1 Kingman's Coalescent

a) The only possible transition at time t is to go from N_t to $N_t - 1$, so the rate as a function of the number of particles k is

$$g(k, k-1) = \begin{cases} \binom{k}{2} & \text{for } 1 < k \leq L \\ 0 & \text{otherwise.} \end{cases}$$

The generator is given by

$$\begin{aligned} (\mathcal{L}f)(k) &= \sum_{\substack{j \in S \\ j \neq k}} g(k, j)[f(j) - f(k)] \\ &= g(k, k-1)[f(k-1) - f(k)] \\ &= \binom{k}{2} \Big[f(k-1) - f(k) \Big], \end{aligned}$$

and the master equation

$$\begin{aligned} \frac{d}{dt} \pi_{\mathrm{t}}(k) &= \sum_{j \neq k} \pi_{\mathrm{t}}(j) g(j, k) - \sum_{j \neq k} \pi_{\mathrm{t}}(k) g(k, j) \\ &= \pi_{\mathrm{t}}(k+1) g(k+1, k) \\ &= \pi_{\mathrm{t}}(k+1) \binom{k+1}{2} - \pi_{\mathrm{t}}(k) \binom{k}{2}. \end{aligned}$$

The process has one absorbing state at $k = 1$ since it needs two particle to coalesce, so once it reaches the one particle state nothing will happen. This also describes the stationary distribution given by $(1, 0, 0, \dots)$, and since the process converges to this stationary distribution from any starting state, the process is ergodic.

b) Since the expected staying time in state k is $1/g(k, k-1)$, the total time to go from $k = L$ to $k = 1$ is the sum of all the expected times, i.e.,

$$\begin{aligned} \mathbb{E}(T) &= \sum_{k=2}^{L} \left[\binom{k}{2} \right]^{-1} = \sum_{k=2}^{L} \frac{2!\,(k-2)!}{k!} \\ &= 2 \sum_{k=2}^{L} \frac{1}{k(k-1)} = 2 \sum_{k=2}^{L} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\ &= 2 \left[\left(\frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \cdots + \left(\cancel{\frac{1}{1-L}} - \frac{1}{L} \right) \right] \\ &= 2 \left(1 - \frac{1}{L} \right) \end{aligned}$$

$\square$

c) The generator for the rescaled process N_t/L is given by

$$(\mathcal{L}f)(x) = \binom{xL}{2} \left[f\left(x - \frac{1}{L} \right) - f(x) \right],$$

where $x = k/L$. Using Taylor expansion for $f(x - 1/L)$ around $f(x)$ gives

$$
\begin{aligned}
f\left(x - \frac{1}{L}\right) &= f(x) - \frac{1}{L}f'(x) + \frac{1}{2L^2}f''(x) + \ldots, \text{ so} \\
(\mathcal{L}f)(x) &= \binom{xL}{2}\left[\frac{1}{2L^2}f''(x) - \frac{1}{L}f'(x)\right]
\end{aligned}
$$

Slowing the process down just applies a factor of $1/L$ to the rates. The generator for $X_t^L = \frac{1}{L}N_{\mathrm{t/L}}$ is then

$$
\begin{aligned}
(\mathcal{L}f)(x) &= \frac{1}{L}\cdot\frac{(Lx)!}{2!\,(Lx-2)!}\left[\frac{1}{2L^2}f''(x) - \frac{1}{L}f'(x)\right] \\
&= \frac{(\cancel{L}x)(Lx-1)}{2\cancel{L}}\left[\frac{1}{2L^2}f''(x) - \frac{1}{L}f'(x)\right] \\
&= \frac{1}{2}\left[\cancel{\frac{x^2}{2L}f''(x)} - x^2 f'(x) - \cancel{\frac{x^2}{2L}f''(x)} + \cancelto{0}{\frac{x}{L}f'(x)}\right] \\
&= -\frac{x^2}{2}f'(x) \text{ as } L \to \infty.
\end{aligned}
$$

Since the process X_t^L starts at $x = L/L = 1$ and always terminates at $x = 1/L \neq 0$, the state space $(0, 1]$ is open at 0 and $X_0 = 1$.

We see that this corresponds to the generator of a diffusion process with only the drift term but no diffusion which specifies the randomness. Therefore the process is deterministic because there is no randomness.

In order to compute X_t explicitly, we start with an SDE with no diffusion term and integrate both sides

$$
\begin{aligned}
dX_t &= a(X_t, t)dt \\
\int_0^t dX_s &= \int_0^t \frac{-X_s^2}{2}ds \\
\int_0^t X_s^{-2}dX_s &= -\int_0^t \frac{1}{2}ds \\
X_s^{-1}\Big|_0^t &= \frac{1}{2}s\Big|_0^t
\end{aligned}
$$

By evaluating this integral and substituting $X_0 = 1$, we arrive at the explicit solution

$$
X_t = \frac{2}{2+t}\,.
$$

This is compatible with the result from part b) because the results are for differently scaled processes. i.e., $2/(2+t)$ describes the path of X_t^L which we have defined to be $\frac{1}{L}N_{t/L}$, whereas the mean time to absorption calculated in part b) is for the unscaled process N_t. By applying the appropriate scaling, we have

