量 子 场 论

第 5 章　量子场的相互作用

5.3 节和 5.4 节

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5.3 节 Wick 定理

- 5.2 节借助时序乘积把 ***S* 算符**写成紧凑的级数形式

$$S=\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}\int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_n\,\mathsf{T}[\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)]$$

- 如何适当地处理级数每一项中的**时序乘积** $\mathsf{T}[\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)]$ 呢?
- 在量子场论中，相互作用哈密顿量密度 $\mathcal{H}_1(x)$ 是由若干个场算符构成的，因而我们需要处理的是**多个场算符**的**时序乘积**
- 这不是一个简单问题，接下来将要介绍的 **Wick 定理**提供了一种简便的处理方法

5.3 节　Wick 定理

- 5.2 节借助时序乘积把 S 算符写成紧凑的级数形式

$$S = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \mathrm{d}^4 x_1 \cdots \mathrm{d}^4 x_n \, \mathsf{T}[\mathcal{H}_1(x_1) \cdots \mathcal{H}_1(x_n)]$$

- 如何适当地处理级数每一项中的时序乘积 $\mathsf{T}[\mathcal{H}_1(x_1) \cdots \mathcal{H}_1(x_n)]$ 呢？
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- 这不是一个简单问题，接下来将要介绍的 Wick 定理提供了一种简便的处理方法
- 在相互作用绘景中，实标量场 $\phi(x)$ 的平面波展开式可以分解成两个部分

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

- 正能解部分 $\phi^{(+)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} a_\mathbf{p} e^{-ip\cdot x}$ 包含湮灭算符和 $e^{-ip\cdot x}$ 因子
- 负能解部分 $\phi^{(-)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} a_\mathbf{p}^\dagger e^{ip\cdot x}$ 包含产生算符和 $e^{ip\cdot x}$ 因子

矢量场和旋量场的正负能解

同样，可以把**相互作用绘景**中的**有质量矢量场 $\boldsymbol{A^\mu(x)}$** 分解为

$$A^\mu(x) = A^{\mu(+)}(x) + A^{\mu(-)}(x)$$

正能解部分为 $A^{\mu(+)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \varepsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} \mathrm{e}^{-\mathrm{i}p\cdot x}$

负能解部分为 $A^{\mu(-)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \varepsilon^{\mu *}(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} \mathrm{e}^{\mathrm{i}p\cdot x}$

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- **正能解**部分为 $A^{\mu(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \varepsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x}$

- **负能解**部分为 $A^{\mu(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \varepsilon^{\mu*}(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x}$

- **相互作用绘景**中 **Dirac 旋量场 $\psi_a(x)$** 的平面波展开式也具有 **Heisenberg 绘景**中自由场展开式的形式，即

$$\psi_a(x) = \psi_a^{(+)}(x) + \psi_a^{(-)}(x)$$

- **正能解**部分为 $\psi_a^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} u_a(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x}$

- **负能解**部分为 $\psi_a^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} v_a(\mathbf{p},\lambda) b^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x}$

- 注意：各类量子场的**正能解**部分只包含**湮灭算符**，而**负能解**部分只包含**产生算符**

正规乘积

- 引入**正规乘积** (normal product) 的概念，以 N 为记号，它的作用是将乘积中的**所有湮灭算符**移动到**所有产生算符**的**右边**，形成**正规排序** (normal order)
- 正规排序对于**多个产生** (或湮灭) **算符之间谁先谁后**没有规定，每种排列都**等价**
- 移动算符的过程中如果涉及**奇数次费米子算符**之间的**交换**，就应该额外添加一个**负号**，这个规定与费米子产生湮灭算符的**反对易性**相匹配

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- 对于**标量场**的产生湮灭算符，有 $\mathrm{N}(a_{\mathrm{p}}a_{\mathrm{q}}^{\dagger}a_{\mathrm{k}}a_{\mathrm{l}}^{\dagger}) = a_{\mathrm{q}}^{\dagger}a_{\mathrm{l}}^{\dagger}a_{\mathrm{p}}a_{\mathrm{k}} = a_{\mathrm{l}}^{\dagger}a_{\mathrm{q}}^{\dagger}a_{\mathrm{p}}a_{\mathrm{k}} = a_{\mathrm{l}}^{\dagger}a_{\mathrm{q}}^{\dagger}a_{\mathrm{k}}a_{\mathrm{p}}$
- 第二步写出两个产生算符的另一种次序，第三步写出两个湮灭算符的另一种次序，而**对易关系** $[a_{\mathrm{q}}^{\dagger}, a_{\mathrm{l}}^{\dagger}] = [a_{\mathrm{p}}, a_{\mathrm{k}}] = 0$ 保证这些表达式是**等价**的

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- 对于**旋量场**的产生湮灭算符，有 $\mathrm{N}(b_{\mathbf{p},\lambda_1}a_{\mathbf{q},\lambda_2}^\dagger a_{\mathbf{k},\lambda_3}b_{\mathbf{l},\lambda_4}^\dagger) = -a_{\mathbf{q},\lambda_2}^\dagger b_{\mathbf{l},\lambda_4}^\dagger b_{\mathbf{p},\lambda_1}a_{\mathbf{k},\lambda_3}$
$$= b_{\mathbf{l},\lambda_4}^\dagger a_{\mathbf{q},\lambda_2}^\dagger b_{\mathbf{p},\lambda_1}a_{\mathbf{k},\lambda_3} = -b_{\mathbf{l},\lambda_4}^\dagger a_{\mathbf{q},\lambda_2}^\dagger a_{\mathbf{k},\lambda_3}b_{\mathbf{p},\lambda_1}$$
- 第二步写出两个产生算符的另一种次序，与第一步相差一个**负号**
- 第三步写出两个湮灭算符的另一种次序，与第二步相差一个**负号**
- **反对易关系** $\{a_{\mathbf{q},\lambda_2}^\dagger, b_{\mathbf{l},\lambda_4}^\dagger\} = \{b_{\mathbf{p},\lambda_1}, a_{\mathbf{k},\lambda_3}\} = 0$ 保证这些表达式是**等价**的

正规乘积的真空期待值

- 正能解包含湮灭算符，负能解包含产生算符，正规排序中正能解处于负能解右边
- 两个标量场算符的正规乘积为

$$\mathsf{N}[\phi(x)\phi(y)] = \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x)$$

- 最后一项中 $\phi^{(+)}(x)$ 被正规排序移动到 $\phi^{(-)}(y)$ 的右边
- 两个旋量场算符的正规乘积为

$$\mathsf{N}[\psi_a(x)\psi_b(y)] = \psi_a^{(-)}(x)\psi_b^{(-)}(y)+\psi_a^{(-)}(x)\psi_b^{(+)}(y)+\psi_a^{(+)}(x)\psi_b^{(+)}(y)-\psi_b^{(-)}(y)\psi_a^{(+)}(x)$$

- 最后一项中 $\psi_a^{(+)}(x)$ 被正规排序移动到 $\psi_b^{(-)}(y)$ 的右边，并出现一个负号

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- 两个旋量场算符的正规乘积为

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- 最后一项中 $\psi_a^{(+)}(x)$ 被正规排序移动到 $\psi_b^{(-)}(y)$ 的右边，并出现一个负号
- 湮灭算符对真空态 $|0\rangle$ 的作用为零，如 $a_{\mathbf{p}}|0\rangle = 0$，相应地，$\langle 0| a_{\mathbf{p}}^\dagger = 0$
- 因此，对一组产生湮灭算符的任意乘积取正规排序之后，真空期待值为零，即

$$\langle 0|\,\mathrm{N}\,(\text{产生湮灭算符的乘积})\,|0\rangle = 0$$

一般场算符的正负能解

用**统一记号** $\Phi_a(x)$ 代表**一般场算符**，它可以是标量场 $\phi(x)$ 或 $\phi^\dagger(x)$，也可以是矢量场 $A^\mu(x)$ 的一个分量，还可以是旋量场 $\psi_a(x)$ 或 $\bar{\psi}_a(x)$ 的一个分量

比如，$\Phi_a(x)\Phi_b(x)\Phi_c(x)$ 可以表示 $\phi(x)\phi(x)\phi(x)$，也可以表示 $A_\mu(x)\bar{\psi}_a(x)\psi_b(x)$

