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# Improved delay-dependent stability criteria for T–S fuzzy systems with time-varying delay

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ABSTRACT

This paper is concerned with the robust stability of uncertain T–S fuzzy systems with time-varying delay. A novel Lyapunov–Krasovskii functional is established by employing the idea of combining delay-decomposition with state vector augmentation. Then, by employing some integral inequalities and the reciprocally convex approach, some less conservative delay-dependent stability criteria are obtained. The proposed stability conditions are formulated in the form of linear matrix inequalities (LMIs), which can be solved efficiently with Semi-Definite Programming (SDP) solvers. Finally, four numerical examples are provided to show that the proposed conditions are less conservative than existing ones.



## 1. Introduction

During the past two decades, Takagi–Sugeno (T–S) fuzzy systems[1] have been an active topic due to the fact that it can combine the flexibility of fuzzy logic theory and fruitful linear system theory into a unified framework to approximate complex nonlinear systems [2–4]. On the other hand, time-delays often occur in many dynamic systems such as biological systems, chemical processes,communication networks and so on, which is usually a source of instability and deteriorated performance. Therefore, stability analysis for T–S fuzzy systems with time-delay has received more interest and achieved fruitful results [5–30].

The stability criteria can be classified into two types: one is delay-dependent and the other is delay-independent. The delay-dependent criteria are less conservative than delay-independent ones since they consider the length information of the delay. In [21], the delay-dependent stability problem of T–S fuzzy systems with time-varying delay was addressed. Recently, a free-weighting matrix approach, which was proposed in [31], has been employed to derive some delay-dependent results in stability analysis or stabilization for T–S fuzzy systems in [22–24]. By retaining ignored terms in estimating the upper bound of derivative of Lyapunov–Krasovskii functional and employing an improved free-weighting matrix approach to consider the relationship between the time-varying delay and its upper bound, some less conservative conditions are established in [26]. To further reduce the conservativeness of the derived results, the other efforts are paid to the construction of Lyapunov–Krasovskii functionals. In [28], an augmented Lyapunov–Krasovskii functional approach that introduces a triple integral and some augmented vectors was employed to investigate the stability problem of T–S fuzzy systems with time-varying delay. Nevertheless, there are too many decision variables included in the derived condition, which leads to

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a large amount of computing time. In addition, some delay-partitioning-based approaches were adopted to deal with the stability of T–S fuzzy systems with time-varying delay in [10,29,30]. However, as pointed out in [32], the integral terms including the upper bound of the time-varying delay are decomposed while the integral term including the time-varying delay is not taken fully into account, which inevitably leads to conservativeness. Therefore, there is still plenty of room for improvement.

In this paper, we discuss the stability of T–S fuzzy systems with time-varying delay. A novel Lyapunov–Krasovskii functional is established by employing the idea of combining delay-decomposition with state vector augmentation. Some LMI-based stability criteria are obtained via introducing some fuzzy-weighting matrixes to express the relationship of the T–S fuzzy models and some new bounding techniques to estimate the derivative of Lyapunov–Krasovskii functional. The effectiveness and the improvements of the proposed method are demonstrated by four numerical examples.

*Notations*. Through this paper, $N^T$ and $N^{-1}$ stands for the transpose and the inverse of the matrix $N$, respectively; $I$ is the identity matrix of appropriate dimensions; $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space with vector norm $\|\cdot\|$; $P>0$ ($P\geqslant 0$) means that $P$ is symmetric and positive definite (semi-positive definite) matrix; diag$\{\cdots\}$ denotes a block-diagonal matrix; the symmetric terms in a symmetric matrix are denoted by $*$, e.g., $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$.

## 2. System description

Consider an uncertain T–S fuzzy system with time-varying delay, which is represented by a T–S fuzzy model composed of a set of fuzzy implications. The $i$th rule of the system is of the following IF-THEN form:

Rule $i$:

If $\theta_1(t)$ is $W_1^i$ and $\cdots$ and $\theta_p(t)$ is $W_p^i$ then

$$\begin{cases} \dot{x}(t) = (A_i + \Delta A_i(t))x(t) + (A_{di} + \Delta A_{di}(t))x(t-d(t)), \\ x(t) = \phi(t) \quad t \in [-h, 0] \quad i = 1, 2, \ldots, r, \end{cases} \tag{1}$$

where $\theta_1(t), \theta_2(t), \ldots, \theta_p(t)$ are the premise variables; $W_j^i$ ($j = 1, 2, \ldots, p$; $i = 1, 2, \ldots, r$) is the fuzzy set; the scalar $r$ is the number of IF-Then rules; $x(t) \in \mathbb{R}^n$ is the state vector; $A_i$ and $A_{di}$ are constant real matrices with appropriate dimensions; the delay, $d(t)$, is a time-varying functional satisfying