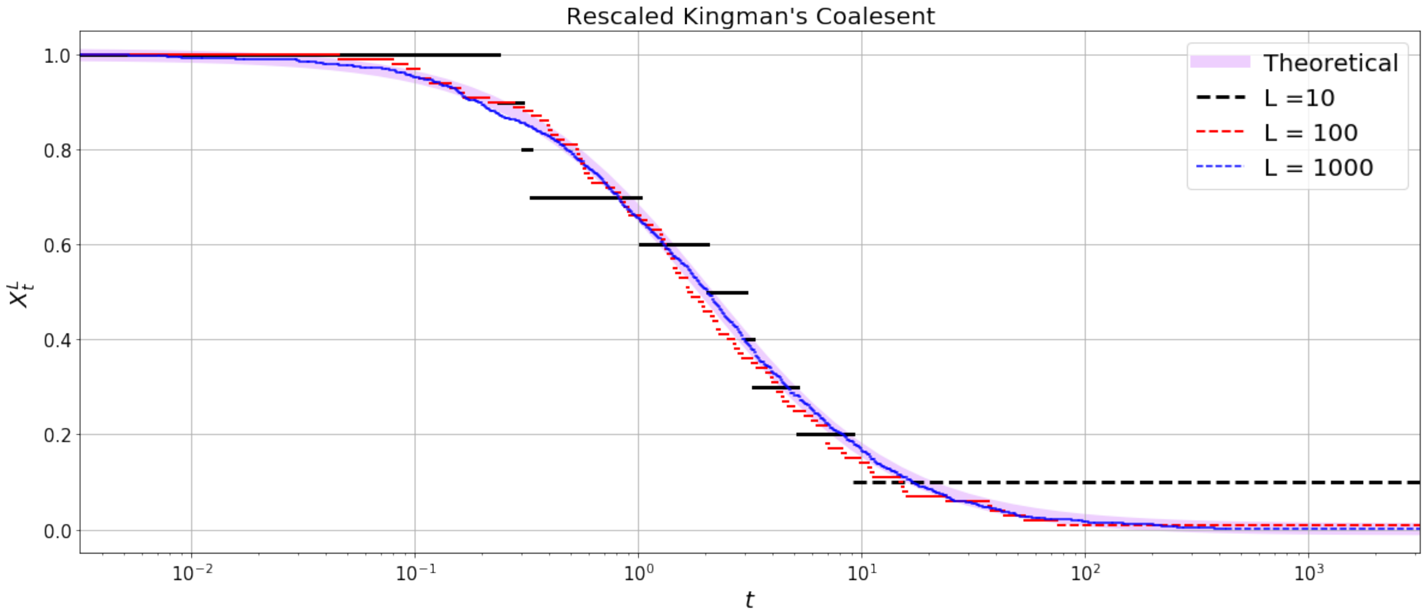
$$
N_t = L \cdot \frac{2}{2+tL}\,,
$$

from which we can deduce that

$$N_t = 1 \implies t = \mathbb{E}(T) = 2\left(1 - \frac{1}{L}\right),$$

which is in line with our results from part b).

d)



2 Orstein-Uhlenbeck process

a+b) Let $f(X_t) = X_t$, then $f'(X_t) = 1$ and $f''(X_t) = 0$, and using the evolution equation,

$$\frac{d}{dt}\mathbb{E}[X_t] = \mathbb{E}[-\alpha X_t] = -\alpha \mathbb{E}[X_t],$$

so

$$\frac{d}{dt}\mathbb{E}[X_t] + \alpha \mathbb{E}[X_t] = 0$$

is the required ODE. Solving this gives the general solution $m(t) := \mathbb{E}[X_t] = Ae^{-\alpha t}$, where A is an arbitrary constant. We can then substitute in $m(0) = x_0$ to obtain

$$m(t) = x_0 e^{-\alpha t}.$$

Similarly, let $f(X_t) = X_t^2$, then $f'(X_t) = 2X_t$ and $f''(X_t) = 2$, and

$$\frac{d}{dt}\mathbb{E}[X_t^2] = \mathbb{E}[-\alpha X_t \cdot 2X_t + \frac{1}{2}\sigma^2 \cdot 2] = -\alpha \mathbb{E}[X_t^2] + \sigma^2,$$

so

$$\frac{d}{dt}\mathbb{E}[X_t^2] + 2\alpha \mathbb{E}[X_t^2] = \sigma^2,$$

which can be solved to give $\mathbb{E}[X_t^2] = \sigma^2/2\alpha + Be^{-2\alpha t}$, where B is an arbitrary constant. Therefore,

$$
\begin{aligned}
v(t) = \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2 &= \frac{\sigma^2}{2\alpha} + Be^{-2\alpha t} - (Ae^{-\alpha t})^2 \\
&= \frac{\sigma^2}{2\alpha} + Ce^{-2\alpha t}.
\end{aligned}
$$

