mathbf{k} \rangle \end{aligned}$$



$$\begin{aligned}
&= +i\kappa \int d^4x \left\langle \overline{\mathbf{q}}, \lambda' \right| \mathbf{N}[\psi_a(x)\phi(x)\bar{\psi}_a(x)] \left| \overline{\mathbf{p}}, \lambda; \mathbf{k} \right\rangle \\
&= +i\kappa \int d^4x v_a(\mathbf{q}, \lambda') \bar{v}_a(\mathbf{p}, \lambda) e^{-i(\mathbf{p} + \mathbf{k} - \mathbf{q}) \cdot \mathbf{x}} \\
&= +i\kappa \int d^4x \bar{v}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') e^{-i(p + k - q) \cdot x} \\
&= +i\kappa \bar{v}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(p + k - q) \\
&= -i\kappa \int d^4x \left\langle \overline{\mathbf{q}}, \lambda' \right| \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)] \left| \overline{\mathbf{p}}, \lambda; \mathbf{k} \right\rangle
\end{aligned}$$

## 内线 Feynman 规则



接下来讨论  $iT_2^{(1)} = -i\kappa \int d^4x N[\phi(x)\bar{\psi}(x)\psi(x)]$



$iT_2^{(1)}$  包含两个场算符之间的缩并，也就是 6.4 节讨论的 **Feynman 传播子**



将 Dirac 旋量场和实标量场 Feynman 传播子在位置空间中的 Feynman 规则设为

$$x \bullet \xrightarrow{p} \bullet y = \overline{\psi(y)\psi}(x) = S_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(p + \textcolor{red}{m}_\psi)}{p^2 - \textcolor{red}{m}_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$

$$x \bullet \overrightarrow{\bullet} \bullet y = \overline{\phi(y)\phi(x)} = D_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_\phi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$



$m_\psi$  是  $\psi$  粒子质量,  $m_\phi$  是  $\phi$  粒子质量



在位置空间中，Feynman 传播子是粒子从  $x$  处顶点传播到  $y$  处顶点的振幅，我们用一条连接两个顶点的粒子线表示，这样的线称为内线 (internal line)



Dirac 费米子的 Feynman 传播子用带箭头的实线表示，动量方向与箭头方向一致



标量玻色子的 Feynman 传播子用虚线表示, 动量方向另外标明

## 虚粒子与实粒子

$$x \bullet \xrightarrow{p} \bullet y = \overline{\psi(y)\bar{\psi}(x)} = S_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(p + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$

$$x \bullet \overrightarrow{\bullet} \bullet y = \overline{\phi(y)\phi(x)} = D_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_\phi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$



在内线 Feynman 规则的表达式中，需要对动量  $p^\mu$  的所有取值积分



内线动量可以是在壳的，但更一般的情况是离壳 (off-shell) 的，即不满足质壳条件  $p^2 = m^2$ ，而且  $p^0$  也不一定为正

## 虚粒子与实粒子

$$x \xrightarrow{p} y = \overline{\psi(y)\psi(x)} = S_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(p + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$

$$x \bullet \overrightarrow{\bullet} \bullet y = \overline{\phi(y)\phi(x)} = D_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_\phi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$



在内线 Feynman 规则的表达式中，需要对动量  $p^\mu$  的所有取值积分



内线动量可以是在壳的，但更一般的情况是离壳 (off-shell) 的，即不满足质壳条件  $p^2 = m^2$ ，而且  $p^0$  也不一定为正



用内线表示的粒子称为虚粒子 (virtual particle)，描述场的传播行为



虚粒子可以是在壳粒子，也可以是离壳粒子



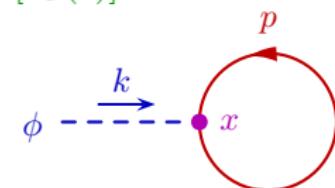
反过来，用外线表示的粒子称为**实粒子** (real particle)，它一定是在壳粒子

## iT<sub>2</sub><sup>(1)</sup> 第 1 种情况

  $iT_2^{(1)}$  剩下一个标量场  $\phi(x)$  未参与缩并, 我们可以让它与初态或末态缩并

考慮初态  $|i\rangle = |k\rangle$  包含 1 个实标量玻色子, 末态  $|f\rangle = |0\rangle$  是真空态, 有

$$\begin{aligned}
& \langle 0 | i T_2^{(1)} | \mathbf{k} \rangle \\
&= -i\kappa \int d^4x \langle 0 | \mathsf{N}[\phi(x) \overline{\psi}(x) \psi(x)] | \mathbf{k} \rangle = -i\kappa \int d^4x \langle 0 | \overline{\psi}_a(x) \psi_a(x) \phi^{(+)}(x) | \mathbf{k} \rangle \\
&= -i\kappa \int d^4x \langle 0 | \mathsf{N}[\overline{\psi}_a(x) \overline{\psi}_a(x) \overline{\phi}(x)] | \mathbf{k} \rangle = +i\kappa \int d^4x \langle 0 | \mathsf{N}[\overline{\psi}_a(x) \overline{\psi}_a(x) \overline{\phi}(x)] | \mathbf{k} \rangle \\
&= +i\kappa \int d^4x S_{F,aa}(x-x) e^{-ik \cdot x} = +i\kappa \int d^4x e^{-ik \cdot x} \text{tr}[S_F(0)] \\
&= +i\kappa (2\pi)^4 \delta^{(4)}(k) \int \frac{d^4p}{(2\pi)^4} \frac{i \text{tr}(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} \\
&= -i\kappa \int d^4x \langle 0 | \mathsf{N}[\phi(x) \overline{\psi}(x) \psi(x)] | \mathbf{k} \rangle
\end{aligned}$$



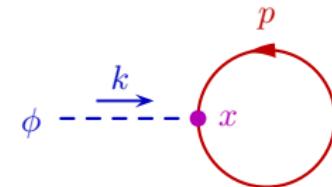
 第四步交换了正规乘积中两个费米子场算符的次序，因而带来一个额外的负号

第六步用到矩阵的迹的定义  $\text{tr}[S_F(0)] = S_{F,aa}(0)$

## 卷图与树图

$$\begin{aligned}
\langle 0 | iT_2^{(1)} | \mathbf{k} \rangle &= -i\kappa \int d^4x \langle 0 | \mathsf{N}[\overbrace{\phi(x)\bar{\psi}(x)\psi(x)}^{\text{Fermion}}] | \mathbf{k} \rangle \\
&= +i\kappa \int d^4x \langle 0 | \mathsf{N}[\overbrace{\psi_a(x)\bar{\psi}_a(x)}^{\text{Fermion}} \overbrace{\phi(x)}^{\text{Scalar}}] | \mathbf{k} \rangle \\
&= +i\kappa \int d^4x S_{F,aa}(0) e^{-i\mathbf{k}\cdot\mathbf{x}} = +i\kappa (2\pi)^4 \delta^{(4)}(k) \int \frac{d^4p}{(2\pi)^4} \frac{i \text{tr}(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon}
\end{aligned}$$

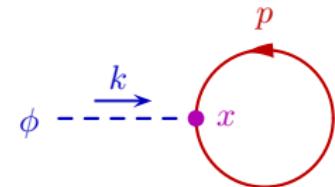

The diagram shows a red circle representing a loop. A point on the loop is marked with a purple dot and labeled 'x'. A red arrow labeled 'k' points from the center of the loop towards the point 'x'.



⌚  $iT_2^{(1)}$  中参与缩并的费米子场算符  $\psi(x)$  和  $\bar{\psi}(x)$  具有相同的时空坐标  $x$ ，因而 Feynman 传播子从  $x$  处的顶点出发，传播回到  $x$  处的顶点，形成一个**封闭的圈**

## 卷图与树图

$$\begin{aligned}
 \langle 0 | iT_2^{(1)} | \mathbf{k} \rangle &= -i\kappa \int d^4x \langle 0 | \mathsf{N}[\overbrace{\phi(x)\bar{\psi}(x)\psi(x)}^{\text{Fermion}}] | \mathbf{k} \rangle \\
 &= +i\kappa \int d^4x \langle 0 | \mathsf{N}[\overbrace{\psi_a(x)\bar{\psi}_a(x)}^{\text{Fermion}} \overbrace{\phi(x)}^{\text{Scalar}}] | \mathbf{k} \rangle \\
 &= +i\kappa \int d^4x S_{F,aa}(0) e^{-ik \cdot x} = +i\kappa (2\pi)^4 \delta^{(4)}(k) \int \frac{d^4p}{(2\pi)^4} \frac{i \text{tr}(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon}
 \end{aligned}$$



⌚  $iT_2^{(1)}$  中参与缩并的费米子场算符  $\psi(x)$  和  $\bar{\psi}(x)$  具有相同的时空坐标  $x$ ，因而 Feynman 传播子从  $x$  处的顶点出发，传播回到  $x$  处的顶点，形成一个**封闭的圈**

这种包含圈结构的 Feynman 图称为圈图 (loop diagram)

相反地，不包含圈结构的 Feynman 图称为 **树图** (tree diagram)

上述与  $iT_1^{(1)}$  相关的 8 种 Feynman 图都是树图

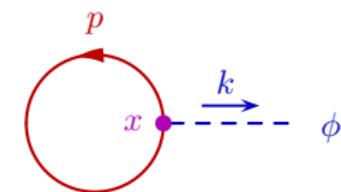
图里出现一个未定的四维动量  $p^\mu$ , 称为圈动量 (loop momentum), 它的取值不能通过初末态四维动量确定, 而是通过积分  $\int d^4p/(2\pi)^4$  考虑所有取值

一个封闭的费米子圈贡献一个额外的负号, 且需要对 Dirac 矩阵 (或其乘积) 求迹

iT<sub>2</sub><sup>(1)</sup> 第 2 种情况

在  $iT_2^{(1)}$  的另一种情况中, 考虑初态  $|i\rangle = |0\rangle$  是真空态, 末态  $\langle f| = \langle k|$  包含 1 个实标量玻色子, 相应的  $T$  矩阵元为

$$\begin{aligned}
\langle \mathbf{k} | i T_2^{(1)} | 0 \rangle &= -i\kappa \int d^4x \langle \mathbf{k} | \mathbf{N}[\phi(x) \bar{\psi}(x) \psi(x)] | 0 \rangle \\
&= -i\kappa \int d^4x \langle \mathbf{k} | \phi^{(-)}(x) \bar{\psi}_a(x) \psi_a(x) | 0 \rangle \\
&= -i\kappa \int d^4x \langle \mathbf{k} | \mathbf{N}[\phi(x) \bar{\psi}_a(x) \psi_a(x)] | 0 \rangle = +i\kappa \int d^4x \langle \mathbf{k} | \mathbf{N}[\phi(x) \psi_a(x) \bar{\psi}_a(x)] | 0 \rangle \\
&= +i\kappa \int d^4x e^{ik \cdot x} S_{F,aa}(x-x) = +i\kappa \int d^4x e^{ik \cdot x} \text{tr}[S_F(0)] \\
&= +i\kappa (2\pi)^4 \delta^{(4)}(k) \int \frac{d^4p}{(2\pi)^4} \frac{i \text{tr}(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon}
\end{aligned}$$

像以上两幅 Feynmann 图一样只包含一条外线的图称为蝌蚪图 (tadpole diagram)

## 动力学与运动学



以上列举的 10 个 Feynman 图对应于 10 个动力学允许的过程。



但是，其中大多数过程在运动学上并不允许，因为初态和末态不能同时满足能量和动量守恒定律，也就是说，不能满足运动学条件

动力学与运动学



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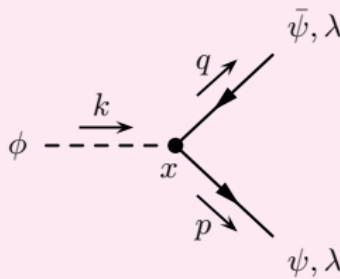
当  $m_\phi > 2m_\psi$  时, 有 2 个过程是例外的, 运动学允许它们发生



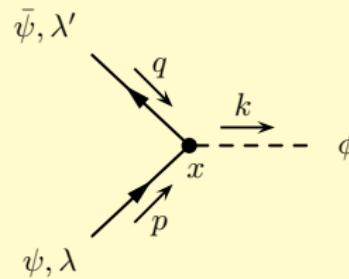
第一个过程是一个  $\phi$  粒子衰变成一对正反  $\psi$  粒子的过程  $\phi \rightarrow \psi\bar{\psi}$



第二个过程是一对正反  $\psi$  粒子融合 (fusion) 成一个  $\phi$  粒子的过程  $\psi\bar{\psi} \rightarrow \phi$



$\phi \rightarrow \psi\bar{\psi}$  衰变过程



### $\psi\bar{\psi} \rightarrow \phi$ 融合过程

$\phi \rightarrow \psi\bar{\psi}$  衰变过程



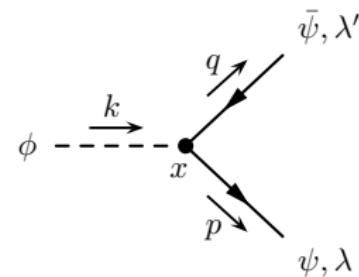
现在计算  $\phi \rightarrow \psi\bar{\psi}$  衰变过程对应的衰变宽度



根据  $\langle f | iT | i \rangle = (2\pi)^4 \delta^{(4)}(p_i - p_f) i\mathcal{M}$ ，将

$$\langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' | iT_1^{(1)} | \mathbf{k} \rangle$$

$$= i\kappa \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(k - p - q)$$



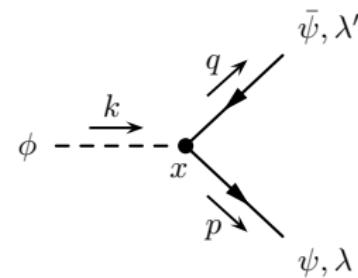
中的  $\delta$  函数因子  $(2\pi)^4 \delta^{(4)}(k - p - q)$  扔掉，就得到  $\phi \rightarrow \psi\bar{\psi}$  衰变过程的**不变振幅**

$$i\mathcal{M} = i\kappa \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda')$$

