Necessary and sufficient stability condition by finite number of mathematical operations for time-delay systems of neutral type

Marco A. Gomez, Alexey V. Egorov and Sabine Mondié

***Abstract*—We present a new necessary and sufficient condition of exponential stability of neutral type LTI systems with one delay. The new stability criterion is given in terms of a block matrix, which exclusively depends on the delay Lyapunov matrix and is of finite dimension.**

***Index Terms*—Time-delay systems, necessary and sufficient stability condition, delay Lyapunov matrix.**

I. Introduction

We consider neutral-type delay systems of the form

$$\frac{d}{dt}\left(x(t) - Dx(t-h)\right) = A_0x(t) + A_1x(t-h), \qquad (1)$$

where D, A_0, $A_1 \in \mathbb{R}^{n\times n}$ and $h > 0$ is the delay.

Neutral type systems are suitable for modeling a variety of phenomena and engineering applications. For instance, the interaction of two species (predator-pray models) [1], thermoacoustic instability phenomenon in the Rijke tube [2], and the stability analysis of the proportional-integral control of a passive linear system with delayed communication [3].

The stability analysis of systems of the form (1) is of major interest not only for practical reasons, but also because the complexity makes them attractive to be studied from a purely theoretical perspective. Both frequency and time domain techniques have been proposed for the stability analysis. The frequency domain techniques basically rely on the continuity property of the roots location with respect to variations of system parameters. See [4]–[8] and Chapter 2 of [9]. Within this approach, results reported in, e.g. [10]–[13], have shown that the roots of the system on the imaginary axis can be determined by studying the spectrum of constant matrices, which has been exploited in order to find time-delay stability intervals, or critical parameters.

Time domain techniques are mostly based on the ideas introduced by Krasovskii [14], which extend Lyapunov stability methods for delay-free systems to the time-delay case. The main difficulty that this approach faces is the adequate choice of functionals. Indeed, the proposal of Lyapunov-Krasovskii candidate functionals have led to obtain only more or less conservative LMI type sufficient stability conditions (see, for instance, Chapter 3 in [15]–[17], and the references therein).

A converse approach for the computation of Lyapunov-Krasovskii functionals that consists in the prescription of a negative quadratic type derivative was first proposed for neutral type systems in [18]. Decades later, the so-called functionals of complete type were introduced [19]–[22]. These functionals, whose derivative capture the whole delay system state, are defined by the so-called delay Lyapunov matrix; see the book [23] for an extensive study of functionals of complete type for different classes of systems with delays. The availability of the Lyapunov matrix for neutral time-delay systems has allowed estimating the exponential decay rate of solutions [20], characterizing the $\mathcal{H}_2$ norm [24], computing critical parameters of the system [13], and obtaining robustness bounds [25], just to mention some useful applications.

Inspired by the results in [26], the following family of necessary stability conditions that uniquely depends on the delay Lyapunov matrix is presented in [27] (see [28] for the multiple delay case):

Theorem 1 ([27]): If system (1) is exponentially stable, then the symmetric matrix

$$\mathcal{K}_r(\tau_1,\ldots,\tau_r) := \left[U(\tau_j - \tau_i)\right]_{i,j=1}^{r}$$

is positive definite, where $U : [-h, h] \mapsto \mathbb{R}^{n\times n}$ is the delay Lyapunov matrix, $r \in \mathbb{N}$, $\tau_1,\ldots,\tau_r$ are real numbers on $[0, h]$ such that $\tau_i \neq \tau_j$, if $i \neq j$, and $\left[U(\tau_j - \tau_i)\right]_{i,j=1}^{r}$ is a block-matrix with i-th row and j-th column element $U(\tau_j - \tau_i)$.

Unlike the classical LMI stability conditions, which are sufficient and given in terms of decision variables, the condition of Theorem 1 is necessary and is determined by the delay Lyapunov matrix. It is because of the latter that the condition can be viewed as the analogue of the Lyapunov stability criterion of delay-free systems, which is expressed in terms of the positivity of P, solution of the Lyapunov equation $A^TP + AP = -Q$. Similar stability conditions have been obtained for different classes of systems with delays; see [29] for the state of the art on this research direction.

In several examples reported in [27] and [30], where stability maps are constructed in the space of the system parameters by using condition of Theorem 1, it is observed that the conservatism of Theorem 1 is reduced by increasing

"This work was funded by RFBR according to the research project No. 19-01-00146, Project SEP-CINVESTAV 155 and Project Conacyt A1S24796"

Marco A. Gomez (e-mail: m.galvarez@outlook.com).
Alexey V. Egorov is with St. Petersburg State University, St. Petersburg, 199034, Russia (e-mail: alexey.egorov@spbu.ru)
S. Mondié is with the Department of Automatic Control, CINVESTAV-IPN, Mexico City, 07360, Mexico (e-mail: smondie@ctrl.cinvestav.mx)

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.3008392, IEEE Transactions on Automatic Control

the number r, i.e. by increasing the dimension of the matrix $\mathcal{K}_r(\tau_1, \ldots, \tau_r)$, and the exact stability zone in the space of parameters is achieved for a sufficiently large number r.

Some queries arise from these observations: does there exist a number r for which the condition of Theorem 1 is necessary and sufficient for the stability of system (1)? If there exists, how can this number be computed? In this paper, we show that it is possible to provide a number $\hat{r}$ such that the positive definiteness of the symmetric matrix $\mathcal{K}_{\hat{r}}(\hat{\tau}_1, \ldots, \hat{\tau}_{\hat{r}})$, with equidistant $\hat{\tau}_1, \ldots, \hat{\tau}_{\hat{r}}$, is a necessary and sufficient stability condition for system (1). A first attempt was presented in [31], but the stability criterion introduced there also depends on the fundamental matrix of system (1) and the strong assumption $\|D\| < 1$ is imposed. The result is obtained by approximating functions from a compact set by a particular class of functions depending on the fundamental matrix of the system, and by introducing an instability theorem that allows us to connect the number r to the stability of the system. A notable feature of the new stability condition is that it is necessary and sufficient, and only requires a finite number of mathematical operations in order to be tested, despite the infinite nature of systems of the form (1). This result generalizes the recent work by the authors [32], where the retarded type case is addressed. Although the key ideas are taken from [32], it is important to mention that the complexity of neutral type systems makes the generalization a non-trivial task.

