JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 103, No. 1, pp. 183-200, OCTOBER 1999

Asymptotic Stability of Neutral Systems with Multiple Delays

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Communicated by G. Leitmann

Abstract. In this paper, the stability analysis problem for linear neutral delay-differential systems with multiple time delays is investigated. Using the Lyapunov method, we present new sufficient conditions for the asymptotic stability of systems in terms of linear matrix inequalities, which can be solved easily by various convex optimization algorithms. Numerical examples are given to illustrate the application of the proposed method.

Key Words. Neutral systems, multiple time delays, asymptotic stability, linear matrix inequalities.

1. Introduction

The stability analysis of neutral delay-differential systems has received considerable attention over the past two decades. In the literature, various techniques have been utilized to develop criteria for the asymptotic stability of systems. These include the Lyapunov technique, characteristic equation approach, and state trajectory approach. Developed stability criteria are classified often into two categories according to their dependence on the size of the delays. Several delay-independent sufficient conditions for the asymptotic stability of systems are presented in Refs. 1-6. Brayton and Willoughby (Ref. 7) and Khusainov and Yunkova (Ref. 8) exploited the delay-dependent sufficient conditions. However, the stability of neutral systems with multiple time delays has been investigated by only a few researchers. Utilizing the characteristic equation, Hale et al. (Ref. 8) studied

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extensively the stability of systems. However, it is difficult to test their stability criteria. The sufficient condition derived by Hui and Hu (Ref. 10) is expressed in terms of norms and measures of the system matrices. Unfortunately, the matrix measure and matrix norm operations usually make the criteria more conservative.

In this paper, new stability criteria for neutral systems with multiple time delays are presented. First, using the Lyapunov second method, we establish a new delay-independent criterion for the asymptotic stability of systems. Next, we extend the proposed method to develop a delay-dependent one, which can give the maximum allowable bound of the time delay h_i. In the criteria, the derived sufficient conditions are expressed in terms of linear matrix inequalities (LMIs) so that the criteria are less conservative. The solutions of the LMIs can be found easily by various effective optimization algorithms (Ref. 11). Three numerical examples are presented to show the applications of the proposed method.

Notation. $\mathscr{R}^n$ denotes the n-dimensional Euclidean space, I denotes the identity matrix of appropriate order, $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm, $\mu(\cdot)$ denotes the matrix measure of a corresponding matrix, and $\operatorname{diag}\{\cdots\}$ denotes a block diagonal matrix. The notation $X \geq Y$ [respectively, $X > Y$], where X and Y are matrices of same dimensions, means that the matrix $X - Y$ is positive semidefinite [respectively, positive definite].

2. Main Results

Consider a neutral delay-differential system with multiple time delays of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} [B_i x(t-h_i) + C_i \dot{x}(t-h_i)], \tag{1}$$

with initial condition function

$$x(t) = \phi(t), \qquad \forall t \in [-\bar{h}, 0], \tag{2}$$

where $x(t) \in \mathscr{R}^n$ is the state vector, A, B_i, C_i are $n \times n$ constant matrices, h_i is the positive constant time-delay, $\bar{h} = \max\{h_1, h_2, \ldots, h_m\}$, $\phi(\cdot)$ is the given continuously differentiable function on $[-\bar{h}, 0]$, and the system matrix A is assumed to be a Hurwitz matrix.

Then, the following theorem gives a sufficient condition for the delay-independent asymptotic stability of the system (1). For the definition of asymptotic stability or characteristics of the system (1), refer to Hale and Verduyn Lunel (Ref. 2).

Theorem 2.1. The system (1) is asymptotically stable, regardless of the size of the time delay h_i, if there exist positive definite matrices P and R_i satisfying the following LMI:

$$
\Omega(P, R_1, \ldots, R_m) = \begin{bmatrix}
\begin{pmatrix} A^T p + PA \\ + A^T A + \sum_{i=1}^{m} R_i \end{pmatrix} & PB_1 + A^T B_1 & PB_2 + A^T B_2 & \cdots & PB_m + A^T B_m & PC_1 + A^T C_1 & PC_2 + A^T C_2 & \cdots & PC_m + A^T C_m \\
B_1^T P + B_1^T A & B_1^T B_1 - R_1 & B_1^T B_2 & \cdots & B_1^T B_m & B_1^T C_1 & B_1^T C_2 & \cdots & B_1^T C_m \\
B_2^T P + B_2^T A & B_2^T B_1 & B_2^T B_2 - R_2 & \cdots & B_2^T B_m & B_2^T C_1 & B_2^T C_2 & \cdots & B_2^T C_m \\
\vdots & & & \ddots & & & & & \vdots \\
B_m^T P + B_m^T A & B_m^T B_1 & B_m^T B_2 & \cdots & B_m^T B_m - R_m & B_m^T C_1 & B_m^T C_2 & \cdots & B_m^T C_m \\
C_1^T P + C_1^T A & C_1^T B_1 & C_1^T B_2 & \cdots & C_1^T B_m & C_1^T C_1 - (1/m)I & C_1^T C_2 & \cdots & C_1^T C_m \\
C_2^T P + C_2^T A & C_2^T B_1 & C_2^T B_2 & \cdots & C_2^T B_m & C_2^T C_1 & C_2^T C_2 - (1/m)I & \cdots & C_2^T C_m \\
\vdots & & & \ddots & & & & \ddots & \vdots \\
C_m^T P + C_m^T A & C_m^T B_1 & C_m^T B_2 & \cdots & C_m^T B_m & C_m^T C_1 & C_m^T C_2 & \cdots & C_m^T C_m - (1/m)I
\end{bmatrix} < 0. \tag{3}
$$