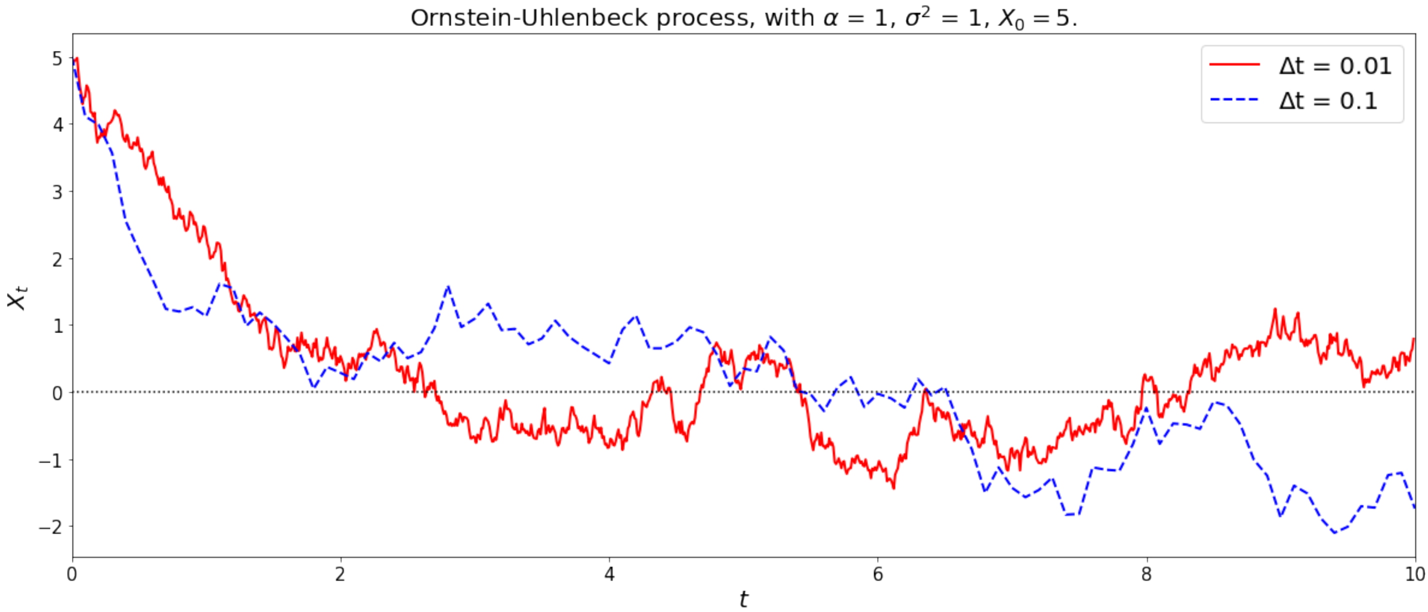
But we know that $v(0) = 0$ because the process is deterministic at $t = 0$, so we can use this fact to obtain

$$
v(t) = \frac{\sigma^2}{2\alpha}\big(1 - e^{-2\alpha t}\big).
$$

Therefore the distribution of X_t is $X_t \sim \mathcal{N}\big(m(t), v(t)\big)$.

As $t \to \infty$, $m(t) \to 0$ and $v(t) \to \frac{\sigma^2}{2\alpha}$, so the stationary distribution of the process is $X_t \sim \mathcal{N}\big(0, \frac{\sigma^2}{2\alpha}\big)$.

c)



3 Moran model and Wright-Fisher diffusion

a) The state space of $(X_t : t \geq 0)$ is $\{(x_1, x_2, \ldots, x_L) : x_i \in \{1, 2, \ldots, L\}\}$. The process is not reducible since once a type has been completely killed off, the process can no longer return to a state where one of the individuals is of that type.

The stationary distributions are $\{(x, x, \ldots, x) : x \in \{1, 2, \ldots, L\}\}$ because the process is only stationary once only a single type survives.

b) $(N_t : t \geq 0)$ is a Markov process and its state space is $\{0, 1, \ldots, L\}$.

The rates of this process for $x \neq 0$ or L is

$$\begin{cases} g(x, x+1) & = x \cdot \frac{L-x}{L-1} \\ g(x, x-1) & = (L-x) \cdot \frac{x}{L-1} \end{cases}$$

The reasoning is that in order to gain an individual, one of the x current individuals must impose its type on someone of a different type. Since it does so randomly, the probability of killing someone of a different type is $\frac{L-x}{L-1}$. We also get $g(x, x-1)$ by applying similar logic.

The generator is

$$(\mathcal{L}f)(x) = \frac{x(L-x)}{L-1}[f(x+1) - 2f(x) + f(x-1)].$$

The process is, once again, not irreducible since $x = 0$ and $x = L$ are absorbing states. This also means that there exist infinitely many stationary distributions of the form

$$(\alpha, 0, \ldots, 1-\alpha) \ , \ \alpha \in [0, 1].$$

The limiting distribution as $t \to \infty$ is

$$\left(\frac{L-1}{L}, 0, \ldots, 0, \frac{1}{L} \right)$$

since the probability of any one individual dominating is exactly the same due to symmetry, and there was only one individual of type k at $t = 0$.

c) By setting $f(N_t) = N_t$ and using the evolution equation, we obtain

$$\frac{d}{dt}\mathbb{E}[N_t] = \mathbb{E}\left[\frac{N_t(L-N_t)}{L-1} \Big((N_t + 1) - 2(N_t) + (N_t - 1) \Big) \right] = 0,$$

which means that N_t is constant in time. Given that $N_0 = n$, we may deduce that $m_1(t) \equiv n$.

