

# 量子场论

## 第 9 章 分立对称性和 Majorana 旋量场

### 9.6 节和 9.7 节

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# 9.6 节 Weyl、Dirac 和 Majorana 旋量

## 9.6.1 小节 左手和右手 Weyl 旋量



Dirac 旋量场和 Majorana 旋量场都可以分解为左手和右手的 Weyl 旋量场



为了更深刻地认识旋量场, 本节进一步研究 Weyl 旋量



用  $\sigma^\mu = (1, \sigma)$  和  $\bar{\sigma}^\mu = (1, -\sigma)$  定义  $2 \times 2$  矩阵

$$s^{\mu\nu} \equiv \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$$



由  $(\sigma^\mu)^\dagger = \sigma^\mu$  和  $(\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu$  推出

$$(s^{\mu\nu})^\dagger = -\frac{i}{4}[(\bar{\sigma}^\nu)^\dagger (\sigma^\mu)^\dagger - (\bar{\sigma}^\mu)^\dagger (\sigma^\nu)^\dagger] = -\frac{i}{4}(\bar{\sigma}^\nu \sigma^\mu - \bar{\sigma}^\mu \sigma^\nu) = \frac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)$$

# 9.6 节 Weyl、Dirac 和 Majorana 旋量

## 9.6.1 小节 左手和右手 Weyl 旋量

 Dirac 旋量场和 Majorana 旋量场都可以分解为左手和右手的 Weyl 旋量场

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 从而将 Weyl 表象中的旋量表示生成元化为

$$\mathcal{S}^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{4} \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix} = \begin{pmatrix} s^{\mu\nu} & \\ & (s^{\mu\nu})^\dagger \end{pmatrix}$$

 也就是说， $4 \times 4$  矩阵  $\mathcal{S}^{\mu\nu}$  是  $2 \times 2$  矩阵  $s^{\mu\nu}$  和  $(s^{\mu\nu})^\dagger$  的直和

 因而  $s^{\mu\nu}$  和  $(s^{\mu\nu})^\dagger$  是两个 Lorentz 群 2 维表示的生成元

## 左手和右手 Weyl 旋量所处 2 维表示

对于 Lorentz 变换  $\Lambda$  的一组变换参数  $\omega_{\mu\nu}$ ，用  $s^{\mu\nu}$  生成固有保时向有限变换

$$d(\Lambda) \equiv \exp \left( -\frac{i}{2} \omega_{\mu\nu} s^{\mu\nu} \right)$$

它属于左手 Weyl 旋量所处的 2 维表示

相应的逆变换矩阵为  $d^{-1}(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}s^{\mu\nu}\right)$ ，取厄米共轭，得

$$d^{-1\dagger}(\Lambda) = \exp \left[ -\frac{i}{2} \omega_{\mu\nu} (s^{\mu\nu})^\dagger \right]$$

这是用  $(s^{\mu\nu})^\dagger$  生成的固有保时向有限变换，属于右手 Weyl 旋量所处的 2 维表示

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这是用  $(s^{\mu\nu})^\dagger$  生成的固有保时向有限变换，属于右手 Weyl 旋量所处的 2 维表示

于是，旋量表示的  $4 \times 4$  Lorentz 变换矩阵分解为

$$D(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{S}^{\mu\nu}\right) = \begin{pmatrix} e^{-i\omega_{\mu\nu} s^{\mu\nu}/2} & \\ & e^{-i\omega_{\mu\nu} (s^{\mu\nu})^\dagger/2} \end{pmatrix} = \begin{pmatrix} d(\Lambda) & \\ & d^{-1\dagger}(\Lambda) \end{pmatrix}$$

因此, 4 维旋量表示  $\{D(\Lambda)\}$  是 2 维表示  $\{d(\Lambda)\}$  和  $\{d^{-1\dagger}(\Lambda)\}$  的直和

## 等价表示

利用  $\sigma^2 \sigma^\mu \sigma^2 = (\bar{\sigma}^\mu)^T$  和  $\sigma^2 \bar{\sigma}^\mu \sigma^2 = (\sigma^\mu)^T$  推出

$$\begin{aligned}\sigma^2 s^{\mu\nu} \sigma^2 &= \frac{i}{4} (\sigma^2 \sigma^\mu \sigma^2 \bar{\sigma}^\nu \sigma^2 - \sigma^2 \sigma^\nu \sigma^2 \bar{\sigma}^\mu \sigma^2) \\ &= \frac{i}{4} [(\bar{\sigma}^\mu)^T (\sigma^\nu)^T - (\bar{\sigma}^\nu)^T (\sigma^\mu)^T] = -(s^{\mu\nu})\end{aligned}$$

$$\begin{aligned} \sigma^2 d(\Lambda) \sigma^2 &= \exp \left( -\frac{i}{2} \omega_{\mu\nu} \sigma^2 s^{\mu\nu} \sigma^2 \right) = \exp \left[ \frac{i}{2} \omega_{\mu\nu} (s^{\mu\nu})^T \right] \\ &= \left[ \exp \left( \frac{i}{2} \omega_{\mu\nu} s^{\mu\nu} \right) \right]^T = d^{-1T}(\Lambda) \end{aligned}$$

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利用  $\sigma^2 \sigma^\mu \sigma^2 = (\bar{\sigma}^\mu)^T$  和  $\sigma^2 \bar{\sigma}^\mu \sigma^2 = (\sigma^\mu)^T$  推出

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$$= \frac{i}{4} [(\bar{\sigma}^\mu)^T (\sigma^\nu)^T - (\bar{\sigma}^\nu)^T (\sigma^\mu)^T] = -(s^{\mu\nu})^T$$

$$\sigma^2 d(\Lambda) \sigma^2 = \exp\left(-\frac{i}{2} \omega_{\mu\nu} \sigma^2 s^{\mu\nu} \sigma^2\right) = \exp\left[\frac{i}{2} \omega_{\mu\nu} (s^{\mu\nu})^T\right]$$

$$= \left[ \exp \left( \frac{i}{2} \omega_{\mu\nu} s^{\mu\nu} \right) \right]^T = d^{-1T}(\Lambda)$$

将 Pauli 矩阵  $\sigma^2$  看作一个幺正变换矩阵, 满足  $(\sigma^2)^{-1} = (\sigma^2)^\dagger = \sigma^2$

则  $d(\Lambda)$  与  $d^{-1T}(\Lambda)$  由一个相似变换联系起来, 相似变换矩阵为  $\sigma^2$

根据 1.4 节定义, 线性表示  $\{d(\Lambda)\}$  和  $\{d^{-1T}(\Lambda)\}$  是等价的

由于  $(\sigma^2)^* = -\sigma^2$ ,  $\sigma^2 d(\Lambda) \sigma^2 = d^{-1T}(\Lambda)$  的复共轭为  $\sigma^2 d^*(\Lambda) \sigma^2 = d^{-1\dagger}(\Lambda)$

  $\{d^*(\Lambda)\}$  是  $\{d(\Lambda)\}$  的复共轭表示, 而线性表示  $\{d^{-1\dagger}(\Lambda)\}$  与  $\{d^*(\Lambda)\}$  等价

## 左手 Weyl 旋量



于是，左手 Weyl 旋量

$$\eta_a = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

的固有保时向 Lorentz 变换为

$$\eta'_a = [d(\Lambda)]_a{}^b \eta_b, \quad a, b = 1, 2$$



$\eta_a$  是  $\{d(\Lambda)\}$  表示空间中的列矢量

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引入反对称的二维 Levi-Civita 符号  $\varepsilon^{ab}$ ，定义为

$$\varepsilon^{12} = -\varepsilon^{21} = 1, \quad \varepsilon^{11} = \varepsilon^{22} = 0$$



它与 Pauli 矩阵  $\sigma^2$  的关系是

$$\varepsilon^{ab} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = i \begin{pmatrix} & -i \\ i & \end{pmatrix} = (i\sigma^2)^{ab}$$

### 等价的左手 Weyl 旋量

通过  $\varepsilon^{ab}$  定义

$$\eta^a \equiv \varepsilon^{ab} \eta_b = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ -\eta_1 \end{pmatrix}$$

则

$$\eta^1 = \eta_2, \quad \eta^2 = -\eta_1$$

等价于  $\sigma^2 d(\Lambda) \sigma^2 = d^{-1T}(\Lambda)$ ，故  $\eta^a$  的 Lorentz 变换为

$$\begin{aligned}\eta'^a &= \varepsilon^{ab} \eta'_b = \varepsilon^{ab} [d(\Lambda)]_b^c \eta_c = i[\sigma^2 d(\Lambda)]^{ac} \eta_c \\ &= i[d^{-1T}(\Lambda) \sigma^2]^{ac} \eta_c = [d^{-1T}(\Lambda)]^a_b \varepsilon^{bc} \eta_c\end{aligned}$$

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即

$$\eta'^a = [d^{-1\top}(\Lambda)]^a{}_b \eta^b$$

可见  $\eta^a$  是  $\{d^{-1T}(\Lambda)\}$  表示空间中的列矢量

由于  $\{d^{-1T}(\Lambda)\}$  等价于  $\{d(\Lambda)\}$ ， $\eta^a$  也是左手 Weyl 旋量

$\varepsilon^{ab}$  和  $\varepsilon_{ab}$

两种左手 Weyl 旋量  $\eta_a$  与  $\eta^a$  是等价的，它们之间的关系类似于 Lorentz 逆变矢量  $A^\mu$  与协变矢量  $A_\mu = g_{\mu\nu} A^\nu$  之间的关系

  $\varepsilon^{ab}$  的作用类似于度规  $g_{\mu\nu}$ ，相当于 2 维旋量空间的“度规”，用于提升旋量指标

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$$\varepsilon_{ab} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -i \begin{pmatrix} & -i \\ i & \end{pmatrix} = (-i\sigma^2)_{ab}$$

  $\varepsilon_{ab}$  是  $\varepsilon^{ab}$  的逆矩阵, 满足

$$\varepsilon_{ab}\varepsilon^{bc} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \delta_a{}^c$$

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于是,  $\eta^1 = \eta_2$  和  $\eta^2 = -\eta_1$  表明

$$\eta_a = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta^2 \\ \eta^1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} = \varepsilon_{ab} \eta^b$$

也就是说,  $\varepsilon_{ab}$  用于下降旋量指标

## 左手 Weyl 旋量的内积

任意两个左手 Weyl 旋量  $\eta_a$  和  $\zeta_a$  的内积

$$\eta^a \zeta_a = \varepsilon^{ab} \eta_b \zeta_a = \varepsilon_{ab} \eta^a \zeta^b$$

在固有保时向 Lorentz 变换下不变，满足

$$\eta'^a \zeta'_a = [d^{-1\mathrm{T}}(\Lambda)]^a{}_b \eta^b [d(\Lambda)]_a{}^c \zeta_c = \eta^b [d^{-1}(\Lambda)]_b{}^a [d(\Lambda)]_a{}^c \zeta_c = \eta^b \delta_b{}^c \zeta_c = \eta^a \zeta_a$$

 第二步用了转置性质  $[d^{-1T}(\Lambda)]^a{}_b = [d^{-1}(\Lambda)]_b{}^a$ ，可见  $\eta^a \zeta_a$  是 Lorentz 标量

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由  $\eta^1 = \eta_2$ 、 $\eta^2 = -\eta_1$ 、 $\zeta^1 = \zeta_2$  和  $\zeta^2 = -\zeta_1$  得

$$\eta^a \zeta_a = \eta^1 \zeta_1 + \eta^2 \zeta_2 = \eta_2 \zeta_1 - \eta_1 \zeta_2 = -\eta_2 \zeta^2 - \eta_1 \zeta^1 = -\eta_1 \zeta^a$$

即参与缩并的旋量指标一升一降会多出一个负号

这种性质与 Lorentz 矢量内积  $A^\mu B_\mu = A_\mu B^\mu$  截然不同

原因在于旋量空间度规  $\varepsilon^{ab}$  是反对称的

## Grassmann 数

羊  $\eta^a \zeta_a = -\eta_a \zeta^a$  表明  $\eta^a \eta_a = -\eta_a \eta^a$ ，若  $\eta_a$  和  $\eta^a$  是普通的复数，则  $\eta^a \eta_a = 0$

为了使  $\eta^a \eta_a \neq 0$ ，必须要求左手 Weyl 旋量  $\eta^a$  与  $\eta_a$  反对易

即它们是 Grassmann 数, 任意两个 Grassmann 数都是反对易的

 若干个 Grassmann 数的线性组合也是 Grassmann 数

因此,  $\eta_a$  是 Grassmann 数意味着  $\eta^a = \varepsilon^{ab}\eta_b$  也是 Grassmann 数

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虽然如此，Grassmann 数是反对易的  $c$  数，不是算符

对 Grassmann 数表达的旋量场进行量子化，才得到旋量场算符，而 Grassmann 数的反对易性质与旋量场算符的反对易关系相匹配

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假设  $\eta_a$  和  $\zeta^a$  都是 Grassmann 数，则  $\eta_a \zeta^a = -\zeta^a \eta_a$ ，相应地，将省略旋量指标的内积写成  $\eta \zeta \equiv \eta^a \zeta_a = -\eta_a \zeta^a = \zeta^a \eta_a = \zeta \eta$ ，即内积  $\eta \zeta$  和  $\zeta \eta$  是相等的

内积  $\eta^a \eta_a$  有等价表达式  $\eta \eta = \eta^a \eta_a = \varepsilon_{ab} \eta^a \eta^b = -\eta^1 \eta^2 + \eta^2 \eta^1 = -2\eta^1 \eta^2$

$$= 2\eta_2\eta_1 = \eta_2\eta_1 - \eta_1\eta_2 = -\varepsilon^{ab}\eta_a\eta_b = -\eta_a\eta^a$$

## 左手 Weyl 旋量的复共轭



将左手 Weyl 旋量  $\eta_a$  的复共轭记为  $\eta_a^\dagger = \begin{pmatrix} \eta_1^\dagger \\ \eta_2^\dagger \end{pmatrix}$



量子化之后，算符  $\eta_a$  和  $\eta_{\dot{a}}^\dagger$  互为厄米共轭



对  $\eta'_a = [d(\Lambda)]_a^b \eta_b$  两边取复共轭, 得到  $\eta_a^\dagger$  的 Lorentz 变换

$$\eta_{\dot{a}}'^{\dagger} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} \eta_{\dot{b}}^{\dagger}$$

# 左手 Weyl 旋量的复共轭

将左手 Weyl 旋量  $\eta_a$  的复共轭记为  $\eta_{\dot{a}}^\dagger = \begin{pmatrix} \eta_i^\dagger \\ \eta_{\dot{2}}^\dagger \end{pmatrix}$

量子化之后, 算符  $\eta_a$  和  $\eta_{\dot{a}}^\dagger$  互为厄米共轭

对  $\eta'_a = [d(\Lambda)]_a^b \eta_b$  两边取复共轭, 得到  $\eta_{\dot{a}}^\dagger$  的 Lorentz 变换

$$\eta_{\dot{a}}'^\dagger = [d^*(\Lambda)]_{\dot{a}}^{\dot{b}} \eta_{\dot{b}}^\dagger$$

引进指标上带着点号的二维 Levi-Civita 符号

$$\varepsilon^{\dot{a}\dot{b}} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = (\mathrm{i}\sigma^2)^{\dot{a}\dot{b}}, \quad \varepsilon_{\dot{a}\dot{b}} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = (-\mathrm{i}\sigma^2)_{\dot{a}\dot{b}}$$

其分量数值与  $\varepsilon^{ab}$  和  $\varepsilon_{ab}$  分别相同

定义  $\eta^{\dagger\dot{a}} \equiv \varepsilon^{\dot{a}\dot{b}} \eta_{\dot{b}}^\dagger = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \eta_i^\dagger \\ \eta_{\dot{2}}^\dagger \end{pmatrix} = \begin{pmatrix} \eta_{\dot{2}}^\dagger \\ -\eta_i^\dagger \end{pmatrix}$ , 则有  $\eta^{\dagger i} = \eta_{\dot{2}}^\dagger$  和  $\eta^{\dagger\dot{2}} = -\eta_i^\dagger$

