

量子场论

第4章 量子旋量场

4.4节和4.5节

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4.4 节 Dirac 旋量场的平面波展开

4.4.1 小节 平面波解的一般形式

本小节讨论与表象选取无关的平面波解一般形式

自由 Dirac 旋量场 $\psi_a(x)$ 满足 Klein-Gordon 方程 $(\partial^2 + m^2)\psi(x) = 0$

因而在无界空间中具有平面波解

对于确定的动量 k ，假设 Dirac 方程具有如下形式的平面波解：

$$\varphi_a(x, \mathbf{k}) = w_a(k^0, \mathbf{k}) e^{-i\mathbf{k} \cdot x}$$

其中，系数 $w_a(k^0, k)$ 是四分量旋量，带着一个旋量指标 a

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篮球 隐去旋量指标，将这个平面波解代入到 Dirac 方程 $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$ 中，得

$$0 = (\mathbf{i}\gamma^\mu \partial_\mu - m)\varphi(x, \mathbf{k}) = (\gamma^\mu k_\mu - m)w(k^0, \mathbf{k})e^{-ik\cdot x} = (k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)w(k^0, \mathbf{k})e^{-ik\cdot x}$$

$$(k^0 \gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) w(k^0, \mathbf{k}) = 0$$

本征值方程

左乘 γ^0 得 $[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0]w(k^0, \mathbf{k}) \equiv 0$

 移项，推出

$$[\gamma^0(\mathbf{k}; \gamma) \pm m\gamma^0]w(k^0, \mathbf{k}) \equiv k^0 w(k^0, \mathbf{k})$$

这是矩阵 $[\gamma^0(k \cdot \gamma) + m\gamma^0]$ 的本征值方程

它具有非平庸解的条件是特征多项式 $\det[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0]$ 为零，即

$$\det[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0] = 0$$

 这个方程的根给出 k^0 的本征值，相应的非平庸解是本征矢量

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 这个方程的根给出 k^0 的本征值，相应的非平庸解是**本征矢量**

利用 $(\gamma^0)^2 = 1$ ，将这个方程化为

$$0 = \det[k^0 \mathbf{1} - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0] = \det[\gamma^0(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)]$$

$$= \det(\gamma^0) \det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)$$

 因而它等价于

$$\det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) = 0$$

$$[\det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)]^2$$

利用 $(\gamma^5)^2 = 1$, 将方程 $\det(k^0 \gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) = 0$ 左边化为

$$\begin{aligned} \det(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m) &= \det[(\gamma^5)^2(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)] = \det[\gamma^5(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)\gamma^5] \\ &= \det[(\gamma^5)^2(-k^0\gamma^0 + \mathbf{k}\cdot\boldsymbol{\gamma} - m)] = \det[-(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma}) - m] \end{aligned}$$

这里第二步用到行列式性质 $\det(AB) = \det(BA)$ ，第三步用到 $\gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu$

$$[\det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)]^2$$

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这里第二步用到行列式性质 $\det(AB) = \det(BA)$ ，第三步用到 $\gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu$

利用 $(k_\mu \gamma^\mu)^2 = k_\mu k_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} k_\mu k_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$
 $= k_\mu k_\nu q^{\mu\nu} \mathbf{1} = k^2 \mathbf{1} = [(k^0)^2 - |\mathbf{k}|^2] \mathbf{1}$

 推出 $[\det(k^0 \gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)]^2$

$$\begin{aligned}
&= \det[(k^0 \gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma}) - m] \det[-(k^0 \gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma}) - m] \\
&= \det(k_\mu \gamma^\mu - m) \det(-k_\nu \gamma^\nu - m) = \det[(k_\mu \gamma^\mu - m)(-k_\nu \gamma^\nu - m)] \\
&= \det[-(k_\mu \gamma^\mu)^2 + m^2] = \det\{[-(k^0)^2 + |\mathbf{k}|^2 + m^2] \mathbf{1}\} \\
&= [-(k^0)^2 + |\mathbf{k}|^2 + m^2]^4 = [E_{\mathbf{k}}^2 - (k^0)^2]^4
\end{aligned}$$

其中 $E_{\mathbf{k}} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$

本征矢量

[$\det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)]^2 = [E_k^2 - (k^0)^2]^4$ 表明

$$\det(k^0 \gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) = [E_{\mathbf{k}}^2 - (k^0)^2]^2 = (E_{\mathbf{k}} + k^0)^2 (E_{\mathbf{k}} - k^0)^2$$

因此方程 $\det(k^0 \gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) = 0$ 有 2 个根 $k^0 = \pm E_{\mathbf{k}}$

这 2 个根都是 2 重根，各自对应于 2 个线性独立的本征矢量

 一共有 4 个线性独立的本征矢量

本征矢量

 $[\det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)]^2 = [E_{\mathbf{k}}^2 - (k^0)^2]^4$ 表明

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 一共有 4 个线性独立的本征矢量

① $k^0 = E_{\mathbf{k}}$ 对应于 2 个本征矢量 $w^{(+)}(E_{\mathbf{k}}, \mathbf{k}, \sigma)$, $\sigma = 1, 2$

因此 $e^{-ik \cdot x} = \exp[-i(E_k t - \mathbf{k} \cdot \mathbf{x})]$ ，平面波解中有 2 个正能解，形式为

$$w^{(+)}(E_{\mathbf{k}}, \mathbf{k}, \sigma) \exp[-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})], \quad \sigma = 1, 2$$

② $k^0 = -E_{\mathbf{k}}$ 对应于 2 个本征矢量 $w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}, \sigma)$, $\sigma = 1, 2$

 因而 $e^{-ik \cdot x} = \exp[-i(-E_k t - \mathbf{k} \cdot \mathbf{x})]$ ，平面波解中有 2 个负能解，形式为

$$w^{(-)(-E_{\mathbf{k}}, \mathbf{k}, \sigma)} \exp[i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})], \quad \sigma = 1, 2$$

正能解和负能解

 将这 4 个本征矢量的正交归一关系取为

$$w^{(+)\dagger}(E_{\mathbf{k}}, \mathbf{k}, \sigma) w^{(+)}(E_{\mathbf{k}}, \mathbf{k}, \sigma') = 2E_{\mathbf{k}}\delta_{\sigma\sigma'}$$

$$w^{(-)\dagger}(-E_{\mathbf{k}}, \mathbf{k}, \sigma) w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}, \sigma') = 2E_{\mathbf{k}}\delta_{\sigma\sigma'}$$

$$w^{(+)\dagger}(E_{\mathbf{k}}, \mathbf{k}, \sigma) w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}, \sigma') = w^{(-)\dagger}(-E_{\mathbf{k}}, \mathbf{k}, \sigma) w^{(+)}(E_{\mathbf{k}}, \mathbf{k}, \sigma') = 0$$

正能解和负能解

将这 4 个本征矢量的正交归一关系取为

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引入四分量旋量 $u(\mathbf{k}, \sigma)$ 和 $v(\mathbf{k}, \sigma)$ ，定义为

$$u(\mathbf{k}, \sigma) \equiv w^{(+)}(E_{\mathbf{k}}, \mathbf{k}, \sigma), \quad v(\mathbf{k}, \sigma) \equiv w^{(-)}(-E_{\mathbf{k}}, -\mathbf{k}, \sigma), \quad \sigma = 1, 2$$

于是， Dirac 方程的正能解和负能解可以分别写作

$$\varphi^{(+)}(x, \mathbf{k}, \sigma) \equiv w^{(+)}(E_{\mathbf{k}}, \mathbf{k}, \sigma) \exp[-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})] = u(\mathbf{k}, \sigma) \exp[-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})]$$

$$\varphi^{(-)}(x, \mathbf{k}, \sigma) \equiv w^{(-)}(-E_{\mathbf{k}}, -\mathbf{k}, \sigma) \exp[i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})] = v(\mathbf{k}, \sigma) \exp[i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})]$$

 替换动量记号，得到 $\varphi^{(+)}(x, \mathbf{p}, \sigma) = u(\mathbf{p}, \sigma) e^{-i\mathbf{p} \cdot x}$ 和 $\varphi^{(-)}(x, \mathbf{p}, \sigma) = v(\mathbf{p}, \sigma) e^{i\mathbf{p} \cdot x}$

其中 $p^0 = E_p \equiv \sqrt{|\mathbf{p}|^2 + m^2} > 0$

平面波展开

从而, Dirac 旋量场算符 $\psi(\mathbf{x}, t)$ 的平面波展开式可写作

$$\begin{aligned}\psi(\mathbf{x}, t) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\sigma=1}^2 \left[\varphi^{(+)}(x, \mathbf{p}, \sigma) c_{\mathbf{p}, \sigma} + \varphi^{(-)}(x, \mathbf{p}, \sigma) d_{\mathbf{p}, \sigma}^\dagger \right] \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\sigma=1}^2 \left[u(\mathbf{p}, \sigma) c_{\mathbf{p}, \sigma} e^{-ip \cdot x} + v(\mathbf{p}, \sigma) d_{\mathbf{p}, \sigma}^\dagger e^{ip \cdot x} \right]\end{aligned}$$

其中, $c_{\mathbf{p}, \sigma}$ 是湮灭算符, $d_{\mathbf{p}, \sigma}^\dagger$ 是产生算符, 而且 $c_{\mathbf{p}, \sigma} \neq d_{\mathbf{p}, \sigma}$

平面波旋量系数 $u(\mathbf{p}, \sigma)$ 和 $v(\mathbf{p}, \sigma)$ 的正交归一关系为

$$u^\dagger(\mathbf{p}, \sigma) u(\mathbf{p}, \sigma') = w^{(+)\dagger}(E_p, \mathbf{p}, \sigma) w^{(+)}(E_p, \mathbf{p}, \sigma') = 2E_p \delta_{\sigma \sigma'}$$

$$v^\dagger(\mathbf{p}, \sigma) v(\mathbf{p}, \sigma') = w^{(-)\dagger}(-E_p, -\mathbf{p}, \sigma) w^{(-)}(-E_p, -\mathbf{p}, \sigma') = 2E_p \delta_{\sigma \sigma'}$$

$$u^\dagger(\mathbf{p}, \sigma) v(-\mathbf{p}, \sigma') = w^{(+)\dagger}(E_p, \mathbf{p}, \sigma) w^{(-)}(-E_p, \mathbf{p}, \sigma') = 0$$

