IET Research Journals

Journals
The Institution of Engineering and Technology

An overview of recent advances in stability of linear systems with time-varying delays

ISSN 1751-8644
doi: 0000000000
www.ietdl.org

Xian-Ming Zhang[1], *Qing-Long Han*[1*], *Alexandre Seuret*[2], *Frédéric Gouaisbaut*[2], *Yong He*[3]

[1] *School of Software and Electrical Engineering, Swinburne University of Technology, Melbourne, VIC 3122, Australia*
[2] *LAAS-CNRS, Université de Toulouse, CNRS, UPS, Toulouse, France*
[3] *School of Automation, China University of Geosciences, Wuhan 430074, P. R. China*
* *E-mail: qhan@swin.edu.au*

Abstract: This paper provides an overview and in-depth analysis of recent advances in stability of linear systems with time-varying delays. First, recent developments of a delay convex analysis approach, a reciprocally convex approach and the construction of Lyapunov-Krasovskii functionals are reviewed insightfully. Second, in-depth analysis of the Bessel-Legendre inequality and some affine integral inequalities is made, and recent stability results are also summarized, including stability criteria for three cases of a time-varying delay, where information on the bounds of the time-varying delay and its derivative is totally known, partly known and completely unknown, respectively. Third, a number of stability criteria are developed for the above three cases of the time-varying delay by employing canonical Bessel-Legendre inequalities, together with augmented Lyapunov-Krasovskii functionals. It is shown through numerical examples that these stability criteria outperform some existing results. Finally, several challenging issues are pointed out to direct the near future research.

1 Introduction

A time-delay system is also called a system with aftereffect or dead-time [1]. A defining feature of a time-delay system is that its future evolution is related not only to the current state but also to the past state of the system. Time-delay systems are a particular class of infinite dimensional systems, which have complicated dynamic properties compared with delay-free systems. A large number of practical systems encountered in areas, such as engineering, physics, biology, operation research and economics, can be modelled as time-delay systems [2, 3]. Therefore, time-delay systems have attracted continuous interest of researchers in a wide range of fields in natural and social sciences, see, e.g. [4–9].

Stability of time-delay systems is a fundamental issue from both theoretical and practical points of view. Indeed, the presence of time-delays may be either beneficial or detrimental to stability of a practical system. Time-delays are usually regarded as a factor of system destabilization. However, practical engineering applications reveal that, for some dynamical systems, intentional introduction of a specific time-delay may stabilize an unstable system [4]. Hence, one main concern about a time-delay system is to determine the *maximum delay interval* on which the system remains stable. For nonlinear time-delay systems, it is quite challenging due to complicated dynamical properties. For linear time-delay systems, a lot of effort has been made on delay-dependent stability analysis in the last two decades, see, e.g. [7–11]. In this paper, we focus on the following linear system with a time-varying delay described by

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)) \\ x(\theta) = \phi(\theta), \ \ \theta \in [-\bar{h}, 0] \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state; A and A_d are real $n \times n$ constant matrices; $\phi(\theta)$ is the initial condition; and $h(t)$ is a time-delay satisfying $0 \le h_0 \le h(t) \le \bar{h} < \infty$. If $h_0 > 0$, the system (1) is called a linear system with an *interval* time-varying delay [12], which allows the system to be unstable at $h(t) = 0$. Networked control systems and event-triggered control systems can be modelled as such a system with an interval time-varying delay [13–20]. For simplicity of presentation, in what follows, let $h_0 = 0$ if no specific declaration is made.

Recalling some existing results, there are two types of approaches to delay-dependent stability of linear time-delay systems: frequency domain approach and time domain approach. Frequency domain approach-based stability criteria have been long in existence [1, 9]. For some recent developments in the frequency domain, we mention an integral quadratic constraint framework [21–23], which describes the stability of a system in the frequency domain in terms of an integral constraint on the Fourier transform of the input/output signals [24]. In the time domain approach, the direct Lyapunov method is a powerful tool for studying stability of linear time-delay systems [1, 9]. Specifically, there are *complete* Lyapunov functional methods and *simple* Lyapunov-Krasovskii functional methods for estimating the maximum admissible delay upper bound that the system can tolerate and still maintain stability. Complete Lyapunov functional methods can provide necessary and sufficient conditions on stability of linear systems with a constant time-delay [25–28]. Simple Lyapunov-Krasovskii functional methods only provide sufficient conditions on stability of linear time-delay systems. Compared with stability criteria based on complete Lyapunov functional methods, although stability criteria based on simple Lyapunov-Krasovskii functional methods are more conservative, they can be applied easily to control synthesis and filter design of linear time-delay systems [29].

A simple Lyapunov-Krasovskii functional for estimating the maximum delay upper bound for linear time-delay systems is based on a proper simple Lyapunov-Krasovskii functional candidate, which usually includes a double integral term as

$$\int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta \tag{2}$$

where $R > 0$ and $h > 0$ being either a constant or a time-varying function. A delay-dependent stability criterion can be derived based on an estimate of the time derivative of the Lyapunov-Krasovskii functional, in which such an integral term is included

$$\mathscr{I}(\dot{x}) := -\int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s) ds \tag{3}$$

The key to the stability criterion is how to deal with the quadratic integral term $\mathscr{I}(\dot{x})$. Typically, there are several approaches reported in the literature.

i) *Model transformation approach.* The model transformation approach employs the Leibniz-Newton formula $x(t-h) = x(t) - \int_{t-h}^{t} \dot{x}(s)ds$ to transform the system (1) to a system such that a cross-term $\eta(t) := -\int_{t-h}^{t} 2x^T(t)PA_d\dot{x}(s)ds$ is introduced in the derivative of the Lyapunov-Krasovskii functional. Then using the basic inequality (or called Young's inequality)

$$2a^Tb \leq a^TRa + b^TR^{-1}b, \quad a, b \in \mathbb{R}^n \tag{4}$$

or the improved basic inequality [30], for $M \in \mathbb{R}^{n\times n}$

$$2a^Tb \leq (a+Mb)^TR(a+Mb) + b^TR^{-1}b + 2b^TMb$$

or the general basic inequality [31]: if $\left[\begin{smallmatrix} R & M \\ M^T & Z \end{smallmatrix}\right] \geq 0$

$$2a^Tb \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} R & M-I \\ M^T-I & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{5}$$

the cross-term $\eta(t)$ can be bounded by $\eta(t) \leq \mathscr{F}(x(t), x(t-h)) + \int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds$, which exactly *"offsets"* the quadratic integral term $\mathscr{I}(\dot{x})$ in the derivative of the Lyapunov-Krasovskii functional, where $\mathscr{F}(x(t), x(t-h))$ is a proper quadratic function on $x(t)$ and $x(t-h)$. As a result, a delay-dependent stability criterion can be derived in terms of linear matrix inequalities. There are a number of model transformation approaches proposed in the literature, namely, "first-order transformation" [32], "parametrized first-order transformation" [33], "second-order transformation" [34], "neutral transformation" [35] and "descriptor model transformation" [36, 37]. As pointed out in [38, 39], under the first-order transformation, or the parametrized first-order transformation, or the second-order transformation, the transformed system is *not* equivalent to the original one due to the fact that additional eigenvalues are introduced into the transformed system. Under the neutral transformation, although no explicit additional eigenvalue is introduced, some additional eigenvalue constraints for the stability of an appropriate operator should be satisfied [39]. The descriptor model transformation delivers some larger delay upper bounds since the transformed system is equivalent to the original one.

ii) *Free-weighting matrix approach*. Compared with model transformation approaches, a free-weighting matrix approach can provide an easier way to deal with the quadratic integral term $\mathscr{I}(\dot{x})$. By introducing some proper zero-valued terms as [40, 41]

$$2[x^T(t)Y_1 + x^T(t-h)Y_2]\left[x(t) - x(t-h) - \int_{t-h}^{t} \dot{x}(s)ds\right]$$

where Y_1 and Y_2 are called free-weighting matrices, the derivative of the Lyapunov-Krasovskii functional can be expressed as $\int_{t-h}^{t} \zeta^T(t,s)\Phi\zeta(t,s)ds$, where $\zeta(t,s) = \mathrm{col}\{x(t), x(t-h), \dot{x}(s)\}$ and Φ is a certain matrix. Then a delay-dependent stability criterion is obtained. It is clear that the model transformation and the bounding of cross-terms are obviated by the free-weighting matrix approach. Moreover, simulation results have shown that free-weighting matrix approaches can produce larger delay upper bounds than model transformation approaches.

iii) *Integral inequality approach*. An integral inequality approach directly provides an upper bound for the quadratic integral term $\mathscr{I}(\dot{x})$ [42–44]. By using the Leibniz-Newton formula, an integral inequality for $\mathscr{I}(\dot{x})$ is proposed, which reads as

$$\mathscr{I}(\dot{x}) \leq \xi^T(t)(E^TM + M^TE + hM^TR^{-1}M)\xi(t), \tag{6}$$

where $\xi(t) = \mathrm{col}\{x(t), x(t-h)\}$, $E = [I \;\; -I]$ and $M \in \mathbb{R}^{n\times 2n}$ is a free matrix. In [45]), Jensen integral inequality is introduced *for the first time* in the stability of time-delay systems. In [46], Jensen integral inequality is used to derive a different upper bound for $\mathscr{I}(\dot{x})$ as

$$h\mathscr{I}(\dot{x}) \leq -\xi^T(t)E^TRE\xi(t). \tag{7}$$

This inequality can be also derived from (6), where the free weighting matrix M is selected as $-(1/h)RE$. The integral inequality approach also employs neither model transformations nor cross-term bounding. It is verified that, a stability criterion based on the integral inequality (6) or (7) can obtain the *same* delay upper bound *as* that based on the free-weighting matrix approach. Notice that

$$\begin{aligned} &E^TM + M^TE + hM^TR^{-1}M + (1/h)E^TRE \\ &\quad = (1/h)(hM+RE)^TR^{-1}(hM+RE) \geq 0. \end{aligned} \tag{8}$$

Thus, the relationship between the integral inequalities (6) and (7) can be disclosed by

$$\begin{aligned} &h\mathscr{I}(\dot{x}) \leq -\xi^T(t)E^TRE\xi(t) \\ &\quad = \inf_{M\in\mathbb{R}^{n\times 2n}} \left\{ h\xi^T(t)(E^TM + M^TE + hM^TR^{-1}M)\xi(t) \right\} \end{aligned}$$

which means that the integral inequality (7) provides a minimum upper bound for $\mathscr{I}(\dot{x})$ among the set of upper bounds given by the integral inequality (6). Therefore, the integral inequality (7) has been attractive in the stability analysis since it can derive some stability criteria without introducing any extra matrix variable, if compared with the integral inequality (6) and the free-weighting matrix approach.

In [47], a *tight* bound for $\mathscr{I}(\dot{x})$ is obtained as

$$h\mathscr{I}(\dot{x}) \leq -\xi^T(t)E^TRE\xi(t) - 3\xi_1^T(t)R\xi_1(t), \tag{9}$$

where $\xi_1(t) := x(t) + x(t-h) - \frac{2}{h}\int_{t-h}^{t} x(s)ds$. Based on the tight bound, a less conservative stability criterion for linear systems with time-varying delay is expected incorporating with the *convex delay analysis* method [48], the *reciprocally convex* approach [49] and a proper Lyapunov-Krasovskii functional. It is shown through a number of numerical examples that the obtained stability criterion can produce an admissible maximum upper bound closely approaching to the system analytical value [50]. Boosted by the Wirtinger-based integral inequality, increasing attention is paid to integral inequality approaches, and a great number of results have been reported in the open literature, see, e.g. [51–60].

