

Majorana 旋量场专题

第二节 二分量旋量与四分量旋量

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二分量旋量与四分量旋量

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旋量表示生成元的约化

Dirac 旋量场和 Majorana 旋量场都可以分解为左手和右手的 Weyl 旋量场

为了更深刻地认识旋量场，本节进一步研究 Weyl 旋量

用 $\sigma^\mu = (1, \boldsymbol{\sigma})$ 和 $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$ 定义 2×2 矩阵

$$s^{\mu\nu} \equiv \frac{i}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$$

由 $(\sigma^\mu)^\dagger = \sigma^\mu$ 和 $(\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu$ 推出

$$(s^{\mu\nu})^\dagger = -\frac{i}{4} [(\bar{\sigma}^\nu)^\dagger (\sigma^\mu)^\dagger - (\bar{\sigma}^\mu)^\dagger (\sigma^\nu)^\dagger] = -\frac{i}{4} (\bar{\sigma}^\nu \sigma^\mu - \bar{\sigma}^\mu \sigma^\nu) = \frac{i}{4} (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)$$

从而将 Weyl 表象中的旋量表示生成元约化为

$$\mathcal{S}^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] = \frac{i}{4} \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & s^{\mu\nu} \\ \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu & (s^{\mu\nu})^\dagger \end{pmatrix} = \begin{pmatrix} s^{\mu\nu} & \\ & (s^{\mu\nu})^\dagger \end{pmatrix}$$

也就是说， 4×4 矩阵 $\mathcal{S}^{\mu\nu}$ 是 2×2 矩阵 $s^{\mu\nu}$ 和 $(s^{\mu\nu})^\dagger$ 的直和

因而 $s^{\mu\nu}$ 和 $(s^{\mu\nu})^\dagger$ 是两个 Lorentz 群 2 维表示的生成元

左手和右手 Weyl 旋量所处 2 维表示

对于 Lorentz 变换 Λ 的一组变换参数 $\omega_{\mu\nu}$ ，用 $s^{\mu\nu}$ 生成固有保时向有限变换

$$d(\Lambda) \equiv \exp \left(-\frac{i}{2} \omega_{\mu\nu} s^{\mu\nu} \right)$$

它属于左手 Weyl 旋量所处的 2 维表示

相应的逆变换矩阵为 $d^{-1}(\Lambda) = \exp \left(\frac{i}{2} \omega_{\mu\nu} s^{\mu\nu} \right)$ ，取厄米共轭，得

$$d^{-1\dagger}(\Lambda) = \exp \left[-\frac{i}{2} \omega_{\mu\nu} (s^{\mu\nu})^\dagger \right]$$

这是用 $(s^{\mu\nu})^\dagger$ 生成的固有保时向有限变换，属于右手 Weyl 旋量所处的 2 维表示

左手和右手 Weyl 旋量所处 2 维表示

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这是用 $(s^{\mu\nu})^\dagger$ 生成的固有保时向有限变换，属于右手 Weyl 旋量所处的 2 维表示

于是，旋量表示的 4×4 Lorentz 变换矩阵分解为

$$D(\Lambda) = \exp\left(-\frac{i}{2} \omega_{\mu\nu} S^{\mu\nu}\right) = \begin{pmatrix} e^{-i\omega_{\mu\nu} s^{\mu\nu}/2} & \\ & e^{-i\omega_{\mu\nu} (s^{\mu\nu})^\dagger/2} \end{pmatrix} = \begin{pmatrix} d(\Lambda) & \\ & d^{-1\dagger}(\Lambda) \end{pmatrix}$$

因此，4 维旋量表示 $\{D(\Lambda)\}$ 是 2 维表示 $\{d(\Lambda)\}$ 和 $\{d^{-1\dagger}(\Lambda)\}$ 的直和

等价表示

利用 $\sigma^2 \sigma^\mu \sigma^2 = (\bar{\sigma}^\mu)^T$ 和 $\sigma^2 \bar{\sigma}^\mu \sigma^2 = (\sigma^\mu)^T$ 推出

$$\begin{aligned}\sigma^2 s^{\mu\nu} \sigma^2 &= \frac{i}{4} (\sigma^2 \sigma^\mu \sigma^2 \sigma^2 \bar{\sigma}^\nu \sigma^2 - \sigma^2 \sigma^\nu \sigma^2 \sigma^2 \bar{\sigma}^\mu \sigma^2) \\ &= \frac{i}{4} [(\bar{\sigma}^\mu)^T (\sigma^\nu)^T - (\bar{\sigma}^\nu)^T (\sigma^\mu)^T] = -(s^{\mu\nu})^T\end{aligned}$$

$$\begin{aligned}\sigma^2 d(\Lambda) \sigma^2 &= \exp \left(-\frac{i}{2} \omega_{\mu\nu} \sigma^2 s^{\mu\nu} \sigma^2 \right) \\ &= \exp \left[\frac{i}{2} \omega_{\mu\nu} (s^{\mu\nu})^T \right] = \left[\exp \left(\frac{i}{2} \omega_{\mu\nu} s^{\mu\nu} \right) \right]^T = d^{-1T}(\Lambda)\end{aligned}$$

这里 $d^{-1T}(\Lambda) = [d^{-1\dagger}(\Lambda)]^*$ ，线性表示 $\{d^{-1T}(\Lambda)\}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ 的复共轭表示

等价表示

利用 $\sigma^2 \sigma^\mu \sigma^2 = (\bar{\sigma}^\mu)^T$ 和 $\sigma^2 \bar{\sigma}^\mu \sigma^2 = (\sigma^\mu)^T$ 推出

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$$\begin{aligned}\sigma^2 d(\Lambda) \sigma^2 &= \exp \left(-\frac{i}{2} \omega_{\mu\nu} \sigma^2 s^{\mu\nu} \sigma^2 \right) \\ &= \exp \left[\frac{i}{2} \omega_{\mu\nu} (s^{\mu\nu})^T \right] = \left[\exp \left(\frac{i}{2} \omega_{\mu\nu} s^{\mu\nu} \right) \right]^T = d^{-1T}(\Lambda)\end{aligned}$$

$$\begin{array}{ccc} \{d(\Lambda)\} & \longleftrightarrow & \{d^*(\Lambda)\} \\ \downarrow \text{等价} & & \uparrow \text{等价} \\ \{d^{-1T}(\Lambda)\} & \longleftrightarrow & \{d^{-1\dagger}(\Lambda)\} \end{array}$$

这里 $d^{-1T}(\Lambda) = [d^{-1\dagger}(\Lambda)]^*$ ，线性表示 $\{d^{-1T}(\Lambda)\}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ 的复共轭表示

将 Pauli 矩阵 σ^2 看作一个幺正变换矩阵，满足 $(\sigma^2)^{-1} = (\sigma^2)^\dagger = \sigma^2$

则 $d(\Lambda)$ 与 $d^{-1T}(\Lambda)$ 由一个相似变换联系起来，相似变换矩阵为 σ^2

因此，线性表示 $\{d(\Lambda)\}$ 和 $\{d^{-1T}(\Lambda)\}$ 是等价的

由于 $(\sigma^2)^* = -\sigma^2$ ， $\sigma^2 d(\Lambda) \sigma^2 = d^{-1T}(\Lambda)$ 的复共轭为 $\sigma^2 d^*(\Lambda) \sigma^2 = d^{-1\dagger}(\Lambda)$

可见，线性表示 $\{d(\Lambda)\}$ 的复共轭表示 $\{d^*(\Lambda)\}$ 与 $\{d^{-1\dagger}(\Lambda)\}$ 等价

