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On robust stabilization for neutral delay-differential systems with parametric uncertainties and its application

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Abstract

This paper considers the problem of robust stabilization for uncertain neutral delay-differential systems. Based on the Lyapunov method, a delay-dependent criterion for determining the stability of systems is obtained in terms of matrix inequalities, which can be easily solved by efficient convex optimization algorithms. Then, the method presented is extended to systems with controller gain variations, that is, the design problem for the nonfragile controller is considered. A numerical example is included to illustrate the design procedure of the proposed method.

Keywords: Robust stabilization; Neutral system; Lyapunov method; Nonfragile controller

1. Introduction

Many physical systems have time-delay in their mathematical model [1,2]. Since the existence of delay frequently is a source of instability and performance degradation in many control systems, the stability and stabilization in such systems are of theoretical and practical importance. Therefore, these problems have received considerable attention, and many results on the problems have been reported during the three decades. Recently, more interests is focused on

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Nomenclature

$\mathfrak{R}^n$	n-dimensional real space
$\mathfrak{R}^{m\times n}$	set of all real m by n matrices
A^{T}	transpose of matrix A
$P > 0$	(respectively $P < 0$) matrix P is symmetric positive (respectively negative) definite
I	identity matrix with appropriate dimension
$\star$	the elements below the main diagonal of a symmetric block matrix
$\mathfrak{C}_{n,h}$	the Banach space of continuous functions mapping the interval $[-h, 0]$ into $\mathfrak{R}^n$, with the topology of uniform convergence
$\mathrm{diag}\{\cdots\}$	block diagonal matrix
$\lambda_{\mathrm{M}}(A)$	maximum eigenvalue of matrix A

neutral systems, which are the general form of time-delay systems and contain delay on the derivatives of some system variables [3–10]. Using the Lyapunov method, characteristic equation approach, and state solution approach, many stability criteria have been developed in the literature; see, e.g., [3–8] and the references therein. However, only a few works consider the design problem of controllers for stabilization of neutral systems [9,10]. For example, Ma et al. [9] developed a control method for a class of neutral system with constraint on system matrices utilizing a delay-differential inequality and a transformation technique. Park [10] presents a control method for guaranteeing an adequate level of performance of neutral systems using linear matrix inequalities.

The present paper considers the further investigation of robustness of the feedback stabilization of uncertain neutral delay-differential systems. The problem addressed is twofold: the first is to derive the stabilization criterion of the system, the second is to extend the result to the system with controller gain variations. For stability analysis, the Lyapunov method with a new model transformation technique is utilized. The proposed methods employs free weighting matrices to obtain less conservative stability criterion. The criterion is derived in terms of matrix inequalities, which can be easily solved by using various convex optimization algorithms [13]. A numerical example is given to show the superiority of the present result to those in the literature.

2. Problem formulation

Consider the neutral control systems described by the following state equation:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - A_2 x(t-h)] &= (A + \Delta A(t))x(t) + (A_1 + \Delta A_1(t))x(t-h) \\
&\quad + (B + \Delta B(t))u(t), \\
x(s) &= \phi(s), \quad s \in [-h, 0],
\end{aligned} \tag{1}
$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control input, $A, A_1, A_2 \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ are known real parameter matrices, $h > 0$ is a constant delay, $\Delta A, \Delta A_1$ and ΔB are the parametric uncertainties in the system, $\phi(s) \in \mathfrak{C}_{n,h}$ is a given continuous vector valued initial function.

In this paper, it is assumed that the uncertainties are of the form

$$
\begin{aligned}
&\Delta A(t) = D_1 F_1(t) E_1, \quad \Delta A_1(t) = D_2 F_2(t) E_2, \\
&\Delta B(t) = D_3 F_3(t) E_3,
\end{aligned} \tag{2}
$$

where D_i, E_i $(i = 1, 2, 3)$ are known real constant matrices of appropriate dimensions, and $F_i(t) \in \Re^{k_i \times l_i}$ are unknown matrices, which satisfy

$$F_i^{\mathrm{T}}(t) F_i(t) \leqslant I \quad (i = 1, 2, 3).$$

Here, we introduce a definition of robust stabilizability for systems (1).

Definition 1. The uncertain system (1) is said to be robustly stabilizable if there exists a feedback controller $u(t)$ such that the resulting closed-loop systems is asymptotically stable for all admissible uncertainties.

Then, the goal of this paper is to develop a procedure to design a memoryless state-feedback controller of the form

$$u(t) = Kx(t), \tag{3}$$

where $K \in \Re^{m\times n}$ is a gain matrix to be determined.

3. Robust stabilization

Now, define an operator $\mathfrak{D}(x_t) : \mathfrak{C}_{n,h} \to \Re^n$ as

$$\mathfrak{D}(x_t) = x(t) + \int_{t-h}^{t} Gx(s)\,\mathrm{d}s - A_2 x(t-h), \tag{4}$$

where $x_t = x(t+s), s \in [-h, 0]$ and $G \in \Re^{n\times n}$ is a constant matrix which will be chosen to make the system asymptotically stable.