$$0 \leqslant d(t) \leqslant h, \tag{2}$$

$$\dot{d}(t) \leqslant \mu, \tag{3}$$

where $\mu$ and $h$ are constants; the matrices $\Delta A_i(t)$ and $\Delta A_{di}(t)$ denote the uncertainties in the system and are defined as

$$[\Delta A_i(t)\ \Delta A_{di}(t)] = DF(t)[E_i\ E_{di}], \tag{4}$$

where $D$, $E_i$ and $E_{di}$ are known constant matrices and $F(t)$ is an unknown matrix function satisfying

$$F^T(t)F(t) \leqslant I, \quad \forall t. \tag{5}$$

By using a center-average defuzzifier, product inference and singleton fuzzifier, the overall fuzzy model is inferred as follows:

$$\begin{cases} \dot{x}(t) & = \frac{\sum_{i=1}^r w_i(\theta(t))[(A_i + \Delta A_i(t))x(t) + (A_{di} + \Delta A_{di}(t))x(t-d(t))]}{\sum_{i=1}^r w_i(\theta(t))} \\ & = \sum_{i=1}^r \rho_i(\theta(t))[(A_i + \Delta A_i(t))x(t) + (A_{di} + \Delta A_{di}(t))x(t-d(t))] \\ & = \hat{A}_i x(t) + \hat{A}_d x(t-d(t)), \\ x(t) & = \phi(t), \quad t \in [-h, 0], \end{cases} \tag{6}$$

where $\theta = [\theta_1, \theta_2, \ldots, \theta_p]$; $w_i : \mathbb{R}^p \to [01]$, $i = 1, \ldots, r$, is the membership function of the system with respect to the plant rule $i$; $\rho_i(\theta(t)) = w_i(\theta(t))/\sum_{i=1}^r w_i(\theta(t))$; and $\hat{A}_i = \sum_{i=1}^r \rho_i(\theta(t))(A_i + \Delta A_i(t))$, $\hat{A}_d = \sum_{i=1}^r \rho_i(\theta(t))(A_{di} + \Delta A_{di}(t))$. It is obvious that the fuzzy weighting functions $\rho_i(\theta(t))$ satisfy $\rho_i(\theta(t)) \geqslant 0$, $\sum_{i=1}^r \rho_i(\theta(t)) = 1$.

Next, we introduce the following lemmas, which are indispensable to derive our main results.

**Lemma 1** [33]. *Let $M = M^T > 0$ be a constant real $n \times n$ matrix, and suppose $\dot{x} : [-\tau, 0] \mapsto \mathbb{R}^n$ with $\tau > 0$ such that the subsequent integration is well defined, then*

$$-\tau \int_{t-\tau}^{t} \dot{x}^T(s) M \dot{x}(s) ds \leqslant \zeta^T(t) \begin{bmatrix} -M & M \\ * & -M \end{bmatrix} \zeta(t),$$

*where $\zeta(t) = [x^T(t), x^T(t-\tau)]^T$.*

**Lemma 2** [34]. *For any matrix* $\begin{bmatrix} M & S \\ * & M \end{bmatrix} \geqslant 0$, *scalars* $\tau > 0$, $\tau(t) > 0$ *satisfying* $0 < \tau(t) < \tau$, *vector function* $\dot{x} : [-\tau, 0] \to \mathbb{R}^n$ *such that the concerned integrations are well defined, then*

$$-\tau \int_{t-\tau}^{t} \dot{x}^T(\alpha) M \dot{x}(\alpha) d\alpha \leqslant \varpi^T(t) \Omega \varpi(t),$$

*where*

$$\varpi(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) & x^T(t-\tau) \end{bmatrix}^T,$$

$$\Omega = \begin{bmatrix} -M & M-S & S \\ * & -2M+S+S^T & -S+M \\ * & * & -M \end{bmatrix}.$$

**Lemma 3** [36]. *For any positive matrix Z, and for differentiable signal x in* $[\alpha, \beta] \to \mathbb{R}^n$ , *the following inequality holds:*

$$\int_{\alpha}^{\beta} \dot{x}^T(u) Z \dot{x}(u) du \geqslant \frac{1}{\beta-\alpha} \begin{bmatrix} x(\beta) \\ x(\alpha) \\ \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} x(u) du \end{bmatrix}^T \hat{\Omega} \begin{bmatrix} x(\beta) \\ x(\alpha) \\ \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} x(u) du \end{bmatrix}, \tag{7}$$