Proof. Let us choose a Lyapunov function as

$$V = x^T(t) P x(t) + W_1 + W_2, \tag{4}$$

where

$$W_1 = (1/m) \sum_{i=1}^{m} \int_{-h_i}^{0} \dot{x}^T(t+s) \dot{x}(t+s)\, ds, \tag{5}$$

$$W_2 = \sum_{i=1}^{m} \int_{-h_i}^{0} x^T(t+s) R_i x(t+s)\, ds. \tag{6}$$

Then, the time derivative of V along the solution of (1) is

$$\dot{V} = x^T(A^TP + PA)x + 2x^TP \sum_{i=1}^{m} [B_i x_{hi} + C_i \dot{x}_{hi}] + \dot{W}_1 + \dot{W}_2, \tag{7}$$

where x, x_{hi}, $\dot{x}_{hi}$ denote respectively $x(t)$, $x(t-h_i)$, $\dot{x}(t-h_i)$. For (5) and (6), we obtain

$$\begin{aligned}
\dot{W}_1 &= (1/m)\{(\dot{x}^T\dot{x} - \dot{x}_{h1}^T\dot{x}_{h1}) + (\dot{x}^T\dot{x} - \dot{x}_{h2}^T\dot{x}_{h2}) + \cdots + (\dot{x}^T\dot{x} - \dot{x}_{hm}^T\dot{x}_{hm})\}\\
&= \dot{x}^T\dot{x} - (1/m) \sum_{i=1}^{m} \dot{x}_{hi}^T\dot{x}_{hi}\\
&= \left(Ax(t) + \sum_{i=1}^{m} [B_i x(t-h_i) + C_i \dot{x}(t-h_i)]\right)^T\\
&\quad \times \left(Ax(t) + \sum_{i=1}^{m} [B_i x(t-h_i) + C_i \dot{x}(t-h_i)]\right) - (1/m) \sum_{i=1}^{m} \dot{x}_{hi}^T\dot{x}_{hi},
\end{aligned} \tag{8}$$

$$\dot{W}_2 = \sum_{i=1}^{m} (x^TR_ix - x_{hi}^TR_ix_{hi}). \tag{9}$$

Substituting (8) and (9) into (7), we have

$$\dot{V} = X^T\Omega(P, R_1, \ldots, R_m)X, \tag{10}$$

where

$$X = [x^T x_{h1}^T, \ldots, x_{hm}^T \dot{x}_{h1}^T, \ldots, \dot{x}_{hm}^T]^T.$$

Therefore, $\dot{V}$ is negative if Inequality (3) is satisfied. This completes the proof. □

Remark 2.1. Hui and Hu (Ref. 10) derived a delay-independent sufficient condition for the asymptotic stability of the system (1) as

$$\mu(A) + \sum_{i=1}^{m} \|B_i\| + \left[\sum_{i=1}^{m} \|C_iA\| + \sum_{i=1}^{m} \left(\sum_{k=1}^{m} \|C_iB_k\|\right)\right] \Big/ \left[1 - \sum_{i=1}^{m} \|C_i\|\right] < 0, \tag{11}$$

under the assumption

$$\sum_{i=1}^{m} \|C_i\| < 1.$$

This condition may be relatively more conservative due to the matrix measure and matrix norm operations. Also note that (11) implies

$$\mu(A) < 0.$$

Example 2.1. Consider the following system:

$$\dot{x}(t)=Ax(t)+B_1x(t-h_1)+B_2x(t-h_2)+C_1\dot{x}(t-h_1)+C_2\dot{x}(t-h_2),$$

where

$$A=\begin{bmatrix}-3 & -2\\ 1 & 0\end{bmatrix},\qquad B_1=\alpha\begin{bmatrix}0.2 & 0.1\\ 0.1 & 0.2\end{bmatrix},\qquad B_2=\begin{bmatrix}0 & 0.1\\ 0.1 & 0\end{bmatrix},$$

$$C_1=\begin{bmatrix}0.1 & 0.05\\ 0.05 & 0.1\end{bmatrix},\qquad C_2=\begin{bmatrix}0.05 & 0\\ 0 & 0.05\end{bmatrix}$$

and α is a nonzero constant.

We now determine the stability bound of the system in terms of α. Since

$$\mu(A)=0.0811>0,$$

the criterion (11) of Hui and Hu (Ref. 10) cannot be applicable. However, the system matrix A is Hurwitz, so that Theorem 2.1 can be applied.

