SIAM J. CONTROL OPTIM.
Vol. 59, No. 1, pp. 24–49

STABILITY ANALYSIS FOR STOCHASTIC NEUTRAL SWITCHED SYSTEMS WITH TIME-VARYING DELAY*

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Abstract. This paper addresses the problems of the input-to-state stochastic stability for neutral switched stochastic delay systems. Two switching signals in the input controller, including the external input disturbance, are considered: (1) the synchronous switching signal, which indicates that the switching signal of the input controller coincides with that of the controlled subsystems, and (2) the asynchronous switching signal, which signifies that two switching instants for the input controller and the subsystem are not identical. Irrespective of which switching signal exists in the input controller and the subsystem, multiple Lyapunov–Krasovskii functions, generalized delay integral inequality, and mode-dependent average dwell time are incorporated to analyze the underlying problem. Sufficient conditions based on two integral inequalities are obtained for these two cases, respectively. Also new and more relaxed Lyapunov monotonicity conditions are introduced. One simulation example is provided to demonstrate the effectiveness of the theoretical results.

Key words. neutral switched stochastic systems, time-varying delay, input-to-state stability, synchronous switching, asynchronous switching

AMS subject classifications. 34K20, 34K40, 34K50, 93B12, 93C10, 93D25, 93D30, 93E03, 93E15, 94C10

DOI. 10.1137/19M1307974

1. Introduction.

Since neutral stochastic delay systems (NSDSs) can be applied in many diverse fields such as economics, ecology, biological, physical, and social sciences [1], the stability analysis has been widely investigated, and some classical methodologies, for example, the Lyapunov–Krasovskii function [2], the Lyapunov–Krasovskii functional [3, 4], and the Razumikhin-type theorem [1, 5, 6], have been proposed. When comparing with stochastic delay systems (SDSs) without neutral term [1, 7], the approach to analyze the stability of NSDSs lacks flexibility, in particular when the time-varying delay is a bounded function. More specifically, when time-varying delay is a bounded function, differentiable delay inequality (Halanay inequality) and comparison principle are effective to analyze the stability of SDSs, but these two approaches cannot be easily applied for stability analysis of NSDSs due to the simultaneous presence of neutral term and random noise. Recently, in [8, 9], the authors developed the delay integral inequality incorporating the Lyapunov–Krasovskii function to discuss the stability of NSDSs.

Continuous-time switched systems (CTSSs), in which the continuous dynamics and the jump phenomena simultaneously exist [10, 11, 12], are complex. The dynamical behavior of CTSSs not only depends on the continuous-time systems but also relies on a switching signal with a discrete value taking the switching logic [12]. With CTSSs being capable of offering a unified framework for mathematical modeling of

*Received by the editors December 20, 2019; accepted for publication (in revised form) October 26, 2020; published electronically January 4, 2021.
https://doi.org/10.1137/19M1307974

Funding: This work is partly is partially supported by the Australian Research Council (DP170102644), by the Natural Science Foundation of Jiangxi Province of China (20171BAB201007, 20171BCB23001), and by the Foundation of Jiangxi Provincial Educations of China (GJJ14155).

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many large scale physical systems, the stability analysis of such systems has attracted many researchers' attention [12, 15, 16, 17, 18, 19, 20, 21, 22, 23]. For example, in [12], some switching control methodologies and techniques were systematically introduced on the stability and stabilization of CTSSs. Note that although CTSSs are also time-varying, the stability analysis of CTSSs is very complex since their solutions depend on not only the initial condition but also the switching signal. It is because of this characteristic that the stability analysis of CTSSs has always been a challenging issue. In addition, time delay usually exists in some realistic models [13, 14], which can cause poor system performance. There exist two main approaches for analyzing the stability of CTSSs with time delay: (1) the Lyapunov–Razumikhin theorem [17] and (2) Lyapunov–Krasovskii methodology [18, 19, 20, 21, 22, 23]. In [17], the Lyapunov–Razumikhin theorem was introduced to analyze the stability of CTSSs with time delay. In [18], the common Lyapunov–Krasovskii function was used to analyze the stability of CTSSs with time delay. One issue with this approach is its conservatism, in that it is usually difficult to find a common Lyapunov–Krasovskii function, especially when time delay is involved. In order to reduce the conservatism, in [19, 20], the dwell time and the average dwell time (ADT) were successively proposed to analyze the stability of CTSSs with time delay by using multiple Lyapunov–Krasovskii functions. In [22], by using multiple Lyapunov–Krasovskii functions, Zhao et al. put forward the mode-dependent average dwell time (MDADT) to give some easily checked sufficient conditions that guarantee the stability of CTSSs.

In the controlled CTSSs [12], there are two possible control scenarios: the synchronous switching [24, 25, 26] and the asynchronous switching [27, 28, 29, 30, 31, 32, 33]. In [25], multiple Lyapunov–Krasovskii functionals and the ADT were used to establish a theoretical framework for the fault-tolerant control of switched time-delay systems. In [26], by using multiple Lyapunov–Krasovskii functionals and the ADT, the stability and state feedback control for switched SDSs have been investigated. Vu and Morgansen in [27] explored multiple Lyapunov–Krasovskii functions and the ADT as well as the merging switching signal technique to study the stability and stabilization of time-delay feedback switched linear systems. By utilizing multiple Lyapunov–Krasovskii functionals and the ADT, the stabilization for neutral switched SDSs [28] and deterministic NDSs [29] both under asynchronous switching were investigated, respectively. Recently, some important theoretical analysis methodologies for some advanced control strategies, such as the event-triggered sampling control, dwell-time-based H_∞ control, and sampled-data-based control for CTSSs, were proposed in [31, 32, 33].

The concepts of input-to-state stability (ISS) and integral input-to-state stability (iISS) are formulated by Sontag in [34, 35]. These two concepts can describe the characteristic of the continuity of the state trajectories, which depend on the initial condition and the external input perturbation [36]. Thus, it is necessary to consider the ISS/iISS of the controlled systems with an external input perturbation, in particular when analyzing the synthesis and design for nonlinear controlled systems. Some excellent methodologies, such as the Lyapunov–Krasovskii function, the Lyapunov–Razumikhin theorem, and the Lyapunov–Krasovskii functional, have been used to investigate the ISS/iISS of nonlinear time–delay systems; see [37, 38, 39] and the references therein. For example, in [39], by using the MDADT, multiple Lyapunov–Krasovskii functionals were constructed to establish some sufficient conditions for the ISS/iISS of nonlinear switched input delay systems with the asynchronous switching. When the random perturbation is involved, the stochastic ISS/iSS of nonlinear switched SDSs have been analyzed in [40, 41].

CTSSs can be regarded as one special case of time-varying systems. The stability analysis and synthesis of time-varying systems with or without time delay have been studied, and some meaningful sufficient conditions were given in [42, 43, 44, 45] and the references therein. In these useful references, some sufficient conditions were presented in terms of integral inequality, in which the crucial time-varying coefficients are permitted to have the indefinite sign over the whole time horizon. Time-varying switched systems are synthesized by time-varying systems and switched systems, which have considerable practical significance in a wide range of engineering fields [46]. In [47], by using the Lyapunov–Razumikhin theorem and the comparison principle as well as the MDADT, the ISS/iISS of nonlinear time-varying switched time-delay systems with the synchronous switching were considered, and the crucial time-varying coefficient in the Lyapunov–Razumikhin condition is permitted to have the indefinite sign. Note that the methodology and technique proposed in [47] may not be directly generalized to handle the ISS/iISS of nonlinear time-varying switched SDSs under the asynchronous switching. It would be even more challenging for investigating the ISS/iISS of neutral switched SDSs since both neutral term and stochastic perturbation are present.

In this paper, the problems regarding the stochastic ISS/iISS of neutral switched SDSs under the synchronous switching and the asynchronous switching are investigated, respectively. The delay integral inequality is used, which differs from our previous approach used in [8, 9]. Multiple Lyapunov–Krasovskii functions and the MDADT as well as the delay integral inequality are incorporated to analyze the stochastic ISS/iISS for neutral switched SDSs under the synchronous switching. For the asynchronous switching case, apart from using these three techniques, the merging switching signal technique is used to produce one new mixed switching signal. Several byproducts are also obtained for these two results on the stochastic ISS/iISS of neutral switched SDSs under these two switching signals, respectively. Finally, one example is given to examine the effectiveness and potential of the theoretic results obtained.

The contribution of this paper is threefold:

(i) Sufficient conditions are obtained to guarantee the stochastic ISS/iISS of neutral switched SDSs under the synchronous switching signal and the asynchronous switching signal, respectively, which have not been derived in the available literature yet. The Lyapunov–Razumikhin theorem has been used to solve the problems regarding the stability analysis of some general NSDSs [1, 5, 6] and the ISS/iISS of time-delay systems [38, 44, 45, 47]. As far as we are aware, there is no published result using the Lyapunov–Razumikhin approach to analyze the ISS/iISS of deterministic switched NDSs, let alone the ISS/iISS of neutral switched SDSs.

(ii) Under some new and more relaxed Lyapunov monotonicity conditions, the ISS/iISS of neutral switched SDSs are analyzed by using the delay integral inequality, multiple Lyapunov–Krasovskii functions, and the MDADT. What is more, the coefficients of the current state in such Lyapunov monotonicity conditions are not only time-varying and mode-dependent, but also have the indefinite sign.

(iii) Some problems have been solved regarding the stability analysis of time-varying NDSs [48, 49] and the ISS/iISS of deterministic special NDSs [50]. For the methodologies on the stability analysis of deterministic NDSs and NSDSs, they usually differ from each other [1, 13, 14]. Thus, the methodologies used in [48, 49, 50] cannot be easily generalized for the problems addressed in this paper. In this paper, one new methodology is proposed to analyze the stochastic ISS/iISS of neutral switched SDSs.

2. Notation. Throughout this paper, unless otherwise stated, let $|\cdot|$ be the Euclidean norm for a real vector in R^n or the trace norm for a real matrix of appropriate dimension. $\mathbb{N}=\{0,1,2,\ldots\}$. Denote by A^T the transpose of a matrix or a vector A. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ signify the maximum eigenvalue and the minimum eigenvalue of square matrix A, respectively. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq t_0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual condition. Let $\mathcal{B}(t)=(\mathcal{B}_1(t),\mathcal{B}_2(t),\ldots,\mathcal{B}_m(t))^T$ be an m-dimensional Brownian motion on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq t_0},\mathbb{P})$. For any $a,b\in R$ $(a<b)$, $\mathcal{PC}([a,b];R^n)$ denotes the family of all bounded piecewise continuous R^n-valued functions on $[a,b]$ with the norm $\|\varphi\|_{\mathcal{PC}}=\sup\{|\varphi(\theta)|:\ a\leq\theta\leq b\}$ for any $\varphi\in\mathcal{PC}([a,b];R^n)$. For $\tau>0$, $\mathcal{L}^p_{\mathcal{F}_t}([t_0-\tau,t_0];R^n)$ $(p\geq 2)$ denotes the family of all $\mathcal{F}_t$-measurable and $\mathcal{PC}([t_0-\tau,t_0];R^n)$-valued random processes $\xi=\{\xi(\theta):\ t_0-\tau\leq\theta\leq t_0\}$ with its norm $|||\xi|||=(\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\xi(\theta)|^p)^{\frac{1}{p}}$. $\mathbb{E}\{\cdot\}$ stands for the expectation operator. Let $H(a^-)$ be the left-hand limit of function $H(\cdot)$ at a, i.e., $H(a^-)=\lim_{u\to 0^-}H(a+u)$. $\mathcal{K}$ denotes the family of all strictly increasing continuous functions $\alpha(\cdot):\ [t_0,+\infty)\to[0,+\infty)$ with $\alpha(t_0)=0$. Let $\mathcal{K}_\infty$ denote the family of all unbounded $\mathcal{K}$-class functions. For any $a,b\in R$, $\mathcal{L}^1([a,b];[0,+\infty))=\{f(\cdot)|\int_a^b f(t)dt<+\infty\}$. For any real function $\varpi(t)$ defined on $[t_0,+\infty)$, for any $t>t_0$, $\|\varpi\|_{[t_0,t]}=\sup_{s\in[t_0,t]}|\varpi(s)|$. For any $a,b\in R$, $a\vee b=\max\{a,b\}$.

3. Problem statement and preliminaries. We consider the following neutral switched SDSs:

$$\begin{aligned}(3.1)\qquad & d[x(t)-\mathcal{D}(t,x(t-\tau(t)))]\\ &\quad=f_{\xi(t)}(t,x(t),x(t-\tau(t)),u(t))dt+g_{\xi(t)}(t,x(t),x(t-\tau(t)),u(t))d\mathcal{B}(t)\end{aligned}$$

on $t\geq t_0$, with the initial value $\{x(\theta):\ t_0-\tau\leq\theta\leq t_0\}=\varphi\in\mathcal{L}^p_{\mathcal{F}_{t_0}}([t_0-\tau,t_0];R^{n_x})$ $(\tau>0)$, where $x(t)=(x_1(t),\ x_2(t),\ \ldots,\ x_n(t))^T\in R^{n_x}$ is the state vector and $x(t-\tau(t))=(x_1(t-\tau(t)),\ x_2(t-\tau(t)),\ \ldots,\ x_n(t-\tau(t)))^T\in R^{n_x}$ is the delayed state vector. $u(\cdot)\in\mathcal{PC}([t_0,+\infty);R^{n_u})$ represents the input controller including the external input disturbance and the switching signal. The time-varying delay $\tau(\cdot)$: $[t_0,\infty)\to[0,\tau]$ is a bounded function. $\xi(\cdot):\ [t_0,+\infty)\to\Xi=\{1,2,\ldots,l\}$ is a switching signal, where l is a positive integer and Ξ is an index set. It is conventionally assumed that $\xi(\cdot)$ is right continuous and has left limits at the switching instants. For $\xi(t)$, the switching sequence is defined as $\{(\varsigma_0,t_0),(\varsigma_1,t_1),\ldots,(\varsigma_k,t_k),\ldots|\varsigma_k\in\Xi,\ k\in\mathbb{N}\}$, which denotes that the ς_kth subsystem is active when $t\in[t_k,t_{k+1})$. For any $\xi(t)=\varsigma_k$ $(t\in[t_k,t_{k+1});\ k\in\mathbb{N})$, $\mathcal{D}(\cdot,\cdot):\ [t_k,t_{k+1})\times R^{n_x}\to R^{n_x}$ is the neutral term, $f_{\varsigma_k}(\cdot,\cdot,\cdot,\cdot):\ [t_k,t_{k+1})\times R^{n_x}\times R^{n_x}\times R^{n_u}\to R^{n_x}$ is the drift coefficient vector, and $g_{\varsigma_k}(\cdot,\cdot,\cdot,\cdot):\ [t_k,t_{k+1})\times R^{n_x}\times R^{n_x}\times R^{n_u}\to R^{n_x\times m}$ is the diffusion coefficient matrix. Assume that $f_{\varsigma_k}(\cdot,\cdot,\cdot,\cdot)$ and $g_{\varsigma_k}(\cdot,\cdot,\cdot,\cdot)$ are continuous with respect to the second and third variables and uniformly continuous with respect to u. Let $x(t;t_0,\varphi,\xi(t_0))$ be the solution of neutral switched SDSs (3.1). For simplicity, $x(t)=x(t;t_0,\varphi,\xi(t_0))$. For the analysis for the well-posedness on neutral switched SDSs (3.1), it is usually assumed that $\mathcal{D}(t,0)=0$, $f_{\varsigma_k}(t,0,0,u)=0$ and $g_{\varsigma_k}(t,0,0,u)=0$ $(\varsigma_k\in\Xi)$, which admits $x(t)=0$ as the trivial solution. For simplicity, $\bar{x}(t)=x(t)-\mathcal{D}(t,x(t-\tau(t)))$. To guarantee the existence and uniqueness of neutral switched SDSs (3.1) without the input controller, two hypotheses are imposed as follows.

Hypothesis I. Assume that there exist some functions $K_{\varsigma_k}(\cdot)\in\mathcal{L}^1([t_0,T];[0,+\infty))$ $(\varsigma_k\in\Xi;\quad T>t_0)$ such that for all $t\in[t_0,T]$ and $x,\tilde{x},y,\tilde{y}\in R^{n_x}$, $|f_{\varsigma_k}(t,x,y,0)-f_{\varsigma_k}(t,\tilde{x},\tilde{y},0)|\vee|g_{\varsigma_k}(t,x,y,0)-g_{\varsigma_k}(t,\tilde{x},\tilde{y},0)|\leq K_{\varsigma_k}(t)(|x-\tilde{x}|+|y-\tilde{y}|)$, $f_{\varsigma_k}(t,0,0,0)=0$, and $g_{\varsigma_k}(t,0,0,0)=0$.

Hypothesis II. There exists a positive function $\kappa(t)$ satisfying, for any $x, y \in R^{n_x}$, $|\mathcal{D}(t,x) - \mathcal{D}(t,y)| \leq \kappa(t)|x-y|$ and $\kappa = \sup_{t \geq t_0}\{\kappa(t)\} \in (0,1)$.

Remark 1. Under Hypotheses I and II, similar to the proposed methodology in [1], the existence and uniqueness theorem of the solution for neutral switched SDSs (3.1) without the input controller can be proved on the interval $[t_0, T]$. In neutral switched SDSs (3.1), for simplicity, the neutral term $\mathcal{D}(t, x(t-\tau(t)))$ does not depend on the switching signal $\xi(t)$.

In this paper, the input controller $u(t)$ also has a switching signal, and two cases are considered:

(1) The synchronous switching signal. The switching signal available to the input controller is synchronized with the switching signal $\xi(t)$ of the plant. In this case, the input candidate controller is designed as $u(t) = h_{\xi(t)}(t, \bar{x}(t), \varpi(t))$, where $\varpi(\cdot) : [t_0, +\infty) \to R^{n_\varpi}$ denotes the external input disturbance.

(2) The asynchronous switching signal. The switching of the candidate controller does not coincide with the switching signal $\xi(t)$ of the plant. Usually, the controller's switching signal is a delayed version of the plant's switching one. In this case, the candidate controller is designed as $u(t) = h_{\xi(t-\tau_d)}(t, \bar{x}(t), \varpi(t))$, where τ_d is the switching delay with $0 < \tau_d < \inf_{k=0,1,2,\ldots}\{t_{k+1} - t_k\}$.

Along with these two cases, one hypothesis imposed on the input controller $u(t)$ in cases (1) and (2) is given as follows.

Hypothesis III. There exist some positive constants L_{ς_i} ($\varsigma_i \in \Xi$) satisfying, for any $x, y \in R^{n_x}$, $|h_{\varsigma_i}(t,x,0) - h_{\varsigma_i}(t,y,0)| \leq L_{\varsigma_i}|x-y|$ and $h_{\varsigma_i}(t,0,0) = 0$.

Remark 2. For NSDSs, in [3], by using the Lyapunov–Krasovskii functional approach, the state feedback controller was designed as $u(t) = h_{\rho(t)}(t, \bar{x}(t))$, in which $\rho(t)$ is the Markovian switching [3]. In [3], the external input disturbance and two switching signals in the candidate controller $u(t)$ were not considered.

