

量子场论

第 9 章 分立对称性和 Majorana 旋量场

9.6 节和 9.7 节

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9.6 节 Weyl、Dirac 和 Majorana 旋量

9.6.1 小节 左手和右手 Weyl 旋量

 **Dirac 旋量场**和 **Majorana 旋量场**都可以分解为左手和右手的 Weyl 旋量场

 为了更深刻地认识旋量场, 本节进一步研究 **Weyl 旋量**

 用 $\sigma^\mu = (1, \sigma)$ 和 $\bar{\sigma}^\mu = (1, -\sigma)$ 定义 2×2 矩阵
$$s^{\mu\nu} \equiv \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$$

 由 $(\sigma^\mu)^\dagger = \sigma^\mu$ 和 $(\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu$ 推出

$$(s^{\mu\nu})^\dagger = -\frac{i}{4}[(\bar{\sigma}^\nu)^\dagger (\sigma^\mu)^\dagger - (\bar{\sigma}^\mu)^\dagger (\sigma^\nu)^\dagger] = -\frac{i}{4}(\bar{\sigma}^\nu \sigma^\mu - \bar{\sigma}^\mu \sigma^\nu) = \frac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)$$

 从而将 **Weyl 表象**中的旋量表示生成元化为

$$\mathcal{S}^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{4} \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix} = \begin{pmatrix} s^{\mu\nu} & \\ & (s^{\mu\nu})^\dagger \end{pmatrix}$$

 也就是说, 4×4 矩阵 $\mathcal{S}^{\mu\nu}$ 是 2×2 矩阵 $s^{\mu\nu}$ 和 $(s^{\mu\nu})^\dagger$ 的直和

 因而 $s^{\mu\nu}$ 和 $(s^{\mu\nu})^\dagger$ 是两个 Lorentz 群 2 维表示的生成元

左手和右手 Weyl 旋量所处 2 维表示

对于 Lorentz 变换 Λ 的一组变换参数 $\omega_{\mu\nu}$ ，用 $s^{\mu\nu}$ 生成固有保时向有限变换

$$d(\Lambda) \equiv \exp \left(-\frac{i}{2} \omega_{\mu\nu} s^{\mu\nu} \right)$$

它属于左手 Weyl 旋量所处的 2 维表示

相应的逆变换矩阵为 $d^{-1}(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}s^{\mu\nu}\right)$ ，取厄米共轭，得

$$d^{-1\dagger}(\Lambda) = \exp \left[-\frac{i}{2} \omega_{\mu\nu} (s^{\mu\nu})^\dagger \right]$$

这是用 $(s^{\mu\nu})^\dagger$ 生成的固有保时向有限变换，属于右手 Weyl 旋量所处的 2 维表示

左手和右手 Weyl 旋量所处 2 维表示

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这是用 $(s^{\mu\nu})^\dagger$ 生成的固有保时向有限变换，属于右手 Weyl 旋量所处的 2 维表示

于是，旋量表示的 4×4 Lorentz 变换矩阵分解为

$$D(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{S}^{\mu\nu}\right) = \begin{pmatrix} e^{-i\omega_{\mu\nu}s^{\mu\nu}/2} & \\ & e^{-i\omega_{\mu\nu}(s^{\mu\nu})^\dagger/2} \end{pmatrix} = \begin{pmatrix} d(\Lambda) & \\ & d^{-1\dagger}(\Lambda) \end{pmatrix}$$

因此, 4 维旋量表示 $\{D(\Lambda)\}$ 是 2 维表示 $\{d(\Lambda)\}$ 和 $\{d^{-1\dagger}(\Lambda)\}$ 的直和

等价表示

利用 $\sigma^2 \sigma^\mu \sigma^2 = (\bar{\sigma}^\mu)^T$ 和 $\sigma^2 \bar{\sigma}^\mu \sigma^2 = (\sigma^\mu)^T$ 推出

$$\begin{aligned}\sigma^2 s^{\mu\nu} \sigma^2 &= \frac{i}{4} (\sigma^2 \sigma^\mu \sigma^2 \bar{\sigma}^\nu \sigma^2 - \sigma^2 \sigma^\nu \sigma^2 \bar{\sigma}^\mu \sigma^2) \\ &= \frac{i}{4} [(\bar{\sigma}^\mu)^T (\sigma^\nu)^T - (\bar{\sigma}^\nu)^T (\sigma^\mu)^T] = -(s^{\mu\nu})^T\end{aligned}$$

$$\begin{aligned}\sigma^2 d(\Lambda) \sigma^2 &= \exp \left(-\frac{i}{2} \omega_{\mu\nu} \sigma^2 s^{\mu\nu} \sigma^2 \right) \\ &= \exp \left[\frac{i}{2} \omega_{\mu\nu} (s^{\mu\nu})^T \right] = \left[\exp \left(\frac{i}{2} \omega_{\mu\nu} s^{\mu\nu} \right) \right]^T = d^{-1T}(\Lambda)\end{aligned}$$

这里 $d^{-1T}(\Lambda) = [d^{-1\dagger}(\Lambda)]^*$ ，线性表示 $\{d^{-1T}(\Lambda)\}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ 的复共轭表示

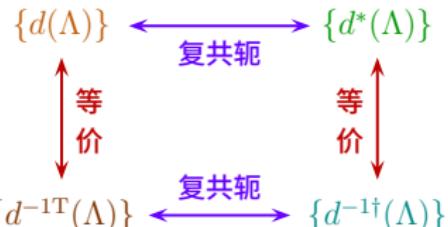
等价表示

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$$\sigma^2 d(\Lambda) \sigma^2 = \exp \left(-\frac{i}{2} \omega_{\mu\nu} \sigma^2 s^{\mu\nu} \sigma^2 \right)$$

$$= \exp \left[\frac{i}{2} \omega_{\mu\nu} (s^{\mu\nu})^T \right] = \left[\exp \left(\frac{i}{2} \omega_{\mu\nu} s^{\mu\nu} \right) \right]^T = d^{-1T}(\Lambda)$$



这里 $d^{-1T}(\Lambda) = [d^{-1\dagger}(\Lambda)]^*$ ，线性表示 $\{d^{-1T}(\Lambda)\}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ 的复共轭表示

将 Pauli 矩阵 σ^2 看作一个幺正变换矩阵, 满足 $(\sigma^2)^{-1} = (\sigma^2)^\dagger = \sigma^2$

则 $d(\Lambda)$ 与 $d^{-1T}(\Lambda)$ 由一个相似变换联系起来, 相似变换矩阵为 σ^2

根据 1.4 节定义, 线性表示 $\{d(\Lambda)\}$ 和 $\{d^{-1T}(\Lambda)\}$ 是等价的

由于 $(\sigma^2)^* = -\sigma^2$, $\sigma^2 d(\Lambda) \sigma^2 = d^{-1T}(\Lambda)$ 的复共轭为 $\sigma^2 d^*(\Lambda) \sigma^2 = d^{-1\dagger}(\Lambda)$

可见, 线性表示 $\{d(\Lambda)\}$ 的复共轭表示 $\{d^*(\Lambda)\}$ 与 $\{d^{-1\dagger}(\Lambda)\}$ 等价

左手 Weyl 旋量



于是，左手 Weyl 旋量

$$\eta_a = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

的固有保时向 Lorentz 变换为

$$\eta'_a = [d(\Lambda)]_a{}^b \eta_b, \quad a, b = 1, 2$$



η_a 是 $\{d(\Lambda)\}$ 表示空间中的列矢量

左手 Weyl 旋量



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引入反对称的二维 Levi-Civita 符号 ε^{ab} ，定义为

$$\varepsilon^{12} = -\varepsilon^{21} = 1, \quad \varepsilon^{11} = \varepsilon^{22} = 0$$



它与 Pauli 矩阵 σ^2 的关系是

$$\varepsilon^{ab} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = i \begin{pmatrix} & -i \\ i & \end{pmatrix} = (i\sigma^2)^{ab}$$

等价的左手 Weyl 旋量

通过 ε^{ab} 定义

$$\eta^a \equiv \varepsilon^{ab} \eta_b = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ -\eta_1 \end{pmatrix}$$

则

$$\eta^1 = \eta_2, \quad \eta^2 = -\eta_1$$

等价于 $\sigma^2 d(\Lambda) \sigma^2 = d^{-1T}(\Lambda)$ ，故 η^a 的 Lorentz 变换为

$$\begin{aligned}\eta'^a &= \varepsilon^{ab} \eta'_b = \varepsilon^{ab} [d(\Lambda)]_b^c \eta_c = i[\sigma^2 d(\Lambda)]^{ac} \eta_c \\ &= i[d^{-1T}(\Lambda) \sigma^2]^{ac} \eta_c = [d^{-1T}(\Lambda)]^a_b \varepsilon^{bc} \eta_c\end{aligned}$$

等价的左手 Weyl 旋量

通过 ε^{ab} 定义

$$\eta^a \equiv \varepsilon^{ab} \eta_b = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ -\eta_1 \end{pmatrix}$$

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即

$$\eta'^a = [d^{-1\top}(\Lambda)]^a{}_b \eta^b$$

可见 η^a 是 $\{d^{-1T}(\Lambda)\}$ 表示空间中的列矢量

由于 $\{d^{-1T}(\Lambda)\}$ 等价于 $\{d(\Lambda)\}$ ， η^a 也是左手 Weyl 旋量

ε^{ab} 和 ε_{ab}

两种左手 Weyl 旋量 η_a 与 η^a 是等价的，它们之间的关系类似于 Lorentz 逆变矢量 A^μ 与协变矢量 $A_\mu = g_{\mu\nu} A^\nu$ 之间的关系

 ε^{ab} 的作用类似于度规 $g_{\mu\nu}$ ，相当于 2 维旋量空间的“度规”，用于提升旋量指标

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用 $\varepsilon_{12} = -\varepsilon_{21} = -1$ 和 $\varepsilon_{11} = \varepsilon_{22} = 0$ 定义 ε_{ab} ，则

$$\varepsilon_{ab} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -i \begin{pmatrix} & -i \\ i & \end{pmatrix} = (-i\sigma^2)_{ab}$$

 ε_{ab} 是 ε^{ab} 的逆矩阵, 满足

$$\varepsilon_{ab}\varepsilon^{bc} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \delta_a^c$$

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于是, $\eta^1 = \eta_2$ 和 $\eta^2 = -\eta_1$ 表明

$$\eta_a = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta^2 \\ \eta^1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} = \varepsilon_{ab} \eta^b$$

也就是说, ε_{ab} 用于下降旋量指标

左手 Weyl 旋量的内积

任意两个左手 Weyl 旋量 η_a 和 ζ_a 的内积

$$\eta^a \zeta_a = \varepsilon^{ab} \eta_b \zeta_a = \varepsilon_{ab} \eta^a \zeta^b$$

在固有保时向 Lorentz 变换下不变，满足

$$\eta'^a \zeta'_a = [d^{-1\mathrm{T}}(\Lambda)]^a{}_b \eta^b [d(\Lambda)]_a{}^c \zeta_c = \eta^b [d^{-1}(\Lambda)]_b{}^a [d(\Lambda)]_a{}^c \zeta_c = \eta^b \delta_b{}^c \zeta_c = \eta^a \zeta_a$$

 第二步用了转置性质 $[d^{-1T}(\Lambda)]^a{}_b = [d^{-1}(\Lambda)]_b{}^a$ ，可见 $\eta^a \zeta_a$ 是 Lorentz 标量

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由 $\eta^1 = \eta_2$ 、 $\eta^2 = -\eta_1$ 、 $\zeta^1 = \zeta_2$ 和 $\zeta^2 = -\zeta_1$ 得

$$\eta^a \zeta_a = \eta^1 \zeta_1 + \eta^2 \zeta_2 = \eta_2 \zeta_1 - \eta_1 \zeta_2 = -\eta_2 \zeta^2 - \eta_1 \zeta^1 = -\eta_1 \zeta^a$$

即参与缩并的旋量指标一升一降会多出一个负号

这种性质与 Lorentz 矢量内积 $A^\mu B_\mu = A_\mu B^\mu$ 截然不同

原因在于旋量空间度规 ε^{ab} 是反对称的

Grassmann 数

羊 $\eta^a \zeta_a = -\eta_a \zeta^a$ 表明 $\eta^a \eta_a = -\eta_a \eta^a$ ，若 η_a 和 η^a 是普通的复数，则 $\eta^a \eta_a = 0$

为了使 $\eta^a \eta_a \neq 0$ ，必须要求左手 Weyl 旋量 η^a 与 η_a 反对易

即它们是 Grassmann 数，任意两个 Grassmann 数都是反对易的

以复数作为组合系数，则若干个 Grassmann 数的线性组合也是 Grassmann 数

因此, η_a 是 Grassmann 数意味着 $\eta^a = \varepsilon^{ab} \eta_b$ 也是 Grassmann 数

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虽然如此，Grassmann 数是反对易的 c 数，不是算符

对 Grassmann 数表达的旋量场进行量子化，才得到旋量场算符，而 Grassmann 数的反对易性质与旋量场算符的反对易关系相匹配

！ 旋量也可以不是 Grassmann 数，旋量系数 $u(p, \lambda)$ 和 $v(p, \lambda)$ 就是普通的复数

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假设 η_a 和 ζ^a 都是 Grassmann 数，则 $\eta_a \zeta^a = -\zeta^a \eta_a$ ，相应地，将省略旋量指标的内积写成 $\eta \zeta \equiv \eta^a \zeta_a = -\eta_a \zeta^a = \zeta^a \eta_a = \zeta \eta$ ，即内积 $\eta \zeta$ 和 $\zeta \eta$ 是相等的

内积 $\eta^a \eta_a$ 有等价表达式 $\eta\eta = \eta^a \eta_a = \varepsilon_{ab} \eta^a \eta^b = -\eta^1 \eta^2 + \eta^2 \eta^1 = -2\eta^1 \eta^2 = 2\eta_2 \eta_1 = \eta_2 \eta_1 - \eta_1 \eta_2 = -\varepsilon^{ab} \eta_a \eta_b = -\eta_a \eta^a$

左手 Weyl 旋量的复共轭

将左手 Weyl 旋量 η_a 的复共轭记为 $\eta_a^\dagger = \begin{pmatrix} \eta_i^\dagger \\ \eta_j^\dagger \end{pmatrix}$

量子化之后，算符 η_a 和 η_a^\dagger 互为厄米共轭

对 $\eta'_a = [d(\Lambda)]_a^b \eta_b$ 两边取复共轭，得到 η_a^\dagger 的 Lorentz 变换

$$\eta_{\dot{a}}'^{\dagger} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} \eta_{\dot{b}}^{\dagger}$$

左手 Weyl 旋量的复共轭

将左手 Weyl 旋量 η_a 的复共轭记为 $\eta_{\dot{a}}^\dagger = \begin{pmatrix} \eta_i^\dagger \\ \eta_{\dot{2}}^\dagger \end{pmatrix}$

量子化之后, 算符 η_a 和 $\eta_{\dot{a}}^\dagger$ 互为厄米共轭

对 $\eta'_a = [d(\Lambda)]_a^b \eta_b$ 两边取复共轭, 得到 $\eta_{\dot{a}}^\dagger$ 的 Lorentz 变换

$$\eta_{\dot{a}}'^\dagger = [d^*(\Lambda)]_{\dot{a}}^{\dot{b}} \eta_{\dot{b}}^\dagger$$

引进指标上带着点号的二维 Levi-Civita 符号

$$\varepsilon^{\dot{a}\dot{b}} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = (\mathrm{i}\sigma^2)^{\dot{a}\dot{b}}, \quad \varepsilon_{\dot{a}\dot{b}} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = (-\mathrm{i}\sigma^2)_{\dot{a}\dot{b}}$$

其分量数值与 ε^{ab} 和 ε_{ab} 分别相同

定义 $\eta^{\dagger\dot{a}} \equiv \varepsilon^{\dot{a}\dot{b}} \eta_{\dot{b}}^\dagger = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \eta_i^\dagger \\ \eta_{\dot{2}}^\dagger \end{pmatrix} = \begin{pmatrix} \eta_{\dot{2}}^\dagger \\ -\eta_i^\dagger \end{pmatrix}$, 则有 $\eta^{\dagger i} = \eta_{\dot{2}}^\dagger$ 和 $\eta^{\dagger\dot{2}} = -\eta_i^\dagger$

