Robust Stabilization of the Distributed Parameter System With Time Delay via Fuzzy Control

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***Abstract*—In this paper, stabilization of the distributed parameter system (DPS) with time delay is studied using Galerkin's method and fuzzy control. With the help of Galerkin's method, the dynamics of DPS with time delay can be first converted into a group of low-order functional ordinary differential equations, which will be used for design of the robust fuzzy controller. The fuzzy controller designed can guarantee exponential stability of the closed-loop DPS. Some sufficient conditions are derived for the stabilization together with the linear matrix inequality design approach. The effectiveness of the proposed control design methodology is demonstrated in numerical simulations.**

***Index Terms*—Distributed parameter system, exponential stability, fuzzy control, Galerkin's method, linear matrix inequality.**

I. Introduction

The distributed parameter system (DPS) is usually described in partial differential equations (PDEs) with mixed or homogeneous boundary conditions. The key characteristic of DPS is that its outputs, inputs, and process states and the relevant parameters may vary temporally as well as spatially. Such characteristic of the DPS model is difficult to control due to its dynamic complexity. Recent research [3]–[14] also shows that the dynamics of the parabolic PDEs can be described approximately in a group of low-order ordinary differential equations (ODEs). In [4], the author proposes a simple but effective modeling method for DPS by integrating the spectral method with neural networks. In [9], the authors stabilize DPS via the Galerkin's method and the geometric control. In [15] and [16], the K-L method is employed to model the distributed parameter system, where a class of DPS modeled by parabolic PDE is considered. The eigenspectrum of parabolic spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite stable fast complement. Based on Galerkin's method, the proposed lower order ODE systems can sufficiently describe dominant dynamics of DPS, and thus can be used as the basis of controller design.

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In [9]–[14], the authors design nonlinear output feedback controllers in combination of the geometric control and the Lyapunov techniques. Since the geometric control is employed to design the controller, this approach is indirect and not convenient to design the controller. In order to simplify the controller design of the nonlinear system, various schemes have been developed in the last two decades, among which fuzzy control is one of successful approaches to obtain nonlinear control systems. It has successful applications not only in consumer products but also in industrial processes (see [18]–[29]). Recently, a nonlocal approach, which is conceptually simple and straightforward, was proposed for nonlinear systems design in the fuzzy control method [17]. The idea of this fuzzy control is using a so-called Takagi–Sugeno (TS) fuzzy model, where local dynamics in different state-space regions are represented by linear models. The overall model of the system is achieved by fuzzy "blending" of these fuzzy models. The controller design is carried out based on the fuzzy model via the so-called parallel distributed compensation scheme. The idea is that for each local linear model, a linear feedback control is designed. The resulting overall controller, which is nonlinear in general, is a fuzzy blending of each individual linear controller. In this paper, we extend the above idea to the slow system that is derived from the DPS with time delay. The nonlinear retarded slow system will be first represented by the TS model with time delay. This fuzzy modeling method is simple and natural.

Being highly nonlinear and infinitely dimensional, the distributed parameter systems are also uncertain. The model uncertainties include unknown or partially known time-varying process parameters and exogenous disturbances. It is well known that the presence of uncertain variables, if not taken into account in the controller design, may lead to severe deterioration of the nominal closed-loop performance or even to the closed-loop instability. Furthermore, in practice, time delays occur due to transportation lag and dead times associated with the measurement sensor and the control actuator. The presence of time delays, if not appropriately taken into account in the controller design, may lead to serious problems in the behavior of the closed-loop system, including poor performance and instability.

Motivated by the above discussion, the aim of this paper is to stabilize distributed parameter systems with time delay using fuzzy control. Employing Galerkin's method, a low-dimensional slow system that contains dominant dynamics of DPS is derived. The slow system is used for the synthesis of fuzzy controllers that guarantee exponential stability and robust stability in the closed-loop system. In this paper, we will focus on the exponential stability instead of the asymptotic stability, because of its robustness to bounded perturbations always present in many practical applications with few presented in the previous studies of fuzzy control. Moreover, this is the first time a T-S type fuzzy control is designed for the distributed parameter system, which expands fuzzy control application. Finally, a numerical example is given to show the effectiveness of the proposed method.

II. Model Formulation and Reduction

Consider the following parabolic partial differential difference equations systems with a state-space representation of the form

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial z^2} + F(T(z,t), T(z,t-\tau)) + G(T, u(t)), \qquad z \in (0,l) \quad (1)$$

subject to the boundary conditions

$$\begin{cases} \left.\frac{\partial T}{\partial z}\right|_{z=0} = f_1(T) \\ \left.\frac{\partial T}{\partial z}\right|_{z=l} = f_2(T) \end{cases} \quad (2)$$

and the initial condition

$$T(z,t) = \varphi(z,t), \qquad t \in [-\tau, 0] \quad (3)$$

where $z \in [0,l]$ is the spatial coordinate, $T(z,t) : [0,l]\times\mathbb{R} \to \mathbb{R}$ denotes state variables, $u(t)$ denotes the manipulated input, k is positive constant, $\tau > 0$ denotes the state delay; the vector function $F(\cdot)$ is a nonlinear locally Lipschitz continuous function, and $F(0) = 0, G(T, u(t)) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The system (1) can be rewritten as the following equivalent system, with its proof given in [4]:

$$\begin{aligned}\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial z^2} &+ F(T(z,t), T(z,t-\tau)) \\ &+ k\delta(z-l)f_2(T) - k\delta(z-0)f_1(T) \\ &+ G(T, u(t)), \quad z \in (0,l)\end{aligned} \quad (4)$$

subject to the homogenous boundary conditions

$$\begin{cases} \left.\frac{\partial T}{\partial z}\right|_{z=0} = 0 \\ \left.\frac{\partial T}{\partial z}\right|_{z=l} = 0 \end{cases} \quad (5)$$

and the initial condition

$$T(z,t) = \varphi(z,t), \qquad t \in [-\tau, 0] \quad (6)$$

where $\delta(\cdot)$ is the Dirac delta function.

After the equivalent parabolic PDE with homogeneous boundary conditions is derived as the above, its low-order model can be further obtained using Galerkin's method, with the detailed derivation given in [14]. A parabolic PDE involves spatial differential operators whose spectrum can be partitioned into a finite-dimensional (slow) and an infinite-dimensional (fast) complement. This implies that the dynamical behavior of such a system can be approximately described by a finite-dimensional ODE system that captures the dynamics of the dominant (slow) modes of the PDE. It is very convenient and effective to use the eigenfunctions of a spatial differential operator to derive a low-order ODE system for such a parabolic PDE system if the boundary conditions are homogenous.

First, the infinite ODE will be derived from (4)–(6) based on the spectral method. As is well known, the eigenvalue problem for operator $\nabla^2 = (\partial^2 T/\partial z^2)$ can be solved as $\lambda_n = -(n^2\pi^2/l^2)$ and $Z_n(z) = \cos(n\pi z/l)$, $n = 0, 1, 2, \ldots$, where λ_n denotes eigenvalue and $Z_n(z)$ denotes the corresponding eigenfunctions.

Next, the solution of system (4)–(6) is expressed in an orthogonally decoupled series

$$T(z,t) = \sum_{n=0}^{\infty} x_n(t)\cos\frac{n\pi z}{l}. \quad (7)$$

Then we can get

$$\begin{aligned}\int_0^l \Bigg[\frac{\partial T}{\partial t} - \Bigg(k\frac{\partial^2 T}{\partial z^2} &+ F(T(z,t), T(z,t-\tau)) \\ &+ k\delta(z-l)f_2(T) - k\delta(z-0)f_1(T) \\ &+ G(T, u(t))\Bigg)\Bigg]\cos\frac{n\pi z}{l}dz = 0.\end{aligned} \quad (8)$$

Substituting (7) into (8) will give

$$\begin{aligned}\dot{x}_n(t) &= k\lambda_n x_n(t) + f_n(x(t), x(t-\tau)) \\ &\quad + g_n(x(t), u(t)) \\ x_n(t) &= \phi_n(t), \quad t \in [-\tau, 0]\end{aligned} \quad (9)$$

where

$$\begin{aligned}f_n(x(t), x(t-\tau)) &= \frac{2}{l}\int_0^l [F(T(z,t), T(z,t-\tau)) \\ &\quad + k\delta(z-l)f_2(T) \\ &\quad - k\delta(z-0)f_1(T)]\cos\frac{n\pi z}{l}dz \\ g_n(x(t), u(t)) &= \frac{2}{l}\int_0^l G(T(z,t), u(t))\cos\frac{n\pi z}{l}dz \\ \phi_n(t) &= \frac{2}{l}\int_0^l \varphi(z,t)\cos\frac{n\pi z}{l}dz.\end{aligned}$$

Then, (9) can be rewritten in a general nonlinear form as follows:

$$\begin{aligned}\dot{x}(t) &= kAx(t) + f(x(t), x(t-\tau)) + g(x(t), u(t)) \\ T &= Cx(t) \\ x(t) &= \phi(t), \qquad t \in [-\tau, 0]\end{aligned} \quad (10)$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)\cdots]^T$, $A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$, $f(x(t), x(t-\tau)) = [f_1(x(t), x(t-\tau)), \ldots, f_n(x(t), x(t-\tau)), \ldots]^T$, $g = [g_1, g_2, \ldots, g_n, \ldots]^T$ $C = [Z_1(z), Z_2(z), \ldots, Z_n(z), \ldots]$, $\phi(t) = [\phi_1(t), \phi_2(t), \ldots, \phi_n(t), \ldots]$.

To reduce the infinite-dimensional nonlinear ODE with time delay (10) to low-order ODE, the following assumption is made for the eigenvalue of the operator $\Gamma = \nabla^2$. For convenience, $\sigma(\Gamma)$ is defined as the set of all eigenvalues of Γ, i.e., $\sigma(\Gamma) = \{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$.

Assumption 1. [11]:

1) $\mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_2) \geq \cdots \geq \mathrm{Re}(\lambda_j) \geq \cdots$, where $\mathrm{Re}(\lambda_j)$ denotes the real part of λ_j.
2) $\sigma(\Gamma)$ can be partitioned as $\sigma(\Gamma) = \sigma_1(\Gamma) + \sigma_2(\Gamma)$, where $\sigma_1(\Gamma)$ consists of the first m eigenvalues, i.e., $\sigma_1(\Gamma) = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, and $(|\mathrm{Re}\lambda_1|/|\mathrm{Re}\lambda_m|) = O(1)$.

3) $\mathrm{Re}(\lambda_{m+1}) < 0$ and $(|\mathrm{Re}\lambda_m|/|\mathrm{Re}\lambda_{m+1}|) = O(\varepsilon)$, where $\varepsilon = (|\mathrm{Re}\lambda_1|/|\mathrm{Re}\lambda_m|) < 1$ is a small positive parameter.

Remark 1: Assumption 1 means that the number of unstable eigenvalues is finite. From [30], the finite number of unstable eigenvalues always satisfies the parabolic PDE systems, while the assumption of existence of only a few dominant modes that describe the dynamics of the parabolic PDE system usually satisfies the majority of diffusion-convection-reaction processes.

Based on Assumption 1, one can use the standard Galerkin's method to perform model reduction. Let H_s, H_f be eigensubspaces of Γ, defined as $H = \mathrm{span}\{Z_1(z), Z_2(z), \ldots\}$, $H_s = \mathrm{span}\{Z_1(z), Z_2(z), \ldots, Z_m(z)\}$, and $H_f = \mathrm{span}\{Z_{m+1}(z), Z_{m+2}(z), \ldots\}$ (the existence of H_s and H_f follows from Assumption 1). Choose the orthogonal projection operators $\bar{P}_s$ and $\bar{P}_f$ such that $x_s = \bar{P}_s x, x_f = \bar{P}_f x$. Applying $\bar{P}_s$, $\bar{P}_f$, $P_s : H \to H_s$, and $P_f : H \to H_f$ to (10), the following equivalent form can be derived:

$$\begin{aligned}\frac{dx_s}{dt} &= A_s x_s(t) + f_s(x_s(t), x_s(t-\tau), x_f(t), x_f(t-\tau))\\ &\quad + g_s(x_s(t), x_f(t), u(t))\\ \frac{\partial x_f}{\partial t} &= A_f x_f(t) + f_f(x_s(t), x_s(t-\tau), x_f(t), x_f(t-\tau))\\ &\quad + g_f(x_s(t), x_f(t), u(t))\\ T &= T_s + T_f\\ x_s(t) &= \phi_s(t) \quad x_f(t) = \phi_f(t), \qquad t \in [-\tau, 0]\end{aligned} \tag{11}$$

where:

1) $A_s = \bar{P}_s A \bar{P}_s$ is an $m \times m$ matrix in the form of $A_s = \mathrm{diag}\{\lambda_1, \ldots, \lambda_m\}$;
2) $A_f = \bar{P}_f A \bar{P}_f = \mathrm{diag}(\lambda_{m+1}, \ldots)$ is an operator that generates a strongly continuous exponentially stable semigroup because the real part of its eigenvalues are negative [3) of Assumption 1];

$$\begin{aligned} f_s &= \bar{P}_s f = [f_1, \ldots, f_m]^T\\ f_f &= \bar{P}_f f = [f_{m+1}, \ldots]^T\end{aligned}$$

are Lipschitz vector functions

$$\begin{aligned}& g_s = \bar{P}_s g = [g_1, \ldots, g_m]^T \quad g_f = \bar{P}_f g = [g_{m+1}, \ldots]^T\\ & \phi_s = \bar{P}_s \phi = [\phi_1, \ldots \phi_m] \quad \phi_f = \bar{P}_f \phi = [\phi_{m+1}, \ldots]\\ & T_s = P_s T = \bar{C}_s x_s(t) \quad T_f = P_f T = \bar{C}_f x_f(t)\\ & \bar{C}_s = \bar{P}_s C = [Z_1 \cdots, Z_m] \quad \bar{C}_f = \bar{P}_f C = [Z_{m+1}, \ldots].\end{aligned}$$

The notation $(\partial x_f/\partial t)$ denotes that the state x_f is in an infinite-dimensional space.