## 右手 Weyl 旋量

山羊  $\sigma^2 d^*(\Lambda) \sigma^2 = d^{-1\dagger}(\Lambda)$  等价于  $\sigma^2 d^*(\Lambda) = d^{-1\dagger}(\Lambda) \sigma^2$

故  $\eta^{\dagger a}$  的 Lorentz 变换为

$$\begin{aligned}\eta'^{\dagger a} &= \varepsilon^{\dot{a}\dot{b}} \eta_b'^{\dagger} = \varepsilon^{\dot{a}\dot{b}} [d^*(\Lambda)]_{\dot{b}}{}^{\dot{c}} \eta_c'^{\dagger} = i[\sigma^2 d^*(\Lambda)]^{\dot{a}\dot{c}} \eta_c'^{\dagger} \\ &= i[d^{-1\dagger}(\Lambda) \sigma^2]^{\dot{a}\dot{c}} \eta_c'^{\dagger} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}} \varepsilon^{\dot{b}\dot{c}} \eta_c'^{\dagger}\end{aligned}$$

即

$$\eta'^{\dagger \dot{a}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{b}} \eta^{\dagger \dot{b}}$$

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山羊  $\sigma^2 d^*(\Lambda) \sigma^2 = d^{-1\dagger}(\Lambda)$  等价于  $\sigma^2 d^*(\Lambda) = d^{-1\dagger}(\Lambda) \sigma^2$

故  $\eta^{\dagger a}$  的 Lorentz 变换为

$$\begin{aligned} \eta^{\dagger \dot{a}} &= \varepsilon^{\dot{a} \dot{b}} \eta_{\dot{b}}^{\dagger} = \varepsilon^{\dot{a} \dot{b}} [d^*(\Lambda)]_{\dot{b}}{}^{\dot{c}} \eta_{\dot{c}}^{\dagger} = i[\sigma^2 d^*(\Lambda)]^{\dot{a} \dot{c}} \eta_{\dot{c}}^{\dagger} \\ &= i[d^{-1\dagger}(\Lambda) \sigma^2]^{\dot{a} \dot{c}} \eta_{\dot{c}}^{\dagger} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}} \varepsilon^{\dot{b} \dot{c}} \eta_{\dot{c}}^{\dagger} \end{aligned}$$

即

$$\eta'^{\dagger \dot{a}} = [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{b}} \eta^{\dagger \dot{b}}$$

可见,  $\eta^{\dagger a}$  是  $\{d^{-1\dagger}(\Lambda)\}$  表示空间中的列矢量, 因而是右手 Weyl 旋量

由于表示  $\{d^*(\Lambda)\}$  等价于  $\{d^{-1\dagger}(\Lambda)\}$ ， $\eta_a^\dagger$  也是右手 Weyl 旋量

因此，在这套符号约定中，不带点的旋量指标对应于左手 Weyl 旋量及其表示

而带点的旋量指标对应于右手 Weyl 旋量及其表示

## 右手 Weyl 旋量的内积

任意两个右手 Weyl 旋量  $\eta^{\dagger a}$  和  $\zeta^{\dagger a}$  的内积

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = \varepsilon_{\dot{a} \dot{b}} \eta^{\dagger \dot{b}} \zeta^{\dagger \dot{a}} = \varepsilon^{\dot{a} \dot{b}} \eta_{\dot{a}}^{\dagger} \zeta_{\dot{b}}^{\dagger}$$

在固有保时向 Lorentz 变换下不变，满足

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dot{c}\dagger \dot{a}} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} \eta_{\dot{b}}^{\dagger} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{b}}^{\dagger} [d^{\dagger}(\Lambda)]^{\dot{b}}_{\dot{a}} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{b}}^{\dagger} \delta^{\dot{b}}_{\dot{c}} \zeta^{\dagger \dot{c}} = \eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}}$$

第二步用了转置性质  $[d^*(\Lambda)]_{\dot{a}}^{\dot{b}} = [d^\dagger(\Lambda)]_{\dot{a}}^{\dot{b}}$ ，可见  $\eta_{\dot{a}}^{\dot{\gamma}} \zeta^{\dot{\gamma}}{}^{\dot{a}}$  是 Lorentz 标量

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第二步用了转置性质  $[d^*(\Lambda)]_{\dot{a}}^{\dot{b}} = [d^\dagger(\Lambda)]^{\dot{b}}_{\dot{a}}$ ，可见  $\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}}$  是 Lorentz 标量

由  $\eta^{\dagger i} = \eta^{\dagger}_2$ 、 $\eta^{\dagger 2} = -\eta^{\dagger}_1$ 、 $\zeta^{\dagger i} = \zeta^{\dagger}_2$  和  $\zeta^{\dagger 2} = -\zeta^{\dagger}_1$  得

$$\eta_{\dot{a}}^\dagger \zeta^{\dagger \dot{a}} = \eta_{\dot{1}}^\dagger \zeta^{\dagger 1} + \eta_{\dot{2}}^\dagger \zeta^{\dagger 2} = -\eta^{\dagger \dot{2}} \zeta^{\dagger 1} + \eta^{\dagger 1} \zeta^{\dagger \dot{2}} = -\eta^{\dagger \dot{2}} \zeta_{\dot{2}}^\dagger - \eta^{\dagger 1} \zeta_{\dot{1}}^\dagger = -\eta^{\dagger \dot{a}} \zeta_{\dot{a}}^\dagger$$

即参与缩并的带点旋量指标一升一降会多出一个负号

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$$\eta_{\dot{a}}^{\dagger} \zeta^{\dagger \dot{a}} = \varepsilon_{\dot{a} \dot{b}} \eta^{\dagger \dot{b}} \zeta^{\dagger \dot{a}} = \varepsilon^{\dot{a} \dot{b}} \eta_{\dot{a}}^{\dagger} \zeta_{\dot{b}}^{\dagger}$$

在固有保时向 Lorentz 变换下不变，满足

$$\eta_{\dot{a}}^{\dagger} \zeta^{\dot{c}\dagger\dot{a}} = [d^*(\Lambda)]_{\dot{a}}{}^{\dot{b}} \eta_{\dot{b}}^{\dagger} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger\dot{c}} = \eta_{\dot{b}}^{\dagger} [d^{\dagger}(\Lambda)]^{\dot{b}}_{\dot{a}} [d^{-1\dagger}(\Lambda)]^{\dot{a}}_{\dot{c}} \zeta^{\dagger\dot{c}} = \eta_{\dot{b}}^{\dagger} \delta^{\dot{b}}_{\dot{c}} \zeta^{\dagger\dot{c}} = \eta_{\dot{a}}^{\dagger} \zeta^{\dagger\dot{a}}$$

第二步用了转置性质  $[d^*(\Lambda)]_{\dot{a}}^{\dot{b}} = [d^\dagger(\Lambda)]_{\dot{a}}^{\dot{b}}$ ，可见  $\eta_{\dot{a}}^{\dot{\gamma}} \zeta^{\dot{\gamma}}{}^{\dot{a}}$  是 Lorentz 标量

由  $\eta^{\dagger i} = \eta_{\dot{2}}^{\dagger}$ 、 $\eta^{\dagger \dot{2}} = -\eta_{\dot{1}}^{\dagger}$ 、 $\zeta^{\dagger i} = \zeta_{\dot{2}}^{\dagger}$  和  $\zeta^{\dagger \dot{2}} = -\zeta_{\dot{1}}^{\dagger}$  得

$$\eta_{\dot{a}}^\dagger \zeta^{\dagger \dot{a}} = \eta_{\dot{i}}^\dagger \zeta^{\dagger i} + \eta_{\dot{j}}^\dagger \zeta^{\dagger \dot{j}} = -\eta^{\dagger \dot{2}} \zeta^{\dagger i} + \eta^{\dagger i} \zeta^{\dagger \dot{2}} = -\eta^{\dagger \dot{2}} \zeta_{\dot{i}}^\dagger - \eta^{\dagger i} \zeta_{\dot{j}}^\dagger = -\eta^{\dagger \dot{a}} \zeta_{\dot{a}}^\dagger$$

即参与缩并的带点旋量指标一升一降会多出一个负号

假设右手 Weyl 旋量  $\eta^{\dagger a}$  和  $\zeta_{\dot{a}}^\dagger$  都是 Grassmann 数，则  $\eta^{\dagger a}\zeta_{\dot{a}}^\dagger = -\zeta_{\dot{a}}^\dagger\eta^{\dagger a}$

将省略带点旋量指标的内积写成  $\eta^\dagger \zeta^\dagger \equiv \eta_{\dot{a}}^\dagger \zeta^{\dagger \dot{a}} = -\eta^{\dagger \dot{a}} \zeta_{\dot{a}}^\dagger = \zeta_{\dot{a}}^\dagger \eta^{\dagger \dot{a}} = \zeta^\dagger \eta^\dagger$

则内积  $\eta^\dagger \zeta^\dagger$  和  $\zeta^\dagger \eta^\dagger$  相等

## Lorentz 不变量和 Weyl 旋量算符

可以看到, 只要将不带点和带点的旋量指标分别缩并完全, 就得到 Lorentz 标量

另一方面，缩并一个不带点的指标和一个带点的指标并不能得到 Lorentz 不变量

比如,  $\eta^a \zeta_{\dot{a}}$  和  $\eta^{\dot{a}} \zeta_a$  都不是 Lorentz 标量

# Lorentz 不变量和 Weyl 旋量算符

🐂 可以看到，只要将不带点和带点的旋量指标分别缩并完毕，就得到 Lorentz 标量

🎩 另一方面，缩并一个不带点的指标和一个带点的指标并不能得到 Lorentz 不变量

🕶 比如， $\eta^a \zeta_{\dot{a}}^\dagger$  和  $\eta^{\dagger \dot{a}} \zeta_a$  都不是 Lorentz 标量

👔 对于 Weyl 旋量算符  $\eta_a$  和  $\zeta_a$ ，有

$$(\eta \zeta)^\dagger = (\eta^a \zeta_a)^\dagger = (\zeta_a)^\dagger (\eta^a)^\dagger = \zeta_{\dot{a}}^\dagger \eta^{\dagger \dot{a}} = \zeta^\dagger \eta^\dagger$$

💼 即  $\zeta^\dagger \eta^\dagger$  是  $\eta \zeta$  的厄米共轭算符

👞 厄米共轭操作将左手和右手 Weyl 旋量算符相互转换

## 9.6.2 小节 Dirac 和 Majorana 旋量场的分解

依照上一小节关于旋量指标的约定, 将 Dirac 旋量场  $\psi(x)$  分解成左手 Weyl 旋量场  $\eta_a(x)$  和右手 Weyl 旋量场  $\zeta^{\dagger a}(x)$ , 形式为

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \zeta^{\dagger a}(x) \end{pmatrix}$$

在量子化之前,  $\eta_a(x)$  和  $\zeta^{\dagger a}(x)$  是 Grassmann 数, 因而  $\psi(x)$  也是 Grassmann 数

这是在 9.2.1 小节中转置两个旋量场必须添加一个额外负号的原因

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根据  $D(\Lambda) = \begin{pmatrix} d(\Lambda) & \\ & d^{-1\dagger}(\Lambda) \end{pmatrix}$ ,  $\psi(x)$  的固有保时向 Lorentz 变换表达成

$$\begin{pmatrix} \eta'_a(x') \\ \zeta'^{\dagger \dot{a}}(x') \end{pmatrix} = \psi'(x') = \textcolor{blue}{D}(\Lambda) \psi(x) = \begin{pmatrix} [d(\Lambda)]_a{}^b \eta_b(x) \\ [d^{-1\dagger}(\Lambda)]^{\dot{a}}{}_{\dot{b}} \zeta^{\dagger \dot{b}}(x) \end{pmatrix}$$

ψ(x) 的 Dirac 共轭是  $\bar{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \eta_{\dot{a}}^\dagger & \zeta^a \end{pmatrix} \begin{pmatrix} & \delta^{\dot{b}}_{\dot{a}} \\ \delta_b^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix}$

## Dirac 矩阵的指标形式

保持旋量指标平衡，则 Dirac 方程  $(i\gamma^\mu \partial_\mu - m)\psi = 0$  化为

$$\begin{pmatrix} -m\delta_a{}^b & i(\sigma^\mu)_{a\dot{b}}\partial_\mu \\ i(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu & -m\delta^{\dot{a}}_{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dagger\dot{b}} \end{pmatrix} = 0$$

因而 Dirac 矩阵的指标形式是

$$\gamma^\mu = \begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}$$

注意,  $\gamma^\mu$  中的  $\gamma^0$  与 Dirac 共轭  $\bar{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \eta_a^\dagger & \zeta^a \end{pmatrix} \begin{pmatrix} & \delta^b_{\dot{a}} \\ \delta_b^a & \end{pmatrix} = \begin{pmatrix} \zeta^a & \eta_a^\dagger \end{pmatrix}$

中的  $\gamma^0$  具有不同的指标结构

两者本质不同，有些书将后者记为  $\beta$  以示区别

### $\sigma^\mu$ 和 $\bar{\sigma}^\mu$ 的 Lorentz 变换规则

于是,  $\gamma^\mu$  的 Lorentz 变换规则  $D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$  左边变成

$$\begin{aligned}
& D^{-1}(\Lambda) \gamma^\mu D(\Lambda) \\
&= \begin{pmatrix} [d^{-1}(\Lambda)]_a{}^c & \\ & [d^\dagger(\Lambda)]^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{c\dot{d}} \\ (\bar{\sigma}^\mu)^{\dot{c}d} & \end{pmatrix} \begin{pmatrix} [d(\Lambda)]_d{}^b & \\ [d^{-1\dagger}(\Lambda)]^{\dot{d}}{}_{\dot{b}} & \end{pmatrix} \\
&= \begin{pmatrix} & [d^{-1}(\Lambda)]_a{}^c (\sigma^\mu)_{c\dot{d}} [d^{-1\dagger}(\Lambda)]^{\dot{d}}{}_{\dot{b}} \\ [d^\dagger(\Lambda)]^{\dot{a}}{}_{\dot{c}} (\bar{\sigma}^\mu)^{\dot{c}d} [d(\Lambda)]_d{}^b & \end{pmatrix}
\end{aligned}$$

 右边化为

$$\Lambda^\mu{}_\nu \gamma^\nu = \begin{pmatrix} & \Lambda^\mu{}_\nu (\sigma^\nu)_{a\dot{b}} \\ \Lambda^\mu{}_\nu (\bar{\sigma}^\nu)^{\dot{a}b} & \end{pmatrix}$$

两相比较，推出

$$[d^{-1}(\Lambda)]_a{}^c (\sigma^\mu)_{cd} [d^{-1\dagger}(\Lambda)]^d{}_b = \Lambda^\mu{}_\nu (\sigma^\nu)_{ab}, \quad [d^\dagger(\Lambda)]^a{}_c (\bar{\sigma}^\mu)_{cd} [d(\Lambda)]_d{}^b = \Lambda^\mu{}_\nu (\bar{\sigma}^\nu)^a{}_b$$

这分别是  $\sigma^\mu$  和  $\bar{\sigma}^\mu$  的 Lorentz 变换规则

# Lorentz 矢量 $\eta\sigma^\mu\zeta^\dagger$ 和 $\eta^\dagger\bar{\sigma}^\mu\zeta$

对任意 Weyl 旋量  $\eta$  和  $\zeta$ , 定义

$$\eta\sigma^\mu\zeta^\dagger \equiv \eta^a(\sigma^\mu)_{ab}\zeta^{\dagger b}, \quad \eta^\dagger\bar{\sigma}^\mu\zeta \equiv \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\zeta_b$$