The paper is organized as follows. In Section II, we introduce basic definitions of the system and the delay Lyapunov matrix. The instrumental results that are used in order to prove the main result are introduced in Section III. In Section IV, the new stability criterion for system (1) is presented. We illustrate this result by one academic example in Section V and conclude with some remarks in Section VI.

Notation: We consider two spaces of functions defined on $[-h, 0]$ with values in $\mathbb{R}^n$: $C^{(1)}([-h,0],\mathbb{R}^n)$ and $PC^{(1)}([-h,0],\mathbb{R}^n)$, which we denote by $C^{(1)}$ and $PC^{(1)}$ for short. They are the spaces of continuously differentiable functions, and piecewise continuous functions with piecewise continuous derivative, respectively. Both spaces are equipped with the sup-norm

$$\|\varphi\|_h := \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|,$$

where $\|\cdot\|$ is the Euclidian norm for vectors. We also consider Euclidian norm for matrices, i.e. the largest singular value of the matrix. We say that a matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if it is symmetric, i.e. $A = A^T$, and $z^T A z > 0$ for all nonzero $z \in \mathbb{R}^n$. Notation $A > 0$ stands for positive definite matrix A. The maximum (minimum) eigenvalue of a symmetric matrix A is denoted by $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$). The notation $j = \overline{1,n}$ means that j takes integer values from 1 to n. The symbol $\lceil\cdot\rceil$ denotes the ceiling function, which maps a real number r to the least integer equal or greater than r.

II. Basic framework

A. The system

We introduce some basic definitions of system (1). Consider initial functions $\varphi \in PC^{(1)}$. The solution $x(t) = x(t,\varphi)$, $t \in [-h,\infty)$, of system (1) is a piecewise continuous function satisfying system (1) almost everywhere on $[0,\infty)$, the initial condition $x(t) = \varphi(t)$ on $[-h, 0]$, and the sewing condition: $x(t) - Dx(t-h)$ is continuous on $[0,\infty)$ (right continuous at zero). The restriction of the solution $x(t,\varphi)$ to the interval $[t-h, t]$, $t \geq 0$, is denoted by

$$x_t(\varphi) : \theta \mapsto x(t+\theta,\varphi), \;\; \theta \in [-h, 0].$$

Definition 1 ([33]): System (1) is exponentially stable if there exist $\eta > 0$ and $\sigma > 0$ such that for any initial function φ

$$\|x(t,\varphi)\| \leq \eta e^{-\sigma t}\|\varphi\|_h, \; t \geq 0.$$

Definition 2 ([23]): Matrix D is Schur stable if all its eigenvalues are inside the unit circle.

It is well known that a necessary condition for the stability of system (1) is the Schur stability of matrix D (see, [15], [34]). A Schur stable matrix admits the following upper bound [35]:

$$\|D^j\| \leq d\rho^j, \;\; j = 1, 2, \ldots, \tag{2}$$

where $\rho \in (0, 1)$ and $d \geq 1$. In particular, one can take $d = \sqrt{\lambda_{\max}(Q)/\lambda_{\min}(Q)}$, where $Q \in \mathbb{R}^{n\times n}$ is a positive definite matrix solution of the inequality

$$D^T QD - \rho^2 Q < 0.$$

The fundamental matrix of system (1), denoted by K, is a solution [33] of the equation

$$\frac{d}{dt}\left(K(t) - DK(t-h)\right) = A_0K(t) + A_1K(t-h)$$

with the initial condition $K(t) = I$ for $t = 0$ and $K(t) = 0$ for $t < 0$. It is noteworthy that

$$K(t) = e^{A_0 t}, \quad t \in [0, h).$$

Hence, one can calculate the number L, used in Section III, such that

$$\left\|\frac{d}{dt}K(t)\right\| = \|A_0e^{A_0 t}\| \leq L, \;\; t \in [0, h).$$

B. Lyapunov matrix

We provide some basic facts on the Lyapunov matrix of system (1). We define the delay Lyapunov matrix U as follows:

Definition 3 ([23]): Let $W \in \mathbb{R}^{n\times n}$ be a positive definite matrix. The delay Lyapunov matrix $U : [-h, h] \mapsto \mathbb{R}^{n\times n}$ is a continuous matrix function, which satisfies the following properties:

1) Dynamic property:

$$U'(\tau) - U'(\tau-h)D = U(\tau)A_0 + U(\tau-h)A_1, \tau \in (0, h).$$

2) Symmetry property:

$$U^T(\tau) = U(-\tau), \; \tau \in [-h, h].$$

3) Algebraic property:

$$\Delta U'(0) - D^T\Delta U'(0)D = -W,$$

where $\Delta U'(0) = U'(+0) - U'(-0)$.

The uniqueness of the delay Lyapunov matrix is established in the next theorem.

Theorem 2 ([23]): System (1) admits a unique Lyapunov matrix if and only if the system satisfies the Lyapunov condition, i.e. if there exists $\varepsilon > 0$ such that any two points s_1 and s_2 of the spectrum of system (1) satisfy $|s_1 + s_2| > \varepsilon$.

