Input–Output Finite-Time Stability of Linear Systems: Necessary and Sufficient Conditions

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***Abstract*—In the recent paper "*Input-output finite-time stabilization of linear systems*," (F. Amato *et al.*) a sufficient condition for input–output finite-time stability (IO-FTS), when the inputs of the system are $\mathcal{L}_2$ signals, has been provided; such condition requires the existence of a feasible solution to an optimization problem involving a certain differential linear matrix inequality (DLMI). Roughly speaking, a system is said to be input–output finite-time stable if, given a class of norm bounded input signals over a specified time interval of length T, the outputs of the system do not exceed an assigned threshold during such time interval. IO-FTS constraints permit to specify quantitative bounds on the controlled variables to be fulfilled during the transient response. In this context, this paper presents several novel contributions. First, by using an approach based on the reachability Gramian theory, we show that the main theorem of F. Amato *et al.* is actually also a *necessary* condition for IO-FTS; at the same time we provide an alternative—still necessary and sufficient—condition for IO-FTS, in this case based on the existence of a suitable solution to a differential Lyapunov equality (DLE). We show that the last condition is computationally more efficient; however, the formulation via DLMI allows to solve the problem of the IO finite-time stabilization via output feedback. The effectiveness and computational issues of the two approaches for the analysis and the synthesis, respectively, are discussed in two examples; in particular, our methodology is used in the second example to minimize the maximum displacement and velocity of a building subject to an earthquake of given magnitude.**

***Index Terms*—Differential linear matrix inequality (DLMI), differential Lyapunov Equation (DLE), input–output finite-time stability (IO-FTS), linear systems, LMI, reachability Gramian, time-varying systems.**

I. INTRODUCTION

IN the recent paper [1], the definition of input-output finite-time stability (IO-FTS) has been introduced; roughly speaking, a system is said to be IO-FTS if, given a class of norm bounded input signals defined over a specified time interval of length T, the outputs of the system do not exceed an assigned threshold during such time interval.

In order to correctly frame the definition of IO-FTS in the current literature, we recall that a system is said to be IO $\mathcal{L}_p$-stable [2, Ch. 5] if for any input of class $\mathcal{L}_p$, the system exhibits a corresponding output which belongs to the same class. IO stability of linear and nonlinear systems has been broadly studied since the early 1960s [3]–[5], and more recently has been extended to the case of networked and hybrid systems [6]. Moreover, a number of results have been proposed in the literature to discuss robustness issues (see for example the recent paper [7] and the bibliography therein).

The main differences between *classic* IO stability and IO-FTS are that the latter involves signals defined over a finite time interval, does not necessarily require the inputs and outputs to belong to the same class, and that *quantitative* bounds on both inputs and outputs must be specified. Therefore, IO stability and IO-FTS are independent concepts. Furthermore, while IO stability deals with the behavior of a system within a sufficiently long (in principle infinite) time interval, IO-FTS is a more practical concept, useful to study the behavior of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the output variables do not exceed a given threshold during the transients, given a certain class of input signals.

Input–output stabilization of time-varying systems on a finite time horizon is tackled also in [8]. However, as for classic IO stability, their concept of IO stability over a finite time horizon does not give explicit bounds on input and output signals, and does not allow the input and output to belong to different classes.

It is important to remark that the definition of IO-FTS given in [1] is fully consistent with the definition of (state) FTS, where the state of a zero-input system, rather than the input and the output, is involved. The concept of FTS dates back to the 1950s, when it was introduced in the Russian literature ([9], [10]); later during the 1960s this concept appeared in the western control literature [11], [12]. Recently, sufficient conditions for FTS and finite-time stabilization (the corresponding design problem) have been provided in the control literature, see for example [13]–[15] in the context of linear systems, and [16]–[18] in the context of impulsive and hybrid systems.

In [1], two sufficient conditions for IO-FTS were provided, for the class of $\mathcal{L}_2$ inputs and the class of $\mathcal{L}_\infty$ inputs, respectively. Both conditions required the solution of a feasibility problem involving differential linear matrix inequalities (DLMIs).

In this paper, the focus is on the class of $\mathcal{L}_2$ inputs. In this context, the novel contribution of this paper goes in several directions; first we show that the condition given in [1] is actually also *necessary*. As said, such condition requires the existence of a feasible positive definite solution to an optimization problem involving DLMIs. Then, we provide an alternative necessary and sufficient condition requiring that a certain Differential Lyapunov Equation (DLE) admits a positive definite solution. The

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efficiency of the two conditions is discussed in the example section; indeed, it is shown that the condition based on the DLE (which is totally new in the context of IO-FTS) is more effective from a numerical point of view. On the other hand, the DLMIs formulation turns out to be useful for design purposes.

To prove these results, a machinery involving the concept of reachability Gramian is used. More precisely, we shall prove that, if a given system is IO-FTS, then a generalized DLE involving the reachability Gramian admits a positive definite solution, which satisfies over the time interval of interest a certain condition on the maximum eigenvalue. This in turn is proven to imply the condition for IO-FTS given in [1]; then we can conclude that the two conditions are equivalent each other and both are equivalent to IO-FTS.

Finally, the analysis condition based on DLMIs is used in the synthesis context; that is, we provide a necessary and sufficient condition for the existence of an output feedback dynamic controller which renders the closed loop system IO-FTS. To derive such condition we use the nonlinear change of matrix variables proposed in [19], which leads to a DLMIs feasibility problem.

For the sake of completeness, it should be mentioned that a different concept of IO-FTS has been recently given for nonlinear systems. In particular, the authors of [20] consider systems with a norm bounded input signal over the interval $[0, +\infty]$ and a nonzero initial condition. In this case, the finite-time input-output stability is related to the property of a system to have a norm bounded output that, after a finite time interval of length T, does not depend anymore on the initial state. Hence, the concept of IO-FTS introduced in [1] and the one in [20] are different. Note, also, that the definition of IO-FTS given in [20] would not be well posed in the context of linear systems, since the output of a zero-input linear system cannot go to zero in finite time.

The paper is organized as follows: in Section II the problem we deal with is precisely stated and some preliminary results are given. In Section III the main result, namely a theorem containing two necessary and sufficient conditions for IO-FTS, is provided. In Section IV, the analysis result is turned into a design condition. In Section V, two examples are provided to illustrate the effectiveness of the proposed approach: the former investigates the numerical efficiency of the analysis conditions, while the latter focuses on the design of a controller to minimize the maximum displacement and velocity of a building subject to an earthquake of given magnitude. Eventually, in Section VI some concluding remarks are given.

Notation: Given a vector $v \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, we will denote with $|v|$ the euclidian norm of v, and with $|A|$ the induced matrix norm

$$|A| = \sup_{v\neq 0} \frac{|Av|}{|v|}.$$

Given the two constants $t_0 \geq 0$ and $T > 0$, we define the bounded interval $\Omega = [t_0, t_0 + T]$. The symbol $\mathcal{L}_p(\Omega)$ denotes the space of vector-valued signals for which[1]

$$s(\cdot) \in \mathcal{L}_p(\Omega) \iff \left(\int_\Omega |s(\tau)|^p d\tau\right)^{\frac{1}{p}} < +\infty.$$

[1]For the sake of brevity, we denote by $\mathcal{L}_p$ the set $\mathcal{L}_p([0, +\infty))$.

Given a symmetric positive definite matrix valued function $R(\cdot)$, bounded on Ω, and a vector-valued signal $s(\cdot) \in \mathcal{L}_p(\Omega)$, the weighted signal norm

$$\left(\int_\Omega \left[s^T(\tau)R(\tau)s(\tau)\right]^{\frac{p}{2}} d\tau\right)^{\frac{1}{p}}$$

will be denoted by $\|s(\cdot)\|_{p,R}$. If $p = \infty$

$$\|s(\cdot)\|_{\infty,R} = \operatorname*{ess\,sup}_{t\in\Omega} \left[s^T(t)R(t)s(t)\right]^{\frac{1}{2}}.$$

When the weighting matrix $R(\cdot)$ is time-invariant and equal to the identity matrix I, we will use the simplified notation $\|s(\cdot)\|_p$.

II. Problem Statement and Preliminary Results

Let us consider a linear time-varying (LTV) system in the form

$$\Gamma : \begin{cases} \dot{x}(t) = A(t)x(t) + G(t)w(t), \quad x(t_0) = 0 \\ y(t) = C(t)x(t) \end{cases} \tag{1}$$

where $A(\cdot) : \Omega \mapsto \mathbb{R}^{n\times n}$, $G(\cdot) : \Omega \mapsto \mathbb{R}^{n\times r}$, and $C(\cdot) : \Omega \mapsto \mathbb{R}^{m\times n}$, are continuous matrix-valued functions; Γ can be viewed as a linear operator mapping input signals ($w(\cdot)$'s) into output signals ($y(\cdot)$'s).