In this paper, we provide an overview and in-depth analysis of integral inequality approaches to stability of linear systems with time-varying delays. First, an insightful overview is made on convex delay analysis approaches, reciprocally convex approaches and the construction of Lyapunov-Krasovskii functionals. Second, in-depth analysis of Bessel-Legendre inequalities and some affine integral inequalities is made, and recent stability results based on these inequalities are reviewed. Specifically, the refined allowable delay sets are discussed with insightful understanding. Third, we develop a number of stability criteria based on a *canonical* Bessel-Legendre inequality recently reported, taking three cases of time-varying delay into account. Simulation results show that the canonical Bessel-Legendre inequality plus an augmented Lyapunov-Krasovskii functional indeed can produce a larger delay upper bound than some existing methods. Finally, some challenging issues are proposed for the near future research.

This paper is organized as follows. Section 2 gives an overview of recent advances in convex and reciprocally convex delay analysis approaches, as well as the construction of Lyapunov-Krasovskii functionals. Recent integral inequalities and their applications to stability of linear systems with time-varying delay are reviewed in Section 3. A canonical Bessel-Legendre inequality and its affine version, together with a proper augmented Lyapunov-Krasovskii functional is developed to derive some stability criteria for three cases of time-varying delays. Section 5 concludes this paper and proposes some challenging problems to be solved in the future research.

Notation: The notations in this paper are standard. $\mathrm{He}\{A\} = A + A^T$. $\mathrm{diag}\{\cdots\}$ and $\mathrm{col}\{\cdots\}$ denote a block-diagonal matrix and a block-column vector, respectively. $\mathrm{Co}\{p_1, p_2\}$ stands for a

IET Research Journals, pp. 1–15

polytope generated by two vertices p_1 and p_2. $\mathbb{H}_N$ represents the set of polynomials of degree less than N, where N is a positive integer; and the notation $\binom{m}{k}$ refers to binomial coefficients given by $\frac{m!}{(m-k)!k!}$. A symmetric term in a symmetric matrix is symbolised by a '$\star$'.

2 Recent advances in convex analysis approaches, reciprocally convex approaches and the construction of Lyapunov-Krasovskii functionals

2.1 Convex delay analysis approach

The convex delay analysis approach provides an effective way to handle the time-varying delay $h(t)$ to ensure less conservatism of stability criteria [48, 61]. Suppose that a stability condition is given in the following form

$$\mathcal{M}_1 + h(t)\mathcal{M}_2 < 0, \ \text{ for } \ h(t) \in [0, \bar{h}] \tag{10}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are real symmetric matrices irrespective of $h(t)$. By exploiting the convex property, the condition (10) is *equivalent* to two boundary linear matrix inequalities (LMIs) as

$$\mathcal{M}_1 < 0, \ \ \mathcal{M}_1 + \bar{h}\mathcal{M}_2 < 0 \tag{11}$$

Some other conditions on stability are of the following form [62]

$$\mathscr{F}(h(t)) := \mathcal{M}_1 + h(t)\mathcal{M}_2 + h^2(t)\mathcal{M}_3 < 0 \quad \text{for } \ h(t) \in [0, \bar{h}] \tag{12}$$

where $\mathcal{M}_i$ $(i = 1, 2, 3)$ are real symmetric matrices independent of $h(t)$. The difference from (10) is that $\mathscr{F}(h(t))$ is a matrix-valued quadratic function of the time-varying delay $h(t)$. If $\mathcal{M}_3 \geq 0$, then $\mathscr{F}(h(t))$ is convex with respect to $h(t)$ on $[0, \bar{h}]$, leading to a necessary and sufficient condition as $\mathscr{F}(0) < 0$ and $\mathscr{F}(\bar{h}) < 0$ [63]. However, if the constraint $\mathcal{M}_3 \geq 0$ is *not* satisfied, the above conclusion is not necessarily true. In this situation, two sufficient conditions are established in [64] and [65], which are given, respectively, as

$$\mathscr{F}(0) < 0, \ \mathscr{F}(\bar{h}) < 0 \ \text{ and } \ -\bar{h}^2\mathcal{M}_3 + \mathcal{M}_1 < 0 \tag{13}$$

$$\mathscr{F}(0) < 0, \ \mathscr{F}(\bar{h}) < 0 \ \text{ and } \ \bar{h}\mathcal{M}_2 + \mathcal{M}_1 < 0 \tag{14}$$

Clearly, these two sufficient conditions described in (13) and (14) are independent. The relationship between them needs to be further investigated. It should be mentioned that Theorem 1 in [66] is *not* correct. In fact, $(\mathscr{B}^{\perp})^T \Omega_{[h(t), h^2(t), \dot{h}(t)]}(\mathscr{B}^{\perp}) < 0$ for $h(t) \in [0, h]$ is *not* equivalent to $(\mathscr{B}^{\perp})^T \Omega_{[0,0,\dot{h}(t)]}(\mathscr{B}^{\perp}) < 0$ and $(\mathscr{B}^{\perp})^T \Omega_{[h, h^2, \dot{h}(t)]}(\mathscr{B}^{\perp}) < 0$ unless the coefficient matrix of $h^2(t)$ is semi-positive definite, where the symbols are defined in [66, Theorem 1].

2.2 Reciprocally convex delay analysis approach

In the stability analysis of the system (1), applying some integral inequalities usually yields a reciprocally convex combination on the time-varying delay $h(t)$ as

$$\aleph(\alpha) := \frac{1}{\alpha}\beta_1^T \mathcal{R}_1 \beta_1 + \frac{1}{1-\alpha}\beta_2^T \mathcal{R}_2 \beta_2 \tag{15}$$

where $\alpha := \frac{h(t)}{h}$; $\mathcal{R}_1$ and $\mathcal{R}_2$ are $n \times n$ definite-positive matrices, and β_1 and β_2 are two n real vectors. A reciprocally convex inequality for $\aleph(\alpha)$ is proposed in [49]: For any $n \times n$ real matrix S satisfying

$$\begin{bmatrix} \mathcal{R}_1 & S \\ S^T & \mathcal{R}_2 \end{bmatrix} \geq 0 \tag{16}$$

the following inequality holds

$$\aleph(\alpha) \geq \beta_1^T \mathcal{R}_1 \beta_1 + \beta_2^T \mathcal{R}_2 \beta_2 + 2\beta_1^T S \beta_2 \tag{17}$$

The reciprocally convex inequality (17) provides a lower bound for $\aleph(\alpha)$, which is independent of α. Its application to stability analysis of the system (1) usually leads to a significant stability criterion in the sense of two aspects: i) it requires less decision variables; and ii) it can derive the same delay upper bound as the one using the free-weighting matrix approach, which is verified through some numerical examples [49]. Nevertheless, the second aspect only holds for stability conditions based on Jensen's inequality. This is indeed not the case anymore when employing for instance the Wirtinger-based integral inequality, which can be seen from [67] or the simulation results in the next section. Moreover, insight analysis of the reciprocally convex approach [49] is made in [68], which points out that the reciprocally convex inequality (17) can be interpreted as a discretized version of the Jensen's inequality.

Recently, an improved reciprocally convex inequality is proposed in [69] and developed in [70], which reads as

$$\begin{aligned} \aleph(\alpha) \geq{}& \beta_1^T \left[\mathcal{R}_1 + (1-\alpha)(\mathcal{R}_1 - S\mathcal{R}_2^{-1}S^T)\right]\beta_1 \\ &+ \beta_2^T\left[\mathcal{R}_2 + \alpha(\mathcal{R}_2 - S^T\mathcal{R}_1^{-1}S)\right]\beta_2 + 2\beta_1^T S\beta_2 \end{aligned} \tag{18}$$

Compared with (17), the significance of (18) lies in two aspects: i) the matrix constraint (16) is *removed* from (18); and ii) the improved reciprocally convex inequality (18) provides a larger lower bound than (17) for $\aleph(\alpha)$, see [70] in detail.

By introducing more slack matrix variables, a general reciprocally convex inequality is proposed in [71]: For any $n \times n$ real matrices X_1, X_2, Y_1 and Y_2 such that

$$\begin{bmatrix} \mathcal{R}_1 - X_1 & Y_1 \\ Y_1^T & \mathcal{R}_2 \end{bmatrix} \geq 0, \ \ \begin{bmatrix} \mathcal{R}_1 & Y_2 \\ Y_2^T & \mathcal{R}_2 - X_2 \end{bmatrix} \geq 0 \tag{19}$$

the following inequality holds

$$\begin{aligned} \aleph(\alpha) \geq{}& \beta_1^T[\mathcal{R}_1 + (1-\alpha)X_1]\beta_1 + \beta_2^T(\mathcal{R}_2 + \alpha X_2)\beta_2 \\ &+ 2\beta_1^T[\alpha Y_1 + (1-\alpha)Y_2]\beta_2 \end{aligned} \tag{20}$$

The reciprocally convex inequality (20) is rather general: i) Taking $X_1 = X_2 = 0$ and $Y_1 = Y_2 = S$, the inequality (20) reduces to (17); ii) Taking $Y_1 = Y_2 = S$, $X_1 = \mathcal{R}_1 - S\mathcal{R}_2^{-1}S^T$ and $X_2 = \mathcal{R}_2 - S^T\mathcal{R}_1^{-1}S$, the inequality (20) becomes (18). However, four slack matrix variables are introduced and two constraints are imposed on them, which lead to higher computation complexity of a stability criterion. Fortunately, following the idea in [70], an improved reciprocally convex inequality for $\aleph(\alpha)$ is obtained in [72]: For any $n \times n$ real matrices Y_1 and Y_2 with appropriate dimensions, one has

$$\begin{aligned} \aleph(\alpha) \geq{}& \beta_1^T[\mathcal{R}_1 + (1-\alpha)(\mathcal{R}_1 - Y_1\mathcal{R}_2^{-1}Y_1^T)]\beta_1 \\ &+ \beta_2^T[\mathcal{R}_2 + \alpha(\mathcal{R}_2 - Y_2^T\mathcal{R}_1^{-1}Y_2)]\beta_2 \\ &+ 2\beta_1^T[\alpha Y_1 + (1-\alpha)Y_2]\beta_2 \end{aligned} \tag{21}$$

Clearly, just two slack matrix variables are introduced and the constraints in (19) are also removed from (21). Combining the improved reciprocally convex inequality (21) with the convex delay analysis approach, some less conservative stability criteria can be derived, which is verified through some numerical examples, see in detail, [72].

If setting $R = \alpha\tilde{R} > 0$ with $\alpha > 0$, the basic inequality (4) immediately reduces to

$$2a^T b \leq \alpha a^T \tilde{R} a + \frac{1}{\alpha} b^T \tilde{R}^{-1} b, \ \ a, b \in \mathbb{R}^n \tag{22}$$

Based on (22), an estimate of $\aleph(\alpha)$ is obtained in [52], which is in a different form as

$$\aleph(\alpha) \geq \beta_0^T \left[\mathrm{He}\{Z\} - \alpha Z_1 \mathcal{R}_1^{-1} Z_1^T - (1-\alpha) Z_2 \mathcal{R}_2^{-1} Z_2^T\right] \beta_0 \quad (23)$$

where $\beta_0 = \mathrm{col}\{\beta_1, \beta_2\}$ and $Z = [Z_1 \ \ Z_2] \in \mathbb{R}^{2n \times 2n}$ is a slack matrix variable. A detailed proof of (23) can be referred to [73]. The link between (21) and (23) is revealed if one considers the following slack variables

$$Z_1 = \mathrm{col}\{\mathcal{R}_1,\ Y_2^T\}, \qquad Z_2 = \mathrm{col}\{Y_1,\ \mathcal{R}_2\}$$

Several calculations allow indeed to recover (21) from (23).