后者不是 Lorentz 不变的，但可以利用 Dirac 矩阵线性地组合出 **Lorentz 不变量** $A_\mu(x)\bar{\psi}_a(x)(\gamma^\mu)_{ab}\psi_b(x) = A_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)$

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令 $\Phi_a(x) = \Phi_a^{(+)}(x) + \Phi_a^{(-)}(x)$，分解为**正能解** $\Phi_a^{(+)}(x)$ 和**负能解** $\Phi_a^{(-)}(x)$，则

$$\Phi_a(x)\Phi_b(y) = \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(-)}(y)$$

由于正能解包含**湮灭算符**，负能解包含**产生算符**，有

$$\Phi_a^{(+)}(x)\,|0\rangle = 0, \quad \langle 0|\,\Phi_a^{(-)}(x) = 0,$$

从而推出

$$\langle 0|\,\Phi_a(x)\Phi_b(y)\,|0\rangle = \langle 0|\,\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)\,|0\rangle$$

两个一般场算符的正规乘积

- 将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的**正规乘积**分解为

$$\begin{aligned}\mathsf{N}[\Phi_a(x)\Phi_b(y)] = {} & \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) \\ & + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x)\end{aligned}$$

- 因子 $\epsilon_{ab} = \pm 1$ 考虑了**费米子算符**的**反对易性**
- 若 $\Phi_a(x)$ 和 $\Phi_b(y)$ **都是费米子算符**，则 $\epsilon_{ab} = -1$；**其余情况** $\epsilon_{ab} = +1$

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将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的**正规乘积**分解为

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利用 ϵ_{ab} 记号，依照两个**产生**（**湮灭**）算符的**对易性**或**反对易性**交换等式右边**第一项**（**第三项**）的两个场算符，得到

$$\begin{aligned}& \mathsf{N}[\Phi_a(x)\Phi_b(y)] \\ = {} & \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \epsilon_{ab}\Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) \\ = {} & \epsilon_{ab}[\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \epsilon_{ab}\Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x)]\end{aligned}$$

即

$$\mathsf{N}[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}\,\mathsf{N}[\Phi_b(y)\Phi_a(x)]$$

交换两个场算符之后，**正规乘积**只相差一个由费米子算符的**反对易性**导致的**符号**

两个一般场算符的时序乘积

- 另一方面，将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的**时序乘积**化为

$$
\begin{aligned}
\mathsf{T}[\Phi_a(x)\Phi_b(y)] &= \Phi_a(x)\Phi_b(y)\theta(x^0-y^0)+\epsilon_{ab}\Phi_b(y)\Phi_a(x)\theta(y^0-x^0) \\
&= \epsilon_{ab}[\epsilon_{ab}\Phi_a(x)\Phi_b(y)\theta(x^0-y^0)+\Phi_b(y)\Phi_a(x)\theta(y^0-x^0)]
\end{aligned}
$$

- 交换两个场算符后，**时序乘积**也只相差一个由费米子算符的**反对易性**导致的**符号**

$$
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- 交换两个场算符后，**时序乘积**也只相差一个由费米子算符的**反对易性**导致的**符号**

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}\,\mathsf{T}[\Phi_b(y)\Phi_a(x)]$$

- **当 $x^0 \geq y^0$ 时**，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积为

$$\begin{aligned}&\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \Phi_a(x)\Phi_b(y)\\ &= \Phi_a^{(-)}(x)\Phi_b^{(-)}(y)+\Phi_a^{(-)}(x)\Phi_b^{(+)}(y)+\Phi_a^{(+)}(x)\Phi_b^{(+)}(y)+\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)\end{aligned}$$

- 把最后一项改写成

$$\begin{aligned}\Phi_a^{(+)}(x)\Phi_b^{(-)}(y) &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x)+\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)-\epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x)\\ &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x)+[\Phi_a^{(+)}(x),\Phi_b^{(-)}(y)]_{\mp}\end{aligned}$$

- 这里 $[\cdot,\cdot]_- = [\cdot,\cdot]$ 代表**对易子**，$[\cdot,\cdot]_+ = \{\cdot,\cdot\}$ 代表**反对易子**
- $\mp$ 号仅当 $\Phi_a(x)$ 和 $\Phi_b(y)$ **都是费米子算符**时取**正号**，其余情况取**负号**

$x^0 \geq y^0$ 时的 $\mathsf{T}[\Phi_a(x)\Phi_b(y)]$

于是推出

$$
\begin{aligned}
\mathsf{T}[\Phi_a(x)\Phi_b(y)] &= \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) \\
&\quad + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp} \\
&= \mathsf{N}[\Phi_a(x)\Phi_b(y)] + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}
\end{aligned}
$$

注意：$[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}$ 必定是一个 **c 数**，因为 $\Phi_a^{(+)}(x)$ 中**湮灭算符**与 $\Phi_b^{(-)}(y)$ 中**产生算符**的**对易子**或**反对易子**并不是算符，而是 c 数

$x^0 \geq y^0$ 时的 $\mathsf{T}[\Phi_a(x)\Phi_b(y)]$

于是推出

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注意：$[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_\mp$ 必定是一个 **c 数**，因为 $\Phi_a^{(+)}(x)$ 中**湮灭算符**与 $\Phi_b^{(-)}(y)$ 中**产生算符**的**对易子**或**反对易子**并不是算符，而是 c 数

由 $\Phi_a^{(+)}(x)\left|0\right\rangle = 0$，$\left\langle 0\right|\Phi_a^{(-)}(x) = 0$，$\left\langle 0\right|\Phi_a(x)\Phi_b(y)\left|0\right\rangle = \left\langle 0\right|\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)\left|0\right\rangle$ 得到

$$\begin{aligned}[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_\mp &= \left\langle 0\right|[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_\mp\left|0\right\rangle = \left\langle 0\right|\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)\left|0\right\rangle \\ &= \left\langle 0\right|\Phi_a(x)\Phi_b(y)\left|0\right\rangle = \left\langle 0\right|\mathsf{T}[\Phi_a(x)\Phi_b(y)]\left|0\right\rangle\end{aligned}$$

因此**当 $x^0 \geq y^0$ 时**，有 $\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \left\langle 0\right|\mathsf{T}[\Phi_a(x)\Phi_b(y)]\left|0\right\rangle$

$x^0 \le y^0$ 时的 $\mathsf{T}[\Phi_a(x)\Phi_b(y)]$

当 $x^0 \le y^0$ 时，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积变成

$$
\begin{aligned}
&\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}\Phi_b(y)\Phi_a(x)\\
&= \epsilon_{ab}[\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(-)}(x)]\\
&= \epsilon_{ab}\{\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x)\\
&\qquad + \epsilon_{ab}\Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + [\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp\}\\
&= \epsilon_{ab}\,\mathsf{N}[\Phi_b(y)\Phi_a(x)] + \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp\\
&= \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp
\end{aligned}
$$

注意到

$$
\begin{aligned}
\epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp &= \epsilon_{ab}\langle 0|\,[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp\,|0\rangle = \epsilon_{ab}\langle 0|\,\Phi_b^{(+)}(y)\Phi_a^{(-)}(x)\,|0\rangle\\
&= \epsilon_{ab}\langle 0|\,\Phi_b(y)\Phi_a(x)\,|0\rangle = \epsilon_{ab}\langle 0|\,\mathsf{T}[\Phi_b(y)\Phi_a(x)]\,|0\rangle\\
&= \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle
\end{aligned}
$$

因此也有 $\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$

场算符的缩并

于是，**无论 x^0 和 y^0 孰大孰小**，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积都可以统一地表达为

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$$

场算符的缩并

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$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$$

- 引入**场算符的缩并** (contraction) 概念，将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的缩并定义为

$$\overbracket{\Phi_a(x)\Phi_b(y)} \equiv \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle = \begin{cases} [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_\mp, & x^0 \geq y^0 \\ \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp, & x^0 < y^0 \end{cases}$$

- 上式仅当 $\Phi_a^{(+)}(x)$ 中的**湮灭算符**与 $\Phi_b^{(-)}(y)$ 中的**产生算符**属于**同一套产生湮灭算符**时**非零**，因而**不同类型**的场算符 (如标量场与旋量场) **的缩并为零**
- 两个场算符的缩并是一个 **c 数**，**不会**受到**正规排序 N** 的影响