With the above operator, the transformed system is

$$
\begin{aligned}
\dot{\mathfrak{D}}(x_t) &= \dot{x}(t) + Gx(t) - Gx(t-h) - A_2 \dot{x}(t-h) \\
&= (A + \Delta A + G)x(t) + (A_1 + \Delta A_1 - G)x(t-h) + (B + \Delta B)Kx(t).
\end{aligned} \tag{5}
$$

We will need the following well-known facts and lemmas to derive the main results.

Fact 1. For given matrices D, E, F with $F^{\mathrm{T}}F \leqslant I$ and scalar $\epsilon > 0$, the following inequality

$$DFE + E^{\mathrm{T}}F^{\mathrm{T}}D^{\mathrm{T}} \leqslant \varepsilon DD^{\mathrm{T}} + \varepsilon^{-1}E^{\mathrm{T}}E$$

is always satisfied.

Fact 2 (Schur complement). The linear matrix inequality

$$\begin{bmatrix} Z(x) & Y(x) \\ Y^{\mathrm{T}}(x) & W(x) \end{bmatrix} > 0 \tag{6}$$

is equivalent to $W(x) > 0$ and $Z(x) - Y(x)W^{-1}(x)Y^{\mathrm{T}}(x)$, where $Z(x) = Z^{\mathrm{T}}(x)$, $W(x) = W^{\mathrm{T}}(x)$ and $Y(x)$ depend affinely on x.

Lemma 1 [14]. *For any positive-definite matrix $M \in \Re^{n\times n}$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \Re^n$ such that the integrations concerned are well defined, then*

$$\left(\int_0^{\gamma} \omega(s)\,\mathrm{d}s\right)^{\mathrm{T}} M\left(\int_0^{\gamma} \omega(s)\,\mathrm{d}s\right) \leqslant \gamma \int_0^{\gamma} \omega^{\mathrm{T}}(s)M\omega(s)\,\mathrm{d}s. \tag{7}$$

Lemma 2 [11]. *For given positive scalar h, the operator $\mathfrak{D}(x_t)$ is stable if there exist a positive definite matrix Γ and positive scalars β_1 and β_2 such that*

$$\beta_1 + \beta_2 < 1, \quad \begin{bmatrix} A_2^{\mathrm{T}}\Gamma A_2 - \beta_1\Gamma & hA_2^{\mathrm{T}}\Gamma G \\ \star & h^2G^{\mathrm{T}}\Gamma G - \beta_2\Gamma_1 \end{bmatrix} < 0. \tag{8}$$

Then, we have the following theorem.

Theorem 1. *For given $\alpha > 1$ and h, the system (1) with controller $u(t) = Kx(t) \equiv MX^{-1}x(t)$ is robustly stabilizable if there exist matrices $X > 0, W > 0$, positive scalars $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_8$, matrices $F_{ii} \geqslant 0 (i = 1, \ldots, 4)$, positive scalars β_1, β_2, and any matrices $Y, M, F_{ij} (i = 1, \ldots, 4, i < j \leqslant 4)$ which satisfy the following matrix inequalities*:

$$\begin{bmatrix} \Xi_1 & \Xi_2 & A_1X-Y+hF_{13} & \Xi_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Xi_4 \\ \star & -h^{-1}(\alpha-1)X & A_1X-Y+F_{23} & F_{24} & D_1 & D_2 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -W+hF_{33} & -XA_1^{\mathrm{T}}+Y^{\mathrm{T}}+hF_{34} & 0 & 0 & 0 & \varepsilon_2XE_2^{\mathrm{T}} & \varepsilon_5XE_2^{\mathrm{T}} & \varepsilon_8XE_2^{\mathrm{T}} & 0 & 0 & 0 \\ \star & \star & \star & -h(\alpha-1)X & 0 & 0 & 0 & 0 & 0 & 0 & D_1 & D_2 & 0 \\ \star & \star & \star & \star & -\varepsilon_4I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\varepsilon_5I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\varepsilon_6I & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\varepsilon_2I & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_5I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_8I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_7I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\varepsilon_8I & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \Xi_5 \end{bmatrix} < 0, \tag{9}$$

$$-X + F_{22} \leqslant 0, \tag{10}$$

$$-X + F_{44} \leqslant 0, \tag{11}$$

$$\beta_1 + \beta_2 < 1, \tag{12}$$

$$\begin{bmatrix} -\beta_1 X & 0 & XA_2^{\mathrm{T}} \\ \star & -\beta_2 X & hY^{\mathrm{T}} \\ \star & \star & -X \end{bmatrix} < 0, \tag{13}$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ \star & F_{22} & F_{23} & F_{24} \\ \star & \star & F_{33} & F_{34} \\ \star & \star & \star & F_{44} \end{bmatrix} \geqslant 0, \tag{14}$$