In the sequel, it will be shown that the inequality (23) is a special case of the inequality obtained from the following lemma (see [74, Lemma 1])

Lemma 1. *Let ξ_1 and ξ_2 be real column vectors with dimensions of n_1 and n_2, respectively. For given real positive symmetric matrices $\mathcal{X} \in \mathbb{R}^{n_1 \times n_1}$ and $\mathcal{Y} \in \mathbb{R}^{n_2 \times n_2}$, the following inequality holds for any scalar $\kappa > 0$ and matrix $\mathcal{S} \in \mathbb{R}^{n_1 \times n_2}$ satisfying $\left[\begin{smallmatrix} \mathcal{X} & \mathcal{S} \\ \mathcal{S}^T & \mathcal{Y} \end{smallmatrix}\right] \geq 0$*

$$2\xi_1^T \mathcal{S} \xi_2 \leq \kappa \xi_1^T \mathcal{X} \xi_1 + \kappa^{-1} \xi_2^T \mathcal{Y} \xi_2. \quad (24)$$

Apply (24) to each term in (15) to obtain

$$\aleph(\alpha) \geq \textstyle\sum_{i=1}^2 \zeta_i^T \mathcal{S}_i \beta_i - \alpha \zeta_1^T \mathcal{X}_1 \zeta_1 - (1-\alpha) \zeta_2^T \mathcal{X}_2 \zeta_2 \quad (25)$$

where ζ_i $(i = 1, 2)$ are two vectors with appropriate dimensions, and

$$\begin{bmatrix} \mathcal{X}_i & \mathcal{S}_i \\ \mathcal{S}_i^T & \mathcal{R}_i \end{bmatrix} \geq 0, \ \ i = 1, 2 \quad (26)$$

If one sets $\zeta_1 = \zeta_2 = \beta_0$, $\mathcal{S}_i = Z_i$ and $\mathcal{X}_i = Z_i \mathcal{R}_i^{-1} Z_i^T$ $(i = 1, 2)$, the inequality (25) immediately reduces to (23).

From the analysis above, it is clear that the inequality (25) is the general form of those reciprocally convex inequalities (17), (18), (20), (21) and (23). Following the idea above, a general inequality for the reciprocally convex combination,

$$\aleph_N(\alpha) := \sum_{j=1}^N \frac{1}{\alpha_j} \beta_j^T \mathcal{R}_j \beta_j, \sum_{j=1}^N \alpha_j = 1, \alpha_j \in (0, 1) \quad (27)$$

is derived, which can be referred to [75, Lemma 4].

2.3 The construction of Lyapunov-Krasovskii functionals

A proper Lyapunov-Krasovskii functional is crucial for deriving less conservative stability criteria for time-delay systems. However, it is still challenging to construct an exact Lyapunov-Krasovskii functional so that a necessary and sufficient stability condition can be derived for the system (1). In general, such a Lyapunov-Krasovskii functional is based on parameters which are solutions to partial differentiable equations, see, e.g. [9]. Hence, many researchers have turned to a simple Lyapunov-Krasovskii functional as

$$V_{\mathrm{usual}}(t, x_t) := x^T(t) P x(t) + \int_{t-h(t)}^t x^T(s) Q x(s) ds + \int_{-\bar{h}}^0 \int_{t+\theta}^t \dot{x}^T(s) R \dot{x}(s) ds d\theta \quad (28)$$

If $P > 0$ and $R > 0$, it is proven that the positive-definiteness of the matrix Q can be weakened by [76]

$$\begin{bmatrix} P + R & R \\ R & \bar{h} Q + R \end{bmatrix} > 0, \quad (29)$$

such that $V_{\mathrm{usual}}(t, x_t)$ is positive-definite. In order to reduce the conservatism of a stability criterion, a great number of Lyapunov-Krasovskii functionals are constructed on the basis of (28). In the following, typically we mention several kinds of Lyapunov-Krasovskii functionals.

2.3.1 Augmented Lyapunov-Krasovskii functionals: An augmented Lyapunov functional is introduced in [77] and [78]. A key feature of it is to augment some terms in (28) such that more information on the delayed states is exploited to derive a stability criterion. For example, in [77, 78], the first two terms $x^T(t)Px(t)$ and $\int_{t-h(t)}^t x^T(s)Qx(s)ds$ in (28) are augmented, respectively, by

$$\begin{bmatrix} x(t) \\ x(t-\bar{h}) \\ \int_{t-\bar{h}}^t x(s)ds \end{bmatrix}^T P \begin{bmatrix} x(t) \\ x(t-\bar{h}) \\ \int_{t-\bar{h}}^t x(s)ds \end{bmatrix}, \ \int_{t-h(t)}^t \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds$$

The augmented Lyapunov-Krasovskii functional can make the system state and some delayed states coupled closely, which possibly enhances the feasibility of the related linear matrix inequalities in a stability criterion. Numerical examples show that such a stability criterion indeed can produce a larger upper bound $\bar{h}$ that the related system remains stable, see, e.g. [79], [62].

The purpose of the augmentation of a Lyapunov-Krasovskii functional is to help provide a tighter estimate on its derivative by introducing some new matrix variables as well as some new state-related vectors. It is true that the estimate of the derivative of a Lyapunov-Krasovskii functional depends mainly on the treatment with some integral terms. However, such an estimate sometimes is not enough for a less conservative stability criterion. In both [72] and [80], it has been proven that the Wirtinger-based inequality can produce a tighter estimate on the derivative of a Lyapunov-Krasovskii functional than Jensen integral inequality, but both the obtained stability criteria are of the same conservatism if the Lyapunov-Krasovskii functional is *not* augmented. Recent research [94, 95] shows that using an augmented Lyapunov-Krasovskii functional plus the N-order Bessel-Legendre inequality indeed can yield nice stability criteria of less conservatism.

2.3.2 Lyapunov-Krasovskii functionals with multiple-integral terms: Another development on constructing a proper Lyapunov-Krasovskii functional is to introduce a triple-integral term as [82]

$$\int_{-\bar{h}}^0 \int_r^0 \int_{t+\theta}^t \dot{x}^T(s) R_1 \dot{x}(s) ds dr d\theta$$

Following this idea, a quadruple-integral term is introduced to the augmented Lyapunov-Krasovskii functional, which is in a different form as [62]

$$\int_{t-\bar{h}}^t (\bar{h} - t + s)^3 \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T R_2 \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds$$

More generally, multiple-integral terms are introduced as [51]

$$\sum_{j=1}^m \int_{t-\bar{h}}^t (\bar{h} - t + s)^j \dot{x}^T(s) \hat{R}_j \dot{x}(s) ds,$$

where m is a certain positive integer. Based on the augmented Lyapunov-Krasovskii functionals with multiple-integral terms, it is shown through numerical examples that the resulting delay-dependent stability conditions for the system (1) are less conservative, see, e.g. [83]. However, the introduction of multiple-integral terms gives rise to some new integral terms to be estimated in the derivative of the Lyapunov-Krasovskii functional [51, 63, 81].

2.3.3 Lyapunov-Krasovskii functionals for linear systems with interval time-varying delays: For a system with an interval time-varying delay $h(t) \in [h_0, \bar{h}]$ with $h_0 > 0$, the stability can be

IET Research Journals, pp. 1–15

analyzed by constructing a proper Lyapunov-Krasovskii functional such as

$$V_{\text{interval}}(t,x_t) := x^T(t)Px(t) + \int_{t-h_0}^{t} x^T(s)Q_1x(s)ds + \int_{t-h(t)}^{t-h_0} x^T(s)Q_2x(s)ds + \int_{-h_0}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta + \int_{-\bar{h}}^{-h_0}\int_{t+\theta}^{t} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta \quad (30)$$

Based on (30), a number of (augmented) Lyapunov-Krasovskii functionals are introduced by exploiting more delayed states such as $x(t-h_0)$ and one can refer to [52, 82–89] and the references therein.

It should be mentioned that, if both lower and upper bounds of $\dot{h}(t)$ are known to be constants, a novel Lyapunov-Krasovskii functional is introduced in [61], where the Lyapunov matrix P is chosen as a convex combination $(\bar{h}-h(t))P_1 + (h(t)-h_0)P_2$ on $h(t) \in [h_0, \bar{h}]$, where P_1 and P_2 are two positive-definite symmetric matrices. This idea is also applicable to the case of $h_0 = 0$.

2.3.4 Lyapunov-Krasovskii functionals based on a delay-fractioning approach: Another kind of Lyapunov-Krasovskii functionals is based on the delay-fractioning approach [90]. The key idea is to introduce fractions $\frac{\bar{h}}{r}$ of $\bar{h}$ so that the following Lyapunov-Krasovskii functional is constructed, where r is a positive integer

$$V_{\text{fractioning}}(t,x_t) := x^T(t)Px(t) + \int_{t-\frac{\bar{h}}{r}}^{t} \jmath^T(s)Q\jmath(s)ds + \int_{-\frac{\bar{h}}{r}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta, \quad (31)$$

where $\jmath(s) := \text{col}\{x(s), x(t-\frac{1}{r}\bar{h}), \cdots, x(t-\frac{r-1}{r}\bar{h})\}$. It is proven [90] that, as the integer r becomes larger, the obtained stability criterion is less conservative. The idea of delay-fractioning is extensively used to construct various Lyapunov-Krasovskii functionals, see, e.g. [8, 29, 91–93].

3 Recent developments of integral inequality approaches to stability of linear systems with time-varying delays

In this section, we focus on the recent developments of integral inequality approaches to stability of the system (1). To begin with, we first give an overview of integral inequalities developed recently.

3.1 Recent integral inequalities

Employing the Wirtinger inequality provides a larger lower bound than the well-used Jensen integral inequality for a non-negative integral term [47]. Soon after, by introducing a proper auxiliary function, an auxiliary-function-based integral inequality is reported in [53]. Both of them are given in the following.

Lemma 2. *For any constant matrix $R > 0$, two scalars a and b with $b > a$, and a vector function $\omega : [a,b] \to \mathbb{R}^n$ such that the integrations below are well defined, the following inequalities hold*

i) Wirtinger-based integral inequality:

$$\int_a^b \omega^T(s)R\omega(s)ds \geq \sum_{i=0}^{1} \frac{2i+1}{b-a}\upsilon_i^T R\upsilon_i \quad (32)$$

ii) Auxiliary-function-based integral inequality:

$$\int_a^b \omega^T(s)R\omega(s)ds \geq \sum_{i=0}^{2} \frac{2i+1}{b-a}\upsilon_i^T R\upsilon_i \quad (33)$$

where

$$\begin{cases} \upsilon_0 := \int_a^b \omega(s)ds, \upsilon_1 := \upsilon_0 - \frac{2}{b-a}\int_a^b\int_s^b \omega(\theta)d\theta ds \\ \upsilon_2 := 3\upsilon_1 - 2\upsilon_0 + \frac{12}{(b-a)^2}\int_a^b\int_s^b\int_r^b \omega(\theta)d\theta dr ds. \end{cases} \quad (34)$$

Clearly, the auxiliary-function-based integral inequality (33) is an improvement over the Wirtinger-based integral inequality (32). A natural inspiration from (33) is to extend the inequality to a general form, which is completed by introducing the Legendre polynomials, leading to the canonical Bessel-Legendre inequality [94, 95].

