PERGAMON

Automatica 36 (2000) 757–764

www.elsevier.com/locate/automatica

Technical Communique

Robust $\mathscr{H}_\infty$ control of linear neutral systems☆

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Received 2 December 1998; revised 7 June 1999; received in final form 14 September 1999

Abstract

This paper investigates the problems of robust stability and robust $\mathscr{H}_\infty$ control for a class of uncertain neutral systems. The class describes linear state models with norm-bounded uncertain system parameters and unknown constant state delay. First, we develop a sufficient condition for robust stability independent of delay. Then, we provide sufficient conditions for designing a memoryless state-feedback controller which stabilizes the uncertain neutral system under consideration and guarantees an $\mathscr{H}_\infty$-norm bound constraint on the disturbance attenuation for all admissible uncertainties and unknown state delay. In both problems, the results are expressed in the form of linear matrix inequalities. It has been established that several earlier results are special cases of the developed theory. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Linear neutral systems; Uncertain systems; Robust stability; Robust $\mathscr{H}_\infty$ control; Norm-bounded uncertainties

1. Introduction

Time delay arises quite naturally in connection with system measurements and/or information flow amongst different parts of dynamical systems (Gorecki, Fuska, Grabowski & Korytowski, 1989). State-space modeling of industrial and engineering systems frequently encounter delay effects in processing state, input or related variables. Considerable research efforts have been undertaken for the past four decades on various aspects of dynamical systems with delays in the states, control inputs or both and in particular the class of uncertain delay systems; see (Dugard & Verriest, 1997; Mahmoud, 1999, 1994; Luo & Van Den Bosch, 1997; Mahmoud, 1996, 1998; Huang & Ren, 1998; Xie & de Souza, 1993; Lee & Leitmann, 1988; Niculescu, et al., 1998; Li & de Souza, 1997; Niculescu, 1998) and their references. A common conclusion drawn from the ensuing results is that time delay has, by and large, a destabilizing effect on the control system. The available methods can be broadly classifed into delay independent (Dugard & Verriest, 1997; Mahmoud, 1999, 1994; Luo & Van Den Bosch, 1997; Mahmoud, 1996, 1998; Huang & Ren, 1998; Xie & de Souza, 1993; Lee & Leitmann, 1988; Niculescu et al., 1998) and delay dependent (Dugard & Verriest, 1997; Mahmoud, 1999; Niculescu et al., 1998; Li & de Souza, 1997; Niculescu, 1998). Despite the fact that delay-independent results tend to be conservative, in several practical applications the developed stability and stabilization methods are proving their great utility and satisfactory performance. These applications (Mahmoud, 1999, 1994; Luo & Van Den Bosch, 1997; Mahmoud, 1996, 1998; Huang & Ren, 1998; Xie & de Souza, 1993; Lee & Leitmann, 1988) include stream water quality, chemical reactors with recycling, vehicle following systems and power systems. Most of the research results however, appear to be heavily focused on retarded functional differential equations (Kolomanovskii & Myshkis, 1992). The topic of uncertain neutral systems has, however, received little attention. Stability measures and stabilization methods of linear neutral systems without uncertainties have been studied in Slemrod and Infante (1972) Logemann and Townley (1996) O'Conner and Tarn (1983) Verriest and Niculescu (1997). On the other hand, $\mathscr{H}_\infty$ attenuation has been proven to be a very useful performance measure. The last decade has witnessed major advances in $\mathscr{H}_\infty$ control theory (Zhou, 1998) of linear dynamical systems. It seems, however, that little results are available so far on $\mathscr{H}_\infty$ control of time-delay systems. An $\mathscr{H}_\infty$ state-feedback controller satisfying some

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Editor Peter Dorato.

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PII: S0005-1098(99)00159-4

α-constraint on the closed-loop eigenvalues is designed in Niculescu (1998). In Shen, Chen and Kung (1991), a frequency-domain approach is used to design an $\mathscr{H}_\infty$ controller when the time delay is constant. A Lyapunov approach is adopted in Mahmoud and Zribi (1999) for the design of state and dynamic output feedback controllers for a class of time-varying delay systems.

This paper contributes to the development of robust stability and stabilization of neutral systems. It treats a class of linear neutral systems with norm-bounded uncertainties and unknown constant state delay. We develop a sufficient condition for robust stability independent of delay. Then, we address the robust $\mathscr{H}_\infty$ control design problem such that the stabilization of the closed-loop feedback system is guaranteed with a prescribed $\mathscr{H}_\infty$-norm bound constraint on disturbance attentuation for all admissible uncertainties and unknown state delay. Throughout the paper, the main thrust stems from a Lyapunov–Krasovskii functional approach (Mahmoud, 1998) which eventually leads to finite-dimensional algebraic Riccati inequalities. These inequalities are cast as linear-matrix inequality (LMI)-feasibility problems which can be effectively be handled by existing software. The contribution of the paper is organized into four theorems: Theorem 1 gives a robust stability independent of delay measure, Theorem 2 describes a sufficient condition for robust stabilization by a memoryless state-feedback, Theorem 3 presents a robust $\mathscr{H}_\infty$ performance analysis and finally Theorem 4 establishes an $\mathscr{H}_\infty$ state-feedback controller. Several corollaries are developed which recover earlier results and/or establish LMI-feasibility conditions.