左手 Weyl 旋量

于是, 左手 Weyl 旋量

$$\eta_a = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

的固有保时向 Lorentz 变换为

$$\eta'_a = [d(\Lambda)]_a^b \eta_b, \quad a, b = 1, 2$$

于是 η_a 是 $\{d(\Lambda)\}$ 表示空间中的列矢量

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引入反对称的二维 Levi-Civita 符号 ε^{ab} , 定义为

$$\varepsilon^{12} = -\varepsilon^{21} = 1, \quad \varepsilon^{11} = \varepsilon^{22} = 0$$

它与 Pauli 矩阵 σ^2 的关系是

$$\varepsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = (i\sigma^2)^{ab}$$

等价的左手 Weyl 旋量

通过 ε^{ab} 定义

$$\eta^a \equiv \varepsilon^{ab} \eta_b = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ -\eta_1 \end{pmatrix}$$

则

$$\eta^1 = \eta_2, \quad \eta^2 = -\eta_1$$

因此 $\sigma^2 d(\Lambda) \sigma^2 = d^{-1T}(\Lambda)$ 等价于 $\sigma^2 d(\Lambda) = d^{-1T}(\Lambda) \sigma^2$ ，故 η^a 的 Lorentz 变换为

$$\begin{aligned}\eta'^a &= \varepsilon^{ab} \eta'_b = \varepsilon^{ab} [d(\Lambda)]_b{}^c \eta_c = i[\sigma^2 d(\Lambda)]^{ac} \eta_c \\ &= i[d^{-1T}(\Lambda) \sigma^2]^{ac} \eta_c = [d^{-1T}(\Lambda)]^a{}_b \varepsilon^{bc} \eta_c\end{aligned}$$

等价的左手 Weyl 旋量

通过 ε^{ab} 定义

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即

$$\eta'^a = [d^{-1\mathrm{T}}(\Lambda)]^a{}_b \eta^b$$

可见 η^a 是 $\{d^{-1}\Lambda\}$ 表示空间中的列矢量

由于 $\{d^{-1T}(\Lambda)\}$ 等价于 $\{d(\Lambda)\}$ ， η^a 也是左手 Weyl 旋量

ε^{ab} 和 ε_{ab}

 两种左手 Weyl 旋量 η_a 与 η^a 是等价的，它们之间的关系类似于 Lorentz 逆变矢量 A^μ 与协变矢量 $A_\mu = g_{\mu\nu} A^\nu$ 之间的关系

 ε^{ab} 的作用类似于度规 $g_{\mu\nu}$ ，相当于 2 维旋量空间的“度规”，用于提升旋量指标

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 用 $\varepsilon_{12} = -\varepsilon_{21} = -1$ 和 $\varepsilon_{11} = \varepsilon_{22} = 0$ 定义 ε_{ab} ，则

$$\varepsilon_{ab} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -i \begin{pmatrix} & -i \\ i & \end{pmatrix} = (-i\sigma^2)_{ab}$$

 ε_{ab} 是 ε^{ab} 的逆矩阵，满足

$$\varepsilon_{ab} \varepsilon^{bc} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \delta_a^c$$

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 于是， $\eta^1 = \eta_2$ 和 $\eta^2 = -\eta_1$ 表明

$$\eta_a = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta^2 \\ \eta^1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} = \varepsilon_{ab} \eta^b$$

 也就是说， ε_{ab} 用于下降旋量指标

左手 Weyl 旋量的内积

任意两个左手 Weyl 旋量 η_a 和 ζ_a 的内积

$$\eta^a \zeta_a = \varepsilon^{ab} \eta_b \zeta_a = \varepsilon_{ab} \eta^a \zeta^b$$

在固有保时向 Lorentz 变换下不变，满足

$$\eta'^a \zeta'_a = [d^{-1T}(\Lambda)]^a_b \eta^b [d(\Lambda)]_a^c \zeta_c = \eta^b [d^{-1}(\Lambda)]_b^a [d(\Lambda)]_a^c \zeta_c = \eta^b \delta_b^c \zeta_c = \eta^a \zeta_a$$

第二步用了转置性质 $[d^{-1T}(\Lambda)]^a_b = [d^{-1}(\Lambda)]_b^a$ ，可见 $\eta^a \zeta_a$ 是 Lorentz 标量

左手 Weyl 旋量的内积

hog 任意两个左手 Weyl 旋量 η_a 和 ζ_a 的内积

$$\eta^a \zeta_a = \varepsilon^{ab} \eta_b \zeta_a = \varepsilon_{ab} \eta^a \zeta^b$$

在固有保时向 Lorentz 变换下不变, 满足

$$\eta'^a \zeta'_a = [d^{-1T}(\Lambda)]^a{}_b \eta^b [d(\Lambda)]_a{}^c \zeta_c = \eta^b [d^{-1}(\Lambda)]_b{}^a [d(\Lambda)]_a{}^c \zeta_c = \eta^b \delta_b{}^c \zeta_c = \eta^a \zeta_a$$

girl 第二步用了转置性质 $[d^{-1T}(\Lambda)]^a{}_b = [d^{-1}(\Lambda)]_b{}^a$, 可见 $\eta^a \zeta_a$ 是 Lorentz 标量

girl 由 $\eta^1 = \eta_2$ 、 $\eta^2 = -\eta_1$ 、 $\zeta^1 = \zeta_2$ 和 $\zeta^2 = -\zeta_1$ 得

$$\eta^a \zeta_a = \eta^1 \zeta_1 + \eta^2 \zeta_2 = \eta_2 \zeta_1 - \eta_1 \zeta_2 = -\eta_2 \zeta^2 - \eta_1 \zeta^1 = -\eta_a \zeta^a$$

girl 即参与缩并的旋量指标一升一降会多出一个负号

girl 这种性质与 Lorentz 矢量内积 $A^\mu B_\mu = A_\mu B^\mu$ 截然不同

girl 原因在于旋量空间度规 ε^{ab} 是反对称的

Grassmann 数

 $\eta^a \zeta_a = -\eta_a \zeta^a$ 表明 $\eta^a \eta_a = -\eta_a \eta^a$ ，若 η_a 和 η^a 是普通的复数，则 $\eta^a \eta_a = 0$

 为了使 $\eta^a \eta_a \neq 0$ ，必须要求左手 Weyl 旋量 η^a 与 η_a 反对易

 即它们是 Grassmann 数，任意两个 Grassmann 数都是反对易的

 以复数作为组合系数，则若干个 Grassmann 数的线性组合也是 Grassmann 数

 因此， η_a 是 Grassmann 数意味着 $\eta^a = \varepsilon^{ab} \eta_b$ 也是 Grassmann 数

Grassmann 数

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巫师 虽然如此，Grassmann 数是反对易的 c 数，不是算符

巫师 对 Grassmann 数表达的旋量场进行量子化，才得到旋量场算符，而 Grassmann 数的反对易性质与旋量场算符的反对易关系相匹配

⚠ 旋量也可以不是 Grassmann 数，旋量系数 $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 就是普通的复数

Grassmann 数

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蜜蜂 假设 η_a 和 ζ^a 都是 Grassmann 数，则 $\eta_a \zeta^a = -\zeta^a \eta_a$ ，相应地，将省略旋量指标的内积写成 $\eta \zeta \equiv \eta^a \zeta_a = -\eta_a \zeta^a = \zeta^a \eta_a = \zeta \eta$ ，即内积 $\eta \zeta$ 和 $\zeta \eta$ 是相等的

蝴蝶 内积 $\eta^a \eta_a$ 有等价表达式 $\eta \eta = \eta^a \eta_a = \varepsilon_{ab} \eta^a \eta^b = -\eta^1 \eta^2 + \eta^2 \eta^1 = -2\eta^1 \eta^2 = 2\eta_2 \eta_1 = \eta_2 \eta_1 - \eta_1 \eta_2 = -\varepsilon^{ab} \eta_a \eta_b = -\eta_a \eta^a$