Lemma 3. *Under the assumption in Lemma 2, the following inequality holds*

$$\int_a^b \omega^T(s)R\omega(s)ds \geq \frac{1}{b-a}\sum_{i=0}^{N}(2i+1)\Omega_i^T R\Omega_i \quad (35)$$

$$= \frac{1}{b-a}\sum_{i=0}^{N}(2i+1)\hat{\Omega}_i^T R\hat{\Omega}_i \quad (36)$$

where

$$\Omega_i := \int_a^b \tilde{L}_i(s)\omega(s)ds,\ \hat{\Omega}_i := \int_a^b \hat{L}_i(s)\omega(s)ds$$

$$\begin{cases} \tilde{L}_i(s) := \sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{k+i}{k}(\frac{b-s}{b-a})^k, \\ \hat{L}_i(s) := \sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{k+i}{k}(\frac{s-a}{b-a})^k. \end{cases} \quad (37)$$

The canonical Bessel-Legendre inequality includes the Wirtinger-based integral inequality and the auxiliary-function-based integral inequality as its special cases. The underlying idea of Bessel-Legendre inequality is to provide a generic and expandable integral inequality which is asymptotically (in the sense that N goes to infinity) not conservative, because of the Parseval's identity [96]. The proof of (35) relies on the expansion of the following non-negative quantity

$$\int_{-h}^{0} \tilde{\omega}_N^T(s)R\tilde{\omega}_N(s)ds,\ \ (h = b-a) \quad (38)$$

where $\tilde{\omega}_N(s) = \omega(s) - \frac{1}{h}\sum_{k=0}^{N}(2k+1)\int_{-h}^{0} L_k(s)\omega(s)ds$ with $L_k(s)$ $(k = 0,1,2,\cdots,N)$ being Legendre polynomials. The principles of using Bessel inequality together with orthogonal polynomials, as for instance Legendre polynomials, can be interpreted as the minimization of the distance between the function ω and the set of polynomials of degree less than N denoted by $\mathbb{H}_N$. Classical theories on Hilbert spaces guarantee that this minimization problem has a unique solution, which is given by an orthogonal projection of the function ω over an orthogonal basis of $\mathbb{H}_N$. This orthogonal projection is unique but can be expressed on different polynomial basis such as the Legendre polynomial basis or a canonical orthogonal basis.

Lemma 3 gives a canonical Bessel-Legendre inequality of two different forms (35) and (36), where $\tilde{L}_i(s)$ is a function of $\frac{b-s}{b-a}$ while $\hat{L}_i(s)$ is a function of $\frac{s-a}{b-a}$. It should be pointed out that the orthogonal polynomials chosen in [64, Lemma 1] can be expressed on the basis of $\{g_i(s) | \ i = 0,1,\cdots,N\}$, where

$$g_i(s) = \sqrt{\tfrac{2i+1}{b-a}}P_i(f(s)),\ \ f(s) := \tfrac{2(b-s)}{b-a} - 1 \quad (39)$$

where $P_i(\cdot)$ $(i=0,1,2,\cdots,N)$ are Legendre polynomials. Thus, the integral inequality in [64] is equivalent to the above canonical Bessel-Legendre inequality if setting $N\to\infty$.

Since Ω_i in (35) depends on the Legendre polynomial, the inequality (35) is not convenient for use. In [95], a useful form of the canonical integral inequality is developed for stability analysis of time-delay systems, which is given in the following.

Lemma 4. *For an integer $N\geq 0$, a real symmetric matrix $R>0$, two scalars a and b with $b>a$, and a vector-valued differentiable function $\omega:[a,b]\to\mathbb{R}^n$ such that the integrations below are well defined, then*

$$-\int_a^b \dot{\omega}^T(s)R\dot{\omega}(s)ds \leq -\frac{1}{b-a}\varpi_N^T\widehat{\Theta}_N^T\mathcal{R}_N\widehat{\Theta}_N\varpi_N, \tag{40}$$

where $\widehat{\Theta}_N=\Theta_N\Lambda_N$, and

$$\mathcal{R}_N := diag\Big\{R, 3R, \cdots, (2N+1)R\Big\} \tag{41}$$

$$\Theta_N := \begin{bmatrix} I & 0 & \cdots & 0\\ I & (-1)^1\binom{1}{1}\binom{1+1}{1}I & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ I & (-1)^1\binom{N}{1}\binom{N+1}{1}I & \cdots & (-1)^N\binom{N}{N}\binom{N+N}{N}I \end{bmatrix} \tag{42}$$

$$\Lambda_N := \begin{bmatrix} I & -I & 0 & 0 & \cdots & 0\\ 0 & -I & I & 0 & \cdots & 0\\ 0 & -I & 0 & 2I & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & -I & 0 & 0 & \cdots & NI \end{bmatrix} \tag{43}$$

$$\varpi_N := col\Big\{\omega(b), \omega(a), \gamma_1, \cdots, \gamma_N\Big\} \tag{44}$$

with

$$\gamma_k := \int_a^b \frac{(b-s)^{k-1}}{(b-a)^k}\omega(s)ds, \ \ (k=1,2,\cdots,N) \tag{45}$$

Notice that γ_k is a k-integral of the vector $\omega(s)$. Thus, the integral inequality (40) discloses an explicit relationship between $\int_a^b \dot{\omega}^T(s)R\dot{\omega}(s)ds$ and the vectors $\omega(a), \omega(b)$ and the multiple integrals γ_k $(k=1,2,\cdots,N)$.

The other class of integral inequalities are called affine integral inequalities, or free-matrix-based integral inequalities, where the coefficient $b-a$ appears linearly rather than in the form of its inverse. An affine version of the integral inequality (40) can be readily obtained based on the fact that the following inequality holds for any real matrix M with compatible dimensions

$$\begin{aligned} -\frac{1}{b-a}(\Theta_N\Lambda_N)^T\mathcal{R}_N(\Theta_N\Lambda_N) &\leq (\Theta_N\Lambda_N)^TM\\ &+ M^T(\Theta_N\Lambda_N)+(b-a)M^T\mathcal{R}_N^{-1}M. \end{aligned}$$

Thus, an affine canonical Bessel-Legendre inequality is given by

$$\begin{aligned} -\int_a^b \dot{\omega}^T(s)R\dot{\omega}(s)ds &\leq \varpi_N^T\Big[\Lambda_N^T\Theta_N^TM\\ &+ M^T\Theta_N\Lambda_N+(b-a)M^T\mathcal{R}_N^{-1}M\Big]\varpi_N. \end{aligned} \tag{46}$$

The affine versions of (32) and (33) can be found in [97], [67] or [98]. As pointed out in [97] and [99], the affine version and its corresponding integral inequality provide an equivalent lower bound for the related integral term. It should be mentioned that those affine integral inequalities can be regarded as special cases of (46). For example, Lemma 1 in [98] is a special case of (46) with $N=2$.

3.2 Recent developments on stability of the system (1) *using recent integral inequalities*

Although the canonical Bessel-Legendre inequality in Lemma 3 provides a lower bound for the integral term as tight as possible if $N\to\infty$, in the recent years, most researchers' interest is focused on its special cases such as $N=1$ [47, 100, 105] and $N=2$ [52, 53, 64, 101]. It is proven in [80] that a tighter bound of the integral term in the derivative of the Lyapunov-Krasovskii functional should *not* be responsible for deriving a less conservative stability criterion. Therefore, although the integral inequality (33) provides a tighter bound than (32), it is not a trivial thing to derive a less conservative stability criterion using the inequality (33). The main difficulty is that the vectors υ_1 and υ_2 in (34) are not easily handled in the stability analysis of the system (1). It is shown from in [72] that the vectors υ_1 and υ_2 should occur in the derivative of the Lyapunov-Krasovskii functional so that a less conservative stability criterion for the system (1) can be obtained using the integral inequality (33). The recent development on this issue is briefly summarized as follows.

For simplicity of presentation, suppose that the time-varying delay $h(t)$ in the system (1) belongs to one of three cases:

Case 1: $h(t)$ is differentiable and satisfies

$$0\leq h(t)\leq \bar{h}, \quad \mu_1\leq \dot{h}(t)\leq\mu_2 \tag{47}$$

Case 2: $h(t)$ is differentiable and satisfies

$$0\leq h(t)\leq \bar{h}, \quad \dot{h}(t)\leq\mu_2 \tag{48}$$

Case 3: $h(t)$ is continuous and satisfies

$$0\leq h(t)\leq \bar{h} \tag{49}$$

where $\bar{h}$, μ_1 and μ_2 are real constants.

In what follows, we consider the three cases.

3.2.1 Case 1: Since information on the upper and lower bounds of the time-varying delay and its time-derivative is available, in order to formulate some less conservative stability criteria, an augmented Lyapunov-Krassovskii functional is introduced in [69] on the basis of the Lyapunov-Krassovskii functional in (28), where the first term $x^T(t)Px(t)$ is augmented with $\eta_1^T(t)P\eta_1(t)$, where $\eta_1(t):=$ col$\{x(t),\ \int_{t-h(t)}^t x(s)ds,\ \int_{t-\bar{h}}^{t-h(t)}x(s)ds,\ \int_{t-h(t)}^t\int_s^t\frac{x(u)}{h(t)}duds,\ \int_{t-\bar{h}}^{t-h(t)}\int_s^{t-h(t)}\frac{x(u)}{\bar{h}-h(t)}duds\}$. Then taking the time-derivative of the term $\eta_1^T(t)P\eta_1(t)$ yields some vectors similar to υ_1 and υ_2 induced by the auxiliary-function-based integral inequality (33). Combining with the extended reciprocally convex inequality (18), some nice results are derived therein.

In [72], an augmented Lyapunov-Krasovskii functional is constructed as

$$\begin{aligned} V_{\rm augm1}(t,x_t) &:= \int_{t-h(t)}^t \eta_2^T(t,s)Q_1\eta_2(t,s)ds\\ &+\int_{t-\bar{h}}^{t-h(t)}\eta_3^T(t,s)Q_2\eta_3(t,s)ds\\ &+\bar{h}\int_{-\bar{h}}^0\int_{t+\theta}^t\dot{x}^T(s)R\dot{x}(s)dsd\theta \end{aligned} \tag{50}$$

where $\eta_2(t,s):=$col$\{\dot{x}(s),x(s),x(t),\int_{t-h(t)}^s x(\theta)d\theta\}$ and $\eta_3(t,s)$ $:=$ col$\{\dot{x}(s),x(s),x(t),\int_{t-\bar{h}}^s x(\theta)d\theta\}$. The significance of (50) lies in two aspects: a) The quadratic term $x^T(t)Px(t)$ in (28) is merged into the first two integral terms such that the vectors $x(t)$, $x(s)$ and $\dot{x}(s)$ are closely coupled by Q_1 on $[t-h(t),t]$ and Q_2 on $[t-\bar{h},t-h(t)]$; and b) The vectors induced by the auxiliary-function-based integral inequality (33) are included in the derivative of $V_{\text{augm1}}(t, x_t)$. By employing the improved reciprocally convex inequality (21), a less conservative stability criterion is presented in [72].

In [94], using the Bessel-Legendre inequality (36), an N-dependent stability criterion is established by choosing the following augmented Lyapunov-Krasovskii functional

$$\begin{aligned} V_N(t, x_t) &= \tilde{x}_N^T(t) P_N \tilde{x}_N(t) + \int_{t-h(t)}^{t} x^T(s) S x(s) ds \\ &+ \int_{t-\bar{h}}^{t-h(t)} x^T(s) Q x(s) ds \\ &+ \bar{h} \int_{-\bar{h}}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta \end{aligned} \tag{51}$$

where $\tilde{x}_N(t) := \text{col}\{x(t),\ h(t)\phi_{1,N}(t),\ (\bar{h} - h(t))\phi_{2,N}(t)\}$,

$$\begin{aligned} \phi_{1,N}(t) &:= \tfrac{1}{h(t)} \int_{-h(t)}^{0} \mathbb{L}_N(\tfrac{s+h(t)}{h(t)}) x(t+s) ds \\ \phi_{2,N}(t) &:= \tfrac{1}{\bar{h}-h(t)} \int_{-\bar{h}}^{-h(t)} \mathbb{L}_N(\tfrac{s+\bar{h}}{\bar{h}-h(t)}) x(t+s) ds \\ \mathbb{L}_N(s) &:= \text{col}\{L_0(s)I,\ L_1(s)I,\ \cdots, L_N(s)I\} \end{aligned}$$

Similar to [54], it is proven that the N-dependent stability criterion presented in [94] also forms a hierarchy, which means that its conservatism will be reduced if N is increased. On the other hand, an observation is that Lyapunov-Krasovskii functional $V_N(t, x_t)$ is *dependent on the Legendre polynomials* $L_k(s)$ $(k = 0, 1, 2, \cdots, N)$, which is inherited from the Bessel-Legedre inequality.