$\phi \rightarrow \psi\bar{\psi}$  衰变过程



现在计算  $\phi \rightarrow \psi\bar{\psi}$  衰变过程对应的衰变宽度



根据  $\langle f | iT | i \rangle \equiv (2\pi)^4 \delta^{(4)}(p_i - p_f) i\mathcal{M}$ ，将

$$\langle \mathbf{p}^+, \lambda; \mathbf{q}^-, \lambda' | iT_1^{(1)} | \mathbf{k} \rangle$$

$$= i\kappa \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(k - p - q)$$

中的  $\delta$  函数因子  $(2\pi)^4 \delta^{(4)}(k - p - q)$  扔掉，就得到  $\phi \rightarrow \psi\bar{\psi}$  衰变过程的**不变振幅**

$$i\mathcal{M} = i\kappa \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda')$$



这是  $\kappa^1$  阶的结果，它是贡献到这个过程的最低阶，即领头阶 (leading order)



当 Yukawa 耦合常数  $\kappa$  比较小时, 领头阶的贡献远大于更高阶的贡献



对  $iM$  取复共轭 (等同于厄米共轭), 得

$$(\mathbf{i}\mathcal{M})^* = [\mathbf{i}\kappa u^\dagger(\mathbf{p}, \lambda)\gamma^0 v(\mathbf{q}, \lambda')]^\dagger = -\mathbf{i}\kappa v^\dagger(\mathbf{q}, \lambda')\gamma^0 u(\mathbf{p}, \lambda) = -\mathbf{i}\kappa \bar{v}(\mathbf{q}, \lambda')u(\mathbf{p}, \lambda)$$



不变振幅的模方是  $|\mathcal{M}|^2 = \kappa^2 \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') \bar{v}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda)$



## Casimir 技巧

 不变振幅模方进一步化为

$$\begin{aligned} |\mathcal{M}|^2 &= \kappa^2 \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') \bar{v}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda) = \kappa^2 \bar{u}_a(\mathbf{p}, \lambda) v_a(\mathbf{q}, \lambda') \bar{v}_b(\mathbf{q}, \lambda') \mathbf{u}_b(\mathbf{p}, \lambda) \\ &= \kappa^2 \mathbf{u}_b(\mathbf{p}, \lambda) \bar{u}_a(\mathbf{p}, \lambda) v_a(\mathbf{q}, \lambda') \bar{v}_b(\mathbf{q}, \lambda') = \kappa^2 \mathbf{tr}[\mathbf{u}(\mathbf{p}, \lambda) \bar{u}(\mathbf{p}, \lambda) \mathbf{v}(\mathbf{q}, \lambda') \bar{v}(\mathbf{q}, \lambda')] \end{aligned}$$

鬃毛  $u_b(\mathbf{p}, \lambda)\bar{u}_a(\mathbf{p}, \lambda)$  和  $v_a(\mathbf{q}, \lambda')\bar{v}_b(\mathbf{q}, \lambda')$  是矩阵  $u(\mathbf{p}, \lambda)\bar{u}(\mathbf{p}, \lambda)$  和  $v(\mathbf{q}, \lambda')\bar{v}(\mathbf{q}, \lambda')$  的  $ba$  分量和  $ab$  分量, 对  $a$  求和表示矩阵相乘, 对  $b$  求和表示求迹

## Casimir 技巧

不变振幅模方进一步化为

$$\begin{aligned} |\mathcal{M}|^2 &= \kappa^2 \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') \bar{v}(\mathbf{q}, \lambda') u(\mathbf{p}, \lambda) = \kappa^2 \bar{u}_a(\mathbf{p}, \lambda) v_a(\mathbf{q}, \lambda') \bar{v}_b(\mathbf{q}, \lambda') \mathbf{u}_b(\mathbf{p}, \lambda) \\ &= \kappa^2 \mathbf{u}_b(\mathbf{p}, \lambda) \bar{u}_a(\mathbf{p}, \lambda) v_a(\mathbf{q}, \lambda') \bar{v}_b(\mathbf{q}, \lambda') = \kappa^2 \mathbf{tr}[\mathbf{u}(\mathbf{p}, \lambda) \bar{u}(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') \bar{v}(\mathbf{q}, \lambda')] \end{aligned}$$

鬃毛  $u_b(\mathbf{p}, \lambda)\bar{u}_a(\mathbf{p}, \lambda)$  和  $v_a(\mathbf{q}, \lambda')\bar{v}_b(\mathbf{q}, \lambda')$  是矩阵  $u(\mathbf{p}, \lambda)\bar{u}(\mathbf{p}, \lambda)$  和  $v(\mathbf{q}, \lambda')\bar{v}(\mathbf{q}, \lambda')$  的  $ba$  分量和  $ab$  分量, 对  $a$  求和表示矩阵相乘, 对  $b$  求和表示求迹

老虎 在计算  $\phi$  的衰变宽度时, 应当包含所有可能的末态, 即包括所有可能的动量取值和螺旋态, 因此需要使用对末态粒子螺旋度求和的非极化不变振幅模方

$$\begin{aligned} |\mathcal{M}|^2 &\equiv \sum_{\lambda\lambda'} |\mathcal{M}|^2 = \kappa^2 \sum_{\lambda\lambda'} \text{tr}[u(\mathbf{p}, \lambda)\bar{u}(\mathbf{p}, \lambda)v(\mathbf{q}, \lambda')\bar{v}(\mathbf{q}, \lambda')] \\ &= \kappa^2 \sum_{\lambda'} \text{tr}[(\not{p} + m_\psi)v(\mathbf{q}, \lambda')\bar{v}(\mathbf{q}, \lambda')] = \kappa^2 \text{tr}[(\not{p} + m_\psi)(\not{q} - m_\psi)] \end{aligned}$$

这里用到**自旋求和关系**  $\sum_{\lambda=+} u(\mathbf{p}, \lambda) \bar{u}(\mathbf{p}, \lambda) = \not{p} + m$  和  $\sum_{\lambda=+} v(\mathbf{p}, \lambda) \bar{v}(\mathbf{p}, \lambda) = \not{p} - m$

这种把旋量计算转化为求迹运算的方法称为 **Casimir 技巧** (Casimir's trick)

## 求迹运算



现在需要对 Dirac 矩阵及其乘积求迹



根据  $(\gamma^5)^2 = 1$ 、 $\gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu$  和矩阵乘积之迹的性质  $\text{tr}(AB) = \text{tr}(BA)$ ，有

$$\text{tr}(\gamma^\mu) = \text{tr}(\gamma^\mu \gamma^5 \gamma^5) = - \text{tr}(\gamma^5 \gamma^\mu \gamma^5) = - \text{tr}(\gamma^5 \gamma^5 \gamma^\mu) = - \text{tr}(\gamma^\mu)$$

$$\text{tr}(\gamma^\mu) = 0$$



$$\text{tr}(\not{p}) = \text{tr}(p_\mu \gamma^\mu) = p_\mu \text{tr}(\gamma^\mu) = 0$$



现在需要对 Dirac 矩阵及其乘积求迹



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$$\text{tr}(p) = \text{tr}(p_\mu \gamma^\mu) = p_\mu \text{tr}(\gamma^\mu) = 0$$



由于  $\gamma^\mu \gamma^\nu = 2g^{\mu\nu} - \gamma^\nu \gamma^\mu$ ，两个 Dirac 矩阵乘积之迹满足

$$\text{tr}(\gamma^\mu \gamma^\nu) = \text{tr}(2g^{\mu\nu} - \gamma^\nu \gamma^\mu) = 2g^{\mu\nu} \text{tr}(1) - \text{tr}(\gamma^\nu \gamma^\mu) = 8g^{\mu\nu} - \text{tr}(\gamma^\mu \gamma^\nu)$$

$$\text{tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$$



$$\text{tr}(\not{p}\not{q}) = p_\mu q_\nu \text{tr}(\gamma^\mu \gamma^\nu) = 4p_\mu q_\nu g^{\mu\nu} = 4p \cdot q$$

## 求迹运算

现在需要对 Dirac 矩阵及其乘积求迹

根据  $(\gamma^5)^2 = 1$ 、 $\gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu$  和矩阵乘积之迹的性质  $\text{tr}(AB) = \text{tr}(BA)$ ，有

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$$\text{tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$$

$$\text{👉 } \text{tr}(\not{p} \not{q}) = p_\mu q_\nu \text{tr}(\gamma^\mu \gamma^\nu) = 4 p_\mu q_\nu g^{\mu\nu} = 4 p \cdot q$$

从而将非极化不变振幅模方化为

$$|\mathcal{M}|^2 = \kappa^2 \text{tr}[(\not{p} - m_\psi \not{p} + m_\psi \not{q} - m_\psi^2) = \kappa^2 [\text{tr}(\not{p}\not{q}) - m_\psi^2 \text{tr}(\mathbf{1})] = 4\kappa^2(p \cdot q - m_\psi^2)$$

$\phi \rightarrow \psi\bar{\psi}$  衰变宽度

根据质壳条件  $k^2 = m_\phi^2$ 、 $p^2 = q^2 = m_\psi^2$  和能动量守恒关系  $k^\mu = p^\mu + q^\mu$ ，有

$$m_\phi^2 = k^2 = (p+q)^2 = p^2 + q^2 + 2p \cdot q = 2(m_\psi^2 + p \cdot q)$$

$$\textcolor{violet}{p} \cdot q = \frac{m_\phi^2}{2} - m_\psi^2, \quad \textcolor{violet}{p} \cdot q - m_\psi^2 = \frac{1}{2}(m_\phi^2 - 4m_\psi^2) = \frac{m_\phi^2}{2} \left(1 - \frac{4m_\psi^2}{m_\phi^2}\right)$$

狮子  $\psi$  与  $\bar{\psi}$  是不全同的正反粒子  末态对称性因子  $S = 1$

代入  $|\mathcal{M}|^2 = 4\kappa^2(p \cdot q - m_\psi^2)$ ,  $\phi \rightarrow \psi\bar{\psi}$  衰变过程的领头阶衰变宽度为

$$\begin{aligned}\Gamma(\phi \rightarrow \psi \bar{\psi}) &= \frac{1}{\mathcal{S}} \frac{\overline{|\mathcal{M}|^2}}{16\pi m_\phi} \sqrt{1 - \frac{4m_\psi^2}{m_\phi^2}} = \frac{\kappa^2}{4\pi m_\phi} \sqrt{1 - \frac{4m_\psi^2}{m_\phi^2}} (p \cdot q - m_\psi^2) \\ &= \frac{\kappa^2 m_\phi}{8\pi} \left(1 - \frac{4m_\psi^2}{m_\phi^2}\right)^{3/2}\end{aligned}$$

如果  $m_\phi = 2m_\psi$ ，则  $\Gamma(\phi \rightarrow \psi\bar{\psi}) = 0$ ， $\phi \rightarrow \psi\bar{\psi}$  衰变过程不能发生

### 7.1.2 小节 $\kappa^2$ 阶 Feynman 图

 在  $iT$  展开式的第 2 阶, 即  $\kappa^2$  阶, 有

$$iT^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{T}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] = \sum_{j=1}^{14} iT_j^{(2)}$$

根据 Wick 定理, 共有 14 个非平庸的项  $iT_i^{(2)}$

首先, 有 1 项**不包含缩并**,  $iT_1^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y N[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)]$

其次，有 5 项包含 1 次缩并，

$$iT_2^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\overline{\phi(x)\bar{\psi}(x)\psi(x)\phi(y)}\bar{\psi}(y)\psi(y)]$$

$$iT_3^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\phi(x)\bar{\psi}(x)\overline{\psi(x)\phi(y)\bar{\psi}(y)}\psi(y)]$$

$$iT_4^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)]$$

$$iT_5^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)]$$

$$iT_6^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)]$$



### 非平庸项 $iT_j^{(2)}$

再次，有 6 项包含 2 次缩并，

$$iT_7^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\overline{\phi(x)\bar{\psi}(x)\psi(x)\phi(y)}\overline{\bar{\psi}(y)\psi(y)}]$$

$$iT_8^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\overbrace{\phi(x)\bar{\psi}(x)\psi(x)\phi(y)}^{\text{}}\overbrace{\bar{\psi}(y)\psi(y)}^{\text{}}]$$

$$iT_9^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\overline{\phi(x)}\overline{\psi(x)}\psi(x)\phi(y)\bar{\psi}(y)\psi(y)]$$

$$iT_{10}^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathcal{N}[\overline{\phi(x)\bar{\psi}(x)\psi(x)\phi(y)}\overline{\bar{\psi}(y)\psi(y)}]$$

$$i T_{11}^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathcal{N} [\phi(x) \bar{\psi}(x) \psi(x) \phi(y) \bar{\psi}(y) \psi(y)]$$

$$i T_{12}^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\phi(x) \bar{\psi}(x) \psi(x) \phi(y) \bar{\psi}(y) \psi(y)]$$

最后，有 2 项包含 3 次缩并，

$$iT_{13}^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)]$$

$$iT_{14}^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N}[\overline{\phi(x)}\overline{\psi(x)}\overline{\psi(x)}\overline{\phi(y)}\overline{\psi(y)}\overline{\psi(y)}]$$

$\psi\bar{\psi} \rightarrow \psi\bar{\psi}$  散射过程

下面讨论几个与  $iT_j^{(2)}$  相关的过程

(1) 首先, 考虑  $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$  散射过程, 初态记为  $|i\rangle = |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2\rangle$ , 末态记为  $\langle f| = \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2|$ ,  $iT_2^{(2)}$  对这个过程贡献的  $T$  矩阵元是

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| i T_2^{(2)} \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right. \right\rangle \\
&= \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathsf{N}[\overline{\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)}] \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right. \right\rangle \\
&= \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathsf{N}[-\psi_a^{(-)}(x)\bar{\psi}_a^{(-)}(x)\overline{\phi(x)\phi(y)\bar{\psi}_b^{(+)}(y)\psi_b^{(+)}(y)} \right. \\
&\quad \left. - \psi_b^{(-)}(y)\bar{\psi}_b^{(-)}(y)\overline{\phi(x)\phi(y)\bar{\psi}_a^{(+)}(x)\psi_a^{(+)}(x)} + \psi_a^{(-)}(x)\bar{\psi}_b^{(-)}(y)\overline{\phi(x)\phi(y)\bar{\psi}_a^{(+)}(x)\psi_b^{(+)}(y)} \right. \\
&\quad \left. + \psi_b^{(-)}(y)\bar{\psi}_a^{(-)}(x)\overline{\phi(x)\phi(y)\bar{\psi}_b^{(+)}(y)\psi_a^{(+)}(x)}] \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right. \right\rangle
\end{aligned}$$



