1. **Spatial Velocity:** A cylinder rolls without slipping in the $\hat{x}_0$ direction on the $\hat{x}_0 - \hat{y}_0$ plane. The cylinder has a radius of r and a constant forward speed of v. Let ${}^0C=[C_x(t),0,0]^T$ be the position of the contact point at time t. Let ${}^0A=[A_x(t),0,0]^T$ be the position of the instantaneous top of the cylinder at time t.

   (a) What is the linear velocity of the point C? (hint: just need to compute $\frac{d}{dt}C_x(t)$)?

   (b) What is the linear velocity of the point A?

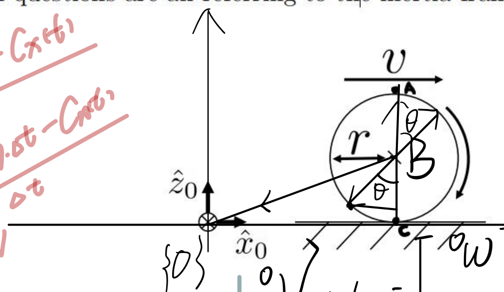
   (c) What is velocity of the body-fixed point currently coincides with C?

   (d) What is velocity of the body-fixed point currently coincides with A?

   (e) What is the spatial velocity of the cylinder in {0}-frame?

   (f) What is the spatial velocity of the cylinder in frame {C}? ({C} has the same orientation as {0}, while its origin is at the contact point C)

Note: The first 4 questions are all referring to the inertia frame {0}



$$\frac{dC_x(t)}{dt} = \lim_{\delta t \to 0} \frac{C_x(t+\delta t) - C_x(t)}{\delta t} = \frac{C_x(t) + v\cdot\delta t - C_x(t)}{\delta t} = v$$

(a) 先取{0}坐标系，圆心B为参考点，

$$V_C = V_B + W_B \times \overrightarrow{BC}$$

$${}^0V_C = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ w \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix} = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -wr \\ 0 \\ 0 \end{bmatrix}$$

由于无滑滚动，所以有 $V_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) $V_A = V_B + W_B \times \overrightarrow{BA}$

$${}^0V_A = 2\begin{bmatrix} 2v \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{dA_x(t)}{dt} = \lim_{\delta t \to 0} \frac{A_x(t+\delta t) - A_x(t)}{\delta t} = v$$

(c) 先取{0}坐标系，圆心B为参考点，C_{bf}代表t时刻在C点的一个b-f点，

$$\cancel{V_{C_{bf}} = V_B + W_B \times \overrightarrow{BC_{bf}} \xrightarrow{\{0\}} \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & w \\ 0 & 0 & 0 \\ -w & 0 & 0 \end{bmatrix} \begin{bmatrix} -r\sin\frac{v}{r}t \\ 0 \\ -r\cos\frac{v}{r}t \end{bmatrix} = \begin{bmatrix} v - v\cos\frac{v}{r}t \\ 0 \\ v\sin\frac{v}{r}t \end{bmatrix}}$$

当与C重合时 $V_{C_{bf}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$V_C = V_B + W \times \overrightarrow{BC} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(d) A_{bf}代表t时刻在A点的一个b-f点.

$$V_{A_{bf}} = V_B + W_B \times \overrightarrow{BA_{bf}} \xrightarrow{\{0\}} \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & w \\ 0 & 0 & 0 \\ -w & 0 & 0 \end{bmatrix} \begin{bmatrix} r\sin\frac{v}{r}t \\ 0 \\ r\cos\frac{v}{r}t \end{bmatrix}$$

$$= \begin{bmatrix} v + v\cos\frac{v}{r}t \\ 0 \\ -v\sin\frac{v}{r}t \end{bmatrix}$$

当与A重合时 $V_{A_{bf}} = \begin{bmatrix} 2v \\ 0 \\ 0 \end{bmatrix}$

$$V_A = V_B + W \times \overrightarrow{BA} = \begin{bmatrix} 2v \\ 0 \\ 0 \end{bmatrix}$$

(e) $${}^0\mathcal{V}_{cylinder} = \begin{bmatrix} {}^0w \\ {}^0V_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \\ {}^0V_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \\ \cancel{2v}\ 0 \\ 0 \\ -\frac{v}{r}C_t \end{bmatrix}$$

$${}^0V_0 = V_B + W \times \overrightarrow{BO} = \begin{bmatrix} 0 \\ 0 \\ C_x\frac{v}{r} \end{bmatrix}$$

$$V_0 = V_B + W \times \overrightarrow{BO} \xrightarrow{\{0\}} {}^0V_0 = {}^0V_B + \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \end{bmatrix} \times \begin{bmatrix} -C_t \\ 0 \\ -r \end{bmatrix}$$

$$= \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{v}{r} \\ 0 & 0 & 0 \\ -\frac{v}{r} & 0 & 0 \end{bmatrix} \begin{bmatrix} -C_t \\ 0 \\ -r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{v}{r}C_t \end{bmatrix}$$

