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Brief paper

Stabilization of switched continuous-time systems with all modes unstable via dwell time switching[☆]

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ABSTRACT

Stabilization of switched systems composed fully of unstable subsystems is one of the most challenging problems in the field of switched systems. In this brief paper, a sufficient condition ensuring the asymptotic stability of switched continuous-time systems with all modes unstable is proposed. The main idea is to exploit the stabilization property of switching behaviors to compensate the state divergence made by unstable modes. Then, by using a discretized Lyapunov function approach, a computable sufficient condition for switched linear systems is proposed in the framework of dwell time; it is shown that the time intervals between two successive switching instants are required to be confined by a pair of upper and lower bounds to guarantee the asymptotic stability. Based on derived results, an algorithm is proposed to compute the stability region of admissible dwell time. A numerical example is proposed to illustrate our approach.

1. Introduction

The stability issue is the main concern in the field of switched systems, which have been extensively studied in the literature (Branicky, 1998; Daafouz, Riedinger, & Iung, 2002; Decarlo, Branicky, Pettersson, & Lennartson, 2000; Geromel & Colaneri, 2006; Lee & Dullerud, 2007; Liberzon, 2003; Lin & Antsaklis, 2009; Margaliot, 2006; Shorten, Wirth, Mason, Wulff, & King, 2007; Sun & Ge, 2005). Most of the reported results are confined to the case where there exist stable subsystems within switched systems. In the early work, the research results mainly focused on the switched systems composed fully of stable modes (Allerhand & Shaked, 2011; Chesi, Colaneri, Geromel, Middleton, & Shorten, 2010; Hespanha, Liberzon, & Morse, 1999; Morse, 1996). In recent years, some advances have been reached to deal with the case when there exist some unstable modes such as Xiang and Xiang (2009), Xiang and Xiao (2012), Xiang, Xiao, and Iqbal (2012), Zhai, Hu, Yasuda, and Michel (2000, 2001, 2002) and Zhang and Shi (2009, 2010), but it should be noted that these results also require the existence of (at least one) stable subsystem to ensure the stability of the whole switched system. The main idea of these results is to activate the stable modes for sufficiently long to absorb the state divergence made by unstable modes. But, when all the subsystems are unstable, this promising idea obviously fails, since there exists no stable period to compensate the state divergence effect.

As is well known, even if all subsystems are unstable, one may carefully switch between unstable modes to make the switched system asymptotically stable, and how to design appropriate switching laws to stabilize the switched system composed fully of unstable subsystems is one of the most interesting and serious challenges for switched systems (Decarlo et al., 2000; Liberzon, 2003; Lin & Antsaklis, 2009; Sun & Ge, 2005). This problem has been extensively studied for years, e.g. Li, Wen, and Soh (2001), Margaliot and Langholz (2003), Pettersson (2003), Pettersson and Lennartson (2001), Wicks, Peleties, and DeCarlo (1998), most of them resort to state-dependent switching strategies such as the min-projection strategy (Pettersson & Lennartson, 2001), largest region function strategy (Pettersson, 2003), etc., but very few results focus on the time-dependent switching law particularly concerned with dwell time, which motivates the present study. Since the previous idea based on the presence of a stable subsystem is not applicable for the case with all subsystems unstable, we have to find another way to establish stability. On the other hand, since an appropriate switching law can stabilize the system, even though all subsystems are unstable, this implies that switching behaviors can also contain a good characteristic of stabilization in some circumstances, e.g. see the examples in Branicky (1998) and Sun and Ge (2005).

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For most of the previous results, the switching behavior has been viewed as a *bad* factor only destroying stability, such as the famous (average) dwell time technique (Hespanha et al., 1999; Morse, 1996). However, since an appropriate switching law can stabilize the system, it implies switching behaviors can also contain a *good* characteristic of stabilization in some circumstances. In this brief paper, when all modes are unstable, a sufficient condition ensuring the switched system asymptotically stable is proposed. Then, in order to derive computable ways to characterize the stabilization property of switching behavior and cover the results in the familiar conception called dwell time, the discretized Lyapunov function technique (Gu, Kharitonov, & Chen, 2003) is applied to the linear case. It is interesting to see that the time interval between two successive switching instants should be confined by a pair of upper and lower bounds to guarantee the asymptotic stability, which can be viewed as an extension of Allerhand and Shaked (2011), from the case composed of stable subsystems to the case fully composed of unstable subsystems. Finally, an algorithm is proposed to compute the admissible upper and lower bounds and determine the stability region for the dwell time.

This paper is organized as follows: Some preliminaries are introduced in Section 2. The stability analysis for a switched system with all subsystems unstable is presented in Section 3, and the main contributions, the computable condition for the switched linear system and computation on the stability region for the admissible dwell time are presented in Section 4. Then, a numerical example is provided in Section 5. Conclusions are given in Section 6.

2. Preliminaries

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}_{\geq 0}$ stand for non-negative real numbers, and $\mathbb{R}^n$ be the n-dimensional real vector space. $|\cdot|$ stands for the Euclidean norm. Class $\mathcal{K}$ is a class of strictly increasing and continuous functions $[0, \infty) \to [0, \infty)$ which is zero at zero. Class $\mathcal{K}_\infty$ denotes the subset of $\mathcal{K}$ consisting of all those functions that are unbounded. The notation $P > 0$ ($P \geq 0$) means P is real symmetric and positive definite (semi-positive definite). I stands for the identity matrix with appropriate dimension.