第二步将场算符分解为正负能解，得到 4 个非零项

## 时空坐标的交换对称性

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| iT_2^{(2)} \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
&= \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathbf{N} \left[ -\psi_a^{(-)}(x) \bar{\psi}_a^{(-)}(x) \phi(x) \phi(y) \bar{\psi}_b^{(+)}(y) \psi_b^{(+)}(y) \right. \right. \\
&\quad \left. \left. - \psi_b^{(-)}(y) \bar{\psi}_b^{(-)}(y) \phi(x) \phi(y) \bar{\psi}_a^{(+)}(x) \psi_a^{(+)}(x) + \psi_a^{(-)}(x) \bar{\psi}_b^{(-)}(y) \phi(x) \phi(y) \bar{\psi}_a^{(+)}(x) \psi_b^{(+)}(y) \right. \right. \\
&\quad \left. \left. + \psi_b^{(-)}(y) \bar{\psi}_a^{(-)}(x) \phi(x) \phi(y) \bar{\psi}_b^{(+)}(y) \psi_a^{(+)}(x) \right] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
&= -(-i\kappa)^2 \int d^4x d^4y \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathbf{N} \left[ \psi_a^{(-)}(x) \bar{\psi}_a^{(-)}(x) \phi(x) \phi(y) \bar{\psi}_b^{(+)}(y) \psi_b^{(+)}(y) \right. \right. \\
&\quad \left. \left. - \psi_a^{(-)}(x) \bar{\psi}_b^{(-)}(y) \phi(x) \phi(y) \bar{\psi}_a^{(+)}(x) \psi_b^{(+)}(y) \right] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle
\end{aligned}$$

由于  $\phi(y)\phi(x) = \phi(x)\phi(y)$ ，对第一步第 2 项交换时空坐标  $x$  和  $y$  得到的结果与第 1 项相同，因而可只保留一项，再乘上  $2!$  因子，它刚好与前面  $1/2!$  因子抵消

类似地, 第3项与第4项也具有这种交换  $x$  和  $y$  的对称性

## 时空坐标的交换对称性

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| iT_2^{(2)} \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
&= \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathbf{N} \left[ -\psi_a^{(-)}(x) \bar{\psi}_a^{(-)}(x) \overline{\phi(x)} \phi(y) \bar{\psi}_b^{(+)}(y) \psi_b^{(+)}(y) \right. \right. \\
&\quad - \psi_b^{(-)}(y) \bar{\psi}_b^{(-)}(y) \overline{\phi(x)} \phi(y) \bar{\psi}_a^{(+)}(x) \psi_a^{(+)}(x) \left. \left. + \psi_a^{(-)}(x) \bar{\psi}_b^{(-)}(y) \overline{\phi(x)} \phi(y) \bar{\psi}_a^{(+)}(x) \psi_b^{(+)}(y) \right. \right. \\
&\quad \left. \left. + \psi_b^{(-)}(y) \bar{\psi}_a^{(-)}(x) \overline{\phi(x)} \phi(y) \bar{\psi}_b^{(+)}(y) \psi_a^{(+)}(x) \right] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
&= -(-i\kappa)^2 \int d^4x d^4y \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathbf{N} \left[ \psi_a^{(-)}(x) \bar{\psi}_a^{(-)}(x) \overline{\phi(x)} \phi(y) \bar{\psi}_b^{(+)}(y) \psi_b^{(+)}(y) \right. \right. \\
&\quad - \psi_a^{(-)}(x) \bar{\psi}_b^{(-)}(y) \overline{\phi(x)} \phi(y) \bar{\psi}_a^{(+)}(x) \psi_b^{(+)}(y) \left. \left. \right] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle
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类似地，第3项与第4项也具有这种交换 $x$ 和 $y$ 的对称性

这是个普遍规律:  $iT^{(n)} = \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \mathcal{T}[\mathcal{H}_1(x_1) \cdots \mathcal{H}_1(x_n)]$

中的  $1/n!$  因子总是恰好与时空坐标的交换对称性引起的  $n!$  因子抵消

## 相对符号

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| i T_2^{(2)} \right| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \rangle \\
&= -(-i\kappa)^2 \int d^4x d^4y \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathbf{N}[\psi_a^{(-)}(x) \bar{\psi}_a^{(-)}(x) \phi(x) \phi(y) \bar{\psi}_b^{(+)}(y) \psi_b^{(+)}(y) \right. \\
&\quad \left. - \psi_a^{(-)}(x) \bar{\psi}_b^{(-)}(y) \phi(x) \phi(y) \bar{\psi}_a^{(+)}(x) \psi_b^{(+)}(y)] \right| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \rangle \\
&= -(-i\kappa)^2 \int d^4x d^4y \left\{ \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathbf{N}[\psi_a(x) \bar{\psi}_a(x) \phi(x) \phi(y) \bar{\psi}_b(y) \psi_b(y)] \right| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right. \\
&\quad \left. - \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathbf{N}[\psi_a(x) \bar{\psi}_b(y) \phi(x) \phi(y) \bar{\psi}_a(x) \psi_b(y)] \right| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right\} \\
&= (-i\kappa)^2 \int d^4x d^4y \left\{ \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathbf{N}[\phi(x) \bar{\psi}(x) \psi(x) \phi(y) \bar{\psi}(y) \psi(y)] \right| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right. \\
&\quad \left. + \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| \mathbf{N}[\phi(x) \bar{\psi}(x) \psi(x) \phi(y) \bar{\psi}(y) \psi(y)] \right| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right\}
\end{aligned}$$

第二步写成场算符与初末态缩并的形式，花括号中两项的相对符号为负

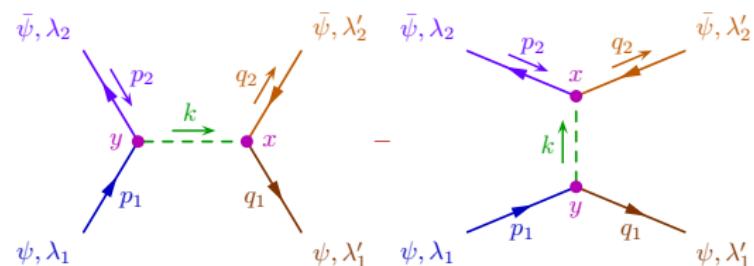
第三步将场算符调回原来的次序，不再出现额外的负号，验证了前述  $T$  矩阵元的缩并规律

### $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$ 散射的 Feynman 图

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda_1'; \mathbf{q}_2^-, \lambda_2' \middle| i T_2^{(2)} \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
&= -(-i\kappa)^2 \int d^4x d^4y \left\{ \left\langle \mathbf{q}_1^+, \lambda_1'; \mathbf{q}_2^-, \lambda_2' \middle| \mathbf{N}[\psi_a(x)\bar{\psi}_a(x)\phi(x)\phi(y)\bar{\psi}_b(y)\psi_b(y)] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right. \\
&\quad \left. - \left\langle \mathbf{q}_1^+, \lambda_1'; \mathbf{q}_2^-, \lambda_2' \middle| \mathbf{N}[\psi_a(x)\bar{\psi}_b(y)\phi(x)\phi(y)\bar{\psi}_a(x)\psi_b(y)] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right\} \\
&= -(-i\kappa)^2 \int d^4x d^4y \left[ v_a(\mathbf{q}_2, \lambda_2') e^{iq_2 \cdot x} \bar{u}_a(\mathbf{q}_1, \lambda_1') e^{iq_1 \cdot x} \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik \cdot (x-y)}}{k^2 - m_\phi^2 + i\epsilon} \right. \\
&\quad \times \bar{v}_b(\mathbf{p}_2, \lambda_2) e^{-ip_2 \cdot y} u_b(\mathbf{p}_1, \lambda_1) e^{-ip_1 \cdot y} \\
&\quad - v_a(\mathbf{q}_2, \lambda_2') e^{iq_2 \cdot x} \bar{u}_b(\mathbf{q}_1, \lambda_1') e^{iq_1 \cdot y} \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik \cdot (x-y)}}{k^2 - m_\phi^2 + i\epsilon} \\
&\quad \times \bar{v}_a(\mathbf{p}_2, \lambda_2) e^{-ip_2 \cdot x} u_b(\mathbf{p}_1, \lambda_1) e^{-ip_1 \cdot y} \left. \right]
\end{aligned}$$

樱桃 Feynman 图包含 2 个子图

 总不变振幅  $i\mathcal{M}$  是 2 个不变振幅的叠加，两者之间的**相**符对号为负，决定了  $|\mathcal{M}|^2$  中干涉项的符号



### $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$ 的 $T$ 矩阵元

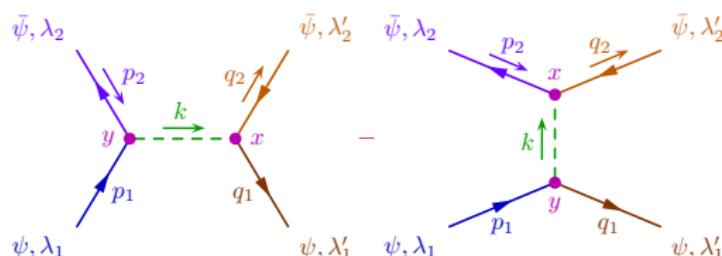
 对  $x$  和  $y$  分别积分，得

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| iT_2^{(2)} \right| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \rangle \\
&= -(-i\kappa)^2 \int \frac{d^4 k}{(2\pi)^4} \left[ \bar{u}(\mathbf{q}_1, \lambda'_1) v(\mathbf{q}_2, \lambda'_2) \frac{i}{k^2 - m_\phi^2 + i\epsilon} \bar{v}(\mathbf{p}_2, \lambda_2) u(\mathbf{p}_1, \lambda_1) \right. \\
&\quad \times (2\pi)^4 \delta^{(4)}(k - q_1 - q_2) (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k) \\
&\quad - \bar{v}(\mathbf{p}_2, \lambda_2) v(\mathbf{q}_2, \lambda'_2) \frac{i}{k^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_1, \lambda_1) \\
&\quad \left. \times (2\pi)^4 \delta^{(4)}(k + p_2 - q_2) (2\pi)^4 \delta^{(4)}(p_1 - q_1 - k) \right]
\end{aligned}$$

对  $x$  和  $y$  积分使方括号中每一项都具有 2 个四维  $\delta$  函数，它们分别代表 2 个顶点处的能动量守恒关系

大蒜 第一项关系为  $k^\mu = q_1^\mu + q_2^\mu$   
和  $k^\mu = p_1^\mu + p_2^\mu$

第二项关系为  $k^\mu = q_2^\mu - p_2^\mu$   
和  $k^\mu = p_1^\mu - q_1^\mu$



### $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$ 的 $T$ 矩阵元

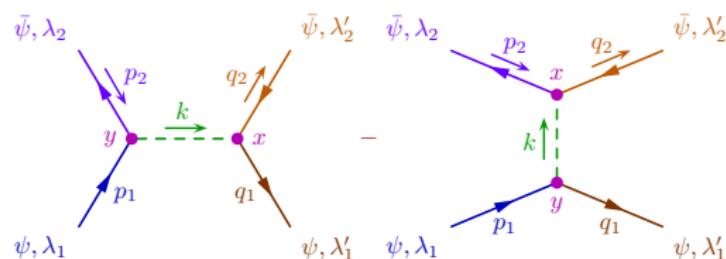
 对  $x$  和  $y$  分别积分，得

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \right| i T_2^{(2)} \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
= & -(-i\kappa)^2 \int \frac{d^4 k}{(2\pi)^4} \left[ \bar{u}(\mathbf{q}_1, \lambda'_1) v(\mathbf{q}_2, \lambda'_2) \frac{i}{k^2 - m_\phi^2 + i\epsilon} \bar{v}(\mathbf{p}_2, \lambda_2) u(\mathbf{p}_1, \lambda_1) \right. \\
& \quad \times (2\pi)^4 \delta^{(4)}(k - q_1 - q_2) (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k) \\
& \quad \left. - \bar{v}(\mathbf{p}_2, \lambda_2) v(\mathbf{q}_2, \lambda'_2) \frac{i}{k^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_1, \lambda_1) \right. \\
& \quad \left. \times (2\pi)^4 \delta^{(4)}(k + p_2 - q_2) (2\pi)^4 \delta^{(4)}(p_1 - q_1 - k) \right]
\end{aligned}$$

对  $x$  和  $y$  积分使方括号中每一项都具有 2 个四维  $\delta$  函数，它们分别代表 2 个顶点处的能动量守恒关系

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第二项关系为  $k^\mu = q_2^\mu - p_2^\mu$   
和  $k^\mu = p_1^\mu - q_1^\mu$



与同一顶点相连的内外线的四维动量应满足**能动量守恒定律**

$\psi\bar{\psi} \rightarrow \psi\bar{\psi}$  不变振幅

对  $k$  积分, 消去 1 个四维  $\delta$  函数, 得

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| iT_2^{(2)} \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right. \right\rangle \\
&= -(-i\kappa)^2 \left[ \bar{u}(\mathbf{q}_1, \lambda'_1) v(\mathbf{q}_2, \lambda'_2) \frac{i}{(p_1 + p_2)^2 - m_\phi^2 + i\epsilon} \bar{v}(\mathbf{p}_2, \lambda_2) u(\mathbf{p}_1, \lambda_1) \right. \\
&\quad \left. - \bar{v}(\mathbf{p}_2, \lambda_2) v(\mathbf{q}_2, \lambda'_2) \frac{i}{(p_1 - q_1)^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_1, \lambda_1) \right] \\
&\times (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2)
\end{aligned}$$

剩下的 1 个四维  $\delta$  函数代表初末态 4 个粒子满足的能动量守恒定律

○○○○○○○○○○

$\psi\bar{\psi} \rightarrow \psi\bar{\psi}$  不变振幅

对  $k$  积分, 消去 1 个四维  $\delta$  函数, 得

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^-, \lambda'_2 \middle| iT_2^{(2)} \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right. \right\rangle \\
&= -(-i\kappa)^2 \left[ \bar{u}(\mathbf{q}_1, \lambda'_1) v(\mathbf{q}_2, \lambda'_2) \frac{i}{(p_1 + p_2)^2 - m_\phi^2 + i\epsilon} \bar{v}(\mathbf{p}_2, \lambda_2) u(\mathbf{p}_1, \lambda_1) \right. \\
&\quad \left. - \bar{v}(\mathbf{p}_2, \lambda_2) v(\mathbf{q}_2, \lambda'_2) \frac{i}{(p_1 - q_1)^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_1, \lambda_1) \right] \\
&\times (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2)
\end{aligned}$$