where

$$\begin{aligned}
\Xi_1 &= AX + XA^{\mathrm{T}} + Y + Y^{\mathrm{T}} + BM + M^{\mathrm{T}}B^{\mathrm{T}} + W + hF_{11}, \\
\Xi_2 &= XA^{\mathrm{T}} + Y^{\mathrm{T}} + M^{\mathrm{T}}B^{\mathrm{T}} + F_{12}, \\
\Xi_3 &= -XA^{\mathrm{T}} - Y^{\mathrm{T}} - M^{\mathrm{T}}B^{\mathrm{T}} + hF_{14}, \\
\Xi_4 &= [\,\alpha hY^{\mathrm{T}} \quad \alpha hXA_2^{\mathrm{T}} \quad D_1 \quad D_2 \quad D_3 \quad \varepsilon_1 XE_1^{\mathrm{T}} \\
&\qquad \varepsilon_4 XE_1^{\mathrm{T}} \quad \varepsilon_7 XE_1^{\mathrm{T}} \quad \varepsilon_3 M^{\mathrm{T}}E_3^{\mathrm{T}} \quad \varepsilon_6 M^{\mathrm{T}}E_3^{\mathrm{T}}\,], \\
\Xi_5 &= \mathrm{diag}\{-\alpha hX, -\alpha hX, -\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_1 I, -\varepsilon_4 I, -\varepsilon_7 I, -\varepsilon_3 I, -\varepsilon_6 I\}.
\end{aligned} \tag{15}$$

Proof. Consider a legitimate Lyapunov function candidate [1] as

$$V = V_1 + V_2 + V_3 + V_4 + V_5, \tag{16}$$

where

$$V_1 = \mathfrak{D}^{\mathrm{T}}(x_t)P\mathfrak{D}(x_t), \tag{17}$$

$$V_2 = \alpha \int_{t-h}^{t} \int_{s}^{t} x^{\mathrm{T}}(u)G^{\mathrm{T}}PGx(u)\,\mathrm{d}u\,\mathrm{d}s, \tag{18}$$

$$V_3 = \int_{t-h}^{t} x^{\mathrm{T}}(s)Qx(s)\,\mathrm{d}s, \tag{19}$$

$$V_4 = \alpha h \int_{t-h}^{t} x^{\mathrm{T}}(s)A_2^{\mathrm{T}}PA_2x(s)\,\mathrm{d}s, \tag{20}$$

$$V_5 = \int_0^t \int_{s-h}^{s} Z^{\mathrm{T}}\overline{P} \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ \star & F_{22} & F_{23} & F_{24} \\ \star & \star & F_{33} & F_{34} \\ \star & \star & \star & F_{44} \end{bmatrix} \overline{P}Z\,\mathrm{d}u\,\mathrm{d}s. \tag{21}$$

where $\alpha > 1$, $Q > 0$, $P > 0$, $\overline{P} = \mathrm{diag}\{P, P, P, P\}$ and

$$Z = \begin{bmatrix} x(s) \\ Gx(u) \\ x(s-h) \\ A_2x(s-h) \end{bmatrix}.$$

Taking the time-derivative of V leads to

$$
\begin{aligned}
\dot{V}_1 = {} & 2\mathfrak{D}^{\mathrm{T}}(x_t)P\dot{\mathfrak{D}}(x_t) \\
= {} & 2\left\{x(t)+\int_{t-h}^{t} Gx(s)\,\mathrm{d}s - A_2x(t-h)\right\}^{\mathrm{T}} P\{(A+\Delta A(t)+G)x(t) \\
& + (A_1+\Delta A_1(t)-G)x(t-h)+(B+\Delta B(t))Kx(t)\} \\
= {} & x^{\mathrm{T}}(t)[P(A+G+BK)+(A+G+BK)^{\mathrm{T}}P]x(t) \\
& + 2x^{\mathrm{T}}(t)P(A_1-G)x(t-h) + 2\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} P(A+G+BK)x(t) \\
& + 2\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} P(A_1-G)x(t-h) - 2x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}P(A+G+BK)x(t) \\
& - 2x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}P(A_1-G)x(t-h) + 2x^{\mathrm{T}}(t)P\Delta A(t)x(t) \\
& + 2x^{\mathrm{T}}(t)P\Delta A_1(t)x(t-h) + 2x^{\mathrm{T}}(t)P\Delta B(t)Kx(t) \\
& + 2\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} P\Delta A(t)x(t) + 2\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} P\Delta A_1(t)x(t-h) \\
& + 2\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} P\Delta B(t)Kx(t) - 2x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}\Delta A(t)x(t) \\
& - 2x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}\Delta A_1(t)x(t-h) - 2x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}\Delta B(t)Kx(t),
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\dot{V}_2 = {} & \alpha h x^{\mathrm{T}}(t)G^{\mathrm{T}}PGx(t) - \alpha\int_{t-h}^{t} x^{\mathrm{T}}(s)G^{\mathrm{T}}PGx(s)\,\mathrm{d}s \\
\leqslant {} & \alpha h x^{\mathrm{T}}(t)G^{\mathrm{T}}PGx(t) - \int_{t-h}^{t} x^{\mathrm{T}}(s)G^{\mathrm{T}}PGx(s)\,\mathrm{d}s \\
& - h^{-1}(\alpha-1)\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} P\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right),
\end{aligned} \tag{23}
$$

$$
\dot{V}_3 = x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-h)Qx(t-h), \tag{24}
$$

$$
\begin{aligned}
\dot{V}_4 = {} & \alpha h x^{\mathrm{T}}(t)A_2^{\mathrm{T}}PA_2x(t) - \alpha h x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}PA_2x(t-h), \\
= {} & \alpha h x^{\mathrm{T}}(t)A_2^{\mathrm{T}}PA_2x(t) - (\alpha-1)h x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}PA_2x(t-h) \\
& - h x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}PA_2x(t-h),
\end{aligned} \tag{25}
$$