场算符的缩并

- 于是，**无论 x^0 和 y^0 孰大孰小**，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积都可以统一地表达为

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- 引入**场算符的缩并** (contraction) 概念，将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的缩并定义为

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- 上式仅当 $\Phi_a^{(+)}(x)$ 中的**湮灭算符**与 $\Phi_b^{(-)}(y)$ 中的**产生算符**属于**同一套产生湮灭算符**时**非零**，因而**不同类型**的场算符（如标量场与旋量场）**的缩并为零**
- 两个场算符的缩并是一个 **c 数**，**不会**受到**正规排序 N** 的影响
- 在正规乘积中出现缩并记号时，参与缩并的一对场算符可以**不相邻**
- 为了使它们相邻，需要**适当地交换**场算符，交换时应计入费米子算符的反对易性引起的**符号差异**，**约定**这样得到的式子与原先的式子**相等**，例如

$$\mathsf{N}(\Phi_a\Phi_b\Phi_c\Phi_d\Phi_e\Phi_f) = \epsilon_{cd}\epsilon_{ef}\mathsf{N}(\Phi_a\Phi_b\Phi_d\Phi_c\Phi_f\Phi_e) = \epsilon_{cd}\epsilon_{ef}\Phi_b\Phi_d\Phi_c\Phi_f\mathsf{N}(\Phi_a\Phi_e)$$

Wick 定理

将 $\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$ 改写为

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y) + \wick{\c\Phi_a(x)\c\Phi_b(y)}]$$

可见，两个场算符的**时序乘积**等于它们的**正规乘积**加上它们的**缩并**

Wick 定理

将 $\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$ 改写为

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可见，两个场算符的**时序乘积**等于它们的**正规乘积**加上它们的**缩并**

这个结论可以**推广**成

Gian Wick (1909–1992)

Wick 定理

一组场算符的**时序乘积**可以分解为它们的**正规乘积**及**所有可能缩并**的正规乘积之和，也就是说，

$$\mathsf{T}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)] = \mathsf{N}\Big[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n) + \big(\Phi_{a_1}\Phi_{a_2}\cdots\Phi_{a_n}\text{的所有可能缩并}\big)\Big]$$

Wick 定理的**证明**见**讲义 5.3.2 小节选读内容**

Wick 定理应用举例

🐷 对于**四个场算符**的情况，**Wick 定理**表明

$$\begin{aligned}\mathsf{T}(\Phi_a\Phi_b\Phi_c\Phi_d) = \mathsf{N}\Big(&\Phi_a\Phi_b\Phi_c\Phi_d + \wick{\c1\Phi_a\c1\Phi_b}\Phi_c\Phi_d + \wick{\c1\Phi_a\Phi_b\c1\Phi_c}\Phi_d + \wick{\c1\Phi_a\Phi_b\Phi_c\c1\Phi_d}\\ &+ \Phi_a\wick{\c1\Phi_b\c1\Phi_c}\Phi_d + \Phi_a\wick{\c1\Phi_b\Phi_c\c1\Phi_d} + \Phi_a\Phi_b\wick{\c1\Phi_c\c1\Phi_d}\\ &+ \wick{\c1\Phi_a\c1\Phi_b\c2\Phi_c\c2\Phi_d} + \wick{\c1\Phi_a\c2\Phi_b\c1\Phi_c\c2\Phi_d} + \wick{\c1\Phi_a\c2\Phi_b\c2\Phi_c\c1\Phi_d}\Big)\end{aligned}$$

🚜 由于 $\langle 0|\,\mathsf{N}\,(\text{产生湮灭算符的乘积})\,|0\rangle = 0$，上式的**真空期待值**为

$$\begin{aligned}&\langle 0|\,\mathsf{T}(\Phi_a\Phi_b\Phi_c\Phi_d)\,|0\rangle\\ =\;& \mathsf{N}(\wick{\c1\Phi_a\c1\Phi_b\c2\Phi_c\c2\Phi_d} + \wick{\c1\Phi_a\c2\Phi_b\c1\Phi_c\c2\Phi_d} + \wick{\c1\Phi_a\c2\Phi_b\c2\Phi_c\c1\Phi_d})\\ =\;& \wick{\c1\Phi_a\c1\Phi_b\c2\Phi_c\c2\Phi_d} + \epsilon_{bc}\wick{\c1\Phi_a\c1\Phi_c\c2\Phi_b\c2\Phi_d} + \epsilon_{cd}\epsilon_{bd}\wick{\c1\Phi_a\c1\Phi_d\c2\Phi_b\c2\Phi_c}\\ =\;& \langle 0|\,\mathsf{T}(\Phi_a\Phi_b)\,|0\rangle\,\langle 0|\,\mathsf{T}(\Phi_c\Phi_d)\,|0\rangle + \epsilon_{bc}\,\langle 0|\,\mathsf{T}(\Phi_a\Phi_c)\,|0\rangle\,\langle 0|\,\mathsf{T}(\Phi_b\Phi_d)\,|0\rangle\\ &+ \epsilon_{cd}\epsilon_{bd}\,\langle 0|\,\mathsf{T}(\Phi_a\Phi_d)\,|0\rangle\,\langle 0|\,\mathsf{T}(\Phi_b\Phi_c)\,|0\rangle\end{aligned}$$

5.4 节　Feynman 传播子

- 在应用 Wick 定理时，两个场算符的缩并是一种基本要素
- 上一节已经指出，仅当参与缩并的场算符中含有同一套产生湮灭算符时，缩并的结果才不为零
- Feynman 传播子 (propagator) 就是这样的非零缩并
- 本节将导出 Feynman 传播子的显式结果

Richard Feynman
(1918–1988)

5.4.1 小节　实标量场的 Feynman 传播子

- **实标量场 $\phi(x)$** 的 **Feynman 传播子** $D_{\mathrm{F}}(x-y)$ 定义为

$$D_{\mathrm{F}}(x-y) \equiv \phi(x)\phi(y) = \langle 0|\, \mathsf{T}[\phi(x)\phi(y)]\, |0\rangle$$

- 根据**正负能解展开式**，**当 $x^0 > y^0$ 时**，有

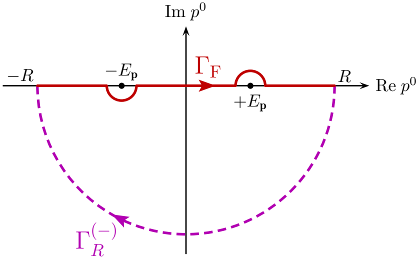
$$\begin{aligned}
&\langle 0|\, \mathsf{T}[\phi(x)\phi(y)]\, |0\rangle = \langle 0|\, \phi(x)\phi(y)\, |0\rangle = \langle 0|\, \phi^{(+)}(x)\phi^{(-)}(y)\, |0\rangle \\
&= \int \frac{d^3p\, d^3q}{(2\pi)^6 \sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \langle 0|\, a_{\mathbf{p}} e^{-ip\cdot x} a^\dagger_{\mathbf{q}} e^{iq\cdot y}\, |0\rangle = \int \frac{d^3p\, d^3q\, e^{-i(p\cdot x - q\cdot y)}}{(2\pi)^6 \sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \langle 0|\, [a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}]\, |0\rangle \\
&= \int \frac{d^3p\, d^3q\, e^{-i(p\cdot x - q\cdot y)}}{(2\pi)^3 \sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \delta^{(3)}(\mathbf{p}-\mathbf{q}) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip\cdot(x-y)}}{2E_{\mathbf{p}}} \\
&= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}
\end{aligned}$$

- 第四步用到**产生湮灭算符的对易关系**

- 借助**复变函数**的知识，可以将最后一行中的因子 $\dfrac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{(2E_{\mathbf{p}})}$ 化为**一维积分**