4.4.2 小节 Weyl 表象中的平面波解

本小节在 Weyl 表象中讨论 Dirac 方程的平面波解

 Dirac 旋量场描述自旋为 $1/2$ 的有质量粒子，根据 3.1.3 小节讨论，这样的粒子具有 2 种自旋极化态，对应于螺旋度的 2 种本征值 $+1/2$ 和 $-1/2$

1 为便于表述, 这里采用归一化的螺旋度本征值 $\lambda = \pm$

类似于矢量场情况, $\lambda = -$ 是左旋极化, $\lambda = +$ 是右旋极化

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类似于矢量场情况, $\lambda = -$ 是左旋极化, $\lambda = +$ 是右旋极化

因此，无论是平面波正能解还是负能解，都能够以 2 种螺旋度本征态作为 2 个线性独立的本征矢量

🎯 按照这个思路，把 2 个正能解表达为

$$\varphi^{(+)}(x, \mathbf{p}, \lambda) = u(\mathbf{p}, \lambda) e^{-i\mathbf{p} \cdot x}, \quad \lambda = \pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$$

根据 Dirac 方程 $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$, 有

$$0 = (\textcolor{blue}{i}\gamma^\mu \partial_\mu - m)\varphi^{(+)}(x, \mathbf{p}, \lambda) = (p_\mu \gamma^\mu - m)\textcolor{red}{u}(\mathbf{p}, \lambda) e^{-i\mathbf{p}\cdot x}$$

$u(p, \lambda)$ 的运动方程

 $(p_\mu \gamma^\mu - m)u(\mathbf{p}, \lambda)e^{-i\mathbf{p} \cdot \mathbf{x}} = 0$ 表明 $u(\mathbf{p}, \lambda)$ 满足运动方程

$$(\mathcal{P} - m)u(\mathbf{p}, \lambda) = 0$$

其中 \not{p} 的定义为 $\not{p} \equiv p_\mu \gamma^\mu$ ，这种记号称为 **Dirac 斜线** (slash)，是 Richard Feynman 引进的



Richard Feynman (1918–1988)

$u(\mathbf{p}, \lambda)$ 的运动方程

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将 $u(p, \lambda)$ 分解为两个二分量旋量 $f_\lambda(p)$ 和 $g_\lambda(p)$ ，

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ g_\lambda(\mathbf{p}) \end{pmatrix}$$

⑧ 根据 Weyl 表象中的 Dirac 矩阵表达式 $\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$ ，运动方程化为

$$0 = (\cancel{p} - m) u(\mathbf{p}, \lambda) = \begin{pmatrix} -m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & -m \end{pmatrix} \begin{pmatrix} f_\lambda(\mathbf{p}) \\ g_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} p_\mu \sigma^\mu g_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p}) \\ p_\mu \bar{\sigma}^\mu f_\lambda(\mathbf{p}) - m g_\lambda(\mathbf{p}) \end{pmatrix}$$

$f_\lambda(\mathbf{p})$ 与 $g_\lambda(\mathbf{p})$ 的关系



从而得到两条方程

$$(p \cdot \sigma)g_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) = 0, \quad (p \cdot \bar{\sigma})f_\lambda(\mathbf{p}) - mg_\lambda(\mathbf{p}) = 0$$

由第二条方程得

$$g_\lambda(\mathbf{p}) = \frac{p \cdot \bar{\sigma}}{m} f_\lambda(\mathbf{p})$$

中 将上式代入到第一条方程左边，得

$$(p \cdot \sigma) \textcolor{red}{g}_{\lambda}(\mathbf{p}) - m f_{\lambda}(\mathbf{p}) = \frac{(p \cdot \sigma)(p \cdot \bar{\sigma})}{m} f_{\lambda}(\mathbf{p}) - m f_{\lambda}(\mathbf{p})$$

$f_\lambda(p)$ 与 $g_\lambda(p)$ 的关系



从而得到两条方程

$$(p \cdot \sigma) q_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p}) = 0, \quad (p \cdot \bar{\sigma}) f_\lambda(\mathbf{p}) - mq_\lambda(\mathbf{p}) = 0$$



由第二条方程得

$$g_\lambda(\mathbf{p}) = \frac{p \cdot \bar{\sigma}}{m} f_\lambda(\mathbf{p})$$



将上式代入到第一条方程左边，得

$$(p \cdot \sigma) \textcolor{red}{g}_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p}) = \frac{(p \cdot \sigma)(p \cdot \bar{\sigma})}{m} f_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p})$$



为化简 $(p \cdot \sigma)(p \cdot \bar{\sigma})$ ，由 $\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$ 得反对易关系

$$2g^{\mu\nu} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \{\gamma^\mu, \gamma^\nu\} = \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu \end{pmatrix}$$



因此 $\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2q^{\mu\nu}$, $\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2q^{\mu\nu}$

$u(\mathbf{p}, \lambda)$ 的形式



从而

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p_\mu p_\nu \sigma^\mu \bar{\sigma}^\nu = \frac{1}{2} p_\mu p_\nu (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu) = \frac{1}{2} p_\mu p_\nu 2g^{\mu\nu} = p^2 = m^2$$



故

$$\begin{aligned}(p \cdot \sigma)g_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) &= \frac{(p \cdot \sigma)(p \cdot \bar{\sigma})}{m} f_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) \\&= \frac{m^2}{m} f_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) = 0\end{aligned}$$

$u(p, \lambda)$ 的形式



从而

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p_\mu p_\nu \sigma^\mu \bar{\sigma}^\nu = \frac{1}{2} p_\mu p_\nu (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu) = \frac{1}{2} p_\mu p_\nu 2g^{\mu\nu} = p^2 = m^2$$



故

$$\begin{aligned}(p \cdot \sigma)g_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) &= \frac{(p \cdot \sigma)(p \cdot \bar{\sigma})}{m} f_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) \\&= \frac{m^2}{m} f_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) = \mathbf{0}\end{aligned}$$



可见关系式 $g_\lambda(\mathbf{p}) = \frac{p \cdot \bar{\sigma}}{m} f_\lambda(\mathbf{p})$ 也符合第一条方程



于是，任取非零 $f_\lambda(p)$ 都能使

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \frac{\mathbf{p} \cdot \bar{\sigma}}{m} f_\lambda(\mathbf{p}) \end{pmatrix}$$

满足运动方程 $(p - m)u(\mathbf{p}, \lambda) = 0$

螺旋度矩阵

在旋量表示中，螺旋度矩阵是自旋角动量矩阵 S 在动量 p 方向上的投影，即 $\hat{p} \cdot S$

对于 Weyl 表象, 由 $S^i = \frac{1}{2} \begin{pmatrix} \sigma^i & \\ & \sigma^i \end{pmatrix}$ 得 $\hat{\mathbf{p}} \cdot \mathbf{S} = \frac{1}{2} \begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & \\ & \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}$

因而归一化螺旋度矩阵为 $2\hat{\mathbf{p}} \cdot \mathbf{S} = \begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & \\ & \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}$

两个分块相同，左手和右手 Weyl 旋量对应的归一化螺旋度矩阵都是 $\hat{p} \cdot \sigma$

代入 Pauli 矩阵 $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 、 $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ 和 $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ，推出

$$\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{|\mathbf{p}|} = \frac{1}{|\mathbf{p}|} \begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix}$$

螺旋态



引入归一化螺旋度矩阵 $\hat{p} \cdot \sigma$ 的本征矢量 $\xi_1(p)$ ，称为螺旋态，满足本征值方程

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) \equiv \lambda \xi_\lambda(\mathbf{p}), \quad \lambda = \pm$$



求解这个方程，得到归一化本征矢量

$$\xi_+(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} |\mathbf{p}| + p^3 \\ p^1 + i p^2 \end{pmatrix}, \quad \xi_-(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} -p^1 + i p^2 \\ |\mathbf{p}| + p^3 \end{pmatrix}$$



这是螺旋态 $\xi_\lambda(p)$ 的具体形式，它们满足正交归一关系

$$\xi_\lambda^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'}$$



和完备性关系

$$\sum_{\lambda=\pm} \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) = 1$$



此外, 由 $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ 得 $(\mathbf{p} \cdot \boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = \lambda|\mathbf{p}|\xi_\lambda(\mathbf{p})$

$u(\mathbf{p}, \lambda)$ 作为螺旋度本征态



为了让 $u(\mathbf{p}, \lambda)$ 作为螺旋度本征态，设 $f_\lambda(\mathbf{p})$ 正比于 $\xi_\lambda(\mathbf{p})$ ， $f_\lambda(\mathbf{p}) = C_\lambda \xi_\lambda(\mathbf{p})$



利用 $(\mathbf{p} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) = \lambda |\mathbf{p}| \xi_\lambda(\mathbf{p})$ ，推出

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \frac{p \cdot \bar{\sigma}}{m} f_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \frac{E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \xi_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \frac{E_{\mathbf{p}} + \lambda |\mathbf{p}|}{m} \xi_\lambda(\mathbf{p}) \end{pmatrix}$$

u(p, λ) 作为螺旋度本征态



为了让 $u(\mathbf{p}, \lambda)$ 作为螺旋度本征态，设 $f_\lambda(\mathbf{p})$ 正比于 $\xi_\lambda(\mathbf{p})$ ， $f_\lambda(\mathbf{p}) = C_\lambda \xi_\lambda(\mathbf{p})$



利用 $(\mathbf{p} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) = \lambda |\mathbf{p}| \xi_\lambda(\mathbf{p})$ ，推出

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \frac{\mathbf{p} \cdot \bar{\boldsymbol{\sigma}}}{m} f_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \frac{E_p + \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \xi_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \frac{E_p + \lambda |\mathbf{p}|}{m} \xi_\lambda(\mathbf{p}) \end{pmatrix}$$



为了使 $u(\mathbf{p}, \lambda)$ 满足归一关系 $u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda) = 2E_p$ ，取

$$C_\lambda = \sqrt{E_p - \lambda |\mathbf{p}|}$$



由 $\sqrt{(E_p + \lambda |\mathbf{p}|)(E_p - \lambda |\mathbf{p}|)} = \sqrt{E_p^2 - \lambda^2 |\mathbf{p}|^2} = \sqrt{E_p^2 - |\mathbf{p}|^2} = m$ 得

$$\begin{aligned} C_\lambda \frac{E_p + \lambda |\mathbf{p}|}{m} &= \sqrt{E_p - \lambda |\mathbf{p}|} \frac{E_p + \lambda |\mathbf{p}|}{m} = \frac{\sqrt{E_p + \lambda |\mathbf{p}|}}{m} \sqrt{(E_p + \lambda |\mathbf{p}|)(E_p - \lambda |\mathbf{p}|)} \\ &= \sqrt{E_p + \lambda |\mathbf{p}|} \end{aligned}$$