Remark 3. In cases (1) and (2), $u(t)$ denotes the input controller which stabilizes systems (3.1). In case (1), the dynamics of the ς_kth subsystem and the ς_kth candidate controller are active in $[t_k, t_{k+1})$ $(k = 0, 1, 2, \ldots)$. The stability for switched systems with the synchronous switching has been analyzed in the available literature, for example, [21, 26] for the stochastic case and [22, 24] for the deterministic case. In a realistic model, there is a frequent phenomenon that the ς_kth candidate controller's action lags behind the dynamics of the ς_kth subsystem, which is as stated in case (2). Recently, the stability analysis for switched systems with the asynchronous switching has attracted considerable attention; see [27, 28, 29, 30, 31, 32, 33] and the references therein.

Denote by $\mathcal{C}^{1,2}([t_k, t_{k+1}) \times R^{n_x}; [0, +\infty))$ $(k \in \mathbb{N})$ the family of all continuous functions $V_{\varsigma_k}(t,x)$ defined on $[t_k, t_{k+1}) \times R^n$, satisfying that for any $\varsigma_k \in \Xi$, they are once continuously differentiable in t $(t \in (t_k, t_{k+1})$ $(k \in \mathbb{N}))$ and twice in x. When $\varsigma_k \in \Xi$, for a given Lyapunov function $V_{\varsigma_k}(t,x) \in \mathcal{C}^{1,2}([t_k, t_{k+1}) \times R^{n_x}; [0, +\infty))$, the Itô operator for neutral switched SDSs (3.1) is presented as $\mathcal{L}V_{\varsigma_k}(\cdot,\cdot,\cdot,\cdot) : [t_k, t_{k+1}) \times R^{n_x} \times R^{n_x} \times R^{n_u} \to R$ with $\mathcal{L}V_{\varsigma_k}(t,x,y,u) = V_{\varsigma_k,t}(t, x - \mathcal{D}(t,y)) + V^T_{\varsigma_k,x}(t, x - \mathcal{D}(t,y)) f_{\varsigma_k}(t,x,y,u) + \frac{1}{2}\mathrm{trace}[g^T_{\varsigma_k}(t,x,y,u) V_{\varsigma_k,xx}(t, x - \mathcal{D}(t,y)) g_{\varsigma_k}(t,x,y,u)]$, where $V_{\varsigma_k,t}\ (t, x - \mathcal{D}(t,y))$, $V_{\varsigma_k,x}(t, x - \mathcal{D}(t,y))$, and $V_{\varsigma_k,xx}(t, x - \mathcal{D}(t,y))$ are given in [21, 28].

In [36], the general definition of the ISS for nonlinear systems was given by using two class functions $\mathcal{K}_\infty$ and $\mathcal{KL}$, where $\mathcal{KL} = \{\psi(\cdot,\cdot) |\ \psi(\cdot,t) \in \mathcal{K}$ for any fixed $t \geq t_0$, and for any fixed $s \geq 0$, as $t \to +\infty$, $\psi(s,t)$ is strictly decreasing to zero$\}$. The definition of the stochastic ISS was presented in [40, 41]. Motivated by [40, 41], the definition about another stochastic version of the ISS is also given as follows.

DEFINITION 3.1. *System* (3.1) *is said to be*

(1) *input-to-state exponentially stable in pth moment (p-ISES) ($p \geq 2$) if there exist two constants $M > 0$, $\alpha > 0$, and $\chi(\cdot) \in \mathcal{K}_\infty$ such that for any $t \geq t_0$, $\mathbb{E}|x(t)|^p \leq M \sup_{\theta\in[t_0-\tau,t_0]} \mathbb{E}|\varphi(\theta)|^p e^{-\alpha(t-t_0)} + \chi(\|\varpi(t)\|_{[t_0,t]})$;*

(2) *integral input-to-state exponentially stable in pth moment (p-iISES) if there exist two constants $M > 0$, $\alpha > 0$, and $\chi(\cdot) \in \mathcal{K}_\infty$ such that for any $t \geq t_0$, $\mathbb{E}|x(t)|^p \leq M \sup_{\theta\in[t_0-\tau,t_0]} \mathbb{E}|\varphi(\theta)|^p e^{-\alpha(t-t_0)} + \int_{t_0}^{t} \chi(|\varpi(s)|)ds$;*

(3) *stochastic input-to-state exponentially stable (SISES) if for any $\epsilon > 0$, there exists two constants $M > 0$, $\alpha > 0$, and $\chi(\cdot) \in \mathcal{K}_\infty$ such that for any $t \geq t_0$, $\mathbb{P}\{|x(t)| \leq M \sup_{\theta\in[t_0-\tau,t_0]} \mathbb{E}|\varphi(\theta)|^p e^{-\alpha(t-t_0)} + \chi(\|\varpi(t)\|_{[t_0,t]})\} \geq 1 - \epsilon$;*

(4) *$e^{\alpha' t}$ ($\alpha' > 0$)-weighted input-to-state exponentially stable in pth moment ($e^{\alpha' t}$-p-ISES) if there exist a positive constant M and $\chi(\cdot) \in \mathcal{K}_\infty$ satisfying $e^{\alpha'(t-t_0)}\mathbb{E}|x(t)|^p \leq M \sup_{\theta\in[t_0-\tau,t_0]} \mathbb{E}|\varphi(\theta)|^p + \sup_{s\in[t_0,t]}\{e^{\alpha'(s-t_0)}\chi(|\varpi(s)|)\}$ for any $t \geq t_0$.*

Remark 4. In Definition 3.1, $\psi(s,t) = se^{-\alpha t} \in \mathcal{KL}$. The definition for the ISS with $\psi(s,t) = se^{-\alpha t}$ was proposed in [38]. Thus, the definition of the ISS given in [38] is generalized to the stochastic version for neutral switched SDSs in Definition 3.1.

Remark 5. The ISS of special deterministic NDSs has been investigated in [50] and the references therein. As we know, while analyzing the stability of NSDSs, the state term and the neutral term are presented in the Lyapunov–Krasovskii function as one term $x(t) - \mathcal{D}(t, x(t-\tau(t)))$; see [1, 2, 3, 4, 5, 6, 8, 9, 21, 28] and the references therein. Thus, those useful techniques widely used in deterministic linear/nonlinear systems cannot be easily employed to analyze the ISS/iISS of NSDSs.

DEFINITION 3.2 (see [22]). *For any $t \in [t_0, T]$, denote by $N_{\xi(t)}(T,t)$ the switching numbers of $\xi(t)$ over the interval (t,T). It is said that $\xi(t)$ has an ADT $\mathcal{T}_a$ if there exist two positive constants N_0 and $\mathcal{T}_a > 0$ such that $N_{\xi(t)}(T,t) \leq N_0 + \frac{T-t}{\mathcal{T}_a}$. Here, N_0 is called a chattering bound.*

DEFINITION 3.3 (see [22]). *For a switching scheme $\xi(t)$ and any $t \in [t_0,T]$, let $N_{\xi(t),\varsigma_i}(T,t)$ ($\varsigma_i \in \Xi$) denote the switching numbers that the ς_ith subsystem is activated over the interval (t,T), and $T_{\varsigma_i}(T,t)$ ($\varsigma_i \in \Xi$) is the total running time of the ς_ith subsystems over the interval (t,T). It is said that $\xi(t)$ has an MDADT $\mathcal{T}_{a,\varsigma_i}$ if there exist some positive constants N_{0,ς_i} and $\mathcal{T}_{a,\varsigma_i}$ such that, for any $t_0 \leq t \leq T$, $N_{\xi(t),\varsigma_i}(T,t) \leq N_{0,\varsigma_i} + \frac{T_{\varsigma_i}(T,t)}{\mathcal{T}_{a,\varsigma_i}}$, where N_{0,ς_i} represents the mode-dependent chattering bound. In addition, $N_{\xi(t)}(T,t) = \sum_{\varsigma_i\in\Xi} N_{\xi(t),\varsigma_i}(T,t)$ and $T - t = \sum_{\varsigma_i\in\Xi} T_{\varsigma_i}(T,t)$. $\mathcal{S}_{ave}[\mathcal{T}_{a,\varsigma_i}, N_{0,\varsigma_i}]$ denotes the family of switching signals with the ς_ith mode ADT $\mathcal{T}_{a,\varsigma_i}$ and the chattering bound N_{0,ς_i}.*

LEMMA 3.4 (see [1]). *For $x, y \in R^n$, $p \geq 2$, $\kappa \in (0,1)$, then $|x|^p \leq \kappa|y|^p + \frac{|x-\mathcal{D}(t,y)|^p}{(1-\kappa)^{p-1}}$.*

4. Main results.

In this section, we analyze the p-ISES, p-iISES, SISES, and $e^{\alpha' t}$ ($\alpha' > 0$)-p-ISES ($p \geq 2$) of neutral switched SDSs (3.1) under the synchronous switching signal and the asynchronous switching signal, respectively. In order to perform these analyses, we are in need of obtaining the following lemma.

LEMMA 4.1. *Let $\tilde{\mu}_1 \in (0,1)$, $\tilde{\mu}_2 > 0$, and $\tilde{\lambda}_i > 0$ ($i = 0,1,2$). Let $\tau(\cdot)$: $[t_0,+\infty) \to [0,\tau]$, and let $y(\cdot)$ be a nonnegative function from $[t_0-\tau,+\infty)$ to R, $\varpi(\cdot)$: $[t_0,+\infty) \to R^{n_\varpi}$ and a $\mathcal{K}_\infty$-class function $\chi(\cdot)$: $[0,+\infty) \to (0,+\infty)$. Assume that there exist two integrable functions $\alpha_0(\cdot)$: $[t_0,+\infty) \to R$ and $\alpha_1(\cdot)$: $[t_0,+\infty) \to (0,+\infty)$, constants $\vartheta > 0$ and $c \in R$ such that for any $t > s \geq t_0$,*

$$\int_s^t [(1-\tilde{\mu}_1)\alpha_0(\theta)+\vartheta+\alpha_1(\theta)]d\theta \le c. \tag{4.1}$$

Furthermore, if $\tilde{\mu}_2 e^{\frac{c}{1-\tilde{\mu}_1}} \in (0,1)$, *and two inequalities* $y(t) \le \tilde{\lambda}_0 e^{\int_{t_0}^t \alpha_0(s)ds} + \tilde{\mu}_1 y(t-\tau(t)) + \tilde{\mu}_2 \int_{t_0}^t e^{\int_s^t \alpha_0(\theta)d\theta}\alpha_1(s)\sup_{\theta\in[-\tau,0]} y(s+\theta)ds + \tilde{\lambda}_1 \int_{t_0}^t e^{\int_s^t \alpha_0(\theta)d\theta}\chi(|\varpi(s)|)ds$, $t \ge t_0$ *and* $y(t) \le \tilde{\lambda}_2$ $(t \in [t_0-\tau, t_0])$ *are satisfied, then* $y(t) \le M_0 e^{-\alpha^*(t-t_0)} + M_1 \int_{t_0}^t e^{-\alpha^*(t-s)}\chi(|\varpi(s)|)ds$ *is satisfied for any* $t \in [t_0-\tau, +\infty)$, *where* $\alpha^* \in (0, \min\{\alpha_1, \alpha_2\})$, $\alpha_1 = \frac{1}{\tau}\log[1/(\tilde{\mu}_1+\tilde{\mu}_2(1-\tilde{\mu}_1)e^{\frac{c}{1-\tilde{\mu}_1}})] > 0$, $\alpha_2 = \vartheta/(1-\tilde{\mu}_1)$, $M_0 = \max\{\tilde{\lambda}_2 e^{\alpha^*\tau}, \tilde{\lambda}_0/(\tilde{\mu}_2(1-\tilde{\mu}_1)e^{\alpha^*\tau})\}$, *and* $M_1 \ge \tilde{\lambda}_1/(1-[\tilde{\mu}_1+(1-\tilde{\mu}_1)\tilde{\mu}_2 e^{\frac{c}{1-\tilde{\mu}_1}}]e^{\alpha^*\tau})$.

Proof. Since $\tilde{\mu}_1 \in (0,1)$ *and* $\tilde{\mu}_2 e^{\frac{c}{1-\tilde{\mu}_1}} \in (0,1)$, *it follows that* $\tilde{\mu}_1+\tilde{\mu}_2(1-\tilde{\mu}_1)e^{\frac{c}{1-\tilde{\mu}_1}} \in (0,1)$. *Furthermore, we have* $\alpha_1 > 0$. *Let* $\alpha^* \in (0, \min\{\alpha_1,\alpha_2\})$; *we have* $[\tilde{\mu}_1+\tilde{\mu}_2(1-\tilde{\mu}_1)e^{\frac{c}{1-\tilde{\mu}_1}}]e^{\alpha^*\tau} < 1$ *and* $\alpha^* \le \frac{\vartheta}{1-\tilde{\mu}_1}$. *For any* $\varepsilon > 0$, *define* $M_{0,\varepsilon} := \max\{(\tilde{\lambda}_2+\varepsilon)e^{\alpha^*\tau}, (\tilde{\lambda}_1+\varepsilon)/(\tilde{\mu}_2(1-\tilde{\mu}_1)e^{\alpha^*\tau})\} > 0$. *In order to obtain the desired result, it is sufficiently shown that for any* $t \ge t_0-\tau$,

$$y(t) \le M_{0,\varepsilon}e^{-\alpha^*(t-t_0)} + M_1\int_{t_0}^t e^{-\alpha^*(t-s)}\chi(|\varpi(s)|)ds. \tag{4.2}$$

For any $t \in [t_0-\tau, t_0]$, $y(t) \le \tilde{\lambda}_2 < M_{0,\varepsilon}$ *is satisfied. Hence,* (4.2) *is satisfied for all* $t \in [t_0-\tau, t_0]$. *If* (4.2) *does not hold for any* $t > t_0$, *then there exists* $t_1^* > t_0$ *such that* $t_1^* = \inf\{t > t_0 : y(t) > M_{0,\varepsilon}e^{-\alpha^*(t-t_0)} + M_1\int_{t_0}^t e^{-\alpha^*(t-s)}\chi(|\varpi(s)|)ds\}$. *Thus, we have* $y(t) \le M_{0,\varepsilon}e^{-\alpha^*(t-t_0)} + M_1\int_{t_0}^t e^{-\alpha^*(t-s)}\chi(|\varpi(s)|)ds$, for all $t \in [t_0-\tau, t_1^*)$, *and* $y(t_1^*) = M_{0,\varepsilon}e^{-\alpha^*(t_1^*-t_0)} + M_1\int_{t_0}^{t_1^*} e^{-\alpha^*(t_1^*-s)}\chi(|\varpi(s)|)ds$.

However, we have

$$\begin{aligned} y(t_1^*) \le{}& \tilde{\lambda}_0 e^{\frac{c}{1-\tilde{\mu}_1}-\frac{1}{1-\tilde{\mu}_1}\int_{t_0}^{t_1^*}\alpha_1(s)ds-\frac{\vartheta}{1-\tilde{\mu}_1}(t_1^*-t_0)} + \tilde{\mu}_1 e^{\alpha^*\tau}M_{0,\varepsilon}e^{-\alpha^*(t_1^*-t_0)} \\ &+ \tilde{\mu}_2 e^{\alpha^*\tau}M_{0,\varepsilon}(1-\tilde{\mu}_1)e^{\frac{c}{1-\tilde{\mu}_1}-\frac{1}{1-\tilde{\mu}_1}\int_{t_0}^{t_1^*}\alpha_1(s)ds-\frac{\vartheta}{1-\tilde{\mu}_1}(t_1^*-t_0)} \\ &\times \int_{t_0}^{t_1^*} e^{\frac{1}{1-\tilde{\mu}_1}\int_{t_0}^s \alpha_1(\theta)d\theta+(\frac{\vartheta}{1-\tilde{\mu}_1}-\alpha^*)(s-t_0)}\left[\frac{\alpha_1(s)}{1-\tilde{\mu}_1}+\frac{\vartheta}{1-\tilde{\mu}_1}-\alpha^*\right]ds \\ &+ \tilde{\mu}_2 M_1 e^{\alpha^*\tau}\int_{t_0}^{t_1^*} e^{\int_s^{t_1^*}\alpha_0(\theta)d\theta}\alpha_1(s)\int_{t_0}^{s-\tau(s)} e^{-\alpha^*(s-\theta)}\chi(|\varpi(\theta)|)d\theta ds \\ &+ \tilde{\mu}_1 M_1 e^{\alpha^*\tau}\int_{t_0}^{t_1^*-\tau(t_1^*)} e^{-\alpha^*(t_1^*-s)}\chi(|\varpi(s)|)ds + \tilde{\lambda}_1\int_{t_0}^{t_1^*} e^{\int_s^{t_1^*}\alpha_0(\theta)d\theta}\chi(|\varpi(s)|)ds. \end{aligned} \tag{4.3}$$

It is derived, from (4.1), *that*

$$\begin{aligned} &\tilde{\mu}_2 M_1 e^{\alpha^*\tau}\int_{t_0}^{t_1^*} e^{\int_s^{t_1^*}\alpha_0(\theta)d\theta}\alpha_1(s)\int_{t_0}^{s-\tau(s)} e^{-\alpha^*(s-\theta)}\chi(|\varpi(\theta)|)d\theta ds \\ &\quad\le \tilde{\mu}_2 M_1 e^{\alpha^*\tau}e^{\frac{c}{1-\tilde{\mu}_1}-\frac{1}{1-\tilde{\mu}_1}\int_{t_0}^{t_1^*}\alpha_1(s)ds-\frac{\vartheta}{1-\tilde{\mu}_1}(t_1^*-t_0)}\int_{t_0}^{t_1^*} e^{\frac{1}{1-\tilde{\mu}_1}\int_{t_0}^s\alpha_1(\theta)d\theta+\frac{\vartheta}{1-\tilde{\mu}_1}(s-t_0)} \\ &\quad\times \alpha_1(s)\int_{t_0}^{s-\tau(s)} e^{-\alpha^*(s-\theta)}\chi(|\varpi(\theta)|)d\theta ds \\ &\quad\le \tilde{\mu}_2 M_1 e^{\alpha^*\tau}(1-\tilde{\mu}_1)e^{\frac{c}{1-\tilde{\mu}_1}}\int_{t_0}^{t_1^*} e^{-\alpha^*(t_1^*-s)}\chi(|\varpi(s)|)ds. \end{aligned} \tag{4.4}$$

Substituting (4.4) *into* (4.3) *yields that* $y(t_1^*) \le [\tilde{\lambda}_0 - \tilde{\mu}_2(1-\tilde{\mu}_1)e^{\alpha^*\tau}M_{0,\varepsilon}]e^{\frac{c}{1-\tilde{\mu}_1}} e^{-\frac{1}{1-\tilde{\mu}_1}\int_{t_0}^{t_1^*}\alpha_1(s)ds - \frac{\vartheta}{1-\tilde{\mu}_1}(t_1^*-t_0)} + [\tilde{\mu}_1 + (1-\tilde{\mu}_1)\tilde{\mu}_2 e^{\frac{c}{1-\tilde{\mu}_1}}]e^{\alpha^*\tau}M_{0,\varepsilon}e^{-\alpha^*(t_1-t_0)} + [\tilde{\mu}_2 M_1 e^{\alpha^*\tau} (1-\tilde{\mu}_1)e^{\frac{c}{1-\tilde{\mu}_1}} + \tilde{\mu}_1 M_1 e^{\alpha^*\tau} + \tilde{\lambda}_1]\int_{t_0}^{t_1^*} e^{-\alpha^*(t_1^*-s)}\chi(|\varpi(s)|)ds$. *The definition of* $M_{0,\varepsilon}$ *gives that* $\tilde{\lambda}_0 - \tilde{\mu}_2(1-\tilde{\mu}_1)e^{\alpha^*\tau}M_{0,\varepsilon} \le \tilde{\lambda}_0 - \tilde{\mu}_2(1-\tilde{\mu}_1)e^{\alpha^*\tau}\frac{\tilde{\lambda}_0+\varepsilon}{\tilde{\mu}_2(1-\tilde{\mu}_1)e^{\alpha^*\tau}} < 0$. *Furthermore, it follows that* $y(t_1^*) < M_{0,\varepsilon}e^{-\alpha^*(t_1^*-t_0)} + M_1\int_{t_0}^{t_1^*} e^{-\alpha^*(t_1^*-s)}\chi(|\varpi(s)|)ds$, *which has a contradiction. Thus, inequality* (4.2) *holds for all* $t \ge t_0$. *When* $\varepsilon \to 0^+$ *in* (4.2), *the desired result is derived.*

Remark 6. Note that in (4.1), the time-varying coefficient $\alpha_0(\cdot):\ [t_0,+\infty) \to R$, which means that $\alpha_0(t)$ takes the positive value on some intervals and the nonpositive value on other intervals. $\varpi(\cdot)$ is the external input perturbation. Several delay integral inequalities have been used in our previous works with $\varpi(\cdot) = 0$, for example, $\alpha_0(t) \equiv \alpha_0 < 0$ for all $t \ge t_0$ in [8], and $\alpha_0(t) < 0$ for all $t \ge t_0$ in [9]. It is seen that such coefficients in [8, 9] have the definite sign. However, in Lemma 4.1, $\alpha_0(t)$ has the indefinite sign on $[t_0,+\infty)$. In addition, it is found from (4.1), Lemma 1 in [8], and Lemma 2.2 in [9] that Lemma 4.1 contains two special cases as Lemma 1 in [8] and Lemma 2.2 in [9].