In general, $w(\cdot) \in \mathcal{L}_p$ does not guarantee that $y(\cdot) \in \mathcal{L}_p$; therefore, it makes sense to give the definition of IO $\mathcal{L}_p$-stability. Roughly speaking (the precise definition is more involved, the interested reader is referred to [2], Ch. 5, or to [21]), system (1) is said to be $\mathcal{L}_p$-stable, if $w(\cdot) \in \mathcal{L}_p$ implies $y(\cdot) \in \mathcal{L}_p$. The most popular cases are the ones with $p = 2$ and $p = \infty$.

The concept of $\mathcal{L}_p$-stability is generally referred to an infinite interval of time. In this paper we are interested to study the input–output behavior of the system over a finite time interval.

In the following, we will denote by $\Phi(t, \tau)$ the state transition matrix of system (1), and by

$$H(t, \tau) = C(t)\Phi(t, \tau)G(\tau)\delta_{-1}(t - \tau)$$

its impulsive response, where $\delta_{-1}(t)$ is the Heaviside step function.

A. Definition of IO-FTS

In this section, we introduce the concept of IO-FTS for the class of systems in the form (1). To this end, let us consider the following definition.

Definition 1 (IO-FTS of LTV Systems): Given a positive scalar T, a class of input signals $\mathcal{W}$ defined over $\Omega = [t_0, t_0 + T]$, a continuous and positive definite matrix-valued function $Q(\cdot)$ defined in Ω, system (1) is said to be IO-FTS with respect to $(\mathcal{W}, Q(\cdot), \Omega)$ if

$$w(\cdot) \in \mathcal{W} \Rightarrow y^T(t)Q(t)y(t) < 1, \quad t \in \Omega.$$

▲

In this paper, we consider the class of norm bounded square integrable signals over Ω, defined as follows:

$$\mathcal{W}_2(\Omega, R(\cdot)) := \{w(\cdot) \in \mathcal{L}_2(\Omega) : \|w\|_{2,R} \leq 1\}$$

where $R(\cdot)$ denotes a continuous positive definite matrix-valued function.

In the rest of the paper, we will drop the dependency of $\mathcal{W}_2$ on Ω and $R(\cdot)$ in order to simplify the notation.

In Section III, two necessary and sufficient conditions for IO-FTS for the class of $\mathcal{W}_2$ input signals are provided. These conditions are then exploited in Section IV to solve the following design problem, namely the problem of *input-output finite-time stabilization via dynamic output feedback*.

Problem 1: Consider the LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t), x(t_0) = 0 \quad (2a)$$
$$y(t) = C(t)x(t) \quad (2b)$$

where $u(\cdot)$ is the control input and $w(\cdot)$ is the exogenous input. Given the class of signals $\mathcal{W}_2$, and a continuous positive definite matrix-valued function $Q(\cdot)$ defined over Ω, find a dynamic output feedback controller in the form

$$\dot{x}_c(t) = A_K(t)x_c(t) + B_K(t)y(t) \quad (3a)$$
$$u(t) = C_K(t)x_c(t) + D_K(t)y(t) \quad (3b)$$

where $x_c(t)$ has the same dimension of $x(t)$, such that the closed loop system obtained by the connection of (2) and (3) is IO-FTS with respect to $(\mathcal{W}_2, Q(\cdot), \Omega)$. In particular, the closed loop system is in the form

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{pmatrix} &= \begin{pmatrix} A(t) + B(t)D_K(t)C(t) & B(t)C_K(t) \\ B_K(t)C(t) & A_K(t) \end{pmatrix} \\ &\quad \times \begin{pmatrix} x(t) \\ x_c(t) \end{pmatrix} + \begin{pmatrix} G(t) \\ 0 \end{pmatrix} w(t) \\ &=: A_{\mathrm{CL}}(t)x_{\mathrm{CL}}(t) + G_{\mathrm{CL}}(t)w(t) \quad (4a) \\ y(t) &= \begin{pmatrix} C(t) & 0 \end{pmatrix} x_{\mathrm{CL}}(t) =: C_{\mathrm{CL}}(t)\, x_{\mathrm{CL}}(t) \quad (4b) \end{aligned}$$

where all the considered matrices depends on time, even when not explicitly written. ▲

B. Reachability Gramian

We now introduce some preliminary results concerning the reachability Gramian of LTV systems, which are then exploited in this paper to prove the main result. More details can be found in [22].

Definition 2 (Reachability Gramian): The *reachability Gramian* of system (1) is defined as

$$W_r(t, t_0) \triangleq \int_{t_0}^{t} \Phi(t, \tau)G(\tau)G^T(\tau)\Phi^T(t, \tau)d\tau.$$

Note that $W_r(t, t_0)$ is symmetric and positive semidefinite for all $t \geq t_0$.

Remark 1: If the pair (A, G) is *controllable*, then $W_r(t, t_0)$ is positive definite for all $t > t_0$. Furthermore, if system (1) is time-invariant, then $W_r(t, t_0) = W_r(t - t_0)$ and (see [22])

$$W_r(t_2 - t_0) \geq W_r(t_1 - t_0), \quad t_2 \geq t_1 \geq t_0.$$

▲

Lemma 1 ([22]): Given system (1), $W_r(t, t_0)$ is the unique solution of the matrix differential equation

$$\begin{aligned} \dot{W}_r(t, t_0) &= A(t)W_r(t, t_0) + W_r(t, t_0)A^T(t) \\ &\quad + G(t)G^T(t) \quad (5a) \\ W_r(t_0, t_0) &= 0. \quad (5b) \end{aligned}$$

■

C. Preliminary Results

In this section we state an equivalent definition of IO-FTS that can be easily derived when the LTV system (1) is regarded as a linear operator that maps signals from the space $\mathcal{L}_2(\Omega)$ to the space $\mathcal{L}_\infty(\Omega)$, i.e.,

$$\Gamma : w(\cdot) \in \mathcal{L}_2(\Omega) \mapsto y(\cdot) \in \mathcal{L}_\infty(\Omega). \quad (6)$$

Furthermore, if we equip the $\mathcal{L}_2(\Omega)$ and $\mathcal{L}_\infty(\Omega)$ spaces with the weighted norms $\|\cdot\|_{2,R}$ and $\|\cdot\|_{\infty,Q}$, respectively, the induced norm of the linear operator (6) is given by

$$\|\Gamma\| = \sup_{\|w(\cdot)\|_{2,R}=1} [\|y(\cdot)\|_{\infty,Q}]$$

that is the norm of Γ is computed considering the input signals in $\mathcal{W}_2$. Hence, to require system (1) being IO-FTS with respect to $(\mathcal{W}_2, Q(\cdot), \Omega)$ is equivalent to require that $\|\Gamma\| < 1$; the following theorem holds.

Theorem 1: Given a time interval Ω, the class of input signals $\mathcal{W}_2$, and a continuous positive definite matrix-valued function $Q(\cdot)$, system (1) is IO-FTS with respect to $(\mathcal{W}_2, Q(\cdot), \Omega)$ *if and only if* Γ is a bounded linear operator with $\|\Gamma\| < 1$. ■

Given the linear operator (6), its *dual operator* is

$$\overline{\Gamma} : z(\cdot) \in \mathcal{L}_1(\Omega) \mapsto v(\cdot) \in \mathcal{L}_2(\Omega)$$

corresponding to the dual system [22, p. 236]

$$\overline{\Gamma} : \begin{cases} \dot{\tilde{x}}(t) = -A^T(t)\tilde{x}(t) - C^T(t)z(t) \\ v(t) = G^T(t)\tilde{x}(t). \end{cases} \quad (7)$$

According to duality, the norm of $\overline{\Gamma}$ is defined

$$\|\overline{\Gamma}\| = \sup_{\|z(\cdot)\|_{1,Q}=1} \left[\|v(\cdot)\|_{2,R}\right].$$

Moreover, by definition of dual operator ([23]) we have that, given $z(\cdot) \in \mathcal{L}_1(\Omega)$ and $w(\cdot) \in \mathcal{L}_2(\Omega)$,

$$\langle z, \Gamma w \rangle = \langle \overline{\Gamma} z, w \rangle \quad (8)$$

where, given two signals $u(\cdot)$ and $v(\cdot)$, we have

$$\langle u, v \rangle = \int_{\Omega} u^T(t)v(t)dt.$$

Therefore, (8) reads

$$\begin{aligned}\langle z, \Gamma w\rangle &= \int_\Omega z^T(t) \int_\Omega H(t,\tau)w(\tau)d\tau dt \\ &= \int_\Omega \left(\int_\Omega z^T(t)H(t,\tau)dt \right) w(\tau)d\tau \\ &= \int_\Omega \left(\int_\Omega z^T(t)\overline{H}^T(\tau,t)dt \right) w(\tau)d\tau = \langle \overline{\Gamma} z, w\rangle\end{aligned}$$

where

$$\overline{H}(t,\tau) = H^T(\tau,t) = G^T(t)\Phi^T(\tau,t)C^T(\tau)\delta_{-1}(\tau-t) \quad (9)$$

is the impulsive response of the dual system (7).