Notations and facts. In the sequel, we denote by W^t and W^{-1} the transpose and the inverse of any square matrix W. We use $W > 0$ ($W < 0$) to denote a positive- (negative-) definite matrix W; and I is used to denote the $n \times n$ identity matrix. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

Fact 1 (Schur complement). *Given constant matrices $\Omega_1, \Omega_2, \Omega_3$ where $0 < \Omega_1 = \Omega_1^t$ and $0 < \Omega_2 = \Omega_2^t$ then $\Omega_1 + \Omega_3^t \Omega_2^{-1} \Omega_3 < 0$ if and only if*

$$\begin{bmatrix} \Omega_1 & \Omega_3^t \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0, \qquad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^t & \Omega_1 \end{bmatrix} < 0.$$

Fact 2. *For any real matrices Σ_1, Σ_2 and Σ_3 with appropriate dimensions and $\Sigma_3^t \Sigma_3 \le I$, it follows that*

$$\Sigma_1 \Sigma_3 \Sigma_2 + \Sigma_2^t \Sigma_3^t \Sigma_1^t \le \alpha^{-1} \Sigma_1 \Sigma_1^t + \alpha \Sigma_2^t \Sigma_2, \quad \alpha > 0.$$

Fact 3. *Let $\Sigma_1, \Sigma_2, \Sigma_3$ be real constant matrices of compatible dimensions and $H(t)$ be a real matrix function satisfying $H^t(t)H(t) \le I$. Then $\forall\ \rho > 0$ and any matrix $0 < R = R^t$ such that $\rho \Sigma_2^t \Sigma_2 < R$ we have*

$$\begin{aligned}(\Sigma_3 + \Sigma_1 H(t) \Sigma_2) R^{-1} (\Sigma_3^t + \Sigma_2^t H^t(t) \Sigma_1^t) \\ \le \rho^{-1} \Sigma_1 \Sigma_1^t + \Sigma_3 [R - \rho \Sigma_2^t \Sigma_2]^{-1} \Sigma_3^t.\end{aligned}$$

2. A class of neutral systems

In this paper, we consider a class of neutral functional differential equation (NFDE) described by a linear model with parameteric uncertainties:

$$\begin{aligned}(\Sigma_{\Delta n}):\quad \dot{x}(t) - D\dot{x}(t-\tau) &= (A + \Delta A)x(t) \\ &\quad + (A_d + \Delta A_d)x(t-\tau) \\ &= A_\Delta x(t) + A_{d\Delta} x(t-\tau), \qquad (1)\end{aligned}$$

$$x(t_o + \eta) = \phi(\eta), \quad \forall \eta \in [-\tau, 0] \qquad (2)$$

where $x \in \Re^n$ is the state, $A \in \Re^{n \times n}$ and $A_d \in \Re^{n \times n}$ are known real constant matrices, $\tau > 0$ is an unknown constant delay factor and $\Delta A \in \Re^{n \times n}$ and $\Delta A_d \in \Re^{n \times n}$ are matrices of uncertain parameters represented by

$$[\Delta A(t) \Delta A_d(t)] = H \Delta(t) [E E_d], \qquad \Delta^t(t) \Delta(t) \le I, \quad \forall t \qquad (3)$$

where $H \in \Re^{n \times \alpha}, E \in \Re^{\beta \times n}, E_d \in \Re^{\beta \times n}$ are known real constant matrices and $\Delta(t) \in \Re^{\alpha \times \beta}$ is unknown matrix with Lebesgue measurable elements. The initial condition is specified as $\langle x(t_o), x(s) \rangle = \langle x_o, \phi(s) \rangle$, where $\phi(\cdot) \in \mathscr{L}_2[-\tau, t_o]$. Note that when $\Delta A \equiv 0, \Delta A_d \equiv 0$, system (1) reduces to the standard linear neutral systems (Mahmoud, 1998). For system $(\Sigma_{\Delta n})$, we assume that:

Assumption 1. $\lambda(A) < 0$.

Assumption 2. $\lambda(D)| < 1$.

We remark that (1) is a continuous-time model for which Assumption 1 is quite standard. However, Assumption 2 gives a condition in the discrete-time sense and its role will be clarified in the subsequent analysis. In the following sections, we examine closely the problems of robust stability and $\mathscr{H}_\infty$ stabilization of system (1). Frequently, the term $\mathscr{M}(x_t) := x(t) - Dx(t-\tau)$ is called the difference operator (Mahmoud, 1998), which will play a major role in the subsequent analysis.

3. Robust stability

In this section, we focus attention on the stability of system $(\Sigma_{\Delta n})$ and establish the following result.

Theorem 1. *Subject to Assumptions* 1 *and* 2, *the neutral system $(\Sigma_{\Delta n})$ is robustly asymptotically stable independent of delay if there exist matrices $0 < P = P^t \in \Re^{n \times n}$,*

$0 < S = S^t \in \Re^{n\times n}$ *and* $0 < R = R^t \in \Re^{n\times n}$ *and scalars* $\varepsilon > 0, \rho > 0$ *such that the following algebraic Riccati inequality* (*ARI*) *and Lyapunov equation* (*LE*) *are satisfied*:

$$PA + A^tP + (\varepsilon+\rho)PHH^tP + \rho^{-1}EE^t + S + [P(AD + A_d) + SD][R - \varepsilon^{-1}(D^tE^tED + E_d^tE_d)]^{-1} \times [P(AD + A_d) + SD]^t < 0, \tag{4}$$

$$D^tSD - S + R = 0. \tag{5}$$

Proof. Introduce a Lyapunov–Krasovskii functional $V(x_t)$ of the form

$$V(x_t) = [x(t) - Dx(t-\tau)]^tP[x(t) - Dx(t-\tau)] + \int_{-\tau}^{0} x^t(t+\theta)Sx(t+\theta)\,d\theta. \tag{6}$$