它们都是 Lorentz 矢量, 相应的固有保时向 Lorentz 变换为

$$\begin{aligned}\eta'\sigma^\mu\zeta'^\dagger &= [d^{-1T}(\Lambda)]^a{}_c\eta^c(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^{\dot{b}}{}_d\zeta^{\dagger d} = \eta^c[d^{-1}(\Lambda)]_c{}^a(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^{\dot{b}}{}_d\zeta^{\dagger d} \\ &= \eta^c\Lambda^\mu{}_\nu(\sigma^\nu)_{cd}\zeta^{\dagger d} = \Lambda^\mu{}_\nu\eta\sigma^\nu\zeta^\dagger\end{aligned}$$

$$\begin{aligned}\eta'^\dagger\bar{\sigma}^\mu\zeta' &= [d^*(\Lambda)]_{\dot{a}}{}^{\dot{c}}\eta^\dagger_{\dot{c}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b{}^d\zeta_d = \eta^\dagger_{\dot{c}}[d^\dagger(\Lambda)]^{\dot{c}}{}_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}[d(\Lambda)]_b{}^d\zeta_d \\ &= \eta^\dagger_{\dot{c}}\Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot{c}d}\zeta_d = \Lambda^\mu{}_\nu\eta^\dagger\bar{\sigma}^\mu\zeta\end{aligned}$$

# Lorentz 矢量 $\eta\sigma^\mu\zeta^\dagger$ 和 $\eta^\dagger\bar{\sigma}^\mu\zeta$

duck 对任意 Weyl 旋量  $\eta$  和  $\zeta$ , 定义

$$\eta\sigma^\mu\zeta^\dagger \equiv \eta^a(\sigma^\mu)_{ab}\zeta^{b\dagger}, \quad \eta^\dagger\bar{\sigma}^\mu\zeta \equiv \eta^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}\dot{b}}\zeta_{\dot{b}}$$

它们都是 Lorentz 矢量, 相应的固有保时向 Lorentz 变换为

$$\begin{aligned}\eta'\sigma^\mu\zeta'^\dagger &= [d^{-1T}(\Lambda)]^a_c\eta^c(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^{\dot{b}}_{\dot{d}}\zeta^{b\dagger} = \eta^c[d^{-1}(\Lambda)]_c{}^a(\sigma^\mu)_{ab}[d^{-1\dagger}(\Lambda)]^{\dot{b}}_{\dot{d}}\zeta^{b\dagger} \\ &= \eta^c\Lambda^\mu{}_\nu(\sigma^\nu)_{c\dot{d}}\zeta^{b\dagger} = \Lambda^\mu{}_\nu\eta\sigma^\nu\zeta^\dagger\end{aligned}$$

$$\begin{aligned}\eta'^\dagger\bar{\sigma}^\mu\zeta' &= [d^*(\Lambda)]_{\dot{a}}{}^{\dot{c}}\eta_{\dot{c}}^\dagger(\bar{\sigma}^\mu)^{\dot{a}\dot{b}}[d(\Lambda)]_b{}^d\zeta_d = \eta_{\dot{c}}^\dagger[d^\dagger(\Lambda)]^{\dot{c}}_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}\dot{b}}[d(\Lambda)]_b{}^d\zeta_d \\ &= \eta_{\dot{c}}^\dagger\Lambda^\mu{}_\nu(\bar{\sigma}^\nu)^{\dot{c}\dot{d}}\zeta_d = \Lambda^\mu{}_\nu\eta^\dagger\bar{\sigma}^\mu\zeta\end{aligned}$$

由  $\sigma^2\sigma^\mu\sigma^2 = (\bar{\sigma}^\mu)^T$  得  $(i\sigma^2)\sigma^\mu(i\sigma^2) = -(\bar{\sigma}^\mu)^T$ , 相应的指标形式为

$$\varepsilon^{ac}(\sigma^\mu)_{cd}\varepsilon^{\dot{d}\dot{b}} = -[(\bar{\sigma}^\mu)^T]^{ab} = -(\bar{\sigma}^\mu)^{ba}$$

对于 Weyl 旋量场  $\eta_a(x)$  和  $\zeta^{\dagger\dot{a}}(x)$ , 有

$$\begin{aligned}[\eta^a(\sigma^\mu)_{ab}\zeta^{b\dagger}]^\dagger &= \zeta^b(\sigma^\mu)_{ba}\eta^{\dagger\dot{a}} = -\eta^{\dagger\dot{a}}(\sigma^\mu)_{b\dot{a}}\zeta^b = -\varepsilon^{\dot{a}\dot{c}}\eta_{\dot{c}}^\dagger(\sigma^\mu)_{b\dot{a}}\varepsilon^{bd}\zeta_d \\ &= \eta_{\dot{c}}^\dagger\varepsilon^{db}(\sigma^\mu)_{b\dot{a}}\varepsilon^{\dot{a}\dot{c}}\zeta_d = -\eta_{\dot{c}}^\dagger(\bar{\sigma}^\mu)^{\dot{c}\dot{d}}\zeta_d = -[\zeta_{\dot{d}}^\dagger(\bar{\sigma}^\mu)^{\dot{d}\dot{c}}\eta_c]^\dagger\end{aligned}$$

即

$$(\eta\sigma^\mu\zeta^\dagger)^\dagger = \zeta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\zeta = -(\zeta^\dagger\bar{\sigma}^\mu\eta)^\dagger$$

Lorentz 张量  $\eta\sigma^\mu\bar{\sigma}^\nu\zeta$  和  $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger$

类似地,  $\eta\sigma^\mu\bar{\sigma}^\nu\zeta \equiv \eta^a(\sigma^\mu)_{ab}(\bar{\sigma}^\nu)^{bc}\zeta_c$  和  $\eta^\dagger\bar{\sigma}^\mu\sigma^\nu\zeta^\dagger \equiv \eta^\dagger_a(\bar{\sigma}^\mu)^{ab}(\sigma^\nu)_{bc}\zeta^{\dagger c}$  都是二阶 Lorentz 张量

由  $\sigma^2 \bar{\sigma}^\mu \sigma^2 = (\sigma^\mu)^T$  得  $(-\mathrm{i}\sigma^2) \bar{\sigma}^\mu (-\mathrm{i}\sigma^2) = -(\sigma^\mu)^T$ ，相应的指标形式为

$$\varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\mu)^{\dot{c}d}\varepsilon_{db} = -[(\sigma^\mu)^T]_{\dot{a}b} \equiv -(\sigma^\mu)_{b\dot{a}}$$

再利用  $\varepsilon_{ab}\varepsilon^{bc} = \delta_a^c$  和  $\varepsilon^{ac}(\sigma^\mu)_{cd}\varepsilon^{db} = -[(\bar{\sigma}^\mu)^T]^{ab} = -(\bar{\sigma}^\mu)^{ba}$  推出

$$\begin{aligned} \varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}(\sigma^\mu)_{d\dot{e}}\varepsilon^{\dot{e}\dot{b}} &= \varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}\delta_d^{\phantom{d}f}(\sigma^\mu)_{f\dot{e}}\varepsilon^{\dot{e}\dot{b}} = \varepsilon_{\dot{a}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}d}\varepsilon_{dg}\varepsilon^{gf}(\sigma^\mu)_{f\dot{e}}\varepsilon^{\dot{e}\dot{b}} \\ &= (-\sigma^\nu)_{g\dot{a}}(-\bar{\sigma}^\mu)^{\dot{b}g} = (\bar{\sigma}^\mu)^{\dot{b}g}(\sigma^\nu)_{g\dot{a}} \end{aligned}$$

$$\begin{aligned}
\text{故 } [\eta^a(\sigma^\mu)_{ab}(\bar{\sigma}^\nu)^{\dot{b}c}\zeta_c]^\dagger &= \zeta_{\dot{c}}^\dagger(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\eta^{\dagger\dot{a}} = -\eta^{\dagger\dot{a}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\zeta_{\dot{c}}^\dagger \\
&= -\varepsilon^{\dot{a}\dot{d}}\eta_{\dot{d}}^\dagger(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\varepsilon_{\dot{c}\dot{e}}\zeta^{\dagger\dot{e}} = \eta_{\dot{d}}^\dagger\varepsilon_{\dot{e}\dot{c}}(\bar{\sigma}^\nu)^{\dot{c}b}(\sigma^\mu)_{b\dot{a}}\varepsilon^{\dot{a}\dot{d}}\zeta^{\dagger\dot{e}} \\
&= \eta_{\dot{d}}^\dagger(\bar{\sigma}^\mu)^{\dot{d}g}(\sigma^\nu)_{g\dot{e}}\zeta^{\dagger\dot{e}} = [\zeta^e(\sigma^\nu)_{e\dot{g}}(\bar{\sigma}^\mu)^{\dot{g}d}\eta_d]^\dagger
\end{aligned}$$

即

$$(\eta \sigma^\mu \bar{\sigma}^\nu \zeta)^\dagger = \zeta^\dagger \bar{\sigma}^\nu \sigma^\mu \eta^\dagger = \eta^\dagger \bar{\sigma}^\mu \sigma^\nu \zeta^\dagger = (\zeta \sigma^\nu \bar{\sigma}^\mu \eta)^\dagger$$

# 旋量双线性型的分解



将 Dirac 旋量双线性型分解成由 Weyl 旋量表达的 Lorentz 张量，有

$$\bar{\psi}\psi = \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} \eta_a \\ \zeta^{\dot{a}} \end{pmatrix} = \zeta^a \eta_a + \eta_{\dot{a}}^\dagger \zeta^{\dot{a}} = \zeta \eta + \eta^\dagger \zeta^\dagger$$

$$\bar{\psi}\gamma^5\psi = \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} -\delta_a{}^b & \\ & \delta^{\dot{a}}{}_{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dot{b}} \end{pmatrix} = -\zeta^a \eta_a + \eta_{\dot{a}}^\dagger \zeta^{\dot{a}} = -\zeta \eta + \eta^\dagger \zeta^\dagger$$

$$\begin{aligned} \bar{\psi}\gamma^\mu\psi &= \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} & \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dot{b}} \end{pmatrix} = \zeta^a (\sigma^\mu)_{ab} \zeta^{\dot{b}} + \eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} \eta_b \\ &= \zeta \sigma^\mu \zeta^\dagger + \eta^\dagger \bar{\sigma}^\mu \eta \end{aligned}$$

$$\begin{aligned} \bar{\psi}\gamma^\mu\gamma^5\psi &= \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} & \end{pmatrix} \begin{pmatrix} -\delta_b{}^c & \\ & \delta^{\dot{b}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \eta_c \\ \zeta^{\dot{c}} \end{pmatrix} \\ &= \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} & (\sigma^\mu)_{ab} \\ (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} & \end{pmatrix} \begin{pmatrix} -\eta_b \\ \zeta^{\dot{b}} \end{pmatrix} = \zeta^a (\sigma^\mu)_{ab} \zeta^{\dot{b}} - \eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} \eta_b \\ &= \zeta \sigma^\mu \zeta^\dagger - \eta^\dagger \bar{\sigma}^\mu \eta \end{aligned}$$

## 旋量双线性型的分解



还有

$$\begin{aligned}\bar{\psi} \sigma^{\mu\nu} \psi &= \frac{i}{2} \begin{pmatrix} \zeta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_a{}^b & \\ & (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)_{\dot{a}}{}^{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dot{b}} \end{pmatrix} \\ &= \frac{i}{2} \zeta^a (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_a{}^b \eta_b + \frac{i}{2} \eta_{\dot{a}}^\dagger (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)_{\dot{a}}{}^{\dot{b}} \zeta^{\dot{b}} \\ &= \frac{i}{2} \zeta (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \eta + \frac{i}{2} \eta^\dagger (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \zeta^\dagger\end{aligned}$$



## 进一步推出

$$\bar{\psi}_R \psi_L = \frac{1}{2} \bar{\psi} (1 - \gamma^5) \psi = \zeta \eta$$

$$\bar{\psi}_L \psi_R = \frac{1}{2} \bar{\psi} (1 + \gamma^5) \psi = \eta^\dagger \zeta^\dagger$$

$$\bar{\psi}_L \gamma^\mu \psi_L = \frac{1}{2} \bar{\psi} (\gamma^\mu - \gamma^\mu \gamma^5) \psi = \eta^\dagger \bar{\sigma}^\mu \eta$$

$$\bar{\psi}_L \gamma^\mu \psi_R = \frac{1}{2} \bar{\psi} (\gamma^\mu + \gamma^\mu \gamma^5) \psi = \zeta \sigma^\mu \zeta^\dagger$$

# 拉氏量的分解



另一方面，自由 Dirac 旋量场的拉氏量分解为

$$\begin{aligned}
 \mathcal{L} &= \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi = \begin{pmatrix} \zeta^a & \eta_a^\dagger \end{pmatrix} \begin{pmatrix} -m\delta_a^b & i(\sigma^\mu)_{ab} \partial_\mu \\ i(\bar{\sigma}^\mu)^{ab} \partial_\mu & -m\delta_b^a \end{pmatrix} \begin{pmatrix} \eta_b \\ \zeta^{\dagger b} \end{pmatrix} \\
 &= -m\zeta^a \eta_a + i\zeta^a (\sigma^\mu)_{ab} \partial_\mu \zeta^{\dagger b} + i\eta_a^\dagger (\bar{\sigma}^\mu)^{ab} \partial_\mu \eta_b - m\eta_a^\dagger \zeta^{\dagger a} \\
 &= i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta + i\zeta \sigma^\mu \partial_\mu \zeta^\dagger - m(\zeta \eta + \eta^\dagger \zeta^\dagger)
 \end{aligned}$$

这里的质量项涉及两个不同的 Weyl 旋量场  $\eta_a(x)$  和  $\zeta_a(x)$ ，称为 Dirac 质量项

如果质量  $m = 0$ ，则

$$\mathcal{L}_L = i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta \quad \text{和} \quad \mathcal{L}_R = i\zeta \sigma^\mu \partial_\mu \zeta^\dagger$$

分别描述自由的左手 Weyl 旋量场  $\eta_a(x)$  和右手 Weyl 旋量场  $\zeta^{\dagger a}(x)$



相应的运动方程是 Weyl 方程

$$i\bar{\sigma}^\mu \partial_\mu \eta = 0, \quad i\sigma^\mu \partial_\mu \zeta^\dagger = 0$$

## Weyl 旋量场的 $C$ 变换

下面讨论 Weyl 旋量场的分立变换

首先，**电荷共轭矩阵**的指标形式为  $C = \begin{pmatrix} -i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}$

将  $\psi(x)$  的电荷共轭场  $\psi^C(x)$  分解成 Weyl 旋量场, 得到

$$\psi^C(x) = \mathcal{C}\bar{\psi}^T(x) = \mathcal{C} \begin{pmatrix} \zeta^b(x) & \eta_b^\dagger(x) \end{pmatrix}^T = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \zeta^b(x) \\ \eta_b^\dagger(x) \end{pmatrix} = \begin{pmatrix} \zeta_a(x) \\ \eta_b^\dagger(x) \end{pmatrix}$$

从而, Dirac 旋量场  $\psi(x)$  的  $C$  变换化为

$$\begin{pmatrix} C^{-1} \eta_a(x) C \\ C^{-1} \zeta^{\dagger a}(x) C \end{pmatrix} = C^{-1} \psi(x) C = \zeta_C^* \psi^C(x) = \begin{pmatrix} \zeta_C^* \zeta_a(x) \\ \zeta_C^* \eta^{\dagger a}(x) \end{pmatrix}$$

## Weyl 旋量场的 $C$ 变换

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首先，电荷共轭矩阵的指标形式为  $C = \begin{pmatrix} -i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}$

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$$\psi^C(x) = \mathcal{C}\bar{\psi}^T(x) = \mathcal{C} \begin{pmatrix} \zeta^b(x) & \eta_b^\dagger(x) \end{pmatrix}^T = \begin{pmatrix} \varepsilon_{ab} & \\ & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \zeta^b(x) \\ \eta_b^\dagger(x) \end{pmatrix} = \begin{pmatrix} \zeta_a(x) \\ \eta_b^\dagger(x) \end{pmatrix}$$

从而, Dirac 旋量场  $\psi(x)$  的  $C$  变换化为

$$\begin{pmatrix} C^{-1} \eta_a(x) C \\ C^{-1} \zeta^{\dagger a}(x) C \end{pmatrix} = C^{-1} \psi(x) C = \zeta_C^* \psi^C(x) = \begin{pmatrix} \zeta_C^* \zeta_a(x) \\ \zeta_C^* \eta^{\dagger a}(x) \end{pmatrix}$$

