1 Kingman's Coalescent

a

The only possible transition at time t is to go from N_t to $N_t - 1$, so the rate as a function of the number of particles k is

$$g(k, k-1) = \begin{cases} \binom{k}{2} & \text{for } 1 < k \leq L \\ 0 & \text{otherwise.} \end{cases}$$

The generator is given by

$$\begin{aligned}
(\mathcal{L}f)(k) &= \sum_{\substack{j \in S \\ j \neq k}} g(k, j)[f(j) - f(k)] \\
&= g(k, k-1)[f(k-1) - f(k)] \\
&= \binom{k}{2} \Big[f(k-1) - f(k) \Big],
\end{aligned}$$

and the master equation

$$\begin{aligned}
\frac{d}{dt}\pi_{\mathrm{t}}(k) &= \sum_{j \neq k} \pi_{\mathrm{t}}(j) g(j, k) - \sum_{j \neq k} \pi_{\mathrm{t}}(k) g(k, j) \\
&= \pi_{\mathrm{t}}(k+1) g(k+1, k) \\
&= \pi_{\mathrm{t}}(k+1) \binom{k+1}{2} - \pi_{\mathrm{t}}(k) \binom{k}{2}.
\end{aligned}$$

The process has one absorbing state at $k = 1$ since it needs two particle to coalesce, so once it reaches the one particle state nothing will happen. This also describes the stationary distribution given by $(1, 0, 0, \dots)$, and since the process converges to this stationary distribution from any starting state, the process is ergodic.

b

Since the expected staying time in state k is $1/g(k,k-1)$, the total time to go from $k=L$ to $k=1$ is the sum of all the expected times, i.e.,

$$\begin{aligned}
\mathbb{E}(T) &= \sum_{k=2}^{L}\left[\binom{k}{2}\right]^{-1} = \sum_{k=2}^{L}\frac{2!\,(k-2)!}{k!} \\
&= 2\sum_{k=2}^{L}\frac{1}{k(k-1)} = 2\sum_{k=2}^{L}\left(\frac{1}{k-1}-\frac{1}{k}\right) \\
&= 2\left[\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\cdots+\left(\cancel{\frac{1}{1-L}}-\frac{1}{L}\right)\right] \\
&= 2\left(1-\frac{1}{L}\right)
\end{aligned}$$

□

c

The generator for the rescaled process N_t/L is given by

$$(\mathcal{L}f)(x) = \binom{xL}{2}\left[f\left(x-\frac{1}{L}\right)-f(x)\right],$$

where $x=k/L$. Using Taylor expansion for $f(x-1/L)$ around $f(x)$ gives

$$\begin{aligned}
f\left(x-\frac{1}{L}\right) &= f(x)-\frac{1}{L}f'(x)+\frac{1}{2L^2}f''(x)+\ldots, \text{ so} \\
(\mathcal{L}f)(x) &= \binom{xL}{2}\left[\frac{1}{2L^2}f''(x)-\frac{1}{L}f'(x)\right]
\end{aligned}$$

Slowing the process down just applies a factor of $1/L$ to the rates. The generator for $X_t^L = \frac{1}{L}N_{\mathrm{t/L}}$ is then

$$\begin{aligned}
(\mathcal{L}f)(x) &= \frac{1}{L}\cdot\frac{(Lx)!}{2!\,(Lx-2)!}\left[\frac{1}{2L^2}f''(x)-\frac{1}{L}f'(x)\right] \\
&= \frac{(\cancel{L}x)(Lx-1)}{2\cancel{L}}\left[\frac{1}{2L^2}f''(x)-\frac{1}{L}f'(x)\right] \\
&= \frac{1}{2}\left[\cancel{\frac{x^2}{2L}f''(x)}-x^2f'(x)-\cancel{\frac{x^2}{2L}f''(x)}+\cancel{\frac{x}{L}f'(x)}^{\,0}\right] \\
&= -\frac{x^2}{2}f'(x) \text{ as } L\to\infty.
\end{aligned}$$

Since the process X_t^L starts at $x = L/L = 1$ and always terminates at $x = 1/L \neq 0$, the state space $(0, 1]$ is open at 0 and $X_0 = 1$.

We see that this corresponds to the generator of a diffusion process with only the drift term but no diffusion which specifies the randomness. Therefore the process is deterministic because there is no randomness.

In order to compute X_t explicitly, we start with an SDE with no diffusion term and integrate both sides

$$\begin{aligned} dX_t &= a(X_t, t)dt \\ \int_0^t dX_s &= \int_0^t \frac{-X_s^2}{2} ds \\ \int_0^t X_s^{-2} dX_s &= -\int_0^t \frac{1}{2} ds \\ X_s^{-1}\Big|_0^t &= \frac{1}{2} s\Big|_0^t \end{aligned}$$

By evaluating this integral and substituting $X_0 = 1$, we arrive at the explicit solution

$$X_t = \frac{2}{2+t} .$$

This is compatible with the result from part b) because the results are for differently scaled processes. i.e., $2/(2+t)$ describes the path of X_t^L which we have defined to be $\frac{1}{L} N_{t/L}$, whereas the mean time to absorption calculated in part b) is for the unscaled process N_t. By applying the appropriate scaling, we have

$$N_t = L \cdot \frac{2}{2+tL} ,$$

from which we can deduce that

$$N_t = 1 \implies t = \mathbb{E}(T) = 2\left(1 - \frac{1}{L}\right) ,$$

which is in line with our results from part b).