This paper is devoted to the study of switched nonlinear systems in the form of

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) \tag{1}$$

where $x(t) \in \mathcal{X} \in \mathbb{R}^n$ is the state vector. Define index set $\mathcal{M} := \{1, 2, \ldots, N\}$, where N is the number of modes. $\sigma(t) : [0, \infty) \to \mathcal{M}$ denotes the switching function, which is assumed to be a piecewise constant function continuous from the right. $f_i : \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions with $f_i(0) = 0,\ \forall i \in \mathcal{M}$, without loss of generality, the origin is not a stable (attractive) equilibrium for any modes $i \in \mathcal{M}$. The switching instants are expressed by a sequence $\mathcal{S} := \{t_0, t_1, t_2, \ldots, t_n, \ldots\}$ where t_0 denotes the initial time and t_n denotes the nth switching instant. The length between successive switching instants is denoted the dwell time $\tau_n = t_{n+1} - t_n$, $n = 0, 1, 2, \ldots$. In this work, we always assume that (1) is forward complete meaning for each $x_0 \in \mathcal{X} \in \mathbb{R}^n$ there exists a unique trajectory $x(t; x_0)$ for (1) satisfying $x(t_0) = x_0$, and we only consider non-zeno switchings (i.e., switching occurs finite times in a finite time interval). With respect to switching law $\sigma(t)$, the following stability notions are given.

Definition 1. Switched system (1) with switching law $\sigma(t)$ is said to be uniformly stable (US) with respect to $\sigma(t)$ if for $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that $|x(t)| < \varepsilon,\ \forall t \in [0, \infty)$ whenever $|x(0)| < \delta$. When for $\forall \delta > 0$ we have $|x(t)| < \varepsilon,\ \forall t \in [0, \infty)$ then system (1) is globally uniformly stable(GUS) with respect to $\sigma(t)$. Furthermore if system (1) is GUS and satisfies $\lim_{t\to\infty} x(t) = 0$, switched system (1) is globally uniformly asymptotically stable (GUAS) with respect to $\sigma(t)$.

The objective of this work is to propose a sufficient condition that guarantees the switched system (1) is GUAS with respect to switching law $\sigma(t)$ when all modes of (1) are unstable. Furthermore, particularly concerned with the linear case of system (1), the set of admissible switching laws that can stabilize the switched system will be ascertained in the framework of dwell time.

3. Stability analysis

It has been well recognized that the multiple Lyapunov function (MLF) $V_i(t),\ i \in \mathcal{M}$ is a popular stability analysis tool for switched systems, especially under dwell time constrained switching (Hespanha et al., 1999; Morse, 1996). At each switching instant t_n from mode i to j, the switching always causes a bounded increment of $V_i(t)$ which is described by $V_j(t_n) < \mu V_i(t_n),\ i \neq j,\ \forall i, j \in \mathcal{M}$, where $\mu > 1$. When unstable subsystems are involved, a class of Lyapunov functions $V_i(t)$ are allowed to increase with bounded increase rate as $L_{f_i}V_i(t) < \eta V_i(t)$, where $\eta > 0$ as unstable modes work. Then, both increment of $V_i(t)$ caused by activation of unstable modes and occurrence of switching will be compensated by the decrement produced by stable subsystems with a decrease rate of $L_{f_i}V_i(t) < -\lambda V_i(t)$, where $\lambda > 0$ (Xiang & Xiang, 2009; Xiang & Xiao, 2012; Xiang et al., 2012; Zhai et al., 2000, 2001, 2002; Zhang & Shi, 2009, 2010).

The above idea requires that there exists at least one stable subsystem satisfying $L_{f_i}V_i(t) < -\lambda V_i(t),\ \lambda > 0$ to compensate the increment of $V_i(t)$. But, this promising idea is obviously not applicable when all subsystems are unstable, i.e., $L_{f_i}V_i(t) < \eta V_i(t)$, $\eta > 0,\ \forall i \in \mathcal{M}$, since there exists no stable mode to be activated to compensate the increment of Lyapunov function. In this brief paper, we turn to the idea that increment of the Lyapunov function is compensated by switching behavior. Before presenting the main results, some useful functions are introduced in advance. Suppose there exists a set of continuous non-negative functions $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a scalar $\eta > 0$ that satisfies

$$\alpha_1(|x|) \leq V_i(t, x) \leq \alpha_2(|x|), \quad \forall i \in \mathcal{M} \tag{2}$$

$$L_{f_i}V_i(t) \leq \eta V_i(t), \quad \forall i \in \mathcal{M}. \tag{3}$$

Since each subsystem is unstable, we cannot find Lyapunov functions which are monotonically decreasing for each mode. Thus, we have to resort to Lyapunov functions allowed to increase. Formulations (2) and (3) cover various divergences of unstable modes, e.g. see Xiang and Xiang (2009), Zhai et al. (2000, 2001, 2002) and Zhang and Shi (2009, 2010). Particularly by inequality (3), this implies the value of $V_i(t)$ may increase with a bounded rate $\eta > 0$ as each unstable mode is activated. Finally, the activated mode indication functions $\theta_i(\cdot) : [0, \infty) \to \{0, 1\}$ are defined as

$$\theta_i(t) = \begin{cases} 1, & \text{if } \sigma(t) = i \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

And we define notations $V_i(t_n^-) = \lim_{t\to t_n^-} V_i(t)$, $V_i(t_n^+) = \lim_{t\to t_n^+} V_i(t)$. Now, we are ready to propose the first result in this paper.