When $\tilde{\mu}_1$ is not active in Lemma 4.1, we have the following.

COROLLARY 4.2. *Let* $\tilde{\mu} > 0$ *and* $\tilde{\lambda}_i > 0$ $(i = 0,1,2)$. *Let* $\tau(\cdot):\ [t_0,+\infty) \to [0,\tau]$, $y(\cdot)$ *be a nonnegative function on* $[t_0-\tau,+\infty)$, *and a* $\mathcal{K}_\infty$ *function* $\chi(\cdot):\ [0,+\infty) \to (0,+\infty)$. *Assume that there exist two integrable functions* $\alpha_0(\cdot):\ [t_0,+\infty) \to R$ *and* $\alpha_1(\cdot):\ [t_0,+\infty) \to (0,+\infty)$ *and constants* $\vartheta > 0$ *and* $c \in R$ *satisfying, for any* $t > s \ge t_0$, $\int_s^t[\alpha_0(\theta)+\vartheta+\alpha_1(\theta)]d\theta \le c$. *Furthermore, if* $\tilde{\mu}e^c \in (0,1)$, *and two inequalities* $y(t) \le \tilde{\lambda}_0 e^{\int_{t_0}^t \alpha_0(s)ds} + \tilde{\mu}\int_{t_0}^t e^{\int_s^t \alpha_0(\theta)d\theta}\alpha_1(s)\sup_{\theta\in[-\tau,0]} y(s+\theta)ds + \tilde{\lambda}_1\int_{t_0}^t e^{\int_s^t \alpha_0(\theta)d\theta}\chi(|\varpi(s)|)ds$ $(t \ge t_0)$ *and* $y(t) \le \tilde{\lambda}_2$ $(t \in [t_0-\tau,t_0])$ *hold, then* $y(t) \le M_0 e^{-\alpha^*(t-t_0)} + M_1\int_{t_0}^t e^{-\alpha^*(t-s)}\ \chi(|\varpi(s)|)ds$ *is satisfied for any* $t \in [t_0-\tau,+\infty)$, *where* $\alpha^* \in (0,\min\{\alpha_1,\alpha_2\})$, $\alpha_1 = \frac{1}{\tau}\log(1/(\tilde{\mu}e^c)) > 0$, $\alpha_2 = \vartheta$, $M_0 = \max\{\tilde{\lambda}_2 e^{\alpha^*\tau}, \tilde{\lambda}_0/(\tilde{\mu}e^{\alpha^*\tau})\}$, *and* $M_1 \ge \tilde{\lambda}_1/(1-\tilde{\mu}_2 e^c e^{\alpha^*\tau}) > 0$.

4.1. Input-to-state stochastic stability of neutral switched SDSs (3.1) **with the synchronous switching.** In this subsection, the p-ISES, p-iISES, SISES, and $e^{\alpha' t}$ $(\alpha' > 0)$-p-ISES $(p \ge 2)$ of neutral switched SDSs (3.1) under the synchronous switching are discussed, and some sufficient conditions are obtained based on multiple Lyapunov–Krasovskii functions, the MDADT, and Lemma 4.1.

THEOREM 4.3. *Suppose that there exist some Lyapunov–Krasovskii functions* $V_{\varsigma_k}(\cdot,\cdot) \in \mathcal{C}^{1,2}([t_k,t_{k+1}) \times R^{n_x};[0,+\infty))$ $(k \in \mathbb{N})$, *two constants* $c_1 > 0$, $c_2 > 0$, *some integrable functions* $\lambda_{0,\varsigma_k}(\cdot):\ [t_0,+\infty) \to R$, $\lambda_{1,\varsigma_k}(\cdot), \lambda_{2,\varsigma_k}(\cdot):\ [t_0,+\infty) \to [0,+\infty)$, *and some* $\mathcal{K}_\infty$*-class functions* $\chi_{\varsigma_k}(\cdot):\ [0,+\infty) \to (0,+\infty)$ $(\varsigma_k \in \Xi)$ *such that*

(C_1) *for any* $x,y \in R^{n_x}$, $t \ge t_0$, *and* $p \ge 2$, $c_1|x-\mathcal{D}(t,y)|^p \le V_{\xi(t)}(t,x-\mathcal{D}(t,y)) \le c_2|x-\mathcal{D}(t,y)|^p$;

(C_2) *for any* $x,y \in R^{n_x}$, $\varpi \in R^{n_\varpi}$, $t \in [t_k,t_{k+1})$ $(k \in \mathbb{N})$, *and* $p \ge 2$, $\mathcal{L}V_{\varsigma_k}(t,x,y,\varpi) \le \lambda_{0,\varsigma_k}(t)V_{\varsigma_k}(t,x-\mathcal{D}(t,y)) + \lambda_{1,\varsigma_k}(t)|x|^p + \lambda_{2,\varsigma_k}(t)|y|^p + \chi_{\varsigma_k}(|\varpi|)$;

(C_3) *for any* $x,y \in R^{n_x}$, $\varsigma_i,\varsigma_j \in \Xi$ $(i \ne j)$, *and* $\mu_{\varsigma_i} > 1$, $V_{\varsigma_i}(t,x-\mathcal{D}(t,y)) \le \mu_{\varsigma_i}V_{\varsigma_j}(t,x-\mathcal{D}(t,y))$;

(C_4) *there exist some constants* $\gamma > 0$, $\beta > 0$, *and* $c' \in R$ *such that* $\Delta_1 \in (0,1)$, *where* $\Delta_1 = \beta e^{\sum_{i=1}^l N_{0,i}\log(\mu_i)}e^{\frac{c'}{1-\kappa}}/(c_1(1-\kappa)^{p-1})$, *and for any* $t_0 \le s < t$, $\int_s^t[(1-\kappa)\lambda_{0,\xi(\theta)}(\theta) + \gamma + (\lambda_{1,\xi(\theta)}(\theta)+\lambda_{2,\xi(\theta)}(\theta))/\beta]d\theta \le c'$.

Then, the p-ISES, p-iISES, SISES, and $e^{\alpha t}$-p-ISES ($\alpha > 0$) of neutral switched SDSs (3.1) *with the synchronous switching are guaranteed over $\mathcal{S}_{ave}[\mathcal{T}_{a,i}, N_{0,i}]$, respectively, where the ith mode ADT $\mathcal{T}_{a,i} > \mathcal{T}^*_{a,i} = \frac{(1-\kappa)\log\mu_i}{\gamma}$ ($i \in \Xi$).*

Proof. The proof of this theorem is divided into three steps as follows:

Step 1. For any $t \in [t_0, t_1)$, defining a Lyapunov function $e^{-\int_{t_0}^t \lambda_{0,\varsigma_0}(s)ds} V_{\varsigma_0}(t, \bar{x}(t))$, using the Itô formula, and taking the mathematical expectation in sequence, from (C_2), yields that $\mathbb{E}\{V_{\varsigma_0}(t,\bar{x}(t))\} \leq \mathbb{E}\{V_{\varsigma_0}(t_0,\bar{x}(t_0))\}e^{\int_{t_0}^t \lambda_{0,\varsigma_0}(s)ds} + \int_{t_0}^t e^{\int_s^t \lambda_{0,\varsigma_0}(\theta)d\theta} \lambda_{1,\varsigma_0}(s)\ \mathbb{E}\{|x(s)|^p\}ds + \int_{t_0}^t e^{\int_s^t \lambda_{0,\varsigma_0}(\theta)d\theta}\lambda_{2,\varsigma_0}(s)\mathbb{E}\{|x(s-\tau(s))|^p\}ds + \int_{t_0}^t e^{\int_s^t \lambda_{0,\varsigma_0}(\theta)d\theta} \chi_{\varsigma_0}(|\varpi(s)|)ds$. When $t = t_1^-$, (C_3) yields that $\mathbb{E}\{V_{\varsigma_1}(t_1,\bar{x}(t_1))\} \leq \mu_{\varsigma_1}\mathbb{E}\{V_{\varsigma_0}(t_0,\ \bar{x}(t_0))\} e^{\int_{t_0}^{t_1} \lambda_{0,\varsigma_0}(s)ds} + \mu_{\varsigma_1}\int_{t_0}^{t_1} e^{\int_s^{t_1} \lambda_{0,\varsigma_0}(\theta)d\theta}\lambda_{1,\varsigma_0}(s)\mathbb{E}\{|x(s)|^p\}ds + \mu_{\varsigma_1}\int_{t_0}^{t_1} e^{\int_s^{t_1} \lambda_{0,\varsigma_0}(\theta)d\theta}\ \lambda_{2,\varsigma_0}(s) \mathbb{E}\{|x(s-\tau(s))|^p\}ds + \mu_{\varsigma_1}\int_{t_0}^{t_1} e^{\int_s^{t_1} \lambda_{0,\varsigma_0}(\theta)d\theta}\chi_{\varsigma_0}(|\varpi(s)|)ds$.

By using a similar derivation, when $t \in [t_1, t_2)$, it gives that $\mathbb{E}\{V_{\varsigma_1}(t,\bar{x}(t))\} \leq \mu_{\varsigma_1}\mathbb{E}\{V_{\varsigma_0}(t_0,\bar{x}(t_0))\}e^{\int_{t_0}^{t_1} \lambda_{0,\varsigma_0}(s)ds}e^{\int_{t_1}^t \lambda_{0,\varsigma_1}(s)ds} + \mu_{\varsigma_1}\int_{t_0}^{t_1} e^{\int_s^{t_1} \lambda_{0,\varsigma_0}(\theta)d\theta}\lambda_{1,\varsigma_0}(s)\mathbb{E}\{|x(s)|^p\} ds\ e^{\int_{t_1}^t \lambda_{0,\varsigma_1}(s)ds} + \mu_{\varsigma_1}\int_{t_0}^{t_1} e^{\int_s^{t_1} \lambda_{0,\varsigma_0}(\theta)d\theta}\lambda_{2,\varsigma_0}(s)\mathbb{E}\{|x(s-\tau(s))|^p\}ds e^{\int_{t_1}^t \lambda_{0,\varsigma_1}(s)ds} + \mu_{\varsigma_1}\int_{t_0}^{t_1} e^{\int_s^{t_1} \lambda_{0,\varsigma_0}(\theta)d\theta}\chi_{\varsigma_0}(|\varpi(s)|)ds e^{\int_{t_1}^t \lambda_{0,\varsigma_1}(s)ds} + \int_{t_1}^t e^{\int_s^t \lambda_{0,\varsigma_1}(\theta)d\theta}\lambda_{1,\varsigma_1}(s)\mathbb{E}\{|x(s)|^p\}ds + \int_{t_1}^t e^{\int_s^t \lambda_{0,\varsigma_1}(\theta)d\theta}\lambda_{2,\varsigma_1}(s)\mathbb{E}\{|x(s-\tau(s))|^p\}ds + \int_{t_1}^t e^{\int_s^t \lambda_{0,\varsigma_1}(\theta)d\theta}\chi_{\varsigma_1}(|\varpi(s)|)ds$.

By repeating the process above, when $t \in [t_k, t_{k+1})$, it follows that $\mathbb{E}\{V_{\varsigma_k}(t,\bar{x}(t))\} \leq \prod_{i=1}^k \mu_{\varsigma_i}\mathbb{E}\{V_{\varsigma_0}(t_0,\bar{x}(t_0))\}e^{\int_{t_0}^t \lambda_{0,\xi(s)}(s)ds} + \sum_{i=1}^k \prod_{j=i}^k \mu_{\varsigma_j}\int_{t_{j-1}}^{t_j} e^{\int_s^{t_j} \lambda_{0,\xi(\theta)}(\theta)d\theta}\lambda_{1,\xi(s)}(s) \mathbb{E}|x(s)|^p ds e^{\int_{t_k}^t \lambda_{0,\varsigma_k}(s)ds} + \sum_{i=1}^k \prod_{j=i}^k \mu_{\varsigma_j}\int_{t_{j-1}}^{t_j} e^{\int_s^t \lambda_{0,\xi(\theta)}(\theta)d\theta}\lambda_{2,\xi(s)}(s)\mathbb{E}|x(s-\tau(s))|^p ds + \sum_{i=1}^k \prod_{j=i}^k \mu_{\varsigma_j}\int_{t_{j-1}}^{t_j} e^{\int_s^{t_j} \lambda_{0,\xi(\theta)}(\theta)d\theta}\chi_{\xi(s)}(s)(|\varpi(s)|)ds e^{\int_{t_k}^t \lambda_{0,\varsigma_k}(s)ds} + \int_{t_k}^t e^{\int_s^t \lambda_{0,\varsigma_k}(\theta)d\theta} \lambda_{1,\varsigma_k}(s)\ \mathbb{E}\{|x(s)|^p\}ds + \int_{t_k}^t e^{\int_s^t \lambda_{0,\varsigma_k}(\theta)d\theta}\lambda_{2,\varsigma_k}(s)\mathbb{E}\{|x(s-\tau(s))|^p\}ds + \int_{t_k}^t e^{\int_s^t \lambda_{0,\varsigma_k}(\theta)d\theta} \chi_{\varsigma_k}(|\varpi(s)|)ds$, where $\prod_{i=1}^0 \mu_{\varsigma_i} \equiv 1$ and $\sum_{i=1}^0(\cdot) \equiv 0$.

Step 2. For any $t \in [t_0, +\infty)$, it is computed that

$$
\begin{aligned}
&\mathbb{E}\{V_{\xi(t)}(t,\bar{x}(t))\} \\
&\quad \leq \prod_{i=1}^l \mu_i^{N_{\xi(t)}(t,t_0)}\mathbb{E}\{V_{\varsigma_0}(t_0,\bar{x}(t_0))\}e^{\int_{t_0}^t \lambda_{0,\xi(s)}(s)ds} + \sum_{k=1}^{N_{\xi(t)}(t,t_0)} \prod_{i=1}^l \mu_i^{N_{\xi(t),i}(t,t_k)} \\
&\qquad \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} \lambda_{0,\xi(\theta)}(\theta)d\theta}\lambda_{1,\xi(s)}(s)\mathbb{E}|x(s)|^p ds e^{\int_{t_k}^t \lambda_{0,\xi(s)}(s)ds} + \sum_{k=1}^{N_{\xi(t)}(t,t_0)} \prod_{i=1}^l \mu_i^{N_{\xi(t),i}(t,t_k)} \\
&\qquad \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} \lambda_{0,\xi(\theta)}(\theta)d\theta}\lambda_{2,\xi(s)}(s)\mathbb{E}|x(s-\tau(s))|^p ds e^{\int_{t_k}^t \lambda_{0,\xi(s)}(s)ds} \\
&\qquad + \sum_{k=1}^{N_{\xi(t)}(t,t_0)} \prod_{i=1}^l \mu_i^{N_{\xi(t),i}(t,t_k)} \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} \lambda_{0,\xi(\theta)}(\theta)d\theta}\chi_{\xi(s)}(|\varpi(s)|)ds e^{\int_{t_k}^t \lambda_{0,\xi(s)}(s)ds} \\
&\qquad + \int_{t_{N_{\xi(t)}(t,t_0)}}^t e^{\int_s^t \lambda_{0,\xi(\theta)}(\theta)d\theta}\lambda_{1,\xi(s)}(s)\mathbb{E}\{|x(s)|^p\}ds \\
&\qquad + \int_{t_{N_{\xi(t)}(t,t_0)}}^t e^{\int_s^t \lambda_{0,\xi(\theta)}(\theta)d\theta}\lambda_{2,\xi(s)}(s)\mathbb{E}\{|x(s-\tau(s))|^p\}ds \\
&\qquad + \int_{t_{N_{\xi(t)}(t,t_0)}}^t e^{\int_s^t \lambda_{0,\xi(\theta)}(\theta)d\theta}\chi_{\xi(s)}(|\varpi(s)|)ds = J_1 + J_2 + J_3 + J_4,
\end{aligned} \tag{4.5}
$$

where

$$J_1 \le \mathbb{E}\{V_{\varsigma_0}(t_0,\bar{x}(t_0))\}e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)}e^{\sum_{i=1}^{l}\frac{\log(\mu_i)}{\mathcal{T}_{a,i}}T_i(t,t_0)+\int_{t_0}^{t}\lambda_{0,\xi(s)}(s)ds}, \tag{4.6}$$

$$J_2 \le e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)}\int_{t_0}^{t} e^{\sum_{i=1}^{l}\frac{\log(\mu_i)}{\mathcal{T}_{a,i}}T_i(t,s)+\int_s^t\lambda_{0,\xi(\theta)}(\theta)d\theta}\lambda_{1,\xi(s)}(s)\mathbb{E}|x(s)|^p ds, \tag{4.7}$$

$$\begin{aligned} J_3 \le{}& e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)}\int_{t_0}^{t} e^{\sum_{i=1}^{l}\frac{\log(\mu_i)}{\mathcal{T}_{a,i}}T_i(t,s)+\int_s^t\lambda_{0,\xi(\theta)}(\theta)d\theta} \\ &\times \lambda_{2,\xi(s)}(s)\mathbb{E}|x(s-\tau(s))|^p ds, \end{aligned} \tag{4.8}$$

and

$$J_4 \le e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)}\int_{t_0}^{t} e^{\sum_{i=1}^{l}\frac{\log(\mu_i\varsigma_i)}{\mathcal{T}_{a,i}}T_i(t,s)+\int_s^t\lambda_{0,\xi(\theta)}(\theta)d\theta}\chi_{\xi(s)}(|\varpi(s)|)ds, \tag{4.9}$$

where Definition 3.3 has been used.