左手 Weyl 旋量的复共轭

将左手 Weyl 旋量 η_a 的复共轭记为 $\eta_{\dot{a}}^\dagger = \begin{pmatrix} \eta_i^\dagger \\ \eta_{\dot{i}}^\dagger \end{pmatrix}$

量子化之后，算符 η_a 和 $\eta_{\dot{a}}^\dagger$ 互为厄米共轭

对 $\eta'_a = [d(\Lambda)]_a^b \eta_b$ 两边取复共轭，得到 $\eta_{\dot{a}}^\dagger$ 的 Lorentz 变换

$$\eta'^{\dagger}_{\dot{a}} = [d^*(\Lambda)]_{\dot{a}}^{\dot{b}} \eta_{\dot{b}}^\dagger$$

左手 Weyl 旋量的复共轭



将左手 Weyl 旋量 η_a 的复共轭记为 $\eta_{\dot{a}}^\dagger = \begin{pmatrix} \eta_i^\dagger \\ \eta_{\dot{2}}^\dagger \end{pmatrix}$

量子化之后, 算符 η_a 和 $\eta_{\dot{a}}^\dagger$ 互为厄米共轭

对 $\eta'_a = [d(\Lambda)]_a^{\dot{b}} \eta_{\dot{b}}$ 两边取复共轭, 得到 $\eta_{\dot{a}}^\dagger$ 的 Lorentz 变换

$$\eta_{\dot{a}}'^\dagger = [d^*(\Lambda)]_{\dot{a}}^{\dot{b}} \eta_{\dot{b}}^\dagger$$

引进指标上带着点号的二维 Levi-Civita 符号

$$\varepsilon^{\dot{a}\dot{b}} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = (\mathrm{i}\sigma^2)^{\dot{a}\dot{b}}, \quad \varepsilon_{\dot{a}\dot{b}} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = (-\mathrm{i}\sigma^2)_{\dot{a}\dot{b}}$$

其分量数值与 ε^{ab} 和 ε_{ab} 分别相同

定义 $\eta^{\dagger\dot{a}} \equiv \varepsilon^{\dot{a}\dot{b}} \eta_{\dot{b}}^\dagger = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \eta_i^\dagger \\ \eta_{\dot{2}}^\dagger \end{pmatrix} = \begin{pmatrix} \eta_{\dot{2}}^\dagger \\ -\eta_i^\dagger \end{pmatrix}$, 则有 $\eta^{\dagger i} = \eta_{\dot{2}}^\dagger$ 和 $\eta^{\dagger\dot{2}} = -\eta_i^\dagger$

右手 Weyl 旋量

🐐 $\sigma^2 d^*(\Lambda) \sigma^2 = d^{-1\dagger}(\Lambda)$ 等价于 $\sigma^2 d^*(\Lambda) = d^{-1\dagger}(\Lambda) \sigma^2$

.ribbon 故 $\eta'^{\dot{a}}$ 的 Lorentz 变换为

$$\begin{aligned}\eta'^{\dagger \dot{a}} &= \varepsilon^{\dot{a} \dot{b}} \eta'^{\dagger}_{\dot{b}} = \varepsilon^{\dot{a} \dot{b}} [d^*(\Lambda)]_{\dot{b}}^{\dot{c}} \eta^{\dagger}_{\dot{c}} = i[\sigma^2 d^*(\Lambda)]^{\dot{a} \dot{c}} \eta^{\dagger}_{\dot{c}} \\ &= i[d^{-1\dagger}(\Lambda) \sigma^2]^{\dot{a} \dot{c}} \eta^{\dagger}_{\dot{c}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{b}} \varepsilon^{\dot{b} \dot{c}} \eta^{\dagger}_{\dot{c}}\end{aligned}$$

ribbon 即

$$\boxed{\eta'^{\dagger \dot{a}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{b}} \eta^{\dagger \dot{b}}}$$

右手 Weyl 旋量

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🎀 即

$$\eta'^{\dagger a} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{b}} \eta^{\dagger \dot{b}}$$

👗 可见, $\eta^{\dagger a}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ 表示空间中的列矢量, 因而是右手 Weyl 旋量

⌚ 由于表示 $\{d^*(\Lambda)\}$ 等价于 $\{d^{-1\dagger}(\Lambda)\}$, $\eta^{\dagger a}$ 也是右手 Weyl 旋量

👛 因此, 在这套符号约定中, 不带点的旋量指标对应于左手 Weyl 旋量及其表示

👠 而带点的旋量指标对应于右手 Weyl 旋量及其表示

右手 Weyl 旋量的内积

任意两个右手 Weyl 旋量 $\eta^{\dagger a}$ 和 $\zeta^{\dagger a}$ 的内积

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = \varepsilon_{\dot{a} \dot{b}} \eta^{\dagger \dot{b}} \zeta^{\dagger \dot{a}} = \varepsilon^{\dot{a} \dot{b}} \eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{b}}$$

在固有保时向 Lorentz 变换下不变，满足

$$\eta_{\dot{a}}'^{\dagger} \zeta'^{\dagger \dot{a}} = [d^*(\Lambda)]_{\dot{a}}^{\dot{b}} \eta_{\dot{b}}^{\dagger} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{b}}^{\dagger} [d^{\dagger}(\Lambda)]^{\dot{b}}_{\dot{a}} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{b}}^{\dagger} \delta^{\dot{b}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}}$$

第二步用了转置性质 $[d^*(\Lambda)]_{\dot{a}}^{\dot{b}} = [d^{\dagger}(\Lambda)]^{\dot{b}}_{\dot{a}}$ ，可见 $\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}}$ 是 Lorentz 标量

右手 Weyl 旋量的内积

任意两个右手 Weyl 旋量 $\eta^{\dagger a}$ 和 $\zeta^{\dagger a}$ 的内积

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = \varepsilon_{\dot{a} \dot{b}} \eta^{\dagger \dot{b}} \zeta^{\dagger \dot{a}} = \varepsilon^{\dot{a} \dot{b}} \eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{b}}$$

在固有保时向 Lorentz 变换下不变，满足

$$\eta_{\dot{a}}'^{\dagger} \zeta'^{\dagger \dot{a}} = [d^*(\Lambda)]_{\dot{a}}^{\dot{b}} \eta_{\dot{b}}^{\dagger} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{b}}^{\dagger} [d^*(\Lambda)]^{\dot{b}}_{\dot{a}} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{b}}^{\dagger} \delta^{\dot{b}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}}$$

第二步用了转置性质 $[d^*(\Lambda)]_{\dot{a}}^{\dot{b}} = [d^*(\Lambda)]^{\dot{b}}_{\dot{a}}$ ，可见 $\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}}$ 是 Lorentz 标量

由 $\eta^{\dagger 1} = \eta_{\dot{2}}^{\dagger}$ 、 $\eta^{\dagger 2} = -\eta_{\dot{1}}^{\dagger}$ 、 $\zeta^{\dagger 1} = \zeta_{\dot{2}}^{\dagger}$ 和 $\zeta^{\dagger 2} = -\zeta_{\dot{1}}^{\dagger}$ 得

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = \eta_{\dot{1}}^{\dagger} \zeta^{\dagger 1} + \eta_{\dot{2}}^{\dagger} \zeta^{\dagger 2} = -\eta^{\dagger 2} \zeta^{\dagger 1} + \eta^{\dagger 1} \zeta^{\dagger 2} = -\eta^{\dagger 2} \zeta_{\dot{2}}^{\dagger} - \eta^{\dagger 1} \zeta_{\dot{1}}^{\dagger} = -\eta^{\dagger \dot{a}} \zeta_{\dot{a}}^{\dagger}$$

