

1. **Probabilistic modeling of uncertainties:** Two outstanding students S_A and S_B are deciding whether to accept the offer for the SUSTech PhD program.
 - (a) Suppose you believe (i) S_A and S_B make decisions independently; (ii) the chance that S_A accepts the offer is 0.8; (iii) the chance S_B accepts the offer is 0.6. Please construct a probability space to represent your uncertain belief, namely, find the sample space Ω (all the possible outcomes) and the probability mass function (probability mass for each outcome) so that the above three conditions are satisfied.
 - (b) Now assume the two students are good friends, you know that (i) if S_B accepts the offer, then S_A will for sure accept the offer ; (ii) if S_B does not accept the offer, then S_A only has 30% chance to accept the offer (also implies there is 70% chance S_A will not accept the offer given the fact that S_B does not accept the offer); (iii) the chance that neither of them accepts the offer is 35%. Please construct a probability space to represent your uncertain knowledge in this case.

2. **Conditional Probability and Expectation:** Suppose X and Y are discrete random variables. X is uniformly distributed on the set $\{0, 1, \dots, n\}$, while Y is conditionally uniform on 0 through i given $X = i$, for each $i = 0, \dots, n$.

- (a) Compute the conditional mean $E(Y|X = i)$ for a general $i \leq n$.
- (b) Compute $E(Y)$ by conditioning on the values of X , namely, using the formula $E(Y) = \sum_{i=0}^n E(Y|X = i)p_X(i)$, where $p_X(i) = \text{Prob}(X = i)$.
- (c) Find the joint probability mass function (pmf) $p(i, j) \text{Prob}(X = i, Y = j)$, for $i = 0, \dots, n$ and $j = 0, \dots, n$. (hint: for some pair (i, j) the joint pmf is zero. Make sure you clearly identify those).
- (d) Compute the marginal $p_Y(j) = \text{Prob}(Y = j)$ for $j = 0, \dots, n$.
- (e) Write a matlab function to compute the mean $E(Y)$ of Y using p_Y for $n = 100$, and compare the result with (b).
- (f) Assume $n \geq 1$. Let $g(X) = 2$, if $X = 1$ or n , and $g(X) = 0$ otherwise. Compute $E(g(X)Y)$ through conditional expectation.

3. Conditional Density and Expectation

- (a) Suppose that (X, Y) is uniformly distributed on the triangle $S = \{(x, y) : -6 < y < x < 6\}$. Find $E(Y|X = x)$.

- (b) Let (X, Y) be two random variables with joint density function:

$$f(x, y) = \begin{cases} \frac{1}{4}(2x + y) & x \in [0, 1], y \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X)$ and $E(X|Y = 1/2)$.

- (c) Let X is an arbitrary 3D random vector with density $f(x_1, x_2, x_3)$. Show that if X_1 is independent of both X_2 and X_3 , then $X_1|X_3$ is independent of $X_2|X_3$.
(hint: show $f(x_1, x_2|x_3) = f(x_1|x_3)f(x_2|x_3)$).