Brief paper

Complete quadratic Lyapunov functionals for distributed delay systems☆

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ABSTRACT

This paper is concerned with the stability analysis of distributed delay systems using complete-Lyapunov functionals. Numerous articles aim at approximating their parameters thanks to a discretization method or polynomial modeling. The interest of such approximations is the design of tractable sufficient stability conditions. In the present article, we provide an alternative method based on polynomial approximation which takes advantages of the Legendre polynomials and their properties. The resulting stability conditions are scalable with respect to the maximum degree of the polynomials and are expressed in terms of tractable linear matrix inequalities. Several examples of delayed systems are tested to show the effectiveness of the method.

1. Introduction

The stability analysis of time-delay systems has attracted many researchers during the last decades (see for instance the survey article Sipahi, Niculescu, Abdallah, Michiels, & Gu, 2011). In particular, some of them focused on the class of distributed time delay systems, for which the delay operator, via an integral action, has a cumulative effect over the past values of the dynamics. For specific integral kernels, including constant, rationals or γ distribution function, stability of such systems can be tackled by a direct method based on pole location (Breda, Maset, & Vermiglio, 2009; Morarescu, Niculescu, & Gu, 2007; Ozbay, Bonnet, & Clairambault, 2008; Sipahi, Atay, & Niculescu, 2008). For kernels which can be interpreted as impulse responses of a linear system, the system can be modeled as a linear pointwise delay system and hence classical techniques on pole location can be used (see Ozbay et al., 2008). Nevertheless, these techniques are often restricted to the case of nominal systems without uncertainties (Sipahi et al., 2011) and for a constant delay.

Another widely used technique for stability analysis relies on Lyapunov Theory. The construction of Lyapunov–Krasovskii functionals is modified in order to take into account the distributed nature of the delay. It results some Lyapunov structure which depends on double and triple integral terms (Chen & Zheng, 2007; Fridman & Shaked, 2002; Han, 2004; Kolmanovskii & Shaikhet, 1996; Li & Zhu, 2008; Xie, Fridman, & Shaked, 2001). Different bounding techniques are then employed to derive numerically efficient stability tests. In general, the kernel is then restricted to be a constant function over the delay interval. In order to avoid this strong constraint, another possibility, described by Gu, Han, Luo, and Niculescu (2001), is to use a very general class of Lyapunov–Krasovskii functional (see for instance Santos, Mondie, & Kharitonov, 2006) and a discretization scheme to obtain numerically tractable stability conditions. This technique designed for piecewise constant delay kernel leads to a reduction of conservatism at the expense of the numerical burden. The case of general kernels has been studied successfully by Solomon and Fridman (2013) where the use of new integral inequalities is proposed. These new bounding techniques which were especially designed for distributed delay systems led to new Lyapunov–Krasovskii functional structures and efficient results on simulations. Another widely used approach in the literature stems from the robust analysis using the same driving forces namely the comparison systems

and integral inequalities (Niculescu, 2001). In Münz, Rieber, and Allgöwer (2009), a combined full block S-Procedure and Lyapunov analysis is performed to prove the stability of a distributed delay system with a rational kernel. More recently, using quadratic separation approach, Gouaisbaut and Ariba (2011) have provided some Linear Matrix inequality (LMI) for polynomial kernels.

In this paper, we aim at presenting a new method for the stability analysis of distributed delay systems using a Lyapunov–Krasovskii approach. The problem consists in selecting a particular structure for the Lyapunov–Krasovskii functional, which is composed of the sum of several classical terms (Gu, Kharitonov, & Chen, 2003), including a quadratic function of the instantaneous state and an integral of quadratic functional of the entire delay state. In general, all these functionals are particular cases of the well-known complete Lyapunov–Krasovskii functionals. Indeed Theorem 5.9 from Gu et al. (2003) ensures that, if the solutions of a time delay system is asymptotically stable, this functional is a Lyapunov–Krasovskii functional, provided that its parameters satisfy some partial differential equations, which are not easy to verify, especially for high dimensional delay systems. In practice, numerically checking the existence of such functionals often requires an approximation of these matrix functions in an appropriate manner using the discretization method from Gu et al. (2003) or polynomial approximation (Peet, Papachristodoulou, & Lall, 2009).

In the present article, we propose an alternative method also based on a polynomial approximation but using the particular setup of the Legendre polynomials. Thanks to these polynomials, we are able to provide a new integral inequality whose conservatism can be made arbitrarily small. This is the core tool for developing a set of new sufficient conditions indexed by N, equal to the degree of the polynomials modeling the parameters of the complete Lyapunov–Krasovskii functional. It is also proven that this set forms a hierarchy with respect to the pair (h, N) in the sense that increasing N improves the result. Finally, several examples will show the effectiveness of the method.

Notations: Throughout the paper $\mathbb{N}$ and $\mathbb{R}^n$ denote, the set of positive integer and the n-dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices. The notation $P \succ 0$, for $P \in \mathbb{R}^{n \times n}$, means that P is symmetric and positive definite. The set $\mathbb{S}_n^+$ represents the set of symmetric positive definite matrices of $\mathbb{R}^{n \times n}$. The set of continuous functions from an interval $\mathit{l} \subset \mathbb{R}$ to $\mathbb{R}^n$ which are square integrable is denoted as space $\mathcal{L}_2(\mathit{l} \rightarrow \mathbb{R}^n)$. The symmetric matrix $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ stands for $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ and diag(A, B) stands for the diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Moreover, for any square matrix $A \in \mathbb{R}^{n \times n}$, we define He$(A) = A + A^T$. The matrix I represents the identity matrix of appropriate dimension. The notation $0_{n,m}$ stands for the matrix in $\mathbb{R}^{n \times m}$ whose entries are zero and, when no confusion is possible, the subscript will be omitted. The notation $\binom{k}{l}$ refers to the binomial coefficients given by $\frac{k!}{(k-l)!l!}$.

2. Problem formulation

2.1. Definition of the system

Consider a linear time-delay system of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d \int_{-h}^{0} f(\theta)x(t + \theta)d\theta, & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, ϕ is a continuous function, representing the initial conditions, A and A_d are constant matrices

and f denotes a known continuous function of $\mathcal{L}_2([-h, 0] \rightarrow \mathbb{R})$ and represents the kernel of the distributed delay. The delay h is assumed to be constant. The goal of this paper is to provide a generic analysis which is able to assess stability of system (1) for any continuous scalar kernel f and to derive sufficient stability conditions. This paper exploits the properties of the Legendre polynomials to provide new stability conditions for systems with distributed delay. Let us first recall the definition of the Legendre polynomials and their basic properties.

2.2. Legendre polynomials and kernel approximation

Let us define the Legendre polynomials considered over the interval $[-h, 0]$

$$\forall k \in \mathbb{N}, \quad L_k(u) = (-1)^k \sum_{l=0}^{k} p_l^k \left(\frac{u+h}{h} \right)^l, \quad (2)$$

with $p_l^k = (-1)^l \binom{k}{l} \binom{k+l}{l}$. The main idea for employing these polynomials comes from their nice properties that are summarized below.