即参与缩并的带点旋量指标一升一降会多出一个负号

右手 Weyl 旋量的内积

任意两个右手 Weyl 旋量 $\eta^{\dagger a}$ 和 $\zeta^{\dagger a}$ 的内积

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = \varepsilon_{\dot{a} \dot{b}} \eta^{\dagger \dot{b}} \zeta^{\dagger \dot{a}} = \varepsilon^{\dot{a} \dot{b}} \eta_{\dot{a}}^{\dagger} \zeta_{\dot{b}}^{\dagger}$$

在固有保时向 Lorentz 变换下不变，满足

$$\eta_{\dot{a}}'^{\dagger} \zeta'^{\dagger \dot{a}} = [d^*(\Lambda)]_{\dot{a}}^{\dot{b}} \eta_{\dot{b}}^{\dagger} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{b}}^{\dagger} [d^*(\Lambda)]^{\dot{b}}_{\dot{a}} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{b}}^{\dagger} \delta^{\dot{b}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}}$$

第二步用了转置性质 $[d^*(\Lambda)]_{\dot{a}}^{\dot{b}} = [d^*(\Lambda)]^{\dot{b}}_{\dot{a}}$ ，可见 $\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}}$ 是 Lorentz 标量

由 $\eta^{\dagger 1} = \eta_{\dot{2}}^{\dagger}$ 、 $\eta^{\dagger 2} = -\eta_{\dot{1}}^{\dagger}$ 、 $\zeta^{\dagger 1} = \zeta_{\dot{2}}^{\dagger}$ 和 $\zeta^{\dagger 2} = -\zeta_{\dot{1}}^{\dagger}$ 得

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = \eta_{\dot{1}}^{\dagger} \zeta^{\dagger 1} + \eta_{\dot{2}}^{\dagger} \zeta^{\dagger 2} = -\eta^{\dagger 2} \zeta^{\dagger 1} + \eta^{\dagger 1} \zeta^{\dagger 2} = -\eta^{\dagger 2} \zeta_{\dot{2}}^{\dagger} - \eta^{\dagger 1} \zeta_{\dot{1}}^{\dagger} = -\eta^{\dagger \dot{a}} \zeta_{\dot{a}}^{\dagger}$$

即参与缩并的带点旋量指标一升一降会多出一个负号

假设右手 Weyl 旋量 $\eta^{\dagger \dot{a}}$ 和 $\zeta_{\dot{a}}^{\dagger}$ 都是 Grassmann 数，则 $\eta^{\dagger \dot{a}} \zeta_{\dot{a}}^{\dagger} = -\zeta_{\dot{a}}^{\dagger} \eta^{\dagger \dot{a}}$

将省略带点旋量指标的内积写成 $\eta^{\dagger} \zeta^{\dagger} \equiv \eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = -\eta^{\dagger \dot{a}} \zeta_{\dot{a}}^{\dagger} = \zeta_{\dot{a}}^{\dagger} \eta^{\dagger \dot{a}} = \zeta^{\dagger} \eta^{\dagger}$

则内积 $\eta^{\dagger} \zeta^{\dagger}$ 和 $\zeta^{\dagger} \eta^{\dagger}$ 相等

Lorentz 不变量和 Weyl 旋量算符

🐂 可以看到，只要将不带点和带点的旋量指标分别缩并完毕，就得到 Lorentz 标量

🎩 另一方面，缩并一个不带点的指标和一个带点的指标并不能得到 Lorentz 不变量

👓 比如， $\eta^a \zeta_{\dot{a}}^\dagger$ 和 $\eta^{\dagger \dot{a}} \zeta_a$ 都不是 Lorentz 标量

Lorentz 不变量和 Weyl 旋量算符

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🕶 比如， $\eta^a \zeta_{\dot{a}}^\dagger$ 和 $\eta^{\dagger \dot{a}} \zeta_a$ 都不是 Lorentz 标量

👔 对于 Weyl 旋量算符 η_a 和 ζ_a ，有

$$(\eta \zeta)^\dagger = (\eta^a \zeta_a)^\dagger = (\zeta_a)^\dagger (\eta^a)^\dagger = \zeta_{\dot{a}}^\dagger \eta^{\dagger \dot{a}} = \zeta^\dagger \eta^\dagger$$

กระเป๋า 即 $\zeta^\dagger \eta^\dagger$ 是 $\eta \zeta$ 的厄米共轭算符

👞 厄米共轭操作将左手和右手 Weyl 旋量算符相互转换

Dirac 旋量场的分解

依照上述关于旋量指标的约定，将 **Dirac 旋量场** $\psi(x)$ 分解成 **左手 Weyl 旋量场** $\eta_a(x)$ 和 **右手 Weyl 旋量场** $\zeta^{\dagger a}(x)$ ，形式为

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \zeta^{\dagger a}(x) \end{pmatrix}$$

在量子化之前， $\eta_a(x)$ 和 $\zeta^{\dagger a}(x)$ 是 Grassmann 数，因而 $\psi(x)$ 也是 Grassmann 数

这是前面 **转置两个旋量场** 必须添加一个额外 **负号** 的原因

根据 $D(\Lambda) = \begin{pmatrix} d(\Lambda) & \\ & d^{-1\dagger}(\Lambda) \end{pmatrix}$ ， $\psi(x)$ 的 **固有保时向 Lorentz 变换** 表达成

$$\begin{pmatrix} \eta'_a(x') \\ \zeta'^{\dagger a}(x') \end{pmatrix} = \psi'(x') = D(\Lambda)\psi(x) = \begin{pmatrix} [d(\Lambda)]_a^b \eta_b(x) \\ [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{b}} \zeta^{\dagger b}(x) \end{pmatrix}$$

$\psi(x)$ 的 **Dirac 共轭** 是 $\bar{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \eta^\dagger_{\dot{b}} & \zeta^b \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}_{\dot{a}} \\ \delta_b^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & \eta^\dagger_{\dot{a}} \end{pmatrix}$

Dirac 矩阵的指标形式

保持旋量指标平衡，则 Dirac 方程 $(i\gamma^\mu \partial_\mu - m)\psi = 0$ 化为

$$\begin{pmatrix} -m\delta_a{}^b & i(\sigma^\mu)_{a\dot{b}} \partial_\mu \\ i(\bar{\sigma}^\mu)^{\dot{a}b} \partial_\mu & -m\delta^{\dot{a}}{}_b \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dagger\dot{b}} \end{pmatrix} = 0$$

因而 Dirac 矩阵的指标形式是

$$\gamma^\mu = \begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}$$

注意， γ^μ 中的 γ^0 与 Dirac 共轭 $\bar{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \eta_{\dot{a}}^\dagger & \zeta^a \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}{}_{\dot{a}} \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix}$

中的 γ^0 具有不同的指标结构

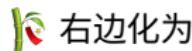
两者本质不同，有些书将后者记为 β 以示区别

σ^μ 和 $\bar{\sigma}^\mu$ 的 Lorentz 变换规则



于是, γ^μ 的 Lorentz 变换规则 $D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$ 左边变成

$$\begin{aligned} & D^{-1}(\Lambda)\gamma^\mu D(\Lambda) \\ &= \begin{pmatrix} [d^{-1}(\Lambda)]_a{}^c & \\ & [d^\dagger(\Lambda)]^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} (\sigma^\mu)_{c\dot{d}} \\ (\bar{\sigma}^\mu)^{\dot{c}d} \end{pmatrix} \begin{pmatrix} [d(\Lambda)]_d{}^b & \\ & [d^{-1\dagger}(\Lambda)]^{\dot{d}}{}_{\dot{b}} \end{pmatrix} \\ &= \begin{pmatrix} & [d^{-1}(\Lambda)]_a{}^c (\sigma^\mu)_{c\dot{d}} [d^{-1\dagger}(\Lambda)]^{\dot{d}}{}_{\dot{b}} \\ [d^\dagger(\Lambda)]^{\dot{a}}{}_{\dot{c}} (\bar{\sigma}^\mu)^{\dot{c}d} [d(\Lambda)]_d{}^b & \end{pmatrix} \end{aligned}$$