即左右手 Weyl 旋量场的  $C$  变换是

$$C^{-1} \eta_a(x) C = \zeta_C^* \zeta_a(x), \quad C^{-1} \zeta^{\dagger a}(x) C = \zeta_C^* \eta^{\dagger a}(x)$$

可见，电荷共轭变换将  $\eta$  和  $\zeta$  相互转换。取厄米共轭，得  $C^{-1}\eta_b^\dagger(x)C = \zeta_C\zeta_b^\dagger(x)$  及  $C^{-1}\zeta^b(x)C = \zeta_C\eta^b(x)$ ，分别与  $\varepsilon^{ab}$  和  $\varepsilon_{ab}$  缩并，推出

$$C^{-1}\eta^{\dagger a}(x)C = \zeta_C \zeta^{\dagger a}(x), \quad C^{-1}\zeta_a(x)C = \zeta_C \eta_a(x)$$

## Weyl 旋量场的 $P$ 变换

其次, Dirac 旋量场  $\psi(x)$  的  $P$  变换表达为

$$\begin{aligned} \begin{pmatrix} P^{-1} \eta_a(x) P \\ P^{-1} \zeta^{\dagger a}(x) P \end{pmatrix} &= P^{-1} \psi(x) P = \zeta_P^* \gamma^0 \psi(\mathcal{P}x) \\ &= \zeta_P^* \begin{pmatrix} \delta^{\dot{a}}_{\dot{b}} \\ \delta_a^b \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{P}x) \\ \zeta^{\dagger b}(\mathcal{P}x) \end{pmatrix} = \begin{pmatrix} \zeta_P^* \zeta^{\dagger a}(\mathcal{P}x) \\ \zeta_P^* \eta_a(\mathcal{P}x) \end{pmatrix} \end{aligned}$$

注意此处  $\gamma^0$  的指标结构与  $\bar{\psi} = \psi^\dagger \gamma^0$  中一样

# Weyl 旋量场的 $P$ 变换

骆驼 其次, Dirac 旋量场  $\psi(x)$  的  $P$  变换表达为

$$\begin{aligned} \begin{pmatrix} P^{-1} \eta_a(x) P \\ P^{-1} \zeta^{\dagger a}(x) P \end{pmatrix} &= P^{-1} \psi(x) P = \zeta_P^* \gamma^0 \psi(\mathcal{P}x) \\ &= \zeta_P^* \begin{pmatrix} \delta^{\dot{a}}_{\dot{b}} \\ \delta_a{}^b \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{P}x) \\ \zeta^{\dagger b}(\mathcal{P}x) \end{pmatrix} = \begin{pmatrix} \zeta_P^* \zeta^{\dagger a}(\mathcal{P}x) \\ \zeta_P^* \eta_a(\mathcal{P}x) \end{pmatrix} \end{aligned}$$

注意此处  $\gamma^0$  的指标结构与  $\bar{\psi} = \psi^\dagger \gamma^0$  中一样

♠ 于是得到左手 Weyl 旋量场的  $P$  变换

$$P^{-1} \eta_a(x) P = \zeta_P^* \zeta^{\dagger a}(\mathcal{P}x), \quad P^{-1} \zeta^{\dagger a}(x) P = \zeta_P^* \eta_a(\mathcal{P}x)$$

❤ 也就是说, 宇称变换将左手和右手 Weyl 旋量场相互转换

♣ 取厄米共轭得  $P^{-1} \eta_b^\dagger(x) P = \zeta_P \zeta^b(\mathcal{P}x)$  和  $P^{-1} \zeta^b(x) P = \zeta_P \eta_b^\dagger(\mathcal{P}x)$

♦ 两边与  $i\sigma^2 = \epsilon^{\dot{a}\dot{b}} = -\epsilon_{ab}$  缩并, 推出

$$P^{-1} \eta^{\dagger a}(x) P = -\zeta_P \zeta_a(\mathcal{P}x), \quad P^{-1} \zeta_a(x) P = -\zeta_P \eta^{\dagger a}(\mathcal{P}x)$$

## Weyl 旋量场的 $T$ 变换

骆驼 最后, 矩阵  $\mathcal{C}\gamma^5$  的指标形式是  $\mathcal{C}\gamma^5 = \begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon^{ab} & \\ & -\varepsilon_{ab} \end{pmatrix}$

Dirac 旋量场  $\psi(x)$  的  $T$  变换化为

$$\begin{aligned} \begin{pmatrix} T^{-1} \eta_a(x) T \\ T^{-1} \zeta^{\dagger a}(x) T \end{pmatrix} &= T^{-1} \psi(x) T = \zeta_T^* \mathcal{C} \gamma^5 \psi(\mathcal{T}x) \\ &= \zeta_T^* \begin{pmatrix} \varepsilon^{ab} \\ -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{T}x) \\ \zeta^{\dagger \dot{b}}(\mathcal{T}x) \end{pmatrix} = \begin{pmatrix} \zeta_T^* \eta^a(\mathcal{T}x) \\ -\zeta_T^* \zeta^{\dagger \dot{a}}(\mathcal{T}x) \end{pmatrix} \end{aligned}$$

## Weyl 旋量场的 $T$ 变换

骆驼 最后, 矩阵  $\mathcal{C}\gamma^5$  的指标形式是  $\mathcal{C}\gamma^5 = \begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon^{ab} & \\ & -\varepsilon_{ab} \end{pmatrix}$

Dirac 旋量场  $\psi(x)$  的  $T$  变换化为

$$\begin{aligned} \begin{pmatrix} T^{-1} \eta_a(x) T \\ T^{-1} \zeta^{\dagger a}(x) T \end{pmatrix} &= T^{-1} \psi(x) T = \zeta_T^* \mathcal{C} \gamma^5 \psi(\mathcal{T}x) \\ &= \zeta_T^* \begin{pmatrix} \varepsilon^{ab} \\ -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b(\mathcal{T}x) \\ \zeta^{\dagger \dot{b}}(\mathcal{T}x) \end{pmatrix} = \begin{pmatrix} \zeta_T^* \eta^a(\mathcal{T}x) \\ -\zeta_T^* \zeta_{\dot{a}}^{\dagger}(\mathcal{T}x) \end{pmatrix} \end{aligned}$$

则左手 Weyl 旋量场的  $T$  变换是

$$T^{-1} \eta_a(x) T = \zeta_T^* \eta^a(\mathcal{T}x), \quad T^{-1} \zeta^{\dagger a}(x) T = -\zeta_T^* \zeta_{\dot{a}}^\dagger(\mathcal{T}x)$$

取厄米共轭, 有  $T^{-1}\eta_b^\dagger(x)T = \zeta_T \eta^{\dagger b}(\mathcal{T}x)$  和  $T^{-1}\zeta^b(x)T = -\zeta_T \zeta_b(\mathcal{T}x)$

与  $i\sigma^2 = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{ab} = \varepsilon^{ab}$  缩并, 得

$$T^{-1} \eta^{\dagger a}(x) T = -\zeta_T \eta_{\dot{a}}^{\dagger}(\mathcal{T}x), \quad T^{-1} \zeta_a(x) T = \zeta_T \zeta^a(\mathcal{T}x)$$

## Majorana 旋量场的分解

下面讨论 Majorana 旋量场, Majorana 条件意味着  $\begin{pmatrix} \eta_a \\ \zeta^{\dagger a} \end{pmatrix} = \psi = \mathcal{C} \bar{\psi}^T = \begin{pmatrix} \zeta_a \\ \eta^{\dagger a} \end{pmatrix}$

即  $\eta = \zeta$ ，这表明 Majorana 旋量场中的左手和右手 Weyl 旋量场是相关的

因此，可以将 Majorana 旋量场  $\psi(x)$  分解成

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \eta^{\dagger a}(x) \end{pmatrix}$$

## Majorana 旋量场的分解

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即  $\eta = \zeta$ ，这表明 Majorana 旋量场中的左手和右手 Weyl 旋量场是相关的

因此，可以将 Majorana 旋量场  $\psi(x)$  分解成

而自由 Majorana 旋量场的拉氏量分解为

$$\psi(x) = \begin{pmatrix} \eta_a(x) \\ \eta^{\dagger a}(x) \end{pmatrix}$$

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi = \frac{1}{2} \begin{pmatrix} \eta^a & \eta_{\dot{a}}^\dagger \end{pmatrix} \begin{pmatrix} -m\delta_a{}^b & i(\sigma^\mu)_{a\dot{b}} \partial_\mu \\ i(\bar{\sigma}^\mu)^{\dot{a}b} \partial_\mu & -m\delta_{\dot{a}}{}^{\dot{b}} \end{pmatrix} \begin{pmatrix} \eta_b \\ \eta^{\dot{b}} \end{pmatrix} \\ &= \frac{1}{2} [i\eta^\dagger \bar{\sigma}^\mu \partial_\mu \eta + i\eta \sigma^\mu \partial_\mu \eta^\dagger - m(\eta \eta + \eta^\dagger \eta^\dagger)]\end{aligned}$$

⑧ 利用  $\zeta \sigma^\mu \eta^\dagger = -\eta^\dagger \bar{\sigma}^\mu \zeta$  将方括号中第二项化为

$$i\eta\sigma^\mu\partial_\mu\eta^\dagger = i\partial_\mu(\eta\sigma^\mu\eta^\dagger) - i(\partial_\mu\eta)\sigma^\mu\eta^\dagger = i\partial_\mu(\eta\sigma^\mu\eta^\dagger) + i\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta$$

🎭 扔掉全散度项  $i\partial_\mu(\eta\sigma^\mu\eta^\dagger)$ ，拉氏量变成  $\mathcal{L} = i\eta^\dagger\bar{\sigma}^\mu\partial_\mu\eta - \frac{1}{2}m(\eta\eta + \eta^\dagger\eta^\dagger)$

这里的质量项只涉及一个 Weyl 旋量场  $\eta_a(x)$ ，称为 Majorana 质量项

## Majorana 旋量场的 $\bar{\psi}\gamma^\mu\psi$ 和 $\bar{\psi}\sigma^{\mu\nu}\psi$

袋鼠  $\zeta \sigma^\mu \eta^\dagger = -\eta^\dagger \bar{\sigma}^\mu \zeta$ 、 $\eta \sigma^\mu \bar{\sigma}^\nu \zeta = \zeta \sigma^\nu \bar{\sigma}^\mu \eta$  和  $\eta^\dagger \bar{\sigma}^\mu \sigma^\nu \zeta^\dagger = \zeta^\dagger \bar{\sigma}^\nu \sigma^\mu \eta^\dagger$  意味着

$$\eta\sigma^\mu\eta^\dagger = -\eta^\dagger\bar{\sigma}^\mu\eta, \quad \eta\sigma^\mu\bar{\sigma}^\nu\eta = \eta\sigma^\nu\bar{\sigma}^\mu\eta, \quad \eta^\dagger\bar{\sigma}^\mu\sigma^\nu\eta^\dagger = \eta^\dagger\bar{\sigma}^\nu\sigma^\mu\eta^\dagger$$

对于 Majorana 旋量场,  $\eta = \zeta$ ,  $\bar{\psi} \gamma^\mu \psi = \zeta \sigma^\mu \zeta^\dagger + \eta^\dagger \bar{\sigma}^\mu \eta$  化为

$$\bar{\psi} \gamma^\mu \psi = \eta \sigma^\mu \eta^\dagger + \eta^\dagger \bar{\sigma}^\mu \eta = -\eta^\dagger \bar{\sigma}^\mu \eta + \eta^\dagger \bar{\sigma}^\mu \eta = 0$$

$$\bar{\psi} \sigma^{\mu\nu} \psi = \frac{i}{2} \zeta (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \eta + \frac{i}{2} \eta^\dagger (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \zeta^\dagger$$

$$\bar{\psi} \sigma^{\mu\nu} \psi = \frac{i}{2} (\eta \sigma^\mu \bar{\sigma}^\nu \eta - \eta \sigma^\nu \bar{\sigma}^\mu \eta) + \frac{i}{2} (\eta^\dagger \bar{\sigma}^\mu \sigma^\nu \eta^\dagger - \eta^\dagger \bar{\sigma}^\nu \sigma^\mu \eta^\dagger) = 0$$

这样就验证了 9.2.2 小节的结论

## 9.7 节 Majorana 旋量场相关 Feynman 规则

 7.1.1 小节提到, 由于 Dirac 旋量场可以携带某种  $U(1)$  荷, 相应费米子线上的箭头代表  $U(1)$  荷流动的方向, 或者说费米子数流动的方向

 另一方面, Majorana 旋量场不能携带任何  $U(1)$  荷, 不存在费米子数流动的方向, 相应的费米子线则不应该具备箭头

 如果相互作用过程涉及到 Majorana 旋量场与 Dirac 旋量场的耦合, 带箭头与不带箭头的费米子线将在顶点处交汇, 导致费米子数破坏 (fermion-number violation)

 我们需要研究适用于这种情况的 Feynman 规则

 本节讨论一个简单例子, 更一般的情况可参考文献

- A. Denner, H. Eck, O. Hahn, and J. Kublbeck, "Feynman rules for fermion number violating interactions," Nucl. Phys. B 387 (1992) 467–481

### 9.7.1 小节 拉氏量和 $CP$ 对称性

考虑复标量场  $\phi(x)$ 、Dirac 旋量场  $\psi(x)$  和 Majorana 旋量场  $\chi(x)$  构成拉氏量

$$\mathcal{L} = (\partial^\mu \phi^\dagger) \partial_\mu \phi - m_\phi^2 \phi^\dagger \phi + \bar{\psi} (i \gamma^\mu \partial_\mu - m_\psi) \psi + \frac{1}{2} \bar{\chi} (i \gamma^\mu \partial_\mu - m_\chi) \chi + \mathcal{L}_{\text{int}}$$

相互作用拉氏量为  $\mathcal{L}_{\text{int}} = \kappa \phi^\dagger \bar{\chi} P_R \psi + \kappa^* \phi \bar{\psi} P_L \chi$

⚠  $\kappa$  是一个复耦合常数,  $\mathcal{L}_{\text{int}}$  是厄米的, 因为  $\mathcal{L}_{\text{int}}$  中两项互为厄米共轭,

$$(\kappa \phi^\dagger \bar{\chi} P_R \psi)^\dagger = \kappa^* \psi^\dagger P_R \gamma^0 \chi = \kappa^* \psi^\dagger \gamma^0 P_L \chi = \kappa^* \phi \bar{\psi} P_L \chi$$

这样的相互作用涉及一个标量场和两个旋量场，属于 Yukawa 相互作用

### 9.7.1 小节 拉氏量和 $CP$ 对称性

考虑复标量场  $\phi(x)$ 、Dirac 旋量场  $\psi(x)$  和 Majorana 旋量场  $\chi(x)$  构成拉氏量

$$\mathcal{L} = (\partial^\mu \phi^\dagger) \partial_\mu \phi - m_\phi^2 \phi^\dagger \phi + \bar{\psi} (i \gamma^\mu \partial_\mu - m_\psi) \psi + \frac{1}{2} \bar{\chi} (i \gamma^\mu \partial_\mu - m_\chi) \chi + \mathcal{L}_{\text{int}}$$

相互作用拉氏量为  $\mathcal{L}_{\text{int}} = \kappa \phi^\dagger \bar{\chi} P_R \psi + \kappa^* \phi \bar{\psi} P_L \chi$

是复耦合常数， $\mathcal{L}_{\text{int}}$  是厄米的，因为  $\mathcal{L}_{\text{int}}$  中两项互为厄米共轭，

$$(\kappa \phi^\dagger \bar{\chi} P_R \psi)^\dagger = \kappa^* \psi^\dagger P_R \gamma^0 \chi = \kappa^* \psi^\dagger \gamma^0 P_L \chi = \kappa^* \phi \bar{\psi} P_L \chi$$

这样的相互作用涉及一个标量场和两个旋量场，属于 Yukawa 相互作用

作  $U(1)$  整体变换  $\phi'(x) = e^{i\varphi\theta}\phi(x)$  和  $\psi'(x) = e^{i\varphi\theta}\psi(x)$ ，则拉氏量  $\mathcal{L}$  不变

可见，这个理论具有一个  $U(1)$  整体对称性，而复标量场  $\phi(x)$  和 Dirac 旋量场  $\psi(x)$  的  $U(1)$  荷相同，均为  $q$