右手 Weyl 旋量

山羊 $\sigma^2 d^*(\Lambda) \sigma^2 = d^{-1\dagger}(\Lambda)$ 等价于 $\sigma^2 d^*(\Lambda) = d^{-1\dagger}(\Lambda) \sigma^2$

故 $\eta^{\dagger a}$ 的 Lorentz 变换为

$$\begin{aligned}\eta'^{\dagger a} &= \varepsilon^{\dot{a}\dot{b}} \eta_{\dot{b}}'^{\dagger} = \textcolor{blue}{\varepsilon^{\dot{a}\dot{b}}} [d^*(\Lambda)]_{\dot{b}}^{\dot{c}} \eta_{\dot{c}}'^{\dagger} = \text{i} [\sigma^2 d^*(\Lambda)]^{\dot{a}\dot{c}} \eta_{\dot{c}}'^{\dagger} \\ &= \text{i} [d^{-1\dagger}(\Lambda) \textcolor{teal}{\sigma^2}]^{\dot{a}\dot{c}} \eta_{\dot{c}}'^{\dagger} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{b}} \textcolor{blue}{\varepsilon^{\dot{b}\dot{c}}} \eta_{\dot{c}}'^{\dagger}\end{aligned}$$

即

$$\eta'^{\dagger a} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}} \eta^{\dagger \dot{b}}$$

右手 Weyl 旋量

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故 $\eta^{\dagger a}$ 的 Lorentz 变换为

$$\begin{aligned}\eta'^{\dagger\dot{a}} &= \varepsilon^{\dot{a}\dot{b}}\eta_b'^{\dagger} = \varepsilon^{\dot{a}\dot{b}}[d^*(\Lambda)]_{\dot{b}}{}^{\dot{c}}\eta_{\dot{c}}^{\dagger} = \mathrm{i}[\sigma^2 d^*(\Lambda)]^{\dot{a}\dot{c}}\eta_{\dot{c}}^{\dagger} \\ &= \mathrm{i}[d^{-1\dagger}(\Lambda)\sigma^2]^{\dot{a}\dot{c}}\eta_{\dot{c}}^{\dagger} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}}\varepsilon^{\dot{b}\dot{c}}\eta_{\dot{c}}^{\dagger}\end{aligned}$$

即

$$\eta'^{\dagger a} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}} \eta^{\dagger b}$$

可见, $\eta^{\dagger a}$ 是 $\{d^{-1\dagger}(\Lambda)\}$ 表示空间中的列矢量, 因而是右手 Weyl 旋量

由于表示 $\{d^*(\Lambda)\}$ 等价于 $\{d^{-1\dagger}(\Lambda)\}$ ， η_a^\dagger 也是右手 Weyl 旋量

因此，在这套符号约定中，不带点的旋量指标对应于左手 Weyl 旋量及其表示

而带点的旋量指标对应于右手 Weyl 旋量及其表示

右手 Weyl 旋量的内积

任意两个右手 Weyl 旋量 $\eta^{\dagger a}$ 和 $\zeta^{\dagger a}$ 的内积

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = \varepsilon_{\dot{a} \dot{b}} \eta^{\dagger \dot{b}} \zeta^{\dagger \dot{a}} = \varepsilon^{\dot{a} \dot{b}} \eta_{\dot{a}}^{\dagger} \zeta_{\dot{b}}^{\dagger}$$

在固有保时向 Lorentz 变换下不变，满足

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dot{c}\dagger\dot{a}} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} \eta_{\dot{b}}^{\dagger} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dot{c}\dagger\dot{a}} = \eta_{\dot{b}}^{\dagger} [d^{\dagger}(\Lambda)]^{\dot{b}}_{\dot{a}} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dot{c}\dagger\dot{a}} = \eta_{\dot{b}}^{\dagger} \delta^{\dot{b}}_{\dot{c}} \zeta^{\dot{c}\dagger\dot{a}} = \eta_{\dot{a}}^{\dagger} \zeta^{\dot{c}\dagger\dot{a}}$$

第二步用了转置性质 $[d^*(\Lambda)]_{\dot{a}}^{\dot{b}} = [d^\dagger(\Lambda)]_{\dot{a}}^{\dot{b}}$ ，可见 $\eta_{\dot{a}}^\dagger \zeta^{\dagger \dot{a}}$ 是 Lorentz 标量

右手 Weyl 旋量的内积

任意两个右手 Weyl 旋量 $\eta^{\dagger a}$ 和 $\zeta^{\dagger a}$ 的内积

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = \varepsilon_{\dot{a} \dot{b}} \eta^{\dagger \dot{b}} \zeta^{\dagger \dot{a}} = \varepsilon^{\dot{a} \dot{b}} \eta_{\dot{a}}^{\dagger} \zeta_{\dot{b}}^{\dagger}$$

在固有保时向 Lorentz 变换下不变，满足

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dot{c}\dagger \dot{a}} = [d^*(\Lambda)]_{\dot{a}}^{\dot{b}} \eta_{\dot{b}}^{\dagger} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dot{c}\dagger \dot{a}} = \eta_{\dot{b}}^{\dagger} [d^{\dagger}(\Lambda)]^{\dot{b}}_{\dot{a}} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dot{c}\dagger \dot{a}} = \eta_{\dot{b}}^{\dagger} \delta^{\dot{b}}_{\dot{c}} \zeta^{\dot{c}\dagger \dot{a}} = \eta_{\dot{a}}^{\dagger} \zeta^{\dot{c}\dagger \dot{a}}$$

第二步用了转置性质 $[d^*(\Lambda)]_{\dot{a}}^{\dot{b}} = [d^\dagger(\Lambda)]_{\dot{a}}^{\dot{b}}$ ，可见 $\eta_{\dot{a}}^{\dot{\dagger}} \zeta^{\dot{\dagger} \dot{a}}$ 是 Lorentz 标量

由 $\eta^{\dagger i} = \eta_{\dot{2}}^\dagger$ 、 $\eta^{\dagger 2} = -\eta_{\dot{1}}^\dagger$ 、 $\zeta^{\dagger i} = \zeta_{\dot{2}}^\dagger$ 和 $\zeta^{\dagger 2} = -\zeta_{\dot{1}}^\dagger$ 得

$$\eta_{\dot{a}}^\dagger \zeta^{\dagger \dot{a}} = \eta_{\dot{1}}^\dagger \zeta^{\dagger 1} + \eta_{\dot{2}}^\dagger \zeta^{\dagger 2} = -\eta^{\dagger \dot{2}} \zeta^{\dagger 1} + \eta^{\dagger 1} \zeta^{\dagger \dot{2}} = -\eta^{\dagger \dot{2}} \zeta_{\dot{2}}^\dagger - \eta^{\dagger 1} \zeta_{\dot{1}}^\dagger = -\eta^{\dagger \dot{a}} \zeta_{\dot{a}}^\dagger$$

即参与缩并的带点旋量指标一升一降会多出一个负号

右手 Weyl 旋量的内积

任意两个右手 Weyl 旋量 $\eta^{\dagger a}$ 和 $\zeta^{\dagger a}$ 的内积

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = \varepsilon_{\dot{a} \dot{b}} \eta^{\dagger \dot{b}} \zeta^{\dagger \dot{a}} = \varepsilon^{\dot{a} \dot{b}} \eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{b}}$$

在固有保时向 Lorentz 变换下**不变**，满足

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dot{c}\dagger\dot{a}} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} \eta_{\dot{b}}^{\dagger} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger\dot{c}} = \eta_{\dot{b}}^{\dagger} [d^{\dagger}(\Lambda)]^{\dot{b}}_{\dot{a}} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger\dot{c}} = \eta_{\dot{b}}^{\dagger} \delta^{\dot{b}}_{\dot{c}} \zeta^{\dagger\dot{c}} = \eta_{\dot{a}}^{\dagger} \zeta^{\dagger\dot{a}}$$

第二步用了转置性质 $[d^*(\Lambda)]_{\dot{a}}^{\dot{b}} = [d^\dagger(\Lambda)]_{\dot{a}}^{\dot{b}}$ ，可见 $\eta_{\dot{a}}^{\dot{\gamma}} \zeta^{\dot{\gamma}}{}^{\dot{a}}$ 是 Lorentz 标量

由 $\eta^{\dagger i} = \eta_{\dot{2}}^{\dagger}$ 、 $\eta^{\dagger 2} = -\eta_{\dot{1}}^{\dagger}$ 、 $\zeta^{\dagger i} = \zeta_{\dot{2}}^{\dagger}$ 和 $\zeta^{\dagger 2} = -\zeta_{\dot{1}}^{\dagger}$ 得

$$\eta_{\dot{a}}^\dagger \zeta^{\dagger \dot{a}} = \eta_{\dot{i}}^\dagger \zeta^{\dagger i} + \eta_{\dot{j}}^\dagger \zeta^{\dagger \dot{j}} = -\eta^{\dagger \dot{2}} \zeta^{\dagger i} + \eta^{\dagger i} \zeta^{\dagger \dot{2}} = -\eta^{\dagger \dot{2}} \zeta_{\dot{i}}^\dagger - \eta^{\dagger i} \zeta_{\dot{j}}^\dagger = -\eta^{\dagger \dot{a}} \zeta_{\dot{a}}^\dagger$$

即参与缩并的带点旋量指标一升一降会多出一个负号

⌚ 假设右手 Weyl 旋量 $\eta^{\dagger a}$ 和 $\zeta_{\dot{a}}^\dagger$ 都是 Grassmann 数，则 $\eta^{\dagger a}\zeta_{\dot{a}}^\dagger = -\zeta_{\dot{a}}^\dagger\eta^{\dagger a}$

将省略带点旋量指标的内积写成 $\eta^\dagger \zeta^\dagger \equiv \eta_{\dot{a}}^\dagger \zeta^{\dagger \dot{a}} = -\eta^{\dagger \dot{a}} \zeta_{\dot{a}}^\dagger = \zeta_{\dot{a}}^\dagger \eta^{\dagger \dot{a}} = \zeta^\dagger \eta^\dagger$

则内积 $\eta^\dagger \zeta^\dagger$ 和 $\zeta^\dagger \eta^\dagger$ 相等

Lorentz 不变量和 Weyl 旋量算符

可以看到, 只要将不带点和带点的旋量指标分别缩并完全, 就得到 Lorentz 标量

另一方面，缩并一个不带点的指标和一个带点的指标并不能得到 Lorentz 不变量

比如, $\eta^a \zeta_{\dot{a}}$ 和 $\eta^{\dot{a}} \zeta_a$ 都不是 Lorentz 标量

Lorentz 不变量和 Weyl 旋量算符

🐂 可以看到，只要将不带点和带点的旋量指标分别缩并完毕，就得到 Lorentz 标量

🎩 另一方面，缩并一个不带点的指标和一个带点的指标并不能得到 Lorentz 不变量

🕶 比如， $\eta^a \zeta_{\dot{a}}^\dagger$ 和 $\eta^{\dagger \dot{a}} \zeta_a$ 都不是 Lorentz 标量

👔 对于 Weyl 旋量算符 η_a 和 ζ_a ，有

$$(\eta \zeta)^\dagger = (\eta^a \zeta_a)^\dagger = (\zeta_a)^\dagger (\eta^a)^\dagger = \zeta_{\dot{a}}^\dagger \eta^{\dagger \dot{a}} = \zeta^\dagger \eta^\dagger$$

💼 即 $\zeta^\dagger \eta^\dagger$ 是 $\eta \zeta$ 的厄米共轭算符

👞 厄米共轭操作将左手和右手 Weyl 旋量算符相互转换

9.6.2 小节 Dirac 和 Majorana 旋量场的分解

依照上一小节关于旋量指标的约定, 将 Dirac 旋量场 $\psi(x)$ 分解成左手 Weyl 旋量场 $\eta_a(x)$ 和右手 Weyl 旋量场 $\zeta^{\dagger a}(x)$, 形式为

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \zeta^{\dagger a}(x) \end{pmatrix}$$

在量子化之前, $\eta_a(x)$ 和 $\zeta^{\dagger a}(x)$ 是 Grassmann 数, 因而 $\psi(x)$ 也是 Grassmann 数

这是在 9.2.1 小节中转置两个旋量场必须添加一个额外负号的原因

根据 $D(\Lambda) = \begin{pmatrix} d(\Lambda) & \\ & d^{-1\dagger}(\Lambda) \end{pmatrix}$, $\psi(x)$ 的固有保时向 Lorentz 变换表达成

$$\begin{pmatrix} \eta'_a(x') \\ \zeta'^{\dagger \dot{a}}(x') \end{pmatrix} = \psi'(x') = \textcolor{blue}{D}(\Lambda) \psi(x) = \begin{pmatrix} [d(\Lambda)]_a{}^b \eta_b(x) \\ [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}} \zeta^{\dagger \dot{b}}(x) \end{pmatrix}$$

ψ(x) 的 Dirac 共轭是 $\bar{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \eta_b^\dagger & \zeta^b \end{pmatrix} \begin{pmatrix} & \delta^b{}_a \\ \delta_b{}^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & \eta_a^\dagger \end{pmatrix}$

Dirac 矩阵的指标形式

保持旋量指标平衡，则 Dirac 方程 $(i\gamma^\mu \partial_\mu - m)\psi = 0$ 化为

$$\begin{pmatrix} -m\delta_a{}^b & i(\sigma^\mu)_{a\dot{b}}\partial_\mu \\ i(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu & -m\delta^{\dot{a}}{}_{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dagger\dot{b}} \end{pmatrix} = 0$$

因而 Dirac 矩阵的指标形式是

$$\gamma^\mu = \begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}$$

注意, γ^μ 中的 γ^0 与 Dirac 共轭 $\bar{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \eta_a^\dagger & \zeta^a \end{pmatrix} \begin{pmatrix} & \delta^b_{\dot{a}} \\ \delta_b^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & \eta_a^\dagger \end{pmatrix}$

中的 γ^0 具有不同的指标结构

两者本质不同，有些书将后者记为 β 以示区别

σ^μ 和 $\bar{\sigma}^\mu$ 的 Lorentz 变换规则

于是, γ^μ 的 Lorentz 变换规则 $D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$ 左边变成

$$\begin{aligned}
& D^{-1}(\Lambda) \gamma^\mu D(\Lambda) \\
&= \begin{pmatrix} [d^{-1}(\Lambda)]_a{}^c & \\ & [d^\dagger(\Lambda)]^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{c\dot{d}} \\ (\bar{\sigma}^\mu)^{\dot{c}d} & \end{pmatrix} \begin{pmatrix} [d(\Lambda)]_d{}^b & \\ [d^{-1\dagger}(\Lambda)]^{\dot{d}}{}_{\dot{b}} & \end{pmatrix} \\
&= \begin{pmatrix} & [d^{-1}(\Lambda)]_a{}^c (\sigma^\mu)_{c\dot{d}} [d^{-1\dagger}(\Lambda)]^{\dot{d}}{}_{\dot{b}} \\ [d^\dagger(\Lambda)]^{\dot{a}}{}_{\dot{c}} (\bar{\sigma}^\mu)^{\dot{c}d} [d(\Lambda)]_d{}^b & \end{pmatrix}
\end{aligned}$$

 右边化为

$$\Lambda^\mu{}_\nu \gamma^\nu = \begin{pmatrix} & \Lambda^\mu{}_\nu (\sigma^\nu)_{ab} \\ \Lambda^\mu{}_\nu (\bar{\sigma}^\nu)^{\dot{a}\dot{b}} & \end{pmatrix}$$

两相比较，推出

$$[d^{-1}(\Lambda)]_a{}^c (\sigma^\mu)_{cd} [d^{-1\dagger}(\Lambda)]^d{}_b = \Lambda^\mu{}_\nu (\sigma^\nu)_{ab}, \quad [d^\dagger(\Lambda)]^a{}_c (\bar{\sigma}^\mu)_{cd} [d(\Lambda)]_d{}^b = \Lambda^\mu{}_\nu (\bar{\sigma}^\nu)^{ab}$$

这分别是 σ^μ 和 $\bar{\sigma}^\mu$ 的 Lorentz 变换规则

Lorentz 矢量 $\eta\sigma^\mu\zeta^\dagger$ 和 $\eta^\dagger\bar{\sigma}^\mu\zeta$

对任意 Weyl 旋量 η 和 ζ , 定义

$$\eta\sigma^\mu\zeta^\dagger \equiv \eta^a(\sigma^\mu)_{ab}\zeta^{\dagger b}, \quad \eta^\dagger\bar{\sigma}^\mu\zeta \equiv \eta_{\dot{a}}^\dagger(\bar{\sigma}^\mu)^{\dot{a}\dot{b}}\zeta_{\dot{b}}$$