Substituting (4.6), (4.7), (4.8), and (4.9) into (4.5) yields that for any $t\in[t_0,+\infty)$,

$$\begin{aligned} &\mathbb{E}\{V_{\xi(t)}(t,\bar{x}(t))\} \\ &\le \mathbb{E}\{V_{\varsigma_0}(t_0,\bar{x}(t_0))\}e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)}e^{\sum_{i=1}^{l}\frac{\log(\mu_i)}{\mathcal{T}_{a,i}}T_i(t,t_0)+\int_{t_0}^{t}\lambda_{0,\xi(s)}(s)ds} \\ &\quad + e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)}\int_{t_0}^{t} e^{\sum_{i=1}^{l}\frac{\log(\mu_i)}{\mathcal{T}_{a,i}}T_i(t,s)+\int_s^t\lambda_{0,\xi(\theta)}(\theta)d\theta}\lambda_{1,\xi(s)}(s)\mathbb{E}|x(s)|^p ds \\ &\quad + e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)}\int_{t_0}^{t} e^{\sum_{i=1}^{l}\frac{\log(\mu_i)}{\mathcal{T}_{a,i}}T_i(t,s)+\int_s^t\lambda_{0,\xi(\theta)}(\theta)d\theta}\lambda_{2,\xi(s)}(s)\mathbb{E}|x(s-\tau(s))|^p ds \\ &\quad + e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)}\int_{t_0}^{t} e^{\sum_{i=1}^{l}\frac{\log(\mu_i)}{\mathcal{T}_{a,i}}T_i(t,s)+\int_s^t\lambda_{0,\xi(\theta)}(\theta)d\theta}\chi_{\xi(s)}(|\varpi(s)|)ds. \end{aligned} \tag{4.10}$$

Step 3. By using condition (C_1) and Lemma 3.4 in turn, it follows from (4.10) that for any $t\in[t_0,+\infty)$,

$$\begin{aligned} \mathbb{E}|x(t)|^p \le{}& \frac{c_2(1+\kappa)^p\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p\Delta_1}{\beta}e^{\sum_{i=1}^{l}\frac{\log(\mu_i)}{\mathcal{T}_{a,i}}T_i(t,t_0)+\int_{t_0}^{t}\lambda_{0,\xi(s)}(s)ds} \\ &+\kappa\mathbb{E}|x(t-\tau(t))|^p+\frac{\Delta_1}{\beta}\int_{t_0}^{t} e^{\sum_{i=1}^{l}\frac{\log(\mu_i)}{\mathcal{T}_{a,i}}T_i(t,s)+\int_s^t\lambda_{0,\xi(\theta)}(\theta)d\theta} \\ &\times[\lambda_{1,\xi(s)}(s)+\lambda_{2,\xi(s)}(s)]\sup_{\theta\in[-\tau,0]}\mathbb{E}|x(s+\theta)|^p ds \\ &+\frac{\Delta_1}{\beta}\int_{t_0}^{t} e^{\sum_{i=1}^{l}\frac{\log(\mu_i)}{\mathcal{T}_{a,i}}T_i(t,s)+\int_s^t\lambda_{0,\xi(\theta)}(\theta)d\theta}\chi_{\xi(s)}(|\varpi(s)|)ds. \end{aligned} \tag{4.11}$$

Since $\mathcal{T}_{a,i} > \mathcal{T}^*_{a,i} = \frac{(1-\kappa)\log(\mu_i)}{\gamma_i}$ $(i\in\Xi)$ and $t-s=\sum_{i=1}^{l}T_i(t,s)$, (4.11) yields that $\mathbb{E}|x(t)|^p \le \frac{c_2(1+\kappa)^p\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p\Delta_1}{\beta}e^{\int_{t_0}^{t}\tilde{\lambda}_{0,\xi(s)}(s)ds}+\kappa\mathbb{E}|x(t-\tau(t))|^p+\frac{\Delta_1}{\beta}\int_{t_0}^{t}e^{\int_s^t\tilde{\lambda}_{0,\xi(\theta)}(\theta)d\theta}[\lambda_{1,\xi(s)}(s)+\lambda_{2,\xi(s)}(s)]\sup_{\theta\in[-\tau,0]}\mathbb{E}|x(s+\theta)|^p ds+\frac{\Delta_1}{\beta}\int_{t_0}^{t}e^{\int_s^t\tilde{\lambda}_{0,\xi(\theta)}(\theta)d\theta}\chi(|\varpi(s)|)ds$, where $\tilde{\lambda}_{0,\xi(t)}(t)=\lambda_{0,\xi(t)}(t)+\frac{\gamma}{1-\kappa}$ and $\chi(|\varpi(t)|)=\max_{\varsigma_i\in\Xi}\{\chi_{\varsigma_i}(|\varpi(t)|)\}$. In addition, $\mathbb{E}|x(t)|^p\le\hat{M}\sup_{\theta\in[t_0-\tau,t_0]}|\varphi(\theta)|^p$ is satisfied for all $t\in[t_0-\tau,t_0]$, where $\hat{M}\ge 1$.

From Lemma 4.1, we have

$$\mathbb{E}|x(t)|^p \le \tilde{M} \sup_{\theta\in[t_0-\tau,t_0]} \mathbb{E}|\varphi(\theta)|^p e^{-\alpha(t-t_0)} + \tilde{M}_1 \int_{t_0}^{t} e^{-\alpha(t-s)}\chi(|\varpi(s)|)ds, \tag{4.12}$$

where $\alpha \in (0, \min\{\gamma/(1-\kappa), \frac{1}{\tau}\log(1/[\kappa+\beta(1-\kappa)e^{\frac{c}{1-\kappa}}])\})$, $\tilde{M} = \max\{\hat{M}, c_2(1+\kappa)^p e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)+\frac{c}{1-\kappa}}/[c_1\beta(1-\kappa)^p e^{\alpha\tau}]\}$, and $\tilde{M}_1 \ge e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)}/[c_1(1-\kappa)^{p-1}(1-[\kappa+(1-\kappa)\beta e^{\frac{c}{1-\kappa}}]e^{\alpha\tau})] > 0$.

Then, for all $t \ge t_0$, $\mathbb{E}|x(t)|^p \le \tilde{M}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p e^{-\alpha(t-t_0)} + \frac{\tilde{M}_1}{\gamma}\chi(\|\varpi\|_{[t_0,t]})$, which implies that neutral switched SDSs (3.1) with the synchronous switching are p-ISES.

From (4.12), it is obtained that $\mathbb{E}|x(t)|^p \le \tilde{M}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p e^{-\alpha(t-t_0)} + \tilde{M}_1\int_{t_0}^{t}\chi(|\varpi(s)|)ds$, which means that neutral switched SDSs (3.1) with the synchronous switching are p-SISES.

For any $\varepsilon > 0$, let $\tilde{M}' = \frac{\tilde{M}}{\varepsilon}$ and $\tilde{M}_1' = \frac{\tilde{M}_1}{\varepsilon}$. Then, the Markov inequality [1] and (4.12) give that $\mathbb{P}\{|x(t)|^p \ge \tilde{M}'\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p e^{-\alpha(t-t_0)} + \frac{\tilde{M}_1'}{\gamma}\chi(\|\varpi\|_{[t_0,t]})\} \le \mathbb{E}|x(t)|^p/(\tilde{M}'\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p e^{-\alpha(t-t_0)} + \frac{\tilde{M}_1'}{\gamma}\chi(\|\varpi\|_{[t_0,t]})) \le \varepsilon$, yielding $\mathbb{P}\{|x(t)| < (\tilde{M}')^{\frac{1}{p}}\sup_{\theta\in[t_0-\tau,t_0]}(\mathbb{E}|\varphi(\theta)|^p)^{\frac{1}{p}}e^{-\frac{\alpha}{p}(t-t_0)} + (\frac{\tilde{M}_1'}{\gamma})^{\frac{1}{p}}\chi(\|\varpi\|_{[t_0,t]})^{\frac{1}{p}}\} > 1-\varepsilon$. Thus, neutral switched SDSs (3.1) with the synchronous switching are SISES.

Let $\alpha = \varepsilon+\lambda$, where $\varepsilon > 0$ and $\lambda > 0$. From (4.12) again, we have $e^{\lambda(t-t_0)}\mathbb{E}|x(t)|^p \le \tilde{M}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p e^{-\varepsilon(t-t_0)} + \tilde{M}_1 e^{-\varepsilon(t-t_0)}\int_{t_0}^{t} e^{(\varepsilon+\lambda)(s-t_0)}\,\chi(|\varpi(s)|)ds \le \tilde{M}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p + \frac{\tilde{M}_1}{\varepsilon}\sup_{s\in[t_0,t]}\{e^{\lambda(s-t_0)}\chi(|\varpi(s)|)\}$, which implies that neutral switched SDSs (3.1) with the synchronous switching are $e^{\lambda t}$-p-ISES. ∎

Remark 7. Note that κ ($\kappa \in (0,1)$) and $\beta\sum_{i=1}^{l} N_{0,i}\log(\mu_i)/[c_1(1-\kappa)^{p-1}]$ in Theorem 4.3 are equivalent to $\tilde{\mu}_1 \in (0,1)$ and $\tilde{\mu}_2$ in Lemma 4.1, respectively. Note that the condition $\Delta_1 = \frac{\beta\sum_{i=1}^{l} N_{0,i}\log(\mu_i)e^{\frac{c'}{1-\kappa}}}{c_1(1-\kappa)^{p-1}} \in (0,1)$ is somewhat conservative when $e^{\frac{c'}{1-\kappa}} > 1$. Thus, under this case, a flexible and adjustable parameter β in condition (C_4) is generally chosen to be small such that $\Delta_1 \in (0,1)$ and, for any $t_0 \le s < t$, $\int_s^t[(1-\kappa)\lambda_{0,\xi(\theta)}(\theta)+\gamma+(\lambda_{1,\xi(\theta)}(\theta)+\lambda_{2,\xi(\theta)}(\theta))/\beta]d\theta \le c'$ are satisfied. How to reduce the conservatism of this condition is the focus of our future work.

Remark 8. Condition (C_1) commonly exists in the available literature [2, 8, 9, 21]. Condition (C_3) is also seen in [22, 23] and the references therein. Condition (C_4) is presented that, for any $t_0 \le s < t$, $\int_s^t[(1-\kappa)\lambda_{0,\xi(\theta)}(\theta)+(\lambda_{1,\xi(\theta)}(\theta)+\lambda_{2,\xi(\theta)}(\theta))/\beta]d\theta \le c'-\gamma(t-s)$, which means that $(1-\kappa)\lambda_{0,\xi(t)}(t)+(\lambda_{1,\xi(t)}(t)+\lambda_{2,\xi(t)}(t))/\beta$ is said to be a uniformly exponentially stable function with a guaranteed decay rate γ [42], where the coefficients $\lambda_{i,\xi(t)}(t)$ $(i = 0,1,2)$ are not only time-varying but also mode-dependent. This concept was proposed in [42], which has since been widely used to analyze the stability of time-varying time-delay systems and time-varying SDSs without neutral term; see [42, 43, 44, 45]. Condition (C_2) is a more relaxed Lyapunov monotonicity one since it is seen from condition (C_4) that the crucial time-varying coefficients $\lambda_{0,\varsigma_k}(t)$ in condition (C_2) can take some positive values in some subintervals and some non-positive values in other subintervals for every interval $[t_k, t_{k+1})$ $(k \in \mathbb{N})$.

Remark 9. Theorem 4.3 investigates the problems regarding the p-ISES, p-iISES, SISES, and $e^{\alpha' t}$-p-ISES $(\alpha' > 0)$ of neutral switched SDSs (3.1) under the synchronous switching, and one sufficient condition is given in terms of condition (C_4). Although some results on the stability analysis and the stabilization for neutral switched SDSs with the synchronous switching have been given in [21, 26] and the references therein, Theorem 4.3 has three advantages: (1) the mode-dependent coefficients $\lambda_{0,\xi(t)}(t)$ in the Lyapunov monotonicity condition (C_2) are not only time-varying, but also have the indefinite sign; (2) the time-varying delay in system (3.1) is a bounded function; and (3) the ISS/iISS of neutral switched SDSs (3.1) are analyzed.

Remark 10. In [47], the Lyapunov–Razumikhin theorem has been used to analyze the ISS/iISS of time-varying switched time-delay systems with the synchronous switching, and the mode-dependent time-varying coefficients are presented in integral inequality. Since the neutral term and the random disturbance are simultaneously present, some difficulties may be encountered when using the Lyapunov–Razumikhin theorem to obtain such integral inequality to guarantee the ISS/iISS of neutral switched SDSs (3.1). It is found that even if only the neutral term is present, there is no result on the ISS/iISS of time-varying switched NDSs with the synchronous switching by using the Lyapunov–Razumikhin theorem. However, the methodology used in Theorem 4.3 can consider the ISS/iISS of time-varying switched NDSs with the synchronous switching.

When $\mathcal{D}(t, x(t-\tau(t)))$ in neutral switched SDSs (3.1) is absent, by Corollary 4.2, multiple Lyapunov–Krasovskii functions, and the MDADT, the ISS/iISS of the corresponding switched SDSs under the synchronous switching can be discussed.

COROLLARY 4.4. *Suppose that there exist some Lyapunov–Krasovskii functions* $V_{\varsigma_k}(\cdot,\cdot) \in \mathcal{C}^{1,2}([t_k, t_{k+1}) \times R^n; [0,+\infty))$, *two constants* $c_1 > 0$, $c_2 > 0$, *and two integrable functions* $\lambda_{0,\xi(\cdot)}(\cdot) : [t_0,+\infty) \to R$, $\lambda_{1,\xi(\cdot)}(\cdot) : [t_0,+\infty) \to [0,+\infty)$ *such that*

(C_5) *for any* $x \in R^{n_x}$, $t \geq t_0$, *and* $p \geq 2$, $c_1|x|^p \leq V_{\xi(t)}(t,x) \leq c_2|x|^p$;

(C_6) *for any* $x, y \in R^{n_x}$, $\varpi \in R^{n_\varpi}$, $t \in [t_k, t_{k+1})$ $(k \in \mathbb{N};\ \varsigma_k \in \Xi)$, *and* $p \geq 2$, $\mathcal{L}V_{\varsigma_k}(t,x,y,\varpi) \leq \lambda_{0,\varsigma_k}(t)V_{\varsigma_k}(t,x) + \lambda_{1,\varsigma_k}(t)|y|^p + \chi_{\varsigma_k}(|\varpi|)$;

(C_7) *for any* $x \in R^{n_x}$ *and* $\varsigma_i, \varsigma_j \in \Xi$ $(i \neq j)$, $V_{\varsigma_i}(t,x) \leq \mu_{\varsigma_i} V_{\varsigma_j}(t,x)$, *where* $\mu_{\varsigma_i} > 1$;

(C_8) *there exist three constants* $\gamma > 0$, $\beta > 0$, *and* $c \in R$ *satisfying* $\beta e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)} e^{c}/c_1 \in (0,1)$, *and for any* $t_0 \leq s < t$, $\int_s^t [\lambda_{0,\xi(\theta)}(\theta) + \gamma + \lambda_{1,\xi(\theta)}(\theta)/\beta] d\theta \leq c$.

Then, switched SDSs (3.1) *with the synchronous switching when* $\mathcal{D}(t, x(t-\tau(t))) = 0$ *are p-ISES, p-iISES, SISES, and* $e^{\alpha t}$*-p-ISES* $(\alpha > 0)$ *over* $\mathcal{S}_{ave}[\mathcal{T}_{a,i}, N_{0,i}]$, *respectively, where the ith mode ADT* $\mathcal{T}_{a,i} > \mathcal{T}^*_{a,i} = \frac{\log \mu_i}{\gamma}$ $(i \in \Xi)$.

Proof. The proof of this corollary is similar to that given in Theorem 4.3, and hence is omitted. ∎

Before giving the following theorem, one condition imposed on $\lambda_{1,\xi(t)}(t)+\lambda_{2,\xi(t)}(t)$ is presented with $\upsilon = \sup_{t\geq t_0} \int_{t-\tau}^{t} [\lambda_{1,\xi(s)}(s) + \lambda_{2,\xi(s)}(s)] ds < +\infty$.

THEOREM 4.5. *Assume that conditions* (C_1)*–*(C_3) *of Theorem* 4.3 *hold and that condition* (C_4) *in Theorem* 4.3 *is replaced by the following:*

(C_4') *there exist two constants* $c' \in R$ *and* $\beta > 0$ *satisfying* $\beta e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i) + \frac{c'}{1-\kappa}} / [c_1(1-\kappa)^p] \in (0,1)$, *and for any* $t_0 \leq s < t$,

$$(4.13)\quad \int_s^t [(1-\kappa)(\lambda_{0,\xi(\theta)}(\theta) + \log(\mu_{\xi(\theta)})/\mathcal{T}_{a,\xi(\theta)}) + [\lambda_{1,\xi(s)}(\theta) + \lambda_{2,\xi(\theta)}(\theta)]/\beta] d\theta \leq c',$$

where $\mathcal{T}_{a,i}$ *denotes the ith mode ADT. Then, neutral switched SDSs* (2.1) *with the synchronous switching are p-iISES over* $\mathcal{S}_{ave}[\mathcal{T}_{a,i}, N_{0,i}]$.