Theorem 1. *Consider switched system (1) given a switching sequence $\mathcal{S} := \{t_0, t_1, t_2, \ldots, t_n, \ldots\}$ generated by $\sigma(t)$. If there exists a set of continuous non-negative functions $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying (2) and (3), and a constant $0 < \mu < 1$ such that*

$$V_j(t_n^+) \leq \mu V_i(t_n^-), \quad i \neq j,\ \forall i, j \in \mathcal{M} \tag{5}$$

$$\ln \mu + \eta \tau_n < 0, \quad \forall n = 0, 1, 2, \ldots \tag{6}$$

where $\tau_n = t_{n+1} - t_n$, $n = 0, 1, 2, \ldots$, then, switched system (1) is GUAS with respect to switching law $\sigma(t)$.

Proof. For switched nonlinear system (1), choosing a Lyapunov function candidate defined as $V(t) = \sum_{i=1}^{N} \theta_i(t)V_i(t,x)$, where $\theta_i(t),\ i \in \mathcal{M}$ are defined in (4), and $V_i(t,x),\ i \in \mathcal{M}$ satisfy (2) and (3). Assuming $\theta_i(t) = 1$ and $\theta_j(t) = 0$, when $t \in [t_n, t_{n+1})$, from (3), we can derive $V(t) \le e^{\eta(t-t_n)}V(t_n),\ t \in [t_n, t_{n+1})$. Then, supposing system (1) switches from subsystem i to j at switching instant t_{n+1}, since $\sigma(t)$ is continuous from the right, we have $V(t_{n+1}) \le \mu e^{\eta(t_{n+1}-t_n)}V(t_n)$. $\tau_n = t_{n+1} - t_n$ and $\ln \mu + \eta\tau_n < 0$ which implies $\mu e^{\eta(t_{n+1}-t_n)} < 1$. Thus, we can see $V(t_{n+1}) < \rho V(t_n)$, where $\rho = \mu e^{\eta(t_{n+1}-t_n)} < 1$. Moreover, if (6) is established, there exists a $\tau_{\max} = \sup_{n=0,1,2,\ldots} \tau_n$. Then, for any $\varepsilon > 0$, we can choose $|x(t_0)| < \delta(\varepsilon) = \alpha_2^{-1}(e^{-\eta\tau_{\max}}\alpha_1(\varepsilon))$. Thus, it yields $V(t_0) < \alpha_2(|x(t_0)|) < e^{-\eta\tau_{\max}}\alpha_1(\varepsilon)$. Since $V(t_n)$ is strictly decreasing, we have $V(t_n) < e^{-\eta\tau_{\max}}\alpha_1(\varepsilon)$. Then, one has $V(t) \le e^{\eta\tau_{\max}}V(t_n),\ t \in [t_n, t_{n+1})$. Hereby, it results in $V(t) < \alpha_1(\varepsilon)$. Furthermore, from (2), we can conclude $|x(t)| < \varepsilon$. Obviously, for $\forall\delta > 0$ we have $|x(t)| < \varepsilon$. And due to the fact that the sequence $V(t_n),\ n = 0, 1, 2, \ldots$ is strictly decreasing, we see $\lim_{t\to\infty} |x(t)| = 0$. Therefore, we can conclude switched system (1) is GUAS with respect to switching law $\sigma(t)$. □

Remark 1. In some previous results based on MLF, e.g. Hespanha et al. (1999), Morse (1996), the switching behaviors are always viewed to increase the value of the Lyapunov function. However, this viewpoint is too restrictive without considering the stabilization characteristic of switchings, see Sun and Ge (2005, Example 1.1) and Branicky (1998, Example 2.1) with negated vector fields. In Theorem 1, even if all modes are unstable, stabilization can be also achieved by switching behaviors due to the reducing value of the Lyapunov function described by (5).

However, in many actual applications, Theorem 1 is trivial since the sequence $\mathcal{S} := \{t_0, t_1, t_2, \ldots, t_n, \ldots\}$ cannot be specified in advance, so it is impossible to check condition (5) for all switching instants t_n as $n \to \infty$. To make our results applicable, by considering the switched linear system, we convert Theorem 1 into the framework of dwell time. It is well known that switching logics generated from dwell time are easy for the implementation of switched controllers (Morse, 1996, Section 3).

4. Stabilization for switched linear system

4.1. Problem description

Consider the switched linear system in the form of

$$\dot{x}(t) = A_{\sigma(t)}x(t) \tag{7}$$

where $x(t)$ and $\sigma(t)$ are described the same as in (1). Without loss of generality, all the subsystem matrices $A_i,\ i \in \mathcal{M}$ are supposed to have eigenvalues located in the right half-plane. In the framework of dwell time, the following intuitive observations can be seen.