Property 1. *The Legendre polynomials described in* (2) *represent an orthogonal sequence with respect to the inner product* $\int_{-h}^{0} \phi(s) \psi(s)ds$, *for any* ϕ, ψ *in* $\mathcal{L}_2([-h, 0] \rightarrow \mathbb{R})$ *and verify for all* (k, l) *in* $\mathbb{N}^2$

$$\int_{-h}^{0} L_k(u)L_l(u)du = \begin{cases} 0, & k \neq l, \\ \dfrac{h}{2k+1}, & k = l. \end{cases} \quad (3)$$

The proof can be found in Gautschi (2004). In light of this property, we will consider the polynomial approximation of the kernel f in $\mathcal{L}_2([-h, 0] \rightarrow \mathbb{R})$. For a given integer N, f is rewritten as

$$\forall \theta \in [-h, 0], \quad f(\theta) = \sum_{k=0}^{N} \frac{2k+1}{h} f_k L_k(\theta) + r_N(f, \theta), \quad (4)$$

where

$$\begin{aligned} f_k &= \int_{-h}^{0} f(u)L_k(u)du, \quad \forall k = 1, \ldots, N, \\ r_N(f, \theta) &= f(\theta) - \sum_{k=0}^{N} \frac{2k+1}{h} f_k L_k(\theta). \end{aligned} \quad (5)$$

Since Legendre polynomials are orthogonal with respect to the inner product under consideration, this manipulation can be interpreted as the Graam–Schmidt normalization of the function f with respect to the N first Legendre polynomials. Indeed, it can be easily verified that the resulting function r_N is orthogonal to the N first Legendre polynomials and represents the remainder of the approximation.

Remark 1. If the kernel f is a polynomial function of degree N^*, then r_{N^*+1} is the null function.

Define ρ_N as the square of norm of r_N associated with the inner product, i.e.

$$\rho_N(f) = \int_{-h}^{0} r_N^2(f, \theta)d\theta. \quad (6)$$

The next lemma provides some properties of $\rho_N(f)$.

Lemma 1. *The following inequalities hold for any f in $\mathcal{L}_2([-h, 0] \to \mathbb{R})$ and for any integer $N \geq 0$*

$$\rho_N(f) = \int_{-h}^{0} f^2(\theta)\mathrm{d}\theta - \frac{1}{h}\sum_{k=0}^{N}(2k+1)f_k^2 \geq 0, \tag{7}$$

$$\rho_{N+1}(f) - \rho_N(f) = -\frac{2N+3}{h}f_{N+1}^2 \leq 0. \tag{8}$$

Proof. The positivity of ρ_N results from its definition. Eq. (7) is obtained from the properties of the Legendre polynomials. The proof of (8) is a direct extension of (7).   ◇

2.3. Modelling of the distributed delay system

Exploiting the polynomial approximation of the kernel f, the time-delay system (1) can be rewritten as

$$\dot{x}(t) = Ax(t) + A_d\sum_{k=0}^{N}(2k+1)f_k\Omega_k(x_t) + \rho_N(f)A_d\Theta_N(f, x_t), \tag{9}$$

where

$$\Omega_k(x_t) = \frac{1}{h}\int_{-h}^{0}L_k(\theta)x_t(\theta)\mathrm{d}\theta,$$

$$\Theta_N(f, x_t) = \begin{cases} 0, & \text{if } \rho_N(f) = 0, \\ \dfrac{1}{\rho_N(f)}\displaystyle\int_{-h}^{0}r_N(f, \theta)x_t(\theta)\mathrm{d}\theta, \\ & \text{if } \rho_N(f) > 0. \end{cases} \tag{10}$$

The objective is to provide a stability analysis able to take into account these additional signals.

3. Bessel–Legendre inequality

Before stating the main result of the paper, we provide some additional properties of the Legendre polynomials that will be helpful in the remainder of the paper.

Property 2. *The Legendre polynomials described in (2) satisfy the following boundary conditions*

$$\forall k \in \mathbb{N}, \quad L_k(0) = 1, \qquad L_k(-h) = (-1)^k.$$

Property 3. *The Legendre polynomials described in Definition (2) satisfy the following differentiation rule:*

$$\frac{\mathrm{d}}{\mathrm{d}u}L_k(u) = \begin{cases} 0, & \text{if } k = 0, \\ \displaystyle\sum_{i=0}^{k-1}\frac{(2i+1)}{h}(1-(-1)^{k+i})L_i(u), \\ & \text{if } k \geq 1. \end{cases}$$

Proofs of these properties can be found in Gautschi (2004) and are thus omitted. Based on the Legendre polynomials and an application of Bessel's inequality (Gautschi, 2004), we obtain the following lemma.

Lemma 2. *Let $x \in \mathcal{L}_2([-h, 0] \to \mathbb{R}^n)$, f be a continuous function in $\mathcal{L}_2([-h, 0] \to \mathbb{R})$ and $R \in \mathbb{S}_n^+$. The integral inequality*

$$\int_{-h}^{0}x^T(u)Rx(u)\mathrm{d}u \geq h\sum_{k=0}^{N}(2k+1)\Omega_k^T(x)R\Omega_k(x)$$

$$+ \sigma(\rho_N(f))\Theta_N^T(f, x)R\Theta_N(f, x) \tag{11}$$

holds, for all $N \in \mathbb{N}$, where ρ_N, Ω_k and Θ_N have been defined in (6) and (10) respectively, and where

$$\sigma(\rho_N(f)) = \begin{cases} \rho_N(f) & \text{if } \rho_N(f) > 0, \\ 1 & \text{if } \rho_N(f) = 0. \end{cases} \tag{12}$$

Proof. Consider a function x in $\mathcal{L}_2([-h, 0] \to \mathbb{R}^n)$, f in $\mathcal{L}_2([-h, 0] \to \mathbb{R})$ and a matrix R in $\mathbb{S}_n^+$. Define the function z_N as follows

$$z_N(u) = x(u) - \sum_{k=0}^{N}(2k+1)L_k(u)\Omega_k(x) - r_N(f, u)\Theta_N(f, x).$$

From its definition, z_N is in $\mathcal{L}_2([-h, 0] \to \mathbb{R}^n)$ and the integral $\int_{-h}^{0}z_N^T(u)Rz_N(u)\mathrm{d}u$ exists. For the sake of simplicity the arguments of ρ_N, Ω_k and Θ_N will be omitted in the following developments. From the orthogonal property of the Legendre polynomials, one has

$$\int_{-h}^{0}z_N^T(u)Rz_N(u)\mathrm{d}u = \int_{-h}^{0}x^T(u)Rx(u)\mathrm{d}u$$

$$- 2\sum_{k=0}^{N}(2k+1)\left(\int_{-h}^{0}L_k(u)x(u)\mathrm{d}u\right)^T R\Omega_k$$

$$+ \sum_{k=0}^{N}(2k+1)^2\left(\int_{-h}^{0}L_k^2(u)\mathrm{d}u\right)\Omega_k^T R\Omega_k$$

$$+ \left(\int_{-h}^{0}r_N^2(f, u)\mathrm{d}u\right)\Theta_N^T R\Theta_N - 2\left(\int_{-h}^{0}r_N(f, u)x(u)\mathrm{d}u\right)^T R\Theta_N$$

$$+ 2\sum_{k=0}^{N}(2k+1)\left(\int_{-h}^{0}r_N(f, u)L_k(u)\mathrm{d}u\right)\Omega_k^T R\Theta_N.$$