Neglecting the fast modes in (11), the following finite-dimensional nonlinear system with delay is obtained:

$$\begin{aligned}\frac{dx_s}{dt} &= A_s x_s(t) + f_s(x_s(t), x_s(t-\tau), 0, 0) + g_s(x_s(t), 0, u)\\ T_s &= \bar{C}_s x_s(t)\\ x_s(t) &= \phi_s(t), \qquad t \in [-\tau, 0].\end{aligned} \tag{12}$$

Remark 2: Based on the separation of the eigenvalue of the operator ∇^2, nonlinear PDE (1) can be separated into a finite-dimensional (slow) and an infinite dimensional stable (fast) complement. The separation can be realized because the dynamical behavior of the system can be approximately described by a finite-dimensional ODE. These ODE systems will be used as the basis for the synthesis of nonlinear low-order output feedback controllers that can guarantee stability in the closed-loop distributed parameter system.

To get the main results, the following lemmas are given.

Lemma 1: (Schur complement [1]) For a given matrix

$$S = \begin{bmatrix} S_{11} & S_{12}\\ S_{12}^T & S_{22}\end{bmatrix} < 0$$

where $S_{11} = S_{11}^T$ and $S_{22} = S_{22}^T$, is equivalent to any one of the following conditions:

i) $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$;
ii) $S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$.

Lemma 2 [2]: If U, V, W are real matrices of appropriate dimension with M satisfying $M = M^T$, then

$$M + UVW + W^T V^T U^T < 0$$

for all $VV^T \le I$, if and only if there exists a positive constant ϵ such that

$$M + \epsilon^{-1} U U^T + \epsilon W^T W < 0.$$

III. Fuzzy Control

A. Stabilization via Fuzzy Control

In this section, the slow system (12) can be presented with the T–S model by fuzzy modeling. If f_s and g_s are known, the T–S fuzzy model can be derived directly; if f_s and g_s are unknown in the slow system (12), the fuzzy identification is required to get the following T–S fuzzy model. To make it simple and without loss of generality, we assume the function f_s and g_s to be known. Then based on T–S fuzzy model, fuzzy observer-based controller is designed to ensure the exponential stability of the slow system (12). The process of model reduction and controller design are shown in Figs. 11 and 12, respectively.

Plant Rule i:
If $\theta_1(t)$ is $F_{i1}, \ldots$, and $\theta_p(t)$ is F_{ip}, then

$$\begin{aligned}\dot{x}_s(t) &= A_i x_s(t) + A_{di} x_s(t-\tau) + B_i u(t)\\ T_s &= C_s x_s(t)\\ x_s(t) &= \phi_s(t), \qquad t \in [-\tau, 0]; \qquad i = 1, 2, \ldots, n\end{aligned} \tag{13}$$

where $\theta_1(t), \theta_2(t), \ldots, \theta_p(t)$ are the premise variables and F_{ij} $(j = 1, \ldots, p)$ is the fuzzy set; $x_s(t) \in \mathbb{R}^m$ is the state vector; $u(t) \in \mathbb{R}^n$ is the control input; $A_i, A_{di} \in \mathbb{R}^{m \times m}$

$$B_i \in \mathbb{R}^{m \times n} C_s = \begin{bmatrix} \bar{C}_s|_{z=z_1^*}\\ \bar{C}_s|_{z=z_2^*}\\ \cdots\cdots\\ \bar{C}_s|_{z=z_k^*}\end{bmatrix} \in \mathbb{R}^{k \times m} (k < m).$$

Furthermore we have $A_i = A_s + A_i^1 + A_i^2$, obviously

$$\begin{aligned} f_s(x_s(t), x_s(t-\tau), 0, 0) &= \sum_{i=1}^{n} h_i(\theta(t)) \left(A_i^1 x_s(t) \right. \\ &\quad \left. + A_{di} x_s(t-\tau)\right) \\ g_s(x_s(t), 0, u(t)) &= \sum_{i=1}^{n} h_i(\theta(t)) \left(A_i^2 x_s(t) \right. \\ &\quad \left. + B_i u(t)\right). \end{aligned}$$

The overall fuzzy system is inferred as follows:

$$\begin{aligned} \dot{x}_s(t) &= \sum_{i=1}^{n} h_i(\theta(t))[A_i x_s(t) + A_{di} x_s(t-\tau) + B_i u(t)] \\ T_s &= C_s x_s(t) \end{aligned} \tag{14}$$

where

$$\begin{aligned} h_i(\theta(t)) &= \frac{\mu_i(\theta(t))}{\sum_{i=1}^{n} \mu_i(\theta(t))} \\ h_i(\theta(t)) &\geq 0, \qquad \sum_{i=1}^{n} h_i(\theta(t)) = 1 \\ \mu_i(\theta(t)) &= \prod_{j=1}^{p} F_{ij}(\theta_j(t)) \\ \theta(t) &= [\theta_1(t), \theta_2(t), \ldots, \theta_p(t)] \end{aligned}$$

and $F_{ij}(\theta_j(t))$ is the grade of membership of $\theta_j(t)$ in F_{ij}.

From the choice of matrix C_s, all the states of an m-dimensional slow system cannot be completely measured because only k ($<m$) sensors are available. To estimate the slow system state, the following observer is used. Moreover, based on the observer, an output feedback controller is designed to ensure the exponential stability of the closed-loop slow system.

The overall fuzzy observer is represented by

$$\begin{aligned} \dot{\hat{x}}_s(t) &= \sum_{i=1}^{n} h_i(\theta(t))[A_i \hat{x}_s(t) + A_{di} \hat{x}_s(t-\tau) + B_i u(t) \\ &\quad + L_i(T(t) - \hat{T}_s(t))] \\ \hat{T}_s(t) &= C_s \hat{x}_s(t) \\ \dot{\hat{x}}_s(t) &= \sum_{i=1}^{n} h_i(\theta(t))[A_i \hat{x}_s(t) + A_{di} \hat{x}_s(t-\tau) + B_i u(t) \\ &\quad + L_i(T_s(t) - \hat{T}_s(t))] + v \\ \hat{T}_s(t) &= C_s \hat{x}_s(t) \end{aligned} \tag{15}$$

where $v = \sum_{i=1}^{n} h_i(\theta(t)) L_i C_f x_f(t)$ and

$$C_f = \begin{bmatrix} \bar{C}_f\,|_{z=z_1^*} \\ \cdots\cdots \\ \bar{C}_f|_{z=z_k^*} \end{bmatrix}.$$

Next the controller will be designed to stabilize the system (15) with $v = 0$. With the above fuzzy observer, the following fuzzy controller is employed to deal with the fuzzy control system (14).

Control Rule i:
If $\theta_1(t)$ is $F_{i1}, \ldots$, and $\theta_p(t)$ is F_{ip}, then

$$u(t) = K_i \hat{x}_s(t), \qquad i = 1, 2, \ldots, n. \tag{16}$$

Hence, the overall fuzzy control law is represented by

$$u(t) = \sum_{i=1}^{n} h_i(\theta(t)) K_i \hat{x}_s(t) \tag{17}$$

where K_i $(i = 1, 2, \ldots, n)$ are the local control gains. With the control law (17), the overall closed-loop system can be written as

$$\begin{aligned} \dot{x}_s(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} h_i(\theta(t)) h_j(\theta(t))[(A_i + B_i K_j) x_s(t) \\ &\qquad + A_{di} x_s(t-\tau) - B_i K_j e(t)] \\ \dot{e}(t) &= \sum_{i=1}^{n} h_i(\theta(t))[(A_i - L_i C_s) e(t) + A_{di} e(t-\tau)] \end{aligned} \tag{18}$$

where $e(t) = x_s(t) - \hat{x}_s(t)$. Hence, the closed-loop system can be written as

$$\dot{\tilde{x}}_s(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} h_i(\theta(t)) h_j(\theta(t))[G_{ij} \tilde{x}_s(t) + M_i \tilde{x}_s(t-\tau)] \tag{19}$$

where

$$\tilde{x}_s(t) = \begin{bmatrix} x_s(t) \\ e(t) \end{bmatrix} \quad M_i = \begin{bmatrix} A_{di} & 0 \\ 0 & A_{di} \end{bmatrix}$$
$$G_{ij} = \begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ 0 & A_i - L_i C_s \end{bmatrix}.$$

Proposition 1: Suppose the system (19) is exponentially stable. The distributed parameter system (11) with the output feedback controller (15) and (17) is exponentially stable if there exist positive real numbers μ_1, μ_2, ϵ_*, and ϵ^* such that $\|x_{s0}\| \leq \mu_1, \|x_{f0}\| \leq \mu_2$ and $\epsilon \in (\epsilon_*, \epsilon^*)$ such that the matrix (80) defined in the Appendix is negative definite.

The proof of Proposition 1 is given in the Appendix.

Remark 3: Proposition 1 is important because it allows establishing exponential stability of the closed-loop infinite dimensional system by performing a stability analysis on a low-order finite-dimensional system.

Next, the exponential stability will be analyzed for the closed system (19) and the output feedback controller will be designed.

Theorem 1: For the fuzzy delay system (14), its closed-loop fuzzy system (19) with observer-based control law (17) is exponentially stable with exponential decay rate $k(>0)$ if there exist matrices $X_1 > 0, X_2 > 0 S_1 > 0, S_2 > 0, Y_i$, and R_i satisfying the following linear matrix inequalities (LMIs) for all i and j except the pairs (i, j), such that $h_i(\theta(t)) h_j(\theta(t)) = 0$

$$\begin{bmatrix} Z_{1ij} & * \\ X_1 A_{di}^T & -S_1 \end{bmatrix} \leq 0 \tag{21}$$

$$\begin{bmatrix} Z_{2ij} & * \\ A_{di}^T X_2 & -S_2 \end{bmatrix} \leq 0 \tag{22}$$

where $*$ represents blocks that are readily inferred by symmetry, $Z_{1ij} = 2kX_1 + X_1A_i^T + A_iX_1 + B_iY_j + Y_j^TB_i^T + e^{2k\tau}S_1$, and $Z_{2ij} = 2kX_2 + A_i^TX_2 + X_2A_i - R_iC_s - C_s^TR_i^T + e^{2k\tau}S_2$.

Then the state feedback gains and observer gains can be constructed as

$$K_j = Y_jX_1^{-1} \quad L_i = X_2^{-1}R_i$$

for $i, j = 1, 2, \ldots, n$, respectively.

Proof: Consider the following Lyanpunov functional:

$$V(\tilde{x}_s) = e^{2kt}\tilde{x}_s^T(t)P\tilde{x}_s(t) + \int_{t-\tau}^{t} e^{2k(\alpha+\tau)}\tilde{x}_s^T(\alpha)Q\tilde{x}_s(\alpha)d\alpha.$$

Calculating the derivative of V along the trajectory of (19), we obtain

$$\begin{aligned}
\dot{V}(\tilde{x}_s) &= 2ke^{2kt}\tilde{x}_s^T(t)P\tilde{x}_s(t) \\
&\quad + e^{2kt}\sum_{i=1}^{n}\sum_{j=1}^{n} h_ih_j\left[\tilde{x}_s^T(t)\left(PG_{ij} + G_{ij}^TP\right)\tilde{x}_s(t)\right. \\
&\qquad\qquad \left. + 2\tilde{x}_s^T(t)PM_i\tilde{x}_s(t-\tau)\right] \\
&\quad + e^{2kt}[\tilde{x}_s^T(t)e^{2k\tau}Q\tilde{x}_s(t) - \tilde{x}_s^T(t-\tau)Q\tilde{x}_s(t-\tau)] \\
&\le e^{2kt}\sum_{i=1}^{n}\sum_{j=1}^{n} h_ih_j\left[\tilde{x}_s^T(t)\left(2kP + PG_{ij} + G_{ij}^TP\right)\tilde{x}_s(t)\right. \\
&\quad \left. \times\, \tilde{x}_s^T(t)PM_iQ^{-1}M_i^TP\tilde{x}_s(t) + \tilde{x}_s^T(t-\tau)Q\tilde{x}_s(t-\tau)\right] \\
&\quad + e^{2kt}\left[\tilde{x}_s^T(t)e^{2k\tau}Q\tilde{x}_s(t) - \tilde{x}_s^T(t-\tau)Q\tilde{x}_s(t-\tau)\right] \\
&= e^{2kt}\sum_{i=1}^{n}\sum_{j=1}^{n} h_ih_j\tilde{x}_s^T(t)(2kP + PG_{ij} + G_{ij}^TP \\
&\quad + PM_iQ^{-1}M_i^TP + e^{2k\tau}Q)\tilde{x}_s(t) \\
&= e^{2kt}\sum_{i=1}^{n}\sum_{j=1}^{n} h_ih_j\tilde{x}_s^T(t)\Omega_{ij}\tilde{x}_s(t) \le 0
\end{aligned} \tag{23}$$

where

$$\Omega_{ij} = 2kP + PG_{ij} + G_{ij}^TP + PM_iQ^{-1}M_i^TP + e^{2k\tau}Q \le 0. \tag{24}$$

From (23), it is easy to see that

$$\begin{aligned}
\lambda_{\min}(P)e^{2kt}\|\tilde{x}_s(t)\|^2 \le V(\tilde{x}_s(t)) &\le V(\tilde{x}_s(0)) \\
&= \tilde{x}_s^T(0)P\tilde{x}_s(0) + \int_{-\tau}^{0}\tilde{x}_s^T(\alpha)Q\tilde{x}_s(\alpha)d\alpha.
\end{aligned} \tag{25}$$

Therefore, it follows from (25) that

$$\|\tilde{x}_s(t)\| \le \sqrt{\frac{\lambda_{\max}(P) + \tau\lambda_{\max}(Q)}{\lambda_{\min}(P)}}\, e^{-kt} \sup_{-\tau\le\theta\le 0}\|\tilde{x}_s(\theta)\|. \tag{26}$$

Thus, from (26), the closed-loop system (19) is globally exponentially stable if (24) holds.