剩下的 1 个四维  $\delta$  函数代表初末态 4 个粒子满足的能动量守恒定律

这符合  $\langle f | iT | i \rangle = (2\pi)^4 \delta^{(4)}(p_i - p_f) i\mathcal{M}$  式的形式，相应的**不变振幅**为

$$\begin{aligned} i\mathcal{M} = & -(-i\kappa)^2 \left[ \bar{u}(\mathbf{q}_1, \lambda'_1) v(\mathbf{q}_2, \lambda'_2) \frac{i}{(p_1 + p_2)^2 - m_\phi^2 + i\epsilon} \bar{v}(\mathbf{p}_2, \lambda_2) u(\mathbf{p}_1, \lambda_1) \right. \\ & \left. - \bar{v}(\mathbf{p}_2, \lambda_2) v(\mathbf{q}_2, \lambda'_2) \frac{i}{(p_1 - q_1)^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_1, \lambda_1) \right] \end{aligned}$$

 这个表达式不包含积分，内线动量由外线动量完全确定，这是树图的特征



### $\psi\psi \rightarrow \psi\psi$ 散射过程

(2)  $iT_2^{(2)}$  也可以贡献到  $\psi\psi \rightarrow \psi\psi$  散射过程, 记初态为  $|i\rangle = |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2\rangle$ , 末态为  $\langle f| = \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2|$ , 相应的  $T$  矩阵元为

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 \middle| i T_2^{(2)} \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \right\rangle \\
&= \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 \middle| \mathbf{N}[\overline{\phi(x)\bar{\psi}(x)\psi(x)}\phi(y)\bar{\psi}(y)\psi(y)] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \right\rangle \\
&= -\frac{(-i\kappa)^2}{2!} \int d^4x d^4y \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 \middle| \mathbf{N}[\bar{\psi}_a^{(-)}(x)\bar{\psi}_b^{(-)}(y) \right. \\
&\quad \left. \times \overline{\phi(x)\phi(y)}\psi_a^{(+)}(x)\psi_b^{(+)}(y)] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \right\rangle
\end{aligned}$$



这里出现了  两个正能解旋量场算符对全同费米子初态的作用



## 两个负能解旋量场算符对全同费米子末态的作用

## 两个正能解旋量场算符对全同费米子初态的作用

类似于 5.5.4 小节计算双费米子态内积的过程，多次运用产生湮灭算符的反对易关系，得到两个正能解旋量场算符对全同费米子初态的作用为

$$\begin{aligned}
& \psi_a^{(+)}(x)\psi_b^{(+)}(y) |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2\rangle \\
&= \int \frac{d^3k_1 d^3k_2}{(2\pi)^6 \sqrt{4E_{\mathbf{k}_1}E_{\mathbf{k}_2}}} \sum_{\sigma_1\sigma_2} u_a(\mathbf{k}_1, \sigma_1) e^{-i k_1 \cdot x} u_b(\mathbf{k}_2, \sigma_2) e^{-i k_2 \cdot y} \\
&\quad \times \sqrt{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}} a_{\mathbf{k}_1, \sigma_1} \textcolor{red}{a_{\mathbf{k}_2, \sigma_2}} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger |0\rangle \\
&= \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \sqrt{\frac{E_{\mathbf{p}_1}E_{\mathbf{p}_2}}{E_{\mathbf{k}_1}E_{\mathbf{k}_2}}} e^{-i(k_1 \cdot x + k_2 \cdot y)} \sum_{\sigma_1\sigma_2} u_a(\mathbf{k}_1, \sigma_1) u_b(\mathbf{k}_2, \sigma_2) \\
&\quad \times a_{\mathbf{k}_1, \sigma_1} [(\textcolor{red}{2\pi})^3 \delta_{\sigma_2 \lambda_1} \delta^{(3)}(\mathbf{k}_2 - \mathbf{p}_1) - a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{k}_2, \sigma_2}] a_{\mathbf{p}_2, \lambda_2}^\dagger |0\rangle \\
&= \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \sqrt{\frac{E_{\mathbf{p}_1}E_{\mathbf{p}_2}}{E_{\mathbf{k}_1}E_{\mathbf{k}_2}}} e^{-i(k_1 \cdot x + k_2 \cdot y)} \sum_{\sigma_1\sigma_2} u_a(\mathbf{k}_1, \sigma_1) u_b(\mathbf{k}_2, \sigma_2) \\
&\quad \times (2\pi)^6 [\delta_{\sigma_2 \lambda_1} \delta^{(3)}(\mathbf{k}_2 - \mathbf{p}_1) \delta_{\sigma_1 \lambda_2} \delta^{(3)}(\mathbf{k}_1 - \mathbf{p}_2) \\
&\quad - \delta_{\sigma_1 \lambda_1} \delta^{(3)}(\mathbf{k}_1 - \mathbf{p}_1) \delta_{\sigma_2 \lambda_2} \delta^{(3)}(\mathbf{k}_2 - \mathbf{p}_2)] |0\rangle \\
&= [u_a(\mathbf{p}_2, \lambda_2) u_b(\mathbf{p}_1, \lambda_1) e^{-i(\textcolor{brown}{p}_2 \cdot x + \textcolor{blue}{p}_1 \cdot y)} - u_a(\mathbf{p}_1, \lambda_1) u_b(\mathbf{p}_2, \lambda_2) e^{-i(\textcolor{blue}{p}_1 \cdot x + \textcolor{brown}{p}_2 \cdot y)}] |0\rangle
\end{aligned}$$

### 场算符与初态的两种可能缩并

$$\begin{aligned}
& \psi_a^{(+)}(x)\psi_b^{(+)}(y) |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2\rangle \\
&= [u_a(\mathbf{p}_2, \lambda_2)u_b(\mathbf{p}_1, \lambda_1)e^{-i(p_2 \cdot x + p_1 \cdot y)} - u_a(\mathbf{p}_1, \lambda_1)u_b(\mathbf{p}_2, \lambda_2)e^{-i(p_1 \cdot x + p_2 \cdot y)}] |0\rangle \\
&= \mathsf{N}[\psi_a(x)\psi_b(y)] |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2\rangle - \mathsf{N}[\psi_b(y)\psi_a(x)] |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2\rangle \\
&= \mathsf{N}[\psi_a(x)\psi_b(y)] |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2\rangle + \mathsf{N}[\psi_a(x)\psi_b(y)] |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2\rangle
\end{aligned}$$

第一、二步中第二项前面的负号体现了交换全同费米子的反对称性

🏀 交换第二项两个场算符恢复原来的次序之后，这个负号没有出现在第三步中，此时表示缩并的线纠缠起来

可见，两个正能解旋量场算符对全同费米子初态的作用对应于场算符与初态的两种可能缩并

## 两个负能解旋量场算符对全同费米子末态的作用



两个负能解旋量场算符对全同费米子末态的作用为

$$\begin{aligned}
& \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \bar{\psi}_a^{(-)}(x) \bar{\psi}_b^{(-)}(y) \\
&= \int \frac{d^3 k_1 d^3 k_2}{(2\pi)^6 \sqrt{4E_{\mathbf{k}_1} E_{\mathbf{k}_2}}} \sum_{\sigma_1 \sigma_2} \bar{u}_a(\mathbf{k}_1, \sigma_1) e^{ik_1 \cdot x} \bar{u}_b(\mathbf{k}_2, \sigma_2) e^{ik_2 \cdot y} \\
&\quad \times \sqrt{4E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \langle 0 | a_{\mathbf{q}_1, \lambda'_1} a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{k}_1, \sigma_1}^\dagger a_{\mathbf{k}_2, \sigma_2}^\dagger \\
&= \dots = \langle 0 | [\bar{u}_a(\mathbf{q}_2, \lambda'_2) \bar{u}_b(\mathbf{q}_1, \lambda'_1) e^{i(q_2 \cdot x + q_1 \cdot y)} - \bar{u}_a(\mathbf{q}_1, \lambda'_1) \bar{u}_b(\mathbf{q}_2, \lambda'_2) e^{i(q_1 \cdot x + q_2 \cdot y)}] \\
&= \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathsf{N}[\bar{\psi}_a(x) \bar{\psi}_b(y)] - \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathsf{N}[\bar{\psi}_b(y) \bar{\psi}_a(x)] \\
&= \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathsf{N}[\bar{\psi}_a(x) \bar{\psi}_b(y)] + \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathsf{N}[\bar{\psi}_a(x) \bar{\psi}_b(y)]
\end{aligned}$$



倒数第二、三步中第二项前面的负号体现了交换全同费米子的反对称性



可见，这种作用对应于场算符与末态的两种可能缩并

### 与全同费米子初末态的缩并

根据上述作用结果, 得到花括号中的 4 项, 对应于初态和末态各自的 2 种缩并

$$\begin{aligned}
& \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | i T_2^{(2)} | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \\
&= -\frac{(-i\kappa)^2}{2!} \int d^4x d^4y \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathbf{N}[\bar{\psi}_a^{(-)}(x)\bar{\psi}_b^{(-)}(y) \\
&\quad \times \phi(x)\phi(y)\psi_a^{(+)}(x)\psi_b^{(+)}(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \\
&= -\frac{(-i\kappa)^2}{2!} \int d^4x d^4y \\
&\quad \times \left\{ \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathbf{N}[\bar{\psi}_a(x)\bar{\psi}_b(y)\phi(x)\phi(y)\psi_a(x)\psi_b(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \right. \\
&\quad - \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathbf{N}[\bar{\psi}_b(y)\bar{\psi}_a(x)\phi(x)\phi(y)\psi_a(x)\psi_b(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \\
&\quad - \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathbf{N}[\bar{\psi}_a(x)\bar{\psi}_b(y)\phi(x)\phi(y)\psi_b(y)\psi_a(x)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \\
&\quad \left. + \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathbf{N}[\bar{\psi}_b(y)\bar{\psi}_a(x)\phi(x)\phi(y)\psi_b(y)\psi_a(x)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \right\}
\end{aligned}$$



## 时空坐标的交换对称性

$$\begin{aligned}
& \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | i T_2^{(2)} | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \\
&= -\frac{(-i\kappa)^2}{2!} \int d^4x d^4y \\
&\quad \times \left\{ \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | N[\bar{\psi}_a(x)\bar{\psi}_b(y)\phi(x)\phi(y)\psi_a(x)\psi_b(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \right. \\
&\quad - \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | N[\bar{\psi}_b(y)\bar{\psi}_a(x)\phi(x)\phi(y)\psi_a(x)\psi_b(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \\
&\quad - \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | N[\bar{\psi}_a(x)\bar{\psi}_b(y)\phi(x)\phi(y)\psi_b(y)\psi_a(x)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \\
&\quad \left. + \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | N[\bar{\psi}_b(y)\bar{\psi}_a(x)\phi(x)\phi(y)\psi_b(y)\psi_a(x)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \right\} \\
&= -(-i\kappa)^2 \int d^4x d^4y \\
&\quad \times \left\{ \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | N[\bar{\psi}_a(x)\bar{\psi}_b(y)\phi(x)\phi(y)\psi_a(x)\psi_b(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \right. \\
&\quad \left. - \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | N[\bar{\psi}_b(y)\bar{\psi}_a(x)\phi(x)\phi(y)\psi_a(x)\psi_b(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \right\}
\end{aligned}$$

第一步第 1、4 项和第 2、3 项分别具有交换时空坐标  $x$  和  $y$  的对称性，贡献相等  
在第二步中只保留第 1 项和第 2 项，并消去最前面的  $1/2!$  因子

## $T$ 矩阵元的缩并规律

$$\begin{aligned}
& \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | i T_2^{(2)} | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \\
&= -(-i\kappa)^2 \int d^4x d^4y \\
&\quad \times \left\{ \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathbf{N}[\bar{\psi}_a(x)\bar{\psi}_b(y)\phi(x)\phi(y)\psi_a(x)\psi_b(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \right. \\
&\quad \left. - \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathbf{N}[\bar{\psi}_b(y)\bar{\psi}_a(x)\phi(x)\phi(y)\psi_a(x)\psi_b(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \right\} \\
&= (-i\kappa)^2 \int d^4x d^4y \\
&\quad \times \left\{ \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \right. \\
&\quad \left. + \langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 | \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \right\}
\end{aligned}$$

第二步将场算符调回原来的次序，不再有额外负号，符合  $T$  矩阵元的缩并规律

根据这个规律，也可以直接写出第二步的结果，接着把纠缠的缩并线解开成第一步的形式，确定**相对符号**，再往下计算

### $\psi\psi \rightarrow \psi\psi$ 散射的 Feynman 图

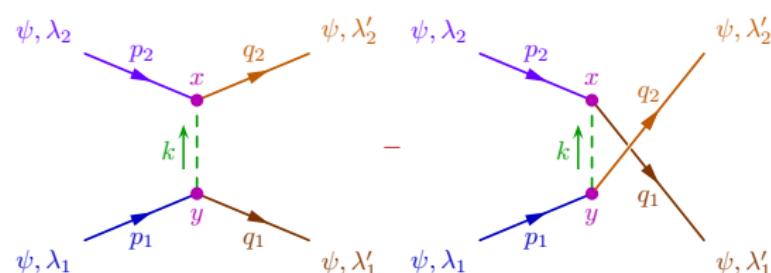
$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 \middle| i T_2^{(2)} \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \right\rangle \\
&= -(-i\kappa)^2 \int d^4x d^4y \\
&\quad \times \left\{ \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 \middle| \mathsf{N}[\bar{\psi}_a(x)\bar{\psi}_b(y)\phi(x)\phi(y)\psi_a(x)\psi_b(y)] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \right\rangle \right. \\
&\quad \left. - \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 \middle| \mathsf{N}[\bar{\psi}_b(y)\bar{\psi}_a(x)\phi(x)\phi(y)\psi_a(x)\psi_b(y)] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \right\rangle \right\} \\
&= -(-i\kappa)^2 \int d^4x d^4y [\bar{u}_a(\mathbf{q}_2, \lambda'_2)\bar{u}_b(\mathbf{q}_1, \lambda'_1)e^{i(\mathbf{q}_2 \cdot \mathbf{x} + q_1 \cdot \mathbf{y})} - \bar{u}_b(\mathbf{q}_2, \lambda'_2)\bar{u}_a(\mathbf{q}_1, \lambda'_1)e^{i(\mathbf{q}_2 \cdot \mathbf{y} + q_1 \cdot \mathbf{x})}] \\
&\quad \times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}}{k^2 - m_\phi^2 + i\epsilon} u_a(\mathbf{p}_2, \lambda_2)u_b(\mathbf{p}_1, \lambda_1)e^{-i(\mathbf{p}_1 \cdot \mathbf{y} + \mathbf{p}_2 \cdot \mathbf{x})}
\end{aligned}$$