$u(\mathbf{p}, \lambda)$ 的螺旋态表达式

tractor 于是得到 $u(\mathbf{p}, \lambda)$ 的螺旋态表达式

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|} \xi_{\lambda}(\mathbf{p}) \\ \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|} \xi_{\lambda}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) \xi_{\lambda}(\mathbf{p}) \\ \omega_{\lambda}(\mathbf{p}) \xi_{\lambda}(\mathbf{p}) \end{pmatrix}$$

fan 其中, $\omega_{\lambda}(\mathbf{p})$ 定义为

$$\omega_{\lambda}(\mathbf{p}) \equiv \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}$$

matryoshka 它是关于 \mathbf{p} 的偶函数, $\omega_{\lambda}(-\mathbf{p}) = \omega_{\lambda}(\mathbf{p})$

balloon 根据 $2\hat{\mathbf{p}} \cdot \mathbf{S} = \begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & \\ & \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}$, $u(\mathbf{p}, \lambda)$ 是螺旋度本征态, 本征值为 λ :

$$(2\hat{\mathbf{p}} \cdot \mathbf{S})u(\mathbf{p}, \lambda) = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_{\lambda}(\mathbf{p}) \\ \omega_{\lambda}(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_{\lambda}(\mathbf{p}) \end{pmatrix} = \lambda \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) \xi_{\lambda}(\mathbf{p}) \\ \omega_{\lambda}(\mathbf{p}) \xi_{\lambda}(\mathbf{p}) \end{pmatrix} = \lambda u(\mathbf{p}, \lambda)$$

$v(\mathbf{p}, \lambda)$ 的运动方程

另一方面，将 2 个**负能解**表达为

$$\varphi^{(-)}(x, \mathbf{p}, \lambda) = v(\mathbf{p}, \lambda) e^{i\mathbf{p} \cdot x}, \quad \lambda = \pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$$

根据 **Dirac 方程** $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$ ，有

$$0 = (i\gamma^\mu \partial_\mu - m)\varphi^{(-)}(x, \mathbf{p}, \lambda) = (-\mathbf{p}_\mu \gamma^\mu - m)v(\mathbf{p}, \lambda)e^{i\mathbf{p} \cdot x}$$

即 $v(\mathbf{p}, \lambda)$ 满足运动方程

$$(\not{p} + m)v(\mathbf{p}, \lambda) = 0$$

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即 $v(\mathbf{p}, \lambda)$ 满足运动方程

$$(\not{p} + m)v(\mathbf{p}, \lambda) = 0$$

同样将 $v(\mathbf{p}, \lambda)$ 分解为两个**二分量旋量** $\tilde{f}_\lambda(\mathbf{p})$ 和 $\tilde{g}_\lambda(\mathbf{p})$ ， $v(\mathbf{p}, \lambda) = \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ \tilde{g}_\lambda(\mathbf{p}) \end{pmatrix}$ ，则

$$0 = (\not{p} + m)v(\mathbf{p}, \lambda) = \begin{pmatrix} m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & m \end{pmatrix} \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ \tilde{g}_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} p_\mu \sigma^\mu \tilde{g}_\lambda(\mathbf{p}) + m \tilde{f}_\lambda(\mathbf{p}) \\ p_\mu \bar{\sigma}^\mu \tilde{f}_\lambda(\mathbf{p}) + m \tilde{g}_\lambda(\mathbf{p}) \end{pmatrix}$$

$v(\mathbf{p}, \lambda)$ 的形式

从而得到两个方程

$$(p \cdot \sigma) \tilde{g}_\lambda(\mathbf{p}) + m \tilde{f}_\lambda(\mathbf{p}) = 0, \quad (p \cdot \bar{\sigma}) \tilde{f}_\lambda(\mathbf{p}) + m \tilde{g}_\lambda(\mathbf{p}) = 0$$

由第二条方程得

$$\tilde{g}_\lambda(\mathbf{p}) = -\frac{p \cdot \bar{\sigma}}{m} \tilde{f}_\lambda(\mathbf{p})$$

代入到第一条方程左边，由 $(p \cdot \sigma)(p \cdot \bar{\sigma}) = m^2$ 式推出

$$(p \cdot \sigma) \tilde{g}_\lambda(\mathbf{p}) + m \tilde{f}_\lambda(\mathbf{p}) = -\frac{(p \cdot \sigma)(p \cdot \bar{\sigma})}{m} \tilde{f}_\lambda(\mathbf{p}) + m \tilde{f}_\lambda(\mathbf{p}) = -\frac{m^2}{m} \tilde{f}_\lambda(\mathbf{p}) + m \tilde{f}_\lambda(\mathbf{p}) = 0$$

可见，关系式 $\tilde{g}_\lambda(\mathbf{p}) = -\frac{p \cdot \bar{\sigma}}{m} \tilde{f}_\lambda(\mathbf{p})$ 符合第一条方程

于是，任取非零 $\tilde{f}_\lambda(\mathbf{p})$ 都能使

$$v(\mathbf{p}, \lambda) = \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ -\frac{p \cdot \bar{\sigma}}{m} \tilde{f}_\lambda(\mathbf{p}) \end{pmatrix}$$

满足运动方程 $(\not{p} + m)v(\mathbf{p}, \lambda) = 0$

$v(\mathbf{p}, \lambda)$ 作为螺旋度本征态

为了让 $v(\mathbf{p}, \lambda)$ 作为螺旋度本征态，设 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$ ， $\tilde{f}_\lambda(\mathbf{p}) = \tilde{C}_\lambda \xi_{-\lambda}(\mathbf{p})$

这里没有选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_\lambda(\mathbf{p})$ ，原因将在 4.5.4 小节中说明

现在姑且接受这种选择，从而

$$v(\mathbf{p}, \lambda) = \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ -\frac{\mathbf{p} \cdot \bar{\sigma}}{m} \tilde{f}_\lambda(\mathbf{p}) \end{pmatrix} = \tilde{C}_\lambda \begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\frac{E_\mathbf{p} + \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \tilde{C}_\lambda \begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\frac{E_\mathbf{p} - \lambda|\mathbf{p}|}{m} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix}$$

$v(\mathbf{p}, \lambda)$ 作为螺旋度本征态

为了让 $v(\mathbf{p}, \lambda)$ 作为螺旋度本征态，设 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$ ， $\tilde{f}_\lambda(\mathbf{p}) = \tilde{C}_\lambda \xi_{-\lambda}(\mathbf{p})$

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现在姑且接受这种选择，从而

$$v(\mathbf{p}, \lambda) = \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ -\frac{\mathbf{p} \cdot \bar{\sigma}}{m} \tilde{f}_\lambda(\mathbf{p}) \end{pmatrix} = \tilde{C}_\lambda \begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\frac{E_p + \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \tilde{C}_\lambda \begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\frac{E_p - \lambda|\mathbf{p}|}{m} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix}$$

为了使 $v(\mathbf{p}, \lambda)$ 满足归一关系 $v^\dagger(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda) = 2E_p$ ，取

$$\tilde{C}_\lambda = \lambda \sqrt{E_p + \lambda|\mathbf{p}|}$$

由 $\sqrt{(E_p + \lambda|\mathbf{p}|)(E_p - \lambda|\mathbf{p}|)} = m$ 得

$$\begin{aligned} -\tilde{C}_\lambda \frac{E_p - \lambda|\mathbf{p}|}{m} &= -\lambda \sqrt{E_p + \lambda|\mathbf{p}|} \frac{E_p - \lambda|\mathbf{p}|}{m} \\ &= -\lambda \frac{\sqrt{E_p - \lambda|\mathbf{p}|}}{m} \sqrt{(E_p + \lambda|\mathbf{p}|)(E_p - \lambda|\mathbf{p}|)} = -\lambda \sqrt{E_p - \lambda|\mathbf{p}|} \end{aligned}$$

$v(\mathbf{p}, \lambda)$ 的螺旋态表达式

于是得到 $v(\mathbf{p}, \lambda)$ 的螺旋态表达式

$$v(\mathbf{p}, \lambda) = \begin{pmatrix} \lambda \sqrt{E_{\mathbf{p}} + \lambda |\mathbf{p}|} \xi_{-\lambda}(\mathbf{p}) \\ -\lambda \sqrt{E_{\mathbf{p}} - \lambda |\mathbf{p}|} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \lambda \omega_{\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \\ -\lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix}$$

这样一来, $v(\mathbf{p}, \lambda)$ 是螺旋度本征态, 本征值为 $-\lambda$:

$$\begin{aligned} (2 \hat{\mathbf{p}} \cdot \mathbf{S}) v(\mathbf{p}, \lambda) &= \begin{pmatrix} \lambda \omega_{\lambda}(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_{-\lambda}(\mathbf{p}) \\ -\lambda \omega_{-\lambda}(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} \\ &= -\lambda \begin{pmatrix} \lambda \omega_{\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \\ -\lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = -\lambda v(\mathbf{p}, \lambda) \end{aligned}$$

平面波旋量系数的关系



可以验证，以上平面波旋量系数 $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 满足正交归一关系

$$u^\dagger(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = v^\dagger(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') = 2E_p\delta_{\lambda\lambda'}$$

$$u^\dagger(\mathbf{p}, \lambda)v(-\mathbf{p}, \lambda') = v^\dagger(-\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = 0$$



也满足 Lorentz 不变的关系式

$$\bar{u}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = 2m\delta_{\lambda\lambda'}, \quad \bar{v}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') = -2m\delta_{\lambda\lambda'}$$

$$\bar{u}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') = \bar{v}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = 0$$



另一方面，考虑螺旋度求和式

$$\begin{aligned} \sum_{\lambda=\pm} u(\mathbf{p}, \lambda)\bar{u}(\mathbf{p}, \lambda) &= \sum_{\lambda=\pm} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \\ &= \sum_{\lambda=\pm} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \\ \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \end{aligned}$$

螺旋度求和



利用等式

$$(p \cdot \bar{\sigma})\xi_\lambda(\mathbf{p}) = (E_p + \mathbf{p} \cdot \boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_p + \lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})$$

$$(p \cdot \sigma)\xi_\lambda(\mathbf{p}) = (E_p - \mathbf{p} \cdot \boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_p - \lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p})$$