*where*

$$\hat{\Omega} = \begin{bmatrix} 4Z & 2Z & -6Z \\ * & 4Z & -6Z \\ * & * & 12Z \end{bmatrix}$$

**Lemma 4** [35]. *Let* $Q = Q^T, H, E$ *and* $F(t)$ *satisfying* $F^T(t)F(t) \leqslant I$ *are appropriately dimensional matrices, then the following inequality*

$$Q + HF(t)E + E^T F^T(t) H^T < 0$$

*is true, if and only if the following inequality holds for any* $\varepsilon > 0$,

$$Q + \varepsilon^{-1} H H^T + \varepsilon E^T E < 0.$$

## 3. Main results

In this section, we shall obtain the stability criteria for T–S fuzzy systems with time-varying delay. First, the following nominal system will be addressed:

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) + \bar{A}_d x(t-d(t)), \\ x(t) = \phi(t), \quad t \in [-h, 0], \end{cases} \tag{8}$$

where $\bar{A} = \sum_{i=1}^{r} \rho_i(\theta(t)) A_i$, $\bar{A}_d = \sum_{i=1}^{r} \rho_i(\theta(t)) A_{di}$.

Based on the Lyapunov–Krasovskii stability theorem, the following result is obtained.

**Theorem 1.** *Given a integer m, scalars* $\delta = \frac{h}{m} > 0$ *and* $\mu$, *the system* (8) *with a time-delay* $d(t)$ *satisfying* (2) *and* (3) *is stable if there exist matrices* $P_a = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} > 0$, $Q_j > 0$, $Z_0 > 0$, $R_l = \begin{bmatrix} R_{1l} & R_{2l} \\ * & R_{3l} \end{bmatrix} > 0$, $Z_j > 0$, *and any matrices* $G_{ij}(i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, m;\ l = 1, 2, \ldots, m-1)$ *such that the LMIs* (9) *and* (10) *are feasible for* $i = 1, 2, \ldots, r$ *and* $k = 1, 2, \ldots, m$

$$\hat{\Pi}_{ik} = \begin{bmatrix} \Phi_i + \Psi_{ik} + \Lambda_k & \delta \Gamma_i^T \bar{Z} \\ * & -\bar{Z} \end{bmatrix} < 0, \tag{9}$$

$$\bar{\Pi}_{ik} = \begin{bmatrix} Z_k & G_{ik} \\ * & Z_k \end{bmatrix} > 0, \tag{10}$$

*where*

$$\Phi_i = \begin{bmatrix} 0 & A_{di}^T P_1 & 0 & \cdots & 0 & 0 & \gamma_1 \\ * & \varphi_1 & \beta_1 & \cdots & 0 & 0 & \gamma_2 \\ * & * & \varphi_2 & \cdots & 0 & 0 & \gamma_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \varphi_m & \beta_m & \gamma_{m+1} \\ * & * & * & \cdots & * & \varphi_{m+1} & 0 \\ * & * & * & \cdots & * & * & -12Z_0 \end{bmatrix},$$

$$\Psi_{ik} = (\psi_{lj})_{(m+3)\times(m+3)} + (\psi_{lj})^T_{(m+3)\times(m+3)}$$

$$\bar{Z} = \sum_{j=0}^{m} Z_j,$$

$$\Gamma_i = [A_{di}\ A_i\ 0_{n\times(m+1)n}],$$

$$\Lambda_k = \mathrm{diag}\{\Lambda_{k1}, \Lambda_{k2}, \ldots, \Lambda_{k(m+3)}\},$$

*with*

$$\psi_{lj} = \begin{cases} -Z_k + G_{ik}, & l = j = 1, \\ Z_k - G_{ik}^T, & l = 1,\ j = k+1, \\ Z_k - G_{ik}, & l = 1,\ j = k+2, \\ -Z_k + G_{ik}, & l = k+1,\ j = k+2, \\ 0, & \text{otherwise}, \end{cases}$$

$$\varphi_j = \begin{cases} PA_i + A_i^T P_1 + R_{11} - Z_1 - 4Z_0 + P_2 + P_2^T, & j = 1, \\ R_{31} + R_{12} - R_{11} - Z_1 - Z_2 - 4Z_0, & j = 2, \\ R_{3(j-2)} + R_{1(j-1)} - R_{3(j-3)} - R_{1(j-2)} - Z_j - Z_{j-1}, & j = 3, \ldots, m-1, \\ R_{3(m-1)} - R_{3(m-2)} - R_{1(m-1)} - Z_m - Z_{m-1}, & j = m, \\ -R_{3(m-1)} - Z_m, & j = m+1, \end{cases}$$