将耦合常数分解为实部和虚部,  $\kappa = \kappa_R + i\kappa_I$ , 则相互作用拉氏量化为

$$\mathcal{L}_{\text{int}} = \kappa_{\text{R}} (\phi^\dagger \bar{\chi} P_{\text{R}} \psi + \phi \bar{\psi} P_{\text{L}} \chi) + \kappa_{\text{I}} (\text{i} \phi^\dagger \bar{\chi} P_{\text{R}} \psi - \text{i} \phi \bar{\psi} P_{\text{L}} \chi)$$

### $C$ 破坏和 $P$ 破坏

假设三个量子场的  $C$ 、 $P$  变换为

$$C^{-1}\phi(x)C = \eta_C^* \phi^\dagger(x), \quad C^{-1}\psi(x)C = \zeta_C^* \mathcal{C}\bar{\psi}^T(x), \quad C^{-1}\chi(x)C = \tilde{\zeta}_C^* \chi(x)$$

$$P^{-1}\phi(x)P = \eta_P^*\phi(\mathcal{P}x), \quad P^{-1}\psi(x)P = \zeta_P^*\gamma^0\psi(\mathcal{P}x), \quad P^{-1}\chi(x)P = \tilde{\zeta}_P^*\gamma^0\chi(\mathcal{P}x)$$

推出算符  $\phi^\dagger \bar{\chi} P_R \psi$  的  $C$ 、 $P$  变换

$$C^{-1}\phi^\dagger(x)\bar{\chi}(x)P_R\psi(x)C = \eta_C\zeta_C^*\tilde{\zeta}_C\phi(x)\bar{\psi}(x)P_R\chi(x)$$

$$P^{-1} \phi^\dagger(x) \bar{\chi}(x) P_{\text{R}} \psi(x) P = \eta_P \zeta_P^* \tilde{\zeta}_P \phi^\dagger(\mathcal{P}x) \bar{\chi}(\mathcal{P}x) P_{\text{L}} \psi(\mathcal{P}x)$$

而算符  $\phi\bar{\psi}P_L\chi$  的  $C$ 、 $P$  变换为

$$C^{-1}\phi(x)\bar{\psi}(x)P_L\chi(x)C = \eta_C^*\zeta_C\tilde{\zeta}_C^*\phi^\dagger(x)\bar{\chi}(x)P_L\psi(x)$$

$$P^{-1}\phi(x)\bar{\psi}(x)P_L\chi(x)P = \eta_P^*\zeta_P\tilde{\zeta}_P^*\phi(\mathcal{P}x)\bar{\psi}(\mathcal{P}x)P_R\chi(\mathcal{P}x)$$

### **$C$ 破坏和 $P$ 破坏**

假设三个量子场的  $C$ 、 $P$  变换为

$$C^{-1}\phi(x)C = \eta_C^* \phi^\dagger(x), \quad C^{-1}\psi(x)C = \zeta_C^* \mathcal{C}\bar{\psi}^T(x), \quad C^{-1}\chi(x)C = \tilde{\zeta}_C^* \chi(x)$$

$$P^{-1}\phi(x)P = \eta_P^*\phi(\mathcal{P}x), \quad P^{-1}\psi(x)P = \zeta_P^*\gamma^0\psi(\mathcal{P}x), \quad P^{-1}\chi(x)P = \tilde{\zeta}_P^*\gamma^0\chi(\mathcal{P}x)$$

推出算符  $\phi^\dagger \bar{\chi} P_R \psi$  的  $C$ 、 $P$  变换

$$C^{-1}\phi^\dagger(x)\bar{\chi}(x)P_R\psi(x)C = \eta_C\zeta_C^*\tilde{\zeta}_C\phi(x)\bar{\psi}(x)P_R\chi(x)$$

$$P^{-1} \phi^\dagger(x) \bar{\chi}(x) P_{\text{R}} \psi(x) P = \eta_P \zeta_P^* \tilde{\zeta}_P \phi^\dagger(\mathcal{P}x) \bar{\chi}(\mathcal{P}x) P_{\text{L}} \psi(\mathcal{P}x)$$

而算符  $\phi\bar{\psi}P_L\chi$  的  $C$ 、 $P$  变换为

$$C^{-1}\phi(x)\bar{\psi}(x)P_{\text{L}}\chi(x)C = \eta_C^*\zeta_C\tilde{\zeta}_C^*\phi^\dagger(x)\bar{\chi}(x)P_{\text{L}}\psi(x)$$

$$P^{-1}\phi(x)\bar{\psi}(x)P_L\chi(x)P = \eta_P^*\zeta_P\tilde{\zeta}_P^*\phi(\mathcal{P}x)\bar{\psi}(\mathcal{P}x)P_R\chi(\mathcal{P}x)$$

无论作  $C$  变换还是  $P$  变换，相互作用拉氏量  $\mathcal{L}_{\text{int}} = \kappa \phi^\dagger \bar{\chi} P_R \psi + \kappa^* \phi \bar{\psi} P_L \chi$  都不能保持不变，因此理论不具有电荷共轭对称性和空间反射对称性

换言之，这个理论既是 **C** 破坏 (*C*-violation) 的，又是 **P** 破坏 (*P*-violation) 的

## CP 破坏?

进一步, 算符  $\phi^\dagger \bar{\chi} P_R \psi$  和  $\phi \bar{\psi} P_L \chi$  的  $CP$  变换为

$$(CP)^{-1} \phi^\dagger(x) \bar{\chi}(x) P_R \psi(x) CP = \eta_{CP} \phi(\mathcal{P}x) \bar{\psi}(\mathcal{P}x) P_L \chi(\mathcal{P}x)$$

$$(CP)^{-1}\phi(x)\bar{\psi}(x)P_L\chi(x)CP = \eta_{CP}^*\phi^\dagger(\mathcal{P}x)\bar{\chi}(\mathcal{P}x)P_R\psi(\mathcal{P}x)$$

如果适当选取相位因子的值, 使得  $\eta_{CP} = \eta_{CP}^* = +1$

则算符  $\phi^\dagger \bar{\chi} P_B \psi + \phi \bar{\psi} P_L \chi$  在  $CP$  变换下不变

而相互作用拉氏量  $\mathcal{L}_{\text{int}} = \kappa_R (\phi^\dagger \bar{\chi} P_R \psi + \phi \bar{\psi} P_L \chi) + \kappa_I (i\phi^\dagger \bar{\chi} P_R \psi - i\phi \bar{\psi} P_L \chi)$  中  $\kappa_R$  对应的项具有  $CP$  对称性,  $\kappa_I$  对应的项引起  $CP$  破坏 ( $CP$ -violation)

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如果相位因子的取值使得  $\eta_{CP} = \eta_{CP}^* = -1$

则算符  $i\phi^\dagger \bar{\chi} P_R \psi - i\phi \bar{\psi} P_L \chi$  在  $CP$  变换下不变

而  $\kappa_I$  对应的项具有  $CP$  对称性,  $\kappa_R$  对应的项引起  $CP$  破坏

因此, 当  $\kappa_R \neq 0$  且  $\kappa_I \neq 0$  时, 相互作用拉氏量  $\mathcal{L}_{\text{int}}$  看起来会破坏  $CP$  对称性

## CP 对称性

不过, Dirac 旋量场  $\psi(x)$  是复的量子场, 即 Hilbert 空间中的非自共轭算符, 它的相位具有任意性, 可用于吸收耦合常数  $\kappa \equiv |\kappa|e^{-i\varphi}$  的相位  $\varphi$

如果将 Dirac 旋量场重新定义为  $\psi'(x) = e^{-i\varphi}\psi(x)$ ，则  $\bar{\psi}'(x) = e^{i\varphi}\bar{\psi}(x)$ ，于是  $\mathcal{L}_{\text{int}} = |\kappa|e^{-i\varphi}\phi^\dagger\bar{\chi}P_R\psi + |\kappa|e^{i\varphi}\phi\bar{\psi}P_L\chi = |\kappa|(\phi^\dagger\bar{\chi}P_R\psi' + \phi\bar{\psi}'P_L\chi)$  描述同一个理论

但此时耦合常数  $|\kappa|$  是实数，不会引起  $CP$  破坏

! 因此, 这个理论实际上是具有  $CP$  对称性的

当理论中所有复耦合常数的相位不能完全被复场吸收时，才会出现  $CP$  破坏

### CP 对称性

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但此时耦合常数  $|\kappa|$  是实数，不会引起  $CP$  破坏

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当理论中所有复耦合常数的相位不能完全被复场吸收时，才会出现 *CP* 破坏

另一方面，像实标量场、实矢量场和 Majorana 旋量场这样的实场必须满足自共轭条件，这导致它不具有相位任意性

在下面的讨论中, 不失一般性, 将耦合常数  $\kappa$  取为实数, 相互作用拉氏量表达为

$$\mathcal{L}_{\text{int}} = \kappa(\phi^\dagger \bar{\chi} \Gamma_1 \psi + \phi \bar{\psi} \Gamma_2 \chi)$$

这里引入了  $\Gamma_1 = P_R$  和  $\Gamma_2 = P_L$ ，下面许多结论与  $\Gamma_1$  和  $\Gamma_2$  的具体形式无关

### 9.7.2 小节 Feynman 规则

将 Dirac 旋量场、复标量场和 Majorana 旋量场的平面波展开式表达为

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda} [u(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x}]$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (\textcolor{brown}{c}_p e^{-ip \cdot x} + d_p^\dagger e^{ip \cdot x})$$

$$\chi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda} [u(\mathbf{p}, \lambda) \mathbf{f}_{\mathbf{p}, \lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) \mathbf{f}_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x}]$$

相应地, 引入以下单粒子态,

Dirac 正费米子  $\psi$  的单粒子态  $|\mathbf{p}^+, \lambda\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}, \lambda}^\dagger |0\rangle$

Dirac 反费米子  $\bar{\psi}$  的单粒子态  $|\mathbf{p}^-, \lambda\rangle = \sqrt{2E_{\mathbf{p}}} b_{\mathbf{p}, \lambda}^\dagger |0\rangle$

正标量玻色子  $\phi$  的单粒子态  $|\mathbf{p}^+\rangle = \sqrt{2E_{\mathbf{p}}} c_{\mathbf{p}}^\dagger |0\rangle$

反标量玻色子  $\bar{\phi}$  的单粒子态  $|\mathbf{p}^-\rangle = \sqrt{2E_{\mathbf{p}}} d_{\mathbf{p}}^\dagger |0\rangle$

Majorana 费米子  $\chi$  的单粒子态  $|p, \lambda\rangle = \sqrt{2E_p} f_{p,\lambda}^\dagger |0\rangle$

注意, Majorana 费米子  $\chi$  是纯中性的, 动量记号的右上角没有正负号

$S$  算符  $n = 1$  阶

Dirac 旋量场和复标量场与初末态的缩并结果见第 7 章

## Majorana 旋量场与初末态的缩并定义为

$$\langle 0 | \overline{\chi(x)} | \mathbf{p}, \lambda \rangle \equiv \langle 0 | \chi^{(+)}(x) | \mathbf{p}, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\langle 0 | \bar{\chi}(x) | \mathbf{p}, \lambda \rangle \equiv \langle 0 | \bar{\chi}^{(+)}(x) | \mathbf{p}, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\langle \bar{\mathbf{p}}, \lambda | \bar{\chi}(x) | 0 \rangle \equiv \langle \mathbf{p}, \lambda | \bar{\chi}^{(-)}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{i p \cdot x}$$

$$\langle \mathbf{p}, \lambda | \chi(x) | 0 \rangle \equiv \langle \mathbf{p}, \lambda | \chi^{(-)}(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{i p \cdot x}$$

$S$  算符  $n = 1$  阶

 Dirac 旋量场 和复标量场 与初末态的缩并结果见第 7 章

Majorana 旋量场与初末态的缩并定义为

$$\begin{aligned}
\langle 0 | \overline{\chi}(x) | \mathbf{p}, \lambda \rangle &\equiv \langle 0 | \chi^{(+)}(x) | \mathbf{p}, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i p \cdot x} \\
\langle 0 | \bar{\chi}(x) | \mathbf{p}, \lambda \rangle &\equiv \langle 0 | \bar{\chi}^{(+)}(x) | \mathbf{p}, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i p \cdot x} \\
\langle \mathbf{p}, \lambda | \bar{\chi}(x) | 0 \rangle &\equiv \langle \mathbf{p}, \lambda | \bar{\chi}^{(-)}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{i p \cdot x} \\
\langle \mathbf{p}, \lambda | \overline{\chi}(x) | 0 \rangle &\equiv \langle \mathbf{p}, \lambda | \chi^{(-)}(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{i p \cdot x}
\end{aligned}$$

由于相互作用哈密顿量密度  $\mathcal{H}_1 = -\mathcal{L}_{\text{int}}$ ， $S$  算符展开式中  $n = 1$  的项为

$$\begin{aligned} iT^{(1)} &= -i \int d^4x \mathsf{T}[\mathcal{H}_1(x)] = i \int d^4x \mathsf{T}[\mathcal{L}_{\text{int}}(x)] \\ &= i\kappa \int d^4x \mathsf{T}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x) + \phi(x)\bar{\psi}(x)\Gamma_2\chi(x)] \end{aligned}$$

根据 Wick 定理,  $iT^{(1)}$  只包含下面两项,

$$i\mathbf{T}_1^{(1)} = i\kappa \int d^4x \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)], \quad i\mathbf{T}_2^{(1)} = i\kappa \int d^4x \mathsf{N}[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)]$$

$\psi \rightarrow \chi\phi$  衰变过程

考虑  $\psi \rightarrow \chi\phi$  衰变, 初末态为  $|p^+, \lambda\rangle$  和  $|q, \lambda'; k^+\rangle$ ,  $iT_1^{(1)}$  贡献的  $T$  矩阵元是

$$\begin{aligned}
\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)] | \mathbf{p}^+, \lambda \rangle \\
&= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)] | \mathbf{p}^+, \lambda \rangle \\
&= i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
&= i\kappa \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p-q-k)
\end{aligned}$$

这是计算  $T$  矩阵元的**第一种方法**，与 7.1 节介绍的方法一样

$\psi \rightarrow \chi\phi$  衰变过程

考慮  $\psi \rightarrow \chi\phi$  衰變，初末態為  $|p^+, \lambda\rangle$  和  $|q, \lambda'; k^+\rangle$ ， $iT_1^{(1)}$  貢獻的  $T$  矩陣元是

$$\begin{aligned}
\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)] | \mathbf{p}^+, \lambda \rangle \\
&= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \overline{\mathsf{N}[\phi^\dagger(x)\bar{\chi}(x)\Gamma_1\psi(x)]} | \mathbf{p}^+, \lambda \rangle \\
&= i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
&= i\kappa \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p-q-k)
\end{aligned}$$

这是计算  $T$  矩阵元的**第一种方法**，与 7.1 节介绍的方法一样

利用电荷共轭变换，可以引进第二种计算方法

将相互作用算符  $\bar{\chi}\Gamma_1\psi$  化为

$$\begin{aligned}\bar{\chi}\Gamma_1\psi &= (\bar{\chi}\Gamma_1\psi)^T = -\psi^T\Gamma_1^T\bar{\chi}^T = -\psi^T\mathcal{C}^{-1}\mathcal{C}\Gamma_1^T\mathcal{C}^{-1}\mathcal{C}\bar{\chi}^T \\ &= \psi^T\mathcal{C}\Gamma_1^T\mathcal{C}^{-1}\mathcal{C}\bar{\chi}^T = \bar{\psi}^C\Gamma_1^C\chi^C\end{aligned}$$

同理推出  $\bar{\psi}\Gamma_2\chi = \bar{\chi}^C\Gamma_2^C\psi^C$

## 第二种计算方法

老虎 通过 Majorana 条件  $\chi = \chi^C$  将  $\bar{\chi}\Gamma_1\psi = \bar{\psi}^C\Gamma_1^C\chi^C$  和  $\bar{\psi}\Gamma_2\chi = \bar{\chi}^C\Gamma_2^C\psi^C$  化为

$$\bar{\chi}\Gamma_1\psi = \bar{\psi}^C\Gamma_1^C\chi, \quad \bar{\psi}\Gamma_2\chi = \bar{\chi}\Gamma_2^C\psi^C$$

