Conditions for Consensus of Multi-Agent Systems with Time-Delays and Uncertain Switching Topology

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***Abstract*—This paper proposes a new approach for the analysis of consensus of multi-agent systems subject to time-varying delayed control inputs and switching topology. The main contribution is a condition for consensus for a networked system based on linear matrix inequalities that takes into account the joint effect of time-varying delays and switching network topology. Topology changes are modeled using Markov jumps with uncertain rates of transitions. A practical example is shown to illustrate the main result in various scenarios.**

***Index Terms*—Consensus analysis, multi-agent system, time-varying delay, switching topology, linear dynamics.**

I. Introduction

MULTI-AGENT systems play an important role in the context of networked systems. These systems consist of a set of linked agents, i.e. nodes, which share and act upon information exchanged through a communication network. Applications span coordinated control of unmanned aerial vehicles [1], flight formation [2], [3], cooperative control of autonomous robots [4], and alignment of satellites [5]. An overview of recent applications of multi-agent systems can be found in [6]. A characteristic of this type of networked system is that agents share information only with their neighbors and usually no global information or centralized processing is available. There is a rich and growing literature on establishing conditions for consensus of multi-agent systems. These conditions seek for guarantees that enable all the agents to reach an agreement, which means that some variables must converge to the same state. The goal is to establish a control law, i.e. a consensus protocol, that based on locally exchanged information can produce a distributed control action ensuring that all agents will agree on some quantity of interest, for example, their velocity and/or position. Often, the information exchange is affected by time-delays due to, for example, finite-time information processing, physical limitations in communication channels, time-response of actuators, etc. [7], [8], to which the consensus protocol must be made robust. Furthermore, it is often the case that the communication topology changes over time due to temporary communication losses, agent failure, or changes in the arrangement of the agents that lead to changes in the action of the consensus protocol. The main objective of this paper is to propose a new condition for consensus of multi-agent systems in the presence of time-varying delays and uncertain switching topology.

Regarding the existing literature, some pioneering works that established the study of consensus of multi-agent systems are: [9] where a distributed control law based on the interaction between neighbors was introduced to govern all agents of a particle system into a common direction; [10] was the first to apply concepts of algebraic graph theory to represent the relations between neighbors in consensus problems; and [11], where many particularities of the consensus problem were treated, such as directed and undirected networks, fixed or variable topologies, time-delay effects, etc. In [12], an event-triggered consensus for multi-agent systems is proposed to determine when the agents must update the control inputs and transmit the computed average state informations of their local neighborhoods to their neighbors. The case of potential communication failures has been considered in [13] where a rendezvous algorithm for multiple agents with limited communication was presented. Concerning consensus problems with time-delays in the agent interactions, the most common strategies are: to consider constant delays for all the agent interactions [14]; to consider constant and nonuniform time-delays [15]; or, to consider nonuniform time-varying delays [16]. The case of multi-agent systems subject to switching topology has been addressed in the literature using diverse methodologies. For example, [17] uses Markov processes to formulate a mean square consensus problem for second-order multi-agent system with time-delays and unknown transition probabilities; in [18] communication constraints and linear system dynamics are considered, and in [1] the switching topology is considered non-stochastic. For a survey in multi-agent systems with constraints see [19]. Finally, high-dimensional linear dynamics has been considered in [3], [18], [20], generalizing second or high-order integrators as in [15], [17], [21]. In this scenario, many practical applications can be analyzed as consensus problems, for example, flight formation of quadrotors as presented in [22], [23].

The main contribution of this paper is to present a new

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consensus analysis methodology for multi-agent systems taking into account: (i) agents described by any time-invariant, linear or linearizable model, (ii) time-varying delays that can be nonuniform and non-differentiable in the control input of each agent, representing the information processing and time-response of actuators, and (iii) switching topology with uncertain transition rates, describing failures in the communication channels of a networked system and changes in the neighboring relations. The topology changes, where a topology state may be absent of spanning trees, behave as a continuous time Markov chain with uncertain transition rates, thus the multi-agent system is transformed into a Markov jump linear system (MJLS) and the consensus analysis is carried out by providing sufficient conditions written as Linear Matrix Inequalities (LMIs) that guarantee the stability of the transformed system. In order to deal with the uncertain transition rates, we take inspiration on the ideas presented in [24], [25], [26], and [27].

Throughout the paper the following notation is used: I_n is the identity matrix of order n; 0_n and 1_n are column vectors of zeros and ones, respectively, of dimension n; $\mathbf{0}_{m\times n}$ is a null matrix of order $m \times n$; $M > 0$ (< 0) means that M is a positive (negative) definite matrix; $\otimes$ is the Kronecker product; M^T means the transpose matrix of M; and $*$ denotes the symmetric block in a matrix.

II. Preliminaries from Algebraic Graph Theory

A simple directed graph is denoted by $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, ..., v_m\}$ is the set of m vertices, and $\mathcal{E}$ represents the set of directed edges connecting them, given by $e_{ij} = (v_i, v_j)$, where the first element, v_i, is the parent node and the second element, v_j, is the child node. The adjacency matrix $\mathcal{A} = [a_{ij}]$ associated with a graph is given by:

$$a_{ij} = \begin{cases} 0, & \text{if } i = j \text{ or } \nexists\, e_{ji}, \\ 1, & \text{iff } \exists\, e_{ji}. \end{cases} \tag{1}$$

The degree matrix $\mathcal{D} = [d_{ii}]$ is a diagonal matrix with elements $d_{ii} = \sum_{j=1}^{m} a_{ij}$. Using this definition, we calculate the Laplacian matrix:

$$L = \mathcal{D} - \mathcal{A}, \tag{2}$$

or, $L = [l_{ij}]$ with $l_{ii} = \sum_{j=1}^{m} a_{ij}$ and $l_{ij} = -a_{ij}$, for $i \neq j$. An important property of the Laplacian matrix is the following:

$$L 1_m = 0_m. \tag{3}$$

III. Problem Formulation

A. Consensus Problem

In this paper the multi-agent system is modeled by a graph where each node represents an agent, and the edges represent communication channels. We consider m agents in a directed network, each with dynamic model:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(\theta_t, t - \tau_i(t)), \quad i = 1, 2, \dots, m, \tag{4}$$

where $x_i \in \mathbb{R}^n$ are the state variables of the i-th agent, with n determining the order of the agent dynamics, and $u_i \in \mathbb{R}^n$ is the control input of the i-th agent. The time-varying variable θ_t represents the connected agents of the multi-agent system at each instant of time t and $\tau_i(t)$ is a time-varying delay affecting the control input of the i-th agent. We assume that $\tau_i(t) = \tau + \mu_i(t)$, where τ is a constant nominal delay value and $\mu_i(t)$ are time-varying perturbations satisfying $|\mu_i(t)| \leq \bar{\mu} < \tau$ such that $\tau_i(t) \in [\tau - \bar{\mu}, \tau + \bar{\mu}]$ for all i.