(f) $${}^C\mathcal{V}_{cylinder} = \begin{bmatrix} {}^Cw \\ {}^CV_C \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \\ {}^CV_C \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_C = V_B + W \times \overrightarrow{BC} = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^CV_C = {}^CV_B + {}^CW \times \overrightarrow{BC} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

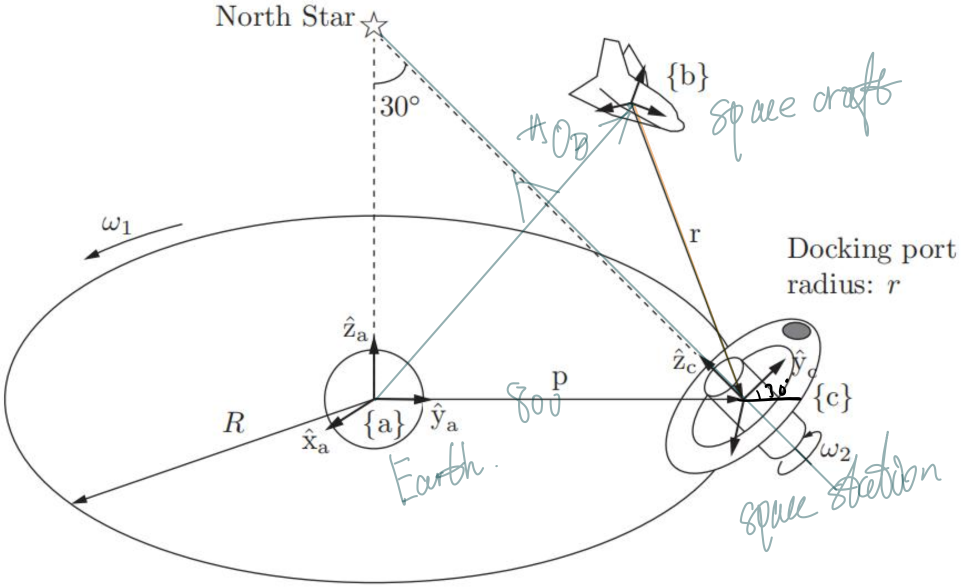


Figure 3.27: A spacecraft and space station.

Exercise 3.21 The space station of Figure 3.27 moves in circular orbit around the Earth, and at the same time rotates about an axis always pointing toward the North Star. Owing to an instrument malfunction, a spacecraft heading toward the space station is unable to locate the docking port. An Earth-based ground station sends the following information to the spacecraft:

$$\underbrace{}_{^AR_B}\quad \underbrace{}_{^AO_B}$$

$$T_{ab} = \begin{bmatrix} 0 & -1 & 0 & -100 \\ 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad p_a = \begin{bmatrix} 0 \\ 800 \\ 0 \end{bmatrix},$$

where p_a is the vector p expressed in {a}-frame coordinates.

(a) From the given information, find r_b, the vector r expressed in {b}-frame coordinates.

(b) Determine T_{bc} at the instant shown in the figure. Assume here that the $\hat{y}$- and $\hat{z}$-axes of the {a} and {c} frames are coplanar with the docking port.

(a). $${}^A r = {}^A p - {}^A O_B = \begin{bmatrix} 0 \\ 800 \\ 0 \end{bmatrix} - \begin{bmatrix} -100 \\ 300 \\ 500 \end{bmatrix} = \begin{bmatrix} 100 \\ 500 \\ -500 \end{bmatrix}$$

$${}^B r = {}^B R_A {}^A r = ({}^A R_B)^{-1} {}^A r = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 500 \\ -500 \end{bmatrix} = \begin{bmatrix} 500 \\ -100 \\ -500 \end{bmatrix}$$