从而将  $iT_1^{(1)}$  和  $iT_2^{(1)}$  改写为

$$iT_1^{(1)} = i\kappa \int d^4x \mathsf{N}[\phi^\dagger(x) \bar{\chi}(x) \Gamma_1 \psi(x)] = i\kappa \int d^4x \mathsf{N}[\phi^\dagger(x) \bar{\psi}^C(x) \Gamma_1^C \chi(x)]$$

$$iT_2^{(1)} = i\kappa \int d^4x \mathsf{N}[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)] = i\kappa \int d^4x \mathsf{N}[\phi(x)\bar{\chi}(x)\Gamma_2^C\psi^C(x)]$$



注意，此时旋量场算符排列的次序与原来相反



现在,  $iT_1^{(1)}$  贡献的  $\psi \rightarrow \chi\phi$  过程  $T$  矩阵元也可以表达成

$$\begin{aligned}
& \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | i T_1^{(1)} | \mathbf{p}^+, \lambda \rangle \\
&= i \kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathbf{N}[\phi^\dagger(x) \bar{\psi}_a^C(x) (\Gamma_1^C)_{ab} \chi_b(x)] | \mathbf{p}^+, \lambda \rangle \\
&= -i \kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \phi^{\dagger(-)}(x) \chi_b^{(-)}(x) (\Gamma_1^C)_{ab} \bar{\psi}_a^C(x) | \mathbf{p}^+, \lambda \rangle
\end{aligned}$$

## 电荷共轭场 $\psi^c(x)$ 的平面波展开和初末态缩并

 Dirac 旋量场  $\psi(x)$  的电荷共轭场  $\psi^C(x)$  的平面波展开式是

$$\begin{aligned}\psi^C(x) &= \mathcal{C}\bar{\psi}^T = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda} \left[ \mathcal{C}\bar{v}^T(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} + \mathcal{C}\bar{u}^T(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^{\dagger} e^{ip \cdot x} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda} [u(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^{\dagger} e^{ip \cdot x}]\end{aligned}$$

跟  $\psi(x)$  展开式的差异只在于  $a$  与  $b$  互换, 相应 Dirac 共轭的展开式为

$$\bar{\psi}^C(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda} [\bar{u}(\mathbf{p}, \lambda) \mathbf{b}_{\mathbf{p}, \lambda}^{\dagger} e^{ip \cdot x} + \bar{v}(\mathbf{p}, \lambda) \mathbf{a}_{\mathbf{p}, \lambda} e^{-ip \cdot x}]$$

据此, 将电荷共轭场  $\psi^C(x)$  和  $\bar{\psi}^C(x)$  与初末态的缩并定义成

$$\langle 0 | \overline{\psi^C(x)} | \mathbf{p}^-, \lambda \rangle \equiv \langle 0 | \psi^{C(+)}(x) | \mathbf{p}^-, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\langle 0 | \bar{\psi}^C(x) | \mathbf{p}^+, \lambda \rangle \equiv \langle 0 | \bar{\psi}^{C(+)}(x) | \mathbf{p}^+, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\langle \left[ \mathbf{p}^-, \lambda \right] \bar{\psi}^C(x) | 0 \rangle \equiv \langle \mathbf{p}^-, \lambda | \bar{\psi}^{C(-)}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{i \mathbf{p} \cdot \mathbf{x}}$$

$$\langle \overline{\mathbf{p}^+, \lambda} | \psi^C(x) | 0 \rangle \equiv \langle \mathbf{p}^+, \lambda | \psi^{C(-)}(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{i p \cdot x}$$

## 第二种方法的计算结果

猩猩  $\psi \rightarrow \chi\phi$  的  $T$  矩阵元变成

$$\begin{aligned}
\langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= -i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathcal{N}[\phi^\dagger(x) \chi_b(x) (\Gamma_1^C)_{ab} \bar{\psi}_a^C(x)] | \mathbf{p}^+, \lambda \rangle \\
&= -i\kappa \int d^4x v_b(\mathbf{q}, \lambda') (\Gamma_1^C)_{ab} \bar{v}_a(\mathbf{p}, \lambda) e^{-i(p-q-k) \cdot x} \\
&= -i\kappa \int d^4x \bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') e^{-i(p-q-k) \cdot x} \\
&= -i\kappa \bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(p-q-k) \\
&= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \mathcal{N}[\phi^\dagger(x) \bar{\psi}_a^C(x) \Gamma_1^C \chi_b(x)] | \mathbf{p}^+, \lambda \rangle
\end{aligned}$$

# 第二种方法的计算结果



$\psi \rightarrow \chi\phi$  的  $T$  矩阵元变成

$$\begin{aligned}
 \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= -i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \overline{N}[\phi^\dagger(x)\chi_b(x)(\Gamma_1^C)_{ab}\bar{\psi}_a^C(x)] | \mathbf{p}^+, \lambda \rangle \\
 &= -i\kappa \int d^4x v_b(\mathbf{q}, \lambda') (\Gamma_1^C)_{ab} \bar{v}_a(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
 &= -i\kappa \int d^4x \bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') e^{-i(p-q-k)\cdot x} \\
 &= -i\kappa \bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(p - q - k) \\
 &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | \overline{N}[\phi^\dagger(x)\bar{\psi}^C(x)\Gamma_1^C\chi(x)] | \mathbf{p}^+, \lambda \rangle
 \end{aligned}$$

倒数第二行是第二种方法的计算结果，有

$$\begin{aligned}
 -\bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') &= -u^T(\mathbf{p}, \lambda) \mathcal{C} \Gamma_1^C \mathcal{C} \bar{u}^T(\mathbf{q}, \lambda') = u^T(\mathbf{p}, \lambda) \mathcal{C} \mathcal{C}^{-1} \Gamma_1^T \mathcal{C} \mathcal{C}^{-1} \bar{u}^T(\mathbf{q}, \lambda') \\
 &= [u^T(\mathbf{p}, \lambda) \Gamma_1^T \bar{u}^T(\mathbf{q}, \lambda')]^T = \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda)
 \end{aligned}$$



第二种方法结果与第一种方法结果  $i\kappa \bar{u}(\mathbf{q}, \lambda') \Gamma_1 u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p - q - k)$  相等

# $\bar{\psi} \rightarrow \chi \bar{\phi}$ 衰变过程：第一种方法

🐒 另一方面，考虑  $\bar{\psi} \rightarrow \chi \bar{\phi}$  衰变过程，初态为  $|\mathbf{p}^-, \lambda\rangle$ ，末态为  $|\mathbf{q}, \lambda'; \mathbf{k}^-\rangle$

🌋 根据  $iT_2^{(1)} = i\kappa \int d^4x N[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)]$  按第一种方法计算

⛩  $iT_2^{(1)}$  贡献的  $T$  矩阵元是

$$\begin{aligned}
 \langle \mathbf{q}, \lambda'; \mathbf{k}^- | iT_2^{(1)} | \mathbf{p}^-, \lambda \rangle &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^- | N[\phi(x)\bar{\psi}(x)\Gamma_2\chi(x)] | \mathbf{p}^-, \lambda \rangle \\
 &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^- | \overline{N[\phi(x)\bar{\psi}_a(x)(\Gamma_2)_{ab}\chi_b(x)]} | \mathbf{p}^-, \lambda \rangle \\
 &= -i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^- | \overline{N[\phi(x)\chi_b(x)(\Gamma_2)_{ab}\bar{\psi}_a(x)]} | \mathbf{p}^-, \lambda \rangle \\
 &= -i\kappa \int d^4x v_b(\mathbf{q}, \lambda') (\Gamma_2)_{ab} \bar{v}_a(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
 &= -i\kappa \bar{v}(\mathbf{p}, \lambda) \Gamma_2 v(\mathbf{q}, \lambda') (2\pi)^4 \delta^{(4)}(p-q-k)
 \end{aligned}$$

## $\bar{\psi} \rightarrow \chi \bar{\phi}$ 衰变过程：第二种方法

根据  $iT_2^{(1)} = i\kappa \int d^4x N[\phi(x)\bar{\chi}(x)\Gamma_2^C \psi^C(x)]$  按**第二种方法**计算

  $iT_2^{(1)}$  贡献的  $T$  矩阵元为

$$\begin{aligned}
\langle \mathbf{q}, \lambda'; \mathbf{k}^- | iT_2^{(1)} | \mathbf{p}^-, \lambda \rangle &= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^- | \mathsf{N}[\phi(x)\bar{\chi}(x)\Gamma_2^C\psi^C(x)] | \mathbf{p}^-, \lambda \rangle \\
&= i\kappa \int d^4x \langle \mathbf{q}, \lambda'; \mathbf{k}^- | \mathsf{N}[\phi(x)\bar{\chi}(x)\Gamma_2^C\psi^C(x)] | \mathbf{p}^-, \lambda \rangle \\
&= i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda')\Gamma_2^C u(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x} \\
&= i\kappa \bar{u}(\mathbf{q}, \lambda')\Gamma_2^C u(\mathbf{p}, \lambda) (2\pi)^4 \delta^{(4)}(p-q-k)
\end{aligned}$$

由于

$$\begin{aligned}\bar{u}(\mathbf{q}, \lambda') \Gamma_2^C u(\mathbf{p}, \lambda) &= v^T(\mathbf{q}, \lambda') \mathcal{C} \Gamma_2^C \mathcal{C} \bar{v}^T(\mathbf{p}, \lambda) = -v^T(\mathbf{q}, \lambda') \mathcal{C} \mathcal{C}^{-1} \Gamma_2^T \mathcal{C} \mathcal{C}^{-1} \bar{v}^T(\mathbf{p}, \lambda) \\ &= -[v^T(\mathbf{q}, \lambda') \Gamma_2^T \bar{v}^T(\mathbf{p}, \lambda)]^T = -\bar{v}(\mathbf{p}, \lambda) \Gamma_2 v(\mathbf{q}, \lambda')\end{aligned}$$

两种方法的计算结果相等

# 费米子流方向

-  以上计算表明，这两种方法都是有效的，在实际计算中可采用任意一种方法
-  现在需要归纳出一套与这两种方法同时相容的 Feynman 规则，这样的规则将特别适用于处理费米子数破坏过程
-  为此，在每条连续费米子线附近添加一条带箭头的点划线，表示费米子流 (fermion flow) 的方向
-  费米子流的两种方向分别对应于上述两种计算方法

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-  费米子流的两种方向分别对应于上述两种计算方法
-  当费米子流方向与 Dirac 费米子线上箭头方向相同时，采用第一种计算方法
-  当费米子流方向与 Dirac 费米子线上箭头方向相反时，采用与电荷共轭场有关的第二种计算方法
-  这样一来，两种费米子流方向是等价的，对每条连续费米子线可采取任意一种方向进行计算

## 位置空间外线规则

于是，位置空间中费米子的外线规则如下，带箭头的点划线表示费米子流方向

### ① Dirac 正费米子 $\psi$ 入射光线:

$$\psi, \lambda \xrightarrow[p]{\quad} x = \langle 0 | \overline{\psi(x)} | \mathbf{p}^+ \rangle, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i p \cdot x}$$

$$\psi, \lambda \xrightarrow[p]{\bullet} x = \langle 0 | \overline{\psi^C(x)} | \mathbf{p}^+, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

## ② Dirac 反费米子 $\bar{\psi}$ 入射光线:

$$\bar{\psi}, \lambda \xrightarrow{\quad \xrightarrow{p} \quad} \bullet x = \langle 0 | \bar{\psi}(x) | \mathbf{p}^-, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i p \cdot x}$$

$$\bar{\psi}, \lambda \xrightarrow[p]{\quad} \bullet x = \langle 0 | \overline{\psi^C(x)} | \mathbf{p}^- \rangle, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

### ③ Dirac 正费米子 $\psi$ 出射光线:

$$x \bullet \xrightarrow{p} \psi, \lambda = \langle \overline{\mathbf{p}^+, \lambda} | \bar{\psi}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{ip \cdot x}$$

$$x \bullet \xrightarrow{p} \psi, \lambda = \langle \overline{\mathbf{p}^+, \lambda} | \psi^C(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{ip \cdot x}$$

## 位置空间外线规则

### ④ Dirac 反费米子 $\bar{\psi}$ 出射光线:

$$x \bullet \xrightarrow{p} \bar{\psi}, \lambda = \langle \overline{\mathbf{p}}, \lambda | \psi(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{i p \cdot x}$$

$$x \bullet \overbrace{\psi, \lambda}^p = \langle \overbrace{\mathbf{p}^-, \lambda}^p | \bar{\psi}^C(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{ip \cdot x}$$

## ⑤ Majorana 费米子 $\chi$ 入射光线:

$$\chi, \lambda \xrightarrow{\quad p \quad} \bullet x = \langle 0 | \overline{\chi(x)} | \mathbf{p}, \lambda \rangle = u(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

$$\chi, \lambda \xrightarrow{\quad p \quad} \bullet x = \langle 0 | \bar{\chi}(x) | \mathbf{p}, \lambda \rangle = \bar{v}(\mathbf{p}, \lambda) e^{-i \mathbf{p} \cdot x}$$

### 6 Majorana 费米子 $\chi$ 出射外线:

$$x \xrightarrow{p} \chi, \lambda = \langle \bar{\chi}(x) | 0 \rangle = \bar{u}(\mathbf{p}, \lambda) e^{i \mathbf{p} \cdot x}$$

$$x \xrightarrow{p} \chi, \lambda = \langle \mathbf{p}, \lambda | \chi(x) | 0 \rangle = v(\mathbf{p}, \lambda) e^{i p \cdot x}$$

Badger Majorana 费米子线

上没有箭头, Feynman 规则依赖于费米子流方向与动量方向之间的异同

从每条连续费米子线写出散射振幅时，总是逆着费米子流方向逐项写下费米子的贡献

## 第一种方法 Feynman 图

对于上述  $\psi \rightarrow \chi\phi$  和  $\bar{\psi} \rightarrow \chi\bar{\phi}$  过程，第一种计算方法对应于

$$\begin{aligned}
 \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | i T_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= \psi, \lambda \begin{array}{c} p \\ \xrightarrow{\hspace{1.5cm}} \end{array} \bullet \begin{array}{c} x \\ \nearrow k \\ \searrow q \\ \xrightarrow{\hspace{1.5cm}} \end{array} \chi, \lambda' \\
 &= i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda') \mathbf{\Gamma}_1 u(\mathbf{p}, \lambda) e^{-i(p-q-k)\cdot x}
 \end{aligned}$$

$$\begin{aligned}
 \langle \mathbf{q}, \lambda'; \mathbf{k}^- | i T_2^{(1)} | \mathbf{p}^-, \lambda \rangle &= \bar{\psi}, \lambda \quad \text{Diagram:} \\
 &= -i\kappa \int d^4x \bar{v}(\mathbf{p}, \lambda) \Gamma_2 v(\mathbf{q}, \lambda') e^{-i(p-q-k)\cdot x}
 \end{aligned}$$

## 第二种方法 Feynman 图

第二种计算方法对应于

$$\begin{aligned}
 \langle \mathbf{q}, \lambda'; \mathbf{k}^+ | iT_1^{(1)} | \mathbf{p}^+, \lambda \rangle &= \psi, \lambda \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\
 &= -i\kappa \int d^4x \bar{v}(\mathbf{p}, \lambda) \Gamma_1^C v(\mathbf{q}, \lambda') e^{-i(p-q-k) \cdot x} \\
 \langle \mathbf{q}, \lambda'; \mathbf{k}^- | iT_2^{(1)} | \mathbf{p}^-, \lambda \rangle &= \bar{\psi}, \lambda \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\
 &= i\kappa \int d^4x \bar{u}(\mathbf{q}, \lambda') \Gamma_2^C u(\mathbf{p}, \lambda) e^{-i(p-q-k) \cdot x}
 \end{aligned}$$