Analysis is based on the following consensus protocol:

$$u_i(\theta_t, t) = -\sum_{j\neq i, j=1}^{m} a_{ij}(\theta_t) K\big(x_i(t) - x_j(t)\big), \tag{5}$$

where $K \in \mathbb{R}^{n\times n}$ is a constant matrix gain, the current topology arrangement is taken into account through $a_{ij}(\theta_t)$, e.g. $a_{ij}(\theta_t) = 0$ if nodes (i, j) do not communicate, and $u_i(\theta_t, t)$ indicates the control input based on the current connected neighbors of agent i. Initial conditions for the agents' states are denoted by:

$$x_i(\iota) = \phi_i(\iota), \quad \forall \iota \in [-\tau - \bar{\mu}, 0], \tag{6}$$

where the functions ϕ_i are arbitrary and correspond to the sets of initial conditions considered over the interval $[-\tau - \bar{\mu}, 0]$.

The dynamics of the parameter θ_t is given by a continuous time Markov chain (see [28]) with discrete states given by a set $\mathcal{S} = 1, 2, ..., s$ (s being the number of the different topologies of the system) and a probability transition matrix $\Psi = [\psi_{pq}]$ defined by:

$$\psi_{pq} = \mathbb{P}\{\theta_{t+\Delta} = q | \theta_t = p\} = \begin{cases} (\pi_{pq} + \epsilon_{pq})\Delta + o(\Delta) & p \neq q, \\ 1 + (\pi_{pp} + \epsilon_{pp})\Delta + o(\Delta) & p = q, \end{cases} \tag{7}$$

in which ψ_{pq} represents the probability of switching from topology p to q in an interval $\Delta > 0$ at given time t, for $p, q \in \mathcal{S}$, and where $\lim_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$, and $(\pi_{pq} + \epsilon_{pq})$ are elements of the uncertain transition rate matrix

$$\Pi = \begin{bmatrix} \pi_{11} + \epsilon_{11} & \pi_{12} + \epsilon_{12} & \cdots & \pi_{1s} + \epsilon_{1s} \\ \pi_{21} + \epsilon_{21} & \pi_{22} + \epsilon_{22} & \cdots & \pi_{2s} + \epsilon_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{s1} + \epsilon_{s1} & \pi_{s2} + \epsilon_{s2} & \cdots & \pi_{ss} + \epsilon_{ss} \end{bmatrix}. \tag{8}$$

The value $\pi_{pq}$is an estimate of the transition rate from state p to state q, and ϵ_{pq} represents the error (uncertainty) of this estimate. We assume that ϵ_{pq} is an unknown constant scalar taking values within a known interval $\epsilon_{pq} \in [-\delta_{pq},\ \delta_{pq}]$, $\delta_{pq} > 0$, $\delta_{pq} < \pi_{pq}$. Note that $\pi_{pq} + \delta_{pq} > 0$, $\pi_{pp} = -\sum_{q=1,q\neq p}^{s} \pi_{pq}$, $\epsilon_{pp} = -\sum_{q=1,q\neq p}^{s} \epsilon_{pq}$, and consequently $\sum_{q=1}^{s}(\pi_{pq} + \epsilon_{pq}) = 0$. Finally, denote $\nu = (\nu_1, \dots, \nu_s)$ the initial distribution of the Markov chain. In order to obtain a compact representation of the multi-agent system, it is rewritten as a unique augmented system comprising all the agents dynamics in (4) and the consensus protocol in (5), as

$$\begin{aligned} \dot{x}(t) &= (I_m \otimes A)x(t) + (I_m \otimes B)\sum_{k=1}^{m} \bar{u}_k(\theta_t, t - \tau_k(t)) \\ \bar{u}_k(\theta_t, t) &= -(L_k(\theta_t) \otimes K)x(t) \end{aligned} \tag{9}$$

where $x(t) = [x_1(t)^T\ x_2(t)^T\ \dots\ x_m(t)^T]^T$ is a complete stacked state vector, $\bar{u}_k(\theta_t, t)$ is the current input vector of the k-th agent arranged in the form $\vartheta_k \otimes u_k(\theta_t, t)$, with ϑ_k a nm-dimensional column vector with the k-th entry being one and the others being zero, and $L_k(\theta_t)$ is the current Laplacian matrix of the subgraph considering only the edges pointing to the vertex v_k. Notice that $\sum_{k=1}^{m} L_k(\theta_t) = L(\theta_t)$, with $L(\theta_t)$ being the Laplacian matrix containing all the edges in the current topology θ_t.

Then, the closed-loop form of (9) is given by the following

$$\dot{x}(t) = \underline{A}x(t) - \sum_{k=1}^{m} \underline{B}_k(\theta_t)x(t-\tau_k(t)), \tag{10}$$

where

$$\underline{A} = I_m \otimes A, \quad \underline{B}_k(\theta_t) = L_k(\theta_t) \otimes (BK). \tag{11}$$

With these definitions we are ready to formalize the definition of consensus:

Definition 1: ([29]) Under stochastic switching topology, the multi-agent system (4) in closed-loop with the consensus protocol (5), represented by system (10), reaches mean square consensus if, for all $i \neq j$, $\lim_{t\to\infty} \mathbb{E}||x_i(t) - x_j(t)||^2 \to 0$ hold in the mean square sense for any initial distribution $\nu = (\nu_1, \ldots, \nu_s)$ of θ_t and initial state conditions.

In the next sections we propose conditions to verify whether the multi-agents system in (10) achieves consensus according to Definition 1.