$$\beta_j = \begin{cases} R_{21} + Z_1 - 2Z_0 - P_2, & j = 1, \\ R_{2j} - R_{2(j-1)} + Z_j, & j = 2, 3, \ldots, m-1, \\ -R_{2(m-1)} + Z_m, & j = m, \end{cases}$$

$$\gamma_j = \begin{cases} \delta A_{di}^T P_2, & j = 1, \\ 6Z_0 + \delta A_i^T P_2 + \delta P_3^T, & j = 2, \\ 6Z_0 - \delta P_3^T, & j = 3, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Lambda_{kj} = \begin{cases} -(1-\mu)Q_k, & j = 1, \\ Q_1, & j = 2, \\ Q_{j-1} - Q_{j-2}, & 3 \leqslant j \leqslant k+1, \\ 0, & \text{otherwise}. \end{cases}$$

**Proof.** Decompose the delay interval $[0, h]$ into $m$ uniform subintervals, i.e., $[0, h] = \bigcup_{j=1}^{m} [(j-1)\delta, j\delta]$ with $\delta = h/m$, where $m$ is a given integer, $m \geqslant 2$. Thus, for any $t \geqslant 0$, there should exist an integer $k \in \{1, 2, \ldots, m\}$, such that $d(t) \in [(k-1)\delta, k\delta]$. Choose the following Lyapunov–Krasovskii functional candidate:

$$V(x_t)|_{d(t)\in[(k-1)\delta,k\delta]} = V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(x_t), \tag{11}$$

where

$$V_1(x_t) = \eta^T(t) P_a \eta(t),$$

$$V_2(x_t) = \sum_{j=1}^{k-1}\int_{t-j\delta}^{t-(j-1)\delta} x^T(s)Q_jx(s)ds + \int_{t-d(t)}^{t-(k-1)\delta} x^T(s)Q_kx(s)ds,$$

$$V_3(x_t) = \sum_{j=1}^{m-1}\int_{t-j\delta}^{t-(j-1)\delta} \eta_2^T(s)R_j\eta_2(s)ds,$$

$$V_4(x_t) = \sum_{j=1}^{m}\delta\int_{-j\delta}^{-(j-1)\delta}\int_{t+\theta}^{t} \dot{x}^T(s)Z_j\dot{x}(s)ds\,d\theta + \delta\int_{-\delta}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z_0\dot{x}(s)ds\,d\theta$$

and $\eta_1(t) = \left[x^T(t) \quad \int_{t-\delta}^{t} x^T(s)ds\right]^T$, $\eta_2(s) = [x^T(s)\ x^T(s-\delta)]^T$; $P_a = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} > 0$, $R_l = \begin{bmatrix} R_{1l} & R_{2l} \\ * & R_{3l} \end{bmatrix} > 0$, $Z_0 > 0$, $Z_j > 0$ and $Q_j > 0$ $(l = 1,2,\ldots,m-1;\ j = 1,2,\ldots,m)$ are to be determined.

In addition, we define

$$\dot{V}(x_t) = \limsup_{\Delta t\to 0^+}\frac{1}{\Delta t}[V(x_{t+\Delta t}) - V(x_t)].$$

The function $\dot{V}(x_t)$ is the upper right-hand derivative of $\dot{V}(x_t)$ with respect to $t$ along the trajectory of system (8). Consequently, we have

$$\dot{V}_1(x_t) = 2\eta_1^T(t)P_a\dot{\eta}_1(t),$$

$$\dot{V}_2(x_t) \leqslant x^T(t)Q_1x(t) + \sum_{j=1}^{k-1}x^T(t-j\delta)(Q_{j+1}-Q_j)x(t-j\delta) - (1-\mu)x^T(t-d(t))Q_kx(t-d(t)),$$

$$\dot{V}_3(x_t) = \sum_{j=1}^{m-1}[\eta_2^T(t-(j-1)\delta)R_j\eta_2(t-(j-1)\delta) - \eta_2^T(t-j\delta)R_j\eta_2(t-j\delta)],$$

$$\dot{V}_4(x_t) = \delta^2\dot{x}^T(t)\sum_{j=0}^{m}Z_j\dot{x}(t) - \delta\sum_{j=1}^{m}\int_{t-j\delta}^{t-(j-1)\delta}\dot{x}^T(s)Z_j\dot{x}(s)ds - \delta\int_{t-\delta}^{t}\dot{x}^T(s)Z_0\dot{x}(s)ds.$$