(b)

$${}^B T_C = \begin{bmatrix} {}^B R_C & {}^B O_C \\ 0 & 1 \end{bmatrix}$$

$${}^B O_C = {}^B r = \begin{bmatrix} 500 \\ -100 \\ -500 \end{bmatrix}$$

$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} {}^A$

$${}^BR_C = {}^BR_A\,{}^AR_C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^AR_C$$

$${}^AR_C = \begin{bmatrix} {}^A\hat{x}_C & {}^A\hat{y}_C & {}^A\hat{z}_C \end{bmatrix}$$

$${}^A\hat{z}_C = \begin{bmatrix} 0 \\ \sin 30^\circ \\ \cos 30^\circ \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \quad {}^A\hat{y}_C = \begin{bmatrix} 0 \\ \cos 30^\circ \\ \sin 30^\circ \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \quad \therefore {}^A\hat{x}_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore {}^AR_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \qquad {}^BR_C = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

$${}^BT_C = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 500 \\ -1 & 0 & 0 & -100 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & -500 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 3.28 Assume that the space-frame angular velocity is $\omega_s = (1, 2, 3)$ for a moving body with frame {b} at

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

relative to the space frame {s}. Calculate the body's angular velocity ω_b in {b}.

$$^{s}R_b = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$^{b}W = {}^{b}R_s \, W_s = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Exercise 5.5 Referring to Figure 5.17, a rigid body, shown at the top right, rotates about the point (L, L) with angular velocity $\dot{\theta} = 1$.

(a) Find the position of point P on the moving body relative to the fixed reference frame {s} in terms of θ.

(b) Find the velocity of point P in terms of the fixed frame.

(c) What is T_{sb}, the configuration of frame {b}, as seen from the fixed frame {s}?

(d) Find the twist of T_{sb} in body coordinates.

(e) Find the twist of T_{sb} in space coordinates.

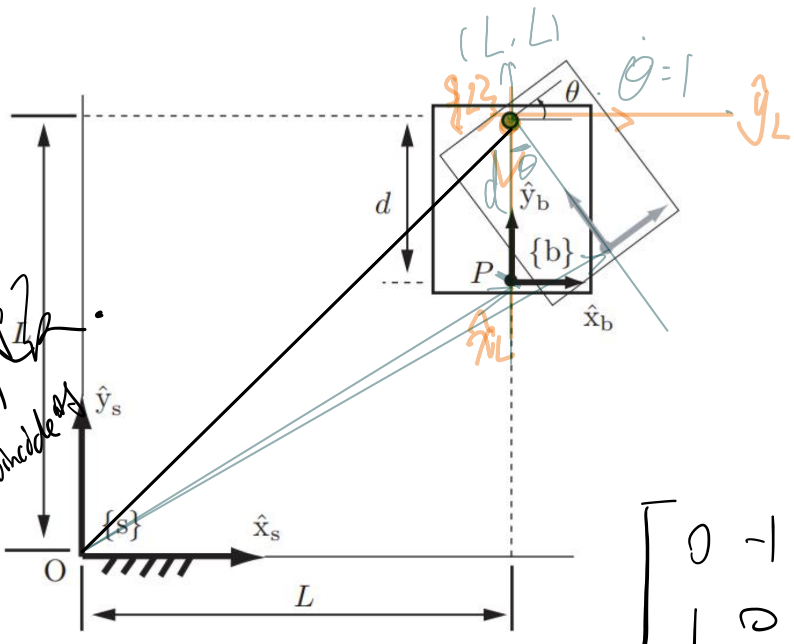
(f) What is the relationship between the twists from (d) and (e)?

(g) What is the relationship between the twist from (d) and $\dot{P}$ from (b)?

fixed frame || space frame

body frame.

人们所说的惯性系，不是固定的，是以[illegible]时刻的惯性系，[illegible] currently coincide的[illegible]



$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Figure 5.17: A rigid body rotating in the plane.

(h) What is the relationship between the twist from (e) and $\dot{P}$ from (b)? $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(a) 选取(L, L)点处的一个坐标系{L}如图，可知(L,L)的角速度 $\hat{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

${}^Lp(t) = e^{[\hat{\omega}]t}\,{}^Lp(0)$ 由于 $\dot{\theta} = 1$ 所以 $\dot{\theta} \cdot t = t = \theta$

${}^Lp(\theta) = e^{[{}^L\hat{\omega}]\theta}\,{}^Lp(0) = e^{[{}^L\hat{\omega}]\theta} \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} = {}^L\mathrm{Rot}(\hat{\omega}, \theta) \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}$

$\hat{x}_b = (\cos\theta)\,\hat{x}_s + (\sin\theta)\,\hat{y}_s$

$\hat{y}_b = (-\sin\theta)\,\hat{x}_s + (\cos\theta)\,\hat{y}_s$

${}^sp(\theta) = {}^sR_L\,{}^Lp(\theta) + \begin{bmatrix} L \\ L \\ 0 \end{bmatrix}$ ${}^sR_L = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP}$