Solving the LMI given in (3), we obtain the bound for asymptotic stability as

$$|\alpha|\leq 4.6808,$$

and the solution of the LMI as

$$P=\begin{bmatrix}9.6932 & 7.6257\\ 7.6257 & 7.2555\end{bmatrix},$$

$$R_1=\begin{bmatrix}8.6695 & 8.8667\\ 8.8667 & 9.6590\end{bmatrix},\qquad R_2=\begin{bmatrix}1.2994 & 0.8513\\ 0.8513 & 0.6203\end{bmatrix}.$$

In order to apply the Hui and Hu criterion, we transform the state as

$$x(t)=Tz(t).$$

Then, the system can be expressed as

$$\begin{aligned}\dot{z}(t)&=T^{-1}ATz(t)+T^{-1}B_1Tz(t-h_1)+T^{-1}B_2Tz(t-h_2)\\ &\quad+T^{-1}C_1T\dot{z}(t-h_1)+T^{-1}C_2T\dot{z}(t-h_2)\\ &=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}z(t)+\alpha\begin{bmatrix}0.3 & 0\\ 0.3 & 0.1\end{bmatrix}z(t-h_1)+\begin{bmatrix}0.1 & 0\\ 0.3 & -0.1\end{bmatrix}z(t-h_2)\\ &\quad+\begin{bmatrix}0.15 & 0\\ 0.15 & 0.05\end{bmatrix}\dot{z}(t-h_1)+\begin{bmatrix}0.05 & 0\\ 0 & 0.05\end{bmatrix}\dot{z}(t-h_2),\end{aligned}$$

where

$$T=\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Now, $\mu(A)<0$; therefore, both criteria can be applied. By simple calculation, we obtain the bound for asymptotic stability as

$$|\alpha|\leq 0.162, \qquad \text{Hui and Hu (Ref. 10)},$$

$$|\alpha|\leq 5.125, \qquad \text{Theorem 2.1}.$$

The example shows that the proposed criterion is considerably less conservative than that of Hui and Hu.

Next, we establish a delay-dependent stability criterion for neutral systems with multiple delays. The following lemmas, assumption, and definition are necessary to develop the criterion.

Lemma 2.1. See Ref. 12. Let D and E be real matrices of appropriate dimensions. Then, for any scalar $\epsilon>0$,

$$DE+E^TD^T\leq \epsilon DD^T+\epsilon^{-1}E^TE.$$

Lemma 2.2. See Ref. 13. The LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix}>0$$

is equivalent to

$$R(x)>0, \qquad Q(x)-S(x)R(x)^{-1}S(x)^T>0,$$

where

$$Q(x)=Q(x)^T, \qquad R(x)=R(x)^T,$$

and $S(x)$ depends affinely on x.

Assumption 2.1. In connection with the system (1), the matrix $A+\sum_{i=1}^m B_i$ has all of its eigenvalues in the open left-half plane.

Definition 2.1. The matrix functions S_1 and S_{2i} are defined as

$$\begin{aligned} &S_1(P,\epsilon_1,\epsilon_2,\epsilon_3,h_1,\ldots,h_m)\\ &=A_0^TP+PA_0+\mathscr{B}_1+(\epsilon_1^{-1}+m\epsilon_2^{-1})\mathscr{C}+(\epsilon_1+\epsilon_2)P\mathscr{B}_2P\\ &\quad+m\epsilon_3PP+3A^TA+\sum_{i=1}^m h_iU+2\sum_{i=1}^m h_iPB_iB_i^TP, \end{aligned} \tag{12}$$

$$S_{2i}(\epsilon_3)=(3m+\epsilon_3^{-1})C_i^TC_i-(1/m)I, \qquad i=1,2,\ldots,m, \tag{13}$$

where P is a symmetric positive-definite matrix, ϵ_1, ϵ_2, ϵ_3 are positive scalars, and

$$A_0=A+B_1+B_2+\cdots+B_m, \qquad U=A^TA+m\sum_{i=1}^m B_i^TB_i,$$

$$\mathscr{B}_1=3m\sum_{i=1}^m B_i^TB_i, \qquad \mathscr{B}_2=m\sum_{i=1}^m B_iB_i^T, \qquad \mathscr{C}=m\sum_{i=1}^m C_i^TC_i.$$

Note that $S_1(P,\epsilon_1,\epsilon_2,\epsilon_3,h_1,\ldots,h_m)$ is monotonic nondecreasing with respect to h_i, in the sense of positive semi-definiteness.

The following theorem gives a delay-dependent sufficient condition for the asymptotic stability of the system (1).

Theorem 2.2. Let $\sigma_i=1/h_i$, $i=1,2,\ldots,m$. For given h_i and m, the system (1) is asymptotically stable, if there exists a symmetric positive-definite matrix X and positive scalars $\epsilon_1,\epsilon_2,\epsilon_3$ satisfying the following LMIs:

$$Y(X,\epsilon_1,\epsilon_2,\epsilon_3)=\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{12}^T & Y_{22} & 0 & 0 \\ Y_{13}^T & 0 & Y_{33} & 0 \\ Y_{14}^T & 0 & 0 & Y_{44}\end{bmatrix}<0 \tag{14}$$

and

$$\begin{bmatrix} 3mC_i^TC_i-(1/m)I & C_i^T \\ C_i & -\epsilon_3 I\end{bmatrix}<0, \qquad i=1,2,\ldots,m, \tag{15}$$

where the upper triangular entries of the symmetric matrix Y are

$$Y_{11}=XA_0^T+A_0X+(\epsilon_1+\epsilon_2)\mathscr{B}_2+m\epsilon_3 I, \tag{16a}$$

$$Y_{12}=[XA^T \; X\mathscr{B}_1^{1/2} \; X\mathscr{C}^{1/2} \; X\mathscr{C}^{1/2}], \tag{16b}$$

$$Y_{13}=[X\mathscr{U}^{1/2}\cdots X\mathscr{U}^{1/2}], \tag{16c}$$

$$Y_{14}=[B_1\cdots B_m], \tag{16d}$$

$$Y_{22}=\mathrm{diag}\{-(1/3)I, -I, -\epsilon_1 I, -(1/m)\epsilon_2 I\}, \tag{16e}$$