两种方法在 Feynman 图上的差异只是费米子流方向不同，即点划线箭头方向不同

额外的负号来自两个费米子场算符的交换

# 位置空间顶点规则

hog 观察各个 Feynman 图元素与振幅表达式的关系，归纳出**位置空间**中的**顶点规则**

$\phi$

$\psi$

$\chi$

$= i\kappa \int d^4x \Gamma_1,$

$\phi$

$\psi$

$\chi$

$= i\kappa \int d^4x \Gamma_2$

$\phi$

$\psi$

$\chi$

$= i\kappa \int d^4x \Gamma_1^C,$

$\phi$

$\psi$

$\chi$

$= i\kappa \int d^4x \Gamma_2^C$

## Dirac 旋量场的 Feynman 传播子

研究  $iT^{(2)}$  的  $T$  矩阵元时可能遇到像  $N[\bar{\chi}(y)\Gamma_1\psi(y)\bar{\psi}(x)\Gamma_2\chi(x)]$  这样的表达式

如果采用第一种方法进行计算，则 Dirac 旋量场的 Feynman 传播子在位置空间中的 Feynman 规则与 7.1.1 小节规则类似，表达为

$$x \bullet \xrightarrow[p]{} \bullet y = \overline{\psi(y)}\bar{\psi}(x) = S_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$

## Dirac 旋量场的 Feynman 传播子

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如果采用第一种方法进行计算，则 Dirac 旋量场的 Feynman 传播子在位置空间中的 Feynman 规则与 7.1.1 小节规则类似，表达为

$$x \bullet \xrightarrow{p} \bullet y = \overline{\psi(y)\psi}(x) = S_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$

由  $\bar{x}\Gamma_1\psi = \bar{\psi}^C\Gamma_1^C\chi$  和  $\bar{\psi}\Gamma_3\chi = \bar{x}\Gamma_2^C\psi^C$  推出

$$\begin{aligned} \mathsf{N}[\bar{\chi}(y)\Gamma_1 \overline{\psi(y)\psi}(x)\Gamma_2\chi(x)] &= \mathsf{N}[\overline{\bar{\psi}^C(y)\Gamma_1^C\chi(y)}\bar{\chi}(x)\Gamma_2^C\overline{\psi^C(x)}] \\ &= \mathsf{N}[\bar{\chi}(x)\Gamma_2^C\overline{\psi^C(x)}\bar{\psi}^C(y)\Gamma_1^C\chi(y)] \end{aligned}$$

如果采用第二种方法进行计算，则相应的 Feynman 传播子是

$$x \xrightarrow[p]{\quad} y = \overline{\psi^C(x)\psi^C(y)} = \langle 0 | \mathsf{T}[\psi^C(x)\psi^C(y)] | 0 \rangle = \langle 0 | \mathsf{T}[\mathcal{C}\bar{\psi}^T(x)\psi^T(y)\mathcal{C}] | 0 \rangle$$

## Majorana 旋量场的 Feynman 传播子



进一步计算得到

$$\begin{aligned}
x \bullet \xrightarrow{p} \bullet y &= \overline{\psi^C(x)} \bar{\psi}^C(y) = \langle 0 | \mathcal{T}[\mathcal{C} \bar{\psi}^T(x) \psi^T(y) \mathcal{C}] | 0 \rangle \\
&\quad \leftarrow \text{red dashed line} \\
&= -\mathcal{C} \{ \langle 0 | \mathcal{T}[\psi(y) \bar{\psi}(x)] | 0 \rangle \}^T \mathcal{C} = \mathcal{C}^{-1} [\overline{\psi(y) \bar{\psi}(x)}]^T \mathcal{C} \\
&= \mathcal{C}^{-1} S_F^T(y-x) \mathcal{C} = \int \frac{d^4 p}{(2\pi)^4} \frac{\mathcal{C}^{-1} i(\not{p} + m_\psi)^T \mathcal{C}}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)} \\
&= \int \frac{d^4 p}{(2\pi)^4} \frac{i(-\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}
\end{aligned}$$



最后一步用到  $\mathcal{C}^{-1}(\gamma^\mu)^T \mathcal{C} = -\gamma^\mu$

Majorana 旋量场的 Feynman 传播子



进一步计算得到

$$\begin{aligned}
x \bullet \xrightarrow{p} \bullet y &= \overline{\psi^C(x)\bar{\psi}^C(y)} = \langle 0 | \mathcal{T}[\mathcal{C}\bar{\psi}^T(x)\psi^T(y)\mathcal{C}] | 0 \rangle \\
&= -\mathcal{C}\{\langle 0 | \mathcal{T}[\psi(y)\bar{\psi}(x)] | 0 \rangle\}^T \mathcal{C} = \mathcal{C}^{-1}[\overline{\psi(y)\bar{\psi}(x)}]^T \mathcal{C} \\
&= \mathcal{C}^{-1}S_F^T(y-x)\mathcal{C} = \int \frac{d^4 p}{(2\pi)^4} \frac{\mathcal{C}^{-1}i(\not{p} + m_\psi)^T \mathcal{C}}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)} \\
&= \int \frac{d^4 p}{(2\pi)^4} \frac{i(-\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} e^{-ip \cdot (y-x)}
\end{aligned}$$



最后一步用到  $\mathcal{C}^{-1}(\gamma^\mu)^T \mathcal{C} = -\gamma^\mu$



另一方面, Majorana 旋量场的 Feynman 传播子为

$$x \bullet \xrightarrow{p} \bullet y = \overline{\chi(y)\bar{\chi}(x)} = S_F(y-x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(p + m_\chi)}{p^2 - m_\chi^2 + i\epsilon} e^{-ip \cdot (y-x)}$$

## 动量空间 Feynman 规则

 转换到动量空间，推出以下 Feynman 规则

- 1 Dirac 正费米子  $\psi$  入射外线:  $\psi, \lambda \xrightarrow[p]{\text{---}} \bullet = u(\mathbf{p}, \lambda), \quad \psi, \lambda \xleftarrow[p]{\text{---}} \bullet = \bar{v}(\mathbf{p}, \lambda)$

2 Dirac 反费米子  $\bar{\psi}$  入射外线:  $\bar{\psi}, \lambda \xrightarrow[p]{\text{---}} \bullet = \bar{v}(\mathbf{p}, \lambda), \quad \bar{\psi}, \lambda \xleftarrow[p]{\text{---}} \bullet = u(\mathbf{p}, \lambda)$

3 Dirac 正费米子  $\psi$  出射外线:  $\bullet \xrightarrow[p]{\text{---}} \psi, \lambda = \bar{u}(\mathbf{p}, \lambda), \quad \bullet \xleftarrow[p]{\text{---}} \psi, \lambda = v(\mathbf{p}, \lambda)$

4 Dirac 反费米子  $\bar{\psi}$  出射外线:  $\bullet \xrightarrow[p]{\text{---}} \bar{\psi}, \lambda = v(\mathbf{p}, \lambda), \quad \bullet \xleftarrow[p]{\text{---}} \bar{\psi}, \lambda = \bar{u}(\mathbf{p}, \lambda)$

5 Majorana 费米子  $\chi$  入射外线:  $\chi, \lambda \xrightarrow[p]{\text{---}} \bullet = u(\mathbf{p}, \lambda), \quad \chi, \lambda \xleftarrow[p]{\text{---}} \bullet = \bar{v}(\mathbf{p}, \lambda)$

6 Majorana 费米子  $\chi$  出射外线:  $\bullet \xrightarrow[p]{\text{---}} \chi, \lambda = \bar{u}(\mathbf{p}, \lambda), \quad \bullet \xleftarrow[p]{\text{---}} \chi, \lambda = v(\mathbf{p}, \lambda)$

## 动量空间 Feynman 规则

### 7 Dirac 费米子传播子:

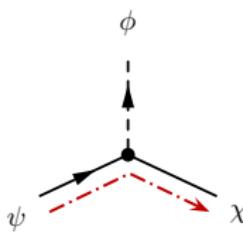
$$\text{---} \xrightarrow{p} \text{---} = \frac{i(p + m_\psi)}{p^2 - m_\psi^2 + i\epsilon}$$

$$\begin{array}{c} p \\ \leftarrow \end{array} = \frac{i(-p + m_\psi)}{p^2 - m_\psi^2 + i\epsilon}$$

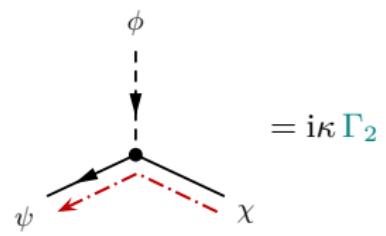
## 8 Majorana 费米子传播子

$$\text{子: } \frac{p}{p^2 - m_\chi^2 + i\epsilon} = \frac{i(p + m_\chi)}{p^2 - m_\chi^2 + i\epsilon}$$

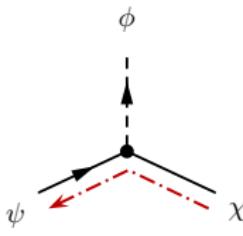
## 9 Yukawa 相互作用顶点:



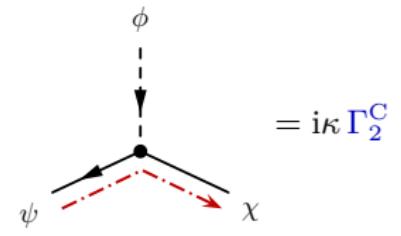
$$= i\kappa \Gamma_1,$$



$$= i\kappa \Gamma_2$$



$$= i\kappa \Gamma_1^C$$



$$= i\kappa \Gamma_2^C$$

# Majorana 旋量场与对称性因子



注意, Majorana 费米子是纯中性粒子

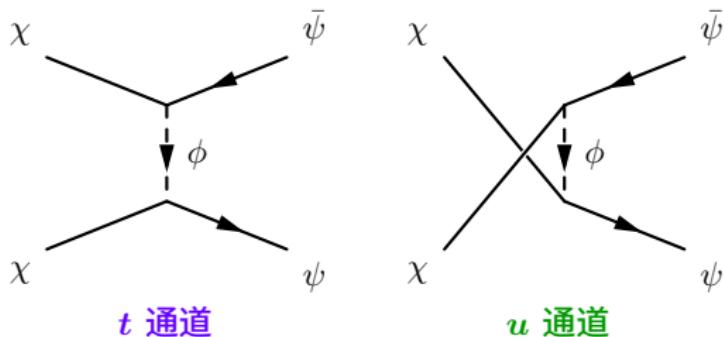
- 🚗 如果末态包含超过 1 个全同的 Majorana 费米子
- 🚙 计算散射截面或衰变宽度时需要考虑末态对称性因子  $S$
- 🚐 假如拉氏量的某个相互作用项包含 2 个或以上全同的 Majorana 旋量场
- 🚚 类似于 7.3 节的讨论, 在导出顶点 Feynman 规则时需要考虑组合因子
- 🚛 计算时还需要留意 Feynman 图的对称性因子

### 9.7.3 小节 应用

下面应用上一小节推导出来的 Feynman 规则进行计算

考虑  $\chi\chi \rightarrow \psi\bar{\psi}$  湮灭过程

摩托车 领头阶 Feynman 图如下图所示, 包含一个  $t$  通道和一个  $u$  通道的 Feynman 图



现在，费米子流方向有多种取法，但各种取法的计算结果应该是等价的

# 费米子流方向第一种取法

天鹅 设初态两个 Majorana 费米子  $\chi$  的四维动量为  $k_1^\mu$  和  $k_2^\mu$ ，末态 Dirac 费米子  $\psi$  和  $\bar{\psi}$  的四维动量为  $p_1^\mu$  和  $p_2^\mu$ ，令  $t = (k_1 - p_1)^2$ ， $u = (k_1 - p_2)^2$

山峰 添加带箭头的点划线表示费米子流方向

火车 应用动量空间 Feynman 规则， $t$  通道和  $u$  通道 Feynman 图贡献的不变振幅是

$$i\mathcal{M}_t = \begin{array}{c} \text{Diagram: Two Majorana fermions } \chi \text{ with momenta } k_1, k_2 \text{ interact via a } \bar{\psi} \text{ loop with momenta } p_1, p_2. \text{ The loop is a dashed line with arrows.} \\ \text{Feynman rule: } = \bar{u}(p_1)(i\kappa\Gamma_2)u(k_1) \frac{i}{(p_1 - k_1)^2 - m_\phi^2} \bar{v}(k_2)(i\kappa\Gamma_1)v(p_2) \end{array}$$

$$= -\frac{i\kappa^2}{t - m_\phi^2} \bar{u}(p_1)\Gamma_2 u(k_1)\bar{v}(k_2)\Gamma_1 v(p_2)$$

$$i\mathcal{M}_u = \begin{array}{c} \text{Diagram: Two Majorana fermions } \chi \text{ with momenta } k_1, k_2 \text{ interact via a } \bar{\psi} \text{ loop with momenta } p_1, p_2. \text{ The loop is a dashed line with arrows.} \\ \text{Feynman rule: } = \bar{v}(k_1)(i\kappa\Gamma_1)v(p_2) \frac{i}{(k_1 - p_2)^2 - m_\phi^2} \bar{u}(p_1)(i\kappa\Gamma_2)u(k_2) \\ = -\frac{i\kappa^2}{u - m_\phi^2} \bar{v}(k_1)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_2) \end{array}$$

# 第一种取法的相对符号

根据

$$\begin{aligned}
 & \langle \mathbf{p}_1^+; \mathbf{p}_2^- | N[\phi(x)\bar{\psi}_a(x)(\Gamma_2)_{ab}\chi_b(x)\phi^\dagger(y)\bar{\chi}_c(y)(\Gamma_1)_{cd}\psi_d(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 & + \langle \mathbf{p}_1^+; \mathbf{p}_2^- | N[\phi(x)\bar{\psi}_a(x)(\Gamma_2)_{ab}\chi_b(x)\phi^\dagger(y)\bar{\chi}_c(y)(\Gamma_1)_{cd}\psi_d(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 = & \langle \mathbf{p}_1^+; \mathbf{p}_2^- | N[\psi_d(y)\bar{\psi}_a(x)(\Gamma_2)_{ab}\phi(x)\phi^\dagger(y)(\Gamma_1)_{cd}\bar{\chi}_c(y)\chi_b(x)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 - & \langle \mathbf{p}_1^+; \mathbf{p}_2^- | N[\psi_d(y)\bar{\psi}_a(x)(\Gamma_2)_{ab}\phi(x)\phi^\dagger(y)(\Gamma_1)_{cd}\chi_b(x)\bar{\chi}_c(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle
 \end{aligned}$$

这两个 Feynman 图的相对符号为负

因而总振幅是  $i\mathcal{M} = i\mathcal{M}_t - i\mathcal{M}_u$

# 费米子流方向第二种取法

当然，也可以选择**其它费米子流方向**进行计算

比如，同时**反转**上述  $t$  通道 Feynman 图中**两条点划线的方向**，则  $t$  通道振幅变成

$$\begin{aligned}
 i\tilde{\mathcal{M}}_t = & \begin{array}{c} \text{Diagram with two dashed lines reversed} \\ \text{Top vertex: } \chi \rightarrow k_2, \bar{\psi} \rightarrow p_2 \\ \text{Bottom vertex: } \chi \rightarrow k_1, \psi \rightarrow p_1 \\ \text{Momentum flow: } k_2 \rightarrow p_2, k_1 \rightarrow p_1 \end{array} = \bar{v}(k_1)(i\kappa\Gamma_2^C)v(p_1) \frac{i}{(p_1 - k_1)^2 - m_\phi^2} \bar{u}(p_2)(i\kappa\Gamma_1^C)u(k_2) \\
 & = -\frac{i\kappa^2}{t - m_\phi^2} \bar{v}(k_1)\Gamma_2^C v(p_1) \bar{u}(p_2)\Gamma_1^C u(k_2)
 \end{aligned}$$