B. Transformed Multi-Agent System

A tree-type transformation will be used to translate the consensus problem into a stability problem, following [21] and [30]. This is done by introducing new variables representing the disagreement of the state variables:

$$z_i(t) = x_1(t) - x_{i+1}(t), \tag{12}$$

for $i = 1, 2, \ldots, m-1$, or simply

$$z(t) = (U \otimes I_n)x(t), \tag{13}$$

and also

$$x(t) = 1_m \otimes x_1(t) + (W \otimes I_n)z(t), \tag{14}$$

with $z = [z_1 \ z_2 \ \ldots \ z_{m-1}]^T$,

$$U = [I_{m-1} \quad -I_{m-1}], \quad \text{and} \quad W = \begin{bmatrix} 0_{m-1}^T \\ -I_{m-1} \end{bmatrix}. \tag{15}$$

Taking the time derivative of equation (13) and considering the system dynamics in (10), it yields:

$$\dot{z}(t) = (U \otimes I_n)(\underline{A}x(t) - \sum_{k=1}^{m} \underline{B}_k(\theta_t)x(t-\tau_k(t))). \tag{16}$$

Now, replacing $x(t)$ in the previous equation by (14), we have:

$$\begin{aligned}
\dot{z}(t) &= (U \otimes I_n)\underline{A}(1_m \otimes x_1(t)) + (U \otimes I_n)\underline{A}(W \otimes I_n)z(t) \\
&\quad - \sum_{k=1}^{m}(U \otimes I_n)\underline{B}_k(\theta_t)\big(1_m \otimes x_1(t-\tau_k(t))\big) \\
&\quad - \sum_{k=1}^{m}(U \otimes I_n)\underline{B}_k(\theta_t)(I_n \otimes W)z(t-\tau_k(t)) \\
&= (U \otimes I_n)\,(I_m \otimes A)\,(1_m \otimes x_1(t)) \\
&\quad + (U \otimes I_n)\,(I_m \otimes A)\,(W \otimes I_n)z(t) \\
&\quad - \sum_{k=1}^{m}(U \otimes I_n)\,(L_k(\theta_t) \otimes (BK))\,\big(1_m \otimes x_1(t-\tau_k(t))\big) \\
&\quad - \sum_{k=1}^{m}(U \otimes I_n)\,(L_k(\theta_t) \otimes (BK))\,(W \otimes I_n)z(t-\tau_k(t)) \\
&= (UI_m 1_m) \otimes (I_n A x_1(t)) + (UI_m W) \otimes (I_n A I_n)\, z(t) \\
&\quad - \sum_{k=1}^{m} (UL_k(\theta_t)1_m) \otimes (I_n BK x_1(t-\tau_k(t))) \\
&\quad - \sum_{k=1}^{m} (UL_k(\theta_t)W) \otimes (I_n BK I_n)\, z(t-\tau_k(t)).
\end{aligned} \tag{17}$$

Then, using the properties $U1_m = 0_{m-1}$ and $L_k 1_m = 0_m$, which eliminate the terms with $x_1(t)$, and noting that $UW = I_{m-1}$, we obtain the following disagreement system:

$$\dot{z}(t) = \bar{A}z(t) - \sum_{k=1}^{m} \bar{B}_k(\theta_t)z(t-\tau_k(t)), \tag{18}$$

with

$$\bar{A} = I_{m-1} \otimes A, \quad \bar{B}_k(\theta_t) = \bar{L}_k(\theta_t) \otimes (BK), \tag{19}$$

and

$$\bar{L}_k(\theta_t) = UL_k(\theta_t)W \in \mathbb{R}^{(m-1)\times(m-1)}. \tag{20}$$

Based on the disagreement of the state variables, we will assess consensus of system (10) by establishing stability of the disagreement system in (18). In this context, Definition 1 can be rewritten in the following way.

Definition 2: Under stochastic switching topology, the multi-agent system (4) under the consensus protocol (5), represented by system (10), reaches mean square consensus if system (18) is stochastically stable in the mean square sense, i.e.:

$$\lim_{t\to\infty} \mathbb{E}[z^T(t)z(t)] \to 0 \tag{21}$$

holds in mean square sense for any initial distribution of the Markov chain and initial state conditions.

Remark 1: For simplicity in the notation of the stochastic variables, the dependency on the argument θ_t will be denoted by the subscript index ℓ, when no confusion is possible. For example, a matrix $M(\theta_t)$ will be denoted simply by M_ℓ. △

IV. Consensus Analysis

The main result of this paper is stated in the next theorem. This theorem gives new sufficient conditions to verify consensus of a specified multi-agent system with linear dynamics and,

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possibly, switching topology and nonuniform time-varying delays. The only restriction about the topologies is that they switch accordingly as a specified Markov chain switches from one state to another. Regarding the time-delays the only constraint is that they vary within a given domain.

Theorem 1: Consider the multi-agent system in (10) with $\tau > 0$, $\tau \geq \bar{\mu} \geq 0$, and Π defined as in (8), whose $\epsilon_{pq} \in [-\delta_{pq},\ \delta_{pq}]$ with $\delta_{pq} > 0\ \forall\ p,q \in \mathcal{S}$, and multiple time-delays $\tau_k(t) \in [\tau-\bar{\mu}, \tau+\bar{\mu}]$, for $k = 1, 2, ..., m$. Then, the multi-agent system (10) achieves consensus in the mean square sense if there exist $n(m-1) \times n(m-1)$ matrices F_ℓ, G_ℓ, $P_{1\ell} = P_{1\ell}^T$, $P_{2\ell}$, $P_{3\ell} = P_{3\ell}^T$, $Q = Q^T > 0$, $R = R^T > 0$, $S = S^T > 0$, and $Z = Z^T > 0$, such that the following LMIs hold $\forall \ell = 1, 2, ..., s$:

$$\begin{bmatrix} P_{1\ell} & P_{2\ell} \\ * & P_{3\ell} \end{bmatrix} > 0, \tag{22}$$

and

$$\begin{bmatrix} \Phi_\ell & \bar{\mu}\Gamma_\ell \\ * & -\bar{\mu}Z \end{bmatrix} < 0, \tag{23}$$

where

$$\Phi_\ell = \begin{bmatrix} \Phi_\ell^{11} & \Phi_\ell^{12} \\ * & \Phi_\ell^{22} \end{bmatrix}, \tag{24}$$

with

$$\Phi_\ell^{11} = \begin{bmatrix} \phi_\ell^{11} & P_{1\ell} - F_\ell + \bar{A}^T G_\ell^T \\ * & \tau R + \frac{\tau^2}{2} S - G_\ell - G_\ell^T + 2\bar{\mu}Z \end{bmatrix}, \tag{25}$$

$$\phi_\ell^{11} = P_{2\ell} + P_{2\ell}^T + \textstyle\sum_{q=1}^{s} \pi_{\ell q} P_{1q} + \sum_{q=1}^{s} \frac{5\delta_{\ell q}}{4} P_{1q} + Q \\ -\frac{4}{\tau} R - 2S + F_\ell \bar{A} + \bar{A}^T F_\ell^T$$

$$\Phi_\ell^{12} = \begin{bmatrix} -P_{2\ell} - \frac{2}{\tau}R - F_\ell \bar{B}_\ell & \phi_\ell^{14} \\ -G_\ell \bar{B}_\ell & P_{2\ell} \end{bmatrix},$$

$$\phi_\ell^{14} = P_{3\ell} + \textstyle\sum_{q=1}^{s} \pi_{\ell q} P_{2q} + \sum_{q=1}^{s} \frac{5\delta_{\ell q}}{4} P_{2q} + \frac{6}{\tau^2} R + \frac{2}{\tau} S,$$

$$\bar{B}_\ell = \bar{L}_\ell \otimes (BK),$$

$$\Phi_\ell^{22} = \begin{bmatrix} -Q - \frac{4}{\tau} R & -P_{3\ell} + \frac{6}{\tau^2} R \\ * & \phi_\ell^{44} \end{bmatrix},$$