它们都是 Lorentz 矢量, 相应的固有保时向 Lorentz 变换为

$$\begin{aligned}\eta'\sigma^\mu\zeta'^\dagger &= [d^{-1T}(\Lambda)]^a{}_c\eta^c(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^{\dot{b}}{}_d\zeta^{\dagger\dot{d}} = \eta^c[d^{-1}(\Lambda)]_c{}^a(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^{\dot{b}}{}_d\zeta^{\dagger\dot{d}} \\ &= \eta^c\Lambda^\mu{}_\nu(\sigma^\nu)_{c\dot{d}}\zeta^{\dagger\dot{d}} = \Lambda^\mu{}_\nu\eta\sigma^\nu\zeta^\dagger\end{aligned}$$

$$\begin{aligned}\eta'^\dagger\bar{\sigma}^\mu\zeta' &= [d^*(\Lambda)]_{\dot{a}}{}^{\dot{c}}\eta_{\dot{c}}^\dagger(\bar{\sigma}^\mu)^{\dot{a}\dot{b}}[d(\Lambda)]_b{}^d\zeta_d = \eta_{\dot{c}}^\dagger[d^\dagger(\Lambda)]_{\dot{c}}{}^{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}\dot{b}}[d(\Lambda)]_b{}^d\zeta_d \\ &= \eta_{\dot{c}}^\dagger\Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot{c}d}\zeta_d = \Lambda^\mu{}_\nu\eta^\dagger\bar{\sigma}^\mu\zeta\end{aligned}$$

Lorentz 矢量 $\eta\sigma^\mu\zeta^\dagger$ 和 $\eta^\dagger\bar{\sigma}^\mu\zeta$

对任意 Weyl 旋量 η 和 ζ ，定义

$$\eta\sigma^\mu\zeta^\dagger \equiv \eta^a(\sigma^\mu)_{ab}\zeta^{b\dagger}, \quad \eta^\dagger\bar{\sigma}^\mu\zeta \equiv \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}\dot{b}}\zeta_{\dot{b}}$$

它们都是 Lorentz 矢量，相应的固有保时向 Lorentz 变换为

$$\begin{aligned}\eta'\sigma^\mu\zeta'^\dagger &= [d^{-1T}(\Lambda)]^a_c\eta^c(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^b_d\zeta^{d\dagger} = \eta^c[d^{-1}(\Lambda)]_c^a(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^b_d\zeta^{d\dagger} \\ &= \eta^c\Lambda^\mu_\nu(\sigma^\nu)_{cd}\zeta^{d\dagger} = \Lambda^\mu_\nu\eta\sigma^\nu\zeta^\dagger\end{aligned}$$

$$\begin{aligned}\eta'^\dagger\bar{\sigma}^\mu\zeta' &= [d^*(\Lambda)]_{\dot{a}}^{\dot{c}}\eta_{\dot{c}}^\dagger(\bar{\sigma}^\mu)^{\dot{a}\dot{b}}[d(\Lambda)]_b^d\zeta_d = \eta_{\dot{c}}^\dagger[d^\dagger(\Lambda)]_{\dot{a}}^{\dot{c}}(\bar{\sigma}^\mu)^{\dot{a}\dot{b}}[d(\Lambda)]_b^d\zeta_d \\ &= \eta_{\dot{c}}^\dagger\Lambda^\mu_\nu(\bar{\sigma}^\nu)^{\dot{c}\dot{d}}\zeta_d = \Lambda^\mu_\nu\eta^\dagger\bar{\sigma}^\mu\zeta\end{aligned}$$

由 $\sigma^2\sigma^\mu\sigma^2 = (\bar{\sigma}^\mu)^T$ 得 $(i\sigma^2)\sigma^\mu(i\sigma^2) = -(\bar{\sigma}^\mu)^T$ ，相应的指标形式为

$$\varepsilon^{ac}(\sigma^\mu)_{cd}\varepsilon^{db} = -[(\bar{\sigma}^\mu)^T]^{ab} = -(\bar{\sigma}^\mu)^{ba}$$

对于 Weyl 旋量场 $\eta_a(x)$ 和 $\zeta^{\dagger\dot{a}}(x)$ ，有



Grassmann 数性质

$$\begin{aligned}[\eta^a(\sigma^\mu)_{ab}\zeta^{b\dagger}]^\dagger &= \zeta^b(\sigma^\mu)_{ba}\eta^{\dagger\dot{a}} = -\eta^{\dagger\dot{a}}(\sigma^\mu)_{b\dot{a}}\zeta^b = -\varepsilon^{\dot{a}\dot{c}}\eta_{\dot{c}}^\dagger(\sigma^\mu)_{b\dot{a}}\varepsilon^{bd}\zeta_d \\ &= \eta_{\dot{c}}^\dagger\varepsilon^{db}(\sigma^\mu)_{b\dot{a}}\varepsilon^{\dot{a}\dot{c}}\zeta_d = -\eta_{\dot{c}}^\dagger(\bar{\sigma}^\mu)^{\dot{c}\dot{d}}\zeta_d = -[\zeta_{\dot{d}}^\dagger(\bar{\sigma}^\mu)^{\dot{d}\dot{c}}\eta_c]^\dagger\end{aligned}$$

即

$$(\eta\sigma^\mu\zeta^\dagger)^\dagger = \zeta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\zeta = -(\zeta^\dagger\bar{\sigma}^\mu\eta)^\dagger$$

Lorentz 张量 $\eta\sigma^\mu\bar{\sigma}^\nu\zeta$ 和 $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger$

类似地, $\eta\sigma^\mu\bar{\sigma}^\nu\zeta \equiv \eta^a(\sigma^\mu)_{ab}(\bar{\sigma}^\nu)^{bc}\zeta_c$ 和 $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger \equiv \eta^\dagger_a(\bar{\sigma}^\mu)^{ab}(\sigma^\nu)_{bc}\zeta^{\dagger c}$ 都是二阶 Lorentz 张量

由 $\sigma^2 \bar{\sigma}^\mu \sigma^2 = (\sigma^\mu)^T$ 得 $(-\mathrm{i}\sigma^2) \bar{\sigma}^\mu (-\mathrm{i}\sigma^2) = -(\sigma^\mu)^T$ ，相应的指标形式为

$$\varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\mu)^{\dot{c}d}\varepsilon_{db} = -[(\sigma^\mu)^T]_{\dot{a}b} = -(\sigma^\mu)_{b\dot{a}}$$

再利用 $\varepsilon_{ab}\varepsilon^{bc} = \delta_a^c$ 和 $\varepsilon^{ac}(\sigma^\mu)_{cd}\varepsilon^{db} = -[(\bar{\sigma}^\mu)^T]^{ab} = -(\bar{\sigma}^\mu)^{ba}$ 推出

$$\begin{aligned} \varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}(\sigma^\mu)_{d\dot{e}}\varepsilon^{\dot{e}\dot{b}} &= \varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}\delta_d{}^f(\sigma^\mu)_{f\dot{e}}\varepsilon^{\dot{e}\dot{b}} = \varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}\varepsilon_{dg}\varepsilon^{gf}(\sigma^\mu)_{f\dot{e}}\varepsilon^{\dot{e}\dot{b}} \\ &= (-\sigma^\nu)_{g\dot{a}}(-\bar{\sigma}^\mu)^{\dot{b}g} = (\bar{\sigma}^\mu)^{\dot{b}g}(\sigma^\nu)_{g\dot{a}} \end{aligned}$$

$$\begin{aligned}
\text{故 } [\eta^a(\sigma^\mu)_{ab}(\bar{\sigma}^\nu)^{\dot{b}c}\zeta_c]^\dagger &= \zeta_{\dot{c}}^\dagger(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\eta^{\dagger\dot{a}} = -\eta^{\dagger\dot{a}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\zeta_{\dot{c}}^\dagger \\
&= -\varepsilon^{\dot{a}\dot{d}}\eta_{\dot{d}}^\dagger(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\varepsilon_{\dot{c}\dot{e}}\zeta^{\dagger\dot{e}} = \eta_{\dot{d}}^\dagger\varepsilon_{\dot{e}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\varepsilon^{\dot{a}\dot{d}}\zeta^{\dagger\dot{e}} \\
&= \eta_{\dot{d}}^\dagger(\bar{\sigma}^\mu)^{\dot{d}g}(\sigma^\nu)_{g\dot{e}}\zeta^{\dagger\dot{e}} = [\zeta^e(\sigma^\nu)_{e\dot{g}}(\bar{\sigma}^\mu)^{\dot{g}d}\eta_d]^\dagger
\end{aligned}$$

即

$$(\eta \sigma^\mu \bar{\sigma}^\nu \zeta)^\dagger = \zeta^\dagger \bar{\sigma}^\nu \sigma^\mu \eta^\dagger = \eta^\dagger \bar{\sigma}^\mu \sigma^\nu \zeta^\dagger = (\zeta \sigma^\nu \bar{\sigma}^\mu \eta)^\dagger$$

旋量双线性型的分解



将 Dirac 旋量双线性型分解成由 Weyl 旋量表达的 Lorentz 张量，有

$$\bar{\psi}\psi = \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} \eta_a \\ \zeta^{\dot{a}} \end{pmatrix} = \zeta^a \eta_a + \eta_{\dot{a}}^\dagger \zeta^{\dot{a}} = \zeta \eta + \eta^\dagger \zeta^\dagger$$

$$\bar{\psi}\gamma^5\psi = \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} -\delta_a{}^b & \\ & \delta^{\dot{a}}{}_{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dot{b}} \end{pmatrix} = -\zeta^a \eta_a + \eta_{\dot{a}}^\dagger \zeta^{\dot{a}} = -\zeta \eta + \eta^\dagger \zeta^\dagger$$

$$\begin{aligned} \bar{\psi}\gamma^\mu\psi &= \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} & \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dot{b}} \end{pmatrix} = \zeta^a (\sigma^\mu)_{ab} \zeta^{\dot{b}} + \eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} \eta_b \\ &= \zeta \sigma^\mu \zeta^\dagger + \eta^\dagger \bar{\sigma}^\mu \eta \end{aligned}$$

$$\begin{aligned} \bar{\psi}\gamma^\mu\gamma^5\psi &= \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} & \end{pmatrix} \begin{pmatrix} -\delta_b{}^c & \\ & \delta^{\dot{b}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \eta_c \\ \zeta^{\dot{c}} \end{pmatrix} \\ &= \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} & \end{pmatrix} \begin{pmatrix} -\eta_b \\ \zeta^{\dot{b}} \end{pmatrix} = \zeta^a (\sigma^\mu)_{ab} \zeta^{\dot{b}} - \eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} \eta_b \\ &= \zeta \sigma^\mu \zeta^\dagger - \eta^\dagger \bar{\sigma}^\mu \eta \end{aligned}$$

旋量双线性型的分解



还有

$$\begin{aligned}\bar{\psi} \sigma^{\mu\nu} \psi &= \frac{i}{2} \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_a{}^b & \\ & (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)_{\dot{a}}{}^{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dot{b}} \end{pmatrix} \\ &= \frac{i}{2} \zeta^a (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_a{}^b \eta_b + \frac{i}{2} \eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)_{\dot{a}}{}^{\dot{b}} \zeta^{\dot{b}} \\ &= \frac{i}{2} \zeta (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \eta + \frac{i}{2} \eta^\dagger (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \zeta^\dagger\end{aligned}$$



进一步推出

$$\bar{\psi}_R \psi_L = \frac{1}{2} \bar{\psi} (1 - \gamma^5) \psi = \zeta \eta$$

$$\bar{\psi}_L \psi_R = \frac{1}{2} \bar{\psi} (1 + \gamma^5) \psi = \eta^\dagger \zeta^\dagger$$

$$\bar{\psi}_L \gamma^\mu \psi_L = \frac{1}{2} \bar{\psi} (\gamma^\mu - \gamma^\mu \gamma^5) \psi = \eta^\dagger \bar{\sigma}^\mu \eta$$

$$\bar{\psi}_R \gamma^\mu \psi_R = \frac{1}{2} \bar{\psi} (\gamma^\mu + \gamma^\mu \gamma^5) \psi = \zeta \sigma^\mu \zeta^\dagger$$

拉氏量的分解

另一方面, **自由 Dirac 旋量场**的**拉氏量**分解为

$$\begin{aligned}
\mathcal{L} &= \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi = \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} -m\delta_a{}^b & i(\sigma^\mu)_{ab} \partial_\mu \\ i(\bar{\sigma}^\mu)^{\dot{a}\dot{b}} \partial_\mu & -m\delta^{\dot{a}}_{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dot{b}} \end{pmatrix} \\
&= -m\zeta^a \eta_a + i\zeta^a (\sigma^\mu)_{ab} \partial_\mu \zeta^{\dot{b}} + i\eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} \partial_\mu \eta_b - m\eta_{\dot{a}}^\dagger \zeta^{\dot{b}} \\
&= i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta + i\zeta \sigma^\mu \partial_\mu \zeta^\dagger - m(\zeta \eta + \eta^\dagger \zeta^\dagger)
\end{aligned}$$

这里的质量项涉及两个不同的 Weyl 旋量场 $\eta_a(x)$ 和 $\zeta_a(x)$ ，称为 Dirac 质量项

如果质量 $m = 0$ ，则

$$\mathcal{L}_L = i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta \quad \text{和} \quad \mathcal{L}_R = i\zeta \sigma^\mu \partial_\mu \zeta^\dagger$$

分别描述自由的左手 Weyl 旋量场 $\eta_a(x)$ 和右手 Weyl 旋量场 $\zeta^{\dagger a}(x)$

相应的运动方程是两个 Weyl 方程：

$$i\bar{\sigma}^\mu \partial_\mu \eta = 0, \quad i\sigma^\mu \partial_\mu \zeta^\dagger = 0$$

Weyl 旋量场的 C 变换

下面讨论 Weyl 旋量场的分立变换

首先，电荷共轭矩阵的指标形式为 $C = \begin{pmatrix} -i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}$

将 $\psi(x)$ 的电荷共轭场 $\psi^C(x)$ 分解成 Weyl 旋量场, 得到

$$\psi^C(x) = \mathcal{C}\bar{\psi}^T(x) = \mathcal{C} \begin{pmatrix} \zeta^b(x) & \eta_b^\dagger(x) \end{pmatrix}^T = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \zeta^b(x) \\ \eta_b^\dagger(x) \end{pmatrix} = \begin{pmatrix} \zeta_a(x) \\ \eta_b^\dagger(x) \end{pmatrix}$$

从而, Dirac 旋量场 $\psi(x)$ 的 C 变换化为

$$\begin{pmatrix} C^{-1} \eta_a(x) C \\ C^{-1} \zeta^{\dagger a}(x) C \end{pmatrix} = C^{-1} \psi(x) C = \zeta_C^* \psi^C(x) = \begin{pmatrix} \zeta_C^* \zeta_a(x) \\ \zeta_C^* \eta^{\dagger a}(x) \end{pmatrix}$$

Weyl 旋量场的 C 变换

下面讨论 Weyl 旋量场的分立变换

首先，电荷共轭矩阵的指标形式为 $C = \begin{pmatrix} -i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}$

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从而, Dirac 旋量场 $\psi(x)$ 的 C 变换化为

$$\begin{pmatrix} C^{-1} \eta_a(x) C \\ C^{-1} \zeta^{\dagger a}(x) C \end{pmatrix} = C^{-1} \psi(x) C = \zeta_C^* \psi^C(x) = \begin{pmatrix} \zeta_C^* \zeta_a(x) \\ \zeta_C^* \eta^{\dagger a}(x) \end{pmatrix}$$

即左右手 Weyl 旋量场的 C 变换是

$$C^{-1} \eta_a(x) C = \zeta_C^* \zeta_a(x), \quad C^{-1} \zeta^{\dagger a}(x) C = \zeta_C^* \eta^{\dagger a}(x)$$

可见，电荷共轭变换将 η 和 ζ 相互转换。取厄米共轭，得 $C^{-1}\eta_b^\dagger(x)C = \zeta_C\zeta_b^\dagger(x)$ 及 $C^{-1}\zeta^b(x)C = \zeta_C\eta^b(x)$ ，分别与 ε^{ab} 和 ε_{ab} 缩并，推出

$$C^{-1}\eta^{\dagger a}(x)C = \zeta_C \zeta^{\dagger a}(x), \quad C^{-1}\zeta_a(x)C = \zeta_C \eta_a(x)$$