Observe that $V(x_t)$ satisfies

$$\lambda_m(P)r^2 \le V(r) \le [\lambda_M(P) + \tau\lambda_M(S)]r^2. \tag{7}$$

By differentiating $V(x_t)$ along the solutions of (1) and arranging terms, we get

$$\dot V(x_t) = [A_\Delta x(t) + A_{d\Delta}x(t-\tau)]^tP[x(t) - Dx(t-\tau)] + [x(t) - Dx(t-\tau)]^tP[A_\Delta x(t) + A_{d\Delta}x(t-\tau)] + x^tSx(t) - x^t(t-\tau)Sx(t-\tau). \tag{8}$$

By algebraic manipulation of (8) using the difference operator $\mathscr{M}(x_t) := x(t) - Dx(t-\tau)$ and arranging terms, we get

$$\dot V(x_t) = \mathscr{M}^t(x_t)[PA_\Delta + A_\Delta^tP + S]\mathscr{M}(x_t) + \mathscr{M}^t(x_t)[PAD + SD + PA_{d\Delta}]x(t-\tau) + x^t(t-\tau)[D^tA^tP + D^tS + A_{d\Delta}^tP]\mathscr{M}(x_t) + x^t(t-\tau)[D^tSD - S]x(t-\tau). \tag{9}$$

In view of (5) and completing the squares, we get

$$\begin{aligned}\dot V(x_t) = {} & \mathscr{M}^t(x_t)[PA_\Delta + A_\Delta^tP + S + (PA_\Delta D + SD + PA_{d\Delta})R^{-1}(PA_\Delta D + SD + PA_{d\Delta})^t]\mathscr{M}^t(x_t) \\ & - [(D^tA_\Delta^tP + A_{d\Delta}^tP + DS)\mathscr{M}(x_t) - Rx(t-\tau)]^tR^{-1} \times [(D^tA_\Delta^tP + A_{d\Delta}^tP + DS)\mathscr{M}(x_t) - Rx(t-\tau)] \\ \le {} & \mathscr{M}^t(x_t)[PA_\Delta + A_\Delta^tP + S + (PA_\Delta D + SD + PA_{d\Delta})R^{-1}(PA_\Delta D + SD + PA_{d\Delta})^t]\mathscr{M}^t(x_t).\end{aligned} \tag{10}$$

By Lyapunov–Krasovskii theorem (Mahmoud, 1998), it is sufficient from (7) and (10) to conclude that system $(\Sigma_{\Delta n})$ is asymptotically stable if

$$PA_\Delta + A_\Delta^tP + S + (PA_\Delta D + SD + PA_{d\Delta})R^{-1} \times (PA_\Delta D + SD + PA_{d\Delta})^t < 0 \tag{11}$$

for all admissible uncertainties satisfying (3). Using Facts 2 and 3, we get for some scalars $\varepsilon > 0, \rho > 0$:

$$PA_\Delta + A_\Delta^tP \le PA + A^tP + \rho PHH^tP + \rho^{-1}E^tE, \tag{12}$$

$$(PA_\Delta D + SD + PA_{d\Delta})R^{-1}(PA_\Delta D + SD + PA_{d\Delta})^t \le \varepsilon PHH^tP + [P(AD + A_d) + SD] \times [R - \varepsilon^{-1}(D^tE^tED + E_d^tE_d)]^{-1}[P(AD + A_d) + SD]^t. \tag{13}$$

By selecting the matrices S and R such that the LE (5) is satisfied and substituting (12)–(13) into (10), one obtains the ARI (4). □

Corollary 1. *Subject to Assumptions* 1 *and* 2, *the neutral system* $(\Sigma_{\Delta n})$ *is asymptotically stable independent of delay if there exist matrices* $0 < Q = Q^t \in \Re^{n\times n}$. $0 < S = S^t \in \Re^{n\times n}$ *and scalars* $\varepsilon > 0, \rho > 0, \mu > 0, v > 0$ *satisfying the system of linear matrix inequalities* (*LMIs*)

$$\begin{bmatrix} AQ + QA^t + \varepsilon HH^t + Q(vEE^t + S)Q & H & AD + A_d + QSD \\ H^t & -vI & 0 \\ D^tA^t + A_d^t + D^tSQ & 0 & -J \end{bmatrix} < 0,$$

$$D^tSD - S < 0, \quad D^tE^tED + E_d^tE_d - \varepsilon(D^tSD - S) < 0,$$

$$\begin{bmatrix} \rho & 1 \\ 1 & v \end{bmatrix} \ge 1, \qquad \begin{bmatrix} \varepsilon & 1 \\ 1 & \mu \end{bmatrix} \ge 1, \tag{14}$$

where

$$J = D^tSD - S + \mu(D^tE^tED + E_d^tE_d).$$

Proof. By Fact 1 under the equality constraints $\rho v = 1$, $\varepsilon\mu = 1$, ARI (4) and LE (5) with $Q = P^{-1}$ are equivalent to the system of LMIs (14). □

Remark 1. Observe that the employment of the coupling constraints $\rho v = 1$, $\varepsilon\mu = 1$ ensures that the conditions of Theorem 1 are convex in $\mu, \varepsilon, v, \rho$ which in turn guarantee the feasibility of LMIs (14). In computer implementation, there are two different approaches. The first approach utilizes a multi-dimensional search procedure for $\mu, \varepsilon, v, \rho$

while solving inequalities (14) for Q, S. The second approach employs the following iterative procedure:

Step 1: Find $Q, S, \mu_o, \varepsilon_o, v_o, \rho_o$ that solve the LMIs (14). If the problem is infeasible, stop. Otherwise, set the iteration index $k = 1$.