$$Y_{33}=\mathrm{diag}\{-\sigma_1 I,\ldots,-\sigma_m I\}, \tag{16f}$$

$$Y_{44}=\mathrm{diag}\{-(1/2)\sigma_1 I,\ldots,-(1/2)\sigma_m I\}. \tag{16g}$$

Proof. Without loss of generality, it is assumed that $x(t)$ is continuously differentiable on the interval $[-\max(h_i+h_j), -\bar{h}]$, $\forall i,j=1,2,\ldots,m$. Rewrite (1) as

$$\begin{aligned}
\dot{x}(t) &= (A+B_1+B_2+\cdots+B_m)x(t) - B_1\int_{t-h_1}^{t} \dot{x}(s)\,ds \\
&\quad - B_2\int_{t-h_2}^{t} \dot{x}(s)\,ds - \cdots - B_m\int_{t-h_m}^{t} \dot{x}(s)\,ds + \sum_{i=1}^{m} C_i\dot{x}(t-h_i) \\
&= A_0x(t) - \sum_{i=1}^{m} B_i\int_{t-h_i}^{t} \dot{x}(s)\,ds + \sum_{i=1}^{m} C_i\dot{x}(t-h_i) \\
&= A_0x(t) - \sum_{i=1}^{m} B_i\int_{t-h_i}^{t} \left\{Ax(s) + \sum_{j=1}^{m} B_jx(s-h_j) + \sum_{j=1}^{m} C_j\dot{x}(s-h_j)\right\} ds \\
&\quad + \sum_{i=1}^{m} C_i\dot{x}(t-h_i) \\
&= A_0x(t) - \eta_1 - \eta_2 - \sum_{i=1}^{m} B_i\left[\sum_{j=1}^{m} C_j\{x(t-h_j) - x(t-h_i-h_j)\}\right] \\
&\quad + \sum_{i=1}^{m} C_i\dot{x}(t-h_i),
\end{aligned} \tag{17}$$

where

$$\eta_1 = \sum_{i=1}^{m} B_i\int_{t-h_i}^{t} Ax(s)\,ds, \tag{18a}$$

$$\eta_2 = \sum_{i=1}^{m} B_i\int_{t-h_i}^{t} \sum_{j=1}^{m} B_jx(s-h_j)\,ds. \tag{18b}$$

Now, consider the following Lyapunov function for the system (17):

$$V = x^TPx + Z_1 + Z_2 + Z_3 + Z_4 + Z_5, \tag{19}$$

where P is the matrix given in (12) and

$$Z_1 = (1/m)\sum_{i=1}^{m}\int_{-h_i}^{0} \dot{x}^T(t+s)\dot{x}(t+s)\,ds, \tag{20}$$

$$Z_2 = \sum_{i=1}^{m}\int_{-h_i}^{0} x^T(t+s)R_ix(t+s)\,ds, \tag{21}$$

$$Z_3=(1/m)\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-h_i-h_j}^{0} x^T(t+s)Q_i x(t+s)\,ds, \tag{22}$$

$$Z_4=\sum_{i=1}^{m}\int_{-h_i}^{0}\left[\int_{t+s}^{t}\|Ax(\theta)\|^2\,d\theta + \int_{t+s}^{t}\left\|\sum_{j=1}^{m} B_j x(\theta-h_j)\right\|^2 d\theta\right] ds, \tag{23}$$

$$Z_5=m\sum_{i=1}^{m}\sum_{j=1}^{m} h_i\int_{-h_j}^{0} x^T(t+s)B_j^T B_j x(t+s)\,ds, \tag{24}$$

and R_i and Q_i are positive semidefinite matrices to be found. Then, the time derivative of V along the solution of (17) is

$$\begin{aligned}\dot{V}&=x^T(A_0^TP+PA_0)x-2x^TP\eta_1-2x^TP\eta_2\\&\quad-2x^TP\sum_{i=1}^{m}\sum_{j=1}^{m}B_iC_jx_{hj}+2x^TP\sum_{i=1}^{m}\sum_{j=1}^{m}B_iC_jx_{hij}\\&\quad+2x^TP\sum_{i=1}^{m}C_i\dot{x}_{hi}+\dot{Z}_1+\dot{Z}_2+\dot{Z}_3+\dot{Z}_4+\dot{Z}_5,\end{aligned} \tag{25}$$

where

$$x_{hij}=x(t-h_i-h_j).$$

From (20) to (23), we have

$$\begin{aligned}\dot{Z}_1&=(1/m)\sum_{i=1}^{m}[\dot{x}^T\dot{x}-\dot{x}_{hi}^T\dot{x}_{hi}]\\&=\left(Ax+\sum_{i=1}^{m}B_ix_{hi}+\sum_{i=1}^{m}C_i\dot{x}_{hi}\right)^T\left(Ax+\sum_{i=1}^{m}B_ix_{hi}+\sum_{i=1}^{m}C_i\dot{x}_{hi}\right)\\&\quad-(1/m)\sum_{i=1}^{m}\dot{x}_{hi}^T\dot{x}_{hi}\\&=x^TA^TAx+2x^TA^T\sum_{i=1}^{m}B_ix_{hi}+2x^TA^T\sum_{i=1}^{m}C_i\dot{x}_{hi}\\&\quad+\sum_{i=1}^{m}\sum_{j=1}^{m}x_{hi}^TB_i^TB_jx_{hj}+2\sum_{i=1}^{m}\sum_{j=1}^{m}x_{hi}^TB_i^TC_j\dot{x}_{hj}\\&\quad+\sum_{i=1}^{m}\sum_{j=1}^{m}\dot{x}_{hi}^TC_i^TC_j\dot{x}_{hj}-(1/m)\sum_{i=1}^{m}\dot{x}_{hi}^T\dot{x}_{hi},\end{aligned} \tag{26}$$