Proof. (4.13) yields that, for any $t_0 \le s < t$,

$$\int_s^t \left[\lambda_{0,\xi(\theta)}(\theta) + \frac{\log(\mu_{\xi(\theta)})}{\mathcal{T}_{a,\xi(\theta)}}\right] d\theta \le \frac{1}{\beta(1-\kappa)}\left[\beta c' - \int_s^t [\lambda_{1,\xi(s)}(\theta) + \lambda_{2,\xi(\theta)}(\theta)]d\theta\right]. \tag{4.14}$$

From (4.11) and (4.14), it follows that for any $t \ge t_0$,

$$\begin{aligned} \mathbb{E}|x(t)|^p \le{}& \tilde{\lambda}_0 \sup_{\theta\in[-\tau,0]} \mathbb{E}|x(t_0+\theta)|^p e^{-\phi(t,t_0)} + \kappa\mathbb{E}|x(t-\tau(t))|^p + \tilde{\lambda}_1 \int_{t_0}^t e^{-\phi(t,s)} \\ & \times [\lambda_{1,\xi(s)}(s)+\lambda_{2,\xi(s)}(s)] \sup_{\theta\in[-\tau,0]} \mathbb{E}|x(s+\theta)|^p ds + \tilde{\lambda}_1 \int_{t_0}^t e^{-\phi(t,s)}\chi(|\varpi(s)|)ds, \end{aligned} \tag{4.15}$$

where $\tilde{\lambda}_0 = c_2(1+\kappa)^p e^{\sum_{i=1}^l N_{0,i}\log(\mu_i)+\frac{c'}{1-\kappa}}/[c_1(1-\kappa)^{p-1}]$, $\tilde{\lambda}_1 = e^{\sum_{i=1}^l N_{0,i}\log(\mu_i)+\frac{c'}{1-\kappa}}/[c_1(1-\kappa)^{p-1}]$, $\phi(t,s) = \frac{1}{\beta(1-\kappa)}\int_s^t[\lambda_{1,\xi(\theta)}(\theta)+\lambda_{2,\xi(\theta)}(\theta)]d\theta$, and $\chi(|\varpi(t)|) = \sup_{\xi(t)\in\Xi}\{\chi_{\xi(t)}(|\varpi(t)|)\}$. It is seen that $\lim_{t\to+\infty}\phi(t,t_0)=+\infty$, and, for any $t_0 \le s$, $s+\tau < t$, $\phi(t,s) = \phi(t,t-\tau)+\phi(t-\tau,s)$.

In addition, a constant $\bar{K} \ge 1$ exists such that $\mathbb{E}|x(t)|^p \le \bar{K}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p$ for any $t\in[t_0-\tau,t_0]$. Now, we are in a position to show that for any $t \ge t_0-\tau$,

$$\mathbb{E}|x(t)|^p \le \tilde{K}^{\#}_{0,\varepsilon} \sup_{\theta\in[t_0-\tau,t_0]} \mathbb{E}|\varphi(\theta)|^p e^{-\delta\phi(t,t_0)} + \tilde{K}^{\#}\int_{t_0}^t e^{-\delta\phi(t,s)}\chi(|\varpi(s)|)ds, \tag{4.16}$$

where $\delta \in (0,\delta_0)$ with $\delta_0 \in (0,1)$ being a unique positive solution of equation: $[\kappa + \beta e^{\sum_{i=1}^l N_{0,i}\log(\mu_i)+\frac{c'}{1-\kappa}}/[c_1(1-\kappa)^{p-1}(1-\delta)]]e^{\delta\upsilon} = 1$, $\tilde{K}^{\#}_{0,\varepsilon} = \max\{\bar{K}+\varepsilon, [c_2(1+\kappa)^p e^{\sum_{i=1}^l N_{0,i}\log(\mu_i)+\frac{c'}{1-\kappa}} + c_1(1-\kappa)^{p-1}\varepsilon]/[\beta(1-\kappa)e^{\sum_{i=1}^l N_{0,i}\log(\mu_i)+\frac{c'}{1-\kappa}}]\}$, and $\tilde{K}^{\#} \ge (1-\delta)e^{\sum_{i=1}^l N_{0,i}\log(\mu_i)+\frac{c'}{1-\kappa}}/[c_1(1-\kappa)^{p-1}(1-\delta) - [c_1\kappa(1-\kappa)^{p-1}(1-\delta) + \beta(1-\kappa)e^{\sum_{i=1}^l N_{0,i}\log(\mu_i)+\frac{c'}{1-\kappa}}]e^{\delta\upsilon}] > 0$.

Define a function $H(\delta) = [\kappa+\beta e^{\sum_{i=1}^l N_{0,i}\log(\mu_i)+\frac{c'}{1-\kappa}}/[c_1(1-\kappa)^{p-1}(1-\delta)]]e^{\delta\upsilon}-1$, where $H(0) = \kappa+\beta e^{\sum_{i=1}^l N_{0,i}\log(\mu_i)+\frac{c'}{1-\kappa}}/[c_1(1-\kappa)^{p-1}]-1<0$, $\lim_{\delta\to1^-}H(\delta)=\infty$, and $H(\delta)$ is a strictly increasing function on the interval $[0,1]$. Thus, there exists $\delta_0\in(0,1)$ such that $H(\delta_0)=0$.

Since $\mathbb{E}|x(t)|^p \le \bar{K}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p < \tilde{K}^{\#}_{0,\varepsilon}\mathbb{E}|\varphi(\theta)|^p$ is satisfied for all $t\in[t_0-\tau,t_0]$, (4.16) holds for all $t\in[t_0-\tau,t_0]$. If (4.16) is not satisfied for any $t>t_0$, then there exists $t^{\#}>t_0$ such that $t^{\#} = \inf\{t>t_0 | \mathbb{E}|x(t)|^p > \tilde{K}^{\#}_{0,\varepsilon}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p e^{-\delta\phi(t,t_0)} + \tilde{K}^{\#}\int_{t_0}^t e^{-\delta\phi(t,s)}\chi(|\varpi(s)|)ds\}$. Consequently, for all $t\in[t_0-\tau,t^{\#})$, $\mathbb{E}|x(t)|^p \le \tilde{K}^{\#}_{0,\varepsilon}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p e^{-\delta\phi(t,t_0)} + \tilde{K}^{\#}\int_{t_0}^t e^{-\delta\phi(t,s)}\chi(|\varpi(s)|)ds$ and $\mathbb{E}|x(t^{\#})|^p = \tilde{K}^{\#}_{0,\varepsilon}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p e^{-\delta\phi(t^{\#},t_0)} + \tilde{K}^{\#}\int_{t_0}^{t^{\#}} e^{-\delta\phi(t^{\#},s)}\chi(|\varpi(s)|)ds$.

However, (4.15) gives $\mathbb{E}|x(t^{\#})|^p \le [\tilde{\lambda}_0 - \tilde{\lambda}_1\tilde{K}^{\#}_{0,\varepsilon}e^{\delta\upsilon}/(1-\delta)]\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p e^{-\phi(t^{\#},t_0)} + [\kappa e^{\delta\upsilon} + \tilde{\lambda}_1 e^{\delta\upsilon}/(1-\delta)]\tilde{K}^{\#}_{0,\varepsilon}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p\, e^{-\delta\phi(t^{\#},t_0)} + [\kappa\tilde{K}^{\#}e^{\delta\upsilon} + \tilde{K}^{\#}\tilde{\lambda}_1 e^{\delta\upsilon}/(1-\delta)+\tilde{\lambda}_1]\int_{t_0}^{t^{\#}} e^{-\delta\phi(t^{\#},s)}\chi(|\varpi(s)|)ds$. The definitions of $\tilde{K}^{\#}_{0,\varepsilon}$ and $\tilde{K}^{\#}$ yield $\tilde{\lambda}_0 - \tilde{\lambda}_1\tilde{K}^{\#}_{0,\varepsilon}e^{\delta\tau}/(1-\delta) \le \tilde{\lambda}_0 - (\tilde{\lambda}_1 e^{\delta\tau}/(1-\delta))\times(1-\delta)(\tilde{\lambda}_0+\varepsilon)/(\tilde{\lambda}_1 e^{\delta\tau}) < 0$ and $\kappa\tilde{K}^{\#}e^{\delta\upsilon} + \frac{\tilde{K}^{\#}\tilde{\lambda}_1 e^{\delta\upsilon}}{1-\delta} + \tilde{\lambda}_1 \le \tilde{K}^{\#}$. Then, it follows that $\mathbb{E}|x(t^{\#})|^p < \tilde{K}^{\#}_{0,\varepsilon}\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}|\varphi(\theta)|^p\, e^{-\delta\phi(t^{\#},t_0)} + \tilde{K}^{\#}\int_{t_0}^{t^{\#}} e^{-\delta\phi(t^{\#},s)}\chi(|\varpi(s)|)ds$, which has a contradiction. Thus, (4.16) is satisfied for all $t\ge t_0$, which means that neutral switched SDSs (3.1) with the synchronous switching are p-iISES over $\mathcal{S}_{ave}$. $\square$

When the coefficients $\lambda_{i,\xi(t)}(t)$ $(i = 0, 1, 2)$ in condition (C_2) are only mode-dependent, that is, $\lambda_{0,\xi(t)}(t) = \lambda_{0,\xi(t)} \in R$, $\lambda_{1,\xi(t)}(t) = \lambda_{1,\xi(t)} \geq 0$, and $\lambda_{2,\xi(t)}(t) = \lambda_{2,\xi(t)} > 0$, for neutral switched SDSs (3.1) with the synchronous switching, we have the following.

COROLLARY 4.6. *Let conditions (C_1) and (C_3) of Theorem 4.3 be satisfied, and assume that conditions (C_2) and (C_4) in Theorem 4.3 are replaced by*

(C_2'') *for any $x, y \in R^{n_x}$, $\varpi \in R^{n_\varpi}$, $t \in [t_k, t_{k+1})$ $(k \in \mathbb{N};\ \varsigma_k \in \Xi)$, and $p \geq 2$, $\mathcal{L}V_{\varsigma_k}(t, x, y, \varpi) \leq \lambda_{0,\varsigma_k} V_{\varsigma_k}(t, x - \mathcal{D}(t, y)) + \lambda_{1,\varsigma_k}|x|^p + \lambda_{2,\varsigma_k}|y|^p + \chi_{\varsigma_k}(|\varpi|)$;*

(C_4'') *there exist three constants $\gamma > 0$, $\beta > 0$, and $\bar{c} \in R$ satisfying $\beta e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i)+\frac{\bar{c}}{1-\kappa}}/[c_1(1-\kappa)^{p-1}] \in (0,1)$, and for any $t_0 \leq s < t$, $(1-\kappa)\sum_{i=1}^{l} \lambda_{0,i}T_i(t,s) + \sum_{i=1}^{l}(\lambda_{1,i} + \lambda_{2,i})T_i(t,s)/\beta \leq \bar{c} - \gamma(t-s)$. Then, neutral switched SDSs (3.1) with the synchronous switching are p-ISES, p-iISES, SISES, and $e^{\alpha t}$-p-ISES $(\alpha > 0)$ over $\mathcal{S}_{ave}[\mathcal{T}_{a,i}, N_{0,i}]$, respectively, where the ith mode ADT $\mathcal{T}_{a,i} > \mathcal{T}_{a,i}^* = \frac{(1-\kappa)\log\mu_i}{\gamma}$ $(i \in \Xi)$.*

Proof. Since $t - s = \sum_{i=1}^{l} T_i(t,s)$, condition (C_4'') yields that $\int_s^t [(1-\kappa)\lambda_{0,\xi(\theta)}(\theta) + \gamma + [\lambda_{1,\xi(\theta)}(\theta) + \lambda_{2,\xi(\theta)}(\theta)]/\beta]d\theta = (1-\kappa)\sum_{i=1}^{l}\lambda_{0,i}T_i(t,s) + \sum_{i=1}^{l}(\lambda_{1,i} + \gamma(t-s) + \lambda_{2,i})T_i(t,s)/\beta \leq \bar{c}$, which means that condition (C_4) of Theorem 4.3 holds. Thus, from Theorem 4.3, the desired result is obtained. $\square$

4.2. Input-to-state stochastic stability of neutral switched SDSs (3.1) with the asynchronous switching. In this subsection, the switching signal of input control $u(t)$ in neutral switched SDSs (3.1) does not coincide with the switching signal in the plant's state. Thus, the ISS/iISS of neutral switched SDSs (3.1) with the asynchronous switching cannot be studied as above. In order to overcome this difficulty caused by asynchronous switching, the merging switching signal technique proposed in [27] will be used. Define a new switching signal: $\bar{\xi}(\cdot): [t_0, +\infty) \to \Xi \times \Xi$ with $\bar{\xi}(t) = (\xi(t), \xi(t-\tau_d))$. Denote by $'\oplus'$ the merging action with $\bar{\xi}(t) = \xi_1(t) \oplus \xi_2(t)$ [27], where $\xi_1(t) = \xi(t)$ and $\xi_2(t) = \xi(t - \tau_d)$. Before presenting the main results, for convenience, two lemmas established in [39] are given as follows.

LEMMA 4.7 (see [39]). *Assume that $\xi_1(t) \in \mathcal{S}_{sav}[\mathcal{T}_{a,i}, N_{0,i}]$; we have $\xi_2(t) \in \mathcal{S}_{sav}[\mathcal{T}_{a,i},\ N_{0,i} + \frac{\tau_d}{\mathcal{T}_{a,i}}]$. Furthermore, $\bar{\xi}(t) \in \mathcal{S}_{sav}[\frac{\mathcal{T}_{a,i}}{2}, 2N_{0,i} + \frac{\tau_d}{\mathcal{T}_{a,i}}]$.*

LEMMA 4.8 (see [39]). *For an interval (t_0, t), let $\mathcal{M}_i(t, t_0)$ denote the total time for $\xi_1(t) = \xi_2(t) = i$ $(i \in \Xi)$, and let $\bar{\mathcal{M}}_i(t, t_0) = T_i(t, t_0) - \mathcal{M}_i(t, t_0)$. If there exist two positive constants ζ_i and ζ_{1i} such that $\tau_d(\zeta_i + \zeta_{1i}) \leq (\zeta_i - \tilde{\zeta}_i)\mathcal{T}_{a,i}$ holds for some $\tilde{\zeta}_i \in (0, \zeta_i)$, then $-\zeta_i\mathcal{M}_i(t, t_0) + \zeta_{1i}\bar{\mathcal{M}}_i(t, t_0) \leq \tilde{c}_i - \tilde{\zeta}_i T_i(t, t_0)$ is satisfied for all $t \geq t_0$, where $\tilde{c}_i = (\zeta_i + \zeta_{1i})N_{0,i}\tau_d$.*

Note that when $\xi(t) = \varsigma_k$ $(t \in [t_k, t_{k+1}))$, $\xi(t-\tau_d) = \varsigma_k$ $(t \in [t_k+\tau_d, t_{k+1}+\tau_d))$ $(k \in \mathbb{N})$. Thus, from the definition of the new switching signal $\bar{\xi}(t)$, we have $\bar{\xi}(t) \in \Xi \times \Xi$ with $\bar{\xi}(t) = (\varsigma_k, \varsigma_k)$ $(t \in [t_k + \tau_d, t_{k+1}))$ and $\bar{\xi}(t) = (\varsigma_k, \varsigma_{k-1})$ $(t \in [t_k, t_k + \tau_d))$. Thus, the switching signal $\bar{\xi}(t)$ has the switching time instants $t_0, t_0 + \tau_d, t_1, t_1 + \tau_d, \ldots$. In addition, for convenience, $\hat{\xi}(t) = (\xi(t), \xi(t))$.

THEOREM 4.9. *Suppose that there exist some Lyapunov–Krasovskii functions $V_{\bar{\xi}(t)}(\cdot,\cdot) \in \mathcal{C}^{1,2}([t_k, t_{k+1}) \times R^n; [0, +\infty))$ $(k \in \mathbb{N})$, two constants $c_1 > 0$, $c_2 > 0$, some integrable functions $\lambda_{0,\bar{\xi}(\cdot)}(\cdot): [t_0, +\infty) \to R$, $\lambda_{1,\hat{\xi}(\cdot)}(\cdot): [t_0, +\infty) \to [0, +\infty)$, and $\lambda_{2,\hat{\xi}(\cdot)}(\cdot): [t_0, +\infty) \to (0, +\infty)$ such that*

(D_1) *for any $x, y \in R^{n_x}$, $t \geq t_0$, and $p \geq 2$, $c_1|x - \mathcal{D}(t, y)|^p \leq V_{\bar{\xi}(t)}(t, x - \mathcal{D}(t, y)) \leq c_2|x - \mathcal{D}(t, y)|^p$;*

(D_2) for any $x, y \in R^{n_x}$, $\varpi \in R^{n_\varpi}$, $t \in [t_k, t_{k+1})$ $(k \in \mathbb{N})$, and $p \geq 2$, $\mathcal{L}V_{\bar{\xi}(t)}(t, x, y, \varpi) \leq \lambda_{0,\bar{\xi}(t)}(t)V_{\bar{\xi}(t)}(t, x-\mathcal{D}(t, y))+\lambda_{1,\hat{\xi}(t)}(t)|x|^p+\lambda_{2,\hat{\xi}(t)}(t)|y|^p+\chi_{\bar{\xi}(t)}(|\varpi|)$;

(D_3) for any $x, y \in R^{n_x}$ and $(\varsigma_i, \varsigma_j, \varsigma_k, \varsigma_l) \in \Xi \times \Xi \times \Xi \times \Xi$, $\varsigma_i \neq \varsigma_k$, or $\varsigma_j \neq \varsigma_l$, $V_{\varsigma_i,\varsigma_j}(t, x-\mathcal{D}(t, y)) \leq \mu_{\varsigma_i,\varsigma_j} V_{\varsigma_k,\varsigma_l}(t, x-\mathcal{D}(t, y))$, where $\mu_{\varsigma_i,\varsigma_j} = \mu_{\varsigma_i} > 1$ $(\varsigma_i \neq \varsigma_k)$ and $\mu_{\varsigma_i,\varsigma_j} = 1$ $(\varsigma_i = \varsigma_k)$.