Furthermore, it holds that (see [23, p. 195]):

$$\|\Gamma\| = \|\overline{\Gamma}\|. \quad (10)$$

The next theorem is a generalization of a result given in [24] to the case of LTV systems, and it allows us to compute the norm of Γ as a function of the spectral radius of the reachability Gramian defined in Section II-B. In order to prove the theorem we need to introduce the following lemma (see Appendix for the proof).

Lemma 2: If

$$v(t) \triangleq \int_\Omega f(t,\tau)\, d\tau, \quad t \in \Omega$$

with $f(\cdot,\tau)$ $\mathcal{L}_2$-integrable, then the following inequality hold

$$\|v(\cdot)\|_2 \le \int_\Omega \|f(\cdot,\tau)\|_2 \, d\tau. \quad (11)$$

■

Theorem 2: Given the LTV system (1), the norm of the corresponding linear operator (6) is given by

$$\|\Gamma\| = \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}} \left(Q^{\frac{1}{2}}(t)C(t)W(t,t_0)C^T(t)Q^{\frac{1}{2}}(t)\right) \quad (12)$$

for all $t \in \Omega$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue, and $W(t,t_0)$ is the positive semidefinite matrix-valued solution of

$$\begin{aligned}\dot W(t,t_0) &= A(t)W(t,t_0) + W(t,t_0)A^T(t) \\ &\quad + G(t)R(t)^{-1}G^T(t) \quad (13a)\\ W(t_0,t_0) &= 0. \quad (13b)\end{aligned}$$

Proof: For the sake of simplicity, we consider the weighting matrices equal to the identity; hence,

$$R(t) = I \quad \text{and} \quad Q(t) = I, \forall\, t \in \Omega.$$

Note that, given this assumption, the solution of (13) is given by the reachability gramian $W_r(t,t_0)$; we will discuss how to take into account the weighting matrices at the end of the proof.

First note that, in view of (10), proving (12) is equivalent to show that

$$\|\overline{\Gamma}\| = \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}} \left(C(t)W_r(t,t_0)C^T(t)\right)$$

where $\overline{\Gamma}$ is the dual operator of Γ. Taking into account the definition of $\overline{\Gamma}$, letting

$$\Upsilon(t) = \int_\Omega H(t,\sigma)H^T(t,\sigma)d\sigma \quad (14)$$

and denoting by $v(\cdot)$ the output of system (7), we have

$$\begin{aligned}\|v(\cdot)\|_2 &= \left\| \int_\Omega \overline{H}(\cdot,\tau)z(\tau)d\tau \right\|_2 \\ &\le \int_\Omega \left\|\overline{H}(\cdot,\tau)z(\tau)\right\|_2 d\tau \qquad \text{by Lemma 2} \\ &= \int_\Omega \left(z^T(\tau) \int_\Omega \overline{H}^T(t,\tau)\overline{H}(t,\tau)dt, z(\tau) \right)^{\frac{1}{2}} d\tau \\ &= \int_\Omega \left(z^T(\tau)\Upsilon(\tau)z(\tau)\right)^{\frac{1}{2}} d\tau \qquad \text{by (9) and (14)} \\ &= \int_\Omega \left|\Upsilon^{\frac{1}{2}}(\tau)z(\tau)\right| d\tau \\ &\le \int_\Omega \left|\Upsilon^{\frac{1}{2}}(\tau)\right| \cdot |z(\tau)| d\tau \\ &= \int_\Omega \lambda_{\max}^{\frac{1}{2}}\left(\Upsilon(\tau)\right) \cdot |z(\tau)|\, d\tau\end{aligned}$$

where the last equality holds since the matrix-valued function $\Upsilon(\cdot)$ is positive semidefinite.

Now, since all system and weighting matrices are bounded, it follows that the impulsive response of system (1) is bounded, and so it is also $\Upsilon(\cdot)$; therefore,

$$\begin{aligned}\|v(\cdot)\|_2 &\le \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\left(\Upsilon(t)\right) \cdot \int_\Omega |z(\tau)|\, d\tau \\ &= \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\left(\Upsilon(t)\right) \cdot \|z(\cdot)\|_1 .\end{aligned}$$

Thus,

$$\|\overline{\Gamma}\| \le \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\left(\Upsilon(t)\right). \quad (15)$$

From Definition 2 the matrix-valued function $\Upsilon(t)$ is equal to

$$\Upsilon(t) = C(t)W_r(t,t_0)C^T(t).$$

Hence, (15) reads

$$\|\overline{\Gamma}\| \le \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\left(C(t)W_r(t,t_0)C^T(t)\right). \quad (16)$$

The last part of the proof is devoted to show that (16) is actually an equality. To this end, let us denote by γ the right-hand side in (16); therefore, (16) can be rewritten

$$\|\overline{\Gamma}\| \leq \gamma. \tag{17}$$

In the following, we shall build a sequence of inputs to system (7) with unit norm in $\mathcal{L}_1(\Omega)$, such that the sequence of the norms of the corresponding output signals converges to γ.

To this end consider a subset $\Omega' \subset \Omega$, such that, for all $t \in \Omega'$

$$\lambda_{\max}^{\frac{1}{2}}\left(C(t)W_r(t,t_0)C^T(t)\right) \geq \gamma - \varepsilon$$

with $\varepsilon > 0$. Now let $\sigma \in \Omega'$ and consider the sequence of inputs

$$z_{\varepsilon,\alpha}(t) = h(\sigma)u_\alpha(t)$$

where $h(\sigma)$ is the unit eigenvector corresponding to the maximum eigenvalue of $C(\sigma)W_r(\sigma,t_0)C^T(\sigma)$, and u_α is a sequence of positive scalar functions with unit norm in $\mathcal{L}_1(\Omega)$, which approach the Dirac delta function applied in σ as $\alpha \mapsto 0$. Let

$$v_{\varepsilon,\alpha}(t) = \overline{\Gamma} z_{\varepsilon,\alpha}(t) = \int_\Omega \overline{H}(t,\tau) z_{\varepsilon,\alpha}(\tau)d\tau.$$

It is simple to recognize that, as $\alpha \to 0$, we have

$$\begin{aligned} v_{\varepsilon,\alpha}(\cdot) &\to \int_\Omega \overline{H}(t,\tau)h(\sigma)\delta(\tau-\sigma)d\tau \\ &= \overline{H}(t,\sigma)h(\sigma) \quad \text{in } \mathcal{L}_2(\Omega). \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{\alpha\to 0} \|v_{\varepsilon,\alpha}(\cdot)\|_2^2 &= \int_\Omega h^T(\sigma)\overline{H}^T(t,\sigma)\overline{H}(t,\sigma)h(\sigma)dt \\ &= h^T(\sigma)\int_\Omega H(\sigma,t)H^T(\sigma,t)dt\, h(\sigma) \\ &= h^T(\sigma)C(\sigma)W_r(\sigma,t_0)C^T(\sigma)h(\sigma). \end{aligned}$$

We can conclude that

$$\lim_{\alpha\mapsto 0} \|v_{\varepsilon,\alpha}(\cdot)\|_2 = \lambda_{\max}^{\frac{1}{2}}\left(C(\sigma)W_r(\sigma,t_0)C^T(\sigma)\right) \geq \gamma - \varepsilon$$

Therefore, given $\eta > 0$, it is possible to choose a sufficiently small α such that

$$\|v_{\varepsilon,\alpha}(\cdot)\|_2 \geq \gamma - \varepsilon - \eta.$$

Taking into account (17), that the scalars ε and η can be chosen arbitrarily small, and that the set of the signals $z_{\varepsilon,\alpha}$ is a subset of the set of the unit norm signals in $\mathcal{L}_1(\Omega)$, we can conclude that

$$\begin{aligned} \gamma \geq \|\overline{\Gamma}\| &= \sup_{\|z(\cdot)\|_1 = 1} \|v(\cdot)\|_2 \\ &\geq \sup_{z_{\varepsilon,\alpha}(\cdot)} \|v_{\varepsilon,\alpha}(\cdot)\|_2 = \gamma. \end{aligned}$$

From the last chain of inequality the proof follows.