复平面上的曲线积分

- 将 p^0 视作**复变量**，在 p^0 的**复平面**上考虑**函数** $\dfrac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})}$ 的**曲线积分**
- 这个函数具有两个**一阶极点**，$p^0=\pm E_\mathbf{p}$，均位于实轴上
- 右下图画出 p^0 复平面上的几条积分路径
- **路径** Γ_F 在两个极点处分别通过一个**半径无穷小**的**半圆**绕过极点，当 $R\to\infty$ 时，Γ_F 将从 $p^0=-\infty$ 一直延伸到 $p^0=+\infty$
- 将 Γ_F 与**下半平面**上的**半圆弧** $\Gamma_R^{(-)}$ 组成**围线** $C_\mathrm{F}^{(-)}=\Gamma_\mathrm{F}+\Gamma_R^{(-)}$，曲线方向沿**顺时针方向**，即**反方向**



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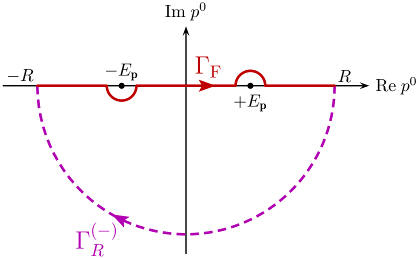
- **由于** $x^0-y^0>0$，$e^{-ip^0(x^0-y^0)}$ 中的因子 $e^{\mathrm{Im}(p^0)(x^0-y^0)}$ 在**下半平面** $(\mathrm{Im}\,p^0<0)$ 随着 R 增大而**指数衰减**
- 根据**复分析**的 **Jordan 引理**，有

$$\lim_{R\to\infty}\int_{\Gamma_R^{(-)}} dp^0\,\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}=0$$

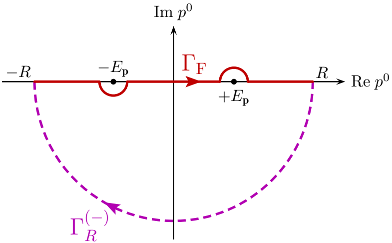


沿 Γ_F 的积分

🍔 当 $R \to \infty$ 时，通过留数定理计算相应的积分主值，得

$$\int_{\Gamma_F} dp^0 \, \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_\mathbf{p})(p^0 + E_\mathbf{p})} = \int_{C_F^{(-)} = \Gamma_F + \Gamma_R^{(-)}} dp^0 \, \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_\mathbf{p})(p^0 + E_\mathbf{p})}$$

$$= -2\pi i \operatorname*{Res}_{p^0 = E_\mathbf{p}} \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_\mathbf{p})(p^0 + E_\mathbf{p})} = -2\pi i \, \frac{e^{-iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}}$$



沿 Γ_F 的积分

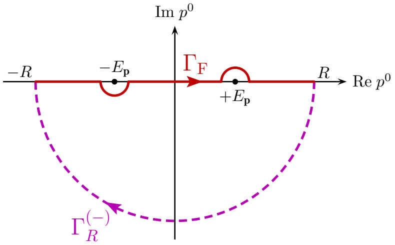
当 $R\to\infty$ 时，通过留数定理计算相应的积分主值，得

$$\int_{\Gamma_\mathrm{F}} dp^0\,\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})}=\int_{C_\mathrm{F}^{(-)}=\Gamma_\mathrm{F}+\Gamma_R^{(-)}} dp^0\,\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})}$$

$$=-2\pi i\,\operatorname*{Res}_{p^0=E_\mathbf{p}}\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})}=-2\pi i\,\frac{e^{-iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}}$$

利用

$$\begin{aligned}(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})&=(p^0)^2-E_\mathbf{p}^2\\&=(p^0)^2-|\mathbf{p}|^2-m^2\\&=p^2-m^2\end{aligned}$$



进一步得到

$$\frac{e^{-iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}}=-\frac{1}{2\pi i}\int_{\Gamma_\mathrm{F}} dp^0\,\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})}=\int_{\Gamma_\mathrm{F}}\frac{dp^0}{2\pi}\,\frac{i\,e^{-ip^0(x^0-y^0)}}{p^2-m^2}$$

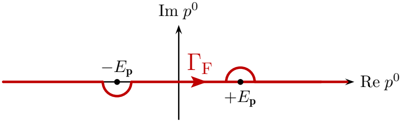
沿实轴积分

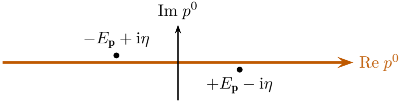
- 如右下图所示，如果将左边极点沿正虚轴方向移动一个**无穷小量** $\eta > 0$，右边极点沿负虚轴方向同样移动无穷小量 η，则**沿实轴积分**将等价于原来**沿 Γ_F 积分**
- 此时**极点**位置为 $p^0 = \pm(E_{\mathbf{p}} - i\eta)$，被积函数中的**分母** $p^2 - m^2$ 应改成

$$[p^0 - (E_{\mathbf{p}} - i\eta)][p^0 + (E_{\mathbf{p}} - i\eta)] = (p^0)^2 - (E_{\mathbf{p}} - i\eta)^2$$
$$= (p^0)^2 - E_{\mathbf{p}}^2 + 2i\eta E_{\mathbf{p}} + \eta^2 \simeq p^2 - m^2 + i\epsilon$$

- 最后一步忽略了 η 的**二阶小量**，而 $\epsilon \equiv 2\eta E_{\mathbf{p}} > 0$ 也是无穷小量，于是得到

$$\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int_{\Gamma_F} \frac{dp^0}{2\pi} \frac{i\, e^{-ip^0(x^0-y^0)}}{p^2 - m^2}$$
$$= \int \frac{dp^0}{2\pi} \frac{i\, e^{-ip^0(x^0-y^0)}}{[p^0 - (E_{\mathbf{p}} - i\eta)][p^0 + (E_{\mathbf{p}} - i\eta)]}$$
$$= \int \frac{dp^0}{2\pi} \frac{i\, e^{-ip^0(x^0-y^0)}}{p^2 - m^2 + i\epsilon}$$





$x^0 > y^0$ 时的 $\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$

将以上 $\dfrac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \displaystyle\int \frac{dp^0}{2\pi}\frac{i\,e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}$

代入 $x^0 > y^0$ 时 $\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$ 的表达式，立即推出

$$
\begin{aligned}
\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} \\
&= \int \frac{d^3p}{(2\pi)^3}\int \frac{dp^0}{2\pi}\,\frac{i\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon} \\
&= \int \frac{d^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}
\end{aligned}
$$

$x^0 < y^0$ 的情况

- 当 $x^0 < y^0$ 时，时间排序将改变 $\phi(x)$ 和 $\phi(y)$ 的次序，有

$$\langle 0| \mathsf{T}[\phi(x)\phi(y)] |0\rangle = \langle 0| \phi(y)\phi(x) |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip\cdot(y-x)}}{2E_{\mathbf{p}}} = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot(x-y)}}{2E_{\mathbf{p}}}$$
$$= \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

- 最后一步把积分变量 $\mathbf{p}$ 替换成 $-\mathbf{p}$

$x^0 < y^0$ 的情况

- 当 $x^0 < y^0$ 时，时间排序将改变 $\phi(x)$ 和 $\phi(y)$ 的次序，有

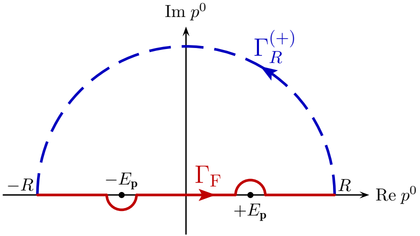
$$\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \langle 0|\,\phi(y)\phi(x)\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{e^{-ip\cdot(y-x)}}{2E_{\mathbf{p}}} = \int \frac{d^3p}{(2\pi)^3}\frac{e^{ip\cdot(x-y)}}{2E_{\mathbf{p}}}$$
$$= \int \frac{d^3p}{(2\pi)^3}\, e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int \frac{d^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

- 最后一步把积分变量 $\mathbf{p}$ 替换成 $-\mathbf{p}$

- 将 Γ_{F} 与上半平面上的半圆弧 $\Gamma_R^{(+)}$ 组成围线 $C_{\mathrm{F}}^{(+)} = \Gamma_{\mathrm{F}} + \Gamma_R^{(+)}$，曲线方向沿逆时针方向，即正方向