Step 2: Find $\mu_k, \varepsilon_k, v_k, \rho_k$ that solve the LMI problem:
min $\quad \phi_k = \varepsilon_{k-1}\mu + \varepsilon\mu_{k-1} + \rho_{k-1}v + \rho v_{k-1}$

s.t. *LMIs* (14).

Step 3: If ϕ_k has reached a stationary point, stop. Otherwise, set $k \leftarrow k+1$ and go to Step 2.

In the sequel, we will adopt the second approach. Our computational experience with several examples indicates that the above procedure, albiet heuristic, does converge after few iterations.

Corollary 2. *Subject to Assumptions* 1 *and* 2, *the neutral system* (Σ_{no})

$$(\Sigma_{no}):\quad \dot{x}(t) - D\dot{x}(t-\tau) = Ax(t) + A_{\mathrm{d}}x(t-\tau), \tag{15}$$

$$x(t_o+\eta) = \phi(\eta), \quad \forall \eta \in [-\tau, 0] \tag{16}$$

is asymptotically stable independent of delay if the following conditions hold:

(1) *There exist matrices* $0 < P = P^{\mathrm{t}} \in \Re^{n\times n}$, $0 < S = S^{\mathrm{t}} \in \Re^{n\times n}$ *and* $0 < R = R^{\mathrm{t}} \in \Re^{n\times n}$ *satisfying the ARI*

$$PA + A^{\mathrm{t}}P + [P(AD + A_{\mathrm{d}}) + SD]R^{-1} \times [P(AD + A_{\mathrm{d}}) + SD]^{\mathrm{t}} + S < 0. \tag{17}$$

(2) *There exist matrices* $0 < S = S^{\mathrm{t}} \in \Re^{n\times n}$ *and* $0 < R = R^{\mathrm{t}} \in \Re^{n\times n}$ *satisfying the LE*

$$D^{\mathrm{t}}SD - S + R = 0. \tag{18}$$

Proof. By Fact 1, ARI (17) and LE (18) are equivalent to the LMIs (19). □

Remark 2. It is important to mention that Corollaries 1 and 2 confirm the results of Niculescu et al. (1998). Interestingly enough, by deleting the matrix D in (1) we obtain the linear retarded system

$$(\Sigma_{\Delta r}):\quad \dot{x}(t) = (A + \Delta A)x(t) + (A_{\mathrm{d}} + \Delta A_{\mathrm{d}})x(t-\tau) = A_{\Delta}x(t) + A_{\mathrm{d}\Delta}x(t-\tau), \tag{20}$$

$$x(t_o+\eta) = \phi(\eta), \quad \forall \eta \in [-\tau, 0] \tag{21}$$

for which Theorem 1, Corollaries 1 and 2 reduce to the following standard results, stated as corollaries without proof.

Corollary 4. *Subject to Assumption* 1, *the retarded system* $(\Sigma_{\Delta r})$ *is robustly asymptotically stable independent of delay if there exist matrices* $0 < P = P^{\mathrm{t}} \in \Re^{n\times n}$, $0 < S = S^{\mathrm{t}} \in \Re^{n\times n}$ *and scalars* $\varepsilon > 0, \rho > 0$ *satisfying the ARI*:

$$PA + A^{\mathrm{t}}P + (\varepsilon + \rho)PHH^{\mathrm{t}}P + \rho^{-1}EE^{\mathrm{t}} + PA_{\mathrm{d}}[S - \varepsilon^{-1}E_{\mathrm{d}}^{\mathrm{t}}E_{\mathrm{d}}]^{-1}A_{\mathrm{d}}^{\mathrm{t}}P + S < 0. \tag{22}$$

Corollary 5. *Subject to Assumption* 1 *the retarded system* $(\Sigma_{\Delta r})$ *is asymptotically stable independent of delay if there exist matrices* $0 < Q = Q^{\mathrm{t}} \in \Re^{n\times n}$, $0 < S = S^{\mathrm{t}} \in \Re^{n\times n}$ *and scalars* $\varepsilon > 0, \rho > 0, \mu > 0, v > 0$ *satisfying the LMIs*

$$\begin{bmatrix} AQ + QA^{\mathrm{t}} + \varepsilon HH^{\mathrm{t}} + Q(vEE^{\mathrm{t}} + S)Q & H & A_{\mathrm{d}} \\ H^{\mathrm{t}} & -vI & 0 \\ A_{\mathrm{d}}^{\mathrm{t}} & 0 & -S + \mu E_{\mathrm{d}}^{\mathrm{t}}E_{\mathrm{d}} \end{bmatrix} < 0, \qquad E_{\mathrm{d}}^{\mathrm{t}}E_{\mathrm{d}} - \varepsilon S < 0,$$

$$\begin{bmatrix} \rho & 1 \\ 1 & v \end{bmatrix} \geq 1, \qquad \begin{bmatrix} \varepsilon & 1 \\ 1 & \mu \end{bmatrix} \geq 1. \tag{23}$$