以及 $\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) = m$ 和 $\xi_\lambda(\mathbf{p})$ 的完备性关系 $\sum_{\lambda=\pm} \xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) = 1$ ，推出

$$\begin{aligned} \sum_{\lambda=\pm} u(\mathbf{p}, \lambda)\bar{u}(\mathbf{p}, \lambda) &= \sum_{\lambda=\pm} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \\ \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \\ &= \sum_{\lambda=\pm} \begin{pmatrix} m\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & (p \cdot \sigma)\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \\ (p \cdot \bar{\sigma})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & m\xi_\lambda^\dagger(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \\ &= \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} = p_\mu \gamma^\mu + m \end{aligned}$$

自旋求和关系

 由 $(p \cdot \bar{\sigma})\xi_{-\lambda}(\mathbf{p}) = (E_p + \mathbf{p} \cdot \boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p}) = (E_p - \lambda|\mathbf{p}|)\xi_{-\lambda}(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p})$

$(p \cdot \sigma)\xi_{-\lambda}(\mathbf{p}) = (E_p - \mathbf{p} \cdot \boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p}) = (E_p + \lambda|\mathbf{p}|)\xi_{-\lambda}(\mathbf{p}) = \omega_{\lambda}^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p})$

 得 $\sum_{\lambda=\pm} v(\mathbf{p}, \lambda) \bar{v}(\mathbf{p}, \lambda)$

$$\begin{aligned} &= \sum_{\lambda=\pm} \begin{pmatrix} \lambda \omega_{\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \\ -\lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} -\lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^{\dagger}(\mathbf{p}) & \lambda \omega_{\lambda}(\mathbf{p}) \xi_{-\lambda}^{\dagger}(\mathbf{p}) \end{pmatrix} \\ &= \sum_{\lambda=\pm} \begin{pmatrix} -\lambda^2 \omega_{\lambda}(\mathbf{p}) \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^{\dagger}(\mathbf{p}) & \lambda^2 \omega_{\lambda}^2(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^{\dagger}(\mathbf{p}) \\ \lambda^2 \omega_{-\lambda}^2(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^{\dagger}(\mathbf{p}) & -\lambda^2 \omega_{-\lambda}(\mathbf{p}) \omega_{\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^{\dagger}(\mathbf{p}) \end{pmatrix} \\ &= \sum_{\lambda=\pm} \begin{pmatrix} -m \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^{\dagger}(\mathbf{p}) & (p \cdot \sigma) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^{\dagger}(\mathbf{p}) \\ (p \cdot \bar{\sigma}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^{\dagger}(\mathbf{p}) & -m \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^{\dagger}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} = p_{\mu} \gamma^{\mu} - m \end{aligned}$$

 整理一下，有如下螺旋度求和关系，或者说，**自旋求和关系**：

$$\sum_{\lambda=\pm} u(\mathbf{p}, \lambda) \bar{u}(\mathbf{p}, \lambda) = \not{p} + m, \quad \sum_{\lambda=\pm} v(\mathbf{p}, \lambda) \bar{v}(\mathbf{p}, \lambda) = \not{p} - m$$

平面波展开

 用 $u(p, \lambda)$ 和 $v(p, \lambda)$ 把 Dirac 旋量场算符 $\psi(x, t)$ 的平面波展开式写作

$$\begin{aligned}\psi(\mathbf{x}, t) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[\varphi^{(+)}(x, \mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} + \varphi^{(-)}(x, \mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger \right] \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[u(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right]\end{aligned}$$

其中 $a_{p,\lambda}$ 是湮灭算符, $b_{p,\lambda}^\dagger$ 是产生算符, 而且 $a_{p,\lambda} \neq b_{p,\lambda}$, 于是

$$\begin{aligned}\psi^\dagger(\mathbf{x}, t) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right] \\ \bar{\psi}(\mathbf{x}, t) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\mp} \left[\bar{u}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + \bar{v}(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right]\end{aligned}$$

4.4.3 小节 哈密顿量和产生湮灭算符

根据拉氏量 $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$, $\psi(x)$ 对应的共轭动量密度是

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger$$

它的平面波展开式为

$$\pi(\mathbf{x}, t) = \mathbf{i}\psi^\dagger(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{i}{\sqrt{2E_p}} \sum_{\lambda=\pm} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right]$$

代入自由旋量场 $\psi(x)$ 满足的 Dirac 方程 $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$ ，拉氏量化为

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = 0$$

因此，自由 Dirac 旋量场的哈密顿量密度为

$$\mathcal{H} = \pi \partial_0 \psi - \mathcal{L} = \pi \partial_0 \psi = i \psi^\dagger \partial_0 \psi$$

哈密顿量算符



从而，哈密顿量算符为

$$\begin{aligned}
H &= \int d^3x \mathcal{H} = \int d^3x \psi^\dagger i\partial_0 \psi \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{4E_p E_q}} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger e^{-ip \cdot x} \right. \\
&\quad \times q_0 \left. \left[u(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} - v(\mathbf{q}, \lambda') b_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \right. \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q q_0}{(2\pi)^6 \sqrt{4E_p E_q}} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} \right. \\
&\quad - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \\
&\quad - u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \\
&\quad \left. \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \right] \right]
\end{aligned}$$

化简哈密顿量

积分，得

$$\begin{aligned}
H &= \sum_{\lambda\lambda'} \int \frac{d^3 p d^3 q E_q}{(2\pi)^3 \sqrt{4E_p E_q}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(E_p - E_q)t} \right. \right. \\
&\quad \left. \left. - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{q}, \lambda'}^\dagger e^{-i(E_p - E_q)t} \right] \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[- u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'}^\dagger e^{i(E_p + E_q)t} \right. \right. \\
&\quad \left. \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(E_p + E_q)t} \right] \right\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda'}^\dagger \right. \\
&\quad \left. - u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_p t} + v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_p t} \right] \\
&\quad = 0 \qquad \qquad \qquad = 0 \\
&= \sum_{\lambda\lambda'} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} (2E_p \delta_{\lambda\lambda'} a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} - 2E_p \delta_{\lambda\lambda'} b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda'}^\dagger) \quad \text{正交归一关系} \\
&= \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} E_p (a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda} - b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda}^\dagger)
\end{aligned}$$

$a_{p,\lambda}$ 和 $a_{p,\lambda}^\dagger$ 的表达式

另一方面，有

$$\begin{aligned}
& \int d^3x e^{ip \cdot x} \color{orange}{u^\dagger(\mathbf{p}, \lambda)} \psi(\mathbf{x}, t) \\
&= \int \frac{d^3x d^3q}{(2\pi)^3 \sqrt{2E_q}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} + u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \right] \\
&= \int \frac{d^3q}{\sqrt{2E_q}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{i(E_p - E_q)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \right. \\
&\quad \left. + u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q}, \lambda'}^\dagger e^{i(E_p + E_q)t} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \right] \\
&= \frac{1}{\sqrt{2E_p}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda'} + u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') b_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_p t} \right] \\
&= \frac{1}{\sqrt{2E_p}} \sum_{\lambda'=\pm} (2E_p \delta_{\lambda \lambda'} a_{\mathbf{p}, \lambda'}) = \sqrt{2E_p} \color{red}{a_{\mathbf{p}, \lambda}}
\end{aligned}$$

从而将湮灭算符 $a_{p,\lambda}$ 和产生算符 $a_{p,\lambda}^\dagger$ 表示为

$$\mathbf{a}_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip\cdot x} u^\dagger(\mathbf{p},\lambda) \psi(\mathbf{x},t), \quad a_{\mathbf{p},\lambda}^\dagger = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip\cdot x} \psi^\dagger(\mathbf{x},t) u(\mathbf{p},\lambda)$$

$b_{p,\lambda}^\dagger$ 和 $b_{p,\lambda}$ 的表达式

同理推出

$$\begin{aligned}
& \int d^3x e^{-ip \cdot x} v^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t) \\
&= \int \frac{d^3x d^3q}{(2\pi)^3 \sqrt{2E_q}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \right] \\
&= \int \frac{d^3q}{\sqrt{2E_q}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-i(E_p+E_q)t} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \right. \\
&\quad \left. + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q}, \lambda'}^\dagger e^{-i(E_p-E_q)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \right] \\
&= \frac{1}{\sqrt{2E_p}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') a_{-\mathbf{p}, \lambda'} e^{-2iE_p t} + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda'}^\dagger \right] \\
&= \frac{1}{\sqrt{2E_p}} \sum_{\lambda'=\pm} \left(2E_p \delta_{\lambda \lambda'} b_{\mathbf{p}, \lambda'}^\dagger \right) = \sqrt{2E_p} b_{\mathbf{p}, \lambda}^\dagger
\end{aligned}$$

于是将产生算符 $b_{p,\lambda}^\dagger$ 和湮灭算符 $b_{p,\lambda}$ 表示成

$$b_{\mathbf{p}, \lambda}^\dagger = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} v^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t), \quad b_{\mathbf{p}, \lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} \psi^\dagger(\mathbf{x}, t) v(\mathbf{p}, \lambda)$$

4.5 节 Dirac 旋量场的正则量子化

4.5.1 小节 用等时对易关系量子化 Dirac 旋量场的困难

 回顾前面标量场和矢量场的正则量子化程序

 我们先假设场算符与其共轭动量密度算符满足等时对易关系

$$[\Phi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y})$$

$$[\Phi_a(\mathbf{x}, t), \Phi_b(\mathbf{y}, t)] = [\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = 0$$

 然后推导出产生湮灭算符的对易关系，再通过计算给出正定的哈密顿量算符

 对于无质量矢量场，则需要用弱 Lorenz 规范条件来得到正的哈密顿量期待值

 这些结果说明在量子场论中使用正则量子化方法是合理的

 本小节将尝试用类似的等时对易关系对 Dirac 旋量场进行量子化

 不过，我们会发现这种方法并不能给出正定的哈密顿量算符，因而是不可行的

等时对易关系

假设 Dirac 旋量场算符 $\psi(x)$ 与其共轭动量密度算符 $\pi(x)$ 满足等时对易关系

$$[\psi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = [\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = 0.$$

 这里将旋量指标明显地写出来

由于 $\pi = i\psi^\dagger$ ，这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时对易关系

$$[\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] = \delta_{ab} \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = [\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] = 0$$