Similarly, by setting $f(N_t) = N_t^2$,

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[N_t^2] &= \mathbb{E}\left[\frac{N_t(L-N_t)}{L-1} \Big((N_t^2 + 2N_t + 1) - 2(N_t^2) + (N_t^2 - 2N_t + 1) \Big) \right] \\ &= \mathbb{E}\left[\frac{2N_t(L-N_t)}{L-1} \right] \\ &= \frac{2L}{L-1}\mathbb{E}[N_t] - \frac{2}{L-1}\mathbb{E}[N_t^2] \end{aligned}$$

i.e.,

$$\frac{d}{dt}\mathbb{E}[N_t^2] + \frac{2}{L-1}\mathbb{E}[N_t^2] = \frac{2Ln}{L-1}.$$

Solving this gives

$$m_2(t) = \mathbb{E}[N_t^2] = Ln + Ae^{\frac{2t}{1-L}}$$

In the scaling limit $t \to \infty$, the exponential term vanishes so we just have $m_2(t) \to Ln$.

As shown in part b), the probability of any one individual dominating the population is $\frac{1}{L}$, and this is independent of the individual's type. We may then deduce that the probability of a given type, which starts with n individuals, dominating - i.e., being absorbed in $N = L$ - is equal to $\frac{n}{L}$. Since we are considering the scaling limit $t \to \infty$, the process must be in one of the absorbing states, so the probability of being absorbed at 0 is $1 - \frac{n}{L}$.

Using $m_1(t)$, $m_2(t)$, and the fact that the process is deterministic at $t = 0$, we can obtain

$$v(t) = (Ln - n^2)(1 - e^{\frac{-2t}{L-1}}).$$

It follows that when L is large, the variance scales linearly with L so the absorption time is also linear in L.

d) Rescaling the generator gives

$$(\mathcal{L}f)(x) = L^\alpha \cdot \frac{xL(L - xL)}{L-1}\left[f\big(x + \tfrac{1}{L}\big) - 2f(x) + f\big(x - \tfrac{1}{L}\big)\right].$$

By using Taylor expansion around x for terms up to second order, we have

$$\begin{aligned}(\mathcal{L}f)(x) &= L^\alpha \cdot \frac{\cancel{L^2}}{L-1} \cdot x(1-x) \cdot \frac{1}{\cancel{L^2}} f''(x) \\ &= \frac{L^\alpha}{L-1} \cdot x(1-x)f''(x).\end{aligned}$$

We can see that the only α for which M_t^L has a non-trivial scaling limit is when $\alpha = 1$, thus we have the generator for $(M_t : t \geq 0)$ is

$$(\mathcal{L}f)(x) = x(1-x)f''(x),$$

which essentially describes a diffusion process with no drift term. This makes sense as the process is essentially completely random. Substituting $a(x,t) = 0$ and $\frac{\sigma^2}{2}(x,t) = x(1-x)$ into the Fokker-Planck equation gives

$$\frac{\partial}{\partial t} p_t(x,y) = \frac{\partial^2}{\partial y^2}\big(y(1-y)p_t(x,y)\big).$$

e) Let $f(M_t) = M_t$, then $f''(M_t) = 0$ and $\frac{d}{dt}\mathbb{E}[M_t] = \mathbb{E}[0]$, which means $\mathbb{E}[M_t]$ is not dependent on time, so