Next, we prove that (24) holds if LMIs (21) and (22) are satisfied. Pre- and postmultiply P^{-1} to (24) and apply the change of variables such that $X = P^{-1}, S = P^{-1}QP^{-1}$; then

$$2kX + XG_{ij}^T + G_{ij}X + e^{2k\tau}S + M_iXS^{-1}XM_i^T \le 0. \tag{27}$$

Let

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2^{-1} \end{bmatrix} \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & X_2^{-1}S_2X_2^{-1} \end{bmatrix}.$$

Then we have

$$\begin{aligned}
&2kX_1 + X_1A_i^T + A_iX_1 + B_iK_jX_1 + X_1K_j^TB_i^T \\
&\quad + e^{2k\tau}S_1 + A_{di}X_1S_1^{-1}X_1A_{di}^T \\
&\quad \le 0
\end{aligned} \tag{28}$$

$$\begin{aligned}
&2kX_2^{-1} + X_2^{-1}A_i^T + A_iX_2^{-1} - L_iC_sX_2^{-1} \\
&\quad - X_2^{-1}C_s^TL_i^T + e^{2k\tau}X_2^{-1}S_2X_2^{-1} + A_{di}X_2^{-1}A_{di}^T \\
&\quad \le 0.
\end{aligned} \tag{29}$$

Pre- and postmultiply X_2 to (29) and apply the change of variables such that $Y_j = K_jX_1, R_i = X_2L_i$. From Schur complements, we can find (28) and (29) are equivalent to the following LMIs, respectively:

$$\begin{bmatrix} Z_{1ij} & * \\ X_1A_{di}^T & -S_1 \end{bmatrix} \le 0 \tag{30}$$

$$\begin{bmatrix} Z_{2ij} & * \\ A_{di}^TX_2 & -S_2 \end{bmatrix} \le 0 \tag{31}$$

and define

$$X = \begin{bmatrix} \lambda X_1 & 0 \\ 0 & X_2^{-1} \end{bmatrix} \quad S = \begin{bmatrix} \lambda S_1 & 0 \\ 0 & X_2^{-1}S_2X_2^{-1} \end{bmatrix}.$$

where $\lambda > 0$. Similar to the derivation of [1, Ch. 7.6], we can prove that there always exist a real $\lambda > 0$ such that (27) holds. This means that the closed-loop fuzzy system with time delay described by (19) is globally exponentially stable. This completes the proof. □

Theorem 2: For the fuzzy delay system (14), its closed-loop fuzzy system (19) with observer-based control law (17) is exponentially stable and the exponential decay rate b satisfying (40), if there exist matrices $X_1 > 0, X_2 > 0, S_1 > 0, S_2 > 0, Y_i$, and R_i satisfying the following LMIs for all i and j except the pairs (i, j), such that $h_i(\theta(t))h_j(\theta(t)) = 0$:

$$\begin{bmatrix} X_1A_i^T + A_iX_1 + B_iY_i + Y_i^TB_i^T + S_1 & * \\ X_1A_{di}^T & -S_1 \end{bmatrix} < 0 \tag{32}$$

$$\begin{bmatrix} A_i^TX_2 + X_2A_i - R_iC_s - C_s^TR_i^T + S_2 & * \\ A_{di}^TX_2 & -S_2 \end{bmatrix} < 0 \tag{33}$$

$$\begin{bmatrix} \Theta_{1ij} + 2S_1 & * \\ X_1(A_{di} + A_{dj})^T & -2S_1 \end{bmatrix} \le 0, \qquad i < j \tag{34}$$

$$\begin{bmatrix} \Theta_{2ij} + 2S_2 & * \\ (A_{di} + A_{dj})^TX_2 & -2S_2 \end{bmatrix} \le 0, \qquad i < j \tag{35}$$

where $*$ represents blocks that are readily inferred by symmetry and $\Theta_{1ij} = X_1A_i^T + A_iX_1 + X_1A_j^T + A_jX_1 + B_iY_j + Y_j^TB_i^T + B_jY_i + Y_i^TB_j^T$, $\Theta_{2ij} = A_i^TX_2 + X_2A_i + A_j^TX_2 + X_2A_j - R_iC_s - C_s^TR_i^T - R_jC_s - C_s^TR_j^T$.

Then the state feedback gains and observer gains can be constructed as

$$K_i = Y_iX_1^{-1} \quad L_i = X_2^{-1}R_i$$

for $i = 1, 2, \ldots, n$, respectively.

Proof: Consider the following fuzzy Lyanpunov functional:

$$V(\tilde{x}_s) = \tilde{x}_s^T(t)P\tilde{x}_s(t) + \int_{t-\tau}^{t} \tilde{x}_s^T(\alpha)Q\tilde{x}_s(\alpha)d\alpha.$$

Calculating the derivative of V along the trajectory of (19), we obtain

$$\begin{aligned}\dot{V}(\tilde{x}_s) &= \sum_{i=1}^{n}\sum_{j=1}^{n} h_ih_j \left[\tilde{x}_s^T(t)\left(PG_{ij} + G_{ij}^TP\right)\tilde{x}_s(t)\right.\\ &\qquad\qquad \left. + 2\tilde{x}_s^T(t)PM_i\tilde{x}_s(t-\tau)\right]\\ &\quad + \tilde{x}_s^T(t)Q\tilde{x}_s(t) - \tilde{x}_s^T(t-\tau)Q\tilde{x}_s(t-\tau)\\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n} h_ih_j \left[\tilde{x}_s^T(t)\left(PG_{ij} + G_{ij}^TP\right)\tilde{x}_s(t)\right.\\ &\qquad\qquad \tilde{x}_s^T(t)PM_iQ^{-1}M_i^TP\tilde{x}_s(t)\\ &\qquad\qquad \left. + \tilde{x}_s^T(t-\tau)Q\tilde{x}_s(t-\tau)\right]\\ &\quad + \tilde{x}_s^T(t)Q\tilde{x}_s(t) - \tilde{x}_s^T(t-\tau)Q\tilde{x}_s(t-\tau)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} h_ih_j\tilde{x}_s^T(t)(PG_{ij} + G_{ij}^TP\\ &\quad + PM_iQ^{-1}M_i^TP + Q)\tilde{x}_s(t)\\ &= \sum_{i=1}^{n} h_i^2\tilde{x}_s^T(t)\bar{\Omega}_{ii}\tilde{x}_s(t) + \sum_{i,j=1,i<j}^{n} h_ih_j\tilde{x}_s^T(t)\bar{\Omega}_{ij}\tilde{x}_s(t)\\ &< 0.\end{aligned} \tag{36}$$

Obviously, (36) holds if the following inequalities are satisfied:

$$\bar{\Omega}_{ii} = PG_{ii} + G_{ii}^TP + PM_iQ^{-1}M_i^TP + Q < 0 \tag{37}$$

$$\begin{aligned}\bar{\Omega}_{ij} &= P(G_{ij} + G_{ji}) + (G_{ij} + G_{ji})^TP + 2Q\\ &\quad + P(M_i + M_j)(2Q)^{-1}(M_i + M_j)^TP\\ &\le 0, \qquad i < j.\end{aligned} \tag{38}$$

From (36), it is easy to see that there exists a scalar a such that

$$\dot{V}(\tilde{x}_{st}) \le a\|\tilde{x}_s(t)\|^2 \tag{39}$$

where $a = \max_{1\le i\le n}(\lambda_{\max}(\bar{\Omega}_{ii})) < 0$.

Now we choose a scalar $b > 0$ satisfying

$$b[\lambda_{\max}(P) + \lambda_{\max}(Q)\tau e^{b\tau}] + a = 0. \tag{40}$$

Note that

$$V(\tilde{x}_{st}) \le \lambda_{\max}(P)\|\tilde{x}_s(t)\|^2 + \lambda_{\max}(Q)\int_{t-\tau}^{t}\|\tilde{x}_s(\alpha)\|^2 d\alpha. \tag{41}$$

Then, for any scalar $b > 0$, it can be verified that

$$\begin{aligned}\frac{d}{dt}(e^{bt}V(\tilde{x}_{st})) &= e^{bt}[bV(\tilde{x}_{st}) + \dot{V}(\tilde{x}_{st})]\\ &\le e^{bt}\Big[(b\lambda_{\max}(P) + a)\|\tilde{x}_s(t)\|^2\\ &\qquad + b\lambda_{\max}(Q)\int_{t-\tau}^{t}\|\tilde{x}_s(\alpha)\|^2 d\alpha\Big]\end{aligned} \tag{42}$$

Now, integrating both sides of (42) from zero to $T > 0$, we obtain

$$\begin{aligned}&e^{bT}V(\tilde{x}_{sT}) - V(\tilde{x}_{s0})\\ &\quad\le (b\lambda_{\max}(P) + a)\int_0^T e^{bt}\|\tilde{x}_s(t)\|^2 dt\\ &\qquad + b\lambda_{\max}(Q)\int_0^T\int_{t-\tau}^{t} e^{bt}\|\tilde{x}_s(\alpha)\|^2 d\alpha dt.\end{aligned} \tag{43}$$

By some calculations, it can be verified that

$$\begin{aligned}&\int_0^T\int_{t-\tau}^{t} e^{bt}\|\tilde{x}_s(\alpha)\|^2 d\alpha dt\\ &\quad\le \tau\int_{-\tau}^{T} e^{b(t+\tau)}\|\tilde{x}_s(t)\|^2 dt\\ &\quad\le \tau e^{b\tau}\int_{-\tau}^{0}\|\tilde{x}_s(t)\|^2 dt + \tau e^{b\tau}\int_0^T e^{bt}\|\tilde{x}_s(t)\|^2 dt.\end{aligned} \tag{44}$$

Using (40), (43), and (44), we have

$$\begin{aligned}e^{bT}V(\tilde{x}_{sT}) &\le (b\lambda_{\max}(P) + b\lambda_{\max}(Q)\tau e^{b\tau} + a)\\ &\qquad\times\int_0^T e^{bt}\|\tilde{x}_s(t)\|^2 dt + b\lambda_{\max}(Q)\tau e^{b\tau}\\ &\qquad\times\int_{-\tau}^{0}\|\tilde{x}_s(t)\|^2 dt + V(\tilde{x}_{s0})\\ &= b\lambda_{\max}(Q)\tau e^{b\tau}\int_{-\tau}^{0}\|\tilde{x}_s(t)\|^2 dt + V(\tilde{x}_{s0}).\end{aligned} \tag{45}$$

Considering this, it is easy to see that

$$e^{bT}V(\tilde{x}_{sT}) \le c\sup_{-\tau\le\theta\le 0}\|\tilde{x}_s(\theta)\|^2 \tag{46}$$

where $c = b\lambda_{\max}(Q)\tau^2e^{b\tau} + \lambda_{\max}(P) + \tau\lambda_{\max}(Q)$.

Therefore, it follows from (46) that

$$\|\tilde{x}_s(T)\| \le \sqrt{\frac{c}{\lambda_{\min}(P)}}e^{-bT/2}\sup_{-\tau\le\theta\le 0}\|\tilde{x}_s(\theta)\|. \tag{47}$$

Thus, from (47), the closed-loop system (19) is globally exponentially stable if (37) and (38) hold.

Next, we prove that (37) and (38) hold if LMIs (32)–(35) are satisfied.

Pre- and postmultiply P^{-1} to (37) and (38) and apply the change of variables such that $X = P^{-1}, S = P^{-1}QP^{-1}$; then

$$XG_{ii}^T + G_{ii}X + S + M_iXS^{-1}XM_i < 0 \tag{48}$$

$$\begin{aligned} &X(G_{ij}+G_{ji})^T + (G_{ij}+G_{ji})X + 2S \\ &\quad + \frac{1}{2}(M_i+M_j)XS^{-1}X(M_i+M_j)^T \\ &\quad \le 0, \qquad i<j. \end{aligned} \tag{49}$$

Let

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2^{-1} \end{bmatrix} \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & X_2^{-1}S_2X_2^{-1} \end{bmatrix}.$$

Then we have

$$\begin{aligned} &X_1A_i^T + A_iX_1 + B_iK_iX_1 + X_1K_i^TB_i^T + S_1 \\ &\quad + A_{di}X_1S_1^{-1}X_1A_{di}^T < 0 \end{aligned} \tag{50}$$

$$\begin{aligned} &X_2^{-1}A_i^T + A_iX_2^{-1} - L_iC_sX_2^{-1} - X_2^{-1}C_s^TL_i^T \\ &\quad + X_2^{-1}S_2X_2^{-1} + A_{di}X_2^{-1}A_{di}^T < 0 \end{aligned} \tag{51}$$

$$\begin{aligned} &X_1A_i^T + A_iX_1 + X_1A_j^T + A_jX_1 + B_iK_jX_1 \\ &\quad + X_1K_j^TB_i^T + B_jK_iX_1 + X_1K_i^TB_j^T \\ &\quad + 2S_1 + \frac{1}{2}(A_{di}+A_{dj})X_1S_1^{-1}X_1(A_{di}+A_{dj})^T \\ &\le 0, \qquad i<j \end{aligned} \tag{52}$$

$$\begin{aligned} &X_2^{-1}A_i^T + A_iX_2^{-1} + X_2^{-1}A_j^T + A_jX_2^{-1} - L_iC_sX_2^{-1} \\ &\quad - X_2^{-1}C_s^TL_i^T - L_jC_sX_2^{-1} - X_2^{-1}C_s^TL_j^T \\ &\quad + 2X_2^{-1}S_2X_2^{-1} + \frac{1}{2}(A_{di}+A_{dj})S_2^{-1}(A_{di}+A_{dj})^T \\ &\le 0, \qquad i<j. \end{aligned} \tag{53}$$

Pre- and postmultiply X_2 to (51) and (53) and apply the change of variables such that $Y_i = K_iX_1, R_i = X_2L_i$. From Schur complements, we can find (50)–(53) are equivalent to LMIs (32)–(35). The rest of the proof is similar to that of Theorem 1. □

Remark 4: The condition of Theorem 1 is dependent on time delay τ and exponential decay rate k, and the condition of Theorem 2 is independent of them.