$\phi_\ell^{44} = \sum_{q=1}^{s} \pi_{\ell q} P_{3q} + \sum_{q=1}^{s} \frac{5\delta_{\ell q}}{4} P_{3q} - \frac{12}{\tau^3} R - \frac{2}{\tau^2} S$, and

$$\Gamma_\ell = \begin{bmatrix} F_\ell \bar{B}_\ell \\ G_\ell \bar{B}_\ell \\ 0 \\ 0 \end{bmatrix}. \tag{26}$$

Proof: First, we show that if the proposed LMIs hold, then the inequalities $V(z_t, \ell) > 0$ and $\mathcal{L}V(z_t, \ell) < 0$ are satisfied, where $\mathcal{L}$ is the infinitesimal generator operator (see [31] for details), and $V(z_t, \ell)$ is the following Lyapunov–Krasovskii stochastic functional:

$$V(z_t, \ell) = V_1(z_t, \ell) + V_2(z_t) + V_3(z_t) + V_4(z_t) + V_5(z_t) \tag{27}$$

where z_t corresponds to the state vector $z(\rho)$ values for ρ within the interval $[t-\tau-\bar{\mu}, t]$,

$$V_1(z_t, \ell) = \chi^T(t) P_\ell \chi(t), \tag{28}$$

$$V_2(z_t) = \int_{t-\tau}^{t} z^T(\xi) Q z(\xi) d\xi, \tag{29}$$

$$V_3(z_t) = \int_{-\tau}^{0} \int_{t+\zeta}^{t} \dot{z}^T(\xi) R \dot{z}(\xi) d\xi d\zeta, \tag{30}$$

$$V_4(z_t) = \int_{-\tau}^{0} \int_{\theta}^{0} \int_{t+\zeta}^{t} \dot{z}^T(\xi) S \dot{z}(\xi) d\xi d\zeta d\theta, \tag{31}$$

$$V_5(z_t) = \int_{-\bar{\mu}}^{\bar{\mu}} \int_{t-\tau+\zeta}^{t} \dot{z}^T(\xi) Z \dot{z}(\xi) d\xi d\zeta, \tag{32}$$

with $\chi^T(t) = [z^T(t) \quad \int_{t-\tau}^{t} z^T(\xi) d\xi]$, and there exist some real matrices $P_\ell = \begin{bmatrix} P_{1\ell} & P_{2\ell} \\ * & P_{3\ell} \end{bmatrix} = P_\ell^T$, $Q = Q^T$, $R = R^T$, $S = S^T$, and$Z = Z^T$.

To satisfy the condition $V(z_t, \ell) > 0$, we assume each matrix variable in each term of (27) to be positive definite: $P_\ell > 0$, as in LMI (22); $Q > 0$; $R > 0$; $S > 0$; and $Z > 0$. If these conditions are satisfied, then $V(z_t, \ell) > 0$.

Next, we prove the LMI condition to guarantee that $\mathcal{L}V(z_t, \ell) < 0$. Initially, consider the following null term [32], [33], derived from the system's equation (18):

$$\begin{aligned} 0 &= 2\Lambda_\ell(t) \left[-\dot{z} + \bar{A}z(t) - \sum_{k=1}^{m} \bar{B}_{k\ell} z(t-\tau_k(t)) \right] \\ &= 2\Lambda_\ell(t) \Bigg[-\dot{z} + \bar{A}z(t) \\ &\quad - \sum_{k=1}^{m} \bar{B}_{k\ell} \left(z(t-\tau) - \int_{-\tau_k(t)}^{-\tau} \dot{z}(t+\xi) d\xi \right) \Bigg] \\ &= 2\Lambda_\ell(t) \left[-\dot{z}(t) + \bar{A}z(t) - \sum_{k=1}^{m} \bar{B}_{k\ell} z(t-\tau) \right] + v_\ell(t), \end{aligned} \tag{33}$$

with

$$v_\ell(t) = 2\Lambda_\ell(t) \sum_{k=1}^{m} \bar{B}_{k\ell} \int_{-\tau_k(t)}^{-\tau} \dot{z}(t+\xi) d\xi, \tag{34}$$

$\Lambda_\ell(t) = [z^T(t) F_\ell + \dot{z}^T(t) G_\ell]$, and F_ℓ and G_ℓ with appropriate dimensions.

Then, applying the inequality $2a^T b \leq a^T X a + b^T X^{-1} b$ (see note[1]) in (34), where a and b are vectors and X is a positive definite matrix chosen to be $\frac{Z}{m}$, we have

$$\begin{aligned} v_\ell(t) &\leq \sum_{k=1}^{m} \int_{-\tau_k(t)}^{-\tau} (\Lambda_\ell(t)\bar{B}_{k\ell}) m Z^{-1} (\Lambda_\ell(t)\bar{B}_{k\ell})^T d\xi \\ &\quad + \sum_{k=1}^{m} \int_{-\tau_k(t)}^{-\tau} \dot{z}^T(t+\xi) \frac{Z}{m} \dot{z}(t+\xi) d\xi \\ &\leq \sum_{k=1}^{m} (\Lambda_\ell(t)\bar{B}_{k\ell}) \bar{\mu} m Z^{-1} (\Lambda_\ell(t)\bar{B}_{k\ell})^T \\ &\quad + \int_{t-\tau-\bar{\mu}}^{t-\tau+\bar{\mu}} \dot{z}^T(\xi) Z \dot{z}(\xi) d\xi. \end{aligned} \tag{35}$$

[1]Other kinds of upper bounds for matrix cross-products have been tried, as summarized in [34], with no further improvements.