$$\dot{Z}_2 = \sum_{i=1}^{m} x^T R_i x - \sum_{i=1}^{m} x_{hi}^T R_i x_{hi}, \tag{27}$$

$$\begin{aligned}\dot{Z}_3 &= (1/m) \sum_{i=1}^{m} \sum_{j=1}^{m} (x^T Q_i x - x_{hij}^T Q_i x_{hij}) \\ &= \sum_{i=1}^{m} x^T Q_i x - (1/m) \sum_{i=1}^{m} \sum_{j=1}^{m} x_{hij}^T Q_i x_{hij},\end{aligned} \tag{28}$$

$$\begin{aligned}\dot{Z}_4 = \sum_{i=1}^{m} \Bigg[h_i \|Ax(t)\|^2 + h_i \left\| \sum_{j=1}^{m} B_j x(t-h_j) \right\|^2 - \int_{s-h_i}^{t} \|Ax(s)\|^2 \, ds \\ - \int_{t-h_i}^{t} \left\| \sum_{j=1}^{m} B_j x(t-h_j) \right\|^2 ds \Bigg],\end{aligned} \tag{29}$$

$$\dot{Z}_5 = m \sum_{i=1}^{m} \sum_{j=1}^{m} h_i x^T B_j^T B_j x - m \sum_{i=1}^{m} \sum_{j=1}^{m} h_i x_{hj}^T B_j^T B_j x_{hj}. \tag{30}$$

By Lemma 2.1, the terms on the right-hand side of (26) satisfy the following inequalities (see Appendix A, Section 4):

$$\sum_{i=1}^{m} \sum_{j=1}^{m} x_{hi}^T B_i^T B_j x_{hj} \le m \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi}, \tag{31}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \dot{x}_{hi}^T C_i^T C_j \dot{x}_{hj} \le m \sum_{i=1}^{m} \dot{x}_{hi}^T C_i^T C_i \dot{x}_{hi}, \tag{32}$$

$$\begin{aligned}2x^T A^T \sum_{i=1}^{m} B_i x_{hi} &\le x^T A^T A x + \sum_{i=1}^{m} \sum_{j=1}^{m} x_{hi}^T B_i^T B_j x_{hj} \\ &\le x^T A^T A x + m \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi},\end{aligned} \tag{33}$$

$$2x^T A^T \sum_{i=1}^{m} C_i \dot{x}_{hi} \le x^T A^T A x + m \sum_{i=1}^{m} \dot{x}_{hi}^T C_i^T C_i \dot{x}_{hi}, \tag{34}$$

$$2 \sum_{i=1}^{m} \sum_{j=1}^{m} x_{hi}^T B_i^T C_j \dot{x}_{hj} \le m \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} + m \sum_{i=1}^{m} \dot{x}_{hi}^T C_i^T C_i \dot{x}_{hi}. \tag{35}$$

Substituting (31)-(35) into (26), we obtain a bound on $\dot{Z}_1$ as

$$\begin{aligned}\dot{Z}_1 \le 3x^T A^T A x + 3m \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} \\ + 3m \sum_{i=1}^{m} \dot{x}_{hi}^T C_i^T C_i \dot{x}_{hi} - (1/m) \sum_{i=1}^{m} \dot{x}_{hi}^T \dot{x}_{hi}.\end{aligned} \tag{36}$$

Using (31), it can be shown that the term $\sum_{i=1}^{m} h_i \|\sum_{j=1}^{m} B_j x(t-h_j)\|^2$ in (29) satisfies the following inequality:

$$\begin{aligned}
&\sum_{i=1}^{m} h_i \left\| \sum_{j=1}^{m} B_j x(t-h_j) \right\|^2 \\
&= h_1 \sum_{i=1}^{m} \sum_{j=1}^{m} x_{hi}^T B_i^T B_j x_{hj} + \cdots + h_m \sum_{i=1}^{m} \sum_{j=1}^{m} x_{hi}^T B_i^T B_j x_{hj} \\
&\leq h_1 m \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} + h_2 m \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} \\
&\quad + \cdots + h_m m \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} \\
&= m \sum_{i=1}^{m} h_i \sum_{j=1}^{m} x_{hj}^T B_j^T B_j x_{hj}.
\end{aligned} \tag{37}$$

Substituting (37) into (29), we have

$$\begin{aligned}
\dot{Z}_4 \leq \sum_{i=1}^{m} \Bigg[& h_i \|Ax(t)\|^2 + m h_i \sum_{j=1}^{m} x_{hj}^T B_j^T B_j x_{hj} - \int_{t-h_i}^{t} \|Ax(s)\|^2 \, ds \\
& - \int_{t-h_i}^{t} \left\| \sum_{j=1}^{m} B_j x(s-h_j) \right\|^2 ds \Bigg].
\end{aligned} \tag{38}$$