B. Robust Stabilization via Fuzzy Control

If there is a perturbation in the nonlinearity or the boundary conditions of distributed parameter system, the slow system can be represented in the following T–S model.

Plant rule i:

If $\theta_1(t)$ is $F_{i1}, \ldots$, and $\theta_p(t)$ is F_{ip}, then

$$\begin{aligned} \dot{x}_s(t) &= (A_i + \Delta A_i(t))x_s(t) \\ &\quad + (A_{di} + \Delta A_{di}(t))x_s(t-\tau) + B_iu(t) \\ T_s &= C_sx_s(t) \\ x_s(t) &= \phi_s(t), \qquad t\in[-\tau,0]; \qquad i=1,2,\ldots,n \end{aligned}$$

where the matrices $\Delta A_i(t)$ and $\Delta A_{di}(t)$ denote the uncertainties at time t and have the following form:

$$[\Delta A_i(t), \Delta A_{di}(t)] = MH(t)[E_i, E_{di}] \tag{54}$$

where M, E_i, and E_{di} are known constant matrices and $F(t)$ is an unknown matrix function with the property $H^T(t)H(t) \le I$.

The overall fuzzy system is inferred as follows:

$$\begin{aligned} \dot{x}_s(t) &= \sum_{i=1}^n h_i(\theta(t))[(A_i+\Delta A_i)x_s(t) \\ &\quad + (A_{di}+\Delta A_{di})x_s(t-\tau) + B_iu(t)] \\ T_s &= C_sx_s(t). \end{aligned} \tag{55}$$

The overall fuzzy observer can be written as follows:

$$\begin{aligned} \dot{\hat{x}}_s(t) &= \sum_{i=1}^n h_i(\theta(t))[(A_i+\Delta A_i)\hat{x}_s(t) + (A_{di}+\Delta A_{di}) \\ &\qquad \times \hat{x}_s(t-\tau) + L_i(T_s - \hat{T}_s) + B_iu(t)] \\ \hat{T}_s &= C_s\hat{x}_s(t). \end{aligned} \tag{56}$$

For the simplicity, introduce the following notations:

$$\bar{A}_i = A_i + \Delta A_i \quad \bar{A}_{di} = A_{di} + \Delta A_{di}.$$

The overall controller is given in the following:

$$u(t) = \sum_{i=1}^n h_i(\theta(t))K_i\hat{x}_s(t). \tag{57}$$

Similarly, the closed-loop system with observer-based control law can be rewritten as follows

$$\dot{\tilde{x}}_s(t) = \sum_{i=1}^n\sum_{j=1}^n h_i(\theta(t))h_j(\theta(t))[\bar{G}_{ij}x_s(t) + \bar{M}_i\tilde{x}_s(t-\tau)] \tag{58}$$

where

$$\begin{aligned} \tilde{x}_s(t) &= \begin{bmatrix} x_s(t) \\ e(t) \end{bmatrix} \quad M_i = \begin{bmatrix} \bar{A}_{di} & 0 \\ 0 & \bar{A}_{di} \end{bmatrix} \\ \bar{G}_{ij} &= \begin{bmatrix} \bar{A}_i + B_iK_j & -B_iK_j \\ 0 & \bar{A}_i - L_iC_s \end{bmatrix}. \end{aligned} \tag{59}$$

Theorem 3: There exists an observer-based fuzzy control law (57) such that the closed-loop fuzzy system with time delay described by (58) is robust exponentially stable if there exist positive constants ϵ_1, ϵ_2 and the matrices $X_1 > 0, X_2 > 0, S_1 > 0, S_2 > 0, R_i$, and Y_i satisfying the following linear matrix inequalities:

$$\begin{bmatrix} V_{1ij} & A_{di}X_1 & X_1E_i^T & 0 & M & M \\ X_1A_{di}^T & -S_1 & 0 & X_1E_{di}^T & 0 & 0 \\ E_iX_1 & 0 & -\epsilon_1^{-1}I & 0 & 0 & 0 \\ 0 & E_{di}X_1 & 0 & -\epsilon_2^{-1}I & 0 & 0 \\ M^T & 0 & 0 & 0 & -\epsilon_1I & 0 \\ M^T & 0 & 0 & 0 & 0 & -\epsilon_2I \end{bmatrix} \le 0 \tag{60}$$

where

$$\begin{aligned} V_{1ij} &= 2kX_1 + X_1A_i^T + A_iX_1 + B_iY_j \\ &\quad + Y_j^TB_i^T + e^{2k\tau}S_1 \end{aligned}$$

$$\begin{bmatrix} V_{2i} & X_2A_{di} & X_2M & X_2M \\ A_{di}^TX_2 & S_{2i} & 0 & 0 \\ M^TX_2 & 0 & -\epsilon_1I & 0 \\ M^TX_2 & 0 & 0 & -\epsilon_2I \end{bmatrix} \le 0 \tag{61}$$

where

$$\begin{aligned} V_{2i} &= 2kX_2 + A_i^TX_2 + X_2A_i - R_iC_s - C_s^TR_i^T \\ &\quad + e^{2k\tau}S_2 + \epsilon_1E_i^TE_i \\ S_{2i} &= -S_2 + \epsilon_2E_{di}^TE_{di}. \end{aligned}$$

Proof: From the proof of Theorem 1, we can derive that the closed-loop system (58) is robust exponentially stable when the following matrix inequalities hold:

$$\begin{bmatrix} \bar{Z}_{1ij} & * \\ X_1\bar{A}_{di}^T & -S_1 \end{bmatrix} \le 0 \tag{62}$$

$$\begin{bmatrix} \bar{Z}_{2ij} & * \\ \bar{A}_{di}^TX_2 & -S_2 \end{bmatrix} \le 0 \tag{63}$$

where

$$\begin{aligned} \bar{Z}_{1ij} &= 2kX_1 + X_1\bar{A}_i^T + \bar{A}_iX_1 \\ &\quad + B_iY_j + Y_j^TB_i^T + e^{2k\tau}S_1 \\ \bar{Z}_{2ij} &= 2kX_2 + \bar{A}_i^TX_2 + X_2\bar{A}_i \\ &\quad - R_iC_s - C_s^TR_i^T + e^{2k\tau}S_2. \end{aligned}$$

Next, we prove the equivalence of (60) and (61) and (62) and (63), respectively. Equation (62) is exactly

$$\begin{aligned} &\begin{bmatrix} Z_{1ij} & A_{di}X_1 \\ X_1A_{di}^T & -S_1 \end{bmatrix} + \begin{bmatrix} X_1\Delta A_i^T + \Delta A_iX_1 & \Delta A_{di}X_1 \\ X_1\Delta A_{di}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} Z_{1ij} & \\ X_1A_{di}^T & -S_1 \end{bmatrix} + \begin{bmatrix} X_1E_i^T \\ 0 \end{bmatrix} H^T(t)[\,M^T \quad 0\,] \\ &\quad + \begin{bmatrix} M \\ 0 \end{bmatrix} H(t)[\,E_iX_1 \quad 0\,] \\ &\quad + \begin{bmatrix} 0 \\ X_1E_{di}^T \end{bmatrix} H^T(t)[\,M^T \quad 0\,] \\ &\quad + \begin{bmatrix} M \\ 0 \end{bmatrix} H(t)[\,0 \quad E_{di}X_1\,] \\ &< 0. \end{aligned} \tag{64}$$

From Lemma 2, (64) holds for all $H^T(t)H(t) \le I$ if and only if there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$$\begin{aligned} &\begin{bmatrix} Z_{1ij} & A_{di}X_1 \\ X_1A_{di}^T & -S_1 \end{bmatrix} + \frac{1}{\epsilon_1}\begin{bmatrix} M \\ 0 \end{bmatrix}[\,M^T \quad 0\,] \\ &\quad + \epsilon_1\begin{bmatrix} X_1E_i^T \\ 0 \end{bmatrix}[\,E_iX_1 \quad 0\,] \\ &\quad + \frac{1}{\epsilon_2}\begin{bmatrix} M \\ 0 \end{bmatrix}[\,M^T \quad 0\,] \\ &\quad + \epsilon_2\begin{bmatrix} 0 \\ X_1E_{di}^T \end{bmatrix}[\,0 \quad E_{di}X_1\,] \\ &\quad < 0 \end{aligned} \tag{65}$$

which is as shown in (66) at the bottom of the page. By Schur's complement, we know (66) is equivalent to (60). Similarly, we can prove the equivalence of (61) and (63). This completes the proof.

Theorem 4: There exists an observer-based fuzzy control law (57) such that the closed-loop fuzzy system with time delay described by (58) is robust exponentially stable if there exist positive constants ϵ_1, ϵ_2 and the matrices $X_1 > 0, X_2 > 0, S_1 > 0, S_2 > 0, R_i$, and Y_i satisfying the following linear matrix inequalities:

$$\begin{bmatrix} Z_{1i} & A_{di}X_1 & X_1E_i^T & 0 & M & M \\ X_1A_{di}^T & -S_1 & 0 & X_1E_{di}^T & 0 & 0 \\ E_iX_1 & 0 & -\epsilon_1^{-1}I & 0 & 0 & 0 \\ 0 & E_{di}X_1 & 0 & -\epsilon_2^{-1}I & 0 & 0 \\ M^T & 0 & 0 & 0 & -\epsilon_1I & 0 \\ M^T & 0 & 0 & 0 & 0 & -\epsilon_2I \end{bmatrix} < 0 \tag{67}$$

where $Z_{1i} = X_1A_i^T + A_iX_1 + B_iY_i + Y_i^TB_i^T + S_1$

$$\begin{bmatrix} Z_{2i} & X_2A_{di} & X_2M & X_2M \\ A_{di}^TX_2 & S_{2i} & 0 & 0 \\ M^TX_2 & 0 & -\epsilon_1I & 0 \\ M^TX_2 & 0 & 0 & -\epsilon_2I \end{bmatrix} < 0 \tag{68}$$

where $Z_{2i} = A_i^TX_2 + X_2A_i - R_iC_s - C_s^TR_i^T + S_2 + \epsilon_1E_i^TE_i, S_{2i} = -S_2 + \epsilon_2E_{di}^TE_{di}$

$$\begin{bmatrix} Z_{1ij} & A_{dij}X_1 & X_1E_{ij}^T & 0 & M & M \\ * & -2S_1 & 0 & X_1E_{dij}^T & 0 & 0 \\ * & * & -\epsilon_1^{-1}I & 0 & 0 & 0 \\ * & * & * & -\epsilon_2^{-1}I & 0 & 0 \\ * & * & * & * & -\epsilon_1I & 0 \\ * & * & * & * & * & -\epsilon_2I \end{bmatrix} \le 0, \quad i < j \tag{69}$$

where $Z_{1ij} = \Theta_{1ij} + 2S_1, A_{dij} = A_{di} + A_{dj}, E_{ij} = E_i + E_j, E_{dij} = E_{di} + E_{dj}$

$$\begin{bmatrix} Z_{2ij} & X_2A_{dij} & X_2M & X_2M \\ A_{dij}^TX_2 & S_{2ij} & 0 & 0 \\ M^TX_2 & 0 & -\epsilon_1I & 0 \\ M^TX_2 & 0 & 0 & -\epsilon_2I \end{bmatrix} \le 0, \qquad i < j \tag{70}$$

where $Z_{2ij} = \Theta_{2ij} + 2S_2 + \epsilon_1(E_i + E_j)^T(E_i + E_j), S_{2ij} = -2S_2 + \epsilon_2(E_{di} + E_{dj})^T(E_{di} + E_{dj})$.

Proof: Similarly, from the proof of Theorems 2 and 3, the proof of Theorem 4 is derived directly. Here omitted.

IV. Numerical Example

In this section, a numerical example will be given to illustrate the approaches developed in Section III.