$$\dot{V}_5 = hx^{\mathrm{T}}(t)PF_{11}Px(t) + 2x^{\mathrm{T}}(t)PF_{12}P\int_{t-h}^{t} Gx(s)\,\mathrm{d}s + 2hx^{\mathrm{T}}(t)PF_{13}Px(t-h)$$
$$+ 2hx^{\mathrm{T}}(t)PF_{14}PA_2x(t-h) + \int_{t-h}^{t} x^{\mathrm{T}}(s)G^{\mathrm{T}}PF_{22}PGx(s)\,\mathrm{d}s$$
$$+ 2\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} PF_{23}Px(t-h)$$
$$+ 2\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} PF_{24}PA_2x(t-h) + hx^{\mathrm{T}}(t-h)PF_{33}Px(t-h)$$
$$+ 2hx^{\mathrm{T}}(t-h)PF_{34}PA_2x(t-h) + hx^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}PF_{44}PA_2x(t-h), \qquad (26)$$

where Lemma 1 is utilized in (23).

Using Fact 1, we get a bound of some terms which include uncertainties as

$$2x^{\mathrm{T}}(t)P\Delta A(t)x(t) = 2x^{\mathrm{T}}(t)PD_1F_1(t)E_1x(t)$$
$$\leqslant \varepsilon_1^{-1}x^{\mathrm{T}}(t)PD_1D_1^{\mathrm{T}}Px(t) + \varepsilon_1x^{\mathrm{T}}(t)E_1^{\mathrm{T}}E_1x(t), \qquad (27)$$

$$2x^{\mathrm{T}}(t)P\Delta A_1(t)x(t-h) \leqslant \varepsilon_2^{-1}x^{\mathrm{T}}(t)PD_2D_2^{\mathrm{T}}Px(t) + \varepsilon_2x^{\mathrm{T}}(t-h)E_2^{\mathrm{T}}E_2x(t-h), \qquad (28)$$

$$2x^{\mathrm{T}}(t)P\Delta B(t)Kx(t) \leqslant \varepsilon_3^{-1}x^{\mathrm{T}}(t)PD_3D_3^{\mathrm{T}}Px(t)x(t) + \varepsilon_3x^{\mathrm{T}}(t)K^{\mathrm{T}}E_3^{\mathrm{T}}E_3Kx(t), \qquad (29)$$

$$2\left(\int_{t-h}^{t} Gx(s)\mathrm{d}s\right)^{\mathrm{T}} P\Delta A(t)x(t) \leqslant \varepsilon_4^{-1}\left(\int_{t-h}^{t} Gx(s)\mathrm{d}s\right)^{\mathrm{T}} PD_1D_1^{\mathrm{T}}P\left(\int_{t-h}^{t} Gx(s)\mathrm{d}s\right)$$
$$+ \varepsilon_4x^{\mathrm{T}}(t)E_1^{\mathrm{T}}E_1x(t), \qquad (30)$$

$$2\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} P\Delta A_1(t)x(t-h)$$
$$\leqslant \varepsilon_5^{-1}\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} PD_2D_2^{\mathrm{T}}P\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)$$
$$+ \varepsilon_5x^{\mathrm{T}}(t-h)E_2^{\mathrm{T}}E_2x(t-h). \qquad (31)$$

$$2\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} P\Delta B(t)Kx(t)$$
$$\leqslant \varepsilon_6^{-1}\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right)^{\mathrm{T}} PD_3D_3^{\mathrm{T}}P\left(\int_{t-h}^{t} Gx(s)\,\mathrm{d}s\right) + \varepsilon_6x^{\mathrm{T}}(t)KE_3^{\mathrm{T}}E_3Kx(t), \qquad (32)$$

$$-2x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}P\Delta A(t)x(t) \leqslant \varepsilon_7^{-1}x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}PD_1D_1^{\mathrm{T}}PA_2x(t-h)$$
$$+ \varepsilon_7x^{\mathrm{T}}(t)E_1^{\mathrm{T}}E_1x(t), \qquad (33)$$

$$-2x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}P\Delta A_1(t)x(t-h) \leqslant \varepsilon_8^{-1}x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}PD_2D_2^{\mathrm{T}}Px(t-h) + \varepsilon_8 x^{\mathrm{T}}(t-h)E_2^{\mathrm{T}}E_2x(t-h), \tag{34}$$

$$-2x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}P\Delta B(t)Kx(t) \leqslant \varepsilon_9^{-1}x^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}PD_3D_3^{\mathrm{T}}Px(t-h) + \varepsilon_9 x^{\mathrm{T}}(t)K^{\mathrm{T}}E_3^{\mathrm{T}}E_3Kx(t). \tag{35}$$