The basic nature of a stable switched system composed fully of unstable modes is that the state trajectories driven by each activated mode diverge to different (approximately opposite) directions, so the divergence of the state trajectory can be compensated by activating unstable modes alternatively. Hence, if the switching signal $\sigma(t)$ can stabilize switched system (7) with all modes unstable, the dwell time τ_n should be confined by $\tau_n \in [\tau_{\min}, \tau_{\max}]$, where $0 < \tau_{\min} \le \tau_{\max}$, since a too small or too large τ_n would make the activation of unstable modes impossible to compensate or be compensated.

Definition 2. Consider switched system (7), the minimal and maximal dwell times are denoted by $\tau_{\min} = \inf_{n=0,1,2,\ldots} \tau_n$, $\tau_{\max} = \sup_{n=0,1,2,\ldots} \tau_n$, respectively.

We call $\mathcal{D}_{[\tau_{\min},\tau_{\max}]}$ the set of all switching policies with dwell time $\tau_n \in [\tau_{\min}, \tau_{\max}]$, $\forall n = 0, 1, 2, \ldots$. Unlike previous results in the presence of stable subsystems which focus on computation of minimal admissible dwell time, e.g. Allerhand and Shaked (2011), here the admissible switching laws are confined by pairs of lower and upper bounds as $\{\tau_{\min}, \tau_{\max}\}$, a necessary and meaningful task in switching law design is to ascertain the region Ω_s covering the admissible dwell time to ensure the GUAS of switched system (7).

Problem 1. Given switched linear system (7), determine the stability region Ω_s of admissible dwell time guaranteeing (7) GUAS with respect to $\sigma(t) \in \Omega_s$.

4.2. Discretized Lyapunov function

The most popular structure of MLF for a switched linear system is in the quadratic form of $V_i(t) = x^T(t)P_ix(t),\ i \in \mathcal{M}$, where $P_i > 0,\ \forall i \in \mathcal{M}$. However, condition (5) in Theorem 1 turns into $P_j \le \mu P_i,\ i \ne j,\ \forall i, j \in \mathcal{M}$ with $0 < \mu < 1$, which will be never satisfied. Hence, we construct a Lyapunov function in the form of

$$V_i(t) = x^T(t)P_i(t)x(t), \quad i \in \mathcal{M} \tag{8}$$

where $P_i(t),\ i \in \mathcal{M}$ is a time-scheduled positive definite matrix. Then, (5) in Theorem 1 can be expressed as $P_j(t_n^+) \le \mu P_i(t_n^-),\ i \ne j,\ \forall i, j \in \mathcal{M}$. However, in practice, to numerically check the existence of such a matrix function $P_i(t),\ i \in \mathcal{M}$, especially in the continuous-time case, is not an easy task. Thus, in this paper we resort to the discretized Lyapunov function technique which has been widely used in time-delay system analysis problems, e.g. Gu et al. (2003, Section 5.7). The basic idea of the discretized Lyapunov function technique is to divide the domain of definition of matrix function $P_i(t),\ i \in \mathcal{M}$ into finite discrete points or smaller regions, thus reducing the choice of time-scheduled Lyapunov function into choosing a finite number of parameters. The discretized Lyapunov function technique used in this work is given in detail as follows:

We divide the interval $[t_n, t_n + \tau_{\min})$ into L segments described as $\mathcal{N}_{n,q} = [t_n + \theta_q, t_n + \theta_{q+1}),\ q = 0, 1, \ldots, L-1$ of equal length $h = \tau_{\min}/L$, and then $\theta_q = qh = q\tau_{\min}/L,\ q = 0, 1, \ldots, L$. The continuous matrix function $P_i(t),\ t \in [t_n, t_{n+1})$ is chosen to be linear within each segment $\mathcal{N}_{n,q},\ q = 0, 1, \ldots, L-1$. Letting $P_{i,q} = P_i(t_n + \theta_q)$, then since the matrix function $P_i(t),\ i \in \mathcal{M}$ is piecewise linear in the minimal dwell time interval $[t_n, t_n + \tau_{\min})$, it can be expressed in terms of the values at dividing points using a linear interpolation formula, i.e., for $0 \le \alpha \le 1,\ q = 0, 1, \ldots, L-1$, and $t \in \mathcal{N}_{n,q}$

$$P_i(t) = P_i(t_n + \theta_q + \alpha h) = (1-\alpha)P_{i,q} + \alpha P_{i,q+1} = P_i^{(q)}(\alpha)$$

where $\alpha = (t - t_n - \theta_q)/h$. Then the continuous matrix function $P_i(t),\ i \in \mathcal{M}$ is completely determined by $P_{i,q},\ q = 0, 1, \ldots, L,\ i \in \mathcal{M}$ in the minimal dwell time interval $[t_n, t_n + \tau_{\min})$. Afterwards, in the interval $[t_n + \tau_{\min}, t_{n+1})$, matrix function $P_i(t),\ i \in \mathcal{M}$ is fixed as a constant matrix $P_i(t) = P_{i,L},\ i \in \mathcal{M}$. Hence the discretized matrix function $P_i(t),\ i \in \mathcal{M}$ is described as