Eventually, note that when the weighting matrices are taken into account the proof still holds by modifying the model matrices as follows:

$$\widetilde{G}(t) = G(t)R(t)^{-\frac{1}{2}}, \quad \widetilde{C}(t) = Q^{\frac{1}{2}}(t)C(t)$$

and replacing $W_r(t,t_0)$ by $W(t,t_0)$. ■

Remark 2: It is worth to notice that, since all the system matrices in (1) and the weighting matrices $R(\cdot)$ and $Q(\cdot)$ are assumed to be continuous, in the closed time interval Ω the condition (12) is equivalent to

$$\|\Gamma\| = \max_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\left(Q^{\frac{1}{2}}(t)C(t)W(t,t_0)C^T(t)Q^{\frac{1}{2}}(t)\right).$$

▲

We conclude the section with the following technical lemma, which will be useful to prove the main result of the paper in Section III.

Lemma 3: Given $\epsilon > 0$, the solution of the matrix differential equation

$$\begin{aligned} \dot{W}_\epsilon(t,t_0) &= A(t)W_\epsilon(t,t_0) + W_\epsilon(t,t_0)A^T(t) \\ &\quad + G(t)R(t)^{-1}G^T(t) + \epsilon I, \end{aligned} \tag{18a}$$

$$W_\epsilon(t_0,t_0) = \epsilon \tag{18b}$$

is the positive definite matrix

$$\begin{aligned} W_\epsilon(t,t_0) = W(t,t_0) &+ \epsilon\Phi(t,t_0)\Phi^T(t,t_0) \\ &+ \epsilon\int_{t_0}^{t} \Phi(t,\tau)\Phi^T(t,\tau)d\tau \end{aligned} \tag{19}$$

where $W(\cdot,\cdot)$ is the solution of (13).

Proof: The proof follows from direct substitution of $W_\epsilon(\cdot,\cdot)$ in (18), and by the fact that the matrix $\Phi(t,t_0)\Phi^T(t,t_0)$ is positive definite. ■

III. Necessary and Sufficient Conditions for IO-FTS of LTV Systems

The main result of this section is the following theorem that states two necessary and sufficient conditions for the IO-FTS of system (1).

Theorem 3: Given system (1), the class of inputs $\mathcal{W}_2$, a continuous positive definite matrix-valued function $Q(\cdot)$, and the time interval Ω, the following statements are equivalent:

1) System (1) is IO-FTS with respect to $(\mathcal{W}_2, Q(\cdot), \Omega)$.
2) The inequality

$$\lambda_{\max}\left(Q^{\frac{1}{2}}(t)C(t)W(t,t_0)C^T(t)Q^{\frac{1}{2}}(t)\right) < 1 \tag{20}$$

holds for all $t \in \Omega$, where $W(\cdot,\cdot)$ is the positive semidefinite solution of the DLE (13).

3) The coupled DLMI/LMI

$$\begin{pmatrix} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0 \tag{21a}$$

$$P(t) > C^T(t)Q(t)C(t), \tag{21b}$$

admits a positive definite solution $P(\cdot)$ over Ω.

Proof: We will prove the equivalence of the three statements by showing that **i**) $\Rightarrow$ **ii**), **ii**) $\Rightarrow$ **iii**), and **iii**) $\Rightarrow$ **i**).

[**i**) $\Rightarrow$ **ii**)]. The proof readily follows from Theorems 1 and 2, and from Remark 2.

[**ii**) $\Rightarrow$ **iii**)]. Given $\epsilon > 0$, consider the DLE (18), whose solution $W_\epsilon(\cdot,\cdot)$, given by (19), is positive definite and satisfies the DLMI

$$-\dot{W}_\epsilon(t,t_0) + A(t)W_\epsilon(t,t_0) + W_\epsilon(t,t_0)A^T(t) + G(t)R(t)^{-1}G^T(t) < 0. \quad (22)$$

Now letting

$$W_\epsilon(t,t_0) = P^{-1}(t)$$

it follows that $\dot{W}_\epsilon(t,t_0) = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$, and inequality (22) reads

$$P^{-1}(t)\dot{P}(t)P^{-1}(t) + A(t)P^{-1}(t) + P^{-1}(t)A^T(t) + G(t)R^{-1}(t)G^T(t) < 0 \quad (23)$$

for all $t \in \Omega$. By pre- and post-multiply (23) by $P(t)$ we obtain

$$\dot{P}(t)+P(t)A(t)+A^T(t)P(t)+P(t)G(t)R^{-1}(t)G^T(t)P(t) < 0 \quad (24)$$

and (21a) readily follows by applying Schur complements[2]

In order to prove (21b), first note that $W_\epsilon(\cdot,\cdot) \stackrel{\epsilon \mapsto 0}{\longrightarrow} W(\cdot,\cdot)$; hence, by continuity arguments, there exists a sufficiently small ϵ such that

$$\lambda_{\max}\left(Q^{\frac{1}{2}}(t)C(t)W_\epsilon(t,t_0)C^T(t)Q^{\frac{1}{2}}(t)\right) < 1. \quad (25)$$

Furthermore, condition (25) is equivalent to

$$I - Q^{\frac{1}{2}}(t)C(t)P^{-1}(t)C^T(t)Q^{\frac{1}{2}}(t) > 0 \quad (26)$$

that, by applying Schur complements, reads

$$\begin{pmatrix} I & Q^{\frac{1}{2}}(t)C(t) \\ C^T(t)Q^{\frac{1}{2}}(t) & P(t) \end{pmatrix} > 0. \quad (27)$$

From [25, Lemma 5.3] inequality (27) is equivalent to

$$\begin{pmatrix} P(t) & C^T(t)Q^{\frac{1}{2}}(t) \\ Q^{\frac{1}{2}}(t)C(t) & I \end{pmatrix} > 0$$

which yields (21b) by applying again Schur complements.

[**iii**) $\Rightarrow$ **i**)]. In [1] it has been proven that conditions (21) imply IO-FTS when the class of $\mathcal{W}_2$ signals is considered. However, the proof of this result is reported here for the sake of completeness.

We have already proved that, by applying Schur complements, condition (21a) is equivalent to (24). Now, let us consider the quadratic function $V(t,x) = x^T(t)P(t)x(t)$; the derivative with respect to time reads

$$\begin{aligned}\frac{d}{dt}\left(x^T(t)P(t)x(t)\right) &= x^T(t)\dot{P}(t)x(t) + \dot{x}^T(t)P(t)x(t) + x^T(t)P(t)\dot{x}(t) \\ &= x^T(t)\left(\dot{P}(t) + A^T(t)P(t) + P(t)A(t)\right)x(t) \\ &\quad + w^T(t)G^T(t)P(t)x(t) + x^T(t)P(t)G(t)w(t).\end{aligned}$$

Thus, condition (24) implies that

$$\frac{d}{dt}\left(x^T(t)P(t)x(t)\right) < w^T(t)G^T(t)P(t)x(t) + x^T(t)P(t)G(t)w(t) - x^T(t)P(t)G(t)R^{-1}(t)G^T(t)P(t)x(t).$$

Let $v(t) = R^{1/2}(t)w(t) - R^{-1/2}(t)G^T(t)P(t)x(t)$. Then

$$\begin{aligned}v^T(t)v(t) &= w^T(t)R(t)w(t) + x^T(t)P(t)G(t)R^{-1}(t)G^T(t) \\ &\quad \times P(t)x(t) - w^T(t)G^T(t)P(t)x(t) \\ &\quad - x^T(t)P(t)G(t)w(t).\end{aligned}$$

It follows that

$$\frac{d}{dt}\left(x^T(t)P(t)x(t)\right) < w^T(t)R(t)w(t) - v^T(t)v(t) \le w^T(t)R(t)w(t). \quad (28)$$

Integrating (28) between t_0 and $t \in \Omega$, taking into account that $x(t_0) = 0$ and that $w(\cdot)$ belongs to $\mathcal{W}_2$, we obtain

$$\begin{aligned}x^T(t)P(t)x(t) &\le \int_{t_0}^{t} w^T(\sigma)R(\sigma)w(\sigma)d\sigma \\ &\le \|w\|_{2,R}^2 \le 1.\end{aligned}$$

By exploiting condition (21b), it follows that

$$\begin{aligned}y^T(t)Q(t)y(t) &= x^T(t)C^T(t)Q(t)C(t)x(t) \\ &< x^T(t)P(t)x(t) \le 1\end{aligned}$$

for all $t \in \Omega$, hence system (1) is IO-FTS with respect to $(\mathcal{W}_2, Q(\cdot), \Omega)$. ■

As it will be shown in Section V, the necessary and sufficient condition based on the reachability Gramian turns out to be computationally more efficient than the coupled DLMI/LMI when checking IO-FTS. However, the coupled DLMI/LMI can be effectively used to solve Problem 1, as it will be shown in the next section.