By applying Lemmas 1 and 2, it can be deduced for $\begin{bmatrix} Z_k & \bar{G}_k \\ * & Z_k \end{bmatrix} > 0$, that

$$\begin{aligned} -\delta\sum_{j=1}^{m}\int_{t-j\delta}^{t-(j-1)\delta}\dot{x}^T(s)Z_j\dot{x}(s)ds &\leqslant \sum_{j=1}^{m}\begin{bmatrix} x(t-(j-1)\delta) \\ x(t-j\delta)\end{bmatrix}^T\begin{bmatrix} -Z_j & Z_j \\ * & -Z_j\end{bmatrix}\begin{bmatrix} x(t-(j-1)\delta) \\ x(t-j\delta)\end{bmatrix} \\ &+ \xi_1^T(t)\begin{bmatrix} -2Z_k+\bar{G}_k+\bar{G}_k^T & Z_k-\bar{G}_k^T & Z_k-\bar{G}_k \\ * & 0 & G_k - Z_k \\ * & * & 0\end{bmatrix}\xi_1(t), \end{aligned} \tag{12}$$

where $\bar{G}_k = \sum_{i=1}^{r}\rho_i(\theta(t))G_{ik}$, $\xi_1^T(t) = [x^T(t-d(t))\ x^T(t-(k-1)\delta)\ x^T(t-k\delta)]$.

It follows from Lemma 3 that

$$-\delta\int_{t-\delta}^{t}\dot{x}^T(s)Z_0\dot{x}(s)ds \leqslant \xi_2^T(t)\begin{bmatrix} -4Z_0 & -2Z_0 & 6Z_0 \\ * & -4Z_0 & 6Z_0 \\ * & * & -12Z_0\end{bmatrix}\xi_2(t) \tag{13}$$

where $\xi_2^T(t) = [x^T(t)\ x^T(t-\delta)\,\frac{1}{\delta}\int_{t-\delta}^{t}x^T(s)ds]^T$.

Applying (12) and (13) yields

$$\dot{V}(x_t) \leqslant \zeta^T(t)[\bar{\Phi} + \bar{\Psi}_k + \Lambda_k + \delta^2\bar{\Gamma}^T\bar{Z}\bar{\Gamma}]\zeta(t), \tag{14}$$

where

$$\zeta(t) = \left[x^T(t-d(t)) \quad x^T(t) \quad x^T(t-\delta) \quad x^T(t-2\delta) \quad \cdots \quad x^T(t-m\delta) \quad \frac{1}{\delta}\int_{t-\delta}^{t}x^T(s)ds\right]^T,$$

$$\bar{\Phi} = \begin{bmatrix} 0 & \bar{A}_d^T P_1 & 0 & \cdots & 0 & 0 & \bar{\gamma}_1 \\ * & \bar{\varphi}_1 & \beta_1 & \cdots & 0 & 0 & \bar{\gamma}_2 \\ * & * & \varphi_2 & \cdots & 0 & 0 & \gamma_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \varphi_m & \beta_m & \gamma_{m+1} \\ * & * & * & \cdots & * & \varphi_{m+1} & 0 \\ * & * & * & \cdots & * & * & -12Z_0 \end{bmatrix},$$

$$\bar{\Psi}_{ik} = (\bar{\psi}_{lj})_{(m+3)\times(m+3)} + (\bar{\psi}_{lj})^T_{(m+3)\times(m+3)},$$

$$\bar{\Gamma} = [\bar{A}_d \ \bar{A} \ 0_{n\times(m+1)n}]$$

with

$$\bar{\psi}_{lj} = \begin{cases} -Z_k + \bar{G}_k, & l = j = 1, \\ Z_k - \bar{G}_k^T, & l = 1, \ j = k+1, \\ Z_k - \bar{G}_k, & l = 1, \ j = k+2, \\ -Z_k + \bar{G}_k, & l = k+1, \ j = k+2, \\ 0, & \text{otherwise}, \end{cases}$$

$$\bar{\varphi}_1 = P\bar{A} + \bar{A}^T P + R_{11} - Z_1 - 4Z_0,$$

$$\bar{\gamma}_1 = \delta \bar{A}_d^T P_2,$$

$$\bar{\gamma}_2 = 6Z_0 + \delta \bar{A}^T P_2 + \delta P_3^T.$$

Thus, for $k = 1, 2, \ldots, m$, if $\bar{\Phi} + \bar{\Psi}_k + \Lambda_k + \delta^2 \bar{\Gamma}^T \bar{Z} \bar{\Gamma} < 0$, then $\dot{V}(x(t)) < -\varepsilon \|x(t)\|^2$ for a sufficiently small $\varepsilon > 0$. By Schur complement, $\bar{\Phi} + \bar{\Psi}_k + \Lambda_k + \delta^2 \bar{\Gamma}^T \bar{Z} \bar{\Gamma} < 0$ is equivalent to the following inequality

$$\begin{bmatrix} \bar{\Phi} + \bar{\Psi}_k + \Lambda_k & \delta \bar{\Gamma}^T \bar{Z} \\ * & -\bar{Z} \end{bmatrix} < 0. \tag{15}$$