- 由于 $x^0 - y^0 < 0$，$e^{-ip^0(x^0-y^0)}$ 中的因子 $e^{\mathrm{Im}(p^0)(x^0-y^0)}$ 在上半平面 $(\mathrm{Im}\,p^0 > 0)$ 随着 R 增大而指数衰减，由 Jordan 引理得

$$\lim_{R\to\infty}\int_{\Gamma_R^{(+)}} dp^0\,\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = 0$$

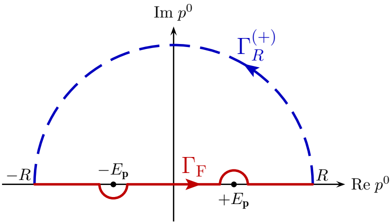


$x^0 < y^0$ 时的 $\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$

从而在 $R \to \infty$ 时推出

$$\int_{\Gamma_\mathrm{F}} \mathrm{d}p^0\, \frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})} = \int_{C_\mathrm{F}^{(+)}=\Gamma_\mathrm{F}+\Gamma_R^{(+)}} \mathrm{d}p^0\, \frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})}$$

$$= 2\pi i \operatorname*{Res}_{p^0=-E_\mathbf{p}} \frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})} = -2\pi i\, \frac{e^{iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}}$$



$x^0 < y^0$ 时的 $\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$

从而在 $R\to\infty$ 时推出

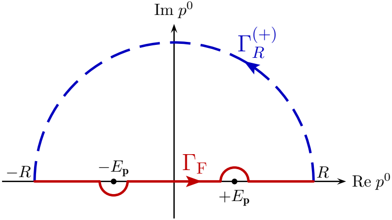
$$\int_{\Gamma_{\mathrm{F}}} \mathrm{d}p^0\, \frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = \int_{C_{\mathrm{F}}^{(+)}=\Gamma_{\mathrm{F}}+\Gamma_R^{(+)}} \mathrm{d}p^0\, \frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}$$

$$= 2\pi i \operatorname*{Res}_{p^0=-E_{\mathbf{p}}} \frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = -2\pi i\, \frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

故

$$\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = -\frac{1}{2\pi i}\int_{\Gamma_{\mathrm{F}}} \mathrm{d}p^0\, \frac{e^{-ip^0(x^0-y^0)}}{p^2-m^2}$$

$$= \int \frac{\mathrm{d}p^0}{2\pi}\, \frac{i\,e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}$$



代入 $x^0 < y^0$ 时 $\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$ 的表达式，即得

$$\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\, \frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, \frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$

实标量场的 Feynman 传播子

- 可见，无论 $x^0 > y^0$ 还是 $x^0 < y^0$，$\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$ 的表达式都是**一样**的
- 因此，**实标量场的 Feynman 传播子**总可以表达为

$$D_{\mathrm{F}}(x-y) = \langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \int \frac{d^4p}{(2\pi)^4}\,\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

实标量场的 Feynman 传播子

- 可见，无论 $x^0 > y^0$ 还是 $x^0 < y^0$，$\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$ 的表达式都是**一样**的
- 因此，**实标量场的 Feynman 传播子**总可以表达为

$$D_{\mathrm{F}}(x-y) = \langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

- 它是 Lorentz 不变的，而且是一个**偶函数**，

$$D_{\mathrm{F}}(y-x) = \int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\epsilon} = \int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\epsilon} = D_{\mathrm{F}}(x-y)$$

- 第二步作了**变量替换** $p^\mu \to -p^\mu$
- 可见，$\phi(y)\phi(x) = \phi(x)\phi(y)$

5.4.2 小节　复标量场的 Feynman 传播子

- 在**相互作用绘景**中，将**复标量场 $\phi(x)$** 的平面波展开式分解为**正负**能解部分，得

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x), \qquad \phi^\dagger(x) = \phi^{\dagger(+)}(x) + \phi^{\dagger(-)}(x)$$

$$\phi^{(+)}(x) \equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\, a_{\mathbf{p}} e^{-ip\cdot x}, \qquad \phi^{\dagger(+)}(x) \equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\, b_{\mathbf{p}} e^{-ip\cdot x}$$

$$\phi^{(-)}(x) \equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\, b_{\mathbf{p}}^\dagger e^{ip\cdot x}, \qquad \phi^{\dagger(-)}(x) \equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\, a_{\mathbf{p}}^\dagger e^{ip\cdot x}$$

- 注意场算符缩并的定义是 $\overbracket{\Phi_a(x)\Phi_b}(y) = \begin{cases} [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_\mp, & x^0 \geq y^0 \\ \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp, & x^0 < y^0 \end{cases}$

- 由于 $(a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger)$ 和 $(b_{\mathbf{p}}, b_{\mathbf{p}}^\dagger)$ 是两套**相互独立**的产生湮灭算符，有

$$\overbracket{\phi(x)\phi}(y) = \langle 0|\, \mathsf{T}[\phi(x)\phi(y)]\, |0\rangle = 0, \qquad \overbracket{\phi^\dagger(x)\phi^\dagger}(y) = \langle 0|\, \mathsf{T}[\phi^\dagger(x)\phi^\dagger(y)]\, |0\rangle = 0$$

- **复标量场的 Feynman 传播子**是**非平庸**的缩并，定义为

$$D_{\mathrm{F}}(x-y) \equiv \overbracket{\phi(x)\phi^\dagger}(y) = \langle 0|\, \mathsf{T}[\phi(x)\phi^\dagger(y)]\, |0\rangle$$

$\langle 0|\,\mathsf{T}[\phi(x)\phi^\dagger(y)]\,|0\rangle$

- 类似于实标量场的计算，利用产生湮灭算符的对易关系，得到

$$\begin{aligned}
\langle 0|\,\phi(x)\phi^\dagger(y)\,|0\rangle &= \langle 0|\,\phi^{(+)}(x)\phi^{\dagger(-)}(y)\,|0\rangle = \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\langle 0|\,a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot x}a^\dagger_{\mathbf{q}}\mathrm{e}^{\mathrm{i}q\cdot y}\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\langle 0|\,[a_{\mathbf{p}},a^\dagger_{\mathbf{q}}]\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}
\end{aligned}$$

$$\begin{aligned}
\langle 0|\,\phi^\dagger(y)\phi(x)\,|0\rangle &= \langle 0|\,\phi^{\dagger(+)}(y)\phi^{(-)}(x)\,|0\rangle = \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\langle 0|\,b_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot y}b^\dagger_{\mathbf{q}}\mathrm{e}^{\mathrm{i}q\cdot x}\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot y-q\cdot x)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\langle 0|\,[b_{\mathbf{p}},b^\dagger_{\mathbf{q}}]\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot y-q\cdot x)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}
\end{aligned}$$

- $$\begin{aligned}
\langle 0|\,\mathsf{T}[\phi(x)\phi^\dagger(y)]\,|0\rangle &= \theta(x^0-y^0)\,\langle 0|\,\phi(x)\phi^\dagger(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\phi^\dagger(y)\phi(x)\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\left[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + \theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right]
\end{aligned}$$

四维动量积分

- 归纳上面实标量场计算中的相关结果，有

$$\theta(x^0-y^0)\,\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}+\theta(y^0-x^0)\,\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}=\int\frac{dp^0}{2\pi}\,\frac{i\,e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}$$

- $\epsilon>0$ 是一个无穷小量，进而推出

$$\begin{aligned}
&\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{ip\cdot(x-y)}]\\
&=\int\frac{d^3p}{(2\pi)^3}\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\left[\theta(x^0-y^0)\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}+\theta(y^0-x^0)\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}\right]\\
&=\int\frac{d^3p}{(2\pi)^3}\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\int\frac{dp^0}{2\pi}\,\frac{i\,e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}=\int\frac{d^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}
\end{aligned}$$

- 第二步将方括号第二项积分变量 $\mathbf{p}$ 替换成 $-\mathbf{p}$，从而把 $e^{ip\cdot(x-y)}$ 中的 $e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$ 因子变成 $e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$ 因子
- 最后一步将三维动量积分和 p^0 积分合成四维动量积分

复标量场的 Feynman 传播子

- 于是，**复标量场的 Feynman 传播子**表达为

$$
\begin{aligned}
D_{\mathrm{F}}(x-y) &= \langle 0|\, \mathsf{T}[\phi(x)\phi^\dagger(y)]\, |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{ip\cdot(x-y)}] \\
&= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)}
\end{aligned}
$$