Let us introduce now the quadratic functional $v_1 : PC^{(1)} \mapsto \mathbb{R}$:

$$\begin{aligned} v_1(\varphi) &= (\varphi(0) - D\varphi(-h))^T U(0) (\varphi(0) - D\varphi(-h)) \\ &\quad + 2 (\varphi(0) - D\varphi(-h))^T \int_{-h}^{0} F_1(-h-\theta)\varphi(\theta)d\theta \\ &\quad + \int_{-h}^{0}\int_{-h}^{0} \varphi^T(\theta_1)F_2(\theta_1-\theta_2)\varphi(\theta_2)d\theta_2 d\theta_1 \\ &\quad - \int_{-h}^{0} \varphi^T(\theta)\Delta U'(0)\varphi(\theta)d\theta, \end{aligned}$$

where

$$\begin{aligned} F_1(\tau) &= U(\tau)A_1 + U'(\tau)D, \quad \tau \in (-h,0), \\ F_2(\tau) &= \begin{cases} A_1^T F_1(\tau) - D^T F_1'(\tau), & \tau \in (-h,0), \\ F_2^T(-\tau), & \tau \in (0,h). \end{cases} \end{aligned}$$

By the formula

$$z(\varphi_1, \varphi_2) = \frac{1}{4}\Big(v(\varphi_1+\varphi_2) - v(\varphi_1 - \varphi_2) \Big)$$

one can construct the corresponding bilinear functional z for $\varphi_1, \varphi_2 \in PC^{(1)}$. Its explicit form can be found in [27]. We provide upper bounds for functionals v_1 and z.

Lemma 1: For any $\varphi, \varphi_1, \varphi_2 \in PC^{(1)}$,

$$\begin{aligned} |v_1(\varphi)| &\le \alpha_2 \|\varphi\|_h^2, \\ |z(\varphi_1,\varphi_2)| &\le \alpha_2 \|\varphi_1\|_h \|\varphi_2\|_h, \end{aligned}$$

where

$$\alpha_2 = (1+\|D\|)^2\|U(0)\| + 2h(1+\|D\|)f_1 + h^2 f_2 + h\|\Delta U'(0)\|,$$

with

$$f_1 = \sup_{\tau\in(-h,0)} \|F_1(\tau)\|, \; f_2 = \sup_{\tau\in(-h,0)} \|F_2(\tau)\|.$$

Proof: The bound on functional $z(\varphi_1, \varphi_2)$ can be easily deduced by applying the Cauchy-Schwartz inequality to each term, and the bound on functional v_1 follows from the equality $v_1(\varphi) = z(\varphi, \varphi)$. ■

III. Instrumental results

We introduce some auxiliary results that enable us to present the main theorem of the paper in the next section. They are based on the following relatively compact set in the space of continuously differentiable functions $C^{(1)}$:

$$\mathcal{S} := \left\{ \varphi \in C^{(1)} \;\middle|\; \|\varphi(0)\| = \|\varphi\|_h = 1, \; \|\varphi'\|_h \le \mu M \right\},$$

where $M = \|A_0\| + \|A_1\|$ and $\mu = \dfrac{d}{1-\rho}$, with the numbers d and ρ satisfying (2).

Consider the function $\psi_r : [-h, 0] \mapsto \mathbb{R}^n$ defined as

$$\psi_r(\theta) = \sum_{k=1}^{r} K(\tau_k + \theta)\gamma_k, \quad \theta \in [-h, 0], \tag{3}$$

where $r \in \mathbb{N}$, $\tau_k \in [0, h]$ and $\gamma_k \in \mathbb{R}^n$, $k = \overline{1,r}$. In [27], by introducing new properties that connect the delay Lyapunov matrix U with the fundamental matrix K, it is shown that

$$v_1(\psi_r) = \gamma^T \mathcal{K}_r(\tau_1, \ldots, \tau_r)\gamma, \tag{4}$$

where $\gamma = \begin{pmatrix}\gamma_1^T & \ldots & \gamma_r^T\end{pmatrix}^T$. Equation (4) along with the fact that the exponential stability of system (1) implies the existence of a positive number $\hat{\alpha}$ such that $v_1(\varphi) \ge \hat{\alpha}\|\varphi(0)\|^2$ for any $\varphi \in PC^{(1)}$ allows proving Theorem 1.

In [31], it is shown that by appropriately choosing τ_k and γ_k, $k = \overline{1,r}$, any arbitrary function from the set $\mathcal{S}$ can be approximated by a function of the form (3). Indeed, by setting

$$\tau_k = \frac{k-1}{r-1}h,$$

where $r \ge 2$, and choosing vectors γ_k, $k = \overline{1,r}$, such that

$$\psi_r(-\tau_i) = \varphi(-\tau_i), \quad i = \overline{1,r}, \tag{5}$$

we can state the following result:

Lemma 2 ([31]): Let matrix D be Schur stable. For every $\varphi \in \mathcal{S}$ there exists a function ψ_r of the form (3), such that

$$\|\varphi - \psi_r\|_h \le \varepsilon_r,$$

where

$$\varepsilon_r = h\frac{(\mu M + L)e^{Lh}}{r-1+Lh}.$$

We next introduce an instability result. The basic idea is inspired by [36] and [32]. For the sake of clarity, the proof is given in the Appendix.

Theorem 3: Assume that matrix D is Schur stable. If system (1) has an eigenvalue with a strictly positive real part, there exists $\bar{\varphi} \in \mathcal{S}$ such that

$$v_1(\bar{\varphi}) \le -\alpha_1,$$

with

$$\alpha_1 = \frac{\lambda_{\min}(W)}{4\mu M} e^{-2\mu M h}\cos^2(b) > 0,$$

where b is the unique root of the function

$$g(b) = \sin^4(b)\left((h\mu M)^2 + b^2\right) - (h\mu M)^2 \tag{6}$$

on $\left[0, \dfrac{\pi}{2}\right]$.

IV. Main result

In this section, we present the main result of the paper. Let us consider the matrices

$$\mathcal{K}_r := \mathcal{K}_r\left(0, \frac{h}{r-1}, \frac{2h}{r-1}, \ldots, h\right), \; r = 2, 3, 4, \ldots.$$

and $\mathcal{K}_1 := U(0)$. With Lemma 2 and Theorem 3 at hand, we are now able to present a stability criterion that requires a finite number of mathematical operations in terms of the matrix $\mathcal{K}_r$.