Consider the following distributed parameter system:

$$\begin{cases} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2} + \alpha T(t) + \beta T(t-\tau)) \\ \quad +\gamma(T(t-\tau))^2 + zu(t), \qquad z \in (0, \pi) \\ \frac{\partial T}{\partial z}\big|_{z=0} = 0, \quad \frac{\partial T}{\partial z}\big|_{z=\pi} = 0 \\ T(t,z) = \cos z + \cos 2z, \qquad t \in [-\tau, 0] \end{cases} \tag{71}$$

where time delay $\tau = 1$. Using Galerkin's method and neglecting the fast system, a three-dimensional ODE system with

$$\begin{bmatrix} V_{1ij} + \epsilon_1X_1E_i^TE_iX_1 + \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)MM^T & A_{di}X_1 \\ XA_{di}^T & -S_1 + \epsilon_2X_1E_{di}^TE_{di}X_1 \end{bmatrix} \tag{66}$$

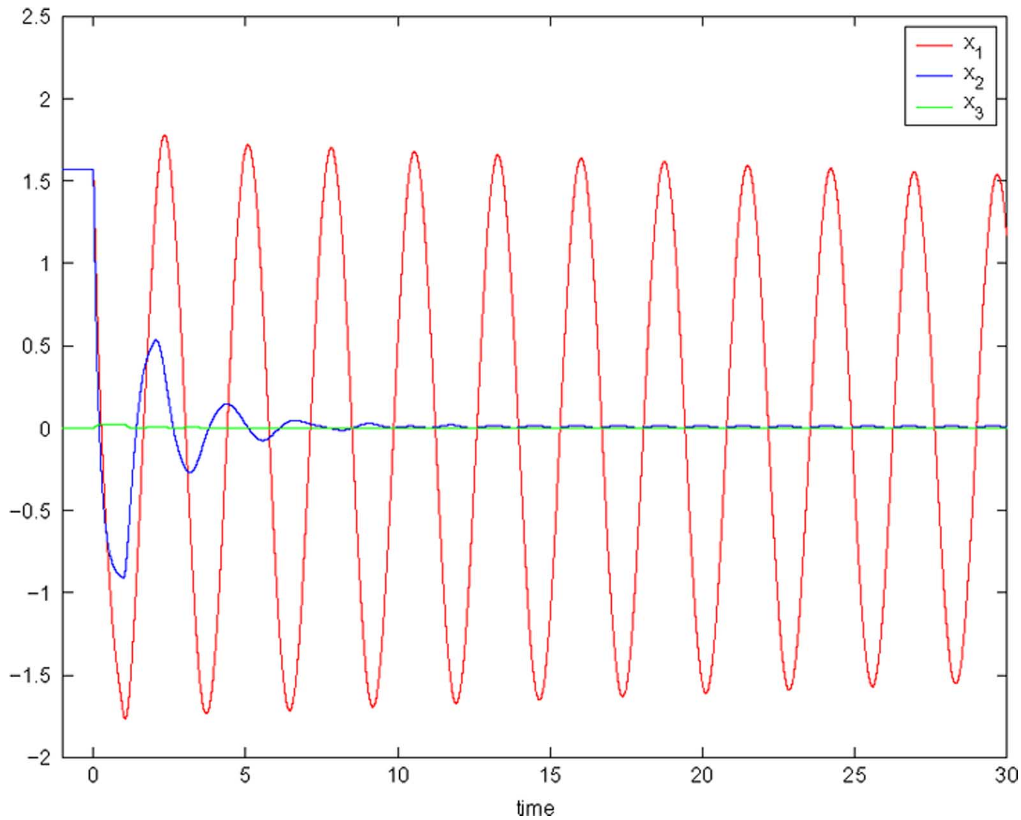

Fig. 1. State response of slow system under $u = 0$.

time delay is derived as follows:

$$\begin{cases} \dot{x}_1(t) = -(1-\alpha)x_1(t) + \beta x_1(t-\tau) \\ \quad +\frac{\gamma}{2}[x_1(t-\tau)x_2(t-\tau) \\ \quad +x_2(t-\tau)x_3(t-\tau)] + \frac{4}{\pi}u(t) \\ \dot{x}_2(t) = -(4-\alpha)x_2(t) + \beta x_2(t-\tau) \\ \quad +\frac{\gamma}{2}\left[x_1^2(t-\tau) + x_1(t-\tau)x_3(t-\tau)\right] \\ \dot{x}_3(t) = -(9-\alpha)x_3(t) + \beta x_3(t-\tau) \\ \quad +\frac{\gamma}{2}x_1(t-\tau)x_2(t-\tau) + \frac{4}{9\pi}u(t) \\ T_s = \bar{C}_s x_s(t) \end{cases} \tag{72}$$

where $\bar{C}_s = [\cos z \quad \cos 2z \quad \cos 3z]$. Let premise variables $\theta_1(t) = (2/\pi)\int_0^\pi T(z,t)\cos z dz = x_1(t), \theta_2(t) = (2/\pi)\int_0^\pi T(z,t)\cos 2z dz = x_2(t), z_1^* = (\pi/2), z_2^* = (2\pi/3)$ and membership functions

$$F_1^1(\theta_1(t)) = \frac{1}{2}\left(1+\frac{\theta_1}{a}\right) \quad F_1^2(\theta_1(t)) = \frac{1}{2}\left(1-\frac{\theta_1}{a}\right)$$
$$F_2^1(\theta_2(t)) = \frac{1}{2}\left(1+\frac{\theta_2}{b}\right) \quad F_2^2(\theta_2(t)) = \frac{1}{2}\left(1-\frac{\theta_2}{b}\right).$$

It is also assumed that $\theta_1(t) \in [-a, a]$ and $\theta_2(t) \in [-b, b]$. By using F_1^1, F_1^2, F_2^1, and F_2^2, (72) can be represented by the following T–S fuzzy model:

Plant Rule 1:

IF $\theta_1(t)$ is F_1^1 and $\theta_2(t)$ is F_2^1, THEN
$\dot{x_s}(t) = A_1 x_s(t) + A_{d1}x_s(t-\tau) + B_s u(t)$;
$T_s = C_s x_s(t)$.

Plant Rule 2:

IF $\theta_1(t)$ is F_1^1 and $\theta_2(t)$ is F_2^2, THEN
$\dot{x_s}(t) = A_2 x_s(t) + A_{d2}x_s(t-\tau) + B_s u(t)$;
$T_s = C_s x_s(t)$.

Plant Rule 3:

IF $\theta_1(t)$ is F_1^2 and $\theta_2(t)$ is F_2^1, THEN
$\dot{x_s}(t) = A_3 x_s(t) + A_{d3}x_s(t-\tau) + B_s u(t)$;
$T_s = C_s x_s(t)$.

Plant Rule 4:

IF $\theta_1(t)$ is F_1^2 and $\theta_2(t)$ is F_2^2, THEN
$\dot{x_s}(t) = A_4 x_s(t) + A_{d4}x_s(t-\tau) + B_s u(t)$;
$T_s = C_s x_s(t)$;

where $x_s(t) = [x_1(t), x_2(t), x_3(t)]^T$ and

$$A_1 = A_2 = A_3 = A_4 = \begin{bmatrix} -(1-\alpha) & 0 & 0 \\ 0 & -(4-\alpha) & 0 \\ 0 & 0 & -(9-\alpha) \end{bmatrix}$$
$$A_{d1} = \begin{bmatrix} \beta & a\gamma/2 & b\gamma/2 \\ a\gamma/2 & \beta & a\gamma/2 \\ a\gamma/2 & 0 & \beta \end{bmatrix}$$
$$A_{d2} = \begin{bmatrix} \beta & a\gamma/2 & -b\gamma/2 \\ a\gamma/2 & \beta & a\gamma/2 \\ a\gamma/2 & 0 & \beta \end{bmatrix}$$
$$A_{d3} = \begin{bmatrix} \beta & -a\gamma/2 & b\gamma/2 \\ -a\gamma/2 & \beta & -a\gamma/2 \\ -a\gamma/2 & 0 & \beta \end{bmatrix}$$
$$A_{d4} = \begin{bmatrix} \beta & -a\gamma/2 & -b\gamma/2 \\ -a\gamma/2 & \beta & -a\gamma/2 \\ -a\gamma/2 & 0 & \beta \end{bmatrix}$$
$$B_s = \left[\tfrac{4}{\pi} \quad 0 \quad \tfrac{4}{9\pi}\right]^T$$
$$C_s = \begin{bmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

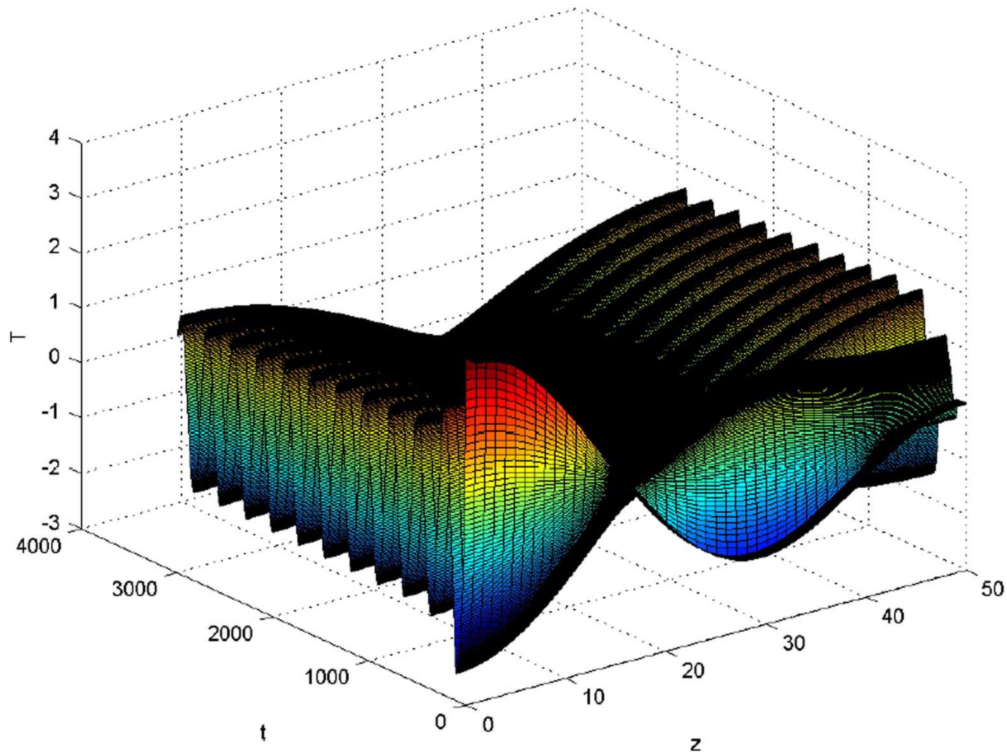

Fig. 2. State response of distributed parameter system under $u = 0$.

A. Fuzzy Control

Let $\alpha = -0.98, \beta = -3, \gamma = 0.1, a = 2$ and $b = 2$. Fig. 1 gives the open-loop state $x_s(t)$ responses and Fig. 2 shows the open-loop state $T(z,t)$ responses of (71). As shown in the simulation, we can find that states $x_s(t)$ of slow system and the state $T(z,t)$ of (71) are not exponentially stable under $u = 0$.

From Theorem 2, the observer and controller gains are constructed as $K_i = Y_i X_1^{-1}$ and $L_i = X_2^{-1} R_i, i = 1, \ldots, 4$. By solving the LMIs (32)–(35), we obtain

$$K1 = [-19.5860 \quad 0.0442 \quad -0.3633]$$
$$K2 = [-19.5840 \quad 0.0447 \quad -0.4061]$$
$$K3 = [-19.5866 \quad -0.0455 \quad -0.3675]$$
$$K4 = [-19.5845 \quad -0.0460 \quad -0.4102]$$
$$L1 = \begin{bmatrix} 102.3362 & -39.5594 \\ -21.9796 & -77.3593 \\ -53.4473 & 11.9687 \end{bmatrix}$$
$$L2 = \begin{bmatrix} -146.8719 & -39.5092 \\ -21.9135 & 111.0422 \\ 76.7123 & 12.0798 \end{bmatrix}$$
$$L3 = \begin{bmatrix} -222.3599 & -39.4988 \\ -21.8936 & 168.0872 \\ 116.0931 & 12.1147 \end{bmatrix}$$
$$L4 = \begin{bmatrix} 24.8920 & -39.4833 \\ -21.9591 & -18.8351 \\ -13.0446 & 12.0048 \end{bmatrix}.$$

Using the above observer-based control law, the closed-loop responses of the slow system are shown in Fig. 3, the observer error is demonstrated in Fig. 4, and the closed-loop state responses of the DPS (71) are shown in Fig. 5.

B. Robust Fuzzy Control

Now, let $\alpha = -0.8 + \sin(0.02t)$ and $\beta = -3 + \sin(0.016t)$. Figs. 6 and 7 give the state responses of slow system and the distributed system under $u = 0$, respectively.