- 可见，复标量场与**实标量场**具有**相同**形式的 Feynman 传播子

- 此外，由 $\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}\,\mathsf{T}[\Phi_b(y)\Phi_a(x)]$ 和 $D_{\mathrm{F}}(x-y)$ 的**偶函数**性质得

$$
\wick{\c{\phi}^\dagger(x)\c{\phi}(y)} = \langle 0|\, \mathsf{T}[\phi^\dagger(x)\phi(y)]\, |0\rangle = \langle 0|\, \mathsf{T}[\phi(y)\phi^\dagger(x)]\, |0\rangle = D_{\mathrm{F}}(y-x) = D_{\mathrm{F}}(x-y)
$$

- 也就是说，$\wick{\c{\phi}^\dagger(x)\c{\phi}(y)}$ **与** $\wick{\c{\phi}(x)\c{\phi}^\dagger(y)}$ **相等**

5.4.3 小节　有质量矢量场的 Feynman 传播子

有质量实矢量场 $A^\mu(x)$ 的 Feynman 传播子 $\Delta_\mathrm{F}^{\mu\nu}(x-y)$ 定义为

$$\Delta_\mathrm{F}^{\mu\nu}(x-y) \equiv \overbrace{A^\mu(x)A^\nu(y)} = \langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle$$

根据平面波展开式、**产生湮灭算符的对易关系**和**极化矢量求和关系**，有

$$\begin{aligned}
\langle 0|\,A^\mu(x)A^\nu(y)\,|0\rangle &= \langle 0|\,A^{\mu(+)}(x)A^{\nu(-)}(y)\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_\mathbf{p}E_\mathbf{q}}} \sum_{\lambda\lambda'} \langle 0|\,\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-\mathrm{i}p\cdot x}\varepsilon^{\nu*}(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{\mathrm{i}q\cdot y}\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,e^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{4E_\mathbf{p}E_\mathbf{q}}} \sum_{\lambda\lambda'} \varepsilon^\mu(\mathbf{p},\lambda)\varepsilon^{\nu*}(\mathbf{q},\lambda')\,\langle 0|\,[a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}]\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,e^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^3\sqrt{4E_\mathbf{p}E_\mathbf{q}}} \sum_{\lambda\lambda'} \varepsilon^\mu(\mathbf{p},\lambda)\varepsilon^{\nu*}(\mathbf{q},\lambda')\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= \int \frac{\mathrm{d}^3p\,e^{-\mathrm{i}p\cdot(x-y)}}{(2\pi)^3 2E_\mathbf{p}} \sum_{\lambda} \varepsilon^\mu(\mathbf{p},\lambda)\varepsilon^{\nu*}(\mathbf{p},\lambda) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{e^{-\mathrm{i}p\cdot(x-y)}}{2E_\mathbf{p}}
\end{aligned}$$

$\langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle$

由上述 $\langle 0|\,A^\mu(x)A^\nu(y)\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{e^{-ip\cdot(x-y)}}{2E_{\mathbf{p}}}$ 得到

$$\langle 0|\,A^\nu(y)A^\mu(x)\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\left(-g^{\nu\mu}+\frac{p^\nu p^\mu}{m^2}\right)\frac{e^{-ip\cdot(y-x)}}{2E_{\mathbf{p}}} = \int \frac{d^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{e^{ip\cdot(x-y)}}{2E_{\mathbf{p}}}$$

从而推出 Feynman 传播子表达式

$$\Delta_F^{\mu\nu}(x-y) = \langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle$$
$$= \theta(x^0-y^0)\,\langle 0|\,A^\mu(x)A^\nu(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,A^\nu(y)A^\mu(x)\,|0\rangle$$
$$= \int \frac{d^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{1}{2E_{\mathbf{p}}}\left[\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{ip\cdot(x-y)}\right]$$

最后一行圆括号中的项 $p^\mu p^\nu/m^2$ **与 p^0 有关**，因此**不能直接应用**前面的

$$\theta(x^0-y^0)\,\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}+\theta(y^0-x^0)\,\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int \frac{dp^0}{2\pi}\,\frac{i\,e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}$$

合成四维动量积分

求导运算

- 为了得到简洁的表达式，需要将 $p^\mu p^\nu/m^2$ 转换为**求导运算**
- 记 $\partial_x^\mu \equiv \partial/\partial x_\mu$，利用阶跃函数与 δ 函数的关系 $\theta'(x) = \delta(x)$，推出

$$
\begin{aligned}
&\partial_x^\mu \partial_x^\nu [\theta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + \theta(y^0 - x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
= {}& \partial_x^\mu [-\mathrm{i}p^\nu \theta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + g^{\nu 0}\delta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + \mathrm{i}p^\nu \theta(y^0 - x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\\
&\quad - g^{\nu 0}\delta(y^0 - x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
= {}& -p^\mu p^\nu \theta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{i}g^{\mu 0}p^\nu \delta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{i}p^\mu g^{\nu 0}\delta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}\\
&+ g^{\mu 0}g^{\nu 0}\partial_x^0 \delta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - p^\mu p^\nu \theta(y^0 - x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)} - \mathrm{i}g^{\mu 0}p^\nu \delta(y^0 - x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\\
&- \mathrm{i}p^\mu g^{\nu 0}\delta(y^0 - x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)} + g^{\mu 0}g^{\nu 0}\partial_x^0 \delta(y^0 - x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\\
= {}& -p^\mu p^\nu [\theta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + \theta(y^0 - x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
&- \mathrm{i}(g^{\mu 0}p^\nu + g^{\nu 0}p^\mu)\delta(x^0 - y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + \mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
&+ g^{\mu 0}g^{\nu 0}\partial_x^0 \delta(x^0 - y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}]
\end{aligned}
$$

分解

从而将包含 $p^\mu p^\nu/m^2$ 的因子化为

$$\begin{aligned}
&\frac{p^\mu p^\nu}{m^2}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
=&-\frac{\partial_x^\mu\partial_x^\nu}{m^2}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
&-\frac{\mathrm{i}}{m^2}(g^{\mu0}p^\nu+g^{\nu0}p^\mu)\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
&+\frac{g^{\mu0}g^{\nu0}}{m^2}\partial_x^0\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]
\end{aligned}$$

由此将 Feynman 传播子**分解**成 $\Delta_\mathrm{F}^{\mu\nu}(x-y)=f_1^{\mu\nu}(x,y)+f_2^{\mu\nu}(x,y)+f_3^{\mu\nu}(x,y)$

$$f_1^{\mu\nu}(x,y)\equiv-\left(g^{\mu\nu}+\frac{\partial_x^\mu\partial_x^\nu}{m^2}\right)\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$

$$f_2^{\mu\nu}(x,y)\equiv-\frac{\mathrm{i}}{m^2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}(g^{\mu0}p^\nu+g^{\nu0}p^\mu)\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$

$$f_3^{\mu\nu}(x,y)\equiv\frac{g^{\mu0}g^{\nu0}}{m^2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\partial_x^0\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$

$f_1^{\mu\nu}(x,y)$ 和 $f_2^{\mu\nu}(x,y)$

- 将**第一项**化为

$$f_1^{\mu\nu}(x,y) = -\left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2}\right)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left[\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}\right]$$

$$= -\left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2}\right)\int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} = \int \frac{d^4p}{(2\pi)^4}\frac{-i(g^{\mu\nu}-p^\mu p^\nu/m^2)}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}$$

- $\delta(x^0-y^0)$ 只在 $x^0-y^0=0$ **处**非零，**该处**有 $e^{-iE_\mathbf{p}(x^0-y^0)} = e^{iE_\mathbf{p}(x^0-y^0)} = 1$，故

$$f_2^{i0}(x,y) = f_2^{0i}(x,y) = -\frac{i}{m^2}\int \frac{d^3p}{(2\pi)^3}\frac{p^i}{2E_\mathbf{p}}\,\delta(x^0-y^0)\left[e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right] = 0$$