$$m(t) := \mathbb{E}[M_t] \equiv n/L.$$

Similarly, by letting $f(M_t) = M_t^2$ we get

$$\frac{d}{dt}\mathbb{E}[M_t^2] = \mathbb{E}[2M_t(1 - M_t)]$$

which can be solved to give

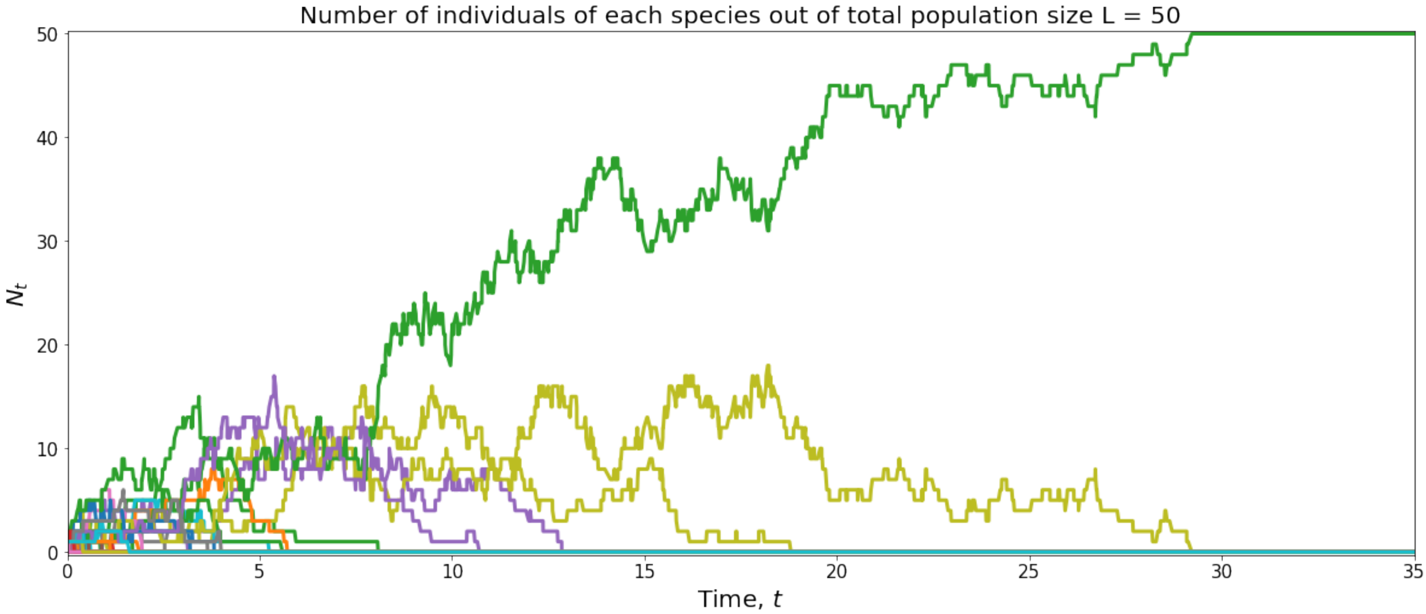
$$\mathbb{E}[M_t^2] = \frac{n}{L} + Ae^{-2t},$$

and hence

$$v(t) := \mathbb{E}[M_t^2] - m(t)^2 = \frac{n}{L} + Ae^{-2t} - \left(\frac{n}{L}\right)^2.$$

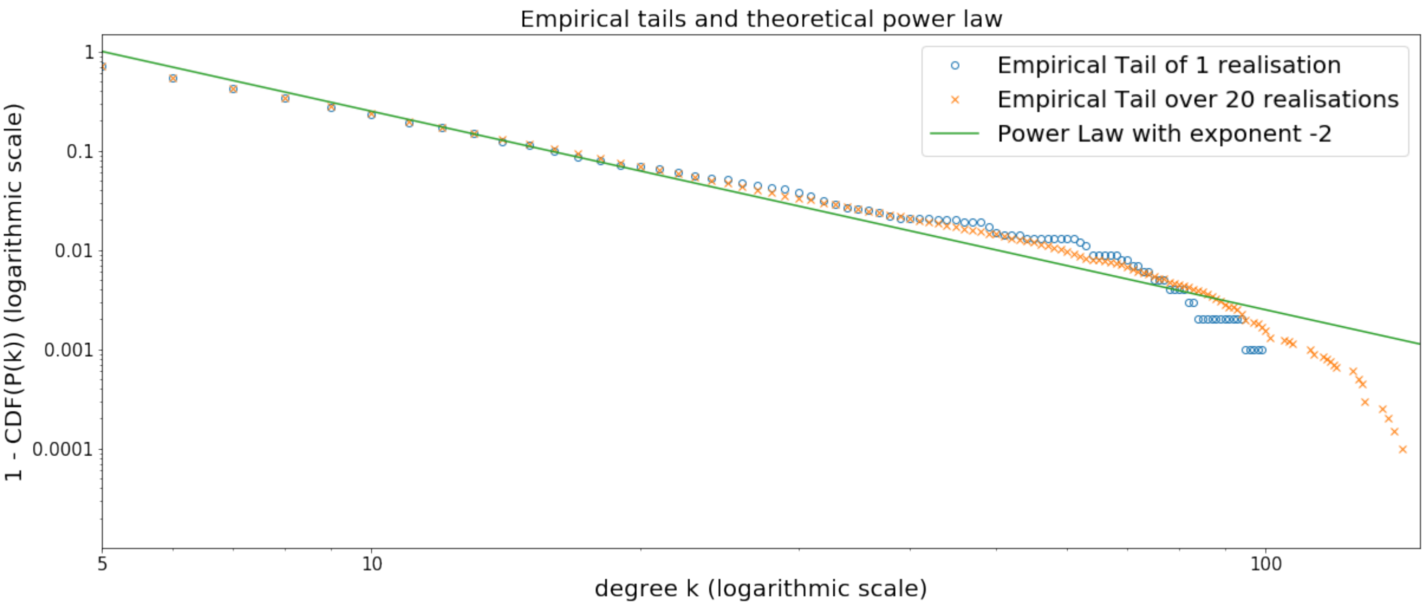
The process is not Gaussian because the state space of this process is bounded, whereas Gaussian distributions have infinitely long tails.

f)