Theorem 4: Assume that matrix D is Schur stable. System (1) is exponentially stable if and only if the Lyapunov condition and the following hold:

$$\mathcal{K}_{\hat{r}} > 0, \tag{7}$$

where

$$\hat{r} = 1 + \left\lceil e^{Lh} h\left(\mu M + L\right)\left(\alpha^\star + \sqrt{\alpha^\star(\alpha^\star+1)}\right) - Lh\right\rceil, \tag{8}$$

with $\alpha^\star = \dfrac{\alpha_2}{\alpha_1}$. Here, α_1 is determined by Theorem 3 and α_2 is given by Lemma 1.

Proof: By Theorem 1, if the system is exponentially stable we have that $\mathcal{K}_r(\tau_1,\dots,\tau_r) > 0$ for any r and pairwise different points $\tau_1,\dots,\tau_r \in [0,h]$, which implies that (7) holds in particular for $\hat{r}$ and $\tau_k = h(k-1)/(\hat{r}-1)$, $k=\overline{1,\hat{r}}$.

We now focus on the sufficiency. We assume by contradiction that system (1) is not exponentially stable. As the Lyapunov condition holds, there exists an eigenvalue with a positive real part, and we can use Theorem 3. Notice that, by Theorem 2, the Lyapunov condition also guarantees the existence and uniqueness of the delay Lyapunov matrix. Consider the function $\bar{\varphi} \in \mathcal{S}$ from Theorem 3 and the corresponding ψ_r, which is constructed as in Lemma 2. It is important to indicate that $\gamma = \begin{pmatrix}\gamma_1^T & \dots & \gamma_r^T\end{pmatrix}^T \neq 0$, as $\|\psi_r(0)\| = \|\bar{\varphi}(0)\| = 1$ by (5). Let $E_r = \bar{\varphi} - \psi_r$, then

$$v_1(\psi_r) = v_1(\bar{\varphi} - E_r) = v_1(\bar{\varphi}) - 2z(\bar{\varphi}, E_r) + v_1(E_r).$$

By Lemmas 1 and 2, and Theorem 3,

$$\begin{aligned} v_1(\psi_r) &\le -\alpha_1 + 2\alpha_2\|\bar{\varphi}\|_h\|E_r\|_h + \alpha_2\|E_r\|_h^2 \\ &\le -\alpha_1\left(1 - 2\alpha^\star\varepsilon_r - \alpha^\star\varepsilon_r^2\right). \end{aligned}$$

By considering $r = \hat{r}$, we have that

$$1 - 2\alpha^\star\varepsilon_{\hat{r}} - \alpha^\star\varepsilon_{\hat{r}}^2 \ge 0.$$

Indeed, for $\hat{r}$

$$\varepsilon_{\hat{r}} = h\frac{(\mu M + L)e^{Lh}}{\hat{r} - 1 + Lh} \le \frac{1}{\alpha^\star + \sqrt{\alpha^\star(\alpha^\star+1)}},$$

and

$$\begin{aligned} 1 - 2\alpha^\star\varepsilon_{\hat{r}} - \alpha^\star\varepsilon_{\hat{r}}^2 \ge 1 &- 2\alpha^\star\left(\alpha^\star + \sqrt{\alpha^\star(\alpha^\star+1)}\right)^{-1} \\ &- \alpha^\star\left(\alpha^\star + \sqrt{\alpha^\star(\alpha^\star+1)}\right)^{-2} = 0. \end{aligned}$$

From the previous inequality and equation (4), we obtain

$$v_1(\psi_{\hat{r}}) = \gamma^T\mathcal{K}_{\hat{r}}\gamma \le 0,$$

which contradicts the initial assumption. ■

Remark 1: Matrix $\mathcal{K}_{\hat{r}}$ of the stability criterion of Theorem 4 have the following form: if $\hat{r} = 2$,

$$\mathcal{K}_2 = \begin{pmatrix} U(0) & U(h) \\ U^T(h) & U(0) \end{pmatrix} > 0,$$

if $\hat{r} = 3$,

$$\mathcal{K}_3 = \begin{pmatrix} U(0) & U\left(\frac{h}{2}\right) & U(h) \\ U^T\left(\frac{h}{2}\right) & U(0) & U\left(\frac{h}{2}\right) \\ U^T(h) & U^T\left(\frac{h}{2}\right) & U(0) \end{pmatrix} > 0,$$

if $\hat{r} = 4$,

$$\mathcal{K}_4 = \begin{pmatrix} U(0) & U\left(\frac{h}{3}\right) & U\left(\frac{2h}{3}\right) & U(h) \\ U^T\left(\frac{h}{3}\right) & U(0) & U\left(\frac{h}{3}\right) & U\left(\frac{2h}{3}\right) \\ U^T\left(\frac{2h}{3}\right) & U^T\left(\frac{h}{3}\right) & U(0) & U\left(\frac{h}{3}\right) \\ U^T(h) & U^T\left(\frac{2h}{3}\right) & U^T\left(\frac{h}{3}\right) & U(0) \end{pmatrix} > 0,$$

and so on.

We highlight next two remarkable features of Theorem 4. First, it provides a new necessary and sufficient stability condition for system (1); and second, the stability criterion can be tested by checking the positive definiteness of a finite dimensional matrix, despite the infinite nature of time-delay systems.

From formula (8), it is evident that $\hat{r}$ depends on the delay h and the system parameters through the constants L, μ, M, $\alpha^\star$. Number $\hat{r}$ has an exponential behavior depending on L and h (except the case $A_0 = 0$) with a multiplicative factor depending on μ, M and $\alpha^\star$. All the coefficients depend on the system parameters, but whereas L, μ, and M do it explicitly, the number $\alpha^\star$ does it through the numbers α_1 and α_2 associated with the functional v_1. One observes that the number $\hat{r}$ is expected to increase as the delay does.