Remark 3: Theorem 3 also holds when the system matrices in (1) are piecewise continuous, provided that there exists an arbitrarily small $\epsilon > 0$ such that, for all $t \in \Omega$

$$\lambda_{\max}\left(Q^{\frac{1}{2}}(t)C(t)W(t,t_0)C^T(t)Q^{\frac{1}{2}}(t)\right) < 1 - \epsilon$$

when checking **ii**), or such that

$$P(t) > \frac{1}{1-\epsilon}C^T(t)Q(t)C(t) \quad (29)$$

[2]The matrix $\begin{pmatrix} J & K \\ K^T & L \end{pmatrix}$ is positive definite *if and only if* L is positive definite and $J - KL^{-1}K^T$ is positive definite. The matrix $J - KL^{-1}K^T$ is called the *Schur complement* of L.

when checking **iii**). Note that (29) is equivalent to the existence of an arbitrary $\xi > 1$ such that the LMI

$$P(t) > \xi C^T(t)Q(t)C(t)$$

is satisfied.

The case of piecewise continuous system matrices allows to capture a condition for IO-FTS in the special case of switching linear systems with known resetting times and without state jumps ([26], [27]). ▲

The following two corollaries deal with the special case in which the linear system (1) is time-invariant and the weighting matrices R and Q are constant. In this special case conditions (20) and (21b) in Theorem 3 need to be checked only for $t = T$.

Corollary 1: Given the time interval $\Omega := [0, T]$, two positive definite matrices $R \in \mathbb{R}^{r\times r}$ and $Q \in \mathbb{R}^{m\times m}$, assume that system (1) is time-invariant; then system (1) is IO-FTS with respect to $(\mathcal{W}_2, Q, \Omega)$ *if and only if*

$$\lambda_{\max}\left(Q^{\frac{1}{2}} C W(T) C^T Q^{\frac{1}{2}}\right) < 1 \tag{30}$$

where W is the positive semidefinite solution of the differential matrix equation

$$-\dot{W}(t) + AW(t) + W(t)A^T + GR^{-1}G^T = 0, \quad W(0) = 0.$$

Proof: The proof readily follows from Theorem 3 taking into account the *monotonicity* of the reachability Gramian in the LTI case (see Remark 1). Indeed, if condition (30) is satisfied then

$$\lambda_{\max}\left(Q^{\frac{1}{2}} C W(t) C^T Q^{\frac{1}{2}}\right) < 1, \quad \forall\, t \in \Omega.$$

■

Corollary 1 can be exploited to prove the following result.

Corollary 2: Given the time interval $\Omega := [0, T]$, two positive definite matrices $R \in \mathbb{R}^{r\times r}$ and $Q \in \mathbb{R}^{m\times m}$, assume that system (1) is time-invariant. System (1) is IO-FTS with respect to $(\mathcal{W}_2, Q, \Omega)$ *if and only if* the DLMI with terminal condition

$$\begin{pmatrix} \dot{P}(t) + A^T P(t) + P(t)A & P(t)G \\ G^T P(t) & -R \end{pmatrix} < 0, \quad t \in \Omega \tag{31a}$$

$$P(T) > C^T Q C \tag{31b}$$

admits a positive definite solution $P(\cdot)$ over Ω. ■

Remark 4: Note that, even when the system is time-invariant, the solution of a DLMI is required in order to check IO-FTS of the given system. This is due to the finite time nature of the problem we are dealing with (see, e.g., the optimal control problem defined over a finite horizon [28]). ▲

IV. IO Finite-Time Stabilization Via Dynamic Output Feedback

In this section, we exploit Theorem 3 to solve Problem 1. In particular, a necessary and sufficient condition for the IO finite-time stabilization of system (1) via dynamic output feedback is provided in terms of a DLMI/LMI feasibility problem.

Theorem 4: Problem 1 is solvable *if and only if* there exist two symmetric matrix-valued functions $S(\cdot), T(\cdot)$, and four matrix-valued functions $\hat{A}_K(\cdot), \hat{B}_K(\cdot), \hat{C}_K(\cdot)$ and $D_K(\cdot)$ such that the following DLMIs are satisfied:

$$\begin{pmatrix} \Theta_{11}(t) & \Theta_{12}(t) & 0 \\ \Theta_{12}^T(t) & \Theta_{22}(t) & T(t)G(t) \\ 0 & G^T(t)T(t) & -R(t) \end{pmatrix} < 0, \quad t \in \Omega \tag{32a}$$

$$\begin{pmatrix} \Psi_{11}(t) & \Psi_{12}(t) & 0 \\ \Psi_{12}^T(t) & S(t) & S(t)C^T(t) \\ 0 & C(t)S(t) & Q^{-1}(t) \end{pmatrix} > 0, \quad t \in \Omega \tag{32b}$$

where

$$\begin{aligned} \Theta_{11}(t) =& -\dot{S}(t) + A(t)S(t) + S(t)A^T(t) + B(t)\hat{C}_K(t) \\ &+ \hat{C}_K^T(t)B^T(t) + G(t)R^{-1}(t)G^T(t) \\ \Theta_{12}(t) =& A(t) + \hat{A}_K^T(t) + B(t)D_K(t)C(t) \\ &+ G(t)R^{-1}(t)G^T(t)T(t) \\ \Theta_{22}(t) =& \dot{T}(t) + T(t)A(t) + A^T(t)T(t) + \hat{B}_K(t)C(t) \\ &+ C^T(t)\hat{B}_K^T(t) \\ \Psi_{11}(t) =& T(t) - C^T(t)Q(t)C(t) \\ \Psi_{12}(t) =& I - C^T(t)Q(t)C(t)S(t). \end{aligned}$$

Proof: From Theorem 3 it readily follows that system (4) is IO-FTS with respect to $(\mathcal{W}_2, Q(\cdot), \Omega)$ if and only if there exists a symmetric matrix-valued function $P(\cdot)$ such that

$$\begin{aligned} &\dot{P}(t) + A_{\mathrm{CL}}^T(t)P(t) + P(t)A_{\mathrm{CL}}(t) \\ &\quad + P(t)G_{\mathrm{CL}}(t)R(t)^{-1}G_{\mathrm{CL}}^T(t)P(t) \\ &\quad < 0,\ t \in \Omega \end{aligned} \tag{33a}$$

$$P(t) > C^T(t)Q(t)C(t), \quad t \in \Omega. \tag{33b}$$

Given two symmetric matrix-valued functions $S(\cdot)$ and $T(\cdot)$, according to [25, Lemma 5.1], consider a symmetric matrix-valued function $U(\cdot)$ and two nonsingular matrix-valued functions $M(\cdot)$ and $N(\cdot)$ such that

$$P(t) = \begin{pmatrix} T(t) & M(t) \\ M^T(t) & U(t) \end{pmatrix},\ P^{-1}(t) = \begin{pmatrix} S(t) & N(t) \\ N^T(t) & \star \end{pmatrix}.$$

Furthermore, we also define

$$\Pi_1(t) = \begin{pmatrix} S(t) & I \\ N^T(t) & 0 \end{pmatrix}, \quad \Pi_2(t) = \begin{pmatrix} I & T(t) \\ 0 & M^T(t) \end{pmatrix}.$$

Note that, by definition

$$T(t)S(t) + M(t)N^T(t) = I \tag{34a}$$

$$\begin{aligned} &S(t)\dot{T}(t)S(t) + N(t)\dot{M}^T(t)S(t) + S(t)\dot{M}(t)N^T(t) \\ &\quad + N(t)\dot{U}(t)N^T(t) = -\dot{S}(t) \end{aligned} \tag{34b}$$

$$P(t)\Pi_1(t) = \Pi_2(t) \tag{34c}$$

where equality (34b) can be easily derived by noticing that $\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$.

We now prove that, with the given choice of $P(t)$, conditions (33) are equivalent to (32). Indeed, by pre- and post-multiplying

(33a)–(33b) by $\Pi_1^T(t)$ and $\Pi_1(t)$, respectively, and taking into account (34) and Lemma 5.1 in [25], the proof follows once we let

$$\begin{pmatrix} S(t) & I \\ I & T(t) \end{pmatrix} > 0 \tag{35a}$$

$$\hat{B}_K(t) = M(t)B_K(t) + T(t)B(t)D_K(t) \tag{35b}$$

$$\hat{C}_K(t) = C_K(t)N^T(t) + D_K(t)C(t)S(t) \tag{35c}$$

$$\begin{aligned}\hat{A}_K(t) = &\dot{T}(t)S(t) + \dot{M}(t)N^T(t) \\ &+ M(t)A_K(t)N^T(t) \\ &+ T(t)B(t)C_K(t)N^T(t) \\ &+ M(t)B_K(t)C(t)S(t) \\ &+ T(t)\left(A(t) + B(t)D_K(t)C(t)\right) \\ &\times S(t).\end{aligned} \tag{35d}$$

Note that (35a) does not need to be explicitly imposed since it is implied by (32b). ■

Remark 5 (Controller Design): Assuming that the hypotheses of Theorem 4 are satisfied, in order to design the controller, the following steps have to be followed:

i) Find $S(\cdot)$, $T(\cdot)$, $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ such that (32) are satisfied.

ii) Let $N(\cdot)$ be any nonsingular matrix-valued function (e.g., $N(t) = I$ for all $t \in \Omega$), and let

$$M(t) = \left[I - T(t)S(t)\right] N^{-T}(t).$$

iii) Obtain $A_K(\cdot)$, $B_K(\cdot)$ and $C_K(\cdot)$ by inverting (35). It is important to remark that, in order to invert (35), we need to preliminarily choose $N(\cdot)$. The only constraint for $N(\cdot)$ is to be a non singular matrix. ▲

As it has been done in Section III, starting from Theorem 4 it is possible to derive the following necessary and sufficient condition for the solution of Problem 1 when the case of linear time-invariant systems and constant weighting matrices is considered.