$$P_i(t) = \begin{cases} P_i^{(q)}(\alpha), & t \in \mathcal{N}_{n,q},\ q = 0, 1, \ldots, L-1 \\ P_{i,L}, & t \in [t_n + \tau_{\min}, t_{n+1}) \end{cases} \tag{9}$$

and the corresponding discretized Lyapunov function for mode $i \in \mathcal{M}$ is

$$V_i(t) = \begin{cases} x^T(t)P_i^{(q)}(\alpha)x(t), & t \in \mathcal{N}_{n,q},\ q = 0, 1, \ldots, L-1 \\ x^T(t)P_{i,L}x(t), & t \in [t_n + \tau_{\min}, t_{n+1}). \end{cases} \tag{10}$$

Obviously, the number of division segments L has to be $L \ge 1$, when the discretized Lyapunov function approach is used. If it is enforced that $L = 0$, by (10), the discretized Lyapunov function is reduced to MLF.

4.3. Stabilization via dwell time switching

At first, it is easy to observe that, for unstable matrices $A_i,\ i \in \mathcal{M}$, there always exists a sufficiently large scalar $\eta^* > 0$ such that matrices $A_i - \frac{\eta^*}{2}I,\ i \in \mathcal{M}$ are Hurwitz stable. Then, we propose the

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stabilization condition for switched linear system (7) on the basis of discretized Lyapunov function (10).

Theorem 2. *Given scalars $\eta \geq \eta^* > 0$, $0 < \mu < 1$, $0 < \tau_{\min} \leq \tau_{\max}$ consider switched linear system (7). If there exists a set of matrices $P_{i,q} > 0$, $q = 0, 1, \ldots, L$, $i \in \mathcal{M}$ such that $\forall q = 0, 1, \ldots, L-1$, $\forall i, j \in \mathcal{M}$, we have*

$$A_i^T P_{i,q} + P_{i,q} A_i + \Psi_i^{(q)} - \eta P_{i,q} < 0 \tag{11}$$

$$A_i^T P_{i,q+1} + P_{i,q+1} A_i + \Psi_i^{(q)} - \eta P_{i,q+1} < 0 \tag{12}$$

$$A_i^T P_{i,L} + P_{i,L} A_i - \eta P_{i,L} < 0 \tag{13}$$

$$P_{j,0} - \mu P_{i,L} \leq 0, \quad i \neq j \tag{14}$$

$$\ln \mu + \eta \tau_{\max} < 0 \tag{15}$$

where $\Psi_i^{(q)} = L(P_{i,q+1} - P_{i,q})/\tau_{\min}$. Then, system (7) is GUAS under any switching law $\sigma(t) \in \mathcal{D}_{[\tau_{\min}, \tau_{\max}]}$.

Proof. Since $A_i - \frac{\eta^*}{2} I$, $\forall i \in \mathcal{M}$ are Hurwitz stable, there must exist a scalar $\eta \geq \eta^* > 0$ satisfying (13). Then, we choose the discretized Lyapunov function for the ith subsystem $V_i(t) = x^T(t) P_i(t) x(t)$, $i \in \mathcal{M}$, where matrix function $P_i(t)$, $i \in \mathcal{M}$ is discretized according to (9). Obviously, (2) is satisfied, then we see

$$\dot{V}_i(t) = x^T(t) \dot{P}_i(t) x(t) + 2\dot{x}^T(t) P_i(t) x(t).$$

In each discretized segment $\mathcal{N}_{n,q}$, the following equation can be derived that for $t \in \mathcal{N}_{n,q}$

$$\dot{P}_i(t) = \dot{P}_i(t_n + \theta_q + \alpha h) = \dot{P}_i^{(q)}(\alpha) = (P_{i,q+1} - P_{i,q})\dot{\alpha}.$$

Due to $\alpha = (t - t_n - \theta_q)/h$, we have $\dot{\alpha} = 1/h$, where $h = \tau_{\min}/L$. Hence $\dot{P}_i(t)$ becomes

$$\dot{P}_i(t) = (P_{i,q+1} - P_{i,q})/h = L(P_{i,q+1} - P_{i,q})/\tau_{\min} = \Psi_i^{(q)}.$$

Moreover, by the linear interpolation relationship of $P_i^{(q)}(\alpha)$, one has

$$\begin{aligned} 2\dot{x}^T(t) P_i(t) x(t) &= 2\dot{x}^T(t) P_i^{(q)}(\alpha) x(t) \\ &= x^T(t)[A_i^T P_i^{(q)}(\alpha) + P_i^{(q)}(\alpha) A_i] x(t). \end{aligned}$$