3.2.2 Case 2: Information on the lower bound of $\dot{h}(t)$ is not known. In this case, an augmented Lyapunov-Krasovskii functional is constructed in [64], in which two augmented terms are given as

$$\eta_4^T(t) P \eta_4(t) + \int_{t-h(t)}^{t} \eta_5^T(t, s) Q \eta_5(t, s) ds \tag{52}$$

where $\eta_4(t) := \text{col}\{x(t), \int_{t-\bar{h}}^{t} x(s)ds, \int_{t-\bar{h}}^{t} (\bar{h} - t + s)x(s)ds\}$ and $\eta_5(t, s) := \text{col}\{x(s), \int_s^t \dot{x}(r)dr, \int_s^t x(r)dr\}$. Applying the integral inequality similar to (33) yields four vectors v_{11}, v_{12}, v_{21} and v_{22} as

$$\begin{aligned} v_{11} &:= \int_{t-h(t)}^{t} \tfrac{x(s)}{h(t)} ds, & v_{12} &:= \int_{t-h(t)}^{t} \tfrac{s-t+h(t)/2}{h(t)} x(s) ds \\ v_{21} &:= \int_{t-\bar{h}}^{t-h(t)} \tfrac{x(s)}{\bar{h}-h(t)} ds, & v_{22} &:= \int_{t-\bar{h}}^{t-h(t)} \tfrac{s-t+\frac{\bar{h}+h(t)}{2}}{\bar{h}-h(t)} x(s) ds \end{aligned}$$

However, taking the derivative of the Lyapunov-Krasovskii functional just gives three vectors v_{11}, v_{12} and v_{21}. In order to ensure that the vector v_{22} also appears in the derivative of the Lyapunov-Krasovskii functional, the following identity is used

$$\begin{aligned} v_3 &= (\bar{h} - h(t))^2 [v_{22} + (1/2)v_{21}] \\ &+ h^2(t) v_{12} + h(t)[\bar{h} - (1/2)h(t)] v_{11} \end{aligned}$$

where $v_3 := \int_{t-\bar{h}}^{t} (\bar{h} - t + s) x(s) ds$. By employing the free-weighting matrix approach and the quadratic convex condition (13), a stability criterion for the system (1) is obtained [64].

3.2.3 Case 3: The time-varying delay is only known to be continuous (possibly not differentiable), which implies that information on the derivative of the time-varying delay is unavailable. Thus, the above Lyapunov-Krasovskii functionals in Cases 1 and 2 can be no longer used to produce the vectors similar to v_1 and v_2 in its time-derivative. In this case, an augmented Lyapunov-Krasovskii functional is constructed in [52], where the quadratic $x^T(t)Px(t)$ in (28) is augmented by $\eta_6^T(t) P \eta_6(t)$ with $\eta_6(t) := \text{col}\{x(t), \int_{-\bar{h}}^{0} x_t(s)ds, \bar{h} \int_{-\bar{h}}^{0} [\frac{2(s+\bar{h})}{\bar{h}} - 1] x_t(s) ds\}$. By dividing the integral interval $[-\bar{h}, 0]$ into two parts as $[-\bar{h}, h(t)]$ and $[-h(t), 0]$,

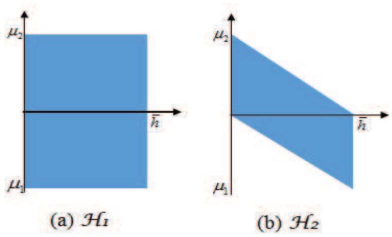

Fig. 1: Allowable delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$ in Case 1

those vectors induced from the integral inequality (35) appear in the derivative of the Lyapunov-Krasovskii functional, see in detail, [52, Theorem 1]. It should be pointed that stability of the system (1) with unknown information on the derivative of $h(t)$ is also investigated in [53] using the auxiliary-function-based integral inequality (33), but those vectors induced from (33) do not exist in the derivative of the chosen Lyapunov-Krasovskii functional. Thus, one can claim that [53, Theorem 1] is of the same conservatism as that using the Wirtinger-based integral inequality (32) instead of (33).

From the above analysis, one can see that, it is still challenging to investigate the stability for the system (1) with time-varying delay based on the recent integral inequalities. When both the time-varying delay and its derivative are bounded from above and from below, most existing stability criteria are based on the Wirtinger-based or the auxiliary-function-based integral inequalities or the second-order Bessel-Legendre inequality. In the other cases where the information on the derivative of the time-varying delay is partly known or completely unknown, *relatively few* results on stability of the system (1) are obtained, even using the second-order Bessel-Legendre inequality.

3.3 Refinement of allowable delay sets

Recently, another development on stability of the system (1) is the *refinement* of allowable delay sets, see, [94, 102]. To make it clear, suppose that a stability condition can be derived from the matrix inequality $\mathcal{F}(h(t), \dot{h}(t)) < 0$, where $\mathcal{F}(h(t), \dot{h}(t))$ is a matrix-valued function that depends *linearly* on both $h(t)$ and $\dot{h}(t)$. In Case 1, $h(t)$ satisfies $0 \le h(t) \le \bar{h}$ and $\mu_1 \le \dot{h}(t) \le \mu_2$, which means that

$$(h(t), \dot{h}(t)) \in \mathcal{H}_1 \triangleq [0, \bar{h}] \times [\mu_1, \mu_2] \tag{53}$$

The set $\mathcal{H}_1$ is called an allowable delay set, which can be described as a polytope (see Fig. 1) with four vertices as

$$\mathcal{H}_1 = \text{Co}\left\{(0, \mu_1), (0, \mu_2), (\bar{h}, \mu_1), (\bar{h}, \mu_2)\right\} \tag{54}$$

A stability condition is thus readily obtained provided that $\mathcal{F}(h(t), \dot{h}(t)) < 0$ is satisfied at four vertices of $\mathcal{H}_1$ [103]. However, it is pointed out [102] that such a stability condition is conservative in some situation. In fact, if $\mu_1 < 0$ and $\mu_2 > 0$, two vertices $(0, \mu_1)$ and $(\bar{h}, \mu_2)$ contradict the fact that 0 and $\bar{h}$ are the minimum and maximum values of $h(t)$. In other words, it is impossible for $(h(t), \dot{h}(t))$ to arrive at these two vertices $(0, \mu_1)$ and $(\bar{h}, \mu_2)$ for any time $t \ge 0$. For example, let [94]

$$h(t) = \frac{\bar{h}}{2}\Big(1 + \cos(\frac{2\mu_2}{\bar{h}} t)\Big). \tag{55}$$

Then $(h(t), \dot{h}(t)) \in [0, \bar{h}] \times [-\mu_2, \mu_2]$ with $\mu_2 > 0$. We now show that the vertex $(\bar{h}, \mu_2)$ will be never reached. Note that

$$\dot{h}(t) = -\mu_2 \sin(\frac{2\mu_2}{\bar{h}} t). \tag{56}$$

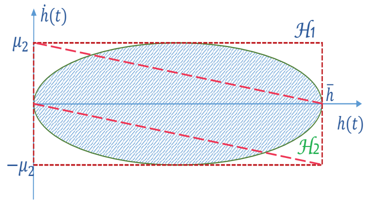

Fig. 2: The allowable delay sets for $h(t)$ in (55)

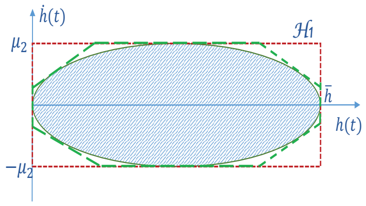

Fig. 3: One refined allowable delay set for $h(t)$ in (55)

Setting $\dot{h}(t) = \mu_2$ gives $t = t_k \triangleq \frac{\bar{h}}{2\mu_2}(\frac{3\pi}{2}+2k\pi),\ (k = 0, 1, \cdots)$. Nevertheless, $h(t_k) = \frac{\bar{h}}{2} \neq \bar{h}$, $(k = 0, 1, 2, \cdots)$. Likewise, it is not difficult to verify that the vertex $(0, -\mu_2)$ is also not reached. Hence, in order to derive a less conservative stability condition, the allowable delay set $\mathcal{H}_1$ should be refined so that all vertices can be reached. In [94, 102], the vertices $(0, \mu_1)$ and $(\bar{h}, \mu_2)$ are replaced with $(0, 0)$ and $(\bar{h}, 0)$, respectively, leading to a new allowable delay set $\mathcal{H}_2$ as

$$\mathcal{H}_2 := \text{Co}\left\{(0, 0), (0, \mu_2), (\bar{h}, 0), (\bar{h}, \mu_1)\right\}. \tag{57}$$

Such an idea is extended to neural networks with time-varying delays [75, 95]. It has been shown through simulation that, a stability criterion based on the new delay set $\mathcal{H}_2$ can produce much larger delay upper bounds than that based on $\mathcal{H}_1$, especially for fast time-varying delays. However, there may exist some issue about those results aforementioned.

On the one hand, the above analysis just keeps an eye on the two vertices $(0, \mu_1)$ and $(\bar{h}, \mu_2)$ in $\mathcal{H}_1$ while no attention is paid to the other vertices $(0, \mu_2)$ and $(\bar{h}, \mu_1)$. In some situation, the vertices $(0, \mu_2)$ and $(\bar{h}, \mu_1)$ can also *not* be reached. Let us still consider the above delay function $h(t)$ in (55). Set $h(t) = 0$. Then $t = \frac{2\mu_2}{\bar{h}}(2k + 1)\pi,\ \ (k = 0, 1, 2, \cdots)$, leading to $\dot{h}(t) = 0 \neq \mu_2$. Similarly, if setting $h(t) = \bar{h}$, then we have that $\dot{h}(t) = 0 \neq -\mu_2$. In a word, these two vertices $(0, \mu_2)$ and $(\bar{h}, -\mu_2)$ for this delay function can never be reached.

On the other hand, an important observation is that the allowable delay set $\mathcal{H}_2$ may be *not* suitable for the description of the time-varying delay function. To reveal such a fact, we stick to $h(t)$ in (55). After simple algebraic manipulations, it is found that

$$\left(\frac{2h(t) - \bar{h}}{\bar{h}}\right)^2 + \left(\frac{\dot{h}(t)}{\mu_2}\right)^2 \leq 1 \tag{58}$$

which means that the function $h(t)$ belongs to a convex ellipsoid (denoted by $\mathcal{H}_0$), see the shadow part in Fig. 2. Unfortunately, the set $\mathcal{H}_2$ can *not* cover the convex ellipsoid $\mathcal{H}_0$ for any finite value of $\mu_2(> 0)$, which implies that a stability criterion based on $\mathcal{H}_2$ cannot ensure the stability of the system (1) with $h(t)$ being defined in (55). Thus, the set $\mathcal{H}_2$ is not suitable for $h(t)$ in (55).

It is a good idea to refine the allowable delay set such that less conservative stability criteria can be obtained. However, based on the above analysis, one cannot claim that the set $\mathcal{H}_2$ in (57) is a refinement of $\mathcal{H}_1$. A set $\mathcal{H}$ is called a refinement of $\mathcal{H}_1$ only if $\mathcal{H}_0 \subset \mathcal{H} \subset \mathcal{H}_1$, where $\mathcal{H}_0$ denotes the real domain of $(h(t), \dot{h}(t))$. Therefore, the refinement of $\mathcal{H}_1$ aims at seeking a possible 'minimum' polytope within $\mathcal{H}_1$ to cover the real domain $\mathcal{H}_0$. How to do it depends on the delay function itself. For the delay $h(t)$ given in (55), one can build a polygon (such as the *octagon* with green dashed lines in Fig. 3) as small as possible to cover the ellipsoid, while for other different $h(t)$ it may be not the case. It should be pointed out that the vertices of a refinement delay set are *not necessarily* reached by $(h(t), \dot{h}(t))$ when a stability criterion is established.