V. Example

We illustrate Theorem 4 by an academic example. We corroborate the validity of the stability test by the location of the rightmost eigenvalue of the system, and point out the relation of the numerical complexity of the criterion with the system parameters. The implementation is carried out in MATLAB.

Example 1: We consider the system

$$\frac{d}{dt}\left(x(t) - \begin{pmatrix} 0 & 0 \\ 0 & 0.25 \end{pmatrix} x(t-h)\right) = \begin{pmatrix} -1 & 0.2 \\ p & 0 \end{pmatrix} x(t-h), \tag{9}$$

where $p \in \mathbb{R}$ and $h > 0$ are free-parameters. We use Theorem 4 to test the stability of system (9) with $h = 0.5$ and several values of p on $[-1.2, 1.4] \setminus \{0\}$. The value $p = 0$ is excluded since in this case the system does not satisfy the Lyapunov condition. In order to compute number $\hat{r}$ and construct matrix $\mathcal{K}_{\hat{r}}$, we compute the Lyapunov matrix associated with $W = I$ via the semianalytic method (see Chapter 6 in [23]). To obtain α_1, we solve equation (6) by the Dekker-Brent algorithm implemented by the function `fzero`. Positiveness of $\mathcal{K}_{\hat{r}}$ is verified by using the function `chol`.

The results are displayed in Table I. The stability test of Theorem 4 is consistent with the spectral abscissa computed via the QPmR [37], depicted in Fig. 1 for $p \in [-1.2, 1.4]$. Elapsed time in the fourth column includes the computation of $\hat{r}$, construction of $\mathcal{K}_{\hat{r}}$ and verification of its positive definiteness. The computation of $\hat{r}$ consumes less than 2 seconds in every case, which shows that the greatest computational effort is in verifying the positiveness condition, as expected. The implementation was done in a Laptop with processor Intel Core i5-8250U, 1.6GHz, 4 cores, 8GB RAM.

Finally, we illustrate the behavior of the number $\hat{r}$ with respect to p and h in Fig. 2. We observe that $\hat{r}$ (and thus the computational complexity of the stability criterion) increases, if either the delay increases or the real part of the rightmost eigenvalue is close to zero . In the latter case, frequency domain techniques introduced in, e.g. [10]–[13] can be useful.

VI. Conclusion

We have introduced a new stability criterion for neutral type systems that only depends on the Lyapunov matrix. A striking

TABLE I
STABILITY TEST (BY THEOREM 4) FOR SYSTEM (9) WITH $h = 0.5$ AND DIFFERENT VALUES OF p

Parameter p	Number $\hat{r}$	Result of stability test	Elapsed time
-1.2	10872	Stable	37 sec
-1.0	5999	Stable	8 sec
-0.8	3553	Stable	3 sec
-0.6	2380	Stable	2 sec
-0.4	1980	Stable	2 sec
-0.2	2544	Stable	2 sec
0.2	2208	Unstable	2 sec
0.4	1517	Unstable	2 sec
0.6	1633	Unstable	2 sec
0.8	2193	Unstable	2 sec
1.0	3326	Unstable	2 sec
1.2	5387	Unstable	3 sec
1.4	9021	Unstable	6 sec

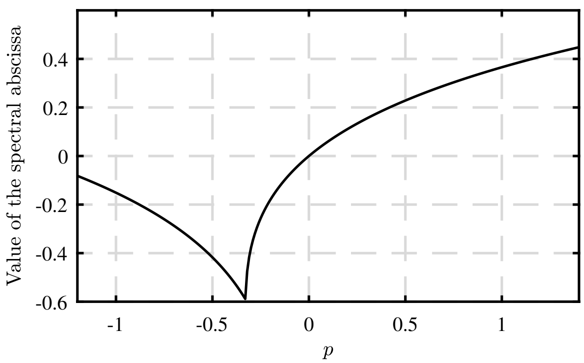

Fig. 1. Value of the spectral abscissa, defined as the real part of the rightmost eigenvalue of system (9), for different values of p on $[-1.2, 1.4]$.

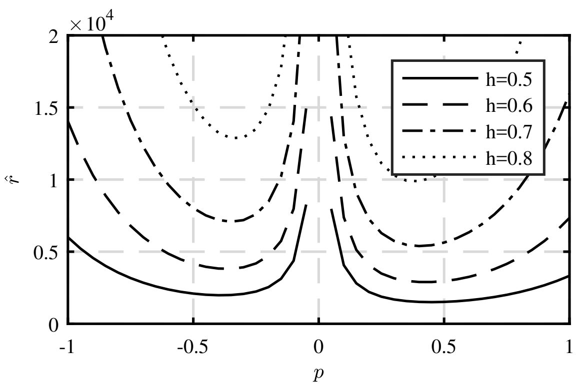

Fig. 2. Behavior of $\hat{r}$ with respect to p and h

fact that the stability criterion shows is that the stability of neutral type systems can be tested by a finite number of mathematical operations, despite their infinite nature. However, it is important to mention that the conservatism in the estimation of number $\hat{r}$ makes the stability test for systems of practical interest a very demanding computational task. Current research is focused on reducing the computational effort of the stability criterion by improving the estimation of $\hat{r}$, which might be carried out by introducing less conservative bounds in Lemmas 1 and 2, and Theorem 3.

REFERENCES

[1] K. Gopalsamy and B. G. Zhang, "On a neutral delay logistic equation," *Dynamics and Stability of Systems*, vol. 2, no. 3-4, pp. 183–195, 1988.

[2] U. Zalluhgolu, A. Kammer, and N. Olgac, "Delay feedback control laws for Rijke tube thermoacoustic instability, synthesis, and experimental validation," *IEEE Trans. on Control Systems Technology*, vol. 24, no. 5, pp. 1861 – 1868, 2016.