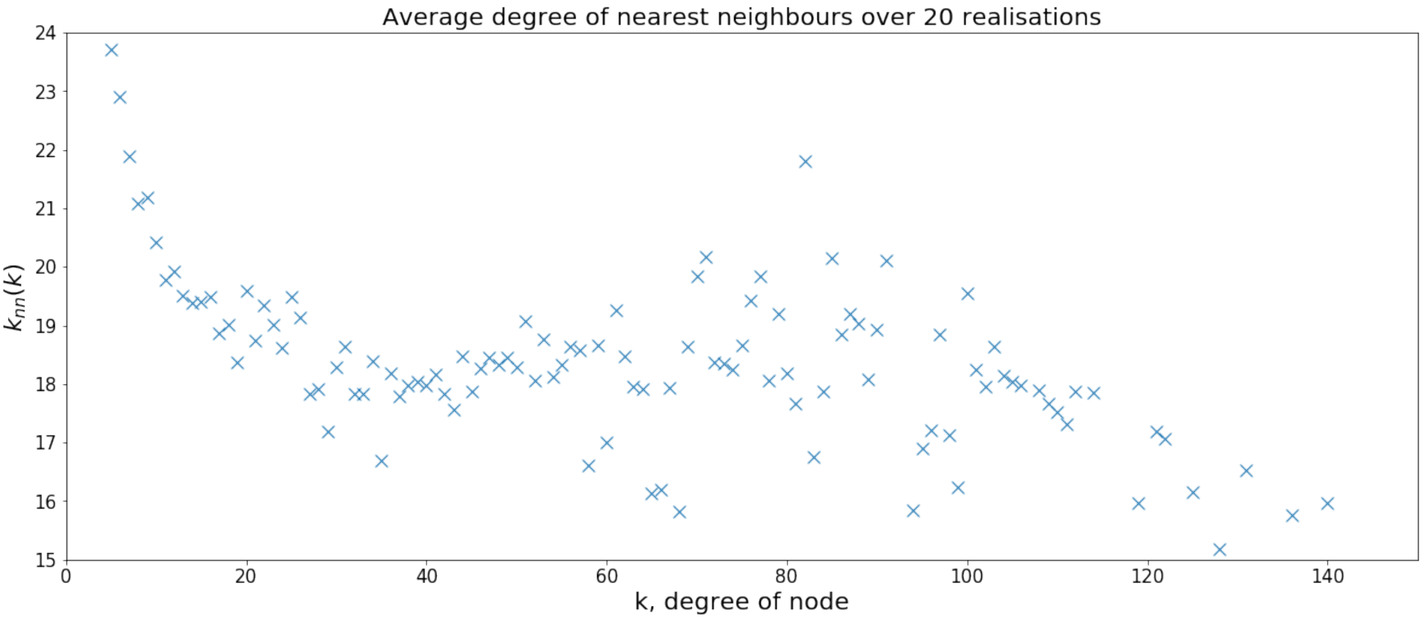


4 Barabási-Albert Model

a)

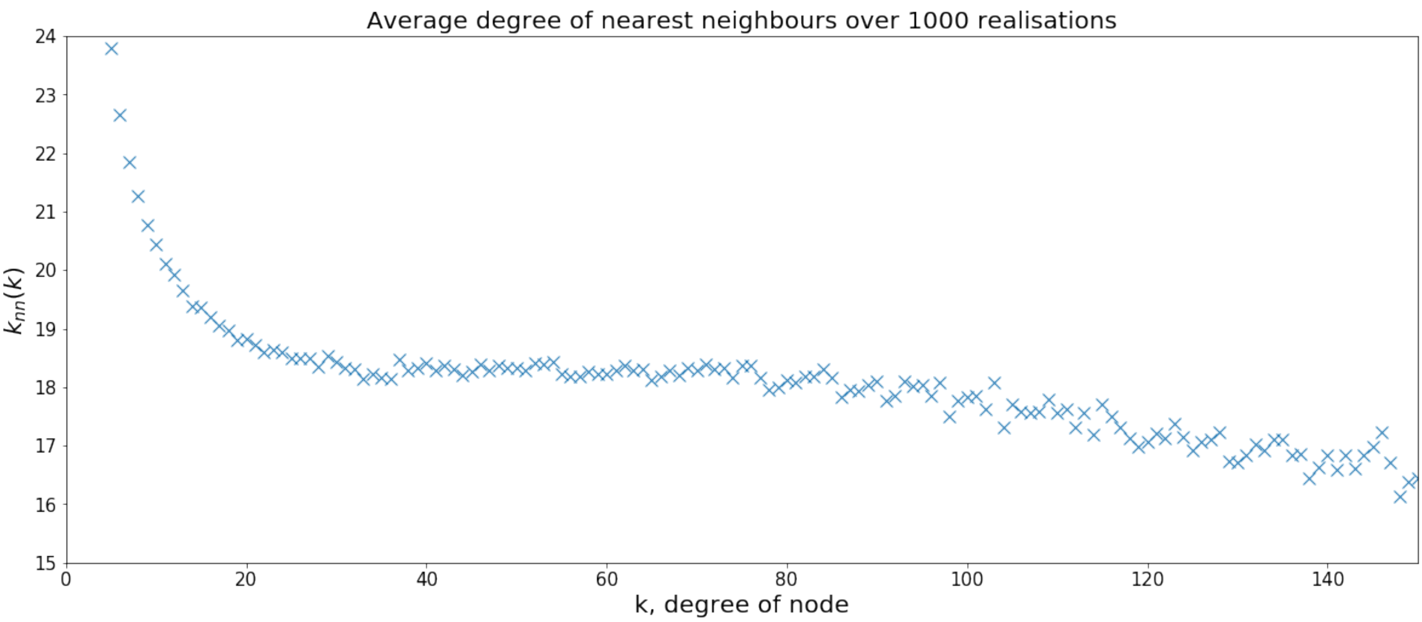


b)