4 Stability criteria based on the canonical Bessel-Legendre inequalities (40) and (46)

In this section, we develop some stability criteria using the canonical Bessel-Legendre inequalities (40) and (46), in order to show the effectiveness of canonical Bessel-Legendre inequalities, and confirm some claims made in the previous sections as well.

4.1 Stability criteria for Case 1

Under Case 1, the time-varying delay $h(t)$ is differentiable satisfying (47). In this situation, we choose the following augmented Lyapunov-Krasovskii functional

$$\begin{aligned} V(t, x_t) = {} & \varsigma_1^T(t)P_N\varsigma_1(t) + \int_{t-h(t)}^{t} \varsigma_2^T(s,t)Q_1\varsigma_2(s,t)ds \\ & + \int_{t-\bar{h}}^{t-h(t)} \varsigma_2^T(s,t)Q_2\varsigma_2(s,t)ds \\ & + \bar{h}\int_{-\bar{h}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta \end{aligned} \tag{59}$$

where $P_N > 0, Q_1 > 0, Q_2 > 0, R > 0$ are Lyapunov matrices to be determined, and

$$\begin{aligned} \varsigma_1(t) &:= \text{col}\{\xi_1(t), \ell(t)\wp_1(t), \ell(t)\wp_2(t), \cdots, \ell(t)\wp_N(t)\} \\ \xi_1(t) &:= \text{col}\{x(t), x(t-h(t)), x(t-\bar{h})\} \\ \ell(t) &:= \text{col}\{h(t)E_1,\ (\bar{h} - h(t))E_2\} \\ \wp_i(t) &:= \text{col}\{\nu_{1i}(t),\ \nu_{2i}(t)\} \\ \varsigma_2(s,t) &:= \text{col}\{\dot{x}(s), x(s), x(t), x(t-h(t)), x(t-\bar{h})\} \end{aligned}$$

with $E_1 = [I\ 0]$ and $E_2 = [0\ I]$, and for $i = 1, 2, \cdots, N$

$$\begin{cases} \nu_{1i}(t) := \dfrac{1}{h^i(t)}\displaystyle\int_{t-h(t)}^{t}(t-s)^{i-1}x(s)ds \\ \nu_{2i}(t) := \dfrac{1}{(\bar{h} - h(t))^i}\displaystyle\int_{t-\bar{h}}^{t-h(t)}(t-h(t)-s)^{i-1}x(s)ds \end{cases} \tag{60}$$

The first augmented term in (59) is motivated from Lemma 4 such that the vectors in (60) induced from the integral inequality (40) appear in the derivative of the Lyapunov-Krasovskii functional $V(t, x_t)$. The second and the third augmented terms are taken from [62]. It should be mentioned that the Lyapunov-Krasovskii functional $V(t, x_t)$ in (59) is different from the one in (51), which is dependent on the Legendre polynomials.

4.1.1 N-dependent stability criteria:

Proposition 1. *For constants μ_1, μ_2, $\bar{h}$ (> 0) and a positive integer N, the system* (1) *subject to $(h(t), \dot{h}(t)) \in \mathcal{H}_1$ is asymptotically stable if there exist real matrices $P_N > 0, Q_1 > 0, Q_2 > 0$ and $R > 0$*

and real matrices Y_{1N} and Y_{2N} with appropriate dimensions such that, for $\mu \in \{\mu_1, \mu_2\}$

$$\Psi_N(0,\mu) := \begin{bmatrix} \Upsilon_{1N}(0,\mu)+\Upsilon_{2N}(0) & \star \\ Y_{1N}^T\mathscr{D}_{1N} & -\mathcal{R}_N \end{bmatrix} < 0 \tag{61}$$

$$\tilde{\Psi}_N(\bar{h},\mu) := \begin{bmatrix} \Upsilon_{1N}(\bar{h},\mu)+\Upsilon_{2N}(\bar{h}) & \star \\ Y_{2N}\mathscr{D}_{2N} & -\mathcal{R}_N \end{bmatrix} < 0 \tag{62}$$

where $\mathcal{R}_N$ is defined in (41), and

$$\begin{aligned}\Upsilon_{1N}(h(t),\dot{h}(t)) &:= \bar{h}^2(Ae_1+A_de_2)^T R(Ae_1+A_de_2) \\ &+(1-\dot{h}(t))\mathscr{C}_4^T(Q_2-Q_1)\mathscr{C}_4 - \mathscr{C}_5^T Q_2\mathscr{C}_5+\mathscr{C}_3^T Q_1\mathscr{C}_3 \\ &+ He\left\{\mathscr{C}_{1N}^T P_N\mathscr{C}_{2N}+(\mathscr{C}_6^T Q_1+\mathscr{C}_7^T Q_2)\mathscr{C}_8\right\} \end{aligned}\tag{63}$$

$$\begin{aligned}\Upsilon_{2N}(h(t)) &:= -(2-\alpha)\mathscr{D}_{1N}^T\mathcal{R}_N\mathscr{D}_{1N} - (1+\alpha)\mathscr{D}_{2N}^T\mathcal{R}_N\mathscr{D}_{2N} \\ &- He\left\{\mathscr{D}_{1N}^T[\alpha Y_{1N}+(1-\alpha)Y_{2N}]\mathscr{D}_{2N}\right\}\end{aligned}\tag{64}$$

with Θ_N being defined in (42), e_i $(i=1,2,\cdots,N+5)$ being the i-th block-row matrix such that $col\{e_1,e_2,\cdots,e_{N+5}\}$ is a $(2N+5)n \times (2N+5)n$ identify matrix, $\alpha = h(t)/\bar{h}$ and

$$\begin{aligned}
\mathscr{C}_{1N} &:= col\{e_1, e_2, e_3, \ell(t)e_6, \cdots, \ell(t)e_{5+N}\}, \\
\mathscr{C}_{2N} &:= col\{Ae_1+A_de_2, (1-\dot{h}(t))e_4, e_5, \Im_1, \Im_2, \cdots, \Im_N\} \\
\mathscr{C}_3 &:= col\{Ae_1+A_de_2, e_1, e_1, e_2, e_3\} \\
\mathscr{C}_4 &:= col\{e_4, e_2, e_1, e_2, e_3\} \\
\mathscr{C}_5 &:= col\{e_5, e_3, e_1, e_2, e_3\} \\
\mathscr{C}_6 &:= col\{e_1 - e_2, h(t)E_1e_6, h(t)e_1, h(t)e_2, h(t)e_3\} \\
\mathscr{C}_7 &:= col\left\{e_2 - e_3, (\bar{h}-h(t))col\{E_2e_6, e_1, e_2, e_3\}\right\} \\
\mathscr{C}_8 &:= col\{0, 0, Ae_1+A_de_2, (1-\dot{h}(t))e_4, e_5\} \\
\Im_1 &:= col\left\{e_1-(1-\dot{h}(t))e_2,\ (1-\dot{h}(t))e_2-e_3\right\} \\
\Im_i &:= \begin{bmatrix} -(1-\dot{h}(t))e_2+(i-1)[E_1e_{5+i-1}-\dot{h}(t)E_1e_{5+i}] \\ -e_3+(i-1)[(1-\dot{h}(t))E_2e_{5+i-1}+\dot{h}(t)E_2e_{5+i}] \end{bmatrix} \\
\mathscr{D}_{1N} &:= \Theta_N\Lambda_N col\{e_1, e_2, E_1e_{5+1}, \cdots, E_1e_{5+N}\} \\
\mathscr{D}_{2N} &:= \Theta_N\Lambda_N col\{e_2, e_3, E_2e_{5+1}, \cdots, E_2e_{5+N}\} \\
E_1 &= [I\ 0],\ E_2 = [0\ I],\ \ i > 1.
\end{aligned}$$

Proof: First, we introduce a vector as $\xi_0(t) := \mathrm{col}\{x(t), x(t-h(t)), x(t-\bar{h}), \dot{x}(t-h(t)), \dot{x}(t-\bar{h}), \wp_1(t), \wp_2(t), \cdots, \wp_N(t)\}$. It is easy to verify that $\varsigma_1(t) = \mathscr{C}_{1N}\xi_0(t)$ and $\dot{\varsigma}_1(t) = \mathscr{C}_{2N}\xi_0(t)$, where $\mathscr{C}_{1N}$ and $\mathscr{C}_{2N}$ are defined in Proposition 1. Taking the derivative of $V(t,x_t)$ in (59) along with the trajectory of the system (1) yields

$$\begin{aligned}\dot{V}(t,x_t) &= \xi_0^T(t)\Upsilon_{1N}(h(t),\dot{h}(t))\xi_0(t) \\ &- \bar{h}\int_{t-\bar{h}}^{t}\dot{x}^T(s)R\dot{x}(s)ds\end{aligned}\tag{65}$$

Now, we estimate the integral term in (65) using the integral inequality (40). Apply the integral inequality (40) to obtain

$$\begin{aligned}-\bar{h}\int_{t-h(t)}^{t}\dot{x}^T(s)R\dot{x}(s)ds &\le -\frac{1}{\alpha}\xi_0^T(t)\mathscr{D}_{1N}^T\mathcal{R}_N\mathscr{D}_{1N}\xi_0(t) \\ -\bar{h}\int_{t-\bar{h}}^{t-h(t)}\dot{x}^T(s)R\dot{x}(s)ds &\le -\frac{1}{1-\alpha}\xi_0^T(t)\mathscr{D}_{2N}^T\mathcal{R}_N\mathscr{D}_{2N}\xi_0(t)\end{aligned}$$

where $\alpha = h(t)/\bar{h}$, $\mathcal{R}_N$ is defined in (41), and $\mathscr{D}_{iN}$ $(i=1,2)$ are defined in Proposition 1. Employing the improved reciprocally convex inequality (21), one has

$$\begin{aligned}&-\bar{h}\int_{t-h(t)}^{t}\dot{x}^T(s)R\dot{x}(s)ds \\ &\le -\tfrac{1}{\alpha}\xi_0^T(t)\mathscr{D}_{1N}^T R_N\mathscr{D}_{1N}\xi_0(t) - \tfrac{1}{1-\alpha}\xi_0^T(t)\mathscr{D}_{2N}^T\mathcal{R}_N\mathscr{D}_{2N}\xi_0(t) \\ &\le \xi_0^T(t)\Big[\Upsilon_{2N}(h(t))+(1-\alpha)\mathscr{D}_{1N}^T Y_{1N}\mathcal{R}_N^{-1}Y_{1N}^T\mathscr{D}_{1N} \\ &\qquad + \alpha\mathscr{D}_{2N}^T Y_{2N}^T\mathcal{R}_N^{-1}Y_{2N}\mathscr{D}_{2N}\Big]\xi_0(t)\end{aligned}\tag{66}$$

where $\Upsilon_{2N}(h(t))$ is defined in (64). Substituting (66) into (65) yields

$$\dot{V}(t,x_t) \le \xi_0^T(t)\Upsilon_N(h(t),\dot{h}(t))\xi_0(t) \tag{67}$$

where

$$\begin{aligned}\Upsilon_N(h(t),\dot{h}(t)) &:= \Upsilon_{1N}(h(t),\dot{h}(t)) + \Upsilon_{2N}(h(t)) \\ &+ (1-\alpha)\mathscr{D}_{1N}^T Y_{1N}\mathcal{R}_N^{-1}Y_{1N}^T\mathscr{D}_{1N} \\ &+ \alpha\mathscr{D}_{2N}^T Y_{2N}^T\mathcal{R}_N^{-1}Y_{2N}\mathscr{D}_{2N}\end{aligned}\tag{68}$$

Note that $\Upsilon_N(h(t),\dot{h}(t))$ is linear on $h(t) \in [0,\bar{h}]$ and also on $\dot{h}(t) \in [\mu_1,\mu_2]$. If the LMIs in (61) and (62) are satisfied, by the Schur complement, one has

$$\Upsilon_N(h(t),\dot{h}(t)) < 0, \text{ for } (h(t),\dot{h}(t)) \in \mathcal{H}_1$$

Thus, from (67), there exists a scalar $\varepsilon > 0$ such that $\dot{V}(t,x_t) \le -\varepsilon\xi_0^T(t)\xi_0(t) \le -\varepsilon x^T(t)x(t)$, which concludes that the system (1) is asymptotically stable for $(h(t),\dot{h}(t)) \in \mathcal{H}_1$. □

Instead of the integral inequality (40), one can also use its affine version (46) to derive another N-dependent stability criterion by slightly modifying the Lyapunov-Krasovskii functional (59), where the term $\bar{h}\int_{-\bar{h}}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R\dot{x}(s)dsd\theta$ is replaced with $\int_{-\bar{h}}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R\dot{x}(s)dsd\theta$. The result is stated in the following.