On the one hand, assume first that ρ_N, and consequently r_N, are not zero. From the definition of Ω_k and Θ_N and noting that $(2k+1)^2\int_{-h}^{0}L_k^2(u)du = (2k+1)h$, it yields

$$\int_{-h}^{0}z_N^T(u)Rz_N(u)\mathrm{d}u = \int_{-h}^{0}x^T(u)Rx(u)\mathrm{d}u$$

$$- \sum_{k=0}^{N}(2k+1)h\Omega_k^T R\Omega_k - \rho_N\Theta_N^T R\Theta_N$$

$$+ 2\sum_{k=0}^{N}\frac{2k+1}{h}\left(\int_{-h}^{0}r_N(f, u)L_k(u)\mathrm{d}u\right)\Omega_k^T R\Theta_N. \tag{13}$$

Moreover, for any $k = 0, \ldots, N$, one has

$$\int_{-h}^{0}r_N(f, u)L_k(u)\mathrm{d}u = \int_{-h}^{0}f(u)L_k(u)\mathrm{d}u$$

$$- \sum_{i=0}^{N}\frac{2i+1}{h}f_i\int_{-h}^{0}L_i(u)L_k(u)\mathrm{d}u.$$

From Property 1, and the definition of r_N, we show that $\int_{-h}^{0}r_N(f, u)L_k(u)\mathrm{d}u = 0$. It thus follows that

$$\int_{-h}^{0}z_N^T(u)Rz_N(u)\mathrm{d}u = \int_{-h}^{0}x^T(u)Rx(u)\mathrm{d}u$$

$$- \sum_{k=0}^{N}(2k+1)h\Omega_k^T R\Omega_k - \rho_N\Theta_N^T R\Theta_N,$$

which concludes the proof for the case $\rho_n \neq 0$. On the other hand, if $\rho_N = 0$ (and $r_N = 0$), the following equation holds

$$\int_{-h}^{0} z_N^T(u) R z_N(u) du = \int_{-h}^{0} x^T(u) R x(u) du$$

$$- \sum_{k=0}^{N} (2k+1) h \Omega_k^T R \Omega_k - \Theta_N^T R \Theta_N$$

because $\Theta_N(f, x_t) = 0$. The result is finally obtained by combining the two previous equations and noting that the left hand side of the previous equations is positive definite. $\quad \diamond$

In the remainder of the paper, the following corollary of the previous lemma will be used. Note that it also corresponds to an inequality already presented in Seuret and Gouaisbaut (2015) and Seuret et al. (2014).

Corollary 3. *Let $x \in \mathcal{L}_2([-h, 0] \to \mathbb{R}^n)$ and $R \in \mathbb{S}_n^+$. The integral inequality*

$$\int_{-h}^{0} x^T(u) R x(u) du \geq h \sum_{k=0}^{N} (2k+1) \Omega_k^T R \Omega_k \qquad (14)$$

holds, for all $N \in \mathbb{N}$, where Ω_k, $k = 0, 1, \ldots, N$ have been defined in Lemma 2.

Proof. The proof is obtained by removing the term $\rho_N(f)\Theta_N^T(x) R \Theta_N(x)$ from (11). $\quad \diamond$

Remark 2. The previous inequality encompasses the Jensen inequality (Gu et al., 2003) and the recent Wirtinger-based integral inequality (Seuret & Gouaisbaut, 2013) as particular cases for $N = 0$ and $N = 1$, respectively. Thus, the set of inequalities provided in Corollary 3 represents a more general formulation than these two inequalities. Additionally, the Parseval identity proves that inequality (14) becomes asymptotically non conservative as N goes to infinity.

4. Application to stability analysis

4.1. Selection of Lyapunov–Krasovskii functionals

Consider the functional given by

$$V(x_t) = x^T(t) P x(t) + 2 x^T(t) \int_{-h}^{0} Q(\theta) x_t(\theta) d\theta$$

$$+ \int_{-h}^{0} x_t^T(\theta) S(\theta) x_t(\theta) d\theta$$

$$+ \int_{-h}^{0} \int_{-h}^{0} x_t(\theta_1) T(\theta_1, \theta_2) x_t(\theta_2) d\theta_1 d\theta_2, \qquad (15)$$

where $x_t(\theta) = x(t + \theta)$ represents the state of the time-delay system and $h > 0$ the delay and where the matrix P is symmetric positive definite and the functions Q, S and T are differentiable (see Section 7.4 in Gu et al. (2003) for more details). Theorem 7.3 from Gu et al. (2003) ensures that if a time-delay system is asymptotically stable, then V is a Lyapunov–Krasovskii functional, provided that Q and T satisfy some specific partial differential equations. In practice, numerically checking the existence of such functionals often requires an approximation of Q, S and T. In Gu et al. (2003) a discretization method was proposed where the functions Q, T and S were chosen piecewise linear and the conditions are presented through the LMI setup. In Peet et al. (2009), these matrices were chosen as polynomial functions and the numerical test was performed using the SOSTOOLS. Here, we

aim at choosing the structure of the Lyapunov–Krasovskii functional based on the use of Legendre polynomials. Considering the functional (15), we propose to model the different matrices P, $Q(\theta)$, $T(\theta, s)$ and $S(\theta)$ as polynomials with respect to the variables θ and s. Contrary to a Sum of Squares (SOS) formulation (Peet et al., 2009), these polynomials are expressed in terms of the Legendre basis as follows:

$$Q(\theta) = \sum_{i=0}^{N} Q_i L_i(\theta), \qquad T(\theta_1, \theta_2) = \sum_{i=0}^{N} \sum_{j=0}^{N} L_i(\theta_1) L_j(\theta_2) T_{ij},$$

where the matrices Q_i, $T_{ij} = T_{ji}^T$, for $i, j = 0, \ldots, N$ have to be optimized. The polynomial matrix $S(\theta)$ is chosen as a linear function with respect to θ and is therefore simply expressed with the canonical basis $S(\theta) = S + (h + \theta)R$. This selection is consistent with the complete Lyapunov–Krasovskii functional defined in Gu et al. (2003) (see Section 5.6.3, p. 181). Hence, we can define the functional V_N (the subscript N denoting the order of the polynomial approximation) as follows:

$$V_N(x_t) = \zeta_N^T(x_t) P_N \zeta_N(x_t) + \int_{-h}^{0} x_t^T(\theta) S x_t(\theta) d\theta$$

$$+ \int_{-h}^{0} (h + \theta) x_t^T(\theta) R x_t(\theta) d\theta, \qquad (16)$$

where $P_N = \begin{bmatrix} P & Q_0 & \cdots & Q_N \\ Q_0^T & T_{00} & \cdots & T_{0N} \\ \vdots & \vdots & & \vdots \\ Q_N^T & T_{N0} & \cdots & T_{NN} \end{bmatrix}$ and

$$\zeta_N(x_t) = \begin{bmatrix} x_t(0) \\ \int_{-h}^{0} L_0(\theta) x_t(\theta) d\theta \\ \vdots \\ \int_{-h}^{0} L_N(\theta) x_t(\theta) d\theta \end{bmatrix} = \begin{bmatrix} x_t(0) \\ h\Omega_0(x_t) \\ \vdots \\ h\Omega_N(x_t) \end{bmatrix}, \quad N \geq 0$$

collects the current state and the projections of the state function x_t onto the N first Legendre polynomials.