等时对易关系

假设 Dirac 旋量场算符 $\psi(x)$ 与其共轭动量密度算符 $\pi(x)$ 满足等时对易关系

$$[\psi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = [\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = 0.$$

 这里将**旋量指标**明显地写出来

由于 $\pi = i\psi^\dagger$ ，这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时对易关系

$$[\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] = \delta_{ab} \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = [\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] = 0$$

根据 $a_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} u^\dagger(\mathbf{p},\lambda) \psi(\mathbf{x},t)$, 推出

$$\begin{aligned}
[a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger] &= \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) [\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] u_b(\mathbf{q}, \lambda') \\
&= \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) u_b(\mathbf{q}, \lambda') \delta_{ab} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \\
&= \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int d^3x e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} u_a^\dagger(\mathbf{p}, \lambda) u_b(\mathbf{q}, \lambda') \\
&= \frac{1}{2E_{\mathbf{p}}} u_a^\dagger(\mathbf{p}, \lambda) u_b(\mathbf{q}, \lambda') (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q})
\end{aligned}$$

$$[b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger]$$



根据

$$b_{\mathbf{p},\lambda}^\dagger = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} v^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t), \quad b_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} \psi^\dagger(\mathbf{x}, t) v(\mathbf{p}, \lambda)$$



以及等时对易关系 $[\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] = \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y})$ ，得到

$$\begin{aligned}
[b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger] &= \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} v_b^\dagger(\mathbf{q}, \lambda') [\psi_a^\dagger(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] v_a(\mathbf{p}, \lambda) \\
&= \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} v_b^\dagger(\mathbf{q}, \lambda') v_a(\mathbf{p}, \lambda) (-\delta_{ba}) \delta^{(3)}(\mathbf{x} - \mathbf{y}) \\
&= -\frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int d^3x e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} v^\dagger(\mathbf{q}, \lambda') v(\mathbf{p}, \lambda) \\
&= -\frac{1}{2E_p} v^\dagger(\mathbf{p}, \lambda') v(\mathbf{p}, \lambda) (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = -(2\pi)^3 \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q})
\end{aligned}$$



这个结果非同寻常地多了一个**负号**

负能量困难

进一步计算，最终通过等时对易关系得到的产生湮灭算符对易关系为

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{q},\lambda'}^\dagger] = 0$$

$$[b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger] = -(2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger] = 0$$

$$[a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger] = [b_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger] = 0$$

利用这样的对易关系，可以把哈密顿量算符化为

$$\begin{aligned} H &= \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} E_p (a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}^\dagger) \\ &= \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} E_p (a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}) + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3 p}{(2\pi)^3} 2E_p \end{aligned}$$

负能量困难

进一步计算，最终通过等时对易关系得到的产生湮灭算符对易关系为

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{q},\lambda'}^\dagger] = 0$$

$$[b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger] = -(2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger] = 0$$

$$[a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger] = [b_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger] = 0$$

利用这样的对易关系，可以把哈密顿量算符化为

$$H = \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} E_p (a_{p,\lambda}^\dagger a_{p,\lambda} - b_{p,\lambda}^\dagger b_{p,\lambda}^\dagger)$$

$$= \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} E_p (\color{red} a_{p,\lambda}^\dagger a_{p,\lambda} - b_{p,\lambda}^\dagger b_{p,\lambda}) + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3 p}{(2\pi)^3} 2E_p$$

 第二项是零点能，第一项中由 $(a_{\mathbf{p},\lambda}, a_{\mathbf{p},\lambda}^\dagger)$ 描述的粒子对总能量的贡献为正

 但第一项中由 $(b_{p,\lambda}, b_{p,\lambda}^\dagger)$ 描述的粒子对总能量的贡献为负

 粒子数密度 $b_{p,\lambda}^\dagger b_{p,\lambda}$ 越大，场的总能量越少，显然是非物理的，出现负能量困难

因此，用等时对易关系对 Dirac 旋量场进行量子化是行不通的

4.5.2 小节 用等时反对易关系量子化 Dirac 旋量场

从以上哈密顿量算符计算过程看出，如果在交换 $b_{p,\lambda}$ 和 $b_{p,\lambda}^\dagger$ 位置的同时能够改变符号，就可以得到正定的哈密顿量算符

因此，需要的不是 $b_{p,\lambda}$ 与 $b_{p,\lambda}^\dagger$ 的对易关系，而是反对易关系

 为了得到合适的 $b_{p,\lambda}$ 与 $b_{p,\lambda}^\dagger$ 的反对易关系，则需要舍弃等时对易关系

4.5.2 小节 用等时反对易关系量子化 Dirac 旋量场

从以上哈密顿量算符计算过程看出，如果在交换 $b_{p,\lambda}$ 和 $b_{p,\lambda}^\dagger$ 位置的同时能够改变符号，就可以得到正定的哈密顿量算符

因此，需要的不是 $b_{p,\lambda}$ 与 $b_{p,\lambda}^\dagger$ 的对易关系，而是反对易关系

为了得到合适的 $b_{p,\lambda}$ 与 $b_{p,\lambda}^\dagger$ 的反对易关系，则需要舍弃等时对易关系，代之以等时反对易关系

$$\{\psi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)\} = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y})$$

$$\{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} \equiv \{\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)\} \equiv 0$$

采用反对易关系进行量子化的方法称为 Jordan-Wigner 量子化

由于 $\pi = i\psi^\dagger$ ，这些关系等价于 ψ 与 ψ^\dagger 的等时反对易关系

$$\{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y})$$

$$\{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = \{\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = 0$$



Pascual Jordan (1902–1980)



Eugene Wigner
(1902–1995)

哈密顿量的正定性

通过等时反对易关系得到的产生湮灭算符反对易关系为

$$\{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}^\dagger\}=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad \{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}\}=\{a_{\mathbf{p},\lambda}^\dagger,a_{\mathbf{q},\lambda'}^\dagger\}=0$$

$$\{b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger\} = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad \{b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}\} = \{b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger\} = 0$$

$$\{a_{p,\lambda}, b_{q,\lambda'}^\dagger\} = \{b_{p,\lambda}, a_{q,\lambda'}^\dagger\} = \{a_{p,\lambda}, b_{q,\lambda'}\} = \{a_{p,\lambda}^\dagger, b_{q,\lambda'}^\dagger\} = 0$$

和 $(b_{p,\lambda}, b_{p,\lambda}^\dagger)$ 各自描述一种粒子

哈密顿量的正定性

 通过等时反对易关系得到的产生湮灭算符反对易关系为

$$\{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}^\dagger\}=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad \{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}\}=\{a_{\mathbf{p},\lambda}^\dagger,a_{\mathbf{q},\lambda'}^\dagger\}=0$$

$$\{b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger\} = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad \{b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}\} = \{b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger\} = 0$$

$$\{a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger\} = \{b_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger\} = \{a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}\} = \{a_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger\} = 0$$

和 $(b_{p,\lambda}, b_{p,\lambda}^\dagger)$ 各自描述一种粒子

利用这样的反对易关系，把哈密顿量算符化为

$$H = \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} E_p (a_{p,\lambda}^\dagger a_{p,\lambda} - b_{p,\lambda}^\dagger b_{p,\lambda}^\dagger)$$

第二项是零点能

$$= \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} E_p (a_{p,\lambda}^\dagger a_{p,\lambda} + b_{p,\lambda}^\dagger b_{p,\lambda}) - (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3 p}{(2\pi)^3} 2E_p$$

第一项是所有动量模式所有螺旋度所有粒子贡献的能量之和，它是正定的

可见，用等时反对易关系对 Dirac 旋量场进行正则量子化是合适的

哈密顿量与产生湮灭算符的对易

 计算哈密顿量 H 与产生湮灭算符的对易子，得到

$$[H, a_{\mathbf{p}, \lambda}^\dagger] = E_{\mathbf{p}} a_{\mathbf{p}, \lambda}^\dagger, \quad [H, a_{\mathbf{p}, \lambda}] = -E_{\mathbf{p}} a_{\mathbf{p}, \lambda}$$

设 $|E\rangle$ 是 H 的**本征态**, 本征值为 E , 则 $H|E\rangle = E|E\rangle$

 从而推出

$$H a_{\mathbf{p}, \lambda}^\dagger |E\rangle = (a_{\mathbf{p}, \lambda}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p}, \lambda}^\dagger) |E\rangle = (E + E_{\mathbf{p}}) a_{\mathbf{p}, \lambda}^\dagger |E\rangle$$

$$H a_{\mathbf{p}, \lambda} |E\rangle = (a_{\mathbf{p}, \lambda} H - E_{\mathbf{p}} a_{\mathbf{p}, \lambda}) |E\rangle = (E - E_{\mathbf{p}}) a_{\mathbf{p}, \lambda} |E\rangle$$

$$H b_{p,\lambda}^\dagger |E\rangle = (b_{p,\lambda}^\dagger H + E_p b_{p,\lambda}^\dagger) |E\rangle = (E + E_p) b_{p,\lambda}^\dagger |E\rangle$$

$$H b_{\mathbf{p},\lambda} |E\rangle = (b_{\mathbf{p},\lambda} H - E_{\mathbf{p}} b_{\mathbf{p},\lambda}) |E\rangle = (E - E_{\mathbf{p}}) b_{\mathbf{p},\lambda} |E\rangle$$

当 $a_{p,\lambda}^\dagger |E\rangle \neq 0$ 和 $b_{p,\lambda}^\dagger |E\rangle \neq 0$ 时, $a_{p,\lambda}^\dagger$ 和 $b_{p,\lambda}^\dagger$ 的作用是使能量本征值增加 E_p

当 $a_{p,\lambda}|E\rangle \neq 0$ 和 $b_{p,\lambda}|E\rangle \neq 0$ 时, $a_{p,\lambda}$ 和 $b_{p,\lambda}$ 的作用是使能量本征值减少 E_p

总动量算符

Dirac 旋量场的总动量算符为

$$\begin{aligned}
\mathbf{P} &= - \int d^3x \pi \nabla \psi = \int d^3x \psi^\dagger (-i\nabla) \psi \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{4E_p E_q}} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger e^{-ip \cdot x} \right] \\
&\quad \times \left[\mathbf{q} u(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} - \mathbf{q} v(\mathbf{q}, \lambda') b_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{4E_p E_q}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(E_p - E_q)t} \right. \right. \\
&\quad \left. \left. - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'}^\dagger e^{-i(E_p - E_q)t} \right] \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[- u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'}^\dagger e^{i(E_p + E_q)t} \right. \right. \\
&\quad \left. \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{-i(E_p + E_q)t} \right] \right\}
\end{aligned}$$