Proposition 2. *For constants $\mu_1, \mu_2, \bar{h}$ (> 0) and an integer $N > 0$, the system (1) subject to $(h(t),\dot{h}(t)) \in \mathcal{H}_1$ is asymptotically stable if there exist real matrices $P_N > 0, Q_1 > 0, Q_2 > 0$ and $R > 0$ and real matrices Y_{1N} and Y_{2N} with appropriate dimensions such that, for $\mu \in \{\mu_1,\mu_2\}$*

$$\begin{bmatrix} \tilde{\Upsilon}_N(0,\mu) & \bar{h}Y_{2N}^T \\ \bar{h}Y_{2N} & -\bar{h}\mathcal{R}_N \end{bmatrix} < 0,\ \begin{bmatrix} \tilde{\Upsilon}_N(\bar{h},\mu) & \bar{h}Y_{1N}^T \\ \bar{h}Y_{1N} & -\bar{h}\mathcal{R}_N \end{bmatrix} < 0,$$

where

$$\begin{aligned}\tilde{\Upsilon}_N(h(t),\dot{h}(t)) &:= \bar{h}(Ae_1+A_de_2)^T R(Ae_1+A_de_2) \\ &+(1-\dot{h}(t))\mathscr{C}_4^T(Q_2-Q_1)\mathscr{C}_4 - \mathscr{C}_5^T Q_2\mathscr{C}_5+\mathscr{C}_3^T Q_1\mathscr{C}_3 \\ &+He\left\{\mathscr{C}_{1N}^T P_N\mathscr{C}_{2N}+(\mathscr{C}_6^T Q_1+\mathscr{C}_7^T Q_2)\mathscr{C}_8\right\} \\ &+\mathscr{D}_{1N}^T Y_{1N}+Y_{1N}^T\mathscr{D}_{1N}+\mathscr{D}_{2N}^T Y_{2N}+Y_{2N}^T\mathscr{D}_{2N}\end{aligned}\tag{69}$$

and the other notations are the same as those in Proposition 1.

Remark 1. *Propositions 1 and 2 deliver two N-dependent stability criteria for the system (1) subject to (47), thanks to the canonical integral inequality (40). The number of required decision variables can be calculated as $(4N^2+12N+32)n^2+(N+7)n$ for Proposition 1 and $(6N^2+20N+40)n^2+(N+7)n$ for Proposition 2. Moreover, the positive definiteness of the matrices P_N, Q_1 and Q_2 can be relaxed if one follows the line in [76] or [94].*

4.1.2 Hierarchy of LMI stability criteria: In [94], it is proven that the stability criterion in terms of LMIs forms a *hierarchy*. In the following, it is shown that such a *hierarchical* characteristic is also hidden in the LMIs of Propositions 1 and 2. Based on Proposition 1, one has

Proposition 3. *For the system* (1) *subject to* (47), *one has that*

$$\mathscr{H}_N \subseteq \mathscr{H}_{N+1} \tag{70}$$

where

$$\mathscr{H}_N := \left\{ \mathcal{H}_1 \;\middle|\; \begin{array}{c} \Psi_N(0,\mu)|_{\mu\in\{\mu_1,\mu_2\}} < 0,\ Q_1(N) > 0 \\ \tilde{\Psi}_N(\bar{h},\mu)|_{\mu\in\{\mu_1,\mu_2\}} < 0,\ Q_2(N) > 0 \\ P_N > 0,\ R(N) > 0,\ Y_{1N},\ Y_{2N} \end{array} \right\} \tag{71}$$

with $\mathcal{H}_1$, Ψ_N *and* $\tilde{\Psi}_N$ *being defined in* (54), (61) *and* (62), *respectively.*

Proof: Without loss generality, suppose that $\mathscr{H}_N$ is not empty. From the definition of $\mathscr{H}_N$, there exist real matrices $P_N > 0$, $Q_1(N) = Q_1 > 0$, $Q_2(N) = Q_2 > 0$ and $R(N) = R > 0$ and real matrices Y_{1N} and Y_{2N} with appropriate dimensions such that $\Psi_N(0,\mu)|_{\mu\in\{\mu_1,\mu_2\}} < 0$ and $\tilde{\Psi}_N(\bar{h},\mu)|_{\mu\in\{\mu_1,\mu_2\}} < 0$, which are equivalent to

$$\Upsilon_N(h(t),\dot{h}(t)) < 0, \text{ for } (h(t),\dot{h}(t)) \in \mathscr{H}_N$$

where $\Upsilon_N(h(t),\dot{h}(t)$ is defined in (68). Let

$$P_{N+1} = \begin{bmatrix} P_N & 0 \\ 0 & \varepsilon_1 I \end{bmatrix}, \quad Y_{1(N+1)} = \begin{bmatrix} Y_{1N} & 0 \\ 0 & \varepsilon_2 I \end{bmatrix}$$
$$Y_{2(N+1)} = \begin{bmatrix} Y_{2N} & 0 \\ 0 & \varepsilon_3 I \end{bmatrix}, \quad \left\{ \begin{array}{c} Q_1(N+1) = Q_1(N) = Q_1 \\ Q_2(N+1) = Q_2(N) = Q_2 \\ R(N+1) = R(N) = R. \end{array} \right.$$

where $\varepsilon_1 > 0$ and $\varepsilon_i \in \mathbb{R}$ $(i = 2, 3)$. Then, we only need to prove that

$$\Upsilon_{N+1}(h(t),\dot{h}(t)) < 0, \text{ for } (h(t),\dot{h}(t)) \in \mathscr{H}_N. \tag{72}$$

In fact, denote

$$\Theta_{N+1} = \begin{bmatrix} \Theta_N & 0 \\ \Sigma_1 & a_1 I \end{bmatrix}, \quad \Lambda_{N+1} = \begin{bmatrix} \Lambda_N & 0 \\ \Sigma_2 & (N+1)I \end{bmatrix}$$
$$\Gamma_N = \text{col}\{e_1, e_2, E_1 e_{5+1}, \cdots, E_1 e_{5+N}\}$$
$$\tilde{\Gamma}_N = \text{col}\{e_2, e_3, E_2 e_{5+1}, \cdots, E_2 e_{5+N}\}$$

where Σ_1 and Σ_2 are proper real matrices and $a_1 := \binom{2N+2}{N+1}$. It follows that

$$\Theta_{N+1}\Lambda_{N+1} = \begin{bmatrix} \Theta_N\Lambda_N & 0 \\ F_0 & a_0 I \end{bmatrix}, \quad \begin{array}{c} F_0 := \Sigma_1\Lambda_N + a_1\Sigma_2 \\ a_0 := (N+1)a_1 \end{array}$$

Let $\tilde{\xi}_0(t) := \text{col}\{\xi_0(t), \wp_{N+1}(t)\}$. Then

$$\mathscr{D}_{1,N+1}\tilde{\xi}_0(t) = \Theta_{N+1}\Lambda_{N+1} \begin{bmatrix} \Gamma_N & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \tilde{\xi}_0(t)$$
$$\mathscr{D}_{2,N+1}\tilde{\xi}_0(t) = \Theta_{N+1}\Lambda_{N+1} \begin{bmatrix} \tilde{\Gamma}_N & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{\xi}_0(t)$$
$$\mathscr{C}_{1,N+1}\tilde{\xi}_0(t) = \begin{bmatrix} \mathscr{C}_{1N} & 0 \\ 0 & \ell(t) \end{bmatrix} \tilde{\xi}_0(t)$$
$$\mathscr{C}_{2,N+1}\tilde{\xi}_0(t) = \begin{bmatrix} \mathscr{C}_{2N} & 0 \\ J_1 & J_2 \end{bmatrix} \tilde{\xi}_0(t),$$

where J_1 and J_2 are some proper real matrices. Thus

$$\mathscr{D}_{1,N+1} = \begin{bmatrix} \mathscr{D}_{1N} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ F_0\Gamma_N & a_0 I & 0 \end{bmatrix}$$
$$\mathscr{D}_{2,N+1} = \begin{bmatrix} \mathscr{D}_{2N} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ F_0\tilde{\Gamma}_N & 0 & a_0 I \end{bmatrix}$$
$$\mathscr{C}_{1,N+1} = \begin{bmatrix} \mathscr{C}_{1N} & 0 \\ 0 & \ell(t) \end{bmatrix}, \quad \mathscr{C}_{2,N+1} = \begin{bmatrix} \mathscr{C}_{2N} & 0 \\ J_1 & J_2 \end{bmatrix}$$

Denote

$$\mathscr{T} := \begin{bmatrix} I & 0 & 0 \\ F_0\Gamma_N & a_0 I & 0 \\ F_0\tilde{\Gamma}_N & 0 & a_0 I \end{bmatrix}$$

After some algebraic manipulations, one has

$$\begin{aligned} \Upsilon_{N+1}(h(t),\dot{h}(t)) = {} & \mathscr{T}^T \begin{bmatrix} \Upsilon_N(h(t),\dot{h}(t)) & 0 & 0 \\ 0 & \varphi_{11} & \varphi_{12} \\ 0 & \varphi_{12}^T & \varphi_{22} \end{bmatrix} \mathscr{T} \\ & + \varepsilon_1 \text{He}\left\{ \begin{bmatrix} 0 & 0 \\ \ell^T(t)J_1 & \ell^T(t)J_2 \end{bmatrix} \right\} \end{aligned} \tag{73}$$

where

$$\varphi_{11} := -(2-\alpha)(2N+3)R + (1-\alpha)\varepsilon_2^2[(2N+3)R]^{-1}$$
$$\varphi_{12} := -\Big(\alpha\varepsilon_2 + (1-\alpha)\varepsilon_3\Big)I$$
$$\varphi_{22} := -(1+\alpha)(2N+3)R + \alpha\varepsilon_3^2[(2N+3)R]^{-1}$$

Since $\mathscr{T}$ is nonsingular and $R > 0$, there exist sufficiently small scalar $\varepsilon_1 > 0$ and ε_i $(i = 2, 3)$ such that $\Upsilon_{N+1}(h(t),\dot{h}(t)) < 0$ if $\Upsilon_N(h(t),\dot{h}(t)) < 0$ for $(h(t),\dot{h}(t)) \in \mathscr{H}_N$, which concludes $\mathscr{H}_N \subseteq \mathscr{H}_{N+1}$. □

Similar to Proposition 3, one can prove that the LMIs in Proposition 2 also form a hierarchy. For given scalars μ_1 and μ_2, we denote by $\bar{h}_N$ the admissible maximum upper bound of the time-varying delay $h(t)$ using Proposition 1 or 2. Then from the hierarchy feature, one can draw a conclusion that $\bar{h}_N \leq \bar{h}_{N+1}$.