4.2. Stability analysis

We provide here a first stability result for distributed delay systems, which is based on the proposed Lyapunov–Krasovskii functional (16) and the use of the inequality provided in Lemma 2.

Theorem 4. *Consider system (1) with a continuous scalar kernel f in $\mathcal{L}_2([-h, 0] \to \mathbb{R})$ and with a given $h > 0$. If, for a given integer $N \geq 0$, there exist a matrix $P_N \in \mathbb{S}_{(N+2)n}^+$, and two matrices $S, R \in \mathbb{S}_n^+$ such that the following LMIs*

$$\Phi_N^+(h) := P_N + \frac{1}{h} S_N \succ 0 \qquad (17)$$

$$\Phi_N^-(h, \rho_N(f)) = \begin{bmatrix} \Phi_{N0}(h) - h R_N & \rho_N(f)\Phi_{N1}(h) \\ \rho_N(f)\Phi_{N1}^T(h) & -\sigma(\rho_N(f))R \end{bmatrix} \prec 0 \qquad (18)$$

are satisfied where the function σ is defined in Lemma 2 and where

$$\Phi_{N0}(h) = \text{He}\left(G_N^T(h) P_N H_N(h)\right) + \Sigma_N(h),$$

$$\Phi_{N1}(h) = G_N^T(h) P_N J_N A_d,$$

$$\Sigma_N(h) = \text{diag}(S + hR, -S, 0_{(N+1)n}),$$

$$S_N = \text{diag}(0, S, 3S, \ldots, (2N+1)S),$$

$$R_N = \text{diag}(0, 0, R, 3R, \ldots, (2N+1)R),$$

$$G_N(h) = \begin{bmatrix} I & 0_n & 0_{n,n(N+1)} \\ 0_{n(N+1),n} & 0_{n(N+1),n} & hI_{n(N+1)} \end{bmatrix},$$

$$H_N = \begin{bmatrix} F_N^T & \Gamma_N^T(0) & \Gamma_N^T(1) & \cdots & \Gamma_N^T(N) \end{bmatrix}^T,$$

$$F_N = \begin{bmatrix} A & 0 & f_0 A_d & 3f_1 A_d & \cdots & (2N+1)f_N A_d \end{bmatrix},$$

$$J_N = \begin{bmatrix} I & 0_{n,n(N+1)} \end{bmatrix}^T,$$

$$\Gamma_N(k) = \begin{bmatrix} I & (-1)^{k+1}I & \gamma_{Nk}^0 I & \cdots & \gamma_{Nk}^N I \end{bmatrix},$$

$$\gamma_{Nk}^i = \begin{cases} -(2i+1)(1-(-1)^{k+i}), & \text{if } i \le k, \\ 0, & \text{if } i \ge k+1, \end{cases}$$

then, the distributed delay system (1) is asymptotically stable for the constant delay h.

Proof. Consider the Lyapunov–Krasovskii functional (16). Following the procedure provided in Gu et al. (2003), Corollary 3 can be applied to the second term of V_N since $S \succ 0$ to obtain a lower bound of V_N. In order to be consistent with the definition of ζ_N, Corollary 3 is considered with the order N. It thus yields

$$\int_{-h}^0 x_t^T(s) S x_t(s) ds \ge h \sum_{k=0}^N \Omega_k^T(x_t) S \Omega_k(x_t)$$
$$= \frac{1}{h} \zeta_N^T(x_t) S_N \zeta_N(x_t).$$

Hence, it holds

$$V_N(x_t) \ge \zeta_N^T(x_t) \Phi_N^+(h) \zeta_N(x_t) + \int_{-h}^0 (h+s) x_t^T(s) R x_t(s) ds.$$

Consequently, if the matrices R and $\Phi_N^+(h)$ are positive definite, there exists a positive scalar $\epsilon_1 > 0$ such that $V_N(x_t) \ge \epsilon_1 |x_t(0)|^2$. Furthermore, there always exists a sufficiently large scalar $\lambda > 0$ such that $P_N \preceq \lambda \text{diag}(I, I, 3I, \ldots, (2N+1)I)$, yielding

$$V_N(x_t) \le \lambda |x_t(0)|^2 + \lambda \sum_{i=0}^{N-1} (2i+1) \Omega_i^T(x_t) \Omega_i(x_t)$$
$$+ \int_{-h}^0 x_t^T(s)(S+hR) x_t(s) ds.$$

Applying Corollary 3 to the second term of the right-hand side ensures that

$$V_N(x_t) \le \lambda |x_t(0)|^2 + \int_{-h}^0 x_t^T(s)(\lambda hI + S + hR) x_t(s) ds$$

which guarantees that there exists a scalar $\epsilon_2 > 0$, such that $V_N(x_t) \le \epsilon_2 |x_t|_h^2$, where the notation $|x_t|_h$ stands for $\sup_{\theta \in [-h \ 0]} |x_t(\theta)|$. Then it holds

$$\epsilon_1 |x_t(0)|^2 \le V_N(x_t) \le \epsilon_2 |x_t|_h^2. \tag{19}$$

The computation of $\dot{V}_N$ leads to

$$\dot{V}_N(x_t) = 2\zeta_N^T(x_t) P_N \dot{\zeta}_N(x_t) + x_t^T(0)(S+hR) x_t(0)$$
$$- x_t^T(-h) S x_t(-h) - \int_{-h}^0 x_t^T(s) R x_t(s) ds. \tag{20}$$

We aim at finding an upper bound of the functional $\dot{V}_N$ using the vector $\Theta_N(f, x_t)$ and the augmented vector ξ_N given, for any integer N, by

$$\xi_N(x_t) = \begin{bmatrix} x_t^T(0) & x_t^T(-h) & \Omega_0^T(x_t) & \Omega_1^T(x_t) & \cdots & \Omega_N^T(x_t) \end{bmatrix}^T.$$

Consider the first term of the right hand side of (20). It is easy to see that

$$\dot{\zeta}_N(x_t) = \begin{bmatrix} \dot{x}_t(0) \\ h\dot{\Omega}_0(x_t) \\ \vdots \\ h\dot{\Omega}_N(x_t) \end{bmatrix} = \begin{bmatrix} \dot{x}_t(0) \\ h\Omega_0(\dot{x}_t) \\ \vdots \\ h\Omega_N(\dot{x}_t) \end{bmatrix}.$$

The definition of F_N and Eq. (9) yield

$$\dot{x}_t(0) = F_N \xi_N(x_t) + \rho_N(f) A_d \Theta_N(f, x_t). \tag{21}$$

Then, an integration by parts ensures that

$$\forall k \le N, \quad h\Omega_k(\dot{x}_t) = L_k(0) x_t(0) - L_k(-h) x_t(-h)$$
$$- \int_{-h}^0 \dot{L}_k(u) x_t(u) du.$$

Thanks to Properties 2 and 3 of the Legendre polynomials, the following expression is derived

$$h\Omega_k(\dot{x}_t) = x_t(0) - (-1)^k x_t(-h) - \sum_{i=0}^{k-1} \gamma_{Nk}^i \Omega_i(x_t)$$
$$= \Gamma_N(k) \xi_N(x_t).$$