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Replace (35) in (33) such that

$$\begin{aligned}0 \leq &- 2z^T(t)F_\ell\dot{z}(t) + 2z^T(t)F_\ell\bar{A}z(t)\\&- 2\dot{z}^T(t)G_\ell\dot{z}(t) + 2\dot{z}^T(t)G_\ell\bar{A}z(t)\\&- 2z^T(t)F_\ell\sum_{k=1}^{m}\bar{B}_{k\ell}z(t-\tau)\\&- 2\dot{z}^T(t)G_\ell\sum_{k=1}^{m}\bar{B}_{k\ell}z(t-\tau)\\&+\sum_{k=1}^{m}\bar{\mu}(\Lambda_\ell(t)\bar{B}_{k\ell})mZ^{-1}(\Lambda_\ell(t)\bar{B}_{k\ell})^T\\&+\int_{t-\tau-\bar{\mu}}^{t-\tau+\bar{\mu}}\dot{z}^T(\xi)Z\dot{z}(\xi)d\xi.\end{aligned} \tag{36}$$

Moreover, invoking the operator of infinitesimal generator in (27), we have

$$\begin{aligned}\mathcal{L}V(z_t,\ell) \;=\; &\mathcal{L}V_1(z_t,\ell) + \mathcal{L}V_2(z_t) + \mathcal{L}V_3(z_t)\\&+\mathcal{L}V_4(z_t) + \mathcal{L}V_5(z_t).\end{aligned} \tag{37}$$

The term $\mathcal{L}V_1(z_t,\ell)$ is given by:

$$\begin{aligned}\mathcal{L}V_1(z_t,\ell) &= 2\dot{\chi}^T(t)P_\ell\chi(t) + \chi^T(t)\left[\sum_{q=1}^{s}(\pi_{\ell q}+\epsilon_{\ell q})P_q\right]\chi(t)\\&= 2\dot{\chi}^T(t)P_\ell\chi(t) + \chi^T(t)\left(\sum_{q=1}^{s}\pi_{\ell q}P_q\right)\chi(t)\\&\quad+\chi^T(t)\left(\sum_{q=1}^{s}\epsilon_{\ell q}P_q\right)\chi(t).\end{aligned} \tag{38}$$

By Lemma 3 (Appendix) we have that :

$$\begin{aligned}\chi^T(t)\left(\sum_{q=1}^{s}\epsilon_{\ell q}P_q\right)\chi(t) &= \chi^T(t)\left[\sum_{q=1}^{s}\frac{1}{2}\epsilon_{\ell q}(P_q+P_q)\right]\chi(t)\\&\leq \chi^T(t)\sum_{q=1}^{s}\left[\left(\frac{\epsilon_{\ell q}}{2}\right)^2 N + P_qN^{-1}P_q\right]\chi(t).\end{aligned} \tag{39}$$

Since $\epsilon_{\ell q} \in [-\delta_{\ell q},\ \delta_{\ell q}]$, we have $\epsilon_{\ell q}^2 \leq \delta_{\ell q}^2$, such that

$$\begin{aligned}&\chi^T(t)\left(\sum_{q=1}^{s}\epsilon_{\ell q}P_q\right)\chi(t)\\&\qquad\leq \chi^T(t)\sum_{q=1}^{s}\left(\frac{\delta_{\ell q}^2}{4}N + P_qN^{-1}P_q\right)\chi(t).\end{aligned} \tag{40}$$

Choosing $N = \frac{1}{\delta_{\ell q}}P_q$, since we assumed P_q to be positive definite $\forall q = 1, ..., s$, it makes

$$\begin{aligned}\chi^T(t)\left(\sum_{q=1}^{s}\epsilon_{\ell q}P_q\right)\chi(t) &\leq \chi^T(t)\sum_{q=1}^{s}\left[\frac{\delta_{\ell q}}{4}P_q + \delta_{\ell q}P_q\right]\chi(t)\\&\leq \chi^T(t)\left(\sum_{q=1}^{s}\frac{5\delta_{\ell q}}{4}P_q\right)\chi(t).\end{aligned} \tag{41}$$

Then, we can write $\mathcal{L}V_1(z(t),\ell)$ as

$$\begin{aligned}\mathcal{L}V_1(z_t,\ell) \leq\; &2\dot{\chi}^T(t)P_\ell\chi(t) + \chi^T(t)\left(\sum_{q=1}^{s}\pi_{\ell q}P_q\right)\chi(t)\\&+\chi^T(t)\left(\sum_{q=1}^{s}\frac{5\delta_{\ell q}}{4}P_q\right)\chi(t).\end{aligned} \tag{42}$$

We have also

$$\mathcal{L}V_2(z_t) = z^T(t)Qz(t) - z^T(t-\tau)Qz(t-\tau), \tag{43}$$

$$\mathcal{L}V_3(z_t) = \tau\dot{z}^T(t)R\dot{z}(t) - \int_{t-\tau}^{t}\dot{z}^T(\xi)R\dot{z}(\xi)d\xi, \tag{44}$$

which considering $R > 0$ we can apply Lemma 2 (Appendix), then

$$\begin{aligned}\mathcal{L}V_3(z_t) \leq\; &\tau\dot{z}^T(t)R\dot{z}(t)\\&-\frac{1}{\tau}\int_{t-\tau}^{t}\dot{z}^T(\xi)d\xi R\int_{t-\tau}^{t}\dot{z}(\xi)d\xi - \frac{3}{\tau}\Omega^TR\Omega.\end{aligned} \tag{45}$$

For $\mathcal{L}V_4(z_t)$, we have

$$\begin{aligned}\mathcal{L}V_4(z_t) = \;&\frac{\tau^2}{2}\dot{z}^T(t)S\dot{z}(t)\\&-\int_{-\tau}^{0}\int_{t+\zeta}^{t}\dot{z}^T(\xi)S\dot{z}(\xi)d\xi d\zeta,\end{aligned} \tag{46}$$

which considering $S > 0$ we can apply Lemma 1 (Appendix), then

$$\begin{aligned}\mathcal{L}V_4(z_t) \leq\; &\frac{\tau^2}{2}\dot{z}^T(t)S\dot{z}(t) - 2z^T(t)Sz(t)\\&+\frac{4}{\tau}z^T(t)S\int_{t-\tau}^{t}z(\xi)d\xi\\&-\frac{2}{\tau^2}\int_{t-\tau}^{t}z^T(\xi)d\xi S\int_{t-\tau}^{t}z(\xi)d\xi,\end{aligned} \tag{47}$$

and

$$\mathcal{L}V_5(z_t) = 2\bar{\mu}\dot{z}^T(t)Z\dot{z}(t) - \int_{t-\tau-\bar{\mu}}^{t-\tau+\bar{\mu}}\dot{z}^T(\xi)Z\dot{z}(\xi)d\xi. \tag{48}$$

We add the null term in (36) to the time derivative of the functional in (37), and replace $\mathcal{L}V_1(z_t)$, $\mathcal{L}V_3(z_t)$, and $\mathcal{L}V_4(z_t)$ by the upper bounds (42), (45), and (47), respectively, such that

$$\mathcal{L}V(z_t,\ell) \leq \Upsilon^T\Phi_\ell\Upsilon + \sum_{k=1}^{m}\bar{\mu}(\Lambda\bar{L}_{k\ell})mZ^{-1}(\Lambda\bar{L}_{k\ell})^T, \tag{49}$$

where $\Upsilon^T = \begin{bmatrix} z^T(t) & \dot{z}^T(t) & z(t-\tau) & \int_{t-\tau}^{t}z(\xi)d\xi\end{bmatrix}$ and Φ_ℓ as defined in (24).