Similarly, the other terms in (25) satisfy the following inequalities [see Appendix B, C, D (Sections 5 to 7)]:

$$-2x^T P \eta_1 \leq \sum_{i=1}^{m} h_i x^T P B_i B_i^T P x + \sum_{i=1}^{m} \int_{t-h_i}^{t} \|Ax(s)\|^2 \, ds, \tag{39}$$

$$-2x^T P \eta_2 \leq \sum_{i=1}^{m} h_i x^T P B_i B_i^T P x + \sum_{i=1}^{m} \int_{t-h_i}^{t} \left\| \sum_{j=1}^{m} B_j x(s- \right. \tag{40}$$

$$-2x^T P \sum_{i=1}^{m} \sum_{j=1}^{m} B_i C_j x_{hj} \leq m\epsilon_1 x^T P \sum_{i=1}^{m} B_i B_i^T P x + m\epsilon_1^{-1} \sum_{i=1}^{m} x_{hi}^t C_i^T C_i x_{hi}, \tag{41}$$

$$2x^T P \sum_{i=1}^{m} \sum_{j=1}^{m} B_i C_j x_{hij} \leq m\epsilon_2 x^T P \sum_{i=1}^{m} B_i B_i^T P x + m\epsilon_2^{-1} \sum_{i=1}^{m} \sum_{j=1}^{m} x_{hij}^T C_i^T C_i x_{hij}, \tag{42}$$

$$2x^T P \sum_{i=1}^{m} C_i \dot{x}_{hi} \leq m\epsilon_3 x^T P P x + \epsilon_3^{-1} \sum_{i=1}^{m} \dot{x}_{hi}^T C_i^T C_i \dot{x}_{hi}. \tag{43}$$

Substituting (27)–(28), (30), (36), (38), and (39)–(43) into (25), we have

$$
\begin{aligned}
\dot{V} \le\, & x^T\left(A_0^T P + PA_0 + \sum_{i=1}^{m} R_i + \sum_{i=1}^{m} Q_i + m\epsilon_3 PP\right)x + 3x^T A^T A x \\
& + 3m \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} + m\epsilon_1^{-1} \sum_{i=1}^{m} x_{hi}^T C_i^T C_i x_{hi} \\
& + m\epsilon_2^{-1} \sum_{i=1}^{m} \sum_{j=1}^{m} x_{hij}^T C_i^T C_i x_{hij} \\
& + 2 \sum_{i=1}^{m} h_i x^T P B_i B_i^T P x + m(\epsilon_1 + \epsilon_2) \sum_{i=1}^{m} x^T P B_i B_i^T P x \\
& + (3m + \epsilon_3^{-1}) \sum_{i=1}^{m} \dot{x}_{hi}^T C_i^T C_i \dot{x}_{hi} - (1/m) \sum_{i=1}^{m} \dot{x}_{hi}^T \dot{x}_{hi} - \sum_{i=1}^{m} x_{hi}^T R_i x_{hi} \\
& - (1/m) \sum_{i=1}^{m} \sum_{j=1}^{m} x_{hij}^T Q_i x_{hij} \\
& + \sum_{i=1}^{m} h_i \|Ax\|^2 + m \sum_{i=1}^{m} h_i \sum_{j=1}^{m} x^T B_j^T B_j x.
\end{aligned} \tag{44}
$$

Now, let us choose Q_i, R_i as

$$
Q_i = m^2 \epsilon_2^{-1} C_i^T C_i, \qquad R_i = 3m B_i^T B_i + m\epsilon_1^{-1} C_i^T C_i.
$$

Then, (44) is simplified to

$$
\begin{aligned}
\dot{V} \le\, & x^T\Bigg[A_0^T P + PA_0 + \sum_{i=1}^{m} \{3m B_i^T B_i + (m\epsilon_1^{-1} + m^2 \epsilon_2^{-1}) C_i^T C_i\} \\
& \quad + m(\epsilon_1 + \epsilon_2) \sum_{i=1}^{m} P B_i B_i^T P + m\epsilon_3 PP + 3A^T A \\
& \quad + \sum_{i=1}^{m} h_i \left\{A^T A + m \sum_{j=1}^{m} B_j^T B_j\right\} + 2 \sum_{i=1}^{m} h_i P B_i B_i^T P\Bigg] x \\
& + \sum_{i=1}^{m} \dot{x}_{hi}^T [(3m + \epsilon_3^{-1}) C_i^T C_i - (1/m) I] \dot{x}_{hi} \\
=\, & x^T S_1(P, \epsilon_1, \epsilon_2, \epsilon_3, h_1, \ldots, h_m) x + \sum_{i=1}^{m} \dot{x}_{hi}^T S_{2i}(\epsilon_3) \dot{x}_{hi} \\
=\, & x^T P P^{-1} S_1(P, \epsilon_1, \epsilon_2, \epsilon_3, h_1, \ldots, h_m) P^{-1} P x \\
& + \sum_{i=1}^{m} \dot{x}_{hi}^T S_{2i}(\epsilon_3) \dot{x}_{hi}.
\end{aligned} \tag{45}
$$