(D_4) there exist some constants $\gamma_{s,\varsigma_k} > 0$, $\gamma_{u,\varsigma_k} > 0$, $\beta > 0$, $c'_{s,\varsigma_k} \in R$, and $c'_{u,\varsigma_k} \in R$ such that $\Delta_2 \in (0, 1)$, with $\Delta_2 = \beta e^{\sum_{i=1}^{l} N_{0,i}[\frac{c'_{u,i}}{1-\kappa}+\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})]+\sum_{i=1}^{l}\frac{\tau_d}{\mathcal{T}_{a,i}}\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}$ $e^{\sum_{i=1}^{l}(\gamma_{s,i}+\gamma_{u,i})N_{0,i}\tau_d}/[c_1(1-\kappa)^{p-1}]$, for any $t_k + \tau_d \leq s < t < t_{k+1}$,

$$\int_s^t [(1-\kappa)\lambda_{0,\bar{\xi}(\theta)}(\theta) + (\lambda_{1,\hat{\xi}(\theta)}(\theta) + \lambda_{2,\hat{\xi}(\theta)}(\theta))/\beta]d\theta \leq c'_{s,\varsigma_k} - \gamma_{s,\varsigma_k}\mathcal{M}_{\varsigma_k}(t, s), \tag{4.17}$$

for any $t_k \leq s < t < t_k + \tau_d$,

$$\int_s^t [(1-\kappa)\lambda_{0,\bar{\xi}(\theta)}(\theta) + (\lambda_{1,\hat{\xi}(\theta)}(\theta) + \lambda_{2,\hat{\xi}(\theta)}(\theta))/\beta]d\theta \leq c'_{u,\varsigma_k} + \gamma_{u,\varsigma_k}\bar{\mathcal{M}}_{\varsigma_k}(t, s). \tag{4.18}$$

Then, neutral switched SDSs (3.1) with the asynchronous switching are p-ISES, p-iISES, SISES, and $e^{\alpha t}$-p-ISES $(\alpha > 0)$ over $\mathcal{S}_{ave}[\mathcal{T}_{a,i}, N_{0,i}]$, respectively, where the ith mode ADT is

$$\mathcal{T}_{a,i} > \mathcal{T}^*_{a,i} = \left[\log\left(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}}\right) + (\gamma_{s,i} + \gamma_{u,i})\,\tau_d\right]/\gamma_{s,i} \quad (i \in \Xi). \tag{4.19}$$

Proof. Denote by $\nu_1, \nu_2, \ldots, \nu_{N_{\bar{\xi}(t)}(T,t_0)}$ the switching times of $\bar{\xi}(t)$ in (t_0, T) with $\nu_0 = t_0$ and $\nu_{N_{\bar{\xi}(t)}(T,t_0)+1} = T^-$. Let $\Lambda_{1,\varsigma_i} = \mu_{\varsigma_i} e^{\frac{c'_{s,\varsigma_i}}{1-\kappa}}$ and $\Lambda_{2,\varsigma_i} = e^{\frac{c'_{u,\varsigma_i}}{1-\kappa}}$ $(\varsigma_i \in \Xi)$. The proof of this theorem is also separated into three steps:

Step 1. For any $t \in [\nu_0, \nu_1)$, by defining a Lyapunov function $e^{-\int_{\nu_0}^t \lambda_{0,\bar{\xi}(\nu_0)}(s)ds}V_{\bar{\xi}(t)}$ $(t, \bar{x}(t))$, using the Itô formula, and taking the mathematical expectation in sequence, condition (D_2) yields that $\mathbb{E}\{V_{\bar{\xi}(\nu_0)}(t, \bar{x}(t))\} \leq \mathbb{E}\{V_{\bar{\xi}(\nu_0)}(\nu_0, \bar{x}(\nu_0))\}\, e^{\int_{\nu_0}^t \lambda_{0,\bar{\xi}(\nu_0)}(s)ds}$ $+ \int_{\nu_0}^t e^{\lambda_{0,\bar{\xi}(\nu_0)}(\theta)}\, \lambda_{1,\hat{\xi}(\nu_0)}(s)\mathbb{E}\{|x(s)|^p\}ds + \int_{\nu_0}^t e^{\int_s^t \lambda_{0,\bar{\xi}(\nu_0)}(\theta)d\theta}\lambda_{2,\hat{\xi}(\nu_0)}(s)\mathbb{E}\{|x(s-\tau(s))|^p\}$ ds $+ \int_{\nu_0}^t e^{\int_s^t \lambda_{0,\bar{\xi}(\nu_0)}(\theta)d\theta}\chi_{\bar{\xi}(\nu_0)}(|\varpi(s)|)ds$. When $t = \nu_1^-$, it follows from condition (D_3) that $\mathbb{E}\{V_{\bar{\xi}(\nu_1)}(\nu_1, \bar{x}(\nu_1))\} \leq \mu_{\bar{\xi}(\nu_1)}\mathbb{E}\{V_{\bar{\xi}(\nu_0)}(\nu_0, \bar{x}(\nu_0))\}\, e^{\int_{\nu_0}^{\nu_1} \lambda_{0,\bar{\xi}(\nu_0)}(s)ds} + \mu_{\bar{\xi}(\nu_1)} \int_{\nu_0}^{\nu_1}$ $e^{\lambda_{0,\bar{\xi}(\nu_0)}(\theta)}\, \lambda_{1,\hat{\xi}(\nu_0)}(s)\mathbb{E}\{|x(s)|^p\}ds + \mu_{\bar{\xi}(\nu_1)} \int_{\nu_0}^{\nu_1} e^{\int_s^t \lambda_{0,\bar{\xi}(\nu_0)}(\theta)d\theta}\, \lambda_{2,\hat{\xi}(\nu_0)}(s)\mathbb{E}\{|x(s-\tau(s))|^p\}$ $ds + \mu_{\bar{\xi}(\nu_1)} \int_{\nu_0}^{\nu_1} e^{\int_s^t \lambda_{0,\bar{\xi}(\nu_0)}(\theta)d\theta}\chi_{\bar{\xi}(\nu_0)}(|\varpi(s)|)ds$.

By using a similar derivation, when $t \in [\nu_1, \nu_2)$, it follows that $\mathbb{E}\{V_{\bar{\xi}(\nu_1)}(t, \bar{x}(t))\}$ $\leq \mu_{\bar{\xi}(\nu_1)}\mathbb{E}\{V_{\bar{\xi}(\nu_0)}(\nu_0, \bar{x}(\nu_0))\}e^{\int_{\nu_0}^{\nu_1} \lambda_{0,\bar{\xi}(\nu_0)}(s)ds}e^{\int_{\nu_1}^t \lambda_{0,\bar{\xi}(\nu_1)}(s)ds} + \mu_{\bar{\xi}(\nu_1)} \int_{\nu_0}^{\nu_1} e^{\int_s^{\nu_1} \lambda_{0,\bar{\xi}(\nu_0)}(\theta)d\theta}$ $\lambda_{1,\hat{\xi}(\nu_0)}(s)\mathbb{E}\{|x(s)|^p\}dse^{\int_{\nu_1}^t \lambda_{0,\bar{\xi}(\nu_1)}(s)ds} + \mu_{\bar{\xi}(\nu_1)} \int_{\nu_0}^{\nu_1} e^{\int_s^{\nu_1} \lambda_{0,\bar{\xi}(\nu_0)}(\theta)d\theta}\lambda_{2,\hat{\xi}(\nu_0)}(s)\mathbb{E}\{|x(s -$ $\tau(s))|^p\}ds\, e^{\int_{\nu_1}^t \lambda_{0,\bar{\xi}(\nu_1)}(s)ds} + \mu_{\bar{\xi}(\nu_1)} \int_{\nu_0}^{\nu_1} e^{\int_s^{\nu_1} \lambda_{0,\bar{\xi}(\nu_0)}(\theta)d\theta}\chi_{\bar{\xi}(\nu_0)}(|\varpi(s)|)dse^{\int_{\nu_1}^t \lambda_{0,\bar{\xi}(\nu_1)}(s)ds} +$ $\int_{\nu_1}^t e^{\int_s^t \lambda_{0,\bar{\xi}(\nu_1)}(\theta)d\theta}\, \lambda_{1,\hat{\xi}(\nu_1)}(s)\mathbb{E}\{|x(s)|^p\}ds + \int_{\nu_1}^t e^{\int_s^t \lambda_{0,\bar{\xi}(\nu_1)}(\theta)d\theta}\lambda_{2,\hat{\xi}(\nu_1)}(s)\mathbb{E}\{|x(s-\tau(s))|^p\}$ $ds + \int_{\nu_1}^t e^{\int_s^t \lambda_{0,\bar{\xi}(\nu_1)}(\theta)d\theta}\, \chi_{\bar{\xi}(\nu_1)}(|\varpi(s)|)ds$.

By repeating the process above, when $t \in [\nu_k, \nu_{k+1})$, we have

$$
\begin{aligned}
&\mathbb{E}\{V_{\bar{\xi}(\nu_k)}(t,\bar{x}(t))\}\\
&\le \prod_{i=1}^{k}\mu_{\bar{\xi}(\nu_i)}\mathbb{E}\{V_{\bar{\xi}(\nu_0)}(\nu_0,\bar{x}(\nu_0))\}e^{\int_{\nu_0}^{\nu_1}\lambda_{0,\bar{\xi}(\nu_0)}(s)ds+\int_{\nu_1}^{\nu_2}\lambda_{0,\bar{\xi}(\nu_1)}(s)ds+\cdots+\int_{\nu_k}^{t}\lambda_{0,\bar{\xi}(\nu_k)}(s)ds}\\
&\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\mu_{\bar{\xi}(\nu_j)}\int_{\nu_{j-1}}^{\nu_j}e^{\int_s^{\nu_j}\lambda_{0,\bar{\xi}(\nu_{j-1})}(\theta)d\theta}\lambda_{1,\hat{\xi}(\nu_{j-1})}(s)\mathbb{E}|x(s)|^p ds e^{\int_{\nu_k}^{t}\lambda_{0,\bar{\xi}(\nu_k)}(s)ds}\\
&\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\mu_{\bar{\xi}(\nu_j)}\int_{\nu_{j-1}}^{\nu_j}e^{\int_s^{\nu_j}\lambda_{0,\bar{\xi}(\nu_{j-1})}(\theta)d\theta}\lambda_{2,\hat{\xi}(\nu_{j-1})}(s)\mathbb{E}|x(s-\tau(s))|^p ds\\
&\quad\times e^{\int_{\nu_k}^{t}\lambda_{0,\bar{\xi}(\nu_k)}(s)ds}+\sum_{i=1}^{k}\prod_{j=i}^{k}\mu_{\bar{\xi}(\nu_j)}\int_{\nu_{j-1}}^{\nu_j}e^{\int_s^{\nu_j}\lambda_{0,\bar{\xi}(\nu_{j-1})}(\theta)d\theta}\chi_{\bar{\xi}(\nu_{j-1})}(|\varpi(s)|)ds\\
&\quad\times e^{\int_{\nu_k}^{t}\lambda_{0,\xi(s)}ds}+\int_{\nu_k}^{t}e^{\int_s^t\lambda_{0,\bar{\xi}(\nu_k)}(\theta)d\theta}\lambda_{1,\hat{\xi}(s)}(s)\mathbb{E}\{|x(s)|^p\}ds\\
&\quad+\int_{\nu_k}^{t}e^{\int_s^t\lambda_{0,\bar{\xi}(\nu_k)}(\theta)d\theta}\lambda_{2,\hat{\xi}(s)}(s)\mathbb{E}\{|x(s-\tau(s))|^p\}ds\\
&\quad+\int_{\nu_k}^{t}e^{\int_s^t\lambda_{0,\bar{\xi}(\nu_k)}(\theta)d\theta}\chi_{\bar{\xi}(s)}(|\varpi(s)|)ds,
\end{aligned}
\tag{4.20}
$$

where $\prod_{i=1}^{0}\mu_{\bar{\xi}(\nu_i)}\equiv 1$ and $\sum_{i=1}^{0}(\cdot)\equiv 0$.

Step 2. As $t = T^-$, (4.20) yields that

$$
\begin{aligned}
&\mathbb{E}\{V_{\bar{\xi}(t_{N_{\bar{\xi}(t)}(T,t_0)}+1)}(T^-,\bar{x}(T^-))\}\\
&\le \prod_{i=1}^{l}\Lambda_{2,i}^{N_{\xi_1(t),i}(T,\nu_0)}\prod_{i=1}^{l}\Lambda_{1,i}^{N_{\xi_2(t),i}(T,\nu_0)}\mathbb{E}\{V_{\bar{\xi}(\nu_0)}(\nu_0,\bar{x}(\nu_0))\}\\
&\quad\times e^{-\frac{1}{\beta(1-\kappa)}\int_{\nu_0}^{T}[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)]ds+\sum_{i=1}^{l}[-\gamma_{s,i}\mathcal{M}_i(T,\nu_0)+\gamma_{u,i}\bar{\mathcal{M}}_i(T,\nu_0)]}\\
&\quad+\sum_{k=0}^{N_{\bar{\xi}(t)}(T,\nu_0)}\int_{\nu_k}^{\nu_{k+1}}\prod_{i=1}^{l}\Lambda_{2,i}^{N_{\xi_1(t),i}(T,s)}\prod_{i=1}^{l}\Lambda_{1,i}^{N_{\xi_2(t),i}(T,s)}\\
&\quad\times e^{-\frac{1}{\beta(1-\kappa)}\int_{s}^{T}[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta+\sum_{i=1}^{l}[-\gamma_{s,i}\mathcal{M}_i(T,s)+\gamma_{u,i}\bar{\mathcal{M}}_i(T,s)]}\\
&\quad\times\left[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)\right]\sup_{\theta\in[-\tau,0]}\mathbb{E}|x(s+\theta)|^p ds\\
&\quad+\sum_{k=0}^{N_{\bar{\xi}(t)}(T,\nu_0)}\int_{\nu_k}^{\nu_{k+1}}\prod_{i=1}^{l}\Lambda_{2,i}^{N_{\xi_1(t),i}(T,s)}\prod_{i=1}^{l}\Lambda_{1,i}^{N_{\xi_2(t),i}(T,s)}\\
&\quad\times e^{-\frac{1}{\beta(1-\kappa)}\int_{s}^{T}[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta+\sum_{i=1}^{l}[-\gamma_{s,i}\mathcal{M}_i(T,s)+\gamma_{u,i}\bar{\mathcal{M}}_i(T,s)]}\\
&\quad\times\chi_{\hat{\xi}(s)}(|\varpi(s)|)ds.
\end{aligned}
\tag{4.21}
$$

From Lemma 4.7, it follows that $\prod_{i=1}^{l}\Lambda_{2,i}^{N_{\xi_1,i}(T,\nu_0)} \leq e^{\sum_{i=1}^{l} N_{\xi_1(t),i}(T,\nu_0)\log(\Lambda_{2,i})} \leq e^{\sum_{i=1}^{l}[N_{0,i}+\frac{T_i(T,\nu_0)}{\mathcal{T}_{a,i}}]\frac{c'_{u,i}}{1-\kappa}}$, and $\prod_{i=1}^{l}\Lambda_{1,i}^{N_{\xi_2,i}(T,t_0)} \leq e^{\sum_{i=1}^{l} N_{\xi_2(t),i}(T,\nu_0)\log(\Lambda_{1,i})}$ $\leq\ e^{\sum_{i=1}^{l}[N_{0,i}+\frac{\tau_d}{\mathcal{T}_{a,i}}+\frac{T_i(T,\nu_0)}{\mathcal{T}_{a,i}}]\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}$. Then, (4.21) yields that

$$
\begin{aligned}
&\mathbb{E}\left\{V_{\bar{\xi}(t_{N_{\bar{\xi}}(T,\nu_0)}+1)}(T^-,\bar{x}(T^-))\right\}\\
&\quad\leq \frac{\Delta_2 c_1(1-\kappa)^{p-1}}{\beta}\mathbb{E}\{V_{\bar{\xi}(\nu_0)}(\nu_0,\bar{x}(\nu_0))\}e^{-\frac{1}{\beta(1-\kappa)}\int_{\nu_0}^{T}[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)]ds}\\
&\qquad\times e^{\sum_{i=1}^{l}\frac{\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}{\mathcal{T}_{a,i}}T_i(T,\nu_0)+\sum_{i=1}^{l}[-\gamma_{s,i}\mathcal{M}_i(T,\nu_0)+\gamma_{u,i}\bar{\mathcal{M}}_i(T,\nu_0)]}\\
&\qquad+\frac{\Delta_2 c_1(1-\kappa)^{p-1}}{\beta}\int_{\nu_0}^{T}e^{-\frac{1}{\beta(1-\kappa)}\int_s^T[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta}\\
&\qquad\times e^{\sum_{i=1}^{l}\frac{\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}{\mathcal{T}_{a,i}}T_i(T,s)+\sum_{i=1}^{l}[-\gamma_{s,i}\mathcal{M}_i(T,s)+\gamma_{u,i}\bar{\mathcal{M}}_i(T,s)]}\\
&\qquad\times[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)]\sup_{\theta\in[-\tau,0]}\mathbb{E}|x(s+\theta)|^p ds\\
&\qquad+\frac{\Delta_2 c_1(1-\kappa)^{p-1}}{\beta}\int_{\nu_0}^{T}e^{-\frac{1}{\beta(1-\kappa)}\int_s^T[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta}\\
&\qquad\times e^{\sum_{i=1}^{l}\frac{\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}{\mathcal{T}_{a,i}}T_i(T,s)+\sum_{i=1}^{l}[-\gamma_{s,i}\mathcal{M}_i(T,s)+\gamma_{u,i}\bar{\mathcal{M}}_i(T,s)]}\chi(|\varpi(s)|)ds,
\end{aligned}
\tag{4.22}
$$

where $\chi(|\varpi(t)|)=\sup_{\xi(t)\in\Xi}\chi_{\xi(t)}(|\varpi(t)|)$.

Step 3. From (4.19), there exists $\gamma_i\in(0,\gamma_{s,i})$ $(i\in\Xi)$ satisfying $\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})/\mathcal{T}_{a,i} < \gamma_i < \gamma_{s,i}-(\gamma_{s,i}+\gamma_{u,i})\tau_d/\mathcal{T}_{a,i}$. Thus, by using Lemma 4.8, (4.22) yields that

$$
\begin{aligned}
&\mathbb{E}\{V_{\bar{\xi}(t_{N_{\bar{\xi}}(T,t_0)}+1)}(T^-,\bar{x}(T^-))\}\\
&\quad\leq \frac{\Delta_2 c_1(1-\kappa)^{p-1}}{\beta}\mathbb{E}\{V_{\bar{\xi}(\nu_0)}(\nu_0,\bar{x}(\nu_0))\}e^{-\frac{1}{\beta(1-\kappa)}\int_{\nu_0}^{T}[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)]ds}\\
&\qquad\times e^{\sum_{i=1}^{l}[\frac{\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}{\mathcal{T}_{a,i}}-\gamma_i]T_i(T,\nu_0)}+\frac{\Delta_2 c_1(1-\kappa)^{p-1}}{\beta}\\
&\qquad\times\int_{\nu_0}^{T}e^{-\frac{1}{\beta(1-\kappa)}\int_s^T[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta+\sum_{i=1}^{l}\left[\frac{\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}{\mathcal{T}_{a,i}}-\gamma_i\right]T_i(T,s)}\\
&\qquad\times[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)]\sup_{\theta\in[-\tau,0]}\mathbb{E}|x(s+\theta)|^p ds\\
&\qquad+\frac{\Delta_2 c_1(1-\kappa)^{p-1}}{\beta}\int_{\nu_0}^{T}e^{-\frac{1}{\beta(1-\kappa)}\int_s^T[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta}\\
&\qquad\times e^{\sum_{i=1}^{l}[\frac{\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}{\mathcal{T}_{a,i}}-\gamma_i]T_i(T,s)}\chi(|\varpi(s)|)ds.
\end{aligned}
\tag{4.23}
$$

From Lemma 3.4, condition (D_1), and (4.23), we have

$$\begin{aligned}\mathbb{E}|x(T)|^p \leq{}& \frac{\Delta_2}{\beta}\mathbb{E}\{V_{\bar{\xi}(\nu_0)}(\nu_0,\bar{x}(\nu_0))\}e^{-\frac{1}{\beta(1-\kappa)}\int_{\nu_0}^{T}[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)]ds}\\&\times e^{\sum_{i=1}^{l}[\frac{\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}{\mathcal{T}_{a,i}}-\gamma_i]T_i(T,\nu_0)}+\kappa\sup_{\theta\in[-\tau,0]}\mathbb{E}|x(t+\theta)|^p\\&+\frac{\Delta_2}{\beta}\int_{\nu_0}^{T}e^{-\frac{1}{\beta(1-\kappa)}\int_s^T[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta}\\&\times e^{\sum_{i=1}^{l}[\frac{\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}{\mathcal{T}_{a,i}}-\gamma_i]T_i(T,s)}[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)]\\&\times\sup_{\theta\in[-\tau,0]}\mathbb{E}|x(s+\theta)|^p ds+\frac{\Delta_2}{\beta}\int_{\nu_0}^{T}e^{-\frac{1}{\beta(1-\kappa)}\int_s^T[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta}\\&\times e^{\sum_{i=1}^{l}[\frac{\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}{\mathcal{T}_{a,i}}-\gamma_i]T_i(T,s)}\chi(|\varpi(s)|)ds.\end{aligned}\tag{4.24}$$

On the other hand, by letting $\gamma=\min_{i=1,2,\ldots,l}\{\gamma_i-\log(\frac{2\mu_i c'_{s,i}}{1-\kappa})/\mathcal{T}_{a,i}\}>0$, it follows that $\sum_{i=1}^{l}[(\log(\frac{2\mu_i c'_{s,i}}{1-\kappa})/\mathcal{T}_{a,i})-\gamma_i]T_i(T,\nu_0)\leq-\gamma(T-\nu_0)$ and $\sum_{i=1}^{l}[(\log(\frac{2\mu_i c'_{s,i}}{1-\kappa})/\mathcal{T}_{a,i})-\gamma_i]T_i(T,s)\leq-\gamma(T-s)$. Then, (4.24) yields that $\mathbb{E}|x(T)|^p\leq\frac{\Delta_2}{\beta}\mathbb{E}\{V_{\bar{\xi}(\nu_0)}(\nu_0,\bar{x}(\nu_0))\}\ e^{-\frac{1}{\beta(1-\kappa)}\int_{\nu_0}^{T}[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)]ds-\gamma(T-\nu_0)}+\kappa\mathbb{E}|x(t-\tau(t))|^p+\frac{\Delta_2}{\beta}\int_{\nu_0}^{T}e^{-\frac{1}{\beta(1-\kappa)}\int_s^T[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta-\gamma(T-s)}[\lambda_{1,\hat{\varrho}(s)}(s)+\lambda_{2,\hat{\varrho}(s)}(s)]\sup_{\theta\in[-\tau,0]}\mathbb{E}|x(s+\theta)|^p ds+\frac{\Delta_2}{\beta}\int_{\nu_0}^{T}e^{-\frac{1}{\beta(1-\kappa)}\int_s^T[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta-\gamma(T-s)}\chi(|\varpi(s)|)ds$.