反转上述  $u$  通道 Feynman 图中**一条点划线的方向**， $u$  通道振幅化为

$$\begin{aligned}
 i\tilde{\mathcal{M}}_u = & \begin{array}{c} \text{Diagram with one dashed line reversed} \\ \text{Top vertex: } \chi \rightarrow k_2, \bar{\psi} \rightarrow p_2 \\ \text{Bottom vertex: } \chi \rightarrow k_1, \psi \rightarrow p_1 \\ \text{Momentum flow: } k_2 \rightarrow p_2, k_1 \rightarrow p_1 \end{array} = \bar{v}(k_1)(i\kappa\Gamma_1)v(p_2) \frac{i}{(k_1 - p_2)^2 - m_\phi^2} \bar{v}(k_2)(i\kappa\Gamma_2^C)v(p_1) \\
 & = -\frac{i\kappa^2}{u - m_\phi^2} \bar{v}(k_1)\Gamma_1 v(p_2) \bar{v}(k_2)\Gamma_2^C v(p_1)
 \end{aligned}$$

# 第二种取法的相对符号

👉 根据

$$\begin{aligned}
 & \langle \mathbf{p}_1^+; \mathbf{p}_2^- | \mathcal{N}[\phi(x)\bar{\chi}_a(x)(\Gamma_2^C)_{ab}\psi_b^C(x)\phi^\dagger(y)\bar{\psi}_c^C(y)(\Gamma_1^C)_{cd}\chi_d(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 & + \langle \mathbf{p}_1^+; \mathbf{p}_2^- | \mathcal{N}[\phi(x)\bar{\chi}_a(x)(\Gamma_2^C)_{ab}\psi_b^C(x)\phi^\dagger(y)\bar{\chi}_c(y)(\Gamma_1)_{cd}\psi_d(y)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 = & \langle \mathbf{p}_1^+; \mathbf{p}_2^- | \mathcal{N}[\bar{\psi}_c^C(y)\psi_b^C(x)(\Gamma_2^C)_{ab}\phi(x)\phi^\dagger(y)(\Gamma_1^C)_{cd}\chi_d(y)\bar{\chi}_a(x)] | \mathbf{k}_1; \mathbf{k}_2 \rangle \\
 & + \langle \mathbf{p}_1^+; \mathbf{p}_2^- | \mathcal{N}[\psi_d(y)\psi_b^C(x)(\Gamma_2^C)_{ab}\phi(x)\phi^\dagger(y)\bar{\chi}_a(x)\bar{\chi}_c(y)(\Gamma_1)_{cd}] | \mathbf{k}_1; \mathbf{k}_2 \rangle
 \end{aligned}$$

👉 这两个 Feynman 图的相对符号为正

👉 因而总振幅是  $i\tilde{\mathcal{M}} = i\tilde{\mathcal{M}}_t + i\tilde{\mathcal{M}}_u$

## 两种取法的等价性

$$\begin{aligned}
\bar{v}(k_1)\Gamma_2^C v(p_1)\bar{u}(p_2)\Gamma_1^C u(k_2) &= u^T(k_1)\mathcal{C}\mathcal{C}^{-1}\Gamma_2^T\mathcal{C}\mathcal{C}\bar{u}^T(p_1)v^T(p_2)\mathcal{C}\mathcal{C}^{-1}\Gamma_1^T\mathcal{C}\mathcal{C}\bar{v}^T(k_2) \\
&= [u^T(k_1)\Gamma_2^T\bar{u}^T(p_1)v^T(p_2)\Gamma_1^T\bar{v}^T(k_2)]^T \\
&= \bar{v}(k_2)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_1) \\
\bar{v}(k_1)\Gamma_1 v(p_2)\bar{v}(k_2)\Gamma_2^C v(p_1) &= \bar{v}(k_1)\Gamma_1 v(p_2)u^T(k_2)\mathcal{C}\mathcal{C}^{-1}\Gamma_2^T\mathcal{C}\mathcal{C}\bar{u}^T(p_1) \\
&= -\bar{v}(k_1)\Gamma_1 v(p_2)[u^T(k_2)\Gamma_2^T\bar{u}^T(p_1)]^T \\
&= -\bar{v}(k_1)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_2)
\end{aligned}$$

→ 
$$\begin{aligned} i\tilde{\mathcal{M}}_t &= -\frac{i\kappa^2}{t-m_\phi^2} \bar{v}(k_1)\Gamma_2^C v(p_1)\bar{u}(p_2)\Gamma_1^C u(k_2) \\ &= -\frac{i\kappa^2}{t-m_\phi^2} \bar{u}(p_1)\Gamma_2 u(k_1)\bar{v}(k_2)\Gamma_1 v(p_2) = i\mathcal{M}_t \\ i\tilde{\mathcal{M}}_u &= -\frac{i\kappa^2}{u-m_\phi^2} \bar{v}(k_1)\Gamma_1 v(p_2)\bar{v}(k_2)\Gamma_2^C v(p_1) \\ &= +\frac{i\kappa^2}{u-m_\phi^2} \bar{v}(k_1)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_2) = -i\mathcal{M}_u \end{aligned}$$

## 两种取法的等价性

$$\begin{aligned}
\bar{v}(k_1)\Gamma_2^C v(p_1)\bar{u}(p_2)\Gamma_1^C u(k_2) &= u^T(k_1)\mathcal{C}\mathcal{C}^{-1}\Gamma_2^T\mathcal{C}\mathcal{C}\bar{u}^T(p_1)v^T(p_2)\mathcal{C}\mathcal{C}^{-1}\Gamma_1^T\mathcal{C}\mathcal{C}\bar{v}^T(k_2) \\
&= [u^T(k_1)\Gamma_2^T\bar{u}^T(p_1)v^T(p_2)\Gamma_1^T\bar{v}^T(k_2)]^T \\
&= \bar{v}(k_2)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_1) \\
\bar{v}(k_1)\Gamma_1 v(p_2)\bar{v}(k_2)\Gamma_2^C v(p_1) &= \bar{v}(k_1)\Gamma_1 v(p_2)u^T(k_2)\mathcal{C}\mathcal{C}^{-1}\Gamma_2^T\mathcal{C}\mathcal{C}\bar{u}^T(p_1) \\
&= -\bar{v}(k_1)\Gamma_1 v(p_2)[u^T(k_2)\Gamma_2^T\bar{u}^T(p_1)]^T \\
&= -\bar{v}(k_1)\Gamma_1 v(p_2)\bar{u}(p_1)\Gamma_2 u(k_2)
\end{aligned}$$

$$\begin{aligned}
 \text{→ } \text{i} \tilde{\mathcal{M}}_t &= -\frac{\text{i} \kappa^2}{t - m_\phi^2} \bar{v}(k_1) \Gamma_2^C v(p_1) \bar{u}(p_2) \Gamma_1^C u(k_2) \\
 &= -\frac{\text{i} \kappa^2}{t - m_\phi^2} \bar{u}(p_1) \Gamma_2 u(k_1) \bar{v}(k_2) \Gamma_1 v(p_2) = \text{i} \mathcal{M}_t \\
 \text{i} \tilde{\mathcal{M}}_u &= -\frac{\text{i} \kappa^2}{u - m_\phi^2} \bar{v}(k_1) \Gamma_1 v(p_2) \bar{v}(k_2) \Gamma_2^C v(p_1) \\
 &= +\frac{\text{i} \kappa^2}{u - m_\phi^2} \bar{v}(k_1) \Gamma_1 v(p_2) \bar{u}(p_1) \Gamma_2 u(k_2) = -\text{i} \mathcal{M}_u
 \end{aligned}$$

可见，根据费米子流方向的不同取法计算出来的结果确实是等价的



因此  $i\tilde{\mathcal{M}} = i\tilde{\mathcal{M}}_t + i\tilde{\mathcal{M}}_u = i\mathcal{M}_t - i\mathcal{M}_u = i\mathcal{M}$

## 非极化振幅模方

接下来计算  $\chi\bar{\chi} \rightarrow \psi\bar{\psi}$  的非极化振幅模方

$$\overline{|\mathcal{M}|^2} = \overline{|\mathcal{M}_t - \mathcal{M}_u|^2} = \overline{|\mathcal{M}_t|^2} + \overline{|\mathcal{M}_u|^2} - (\overline{\mathcal{M}_t^* \mathcal{M}_u} + \text{H.c.})$$

使用具体形式  $\Gamma_1 = P_R$  和  $\Gamma_2 = P_L$ ，由第一种取法的振幅计算结果得到

$$i\mathcal{M}_t = -\frac{i\kappa^2}{t-m_\phi^2} \bar{u}(p_1)P_L u(k_1) \bar{v}(k_2)P_R v(p_2)$$

$$(\mathbf{i}\mathcal{M}_t)^* = \frac{\mathbf{i}\kappa^2}{t - m_\phi^2} \bar{u}(k_1) P_{\mathbf{R}} u(p_1) \bar{v}(p_2) P_{\mathbf{L}} v(k_2)$$

$$i\mathcal{M}_u = -\frac{i\kappa^2}{u - m_\phi^2} \bar{v}(k_1) P_R v(p_2) \bar{u}(p_1) P_L u(k_2)$$

$$(\mathbf{i}\mathcal{M}_u)^* = \frac{\mathbf{i}\kappa^2}{u - m_\phi^2} \bar{v}(p_2) P_{\text{L}} v(k_1) \bar{u}(k_2) P_{\text{R}} u(p_1)$$

单纯  $t$  通道贡献

由  $P_L \gamma^\mu = \gamma^\mu P_R$ 、 $P_R \gamma^\mu = \gamma^\mu P_L$ 、 $P_L^2 = P_L$ 、 $P_R^2 = P_R$  和  $P_L P_R = P_R P_L = 0$  得

$$\begin{aligned} \text{tr}[(\not{p}_1 + m_\psi) P_L (\not{k}_1 + m_\chi) P_R] &= \text{tr}[(\not{p}_1 + m_\psi)(\not{k}_1 P_R + m_\chi P_L) P_R] \\ &= \text{tr}[(\not{p}_1 + m_\psi) \not{k}_1 P_R] = \frac{1}{2} \text{tr}[(\not{p}_1 + m_\psi) \not{k}_1 (1 + \gamma^5)] = \frac{1}{2} \text{tr}(\not{p}_1 \not{k}_1) = 2 \not{k}_1 \cdot \not{p}_1 \\ \text{tr}[(\not{k}_2 - m_\chi) P_R (\not{p}_2 - m_\psi) P_L] &= \frac{1}{2} \text{tr}[(\not{k}_2 - m_\chi) \not{p}_2 (1 - \gamma^5)] = 2 \not{k}_2 \cdot \not{p}_2 \end{aligned}$$

从而, 单纯  $t$  通道对非极化振幅模方的贡献是

$$\begin{aligned}
& \overline{|\mathcal{M}_t|^2} = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_t|^2 \\
&= \frac{\kappa^4}{4(t - m_\phi^2)^2} \sum_{\text{spins}} \bar{u}(p_1) P_L u(k_1) \bar{u}(k_1) P_R u(p_1) \bar{v}(k_2) P_R v(p_2) \bar{v}(p_2) P_L v(k_2) \\
&= \frac{\kappa^4}{2 \cdot 2(t - m_\phi^2)^2} \text{tr}[(\not{p}_1 + m_\psi) P_L (\not{k}_1 + m_\chi) P_R] \text{tr}[(\not{k}_2 - m_\chi) P_R (\not{p}_2 - m_\psi) P_L] \\
&= \frac{\kappa^4 (k_1 \cdot p_1) (k_2 \cdot p_2)}{(t - m_\phi^2)^2}
\end{aligned}$$

### 单纯 $u$ 通道贡献和交叉贡献

另一方面，单纯  $u$  通道的贡献为

$$\begin{aligned}
& \overline{|\mathcal{M}_u|^2} = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_u|^2 \\
&= \frac{\kappa^4}{4(u - m_\phi^2)^2} \sum_{\text{spins}} \bar{v}(k_1) P_R v(p_2) \bar{v}(p_2) P_L v(k_1) \bar{u}(p_1) P_L u(k_2) \bar{u}(k_2) P_R u(p_1) \\
&= \frac{\kappa^4}{2 \cdot 2(u - m_\phi^2)^2} \text{tr}[(\not{k}_1 - m_\chi) P_R (\not{p}_2 - m_\psi) P_L] \text{tr}[(\not{p}_1 + m_\psi) P_L (\not{k}_2 + m_\chi) P_R] \\
&= \frac{\kappa^4 (k_1 \cdot p_2) (k_2 \cdot p_1)}{(u - m_\phi^2)^2}
\end{aligned}$$

而  $t$  和  $u$  通道的交叉贡献是

$$\overline{\mathcal{M}_t^* \mathcal{M}_u} = \frac{1}{4} \sum_{\text{spins}} \mathcal{M}_t^* \mathcal{M}_u$$

$$= \frac{\kappa^4}{4(t - m_\phi^2)(u - m_\phi^2)} \sum_{\text{spins}} \bar{u}(k_1) P_R u(p_1) \bar{u}(p_1) P_L u(k_2) \bar{v}(p_2) P_L v(k_2) \bar{v}(k_1) P_R v(p_2)$$

### $\chi\chi \rightarrow \psi\bar{\psi}$ 非极化振幅模方

$$\begin{aligned}
& \sum_{\text{spins}} \bar{u}(k_1) P_{\text{R}} u(p_1) \bar{u}(p_1) P_{\text{L}} u(k_2) \bar{v}(p_2) P_{\text{L}} v(k_2) \bar{v}(k_1) P_{\text{R}} v(p_2) \\
&= \sum_{\text{spins}} \bar{u}(k_1) P_{\text{R}} u(p_1) \bar{u}(p_1) P_{\text{L}} u(k_2) [u^T(p_2) \mathcal{C} P_{\text{L}} \mathcal{C} \bar{u}^T(k_2)]^T [u^T(k_1) \mathcal{C} P_{\text{R}} \mathcal{C} \bar{u}^T(p_2)]^T \\
&= \sum_{\text{spins}} \bar{u}(k_1) P_{\text{R}} u(p_1) \bar{u}(p_1) P_{\text{L}} u(k_2) \bar{u}(k_2) \mathcal{C}^T P_{\text{L}}^T \mathcal{C}^T u(p_2) \bar{u}(p_2) \mathcal{C}^T P_{\text{R}}^T \mathcal{C}^T u(k_1) \\
&= \text{tr}[(\not{k}_1 + m_\chi) P_{\text{R}} (\not{p}_1 + m_\psi) P_{\text{L}} (\not{k}_2 + m_\chi) \mathcal{C}^{-1} P_{\text{L}}^T \mathcal{C} (\not{p}_2 + m_\psi) \mathcal{C}^{-1} P_{\text{R}}^T \mathcal{C}] \\
&= \text{tr}[(\not{k}_1 + m_\chi) \not{p}_1 P_{\text{L}} (\not{k}_2 + m_\chi) P_{\text{L}} (\not{p}_2 + m_\psi) P_{\text{R}}] = m_\chi \text{tr}[(\not{k}_1 + m_\chi) \not{p}_1 P_{\text{L}} (\not{p}_2 + m_\psi) P_{\text{R}}] \\
&= [(\not{k}_1 + m_\chi) \not{p}_1 \not{p}_2 (1 + \gamma^5)] = \frac{m_\chi^2}{2} \text{tr}(\not{p}_1 \not{p}_2) = 2m_\chi^2 (p_1 \cdot p_2)
\end{aligned}$$

$$\overline{\mathcal{M}_t^* \mathcal{M}_u} + \text{H.c.} = \frac{\kappa^4 m_\chi^2 (p_1 \cdot p_2)}{2(t - m_\phi^2)(u - m_\phi^2)} + \text{H.c.} = \frac{\kappa^4 m_\chi^2 (p_1 \cdot p_2)}{(t - m_\phi^2)(u - m_\phi^2)}$$

于是,  $\chi\bar{\chi} \rightarrow \psi\bar{\psi}$  的非极化振幅模方为

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \overline{|\mathcal{M}_t|^2} + \overline{|\mathcal{M}_u|^2} - (\overline{\mathcal{M}_t^* \mathcal{M}_u} + \text{H.c.}) \\ &= \kappa^4 \left[ \frac{(k_1 \cdot p_1)(k_2 \cdot p_2)}{(t - m_\phi^2)^2} + \frac{(k_1 \cdot p_2)(k_2 \cdot p_1)}{(u - m_\phi^2)^2} - \frac{m_\chi^2(p_1 \cdot p_2)}{(t - m_\phi^2)(u - m_\phi^2)} \right] \end{aligned}$$