Furthermore, (9) and (10) imply $\sum_{i=1}^r \rho_i(\theta(t)) \hat{\Pi}_{ik} < 0$ and $\sum_{i=1}^r \rho_i(\theta(t)) \bar{\Pi}_{ik} > 0$, which is equivalent to (15) and $\begin{bmatrix} Z_k & \bar{G}_k \\ * & Z_k \end{bmatrix} > 0$, respectively. Therefore, if LMIs (9) and (10) are feasible, system (8) is asymptotically stable. This completes the proof. □

**Remark 1.** The Lyapunov–Krasovskii functional (11) is different from those in [23–29]. The integral term $\int_{t-\delta}^t x^T(s) ds$ have been taken into account. In addition, a new term $V_2(x_t)$ that is continuous at $d(t) = k\delta$ and a surplus term $\delta \int_{-\delta}^0 \int_{t+\theta}^t \dot{x}^T(s) Z_0 \dot{x}(s) ds\, d\theta$ are included.

**Remark 2.** Enlightened by Seuret and Gouaisbaut [36], a new integral inequality that is shown less conservative than previous inequalities often based on Jensen's theorem, is adopted to estimate the integral term $-\delta \int_{t-\delta}^t \dot{x}^T(s) Z_0 \dot{x}(s) ds$, which will be helpful to reduce the conservativeness of the derived conditions.

**Remark 3.** In the proof of Theorem 1, some fuzzy-weighting matrices, $\bar{G}_k = \sum_{i=1}^r \rho_i(\theta(t)) G_{ik}$ $(k = 1, 2, \ldots, m)$, is introduced to consider the relationship of the T–S fuzzy model, which will lead to less conservative results. In addition, the relationship between the time-varying delay and its varying-interval has been taken into account by employing a reciprocally convex approach.

**Remark 4.** The previous works such as [23–29], are focused on the derivative of the time-varying delay $\dot{d}(t)$ satisfying (3). As for the case of $\dot{d}(t)$ satisfying

$$\dot{d}(t) \leqslant \mu_k, \quad d(t) \in [(k-1)\delta, k\delta], \quad k = 1, 2, \ldots, m \tag{16}$$

the treatment in [23–26,28,29] means that $\dot{\tau}(t)$ in (16) is enlarged to $\dot{\tau}(t) \leqslant \mu = max\{\mu_1, \mu_2, \ldots, \mu_m\}$, while the derivative information of the time-varying delay can be taken fully into account by replacing $\mu$ with $\mu_k$ in Theorem 1.

For uncertain T–S fuzzy system (1), replacing $A_i$ and $A_{di}$ with $A_i + DF(t)E_i$ and $A_{di} + DF(t)E_{di}$ in (9), the following result can be easily derived by applying Lemma 4 and Schur complement [37].

**Theorem 2.** *Given a integer m, scalars* $\delta = \frac{h}{m} > 0$ *and* $\mu$, *the system (6) with a time-delay* $d(t)$ *satisfying (2) and (3) is robustly stable if there exist matrices* $P_a = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} > 0$, $Q_j > 0$, $R_l = \begin{bmatrix} R_{1l} & R_{2l} \\ * & R_{3l} \end{bmatrix} > 0$, $Z_0 > 0$, $Z_j > 0$, *and any matrices* $G_{ij}$ $(i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, m;\ l = 1, 2, \ldots, m-1)$ *and scalars* $\lambda_{ik} > 0$, *such that the LMIs (10) and (17) are feasible for* $i = 1, 2, \ldots, r$ *and* $k = 1, 2, \ldots, m$

$$\begin{bmatrix} \Phi_i + \Psi_{ik} + \Lambda_{ik} & \delta\Gamma_i^T\bar{Z} & \bar{P}D & \lambda_{ik}\bar{E} \\ * & -\bar{Z} & \delta\bar{Z}D & 0 \\ * & * & -\lambda_{ik}I & 0 \\ * & * & * & -\lambda_{ik}I \end{bmatrix} < 0, \tag{17}$$