 Feynman 图包含 2 个子图

第 2 个子图可通过**交换**第 1 个

子图中末态两条费米子外线得到

两个子图相对符号为负, 体现  
交换末态全同费米子的反对称性



## Feynman 图拓扑结构

交换左上图中初态的两条费米子外线, 得到右下图

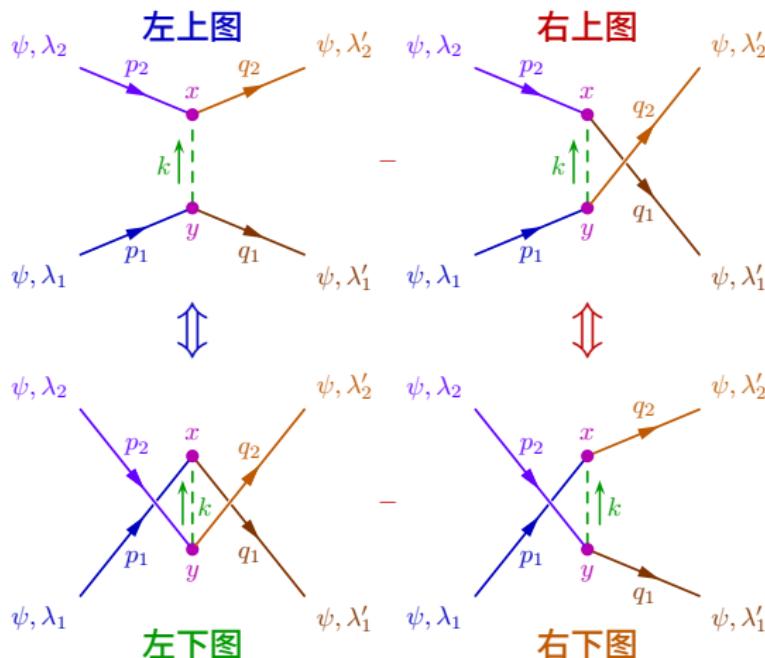
在拓扑结构上, 右下图与右上图唯一的差别是两个顶点上的  $x$  和  $y$  标签位置相反, 故两者具有交换时空坐标的对称性

由于对  $x$  和  $y$  都进行了积分，  
右下图与右上图是拓扑等价的，  
可通过连续变形相互转换

同理，交换右上图中初态两条费米子外线得到的左下图与左上图是拓扑等价的

实际上，左下图和右下图对应于前文利用时空坐标的交换对称性合并掉的两项

左上图和右上图包含了  $iT_2^{(2)}$  贡献到  $\psi\psi \rightarrow \psi\psi$  散射过程的全部可能拓扑结构



$\psi\psi \rightarrow \psi\psi$  的  $T$  矩阵元

 对上述表达式逐步积分，得

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda'_1; \mathbf{q}_2^+, \lambda'_2 \middle| i T_2^{(2)} \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \right. \right\rangle \\
&= -(-i\kappa)^2 \int \mathbf{d}^4x \mathbf{d}^4y [\bar{u}_a(\mathbf{q}_2, \lambda'_2) \bar{u}_b(\mathbf{q}_1, \lambda'_1) e^{i(q_2 \cdot \mathbf{x} + q_1 \cdot \mathbf{y})} - \bar{u}_b(\mathbf{q}_2, \lambda'_2) \bar{u}_a(\mathbf{q}_1, \lambda'_1) e^{i(q_2 \cdot \mathbf{y} + q_1 \cdot \mathbf{x})}] \\
&\quad \times \int \frac{\mathbf{d}^4k}{(2\pi)^4} \frac{i e^{-ik \cdot (\mathbf{x} - \mathbf{y})}}{k^2 - m_\phi^2 + i\epsilon} u_a(\mathbf{p}_2, \lambda_2) u_b(\mathbf{p}_1, \lambda_1) e^{-i(p_1 \cdot \mathbf{y} + p_2 \cdot \mathbf{x})} \\
&= -(-i\kappa)^2 \int \frac{\mathbf{d}^4k}{(2\pi)^4} \left[ \bar{u}(\mathbf{q}_2, \lambda'_2) u(\mathbf{p}_2, \lambda_2) \frac{i}{k^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_1, \lambda_1) \right. \\
&\quad \times (2\pi)^4 \delta^{(4)}(\mathbf{k} + \mathbf{p}_2 - \mathbf{q}_2) (2\pi)^4 \delta^{(4)}(\mathbf{p}_1 - \mathbf{q}_1 - \mathbf{k}) \\
&\quad - \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_2, \lambda_2) \frac{i}{k^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_2, \lambda'_2) u(\mathbf{p}_1, \lambda_1) \\
&\quad \left. \times (2\pi)^4 \delta^{(4)}(\mathbf{k} + \mathbf{p}_2 - \mathbf{q}_1) (2\pi)^4 \delta^{(4)}(\mathbf{p}_1 - \mathbf{q}_2 - \mathbf{k}) \right] \\
&= -(-i\kappa)^2 \left[ \bar{u}(\mathbf{q}_2, \lambda'_2) u(\mathbf{p}_2, \lambda_2) \frac{i}{(p_1 - q_1)^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_1, \lambda_1) \right. \\
&\quad - \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_2, \lambda_2) \frac{i}{(p_1 - q_2)^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_2, \lambda'_2) u(\mathbf{p}_1, \lambda_1) \\
&\quad \left. \times (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \right]
\end{aligned}$$

### $\psi\bar{\psi} \rightarrow \phi\phi$ 湮灭过程

(3) 接着讨论一对正反  $\psi$  费米子湮灭 (annihilation) 成一对全同  $\phi$  玻色子的过程  $\psi\bar{\psi} \rightarrow \phi\phi$ ，初末态分别为  $|i\rangle = |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2\rangle$  和  $|f\rangle = |\mathbf{k}_1; \mathbf{k}_2\rangle$

和  $iT_4^{(2)}$  都会贡献到这个过程，且满足

$$\begin{aligned}
\mathrm{i}T_4^{(2)} &= \frac{(-\mathrm{i}\kappa)^2}{2!} \int \mathrm{d}^4x \mathrm{d}^4y \mathsf{N}[\phi(x)\bar{\psi}_a(x)\psi_a(x)\phi(y)\bar{\psi}_b(y)\psi_b(y)] \\
&= \frac{(-\mathrm{i}\kappa)^2}{2!} \int \mathrm{d}^4x \mathrm{d}^4y \mathsf{N}[\phi(y)\bar{\psi}_b(y)\psi_b(y)\phi(x)\bar{\psi}_a(x)\psi_a(x)] \\
&= \frac{(-\mathrm{i}\kappa)^2}{2!} \int \mathrm{d}^4y \mathrm{d}^4x \mathsf{N}[\phi(x)\bar{\psi}_b(x)\psi_b(x)\phi(y)\bar{\psi}_a(y)\psi_a(y)] = \mathrm{i}T_3^{(2)}
\end{aligned}$$

第二步在正规乘积内移动场算符，第三步交换了时空坐标  $x$  和  $y$ ，故

$$iT_3^{(2)} + iT_4^{(2)} = 2iT_3^{(2)} = (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{k}_1; \mathbf{k}_2 | \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)]$$

可见,  $iT_3^{(2)}$  和  $iT_4^{(2)}$  具有交换时空坐标的对称性, 两项相加刚好抵消  $1/2!$  因子

### $\psi\bar{\psi} \rightarrow \phi\phi$ 的 $T$ 矩阵元

和  $iT_4^{(2)}$  对  $\psi\bar{\psi} \rightarrow \phi\phi$  的  $T$  矩阵元贡献为

$$\begin{aligned}
& \langle \mathbf{k}_1; \mathbf{k}_2 | (iT_3^{(2)} + iT_4^{(2)}) \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
&= (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{k}_1; \mathbf{k}_2 | \mathsf{N}[\phi(x)\bar{\psi}_a(x)\psi_a(x)\phi(y)\bar{\psi}_b(y)\psi_b(y)] \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
&= (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{k}_1; \mathbf{k}_2 | \mathsf{N}[\phi^{(-)}(x)\phi^{(-)}(y)\bar{\psi}_a(x)\psi_a(x)\bar{\psi}_b(y)\psi_b(y)] \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle
\end{aligned}$$

这里出现两个负能解标量场算符对全同玻色子末态的作用，作用结果为

$$\begin{aligned}
& \langle \mathbf{k}_1; \mathbf{k}_2 | \phi^{(-)}(x) \phi^{(-)}(y) \\
&= \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^6 \sqrt{4E_{\mathbf{q}_1} E_{\mathbf{q}_2}}} e^{i(q_1 \cdot x + q_2 \cdot y)} \sqrt{4E_{\mathbf{k}_1} E_{\mathbf{k}_2}} \langle 0 | a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{q}_1}^\dagger a_{\mathbf{q}_2}^\dagger \\
&= \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^6} \sqrt{\frac{E_{\mathbf{k}_1} E_{\mathbf{k}_2}}{E_{\mathbf{q}_1} E_{\mathbf{q}_2}}} e^{i(q_1 \cdot x + q_2 \cdot y)} \langle 0 | a_{\mathbf{k}_1} [(2\pi)^3 \delta^{(3)}(\mathbf{k}_2 - \mathbf{q}_1) + a_{\mathbf{q}_1}^\dagger a_{\mathbf{k}_2}] a_{\mathbf{q}_2}^\dagger \\
&= \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^6} \sqrt{\frac{E_{\mathbf{k}_1} E_{\mathbf{k}_2}}{E_{\mathbf{q}_1} E_{\mathbf{q}_2}}} e^{i(q_1 \cdot x + q_2 \cdot y)} \\
&\quad \times \langle 0 | (2\pi)^6 [\delta^{(3)}(\mathbf{k}_2 - \mathbf{q}_1) \delta^{(3)}(\mathbf{k}_1 - \mathbf{q}_2) + \delta^{(3)}(\mathbf{k}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{k}_2 - \mathbf{q}_2)]
\end{aligned}$$



# 全同正负能解场算符对全同粒子态的作用

$$\langle \mathbf{k}_1; \mathbf{k}_2 | \phi^{(-)}(x) \phi^{(-)}(y) = \langle 0 | [e^{i(k_2 \cdot x + k_1 \cdot y)} + e^{i(k_1 \cdot x + k_2 \cdot y)}]$$

$$= \langle \mathbf{k}_1; \mathbf{k}_2 | N[\phi(x) \phi(y)] + \langle \mathbf{k}_1; \mathbf{k}_2 | N[\phi(y) \phi(x)] = \langle \mathbf{k}_1; \mathbf{k}_2 | N[\phi(x) \phi(y)] + \langle \mathbf{k}_1; \mathbf{k}_2 | N[\phi(x) \phi(y)]$$

蛋糕图标 这对应于场算符与末态的两种可能缩并

寿司图标 第一、二步中第二项前面的正号体现了交换全同玻色子的对称性

毛线球图标 第三步交换第二项两个场算符恢复原来的次序之后，此时表示缩并的线纠缠起来

## 全同正负能解场算符对全同粒子态的作用

$$\begin{aligned} \langle \mathbf{k}_1; \mathbf{k}_2 | \phi^{(-)}(x) \phi^{(-)}(y) &= \langle 0 | [e^{i(k_2 \cdot x + k_1 \cdot y)} + e^{i(k_1 \cdot x + k_2 \cdot y)}] \\ &= \underbrace{\langle \mathbf{k}_1; \mathbf{k}_2 | \mathbf{N}[\phi(x)\phi(y)]}_{\text{purple}} + \underbrace{\langle \mathbf{k}_1; \mathbf{k}_2 | \mathbf{N}[\phi(y)\phi(x)]}_{\text{green}} = \underbrace{\langle \mathbf{k}_1; \mathbf{k}_2 | \mathbf{N}[\phi(x)\phi(y)]}_{\text{purple}} + \underbrace{\langle \mathbf{k}_1; \mathbf{k}_2 | \mathbf{N}[\phi(x)\phi(y)]}_{\text{green}} \end{aligned}$$

这对应于场算符与末态的两种可能缩并

第一、二步中第二项前面的正号体现了交换全同玻色子的对称性

第三步交换第二项两个场算符恢复原来的次序之后，此时表示缩并的线纠缠起来

同理, 两个正能解标量场算符对全同玻色子初态的作用结果为

$$\begin{aligned} \phi^{(+)}(x)\phi^{(+)}(y)|\mathbf{k}_1; \mathbf{k}_2\rangle &= [e^{-i(k_2 \cdot x + k_1 \cdot y)} + e^{-i(k_1 \cdot x + k_2 \cdot y)}] |0\rangle \\ &= \mathsf{N}[\phi(x)\phi(y)]|\mathbf{k}_1; \mathbf{k}_2\rangle + \mathsf{N}[\phi(y)\phi(x)]|\mathbf{k}_1; \mathbf{k}_2\rangle = \mathsf{N}[\phi(x)\phi(y)]|\mathbf{k}_1; \mathbf{k}_2\rangle + \mathsf{N}[\phi(x)\phi(y)]|\mathbf{k}_1; \mathbf{k}_2\rangle \end{aligned}$$

这对于~~场算符与初态的两种可能缩并~~



# 全同正负能解场算符对全同粒子态的作用

$$\langle \mathbf{k}_1; \mathbf{k}_2 | \phi^{(-)}(x) \phi^{(-)}(y) = \langle 0 | [e^{i(k_2 \cdot x + k_1 \cdot y)} + e^{i(k_1 \cdot x + k_2 \cdot y)}]$$

$$= \langle \mathbf{k}_1; \mathbf{k}_2 | N[\phi(x) \phi(y)] + \langle \mathbf{k}_1; \mathbf{k}_2 | N[\phi(y) \phi(x)] = \langle \mathbf{k}_1; \mathbf{k}_2 | N[\phi(x) \phi(y)] + \langle \mathbf{k}_1; \mathbf{k}_2 | N[\phi(x) \phi(y)]$$