Letting $\Xi_i^{(q)}(\alpha) = A_i^T P_i^{(q)}(\alpha) + P_i^{(q)}(\alpha) A_i + \Psi_i^{(q)}$, we have

$$\Xi_i^{(q)}(\alpha) = (1 - \alpha) \Xi_{i,1}^{(q)} + \alpha \Xi_{i,2}^{(q)}$$

where $\Xi_{i,1}^{(q)} = A_i^T P_{i,q} + P_{i,q} A_i + \Psi_i^{(q)}$ and $\Xi_{i,2}^{(q)} = A_i^T P_{i,q+1} + P_{i,q+1} A_i + \Psi_i^{(q)}$. Thus, from (11), (12), we see

$$\begin{aligned} &\dot{V}_i(t) - \eta V_i(t) = x^T(t)[\Xi_i^{(q)}(\alpha) - \eta P_i^{(q)}(\alpha)] x(t) < 0 \\ &\quad t \in \bigcup_{q=0,1,\ldots,L-1} \mathcal{N}_{n,q} = [t_n, t_n + \tau_{\min}). \end{aligned}$$

On the other hand, since $P_i(t) = P_{i,L}$, when $t \in [t_n + \tau_{\min}, t_{n+1})$, this yields $\dot{V}_i(t) = x^T(t)(A_i^T P_i + P_i A_i) x(t)$, $t \in [t_n + \tau_{\min}, t_{n+1})$. By (13), this yields $\dot{V}_i(t) < \eta V_i(t)$, $t \in [t_n + \tau_{\min}, t_{n+1})$. Thus, we arrive at $\dot{V}_i(t) < \eta V_i(t)$, $\forall t \in [t_n, t_{n+1})$, which implies (3) is satisfied. Furthermore, by (14) and the definition of the discretized function (9), we have $P_j(t_n^+) \leq \mu P_i(t_n^-)$, $i \neq j$, $\forall i, j \in \mathcal{M}$, which leads to $V_j(t_n^+) \leq \mu V_i(t_n^-)$, $i \neq j$, $\forall i, j \in \mathcal{M}$, that is to say, (5) can be satisfied. Since $\tau_{\max} \geq \tau_n$, $\forall n = 0, 1, 2, \ldots$, we have that (6) is also satisfied. Therefore, the GUAS of system (7) governed by any switching law $\sigma(t) \in \mathcal{D}_{[\tau_{\min}, \tau_{\max}]}$ can be established based on Theorem 1. □

Remark 2. Here it has to be noted that, the number of discretized matrices $P_{i,q}$, $q = 0, 1, \ldots, L$, $i \in \mathcal{M}$ is $L + 1$ which has to be prescribed in advance, and different choices of L could lead to different analysis results. Roughly speaking, the larger L is chosen, the denser the division of interval $[t_n, t_n + \tau_{\min})$ is therefore produced and, intuitively, a less conservative result can be obtained, which will be demonstrated by a numerical example later.

Remark 3. The following properties for the pair of dwell times $\{\tau_{\min}, \tau_{\max}\}$ hold with fixed constants $\eta \geq \eta^* > 0$ and $0 < \mu < 1$: Given a $\tau_{\min}$, it is easy to see that if Theorem 2 holds for some $\tau_{\max}^*$, then it holds for any $\tau_{\max} < \tau_{\max}^*$ since (15) can be satisfied for any $\tau_{\max} < \tau_{\max}^*$. Thus, if LMIs (11)–(14) are feasible with given η, μ and $\tau_{\min}$, the corresponding maximal admissible dwell time $\tau_{\max}^*$ can be computed by

$$\tau_{\max}^* = \max_{\tau_{\max} > \tau_{\min}} \{\tau_{\max} : (15) \text{ holds}\}. \tag{16}$$

Similarly, given a $\tau_{\max}$, if Theorem 2 holds for some $\tau_{\min}^*$, then it holds for any $\tau_{\min} > \tau_{\min}^*$. To see this, assume that conditions (11)–(14) of Theorem 2 are satisfied for a given $\tau_{\min}^*$, we can still choose $[t_n, t_n + \tau_{\min}^*)$ to be divided into the same segments $\mathcal{N}_{n,q}$, $q = 0, 1, \ldots, L-1$, then, (11)–(12) can guarantee $\dot{V}_i(t) < \eta V_i(t)$, $t \in \bigcup_{q=0,\ldots,L-1} \mathcal{N}_{n,q} = [t_n, t_n + \tau_{\min}^*)$. And (13) has $\dot{V}_i(t) < \eta V_i(t)$, $\forall t \in [t_n + \tau_{\min}^*, t_{n+1}) = [t_n + \tau_{\min}^*, t_n + \tau_{\min}) \cup [t_n + \tau_{\min}, t_{n+1})$. Thus, $\dot{V}_i(t) < \eta V_i(t)$, $\forall t \in [t_n, t_{n+1})$ can be also obtained for $\tau_{\min} > \tau_{\min}^*$, and the admissible minimal dwell time $\tau_{\min}^*$ with given η, μ and $\tau_{\max}$ satisfying (15) can be estimated by

$$\tau_{\min}^* = \min_{\tau_{\min} < \tau_{\max}} \{\tau_{\min} : (11)\text{–}(14) \text{ hold}\}. \tag{17}$$

From (16) and (17), we see that the admissible dwell time is bounded in the section $[\tau_{\min}, \tau_{\max}]$, so Theorem 2 provides us a computable way to obtain the admissible dwell time region for GUAS.