*where*

$$\bar{P} = [0 \quad P_1^T \quad 0_{n\times mn} \quad P_2^T]^T,$$

$$\bar{E} = [E_i \quad E_{di} \quad 0_{n\times(m+1)n}]^T$$

*and* $\Phi_i$, $\Psi_{ik}$, $\Lambda_{ik}$, $\Gamma_i$ *and* $\bar{Z}$ *are defined in Theorem 1.*

Finally, in the case of the time-varying delay $d(t)$ being non-differentiable or unknown $\dot{d}(t)$, setting $Q_j = 0$ $(j = 1, 2, \ldots, m)$ in Theorem 2, we have the following result.

**Corollary 1.** *Given a integer m, scalars* $\delta = \frac{h}{m} > 0$, *the system (6) with a time-delay* $d(t)$ *satisfying (2) is robustly stable if there exist matrices* $P_a = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} > 0$, $R_l = \begin{bmatrix} R_{1l} & R_{2l} \\ * & R_{3l} \end{bmatrix} > 0$, $Z_0 > 0$, $Z_j > 0$, *and any matrices* $G_{ij}$ $(i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, m;\ l = 1, 2, \ldots, m-1)$ *and scalars* $\lambda_{ik} > 0$, *such that the LMIs (10) and (18) are feasible for* $i = 1, 2, \ldots, r$ *and* $k = 1, 2, \ldots, m$

$$\begin{bmatrix} \Phi_i + \Psi_{ik} & \delta\Gamma_i^T\bar{Z} & \bar{P}D & \lambda_{ik}\bar{E} \\ * & -\bar{Z} & \delta\bar{Z}D & 0 \\ * & * & -\lambda_{ik}I & 0 \\ * & * & * & -\lambda_{ik}I \end{bmatrix} < 0, \tag{18}$$

*where and* $\Phi_i$, $\Psi_{ik}$, $\Gamma_i$ *and* $\bar{Z}$ *are defined in Theorem 1,* $\bar{P}$ *and* $\bar{E}$ *are defined in Theorem 2.*

**Remark 5.** Similarly, in the case of time-varying delay $d(t)$ being non-differentiable or unknown $\dot{d}(t)$ in a subinterval, i.e. $d(t) \in [(k-1)\delta, k\delta]$, $k \in \{1, 2, \ldots, m\}$, setting corresponding matrix $Q_k = 0$ in Theorem 1, the corresponding criterion can be readily obtained. For brevity, it is omitted here.

## 4. Numerical examples

In this section, we provide four numerical examples to verify the effectiveness of the proposed method.

**Example 1.** Consider a nominal T–S fuzzy system of the following form:

$$\dot{x}(t) = \sum_{i=1}^{2} \rho_i(A_i x(t) + A_{di}x(t - d(t))),$$

where

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}$$

and the membership functions for rules 1 and 2 are

$$\rho_1(\theta(t)) = \frac{1}{1 + exp(-2\theta(t))}, \quad \rho_2(\theta(t)) = 1 - \rho_1(\theta(t)).$$

This example has been widely discussed in previous works such as [23–28]. For different $\mu$, the upper bounds of the time-varying delay computed by the Theorem 1 with $m = 2$ and $m = 3$ are listed in Table 1. For comparison, the upper bounds

**Table 1**
Allowable upper bound of $h$ for different $\mu$: Example 1.

| $\mu$ | 0 | 0.1 | $\geqslant 1$ |
|---|---|---|---|
| [23] | 1.597 | – | 0.721 |
| [24] | 1.5973 | 1.484 | 0.831 |
| [25] | 1.5974 | 1.4847 | 0.982 |
| [26] | 1.5974 | 1.4957 | 1.2642 |
| [27] | 1.8034 | – | 0.9899 |
| [28] | 1.6609 | 1.5332 | 1.2696 |
| *The proposed* ($m = 2$) | 1.9676 | 1.7871 | 1.3446 |
| *The proposed* ($m = 3$) | 2.0002 | 1.8090 | 1.3631 |

obtained by the conditions in [23–28] are also tabulated in Table 1, where "–" denotes that the results are not provided in these papers. It is clear that the method proposed in this paper is less conservative than those in [23–28]. It is also concluded that the conservatism is gradually reduced with the increase of $m$.