Proof. Set $E = 0, H = 0, E_{\mathrm{d}} = 0$ in ARI (4) and LE (5). □

Corollary 3. *Subject to Assumptions* 1 *and* 2, *the neutral system* (Σ_{no}) *is asymptotically stable independent of delay if there exist matrices* $0 < Q = Q^{\mathrm{t}} \in \Re^{n\times n}$ *and* $0 < S = S^{\mathrm{t}} \in \Re^{n\times n}$ *satisfying the LMIs*

$$\begin{bmatrix} AQ + QA^{\mathrm{t}} + QSQ & AD + A_{\mathrm{d}} + QSD \\ D^{\mathrm{t}}A^{\mathrm{t}} + A_{\mathrm{d}}^{\mathrm{t}} + D^{\mathrm{t}}SQ & D^{\mathrm{t}}SD - S \end{bmatrix} < 0,$$

$$D^{\mathrm{t}}SD - S < 0. \tag{19}$$

Corollary 6. *Subject to Assumption* 1, *the retarded system* (Σ_{ro})

$$(\Sigma_{ro}):\quad \dot{x}(t) = Ax(t) + A_{\mathrm{d}}x(t-\tau), \tag{24}$$

$$x(t_o+\eta) = \phi(\eta), \quad \forall \eta \in [-\tau, 0] \tag{25}$$

is asymptotically stable independent of delay if one of the following conditions hold:

(1) *There exist matrices* $0 < P = P^{\mathrm{t}} \in \Re^{n\times n}$ *and* $0 < S = S^{\mathrm{t}} \in \Re^{n\times n}$ *satisfying the ARI*

$$PA + A^{\mathrm{t}}P + PA_{\mathrm{d}}S^{-1}A_{\mathrm{d}}^{\mathrm{t}}P + S < 0. \tag{26}$$

(2) *There exist matrices* $0 < Q = Q^t \in \Re^{n\times n}$ *and* $0 < S = S^t \in \Re^{n\times n}$ *satisfying the LMI*

$$\begin{bmatrix} AQ + QA^t + QSQ & A_d \\ A_d^t & -S \end{bmatrix} < 0. \tag{27}$$

4. Robust stabilization

Here, we consider a class of uncertain neutral systems represented by

$$(\Sigma_{\Delta nu}):\quad \dot{x}(t) - D\dot{x}(t-\tau)$$
$$= (A + \Delta A)x(t) + (A_d + \Delta A_d)x(t-\tau) + [B + \Delta B(t)]u(t)$$
$$= A_\Delta x(t) + A_{d\Delta}x(t-\tau) + B_\Delta u(t), \tag{28}$$

$$x(t_o + \eta) = \phi(\eta), \quad \forall \eta \in [-\tau, 0], \tag{29}$$

where $u(t) \in \Re^p$ is the control input, $B \in \Re^{n\times p}$ is a real matrix and $\Delta B(t)$ represents time-varying parameteric uncertainties at the input which is of the form

$$\Delta B(t) = H\Delta(t)E_b \tag{30}$$

and $E_b \in \Re^{\beta\times p}$ is a known constant matrix. The remaining matrices are as in (1)–(3). In this section, we consider the problem of robust stabilization of the uncertain neutral system $(\Sigma_{\Delta nu})$ using a linear memoryless state-feedback $u(t) = K_s x(t)$.

Theorem 2. *System* $(\Sigma_{\Delta nu})$ *is robustly stable via memoryless state-feedback* $u(t) = K_s x(t)$ *if there exist scalars* $\varepsilon > 0, \rho > 0, \mu > 0$, *matrices* $0 < Y = Y^t \in \Re^{n\times n}$, $0 < Z = Z^t \in \Re^{n\times n}$, $0 < R = R^t \in \Re^{n\times n}$, $0 < S = S^t \in \Re^{n\times n}$ *and* $X \in \Re^{m\times n}$ *satisfying the LMIs*:

$$\begin{bmatrix} W(X,Y,S) & YE^t + X^tE_b^t & G(X,Y,Z,S) \\ EY + E_bX & -\rho I & 0 \\ G^t(X,Y,Z,S) & 0 & -J_s \end{bmatrix} < 0,$$

$$(E_d + [E + E_bXZ]D)^t(E_d + [E + E_bXZ]D) - \varepsilon R < 0,$$

$$D^tSD - S < 0, \qquad \begin{bmatrix} Y & I \\ I & Z \end{bmatrix} \geq I, \qquad \begin{bmatrix} \varepsilon & 1 \\ 1 & \mu \end{bmatrix} \geq 1. \tag{31}$$

Moreover, the feedback gain is given by

$$K_s = XZ, \tag{32}$$

where

$$W(X,Y,S) = YA^t + AY + (\varepsilon + \rho)HH^t + YSY + BX + X^tB^t,$$

$$G(X,Y,Z,S) = AD + A_d + BXZD + YSD,$$

$$J_s = R - \mu(E_d + [E + E_bXZ]D)^t(E_d + [E + E_bXZ]D). \tag{33}$$

Proof. System $(\Sigma_{\Delta nu})$ with the memoryless feedback control law $u(t) = K_s x(t)$ becomes

$$(\Sigma_{\Delta nu}):\quad \dot{x}(t) - D\dot{x}(t-\tau) = A_{\Delta c}(t)x(t) + A_{d\Delta}(t)x(t-\tau)$$
$$= [A_c + H\Delta(t)M_c]x(t) + A_{d\Delta}(t)x(t-\tau), \tag{34}$$

where

$$A_c = A + BK_s, \qquad M_c = E + E_bK_s. \tag{35}$$

By Theorem 1, system (34)–(35) is robustly asymptotically stable if:

$$PA_{\Delta c} + A_{\Delta c}^tP + S + (PA_{\Delta c}D + SD + PA_{d\Delta})R^{-1}(PA_{\Delta c}D + SD + PA_{d\Delta})^t < 0 \tag{36}$$

for all admissible uncertainties satisfying (3). Applying Facts 2 and 3, it can be shown that (36) reduces for some $(\varepsilon > 0, \rho > 0)$ to:

$$PA + A^tP + (\varepsilon + \rho)PHH^tP + S + PBK_s + K_s^tB^tP + \rho^{-1}(E + E_bK_s)^t(E + E_bK_s)$$
$$+ \{P[(A + BK_s)D + A_d] + SD\}$$
$$\times \{R - \varepsilon^{-1}(E_d + [E + E_bK_s]D)^t(E_d + [E + E_bK_s]D)\}^{-1}$$
$$\times \{P[(A + BK_s)D + A_d] + SD\}^t < 0. \tag{37}$$