右边化为

$$\Lambda^\mu{}_\nu \gamma^\nu = \begin{pmatrix} & \Lambda^\mu{}_\nu (\sigma^\nu)_{a\dot{b}} \\ \Lambda^\mu{}_\nu (\bar{\sigma}^\nu)^{\dot{a}b} & \end{pmatrix}$$



两相比较, 推出

$$[d^{-1}(\Lambda)]_a{}^c (\sigma^\mu)_{c\dot{d}} [d^{-1\dagger}(\Lambda)]^{\dot{d}}{}_{\dot{b}} = \Lambda^\mu{}_\nu (\sigma^\nu)_{a\dot{b}}, \quad [d^\dagger(\Lambda)]^{\dot{a}}{}_{\dot{c}} (\bar{\sigma}^\mu)^{\dot{c}d} [d(\Lambda)]_d{}^b = \Lambda^\mu{}_\nu (\bar{\sigma}^\nu)^{\dot{a}b}$$



这分别是 σ^μ 和 $\bar{\sigma}^\mu$ 的 Lorentz 变换规则

Lorentz 矢量 $\eta\sigma^\mu\zeta^\dagger$ 和 $\eta^\dagger\bar{\sigma}^\mu\zeta$

对任意 Weyl 旋量 η 和 ζ ，定义

$$\eta\sigma^\mu\zeta^\dagger \equiv \eta^a(\sigma^\mu)_{ab}\zeta^{\dagger b}, \quad \eta^\dagger\bar{\sigma}^\mu\zeta \equiv \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\zeta_b$$

它们都是 Lorentz 矢量，相应的固有保时向 Lorentz 变换为

$$\begin{aligned}\eta'\sigma^\mu\zeta'^\dagger &= [d^{-1T}(\Lambda)]^a_c\eta^c(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^{\dot{b}}_d\zeta^{\dagger d} = \eta^c[d^{-1}(\Lambda)]_c{}^a(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^{\dot{b}}_d\zeta^{\dagger d} \\ &= \eta^c\Lambda^\mu{}_\nu(\sigma^\nu)_{c\dot{d}}\zeta^{\dagger d} = \Lambda^\mu{}_\nu\eta\sigma^\nu\zeta^\dagger\end{aligned}$$

$$\begin{aligned}\eta'^\dagger\bar{\sigma}^\mu\zeta' &= [d^*(\Lambda)]_{\dot{a}}{}^{\dot{c}}\eta^\dagger_{\dot{c}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b{}^d\zeta_d = \eta^\dagger_{\dot{c}}[d^\dagger(\Lambda)]^{\dot{c}}{}_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b{}^d\zeta_d \\ &= \eta^\dagger_{\dot{c}}\Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot{c}d}\zeta_d = \Lambda^\mu{}_\nu\eta^\dagger\bar{\sigma}^\mu\zeta\end{aligned}$$

Lorentz 矢量 $\eta\sigma^\mu\zeta^\dagger$ 和 $\eta^\dagger\bar{\sigma}^\mu\zeta$

duck 对任意 Weyl 旋量 η 和 ζ , 定义

$$\eta\sigma^\mu\zeta^\dagger \equiv \eta^a(\sigma^\mu)_{ab}\zeta^{\dagger b}, \quad \eta^\dagger\bar{\sigma}^\mu\zeta \equiv \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\zeta_b$$

它们都是 Lorentz 矢量, 相应的固有保时向 Lorentz 变换为

$$\begin{aligned}\eta'\sigma^\mu\zeta'^\dagger &= [d^{-1T}(\Lambda)]^a_c\eta^c(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^{\dot{b}}_d\zeta^{\dagger\dot{d}} = \eta^c[d^{-1}(\Lambda)]_c^a(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^{\dot{b}}_d\zeta^{\dagger\dot{d}} \\ &= \eta^c\Lambda^\mu_\nu(\sigma^\nu)_{cd}\zeta^{\dagger\dot{d}} = \Lambda^\mu_\nu\eta\sigma^\nu\zeta^\dagger\end{aligned}$$

$$\begin{aligned}\eta'^\dagger\bar{\sigma}^\mu\zeta' &= [d^*(\Lambda)]_{\dot{a}}^{\dot{c}}\eta_{\dot{c}}^\dagger(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b^d\zeta_d = \eta_{\dot{c}}^\dagger[d^\dagger(\Lambda)]_{\dot{a}}^{\dot{c}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b^d\zeta_d \\ &= \eta_{\dot{c}}^\dagger\Lambda^\mu_\nu(\bar{\sigma}^\nu)^{\dot{c}d}\zeta_d = \Lambda^\mu_\nu\eta^\dagger\bar{\sigma}^\mu\zeta\end{aligned}$$

green ball 由 $\sigma^2\sigma^\mu\sigma^2 = (\bar{\sigma}^\mu)^T$ 得 $(i\sigma^2)\sigma^\mu(i\sigma^2) = -(\bar{\sigma}^\mu)^T$, 相应的指标形式为

$$\varepsilon^{ac}(\sigma^\mu)_{cd}\varepsilon^{\dot{d}\dot{b}} = -[(\bar{\sigma}^\mu)^T]^{a\dot{b}} = -(\bar{\sigma}^\mu)^{\dot{b}a}$$

blue kite 对于 Weyl 旋量场 $\eta_a(x)$ 和 $\zeta^{\dagger\dot{a}}(x)$, 有



Grassmann 数性质

$$\begin{aligned}[\eta^a(\sigma^\mu)_{ab}\zeta^{\dagger\dot{b}}]^\dagger &= \zeta^b(\sigma^\mu)_{ba}\eta^{\dagger\dot{a}} = -\eta^{\dagger\dot{a}}(\sigma^\mu)_{b\dot{a}}\zeta^b = -\varepsilon^{\dot{a}\dot{c}}\eta_{\dot{c}}^\dagger(\sigma^\mu)_{b\dot{a}}\varepsilon^{bd}\zeta_d \\ &= \eta_{\dot{c}}^\dagger\varepsilon^{db}(\sigma^\mu)_{b\dot{a}}\varepsilon^{\dot{a}\dot{c}}\zeta_d = -\eta_{\dot{c}}^\dagger(\bar{\sigma}^\mu)^{\dot{c}d}\zeta_d = -[\zeta_{\dot{d}}^\dagger(\bar{\sigma}^\mu)^{\dot{d}c}\eta_c]^\dagger\end{aligned}$$

yellow star 即

$$(\eta\sigma^\mu\zeta^\dagger)^\dagger = \zeta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\zeta = -(\zeta^\dagger\bar{\sigma}^\mu\eta)^\dagger$$

Lorentz 张量 $\eta \sigma^\mu \bar{\sigma}^\nu \zeta$ 和 $\eta^\dagger \bar{\sigma}^\mu \sigma^\nu \zeta^\dagger$

类似地, $\eta \sigma^\mu \bar{\sigma}^\nu \zeta \equiv \eta^a (\sigma^\mu)_{ab} (\bar{\sigma}^\nu)^{bc} \zeta_c$ 和 $\eta^\dagger \bar{\sigma}^\mu \sigma^\nu \zeta^\dagger \equiv \eta^\dagger_{\dot{a}} (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} (\sigma^\nu)_{\dot{b}\dot{c}} \zeta^{\dot{c}\dagger}$ 都是二阶 Lorentz 张量