Gathering all the components of $\dot{\zeta}_N(t)$, we obtain

$$\dot{\zeta}_N(x_t) = H_N \xi_N(x_t) + \rho_N(f) J_N A_d \Theta_N(f, x_t),$$

where the matrices H_N and J_N are given in the statement of the theorem. Since $\zeta_N(x_t) = G_N(h) \xi_N(x_t)$ and $x_t^T(0)(S+hR) x_t(0) - x_t^T(-h) S x_t(-h) = \xi_N^T(x_t) \Sigma_N(h) \xi_N(x_t)$, we have

$$\dot{V}_N(x_t, \dot{x}_t) = \bar{\xi}_N^T \begin{bmatrix} \Phi_{N0}(h) & \rho_N(f) \Phi_{N1}(h) \\ \rho_N(f) \Phi_{N1}^T(h) & 0 \end{bmatrix} \bar{\xi}_N(x_t)$$
$$- \int_{-h}^0 x_t^T(s) R x_t(s) ds \tag{22}$$

where $\bar{\xi}_N(x_t) = \begin{bmatrix} \xi_N^T(x_t) & \Theta_N^T(f, x_t) \end{bmatrix}^T$. Applying Lemma 2 to the order N, we obtain

$$\dot{V}_N(x_t) \le \bar{\xi}_N^T(x_t) \Phi_N^-(h, \rho_N(f)) \bar{\xi}_N(x_t).$$

Therefore, if the matrix $\Phi_N^-(h, \rho_N(f))$ is negative definite, there exists a positive scalar ϵ_3 such that

$$\dot{V}_N(x_t) \le -\epsilon_3 |x_t(0)|^2. \tag{23}$$

To conclude, if the LMI conditions (17) and (18) are satisfied, the functional meets the requirements of the Lyapunov–Krasovskii theorem, (19) and (23), and the distributed delay system (1) is asymptotically stable for the constant delay h. ◇

Remark 3. A notable aspect of this theorem is the possibility to extend it to robust stability where the matrices A and A_d are uncertain but belong to a given polytope since the $\Psi^-(h)$ is affine with respect to these matrices.

Remark 4. In the stability conditions derived in Theorem 4, the integer N appears as a tuning parameter of the LMIs. As we will show in Section 5, increasing N may lead to a reduction of the conservatism, of course at the price of increasing of the computational cost. Indeed the number of decision variables and the size of the LMIs also increase with N.

4.3. Distributed delay systems with two kernels

Theorem 7 only addresses the case of scalar kernels f. Extensions to matrix kernels corresponding to the following system

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_{-h}^{0} A_d(\theta)x(t+\theta)d\theta, & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases}$$

is not direct since the methodology developed in Theorem 7 relies on the use of the term $\int_{-h}^{0} f(\theta)x_t(\theta)d\theta$. This limitation can be easily undertaken by splitting the matrix kernel $A_d(\theta)$ into several pieces which can be treated following the first theorem. In this subsection, we present the simplest extension which consists of the following distributed delay systems with two distributed terms:

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_{-h}^{0} [f(\theta)A_{df} + g(\theta)A_{dg}]x(t+\theta)d\theta, \\ \quad \forall t \geq 0, \\ x(t) = \phi(t), \quad \forall t \in [-h, 0], \end{cases} \tag{24}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, ϕ is a continuous initial condition function, A, A_{df} and A_{dg} are constant matrices and f, g are two known continuous functions over the interval $[-h, 0]$. The delay h is assumed to be constant. Following the procedure as in the simple kernel case, system (24) is rewritten as

$$\begin{cases} \dot{x}(t) = Ax(t) + \sum_{k=0}^{N} (2k+1)A_d^k \Omega_k(x_t) \\ \quad + [\rho_N(f)A_{df} \ \rho_N(g)A_{dg}]\bar{\Theta}_N(x_t), & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases} \tag{25}$$

where $A_d^k = f_k A_{df} + g_k A_{dg}$ and

$$g_k = \int_{-h}^{0} g(\theta)L_k(\theta)d\theta, \qquad \bar{\Theta}_N(x_t) = \begin{bmatrix} \Theta_N(f, \theta) \\ \Theta_N(g, \theta) \end{bmatrix}.$$

The following lemma and theorem are adapted from Section 4.2.

Lemma 5. *Let $x \in \mathcal{L}_2([-h, 0] \to \mathbb{R}^n)$, f and g be two continuous functions in $[-h, 0] \to \mathbb{R}^n$ and $R \in \mathbb{S}_n^+$. The integral inequality*

$$\int_{-h}^{0} x(u)Rx(u)du \geq h \sum_{k=0}^{N} (2k+1)\Omega_k^T(x)R\Omega_k(x)$$

$$+ \bar{\Theta}_N^T(x) \begin{bmatrix} \sigma(\rho_N(f))R & \beta_N(f, g)R \\ \beta_N(f, g)R & \sigma(\rho_N(g))R \end{bmatrix} \bar{\Theta}_N(x), \tag{26}$$

holds, for all $N \in \mathbb{N}$, where

$$\beta_N(f, g) = \rho_N(f)\rho_N(g)\int_{-h}^{0} r_N(f, \theta)r_N(g, \theta)d\theta$$

and where Ω_k, $\bar{\Theta}_N$, ρ_N and the function σ have been defined earlier.

Theorem 6. *For a given integer $N \geq 0$ and a constant delay h, assume that there exist a matrix $P_N \in \mathbb{S}_{(N+2)n}^+$, and two matrices $S, R \in \mathbb{S}_n^+$ such that $\Phi_N^+(h) \succ 0$ and*

$$\Phi_N^+(h) := P_N + \frac{1}{h}S_N \succ 0, \tag{27}$$

where

$$\Phi_N^-(h, \rho_N)$$

$$= \begin{bmatrix} \bar{\Phi}_{N0}(h) - hR_N & \rho_N(f)\Phi_{N1}(h) & \rho_N(g)\Phi_{N2}(h) \\ \rho_N\Phi_{N1}^T(h) & -\sigma(\rho_N(f))R & -\beta_N(f, g)R \\ \gamma_N\Phi_{N2}^T(h) & -\beta_N(f, g)R & -\sigma(\rho_N(g))R \end{bmatrix} \prec 0, \tag{28}$$

$$\bar{\Phi}_{N0}(h) = \text{He}\left(G_N^T(h)P_N\bar{H}_N\right) + \Sigma_N(h),$$

$$\Phi_{N1}(h) = G_N^T(h)P_N J_N A_{df},$$

$$\Phi_{N2}(h) = G_N^T(h)P_N J_N A_{dg},$$

$$\bar{H}_N = \begin{bmatrix} \bar{F}_N^T & \Gamma_N^T(0) & \Gamma_N^T(1) & \cdots & \Gamma_N^T(N) \end{bmatrix}^T,$$

$$\bar{F}_N = \begin{bmatrix} A & 0 & A_d^1 & 3A_d^2 & \cdots & (2N+1)A_d^N \end{bmatrix},$$

with $\Sigma_N(h)$, S_N, R_N, J_N, $\Gamma_N(k)$ and $G_N(h)$ defined in Theorem 4. Then system (24) is asymptotically stable for the constant delay h.