In Theorem 4, by solving LMIs (67)–(70), we get $\epsilon_1 = 0.7$ and $\epsilon_2 = 0.6$, and the gains of the observer and controller are constructed as $K_i = Y_i X_1^{-1}$ and $L_i = X_2^{-1} R_i, i = 1, \ldots, 4$. By calculation, we obtain

$$K1 = [-14.6708 \quad 0.2351 \quad -0.2249]$$
$$K2 = [-14.6578 \quad 0.2475 \quad -0.4647]$$
$$K3 = [-14.6752 \quad -0.2552 \quad -0.2652]$$
$$K4 = [-14.6619 \quad -0.2675 \quad -0.5011]$$
$$L1 = 1.0e+003 * \begin{bmatrix} 1.0597 & -0.0330 \\ -0.0190 & -0.9667 \\ -0.5951 & -0.0080 \end{bmatrix}$$
$$L2 = 1.0e+003 * \begin{bmatrix} -1.0875 & -0.0166 \\ -0.0117 & 0.9916 \\ 0.6101 & 0.0187 \end{bmatrix}$$
$$L3 = 1.0e+003 * \begin{bmatrix} 2.1485 & -0.0411 \\ -0.0227 & -1.9600 \\ -1.2065 & -0.0216 \end{bmatrix}$$
$$L4 = 1.0e+003 * \begin{bmatrix} -2.3838 & -0.0065 \\ -0.0072 & 2.1738 \\ 1.3374 & 0.0349 \end{bmatrix}.$$

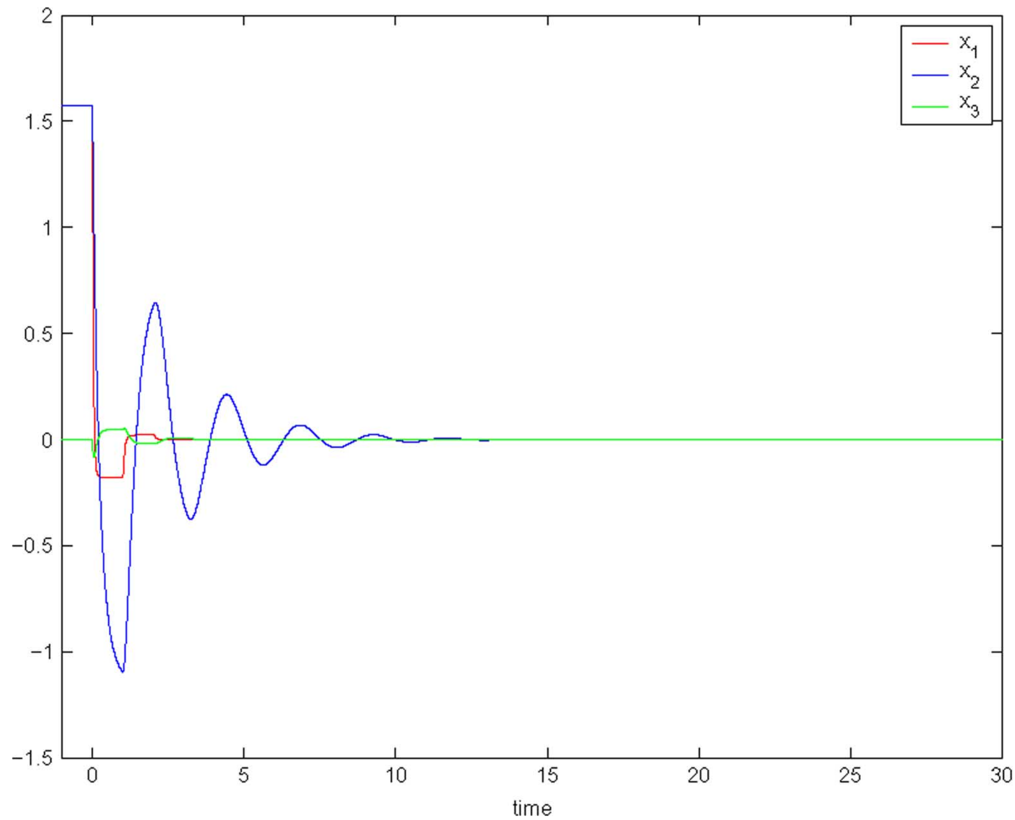

Fig. 3. State of slow closed-loop system.

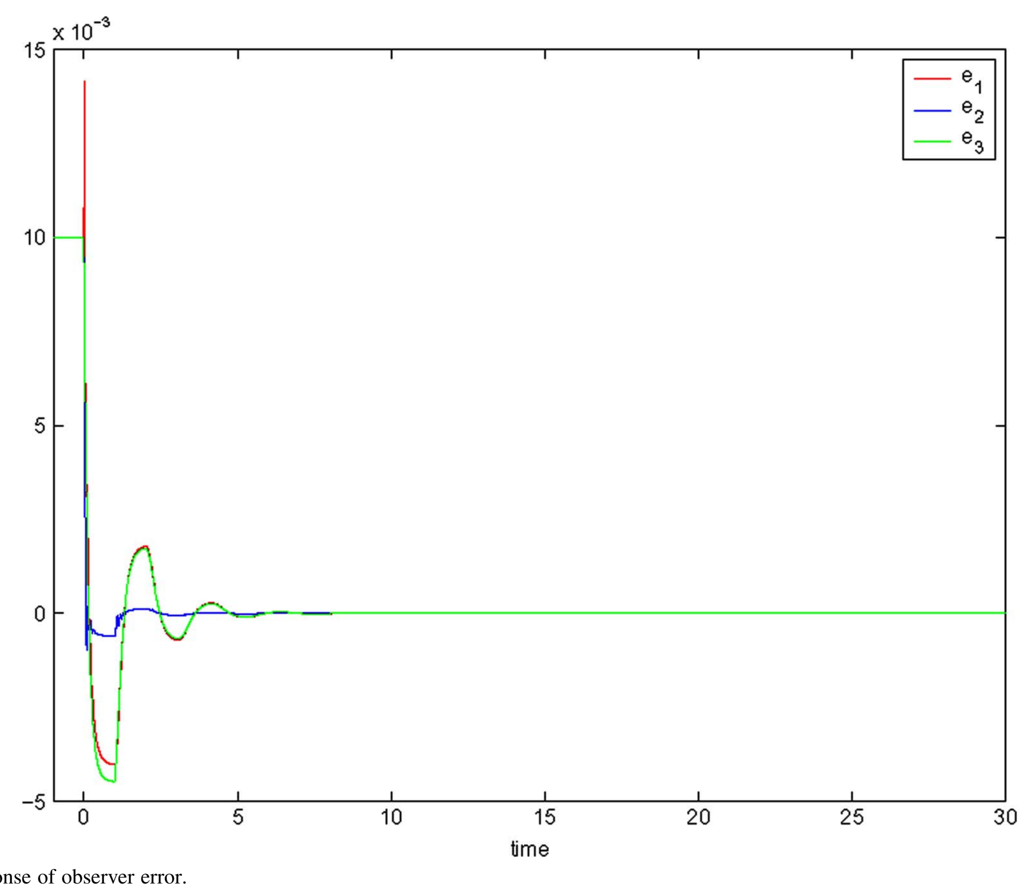

Fig. 4. Time response of observer error.

Using the above observer-based control law, the closed-loop responses of the slow system with uncertainty are shown in Fig. 8, the observer error is disclosed in Fig. 9, and the closed-loop state responses of the DPS with uncertainty are presented in Fig. 10.

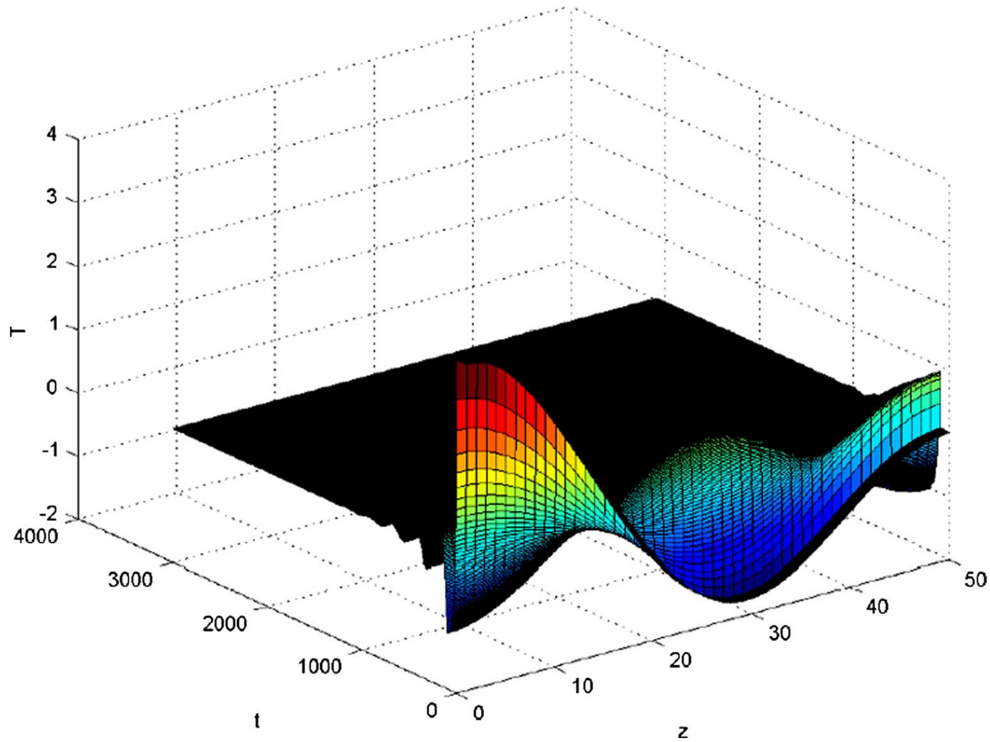

Fig. 5. State of closed-loop distributed parameter system.

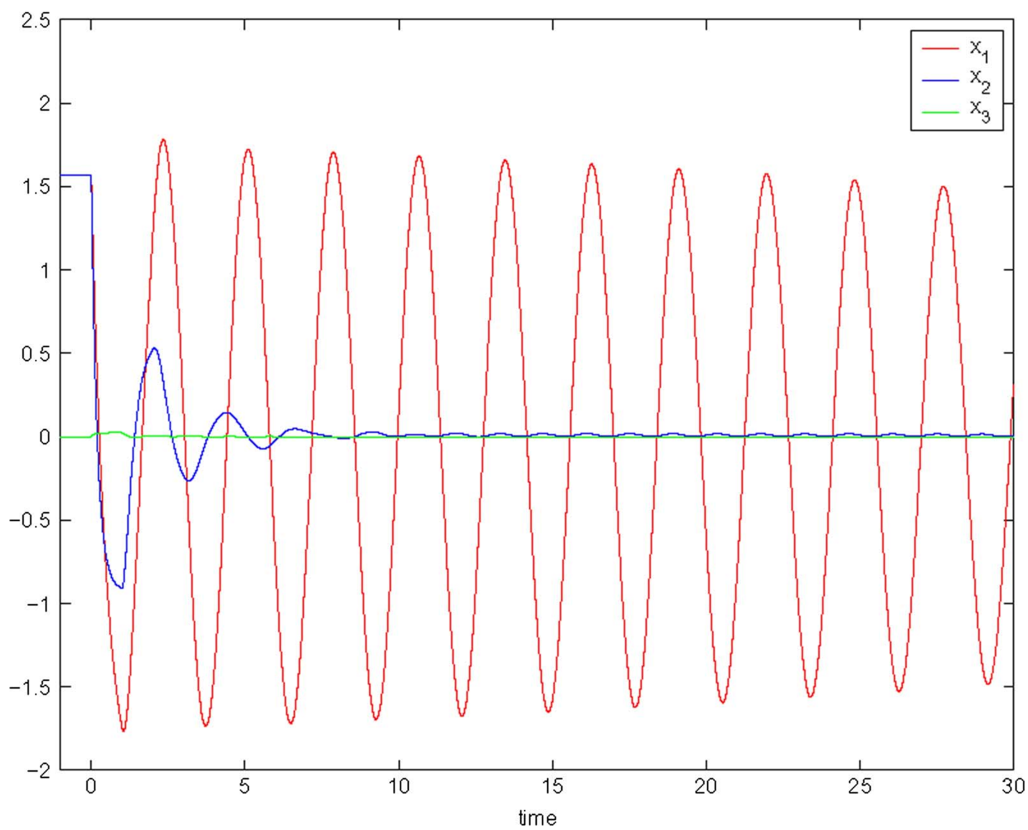

Fig. 6. Time response of slow system state under $u = 0$.

V. Conclusion

In this paper, we have studied stabilization for a class of distributed parameter system with time delay based on Galerkin's method and the fuzzy control approach. Using Galerkin's method, the low-order functional ODE is first derived to approximate the dynamics of DPS with delay. This low-order functional ODE can be used directly for the subsequent fuzzy

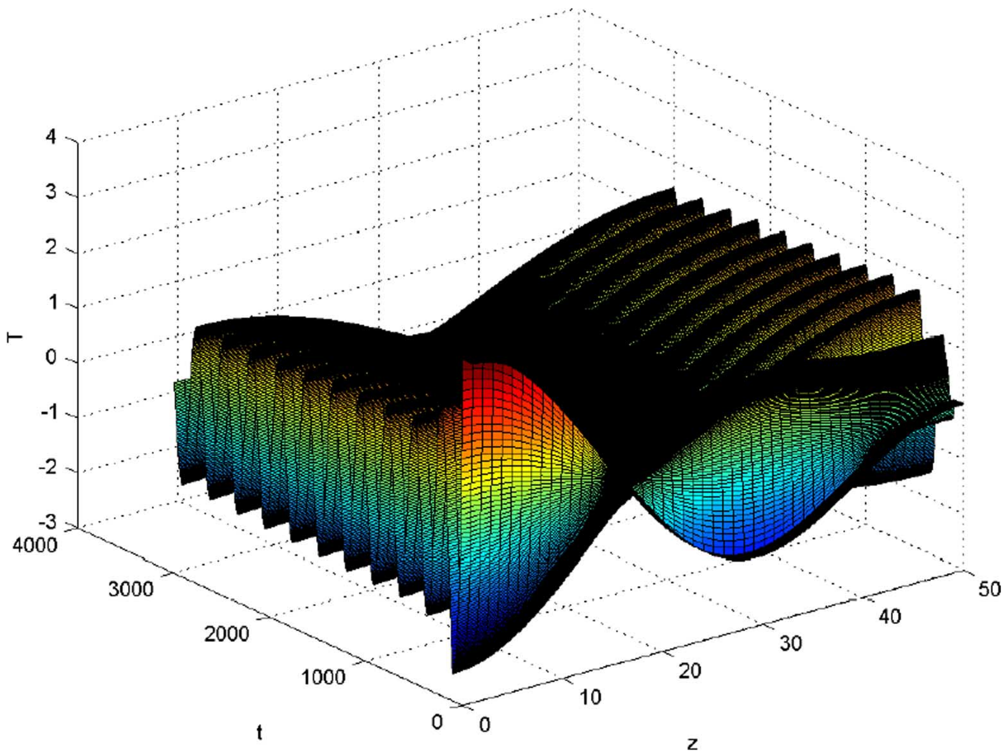

Fig. 7. State of distributed parameter system under $u = 0$.