[3] F. Castaños, E. Estrada, S. Mondié, and A. Ramírez, "Passivity-based PI control of first-order systems with I/O communication delays: A frequency domain analysis," *Int. J. of Control*, vol. 91, no. 11, pp. 2549–2562, 2018.

[4] J. Neimark, "D-subdivisions and spaces of quasipolynomials," *Prikladnaya Matematika i Mekhanika*, vol. 13, pp. 349–380, 1949.

[5] F. G. Boese, "Stability with respect to the delay: on a paper of K. L. Cooke and P. Van Den Driessche," *J. of Mathematical Analysis and Applications*, vol. 228, no. 2, pp. 293–321, 1998.

[6] K. Gu and M. Naghnaeian, "Stability crossing set for systems with three delays," *IEEE Transactions on Automatic Control*, vol. 56, no. 1, pp. 11–26, 2011.

[7] N. Olgac and R. Sipahi, "A practical method for analyzing the stability of neutral type LTI-time delay systems," *Automatica*, vol. 40, no. 5, pp. 847–853, 2004.

[8] W. Michiels and T. Vyhlídal, "An eigenvalue based approach for the stabilization of linear time-delay systems of neutral type," *Automatica*, vol. 41, no. 6, pp. 991–998, 2005.

[9] W. Michiels and S. I. Niculescu, *Stability, control, and computation for time-delay systems: An eigenvalue-based approach*. Philadelphia: SIAM, 2014.

[10] J. Chen, "On computing the maximal delay intervals for stability of linear delay systems," *IEEE Trans. on Automatic Control*, vol. 40, no. 6, pp. 1087–1093, 1995.

[11] J. Louisell, "A matrix method for determining the imaginary axis eigenvalues of a delay system," *IEEE Transactions on Automatic Control*, vol. 46, no. 12, pp. 2008–2012, 2001.

[12] F. de Oliveira Souza, "Imaginary characteristic roots of neutral systems with commensurate delays," *Systems & Control Letters*, vol. 127, pp. 19–24, 2019.

[13] G. Ochoa, V. L. Kharitonov, and S. Mondié, "Critical frequencies and parameters for linear delay systems: A Lyapunov matrix approach," *Systems & Control Letters*, vol. 62, no. 9, pp. 781–790, 2013.

[14] N. N. Krasovskii, "On the application of the second method of Lyapunov for equations with time delays," *Prikladnaya Matematika i Mekhanika*, vol. 20, pp. 315–327, 1956, in Russian.

[15] E. Fridman, *Introduction to time-delay systems: Analysis and control*. Basel: Birkhäuser, 2014.

[16] ——, "New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems," *Systems & control letters*, vol. 43, no. 4, pp. 309–319, 2001.

[17] J. Sun, G. Liu, and J. Chen, "Delay-dependent stability and stabilization of neutral time-delay systems," *Int. J. of Robust and Nonlinear Control*, vol. 19, no. 12, pp. 1364–1375, 2009.

[18] W. B. Castelan and E. F. Infante, "A Liapunov functional for a matrix neutral difference-differential equation with one delay," *J. of Mathematical Analysis and Applications*, vol. 71, no. 1, pp. 105–130, 1979.

[19] S. A. Rodriguez, V. L. Kharitonov, J. M. Dion, and L. Dugard, "Robust stability of neutral systems: a Lyapunov-Krasovskii constructive

approach," *Int.l J. of Robust and Nonlinear Control*, vol. 14, no. 16, pp. 1345–1358, 2004.

[20] V. L. Kharitonov, "Lyapunov functionals and Lyapunov matrices for neutral type time delay systems: A single delay case," *Int. J. of Control*, vol. 78, no. 11, pp. 783–800, 2005.

[21] J. E. Velázquez-Velázquez and V. L. Kharitonov, "Lyapunov- Krasovskii functionals for scalar neutral type time delay equations," *Systems & Control Letters*, vol. 58, no. 1, pp. 17–25, 2009.

[22] G. Ochoa, S. Mondié, and V. L. Kharitonov, "Computation of imaginary axis eigenvalues and critical parameters for neutral time delay systems," in *Time Delay Systems: Methods, Applications and New Trends*, ser. Lecture Notes in Control and Information Sciences, R. Sipahi, T. Vyhlídal, S. I. Niculsecu, and P. Pepe, Eds. Berlin: Springer, 2012, vol. 423, pp. 61–72.

[23] V. L. Kharitonov, *Time-delay systems: Lyapunov functionals and matrices*. Basel: Birkhäuser, 2013.

[24] E. Jarlebring, J. Vanbiervliet, and W. Michiels, "Characterizing and computing the $\mathcal{H}_2$ norm of time-delay systems by solving the delay Lyapunov equation," *IEEE Trans. on Automatic Control*, vol. 56, no. 4, pp. 814–825, 2011.

[25] I. V. Alexandrova, "New robustness bounds for neutral type delay systems via functionals with prescribed derivative," *Applied Mathematics Letters*, vol. 76, pp. 34–39, 2018.

[26] A. V. Egorov and S. Mondié, "Necessary stability conditions for linear delay systems," *Automatica*, vol. 50, no. 12, pp. 3204–3208, 2014.

[27] M. A. Gomez, A. V. Egorov, and S. Mondié, "Necessary stability conditions for neutral type systems with a single delay," *IEEE Trans. on Automatic Control*, vol. 62, no. 9, pp. 4691–4697, 2017.

[28] M. A. Gomez, A. V. Egorov, and S. Mondié, "Necessary stability conditions for neutral-type systems with multiple commensurate delays," *Int. J. of Control*, vol. 92, no. 5, pp. 1155–1166, 2019.

[29] S. Mondié, A. V. Egorov, and M. A. Gomez, "Stability conditions for time delay systems in terms of the Lyapunov matrix," in *Proceedings of the 14th IFAC Workshop on Time Delay Systems*, Budapest, Hungary, 2018, pp. 136–141.