由 $\sigma^2 \bar{\sigma}^\mu \sigma^2 = (\sigma^\mu)^T$ 得 $(-i\sigma^2) \bar{\sigma}^\mu (-i\sigma^2) = -(\sigma^\mu)^T$, 相应的指标形式为

$$\varepsilon_{\dot{a}\dot{c}} (\bar{\sigma}^\mu)^{\dot{c}\dot{d}} \varepsilon_{\dot{d}\dot{b}} = -[(\sigma^\mu)^T]_{\dot{a}\dot{b}} = -(\sigma^\mu)_{\dot{b}\dot{a}}$$

再利用 $\varepsilon_{ab} \varepsilon^{bc} = \delta_a^c$ 和 $\varepsilon^{ac} (\sigma^\mu)_{cd} \varepsilon^{db} = -[(\bar{\sigma}^\mu)^T]^{ab} = -(\bar{\sigma}^\mu)^{\dot{b}\dot{a}}$ 推出

$$\begin{aligned} \varepsilon_{\dot{a}\dot{c}} (\bar{\sigma}^\nu)^{\dot{c}\dot{d}} (\sigma^\mu)_{d\dot{e}} \varepsilon^{\dot{e}\dot{b}} &= \varepsilon_{\dot{a}\dot{c}} (\bar{\sigma}^\nu)^{\dot{c}\dot{d}} \delta_d^f (\sigma^\mu)_{f\dot{e}} \varepsilon^{\dot{e}\dot{b}} = \varepsilon_{\dot{a}\dot{c}} (\bar{\sigma}^\nu)^{\dot{c}\dot{d}} \varepsilon_{dg} \varepsilon^{gf} (\sigma^\mu)_{f\dot{e}} \varepsilon^{\dot{e}\dot{b}} \\ &= (-\sigma^\nu)_{g\dot{a}} (-\bar{\sigma}^\mu)^{\dot{b}\dot{g}} = (\bar{\sigma}^\mu)^{\dot{b}\dot{g}} (\sigma^\nu)_{g\dot{a}} \end{aligned}$$

故 $[\eta^a (\sigma^\mu)_{ab} (\bar{\sigma}^\nu)^{bc} \zeta_c]^\dagger = \zeta_{\dot{c}}^\dagger (\bar{\sigma}^\nu)^{\dot{c}\dot{b}} (\sigma^\mu)_{b\dot{a}} \eta^{\dot{a}\dagger} = -\eta^{\dot{a}\dagger} (\bar{\sigma}^\nu)^{\dot{c}\dot{b}} (\sigma^\mu)_{b\dot{a}} \zeta_{\dot{c}}^\dagger$
 $= -\varepsilon^{\dot{a}\dot{d}} \eta_{\dot{d}}^\dagger (\bar{\sigma}^\nu)^{\dot{c}\dot{b}} (\sigma^\mu)_{b\dot{a}} \varepsilon_{\dot{c}\dot{e}} \zeta^{\dot{e}\dagger} = \eta_{\dot{d}}^\dagger \varepsilon_{\dot{e}\dot{c}} (\bar{\sigma}^\nu)^{\dot{c}\dot{b}} (\sigma^\mu)_{b\dot{a}} \varepsilon^{\dot{a}\dot{d}} \zeta^{\dot{e}\dagger}$
 $= \eta_{\dot{d}}^\dagger (\bar{\sigma}^\mu)^{\dot{d}\dot{g}} (\sigma^\nu)_{g\dot{e}} \zeta^{\dot{e}\dagger} = [\zeta^e (\sigma^\nu)_{e\dot{g}} (\bar{\sigma}^\mu)^{\dot{g}\dot{d}} \eta_d]^\dagger$

即

$$(\eta \sigma^\mu \bar{\sigma}^\nu \zeta)^\dagger = \zeta^\dagger \bar{\sigma}^\nu \sigma^\mu \eta^\dagger = \eta^\dagger \bar{\sigma}^\mu \sigma^\nu \zeta^\dagger = (\zeta \sigma^\nu \bar{\sigma}^\mu \eta)^\dagger$$

旋量双线性型的分解



将 Dirac 旋量双线性型分解成由 Weyl 旋量表达的 Lorentz 张量，有

$$\bar{\psi}\psi = \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} \eta_a \\ \zeta^{\dot{a}} \end{pmatrix} = \zeta^a \eta_a + \eta_{\dot{a}}^\dagger \zeta^{\dot{a}} = \zeta \eta + \eta^\dagger \zeta^\dagger$$

$$\bar{\psi}\gamma^5\psi = \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} -\delta_a{}^b & \\ & \delta^{\dot{a}}{}_{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dot{b}} \end{pmatrix} = -\zeta^a \eta_a + \eta_{\dot{a}}^\dagger \zeta^{\dot{a}} = -\zeta \eta + \eta^\dagger \zeta^\dagger$$

$$\begin{aligned} \bar{\psi}\gamma^\mu\psi &= \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} & \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dot{b}} \end{pmatrix} = \zeta^a (\sigma^\mu)_{ab} \zeta^{\dot{b}} + \eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} \eta_b \\ &= \zeta \sigma^\mu \zeta^\dagger + \eta^\dagger \bar{\sigma}^\mu \eta \end{aligned}$$

$$\begin{aligned} \bar{\psi}\gamma^\mu\gamma^5\psi &= \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} & \end{pmatrix} \begin{pmatrix} -\delta_b{}^c & \\ & \delta^{\dot{b}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \eta_c \\ \zeta^{\dot{c}} \end{pmatrix} \\ &= \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} & \end{pmatrix} \begin{pmatrix} -\eta_b \\ \zeta^{\dot{b}} \end{pmatrix} = \zeta^a (\sigma^\mu)_{ab} \zeta^{\dot{b}} - \eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} \eta_b \\ &= \zeta \sigma^\mu \zeta^\dagger - \eta^\dagger \bar{\sigma}^\mu \eta \end{aligned}$$

旋量双线性型的分解

还有

$$\begin{aligned}
 \bar{\psi} \sigma^{\mu\nu} \psi &= \frac{i}{2} \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_a{}^b & \\ & (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)^{\dot{a}}{}_{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dagger \dot{b}} \end{pmatrix} \\
 &= \frac{i}{2} \zeta^a (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_a{}^b \eta_b + \frac{i}{2} \eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)^{\dot{a}}{}_{\dot{b}} \zeta^{\dagger \dot{b}} \\
 &= \frac{i}{2} \zeta (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \eta + \frac{i}{2} \eta^\dagger (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \zeta^\dagger
 \end{aligned}$$

进一步推出

$$\bar{\psi}_R \psi_L = \frac{1}{2} \bar{\psi} (1 - \gamma^5) \psi = \zeta \eta$$

$$\bar{\psi}_L \psi_R = \frac{1}{2} \bar{\psi} (1 + \gamma^5) \psi = \eta^\dagger \zeta^\dagger$$

$$\bar{\psi}_L \gamma^\mu \psi_L = \frac{1}{2} \bar{\psi} (\gamma^\mu - \gamma^\mu \gamma^5) \psi = \eta^\dagger \bar{\sigma}^\mu \eta$$

$$\bar{\psi}_R \gamma^\mu \psi_R = \frac{1}{2} \bar{\psi} (\gamma^\mu + \gamma^\mu \gamma^5) \psi = \zeta \sigma^\mu \zeta^\dagger$$

拉氏量的分解

 另一方面，自由 Dirac 旋量场的拉氏量分解为

$$\begin{aligned}
 \mathcal{L} &= \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi = \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} -m\delta^a{}_b & i(\sigma^\mu)_{a\dot{b}} \partial_\mu \\ i(\bar{\sigma}^\mu)^{\dot{a}b} \partial_\mu & -m\delta^{\dot{a}}_{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dagger\dot{b}} \end{pmatrix} \\
 &= -m\zeta^a \eta_a + i\zeta^a (\sigma^\mu)_{a\dot{b}} \partial_\mu \zeta^{\dagger\dot{b}} + i\eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu)^{\dot{a}b} \partial_\mu \eta_b - m\eta_{\dot{a}}^\dagger \zeta^{\dagger\dot{a}} \\
 &= i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta + i\zeta \sigma^\mu \partial_\mu \zeta^\dagger - m(\zeta \eta + \eta^\dagger \zeta^\dagger)
 \end{aligned}$$