Therefore, $\dot{V}$ is negative if the following two inequalities are satisfied:

$$P^{-1}S_1(P, \epsilon_1, \epsilon_2, \epsilon_3, h_1, \ldots, h_m)P^{-1}<0, \tag{46}$$

$$S_{2i}(\epsilon_3)<0, \qquad i=1, 2, \ldots, m. \tag{47}$$

Let

$$P^{-1}=X;$$

substituting (12)–(13) into (46)–(47), we obtain

$$\begin{aligned} &XA_0^T+A_0X+X\mathscr{B}_1X+(\epsilon_1^{-1}+m\epsilon_2^{-1})X\mathscr{C}X+(\epsilon_1+\epsilon_2)\mathscr{B}_2 \\ &+m\epsilon_3I+3XA^TAX+\sum_{i=1}^m h_iXUX+2\sum_{i=1}^m h_iB_iB_i^T<0, \end{aligned} \tag{48}$$

$$(3m+\epsilon_3^{-1})C_i^TC_i-(1/m)I<0, \qquad i=1, 2, \ldots, m. \tag{49}$$

Then, by Lemma 2.2, Inequalities (48) and (49) are equivalent to (14) and (15), respectively. This completes the proof. □

Remark 2.2. Since the matrix function $S_1(\cdot)$ is monotonic nondecreasing with respect to h_i, we have the relation

$$S_1(P, \epsilon_1, \epsilon_2, \epsilon_3, h_1, \ldots, h_m)\leq S_1(P, \epsilon_1, \epsilon_2, \epsilon_3, \bar{h}, \ldots, \bar{h}).$$

Therefore, if we modify the stability condition (46) as

$$P^{-1}S_1(P, \epsilon_1, \epsilon_2, \epsilon_3, \bar{h}, \ldots, \bar{h})P^{-1}<0, \tag{50}$$

then the LMI (14) in Theorem 2.2 becomes

$$Y(X, \epsilon_1, \epsilon_2, \epsilon_3)=\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{12}^T & Y_{22} & 0 & 0 \\ Y_{13}^T & 0 & Y_{33} & 0 \\ Y_{14}^T & 0 & 0 & Y_{44} \end{bmatrix}<0, \tag{51}$$

where

$$Y_{33}=\mathrm{diag}\{-\sigma I, \ldots, -\sigma I\}, \tag{52a}$$

$$Y_{44}=\mathrm{diag}\{-(1/2)\sigma I, \ldots, -(1/2)\sigma I\}, \qquad \sigma=1/\bar{h}, \tag{52b}$$

and the other entries of the matrix $Y(\cdot)$ are the same as those in (16). Then, we can find the maximum allowable delay bound $\bar{h}_{\max}=1/\sigma$ by solving the convex optimization problem for minimization of σ subjected to

$$X>0, \qquad \epsilon_1>0, \qquad \epsilon_2>0, \qquad \epsilon_3>0, \qquad \sigma>0,$$

and the two LMIs (15) and (51). For the details of the convex optimization algorithm, see Ref. 11.

Now, we present two examples for application of the proposed delay-dependent criterion. Example 2.2 refers to the investigation of the stability for given time delays h_i'; Example 2.3 refers to finding the maximum allowable delay bound.

Example 2.2. Consider the system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -2 & 0.1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0.2 & 0.1 \\ -0.1 & 0.2 \end{bmatrix} x(t-0.3) \\ &\quad + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} x(t-0.6) + \begin{bmatrix} 0.1 & 0.05 \\ 0.02 & 0.1 \end{bmatrix} \dot{x}(t-0.3) \\ &\quad + \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix} \dot{x}(t-0.6).\end{aligned}$$

Now, by solving the LMIs (14) and (15) of Theorem 2.2, we have

$$\epsilon_1 = 0.0906, \qquad \epsilon_2 = 0.11, \qquad \epsilon_3 = 0.0493,$$

$$X = \begin{bmatrix} 0.1075 & -0.0013 \\ -0.0013 & 0.0953 \end{bmatrix}.$$

This gives the asymptotic stability of the above system for the given time delays $h_1 = 0.3$ and $h_2 = 0.6$.

Example 2.3. Consider the system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0.3 \\ -0.3 & 0 \end{bmatrix} x(t-h_1) \\ &\quad + \begin{bmatrix} 0.1 & -0.05 \\ 0.05 & 0.1 \end{bmatrix} x(t-h_2) + \begin{bmatrix} 0 & -0.1 \\ -0.1 & 0 \end{bmatrix} \dot{x}(t-h_1) \\ &\quad + \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix} \dot{x}(t-h_2).\end{aligned}$$

In light of Remark 2.2, we can compute the maximum allowable bound $\bar{h}_{\max}$ for the asymptotic stability of the system by solving the LMIs (51) and (15) such that σ is minimized. The solution is

$$\epsilon_1 = 0.0356, \qquad \epsilon_2 = 0.0503, \qquad \epsilon_3 = 0.0227, \qquad \sigma = 1.8117,$$

$$X = \begin{bmatrix} 0.1122 & 0.0123 \\ 0.0123 & 0.0813 \end{bmatrix}.$$

and the maximum allowable bound is

$$\bar{h}_{\max} = 1/\sigma = 0.552.$$

3. Concluding Remarks

In this paper, the asymptotic stability of linear neutral delay-differential systems with multiple time delays is investigated. According to their dependence on the size of the delays, we present two new stability criteria which are expressed in terms of LMIs. There is a good possibility that the proposed delay-independent criterion is less conservative than the criteria in the literature, which use the matrix norms and matrix measure. Furthermore, the delay-dependent criterion can give the maximum allowable bound of the delay for asymptotic stability. The numerical examples show the effectiveness of the proposed criteria.