Weyl 旋量场的 P 变换

其次, Dirac 旋量场 $\psi(x)$ 的 P 变换表达为

$$\begin{aligned} \begin{pmatrix} P^{-1} \eta_a(x) P \\ P^{-1} \zeta^{\dagger a}(x) P \end{pmatrix} &= P^{-1} \psi(x) P = \zeta_P^* \gamma^0 \psi(\mathcal{P}x) \\ &= \zeta_P^* \begin{pmatrix} \delta^{\dot{a}}_{\dot{b}} \\ \delta_a^b \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{P}x) \\ \zeta^{\dagger b}(\mathcal{P}x) \end{pmatrix} = \begin{pmatrix} \zeta_P^* \zeta^{\dagger a}(\mathcal{P}x) \\ \zeta_P^* \eta_a(\mathcal{P}x) \end{pmatrix} \end{aligned}$$

注意此处 γ^0 的指标结构与 $\bar{\psi} = \psi^\dagger \gamma^0$ 中一样

Weyl 旋量场的 P 变换

其次, Dirac 旋量场 $\psi(x)$ 的 P 变换表达为

$$\begin{aligned} \begin{pmatrix} P^{-1} \eta_a(x) P \\ P^{-1} \zeta^{\dagger a}(x) P \end{pmatrix} &= P^{-1} \psi(x) P = \zeta_P^* \gamma^0 \psi(\mathcal{P}x) \\ &= \zeta_P^* \begin{pmatrix} \delta^a{}_b \\ \delta_a{}^b \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{P}x) \\ \zeta^{\dagger b}(\mathcal{P}x) \end{pmatrix} = \begin{pmatrix} \zeta_P^* \zeta^{\dagger a}(\mathcal{P}x) \\ \zeta_P^* \eta_a(\mathcal{P}x) \end{pmatrix} \end{aligned}$$

注意此处 γ^0 的指标结构与 $\bar{\psi} = \psi^\dagger \gamma^0$ 中一样

于是得到左右手 Weyl 旋量场的 P 变换

$$P^{-1} \eta_a(x) P = \zeta_P^* \zeta^{\dagger \dot{a}}(\mathcal{P}x), \quad P^{-1} \zeta^{\dagger \dot{a}}(x) P = \zeta_P^* \eta_a(\mathcal{P}x)$$

也就是说，宇称变换将左手和右手 Weyl 旋量场相互转换

♣ 取厄米共轭得 $P^{-1}\eta_b^\dagger(x)P = \zeta_P\zeta^b(\mathcal{P}x)$ 和 $P^{-1}\zeta^b(x)P = \zeta_P\eta_b^\dagger(\mathcal{P}x)$

◆ 两边与 $i\sigma^2 = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{ab}$ 缩并, 推出

$$P^{-1} \eta^{\dagger a}(x) P = -\zeta_P \zeta_a(\mathcal{P}x), \quad P^{-1} \zeta_a(x) P = -\zeta_P \eta^{\dagger a}(\mathcal{P}x)$$

Weyl 旋量场的 T 变换

骆驼 最后, **时间反演矩阵**的指标形式是 $D(\mathcal{T}) = \mathcal{C}\gamma^5 = \begin{pmatrix} i\sigma^2 & & \\ & i\sigma^2 & \\ & & \end{pmatrix} = \begin{pmatrix} \varepsilon^{ab} & & \\ & & -\varepsilon_{ab} \end{pmatrix}$

Dirac 旋量场 $\psi(x)$ 的 T 变换化为

$$\begin{aligned} \begin{pmatrix} T^{-1} \eta_a(x) T \\ T^{-1} \zeta^{\dagger a}(x) T \end{pmatrix} &= T^{-1} \psi(x) T = \zeta_T^* \mathcal{C} \gamma^5 \psi(\mathcal{T}x) \\ &= \zeta_T^* \begin{pmatrix} \varepsilon^{ab} & \\ & -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{T}x) \\ \zeta^{\dagger \dot{b}}(\mathcal{T}x) \end{pmatrix} = \begin{pmatrix} \zeta_T^* \eta^a(\mathcal{T}x) \\ -\zeta_T^* \zeta_{\dot{a}}^{\dagger}(\mathcal{T}x) \end{pmatrix} \end{aligned}$$

Weyl 旋量场的 T 变换

骆驼 最后, **时间反演矩阵**的指标形式是 $D(\mathcal{T}) = \mathcal{C}\gamma^5 = \begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon^{ab} & \\ & -\varepsilon_{ab} \end{pmatrix}$

Dirac 旋量场 $\psi(x)$ 的 T 变换化为

$$\begin{aligned} \begin{pmatrix} T^{-1} \eta_a(x) T \\ T^{-1} \zeta^{\dagger a}(x) T \end{pmatrix} &= T^{-1} \psi(x) T = \zeta_T^* \mathcal{C} \gamma^5 \psi(\mathcal{T}x) \\ &= \zeta_T^* \begin{pmatrix} \varepsilon^{ab} & \\ & -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{T}x) \\ \zeta^{\dagger \dot{b}}(\mathcal{T}x) \end{pmatrix} = \begin{pmatrix} \zeta_T^* \eta^a(\mathcal{T}x) \\ -\zeta_T^* \zeta_{\dot{a}}^{\dagger}(\mathcal{T}x) \end{pmatrix} \end{aligned}$$

则左手 Weyl 旋量场的 T 变换是

$$T^{-1} \eta_a(x) T = \zeta_T^* \eta^a(\mathcal{T}x), \quad T^{-1} \zeta^{\dagger a}(x) T = -\zeta_T^* \zeta_{\dot{a}}^\dagger(\mathcal{T}x)$$

取厄米共轭, 有 $T^{-1}\eta_b^\dagger(x)T = \zeta_T\eta^{\dagger b}(\mathcal{T}x)$ 和 $T^{-1}\zeta^b(x)T = -\zeta_T\zeta_b(\mathcal{T}x)$

与 $i\sigma^2 = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{ab} = \varepsilon^{ab}$ 缩并, 得

$$T^{-1} \eta^{\dagger a}(x) T = -\zeta_T \eta_{\dot{a}}^\dagger(\mathcal{T}x), \quad T^{-1} \zeta_a(x) T = \zeta_T \zeta^a(\mathcal{T}x)$$

Majorana 旋量场的分解

下面讨论 Majorana 旋量场, Majorana 条件意味着 $\begin{pmatrix} \eta_a \\ \zeta^{\dagger a} \end{pmatrix} = \psi = \mathcal{C} \bar{\psi}^T = \begin{pmatrix} \zeta_a \\ \eta^{\dagger a} \end{pmatrix}$

即 $\eta = \zeta$ ，这表明 Majorana 旋量场中的左手和右手 Weyl 旋量场是相关的

因此，可以将 Majorana 旋量场 $\psi(x)$ 分解成

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \eta^{\dagger a}(x) \end{pmatrix}$$

Majorana 旋量场的分解

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即 $\eta = \zeta$ ，这表明 Majorana 旋量场中的左手和右手 Weyl 旋量场是相关的

因此，可以将 Majorana 旋量场 $\psi(x)$ 分解成

而自由 Majorana 旋量场的拉氏量分解为

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \eta^{\dagger a}(x) \end{pmatrix}$$

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi = \frac{1}{2} \begin{pmatrix} \eta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} -m\delta_a{}^b & i(\sigma^\mu)_{a\dot{b}} \partial_\mu \\ i(\bar{\sigma}^\mu)_{\dot{a}b} \partial_\mu & -m\delta_{\dot{a}}{}^{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \eta^{\dot{b}} \end{pmatrix} \\ &= \frac{1}{2} [i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta + i\eta \sigma^\mu \partial_\mu \eta^\dagger - m(\eta \eta + \eta^\dagger \eta^\dagger)]\end{aligned}$$

⑧ 利用 $\zeta \sigma^\mu \eta^\dagger = -\eta^\dagger \bar{\sigma}^\mu \zeta$ 将方括号中第二项化为

$$i\eta\sigma^\mu\partial_\mu\eta^\dagger = i\partial_\mu(\eta\sigma^\mu\eta^\dagger) - i(\partial_\mu\eta)\sigma^\mu\eta^\dagger = i\partial_\mu(\eta\sigma^\mu\eta^\dagger) + i\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta$$

🎭 扔掉全散度项 $i\partial_\mu(\eta\sigma^\mu\eta^\dagger)$ ，拉氏量变成 $\mathcal{L} = i\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta - \frac{1}{2}m(\eta\eta + \eta^\dagger\eta^\dagger)$

这里的质量项只涉及一个 Weyl 旋量场 $\eta_a(x)$ ，称为 Majorana 质量项

Majorana 旋量场的 $\bar{\psi}\gamma^\mu\psi$ 和 $\bar{\psi}\sigma^{\mu\nu}\psi$

袋鼠 $\zeta \sigma^\mu \eta^\dagger = -\eta^\dagger \bar{\sigma}^\mu \zeta$ 、 $\eta \sigma^\mu \bar{\sigma}^\nu \zeta = \zeta \sigma^\nu \bar{\sigma}^\mu \eta$ 和 $\eta^\dagger \bar{\sigma}^\mu \sigma^\nu \zeta^\dagger = \zeta^\dagger \bar{\sigma}^\nu \sigma^\mu \eta^\dagger$ 意味着

$$\eta \sigma^\mu \eta^\dagger = -\eta^\dagger \bar{\sigma}^\mu \eta, \quad \eta \sigma^\mu \bar{\sigma}^\nu \eta = \eta \sigma^\nu \bar{\sigma}^\mu \eta, \quad \eta^\dagger \bar{\sigma}^\mu \sigma^\nu \eta^\dagger = \eta^\dagger \bar{\sigma}^\nu \sigma^\mu \eta^\dagger$$

对于 Majorana 旋量场, $\eta = \zeta$, $\bar{\psi} \gamma^\mu \psi = \zeta \sigma^\mu \zeta^\dagger + \eta^\dagger \bar{\sigma}^\mu \eta$ 化为

$$\bar{\psi} \gamma^\mu \psi = \eta \sigma^\mu \eta^\dagger + \eta^\dagger \bar{\sigma}^\mu \eta = -\eta^\dagger \bar{\sigma}^\mu \eta + \eta^\dagger \bar{\sigma}^\mu \eta = 0$$

$$\bar{\psi} \sigma^{\mu\nu} \psi = \frac{i}{2} \zeta (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \eta + \frac{i}{2} \eta^\dagger (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \zeta^\dagger$$

$$\bar{\psi} \sigma^{\mu\nu} \psi = \frac{i}{2} (\eta \sigma^\mu \bar{\sigma}^\nu \eta - \eta \sigma^\nu \bar{\sigma}^\mu \eta) + \frac{i}{2} (\eta^\dagger \bar{\sigma}^\mu \sigma^\nu \eta^\dagger - \eta^\dagger \bar{\sigma}^\nu \sigma^\mu \eta^\dagger) = 0$$

这样就验证了 9.2.2 小节的结论

9.7 节 Majorana 旋量场相关 Feynman 规则

 7.1.1 小节提到, 由于 Dirac 旋量场可以携带某种 $U(1)$ 荷, 相应费米子线上的箭头代表 $U(1)$ 荷流动的方向, 或者说费米子数流动的方向

 另一方面, Majorana 旋量场不能携带任何 $U(1)$ 荷, 不存在费米子数流动的方向, 相应的费米子线则不应该具备箭头

 如果相互作用过程涉及到 Majorana 旋量场与 Dirac 旋量场的耦合, 带箭头与不带箭头的费米子线将在顶点处交汇, 导致费米子数破坏 (fermion-number violation)

 我们需要研究适用于这种情况的 Feynman 规则

 本节讨论一个简单例子, 更一般的情况可参考文献

- A. Denner, H. Eck, O. Hahn, and J. Kublbeck, "Feynman rules for fermion number violating interactions," Nucl. Phys. B 387 (1992) 467–481

9.7.1 小节 拉氏量和 CP 对称性

考虑复标量场 $\phi(x)$ 、Dirac 旋量场 $\psi(x)$ 和 Majorana 旋量场 $\chi(x)$ 构成拉氏量

$$\mathcal{L} = (\partial^\mu \phi^\dagger) \partial_\mu \phi - m_\phi^2 \phi^\dagger \phi + \bar{\psi} (i \gamma^\mu \partial_\mu - m_\psi) \psi + \frac{1}{2} \bar{\chi} (i \gamma^\mu \partial_\mu - m_\chi) \chi + \mathcal{L}_{\text{int}}$$

相互作用拉氏量为 $\mathcal{L}_{\text{int}} = \kappa \phi^\dagger \bar{\chi} P_R \psi + \kappa^* \phi \bar{\psi} P_L \chi$

⚠ κ 是一个复耦合常数, \mathcal{L}_{int} 是厄米的, 因为 \mathcal{L}_{int} 中两项互为厄米共轭,

$$(\kappa \phi^\dagger \bar{\chi} P_R \psi)^\dagger = \kappa^* \psi^\dagger P_R \gamma^0 \chi = \kappa^* \psi^\dagger \gamma^0 P_L \chi = \kappa^* \phi \bar{\psi} P_L \chi$$

这样的相互作用涉及一个标量场和两个旋量场，属于 Yukawa 相互作用

9.7.1 小节 拉氏量和 CP 对称性

考虑复标量场 $\phi(x)$ 、Dirac 旋量场 $\psi(x)$ 和 Majorana 旋量场 $\chi(x)$ 构成拉氏量

$$\mathcal{L} = (\partial^\mu \phi^\dagger) \partial_\mu \phi - m_\phi^2 \phi^\dagger \phi + \bar{\psi} (i \gamma^\mu \partial_\mu - m_\psi) \psi + \frac{1}{2} \bar{\chi} (i \gamma^\mu \partial_\mu - m_\chi) \chi + \mathcal{L}_{\text{int}}$$

相互作用拉氏量为 $\mathcal{L}_{\text{int}} = \kappa \phi^\dagger \bar{\chi} P_R \psi + \kappa^* \phi \bar{\psi} P_L \chi$

是复耦合常数， \mathcal{L}_{int} 是厄米的，因为 \mathcal{L}_{int} 中两项互为厄米共轭，

$$(\kappa \phi^\dagger \bar{\chi} P_R \psi)^\dagger = \kappa^* \psi^\dagger P_R \gamma^0 \chi = \kappa^* \psi^\dagger \gamma^0 P_L \chi = \kappa^* \phi \bar{\psi} P_L \chi$$

这样的相互作用涉及一个标量场和两个旋量场，属于 Yukawa 相互作用

作 $U(1)$ 整体变换 $\phi'(x) = e^{i\eta\theta}\phi(x)$ 和 $\psi'(x) = e^{i\eta\theta}\psi(x)$ ，则拉氏量 \mathcal{L} 不变

可见，这个理论具有一个 $U(1)$ 整体对称性，而复标量场 $\phi(x)$ 和 Dirac 旋量场 $\psi(x)$ 的 $U(1)$ 荷相同，均为 q

将耦合常数分解为实部和虚部, $\kappa = \kappa_R + i\kappa_I$, 则相互作用拉氏量化为

$$\mathcal{L}_{\text{int}} = \kappa_{\text{R}} (\phi^\dagger \bar{\chi} P_{\text{R}} \psi + \phi \bar{\psi} P_{\text{L}} \chi) + \kappa_{\text{I}} (\text{i} \phi^\dagger \bar{\chi} P_{\text{R}} \psi - \text{i} \phi \bar{\psi} P_{\text{L}} \chi)$$

C 破坏和 P 破坏

假设三个量子场的 C 、 P 变换为

$$C^{-1}\phi(x)C = \eta_C^* \phi^\dagger(x), \quad C^{-1}\psi(x)C = \zeta_C^* \mathcal{C}\bar{\psi}^T(x), \quad C^{-1}\chi(x)C = \tilde{\zeta}_C^* \chi(x)$$

$$P^{-1}\phi(x)P = \eta_P^*\phi(\mathcal{P}x), \quad P^{-1}\psi(x)P = \zeta_P^*\gamma^0\psi(\mathcal{P}x), \quad P^{-1}\chi(x)P = \tilde{\zeta}_P^*\gamma^0\chi(\mathcal{P}x)$$