 这里的质量项涉及两个不同的 Weyl 旋量场 $\eta_a(x)$ 和 $\zeta_a(x)$ ，称为 Dirac 质量项

 如果质量 $m = 0$ ，则

$$\mathcal{L}_L = i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta$$

和

$$\mathcal{L}_R = i\zeta \sigma^\mu \partial_\mu \zeta^\dagger$$

分别描述自由的左手 Weyl 旋量场 $\eta_a(x)$ 和右手 Weyl 旋量场 $\zeta^{\dagger\dot{a}}(x)$

 相应的运动方程是两个 Weyl 方程：

$$i\bar{\sigma}^\mu \partial_\mu \eta = 0, \quad i\sigma^\mu \partial_\mu \zeta^\dagger = 0$$

Weyl 旋量场的 C 变换

接下来讨论 Weyl 旋量场的 C 变换

首先, 电荷共轭矩阵的指标形式为 $C = \begin{pmatrix} -i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}$

将 $\psi(x)$ 的电荷共轭场 $\psi^C(x)$ 分解成 Weyl 旋量场, 得到

$$\psi^C(x) = C\bar{\psi}^T(x) = C \begin{pmatrix} \zeta^b(x) & \eta_b^\dagger(x) \end{pmatrix}^T = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \zeta^b(x) \\ \eta_b^\dagger(x) \end{pmatrix} = \begin{pmatrix} \zeta_a(x) \\ \eta_b^\dagger(x) \end{pmatrix}$$

从而, Dirac 旋量场 $\psi(x)$ 的 C 变换化为

$$\begin{pmatrix} C^{-1} \eta_a(x) C \\ C^{-1} \zeta^{\dot{a}}(x) C \end{pmatrix} = C^{-1} \psi(x) C = \zeta_C^* \psi^C(x) = \begin{pmatrix} \zeta_C^* \zeta_a(x) \\ \zeta_C^* \eta^{\dot{a}}(x) \end{pmatrix}$$

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从而, Dirac 旋量场 $\psi(x)$ 的 C 变换化为

$$\begin{pmatrix} C^{-1}\eta_a(x)C \\ C^{-1}\zeta^{\dot{a}}(x)C \end{pmatrix} = C^{-1}\psi(x)C = \zeta_C^* \psi^C(x) = \begin{pmatrix} \zeta_C^* \zeta_a(x) \\ \zeta_C^* \eta^{\dot{a}}(x) \end{pmatrix}$$

即左手 Weyl 旋量场的 C 变换是

$$C^{-1}\eta_a(x)C = \zeta_C^* \zeta_a(x), \quad C^{-1}\zeta^{\dot{a}}(x)C = \zeta_C^* \eta^{\dot{a}}(x)$$

可见, 电荷共轭变换将 η 和 ζ 相互转换。取厄米共轭, 得 $C^{-1}\eta_b^\dagger(x)C = \zeta_C \zeta_b^\dagger(x)$ 及 $C^{-1}\zeta^b(x)C = \zeta_C \eta^b(x)$, 分别与 $\varepsilon^{\dot{a}\dot{b}}$ 和 ε_{ab} 缩并, 推出

$$C^{-1}\eta^{\dot{a}}(x)C = \zeta_C \zeta^{\dot{a}}(x), \quad C^{-1}\zeta_a(x)C = \zeta_C \eta_a(x)$$

Majorana 旋量场的分解

🦙 下面讨论 Majorana 旋量场, Majorana 条件 意味着 $\begin{pmatrix} \eta_a \\ \zeta^{\dagger a} \end{pmatrix} = \psi = \mathcal{C}\bar{\psi}^T = \begin{pmatrix} \zeta_a \\ \eta^{\dagger a} \end{pmatrix}$

🎰 即 $\eta = \zeta$ ，这表明 Majorana 旋量场中的左手和右手 Weyl 旋量场是相关的

🎮 因此，可以将 Majorana 旋量场 $\psi(x)$ 分解成

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \eta^{\dagger a}(x) \end{pmatrix}$$

Majorana 旋量场的分解

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因此, 可以将 Majorana 旋量场 $\psi(x)$ 分解成

而自由 Majorana 旋量场的拉氏量分解为

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \eta^{\dagger a}(x) \end{pmatrix}$$

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi = \frac{1}{2} \begin{pmatrix} \eta^a & \eta^{\dagger a} \end{pmatrix} \begin{pmatrix} -m\delta_a^b & i(\sigma^\mu)_{ab} \partial_\mu \\ i(\bar{\sigma}^\mu)^{ab} \partial_\mu & -m\delta^a_b \end{pmatrix} \begin{pmatrix} \eta_b \\ \eta^{\dagger b} \end{pmatrix} \\ &= \frac{1}{2} [i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta + i\eta \sigma^\mu \partial_\mu \eta^\dagger - m(\eta \eta + \eta^\dagger \eta^\dagger)] \end{aligned}$$

⑧ 利用 $\zeta \sigma^\mu \eta^\dagger = -\eta^\dagger \bar{\sigma}^\mu \zeta$ 将方括号中第二项化为

$$i\eta \sigma^\mu \partial_\mu \eta^\dagger = i\partial_\mu (\eta \sigma^\mu \eta^\dagger) - i(\partial_\mu \eta) \sigma^\mu \eta^\dagger = i\partial_\mu (\eta \sigma^\mu \eta^\dagger) + i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta$$

扔掉全散度项 $i\partial_\mu (\eta \sigma^\mu \eta^\dagger)$, 拉氏量变成 $\mathcal{L} = i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta - \frac{1}{2} m(\eta \eta + \eta^\dagger \eta^\dagger)$

这里的质量项只涉及一个 Weyl 旋量场 $\eta_a(x)$, 称为 Majorana 质量项

Majorana 旋量场的 $\bar{\psi}\gamma^\mu\psi$ 和 $\bar{\psi}\sigma^{\mu\nu}\psi$

老虎 $\zeta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\zeta$ 、 $\eta\sigma^\mu\bar{\sigma}^\nu\zeta = \zeta\sigma^\nu\bar{\sigma}^\mu\eta$ 和 $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger = \zeta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger$ 意味着

$$\eta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\eta, \quad \eta\sigma^\mu\bar{\sigma}^\nu\eta = \eta\sigma^\nu\bar{\sigma}^\mu\eta, \quad \eta^\dagger\bar{\sigma}^\mu\sigma^\nu\eta^\dagger = \eta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger$$

对于 Majorana 旋量场， $\eta = \zeta$ ， $\bar{\psi}\gamma^\mu\psi = \zeta\sigma^\mu\zeta^\dagger + \eta^\dagger\bar{\sigma}^\mu\eta$ 化为

$$\bar{\psi}\gamma^\mu\psi = \eta\sigma^\mu\eta^\dagger + \eta^\dagger\bar{\sigma}^\mu\eta = -\eta^\dagger\bar{\sigma}^\mu\eta + \eta^\dagger\bar{\sigma}^\mu\eta = 0$$

○ $\bar{\psi}\sigma^{\mu\nu}\psi = \frac{i}{2}\zeta(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)\eta + \frac{i}{2}\eta^\dagger(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)\zeta^\dagger$ 化为

$$\bar{\psi}\sigma^{\mu\nu}\psi = \frac{i}{2}(\eta\sigma^\mu\bar{\sigma}^\nu\eta - \eta\sigma^\nu\bar{\sigma}^\mu\eta) + \frac{i}{2}(\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\eta^\dagger - \eta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger) = 0$$