Corollary 3: Given the time interval $\Omega := [0, T]$, two positive definite matrices $R \in \mathbb{R}^{r\times r}$ and $Q \in \mathbb{R}^{m\times m}$, assume that system (1) is time-invariant. Problem 1 is solvable *if and only if* there exist two symmetric matrix-valued functions $S(\cdot)$, $T(\cdot)$, and four matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ such that the following DLMI (32a) with terminal condition is satisfied:

$$\begin{pmatrix} \widehat{\Theta}_{11}(t) & \widehat{\Theta}_{12}(t) & 0 \\ \widehat{\Theta}_{12}^T(t) & \widehat{\Theta}_{22}(t) & T(t)G \\ 0 & G^T T(t) & -R \end{pmatrix} < 0, \quad t \in \Omega$$

$$\begin{pmatrix} \widehat{\Psi}_{11}(T) & \widehat{\Psi}_{12}(T) & 0 \\ \widehat{\Psi}_{12}^T(T) & S(T) & S(T)C^T \\ 0 & CS(T) & Q^{-1} \end{pmatrix} > 0$$

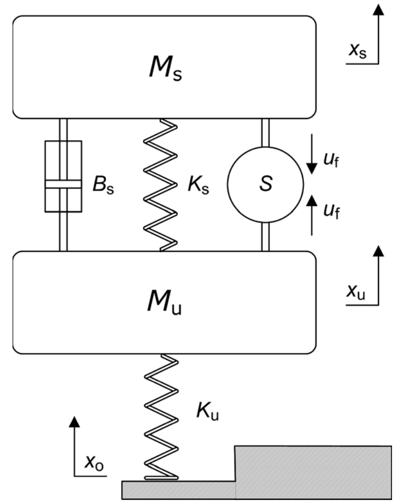

Fig. 1. Schematic representation of the active suspension system.

where

$$\begin{aligned}\widehat{\Theta}_{11}(t) =& -\dot{S}(t) + AS(t) + S(t)A^T + B\hat{C}_K(t) \\ &+ \hat{C}_K^T(t)B^T + GR^{-1}G^T \\ \widehat{\Theta}_{12}(t) =& A + \hat{A}_K^T(t) + BD_K(t)C + GR^{-1}G^T T(t) \\ \widehat{\Theta}_{22}(t) =& \dot{T}(t) + T(t)A + A^T T(t) + \hat{B}_K(t)C + C^T\hat{B}_K^T(t) \\ \widehat{\Psi}_{11}(t) =& T(t) - C^T QC \\ \widehat{\Psi}_{12}(t) =& I - C^T QCS(t).\end{aligned}$$

Proof: The proof readily follows from Theorem 4 and Corollary 2.

V. EXAMPLES

This section shows the effectiveness of the proposed results by means of two examples. We start in Example 1, by discussing some computational issues. In particular, we compare the numerical efficiency when applying the two necessary and sufficient conditions of Theorem 3 to check IO-FTS of LTV systems.

Afterwards, in Example 2, we apply the results of Section IV in order to propose a controller which guarantees that the maximum oscillation of an N-story building is bounded during an earthquake.

Example 1: A two-degree-of-freedom quarter-car model is considered in this example, whose scheme is reported in Fig. 1. The system comprises the sprung mass M_s, the unsprung mass M_u, the suspension damper with damping coefficient B_s, the suspension spring with elastic coefficient K_s, the elastic effect caused by the tire deflection, modeled by means of a spring with elastic coefficient K_u. The system model has been taken from [29] and it has been used also to test robust (state) FTS in [30]. The state variables are the suspension stroke $x_s - x_u$, the tire deflection $x_u - x_o$ and their derivatives with respect to time, that is

$$\begin{aligned} x_1 &= x_s - x_u \\ x_2 &= \dot{x}_s \\ x_3 &= x_u - x_o \\ x_4 &= \dot{x}_u \end{aligned}$$

TABLE I
MAXIMUM VALUES OF Q SATISFYING THEOREM 3 FOR THE LTV SYSTEM (36)

IO-FTS condition	Sample Time (T_s)	Estimate of Q_{max}	Computation time [s]
DLMI (21)	0.05	$1.93 \cdot 10^{-4}$	12.5
	0.025	$1.99 \cdot 10^{-4}$	106
	0.0125	$2.04 \cdot 10^{-4}$	1150
	0.01	$2.05 \cdot 10^{-4}$	2005
	0.008	$2.06 \cdot 10^{-4}$	4915
Solution of (13) and inequality (20)	$2 \cdot 10^{-5}$	$2.12 \cdot 10^{-4}$	9

where x_s and x_u are the vertical displacement of the sprung and unsprung masses, respectively, and x_o is the vertical ground displacement caused by road unevenness. As exogenous input, we chose the vertical ground velocity $w = \dot{x}_0$, while the considered output is the acceleration of the sprung mass M_s.

The resulting open-loop dynamical model reads

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{K_s}{M_s} & -\frac{B_s}{M_s} & 0 & \frac{B_s}{M_s} \\ 0 & 0 & 0 & 1 \\ \frac{K_s}{M_u} & \frac{B_s}{M_u} & -\frac{K_u}{M_u} & -\frac{B_s}{M_u} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} -\frac{K_s}{M_s} & -\frac{B_s}{M_s} & 0 & \frac{B_s}{M_s} \end{bmatrix} x(t) \tag{36}$$

where the values of the model parameters have been chosen as

$$M_s = 320 \text{ kg}, \quad K_s = 18 \, \frac{kN}{m}, \quad B_s = 1000 \, \frac{N \cdot s}{m}$$
$$K_u = 200 \, \frac{kN}{m}, \quad M_u = 40 \text{ kg}, \quad x_{1\,\max} = 0.08 \text{ m}$$
$$u_{\max} = 1.5 \text{ kN}.$$

The following IO-FTS parameters have been considered:

$$R = 0.25, \ \Omega = [0, 0.2].$$

For the considered example, if the system is IO-FTS then the acceleration of the sprung mass is bounded over the given time interval, for all the possible square integrable *time profiles* of the ground velocity.

The conditions stated in Theorem 3 are, in principle, necessary and sufficient. However, due to the time-varying nature of the involved matrices, the numerical implementation of such conditions introduces some conservativeness.

In order to compare each other, from the computational point of view, the conditions stated in Theorem 3, the output weighting matrix is left as a free parameter. More precisely, we introduce the parameter Q_{max}, defined as the maximum value of the matrix Q such that system (36) is IO-FTS, and use the conditions stated in Theorem 3 to obtain an estimate of Q_{max}.

To recast the DLMI condition (21) in terms of LMIs, the matrix-valued functions $P(\cdot)$ has been assumed piecewise linear. In particular, the time interval Ω has been divided in $n = T/T_s$ subintervals, and the time derivatives of $P(t)$ have been considered constant in each subinterval. It is straightforward to recognize that such a piecewise linear function can approximate at will a given continuous matrix function, provided that T_s is sufficiently small.

Given a piecewise linear function $P(\cdot)$, the feasibility problem (21) has been solved by exploiting standard optimization tools such as the Matlab LMI Toolbox [31].

Since the equivalence between IO-FTS and condition (21) holds when $T_s \mapsto 0$, the maximum value of Q satisfing condition (21), namely Q_{max}, has been evaluated for different values of T_s. The obtained estimates of Q_{max}, the corresponding values of T_s and of the computation time are shown in Table I. These results have been obtained by using a PC equipped with an Intel i7-720QM processor and 4 GB of RAM.

We have then considered the problem of finding the maximum value of Q satisfying condition (20), where $W(\cdot,\cdot)$ is the positive semidefinite solution of (13). In particular, (13) has been first integrated, with a sample time $T_s = 2 \cdot 10^{-5} s$, by using the Euler forward method, and then the maximum value of Q satisfying condition (20) has been evaluated by means of a linear search. As a result, it has been found the estimate $Q_{max} = 2.12 \cdot 10^{-4}$, with a computation time of about 9 s, as it is shown in the last row of Table I.