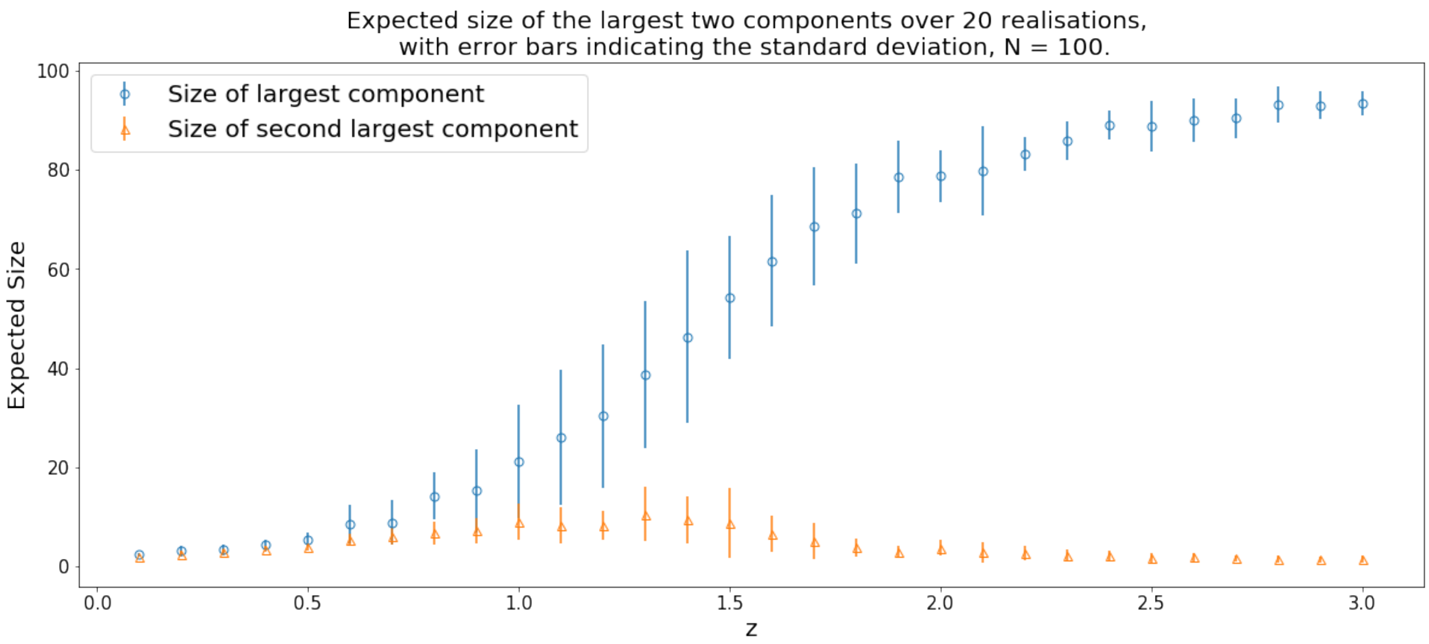
The plots, and therefore the graphs, appear to be mostly uncorrelated when we only allow 20 realisations of the simulation. There is a very large spread in the data.