○ 这样就验证了上一节的结论

9.6.3 小节 手征旋量场

本小节从手征旋量场的角度分析 Dirac 质量项和 Majorana 质量项的构造过程

用 Weyl 旋量场 $\eta_a(x)$ 和 $\zeta^{\dagger a}(x)$ 将四分量左手旋量场 $\psi_L(x)$ 和右手旋量场 $\psi_R(x)$ 表达为

$$\psi_L = \begin{pmatrix} \eta_a \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ \zeta^{\dagger a} \end{pmatrix}$$

$$\bar{\psi}_L = (\psi_L)^\dagger \gamma^0 = \begin{pmatrix} \eta_b^\dagger & 0 \end{pmatrix} \begin{pmatrix} & \delta^b{}_a \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix} 0 & \eta_a^\dagger \end{pmatrix}$$

$$\bar{\psi}_R = (\psi_R)^\dagger \gamma^0 = \begin{pmatrix} 0 & \zeta^b \end{pmatrix} \begin{pmatrix} & \delta^b{}_a \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & 0 \end{pmatrix}$$

从而，拉氏量

$$\mathcal{L}_L = i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta = i \begin{pmatrix} 0 & \eta_a^\dagger \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)_{ab} & \end{pmatrix} \partial_\mu \begin{pmatrix} \eta_b \\ 0 \end{pmatrix} = i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L$$

$$\mathcal{L}_R = i\zeta^\mu \bar{\sigma}^\mu \zeta^\dagger = i \begin{pmatrix} \zeta^a & 0 \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)_{ab} & \end{pmatrix} \partial_\mu \begin{pmatrix} 0 \\ \zeta^{\dagger b} \end{pmatrix} = i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R$$

分别描述自由的无质量左手旋量场 ψ_L 和右手旋量场 ψ_R

构造 Dirac 质量项

如果要构造 Dirac 质量项, 需采用前面给出的 $\bar{\psi}_R \psi_L = \zeta \eta$ 和 $\bar{\psi}_L \psi_R = \eta^\dagger \zeta^\dagger$, 得到自由 Dirac 旋量场 $\psi(x) = \psi_L(x) + \psi_R(x)$ 的拉氏量

$$\begin{aligned}\mathcal{L}_D &= i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta + i\zeta \sigma^\mu \partial_\mu \zeta^\dagger - m(\zeta \eta + \eta^\dagger \zeta^\dagger) \\ &= i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R - m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)\end{aligned}$$

如果要构造 Majorana 质量项, 需用到 ψ_L 和 ψ_R 的电荷共轭场:

$$(\psi_L)^C = \mathcal{C}(\bar{\psi}_L)^T = \mathcal{C} \begin{pmatrix} 0 & \eta_b^\dagger \end{pmatrix}^T = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} 0 \\ \eta_b^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ \eta^{\dagger a} \end{pmatrix}$$

$$(\psi_R)^C = \mathcal{C}(\bar{\psi}_R)^T = \mathcal{C} \begin{pmatrix} \zeta^b & 0 \end{pmatrix}^T = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \zeta^b \\ 0 \end{pmatrix} = \begin{pmatrix} \zeta^a \\ 0 \end{pmatrix}$$

$$\overline{(\psi_L)^C} = [(\psi_L)^C]^\dagger \gamma^0 = \begin{pmatrix} 0 & \eta^b \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}_{\dot{a}} \end{pmatrix} = \begin{pmatrix} \eta^a & 0 \end{pmatrix}$$

$$\overline{(\psi_R)^C} = [(\psi_R)^C]^\dagger \gamma^0 = \begin{pmatrix} \zeta_b^\dagger & 0 \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}_{\dot{a}} \end{pmatrix} = \begin{pmatrix} 0 & \zeta_{\dot{a}}^\dagger \end{pmatrix}$$

构造 Majorana 质量项

hog 由此推出

$$\begin{aligned}
 \overline{(\psi_L)^C}(\psi_L)^C &= \begin{pmatrix} \eta^a & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta^{\dagger a} \end{pmatrix} = 0, & \overline{(\psi_R)^C}(\psi_R)^C &= \begin{pmatrix} 0 & \zeta_a^\dagger \end{pmatrix} \begin{pmatrix} \zeta_a \\ 0 \end{pmatrix} = 0 \\
 \overline{(\psi_R)^C}\psi_L &= \begin{pmatrix} 0 & \zeta_a^\dagger \end{pmatrix} \begin{pmatrix} \eta_a \\ 0 \end{pmatrix} = 0, & \overline{(\psi_L)^C}\psi_R &= \begin{pmatrix} \eta^a & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \zeta_a^\dagger \end{pmatrix} = 0 \\
 \bar{\psi}_R(\psi_L)^C &= \begin{pmatrix} \zeta^a & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta^{\dagger a} \end{pmatrix} = 0, & \bar{\psi}_L(\psi_R)^C &= \begin{pmatrix} 0 & \eta_a^\dagger \end{pmatrix} \begin{pmatrix} \zeta_a \\ 0 \end{pmatrix} = 0
 \end{aligned}$$

pineapple 以上 6 个算符不能用于构造质量项。可用的 Majorana 质量项算符是

$$\begin{aligned}
 \overline{(\psi_L)^C}\psi_L &= \begin{pmatrix} \eta^a & 0 \end{pmatrix} \begin{pmatrix} \eta_a \\ 0 \end{pmatrix} = \eta\eta, & \bar{\psi}_L(\psi_L)^C &= \begin{pmatrix} 0 & \eta_a^\dagger \end{pmatrix} \begin{pmatrix} 0 \\ \eta^{\dagger a} \end{pmatrix} = \eta^\dagger\eta^\dagger \\
 \overline{(\psi_R)^C}\psi_R &= \begin{pmatrix} 0 & \zeta_a^\dagger \end{pmatrix} \begin{pmatrix} 0 \\ \zeta^{\dagger a} \end{pmatrix} = \zeta^\dagger\zeta^\dagger, & \bar{\psi}_R(\psi_R)^C &= \begin{pmatrix} \zeta^a & 0 \end{pmatrix} \begin{pmatrix} \zeta_a \\ 0 \end{pmatrix} = \zeta\zeta
 \end{aligned}$$

apple $\overline{(\psi_L)^C}\psi_L$ 与 $\bar{\psi}_L(\psi_L)^C$ 互为厄米共轭, 而 $\overline{(\psi_R)^C}\psi_R$ 与 $\bar{\psi}_R(\psi_R)^C$ 互为厄米共轭

组合成 Majorana 旋量场

将自由 Majorana 旋量场的拉氏量改写为

$$\mathcal{L}_{M1} = i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta - \frac{m}{2} (\eta\eta + \eta^\dagger \eta^\dagger) = i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L - \frac{m}{2} \left[(\overline{\psi_L})^C \psi_L + \bar{\psi}_L (\psi_L)^C \right]$$

它描述 Majorana 旋量场 $\psi_1 \equiv \psi_L + (\psi_L)^C = \begin{pmatrix} \eta_a \\ \eta^\dagger \dot{a} \end{pmatrix}$ 的自由运动

另一方面，描述 Majorana 旋量场 $\psi_2 \equiv \psi_R + (\psi_R)^C = \begin{pmatrix} \zeta_a \\ \zeta^\dagger \dot{a} \end{pmatrix}$ 自由运动的拉氏量是

$$\mathcal{L}_{M2} = i\zeta \sigma^\mu \partial_\mu \zeta^\dagger - \frac{m}{2} (\zeta\zeta + \zeta^\dagger \zeta^\dagger) = i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R - \frac{m}{2} \left[\bar{\psi}_R (\psi_R)^C + \overline{(\psi_R)^C} \psi_R \right]$$