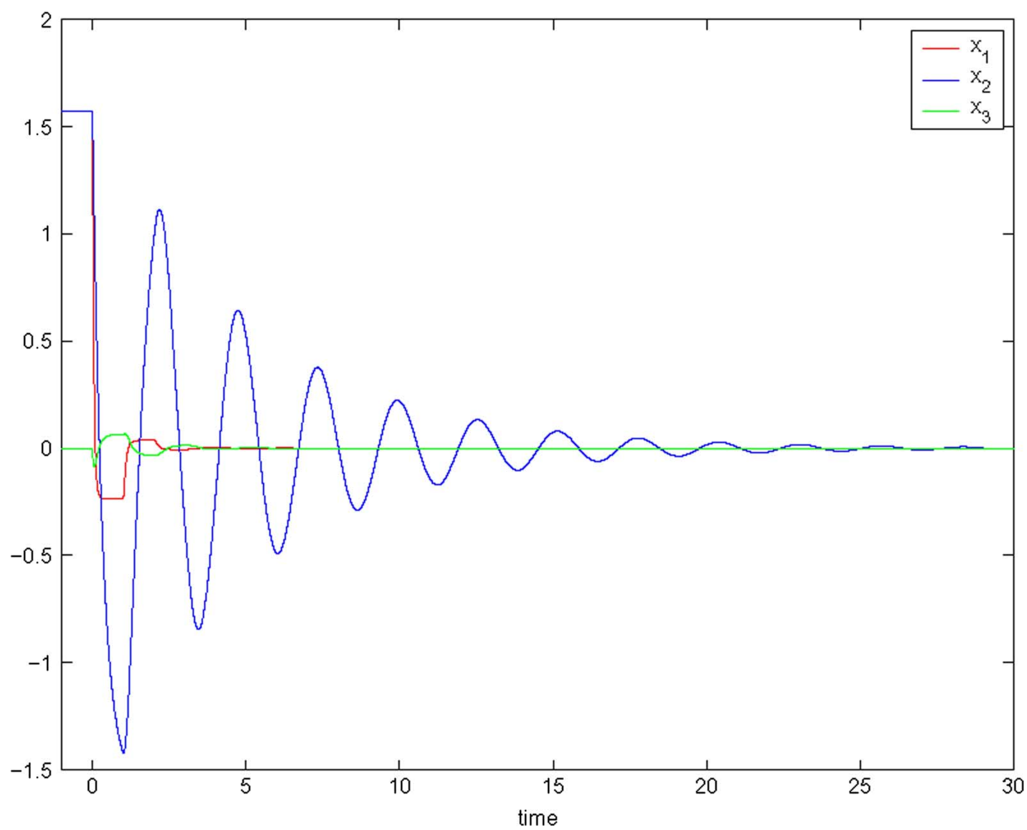

Fig. 8. State of slow closed-loop system.

controller design. Sufficient conditions on the existence of fuzzy observer-based control law have been given in terms of LMIs to ensure the exponential stability of the closed-loop distributed parameter system. The advantage of the proposed approach is that the resulting criteria can be performed efficiently via numerical algorithms such as interior-point algorithms for solving LMIs. A numerical example is also provided to demonstrate the effectiveness of the proposed method.

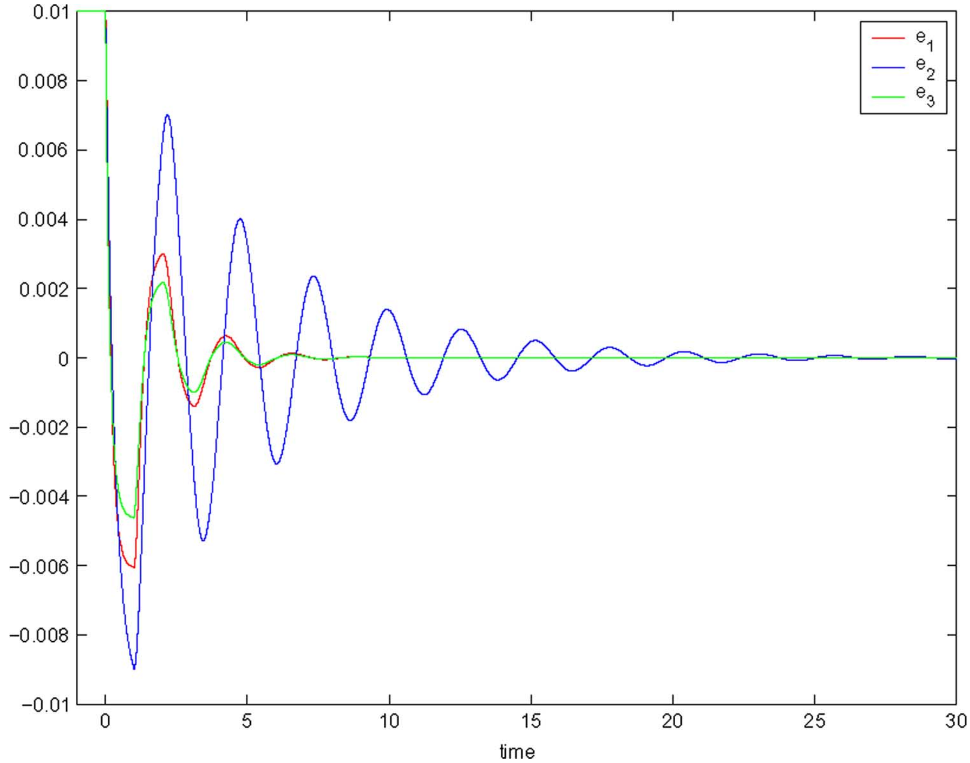

Fig. 9. Time response of observer error.

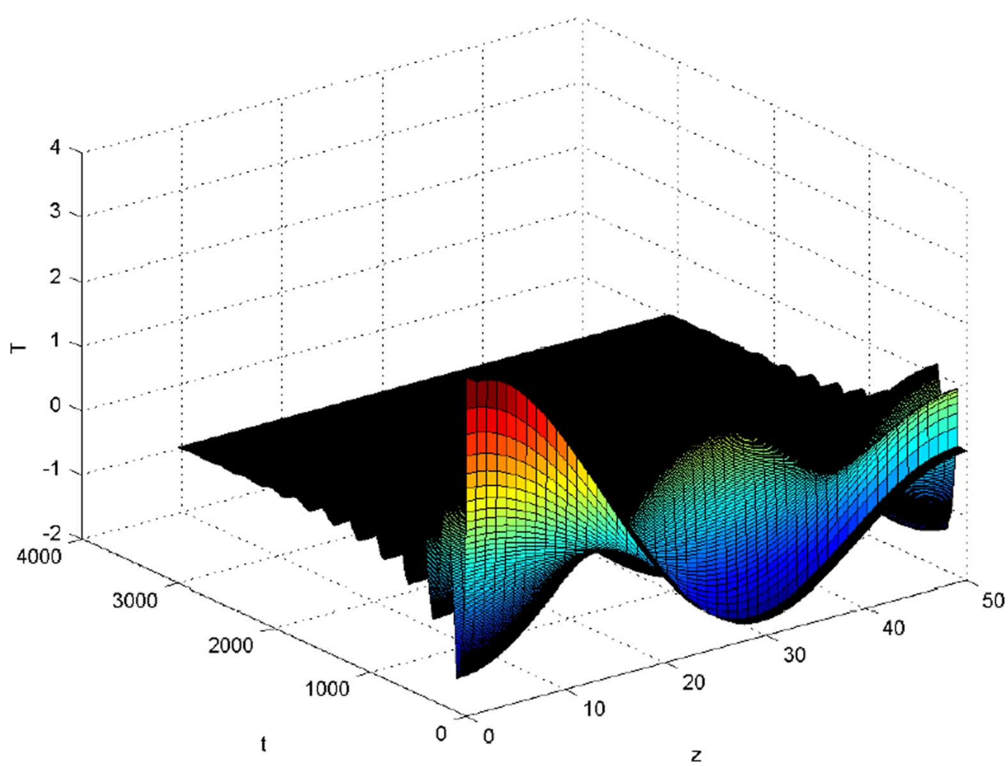

Fig. 10. State of closed-loop distributed parameter system.

APPENDIX

Proof of Proposition 1: First, denote

$$x_{st} = x_s(t+\theta) \quad x_{ft} = x_f(t+\theta)$$
$$e_t = e(t+\theta), \qquad \theta \in [-\tau, 0]$$

and $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$, where A is a real matrix.

The closed-loop slow system (19) with output feedback control can be equivalently written as

$$\begin{cases} \dot{x}_s(t) = A_s x_s(t) + f_s(x_s(t), x_s(t-\tau), 0, 0) \\ \quad +B_s \sum_{i=1}^{n} h_i K_i (x_s(t) - e(t)) \\ \dot{e}(t) = A_s e(t) + f_s(e(t), e(t-\tau), 0, 0) \\ \quad -C_s \sum_{i=1}^{n} h_i L_i e(t) \end{cases} . \tag{73}$$

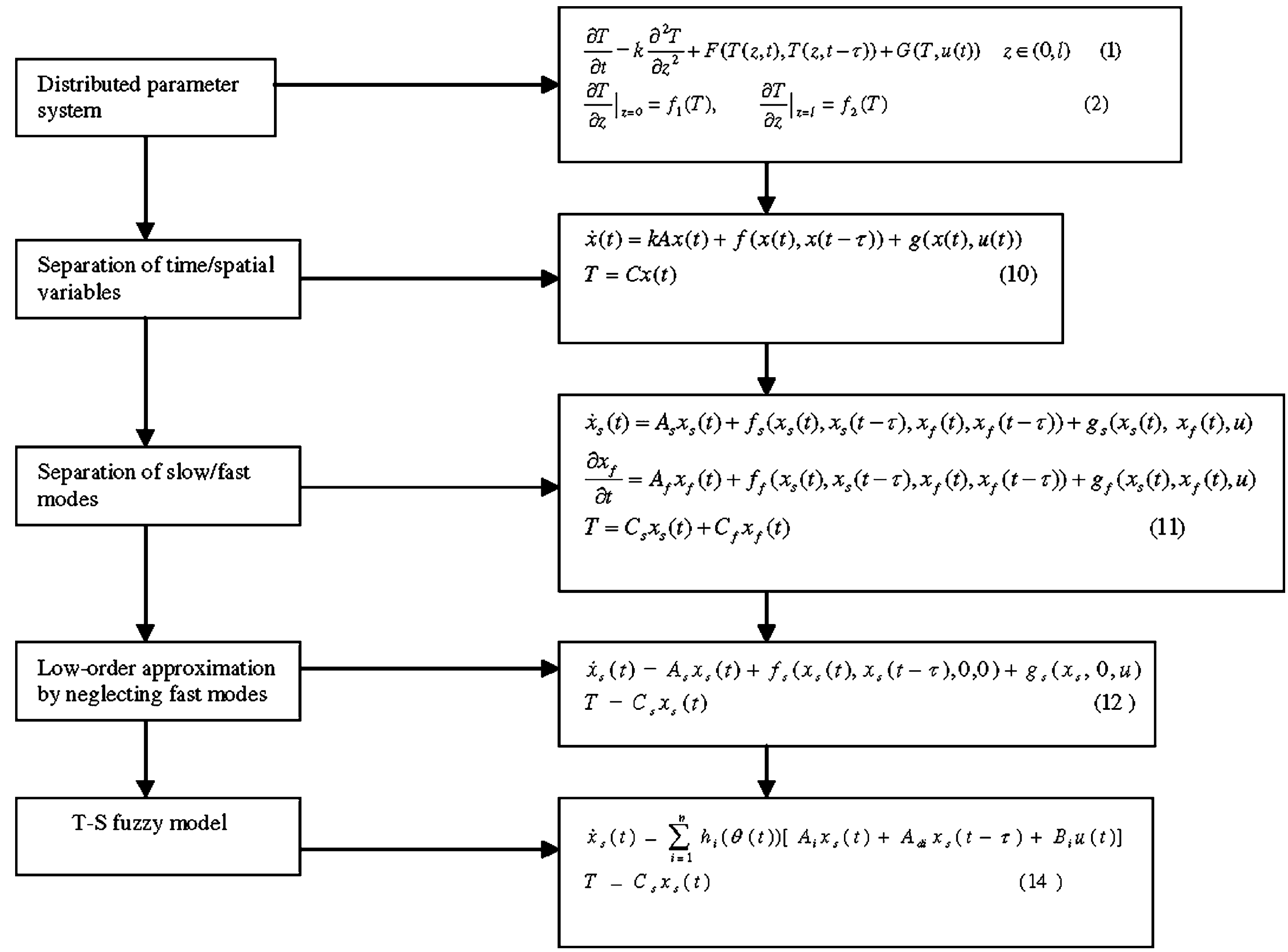

Fig. 11. Model reduction procedure.