$\overrightarrow{LP} = -d\,\hat{y}_b$

$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^L\mathrm{Rot}(\hat{\omega}, \theta) \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} L \\ L \\ 0 \end{bmatrix} = \begin{bmatrix} L + d\sin\theta \\ L - d\cos\theta \\ 0 \end{bmatrix}$

(b) ${}^sV_p = {}^sV_L + {}^s\omega \times {}^s\overrightarrow{LP} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} d\sin\theta \\ -d\cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} d\cos\theta \\ d\sin\theta \\ 0 \end{bmatrix}$ $\dot{P}$

(c) $\begin{bmatrix} {}^sR & {}^sp \end{bmatrix}$ [illegible]

(c) $${}^S T_B = \begin{bmatrix} {}^S R_B & {}^S O_B \\ 0 & 1 \end{bmatrix} \qquad {}^S O_B = {}^S P = \begin{bmatrix} L + d\sin\theta \\ L - d\cos\theta \\ 0 \end{bmatrix}$$

$${}^S T_B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & L + d\sin\theta \\ \sin\theta & \cos\theta & 0 & L - d\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) $${}^B V_{Top} = \begin{bmatrix} {}^B\omega \\ {}^B V_B \end{bmatrix} \qquad {}^B V_B = {}^B V_L + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times {}^B\overrightarrow{LB} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -d \\ 0 \end{bmatrix} = \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}$$

$${}^B V_{Top} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ d \\ 0 \\ 0 \end{bmatrix}$$

(e) $${}^S V_{Top} = \begin{bmatrix} {}^S\omega \\ {}^S V_O \end{bmatrix} \qquad {}^S V_O = {}^S V_L + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times {}^S\overrightarrow{LO} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -L \\ -L \\ 0 \end{bmatrix} = \begin{bmatrix} L \\ -L \\ 0 \end{bmatrix}$$

$${}^S V_{Top} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ L \\ -L \\ 0 \end{bmatrix}$$

(f) $${}^B\omega = {}^S\omega$$

$${}^S R_B\, {}^B V_B = \frac{d}{L}\, {}^S Rot(1, \arctan\frac{L}{d})\, {}^S V_O$$

(g) $${}^B V_B = \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} \qquad {}^S V_P = \begin{bmatrix} d\cos\theta \\ d\sin\theta \\ 0 \end{bmatrix}$$

$${}^S R_B\, {}^B V_B = {}^S V_P$$

(h) $${}^S V_P = \begin{bmatrix} d\cos\theta \\ d\sin\theta \\ 0 \end{bmatrix} \qquad {}^S V_O = \begin{bmatrix} L \\ -L \\ 0 \end{bmatrix}$$

$${}^S V_P = \frac{d}{L}\, {}^S Rot(1, \arctan\frac{L}{d})\, {}^S V_O$$

$V_p = \frac{d}{L} Rot(1, \arctan \frac{\ }{d}) V_O$

Exercise 5.6 Figure 5.18 shows a design for a new amusement park ride. A rider sits at the location indicated by the moving frame {b}. The fixed frame {s} is attached to the top shaft as shown. The dimensions indicated in the figure are $R = 10$ m and $L = 20$ m, and the two joints each rotate at a constant angular velocity of 1 rad/s.

(a) Suppose $t = 0$ at the instant shown in the figure. Find the linear velocity v_b and angular velocity ω_b of the rider as functions of time t. Express your answer in frame-{b} coordinates.

(b) Let p be the linear coordinates expressing the position of the rider in {s}. Find the linear velocity $\dot{p}(t)$.