化简总动量



积分，得

$$\begin{aligned}
\mathbf{P} &= \sum_{\lambda\lambda'} \int \frac{d^3 p \, \mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{p}, \lambda'} \right. \\
&\quad \left. + u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} - v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3 p \, \mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} (2E_{\mathbf{p}} \delta_{\lambda\lambda'} a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} - 2E_{\mathbf{p}} \delta_{\lambda\lambda'} b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{p}, \lambda'}) \\
&= \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda} - b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{p}, \lambda}) \quad \text{指向} \quad \{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\
&= \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda} + b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{p}, \lambda}) - 2\delta^{(3)}(\mathbf{0}) \int d^3 p \, \mathbf{p} \\
&= \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (\color{red}a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda} + \color{blue}b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{p}, \lambda})
\end{aligned}$$



总动量是所有动量模式所有螺旋度所有粒子贡献的动量之和

4.5.3 小节 U(1) 整体对称性

类似于复标量场，Dirac 旋量场也具有 $U(1)$ 整体对称性

对 Dirac 旋量场 $\psi(x)$ 作 $U(1)$ 整体变换 $\psi'(x) = e^{i\eta\theta}\psi(x)$

则 $\psi^\dagger(x)$ 和 $\bar{\psi}(x)$ 的相应变换为

$$[\psi^\dagger(x)]' = [\psi'(x)]^\dagger = \psi^\dagger(x) e^{-i\eta\theta}, \quad [\bar{\psi}(x)]' = \bar{\psi}'(x) = [\psi'(x)]^\dagger \gamma^0 = \bar{\psi}(x) e^{-i\eta\theta}$$

 在此变换下，拉氏量不变，

$$\begin{aligned}\mathcal{L}'(x) &= \bar{\psi}'(x)(i\gamma^\mu\partial_\mu - m)\psi'(x) = \bar{\psi}(x)e^{-iq\theta}(i\gamma^\mu\partial_\mu - m)e^{iq\theta}\psi(x) \\ &= \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) = \mathcal{L}(x)\end{aligned}$$

4.5.3 小节 U(1) 整体对称性

类似于复标量场, Dirac 旋量场也具有 $U(1)$ 整体对称性

对 Dirac 旋量场 $\psi(x)$ 作 $U(1)$ 整体变换 $\psi'(x) = e^{i\varphi(x)}\psi(x)$

则 $\psi^\dagger(x)$ 和 $\bar{\psi}(x)$ 的相应变换为

$$[\psi^\dagger(x)]' = [\psi'(x)]^\dagger = \psi^\dagger(x) e^{-\textcolor{brown}{i} q \theta}, \quad [\bar{\psi}(x)]' = \bar{\psi}'(x) = [\psi'(x)]^\dagger \gamma^0 = \bar{\psi}(x) e^{-\textcolor{brown}{i} q \theta}$$

 在此变换下，拉氏量不变，

$$\begin{aligned}\mathcal{L}'(x) &= \bar{\psi}'(x)(i\gamma^\mu\partial_\mu - m)\psi'(x) = \bar{\psi}(x)e^{-iq\theta}(i\gamma^\mu\partial_\mu - m)e^{iq\theta}\psi(x) \\ &= \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) = \mathcal{L}(x)\end{aligned}$$

 于是，跟复标量场一样，Dirac 旋量场具有相位任意性

相差一个任意整体相位因子 $e^{i\varphi}$ 的 Dirac 旋量场描述相同的物理

 容易验证，前面列举的旋量双线性型都在这种 $U(1)$ 整体变换下不变

因此，用这些旋量双线性型构造的拉氏量都具有 U(1) 整体对称性

U(1) 守恒流

 U(1) 整体变换的无穷小形式为

$$\psi'(x) = \psi(x) + i q \theta \psi(x)$$

由于 $\delta x^\mu = 0$, $\bar{\delta}\psi = \delta\psi = iq\theta\psi$

按照 $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \bar{\delta} \Phi_a + \mathcal{L} \delta x^\mu$ ，相应的 **Noether** 守恒流为

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \bar{\delta}\psi = i\bar{\psi}\gamma^\mu(iq\theta\psi) = -q\theta\bar{\psi}\gamma^\mu\psi$$

 扔掉无穷小参数 $-\theta$, 定义 U(1) 守恒流算符

$$J^\mu \equiv q\bar{\psi}\gamma^\mu\psi$$

 则 Noether 定理给出

$$\partial_\mu J^\mu = 0$$

U(1) 守恒荷

相应的 U(1) 守恒荷算符为

$$\begin{aligned}
Q &= \int d^3x \mathcal{J}^0 = q \int d^3x \bar{\psi} \gamma^0 \psi = q \int d^3x \psi^\dagger \psi \\
&= q \sum_{\lambda \lambda'} \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{4E_p E_k}} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right] \\
&\quad \times \left[u(\mathbf{k}, \lambda') a_{\mathbf{k}, \lambda'} e^{-ik \cdot x} + v(\mathbf{k}, \lambda') b_{\mathbf{k}, \lambda'}^\dagger e^{ik \cdot x} \right] \\
&= q \sum_{\lambda \lambda'} \int \frac{d^3p d^3k}{(2\pi)^3 \sqrt{4E_p E_k}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{k}) \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{k}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{k}, \lambda'} e^{i(E_p - E_k)t} \right. \right. \\
&\quad \left. \left. + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{k}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{k}, \lambda'}^\dagger e^{-i(E_p - E_k)t} \right] \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{k}) \left[u^\dagger(\mathbf{p}, \lambda) v(\mathbf{k}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{k}, \lambda'}^\dagger e^{i(E_p + E_k)t} \right. \right. \\
&\quad \left. \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{k}, \lambda') b_{\mathbf{p}, \lambda} a_{\mathbf{k}, \lambda'} e^{-i(E_p + E_k)t} \right] \right\}
\end{aligned}$$

正粒子和反粒子



积分，得

$$\begin{aligned}
Q &= q \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3 2E_p} \left[\textcolor{brown}{u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda'}^\dagger} \right. \\
&\quad \left. + \textcolor{brown}{u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_p t} + v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_p t}} \right] \\
&= q \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3 2E_p} (\textcolor{brown}{2E_p \delta_{\lambda \lambda'} a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'}} + \textcolor{teal}{2E_p \delta_{\lambda \lambda'} b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda'}^\dagger}) \\
&= q \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} (a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda} + \textcolor{blue}{b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda}^\dagger}) \quad \text{指向} \quad \{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} = (2\pi)^3 \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\
&= \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} (\textcolor{red}{q a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda}} - \textcolor{blue}{q b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{p}, \lambda}}) + 2\delta^{(3)}(\mathbf{0}) \int d^3 p \ q \ (\text{零点荷})
\end{aligned}$$

正粒子和反粒子



积分，得

$$\begin{aligned}
Q &= q \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3 2E_p} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda'}^\dagger \right. \\
&\quad \left. + u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_p t} + v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_p t} \right] \\
&= q \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3 2E_p} (2E_\lambda \delta_{\lambda \lambda'} a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} + 2E_p \delta_{\lambda \lambda'} b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda'}^\dagger) \\
&= q \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} (a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda} + b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda}^\dagger) \quad \text{指向} \quad \{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} = (2\pi)^3 \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\
&= \sum_{\lambda=\pm} \int \frac{d^3 p}{(2\pi)^3} (\cancel{q} a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda} - \cancel{q} b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{p}, \lambda}) + 2\delta^{(3)}(\mathbf{0}) \int d^3 p q \quad (\text{零点荷})
\end{aligned}$$



从第一项可以看出，由 $(a_{\mathbf{p}, \lambda}, a_{\mathbf{p}, \lambda}^\dagger)$ 描述的粒子是正粒子，携带的 U(1) 荷为 q



由 $(b_{p,\lambda}, b_{n,\lambda}^\dagger)$ 描述的粒子是反粒子，携带的 U(1) 荷为 $-q$



除去零点荷，总荷是所有动量模式所有螺旋度所有正反粒子贡献的 U(1) 荷之和

4.5.4 小节 粒子态

对于自由 Dirac 旋量场，真空态 $|0\rangle$ 定义为被任意 a_{p_λ} 和任意 b_{p_λ} 涅灭的态，

$$a_{p,\lambda} |0\rangle = b_{p,\lambda} |0\rangle = 0$$



满足

$$\langle 0|0\rangle = 1, \quad H|0\rangle = E_{\text{vac}}|0\rangle, \quad E_{\text{vac}} = -2\delta^{(3)}(\mathbf{0}) \int d^3 p E_{\mathbf{p}}$$



动量为 p 、螺旋度为 λ 的单个正粒子态和单个反粒子态分别定义为

$$|\mathbf{p}^+, \lambda\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}, \lambda}^\dagger |0\rangle, \quad |\mathbf{p}^-, \lambda\rangle \equiv \sqrt{2E_{\mathbf{p}}} b_{\mathbf{p}, \lambda}^\dagger |0\rangle$$

4.5.4 小节 粒子态

对于自由 Dirac 旋量场，真空态 $|0\rangle$ 定义为被任意 $a_{p,\lambda}$ 和任意 $b_{p,\lambda}$ 涫灭的态，

$$a_{p,\lambda} |0\rangle = b_{p,\lambda} |0\rangle = 0$$



满足

$$\langle 0|0\rangle = 1, \quad H|0\rangle = E_{\text{vac}}|0\rangle, \quad E_{\text{vac}} = -2\delta^{(3)}(\mathbf{0}) \int d^3 p E_{\mathbf{p}}$$



动量为 p 、螺旋度为 λ 的单个正粒子态和单个反粒子态分别定义为

$$|\mathbf{p}^+, \lambda\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}, \lambda}^\dagger |0\rangle, \quad |\mathbf{p}^-, \lambda\rangle \equiv \sqrt{2E_{\mathbf{p}}} b_{\mathbf{p}, \lambda}^\dagger |0\rangle$$



根据产生湮灭算符的反对易关系，单粒子态的内积是

$$\begin{aligned}\langle \mathbf{q}^+, \lambda' | \mathbf{p}^+, \lambda \rangle &= \sqrt{4E_{\mathbf{q}} E_{\mathbf{p}}} \langle 0 | a_{\mathbf{q}, \lambda'} a_{\mathbf{p}, \lambda}^\dagger | 0 \rangle \\ &= \sqrt{4E_{\mathbf{q}} E_{\mathbf{p}}} \langle 0 | [(2\pi)^3 \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}] | 0 \rangle \\ &= 2E_{\mathbf{p}} (2\pi)^3 \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q})\end{aligned}$$

$$\langle \mathbf{q}^-, \lambda' | \mathbf{p}^-, \lambda \rangle = \sqrt{4E_{\mathbf{q}} E_{\mathbf{p}}} \langle 0 | b_{\mathbf{q}, \lambda'} b_{\mathbf{p}, \lambda}^\dagger | 0 \rangle = 2E_{\mathbf{p}} (2\pi)^3 \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q})$$

$$\langle \mathbf{q}^-, \lambda' | \mathbf{p}^+, \lambda \rangle = \sqrt{4E_{\mathbf{q}} E_{\mathbf{p}}} \langle 0 | b_{\mathbf{q}, \lambda'} a_{\mathbf{p}, \lambda}^\dagger | 0 \rangle = -\sqrt{4E_{\mathbf{q}} E_{\mathbf{p}}} \langle 0 | a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'} | 0 \rangle = 0$$

A horizontal row of 15 small white circles, evenly spaced, representing a sequence or a set of items.