Note that $\Lambda\bar{L}_{k\ell} = \Upsilon^T\Gamma_{k\ell}$ with

$$\Gamma_{k\ell} = \begin{bmatrix} F_\ell\bar{B}_{k\ell}\\ G_\ell\bar{B}_{k\ell}\\ 0\\ 0\end{bmatrix}. \tag{50}$$

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Thus, we can write (49) as

$$\mathcal{L}V(z_t,\ell) \le \Upsilon^T\Phi_\ell\Upsilon + \sum_{k=1}^{m}\bar{\mu}\Upsilon^T\Gamma_{k\ell}mZ^{-1}\Gamma_{k\ell}^T\Upsilon$$
$$\le \Upsilon^T\left[\sum_{k=1}^{m}\left(\frac{1}{m}\Phi_\ell + \bar{\mu}\Gamma_{k\ell}mZ^{-1}\Gamma_{k\ell}^T\right)\right]\Upsilon. \quad (51)$$

In order to guarantee $\mathcal{L}V(z_t,\ell) < 0$ for any $\Upsilon \neq 0$, the matrix between parentheses in (51) must be imposed negative. Thus, applying Schur's complement on it we obtain:

$$\sum_{k=1}^{m}\begin{bmatrix} \frac{1}{m}\Phi_\ell & \bar{\mu}\Gamma_{k\ell} \\ * & -\frac{\bar{\mu}}{m}Z \end{bmatrix} = \begin{bmatrix} \Phi_\ell & \bar{\mu}\Gamma_\ell \\ * & -\bar{\mu}Z \end{bmatrix} < 0 \quad (52)$$

where $\Gamma_\ell = \sum_{k=1}^{m}\Gamma_{k\ell}$, leading to the inequality in (23). If $R > 0$, $S > 0$, and the LMI in (23) holds, then the LKF time-derivative condition $\mathcal{L}V(z_t,\ell) < 0$ is satisfied.

Finally, assume that the LMIs (22) and (23) hold, then $V(z_t,\ell) > 0$ and:

$$\mathcal{L}V(z_t,\ell) \le -\sigma z^T(t)z(t), \quad (53)$$

for some sufficiently small $\sigma > 0$. Applying the expectancy, we obtain: $\mathbb{E}[V(z_t,\ell)] > 0$ and

$$\mathbb{E}[\mathcal{L}V(z_t,\ell)] \le -\sigma\mathbb{E}[z^T(t)z(t)]. \quad (54)$$

Then, applying the generalized Itô's formula, it yields

$$\mathbb{E}\left[V(z_t,\ell)\right] - V(z(0),\ell_0) = \int_0^t \mathbb{E}\left[\mathcal{L}V(z(\xi),\ell)\right]d\xi$$
$$\le -\sigma\int_0^t \mathbb{E}\left[z^T(\xi)z(\xi)\right]d\xi, \quad (55)$$

where ℓ_0 is the arbitrary initial topology at $t = 0$. Thus,

$$\sigma\int_0^t \mathbb{E}\left[z^T(\xi)z(\xi)\right]d\xi < V(z(0),\ell_0) - \mathbb{E}\left[V(z_t,\ell)\right]$$
$$\int_0^t \mathbb{E}\left[z^T(\xi)z(\xi)\right]d\xi < \frac{1}{\sigma}V(z(0),\ell_0). \quad (56)$$

which implies $\lim_{t\to\infty}\mathbb{E}[z^T(t)z(t)] \to 0$, meaning that (18) is stochastic asymptotically stable in the mean square sense. Consequently, system (10) achieves consensus according to Definition 2. This completes the proof. ■

It is relevant to mention that some steps in this proof are inspired from [35] and [36], where it is also assumed the same time-delay description adopted here. Such description is a very general one involving uniform, nonuniform, constant, time-varying, differentiable, and non-differentiable time-delays. On the other hand, in [35] it is presented a condition specific to deal with differentiable time-delays, where the upper bound for its time-derivative should be known. However, in despite of a greater knowledge about the time-delay be needed and of that the obtained condition contains a few more variables, it does not always lead to less conservative results. See [35] for further comments.

Moreover, it is worth to mention that in the proposed condition in Theorem 1, the Wirtinger integral inequality in Lemma 2 is applied in (45) instead of the Jensen inequality, which does not contain the Ω term responsible for delivering a more accurate lower bound. To make clearer the importance of the Wirtinger integral inequality in the proposed criterion consider the following remark.

Remark 2: A particular case of Theorem 1 can be obtained when the Wirtinger integral inequality is not used in its proof but the Jensen inequality is applied instead. It thus results in replacing matrix Φ_ℓ in LMI (23) by:

$$\bar{\Phi}_\ell = \Phi_\ell + \left[\begin{array}{cc|cc} \frac{3}{\tau}R & 0 & \frac{3}{\tau}R & -\frac{6}{\tau^2}R \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{3}{\tau}R & -\frac{6}{\tau^2}R \\ 0 & 0 & 0 & \frac{12}{\tau^3}R \end{array}\right]. \quad (57)$$

It is worth to note that this replacement does not add nor remove any LMI variable from (23), however it may increase the conservativeness of the method. For illustration, see the tests performed in the next section. △

As a final comment in this section, we want to stress that the numerical complexity of the proposed conditions does not increase as the number of time-delays increases, since no LMI row/column or variable is added when the number of time-delays increase. It is an advantage compared to usual results for systems with multiple time-delays [37]. The numerical complexity of LMI conditions depend on the LMI solver to be used, e.g., the commercial LMI Toolbox for Matlab [38] has the worst-case complexity $\mathcal{O}\left(v^3 l\log\left(\varepsilon^{-1}\right)\right)$ and the solver SeDuMi [39] has worst-case complexity $\mathcal{O}\left((v^2 l^{2.5} + l^{3.5}v)\log\left(\varepsilon^{-1}\right)\right)$, where ε is the required relative accuracy, v is the number of scalar variables, and l is the number of LMI lines.