Substituting (27)–(35) into (22) and using (23)–(26) gives that the time-derivative of V has new upper bound as follows:

$$\dot{V} \leqslant \begin{bmatrix} x(t) \\ \int_{t-h}^{t} Gx(s)\,\mathrm{d}s \\ x(t-h) \\ A_2x(t-h) \end{bmatrix}^{\mathrm{T}} \Omega \begin{bmatrix} x(t) \\ \int_{t-h}^{t} Gx(s)\,\mathrm{d}s \\ x(t-h) \\ A_2x(t-h) \end{bmatrix} + \int_{t-h}^{t} x^{\mathrm{T}}(s)G^{\mathrm{T}}(-P+PF_{22}P)Gx(s)\,\mathrm{d}s + hx^{\mathrm{T}}(t-h)A_2^{\mathrm{T}}(-P+PF_{44}P)A_2x(t-h), \tag{36}$$

where

$$\Omega = \begin{bmatrix} \Sigma_1 & \Sigma_5 & P(A_1-G)+hPF_{13}P & \Sigma_6 \\ \star & \Sigma_2 & P(A_1-G)+PF_{23}P & PF_{24}P \\ \star & \star & \Sigma_3 & -(A_1-G)^{\mathrm{T}}P+hPF_{34}P \\ \star & \star & \star & \Sigma_4 \end{bmatrix},$$

$$\begin{aligned}
\Sigma_1 &= P(A+G+BK)+(A+G+BK)^{\mathrm{T}}P+\alpha hG^{\mathrm{T}}PG+\alpha hA_2^{\mathrm{T}}PA_2+Q \\
&\quad + hPF_{11}P+\varepsilon_1^{-1}PD_1D_1^{\mathrm{T}}P+\varepsilon_1E_1^{\mathrm{T}}E_1+\varepsilon_2^{-1}PD_2D_2^{\mathrm{T}}P+\varepsilon_3^{-1}PD_3D_3^{\mathrm{T}}P \\
&\quad + \varepsilon_3K^{\mathrm{T}}E_3^{\mathrm{T}}E_3K+\varepsilon_4E_1^{\mathrm{T}}E_1+\varepsilon_6K^{\mathrm{T}}E_3^{\mathrm{T}}E_3K+\varepsilon_7E_1^{\mathrm{T}}E_1, \\
\Sigma_2 &= -h^{-1}(\alpha-1)P+\varepsilon_4^{-1}PD_1D_1^{\mathrm{T}}P+\varepsilon_5^{-1}PD_2D_2^{\mathrm{T}}P+\varepsilon_6^{-1}PD_3D_3^{\mathrm{T}}P \\
\Sigma_3 &= -Q+hPF_{33}P+\varepsilon_2E_2^{\mathrm{T}}E_2+\varepsilon_5E_2^{\mathrm{T}}E_2+\varepsilon_8E_2^{\mathrm{T}}E_2 \\
\Sigma_4 &= -h(\alpha-1)P+\varepsilon_7^{-1}PD_1D_1^{\mathrm{T}}P+\varepsilon_8^{-1}PD_2D_2^{\mathrm{T}}P, \\
\Sigma_5 &= (A+G+BK)^{\mathrm{T}}P+PF_{12}P, \\
\Sigma_6 &= -(A+G+BK)^{\mathrm{T}}P+hPF_{14}P.
\end{aligned} \tag{37}$$

Hence, if $\Omega < 0$, then a positive scalar γ exists which satisfies

$$\dot{V} < -\gamma\|x(t)\|^2. \tag{38}$$

Let

$$X = P^{-1}, \quad W = XQX, \quad Y = GX, \quad M = KX. \tag{39}$$

By pre- and post-multiplying the inequalities $-P + PF_{22}P \leqslant 0$, $-P + PF_{44}P \leqslant 0$ and $\Omega < 0$ by X, X, and $\mathrm{diag}\{X,X,X,X\}$, respectively, the resulting inequalities are equivalent to (10) and (11) and the following inequality:

$$\Omega_1 = \begin{bmatrix} \Gamma_1 & \Gamma_5 & A_1X - Y + hF_{13} & -XA^{\mathrm{T}} - Y^{\mathrm{T}} - M^{\mathrm{T}}B^{\mathrm{T}} + hF_{14} \\ \star & \Gamma_2 & A_1X - Y + F_{23} & F_{24} \\ \star & \star & \Gamma_3 & -XA_1^{\mathrm{T}} + Y^{\mathrm{T}} + hF_{34} \\ \star & \star & \star & \Gamma_4 \end{bmatrix} < 0, \tag{40}$$

where

$$\begin{aligned} \Gamma_1 &= AX + Y + BM + XA + Y^{\mathrm{T}} + M^{\mathrm{T}}B^{\mathrm{T}} + W + hF_{11} + \alpha hY^{\mathrm{T}}X^{-1}Y \\ &\quad + \alpha hXA_2^{\mathrm{T}}X^{-1}A_2X + \varepsilon_1^{-1}D_1D_1^{\mathrm{T}} + \varepsilon_2^{-1}D_2D_2^{\mathrm{T}} + \varepsilon_3^{-1}D_3D_3^{\mathrm{T}} + \varepsilon_1XE_1^{\mathrm{T}}E_1X \\ &\quad + \varepsilon_4XE_1^{\mathrm{T}}E_1X + \varepsilon_7XE_1^{\mathrm{T}}E_1X + \varepsilon_3M^{\mathrm{T}}E_3^{\mathrm{T}}E_3M + \varepsilon_6M^{\mathrm{T}}E_3^{\mathrm{T}}E_3M \\ \Gamma_2 &= -h^{-1}(\alpha - 1)X + \varepsilon_4^{-1}D_1D_1^{\mathrm{T}} + \varepsilon_5^{-1}D_2D_2^{\mathrm{T}} + \varepsilon_6^{-1}D_3D_3^{\mathrm{T}} \\ \Gamma_3 &= -W + hF_{33} + \varepsilon_2XE_2^{\mathrm{T}}E_2X + \varepsilon_5XE_2^{\mathrm{T}}E_2X + \varepsilon_8XE_2^{\mathrm{T}}E_2X \\ \Gamma_4 &= -h(\alpha - 1)X + \varepsilon_7^{-1}D_1D_1^{\mathrm{T}} + \varepsilon_8^{-1}D_2D_2^{\mathrm{T}}, \\ \Gamma_5 &= XA^{\mathrm{T}} + Y^{\mathrm{T}} + M^{\mathrm{T}}B^{\mathrm{T}} + F_{12}. \end{aligned} \tag{41}$$