 根据 $Ha_{p,\lambda}^\dagger |E\rangle = (E + E_p)a_{p,\lambda}^\dagger |E\rangle$ 和 $Hb_{p,\lambda}^\dagger |E\rangle = (E + E_p)b_{p,\lambda}^\dagger |E\rangle$, 有

$$H |\mathbf{p}^+, \lambda\rangle = (E_{\text{vac}} + E_p) |\mathbf{p}^+, \lambda\rangle, \quad H |\mathbf{p}^-, \lambda\rangle = (E_{\text{vac}} + E_p) |\mathbf{p}^-, \lambda\rangle$$

可见, $|p^+, \lambda\rangle$ 和 $|p^-, \lambda\rangle$ 都比真空态多了一份能量 $E_p = \sqrt{|\mathbf{p}|^2 + m^2}$

单粒子态的能量本征值

 根据 $Ha_{p,\lambda}^\dagger |E\rangle = (E + E_p)a_{p,\lambda}^\dagger |E\rangle$ 和 $Hb_{p,\lambda}^\dagger |E\rangle = (E + E_p)b_{p,\lambda}^\dagger |E\rangle$, 有

$$H |\mathbf{p}^+, \lambda\rangle = (E_{\text{vac}} + E_p) |\mathbf{p}^+, \lambda\rangle, \quad H |\mathbf{p}^-, \lambda\rangle = (E_{\text{vac}} + E_p) |\mathbf{p}^-, \lambda\rangle$$

可见, $|p^+, \lambda\rangle$ 和 $|p^-, \lambda\rangle$ 都比真空态多了一份能量 $E_p = \sqrt{|\mathbf{p}|^2 + m^2}$

 将 $\psi(x)$ 的平面波解代入 $[\psi(x), \mathbf{J}] = (\mathbf{L} + \mathbf{S})\psi(x)$ 左边，得

$$[\psi(x), \mathbf{J}] = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm} \left\{ u(\mathbf{p}, \lambda) [a_{\mathbf{p}, \lambda}, \mathbf{J}] e^{-ip \cdot x} + v(\mathbf{p}, \lambda) [b_{\mathbf{p}, \lambda}^\dagger, \mathbf{J}] e^{ip \cdot x} \right\}$$

 代入右边，得

$$\begin{aligned}
& (\mathbf{L} + \mathbf{S})\psi(\mathbf{x}) \\
&= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} (-i\mathbf{x} \times \nabla + \mathbf{S}) \left[u(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right] \\
&= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[(\mathbf{x} \times \mathbf{p} + \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + (-\mathbf{x} \times \mathbf{p} + \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right]
\end{aligned}$$

[$a_{\mathbf{p},\lambda}$, $2\hat{\mathbf{p}} \cdot \mathbf{J}$] 和 [$b_{\mathbf{p},\lambda}^\dagger$, $2\hat{\mathbf{p}} \cdot \mathbf{J}$]

两相比较，对于动量模式 p 和螺旋度 λ ，有

$$u(\mathbf{p}, \lambda)[a_{\mathbf{p}, \lambda}, \mathbf{J}] = (\mathbf{x} \times \mathbf{p} + \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}, \quad v(\mathbf{p}, \lambda)[b_{\mathbf{p}, \lambda}^\dagger, \mathbf{J}] = (-\mathbf{x} \times \mathbf{p} + \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger$$

 按照前面讨论, $u(p, \lambda)$ 和 $v(p, \lambda)$ 分别是本征值为 λ 和 $-\lambda$ 的螺旋度本征态, 故

$$\begin{aligned} u(\mathbf{p}, \lambda)[a_{\mathbf{p}, \lambda}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] &= 2\hat{\mathbf{p}} \cdot (\mathbf{x} \times \mathbf{p} + \mathbf{S})u(\mathbf{p}, \lambda)a_{\mathbf{p}, \lambda} \\ &= (2\hat{\mathbf{p}} \cdot \mathbf{S})u(\mathbf{p}, \lambda)a_{\mathbf{p}, \lambda} = \lambda u(\mathbf{p}, \lambda)a_{\mathbf{p}, \lambda} \end{aligned}$$

[$a_{\mathbf{p},\lambda}$, $2\hat{\mathbf{p}} \cdot \mathbf{J}$] 和 [$b_{\mathbf{p},\lambda}^\dagger$, $2\hat{\mathbf{p}} \cdot \mathbf{J}$]

两相比较，对于动量模式 p 和螺旋度 λ ，有

$$u(\mathbf{p}, \lambda)[a_{\mathbf{p}, \lambda}, \mathbf{J}] = (\mathbf{x} \times \mathbf{p} + \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}, \quad v(\mathbf{p}, \lambda)[b_{\mathbf{p}, \lambda}^\dagger, \mathbf{J}] = (-\mathbf{x} \times \mathbf{p} + \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger$$

 按照前面讨论, $u(p, \lambda)$ 和 $v(p, \lambda)$ 分别是本征值为 λ 和 $-\lambda$ 的螺旋度本征态, 故

$$\begin{aligned} u(\mathbf{p}, \lambda)[a_{\mathbf{p}, \lambda}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] &= 2\hat{\mathbf{p}} \cdot (\mathbf{x} \times \mathbf{p} + \mathbf{S})u(\mathbf{p}, \lambda)a_{\mathbf{p}, \lambda} \\ &= (2\hat{\mathbf{p}} \cdot \mathbf{S})u(\mathbf{p}, \lambda)a_{\mathbf{p}, \lambda} = \lambda u(\mathbf{p}, \lambda)a_{\mathbf{p}, \lambda} \end{aligned}$$

$$\begin{aligned} v(\mathbf{p}, \lambda)[b_{\mathbf{p}, \lambda}^\dagger, 2\hat{\mathbf{p}} \cdot \mathbf{J}] &= 2\hat{\mathbf{p}} \cdot (-\mathbf{x} \times \mathbf{p} + \mathbf{S})v(\mathbf{p}, \lambda)b_{\mathbf{p}, \lambda}^\dagger \\ &= (2\hat{\mathbf{p}} \cdot \mathbf{S})v(\mathbf{p}, \lambda)b_{\mathbf{p}, \lambda}^\dagger = -\cancel{\lambda} v(\mathbf{p}, \lambda)b_{\mathbf{p}, \lambda}^\dagger \end{aligned}$$

因而

$$[a_{\mathbf{p},\lambda}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] = \lambda a_{\mathbf{p},\lambda}, \quad [b_{\mathbf{p},\lambda}^\dagger, 2\hat{\mathbf{p}} \cdot \mathbf{J}] = -\lambda b_{\mathbf{p},\lambda}^\dagger$$

由于 J 是厄米算符，对第一式取厄米共轭得

$$\lambda \color{red}{a_{\mathbf{p},\lambda}^\dagger} = [a_{\mathbf{p},\lambda}, 2\hat{\mathbf{p}} \cdot \mathbf{J}]^\dagger = (2\hat{\mathbf{p}} \cdot \mathbf{J})a_{\mathbf{p},\lambda}^\dagger - a_{\mathbf{p},\lambda}^\dagger(2\hat{\mathbf{p}} \cdot \mathbf{J}) = [2\hat{\mathbf{p}} \cdot \mathbf{J}, \color{red}{a_{\mathbf{p},\lambda}^\dagger}]$$

单粒子态的螺旋度本征值

于是, $[2\hat{\mathbf{p}} \cdot \mathbf{J}, a_{\mathbf{p}, \lambda}^\dagger] = \lambda a_{\mathbf{p}, \lambda}^\dagger$, $[2\hat{\mathbf{p}} \cdot \mathbf{J}, b_{\mathbf{p}, \lambda}^\dagger] = \lambda b_{\mathbf{p}, \lambda}^\dagger$

粉色的 J 是总角动量算符，真空态 $|0\rangle$ 满足 $\text{J}|0\rangle = 0$ ，由此得到

$$(2\hat{\mathbf{p}} \cdot \mathbf{J})a_{\mathbf{p},\lambda}^\dagger |0\rangle = [a_{\mathbf{p},\lambda}^\dagger (2\hat{\mathbf{p}} \cdot \mathbf{J}) + \lambda a_{\mathbf{p},\lambda}^\dagger] |0\rangle = \lambda a_{\mathbf{p},\lambda}^\dagger |0\rangle$$

$$(2\hat{\mathbf{p}} \cdot \mathbf{J}) b_{\mathbf{p},\lambda}^\dagger |0\rangle = [b_{\mathbf{p},\lambda}^\dagger (2\hat{\mathbf{p}} \cdot \mathbf{J}) + \lambda b_{\mathbf{p},\lambda}^\dagger] |0\rangle = \lambda b_{\mathbf{p},\lambda}^\dagger |0\rangle$$

 自由的单粒子态没有轨道角动量，而 $\hat{\mathbf{p}} \cdot \hat{\mathbf{J}}$ 相当于归一化的螺旋度算符

单粒子态的螺旋度本征值

于是, $[2\hat{\mathbf{p}} \cdot \mathbf{J}, a_{\mathbf{p}, \lambda}^\dagger] = \lambda a_{\mathbf{p}, \lambda}^\dagger$, $[2\hat{\mathbf{p}} \cdot \mathbf{J}, b_{\mathbf{p}, \lambda}^\dagger] = \lambda b_{\mathbf{p}, \lambda}^\dagger$

