1. **Probabilistic modeling of uncertainties:** Two outstanding students S_A and S_B are deciding whether to accept the offer for the SUSTech PhD program.

   (a) Suppose you believe (i) S_A and S_B make decisions independently; (ii) the chance that S_A accepts the offer is 0.8; (iii) the chance S_B accepts the offer is 0.6. Please construct a probability space to represent your uncertain belief, namely, find the sample space Ω (all the possible outcomes) and the probability mass function (probability mass for each outcome) so that the above three conditions are satisfied.

   (b) Now assume the two students are good friends, you know that (i) if S_B accepts the offer, then S_A will for sure accept the offer ; (ii) if S_B does not accept the offer, then S_A only has 30% chance to accept the offer (also implies there is 70% chance S_A will not accept the offer given the fact that S_B does not accept the offer); (iii) the chance that neither of them accepts the offer is 35%. Please construct a probability space to represent your uncertain knowledge in this case.

(a). 设. S_A接受为A_a, 拒绝为A_r. S_B接受为B_a, 拒绝为B_r.

Sample space $\Omega = \{(A_a, B_a), (A_r, B_a), (A_a, B_r), (A_r, B_r)\}$.

$P(A_a, B_a) = 0.48$. $P(A_r, B_a) = 0.12$ $P(A_a, B_r) = 0.32$ $P(A_r, B_r) = 0.08$.

(b). $P(A_r, B_r) = P(B_r) \cdot P(A_r | B_r) = P(B_r) \cdot 0.7 = 0.35$ $\therefore P(B_r) = 0.5 = P(B_a)$

$P(A_a, B_r) = P(B_r) \cdot P(A_a | B_r) = 0.5 \cdot 0.3 = 0.15$

$P(A_a, B_a) = P(B_a) \cdot P(A_a | B_a) = 0.5$

$P(A_r, B_a) = 0$.

2. **Conditional Probability and Expectation:** Suppose X and Y are discrete random variables. X is uniformly distributed on the set $\{0, 1, \ldots n\}$, while Y is conditionally uniform on 0 through i given $X = i$, for each $i = 0, \ldots, n$.

   (a) Compute the conditional mean $E(Y|X = i)$ for a general $i \leq n$.

   (b) Compute $E(Y)$ by conditioning on the values of X, namely, using the formula $E(Y) = \sum_{i=0}^{n} E(Y|X = i)p_X(i)$, where $p_X(i) = Prob(X = i)$.

   (c) Find the joint probability mass function (pmf) $p(i, j)\, Prob(X = i, Y = j)$, for $i = 0, \ldots, n$ and $j = 0, \ldots, n$. (hint: for some pair (i, j) the joint pmf is zero. Make sure you clearly identify those).

   (d) Compute the marginal $p_Y(j) = Prob(Y = j)$ for $j = 0, \ldots n$.

   (e) Write a matlab function to compute the mean $E(Y)$ of Y using p_Y for $n = 100$, and compare the result with (b).

   (f) Assume $n \geq 1$. Let $g(X) = 2$, if $X = 1$ or n, and $g(X) = 0$ otherwise. Compute $E(g(X)Y)$ through conditional expectation.

(a). $E(Y|X=i) = \sum_{y=0}^{i} y \cdot P(y|X=i) = \sum_{y=0}^{i} y \cdot \frac{1}{i+1} = \frac{1}{i+1} \frac{(i+0)(i+1)}{2} = \frac{i}{2}$.

(b). $E(Y) = \sum_{i=0}^{n} E(Y|X=i)\, p_X(i) = \sum_{i=0}^{n} \frac{i}{2} \frac{1}{n+1} = \frac{1}{2(n+1)} \cdot \frac{(0+n)(n+1)}{2} = \frac{n}{4}$.

(c). $p(i,j) = Prob(X=i, Y=j) = prob(Y=j|X=i) \cdot Prob(X=i) = \begin{cases} \frac{1}{i+1} \frac{1}{n+1}, & j \in [0, i] \\ 0, & \text{others}. \end{cases}$

d). $P_Y(j) = Prob(Y=j) = \sum_{i=j}^{n} p(i,j) = \sum_{i=j}^{n} \left(\frac{1}{i+1} \frac{1}{n+1}\right) = \frac{1}{n+1}\left(\frac{1}{j+1} + \cdots + \frac{1}{n+1}\right)\ j \in [0, n]$

(e). $E(Y) = \sum_{j=0}^{n} \left[\sum_{i=j}^{n} \left(\frac{1}{i+1} \frac{1}{n+1}\right)\right] \cdot j = \frac{1}{n+1} \sum_{j=0}^{n} j \left(\sum_{i=j}^{n} \frac{1}{i+1}\right)$.

for $n = 100$. $E(Y) = \frac{1}{101} \cdot \sum_{j=0}^{100} j \left(\sum_{i=j}^{100} \frac{1}{i+1}\right) = 25.000$.