- 上式中被积函数是**关于 $\mathbf{p}$ 的奇函数**，因而对整个三维动量空间积分为零
- 利用 **Fourier 变换公式**导出

$$\begin{aligned} f_2^{00}(x,y) &= -\frac{i}{m^2}\int \frac{d^3p}{(2\pi)^3}\frac{2p^0}{2E_\mathbf{p}}\,\delta(x^0-y^0)\left[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right] \\ &= -\frac{2i}{m^2}\,\delta(x^0-y^0)\delta^{(3)}(\mathbf{x}-\mathbf{y}) = -\frac{2i}{m^2}\,\delta^{(4)}(x-y) \end{aligned}$$

- 归纳得到**第二项**表达式 $f_2^{\mu\nu}(x,y) = -\dfrac{2i}{m^2}\,g^{\mu 0}g^{\nu 0}\delta^{(4)}(x-y)$

$f_3^{\mu\nu}(x,y)$

- 另一方面，根据 δ 函数的导数的定义，有

$$\int \mathrm{d}x\, f(x)\delta'(x-a) = -f'(a) = -\int \mathrm{d}x\, f'(x)\delta(x-a)$$

- 因而对第三项中的被积函数可作替换

$$\partial_x^0\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \to -\delta(x^0-y^0)\partial_x^0[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$

- 由此将第三项化为

$$\begin{aligned}
f_3^{\mu\nu}(x,y) &= \frac{g^{\mu 0}g^{\nu 0}}{m^2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\,\partial_x^0\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \\
&= -\frac{g^{\mu 0}g^{\nu 0}}{m^2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\,\delta(x^0-y^0)\partial_x^0[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \\
&= -\frac{g^{\mu 0}g^{\nu 0}}{m^2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\,\delta(x^0-y^0)[-\mathrm{i}p^0\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{i}p^0\mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \\
&= \frac{\mathrm{i}}{2m^2}\,g^{\mu 0}g^{\nu 0}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\delta(x^0-y^0)[\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}] \\
&= \frac{\mathrm{i}}{m^2}\,g^{\mu 0}g^{\nu 0}\delta^{(4)}(x-y)
\end{aligned}$$

有质量矢量场的 Feynman 传播子

- 综合起来，**有质量矢量场 Feynman 传播子**的表达式为

$$\Delta_F^{\mu\nu}(x-y) = f_1^{\mu\nu}(x,y) + f_2^{\mu\nu}(x,y) + f_3^{\mu\nu}(x,y)$$
$$= \int \frac{d^4p}{(2\pi)^4} \frac{-i(g^{\mu\nu} - p^\mu p^\nu/m^2)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} - \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x-y)$$

- 第一项是 **Lorentz 协变**的，但第二项是**非协变**的
- 幸好，6.4 节将证明这个非协变项在微扰论中的贡献被**相互作用哈密顿量密度**中**非协变项** $\mathcal{H}^{\mathrm{I}}_{J0} = \dfrac{g^2}{2m^2}(J^{\mathrm{I},0})^2$ 的贡献**精确抵消**，从而**理论是 Lorentz 协变的**
- 因此，在实际计算中可以**只保留协变项**

$$\Delta_F^{\mu\nu}(x-y) = \langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle \to \int \frac{d^4p}{(2\pi)^4} \frac{-i(g^{\mu\nu} - p^\mu p^\nu/m^2)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

5.4.4 小节　无质量矢量场的 Feynman 传播子

- 无质量实矢量场 $A^\mu(x)$ 的 **Feynman 传播子依赖于规范的选择**
- 这里取 **Feynman 规范** $(\xi = 1)$
- 在相互作用绘景中，将无质量矢量场 $A^\mu(x)$ 的平面波展开式分解为

$$\text{正能解}\quad A^{\mu(+)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma) b_{\mathbf{p},\sigma} \mathrm{e}^{-\mathrm{i}p\cdot x}$$

$$\text{负能解}\quad A^{\mu(-)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma) b^\dagger_{\mathbf{p},\sigma} \mathrm{e}^{\mathrm{i}p\cdot x}$$

- 相应的 **Feynman 传播子**定义为

$$\Delta_\mathrm{F}^{\mu\nu}(x-y) \equiv A^\mu(x)A^\nu(y) = \langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle$$

$\langle 0|\, A^\mu(x) A^\nu(y)\, |0\rangle$

根据产生湮灭算符的对易关系和极化矢量的完备性关系，得到

$$
\begin{aligned}
&\langle 0|\, A^\mu(x)A^\nu(y)\,|0\rangle = \langle 0|\, A^{\mu(+)}(x)A^{\nu(-)}(y)\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_\mathbf{p}E_\mathbf{q}}} \sum_{\sigma\sigma'} \langle 0|\, e^\mu(\mathbf{p},\sigma) b_{\mathbf{p},\sigma} \mathrm{e}^{-\mathrm{i}p\cdot x} e^\nu(\mathbf{q},\sigma') b^\dagger_{\mathbf{q},\sigma'} \mathrm{e}^{\mathrm{i}q\cdot y}\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x - q\cdot y)}}{(2\pi)^6\sqrt{4E_\mathbf{p}E_\mathbf{q}}} \sum_{\sigma\sigma'} e^\mu(\mathbf{p},\sigma)e^\nu(\mathbf{q},\sigma')\,\langle 0|\,[b_{\mathbf{p},\sigma}, b^\dagger_{\mathbf{q},\sigma'}]\,|0\rangle \\
&= -\int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x - q\cdot y)}}{(2\pi)^3\sqrt{4E_\mathbf{p}E_\mathbf{q}}} \sum_{\sigma\sigma'} e^\mu(\mathbf{p},\sigma)e^\nu(\mathbf{q},\sigma') g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= -\int \frac{\mathrm{d}^3p\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{(2\pi)^3 2E_\mathbf{p}} \sum_{\sigma} g_{\sigma\sigma} e^\mu(\mathbf{p},\lambda)e^\nu(\mathbf{p},\lambda) = -g^{\mu\nu}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{2E_\mathbf{p}}
\end{aligned}
$$

进而推出

$$
\langle 0|\, A^\nu(y)A^\mu(x)\,|0\rangle = -g^{\nu\mu}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(y-x)}}{2E_\mathbf{p}} = -g^{\mu\nu}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{2E_\mathbf{p}}
$$

无质量矢量场的 Feynman 传播子

从而得到

$$\begin{aligned}\Delta_{\mathrm{F}}^{\mu\nu}(x-y) &= \langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle \\ &= \theta(x^0-y^0)\,\langle 0|\,A^\mu(x)A^\nu(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,A^\nu(y)A^\mu(x)\,|0\rangle \\ &= -g^{\mu\nu}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\end{aligned}$$

当**质量** $m=0$ 时，有

$$\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}] = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2+\mathrm{i}\epsilon}$$

于是，**Feynman 规范**下**无质量矢量场的 Feynman 传播子**可以表达为

$$\Delta_{\mathrm{F}}^{\mu\nu}(x-y) = \langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{-\mathrm{i}g^{\mu\nu}}{p^2+\mathrm{i}\epsilon}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

5.4.5 小节 Dirac 旋量场的 Feynman 传播子

Dirac 旋量场 $\psi_a(x)$ 的 **Feynman 传播子** $S_{\mathrm{F},ab}(x-y)$ 定义为

$$S_{\mathrm{F},ab}(x-y) \equiv \wick{\c\psi_a(x)\c{\bar\psi}_b(y)} = \langle 0|\,\mathsf{T}[\psi_a(x)\bar\psi_b(y)]\,|0\rangle$$

在**相互作用绘景**中，将 $\psi_a(x)$ 和 $\bar\psi_a(x)$ 的平面波展开式分解为**正负**能解部分，得

$$\psi_a(x) = \psi_a^{(+)}(x) + \psi_a^{(-)}(x), \quad \bar\psi_a(x) = \bar\psi_a^{(+)}(x) + \bar\psi_a^{(-)}(x)$$

$$\psi_a^{(+)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm} u_a(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}$$

$$\psi_a^{(-)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm} v_a(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}$$

$$\bar\psi_a^{(+)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm} \bar v_a(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}$$

$$\bar\psi_a^{(-)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm} \bar u_a(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}$$