Remark 4. By revisiting the main idea of the well-known dwell time technique (Hespanha et al., 1999; Morse, 1996), it requires the switched system stays in stable modes sufficiently long, when it is extended to the case containing unstable systems, the activation time of unstable modes naturally needs to be small enough to avoid overly producing increment of the Lyapunov function, e.g. Xiang and Xiao (2012) and Zhai et al. (2000, 2001, 2002). But when it is concerned with the extreme case with all modes unstable, according to Theorem 2 and Remark 3, the stability of switched system (7) is related to the pair of $\{\tau_{\min}, \tau_{\max}\}$ which means the activation time of each mode is required to be neither too long nor too small and confined in the section $[\tau_{\min}, \tau_{\max}]$.

Theorem 2 provides us a sufficient condition ensuring GUAS of switched linear system (7) composed fully by unstable subsystems, but the conditions are not easy to check, and especially not convenient to solve Problem 1 for computation on the stability region for admissible dwell time. Thus, we are going to derive the following conditions which are much easier to apply to solve Problem 1. First, for $\forall q = 0, 1, \ldots, L$, $\forall i \in \mathcal{M}$, we let

$$P_{i,q} > I, \quad \forall q = 0, 1, \ldots, L, \ \forall i \in \mathcal{M}. \tag{18}$$

Hereby, (11)–(13) can be rewritten, for $\forall q = 0, 1, \ldots, L-1$ such that

$$A_i^T P_{i,q} + P_{i,q} A_i + \Psi_i^{(q)} - \eta I < 0 \tag{19}$$

$$A_i^T P_{i,q+1} + P_{i,q+1} A_i + \Psi_i^{(q)} - \eta I < 0 \tag{20}$$

$$A_i^T P_{i,L} + P_{i,L} A_i - \eta I < 0. \tag{21}$$

With a fixed μ and given a specific maximal dwell time $\tau_{\max}$, sufficient conditions (14), (15) and (18)–(21) become a feasibility problem of LMIs, and the lower bound for admissible minimal dwell time $\tau_{\min}$ can be easily figured out as

$$\tau_{\min}^* = \min_{\tau_{\min} < \tau_{\max}} \{\tau_{\min} : (14), (15), (18)\text{–}(21) \text{ hold}\}. \tag{22}$$

Then, switched linear system (7) is GUAS under switching signal any switching law $\sigma(t) \in \mathcal{D}_{[\tau^*_{\min}, \tau_{\max}]}$.

Since $0 < \mu < 1$, it is possible to set a variation $\Delta\mu$ and check all the values in (0, 1) with discretized step $\Delta\mu$. Thus, given a maximal dwell time $\tau_{\max}$, the following Algorithm 1 is proposed to compute the minimal admissible value of $\tau^*_{\min}$ guaranteeing the GUAS of system (7).

Algorithm 1 Computation on admissible $\tau^*_{\min}$ with a prescribed $\tau_{\max} > 0$

```
Initialize μ = 0, loop counter N = 0, and set a variation Δμ > 0;
while μ < 1 do
   Set N = N + 1 and μ = μ + Δμ;
   Solve (22) to obtain τ*_min;
   if (22) is feasible then
      Record D(N) = τ*_min;
   else
      Record D(N) = τ_max + 1;
   end if
end while
if ∃n = 1, 2, ..., N such that D(n) ≤ τ_max then
   τ*_min = min_{n=1,2,...,N} {D(n)};
else
   The GUAS cannot be established with τ_max;
end if
```

Obviously, given a switched system (7), the computation precision and cost of Algorithm 1 totally depend on the value of $\Delta\mu$, the choice of a smaller $\Delta\mu$ leads to a result with higher computation precision but larger computation cost. With the aid of Algorithm 1, Problem 1 can be solved by running it with every $\tau_{\max}$. The stability region Ω_s described by the pairs of $\{\tau^*_{\min}, \tau_{\max}\}$ can be determined for admissible dwell time.

5. Example

Consider system (7) composed of two subsystems as:

$$A_1 = \begin{bmatrix} -1.9 & 0.6 \\ 0.6 & -0.1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.1 & -0.9 \\ 0.1 & -1.4 \end{bmatrix}.$$

The eigenvalues of A_1 are $\lambda_{1,1} = 0.0817$ and $\lambda_{1,2} = -2.0817$, and eigenvalues of A_2 are $\lambda_{2,1} = 0.0374$ and $\lambda_{2,2} = -1.3374$. Obviously, they are unstable since both of them have eigenvalues located in the right half-plane.

(1) Given the initial state as $x(0) = [3\ 5]^T$ and a periodical switching sequence $\mathcal{S}$ satisfies $t_{n+1} - t_n = 2$, $n = 0, 1, 2, \ldots$. Therefore, we see that $\tau_{\min} = \tau_{\max} = 2$. Then, by (11)–(15), if we fix $L = 1$, $\mu = 0.5$ and $\eta = 0.3$, the following feasible solution can be obtained.

$$P_{1,0} = \begin{bmatrix} 7.3707 & 2.2116 \\ 2.2116 & 4.3990 \end{bmatrix}, \qquad P_{1,1} = \begin{bmatrix} 45.2459 & -12.3623 \\ -12.3623 & 28.8193 \end{bmatrix}$$

$$P_{2,0} = \begin{bmatrix} 18.5432 & -6.2729 \\ -6.2729 & 13.0147 \end{bmatrix}, \qquad P_{2,1} = \begin{bmatrix} 17.0657 & 0.9100 \\ 0.9100 & 59.5486 \end{bmatrix}.$$

The state trajectory is shown in Fig. 1. Obviously, the switched system can be stabilized by the switching signal $\sigma(t) \in \mathcal{D}_{[2,2]}$. The corresponding value of Lyapunov function is illustrated in Fig. 2, in which we can find that the value of the Lyapunov function may increase during the evolution time (e.g., in [0, 2)), but the increments are compensated by the switching behaviors (e.g. the Lyapunov function decreases at switching instant t_1) and the value of the Lyapunov function finally converges to zero.