If (73) is exponentially stable, the assumptions hold

$$\begin{cases} a_1(\|x_{st}\|^2+\|e_t\|^2) \\ \le V(x_{st},e_t) \le a_2(\|x_{st}\|^2+\|e_t\|^2) \\ \dot V(x_{st},e_t)|_{(73)} \\ \le -a_3(\|x_s(t)\|^2+\|e(t)\|^2) \\ -a_4(\|x_s(t-\tau)\|^2+\|e(t-\tau)\|^2) \\ \left\|\frac{\partial V}{\partial x_s}\right\| \le a_5\|x_s(t)\|+a_6\|x_s(t-\tau)\| \\ +a_7\|e(t)\|+a_8\|e(t-\tau)\| \\ \left\|\frac{\partial V}{\partial e}\right\| \le a_9\|x_s(t)\|+a_{10}\|x_s(t-\tau)\| \\ +a_{11}\|e(t)\|+a_{12}\|e(t-\tau)\| \end{cases} . \tag{74}$$

Next, we will show that the following closed-loop system is exponentially stable under the assumption (74):

$$\begin{cases} \dot x_s(t) = A_s x_s(t) \\ \quad +f_s(x_s(t),x_s(t-\tau),x_f(t),x_f(t-\tau)) \\ \quad +\sum_{i=1}^{n} h_i h_j B_i K_j(x_s(t)-e(t)) \\ \dot e(t) = A_s e(t) + f_s(e(t),e(t-\tau),0,0) \\ \quad -\sum_{i=1}^{n} h_i L_i C_s e(t) \\ \quad -\sum_{i=1}^{n} h_i L_i C_f x_f(t) \\ \epsilon\frac{\partial x_f}{\partial t} = \epsilon A_f x_f(t) \\ \quad +\epsilon[f_f(x_s(t),x_s(t-\tau),x_f(t),x_f(t-\tau)) \\ \quad +g_f(x_s(t),x_f(t) \\ \sum_{i=1}^{n} h_i K_i(x_s(t)-e(t)))] \end{cases} \tag{75}$$

where $\epsilon = (|\mathrm{Re}(\lambda_1)|/|\mathrm{Re}(\lambda_m)|)$. Let μ_1, μ_2^*, and μ_3^* be positive real numbers such that if $\|x_s(t)\| \le \mu_1^*, \|x_f(t)\| \le \mu_2^*$, and $e(t) \le \mu_3^*$, then there exist positive real numbers k_i $(i = 1,\ldots,9)$ such that

$$\begin{cases} \|f_s(x_s(t),x_s(t-\tau),x_f(t),x_f(t-\tau)) \\ \quad -f_s(x_s(t),x_s(t-\tau),0,0)\| \\ \le k_1\|x_f(t)\|+k_2\|x_f(t-\tau)\| \\ \|f_f(x_s(t),x_s(t-\tau),x_f(t),x_f(t-\tau))\| \\ \le k_3\|x_s(t)\|+k_4\|x_s(t-\tau)\| \\ \quad +k_5\|x_f(t)\|+k_6\|x_f(t-\tau)\| \\ \|g_f(x_s(t),x_f(t),\sum_{i=1}^{n} h_i K_i(x_s(t)-e(t)))\| \\ \le k_7\|x_s(t)\|+k_8\|x_f(t)\|+k_9\|e(t)\| \end{cases} . \tag{76}$$

Furthermore, since the fast subsystem

$$\epsilon\frac{\partial x_f}{\partial t} = A_{f\epsilon} x_f \tag{77}$$

is exponentially stable, there exist a smooth Lyapunov functional $W(\cdot)$ and positive real numbers b_i $(i = 1,\ldots,7)$ such that for all $x_f \in H_f$ that satisfy $\|x_f(t)\| \le b_7$, the following

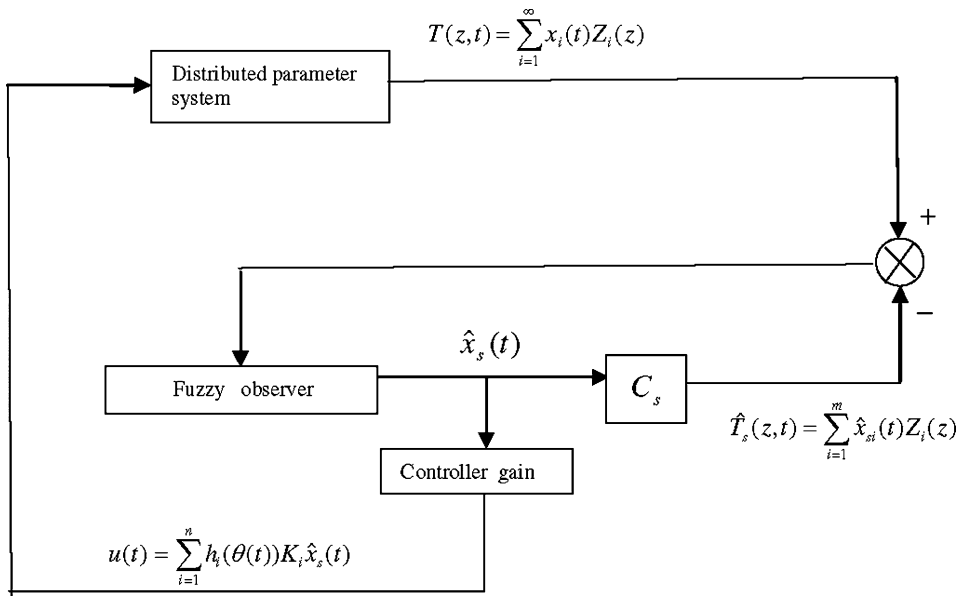

Fig. 12. Output feedback control.

conditions hold:

$$\begin{cases} b_1\|x_{ft}\|^2 \le W(x_{ft}) \le b_2\|x_{ft}\|^2 \\ \dot{W}(x_{ft})|_{(77)} \le -\frac{b_3}{\epsilon}\|x_f(t)\|^2 - b_4\|x_f(t-\tau)\|^2 \\ \left\|\frac{\partial W}{\partial x_f}\right\| \le b_5\|x_f(t)\| + b_6\|x_f(t-\tau)\| \end{cases} . \tag{78}$$

Pick $\mu_2 = \min\{b_7, \mu_2^*\}$ and consider the Lyapunov functional

$$L(x_{st}, e_t, x_{ft}) = V(x_{st}, e_t) + W(x_{ft}).$$

Calculating the derivative of L along the trajectory of (75), and using the bounds of (74), (78), and the estimates of (76), the following expressions can be easily obtained:

$$\begin{aligned}
&\dot{L}(x_{st}, e_t, x_{ft})|_{(75)} \\
&\quad = \frac{\partial V}{\partial x_s}\left[A_s x_s(t) + f_s(x_s(t), x_s(t-\tau), 0, 0)\right. \\
&\qquad \left. + B_s \sum_{i=1}^{n} h_i K_i (x_s(t) - e(t))\right] \\
&\qquad + \frac{\partial V}{\partial e}\left[A_s e(t) + f_s(e(t), e(t-\tau), 0, 0)\right. \\
&\qquad \left. - \sum_{i=1}^{n} h_i L_i C_s e(t)\right] - \frac{\partial V}{\partial e}\sum_{i=1}^{n} h_i L_i C_f x_f \\
&\qquad + \frac{\partial V}{\partial x_s}[f_s(x_s(t), x_s(t-\tau), x_f(t), x_f(t-\tau)) \\
&\qquad - f_s(x_s(t), x_s(t-\tau), 0, 0)] + \frac{1}{\epsilon}\frac{\partial W}{\partial x_f} A_{f\epsilon} x_f \\
&\qquad + \frac{\partial W}{\partial x_f}[f_f(x_s(t), x_s(t-\tau), x_f(t), x_f(t-\tau)) \\
&\qquad + B_f \sum_{i=1}^{n} h_i K_i (x_s(t) - e(t))] \\
&\quad \le -a_3(\|x_s(t)\|^2 + \|e(t)\|^2) \\
&\qquad - a_4(\|x_s(t-\tau)\|^2 + \|e(t-\tau)\|^2) \\
&\qquad + [a_5\|x_s(t)\| + a_6\|x_s(t-\tau)\| + a_7\|e(t)\| \\
&\qquad + a_8\|e(t-\tau)\|](k_1\|x_f(t)\| + k_2\|x_f(t-\tau)\|) \\
&\qquad + [a_9\|x_s(t)\| + a_{10}\|x_s(t-\tau)\| + a_{11}\|e(t)\| \\
&\qquad + a_{12}\|e(t-\tau)\|]l\|C_f\|\|x_f\| \\
&\qquad - \frac{b_3}{\epsilon}\|x_f(t)\|^2 - b_4\|x_f(t-\tau)\|^2 \\
&\qquad + (b_5\|x_f(t)\| + b_6\|x_f(t-\tau)\|)[k_3\|x_s(t)\| \\
&\qquad + k_4\|x_s(t-\tau)\| + k_5\|x_f(t)\| + k_6\|x_f(t-\tau)\| \\
&\quad + k_7\|x_s(t)\| + k_8\|x_f(t)\| + k_9\|e(t)\|)] \\
&\quad \le -a_3\|x_s(t)\|^2 - a_4\|x_s(t-\tau)\|^2 \\
&\qquad - a_3\|e(t)\|^2 - a_4\|e(t-\tau)\|^2 + [(k_5 + k_8)b_5 \\
&\qquad - \frac{b_3}{\epsilon}]\|x_f(t)\|^2 + (k_6 b_6 - b_4)\|x_f(t-\tau)\|^2 \\
&\qquad + [k_1 a_5 + (k_3 + k_7)b_5 + l a_9\|C_f\|] \\
&\qquad \times \|x_s(t)\|\|x_f(t)\| \\
&\qquad + [k_2 a_5 + (k_3 + k_7)b_6]\|x_s(t)\|\|x_f(t-\tau)\| \\
&\qquad + (k_1 a_6 + k_4 b_5 + l a_{10}\|C_f\|)\|x_s(t-\tau)\|\|x_f(t)\| \\
&\qquad + (k_2 a_6 + b_6 k_4)\|x_s(t-\tau)\|\|x_f(t-\tau)\| \\
&\qquad + [(k_5 + k_8)b_6 + k_6 b_5]\|x_f(t)\|\|x_f(t-\tau)\| \\
&\qquad + (a_7 k_1 + b_5 k_9 + l a_{11}\|C_f\|)\|x_f(t)\|\|e(t)\| \\
&\qquad + (a_8 k_1 + l a_{12}\|C_f\|)\|x_f(t)\|\|e(t-\tau)\| \\
&\qquad + (a_7 k_2 + b_6 k_9)\|x_f(t-\tau)\|\|e(t)\| \\
&\qquad + a_8 k_2\|x_f(t-\tau)\|\|e(t-\tau)\| \\
&\quad \le \xi\Omega\xi^T
\end{aligned} \tag{79}$$

where $\xi = [\|x_s(t)\|\|x_s(t-\tau)\|\|x_f(t)\|\|x_f(t-\tau)\|\|e(t)\|\|e(t-\tau)\|], l = \max_{i\le 1\le n}\|L_i\|$ and (80) as shown at the top of the next page. Picking $\epsilon_* = (b_3/e_1)$ and $\epsilon^* = (b_3 e_3/c_1 e_2 - e_3^2)$ such that when $\epsilon \in (\epsilon_*, \epsilon^*)$ the matrix Ω is negative definite, where $e_1 = (c_1^2 + d_1^2/4a_3) + (d_1^2 + d_2^2/4a_4) + (k_5 + k_8)b_5, e_2 = (c_1 c_2 + d_1 d_3/4a_3) +$

$$\begin{bmatrix} -a_3 & 0 & \frac{k_1a_5+(k_3+k_7)b_5+la_9\|C_f\|}{2} & \frac{k_2a_5+(k_3+k_7)b_6}{2} & 0 & 0 \\ 0 & -a_4 & \frac{k_1a_6+k_4b_5+la_{10}\|C_f\|}{2} & \frac{k_2a_6+b_6k_4}{2} & 0 & 0 \\ * & * & (k_5+k_8)b_5-\frac{b_3}{\epsilon} & \frac{(k_5+k_8)b_6+k_6b_5}{2} & \frac{a_7k_1+b_5k_9+la_{11}\|C_f\|}{2} & \frac{a_8k_1+la_{12}\|C_f\|}{2} \\ * & * & * & k_6b_6-b_4 & \frac{a_7k_2+b_6k_9}{2} & \frac{a_8k_2}{2} \\ * & * & * & * & -a_3 & 0 \\ * & * & * & * & * & -a_4 \end{bmatrix} = \Omega < 0 \tag{80}$$

$(c_3c_4 + d_2d_4/4a_4) + (k_5b_6 + b_6(k_5 + k_8)/2), e_3 = ((c_2^2 + d_3^2/4a_3) + (c_4^2 + d_4^2/4a_4) + k_6b_6 - b_4$ and $c_1 = k_1a_5 + (k_3 + k_7)b_5 + la_9\|C_f\|, c_2 = k_2a_5 + (k_3 + k_7)b_6, c_3 = k_1a_6 + k_4b_5 + la_{10}\|C_f\|, c_4 = k_2a_6 + b_6k_4, d_1 = a_7k_1 + b_5k_9 + la_{11}\|C_f\|, d_2 = a_8k_1 + la_{12}\|C_f\|, d_3 = a_7k_2 + b_6k_9, d_4 = a_8k_2$.

Furthermore, we have

$$\begin{aligned} \dot{L}(x_{st}, e_t, x_{ft}) \le \lambda_{\max}(\Omega)(&\|x_s(t)\|^2 + \|x_s(t-\tau)\|^2 \\ &+ \|x_f(t)\|^2 + \|x_f(t-\tau)\|^2 \\ &+ \|e(t)\|^2 + \|e(t-\tau)\|^2) \end{aligned}$$

which implies the system (75) is exponentially stable. □

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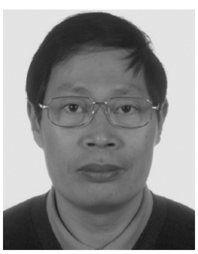

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