2 Orstein-Uhlenbeck process

a

Let $f(X_t) = X_t$, then $f'(X_t) = 1$ and $f''(X_t) = 0$, and using the evolution equation,

$$\frac{d}{dt}\mathbb{E}[X_t] = \mathbb{E}[-\alpha X_t] = -\alpha\mathbb{E}[X_t],$$

so

$$\frac{d}{dt}\mathbb{E}[X_t] + \alpha\mathbb{E}[X_t] = 0$$

is the required ODE. Solving this gives $m(t) := \mathbb{E}[X_t] = Ae^{-\alpha t}$, where A is an arbitrary constant.
Similarly, let $f(X_t) = X_t^2$, then $f'(X_t) = 2X_t$ and $f''(X_t) = 2$, and

$$\frac{d}{dt}\mathbb{E}[X_t^2] = \mathbb{E}[-\alpha X_t \cdot 2X_t + \frac{1}{2}\sigma^2 \cdot 2] = -\alpha\mathbb{E}[X_t^2] + \sigma^2,$$

so

$$\frac{d}{dt}\mathbb{E}[X_t^2] + 2\alpha\mathbb{E}[X_t^2] = \sigma^2,$$

which can be solved to give $\mathbb{E}[X_t^2] = \sigma^2/2\alpha + Be^{-2\alpha t}$, where B is an arbitrary constant. Therefore,

$$\begin{aligned} v(t) = \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2 &= \frac{\sigma^2}{2\alpha} + Be^{-2\alpha t} - (Ae^{-\alpha t})^2 \\ &= \frac{\sigma^2}{2\alpha} + Ce^{-2\alpha t} \end{aligned}$$

3 Moran model and Wright-Fisher diffusion

a

The state space of $(X_t : t \geq 0)$ is $\{(x_1, x_2, \ldots, x_L) : x_i \in \{1, 2, \ldots, L\}\}$. The process is not reducible since once a type has been completely killed off, the process can no longer return to a state where one of the individuals is of that type.

The stationary distributions are $\{(x, x, \ldots, x) : x \in \{1, 2, \ldots, L\}\}$ because the process is only stationary once only a single type survives.

b

$(N_t : t \geq 0)$ is a Markov process and its state space is $\{0, 1, \ldots, L\}$.

The rates of this process for $x \neq 0$ or L is

$$\begin{cases} g(x, x+1) & = x \cdot \frac{L-x}{L-1} \\ g(x, x-1) & = (L-x) \cdot \frac{x}{L-1} \end{cases} \text{..28?????}$$

The reasoning is that in order to gain an individual, one of the x current individuals must impose its type on someone of a different type. Since it does so randomly, the probability of killing someone of a different type is $\frac{L-x}{L-1}$. We also get $g(x, x-1)$ by applying similar logic.

The generator is

$$(\mathcal{L}f)(x) = \frac{x(L-x)}{L-1}[f(x+1) - 2f(x) + f(x-1)].$$

The process is, once again, not irreducible since $x = 0$ and $x = L$ are absorbing states. This also means that there exist infinitely many stationary distributions of the form

$$(\alpha, 0, \ldots, 1-\alpha) \ , \ \alpha \in [0, 1].$$

The limiting distribution as $t \to \infty$ is

$$\left(\frac{L-1}{L}, 0, \ldots, 0, \frac{1}{L}\right)$$

since the probability of any one individual dominating is exactly the same due to symmetry, and there was only one individual of type k at $t = 0$.

c

By setting $f(N_t) = N_t$ and using the evolution equation, we obtain

$$\frac{d}{dt}\mathbb{E}[N_t] = \mathbb{E}\left[\frac{N_t(L-N_t)}{L-1}\Big((N_t+1) - 2(N_t) + (N_t-1)\Big)\right] = 0,$$

which means that N_t is constant in time. Given that $N_0 = n$, we may deduce that $m_1(t) \equiv n$.

Similarly, by setting $f(N_t) = N_t^2$,

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}[N_t^2] &= \mathbb{E}\left[\frac{N_t(L-N_t)}{L-1}\Big((N_t^2+2N_t+1) - 2(N_t^2) + (N_t^2-2N_t+1)\Big)\right] \\
&= \mathbb{E}\left[\frac{2N_t(L-N_t)}{L-1}\right] \\
&= \frac{2L}{L-1}\mathbb{E}[N_t] - \frac{2}{L-1}\mathbb{E}[N_t^2]
\end{aligned}$$

i.e.,

$$\frac{d}{dt}\mathbb{E}[N_t^2] + \frac{2}{L-1}\mathbb{E}[N_t^2] = \frac{2Ln}{L-1}.$$

Solving this gives

$$m_2(t) = \mathbb{E}[N_t^2] = Ln + Ae^{\frac{2t}{1-L}}$$

In the scaling limit $t \to \infty$, the exponential term vanishes so we just have $m_2(t) \to Ln$.