蛋糕 这对应于场算符与末态的两种可能缩并

寿司 第一、二步中第二项前面的正号体现了交换全同玻色子的对称性

毛线 第三步交换第二项两个场算符恢复原来的次序之后，此时表示缩并的线纠缠起来

蛋挞 同理，两个正能解标量场算符对全同玻色子初态的作用结果为

$$\phi^{(+)}(x) \phi^{(+)}(y) | \mathbf{k}_1; \mathbf{k}_2 \rangle = [e^{-i(k_2 \cdot x + k_1 \cdot y)} + e^{-i(k_1 \cdot x + k_2 \cdot y)}] | 0 \rangle$$

$$= N[\phi(x) \phi(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle + N[\phi(y) \phi(x)] | \mathbf{k}_1; \mathbf{k}_2 \rangle = N[\phi(x) \phi(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle + N[\phi(x) \phi(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle$$

金币 这对应于场算符与初态的两种可能缩并

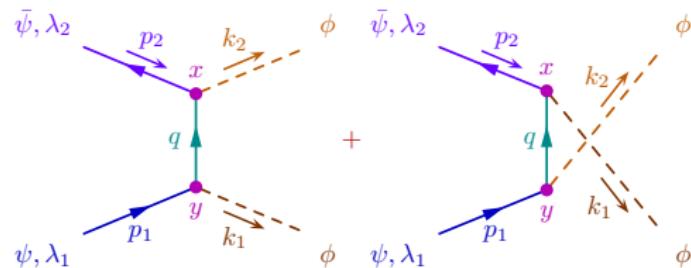


类似地可以证明， $n$  个全同正（负）能解场算符对  $n$  个相应全同粒子初（末）态的作用等价于这些场算符与初（末）态的  $n!$  种缩并

### $\psi\bar{\psi} \rightarrow \phi\phi$ 湮灭过程的 Feynman 图

$$\begin{aligned}
& \langle \mathbf{k}_1; \mathbf{k}_2 | (iT_3^{(2)} + iT_4^{(2)}) \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right. \\
&= (-i\kappa)^2 \int d^4x d^4y \left\{ \left. \langle \mathbf{k}_1; \mathbf{k}_2 | \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right. \right. \\
&\quad + \left. \left. \langle \mathbf{k}_1; \mathbf{k}_2 | \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right\} \right. \\
&= (-i\kappa)^2 \int d^4x d^4y \left\{ \left. \langle \mathbf{k}_1; \mathbf{k}_2 | \mathsf{N}[\phi(x)\phi(y)\psi_a(x)\bar{\psi}_b(y)\bar{\psi}_a(x)\psi_b(y)] \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right. \right. \\
&\quad + \left. \left. \langle \mathbf{k}_1; \mathbf{k}_2 | \mathsf{N}[\phi(y)\phi(x)\psi_a(x)\bar{\psi}_b(y)\bar{\psi}_a(x)\psi_b(y)] \left| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \right\} \right. \\
&= (-i\kappa)^2 \int d^4x d^4y [e^{i(k_2 \cdot x + k_1 \cdot y)} + e^{i(k_1 \cdot x + k_2 \cdot y)}] \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q} + m_\psi)_{ab}}{q^2 - m_\psi^2 + i\epsilon} e^{-iq \cdot (x-y)}
\end{aligned}$$

Feynman 图包含 2 个拓扑不等价的子图，相对符号为正，体现交换末态两个全同玻色子的对称性





### $\psi\bar{\psi} \rightarrow \phi\phi$ 的 $T$ 矩阵元



逐步积分，得

$$\begin{aligned}
& \langle \mathbf{k}_1; \mathbf{k}_2 | (i T_3^{(2)} + i T_4^{(2)}) | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \rangle \\
&= (-i\kappa)^2 \int d^4x d^4y [e^{i(k_2 \cdot \mathbf{x} + k_1 \cdot \mathbf{y})} + e^{i(k_1 \cdot \mathbf{x} + k_2 \cdot \mathbf{y})}] \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q} + m_\psi)_{ab}}{q^2 - m_\psi^2 + i\epsilon} e^{-iq \cdot (\mathbf{x} - \mathbf{y})} \\
&\quad \times \bar{v}_a(\mathbf{p}_2, \lambda_2) e^{-ip_2 \cdot \mathbf{x}} u_b(\mathbf{p}_1, \lambda_1) e^{-ip_1 \cdot \mathbf{y}} \\
&= (-i\kappa)^2 \int \frac{d^4q}{(2\pi)^4} \left[ \bar{v}(\mathbf{p}_2, \lambda_2) \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}_1, \lambda_1) \right. \\
&\quad \times (2\pi)^4 \delta^{(4)}(\not{q} + p_2 - k_2) (2\pi)^4 \delta^{(4)}(p_1 - k_1 - \not{q}) \\
&\quad + \bar{v}(\mathbf{p}_2, \lambda_2) \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}_1, \lambda_1) \\
&\quad \left. \times (2\pi)^4 \delta^{(4)}(\not{q} + p_2 - k_1) (2\pi)^4 \delta^{(4)}(p_1 - k_2 - \not{q}) \right] \\
&= (-i\kappa)^2 \left[ \bar{v}(\mathbf{p}_2, \lambda_2) \frac{i(\not{p}_1 - \not{k}_1 + m_\psi)}{(p_1 - k_1)^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}_1, \lambda_1) \right. \\
&\quad \left. + \bar{v}(\mathbf{p}_2, \lambda_2) \frac{i(\not{p}_1 - \not{k}_2 + m_\psi)}{(p_1 - k_2)^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}_1, \lambda_1) \right] (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2)
\end{aligned}$$

$iT_5^{(2)}$  和  $iT_6^{(2)}$

(4)  $iT_5^{(2)}$  和  $iT_6^{(2)}$  也具有交换时空坐标的对称性

$$\begin{aligned}
\mathrm{i}T_5^{(2)} &= \frac{(-\mathrm{i}\kappa)^2}{2!} \int \mathrm{d}^4x \mathrm{d}^4y \mathsf{N}[\phi(x) \bar{\psi}(x) \overline{\psi(x)\phi(y)\bar{\psi}(y)\psi(y)}] \\
&= \frac{(-\mathrm{i}\kappa)^2}{2!} \int \mathrm{d}^4x \mathrm{d}^4y \mathsf{N}[\phi(y) \bar{\psi}(y) \psi(y) \phi(x) \bar{\psi}(x) \overline{\psi(x)\phi(y)\bar{\psi}(y)\psi(y)}] \\
&= \frac{(-\mathrm{i}\kappa)^2}{2!} \int \mathrm{d}^4y \mathrm{d}^4x \mathsf{N}[\phi(x) \bar{\psi}(x) \psi(x) \phi(y) \bar{\psi}(y) \overline{\psi(y)\phi(y)\bar{\psi}(y)\psi(y)}] = \mathrm{i}T_6^{(2)}
\end{aligned}$$

$$iT_5^{(2)} + iT_6^{(2)} = 2iT_5^{(2)} = (-i\kappa)^2 \int d^4x d^4y \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)]$$

考虑初态  $|i\rangle = |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2\rangle$  包含一对正反费米子，末态  $\langle f| = \langle \mathbf{k}_1; \mathbf{k}_2|$  包含一对全同玻色子，则  $iT_5^{(2)} + iT_6^{(2)}$  对  $T$  矩阵元的贡献为

$$\langle \mathbf{k}_1; \mathbf{k}_2 | (iT_5^{(2)} + iT_6^{(2)}) | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \rangle \\ = (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{k}_1; \mathbf{k}_2 | N[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \rangle$$

## 不相连的 Feynman 图

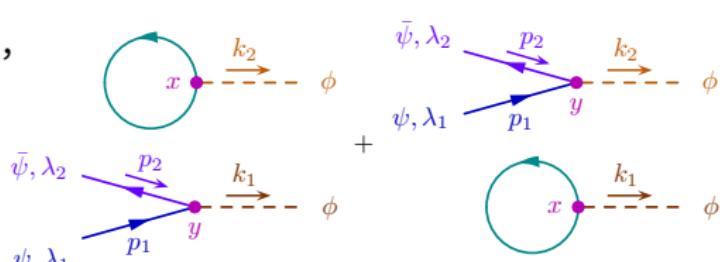
$$\begin{aligned}
& \langle \mathbf{k}_1; \mathbf{k}_2 | (iT_5^{(2)} + iT_6^{(2)}) | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \rangle \\
&= (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{k}_1; \mathbf{k}_2 | \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \rangle \\
&= (-i\kappa)^2 \int d^4x d^4y \{ \langle \mathbf{k}_1; \mathbf{k}_2 | \mathsf{N}[\overbrace{\phi(x)\bar{\psi}(x)}^{\text{orange}} \overbrace{\psi(x)\phi(y)}^{\text{green}} \overbrace{\bar{\psi}(y)\psi(y)}^{\text{blue}}] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \rangle \\
&\quad + \langle \mathbf{k}_1; \mathbf{k}_2 | \mathsf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \rangle \}
\end{aligned}$$

第二步跳过用正负能解表达的步骤，  
直接按照  $T$  矩阵元的缩并规律写下  
场算符与初末态的 2 种可能缩并

相应 Feynman 图包含 2 个子图

每个子图都具有 2 个不相连的部 分，这些部分是在  $\kappa^1$  阶讨论过的，它们之间不会相互影响

由于末态中有一个  $\phi$  粒子没有跟其它粒子连接起来，这种图既不满足能动量守恒定律，也不会贡献到  $\psi\bar{\psi} \rightarrow \phi\phi$  散射过程



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$iT_7^{(2)}$  和  $iT_8^{(2)}$

(5)  $iT_7^{(2)}$  和  $iT_8^{(2)}$  具有交换时空坐标的对称性

$$\begin{aligned}
\mathrm{i}T_7^{(2)} &= \frac{(-\mathrm{i}\kappa)^2}{2!} \int \mathrm{d}^4x \mathrm{d}^4y \mathsf{N}[\overbrace{\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)}^{\text{brown}}] \\
&= \frac{(-\mathrm{i}\kappa)^2}{2!} \int \mathrm{d}^4x \mathrm{d}^4y \mathsf{N}[\overbrace{\phi(\textcolor{violet}{y})\bar{\psi}(\textcolor{violet}{y})\psi(\textcolor{violet}{y})\phi(\textcolor{blue}{x})\bar{\psi}(\textcolor{blue}{x})\psi(\textcolor{blue}{x})}^{\text{brown}}] \\
&= \frac{(-\mathrm{i}\kappa)^2}{2!} \int \mathrm{d}^4y \mathrm{d}^4x \mathsf{N}[\overbrace{\phi(\textcolor{blue}{x})\bar{\psi}(\textcolor{blue}{x})\psi(\textcolor{blue}{x})\phi(\textcolor{violet}{y})\bar{\psi}(\textcolor{violet}{y})\psi(\textcolor{violet}{y})}^{\text{brown}}] = \mathrm{i}T_8^{(2)}
\end{aligned}$$

$$iT_7^{(2)} + iT_8^{(2)} = 2iT_7^{(2)} = (-i\kappa)^2 \int d^4x d^4y \mathsf{N}[\overbrace{\phi(x)\bar{\psi}(x)\psi(x)\phi(y)}^{\text{F}}\overbrace{\bar{\psi}(y)\psi(y)}^{\text{G}}]$$

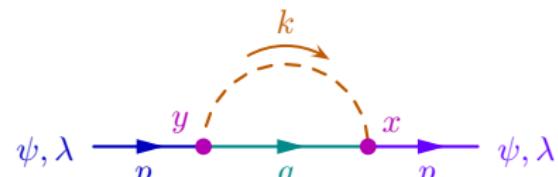
考虑初态和末态均是一个动量为  $\mathbf{p}$ 、螺旋度为  $\lambda$  的  $\psi$  粒子，即  $|i\rangle = |\mathbf{p}^+, \lambda\rangle$ ，  
 $\langle f| = \langle \mathbf{p}^+, \lambda|$ ，则  $iT_7^{(2)} + iT_8^{(2)}$  对  $T$  矩阵元的贡献为

$$\begin{aligned} & \langle \mathbf{p}^+, \lambda | (iT_7^{(2)} + iT_8^{(2)}) | \mathbf{p}^+, \lambda \rangle \\ &= (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{p}^+, \lambda | \mathbf{N}[\overbrace{\phi(x)\bar{\psi}(x)\psi(x)\phi(y)}^{\text{Fermion loop}} \overbrace{\bar{\psi}(y)\psi(y)}^{\text{Fermion loop}}] | \mathbf{p}^+, \lambda \rangle \end{aligned}$$

$$\begin{aligned}
& \langle \mathbf{p}^+, \lambda | (iT_7^{(2)} + iT_8^{(2)}) | \mathbf{p}^+, \lambda \rangle \\
&= (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{p}^+, \lambda | \mathbf{N}[\overbrace{\phi(x)\bar{\psi}(x)\psi(x)}^{\text{green}} \overbrace{\phi(y)\bar{\psi}(y)\psi(y)}^{\text{green}}] | \mathbf{p}^+, \lambda \rangle \\
&= (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{p}^+, \lambda | \mathbf{N}[\overbrace{\phi(x)\bar{\psi}(x)}^{\text{green}} \overbrace{\psi(x)\phi(y)}^{\text{green}} \overbrace{\bar{\psi}(y)\psi(y)}^{\text{green}}] | \mathbf{p}^+, \lambda \rangle \\
&= (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{p}^+, \lambda | \mathbf{N}[\overbrace{\bar{\psi}(x)}^{\text{purple}} \overbrace{\phi(x)\phi(y)}^{\text{orange}} \overbrace{\psi(x)\bar{\psi}(y)}^{\text{green}} \overbrace{\psi(y)}^{\text{green}}] | \mathbf{p}^+, \lambda \rangle \\
&= (-i\kappa)^2 \int d^4x d^4y \bar{u}(\mathbf{p}, \lambda) e^{i\mathbf{p} \cdot \mathbf{x}} \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}}{k^2 - m_\phi^2 + i\epsilon} \\
&\quad \times \int \frac{d^4q}{(2\pi)^4} \frac{i(q + m_\psi) e^{-i\mathbf{q} \cdot (\mathbf{x} - \mathbf{y})}}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda) e^{-i\mathbf{p} \cdot \mathbf{y}}
\end{aligned}$$

相应的 Feynman 图是一个圈图

这种初末态都是同一个粒子的圈图称为该粒子的**自能图** (self-energy diagram)