(2) For every $\tau_{\max}$, the corresponding $\tau^*_{\min}$ can be calculated by Algorithm 1. All the pairs of $\{\tau^*_{\min}, \tau_{\max}\}$ determine the stability region of admissible dwell time, which is shown by Fig. 3 with the choices of $L = 1, 2, 3, 10$, respectively. The dwell time located in the stability region can ensure the GUAS such as the above simulation results under switching signal $\sigma(t) \in \mathcal{D}_{[2,2]}$. In Fig. 3, it is explicit to see that, as stated in Remark 2, the stability region expands as the parameter L increases, which obviously leads to a denser division of time interval $[t_n, t_n + \tau_{\min})$. Thus, we can say a less conservative result can be obtained by the choice of a larger L.

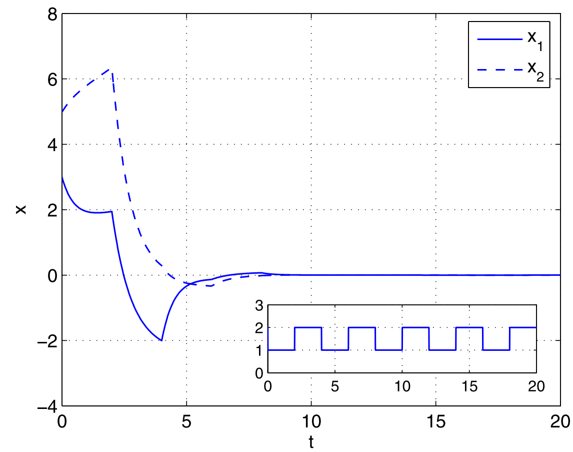

Fig. 1. State trajectories of $x(t)$.

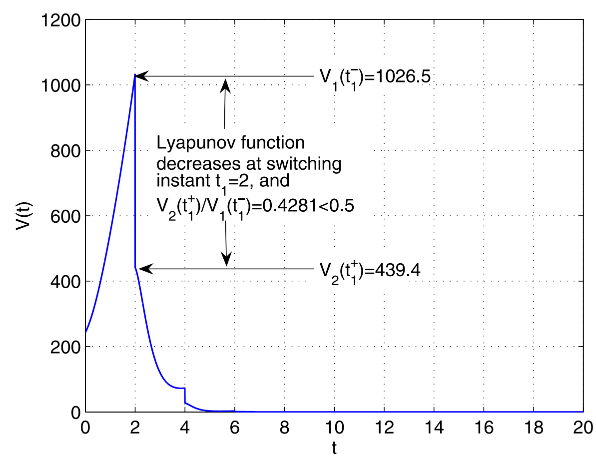

Fig. 2. Evolution of Lyapunov function.

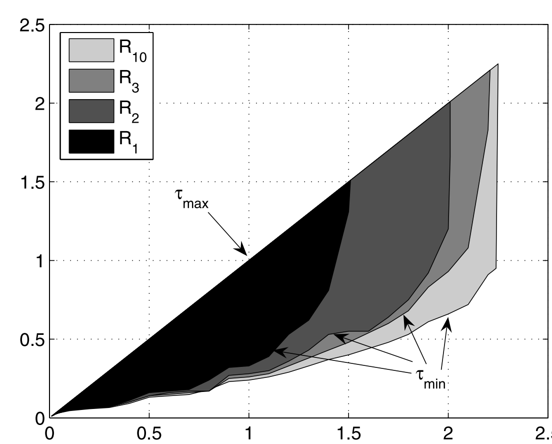

Fig. 3. Stability region for admissible dwell time: R_1 is region Ω_s with $L = 1$, $R_2 \cup R_1$ is Ω_s for $L = 2$, $\bigcup_{i=1,2,3} R_i$ is Ω_s for $L = 3$, $\bigcup_{i=1,2,3,10} R_i$ is Ω_s for $L = 10$, respectively.

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6. Conclusions

The stability problem of a switched continuous-time system with all modes unstable is addressed in this paper. By exploiting the stabilization property of switching behavior, a sufficient condition is proposed. Then, the discretized Lyapunov function technique is introduced to particularly study the linear case, a computable sufficient condition is presented which relates to a pair of upper and lower bounds of dwell time. An algorithm is provided to compute the upper and lower bounds, and the stability region of admissible dwell time. Our approach provides a numerical method to exploit the stabilization role of switching behaviors in the framework of dwell time, thus it is supposed to be of further use to improve most of the previous results.

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