推出算符 $\phi^\dagger \bar{\chi} P_R \psi$ 的 C 、 P 变换

$$C^{-1}\phi^\dagger(x)\bar{\chi}(x)P_{\mathbb{R}}\psi(x)C = \eta_C\zeta_C^*\tilde{\zeta}_C\phi(x)\bar{\psi}(x)P_{\mathbb{R}}\chi(x)$$

$$P^{-1}\phi^\dagger(x)\bar{\chi}(x)P_{\text{R}}\psi(x)P = \eta_P\zeta_P^*\tilde{\zeta}_P\phi^\dagger(\mathcal{P}x)\bar{\chi}(\mathcal{P}x)P_{\text{L}}\psi(\mathcal{P}x)$$

而算符 $\phi\bar{\psi}P_L\chi$ 的 C 、 P 变换为

$$C^{-1}\phi(x)\bar{\psi}(x)P_{\text{L}}\chi(x)C = \eta_C^*\zeta_C\tilde{\zeta}_C^*\phi^\dagger(x)\bar{\chi}(x)P_{\text{L}}\psi(x)$$

$$P^{-1}\phi(x)\bar{\psi}(x)P_L\chi(x)P = \eta_P^*\zeta_P\tilde{\zeta}_P^*\phi(\mathcal{P}x)\bar{\psi}(\mathcal{P}x)P_R\chi(\mathcal{P}x)$$

C 破坏和 P 破坏

假设三个量子场的 C 、 P 变换为

$$C^{-1}\phi(x)C = \eta_C^* \phi^\dagger(x), \quad C^{-1}\psi(x)C = \zeta_C^* \mathcal{C}\bar{\psi}^T(x), \quad C^{-1}\chi(x)C = \tilde{\zeta}_C^* \chi(x)$$

$$P^{-1}\phi(x)P = \eta_P^*\phi(\mathcal{P}x), \quad P^{-1}\psi(x)P = \zeta_P^*\gamma^0\psi(\mathcal{P}x), \quad P^{-1}\chi(x)P = \tilde{\zeta}_P^*\gamma^0\chi(\mathcal{P}x)$$

推出算符 $\phi^\dagger \bar{\chi} P_R \psi$ 的 C 、 P 变换

$$C^{-1}\phi^\dagger(x)\bar{\chi}(x)P_R\psi(x)C = \eta_C\zeta_C^*\tilde{\zeta}_C\phi(x)\bar{\psi}(x)P_R\chi(x)$$

$$P^{-1} \phi^\dagger(x) \bar{\chi}(x) P_{\text{R}} \psi(x) P = \eta_P \zeta_P^* \tilde{\zeta}_P \phi^\dagger(\mathcal{P}x) \bar{\chi}(\mathcal{P}x) P_{\text{L}} \psi(\mathcal{P}x)$$

而算符 $\phi\bar{\psi}P_L\chi$ 的 C 、 P 变换为

$$C^{-1}\phi(x)\bar{\psi}(x)P_{\text{L}}\chi(x)C = \eta_C^*\zeta_C\tilde{\zeta}_C^*\phi^\dagger(x)\bar{\chi}(x)P_{\text{L}}\psi(x)$$

$$P^{-1}\phi(x)\bar{\psi}(x)P_L\chi(x)P = \eta_P^*\zeta_P\tilde{\zeta}_P^*\phi(\mathcal{P}x)\bar{\psi}(\mathcal{P}x)P_R\chi(\mathcal{P}x)$$

无论作 C 变换还是 P 变换，相互作用拉氏量 $\mathcal{L}_{\text{int}} = \kappa \phi^\dagger \bar{\chi} P_R \psi + \kappa^* \phi \bar{\psi} P_L \chi$ 都不能保持不变，因此理论不具有电荷共轭对称性和空间反射对称性

换言之，这个理论既是 **C** 破坏 (*C*-violation) 的，又是 **P** 破坏 (*P*-violation) 的

CP 破坏?

进一步, 算符 $\phi^\dagger \bar{\chi} P_R \psi$ 和 $\phi \bar{\psi} P_L \chi$ 的 CP 变换为

$$(CP)^{-1} \phi^\dagger(x) \bar{\chi}(x) P_R \psi(x) CP = \eta_{CP} \phi(\mathcal{P}x) \bar{\psi}(\mathcal{P}x) P_L \chi(\mathcal{P}x)$$

$$(CP)^{-1} \phi(x) \bar{\psi}(x) P_L \chi(x) CP = \eta_{CP}^* \phi^\dagger(\mathcal{P}x) \bar{\chi}(\mathcal{P}x) P_R \psi(\mathcal{P}x)$$

其中 $\eta_{CP} \equiv \eta_C \eta_P^* \zeta_C^* \zeta_P \tilde{\zeta}_C \tilde{\zeta}_P^*$

9.1.1 小节末提到, 复场的分立变换相位因子的取值是任意的

如果适当选取 $\phi(x)$ 和 $\psi(x)$ 相位因子的值, 使得 $\eta_{CP} = \eta_{CP}^* = +1$

则算符 $\phi^\dagger \bar{\chi} P_R \psi + \phi \bar{\psi} P_L \chi$ 在 CP 变换下不变

而相互作用拉氏量 $\mathcal{L}_{\text{int}} = \kappa_R (\phi^\dagger \bar{\chi} P_R \psi + \phi \bar{\psi} P_L \chi) + \kappa_I (i\phi^\dagger \bar{\chi} P_R \psi - i\phi \bar{\psi} P_L \chi)$ 中 κ_R 对应的项具有 CP 对称性, κ_I 对应的项引起 CP 破坏 (CP -violation)

CP 破坏?

进一步, 算符 $\phi^\dagger \bar{\chi} P_R \psi$ 和 $\phi \bar{\psi} P_L \chi$ 的 CP 变换为

$$(CP)^{-1} \phi^\dagger(x) \bar{\chi}(x) P_R \psi(x) CP = \eta_{CP} \phi(\mathcal{P}x) \bar{\psi}(\mathcal{P}x) P_L \chi(\mathcal{P}x)$$

$$(CP)^{-1} \phi(x) \bar{\psi}(x) P_L \chi(x) CP = \eta_{CP}^* \phi^\dagger(\mathcal{P}x) \bar{\chi}(\mathcal{P}x) P_R \psi(\mathcal{P}x)$$

其中 $\eta_{CP} \equiv \eta_C \eta_P^* \zeta_C^* \zeta_P \tilde{\zeta}_C \tilde{\zeta}_P^*$

9.1.1 小节末提到, 复场的分立变换相位因子的取值是任意的

如果适当选取 $\phi(x)$ 和 $\psi(x)$ 相位因子的值, 使得 $\eta_{CP} = \eta_{CP}^* = +1$

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而相互作用拉氏量 $\mathcal{L}_{\text{int}} = \kappa_R (\phi^\dagger \bar{\chi} P_R \psi + \phi \bar{\psi} P_L \chi) + \kappa_I (i\phi^\dagger \bar{\chi} P_R \psi - i\phi \bar{\psi} P_L \chi)$ 中 κ_R 对应的项具有 CP 对称性, κ_I 对应的项引起 CP 破坏 (CP -violation)

如果相位因子的取值使得 $\eta_{CP} = \eta_{CP}^* = -1$

则算符 $i\phi^\dagger \bar{\chi} P_R \psi - i\phi \bar{\psi} P_L \chi$ 在 CP 变换下不变

而 κ_I 对应的项具有 CP 对称性, κ_R 对应的项引起 CP 破坏

因此, 当 $\kappa_R \neq 0$ 且 $\kappa_I \neq 0$ 时, 相互作用拉氏量 \mathcal{L}_{int} 看起来会破坏 CP 对称性

CP 对称性

不过, Dirac 旋量场 $\psi(x)$ 是 Hilbert 空间中的非自共轭算符, 它的相位具有任意性 ($\psi(x)|\Psi\rangle$ 与 $e^{-i\varphi}\psi(x)|\Psi\rangle$ 描述相同的量子态), 可用于吸收 $\kappa \equiv |\kappa|e^{-i\varphi}$ 的相位 φ

如果将 Dirac 旋量场重新定义为 $\psi'(x) = e^{-i\varphi}\psi(x)$ ，则 $\bar{\psi}'(x) = e^{i\varphi}\bar{\psi}(x)$ ，于是 $\mathcal{L}_{\text{int}} = |\kappa|e^{-i\varphi} \phi^\dagger \bar{\chi} P_R \psi + |\kappa|e^{i\varphi} \phi \bar{\psi} P_L \chi = |\kappa|(\phi^\dagger \bar{\chi} P_R \psi' + \phi \bar{\psi}' P_L \chi)$ 描述同一个理论

但此时耦合常数 $|\kappa|$ 是实数，不会引起 CP 破坏

! 因此, 这个理论实际上是具有 CP 对称性的

当理论中所有复耦合常数的相位不能完全被复场吸收时，才会出现 CP 破坏

CP 对称性

不过, Dirac 旋量场 $\psi(x)$ 是 Hilbert 空间中的非自共轭算符, 它的相位具有任意性 ($\psi(x)|\Psi\rangle$ 与 $e^{-i\varphi}\psi(x)|\Psi\rangle$ 描述相同的量子态), 可用于吸收 $\kappa \equiv |\kappa|e^{-i\varphi}$ 的相位 φ

如果将 Dirac 旋量场重新定义为 $\psi'(x) = e^{-i\varphi}\psi(x)$ ，则 $\bar{\psi}'(x) = e^{i\varphi}\bar{\psi}(x)$ ，于是 $\mathcal{L}_{\text{int}} = |\kappa|e^{-i\varphi}\phi^\dagger\bar{\chi}P_R\psi + |\kappa|e^{i\varphi}\phi\bar{\psi}P_L\chi = |\kappa|(\phi^\dagger\bar{\chi}P_R\psi' + \phi\bar{\psi}'P_L\chi)$ 描述同一个理论

但此时耦合常数 $|\kappa|$ 是实数，不会引起 CP 破坏

! 因此, 这个理论实际上是具有 CP 对称性的

当理论中所有复耦合常数的相位不能完全被复场吸收时，才会出现 CP 破坏

另一方面，像实标量场、实矢量场和 Majorana 旋量场这样的实场必须满足自共轭条件，这导致它不具有相位任意性

在下面的讨论中, 不失一般性, 将耦合常数 κ 取为实数, 相互作用拉氏量表达为

$$\mathcal{L}_{\text{int}} = \kappa(\phi^\dagger \bar{\chi} \Gamma_1 \psi + \phi \bar{\psi} \Gamma_2 \chi)$$

这里引入了 $\Gamma_1 = P_R$ 和 $\Gamma_2 = P_L$ ，下面许多结论与 Γ_1 和 Γ_2 的具体形式无关

9.7.2 小节 Feynman 规则

将 Dirac 旋量场、复标量场和 Majorana 旋量场的平面波展开式表达为

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda} [u(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x}]$$

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (c_{\mathbf{p}} e^{-ip \cdot x} + d_{\mathbf{p}}^\dagger e^{ip \cdot x})$$

$$\chi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda} [u(\mathbf{p}, \lambda) f_{\mathbf{p}, \lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) f_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x}]$$

相应地，引入以下单粒子态，

Dirac 正费米子 ψ 的单粒子态 $|p^+, \lambda\rangle = \sqrt{2E_p} a_{\mathbf{p}, \lambda}^\dagger |0\rangle$

Dirac 反费米子 $\bar{\psi}$ 的单粒子态 $|p^-, \lambda\rangle = \sqrt{2E_p} b_{\mathbf{p}, \lambda}^\dagger |0\rangle$

正标量玻色子 ϕ 的单粒子态 $|p^+\rangle = \sqrt{2E_p} c_{\mathbf{p}}^\dagger |0\rangle$

反标量玻色子 $\bar{\phi}$ 的单粒子态 $|p^-\rangle = \sqrt{2E_p} d_{\mathbf{p}}^\dagger |0\rangle$

Majorana 费米子 χ 的单粒子态 $|p, \lambda\rangle = \sqrt{2E_p} f_{\mathbf{p}, \lambda}^\dagger |0\rangle$

注意，Majorana 费米子 χ 是纯中性的，动量记号的右上角没有正负号

iT 算符 $n = 1$ 阶

 Dirac 旋量场和复标量场与初末态的缩并结果见第 7 章

Majorana 旋量场与初末态的缩并定义为

$$\langle 0 | \overline{\chi(x)} | \mathbf{p}, \lambda \rangle \equiv \langle 0 | \chi^{(+)}(x) | \mathbf{p}, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\langle 0 | \bar{\chi}(x) | \mathbf{p}, \lambda \rangle \equiv \langle 0 | \bar{\chi}^{(+)}(x) | \mathbf{p}, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\langle \bar{\mathbf{p}}, \lambda | \bar{\chi}(x) | 0 \rangle \equiv \langle \mathbf{p}, \lambda | \bar{\chi}^{(-)}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{i p \cdot x}$$

$$\langle \mathbf{p}, \lambda | \chi(x) | 0 \rangle \equiv \langle \mathbf{p}, \lambda | \chi^{(-)}(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{i p \cdot x}$$

iT 算符 $n = 1$ 阶

 Dirac 旋量场和复标量场与初末态的缩并结果见第 7 章

Majorana 旋量场与初末态的缩并定义为

$$\begin{aligned}
 \langle 0 | \overline{\chi}(x) | \mathbf{p}, \lambda \rangle &\equiv \langle 0 | \chi^{(+)}(x) | \mathbf{p}, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i p \cdot x} \\
 \langle 0 | \bar{\chi}(x) | \mathbf{p}, \lambda \rangle &\equiv \langle 0 | \bar{\chi}^{(+)}(x) | \mathbf{p}, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i p \cdot x} \\
 \langle \mathbf{p}, \lambda | \bar{\chi}(x) | 0 \rangle &\equiv \langle \mathbf{p}, \lambda | \bar{\chi}^{(-)}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{i p \cdot x} \\
 \langle \mathbf{p}, \lambda | \overline{\chi}(x) | 0 \rangle &\equiv \langle \mathbf{p}, \lambda | \chi^{(-)}(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{i p \cdot x}
 \end{aligned}$$

由于相互作用哈密顿量密度 $\mathcal{H}_1 = -\mathcal{L}_{\text{int}}$, iT 算符展开式中 $n = 1$ 的项为

$$\begin{aligned} iT^{(1)} &= -i \int d^4x \mathsf{T}[\mathcal{H}_1(x)] = i \int d^4x \mathsf{T}[\mathcal{L}_{\text{int}}(x)] \\ &= i\kappa \int d^4x \mathsf{T}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x) + \phi(x)\bar{\psi}(x)\Gamma_2\chi(x)] \end{aligned}$$

根据 Wick 定理, $iT^{(1)}$ 只包含下面两项,

$$iT_1^{(1)} = i\kappa \int d^4x \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)], \quad iT_2^{(1)} = i\kappa \int d^4x \mathsf{N}[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)]$$

$\psi \rightarrow \chi\phi$ 衰变过程

考慮 $\psi \rightarrow \chi\phi$ 衰變，初末態為 $|p^+, \lambda\rangle$ 和 $|q, \lambda'; k^+\rangle$ ， $iT_1^{(1)}$ 貢獻的 T 矩陣元是

$$\begin{aligned}
\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)] | \mathbf{p}^+, \lambda \rangle \\
&= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)] | \mathbf{p}^+, \lambda \rangle \\
&= i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
&= i\kappa \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p-q-k)
\end{aligned}$$

这是计算 T 矩阵元的**第一种方法**，与 7.1 节介绍的方法一样

$\psi \rightarrow \chi\phi$ 衰变过程

考虑 $\psi \rightarrow \chi\phi$ 衰变, 初末态为 $|p^+, \lambda\rangle$ 和 $|q, \lambda'; k^+\rangle$, $iT_1^{(1)}$ 贡献的 T 矩阵元是

$$\begin{aligned}
\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)] | \mathbf{p}^+, \lambda \rangle \\
&= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \overline{\mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)]} | \mathbf{p}^+, \lambda \rangle \\
&= i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
&= i\kappa \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p-q-k)
\end{aligned}$$

这是计算 T 矩阵元的**第一种方法**，与 7.1 节介绍的方法一样

利用电荷共轭变换，可以引进第二种计算方法

将相互作用算符 $\bar{\chi}\Gamma_1\psi$ 化为

$$\begin{aligned}\bar{\chi}\Gamma_1\psi &= (\bar{\chi}\Gamma_1\psi)^T = -\psi^T\Gamma_1^T\bar{\chi}^T = -\psi^T\mathcal{C}^{-1}\mathcal{C}\Gamma_1^T\mathcal{C}^{-1}\mathcal{C}\bar{\chi}^T \\ &= \psi^T\mathcal{C}\Gamma_1^T\mathcal{C}^{-1}\mathcal{C}\bar{\chi}^T = \bar{\psi}^C\Gamma_1^C\chi^C\end{aligned}$$