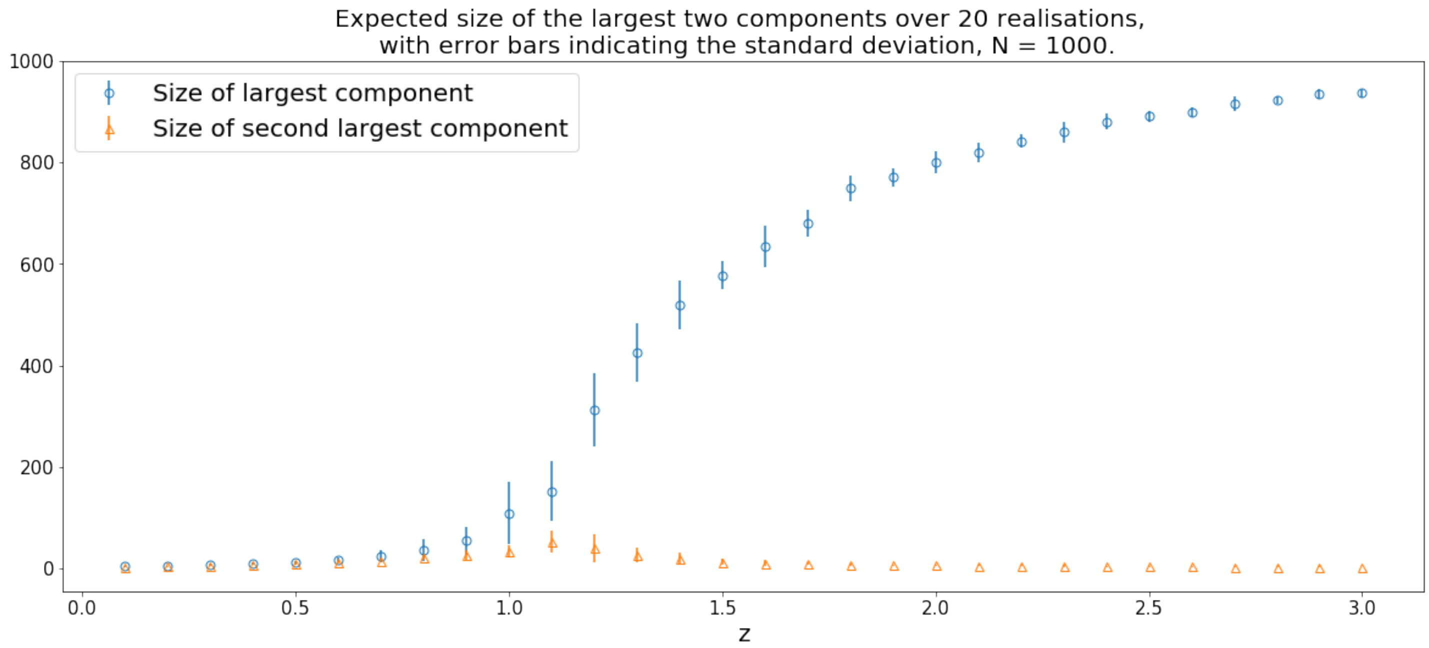
Running the simulation for larger numbers of realisations, $N = 1000$ for example, seemed to help smooth out the plot and highlight the disassortative nature of the graphs that we would expect from the theory - I have included that plot below.



5 Erdös Rényi Random Graphs

a)

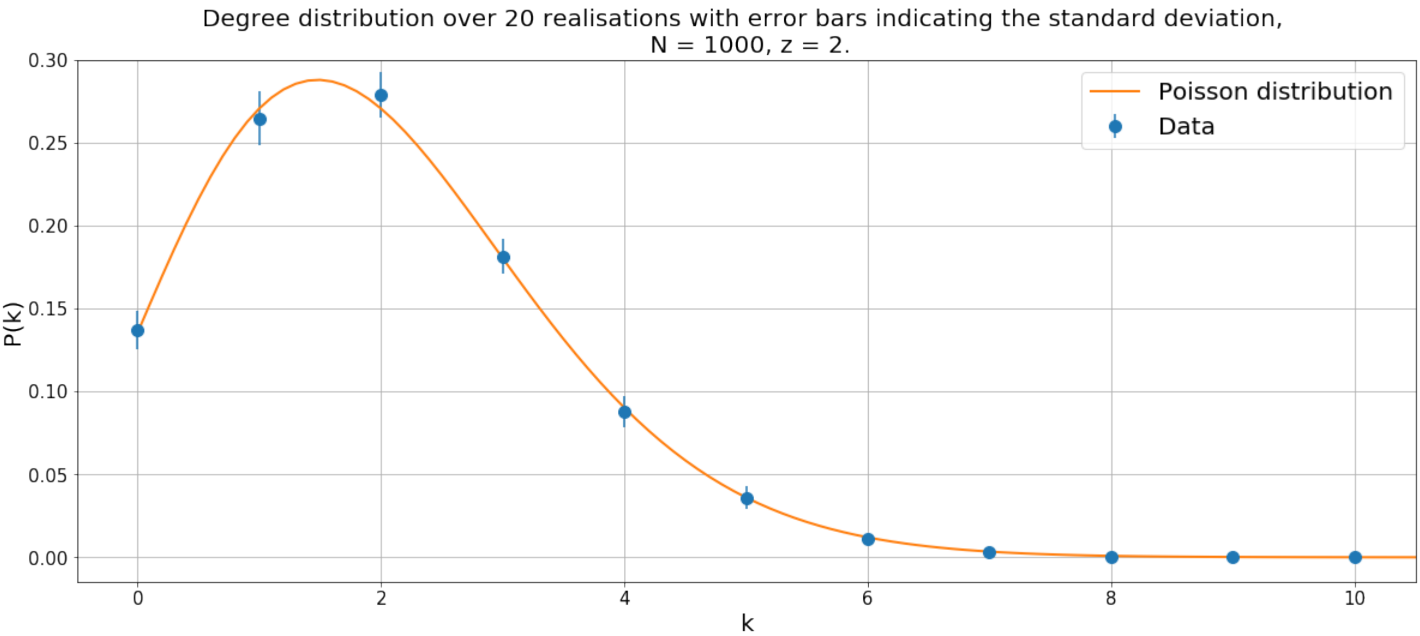




b)

The theoretical limit p line fits a lot better when we allow for higher values of z.

c)



I interpreted this question as collate all 20 realisations together to obtain one single degree distribution $p(k)$ but then discovered that some people have plotted all the realisations separately. Therefore I settled and added some error bars.