Pre-multiplying and post-multiplying (37) by P^{-1}, letting $Y = P^{-1}$ and using (32), we get

$$AY + YA^t + (\varepsilon + \rho)HH^t + YSY + \rho^{-1}(EY + E_bX)^t(EY + E_bX) + BX + X^tB^t$$
$$+ \{(A + BXY^{-1})D + A_d + YSD\}\{R - \varepsilon^{-1}(E_d + [E + E_bXY^{-1}]D)^t(E_d + [E + E_bXY^{-1}]D)\}$$
$$\times \{(A + BXY^{-1})D + A_d + YSD\}^t < 0. \tag{38}$$

Finally by Fact 1 under the equality constraints $\varepsilon\mu = 1, \nu\rho = 1, YZ = I$, LMIs (31) follow from ARE (38). □

Corollary 7. *System* (Σ_{no}) *is robustly stable via memoryless state feedback* $u(t) = K_s x(t)$ *if there exist matrices* $0 < Y = Y^t \in \Re^{n\times n}, 0 < Z = Z^t \in \Re^{n\times n}$, $0 < R = R^t \in \Re^{n\times n}$,

$0 < S = S^t \in \Re^{n\times n}$ *and* $X \in \Re^{m\times n}$ *satisfying the LMIs*:

$$\begin{bmatrix} YA^t + AY + YSY + BX + X^tB^t & AD + A_d + BXZD + YSD \\ D^tS^tY + D^tZX^tB^t + A_d^t + D^tA^t & -R \end{bmatrix} < 0,$$

$$\begin{bmatrix} Y & I \\ I & Z \end{bmatrix} \geq I, \qquad D^tSD - S < 0, \tag{39}$$

Moreover, the feedback gain is given by:

$$K_s = XZ. \tag{40}$$

Proof. Set $E=0$, $H=0$, $E_b=0$ in (38). $\square$

5. Robust $\mathscr{H}_\infty$ performance

Now, we extend the robust stabilization results developed in the previous section to the case of robust $\mathscr{H}_\infty$ performance problem. First, we consider the following system:

$$(\Sigma_{\Delta nw}): \quad \dot{x}(t) - D\dot{x}(t-\tau) = A_\Delta x(t) + A_{d\Delta}x(t-\tau) + Nw(t), \tag{41}$$

$$z(t) = Cx(t), \tag{42}$$

$$x(t_o + \eta) = \phi(\eta), \quad \forall \eta \in [-\tau, 0], \tag{43}$$

where $z \in \Re^p$ is the controlled output, $N \in \Re^{n\times p}$, $C \in \Re^{p\times n}$ are known real constant matrices and $w(t) \in \mathscr{L}_2[0,\infty)$ is the external input.

Theorem 3. *Subject to Assumptions* 1 *and* 2, *the neutral system* $(\Sigma_{\Delta nw})$ *is robustly asymptotically stable independent of delay with disturbance attenuation* γ *if the following conditions hold*:

(1) *There exist matrices* $0 < P = P^t \in \Re^{n\times n}$, $0 < S = S^t \in \Re^{n\times n}$ *and* $0 < R = R^t \in \Re^{n\times n}$ *and scalars* $\varepsilon > 0$, $\rho > 0$ *satisfying the ARI*:

$$\begin{aligned} &PA + A^tP + P[(\varepsilon+\rho)HH^t + \gamma^{-2}NN^t]P + S \\ &\quad + C^tC + \rho^{-1}EE^t + [P(AD + A_d) + (S + C^tC)D] \\ &\quad \times [R_u - \varepsilon^{-1}(D^tE^tED + E_d^tE_d)]^{-1}[P(AD + A_d) \\ &\quad + (S + C^tC)D]^t < 0. \end{aligned} \tag{44}$$

(2) *There exist matrices* $0 < S = S^t \in \Re^{n\times n}$ *and* $0 < R = R^t \in \Re^{n\times n}$ *such that* $R_u = R + C^tC$ *satisfying the LE*

$$D^t(S + C^tC)D - (S + C^tC) + R_u = 0. \tag{45}$$

Proof. In order to show that system $(\Sigma_{\Delta w})$ is robustly stable with a disturbance attenuation γ, it is required that the associated Hamiltonian $H(x,w,t)$ satisfies (Li & de Souza, 1997):

$$H(x,w,t) = \dot{V}(x_t) + z^t(t)z(t) - \gamma^2 w^t(t)w(t) < 0,$$

where $V(x_t)$ is given by (6). By differentiating (6) along the trajectories of (41)–(42), we get

$$\begin{aligned} H(x,w,t) = {} & [A_\Delta x(t) + A_{d\Delta}x(t-\tau)]^tP[x(t) - Dx(t-\tau)] \\ & + [x(t) - Dx(t-\tau)]^tP[A_\Delta x(t) + A_{d\Delta}x(t-\tau)] \\ & + x^tSx(t) - x^t(t-\tau)Sx(t-\tau) \\ & + x^tC^tCx - \gamma^{-2}w^tw \\ & + w^tN^tP[x(t) - Dx(t-\tau)] \\ & + [x(t) - Dx(t-\tau)]^tPNw. \end{aligned} \tag{46}$$