4.4. Comments on systems with a single discrete delay

By a slight modification, the stability conditions from Theorem 4 can be extended to the case of linear systems with a point-wise delay, defined by

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h), & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases} \tag{29}$$

where A, A_d, h and ϕ satisfy the same assumptions of Theorem 4. Following the same procedure as previously, the next theorem is stated.

Theorem 7. *For a given integer $N \geq 0$ and a constant delay h, assume that there exist a matrix $P_N \in \mathbb{S}_{(N+2)n}^+$, and two matrices $S, R \in \mathbb{S}_n^+$ such that $\Phi_N^+(h) \succ 0$ and*

$$\Psi_N^-(h) = \Psi_{N0}(h) - hR_N \prec 0, \tag{30}$$

are satisfied where

$$\Psi_{N0}(h) = \text{He}\left(G_N^T(h)P\tilde{H}_N\right) + \Sigma_N(h),$$

$$\tilde{H}_N = \begin{bmatrix} \tilde{F}_N^T & \Gamma_N^T(0) & \Gamma_N^T(1) & \cdots & \Gamma_N^T(N) \end{bmatrix}^T,$$

$$\tilde{F}_N = \begin{bmatrix} A & A_d & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

with $\Phi_N^+(h)$, R_N, $G_N(h)$, $\Sigma_N(h)$ and $\Gamma_N(i)$, $i = 0, \ldots, N$ defined in Theorem 4. Then system (29) is asymptotically stable for the constant delay h.

Proof. The proof strictly follows the one of Theorem 4 except for equation (21), which is replaced by $\dot{x}(t) = \tilde{F}_N \xi_N(t)$. ◇

Remark 5. The LMI of Theorem 7 are equivalent to the stability condition provided in Seuret et al. (2014), which only copes with pointwise delays. In Seuret and Gouaisbaut (2015), another class of Lyapunov–Krasovskii functionals was employed which contains a double integral term in $\dot{x}(t)$.

5. Hierarchy of LMI stability conditions

This section aims at proving that the previous stability conditions form a hierarchy of LMI conditions. This is formulated in the following theorem based on the stability conditions of Theorem 4.

Theorem 8. *For any time delay system (1), define the set $\mathcal{H}_N$ by*

$$\mathcal{H}_N := \left\{ \begin{array}{c} h > 0 : \exists (P_N, S(N), R(N)) \in \mathbb{S}_{(N+2)n} \times (\mathbb{S}_n^+)^2, \\ s.t. \quad \Phi_N^+(h) \succ 0, \Phi_N^-(h) \prec 0 \end{array} \right\}.$$

Then, the inclusion $\mathcal{H}_N \subset \mathcal{H}_{N+1}$ holds for any $N \geq 0$.

Table 1
Maximal allowable delay $h_{\max}$ for system (33).

Theorems	$h_{\max}$	Number of variables
Breda et al., 2009 ("analytical" bound)	1.498	–
Solomon & Fridman, 2013	1.03	3
Theorem 4 ($N = 0$)	0.95	5
Theorem 4 ($N = 1$)	1.45	8
Theorem 4 ($N = 2$)	1.497	12
Theorem 4 ($N = 3$)	1.498	17

Table 2
Intervals of stability w.r.t. h for system (34). The notation "nv" denotes the number of decision variables.

Theorems	First interval	Second interval	nv
Breda et al., 2009	[0, 0.964]	[1.372, 2.105]	–
Solomon & Fridman, 2013	[0, 0.964]	–	3
Theorem 4 ($N = 1$)	[0, 0.91]	–	8
Theorem 4 ($N = 2$)	[0, 0.963]	[1.382, 2.100]	12
Theorem 4 ($N = 3$)	[0, 0.964]	[1.372, 2.103]	17
Theorem 4 ($N = 4$)	[0, 0.964]	[1.372, 2.105]	23

Proof. Let $N \in \mathbb{N}$. If $\mathcal{H}_N$ is empty, the inclusion is trivial. Assume that $\mathcal{H}_N$ is not empty and consider an element $h \in \mathcal{H}_N$. From the definition of $\mathcal{H}_N$, there exist $P_N = P_N^T, S(N) \succ 0$ and $R(N) \succ 0$ such that $\Phi_N^+(h) \succ 0$ and $\Phi_N^-(h, \rho_N(f)) \prec 0$. Taking advantages of the Lyapunov–Krasovskii functional (16), we suggest the matrices

$$P_{N+1} = \begin{bmatrix} P_N & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{array}{l} S(N+1) = S(N) = S, \\ R(N+1) = R(N) = R. \end{array}$$

Clearly this selection ensures that the functional implies $\Phi_{N+1}^+(h) \succ 0$, because for this selection of P_{N+1}, we have

$$\Phi_{N+1}^+(h) = \begin{bmatrix} P_N + \dfrac{1}{h}S_N & 0 \\ 0 & \dfrac{1}{h}(2N+3)S \end{bmatrix}$$
$$= \begin{bmatrix} \Phi_N^+(h) & 0 \\ 0 & \dfrac{1}{h}(2N+3)S \end{bmatrix}.$$

Assume now that $\Phi_N^-(h) \prec 0$ and $\rho_N(f) \neq 0$. Then, by application of the Schur complement it holds

$$\Phi_{N0}(h) - hR_N + \rho_N(f)\Phi_{N1}(h)R^{-1}\Phi_{N1}^T(h) \prec 0.$$

From (8), we get that $\rho_N(f) = \rho_{N+1}(f) + \dfrac{2N+3}{h}f_{n+1}^2$. Reinjecting this relation into the previous equation and applying twice the Schur complement leads to

$$\Phi_N^-(h, \rho_N(f)) \prec 0 \quad \Leftrightarrow$$
$$\begin{bmatrix} \Phi_{N0}(h) - hR_N & (2N+3)f_{n+1}\Phi_{N1}(h) & \rho_{N+1}\Phi_{N1}(h) \\ (2N+3)f_{n+1}\Phi_{N1}^T(h) & -(2N+3)hR & 0 \\ \rho_{N+1}\Phi_{N1}^T(h) & 0 & -\rho_{N+1}R \end{bmatrix}$$
$$\prec 0. \tag{31}$$

According to the construction of the matrices G_N, H_N, F_N and $\tilde{S}_N$, some matrix computations show that

$$\begin{bmatrix} \Phi_{N0}(h) - hR_N & (2N+3)f_{n+1}\Phi_{N1}(h) \\ (2N+3)f_{n+1}\Phi_{N1}^T(h) & -(2N+3)hR \end{bmatrix}$$
$$= \Phi_{(N+1)0}(h) - hR_{N+1}, \qquad \begin{bmatrix} \Phi_{N1}(h) \\ 0 \end{bmatrix} = \Phi_{(N+1)1}(h),$$

which ensure the following equivalence, under the constraint $P_{N+1} = \begin{bmatrix} P_N & 0 \\ 0 & 0 \end{bmatrix}$

$$\Phi_N^-(h, \rho_N(f)) \prec 0 \Leftrightarrow \Phi_{N+1}^-(h, \rho_{N+1}(f)) \prec 0. \tag{32}$$

The case $\rho_N = 0$ follows straightforwardly because in this situation, it must hold $f_{N+1} = 0$ and $\rho_{N+1} = 0$. Consequently, if there exists a solution to the LMI conditions of Theorem 4 at the order N, then there also exists a solution to the LMIs at the order $N + 1$ which proves that h belongs to $\mathcal{H}_{N+1}$, which allows to conclude that $\mathcal{H}_N \subset \mathcal{H}_{N+1}$. ◇

Since Theorem 4 only provides sufficient stability condition, the sequence of sets $\{\mathcal{H}_N\}_{N\in\mathbb{N}}$ is an increasing sequence of sets

representing an inner approximation of the stability pockets. However, the previous theorem does not prove that the conditions of Theorem 4 will converge to the analytical bounds of the delay. Extensions of this previous theorem to Theorems 6 and 7 are similarly derived.