V. Numerical Example

In this section we present a numerical example that illustrates the applicability of our method. In order to show that the method can indeed provide interesting results in practical problems, we consider the following scenario: a team of three networked quadrotors coordinating themselves with the objective of reaching formation on one of its coordinate axes, x or y, in relation to a given inertial frame, assuming time-varying delayed control inputs and intermittent communication. We assume the same second order model for the quadrotors described in [22], [23], also considering switching topology and time-varying delays:

$$\ddot{p}_i + b\dot{p}_i + cp_i = u_i(\theta_t, t - \tau_i(t)), \quad (58)$$

where p_i is the position of the quadrotor along the axis in which we want to reach formation, and b and c are the damping and spring constants, respectively. Equation (58) can be compactly written in state space format as

$$\begin{bmatrix}\dot{p}_i \\ \ddot{p}_i\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}\begin{bmatrix} p_i \\ \dot{p}_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_i(\theta_t, t - \tau_i(t)). \quad (59)$$

In order to analyze the posed formation problem by the proposed methodology, we rewrite the formation problem as a consensus one using the distributed control law

$$u_i = c\alpha_i - \sum_{j\neq i, j=1}^{m} a_{ij}(\theta_t)k\big(p_i - \alpha_i - p_j + \alpha_j\big), \quad (60)$$

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where α_i and α_j are used to compute the desired constant distance between the j-th and i-th agents as $\alpha_{j,i} = \alpha_j - \alpha_i$, and k is the control gain. Replacing (60) into (59), the system can be reformulated as

$$\begin{bmatrix} \dot{p}_i \\ \ddot{p}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} p_i - \alpha_i \\ \dot{p}_i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \bar{u}_i(\theta_t, t - \tau_i(t)) \tag{61}$$

with

$$\bar{u}_i(\theta_t, t) = -\sum_{j\neq i, j=1}^{m} a_{ij}(\theta_t) \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} p_i - \alpha_i \\ \dot{p}_i \end{bmatrix} - \begin{bmatrix} p_j - \alpha_j \\ \dot{p}_j \end{bmatrix} \right). \tag{62}$$

The previous two equations are in the same format of (4) and (5), respectively, by making $x_i = [p_i - \alpha_i \;\; \dot{p}_i]^T$. Thus we are in position to apply the Theorem 1. In the following we consider $b = 1$, $c = 0$, and $k = 1$.

We model the intermittent communication with a directed network topology switching between two cases: $\mathcal{G}_1$ and $\mathcal{G}_2$. These topologies are illustrated in Figure 1 where the arrows indicate the existing communication channels in each case and $\tau_i(t)$ is the value of the associated input time-varying delay.

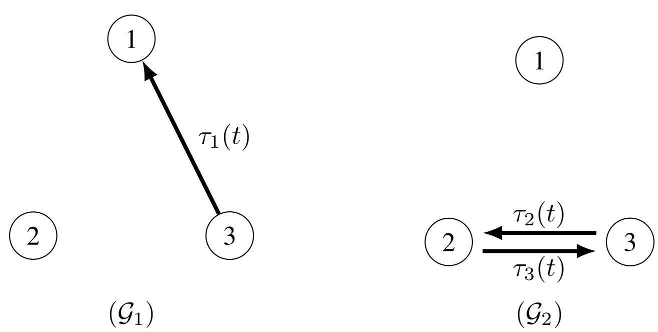

Fig. 1. Multi-agent system subject to time-varying delays $\tau_k(t)$, composed of three agents with switching topologies $\mathcal{G}_1$ and $\mathcal{G}_2$.

The Laplacians in this case are:

$$L_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{63}$$

In order to match the notation $L_{k\ell}$ introduced earlier we extract from the Laplacians the subgraphs related to each vertex v_k, that is:

$$L_{1,1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_{2,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } L_{3,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \tag{64}$$

If there are no edges pointing to v_k in a given topology ℓ, its Laplacian $L_{k\ell}$ is null.

Notice the important fact that this system would never be able to achieve consensus if the topology were fixed at either $\mathcal{G}_1$ or $\mathcal{G}_2$ since none of these graphs contains a spanning tree. However the stochastic switching between the two topologies makes consensus possible.

We suppose the transition rates not precisely known but with $\pi_{pq} = 1$, $\forall (p \neq s)$, $p, q \in S$ and uncertainty $\epsilon_{pq} = \pm 0.2$, $\forall\, p, q \in S$, such that

$$\Pi = \begin{bmatrix} -1 \pm 0.2 & 1 \pm 0.2 \\ 1 \pm 0.2 & -1 \pm 0.2 \end{bmatrix}. \tag{65}$$

In the remainder, we will illustrate the application of Theorem 1 in various scenarios. We also present the results obtained when applying Theorem 1 as proposed in Remark 2 to highlight the conservatism reduction due to the Wirtinger integral inequality.

First consider the case of a constant and uniform time-delay, that is $\bar{\mu} = 0$. We seek to determine the highest value of τ guaranteeing the feasibility of LMIs in Theorem 1. The maximum time-delay that can be achieved is $\tau = 0.324$. On the other hand, when we apply the modification of Theorem 1 as proposed in Remark 2, it is obtained $\tau = 0.242$.

Then, we consider the case when the agents' input time-delays are time-varying. For given values of τ, we seek the largest value of $\bar{\mu}$ satisfying Theorem 1. Results are summarized in Table I, which also presents the results obtained as proposed in Remark 2. Notice that $--$ stands for unfeasibility.

TABLE I
LARGEST $\bar{\mu}$ OBTAINED FOR GIVEN τ

	τ	0.140	0.175	0.200	0.240	0.280	0.300
Thm. 1	$\bar{\mu}$	0.140	0.115	0.092	0.061	0.031	0.017
Rem. 2	$\bar{\mu}$	0.093	0.060	0.037	0.002	$--$	$--$

Finally we fix the values of τ, $\bar{\mu}$, $\pi_{pq} = \bar{\pi}$ $(\forall\, p, q \in S)$, and seek the common largest interval bound, $\delta_{pq} = \bar{\delta}$ $(\forall\, p, q \in S)$, for the uncertain transition rates in (8) that satisfy Theorem 1 and the condition described in Remark 2. We perform this computation for various triplets $(\tau, \bar{\mu}, \bar{\pi})$ and summarize the results in Table II.

TABLE II
LARGEST $\bar{\delta}$ OBTAINED FOR GIVEN $(\tau, \bar{\mu}, \bar{\pi})$

$(\tau, \bar{\mu}, \bar{\pi})$	$(0.15, 0.10, 0.5)$	$(0.15, 0.10, 1)$	$(0.15, 0.10, 2)$
Thm. 1	0.14	0.25	0.47
Rem. 2	0.11	0.18	0.26

For illustration, a simulation of the state trajectories for $p_i(t)$ and $\dot{p}_i(t)$, for all the agents, is shown in Figure 2 considering the desired distances between the agents obtained by $\alpha_1 = 1$, $\alpha_2 = 2$, and $\alpha_3 = 3$, i.e., $\alpha_{2,1} = 1$, $\alpha_{3,1} = 2$ and $\alpha_{3,2} = 1$. The initial conditions are $p_1(0) = 0$, $p_2(0) = 4$, and $p_3(0) = 7$ for the position, and $\dot{p}_1(0) = 3$, $\dot{p}_2(0) = 2$, and $\dot{p}_3(0) = -1$ for the velocity. The bottom plot in Figure 2 shows the switching behavior of θ_t which governs the switching between the topologies $\mathcal{G}_1$ and $\mathcal{G}_2$ (see Figure 1). The topology $\mathcal{G}_1$ is active when $\theta_t = 1$ and the topology $\mathcal{G}_2$ is active when $\theta_t = 2$. The simulation is performed with $\tau = 0.17$ and $\bar{\mu} = 0.11$, such that $\tau_k \in [0.06, 0.28]$, as shown in Figure 3 for $\tau_1(t)$. Π is chosen as in (65).