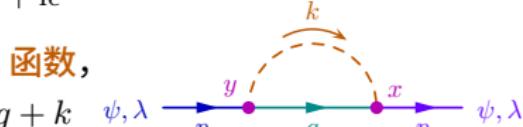


# $iT_7^{(2)} + iT_8^{(2)}$ 贡献的 $\psi$ 粒子单圈自能图

$$\begin{aligned}
 & \langle \mathbf{p}^+, \lambda | (iT_7^{(2)} + iT_8^{(2)}) | \mathbf{p}^+, \lambda \rangle \\
 &= (-i\kappa)^2 \int d^4x d^4y \bar{u}(\mathbf{p}, \lambda) e^{ip \cdot x} \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik \cdot (x-y)}}{k^2 - m_\phi^2 + i\epsilon} \\
 &\quad \times \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q} + m_\psi) e^{-iq \cdot (x-y)}}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda) e^{-ip \cdot y} \\
 &= (-i\kappa)^2 \int \frac{d^4k d^4q}{(2\pi)^8} \bar{u}(\mathbf{p}, \lambda) \frac{i}{\not{k}^2 - m_\phi^2 + i\epsilon} \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda) \\
 &\quad \times (2\pi)^4 \delta^{(4)}(q + \mathbf{k} - p) (2\pi)^4 \delta^{(4)}(p - q - \mathbf{k}) \\
 &= (-i\kappa)^2 \int \frac{d^4q}{(2\pi)^4} \bar{u}(\mathbf{p}, \lambda) \frac{i}{(p - q)^2 - m_\phi^2 + i\epsilon} \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(0)
 \end{aligned}$$

🦀 第二步对  $x$  和  $y$  积分, 得到 2 个相等的四维  $\delta$  函数,

说明 2 个顶点处的能动量守恒关系相同, 都是  $p = q + k$



🦀 最后一步对  $k$  积分, 剩下  $\delta^{(4)}(0) = \delta^{(4)}(p - p)$  体现初末态满足的能动量守恒定律

🦀 最后剩下一个未定的圈动量  $q^\mu$ , 需要对它的所有取值积分, 相应的不变振幅是

$$i\mathcal{M} = (-i\kappa)^2 \int \frac{d^4q}{(2\pi)^4} \bar{u}(\mathbf{p}, \lambda) \frac{i}{(p - q)^2 - m_\phi^2 + i\epsilon} \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda)$$

$iT_7^{(2)} + iT_8^{(2)}$  贡献的  $\psi$  粒子单圈自能图

$$\begin{aligned}
& \langle \mathbf{p}^+, \lambda | (iT_7^{(2)} + iT_8^{(2)}) | \mathbf{p}^+, \lambda \rangle \\
&= (-i\kappa)^2 \int d^4x d^4y \bar{u}(\mathbf{p}, \lambda) e^{ip \cdot x} \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik \cdot (x-y)}}{k^2 - m_\phi^2 + i\epsilon} \\
&\quad \times \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q} + m_\psi) e^{-iq \cdot (x-y)}}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda) e^{-ip \cdot y} \\
&= (-i\kappa)^2 \int \frac{d^4k d^4q}{(2\pi)^8} \bar{u}(\mathbf{p}, \lambda) \frac{i}{\not{k}^2 - m_\phi^2 + i\epsilon} \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda) \\
&\quad \times (2\pi)^4 \delta^{(4)}(q + k - p) (2\pi)^4 \delta^{(4)}(p - q - k) \\
&= (-i\kappa)^2 \int \frac{d^4q}{(2\pi)^4} \bar{u}(\mathbf{p}, \lambda) \frac{i}{(p - q)^2 - m_\phi^2 + i\epsilon} \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda)
\end{aligned}$$

 第二步对  $x$  和  $y$  积分，得到 2 个相等的四维  $\delta$  函数

说明 2 个顶点处的能动量守恒关系相同, 都是  $p = q +$

最后一步对  $k$  积分, 剩下  $\delta^{(4)}(0) = \delta^{(4)}(p - p)$  体现初末态满足的能动量守恒定律

最后剩下最后一个未定的圈动量  $q^\mu$ ，需要对它的所有取值积分，相应的不变振幅是

$$i\mathcal{M} = (-i\kappa)^2 \int \frac{d^4 q}{(2\pi)^4} \bar{u}(\mathbf{p}, \lambda) \frac{i}{(p-q)^2 - m_\phi^2 + i\epsilon} \frac{i(q + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda)$$

余钊煥 (中山大学)

第7章 Feynman图

## 7.1 节和 7.2 节



# $iT_9^{(2)}$ 和 $iT_{10}^{(2)}$ 贡献的 $\psi$ 粒子单圈自能图

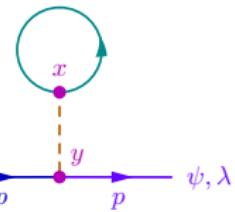
(6)  $iT_9^{(2)}$  和  $iT_{10}^{(2)}$  具有交换时空坐标的对称性

$$\begin{aligned}
 iT_9^{(2)} &= \frac{(-i\kappa)^2}{2!} \int d^4x d^4y N[\overbrace{\phi(x)\bar{\psi}(x)}^{\text{outer}} \overbrace{\psi(x)\phi(y)}^{\text{inner}} \bar{\psi}(y)\psi(y)] \\
 &= \frac{(-i\kappa)^2}{2!} \int d^4x d^4y N[\overbrace{\phi(y)\bar{\psi}(y)}^{\text{outer}} \overbrace{\psi(y)\phi(x)}^{\text{inner}} \bar{\psi}(x)\psi(x)] \\
 &= \frac{(-i\kappa)^2}{2!} \int d^4y d^4x N[\overbrace{\phi(x)\bar{\psi}(x)}^{\text{outer}} \overbrace{\psi(x)\phi(y)}^{\text{inner}} \bar{\psi}(y)\psi(y)] = iT_{10}^{(2)}
 \end{aligned}$$

$$iT_9^{(2)} + iT_{10}^{(2)} = 2iT_9^{(2)} = (-i\kappa)^2 \int d^4x d^4y N[\overbrace{\phi(x)\bar{\psi}(x)}^{\text{outer}} \overbrace{\psi(x)\phi(y)}^{\text{inner}} \bar{\psi}(y)\psi(y)]$$

蜘蛛  $iT_9^{(2)} + iT_{10}^{(2)}$  也会贡献到  $\psi$  粒子的单圈自能图, 对应的  $T$  矩阵元为

$$\begin{aligned}
 &\langle \mathbf{p}^+, \lambda | (iT_9^{(2)} + iT_{10}^{(2)}) | \mathbf{p}^+, \lambda \rangle \\
 &= (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{p}^+, \lambda | N[\overbrace{\phi(x)\bar{\psi}(x)}^{\text{outer}} \overbrace{\psi(x)\phi(y)}^{\text{inner}} \bar{\psi}(y)\psi(y)] | \mathbf{p}^+, \lambda \rangle \\
 &= (-i\kappa)^2 \int d^4x d^4y \langle \mathbf{p}^+, \lambda | N[\overbrace{\phi(x)\bar{\psi}(x)}^{\text{outer}} \overbrace{\psi(x)\phi(y)}^{\text{inner}} \bar{\psi}(y)\psi(y)] | \mathbf{p}^+, \lambda \rangle
 \end{aligned}$$



### iT<sub>11</sub><sup>(2)</sup> 贡献的 T 矩阵元

(7) 考虑初态和末态均是一个动量为  $\mathbf{k}$  的  $\phi$  粒子, 即  $|i\rangle = |\mathbf{k}\rangle$ ,  $\langle f| = \langle \mathbf{k}|$ , 则  $iT_{11}^{(2)}$  对  $T$  矩阵元的贡献为

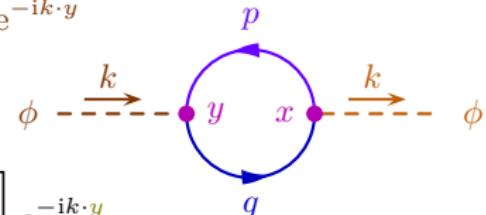
$$\begin{aligned}
\langle \mathbf{k} | i T_{11}^{(2)} | \mathbf{k} \rangle &= \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \langle \mathbf{k} | \mathsf{N}[\phi(x) \overline{\psi}(x) \overline{\psi}(y) \overline{\psi}(y) \psi(y)] | \mathbf{k} \rangle \\
&= \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \{ \langle \mathbf{k} | \mathsf{N}[\phi(x) \overline{\psi}_a(x) \overline{\psi}_a(x) \overline{\psi}_b(y) \overline{\psi}_b(y)] | \mathbf{k} \rangle \\
&\quad + \langle \mathbf{k} | \mathsf{N}[\phi(x) \overline{\psi}_a(x) \overline{\psi}_a(x) \overline{\psi}_b(y) \overline{\psi}_b(y)] | \mathbf{k} \rangle \} \\
&= -\frac{(-i\kappa)^2}{2!} \int d^4x d^4y \{ \langle \mathbf{k} | \mathsf{N}[\phi(\textcolor{red}{x}) \psi_a(\textcolor{red}{x}) \overline{\psi}_b(\textcolor{blue}{y}) \psi_b(\textcolor{blue}{y}) \overline{\psi}_a(\textcolor{red}{x}) \phi(\textcolor{blue}{y})] | \mathbf{k} \rangle \\
&\quad + \langle \mathbf{k} | \mathsf{N}[\phi(\textcolor{red}{y}) \psi_b(\textcolor{blue}{y}) \overline{\psi}_a(\textcolor{blue}{x}) \psi_a(\textcolor{red}{x}) \overline{\psi}_b(\textcolor{blue}{y}) \phi(\textcolor{blue}{x})] | \mathbf{k} \rangle \} \\
&= -(-i\kappa)^2 \int d^4x d^4y \langle \mathbf{k} | \mathsf{N}[\phi(x) \psi_b(y) \overline{\psi}_a(x) \overline{\psi}_a(x) \overline{\psi}_b(y) \phi(y)] | \mathbf{k} \rangle
\end{aligned}$$

 第三步解开缩并线，排列成 Feynman 传播子的形式，产生一个整体的负号

第三步花括号中两项具有交换时空坐标  $x$  和  $y$  的对称性，因而在第四步中合为一项，消去  $1/2!$  因子

### iT<sub>11</sub><sup>(2)</sup> 贡献的 $\phi$ 粒子单圈自能图

$$\begin{aligned}
\langle \mathbf{k} | i T_{11}^{(2)} | \mathbf{k} \rangle &= -(-i\kappa)^2 \int d^4x d^4y \langle \mathbf{k} | \mathbf{N}[\phi(x) \psi_b(y) \bar{\psi}_a(x) \bar{\psi}_a(x) \bar{\psi}_b(y) \phi(y)] | \mathbf{k} \rangle \\
&= -(-i\kappa)^2 \int d^4x d^4y e^{i\mathbf{k} \cdot \mathbf{x}} S_{F,ba}(y-x) S_{F,ab}(x-y) e^{-i\mathbf{k} \cdot \mathbf{y}} \\
&= -(-i\kappa)^2 \int d^4x d^4y e^{i\mathbf{k} \cdot \mathbf{x}} \int \frac{d^4p d^4q}{(2\pi)^8} \\
&\quad \times \text{tr} \left[ \frac{i(\not{p} + m_\psi) e^{-ip \cdot (y-x)}}{p^2 - m_\psi^2 + i\epsilon} \frac{i(\not{q} + m_\psi) e^{-iq \cdot (x-y)}}{q^2 - m_\psi^2 + i\epsilon} \right] e^{-ik \cdot y} \\
&= -(-i\kappa)^2 \int \frac{d^4p d^4q}{(2\pi)^8} \text{tr} \left[ \frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} \right] \\
&\quad \times (2\pi)^4 \delta^{(4)}(\not{q} - p - k) (2\pi)^4 \delta^{(4)}(k + p - \not{q}) \\
&= -(-i\kappa)^2 \int \frac{d^4p}{(2\pi)^4} \text{tr} \left[ \frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} \frac{i(\not{k} + \not{p} + m_\psi)}{(\not{k} + \not{p})^2 - m_\psi^2 + i\epsilon} \right] (2\pi)^4 \delta^{(4)}(0)
\end{aligned}$$



相应的 Feynman 图是  $\phi$  粒子的单圈自能图



# 封闭的费米子圈

鱼 相应**不变振幅**为  $i\mathcal{M} = -(-i\kappa)^2 \int \frac{d^4 p}{(2\pi)^4} \text{tr} \left[ \frac{i(p + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} \frac{i(k + p + m_\psi)}{(k + p)^2 - m_\psi^2 + i\epsilon} \right]$

鱼 类似于前面的**蝌蚪图**，这里验证了一个**普遍的结论**：



一个**封闭的费米子圈**贡献一个额外的**负号**，  
并且需要对 Dirac 矩阵的乘积**求迹**

鱼 这样的**负号**是**重要的**，有可能影响观测量

## 封闭的费米子圈

相应不变振幅为  $i\mathcal{M} = -(-i\kappa)^2 \int \frac{d^4 p}{(2\pi)^4} \text{tr} \left[ \frac{i(p + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} \frac{i(k + p + m_\psi)}{(k + p)^2 - m_\psi^2 + i\epsilon} \right]$

类似于前面的蝌蚪图，这里验证了一个普遍的结论：



一个封闭的费米子圈贡献一个额外的负号，并且需要对 Dirac 矩阵的乘积求迹

这样的**负号**是重要的，有可能影响观测量

一个封闭的费米子圈上的  $n$  个顶点对应着  $n$  条费米子内线，求迹是对  $n$  个传播子乘积进行的；我们已经验证了  $n = 1, 2$  的情形，当  $n = 3$  时，场算符缩并结构为

$$\begin{aligned} \mathsf{N}[\bar{\psi}_a(x)\psi_a(x)\bar{\psi}_b(y)\psi_b(y)\bar{\psi}_c(z)\psi_c(z)] &= -\mathsf{N}[\psi_c(z)\bar{\psi}_a(x)\psi_a(x)\bar{\psi}_b(y)\psi_b(y)\bar{\psi}_c(z)] \\ &= -S_{\mathsf{F},ca}(z-x)S_{\mathsf{F},ab}(x-y)S_{\mathsf{F},bc}(y-z) = -\text{tr}[S_{\mathsf{F}}(z-x)S_{\mathsf{F}}(x-y)S_{\mathsf{F}}(y-z)] \end{aligned}$$