The inequality (40) is equivalent to the inequality (9) by Fact 2. Also, the inequality (13) is equivalent to

$$\begin{bmatrix} A_2^{\mathrm{T}}X^{-1}A_2 - \beta_1P & hA_2^{\mathrm{T}}PG \\ \star & h^2G^{\mathrm{T}}PG - \beta_2P \end{bmatrix} < 0 \tag{42}$$

by pre- and post-multiplying the inequality (13) by $\mathrm{diag}\{X^{-1},X^{-1},X^{-1}\}$. Therefore, from Lemma 2, if the inequality (12) and (13) hold, then operator $\mathfrak{D}(x_t)$ is stable. The inequality (14) means that V_5 is non-negative. According Theorem 9.8.1 in [1], we conclude that if matrix inequalities (9)–(14) holds, then, system (1) is asymptotically stable. This completes our proof. □

Remark 1. In this paper, we use the operator $\mathfrak{D}(x_t)$ to transform the original system. Note that if the free parameter G in $\mathfrak{D}(x_t)$ is A_1, then the transformation is the neutral model transformation one [1]. Since the operator $\mathfrak{D}(x_t)$ has free weighting matrix, it is less conservative than the results obtained by using the neutral model transformation.

Remark 2. The solutions of Theorem 1 can be obtained by solving the generalized eigenvalue problem with respect to solution variables, which is a quasiconvex optimization problem. In this paper, in order to solve the problem, we utilize Matlab's LMI Control Toolbox [15] which implements interior-point algorithms. These algorithms are significantly faster than classical convex optimization algorithms [13].

Remark 3. The method presented in the paper can be easily extended to uncertain time-delay system with $A_2 = 0$ or neutral systems with multiple time delays.

4. Extension to nonfragile controller

In the field of control for uncertain systems, it is generally known that feedback systems designed for robustness with regard to plant parameters may require very accurate controllers. Recently, it is shown that relatively small perturbations in controller parameters could even destabilize the closed-loop system [12]. Therefore, it is necessary that any controller should be able to tolerate some level of controller gain variations. This has motivated the study of nonfragile controller problems.

Since gain perturbations may arise when implementing the controller given in (3), the actual controller will be of the form

$$u(t) = (K + \Delta K(t))x(t), \tag{43}$$

where ΔK is the additive controller gain variations of the form:

$$\Delta K(t) = D_4 F_4(t) E_4,$$

where D_4 and E_4 are known matrices, and $F_4(t)$ is an unknown perturbation matrix satisfying

$$F_4(t)^{\mathrm{T}} F_4(t) \leqslant I.$$

In this case, using Fact 1, the matrix Ω given in (36) is modified as

$$
\begin{aligned}
\overline{\Omega} &= \Omega + \begin{bmatrix} \begin{pmatrix} PB\Delta K(t) + \Delta K^{\mathrm T}(t)B^{\mathrm T}P \\ +\varepsilon_3 K^{\mathrm T}E_3^{\mathrm T}E_3\Delta K(t) \\ +\varepsilon_3\Delta K^{\mathrm T}(t)E_3^{\mathrm T}E_3K \\ +\varepsilon_3\Delta K^{\mathrm T}(t)E_3^{\mathrm T}E_3\Delta K(t) \\ +\varepsilon_6 K^{\mathrm T}E_3^{\mathrm T}E_3\Delta K(t) \\ +\varepsilon_6\Delta K^{\mathrm T}(t)E_3^{\mathrm T}E_3K \\ +\varepsilon_6\Delta K^{\mathrm T}(t)E_3^{\mathrm T}E_3\Delta K(t) \end{pmatrix} & \Delta K^{\mathrm T}(t)B^{\mathrm T}P & 0 & -\Delta K^{\mathrm T}(t)B^{\mathrm T}P \\ \star & 0 & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix} \\
&\leqslant \Omega + \begin{bmatrix} \begin{pmatrix} \varepsilon_9^{-1}PBD_4D_4^{\mathrm T}B^{\mathrm T}P \\ +\varepsilon_9E_4^{\mathrm T}E_4 + \varepsilon_3(E_4^{\mathrm T}E_4 \\ +K^{\mathrm T}E_3^{\mathrm T}E_3D_4D_4^{\mathrm T}E_3^{\mathrm T}E_3K) \\ +\varepsilon_3\beta E_4^{\mathrm T}E_4 + \varepsilon_6(E_4^{\mathrm T}E_4 \\ +K^{\mathrm T}E_3^{\mathrm T}E_3D_4D_4^{\mathrm T}E_3^{\mathrm T}E_3K) \\ +\varepsilon_6\beta E_4^{\mathrm T}E_4 + \varepsilon_{10}^{-1}E_4^{\mathrm T}E_4 \\ +\varepsilon_{11}^{-1}E_4^{\mathrm T}E_4 \end{pmatrix} & 0 & 0 & 0 \\ \star & \varepsilon_{10}PBD_4D_4^{\mathrm T}B^{\mathrm T}P & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \varepsilon_{11}PBD_4D_4^{\mathrm T}B^{\mathrm T}P \end{bmatrix} \\
&\equiv \Omega + \Omega_2,
\end{aligned} \tag{44}
$$