同理推出 $\bar{\psi}\Gamma_2\chi = \bar{\chi}^C\Gamma_2^C\psi^C$

第二种计算方法

老虎 通过 Majorana 条件 $\chi = \chi^c$ 将 $\bar{\chi}\Gamma_1\psi = \bar{\psi}^c\Gamma_1^c\chi^c$ 和 $\bar{\psi}\Gamma_2\chi = \bar{\chi}^c\Gamma_2^c\psi^c$ 化为

$$\bar{\chi}\Gamma_1\psi = \bar{\psi}^C\Gamma_1^C\chi, \quad \bar{\psi}\Gamma_2\chi = \bar{\chi}\Gamma_2^C\psi^C$$

从而将 $iT_1^{(1)}$ 和 $iT_2^{(1)}$ 改写为

$$i T_1^{(1)} = i \kappa \int d^4x \, \mathsf{N}[\phi^\dagger(x) \bar{\chi}(x) \Gamma_1 \psi(x)] = i \kappa \int d^4x \, \mathsf{N}[\phi^\dagger(x) \bar{\psi}^C(x) \Gamma_1^C \chi(x)]$$

$$iT_2^{(1)} = i\kappa \int d^4x \mathsf{N}[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)] = i\kappa \int d^4x \mathsf{N}[\phi(x)\bar{\chi}(x)\Gamma_2^C\psi^C(x)]$$



注意，此时旋量场算符排列的次序与原来相反



现在, $iT_1^{(1)}$ 贡献的 $\psi \rightarrow \chi\phi$ 过程 T 矩阵元也可以表达成

$$\begin{aligned}
& \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | i T_1^{(1)} | \mathbf{p}^+, \lambda \rangle \\
&= i \kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathbf{N}[\phi^\dagger(x) \bar{\psi}_a^C(x) (\Gamma_1^C)_{ab} \chi_b(x)] | \mathbf{p}^+, \lambda \rangle \\
&= -i \kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \phi^{\dagger(-)}(x) \chi_b^{(-)}(x) (\Gamma_1^C)_{ab} \bar{\psi}_a^C(x) | \mathbf{p}^+, \lambda \rangle
\end{aligned}$$

电荷共轭场 $\psi^c(x)$ 的平面波展开和初末态缩并

 Dirac 旋量场 $\psi(x)$ 的电荷共轭场 $\psi^C(x)$ 的平面波展开式是

$$\begin{aligned}\psi^C(x) &= \mathcal{C}\bar{\psi}^T = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda} \left[\mathcal{C}\bar{v}^T(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} + \mathcal{C}\bar{u}^T(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^{\dagger} e^{ip \cdot x} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda} [u(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^{\dagger} e^{ip \cdot x}]\end{aligned}$$

跟 $\psi(x)$ 展开式的差异只在于 a 与 b 互换, 相应 Dirac 共轭的展开式为

$$\bar{\psi}^C(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda} [\bar{u}(\mathbf{p}, \lambda) \textcolor{teal}{b}_{\mathbf{p}, \lambda}^{\dagger} e^{ip \cdot x} + \bar{v}(\mathbf{p}, \lambda) \textcolor{blue}{a}_{\mathbf{p}, \lambda} e^{-ip \cdot x}]$$

据此, 将电荷共轭场 $\psi^C(x)$ 和 $\bar{\psi}^C(x)$ 与初末态的缩并定义成

$$\langle 0 | \psi^C(x) | \mathbf{p}^-, \lambda \rangle \equiv \langle 0 | \psi^{C(+)}(x) | \mathbf{p}^-, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\langle 0 | \bar{\psi}^C(x) | \mathbf{p}^+, \lambda \rangle \equiv \langle 0 | \bar{\psi}^{C(+)}(x) | \mathbf{p}^+, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i p \cdot x}$$

$$\langle \left[\mathbf{p}^-, \lambda \right] \bar{\psi}^C(x) | 0 \rangle \equiv \langle \mathbf{p}^-, \lambda | \bar{\psi}^{C(-)}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{i \mathbf{p} \cdot \mathbf{x}}$$

$$\langle \overline{\mathbf{p}^+, \lambda} | \psi^C(x) | 0 \rangle \equiv \langle \mathbf{p}^+, \lambda | \psi^{C(-)}(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{i \mathbf{p} \cdot \mathbf{x}}$$

第二种方法的计算结果

由 $\psi \rightarrow \chi\phi$ 的 T 矩阵元变成

$$\begin{aligned}
\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= -i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathbf{N}[\phi^\dagger(x) \chi_b(x) (\Gamma_1^C)_{ab} \bar{\psi}_a^C(x)] | \mathbf{p}^+, \lambda \rangle \\
&= -i\kappa \int d^4x v_b(\mathbf{q}, \lambda') (\Gamma_1^C)_{ab} \bar{v}_a(\mathbf{p}, \lambda) e^{-i(p-q-k) \cdot x} \\
&= -i\kappa \int d^4x \bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') e^{-i(p-q-k) \cdot x} \\
&= -i\kappa \bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(p-q-k) \\
&= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathbf{N}[\phi^\dagger(x) \bar{\psi}_a^C(x) \Gamma_1^C \chi_b(x)] | \mathbf{p}^+, \lambda \rangle
\end{aligned}$$

第二种方法的计算结果



$\psi \rightarrow \chi\phi$ 的 T 矩阵元变成

$$\begin{aligned}
 \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= -i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \overline{N}[\phi^\dagger(x)\chi_b(x)(\Gamma_1^C)_{ab}\bar{\psi}_a^C(x)] | \mathbf{p}^+, \lambda \rangle \\
 &= -i\kappa \int d^4x v_b(\mathbf{q}, \lambda') (\Gamma_1^C)_{ab} \bar{v}_a(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
 &= -i\kappa \int d^4x \bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') e^{-i(p-q-k)\cdot x} \\
 &= -i\kappa \bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(p - q - k) \\
 &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \overline{N}[\phi^\dagger(x)\bar{\psi}^C(x)\Gamma_1^C\chi(x)] | \mathbf{p}^+, \lambda \rangle
 \end{aligned}$$

倒数第二行是第二种方法的计算结果，有

$$\begin{aligned}
 -\bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') &= -u^T(\mathbf{p}, \lambda) \mathcal{C} \Gamma_1^C \mathcal{C} \bar{u}^T(\mathbf{q}, \lambda') = u^T(\mathbf{p}, \lambda) \mathcal{C} \mathcal{C}^{-1} \Gamma_1^T \mathcal{C} \mathcal{C}^{-1} \bar{u}^T(\mathbf{q}, \lambda') \\
 &= [u^T(\mathbf{p}, \lambda) \Gamma_1^T \bar{u}^T(\mathbf{q}, \lambda')]^T = \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda)
 \end{aligned}$$

第二种方法结果与第一种方法结果 $i\kappa \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p - q - k)$ 相等

$\bar{\psi} \rightarrow \chi \bar{\phi}$ 衰变过程：第一种方法

🐒 另一方面，考虑 $\bar{\psi} \rightarrow \chi \bar{\phi}$ 衰变过程，初态为 $|\mathbf{p}^-, \lambda\rangle$ ，末态为 $|\mathbf{q}, \lambda'; \mathbf{k}^-\rangle$

🌰 根据 $iT_2^{(1)} = i\kappa \int d^4x N[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)]$ 按第一种方法计算

ट्रॅट $iT_2^{(1)}$ 贡献的 T 矩阵元是

$$\begin{aligned}
 \langle \mathbf{q}, \lambda'; \mathbf{k}^- | iT_2^{(1)} | \mathbf{p}^-, \lambda \rangle &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^- | N[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)] | \mathbf{p}^-, \lambda \rangle \\
 &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^- | \overline{N[\phi(x)\bar{\psi}_a(x)(\Gamma_2)_{ab}\chi_b(x)]} | \mathbf{p}^-, \lambda \rangle \\
 &= -i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^- | \overline{N[\phi(x)\chi_b(x)(\Gamma_2)_{ab}\bar{\psi}_a(x)]} | \mathbf{p}^-, \lambda \rangle \\
 &= -i\kappa \int d^4x v_b(\mathbf{q}, \lambda') (\Gamma_2)_{ab} \bar{v}_a(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
 &= -i\kappa \bar{v}(\mathbf{p}, \lambda) \Gamma_2 v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(p-q-k)
 \end{aligned}$$

$\bar{\psi} \rightarrow \chi \bar{\phi}$ 衰变过程：第二种方法

根据 $iT_2^{(1)} = i\kappa \int d^4x N[\phi(x)\bar{\chi}(x)\Gamma_2^C \psi^C(x)]$ 按**第二种方法**计算

 $iT_2^{(1)}$ 贡献的 T 矩阵元为

$$\begin{aligned}
\langle \mathbf{q}, \lambda'; \mathbf{k}^- | iT_2^{(1)} | \mathbf{p}^-, \lambda \rangle &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^- | \mathsf{N}[\phi(x)\bar{\chi}(x)\Gamma_2^C\psi^C(x)] | \mathbf{p}^-, \lambda \rangle \\
&= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^- | \overline{\mathsf{N}[\phi(x)\bar{\chi}(x)\Gamma_2^C\psi^C(x)]} | \mathbf{p}^-, \lambda \rangle \\
&= i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda') \Gamma_2^C u(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
&= i\kappa \bar{u}(\mathbf{q}, \lambda') \Gamma_2^C u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p-q-k)
\end{aligned}$$

由于

$$\begin{aligned}\bar{u}(\mathbf{q}, \lambda') \Gamma_2^C u(\mathbf{p}, \lambda) &= v^T(\mathbf{q}, \lambda') \mathcal{C} \Gamma_2^C \mathcal{C} \bar{v}^T(\mathbf{p}, \lambda) = -v^T(\mathbf{q}, \lambda') \mathcal{C} \mathcal{C}^{-1} \Gamma_2^T \mathcal{C} \mathcal{C}^{-1} \bar{v}^T(\mathbf{p}, \lambda) \\ &= -[v^T(\mathbf{q}, \lambda') \Gamma_2^T \bar{v}^T(\mathbf{p}, \lambda)]^T = -\bar{v}(\mathbf{p}, \lambda) \Gamma_2 v(\mathbf{q}, \lambda')\end{aligned}$$

两种方法的计算结果相等

费米子流方向

-  以上计算表明，这两种方法都是有效的，在实际计算中可采用任意一种方法
-  现在需要归纳出一套与这两种方法同时相容的 Feynman 规则，这样的规则将特别适用于处理费米子数破坏过程
-  为此，在每条连续费米子线附近添加一条带箭头的点划线，表示费米子流 (fermion flow) 的方向
-  费米子流的两种方向分别对应于上述两种计算方法

费米子流方向

以上计算表明，这两种方法都是有效的，在实际计算中可采用任意一种方法。

现在需要归纳出一套与这两种方法同时相容的 **Feynman 规则**，这样的规则将特别适用于处理 **费米子数破坏** 过程

为此，在每条连续费米子线附近添加一条带箭头的点划线，表示费米子流 (fermion flow) 的方向

费米子流的两种方向分别对应于上述两种计算方法

当费米子流方向与 Dirac 费米子线上箭头方向相同时, 采用第一种计算方法

当费米子流方向与 Dirac 费米子线上箭头方向相反时，采用与电荷共轭场有关的第二种计算方法

这样一来，两种费米子流方向是等价的，对每条连续费米子线可采取任意一种方向进行计算

位置空间外线规则

于是, 位置空间中费米子的外线规则如下, 带箭头的点划线表示费米子流方向

① Dirac 正费米子 ψ 入射光线:

$$\psi, \lambda \xrightarrow{p} x = \langle 0 | \psi(x) | \mathbf{p}^+, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\psi, \lambda \xrightarrow[p]{\bullet} x = \langle 0 | \overline{\psi^C(x)} | \mathbf{p}^+, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

② Dirac 反费米子 $\bar{\psi}$ 入射光线:

$$\bar{\psi}, \lambda \xrightarrow[p]{\quad} \bullet x = \langle 0 | \bar{\psi}(x) | \bar{\mathbf{p}}^-, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\bar{\psi}, \lambda \xrightarrow[p]{\quad} \bullet x = \langle 0 | \overline{\psi^C(x)} | \mathbf{p}^- \rangle, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

③ Dirac 正费米子 ψ 出射光线:

$$x \bullet \xrightarrow{p} \psi, \lambda = \langle \overline{\mathbf{p}^+}, \lambda | \bar{\psi}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{ip \cdot x}$$

$$x \xrightarrow[p]{} \psi, \lambda = \langle \overline{\mathbf{p}^+, \lambda} | \psi^C(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{ip \cdot x}$$

位置空间外线规则

④ Dirac 反费米子 $\bar{\psi}$ 出射光线:

$$x \bullet \overbrace{\psi, \lambda}^{\langle \mathbf{p}^-, \lambda | \psi(x) | 0 \rangle} = v(\mathbf{p}, \lambda) e^{i \mathbf{p} \cdot x}$$

$$x \bullet \overbrace{\psi, \lambda}^p = \langle \overbrace{\mathbf{p}^-, \lambda}^p | \bar{\psi}^C(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{ip \cdot x}$$

⑤ Majorana 费米子 χ 入射光线:

$$\chi, \lambda \xrightarrow{\quad p \quad} \bullet x = \langle 0 | \overline{\chi(x)} | \mathbf{p}, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\chi, \lambda \xrightarrow[p]{\quad} \bullet x = \langle 0 | \bar{\chi}(x) | \mathbf{p}, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

6 Majorana 费米子 χ 出射光线:

$$x \xrightarrow{p} \chi, \lambda = \langle \bar{\chi}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{i \mathbf{p} \cdot x}$$

$$x \xrightarrow{p} \chi, \lambda = \langle \mathbf{p}, \lambda | \chi(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{i p \cdot x}$$

没有箭头, Feynman 规则
依赖于费米子流方向
与动量方向之间的异同

从每条连续费米子线
写出散射振幅时, 总是逆
着用点划线表示的费米子
流方向逐项写下费米子的
贡献

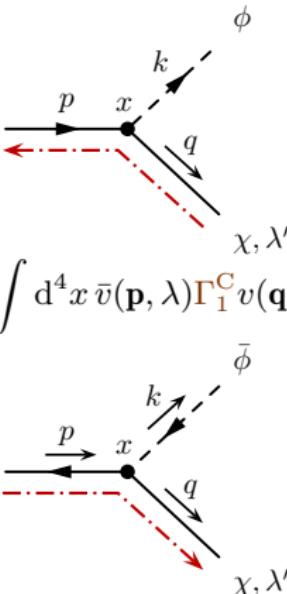
第一种方法 Feynman 图

对于上述 $\psi \rightarrow \chi\phi$ 和 $\bar{\psi} \rightarrow \chi\bar{\phi}$ 过程，第一种计算方法对应于

$$\begin{aligned}
 \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | i T_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= \psi, \lambda \begin{array}{c} p \\ \nearrow x \\ q \end{array} \phi \chi, \lambda' \\
 &= i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda') \mathbf{\Gamma}_1 u(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x}
 \end{aligned}$$