In terms of $\mathscr{M}$, we manipulate (46) to obtain

$$\begin{aligned} H(x,w,t) = {} & \mathscr{M}^t(x_t)[PA_\Delta + A_\Delta^tP + S + C^tC]\mathscr{M}(x_t) \\ & + \mathscr{M}^t(x_t)[PA_\Delta D + (S + C^tC)D \\ & + PA_{d\Delta}]x(t-\tau) + x^t(t-\tau)[D^tA_\Delta^tP \\ & + D^t(S + C^tC) + A_{d\Delta}^tP]\mathscr{M}(x_t) \\ & + x^t(t-\tau)[D^t(S + C^tC)D - S]x(t-\tau) \\ & + w^tN^tP\mathscr{M}(x_t) + \mathscr{M}^t(x_t)PNw - \gamma^{-2}w^tw. \end{aligned} \tag{47}$$

Using $R_u = R + C^tC$, completing the squares in (47) and arranging terms we obtain:

$$\begin{aligned} H(x,w,t) \leq {} & \mathscr{M}^t(x_t)[PA_\Delta + A_\Delta^tP + S + C^tC \\ & + \gamma^{-2}PNN^tP]\mathscr{M}(x_t) + \mathscr{M}^t(x_t)[PA_\Delta D \\ & + (S + C^tC)D + PA_{d\Delta}]R_u^{-1}[PA_\Delta D \\ & + (S + C^tC)D + PA_{d\Delta}]^t\mathscr{M}(x_t). \end{aligned} \tag{48}$$

For asymptotic stability of system $(\Sigma_{\Delta nw})$, it is sufficient that

$$\begin{aligned} & PA_\Delta + A_\Delta^tP + S + C^tC + \gamma^{-2}PNN^tP \\ & \quad + [PA_\Delta D + (S + C^tC)D + PA_{d\Delta}]R_u^{-1}[PA_\Delta D \\ & \quad + (S + C^tC)D + PA_{d\Delta}] < 0. \end{aligned} \tag{49}$$

Using Facts 2 and 3 in (49), it follows for some $\mu > 0, \sigma > 0$ that

$$\begin{aligned} & PA + A^tP + P[(\varepsilon+\rho)HH^t + \gamma^{-2}NN^t]P + S + C^tC \\ & \quad + \rho^{-1}EE^t + [P(AD + A_d) + (S + C^tC)D] \\ & \quad \times [R_u - \varepsilon^{-1}(D^tE^tED + E_d^tE_d)]^{-1}[P(AD + A_d) \\ & \quad + (S + C^tC)D]^t < 0. \end{aligned} \tag{50}$$

Finally, ARI (50) corresponds to (44) such that S and R satisfy (45). $\square$

Corollary 8. *Subject to Assumptions 1 and 2, the neutral system* $(\Sigma_{\Delta nw})$ *is asymptotically stable independent of delay if there exist matrices* $0 < Q = Q^t \in \Re^{n\times n}$, $0 < S = S^t \in \Re^{n\times n}$ *and scalars* $\varepsilon > 0, \rho > 0, \mu > 0, v > 0$ *satisfying the LMIs*

$$\begin{bmatrix} AQ + QA^t + \varepsilon HH^t + Q(vEE^t + S + C^tC)Q & H & AD + A_d + QSD & N \\ H^t & -vI & 0 & 0 \\ D^tA^t + A_d^t + D^tSQ & 0 & -J_w & 0 \\ N^t & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0,$$
$$D^t(S + C^tC)D - (S + C^tC) < 0,$$
$$\varepsilon[D^t(S + C^tC)D - (S + C^tC)] + [D^tE^tED + E_d^tE_d] < 0,$$
$$\begin{bmatrix} \rho & 1 \\ 1 & v \end{bmatrix} \geq 1, \qquad \begin{bmatrix} \varepsilon & 1 \\ 1 & \mu \end{bmatrix} \geq 1, \tag{51}$$

where

$$J_w = D^t(S + C^tC)D - (S + C^tC) + \mu(D^tE^tED + E_d^tE_d).$$

Proof. By Fact 1 under the equality constraints $\rho v = 1, \varepsilon\mu = 1$, ARI (44) and LE (45) with $Q = P^{-1}$ are equivalent to the LMIs (51). $\square$

We now consider the robust synthesis problem for system $(\Sigma_{\Delta nwu})$:

$$(\Sigma_{\Delta nwu}):\quad \dot{x}(t) - D\dot{x}(t-\tau) = A_\Delta x(t) + A_{d\Delta}x(t-\tau) + B_\Delta u(t) + Nw(t), \tag{52}$$

$$x(t_o + \eta) = \phi(\eta), \quad \forall \eta \in [-\tau, 0]. \tag{53}$$

The following theorem establishes the main result.

Theorem 4. *System* $(\Sigma_{\Delta nwu})$ *is robustly stable with a disturbance attenuation* γ *via memoryless state feedback if there exist scalars* $\varepsilon > 0, \rho > 0, \mu > 0, v > 0$, *matrices* $0 < Y = Y^t \in \Re^{n\times n}$, $0 < Z = Z^t \in \Re^{n\times n}$, $0 < R = R^t \in \Re^{n\times n}$, $0 < S = S^t \in \Re^{n\times n}$ *and* $X \in \Re^{m\times n}$ *satisfying the system of LMIs*:

$$\begin{bmatrix} W_*(X,Y,S) & YE^t + X^tE_b^t & G_*(X,Z,Y,S) \\ EY + E_bX & -\rho I & 0 \\ G_*^t(X,Z,Y,S) & 0 & -J_* \end{bmatrix} < 0,$$

$$D^t(S + C^tC)D - (S + C^tC) < 0,$$

$$(E_d + [E + E_bXZ]D)^t(E_d + [E + E_bXZ]D) - \varepsilon(R + C^tC) < 0,$$

$$\begin{bmatrix} Z & I \\ I & Y \end{bmatrix} \geq I, \qquad \begin{bmatrix} \rho & 1 \\ 1 & v \end{bmatrix} \geq 1, \qquad \begin{bmatrix} \varepsilon & 1 \\ 1 & \mu \end{bmatrix} \geq 1. \tag{54}$$