We can conclude that the condition based on the reachability Gramian is much more efficient with respect to the solution of the DLMI when considering the IO-FTS analysis problem; however, as said, the DLMI feasibility problem is necessary in order to solve the stabilization problem discussed in Section IV, as it will be shown by the next example.

Example 2: Let us consider an N-story building subject to an earthquake. The building lumped parameters model is reported in Fig. 2. The control system is made by a base isolator together with an actuator that generates a control force on the base floor.

The aim of the isolator is to produce a dynamic decoupling of the structure from its foundation. If this is the case, the interstory drifts are reduced and the building behavior can be approximated by the one of a rigid body [32]. Furthermore, the description of the system in terms of absolute coordinates, i.e., when the displacement is defined with respect to an inertial reference, ensures that the disturbances act only at the base floor [33].

It turns out that it is sufficient to provide an actuator only on the base floor in order to keep the displacement and velocity of the structure under a specified boundary. Indeed, the goal of the control system is to overcome the forces generated by the isolation system at the base floor, in order to minimize the absolute displacement and velocity of the structure.

The state-space model of the considered system is

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t) \tag{37a}$$
$$y(t) = Cx(t). \tag{37b}$$

If we denote by $s_0(\cdot)$ and $\dot{s_0}(\cdot)$ the displacement and the velocity of the ground and with $s_i(\cdot)$ and $\dot{s_i}(\cdot)$ the displacement and the velocity of the ith floor, then the state vector can be

TABLE II
MODEL PARAMETERS FOR THE CONSIDERED SIX STORIES BUILDING ($N = 6$)

Floor #	Mass [kg]	Spring coefficient [kN/m]	Damping coefficient [kNs/m]
0	–	k_0=1200	c_0=2.4
1	m_1=6800	k_1=33732	c_1=67
2	m_2=5897	k_2=29093	c_2=58
3	m_3=5897	k_3=28621	c_3=57
4	m_4=5897	k_4=24954	c_4=50
5	m_5=5897	k_5=19059	c_5=38
6	m_6=5897	–	–

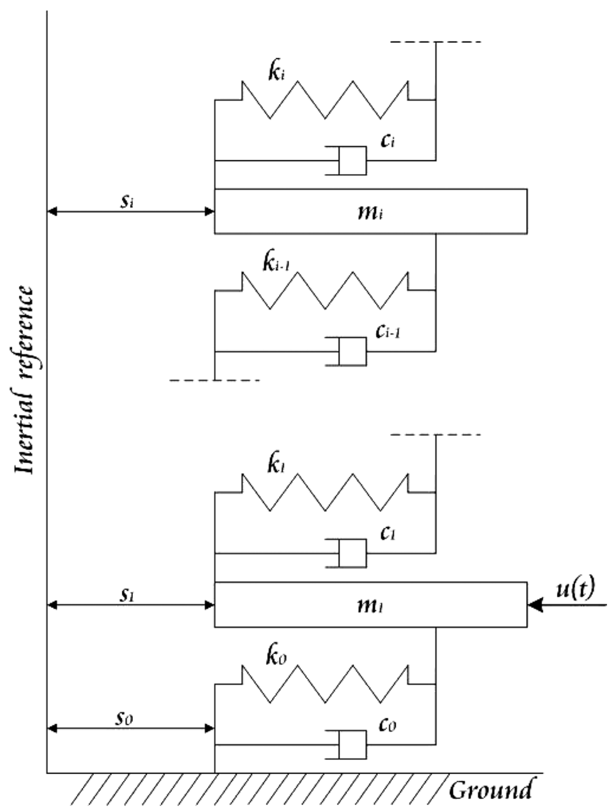

Fig. 2. Lumped parameters model of a N-stories building.

defined as $x(\cdot) = [x_1(\cdot)\ x_2(\cdot)\ \ldots\ x_{2N}(\cdot)]^T$, where $x_i(\cdot) = \dot{s}_i(\cdot)$ and $x_{i+N}(\cdot) = s_i(\cdot)$, $i = 1,\ldots,N$. The vector $w(\cdot) = [s_0(\cdot)\ \dot{s}_0(\cdot)]^T$ represents the exogenous input and $u(t)$ is the control force applied to the base floor. The model matrices in (37) are

$$A = \begin{pmatrix} A_1 & A_2 \\ I & 0 \end{pmatrix},\ B = \begin{pmatrix} 1/m_1 \\ 0 \end{pmatrix}$$
$$G = \begin{pmatrix} k_0/m_1 & c_0/m_1 \\ 0 & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} \frac{-(c_0+c_1)}{m_1} & \frac{c_1}{m_1} & \underbrace{0\ \ldots\ 0}_{N-2} & \frac{-(k_0+k_1)}{m_1} & \frac{k_1}{m_1} & \underbrace{0\ \ldots\ 0}_{N-2} \end{pmatrix}$$

where A_1 and A_2 are $N \times N$ tridiagonal matrices defined as shown in the equation at the bottom of the page.

The model parameters are reported in Table II for the six stories building considered in this example.

Taking into account the presence of the isolator and given the choice of the C matrix, the controlled output is related to the acceleration at the base floor. Concerning the choice of the IO-FTS parameters, for a given geographic area these can be chosen starting from the worst earthquake on record. Indeed, from the time trace of the ground acceleration, velocity and displacement of the *El Centro* earthquake (May 18, 1940) reported in Fig. 3, the following IO-FTS parameters have been considered

$$R = I,\ Q = 0.1,\ \Omega = [0, 35]. \tag{38}$$

Indeed, given the considered output of the building model, the values for the matrices W and Q should assure that

$$|s_1(t)| \le 10\ \text{cm and } |\dot{s}_1(t)| \le 1.5\ \text{cm/s} \tag{39}$$

$$A_1 = \begin{pmatrix} -\frac{(c_0+c_1)}{m_1} & \frac{c_1}{m_1} & 0 & & \cdots & & 0 \\ & & & \cdots\cdots & & & \\ 0 & \cdots & \frac{c_{i-1}}{m_i} & -\frac{(c_{i-1}+c_i)}{m_i} & \frac{c_i}{m_i} & \cdots & 0 \\ & & & \cdots\cdots & & & \\ 0 & & \cdots & & 0 & \frac{c_{N-1}}{m_N} & -\frac{c_{N-1}}{m_N} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -\frac{(k_0+k_1)}{m_1} & \frac{k_1}{m_1} & 0 & & \cdots & & 0 \\ & & & \cdots\cdots & & & \\ 0 & \cdots & \frac{k_{i-1}}{m_i} & -\frac{(k_{i-1}+k_i)}{m_i} & \frac{k_i}{m_i} & \cdots & 0 \\ & & & \cdots\cdots & & & \\ 0 & & \cdots & & 0 & \frac{k_{N-1}}{m_N} & -\frac{k_{N-1}}{m_N} \end{pmatrix}$$

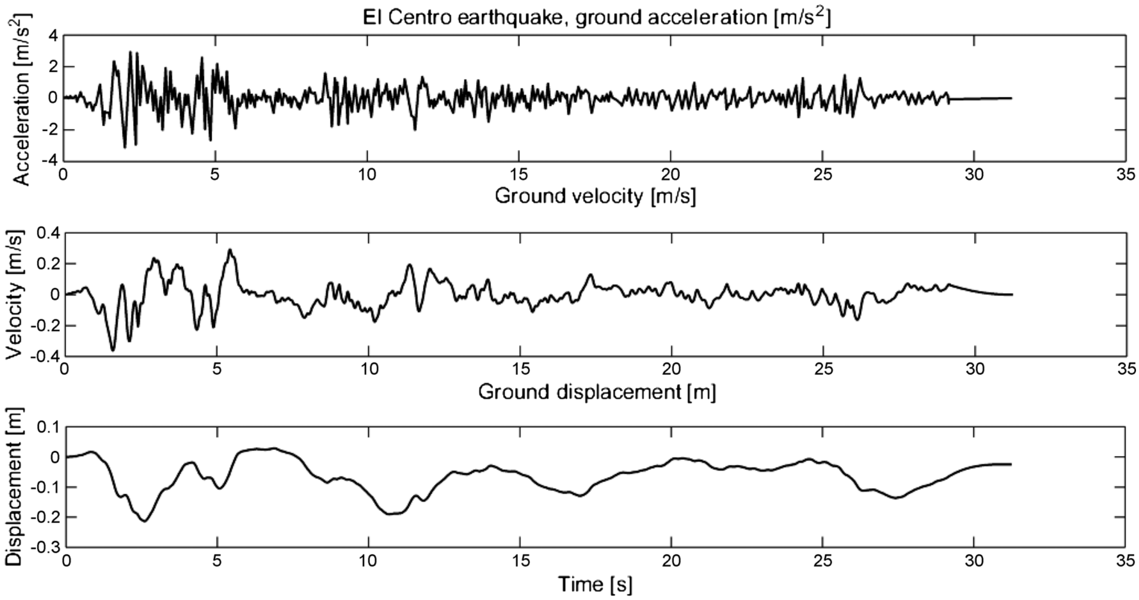

Fig. 3. Ground acceleration, velocity, and displacement of El Centro earthquake.