4. Appendix A: Derivation of (31) and (32)

Using Lemma 2.1, we have

$$\begin{aligned}
&\sum_{i=1}^{m} \sum_{j=1}^{m} x_{hi}^T B_i^T B_j x_{hj} \\
&= \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} + \sum_{i=1}^{m} \sum_{j \neq i}^{m} x_{hi}^T B_i^T B_j x_{hj} \\
&= \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x_{hi}^T B_i^T B_j x_{hj} \\
&\leq \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (x_{hi}^T B_i^T B_i x_{hi} + x_{hj}^T B_j^T B_j x_{hj}) \\
&= \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} + (m-1) \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi} \\
&= m \sum_{i=1}^{m} x_{hi}^T B_i^T B_i x_{hi},
\end{aligned}$$

hence Inequality (31). Similarly, we can also obtain Inequality (32).

5. Appendix B: Derivation of (39) and (40)

Using Lemma 2.1, we have

$$\begin{aligned}
-2x^TP\eta_1 &= -2x^TP\left(B_1\int_{t-h_1}^{t} Ax(s)\,ds + B_2\int_{t-h_2}^{t} Ax(s)\,ds\right.\\
&\qquad\left. +\cdots + B_m\int_{t-h_m}^{t} Ax(s)\,ds\right)\\
&\le (h_1x^TPB_1B_1^TPx + h_1^{-1}\rho_1^T\rho_1)\\
&\quad + (h_2x^TPB_2B_2^TPx + h_2^{-1}\rho_2^T\rho_2)\\
&\quad + \cdots + (h_mx^TPB_mB_m^TPx + h_m^{-1}\rho_m^T\rho_m)\\
&= \sum_{i=1}^{m} h_ix^TPB_iB_i^TPx + h_i^{-1}\sum_{i=1}^{m}\rho_i^T\rho_i,
\end{aligned} \tag{53}$$

where

$$\rho_i = \int_{t-h_i}^{t} Ax(s)\,ds, \qquad i = 1, 2, \ldots, m.$$

Here,

$$\begin{aligned}
\rho_i^T\rho_i &\le \left\|\int_{t-h_i}^{t} Ax(s)\,ds\right\|^2\\
&\le \left[\int_{t-h_i}^{t} \|Ax(s)\|\,ds\right]^2\\
&\le h_i\int_{t-h_i}^{t} \|Ax(s)\|^2\,ds,
\end{aligned} \tag{54}$$

where the third inequality is obtained using the Schwartz inequality. Substituting (54) into (53), we have Inequality (39). Similarly, we can obtain Inequality (40).

6. Appendix C: Derivation of (41) and (42)

Using Lemma 2.1, we have

$$\begin{aligned}
&-2x^TP\sum_{i=1}^{m}\sum_{j=1}^{m} B_iC_jx_{hj}\\
&= -2x^TPB_1(C_1x_{h1} + C_2x_{h2} + \cdots + C_mx_{hm})
\end{aligned}$$

$$
\begin{aligned}
&-2x^T PB_2(C_1 x_{h1} + C_2 x_{h2} + \cdots + C_m x_{hm}) \\
&\quad + \cdots - 2x^T PB_m(C_1 x_{h1} + C_2 x_{h2} + \cdots + C_m x_{hm}) \\
&\leq \left(m\epsilon_1 x^T PB_1 B_1^T Px + \epsilon_1^{-1} \sum_{j=1}^{m} x_{hj}^T C_j^T C_j x_{hj} \right) \\
&\quad + \cdots + \left(m\epsilon_1 x^T PB_m B_m^T Px + \epsilon_1^{-1} \sum_{j=1}^{m} x_{hj}^T C_j^T C_j x_{hj} \right) \\
&= m\epsilon_1 x^T P \sum_{i=1}^{m} B_i B_i^T Px + m\epsilon_1^{-1} \sum_{i=1}^{m} x_{hi}^T C_i^T C_i x_{hi},
\end{aligned}
\tag{55}
$$

where $\epsilon_1 > 0$, hence Inequality (41). Also, Inequality (42) can be easily derived utilizing the above procedure.

7. Appendix D: Derivation of (43)

Using Lemma 2.1, we have

$$
\begin{aligned}
2x^T P \sum_{i=1}^{m} C_i \dot{x}_{hi} &= 2x^T P(C_1 \dot{x}_{h1} + C_2 \dot{x}_{h2} + \cdots + C_m \dot{x}_{hm}) \\
&\leq (\epsilon_3 x^T PPx + \epsilon_3^{-1} \dot{x}_{h1}^T C_1^T C_1 \dot{x}_{h1}) \\
&\quad + (\epsilon_3 x^T PPx + \epsilon_3^{-1} \dot{x}_{h2}^T C_2^T C_2 \dot{x}_{h2}) \\
&\quad + \cdots + (\epsilon_3 x^T PPx + \epsilon_3^{-1} \dot{x}_{hm}^T C_m^T C_m \dot{x}_{hm}) \\
&= m\epsilon_3 x^T PPx + \epsilon_3^{-1} \sum_{i=1}^{m} \dot{x}_{hi}^T C_i^T C_i \dot{x}_{hi},
\end{aligned}
\tag{56}
$$

where $\epsilon_3 > 0$, hence Inequality (43).

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