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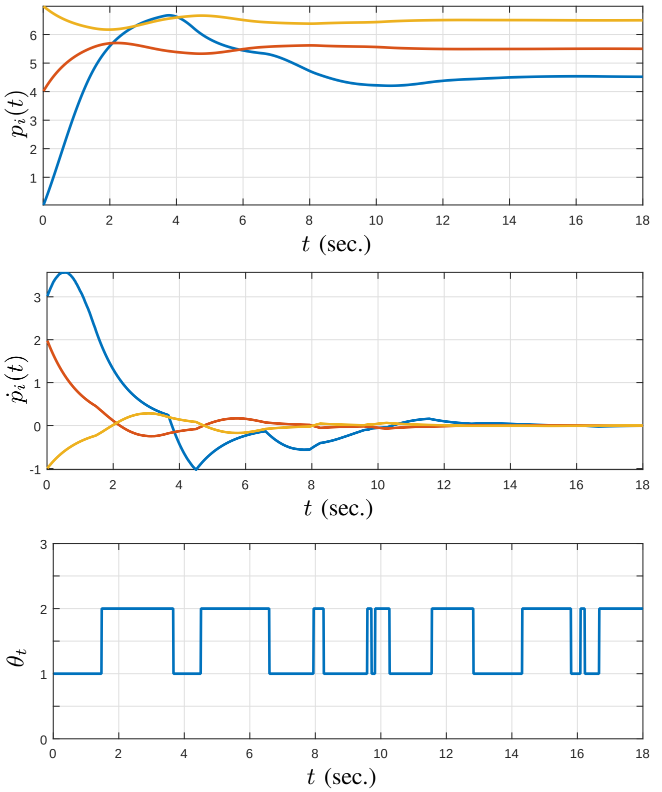

Fig. 2. Simulation with time-varying delay with $\tau = 0.17$ and $\bar{\mu} = 0.11$. Top: state trajectory of position $p_i(t)$ for all agents. Middle: state trajectory of velocity $\dot{p}_i(t)$ for all agents. Bottom: state of the switching topology θ_t, $\mathcal{G}_1$ and $\mathcal{G}_2$, for $\theta_t = 1$ and $\theta_t = 2$, respectively.

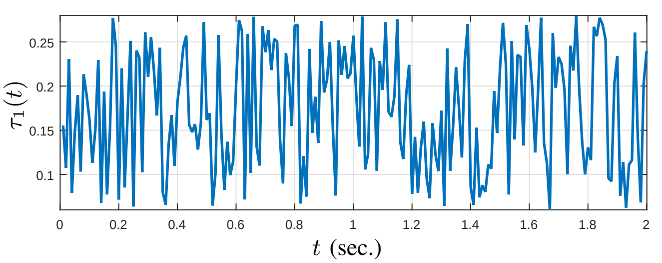

Fig. 3. Time-varying delay $\tau_1(t)$ with random amplitude given by $\tau = 0.17$ and $\bar{\mu} = 0.11$.

This example illustrates that the proposed method can be well applied in the verification of consensus of time-delayed systems with linear dynamics with the advantage that we can consider an upper and a lower bound for the delay variation. Furthermore, the simulation illustrates how the system converges even when switching from two topologies that have no spanning trees, as indicated by the analysis. Finally, since the transition rates are only estimates, this method can analyze consensus while considering uncertainties in the estimation, which allows a more flexible, while still guaranteed, analysis for consensus.

VI. Conclusion

This paper addressed the problem of consensus for multi-agent systems described by any time-invariant, linear or linearizable model subject to time-delays and uncertain switching topology. It is proposed an analysis condition that is capable of verifying consensus in the mean square sense even when each topology contains no spanning tree. The result is of great practical interest since neither the value of the input delays nor the transition rates are assumed to be known. In future works we hope to deal with design of the matrix gain in the consensus protocol as well as the coupling strengths between the agents, in order to achieve consensus with a priori prescribed performance.

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Appendix

The following Lemmas are used to prove the main results for consensus analysis.

Lemma 1: ([40]) For any constant matrix $M = M^T > 0$ and scalars $t > t - \tau \geq 0$ such that the following integrations are well defined, then

$$\int_{-\tau}^{0}\int_{t+\zeta}^{t} z^T(\xi) M z(\xi) d\xi d\zeta \geq \frac{2}{\tau^2}\int_{-\tau}^{0}\int_{t+\zeta}^{t} z^T(\xi) d\xi d\zeta M \int_{-\tau}^{0}\int_{t+\zeta}^{t} z(\xi) d\xi d\zeta. \tag{66}$$

Lemma 2 (Wirtinger inequality): ([41]) For any constant matrix $M = M^T > 0$ and scalars $t > t - \tau \geq 0$ such that the following integrations are well defined, then

$$\int_{t-\tau}^{t} \dot{z}^T(\xi) M \dot{z}(\xi) d\xi \geq \frac{1}{\tau}\int_{t-\tau}^{t} \dot{z}^T(\xi) d\xi M \int_{t-\tau}^{t} \dot{z}(\xi) d\xi + \frac{3}{\tau}\Omega^T M \Omega, \tag{67}$$

with

$$\Omega = z(t-\tau) + z(t) - \frac{2}{\tau}\int_{t-\tau}^{t} z(\xi) d\xi. \tag{68}$$

Lemma 3: [25] Let a real number $\gamma \in \mathbb{R}$, a square matrix $M \in \mathbb{R}^{n \times n}$, and a symmetric positive definite matrix $N \in \mathbb{R}^{n \times n}$ be given. The following inequality is true:

$$\gamma(M + M^T) \leq \gamma^2 N + M N^{-1} M^T. \tag{69}$$

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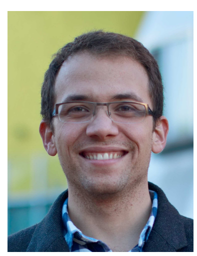

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