$\langle 0|\,\psi_a(x)\bar{\psi}_b(y)\,|0\rangle$

利用产生湮灭算符的反对易关系和自旋求和关系，推出

$$
\begin{aligned}
&\langle 0|\,\psi_a(x)\bar{\psi}_b(y)\,|0\rangle \\
=\;& \langle 0|\,\psi_a^{(+)}(x)\bar{\psi}_b^{(-)}(y)\,|0\rangle \\
=\;& \int \frac{d^3p\,d^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \langle 0|\, u_a(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}\bar{u}_b(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot y}\,|0\rangle \\
=\;& \int \frac{d^3p\,d^3q\,e^{-i(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \sum_{\lambda\lambda'} u_a(\mathbf{p},\lambda)\bar{u}_b(\mathbf{q},\lambda')\,\langle 0|\,\{a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}\}\,|0\rangle \\
=\;& \int \frac{d^3p\,d^3q\,e^{-i(p\cdot x-q\cdot y)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \sum_{\lambda\lambda'} u_a(\mathbf{p},\lambda)\bar{u}_b(\mathbf{q},\lambda')\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
=\;& \int \frac{d^3p\,e^{-ip\cdot(x-y)}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\lambda} u_a(\mathbf{p},\lambda)\bar{u}_b(\mathbf{p},\lambda) \\
=\;& \int \frac{d^3p}{(2\pi)^3}(\gamma_\mu p^\mu+m)_{ab}\,\frac{e^{-ip\cdot(x-y)}}{2E_{\mathbf{p}}}
\end{aligned}
$$

$\langle 0|\,\bar\psi_b(y)\psi_a(x)\,|0\rangle$

再推出

$$
\begin{aligned}
&\langle 0|\,\bar\psi_b(y)\psi_a(x)\,|0\rangle\\
={}&\langle 0|\,\bar\psi_b^{(+)}(y)\psi_a^{(-)}(x)\,|0\rangle\\
={}&\int\frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{\lambda\lambda'}\langle 0|\,\bar v_b(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot y}v_a(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}\mathrm{e}^{\mathrm{i}q\cdot x}\,|0\rangle\\
={}&\int\frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot y-q\cdot x)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{\lambda\lambda'}v_a(\mathbf{q},\lambda')\bar v_b(\mathbf{p},\lambda)\,\langle 0|\,\{b_{\mathbf{p},\lambda},b^\dagger_{\mathbf{q},\lambda'}\}\,|0\rangle\\
={}&\int\frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot y-q\cdot x)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{\lambda\lambda'}v_a(\mathbf{q},\lambda')\bar v_b(\mathbf{p},\lambda)\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q})\\
={}&\int\frac{\mathrm{d}^3p\,\mathrm{e}^{-\mathrm{i}p\cdot(y-x)}}{(2\pi)^3 2E_{\mathbf{p}}}\sum_{\lambda}v_a(\mathbf{p},\lambda)\bar v_b(\mathbf{p},\lambda)\\
={}&\int\frac{\mathrm{d}^3p}{(2\pi)^3}(\gamma^\mu p_\mu-m)_{ab}\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}
\end{aligned}
$$

Dirac 旋量场的 Feynman 传播子

✂ 于是，**Dirac 旋量场的 Feynman 传播子**为

$$
\begin{aligned}
S_{\mathrm{F},ab}(x-y) &= \langle 0|\,\mathsf{T}[\psi_a(x)\bar{\psi}_b(y)]\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,\psi_a(x)\bar{\psi}_b(y)\,|0\rangle - \theta(y^0-x^0)\,\langle 0|\,\bar{\psi}_b(y)\psi_a(x)\,|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\left[\theta(x^0-y^0)(\gamma_\mu p^\mu + m)_{ab}\,\frac{e^{-ip\cdot(x-y)}}{2E_{\mathbf{p}}}\right. \\
&\qquad\qquad \left. -\,\theta(y^0-x^0)(\gamma^\mu p_\mu - m)_{ab}\,\frac{e^{ip\cdot(x-y)}}{2E_{\mathbf{p}}}\right] \\
&= (\gamma_0)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[E_{\mathbf{p}}\theta(x^0-y^0)e^{-ip\cdot(x-y)} - E_{\mathbf{p}}\theta(y^0-x^0)e^{ip\cdot(x-y)}] \\
&\quad + (\gamma_i)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[p^i\theta(x^0-y^0)e^{-ip\cdot(x-y)} - p^i\theta(y^0-x^0)e^{ip\cdot(x-y)}] \\
&\quad + m\delta_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[\theta(x^0-y^0)e^{-ip\cdot(x-y)} + (y^0-x^0)e^{ip\cdot(x-y)}\right]
\end{aligned}
$$

Dirac 旋量场的 Feynman 传播子

类似于前面得到的等式

$$\theta(x^0-y^0)\,\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}+\theta(y^0-x^0)\,\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}=\int\frac{dp^0}{2\pi}\,\frac{i\,e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}$$

$$\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{ip\cdot(x-y)}]=\int\frac{d^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$

对于**任意解析函数** $f(p^0)$，可以推出

$$\theta(x^0-y^0)\,\frac{f(E_{\mathbf{p}})e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}+\theta(y^0-x^0)\,\frac{f(-E_{\mathbf{p}})e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

$$=\int\frac{dp^0}{2\pi}\,\frac{i f(p^0)e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}$$

$$\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}[f(E_{\mathbf{p}})\theta(x^0-y^0)e^{-ip\cdot(x-y)}+f(-E_{\mathbf{p}})\theta(y^0-x^0)e^{ip\cdot(x-y)}]$$

$$=\int\frac{d^4p}{(2\pi)^4}\,\frac{i f(p^0)e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}$$

左边**第一项**对应于 $p^0=E_{\mathbf{p}}$ 处的**留数**，**第二项**对应于 $p^0=-E_{\mathbf{p}}$ 处的**留数**

Dirac 旋量场的 Feynman 传播子

- 取 $f(p^0)=p^0$，把 $S_{\mathrm{F},ab}(x-y)$ 的第一项变成

$$(\gamma_0)_{ab}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[E_{\mathbf{p}}\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-E_{\mathbf{p}}\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$
$$=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(\gamma_0p^0)_{ab}\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2+\mathrm{i}\epsilon}$$

- 再将 $S_{\mathrm{F},ab}(x-y)$ 式的后两项化成

$$(\gamma_i)_{ab}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[p^i\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-p^i\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$
$$+m\delta_{ab}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right]$$
$$=[(\gamma_i)_{ab}\mathrm{i}\partial_x^i+m\delta_{ab}]\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right]$$
$$=[(\gamma_i)_{ab}\mathrm{i}\partial_x^i+m\delta_{ab}]\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(\gamma_ip^i+m)_{ab}}{p^2-m^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}.$$

- 把这三项加起来，得到 **Dirac 旋量场的 Feynman 传播子**

$$S_{\mathrm{F},ab}(x-y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(\not{p}+m)_{ab}}{p^2-m^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

等价形式

将 **Dirac 旋量场的 Feynman 传播子**写成**旋量空间矩阵**的形式，得

$$S_{\mathrm{F}}(x-y)=\langle 0|\,\mathsf{T}[\psi(x)\bar{\psi}(y)]\,|0\rangle=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\mathrm{i}(\not{p}+m)}{p^2-m^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

由 $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$ 有

$$\not{p}\not{p}=p_\mu p_\nu\gamma^\mu\gamma^\nu=\frac{1}{2}\,p_\mu p_\nu(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)=p_\mu p_\nu g^{\mu\nu}=p^2$$

从而得到 $(\not{p}+m)(\not{p}-m)=\not{p}\not{p}-m^2=p^2-m^2$，则

$$(\not{p}+m)(\not{p}-m+\mathrm{i}\epsilon)=p^2-m^2+\mathrm{i}\epsilon(\not{p}+m)$$

$\mathrm{i}\epsilon(\not{p}+m)$ 是**无穷小量**，因而上式右边**与** $p^2-m^2+\mathrm{i}\epsilon$ **等价**，故

$$S_{\mathrm{F}}(x-y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\mathrm{i}(\not{p}+m)\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{(\not{p}+m)(\not{p}-m+\mathrm{i}\epsilon)}=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\mathrm{i}}{\not{p}-m+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

最后的表达式更为**简洁**，但在**矩阵**的意义上不好理解，应将它转化回**原式**来理解