第二种方法 Feynman 图

第二种计算方法对应于

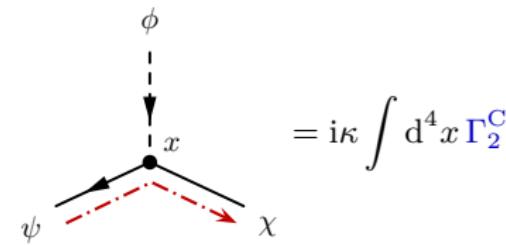
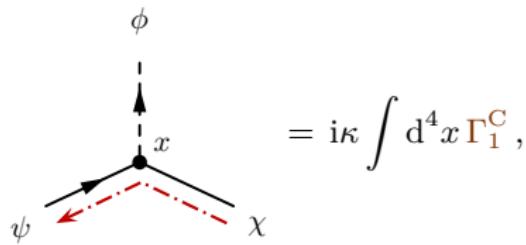
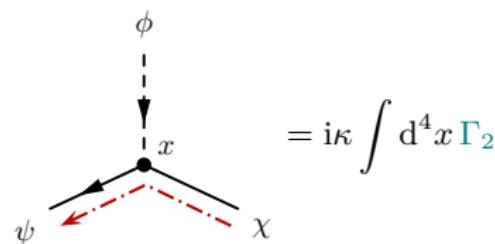
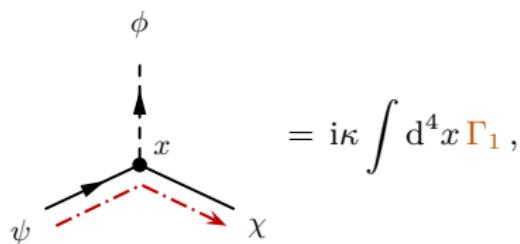
$$\begin{aligned}
 \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= \psi, \lambda \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\
 &= -i\kappa \int d^4x \bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') e^{-i(p-q-k) \cdot x} \\
 \langle \mathbf{q}, \lambda'; \mathbf{k}^- | iT_2^{(1)} | \mathbf{p}^-, \lambda \rangle &= \bar{\psi}, \lambda \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\
 &= i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda') \Gamma_2^C u(\mathbf{p}, \lambda) e^{-i(p-q-k) \cdot x}
 \end{aligned}$$


两种方法在 Feynman 图上的差异只是费米子流方向不同，即点划线箭头方向不同

额外的负号来自两个费米子场算符的交换

位置空间顶点规则

hog 观察各个 Feynman 图元素与振幅表达式的关系，归纳出**位置空间**中的**顶点规则**



这里**实线**和**虚线**上的**箭头**表征着 U(1) 荷流动的方向，U(1) 荷仍然是连续流动的

Dirac 旋量场的 Feynman 传播子

研究 $iT^{(2)}$ 的 T 矩阵元时可能遇到像 $N[\bar{\chi}(y)\Gamma_1 \overline{\psi(y)\psi(x)}\Gamma_2\chi(x)]$ 这样的表达式

如果采用第一种方法进行计算，则 Dirac 旋量场的 Feynman 传播子在位置空间中的 Feynman 规则与 7.1.1 小节规则类似，表达为

$$x \xrightarrow[p]{\quad} y = \overline{\psi(y)\psi(x)} = S_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(p + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$

Dirac 旋量场的 Feynman 传播子

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$$x \bullet \xrightarrow{p} \bullet y = \overline{\psi(y)\psi}(x) = S_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$

由 $\bar{x}\Gamma_1\psi = \bar{\psi}^C\Gamma_1^C\chi$ 和 $\bar{\psi}\Gamma_3\chi = \bar{x}\Gamma_2^C\psi^C$ 推出

$$\begin{aligned} \mathsf{N}[\bar{\chi}(y)\Gamma_1 \overline{\psi(y)\psi}(x)\Gamma_2\chi(x)] &= \mathsf{N}[\overline{\psi^C(y)\Gamma_1^C\chi(y)}\bar{\chi}(x)\Gamma_2^C\psi^C(x)] \\ &= \mathsf{N}[\bar{\chi}(x)\Gamma_2^C\overline{\psi^C(x)}\bar{\psi}^C(y)\Gamma_1^C\chi(y)] \end{aligned}$$

如果采用第二种方法进行计算，则相应的 Feynman 传播子是

$$x \xrightarrow[p]{\quad} y = \overline{\psi^C(x)} \bar{\psi}^C(y) = \langle 0 | \mathsf{T}[\psi^C(x) \bar{\psi}^C(y)] | 0 \rangle = \langle 0 | \mathsf{T}[\mathcal{C} \bar{\psi}^T(x) \psi^T(y) \mathcal{C}] | 0 \rangle$$

Majorana 旋量场的 Feynman 传播子

进一步计算得到

$$\begin{aligned}
 x \bullet \xrightarrow[p]{\quad} \bullet y &= \overbrace{\psi^C(x) \bar{\psi}^C(y)} = \langle 0 | T[\mathcal{C} \bar{\psi}^T(x) \psi^T(y) \mathcal{C}] | 0 \rangle \\
 &= -\mathcal{C} \{ \langle 0 | T[\psi(y) \bar{\psi}(x)] | 0 \rangle \}^T \mathcal{C} = \mathcal{C}^{-1} \overbrace{[\psi(y) \bar{\psi}(x)]^T} \mathcal{C} \\
 &= \mathcal{C}^{-1} S_F^T(y-x) \mathcal{C} = \int \frac{d^4 p}{(2\pi)^4} \frac{\mathcal{C}^{-1} i(\not{p} + m_\psi)^T \mathcal{C}}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)} \\
 &= \int \frac{d^4 p}{(2\pi)^4} \frac{i(-\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}
 \end{aligned}$$

最后一步用到 $\mathcal{C}^{-1}(\gamma^\mu)^T \mathcal{C} = -\gamma^\mu$

Majorana 旋量场的 Feynman 传播子

进一步计算得到

$$\begin{aligned}
 x \bullet \xrightarrow{p} \bullet y &= \overline{\psi^C(x)\psi^C(y)} = \langle 0 | \mathcal{T}[\mathcal{C}\bar{\psi}^T(x)\psi^T(y)\mathcal{C}] | 0 \rangle \\
 &= -\mathcal{C}\{\langle 0 | \mathcal{T}[\psi(y)\bar{\psi}(x)] | 0 \rangle\}^T \mathcal{C} = \mathcal{C}^{-1} \overline{[\psi(y)\bar{\psi}(x)]^T} \mathcal{C} \\
 &= \mathcal{C}^{-1} S_F^T(y-x) \mathcal{C} = \int \frac{d^4 p}{(2\pi)^4} \frac{\mathcal{C}^{-1} i(\not{p} + m_\psi)^T \mathcal{C}}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)} \\
 &= \int \frac{d^4 p}{(2\pi)^4} \frac{i(-\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}
 \end{aligned}$$

最后一步用到 $\mathcal{C}^{-1}(\gamma^\mu)^T \mathcal{C} = -\gamma^\mu$

另一方面, Majorana 旋量场的 Feynman 传播子为

$$x \bullet \xrightarrow{p} \bullet y = \overline{\chi(y)\bar{\chi}(x)} = S_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m_\chi)}{p^2 - m_\chi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$

动量空间 Feynman 规则

转换到动量空间，推出以下 Feynman 规则

- ① Dirac 正费米子 ψ 入射外线: $\psi, \lambda \xrightarrow[p]{\quad} \bullet = u(\mathbf{p}, \lambda), \quad \psi, \lambda \xleftarrow[p]{\quad} \bullet = \bar{v}(\mathbf{p}, \lambda)$

② Dirac 反费米子 $\bar{\psi}$ 入射外线: $\bar{\psi}, \lambda \xrightarrow[p]{\quad} \bullet = \bar{v}(\mathbf{p}, \lambda), \quad \bar{\psi}, \lambda \xleftarrow[p]{\quad} \bullet = u(\mathbf{p}, \lambda)$

③ Dirac 正费米子 ψ 出射外线: $\bullet \xrightarrow[p]{\quad} \psi, \lambda = \bar{u}(\mathbf{p}, \lambda), \quad \bullet \xleftarrow[p]{\quad} \psi, \lambda = v(\mathbf{p}, \lambda)$

④ Dirac 反费米子 $\bar{\psi}$ 出射外线: $\bullet \xrightarrow[p]{\quad} \bar{\psi}, \lambda = v(\mathbf{p}, \lambda), \quad \bullet \xleftarrow[p]{\quad} \bar{\psi}, \lambda = \bar{u}(\mathbf{p}, \lambda)$

⑤ Majorana 费米子 χ 入射外线: $\chi, \lambda \xrightarrow[p]{\quad} \bullet = u(\mathbf{p}, \lambda), \quad \chi, \lambda \xleftarrow[p]{\quad} \bullet = \bar{v}(\mathbf{p}, \lambda)$

⑥ Majorana 费米子 χ 出射外线: $\bullet \xrightarrow[p]{\quad} \chi, \lambda = \bar{u}(\mathbf{p}, \lambda), \quad \bullet \xleftarrow[p]{\quad} \chi, \lambda = v(\mathbf{p}, \lambda)$

动量空间 Feynman 规则

7 Dirac 费米子传播子:

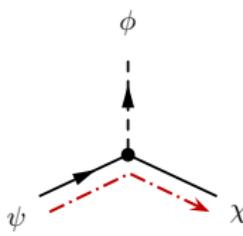
$$\text{---} \overset{p}{\bullet} \text{---} \bullet = \frac{i(p + m_\psi)}{p^2 - m_\psi^2 + i\epsilon}$$

$$\begin{array}{c} p \\ \leftarrow \end{array} = \frac{i(-p + m_\psi)}{p^2 - m_\psi^2 + i\epsilon}$$

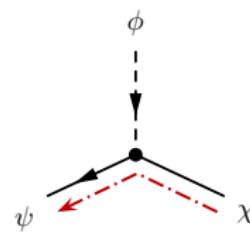
8 Majorana 费米子传播子

$$\text{子: } \frac{p}{p^2 - m_\chi^2 + i\epsilon} = \frac{i(p + m_\chi)}{p^2 - m_\chi^2 + i\epsilon}$$

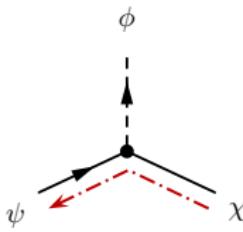
9 Yukawa 相互作用顶点:



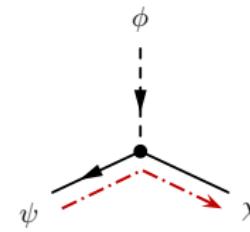
$$= i\kappa \Gamma_1,$$



$$= i\kappa \Gamma_2$$



$$= i\kappa \Gamma_1^C$$



$$= i\kappa \Gamma_2^C$$

Majorana 旋量场与对称性因子



注意, Majorana 费米子是纯中性粒子

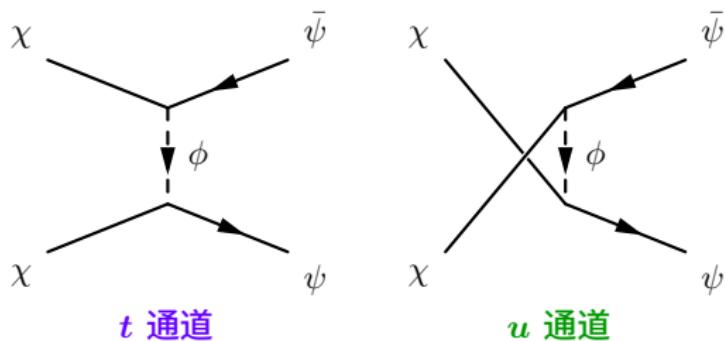
- 🚗 如果末态包含超过 1 个全同的 Majorana 费米子
- 🚙 计算散射截面或衰变宽度时需要考虑末态对称性因子 S
- 🚐 假如拉氏量的某个相互作用项包含 2 个或以上全同的 Majorana 旋量场
- 🚚 类似于 7.3 节的讨论, 在导出顶点 Feynman 规则时需要考虑组合因子
- 🚛 计算时还需要留意 Feynman 图的对称性因子

9.7.3 小节 应用

下面应用上一小节推导出来的 Feynman 规则进行计算

考虑 $\chi\chi \rightarrow \psi\bar{\psi}$ 湮灭过程

领头阶 Feynman 图如下图所示，包含一个 t 通道和一个 u 通道的 Feynman 图



现在，费米子流方向有多种取法，但各种取法的计算结果应该是等价的

在画出拓扑不等价的 Feynman 图时，不需要考虑费米子流的方向

费米子流方向第一种取法

设初态两个 Majorana 费米子 χ 的四维动量为 k_1^μ 和 k_2^μ ，末态 Dirac 费米子 ψ 和 $\bar{\psi}$ 的四维动量为 p_1^μ 和 p_2^μ ，令 $t = (k_1 - p_1)^2$ ， $u = (k_1 - p_2)^2$

添加带箭头的点划线表示费米子流方向

应用动量空间 Feynman 规则， t 通道和 u 通道 Feynman 图贡献的不变振幅是

$$\begin{aligned}
 i\mathcal{M}_t &= \begin{array}{c} \text{Diagram: Two Majorana } \chi \text{ lines (dashed)} \rightarrow \text{Dirac } \psi, \bar{\psi} \text{ lines (solid)} \rightarrow. \\ \text{Momentum flow: } k_2 \rightarrow p_2, k_1 \rightarrow p_1. \\ \text{Feynman rule: } \bar{u}(p_1)(i\kappa\Gamma_2)u(k_1) \frac{i}{(p_1 - k_1)^2 - m_\phi^2} \bar{v}(k_2)(i\kappa\Gamma_1)v(p_2) \end{array} \\
 &= -\frac{i\kappa^2}{t - m_\phi^2} \bar{u}(p_1)\Gamma_2 u(k_1)\bar{v}(k_2)\Gamma_1 v(p_2) \\
 i\mathcal{M}_u &= \begin{array}{c} \text{Diagram: Two Majorana } \chi \text{ lines (dashed)} \rightarrow \text{Dirac } \psi, \bar{\psi} \text{ lines (solid)} \rightarrow. \\ \text{Momentum flow: } k_2 \rightarrow p_2, k_1 \rightarrow k_1 - p_2. \\ \text{Feynman rule: } \bar{v}(k_1)(i\kappa\Gamma_1)v(p_2) \frac{i}{(k_1 - p_2)^2 - m_\phi^2} \bar{u}(p_1)(i\kappa\Gamma_2)u(k_2) \end{array} \\
 &= -\frac{i\kappa^2}{u - m_\phi^2} \bar{v}(k_1)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_2)
 \end{aligned}$$

第一种取法的相对符号

根据

$$\begin{aligned}
 & \langle \mathbf{p}_1^+; \mathbf{p}_2^- | N[\phi(x)\bar{\psi}_a(x)(\Gamma_2)_{ab}\chi_b(x)\phi^\dagger(y)\bar{\chi}_c(y)(\Gamma_1)_{cd}\psi_d(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 & + \langle \mathbf{p}_1^+; \mathbf{p}_2^- | N[\phi(x)\bar{\psi}_a(x)(\Gamma_2)_{ab}\chi_b(x)\phi^\dagger(y)\bar{\chi}_c(y)(\Gamma_1)_{cd}\psi_d(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 = & \langle \mathbf{p}_1^+; \mathbf{p}_2^- | N[\psi_d(y)\bar{\psi}_a(x)(\Gamma_2)_{ab}\phi(x)\phi^\dagger(y)(\Gamma_1)_{cd}\bar{\chi}_c(y)\chi_b(x)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 - & \langle \mathbf{p}_1^+; \mathbf{p}_2^- | N[\psi_d(y)\bar{\psi}_a(x)(\Gamma_2)_{ab}\phi(x)\phi^\dagger(y)(\Gamma_1)_{cd}\chi_b(x)\bar{\chi}_c(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle
 \end{aligned}$$

这两个 Feynman 图的相对符号为负

因而总振幅是 $i\mathcal{M} = i\mathcal{M}_t - i\mathcal{M}_u$

费米子流方向第二种取法

当然，也可以选择**其它费米子流方向**进行计算

比如，同时**反转**上述 t 通道 Feynman 图中**两条点划线的方向**，则 t 通道振幅变成

$$\begin{aligned}
 i\tilde{\mathcal{M}}_t = & \begin{array}{c} \text{Diagram with two dashed lines reversed} \\ \text{Top vertex: } \chi \rightarrow k_2, \bar{\psi} \rightarrow p_2 \\ \text{Bottom vertex: } \chi \rightarrow k_1, \psi \rightarrow p_1 \\ \text{Momentum flow: } k_2 \rightarrow p_2, k_1 \rightarrow p_1 \end{array} = \bar{v}(k_1)(i\kappa\Gamma_2^C)v(p_1) \frac{i}{(p_1 - k_1)^2 - m_\phi^2} \bar{u}(p_2)(i\kappa\Gamma_1^C)u(k_2) \\
 & = -\frac{i\kappa^2}{t - m_\phi^2} \bar{v}(k_1)\Gamma_2^C v(p_1) \bar{u}(p_2)\Gamma_1^C u(k_2)
 \end{aligned}$$