6. Examples

Example 1. Let us consider a first scalar example

$$\dot{x}(t) = -x(t) + \int_{-h}^{0} e^{-\theta} \sin(\theta)x(t + \theta)\mathrm{d}\theta. \tag{33}$$

Using a numerical method (Breda, Maset, & Vermiglio, 2005; Breda et al., 2009), system (33) is shown to be asymptotically stable for all delays less than 1.498. Table 1 shows results obtained with Theorem 4. As expected, better results are obtained as the degree of the polynomial N increases. Moreover, the theoretical upper bound of the delay is recovered with $N = 3$.

Example 2. Consider the distributed delay system

$$\dot{x}(t) = -2x(t) + \int_{-h}^{0} (\theta + 3\cos\theta)x(t + \theta)\mathrm{d}\theta. \tag{34}$$

This example is interesting because it is stable for all h in [0, 0.964] and in [1.372, 2.105] and unstable otherwise (Breda et al., 2009). Table 2 shows results obtained with Theorem 4. Note that even if Solomon and Fridman (2013) provide a very good estimation of the first interval of stability, it is not able to detect the second one.

Example 3. Consider the distributed delay system

$$\dot{x}(t) = -ax(t) - \int_{-h}^{0} \gamma(k, \alpha, -\theta)x(t + \theta)\mathrm{d}\theta, \tag{35}$$

where a is a positive scalar and γ the scalar kernel function of the truncated Gamma Distribution defined by

$$\gamma(k, \alpha, \theta) = \frac{\theta^{k-1}e^{-\theta/\alpha}}{(k-1)!\alpha^k}, \quad \forall (k, \alpha, \theta) \in \mathbb{N}_{>0} \times \mathbb{R}_{>0} \times [-h, 0],$$

and such that $\int_{-\infty}^{0} \gamma(k, \alpha, \theta)\mathrm{d}\theta = 1$. From the theoretical point of view, Gamma distributions are often considered over the interval $(-\infty, 0]$. Since the kernel γ contains an exponential term, it is reasonable to consider the truncated interval $[-h, 0]$ because the main contribution to the distributed term relies on this first interval. Fig. 1 represents the stability regions obtained by solving Theorem 4 for several values of N when $\alpha = 1$ and $k = 1$. The dashed black line represents the theoretical limits resulting from the eigenvalue analysis issued from Breda et al. (2009). Fig. 1 shows that from small values of a in (0, 0.6], Theorem 4 with $N = 0, 1$ delivers good estimations of the stability regions. However for larger values of a, Fig. 1 shows that Theorem 4 with $N = 0, 1$ is conservatism since the stability regions do not match with the theoretical limits drawn by the dashed blue line. However, increasing N in Theorem 4 allows reducing this conservatism and
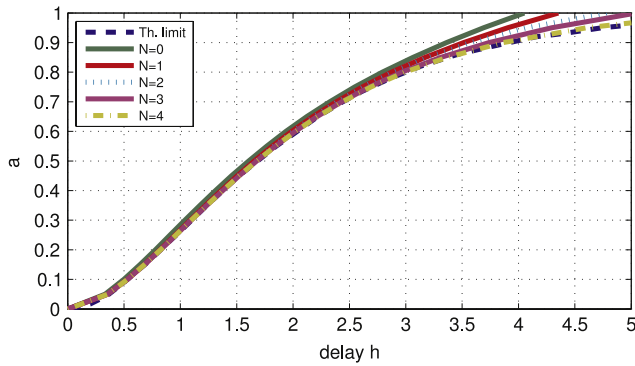
Fig. 1. Stability regions in the plan (a, h) for system (35), obtained using Theorem 4 for several values of N. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

one can see that for $N = 4$, the estimation of the stability region is very close to the theoretical region. This example illustrates the potential of our hierarchical approach to reduce the conservatism by increasing the LMI parameter N, of course at the price of increasing the complexity of the conditions, showing the tradeoff between conservatism and complexity.

Example 4. Consider the linear time-delay systems (1) taken from Chen and Zheng (2007)

$$\dot{x}(t) = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \int_{-h}^{0} x_t(s)\mathrm{d}s. \qquad (36)$$

An eigenvalue analysis provides that the system remains stable for all constant delays in the interval [0.200, 2.04]. In Chen and Zheng (2007), Seuret and Gouaisbaut (2013) and Zheng and Frank (2002), stability is guaranteed for delays over the interval [0.2090, 1.1942], [0.2001, 1.6339] and [0.2001, 1.877], respectively. Using our new inequality, Theorem 4 ensures stability for any constant delay h in the intervals [0.2001, 1.58], [0.2001, 1.83], [0.2001, 1.95], [0.2001, 2.02], for $N = 1, 2, 3, 4$, respectively, which encompasses these previous results. This demonstrates again the potential of Theorem 4.

Example 5. Consider the system with pointwise delay taken from Freitas (2000) given by

$$\ddot{x}(t) + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dot{x}(t - h) + \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \dot{x}(t) = 0.$$

The analysis provided in Freitas (2000) based on a frequency method ensures that this systems has exactly three stable delay intervals [0.4108, 0.7509], [2.054, 2.252] and [3.697, 3.754]. Fig. 2 shows the inner approximations obtained with Theorem 7 for several values of N. The first interval is detected at $N = 4$ and a good[1] estimation of the first stable interval is first obtained at $N = 7$. First values of the delay h in the second stable interval are detected with $N = 9$ and a good estimation of the whole interval is obtained at $N = 11$. Delay values in the third and last stable interval, which is more difficult to detect are found with $N = 14$ and a good estimation of the interval is provided with $N = 16$.