The remaining proof of this theorem is similar to that in Step 3 of the proof of Theorem 4.3, so it is omitted. ∎

Remark 11. In the proof of Theorem 4.9, constants c'_{s,ς_k} and c'_{u,ς_k} in condition (D_4) are absorbed into Λ_{1,ς_k} and Λ_{2,ς_k} ($\varsigma_k\in\Xi$), respectively. This treatment is different from the one given in Theorem 4.3. Note that the value of these two constants Λ_{1,ς_k} and Λ_{2,ς_k} may be larger than one without any constraints. Thus, in Theorem 4.9, β in condition (D_4) is also a flexible and adjustable parameter, which is generally chosen to be small such that $\Delta_2\in(0,1)$, (4.17), and (4.18) are satisfied.

Remark 12. In Theorem 4.9, the merging switching signal technique is first used, and then the new switching signal $\bar{\xi}(t)$ arises. Conditions (D_1) and (D_3) commonly exist in the available literature; see [27, 39] and the references therein. Since the input control in system (3.1) is $u(t)=h_{\xi(t-\tau_d)}(t,\bar{x}(t),\varpi(t))$, the Itô operator is obtained as condition (D_2), not condition (C_2), with the coefficients $\lambda_{0,\bar{\xi}(t)}(t)$, $\lambda_{1,\hat{\xi}(t)}(t)$ and $\lambda_{2,\hat{\xi}(t)}(t)$. Condition (D_4) is simultaneously reflected since $\lambda_{0,\bar{\xi}(t_k+\tau_d)}(t)\neq\lambda_{0,\bar{\xi}(t_k)}(t)$ for all $t\in[t_k,t_{k+1})$ ($k\in\mathbb{N}$). In addition, condition (D_2) is also a more relaxed Lyapunov monotonicity condition since it can be seen from condition (D_4) that the crucial time-varying coefficients $\lambda_{0,\bar{\xi}(t)}(t)$ ($\bar{\xi}(t)\in\Xi\times\Xi$) have the indefinite sign on $[t_0,+\infty)$.

Remark 13. In [28, 29], the stability analysis for neutral switched SDSs and deterministic switched NDSs with the asynchronous switching was considered by using multiple Lyapunov–Krasovskii functionals, respectively. Theorem 4.9 has three distinctive features: (1) in system (3.1), the time-varying delay $\tau(t)$ is a bounded function; (2) the external input disturbance is involved; and (3) the time-varying coefficients $\lambda_{0,\bar{\xi}(t)}(t)$ ($\bar{\xi}(t) \in \Xi \times \Xi$) are permitted to be not only mode-dependent but also to have the indefinite sign.

When the neutral term $\mathcal{D}(t, x(t-\tau(t)))$ in neutral switched SDSs (3.1) is not present, based on Corollary 4.2, multiple Lyapunov–Krasovskii functions, the MDADT, and the merging switching signal technique, the result on the ISS/iISS of the corresponding switched SDSs with the asynchronous switching can be readily stated as follows.

COROLLARY 4.10. *Suppose that there exist some Lyapunov–Krasovskii functions* $V_{\varsigma_k}(\cdot,\cdot) \in \mathcal{C}^{1,2}([t_k, t_{k+1}) \times R^{n_x}; [0, +\infty))$ $(k \in \mathbb{N})$, *two constants* $c_1 > 0$, $c_2 > 0$, *and two integrable functions* $\lambda_{0,\bar{\xi}(\cdot)}(\cdot) : [t_0, +\infty) \times R$ *and* $\lambda_{1,\hat{\xi}(\cdot)}(\cdot) : [t_0, +\infty) \times [0, +\infty)$ *such that*

(D_5) *for any* $x \in R^{n_x}$, $t \geq t_0$, *and* $p \geq 2$, $c_1|x|^p \leq V_{\bar{\xi}(t)}(t, x) \leq c_2|x|^p$;

(D_6) *for any* $x, y \in R^{n_x}$, $\varpi \in R^{n_\varpi}$, *and* $t \in [t_k, t_{k+1})$ $(k \in \mathbb{N})$, $\mathcal{L}V_{\bar{\xi}(\cdot)}(t, x, y, \varpi) \leq \lambda_{0,\bar{\xi}(\cdot)}(t)V_{\bar{\xi}(t)}(t, x) + \lambda_{1,\hat{\xi}(t)}(t)|y|^p + \chi_{\hat{\xi}(t)}(|\varpi(t)|)$;

(D_7) *for any* $x, y \in R^{n_x}$ *and* $(\varsigma_i, \varsigma_j, \varsigma_k, \varsigma_l) \in \Xi \times \Xi \times \Xi \times \Xi$, $\varsigma_i \neq \varsigma_k$, *or* $\varsigma_j \neq \varsigma_l$, $V_{\varsigma_i,\varsigma_j}(t, x) \leq \mu_{\varsigma_i,\varsigma_j} V_{\varsigma_k,\varsigma_l}(t, x)$, *where* $\mu_{\varsigma_i,\varsigma_j} = \mu_{\varsigma_i} > 1$ ($\varsigma_i \neq \varsigma_k$) *and* $\mu_{\varsigma_i,\varsigma_j} = 1$ ($\varsigma_i = \varsigma_k$).

(D_8) *there exist some constants* $\gamma_{s,\varsigma_k} > 0$, $\gamma_{u,\varsigma_k} > 0$, $\beta > 0$, $c'_{s,\varsigma_k} \in R$, *and* $c'_{u,\varsigma_k} \in R$ *satisfying* $\beta e^{\sum_{i=1}^{l} N_{0,i}[c'_{u,i} + \log(\mu_i e^{c'_{s,i}})] + \sum_{i=1}^{l} \frac{\tau_d}{\mathcal{T}_{a,i}} \log(\mu_i e^{c'_{s,i}}) + \sum_{i=1}^{l} (\gamma_{s,i} + \gamma_{u,i}) N_{0,i} \tau_d} / c_1 \in (0, 1)$ *for any* $t_k + \tau_d \leq s < t \leq t_{k+1}$, $\int_s^t [\lambda_{0,\bar{\xi}(\theta)}(\theta) + \lambda_{1,\hat{\xi}(\theta)}(\theta)/\beta] d\theta \leq c'_{s,\varsigma_k} - \gamma_{s,\varsigma_k}\mathcal{M}_{\varsigma_k}(t, s)$ *and for any* $t_k \leq s < t \leq t_k + \tau_d$, $\int_s^t [\lambda_{0,\bar{\xi}(\theta)}(\theta) + \lambda_{1,\hat{\xi}(\theta)}(\theta)/\beta] d\theta \leq c'_{u,\varsigma_k} + \gamma_{u,\varsigma_k}\bar{\mathcal{M}}_{\varsigma_k}(t, s)$. *Then, neutral switched SDSs* (3.1) *with the asynchronous switching when* $\mathcal{D}(t, x(t-\tau(t))) = 0$ *are p-ISES, p-iISES, SISES, and* $e^{\alpha t}$*-p-ISES* ($\alpha > 0$) *over* $\mathcal{S}_{ave}[\mathcal{T}_{a,i}, N_{0,i}]$, *respectively, where the ith mode ADT is*

$$\mathcal{T}_{a,i} > \mathcal{T}^*_{a,i} = \frac{\log(\mu_i e^{c'_{s,i}}) + (\gamma_{s,i} + \gamma_{u,i})\tau_d}{\gamma_{s,i}} \quad (i \in \Xi).$$

Proof. The proof of Corollary 4.10 is similar to that of Theorem 4.9, and it is omitted. ∎

THEOREM 4.11. *Assume that conditions* (D_1)–(D_3) *of Theorem* 4.9 *hold and that condition* (D_4) *in Theorem* 4.9 *is replaced by*

(D'_4) *there exist some constants* $\beta > 0$, $c'_{s,\varsigma_k} \in R$ *and* $c'_{u,\varsigma_k} \in R$ *such that* $[\beta e^{\sum_{i=1}^{l} N_{0,i}[\frac{c'_{u,i}}{1-\kappa} + \log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})] + \sum_{i=1}^{l} \frac{\tau_d}{\mathcal{T}_{a,i}} \log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}]/(c_1(1-\kappa)^{p-1}) \in (0, 1)$ *for any* $t_k + \tau_d \leq s < t < t_{k+1}$, $\int_s^t [(1-\kappa)(\lambda_{0,\bar{\xi}(\theta)}(\theta) + \log(\mu_{\bar{\xi}(\theta)} e^{\frac{c'_{s,\hat{\xi}(\theta)}}{1-\kappa}})/\mathcal{T}_{a,\varsigma_k}) + (\lambda_{1,\hat{\xi}(\theta)}(\theta) + \lambda_{2,\hat{\xi}(\theta)}(\theta))/\beta] d\theta \leq c'_{s,\varsigma_k}$ *and for any* $t_k \leq s < t < t_k + \tau_d$, $\int_s^t [(1-\kappa)(\lambda_{0,\bar{\xi}(\theta)}(\theta) + \log(\mu_{\bar{\xi}(\theta)} e^{\frac{c'_{u,\hat{\xi}(\theta)}}{1-\kappa}})/\mathcal{T}_{a,\varsigma_k}) + (\lambda_{1,\hat{\xi}(\theta)}(\theta) + \lambda_{2,\hat{\xi}(\theta)}(\theta))/\beta] d\theta \leq c'_{u,\varsigma_k}$, *where* $\mathcal{T}_{a,\varsigma_k}$ *is the* ς_k*th mode ADT and* $\upsilon = \sup_{t \geq t_0} \int_{t-\tau}^t [\lambda_{1,\hat{\xi}(s)}(s) + \lambda_{2,\hat{\xi}(s)}(s)] ds < +\infty$. *Then, neutral switched SDSs* (3.1) *with the asynchronous switching are p-iISES over* $\mathcal{S}_{ave}[\mathcal{T}_{a,i}, N_{0,i}]$.

Proof. From (4.21), it follows that $\mathbb{E}\{V_{\bar{\xi}(t_{N_{\bar{\xi}}(T,\nu_0)})}(T^-,\bar{x}(T^-))\} \le e^{\sum_{i=1}^{l} N_{0,i}\frac{c'_{u,i}}{1-\kappa}}$ $e^{\sum_{i=1}^{l} N_{0,i}\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})]+\sum_{i=1}^{l}\frac{\tau_d}{\mathcal{T}_{a,i}}\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})-\frac{1}{\beta(1-\kappa)}\int_{\nu_0}^{T}[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)]ds}\mathbb{E}\{V_{\bar{\xi}(\nu_0)}(\nu_0,$ $\bar{x}(\nu_0))\} + e^{\sum_{i=1}^{l} N_{0,i}[\frac{c'_{u,i}}{1-\kappa}+\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})]+\sum_{i=1}^{l}\frac{\tau_d}{\mathcal{T}_{a,i}}\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}\int_{t_0}^{T}$ $e^{-\frac{1}{\beta(1-\kappa)}\int_s^T[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta}[\lambda_{1,\hat{\xi}(s)}(s)+\lambda_{2,\hat{\xi}(s)}(s)]\sup_{\theta\in[-\tau,0]}\mathbb{E}|x(s+\theta)|^p ds$ $+\ e^{\sum_{i=1}^{l} N_{0,i}[\frac{c'_{u,i}}{1-\kappa}+\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})]}\ e^{\sum_{i=1}^{l}\frac{\tau_d}{\mathcal{T}_{a,i}}\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})}\int_{t_0}^{T} e^{-\frac{1}{\beta(1-\kappa)}\int_s^T[\lambda_{1,\hat{\xi}(\theta)}(\theta)+\lambda_{2,\hat{\xi}(\theta)}(\theta)]d\theta}$ $\chi(|\varpi(s)|)ds$. The remaining proof of this theorem is similar to that of Theorem 4.5 and is therefore omitted. ∎

If $\lambda_{0,\bar{\xi}(t)}(t)=\lambda_{0,\bar{\xi}(t)}$ and $\lambda_{i,\hat{\xi}(t)}(t)=\lambda_{i,\hat{\xi}(t)}$ $(i=1,2)$, then for neutral switched SDSs (3.1) with the asynchronous switching, we obtain the following.

COROLLARY 4.12. *Let conditions* (D_1) *and* (D_3) *of Theorem* 4.9 *be satisfied, and assume that conditions* (D_2) *and* (D_4) *in Theorem* 4.9 *are replaced by*

(D_2'') *for any* $x, y \in R^{n_x}$, $\varpi \in R^{n_\varpi}$, *and* $t \in [t_k, t_{k+1})$ $(k \in \mathbb{N})$, $\mathcal{L}V_{\bar{\xi}(t)}(t,x,y,\varpi) \le \lambda_{0,\bar{\xi}(t)}V_{\bar{\xi}(t)}(t,x-\mathcal{D}(t,y))+\lambda_{1,\hat{\xi}(t)}|x|^p+\lambda_{2,\hat{\xi}(t)}|y|^p+\chi_{\hat{\xi}(t)}(|\varpi|)$;

(D_4'') *there exist some constants* $\gamma_{s,\varsigma_k} > 0$, $\gamma_{u,\varsigma_k} > 0$, $\beta > 0$, $c'_{s,\varsigma_k} \in R$, *and* $c'_{u,\varsigma_k} \in R$ *such that* $\Delta_2 \in (0,1)$, *with* Δ_2 *being given in Theorem* 4.9, *for any* $t_k+\tau_d \le s < t < t_{k+1}$, $[(1-\kappa)\lambda_{0,\hat{\xi}(t_k)}+\gamma_{s,\varsigma_k}+(\lambda_{1,\hat{\xi}(t_k)}+\lambda_{2,\hat{\xi}(t_k)})/\beta]\mathcal{M}_{\varsigma_k}(t,s) \le c'_{s,\varsigma_k}$, *and for any* $t_k \le s < t < t_k+\tau_d$, $[(1-\kappa)\lambda_{0,\bar{\xi}(t_k)}-\gamma_{u,\varsigma_k}+(\lambda_{1,\hat{\xi}(t_k)}+\lambda_{2,\hat{\xi}(t_k)})/\beta]\bar{\mathcal{M}}_{\varsigma_k}(t,s) \le c'_{u,\varsigma_k}$.
Then, neutral switched SDSs (3.1) *with the asynchronous switching are p-ISES, p-iISES, SISES, and* $e^{\alpha t}$*-p-ISES* $(\alpha > 0)$ *over* $\mathcal{S}_{ave}[\mathcal{T}_{a,i}, N_{0,i}]$, *respectively, where the ith mode ADT is*

$$\mathcal{T}_{a,i} > \mathcal{T}^*_{a,i} = \frac{\log\left(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}}\right)+(\gamma_{s,i}+\gamma_{u,i})\tau_d}{\gamma_{s,i}} \quad (i \in \Xi).$$

Proof. When $\lambda_{0,\bar{\xi}(t)}(t)=\lambda_{0,\bar{\xi}(t)}$ and $\lambda_{i,\hat{\xi}(t)}(t)=\lambda_{i,\hat{\xi}(t)}$ $(i=1,2)$, the reasoning processes from condition (D_4'') to condition (D_4) can be realized, respectively. Thus, the proof of this corollary can be completed based on Theorem 4.9. ∎

5. Example. Consider the coupled system consisting of a mass-spring-damper (MSD) model with a pendulum [51]. The mathematical expression of this model is NDSs: $M\ddot{y}(t)+C\dot{y}(t)+Ky(t)+m\ddot{y}(t-\tau)=0$ on $t \ge t_0 = 0$, where M, C, K denote the mass, stiffness, and damping of the MSD model, m is the mass of a pendulum, $\tau > 0$ is the time delay, and $y(t)$, $\dot{y}(t)$, $\ddot{y}(t)$ represent the position, velocity, and acceleration of the MSD model at time t. If this physical model is perturbed by an external random force, then this model is described as

$$M\ddot{y}(t)+C\dot{y}(t)+Ky(t)+m\ddot{y}(t-\tau)+F(t)=0 \tag{5.1}$$

on $t \ge t_0 = 0$, where $F(t)$ denotes the external random force. It is assumed that this external random force is subject to a white noise, the input controller with the external input disturbance, and abrupt changes in the parameters. Then, $F(t)=F_{1,\xi(t)}(t,\dot{y}(t),\dot{y}(t-\tau),u(t))+F_{2,\xi(t)}(t,\dot{y}(t),y(t-\tau),\dot{y}(t-\tau))\dot{\mathcal{B}}(t)$, where $\dot{\mathcal{B}}(t)$ is a scalar white noise ($\mathcal{B}(t)$ is a scalar Brownian motion on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge t_0},\mathbb{P})$) and $\xi(t)$ is a switching signal with its state in $\Xi=\{1,2\}$. Here, $F(t)$ in (5.1) is determined by $F_{1,\xi(t)}(t,\dot{y}(t),\dot{y}(t-\tau),u(t))=a_{1,\xi(t)}\cos^2(2.5t)\dot{y}(t)+a_{2,\xi(t)}\cos^2(2.5t)\dot{y}(t-$

$\tau) + u(t)$ $(\xi(t) = 1)$, $F_{1,\xi(t)}(t, y(t-\tau), \dot{y}(t-\tau), u(t)) = a_{1,\xi(t)} \sin^2(2.5t)y(t-\tau) + a_{2,\xi(t)} \sin^2(2.5t)\dot{y}(t-\tau)+u(t)$ $(\xi(t) = 2)$, $F_{2,\xi(t)}(t, y(t), \dot{y}(t), u(t)) = a_{3,\xi(t)} \cos(2.5t)y(t) + a_{4,\xi(t)} \cos(2.5t)\dot{y}(t)$ $(\xi = 1)$, and $F_{2,\xi(t)}(t, \dot{y}(t), \dot{y}(t-\tau), u(t)) = a_{3,\xi(t)} \sin(2.5t)\dot{y}(t) + a_{4,\xi(t)} \sin(2.5t)\dot{y}(t-\tau)$ $(\xi(t) = 2)$ with $u(t) = h_{\xi(t-\tau_d)}(t, y(t), \dot{y}(t), \dot{y}(t-\tau), \varpi(t))$ $(0 \leq \tau_d < \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\};\ \varpi(t) \in R)$, and t_k $(k \in \mathbb{N})$ are the switching time instants.