反转上述 u 通道 Feynman 图中**一条点划线的方向**， u 通道振幅化为

$$\begin{aligned}
 i\tilde{\mathcal{M}}_u = & \begin{array}{c} \text{Diagram with one dashed line reversed} \\ \text{Top vertex: } \chi \rightarrow k_2, \bar{\psi} \rightarrow p_2 \\ \text{Bottom vertex: } \chi \rightarrow k_1, \psi \rightarrow p_1 \\ \text{Momentum flow: } k_2 \rightarrow p_2, k_1 \rightarrow p_1 \end{array} = \bar{v}(k_1)(i\kappa\Gamma_1)v(p_2) \frac{i}{(k_1 - p_2)^2 - m_\phi^2} \bar{v}(k_2)(i\kappa\Gamma_2^C)v(p_1) \\
 & = -\frac{i\kappa^2}{u - m_\phi^2} \bar{v}(k_1)\Gamma_1 v(p_2) \bar{v}(k_2)\Gamma_2^C v(p_1)
 \end{aligned}$$

第二种取法的相对符号

👉 根据

$$\begin{aligned}
 & \langle \mathbf{p}_1^+; \mathbf{p}_2^- | \mathbf{N}[\phi(x)\bar{\chi}_a(x)(\Gamma_2^C)_{ab}\psi_b^C(x)\phi^\dagger(y)\bar{\psi}_c^C(y)(\Gamma_1^C)_{cd}\chi_d(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 & + \langle \mathbf{p}_1^+; \mathbf{p}_2^- | \mathbf{N}[\phi(x)\bar{\chi}_a(x)(\Gamma_2^C)_{ab}\psi_b^C(x)\phi^\dagger(y)\bar{\chi}_c(y)(\Gamma_1)_{cd}\psi_d(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 = & \langle \mathbf{p}_1^+; \mathbf{p}_2^- | \mathbf{N}[\bar{\psi}_c^C(y)\psi_b^C(x)(\Gamma_2^C)_{ab}\phi(x)\phi^\dagger(y)(\Gamma_1^C)_{cd}\chi_d(y)\bar{\chi}_a(x)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 & + \langle \mathbf{p}_1^+; \mathbf{p}_2^- | \mathbf{N}[\psi_d(y)\psi_b^C(x)(\Gamma_2^C)_{ab}\phi(x)\phi^\dagger(y)\bar{\chi}_a(x)\bar{\chi}_c(y)(\Gamma_1)_{cd}] | \mathbf{k}_1; \mathbf{k}_2 \rangle
 \end{aligned}$$

👉 这两个 Feynman 图的相对符号为正

👉 因而总振幅是 $i\tilde{\mathcal{M}} = i\tilde{\mathcal{M}}_t + i\tilde{\mathcal{M}}_u$

两种取法的等价性

$$\begin{aligned}
\bar{v}(k_1)\Gamma_2^C v(p_1)\bar{u}(p_2)\Gamma_1^C u(k_2) &= u^T(k_1)\mathcal{C}\mathcal{C}^{-1}\Gamma_2^T\mathcal{C}\mathcal{C}\bar{u}^T(p_1)v^T(p_2)\mathcal{C}\mathcal{C}^{-1}\Gamma_1^T\mathcal{C}\mathcal{C}\bar{v}^T(k_2) \\
&= [u^T(k_1)\Gamma_2^T\bar{u}^T(p_1)v^T(p_2)\Gamma_1^T\bar{v}^T(k_2)]^T \\
&= \bar{v}(k_2)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_1) \\
\bar{v}(k_1)\Gamma_1 v(p_2)\bar{v}(k_2)\Gamma_2^C v(p_1) &= \bar{v}(k_1)\Gamma_1 v(p_2)u^T(k_2)\mathcal{C}\mathcal{C}^{-1}\Gamma_2^T\mathcal{C}\mathcal{C}\bar{u}^T(p_1) \\
&= -\bar{v}(k_1)\Gamma_1 v(p_2)[u^T(k_2)\Gamma_2^T\bar{u}^T(p_1)]^T \\
&= -\bar{v}(k_1)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_2)
\end{aligned}$$

$$\begin{aligned}
 \text{[Right Arrow]} \quad \text{i} \tilde{\mathcal{M}}_t &= -\frac{\text{i} \kappa^2}{t - m_\phi^2} \bar{v}(k_1) \Gamma_2^C v(p_1) \bar{u}(p_2) \Gamma_1^C u(k_2) \\
 &= -\frac{\text{i} \kappa^2}{t - m_\phi^2} \bar{u}(p_1) \Gamma_2 u(k_1) \bar{v}(k_2) \Gamma_1 v(p_2) = \text{i} \mathcal{M}_t \\
 \text{i} \tilde{\mathcal{M}}_u &= -\frac{\text{i} \kappa^2}{u - m_\phi^2} \bar{v}(k_1) \Gamma_1 v(p_2) \bar{v}(k_2) \Gamma_2^C v(p_1) \\
 &\equiv +\frac{\text{i} \kappa^2}{u - m_\phi^2} \bar{v}(k_1) \Gamma_1 v(p_2) \bar{u}(p_1) \Gamma_2 u(k_2) = -\text{i} \mathcal{M}_u
 \end{aligned}$$

两种取法的等价性

$$\begin{aligned}
\bar{v}(k_1)\Gamma_2^C v(p_1)\bar{u}(p_2)\Gamma_1^C u(k_2) &= u^T(k_1)\mathcal{C}\mathcal{C}^{-1}\Gamma_2^T\mathcal{C}\mathcal{C}\bar{u}^T(p_1)v^T(p_2)\mathcal{C}\mathcal{C}^{-1}\Gamma_1^T\mathcal{C}\mathcal{C}\bar{v}^T(k_2) \\
&= [u^T(k_1)\Gamma_2^T\bar{u}^T(p_1)v^T(p_2)\Gamma_1^T\bar{v}^T(k_2)]^T \\
&= \bar{v}(k_2)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_1) \\
\bar{v}(k_1)\Gamma_1 v(p_2)\bar{v}(k_2)\Gamma_2^C v(p_1) &= \bar{v}(k_1)\Gamma_1 v(p_2)u^T(k_2)\mathcal{C}\mathcal{C}^{-1}\Gamma_2^T\mathcal{C}\mathcal{C}\bar{u}^T(p_1) \\
&= -\bar{v}(k_1)\Gamma_1 v(p_2)[u^T(k_2)\Gamma_2^T\bar{u}^T(p_1)]^T \\
&= -\bar{v}(k_1)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_2)
\end{aligned}$$

$$\begin{aligned}
 \text{→ } \text{i} \tilde{\mathcal{M}}_t &= -\frac{\text{i} \kappa^2}{t - m_\phi^2} \bar{v}(k_1) \Gamma_2^C v(p_1) \bar{u}(p_2) \Gamma_1^C u(k_2) \\
 &= -\frac{\text{i} \kappa^2}{t - m_\phi^2} \bar{u}(p_1) \Gamma_2 u(k_1) \bar{v}(k_2) \Gamma_1 v(p_2) = \text{i} \mathcal{M}_t \\
 \text{i} \tilde{\mathcal{M}}_u &= -\frac{\text{i} \kappa^2}{u - m_\phi^2} \bar{v}(k_1) \Gamma_1 v(p_2) \bar{v}(k_2) \Gamma_2^C v(p_1) \\
 &= +\frac{\text{i} \kappa^2}{u - m_\phi^2} \bar{v}(k_1) \Gamma_1 v(p_2) \bar{u}(p_1) \Gamma_2 u(k_2) = -\text{i} \mathcal{M}_u
 \end{aligned}$$

可见，根据费米子流方向的不同取法计算出来的结果确实是等价的



因此 $i\tilde{\mathcal{M}} = i\tilde{\mathcal{M}}_t + i\tilde{\mathcal{M}}_u = i\mathcal{M}_t - i\mathcal{M}_u = i\mathcal{M}$

非极化振幅模方

接下来计算 $\chi\chi \rightarrow \psi\bar{\psi}$ 的非极化振幅模方

$$\overline{|\mathcal{M}|^2} = \overline{|\mathcal{M}_t - \mathcal{M}_u|^2} = \overline{|\mathcal{M}_t|^2} + \overline{|\mathcal{M}_u|^2} - (\overline{\mathcal{M}_t^* \mathcal{M}_u} + \text{H.c.})$$

使用具体形式 $\Gamma_1 = P_R$ 和 $\Gamma_2 = P_L$ ，由第一种取法的振幅计算结果得到

$$i\mathcal{M}_t = -\frac{i\kappa^2}{t-m_\phi^2} \bar{u}(p_1)P_L u(k_1) \bar{v}(k_2)P_R v(p_2)$$

$$(\mathbf{i}\mathcal{M}_t)^* = \frac{\mathbf{i}\kappa^2}{t - m_\phi^2} \bar{u}(k_1) P_{\mathbf{R}} u(p_1) \bar{v}(p_2) P_{\mathbf{L}} v(k_2)$$

$$i\mathcal{M}_u = -\frac{i\kappa^2}{u - m_\phi^2} \bar{v}(k_1) P_R v(p_2) \bar{u}(p_1) P_L u(k_2)$$

$$(\mathbf{i}\mathcal{M}_u)^* = \frac{\mathbf{i}\kappa^2}{u - m_\phi^2} \bar{v}(p_2) P_{\text{L}} v(k_1) \bar{u}(k_2) P_{\text{R}} u(p_1)$$

单纯 t 通道贡献

由 $P_L \gamma^\mu = \gamma^\mu P_R$ 、 $P_R \gamma^\mu = \gamma^\mu P_L$ 、 $P_L^2 = P_L$ 、 $P_R^2 = P_R$ 和 $P_L P_R = P_R P_L = 0$ 得

$$\begin{aligned} \text{tr}[(\not{p}_1 + m_\psi) P_L (\not{k}_1 + m_\chi) P_R] &= \text{tr}[(\not{p}_1 + m_\psi)(\not{k}_1 P_R + m_\chi P_L) P_R] \\ &= \text{tr}[(\not{p}_1 + m_\psi) \not{k}_1 P_R] = \frac{1}{2} \text{tr}[(\not{p}_1 + m_\psi) \not{k}_1 (1 + \gamma^5)] = \frac{1}{2} \text{tr}(\not{p}_1 \not{k}_1) = 2 \not{k}_1 \cdot \not{p}_1 \\ \text{tr}[(\not{k}_2 - m_\chi) P_R (\not{p}_2 - m_\psi) P_L] &= \frac{1}{2} \text{tr}[(\not{k}_2 - m_\chi) \not{p}_2 (1 - \gamma^5)] = 2 \not{k}_2 \cdot \not{p}_2 \end{aligned}$$

从而, 单纯 t 通道对非极化振幅模方的贡献是

$$\begin{aligned}
& \overline{|\mathcal{M}_t|^2} = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_t|^2 \\
&= \frac{\kappa^4}{4(t - m_\phi^2)^2} \sum_{\text{spins}} \bar{u}(p_1) P_L u(k_1) \bar{u}(k_1) P_R u(p_1) \bar{v}(k_2) P_R v(p_2) \bar{v}(p_2) P_L v(k_2) \\
&= \frac{\kappa^4}{2 \cdot 2(t - m_\phi^2)^2} \text{tr}[(\not{p}_1 + m_\psi) P_L (\not{k}_1 + m_\chi) P_R] \text{tr}[(\not{k}_2 - m_\chi) P_R (\not{p}_2 - m_\psi) P_L] \\
&= \frac{\kappa^4 (k_1 \cdot p_1) (k_2 \cdot p_2)}{(t - m_\phi^2)^2}
\end{aligned}$$

单纯 u 通道贡献和交叉贡献

另一方面，单纯 u 通道的贡献为

$$\begin{aligned}
& \overline{|\mathcal{M}_u|^2} = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_u|^2 \\
&= \frac{\kappa^4}{4(u - m_\phi^2)^2} \sum_{\text{spins}} \bar{v}(k_1) P_R v(p_2) \bar{v}(p_2) P_L v(k_1) \bar{u}(p_1) P_L u(k_2) \bar{u}(k_2) P_R u(p_1) \\
&= \frac{\kappa^4}{2 \cdot 2(u - m_\phi^2)^2} \text{tr}[(\not{k}_1 - m_\chi) P_R (\not{p}_2 - m_\psi) P_L] \text{tr}[(\not{p}_1 + m_\psi) P_L (\not{k}_2 + m_\chi) P_R] \\
&= \frac{\kappa^4 (\not{k}_1 \cdot \not{p}_2) (\not{k}_2 \cdot \not{p}_1)}{(u - m_\phi^2)^2}
\end{aligned}$$

而 t 和 u 通道的交叉贡献是

$$\overline{\mathcal{M}_t^* \mathcal{M}_u} = \frac{1}{4} \sum_{\text{spins}} \mathcal{M}_t^* \mathcal{M}_u$$

$$= \frac{\kappa^4}{4(t - m_\phi^2)(u - m_\phi^2)} \sum_{\text{spins}} \bar{u}(k_1) P_R u(p_1) \bar{u}(p_1) P_L u(k_2) \bar{v}(p_2) P_L v(k_2) \bar{v}(k_1) P_R v(p_2)$$

$\chi\chi \rightarrow \psi\bar{\psi}$ 非极化振幅模方

$$\begin{aligned}
& \sum_{\text{spins}} \bar{u}(k_1) P_{\text{R}} u(p_1) \bar{u}(p_1) P_{\text{L}} u(k_2) \bar{v}(p_2) P_{\text{L}} v(k_2) \bar{v}(k_1) P_{\text{R}} v(p_2) \\
&= \sum_{\text{spins}} \bar{u}(k_1) P_{\text{R}} u(p_1) \bar{u}(p_1) P_{\text{L}} u(k_2) [\mathbf{u}^T(p_2) \mathcal{C} P_{\text{L}} \mathcal{C} \bar{u}^T(k_2)]^T [\mathbf{u}^T(k_1) \mathcal{C} P_{\text{R}} \mathcal{C} \bar{u}^T(p_2)]^T \\
&= \sum_{\text{spins}} \bar{u}(k_1) P_{\text{R}} u(p_1) \bar{u}(p_1) P_{\text{L}} \mathbf{u}(k_2) \bar{u}(k_2) \mathcal{C}^T P_{\text{L}}^T \mathcal{C}^T u(p_2) \bar{u}(p_2) \mathcal{C}^T P_{\text{R}}^T \mathcal{C}^T \mathbf{u}(k_1) \\
&= \text{tr}[(\not{k}_1 + m_\chi) P_{\text{R}} (\not{p}_1 + m_\psi) P_{\text{L}} (\not{k}_2 + m_\chi) \mathcal{C}^{-1} P_{\text{L}}^T \mathcal{C} (\not{p}_2 + m_\psi) \mathcal{C}^{-1} P_{\text{R}}^T \mathcal{C}] \\
&= \text{tr}[(\not{k}_1 + m_\chi) \not{p}_1 P_{\text{L}} (\not{k}_2 + m_\chi) P_{\text{L}} (\not{p}_2 + m_\psi) P_{\text{R}}] = m_\chi \text{tr}[(\not{k}_1 + m_\chi) \not{p}_1 P_{\text{L}} (\not{p}_2 + m_\psi) P_{\text{R}}] \\
&= \frac{m_\chi}{2} \text{tr}[(\not{k}_1 + m_\chi) \not{p}_1 \not{p}_2 (1 + \gamma^5)] = \frac{m_\chi^2}{2} \text{tr}(\not{p}_1 \not{p}_2) = 2m_\chi^2 (p_1 \cdot p_2)
\end{aligned}$$

$$\overline{\mathcal{M}_t^* \mathcal{M}_u} + \text{H.c.} = \frac{\kappa^4 m_\chi^2 (p_1 \cdot p_2)}{2(t - m_\phi^2)(u - m_\phi^2)} + \text{H.c.} = \frac{\kappa^4 m_\chi^2 (p_1 \cdot p_2)}{(t - m_\phi^2)(u - m_\phi^2)}$$

于是, $\chi\bar{\chi} \rightarrow \psi\bar{\psi}$ 的非极化振幅模方为

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \overline{|\mathcal{M}_t|^2} + \overline{|\mathcal{M}_u|^2} - (\overline{\mathcal{M}_t^* \mathcal{M}_u} + \text{H.c.}) \\ &= \kappa^4 \left[\frac{(k_1 \cdot p_1)(k_2 \cdot p_2)}{(t - m_\phi^2)^2} + \frac{(k_1 \cdot p_2)(k_2 \cdot p_1)}{(u - m_\phi^2)^2} - \frac{m_\chi^2(p_1 \cdot p_2)}{(t - m_\phi^2)(u - m_\phi^2)} \right] \end{aligned}$$