**Example 2.** Consider a nominal T–S fuzzy system of the following form:

$$\dot{x}(t) = \sum_{i=1}^{2} \rho_i (A_i x(t) + A_{di} x(t - d(t))),$$

where $\rho_1(\theta(t)) = 1/[1 + exp(-2\theta(t))]$, $\rho_2(\theta(t)) = 1 - \rho_1(\theta(t))$,

$$A_1 = \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix}.$$

For different $\mu$, The upper bounds of the time-varying delay computed by the method proposed in this paper and those in [26,29] are listed in Table 2. It can be seen that the method proposed in this paper is less conservative than those in [26,29]. Especially, it is worth to mention that the proposed result has great improvements over those in [29], where a delay-partitioning approach is also involved.

**Example 3.** Consider the following T–S fuzzy model with the same membership functions as Example 1.

$$\dot{x}(t) = \sum_{i=1}^{2} \rho_i (A_i x(t) + A_{di} x(t - d(t))),$$

where

$$A_1 = \begin{bmatrix} -3.2 & 0.6 \\ 0 & -2.1 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 1 & 0.9 \\ 0 & 2 \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6 \end{bmatrix}.$$

The upper bounds of the time-varying delay obtained by Theorem 1 and those in [15,16] are listed in Table 3. It can be concluded that, for different $\mu$, Theorem 1 gives larger delay bound than those reported in [15,16].

**Example 4.** Consider the following uncertain fuzzy system

**Table 2**
Allowable upper bound of $h$ for different $\mu$: Example 2.

| $\mu$ | 0 | 0.1 | 0.5 | Unknown |
|---|---|---|---|---|
| [26] | 3.29 | 2.71 | 1.75 | 1.44 |
| [29] | 3.70 | 3.01 | 1.65 | 1.19 |
| *The proposed* ($m = 2$) | 4.29 | 3.35 | 1.93 | 1.71 |
| *The proposed* ($m = 3$) | 4.37 | 3.41 | 1.95 | 1.77 |

**Table 3**
Allowable upper bound of $h$ for different $\mu$: Example 3.

| $\mu$ | 0.03 | 0.1 | 0.5 | 0.9 |
|---|---|---|---|---|
| [15] | 0.5432 | 0.4809 | 0.4752 | 0.4455 |
| [16] | 0.5456 | 0.5030 | 0.4995 | 0.4988 |
| *The proposed* ($m = 2$) | 0.8330 | 0.7278 | 0.7180 | 0.7112 |
| *The proposed* ($m = 3$) | 0.8771 | 0.7687 | 0.7584 | 0.7524 |

**Table 4**
Allowable upper bound of $h$ for different $\mu$: Example 4.

| $\mu$ | 0 | 0.01 | 0.1 | 0.5 | Unknown |
|---|---|---|---|---|---|
| [21] | 0.950 | 0.944 | 0.892 | 0.637 | – |
| [24] | 1.168 | 1.163 | 1.122 | 0.934 | 0.499 |
| [26] | 1.192 | 1.187 | 1.155 | 1.100 | 1.050 |
| *The proposed* ($m = 2$) | 1.390 | 1.382 | 1.318 | 1.132 | 1.127 |

$$\dot{x}(t) = \sum_{i=1}^{2}\rho_i(\theta(t))[(A_i + \Delta A_i(t)) + (A_{di} + \Delta A_{di}(t))x(t - d(t))]x(t), \tag{19}$$

where

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0.5 & -1 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -1.6 & 0 \\ 0 & -1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1.6 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad E_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1.6 & 0 \\ 0 & -0.05 \end{bmatrix}, \quad E_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.03 & 0 \\ 0 & -0.03 \end{bmatrix}$$

and the membership functions for rules 1 and 2 are

$$\rho_1(\theta(t)) = \left(1 - \frac{1}{1 + exp(-3(\theta(t) - 0.5\pi))}\right)\left(\frac{1}{1 + exp(-3(\theta(t) - 0.5\pi))}\right)$$

$$\rho_2(\theta(t)) = 1 - \rho_1(\theta(t)).$$

For various $\mu$, by utilizing Theorem 2 and Corollary 1 and the conditions in [21,24,26], the computed upper bounds that guarantee the robust stability of the considered system are summarized in Table 4. It can be concluded that the result proposed in this paper is better than those in [21,24,26].

## 5. Conclusion

The robust stability has been investigated for uncertain T–S fuzzy systems with time-varying delay. Based on a novel Lyapunov–Krasovskii functional, some improved stability criteria are obtained by employing some new bounding techniques to estimate the derivative of Lyapunov–Krasovskii functional and introducing some fuzzy-weighting matrixes express the relationship of the T–S fuzzy models. Four numerical examples have been given to demonstrate that the proposed result is an improvement over existing ones.

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