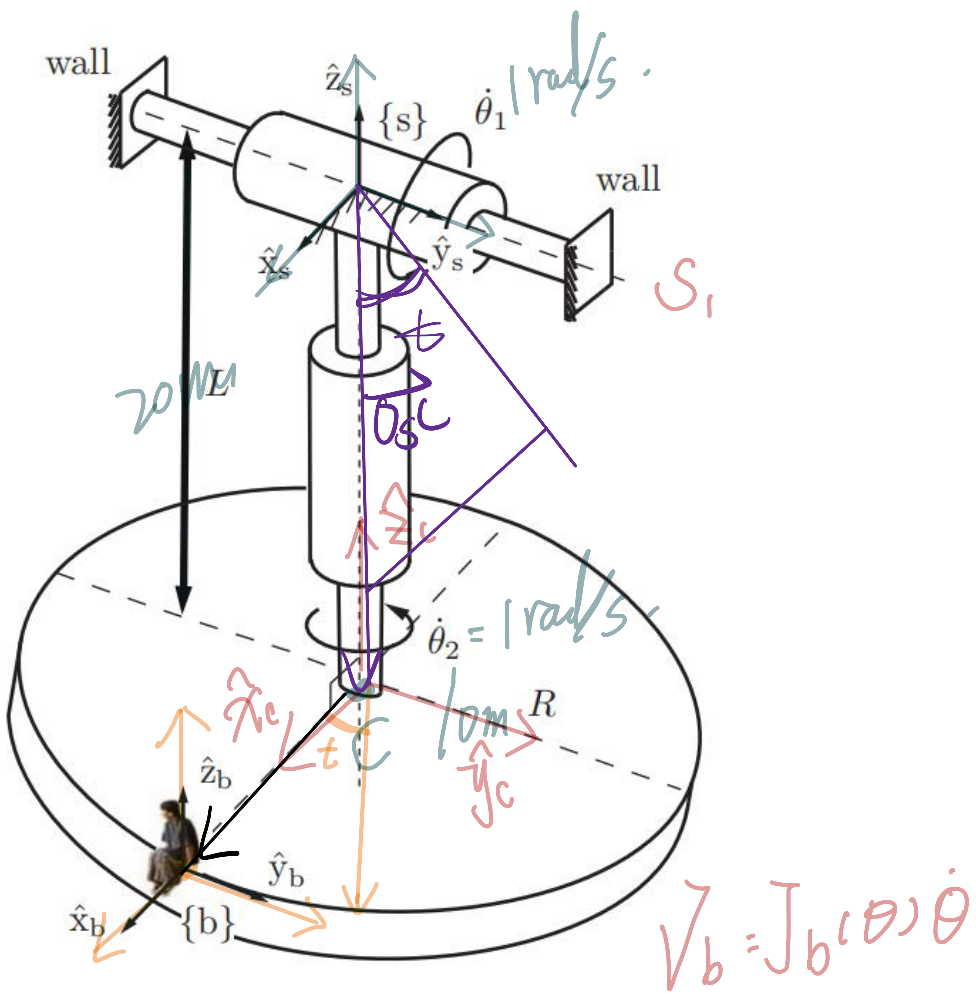


Figure 5.18: A new amusement park ride.

(a). ${}^bV_b = {}^bV_c + {}^bW_c \times {}^b\overrightarrow{CO_b}$

${}^sV_c = {}^sR_b\,{}^bV_c = {}^sR_c\,{}^cR_b\,{}^bV_c = {}^sV_{O_s} + {}^sW_s \times {}^s\overrightarrow{O_sC}$

${}^bW_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ${}^b\overrightarrow{CO_b} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$ ${}^bW_c \times \overrightarrow{CO_b} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$

${}^bV_c = ({}^sR_c\,{}^cR_b)^{-1}(\ldots {}^s\overrightarrow{\ })$

$${}^{b}V_c = ({}^{s}R_c\, {}^{c}R_b)^{-1} ({}^{s}W_s \times {}^{s}O_s C)$$

$$= ({}^{s}R_c\, {}^{c}R_b)^{-1} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -20\sin t \\ 0 \\ -20\cos t \end{bmatrix} \right)$$

$${}^{c}R_b = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad ({}^{s}R_c\, {}^{c}R_b)^{-1} = {}^{c}R_b^{-1}\, {}^{s}R_c^{-1}$$

$${}^{s}R_c = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix} \qquad {}^{c}R_b^{-1} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{s}R_c^{-1} = \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix}$$

$${}^{c}R_b^{-1}\, {}^{s}R_c^{-1} = \begin{bmatrix} \cos^2 t & \sin t & -\sin t\cos t \\ -\sin t\cos t & \cos t & \sin^2 t \\ \sin t & 0 & \cos t \end{bmatrix}$$

$${}^{b}V_c = {}^{c}R_b^{-1}\, {}^{s}R_c^{-1} \begin{bmatrix} -20\cos t \\ 0 \\ 20\sin t \end{bmatrix} = \begin{bmatrix} -20\cos^3 t - 20\sin^2 t\cos t \\ 20\sin t\cos^2 t + 20\sin^3 t \\ -20\cos t\sin t + 20\cos t\sin t \end{bmatrix} = \begin{bmatrix} -20\cos t \\ 20\sin t \\ 0 \end{bmatrix}$$

$${}^{b}V_b = \begin{bmatrix} -20\cos t \\ 10 + 20\sin t \\ 0 \end{bmatrix}$$

(b) $${}^{b}V_b = \begin{bmatrix} -20\cos t \\ 10 + 20\sin t \\ 0 \end{bmatrix} \qquad {}^{s}V_b = \begin{bmatrix} -20\cos t - 20\cos t\sin t \\ 10\cos t \\ 20\sin t\cos t \end{bmatrix}.$$