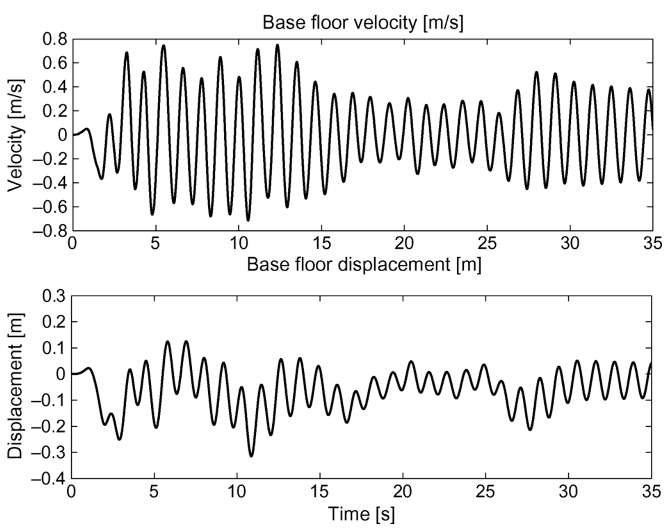

Fig. 4. Uncontrolled base floor velocity and displacement.

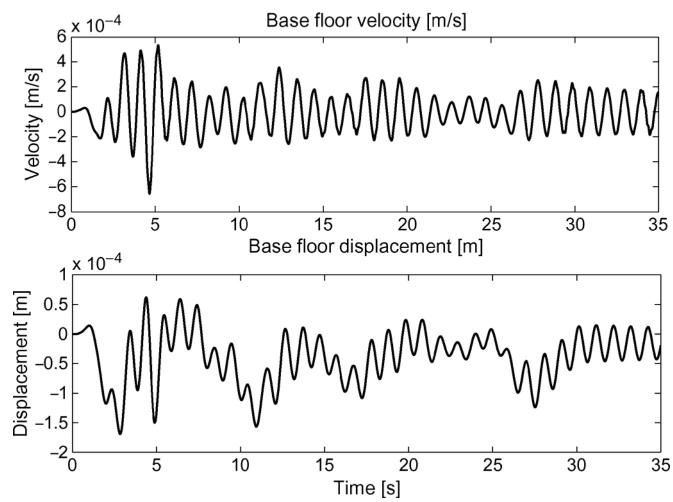

Fig. 5. Controlled base floor velocity and displacement.

when an earthquake having a magnitude and a duration similar to the *El Centro* occurs.

By means of simulation it can be verified that the considered system is not open-loop IO-FTS with respect to the chosen parameters; hence, it does not meet the constraints (39), as shown in Fig. 4.

Exploiting Theorem 4, and assuming for $S(\cdot)$ and $T(\cdot)$ the same structure foreseen for the $P(\cdot)$ matrix-valued function in Example 1, it is possible to find the controller matrix-valued functions $A_k(\cdot), B_k(\cdot), C_k(\cdot)$, and $D_k(\cdot)$ that make system (37) IO-FTS with respect to the parameters given in (38), when $\mathcal{W}_2$ disturbances are considered.

Fig. 4 shows the base floor velocity and displacement time traces for the uncontrolled building with base isolation system, under the assumed earthquake excitation. As it can be seen in Fig. 5, the control system manages to keep very small both the velocity and the displacement of the structure. The relative control force is depicted in Fig. 6.

VI. Conclusion

In this paper, we have dealt with the input–output finite-time stabilization problem for linear time-varying systems. The main results of the work are a couple of necessary and sufficient conditions for IO-FTS of a given system; the former requires the solution of a DLE, the latter solution of a feasibility problem involving a DLMI. Both conditions are shown to be useful; indeed, the one based on the DLE is more efficient from a computational point of view at the analysis stage, while the condition based on the DLMI is useful in the design context. The effectiveness of the proposed approach has been illustrated through two examples, the first one to investigate the computational efficiency of the proposed approach and the second one to discuss the applicability of the proposed technique to an interesting engineering problem.

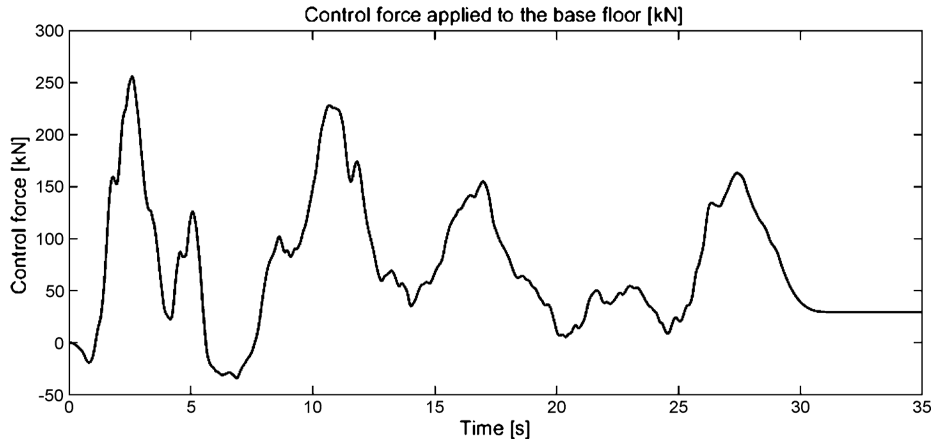

Fig. 6. Control force applied to the base floor.

APPENDIX

In this appendix we provide the proof of Lemma 2 which has been used in Section II-C to prove Theorem 2. Before proving Lemma 2, let us introduce the following result.

Lemma 4: Given a positive scalar k and a signal $v(\cdot)$, we have

$$\|v(\cdot)\|_2 \leq k$$

iff for all $z(\cdot)$ with $\|z(\cdot)\|_2 \leq 1$

$$\langle v, z\rangle \leq k.$$

Proof (only if): Let $\|v(\cdot)\|_2 \leq k$ and assume, *ad absurdum*, that there exists a $z(\cdot)$ such that

$$\|z(\cdot)\|_2 \leq 1$$

and

$$\langle v, z\rangle > k.$$

Since

$$\langle v, z\rangle = \int\limits_{\Omega} v^T(t)z(t)\ dt \leq \int\limits_{\Omega} |v(t)| \cdot |z(t)|\ dt$$

from the Hölder inequality [23, p. 33] it readily follows that

$$\langle v, z\rangle \leq \|v(\cdot)\|_2 \cdot \|z(\cdot)\|_2$$

which contradicts the initial assumption.

(if). Let $\langle v, z\rangle \leq k$ for all $z(\cdot)$ such that $\|z(\cdot)\|_2 \leq 1$, and assume, *ad absurdum*, that $\|v(\cdot)\| > k$. If we pick

$$z(t) := \frac{1}{\|v(\cdot)\|_2} v(t)$$

it obviously turns out that $\|z(\cdot)\|_2 = 1$. We have

$$\langle v, z\rangle = \frac{1}{\|v(\cdot)\|_2} \int\limits_{\Omega} v^T(t)v(t)\ dt = \|v(\cdot)\|_2 > k$$

which contradicts the initial assumption. ■

Proof of Lemma 2: Let $z(t)$ such that $\|z(\cdot)\|_2 \leq 1$, then

$$\dot{x}_c(t) = A_K(t)x_c(t) + B_K(t)y(t), \tag{40a}$$
$$u(t) = C_K(t)x_c(t) + D_K(t)y(t) \tag{40b}$$

where equality (40a) follows from the Fubini's theorem [23, p. 18], while (40b) follows from the Cauchy–Schwartz inequality for vector norms. Applying again the Hölder inequality, we obtain

$$\langle v\,, z\rangle \leq \int_{\Omega} \left\{ \left[\int_{\Omega} f(t\,,\tau)^2\ dt \right]^{1/2} \cdot \left[\int_{\Omega} z(t)^2\ dt \right]^{1/2} \right\} d\tau$$

Since $\|z(\cdot)\|_2 \leq 1$, we have

$$\langle v\,, z\rangle \leq \int_{\Omega} \|f(\cdot\,,\tau)\|_2\ d\tau$$

hence, inequality (11) readily follows from the arbitrariness of $z(\cdot)$ and the application of Lemma 4. ■

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