where $\beta = \lambda_{\mathrm M}(D_4^{\mathrm T}E_3^{\mathrm T}E_3D_4)$ and ε_9, ε_{10} and ε_{11} are positive scalars.

Then, the fact that $(\Omega + \Omega_2) < 0$ is equivalent to $\overline{X}(\Omega + \Omega_2)\overline{X} < 0$, and we have

$$
\overline{X}(\Omega + \Omega_2)\overline{X} = \Omega_1 + \overline{X}\Omega_2\overline{X} < 0, \tag{45}
$$

where $\overline{X} = \mathrm{diag}\{X, X, X, X\}$.

Then we have the following theorem.

Theorem 2. *Consider system* (1) *with given* $\alpha > 1$ *and* h. *the system with controller gain variation* (43) *is robustly stabilizable if there exist matrices* $X > 0$, $W > 0$, *positive scalars* $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_{10}$, *matrices* $F_{ii} \geqslant 0 (i = 1, \ldots, 4)$, *positive scalars* β_1, β_2, *and any matrices* $Y, M, F_{ij} (i = 1, \ldots, 4, i < j \leqslant 4)$ *which satisfy the following matrix inequalities*:

$$
\begin{bmatrix}
\Xi_1 & \Xi_2 & A_1X - Y + hF_{13} & \Xi_3 & 0 & 0 & 0 & 0 & 0 & \Xi_{10} \\
\star & -h^{-1}(\alpha-1)X & A_1X - Y + F_{23} & F_{24} & \Xi_6 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & -W + hF_{33} & -XA_1^{\mathrm{T}} + Y^{\mathrm{T}} + hF_{34} & 0 & \varepsilon_2 XE_2^{\mathrm{T}} & \varepsilon_5 XE_2^{\mathrm{T}} & \varepsilon_8 XE_2^{\mathrm{T}} & 0 & 0 \\
\star & \star & \star & -h(\alpha-1)X & 0 & 0 & 0 & 0 & \Xi_8 & 0 \\
\star & \star & \star & \star & \Xi_7 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -\varepsilon_5 I & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & -\varepsilon_8 I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \Xi_9 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \Xi_{11}
\end{bmatrix} < 0
$$

matrix inequalities (10)–(14), (46)

where Ξ_1, Ξ_2 *and* Ξ_3 *are the same with* (15), *and*

$$
\begin{aligned}
\Xi_6 &= [D_1 \ \ D_2 \ \ D_3 \ \ \varepsilon_{10} BD_4], \\
\Xi_7 &= \mathrm{diag}\{-\varepsilon_4 I, -\varepsilon_5 I, -\varepsilon_6 I, -\varepsilon_{10} I\}, \\
\Xi_8 &= [D_1 \ \ D_2 \ \ \varepsilon_{11} BD_4], \\
\Xi_9 &= \mathrm{diag}\{-\varepsilon_7 I, -\varepsilon_8 I, -\varepsilon_{11} I\}, \\
\Xi_{10} &= [\alpha h Y^{\mathrm{T}} \ \ \alpha h X A_2^{\mathrm{T}} \ \ D_1 \ \ D_2 \ \ D_3 \ \ \varepsilon_1 XE_1^{\mathrm{T}} \ \ \varepsilon_4 XE_1^{\mathrm{T}} \ \ \varepsilon_7 XE_1^{\mathrm{T}} \ \ \varepsilon_3 M^{\mathrm{T}} E_3^{\mathrm{T}} \\
&\qquad \varepsilon_6 M^{\mathrm{T}} E_3^{\mathrm{T}} \ \ BD_4 \ \ \varepsilon_9 E_4^{\mathrm{T}} \ \ \varepsilon_3 XE_4^{\mathrm{T}} \ \ \varepsilon_3 M^{\mathrm{T}} E_3^{\mathrm{T}} E_3 D_4 \ \ \varepsilon_3 \beta E_4^{\mathrm{T}} \ \ \varepsilon_6 XE_4^{\mathrm{T}} \\
&\qquad \varepsilon_6 M^{\mathrm{T}} E_3^{\mathrm{T}} E_3 D_4 \ \ \varepsilon_6 \beta E_4^{\mathrm{T}} \ \ XE_4^{\mathrm{T}} \ \ XE_4^{\mathrm{T}}], \\
\Xi_{11} &= \mathrm{diag}\{-\alpha h X, -\alpha h X, -\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_1 I, -\varepsilon_4 I, -\varepsilon_7 I, -\varepsilon_3 I, -\varepsilon_6 I, -\varepsilon_9 I, \\
&\qquad -\varepsilon_9 I, -\varepsilon_3 I, -\varepsilon_3 I, -\varepsilon_3 \beta I, -\varepsilon_6 I, -\varepsilon_6 I, -\varepsilon_6 \beta I, -\varepsilon_{10} I, -\varepsilon_{11} I\}.
\end{aligned}
\tag{47}
$$