Let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, then (5.1) can be abstracted as

$$\begin{aligned} &d[x(t) - \mathcal{D}x(t-\tau)] \\ &\quad= [A_{\xi(t)}x(t) + F_{\xi(t)}(t, x(t), x(t-\tau), u(t))]dt + G_{\xi(t)}(t, x(t), x(t-\tau))d\mathcal{B}(t) \end{aligned} \tag{5.2}$$

on $t \geq t_0 = 0$, where $x(t) = (x_1(t)\ x_2(t))^T$,

$$\mathcal{D} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{m}{M} \end{bmatrix}, A_{\xi(t)} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{C}{M} \end{bmatrix},$$

$F_1(t, x(t), x(t-\tau), u(t)) = (0\ \ [-a_{1,1} \cos^2(2.5t)x_2(t) - a_{2,1} \cos^2(2.5t)x_2(t-\tau) - u(t)] /M)^T$, $F_2(t, x(t), x(t-\tau), u(t)) = (0\ \ [-a_{1,2} \sin^2(2.5t)x_1(t-\tau) - a_{2,2} \sin^2(2.5t)x_2(t-\tau) - u(t)]/M)^T$, $G_1(t, x(t), x(t-\tau)) = (0\ \ [-a_{3,1} \cos(2.5t)x_1(t) - a_{4,1} \cos(2.5t)x_2(t)] /M)^T$, and $G_2(t, x(t), x(t-\tau)) = (0\ \ [-a_{3,2} \sin(2.5t)x_1(t-\tau) - a_{4,2} \sin(2.5t)x_2(t-\tau)]/M)^T$. Let $M = 10$, $K = 10$, $C = 20$, $m = 1$, $a_{1,1} = 1$, $a_{2,1} = -2$, $a_{3,1} = 1$, $a_{4,1} = -2$, $a_{1,2} = 2$, $a_{2,2} = -1$, $a_{3,2} = -1$, and $a_{4,2} = 1$.

The following discussion is related to Theorems 4.3 and 4.9, which is separated into two cases:

(i) Synchronous switching (i.e., $\tau_d = 0$). Define two Lyapunov–Krasovskii functions $V_1(t, x(t) - \mathcal{D}x(t-\tau)) = \frac{9}{10}|x(t) - \mathcal{D}x(t-\tau)|^2$ and $V_2(t, x(t) - \mathcal{D}x(t-\tau)) = |x(t) - \mathcal{D}x(t-\tau)|^2$ with $c_1 = 0.9$ and $c_2 = 1.0$. Thus, $\mu_1 = 1.01$, and $\mu_2 = 1.1111$. For subsystem 1 with the input controller $u(t) = h_1(t, x(t), x(t-\tau(t)), \varpi(t)) = 10(x_2(t) + 0.1x_2(t-\tau)) - 19 \sin^2(2.5t)\varpi(t)$ for $t \in [t_k, t_{k+1})$, the Itô operator is computed as $\mathcal{L}V_1(t, x(t), x(t-\tau), \varpi(t)) \leq \lambda_{0,1}(t)V_1(t, x(t) - \mathcal{D}x(t-\tau)) + \lambda_{1,1}(t)|x(t)|^2 + \lambda_{2,1}(t)|x(t-\tau)|^2 + \chi_1(|\varpi(t)|)$, where $\lambda_{0,1}(t) = -0.63 - 0.675 \cos(5t)$, $\lambda_{1,1}(t) = 0.063 + 0.063 \cos(5t)$, $\lambda_{2,1}(t) = 0.27 + 0.09 \cos(5t)$, and $\chi_1(|\varpi(t)|) = 1.71 \sin^2(2.5t)|\varpi(t)|^2$.

For subsystem 2 with the input controller $u(t) = h_2(t, x(t), x(t-\tau(t)), \varpi(t)) = 10(x_2(t) + \frac{m}{M}x_2(t-\tau)) - 19 \cos^2(2.5t)\varpi(t)$ for $t \in [t_k, t_{k+1})$, the Itô operator is computed as $\mathcal{L}V_2(t, x(t), x(t-\tau), \varpi(t)) \leq \lambda_{0,2}(t)V_2(t, x(t) - \mathcal{D}x(t-\tau)) + \lambda_{1,2}(t)|x(t)|^2 + \lambda_{2,2}(t)|x(t-\tau)|^2 + \chi_2(|\varpi(t)|)$, where $\lambda_{0,2}(t) = -0.7 + 0.75 \cos(5t)$, $\lambda_{1,2}(t) = 0.15 - 0.15 \cos(5t)$, $\lambda_{2,2}(t) = 0.205 - 0.005 \cos(5t)$, and $\chi_2(|\varpi(t)|) = 1.9 \cos^2(2.5t)|\varpi(t)|^2$.

Taking $\beta = 0.58$ and $N_{0,1} = N_{0,2} = 1$, there exist $\gamma = 0.01$ and $c = 0.1631$ such that $\Delta_1 = \frac{\beta e^{\sum_{i=1}^2 N_{0,i} \log(\mu_i)} e^{\frac{c}{1-\kappa}}}{c_1(1-\kappa)^{p-1}} = 0.9632 \in (0, 1)$, $\mathcal{T}^*_{a,1} = 0.5268$, and $\mathcal{T}^*_{a,2} = 5.5774$. Moreover, for any $t_0 \leq s < t$, $\int_s^t [(1-\kappa)\lambda_{0,\xi(\theta)}(\theta) + \gamma + \frac{\lambda_{1,\xi(\theta)}(\theta) + \lambda_{2,\xi(\theta)}(\theta)}{\beta}]d\theta \leq c$.

Consequently, all conditions of Theorem 4.3 hold, which can guarantee that system (5.2) with the synchronous switching is ISES in mean square, iISES in mean square, SISES, and $e^{\alpha t}$ $(\alpha > 0)$-ISES in mean square. With $\mathcal{T}_{a,1} = 0.6$, $\mathcal{T}_{a,2} = 6.0$, $\tau = 2.3$, and the initial value $(x(t)\ y(t))^T = [-1.0\ 1.0]$ $(t \in [-2.3, 0])$, and $\xi(0) = 1$, Figure 1 illustrates the switching signal $\xi(t)$ in system (5.2). Figures 2 and 3 show the state responses of the resulting system without and with the external input disturbance $(\varpi(t) = \sin(t))$ imposed on the synchronous input control, respectively.

(ii) Asynchronous switching (i.e., $\tau_d \in (0, \inf_{k=0,1,2,\ldots}\{t_{k+1} - t_k\})$). Firstly, we introduce the merge switching signal as $\bar{\xi}(t) = (\xi_1(t), \xi_2(t))$ with $\xi_1(t) = \xi(t)$ and $\xi_2(t) = \xi(t-\tau_d)$. Then, define the Lyapunov–Krasovskii function $V_{\bar{\xi}(t)}(t, x(t) - \mathcal{D}x(t-\tau)) = |x(t) - \mathcal{D}x(t-\tau)|^2$ with $c_1 = c_2 = 1.0$ and $\mu_1 = \mu_2 = 1.001$.

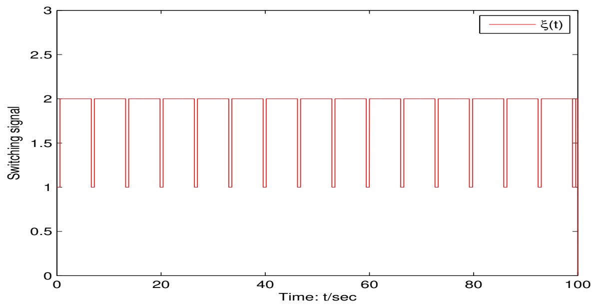

FIG. 1. *Switching signal $\xi(t)$ in* (5.2).

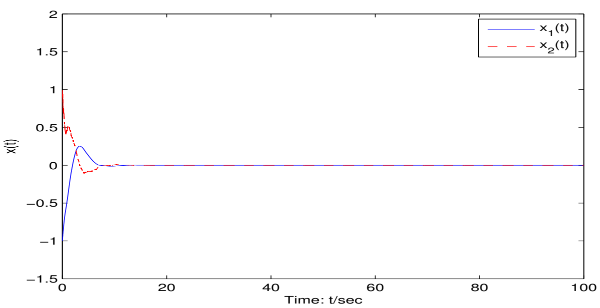

FIG. 2. *State response of iISES for the resulting system without the external input disturbance imposed on the synchronous input control.*

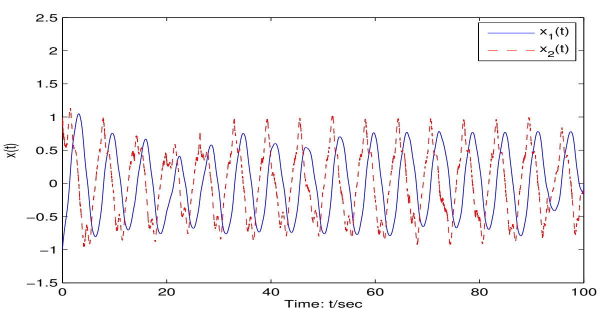

FIG. 3. *State response of iISES for the resulting system with the external input disturbance $\varpi(t) = \sin(t)$ imposed on the synchronous input control.*

On subsystem 1, when the input controller is $u(t) = h_1(t, x(t), x(t-\tau(t)), \varpi(t)) = 10(x_2(t)+0.1x_2(t-\tau))-19\sin^2(2.5t)\varpi(t)$ for $t \in [t_k+\tau_d, t_{k+1})$, the Itô operator is computed with $\mathcal{L}V_{(1,1)}(t, x(t), x(t-\tau), \varpi(t)) \leq [-0.7 - 0.75\cos(5t)]V_{(1,1)}(t, x(t) - \mathcal{D}x(t-\tau)) + [0.07 + 0.07\cos(5t)]|x(t)|^2 + [0.3 + 0.1\cos(5t)]|x(t-\tau)|^2 + 1.9\sin^2(2.5t)|\varpi(t)|^2$, and when the input controller is $u(t) = h_2(t, x(t), x(t-\tau(t)), \varpi(t)) = -(x_2(t) + \frac{m}{M}x_2(t-\tau)) - 18\sin^2(2.5t)\varpi(t)$, $t \in [t_k, t_k + \tau_d)$, the Itô operator is computed with $\mathcal{L}V_{(1,2)}(t, x(t), x(t-\tau), \varpi(t)) \leq [-0.65-0.75\cos(5t)]V_{(1,2)}(t, x(t)-\mathcal{D}x(t-\tau))+[0.07+0.07\cos(5t)]|x(t)|^2 + [0.3 + 0.1\cos(5t)]|x(t-\tau)|^2 + 1.8\sin^2(2.5t)|\varpi(t)|^2$.

On subsystem 2, when the input controller is $u(t) = h_1(t, x(t), x(t-\tau(t)), \varpi(t)) = -(x_2(t) + \frac{m}{M}x_2(t-\tau)) - 18\cos^2(2.5t)\varpi(t)$ for $t \in [t_k, t_k+\tau_d)$, the Itô operator is computed with $\mathcal{L}V_{(2,1)}(t, x(t), x(t-\tau), \varpi(t)) \le [-0.65 + 0.75\cos(5t)]V_{(2,1)}(t, x(t) - \mathcal{D}x(t-\tau)) + [0.15 - 0.15\cos(5t)]|x(t)|^2 + [0.205 - 0.005\cos(5t)]|x(t-\tau)|^2 + 1.8\cos^2(2.5t)|\varpi(t)|^2$, and when the input controller is $u(t) = h_2(t, x(t), x(t-\tau(t)), \varpi(t)) = 10(x_2(t) + \frac{m}{M}x_2(t-\tau)) - 19\cos^2(2.5t)\varpi(t)$ for $t \in [t_k+\tau_d, t_{k+1})$, the Itô operator is calculated by $\mathcal{L}V_{(2,2)}(t, x(t), x(t-\tau), \varpi(t)) \le [-0.7 + 0.75\cos(5t)]V_{(2,2)}(t, x(t) - \mathcal{D}x(t-\tau)) + [0.15 - 0.15\cos(5t)]|x(t)|^2 + [0.205 - 0.005\cos(5t)]|x(t-\tau)|^2 + 1.9\cos^2(2.5t)|\varpi(t)|^2$.

By letting $\beta = 0.60$ and $\tau_d = 0.5$, then $c'_{s,1} = c'_{u,1} = 0.1567$, $c'_{s,2} = c'_{u,2} = 0.1667$, $\gamma_{s,1} = 0.0133$, $\gamma_{s,2} = 0.0383$, $\gamma_{u,1} = 0.0317$, $\gamma_{u,2} = 0.0067$, $\mathcal{T}^*_{a1} = 14.7895$, and $\mathcal{T}^*_{a,2} = 5.4256$. Furthermore, if $N_{0,1} = N_{0,2} = 1.0$, $\mathcal{T}_{a,1} = 14.7896$, and $\mathcal{T}_{a,2} = 5.5$, then we have $\beta e^{\sum_{i=1}^2 N_{0,i}[\frac{c'_{u,i}}{1-\kappa} + \log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}})] + \sum_{i=1}^2 \frac{\tau_d}{\mathcal{T}_{a,i}}\log(\mu_i e^{\frac{c'_{s,i}}{1-\kappa}}) + \sum_{i=1}^2(\gamma_{s,i}+\gamma_{u,i})N_{0,i}\tau_d}/(1-\kappa) = 0.8739 \in (0,1)$. Thus, all conditions of Theorem 4.9 hold, which means that system (5.2) with the asynchronous switching is ISES in mean square, iISES in mean square, SISES, and $e^{\alpha t}$ $(\alpha > 0)$-ISES in mean square. Figure 4 gives the switching signals $\xi(t)$ and $\xi(t-\tau_d)$, which demonstrates that the switching signal in the input controller lags behind that of the controlled subsystems. When $\mathcal{T}_{a,1} = 14.7896$, $\mathcal{T}_{a,2} = 5.5$, $\tau = 2.3$, $\tau_d = 0.5$, and the initial value $(x(t)\ y(t))^T = [-1.0\ 1.0]$ $(t \in [-2.3, 0])$, the state responses of the resulting system without and with the external input disturbance ($\varpi(t) = \sin(t)$) imposed on the asynchronous input control are shown in Figures 5 and 6, respectively.

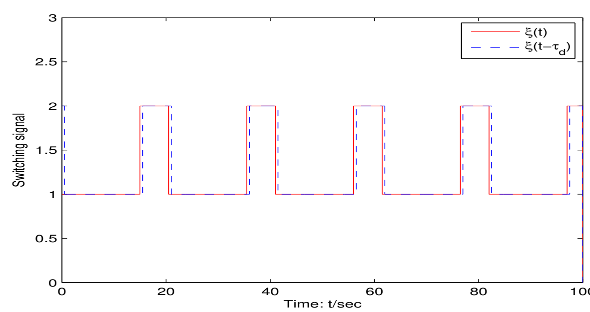

FIG. 4. *Two switching signals* $\xi(t)$ *and* $\xi(t-\tau_d)$ *in system* (5.2).

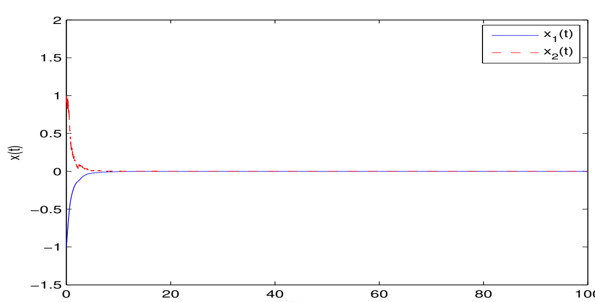

FIG. 5. *State responses of iISES for the resulting system without the external input disturbance imposed on the asynchronous input control.*

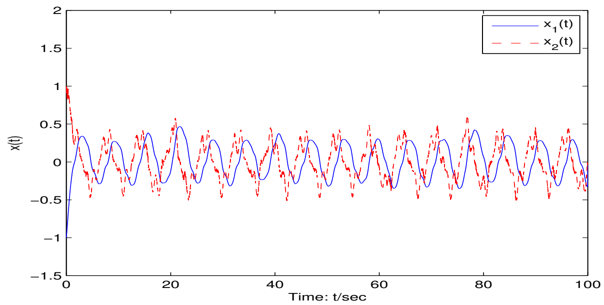

FIG. 6. *State responses of iISES for the resulting system with the external input disturbance $\varpi(t) = \sin(t)$ imposed on the asynchronous input control.*

6. Conclusion. By using multiple Lyapunov–Krasovskii functions, the MDADT, and the generalized delay integral inequality, we investigated the problems of stochastic ISS/iISS of neutral switched SDSs under the synchronous switching and the asynchronous switching, respectively. The merging switching signal technique was used for the asynchronous switching. Conditions are obtained based on integral inequality, which can permit the crucial time-varying coefficients in the Lyapunov monotonicity condition to have the indefinite sign over the whole time horizon.

Future work will involve designing of the adaptive controller algorithm on the input-to-state stabilization and the integral input-to-state stabilization for time-varying neutral delay systems, in particular when the time-varying coefficient of the current state in the Lyapunov monotonicity condition is permitted to have the indefinite sign and the time-varying delay is a bounded function.

Acknowledgments. The authors thank the associate editors and the reviewers for valuable suggestion and comments, which have helped us greatly improve the presentation of the paper.

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