[30] M. A. Gomez, C. Cuvas, S. Mondié, and A. Egorov, "Scanning the space of parameters for stability regions of neutral type delay systems: a Lyapunov matrix approach," in *Proceedings of the 55th IEEE Conference on Decision and Control*, Las Vegas, USA, 2016, pp. 3149–3154.

[31] M. A. Gomez, A. V. Egorov, and S. Mondié, "A new stability criterion for neutral-type systems with one delay," in *Proceedings of the 14th IFAC Workshop on Time Delay Systems*, Budapest, Hungary, 2018, pp. 177–182.

[32] ——, "Lyapunov matrix based necessary and sufficient stability condition by finite number of mathematical operations for retarded type systems," *Automatica*, vol. 108, 2019.

[33] R. E. Bellman and K. L. Cooke, *Differential-difference equations*. New York: Academic Press, 1963.

[34] J. K. Hale and S. M. Verduyn-Lunel, *Introduction to Functional Differential Equations*. New York: Springer Science + Business Media, 1993.

[35] V. Kharitonov, J. Collado, and S. Mondié, "Exponential estimates for neutral time delay systems with multiple delays," *Int. J. of Robust and Nonlinear Control*, vol. 16, pp. 71–84, 2006.

[36] I. V. Alexandrova and A. P. Zhabko, "Stability of neutral type delay systems: A joint Lyapunov–Krasovskii and Razumikhin approach," *Automatica*, vol. 106, pp. 83–90, 2019.

[37] T. Vyhlídal and P. Zítek, "Mapping based algorithm for large-scale computation of quasi-polynomial zeros," *IEEE Trans. on Automatic Control*, vol. 54, no. 1, pp. 171–177, 2009.

[38] M. Gomez, A. V. Egorov, and S. Mondié, "A Lyapunov matrix based stability criterion for a class of time-delay systems," *Vestnik Sankt-Peterburgskogo Universiteta. Prikl. Mat., Inf., Prot. Upr.*, vol. 13, no. 4, pp. 407–416, 2017.

Appendix

In order to make concise the proof of Theorem 3, we first introduce the following auxiliary lemma, which gives some useful properties of a particular solution of system (1).

Lemma 3 ([38]): Let F be a complex matrix in $\mathbb{C}^{n\times n}$. If $\det F = 0$, then there exist two vectors C_1 and C_2 such that

1) $F(C_1 + iC_2) = 0$.
2) $\|C_1\| = 1$.
3) $\|C_2\| \le 1$.
4) $C_1^T C_2 = 0$.

We focus now on the proof of Theorem 3. Let $s_0 = \alpha + i\beta$ ($\alpha > 0$, $\beta \ge 0$) be the eigenvalue of system (1) with a positive real part. Notice that there exist two vectors $C_1, C_2 \in \mathbb{R}^n$ that satisfy the conditions of Lemma 3 with $F = s_0(I - De^{-s_0 h}) - A_0 - A_1 e^{-s_0 h}$. In this case the following expression satisfies system (1) on $(-\infty, \infty)$:

$$\bar{x}(t) = e^{\alpha t}\phi(t), \quad \phi(t) = \cos(\beta t)\, C_1 - \sin(\beta t)\, C_2.$$

First part of the proof. We need to estimate α. The following equality holds:

$$s_0\left(I - e^{-s_0 h} D\right) C = \left(A_0 + A_1 e^{-s_0 h}\right) C, \qquad (10)$$

where $C = C_1 + iC_2$. As matrix D is Schur stable, the inverse of the matrix $I - e^{-s_0 h} D$ exists. Indeed, since every eigenvalue of the matrix D is located inside the unit circle, then

$$\det\left(I - e^{-s_0 h} D\right) = e^{-n s_0 h} \det\left(e^{s_0 h} I - D\right) \neq 0.$$

Notice that

$$\|C\| \le \|(I - e^{-s_0 h} D)^{-1}\| \|(I - e^{-s_0 h} D) C\|,$$

therefore, from equation (10),

$$|s_0| \le \left(\|A_0\| + \|A_1\|\right) \|(I - e^{-s_0 h} D)^{-1}\|.$$

By Neumann series, we have

$$(I - e^{-s_0 h} D)^{-1} = I + \sum_{k=1}^{\infty} e^{-k s_0 h} D^k,$$

and since $|e^{-ksh}| < 1$ and $\|D^k\| \le d\rho^k$ by (2), we get,

$$\|(I - e^{-sh} D)^{-1}\| = \left\| \sum_{k=0}^{\infty} e^{-ksh} D^k \right\| \le \sum_{k=0}^{\infty} d\rho^k = \frac{d}{1-\rho},$$

hence,

$$\alpha \le |s_0| \le \frac{d}{1-\rho}\left(\|A_0\| + \|A_1\|\right) = \mu M.$$

Second part of the proof. The derivative of v_1 along the solutions of system (1) is [27]

$$\frac{d}{dt} v_1(\bar{x}_t) = -\bar{x}^T(t-h) W \bar{x}(t-h).$$

Integrating this equation from 0 to T, where $T = 2\pi/\beta$, if $\beta \neq 0$, and $T = 1$, if $\beta = 0$, we obtain

$$v_1(\bar{x}_0) = v_1(\bar{x}_T) + \int_{-h}^{T-h} \bar{x}^T(t) W \bar{x}(t) dt.$$

Since T is the period of the function ϕ, for $t \in (-\infty, \infty)$ we have $\bar{x}(T+t) = e^{\alpha T}\bar{x}(t)$ and $v_1(\bar{x}_T) = e^{2\alpha T} v_1(\bar{x}_0)$, which implies that,

$$v_1(\bar{x}_0) = -\frac{1}{e^{2\alpha T} - 1} \int_{-h}^{T-h} \bar{x}^T(t) W \bar{x}(t) dt$$
$$\le -\frac{\lambda_{\min}(W)}{e^{2\alpha T} - 1} \int_{-h}^{T-h} \|\bar{x}(t)\|^2 dt.$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.3008392, IEEE Transactions on Automatic Control