Proof. The inequality in (45) is equivalent to (46) by Fact 2. The rest of proof is similar to that of Theorem 1. This completes the proof. □

Remark 4. The method presented in this paper can be easily applied to multiplicative controller gain variation of the form

$$\Delta K(t) = DF(t)EK,$$

where D and E are known matrices and $F(t)$ is an unknown matrix satisfying $F^{\mathrm{T}}(t)F(t) \leqslant I$.

5. Numerical example

We demonstrate the applicability of the proposed approach by a example. Consider the following neutral systems (1) with

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0 & 0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1.1 & 0.2 \\ 0.1 & 1.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_1 = D_2 = I, \quad D_3 = 0.1I,$$

$$E_1 = E_2 = 0.2I, \quad E_3 = 1, \quad F_i^{\mathrm{T}}(t)F_i(t) \leqslant I \quad \text{for } i = 1,2,3.$$

For the case of $h = 0.5$ and $\alpha = 2.6$, by solving the inequalities of Theorem 1, we found the solutions as

$$X = \begin{bmatrix} 12.9506 & -14.4042 \\ -14.4042 & 22.1891 \end{bmatrix}, \quad W = \begin{bmatrix} 8.2874 & -8.7788 \\ -8.7788 & 12.9182 \end{bmatrix},$$

$$Y = \begin{bmatrix} 1.0491 & -2.3915 \\ -1.1702 & 6.1155 \end{bmatrix}, \quad M = [\,29.0232 \quad -49.2531\,],$$

$$F_{11} = \begin{bmatrix} 14.1799 & -15.6437 \\ -15.6437 & 24.2074 \end{bmatrix}, \quad F_{12} = \begin{bmatrix} 9.1104 & -9.4473 \\ -10.6990 & 15.6443 \end{bmatrix},$$

$$F_{13} = \begin{bmatrix} -10.5045 & 9.1388 \\ 13.6963 & -17.1465 \end{bmatrix}, \quad F_{14} = \begin{bmatrix} -12.3706 & 12.4273 \\ 14.9856 & -21.2286 \end{bmatrix},$$

$$F_{22} = \begin{bmatrix} 12.6713 & -14.0754 \\ -14.0754 & 21.7868 \end{bmatrix}, \quad F_{23} = \begin{bmatrix} -6.9249 & 6.2398 \\ 8.8066 & -11.3512 \end{bmatrix},$$

$$F_{24} = \begin{bmatrix} -4.8038 & 5.0393 \\ 5.6044 & -8.2670 \end{bmatrix}, \quad F_{33} = \begin{bmatrix} 8.4743 & -8.9845 \\ -8.9845 & 13.1880 \end{bmatrix},$$

$$F_{34} = \begin{bmatrix} 9.5492 & -11.3467 \\ -9.4074 & 15.6106 \end{bmatrix}, \quad F_{44} = \begin{bmatrix} 12.8363 & -14.2733 \\ -14.2733 & 22.0351 \end{bmatrix},$$

Table 1
Control gains in case of certain h

	$h=1$	$h=2$	$h=3$
$\alpha=9.7$	$K=[-5.7582 \quad -4.6512]$	$K=[-10.1083 \quad -5.1341]$	$K=[-14.277 \quad -5.2132]$
$\alpha=4.2$	$K=[-1.9419 \quad -3.3200]$	$K=[-2.3362 \quad -3.2078]$	Infeasible
$\alpha=2.6$	$K=[-1.0198 \quad -2.8281]$	Infeasible	Infeasible

$$\varepsilon_1 = 10^5 \times 6.9474, \quad \varepsilon_2 = 10^5 \times 7.0083, \quad \varepsilon_3 = 10^6 \times 3.5622,$$
$$\varepsilon_4 = 10^5 \times 7.2417, \quad \varepsilon_5 = 10^5 \times 7.2578, \quad \varepsilon_6 = 10^6 \times 5.6565,$$
$$\varepsilon_7 = 10^5 \times 7.1855, \quad \varepsilon_18 = 10^5 \times 7.5637, \quad \beta_1 = 1/3, \quad \beta_2 = 1/3.$$

Therefore, the free parameter G of operator $\mathfrak{D}(z_t)$ and gain matrix K of controller are as follows:

$$G = \begin{bmatrix} -0.1398 & -0.1985 \\ 0.7777 & 0.7804 \end{bmatrix}, \quad K = [-0.8194 \quad -2.7516].$$

For other h and α, the control gains for above system are given in Table 1.

6. Conclusions

In this article, the problem of finding a state-feedback controller that asymptotically stabilizes a class of neutral delay-differential systems with uncertainties has been investigated. A delay-dependent criterion expressed in terms of matrix inequalities has been derived using a Lyapunov functional. Moreover, the design method is extended to the systems with controller gain variations.

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