Moreover, the feedback gain is given by:

$$K_s = XZ, \tag{55}$$

where

$$\begin{aligned} W_*(X,Y,S) &= YA^t + AY + [(\varepsilon + \rho)HH^t + \gamma^{-2}NN^t] \\ &\quad + YSY + BX + X^tB^t \\ &\quad + Y(S + C^tC + vEE^t)Y, \\ G_*(X,Z,Y,S) &= (A + BXZ)D + A_d + Y(S + C^tC)D, \\ J_* &= (R + C^tC) - \mu(E_d + [E + E_bXZ]D)^t \\ &\quad \times (E_d + [E + E_bXZ]D). \end{aligned} \tag{56}$$

Proof. System $(\Sigma_{\Delta nwu})$ subject to the control law $u(t) = K_* x(t)$ has the form:

$$\begin{aligned} \dot{x}(t) - D\dot{x}(t-\tau) &= A_{\Delta *}(t)x(t) + A_{d\Delta}(t)x(t-\tau) + Nw(t) \\ &= [A_* + H\Delta(t)M_*]x(t) + Nw(t), \end{aligned} \tag{57}$$

$$z(t) = Cx(t), \tag{58}$$

where

$$A_* = A + BK_*, \qquad M_* = E + E_bK_*. \tag{59}$$

In terms of $V(x_t)$ in (6), we evaluate the Hamiltonian $H(x,w,t)$ associated with (57)–(59) in the manner of Theorem 3 to yield

$$\begin{aligned} H(x,w,t) &= [A_{\Delta *}x(t) + A_{d\Delta}x(t-\tau)]^tP[x(t) - Dx(t-\tau)] \\ &\quad + [x(t) - Dx(t-\tau)]^tP[A_{\Delta *}x(t) + A_{d\Delta}x(t-\tau)] \\ &\quad + x^tSx(t) - x^t(t-\tau)Sx(t-\tau) + x^tC^tCx \\ &\quad - \gamma^{-2}w^tw + w^tN^tP[x(t) - Dx(t-\tau)] \\ &\quad + [x(t) - Dx(t-\tau)]^tPNw. \end{aligned} \tag{60}$$

In terms of $\mathscr{M}$, we mainpulate (60) to obtain

$$\begin{aligned} H(x,w,t) \leq\; & \mathscr{M}^{t}(x_t)[PA_{\Delta *} + A_{\Delta}^{t}P + S + C^{t}C \\ & + \gamma^{-2}PNN^{t}P]\mathscr{M}(x_t) + \mathscr{M}^{t}(x_t)[PA_{\Delta *}D \\ & + (S + C^{t}C)D + PA_{d\Delta}]R_u^{-1}[PA_{\Delta}D \\ & + (S + C^{t}C)D + PA_{d\Delta}]^{t}\mathscr{M}(x_t), \end{aligned} \tag{61}$$

from which it suffices for asymptotic stability that

$$\begin{aligned} & PA_{\Delta *} + A_{\Delta *}^{t}P + S + C^{t}C + \gamma^{-2}PNN^{t}P \\ & + [PA_{\Delta *}D + (S + C^{t}C)D + PA_{d\Delta}]R_u^{-1}[PA_{\Delta *}D \\ & + (S + C^{t}C)D + PA_{d\Delta}] < 0. \end{aligned} \tag{62}$$

Using Facts 2 and 3 in (62), it follows for some $\mu > 0, \sigma > 0$ that

$$\begin{aligned} & PA + A^{t}P + PBK_{*} + K_{*}^{t}B^{t}P + P[(\varepsilon + \rho)HH^{t} \\ & + \gamma^{-2}NN^{t}]P + S + C^{t}C + \rho^{-1}EE^{t} \\ & + \rho^{-1}(E + E_{b}K_{*})^{t}(E + E_{b}K_{*})[P(A + BK_{*})D + PA_{d} \\ & + (S + C^{t}C)D][R_u - \varepsilon^{-1}(E_{d} + [E + E_{b}K_{*}]D)^{t}(E_{d} \\ & + [E + E_{b}K_{*}]D)]^{-1}[P(A + BK_{*})D + PA_{d} \\ & + (S + C^{t}C)D]^{t} < 0. \end{aligned} \tag{63}$$

Finally, pre-multiplying (63) and post-multiplying by P^{-1}, letting $Y = P^{-1}, Z = Y^{-1}$, enforcing the equality constraints $\rho v = 1, \varepsilon\mu = 1, YZ = I$ and using (55), the LMIs (54) follow. □

6. Conclusions

We have considered the robust stability and $\mathscr{H}_\infty$ control problems for a class of linear neutral systems with norm-bounded uncertainties and unknown constant state delay. We have developed sufficient conditions for robust stability independent of delay. Then, we established sufficient LMI-based conditions for designing a memoryless state-feedback controller which stabilizes the uncertain time-delay system under consideration and guarantees an $\mathscr{H}_\infty$-norm bound constraint on the disturbance attenuation for all admissible uncertainties and unknown state delay.

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