By Lemma 3,

$$\int_{-h}^{T-h} \|\bar{x}(t)\|^2 dt = \int_{-h}^{T-h} e^{2\alpha t}\|\phi(t)\|^2 dt \geq \int_{-h}^{T-h} e^{2\alpha t}\cos^2(\beta t) dt = \frac{e^{-2\alpha h}\left(e^{2\alpha T}-1\right)}{4\alpha} f(\beta),$$

where

$$f(\beta) = \cos^2(\beta h) + \frac{(\alpha\cos(\beta h) - \beta\sin(\beta h))^2}{\alpha^2+\beta^2} = 1 + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\frac{\alpha\cos(2\beta h) - \beta\sin(2\beta h)}{\sqrt{\alpha^2+\beta^2}}.$$

From these two expressions we can deduce two lower estimates for f:

$$f(\beta) \geq \cos^2(\beta h),$$
$$f(\beta) \geq 1 - \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \geq 1 - \frac{\mu M}{\sqrt{(\mu M)^2+\beta^2}}.$$

Thus, we obtain the resulting estimate

$$f(\beta) \geq \inf_{\beta\in[0,\infty)} \max\left\{\cos^2(\beta h), 1 - \frac{\mu M}{\sqrt{(\mu M)^2+\beta^2}}\right\},$$

which is independent of β. It is not difficult to see that the solution of the minimax problem on the right hand side is achieved at $\beta_0 \in \left(0, \frac{\pi}{2h}\right)$ such that

$$\cos^2(\beta_0 h) = 1 - \frac{\mu M}{\sqrt{(\mu M)^2+\beta_0^2}}.$$

Equation $g(b) = 0$, where g is from (6), follows from the previous equality by setting $b = \beta_0 h$. Since $g(b)$ is an increasing function on $[0, \pi/2]$, and $g(0) < 0$, $g(\pi/2) > 0$, we conclude that the number b on $(0, \pi/2)$ such that $g(b) = 0$ exists and is unique. Therefore, we arrive at

$$v_1(\bar{x}_0) \leq -\frac{\lambda_{\min}(W)e^{-2\mu M h}}{4\mu M}\cos^2(b) = -\alpha_1 < 0.$$

Third part of the proof. It remains to prove that $\bar{x}_0 \in \mathcal{S}$. By Lemma 3, for $t \leq 0$

$$\|\phi(t)\|^2 = \cos^2(\beta t)\|C_1\|^2 - 2\sin(\beta t)\cos(\beta t)C_1^T C_2 + \sin^2(\beta t)\|C_2\|^2 = \cos^2(\beta t) + \sin^2(\beta t)\|C_2\|^2 \leq 1.$$

From this inequality we deduce that $\|\bar{x}(t)\| \leq 1$; moreover, $\|\bar{x}(t)\| \leq e^{\alpha t}$, $t \leq 0$. It also is easy to see that $\|\bar{x}(0)\| = 1$. Now, since $\bar{x}$ satisfies (1), and is continuously differentiable,

$$\|\dot{\bar{x}}(t) - D\dot{\bar{x}}(t-h)\| \leq \|A_0\|\,\|\bar{x}(t)\| + \|A_1\|\,\|\bar{x}(t-h)\| \leq Me^{\alpha t}, \quad t \leq 0.$$

The previous expression means that there is a function ξ that satisfies $\|\xi(t)\| \leq Me^{\alpha t}$ for $t \leq 0$ and

$$\dot{\bar{x}}(t) = D\dot{\bar{x}}(t-h) + \xi(t).$$

Thus, function $z = \dot{\bar{x}}$ satisfies equation

$$z(t) = Dz(t-h) + \xi(t), \quad t \leq 0, \tag{11}$$

and, by definition of $\bar{x}$, it satisfies the additional condition

$$\lim_{t\to-\infty} z(t) = 0. \tag{12}$$

Notice that the function

$$z_1(t) = \sum_{j=0}^{\infty} D^j \xi(t - jh)$$

also does. Indeed, as matrix D is Schur stable, the sum converges and there are constants $\rho \in (0,1)$ and $d \geq 1$ such that $\left\|D^j\right\| \leq d\rho^j$. Then, for $t \leq 0$,

$$\|z_1(t)\| \leq \sum_{j=0}^{\infty} d\rho^j M e^{\alpha(t-jh)} \leq \frac{d}{1-\rho}Me^{\alpha t} = \mu M e^{\alpha t}.$$

Condition (12) is obvious, and

$$z_1(t) - Dz_1(t-h) = \sum_{j=0}^{\infty} D^j\xi(t-jh) - \sum_{j=0}^{\infty} D^{j+1}\xi(t-jh-h) = \sum_{j=0}^{\infty} D^j\xi(t-jh) - \sum_{j=1}^{\infty} D^j\xi(t-jh) = \xi(t).$$

Thus, we have two solutions $\dot{\bar{x}}$ and z_1 of equation (11) that satisfy condition (12). To prove that $\dot{\bar{x}} = z_1$, we need to show the uniqueness of the solution of the homogeneous equation

$$z(t) = Dz(t-h), \quad t \leq 0,$$

under condition (12). In fact, for any $t \leq 0$ and any positive integer k: $z(t) = D^k z(t - kh)$. For sufficiently large k the right hand side is arbitrarily close to zero, therefore, $z(t) = 0$ for any $t \leq 0$. Thus,

$$\|\dot{\bar{x}}(t)\| = \|z_1(t)\| \leq \mu M e^{\alpha t} \leq \mu M, \quad t \leq 0.$$

Hence, we can take $\bar{\varphi} = x_0 \in \mathcal{S}$.