④ \mathbf{J} 是总角动量算符, 真空态 $|0\rangle$ 满足 $\mathbf{J}|0\rangle = \mathbf{0}$, 由此得到

$$(2 \hat{\mathbf{p}} \cdot \mathbf{J}) a_{\mathbf{p}, \lambda}^\dagger |0\rangle = [a_{\mathbf{p}, \lambda}^\dagger (2 \hat{\mathbf{p}} \cdot \mathbf{J}) + \lambda a_{\mathbf{p}, \lambda}^\dagger] |0\rangle = \lambda a_{\mathbf{p}, \lambda}^\dagger |0\rangle$$

$$(2 \hat{\mathbf{p}} \cdot \mathbf{J}) b_{\mathbf{p},\lambda}^\dagger |0\rangle = [b_{\mathbf{p},\lambda}^\dagger (2 \hat{\mathbf{p}} \cdot \mathbf{J}) + \lambda b_{\mathbf{p},\lambda}^\dagger] |0\rangle = \lambda b_{\mathbf{p},\lambda}^\dagger |0\rangle$$

 自由的单粒子态没有轨道角动量，而 $\hat{\mathbf{p}} \cdot \mathbf{J}$ 相当于归一化的螺旋度算符

因此，上面两式说明 $|p^+, \lambda\rangle$ 和 $|p^-, \lambda\rangle$ 都是螺旋度本征态，本征值为 λ ：

$$(2 \hat{\mathbf{p}} \cdot \mathbf{J}) |\mathbf{p}^\pm, \lambda\rangle = \lambda |\mathbf{p}^\pm, \lambda\rangle$$

以上讨论表明，产生算符 $a_{p,\lambda}^\dagger$ 的作用是产生一个动量为 p 、螺旋度为 λ 的正粒子。

另一个产生算符 $b_{p,\lambda}^\dagger$ 的作用是产生一个动量为 p 、螺旋度为 λ 的反粒子

 正粒子和反粒子具有相同的质量 m

湮灭算符的作用

 在 $\tilde{f}_\lambda(\mathbf{p}) = \tilde{C}_\lambda \xi_{-\lambda}(\mathbf{p})$ 中，我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$ ，使得 $v(\mathbf{p}, \lambda)$ 的螺旋度本征值为 $-\lambda$ ，从而得到 $b_{\mathbf{p}, \lambda}^\dagger |0\rangle$ 的螺旋度本征值为 λ 的结果

 如果我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_\lambda(\mathbf{p})$ ，依照上述推导， $b_{\mathbf{p}, \lambda}^\dagger |0\rangle$ 的螺旋度本征值就会变成 $-\lambda$ ，则 $(b_{\mathbf{p}, \lambda}, b_{\mathbf{p}, \lambda}^\dagger)$ 将描述螺旋度为 $-\lambda$ 的反粒子

 这不符合我们的记号，因此，我们将 $\tilde{f}_\lambda(\mathbf{p})$ 取为 $\tilde{f}_\lambda(\mathbf{p}) = \tilde{C}_\lambda \xi_{-\lambda}(\mathbf{p})$

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由产生湮灭算符的反对易关系，有

$$\begin{aligned} a_{\mathbf{p}, \lambda} |\mathbf{q}^+, \lambda' \rangle &= \sqrt{2E_{\mathbf{q}}} a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger |0\rangle = \sqrt{2E_{\mathbf{q}}} [(2\pi)^3 \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - a_{\mathbf{q}, \lambda'}^\dagger a_{\mathbf{p}, \lambda}] |0\rangle \\ &= \sqrt{2E_{\mathbf{q}}} (2\pi)^3 \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle \end{aligned}$$

$$\begin{aligned} b_{\mathbf{p}, \lambda} |\mathbf{q}^-, \lambda' \rangle &= \sqrt{2E_{\mathbf{q}}} b_{\mathbf{p}, \lambda} b_{\mathbf{q}, \lambda'}^\dagger |0\rangle = \sqrt{2E_{\mathbf{q}}} [(2\pi)^3 \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - b_{\mathbf{q}, \lambda'}^\dagger b_{\mathbf{p}, \lambda}] |0\rangle \\ &= \sqrt{2E_{\mathbf{q}}} (2\pi)^3 \delta_{\lambda \lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle \end{aligned}$$

 可以看出，湮灭算符 $a_{p,\lambda}$ 的作用是减少一个动量为 p 、螺旋度为 λ 的正粒子

 涅灭算符 $b_{p,\lambda}$ 的作用是减少一个动量为 p 、螺旋度为 λ 的反粒子

粒子交换

 将包含 2 个正粒子和 2 个反粒子的态记为

$$|\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2; \mathbf{p}_3^-, \lambda_3; \mathbf{p}_4^-, \lambda_4\rangle \equiv \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{p}_3}E_{\mathbf{p}_4}}a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle$$

多次利用反对易关系 $\{a_{p,\lambda}, a_{q,\lambda'}^\dagger\} = \{b_{p,\lambda}, b_{q,\lambda'}^\dagger\} = \{a_{p,\lambda}^\dagger, b_{q,\lambda'}^\dagger\} = 0$

 调换产生算符的位置，可得

$$a_{p_1, \lambda_1}^\dagger a_{p_2, \lambda_2}^\dagger b_{p_3, \lambda_3}^\dagger b_{p_4, \lambda_4}^\dagger |0\rangle = - b_{p_4, \lambda_4}^\dagger a_{p_2, \lambda_2}^\dagger b_{p_3, \lambda_3}^\dagger a_{p_1, \lambda_1}^\dagger |0\rangle$$

 负号源自奇数次反对易，从而

$$|\mathbf{p}_4^-, \lambda_4; \mathbf{p}_2^+, \lambda_2; \mathbf{p}_3^-, \lambda_3; \mathbf{p}_1^+, \lambda_1\rangle = - |\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2; \mathbf{p}_3^-, \lambda_3; \mathbf{p}_4^-, \lambda_4\rangle$$

即交换第 1 和第 4 个粒子得到的态与原来的态相差一个负号

 同理，交换其中任意两个粒子，也会出现一个负号

费米子与 Pauli 不相容原理

对于包含多个全同粒子的态，**交换**任意两个**全同粒子**，需要对**产生算符**进行**奇数次反对易**，得到的态与原来的态相差一个**负号**

也就是说，态对**全同粒子交换**是**反对称的**

这说明 **Dirac 旋量场** 描述的粒子是一种**费米子**，称为 **Dirac 费米子**，服从 **Fermi-Dirac 统计**

得到这个结论的关键在于两个**产生算符反对易**



Enrico Fermi
(1901–1954)

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得到这个结论的关键在于两个产生算符**反对易**

对于两个相同的产生算符 $a_{p_\lambda}^\dagger$ 或 $b_{p_\lambda}^\dagger$ ，反对易关系导致

$$a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda}^\dagger |0\rangle = -a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda}^\dagger |0\rangle, \quad b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{p}, \lambda}^\dagger |0\rangle = -b_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{p}, \lambda}^\dagger |0\rangle$$

 故

$$a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}^\dagger |0\rangle = \mathbf{0}, \quad b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}^\dagger |0\rangle = \mathbf{0}$$

 没有其它自由度时，**不存在动量和螺旋度都相同的两个正费米子或两个反费米子组成的态**，这符合 **Pauli 不相容原理**



Enrico Fermi
(1901–1954)



Wolfgang Ernst Pauli (1900–1958)

自旋—统计定理

 在第 2 章和第 3 章中，我们分别讨论了**自旋为 0 的标量场**和**自旋为 1 的矢量场**，合适的处理方式是通过**对易关系**对它们进行量子化，因而它们都描述**玻色子**

 在本章中，我们需要采用**反对易关系**才能对**自旋为 1/2 的旋量场**进行合适的量子化，因而旋量场描述的粒子是**费米子**

 实际上，这样的状况是普遍的，存在

自旋—统计定理

 整数自旋的物理场必须用**对易关系**进行量子化，对应的粒子是**玻色子**

 半整数自旋的物理场必须用**反对易关系**进行量子化，对应的粒子是**费米子**

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 可从多个角度证明这个定理成立，前面已说明哈密顿量算符的正定性要求它成立

 此外，也可以从交换全同粒子的路径依赖性、散射矩阵的 Lorentz 不变性、因果性的角度加以证明（详细讨论见 M. D. Schwartz 的书 *Quantum Field Theory and the Standard Model* 第 12 章）

双粒子态内积

记两个正费米子组成的双粒子态为 $|\mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2\rangle \equiv \sqrt{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger |0\rangle$

双粒子态的内积关系是

$$\begin{aligned}
& \langle \mathbf{q}_1^+, \lambda_1'; \mathbf{q}_2^+, \lambda_2' | \mathbf{p}_1^+, \lambda_1; \mathbf{p}_2^+, \lambda_2 \rangle \\
&= \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}} \langle 0 | a_{\mathbf{q}_2, \lambda_2'} \textcolor{blue}{a_{\mathbf{q}_1, \lambda_1'}} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \\
&= \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}} \left[(2\pi)^3 \delta_{\lambda_1 \lambda_1'} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2, \lambda_2'} a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \right. \\
&\quad \left. - \langle 0 | a_{\mathbf{q}_2, \lambda_2'} \textcolor{blue}{a_{\mathbf{p}_1, \lambda_1}} a_{\mathbf{q}_1, \lambda_1'} a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \right] \\
&= \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}} \left[(2\pi)^3 \delta_{\lambda_1 \lambda_1'} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2, \lambda_2'} a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \right. \\
&\quad \left. - (2\pi)^3 \delta_{\lambda_2 \lambda_1'} \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2, \lambda_2'} a_{\mathbf{p}_1, \lambda_1}^\dagger | 0 \rangle \right] \\
&= 4E_{\mathbf{p}_1}E_{\mathbf{p}_2}(2\pi)^6 \left[\delta_{\lambda_1 \lambda_1'} \delta_{\lambda_2 \lambda_2'} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2) \right. \\
&\quad \left. - \delta_{\lambda_1 \lambda_2'} \delta_{\lambda_2 \lambda_1'} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \right]
\end{aligned}$$

 最后两行方括号中第二项前面有一个**负号**, 由**产生湮灭算符的反对易关系**引起

这是双费米子态内积关系与双玻色子态内积关系在形式上的不同之处