👉 确实出现了**负号**和**求迹**

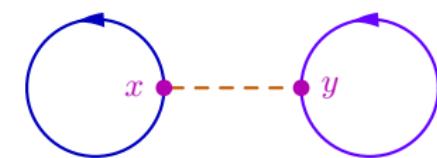
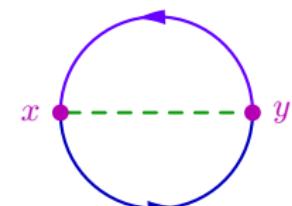
这个结论显然可以推广到任意  $n$  的情形

## 气泡图

(8) 最后,  $iT_{13}^{(2)}$  和  $iT_{14}^{(2)}$  的表达式为

$$i T_{13}^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathsf{N} \left[ \phi(x) \bar{\psi}(x) \psi(x) \phi(y) \bar{\psi}(y) \psi(y) \right]$$

$$iT_{14}^{(2)} = \frac{(-i\kappa)^2}{2!} \int d^4x d^4y \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)]$$



以上正规乘积里面所有场算符都已经参与缩并了，可以直接画出相应 Feynman 图

这种不包含任何外线的圈图称为**气泡图** (bubble diagram)。

由于缺乏需要与初末态缩并的场算符,  $iT_{13}^{(2)}$  和  $iT_{14}^{(2)}$  可贡献到任意  $T$  矩阵元中

不过，这些气泡图只会产生一些相位因子，没有可观测的物理效应

## 7.2 节 动量空间 Feynman 规则

在上一节中，我们利用 Wick 定理计算  $T$  矩阵元  $\langle f | iT | i \rangle$

将计算过程中的各个部分表达成图形，画出 Feynman 图

从中归纳出一套位置空间中的 Feynman 规则

理解这些规律之后，反过来，可以对各个过程画出所有拓扑不等价的 Feynman 图，然后通过 Feynman 规则写出相应  $T$  矩阵元  $\langle f | iT | i \rangle$  的代数表达式

不过,当同一过程存在多个子图且涉及费米子场算符时,需要回到带着缩并的表达式,将缩并线解开,以确定各个子图之间的相对符号

## 7.2 节 动量空间 Feynman 规则

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在位置空间 Feynman 规则中，每个顶点对应于一个时空积分，积分的结果是使得出入顶点的内外线上的四维动量满足能动量守恒关系

最后得到依赖于外线动量、但不依赖于时空坐标的结果，而  $T$  矩阵元  $\langle f | iT | i \rangle$  分解为不变振幅  $i\mathcal{M}$  与表示能动量守恒定律的因子  $(2\pi)^4 \delta^{(4)}(p_i - p_f)$  之积

利用这个规律，将 Feynman 规则改成不依赖于时空坐标的形式，称为动量空间中的 Feynman 规则，然后从 Feynman 图直接给出不变振幅  $iM$  的代数表达式



根据上一节体现的规律, Yukawa 理论在动量空间中的 Feynman 规则如下

- 1 Dirac 正费米子入射外线:  $\psi, \lambda \xrightarrow{p} \bullet = u(\mathbf{p}, \lambda)$
  - 2 Dirac 反费米子入射外线:  $\bar{\psi}, \lambda \xleftarrow{p} \bullet = \bar{v}(\mathbf{p}, \lambda)$
  - 3 Dirac 正费米子出射外线:  $\bullet \xrightarrow{p} \psi, \lambda = \bar{u}(\mathbf{p}, \lambda)$
  - 4 Dirac 反费米子出射外线:  $\bullet \xleftarrow{p} \bar{\psi}, \lambda = v(\mathbf{p}, \lambda)$
  - 5 Dirac 费米子传播子:  $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} = \frac{i}{\not{p} - m_\psi + i\epsilon}$
  - 6 实标量玻色子入射外线:  $\phi \xrightarrow{p} \bullet = 1$
  - 7 实标量玻色子出射外线:  $\bullet \xrightarrow{p} \phi = 1$

## 动量空间中的 Feynman 规则

- 8 实标量玻色子传播子:  $\bullet - \xrightarrow{p} - \bullet = \frac{i}{p^2 - m_\phi^2 + ie}$

- 9 Yukawa 相互作用顶点:

- ⑩ 出入每个顶点的内外线四维动量满足能动量守恒关系

- 11 每个未定的圈动量  $p^\mu$  贡献一个积分  $\int \frac{d^4 p}{(2\pi)^4}$

- ⑫ 每个封闭的费米子圈贡献一个额外的负号，并需要对费米子传播子的乘积求迹

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- 12 每个封闭的费米子圈贡献一个额外的负号，并需要对费米子传播子的乘积求迹

除了顶点规则具有 Yukawa 相互作用特有的形式之外，其它规则具有一般性

 注意：顶点规则与拉氏量中的相互作用项直接对应：剥离 Yukawa 相互作用项

$\mathcal{L}_Y = -\kappa \phi \bar{\psi} \psi$  中的场算符  $\phi$ 、 $\psi$  和  $\bar{\psi}$ ，再乘以  $i$ ，就得到顶点规则的表达式  $-i\kappa$

相机  $\phi$  场对应于顶点上的虚线头,  $\psi$  场对应于指向顶点(背向顶点)的实线头

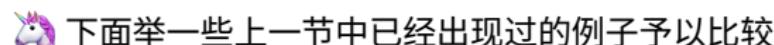
## 动量空间 Feynman 规则的应用

对于某个过程，先画出微扰论某一阶上所有拓扑不等价的 Feynman 图，再应用动量空间 Feynman 规则，逆着费米子线上箭头方向组合出不变振幅  $iM$  的代数表达式

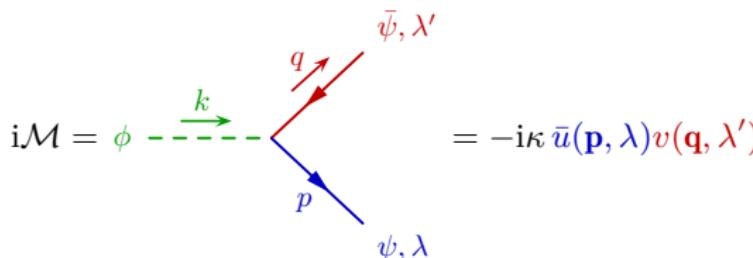
 下面举一些上一节中已经出现过的例子予以比较

## 动量空间 Feynman 规则的应用

对于某个过程，先画出微扰论某一阶上所有拓扑不等价的 Feynman 图，再应用动量空间 Feynman 规则，逆着费米子线上箭头方向组合出不变振幅  $iM$  的代数表达式



(1)  $\phi \rightarrow \psi\bar{\psi}$  衰变过程的领头阶不变振幅为



在不会引起混淆的情况下，可以省略 Feynman 图顶点上的圆点



这个结果与前面的结果整体相差一个负号，这是因为此处没有调换旋量场算符的位置以符合末态中湮灭算符的次序



不过，这个过程只有一个 Feynman 图，没有干涉效应，额外负号不影响衰变宽度



像这样的整体符号差异无关紧要，它不会改变振幅模方，因而不会影响观测量

$\psi\bar{\psi} \rightarrow \psi\bar{\psi}$  散射过程

(2) 在领头阶,  $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$  散射过程具有 2 个拓扑不等价的 Feynman 图, 它们之间的相对符号至关重要, 不变振幅为

$$\begin{aligned}
i\mathcal{M} &= \text{Diagram 1} - \text{Diagram 2} \\
&= (-i\kappa)^2 \left[ \bar{u}(\mathbf{q}_1, \lambda'_1) v(\mathbf{q}_2, \lambda'_2) \frac{i}{(p_1 + p_2)^2 - m_\phi^2 + i\epsilon} \bar{v}(\mathbf{p}_2, \lambda_2) u(\mathbf{p}_1, \lambda_1) \right. \\
&\quad \left. - \bar{v}(\mathbf{p}_2, \lambda_2) v(\mathbf{q}_2, \lambda'_2) \frac{i}{(p_1 - q_1)^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_1, \lambda_1) \right]
\end{aligned}$$

这里利用顶点处的能动量守恒关系，从外线动量决定内线动量

从 Feynman 图本身看不出它们之间的相对符号, 应当写出缩并表达式进行考察

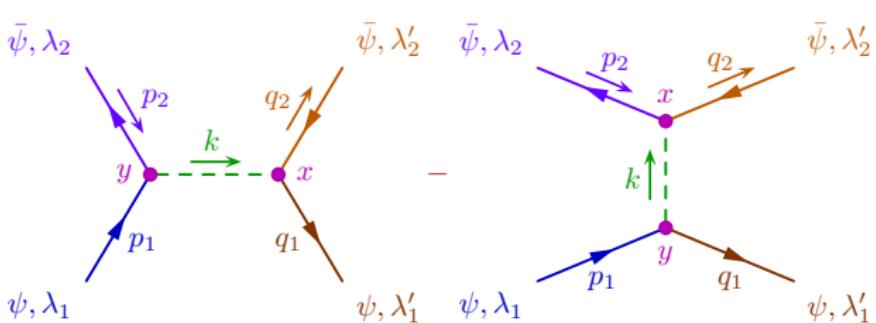
### 从缩并表达式考察相对符号

先保持场算符位置画出 2 种拓扑不等价的缩并方式，再调换场算符将缩并线解开

$$\begin{aligned}
& \left\langle \mathbf{q}_1^+, \lambda_1'; \mathbf{q}_2^-, \lambda_2' \middle| \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
& + \left\langle \mathbf{q}_1^+, \lambda_1'; \mathbf{q}_2^-, \lambda_2' \middle| \mathbf{N}[\phi(x)\bar{\psi}(x)\psi(x)\phi(y)\bar{\psi}(y)\psi(y)] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
= & - \left\langle \mathbf{q}_1^+, \lambda_1'; \mathbf{q}_2^-, \lambda_2' \middle| \mathbf{N}[\psi_a(x)\bar{\psi}_a(x)\phi(x)\phi(y)\bar{\psi}_b(y)\psi_b(y)] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle \\
& + \left\langle \mathbf{q}_1^+, \lambda_1'; \mathbf{q}_2^-, \lambda_2' \middle| \mathbf{N}[\psi_a(x)\bar{\psi}_b(y)\phi(x)\phi(y)\bar{\psi}_a(x)\psi_b(y)] \middle| \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^-, \lambda_2 \right\rangle
\end{aligned}$$

由此可知，两个子振幅的符号相反，从而确定两图之间的相对符号为负

这里  $iM$  与前面的结果在整体上相差一个负号，但不会影响散射截面



### $\psi\psi \rightarrow \psi\psi$ 散射过程

(3)  $\psi\psi \rightarrow \psi\psi$  散射过程在领头阶具有 2 个拓扑不等价的 Feynman 图, 不变振幅为

$$\begin{aligned}
i\mathcal{M} &= \psi, \lambda_2 \quad \psi, \lambda'_2 \quad \psi, \lambda_2 \quad \psi, \lambda'_2 \\
&= (-i\kappa)^2 \left[ \bar{u}(\mathbf{q}_2, \lambda'_2) u(\mathbf{p}_2, \lambda_2) \frac{i}{(p_1 - q_1)^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_1, \lambda_1) \right. \\
&\quad \left. - \bar{u}(\mathbf{q}_1, \lambda'_1) u(\mathbf{p}_2, \lambda_2) \frac{i}{(p_1 - q_2)^2 - m_\phi^2 + i\epsilon} \bar{u}(\mathbf{q}_2, \lambda'_2) u(\mathbf{p}_1, \lambda_1) \right]
\end{aligned}$$

这里画出拓扑不等价 Feynman 图的关键在于注意外线与顶点连接情况的不同

在第一个子图中,  $p_1$  外线与  $q_1$  外线交于同一顶点

在第二个子图中,  $p_1$  外线则与  $q_3$  外线交于同一顶点

此过程具有交换末态全同费米子的反对称性，因而两个子图的相对符号为负

$\psi\bar{\psi} \rightarrow \phi\phi$  湮灭过程

(4) 在领头阶,  $\bar{\psi}\psi \rightarrow \phi\phi$  湮灭过程具有 2 个拓扑不等价的 Feynman 图, 不变振幅为

$$\begin{aligned}
i\mathcal{M} &= \text{Diagram 1} + \text{Diagram 2} \\
&= (-i\kappa)^2 \left[ \bar{v}(\mathbf{p}_2, \lambda_2) \frac{i(\not{p}_1 - \not{k}_1 + m_\psi)}{(p_1 - k_1)^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}_1, \lambda_1) \right. \\
&\quad \left. + \bar{v}(\mathbf{p}_2, \lambda_2) \frac{i(\not{p}_1 - \not{k}_2 + m_\psi)}{(p_1 - k_2)^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}_1, \lambda_1) \right]
\end{aligned}$$

两个子图的费米子线结构相同，不存在相对符号差异

两个子图的相对符号为正，体现交换末态全同玻色子的对称性

### ψ 和 φ 粒子的单圈自能图

(5) 对于  $iT_7^{(2)} + iT_8^{(2)}$  贡献的  $\psi$  粒子单圈自能图, 不变振幅为

$$\begin{aligned}
i\mathcal{M} &= \psi, \lambda \xrightarrow[p]{\quad} \psi, \lambda \xrightarrow[q]{\quad} \psi, \lambda \\
&= (-i\kappa)^2 \int \frac{d^4 q}{(2\pi)^4} \bar{u}(\mathbf{p}, \lambda) \frac{i(q + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda) \frac{i}{(p - q)^2 - m_\phi^2 + i\epsilon}
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&= (-i\kappa)^2 \int \frac{d^4 q}{(2\pi)^4} \bar{u}(\mathbf{p}, \lambda) \frac{i(q + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} u(\mathbf{p}, \lambda) \frac{i}{(p - q)^2 - m_\phi^2 + i\epsilon}
\end{aligned}$$

(6)  $\phi$  粒子单圈自能图的不变振幅为

$$\begin{aligned}
 i\mathcal{M} &= \phi - \xrightarrow[k]{\text{---}} - \text{---} \xrightarrow[k]{\text{---}} - \phi \\
 &= -(-i\kappa)^2 \int \frac{d^4 p}{(2\pi)^4} \text{tr} \left[ \frac{i(p + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} \frac{i(k + p + \not{p} + m_\psi)}{(k + p)^2 - m_\psi^2 + i\epsilon} \right]
 \end{aligned}$$

这个自能图包含一个封闭的费米子圈，上式出现了**负号**和**求迹**