(f).

$g(x) = \begin{cases} 2, & x = 1 \text{ or } n \\ 0, & \text{others}. \end{cases}$

$E(g(X)Y) = E(E(g(x)Y|X))$

$E(g(x)Y|X) = g(x) E(Y|X) = g(x) \frac{X}{2}$

$E(g(x)Y) = E(g(x) \frac{X}{2}) = \sum_{x=0}^{n} (g(x) \frac{x}{2}) P(X=x)$

$= g(1) \frac{1}{2} \frac{1}{n+1} + g(n) \frac{n}{2} \cdot \frac{1}{n+1}$

$= \frac{1}{n+1} + \frac{n}{n+1} = 1$.

3. **Conditional Density and Expectation**

(a) Suppose that (X, Y) is uniformly distributed on the triangle $S = \{(x, y) : -6 < y < x < 6\}$. Find $E(Y|X = x)$.

(b) Let (X, Y) be two random variables with joint density function:

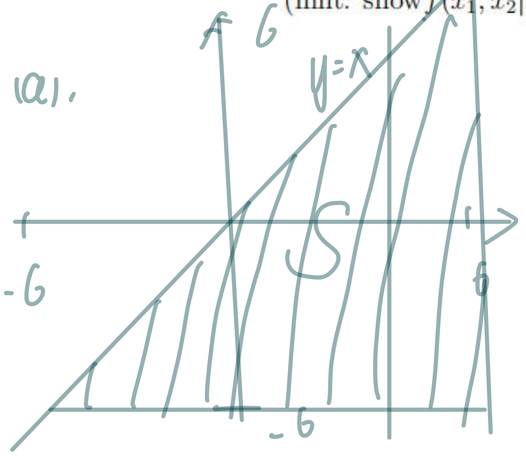
$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y) & x \in [0,1], y \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X)$ and $E(X|Y = 1/2)$.

(c) Let X is an arbitrary 3D random vector with density $f(x_1, x_2, x_3)$. Show that if X_1 is independent of both X_2 and X_3, then $X_1|X_3$ is independent of $X_2|X_3$. (hint: show $f(x_1, x_2|x_3) = f(x_1|x_3)f(x_2|x_3)$).

$f(X_1|X_3, X_2|X_3)$

(a).



$E(Y|X=x) = \int_{-6}^{x} y\, p(Y|X=x)\, dy$.

$p(y|X=x) = \dfrac{\frac{1}{18}}{p(X=x)} = \dfrac{\frac{1}{18}}{\int_{-6}^{x} p(x,y)\,dy} = \dfrac{1}{6+x}$

$E(Y|X=x) = \int_{-6}^{x} \dfrac{y}{6+x}\, dy = \dfrac{1}{6+x} \cdot \dfrac{1}{2} y^2 \Big|_{-6}^{x} = \dfrac{1}{2} \dfrac{x^2-36}{x+6} = \dfrac{x-6}{2}$

(b). $E(X) = \int_x x f_x\, dx$

$f_x = \int_y f(x,y)\,dy = \int_0^2 \frac{1}{4}(2x+y)\,dy = \frac{1}{4}(2xy + \frac{1}{2}y^2)\Big|_0^2 = x + \frac{1}{2}, \; x \in [0,1]$

$E(X) = \int_0^1 x(x+\frac{1}{2})\,dx = \frac{1}{3}x^3 + \frac{1}{4}x^2 \Big|_0^1 = \frac{7}{12}$

$E(X|Y=\frac{1}{2}) = \int_0^1 x f(x|Y=\frac{1}{2})\,dx$

$f(x|Y=\frac{1}{2}) = \dfrac{f(x, Y=\frac{1}{2})}{f(Y=\frac{1}{2})} = \dfrac{\frac{1}{4}(2x+\frac{1}{2})}{\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2}} = \frac{4}{3}x + \frac{1}{3}$

$E(X|Y=\frac{1}{2}) = \frac{1}{3}\int_0^1 4x^2 + x\,dx = \frac{1}{3}\left(\frac{4}{3}x^3 + \frac{1}{2}x^2\Big|_0^1\right) = \frac{11}{18}$

(c) $f(x_1, x_2|x_3) = f(x_2|x_3|x_1) f(x_1) = f(x_2|x_3 \,|\, x_1) f(x_1|x_3)$

$= \dfrac{f(x_1|x_2|x_3) f(x_2|x_3)}{f(x_1)} f(x_1|x_3)$

$$= \frac{f(x_1|x_3 \mid x_2|x_3)\, f(x_2|x_3)}{f(x_1|x_3)}\, f(x_1|x_3)$$

$$= f(x_1|x_3,\ x_2|x_3) = f(x_1|x_3)\, f(x_2|x_3)$$

∴ 得证.