This example shows the efficiency of our method to detect stability intervals even for systems with discrete delay and with dimension that are not small unlike usual examples. In addition, The example and Fig. 2 also illustrate Theorem 8. Indeed, Fig. 2 shows that the stability regions become larger when N increases.
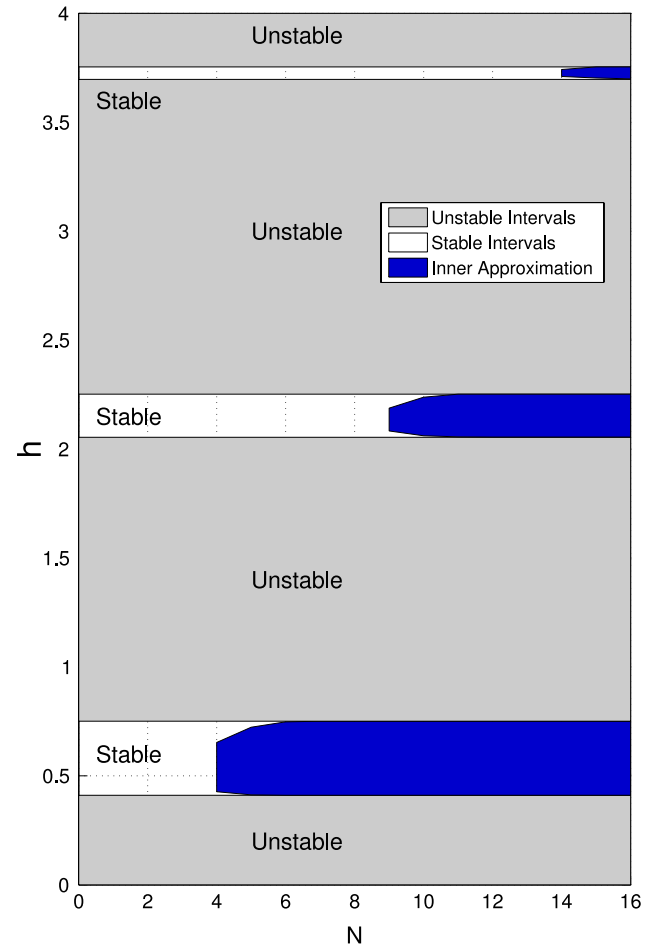

Fig. 2. Stability region in the plan (N, h), obtained using Theorem 7.

Theorem 7 is able to assess stability of the system for all the values of the delay which belongs to the stable intervals.

7. Conclusions

In this paper, we have presented a new Lyapunov–Krasovskii approach to assess stability of distributed delay systems. Our method is based on a polynomial approximation of the kernel of the distributed delay. The stability analysis is performed thanks to a new set of integral inequalities, which includes Jensen and Wirtinger-based inequalities as particular cases of order 0 and 1. Several stability theorems are provided to cope with distributed delay systems with one or two kernels and also with systems with constant pointwise delay. The efficiency of the method has been successfully tested on several examples.

This work opens a new direction of research regarding the various possibilities to exploit the Bessel inequality with other sequences of orthogonal functions such as trigonometric functions, Chebyshev or Laguerre polynomials, which would potentially allow to design a wider class of Lyapunov–Krasovskii functionals.

References

Breda, D., Maset, S., & Vermiglio, R. (2005). Pseudospectral differencing methods for characteristic roots of delay differential equations. *SIAM Journal on Scientific Computing*, 27(2), 482–495.

Breda, D., Maset, S., & Vermiglio, R. (2009). Trace-DDE: a tool for robust analysis and characteristic equations for delay differential equations. In *Lecture notes in control and information sciences*: Vol. 388. *Topics in time delay systems* (pp. 145–155). Berlin, Heidelberg: Springer.

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[1] With a precision of 10^{-4}.

Chen, W.-H., & Zheng, W. X. (2007). Delay-dependent robust stabilization for uncertain neutral systems with distributed delays. *Automatica, 43*(1), 95–104.

Freitas, P. (2000). Delay-induced instabilities in gyroscopic systems. *SIAM Journal on Control and Optimization, 39*(1), 196–207.

Fridman, E., & Shaked, U. (2002). A descriptor system approach to $\mathcal{H}_\infty$ control of linear time-delay systems. *IEEE Transactions on Automatic Control, 47*, 253–270.

Gautschi, W. (2004). *Orthogonal polynomials, computation and approximation.* Oxford Science.

Gouaisbaut, F., & Ariba, Y. (2011). Delay range stability of a class of distributed time delay systems. *Systems & Control Letters, 60*(3), 211–217.

Gu, K., Han, Q. L., Luo, A. C. J., & Niculescu, S. I. (2001). Discretized lyapunov functional for systems with distributed delay and piecewise constant coefficients. *International Journal of Control, 74*(7), 737–744.

Gu, K., Kharitonov, V. L., & Chen, J. (2003). *Control engineering, Stability of time-delay systems.* Boston: Birkhäuser.

Han, Q.-L. (2004). A descriptor system approach to robust stability of uncertain neutral systems with discrete and distributed delays. *Automatica, 40*(10), 1791–1796.

Kolmanovskii, V. B., & Shaikhet, L. E. (1996). *Control of systems with aftereffect.* American Mathematical Society.

Li, X.-G., & Zhu, X.-J. (2008). Stability analysis of neutral systems with distributed delays. *Automatica, 44*(8), 2197–2201.

Morarescu, C.-I., Niculescu, S.-I., & Gu, K. (2007). Stability crossing curves of shifted gamma-distributed delay systems. *SIAM Journal on Applied Dynamical Systems, 6*(2), 475–793.

Münz, U., Rieber, J., & Allgöwer, F. (2009). Robust stabilization and $\mathcal{H}_\infty$ control of uncertain distributed delay systems. In *LNCIS: Vol. 388. Topics in time-delay systems: analysis, algorithms, and control* (pp. 221–231). Berlin: Springer, (Chapter).

Niculescu, S. I. (2001). *Lecture notes in control and information sciences: Vol. 269. Delay effects on stability. a robust control approach.* Heildelberg: Springer-Verlag.

Ozbay, H., Bonnet, C., & Clairambault, J. (2008). Stability analysis of systems with distributed delays and application to hematopoietic cell maturation dynamics. In *Decision and control, 2008. CDC 2008. 47th IEEE conference on.* December (pp. 2050–2055).

Peet, M. M., Papachristodoulou, A., & Lall, S. (2009). Positive forms and stability of linear time-delay systems. *SIAM Journal on Control and Optimization, 47*(6), 3227–3258.

Santos, O., Mondie, S., & Kharitonov, V. L. (2006). Robust stability conditions for systems with distributed delays. In *Decision and control, 2006 45th IEEE conference on.* December (pp. 217–222).

Seuret, A., & Gouaisbaut, F. (2013). Wirtinger-based integral inequality: Application to time-delay systems. *Automatica, 49*(9), 2860–2866.

Seuret, A., & Gouaisbaut, F. (2015). Hierarchy of LMI conditions for the stability of time delay systems. *Systems & Control Letters, 81*, 1–7.

Seuret, A., & Gouaisbaut, F. et al. (2014). Complete quadratic Lyapunov functionals using Bessel-Legendre inequality. In *Proceeding of the 13th European control conference.*

Sipahi, R., Atay, F., & Niculescu, S. (2008). Stability of traffic flow behavior with distributed delays modeling the memory effects of the drivers. *SIAM Journal of Applied Mathematics, 68*(3), 738–759.

Sipahi, R., Niculescu, S., Abdallah, C. T., Michiels, W., & Gu, Keqin (2011). Stability and stabilization of systems with time delay. *IEEE Control Systems, 31*(1), 38–65.

Solomon, O., & Fridman, E. (2013). New stability conditions for systems with distributed delays. *Automatica, 49*(11), 3467–3475.

Xie, L., Fridman, E., & Shaked, U. (2001). A robust $\mathcal{H}_\infty$ control of distributed delay systems with application to combustion control. *IEEE Transactions on Automatic Control, 46*(12), 1930–1935.

Zheng, F., & Frank, P. M. (2002). Robust control of uncertain distributed delay systems with application to the stabilization of combustion in rocket motor chambers. *Automatica, 3*, 487–497.

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