4.2 Stability criteria for Case 2

In the case where the time-delay $h(t)$ satisfies (48), it is challenging to establish an N-dependent stability criterion using Corollary 4 since one cannot exploit the convex property to cope with the derivative of $h(t)$ caused from the vectors in (60) if $\dot{h}(t)$ is unbounded from below. However, for $N = 2$, we can derive some novel results based on the following augmented Lyapunov-Krasovskii functional as

$$\begin{aligned} \widehat{V}(t,x_t) := {} & \vartheta_1^T(t)P\vartheta_1(t) + \int_{t-h(t)}^{t} \vartheta_2^T(s,t)Q_1\vartheta_2(s,t)ds \\ & + \int_{t-\bar{h}}^{t} \vartheta_2^T(s,t)Q_2\vartheta_2(s,t)ds \\ & + \bar{h}\int_{-\bar{h}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta \end{aligned} \tag{74}$$

where $\vartheta_1(t) := \text{col}\big\{x(t), \int_{t-\bar{h}}^{t} x(s)ds, \int_{t-\bar{h}}^{t}(t-s)x(s)ds\big\}$ and $\vartheta_2(s,t) := \text{col}\big\{x(t), x(s), \int_s^t x(\theta)d\theta\big\}$.

Proposition 4. *For constants* μ_2 *and* $\bar{h}$ (> 0), *the system* (1) *subject to* (48) *is asymptotically stable if there exist real matrices* $P > 0$, $Q_1 > 0$, $Q_2 > 0$ *and* $R > 0$ *and real matrices* Y_1, Y_2 *and* M_0 *with appropriate dimensions such that*

$$\begin{bmatrix} \Xi_1(0) + \Xi_2(0) & \mathcal{C}_{11}^T Y_1 \\ Y_1^T \mathcal{C}_{11} & -\tilde{R} \end{bmatrix} < 0 \tag{75}$$

$$\begin{bmatrix} \Xi_1(\bar{h})+\Xi_2(\bar{h}) & \mathcal{C}_{10}^T Y_2^T \\ Y_2\mathcal{C}_{10} & -\tilde{R} \end{bmatrix} < 0 \tag{76}$$

where $\tilde{R} := diag\{R, 3R, 5R\}$, Θ_2 *and* Λ_2 *are defined in* (42) *and* (43) *with* $N = 2$; *and*

$$\Xi_1(h(t)) := \Xi_{11}(h(t)) - (1-\mu_2)\mathcal{C}_4^T Q_1 \mathcal{C}_4 \tag{77}$$

$$\begin{aligned}\Xi_{11}(h(t)) := He\Big\{&\mathcal{C}_1^T P\mathcal{C}_2+(\mathcal{C}_7^T Q_1+\mathcal{C}_8^T Q_2)\mathcal{C}_6+M_0\mathcal{C}_9\Big\}\\ &+\mathcal{C}_3^T(Q_1+Q_2)\mathcal{C}_3-\mathcal{C}_5^T Q_2\mathcal{C}_5\\ &+\bar{h}^2(A\bar{e}_1+A_d\bar{e}_2)^T R(A\bar{e}_1+A_d\bar{e}_2)\end{aligned} \tag{78}$$

$$\begin{aligned}\Xi_2(h(t)) := &-(2-\alpha)\mathcal{C}_{11}^T\tilde{R}\mathcal{C}_{11}-(1+\alpha)\mathcal{C}_{10}^T\tilde{R}\mathcal{C}_{10}\\ &- He\Big\{\mathcal{C}_{11}^T[\alpha Y_1+(1-\alpha)Y_2]\mathcal{C}_{10}\Big\}\end{aligned} \tag{79}$$

with $\bar{e}_i$ $(i = 1, 2, \cdots, 11)$ *being the ith* $n \times 11n$ *row-block of the* $11n \times 11n$ *identity matrix,* $\alpha = h(t)/\bar{h}$*; and*

$$\begin{aligned}
\mathcal{C}_1 &:= col\{A\bar{e}_1+A_d\bar{e}_2, \bar{e}_1-\bar{e}_3, \bar{e}_8+\bar{e}_{10}-\bar{h}\bar{e}_3\}\\
\mathcal{C}_2 &:= col\{\bar{e}_1, \bar{e}_8+\bar{e}_{10}, h(t)(\bar{e}_9+\bar{e}_{10})+(\bar{h}-h(t))\bar{e}_{11}\}\\
\mathcal{C}_3 &:= col\{\bar{e}_1, \bar{e}_1, 0\},\ \ \mathcal{C}_4 := col\{\bar{e}_1, \bar{e}_2, \bar{e}_8\}\\
\mathcal{C}_5 &:= col\{\bar{e}_1, \bar{e}_3, \bar{e}_8+\bar{e}_{10}\},\ \mathcal{C}_6 := col\{A\bar{e}_1+A_d\bar{e}_2, 0, \bar{e}_1\}\\
\mathcal{C}_7 &:= col\{h(t)\bar{e}_1, h(t)\bar{e}_4, h(t)(\bar{e}_8-\bar{e}_9)\}\\
\mathcal{C}_8 &:= \begin{bmatrix} col\{\bar{h}\bar{e}_1, \bar{e}_8+\bar{e}_{10}\} \\ \bar{h}(\bar{e}_8+\bar{e}_{10})-h(t)(\bar{e}_9+\bar{e}_{10})-(\bar{h}-h(t))\bar{e}_{11} \end{bmatrix}\\
\mathcal{C}_9 &:= \begin{bmatrix} col\{\bar{e}_8-h(t)\bar{e}_4,\ \bar{e}_9-h(t)\bar{e}_5\} \\ col\{\bar{e}_{10}-(\bar{h}-h(t))\bar{e}_6, \bar{e}_{11}-(\bar{h}-h(t))\bar{e}_7\} \end{bmatrix}\\
\mathcal{C}_{11} &:= \Theta_2\Lambda_2 col\{\bar{e}_1, \bar{e}_2, \bar{e}_4, \bar{e}_5\}\\
\mathcal{C}_{10} &:= \Theta_2\Lambda_2 col\{\bar{e}_2, \bar{e}_3, \bar{e}_6, \bar{e}_7\}.
\end{aligned}$$

Proof: Taking the time-derivative of $\widehat{V}(t, x_t)$ in (74) yields

$$\begin{aligned}\dot{\widehat{V}}(t, x_t) = \ &2\vartheta_1^T(t)P\dot{\vartheta}_1(t)+\vartheta_2^T(t,t)(Q_1+Q_2)\vartheta_2(t,t)\\ &-(1-\dot{h}(t))\vartheta_2^T(t-h(t),t)Q_1\vartheta_2(t-h(t),t)\\ &+\bar{h}^2\dot{x}^T(t)R\dot{x}(t)-\vartheta_2^T(t-\bar{h},t)Q_2\vartheta_2(t-\bar{h},t)\\ &+\int_{t-h(t)}^{t} 2\vartheta_2^T(s,t)Q_1\frac{\partial\vartheta_2(s,t)}{\partial t}ds\\ &+\int_{t-\bar{h}}^{t} 2\vartheta_2^T(s,t)Q_2\frac{\partial\vartheta_2(s,t)}{\partial t}ds\\ &-\bar{h}\int_{t-\bar{h}}^{t}\dot{x}^T(s)R\dot{x}(s)dsd\theta\end{aligned} \tag{80}$$

Denote $\bar{\xi}(t) := \mathrm{col}\{x(t), x(t-h(t)), x(t-\bar{h}), \nu_{11}(t), \nu_{12}(t)\ \nu_{21}(t), \nu_{22}(t), \nu_{31}(t), \nu_{32}(t), \nu_{41}(t), \nu_{42}(t)\}$, where $\nu_{1i}(t)$ and $\nu_{2i}(t)$ $(i = 1, 2)$ are defined in (60), and

$$\nu_{3i}(t)=h(t)\nu_{1i}(t), \nu_{4i}(t)=(\bar{h}-h(t))\nu_{2i}(t), i = 1, 2 \tag{81}$$

Then for any real matrix M_0 with appropriate dimensions, the following equation holds

$$2\bar{\xi}^T(t)M_0\mathcal{C}_9\bar{\xi}(t) = 0 \tag{82}$$

From (48), (80) and (82), one obtains

$$\dot{\widehat{V}}(t, x_t) \le \bar{\xi}^T(t)\Xi_1(h(t))\bar{\xi}(t)-\bar{h}\int_{t-\bar{h}}^{t}\dot{x}^T(s)R\dot{x}(s)ds \tag{83}$$

where $\Xi_1(h(t))$ is defined in (77). Apply the integral inequality (40) with $N = 2$ to get

$$\begin{aligned}-\bar{h}\int_{t-\bar{h}}^{t}\dot{x}^T(s)R\dot{x}(s)ds \le &-\tfrac{1}{\alpha}\bar{\xi}^T(t)\mathcal{C}_{11}^T\tilde{R}\mathcal{C}_{11}\bar{\xi}(t)\\ &-\tfrac{1}{1-\alpha}\bar{\xi}^T(t)\mathcal{C}_{10}^T\tilde{R}\mathcal{C}_{10}\bar{\xi}(t)\end{aligned}$$

where $\alpha = h(t)/\bar{h}$ and $\tilde{R} := \mathrm{diag}\{R, 3R, 5R\}$. By the improved reciprocally convex inequality (21), one has

$$\begin{aligned}&-\bar{h}\int_{t-\bar{h}}^{t}\dot{x}^T(s)R\dot{x}(s)ds\\ &\le \bar{\xi}^T(t)\Big\{\mathcal{C}_{11}^T[-(2-\alpha)\tilde{R}+(1-\alpha)Y_1\tilde{R}^{-1}Y_1^T]\mathcal{C}_{11}\\ &\quad+\mathcal{C}_{10}^T[-(1+\alpha)\tilde{R}+\alpha Y_2^T\tilde{R}^{-1}Y_2]\mathcal{C}_{10}\\ &\quad-2\mathcal{C}_{11}^T[\alpha Y_1+(1-\alpha)Y_2]\mathcal{C}_{10}\Big\}\bar{\xi}(t)\end{aligned} \tag{84}$$

Substituting (84) into (83) yields

$$\begin{aligned}&\dot{\widehat{V}}(t, x_t) \le \bar{\xi}^T(t)\Xi(h(t))\bar{\xi}(t) \qquad (85)\\ &\Xi(h(t)) := \Xi_1(h(t))+\mathcal{C}_{11}^T(1-\alpha)Y_1\tilde{R}^{-1}Y_1^T\mathcal{C}_{11}\\ &\qquad+\Xi_2(h(t))+\alpha\mathcal{C}_{10}^T Y_2^T\tilde{R}^{-1}Y_2^T\mathcal{C}_{10}\end{aligned}$$

where $\Xi_2(h(t))$ defined in (79). If the LMIs in (75) and (76) are satisfied, $\Xi(h(t)) < 0$ holds by using the Schur complement. Thus, one can conclude that the system (1) subject to (48) is asymptotically stable. □

Similar to Proposition 2, if using the affine integral inequality (46) instead of (40), we have the following result.

Proposition 5. *For constants* μ_2 *and* $\bar{h}$ (> 0)*, the system* (1) *subject to* (48) *is asymptotically stable if there exist real matrices* $P > 0$, $Q_1 > 0$, $Q_2 > 0$ *and* $R > 0$ *and real matrices* Y_1, Y_2 *and* M_0 *with appropriate dimensions such that*

$$\begin{bmatrix} \widehat{\Xi}(0) & \bar{h}Y_2^T \\ \bar{h}Y_2 & -\bar{h}\tilde{R} \end{bmatrix} < 0, \quad \begin{bmatrix} \widehat{\Xi}(\bar{h}) & \bar{h}Y_1^T \\ \bar{h}Y_1 & -\bar{h}\tilde{R} \end{bmatrix} < 0, \tag{86}$$

where

$$\begin{aligned}\widehat{\Xi}(h(t)) := He\Big\{&\mathcal{C}_1^T P\mathcal{C}_2+(\mathcal{C}_7^T Q_1+\mathcal{C}_8^T Q_2)\mathcal{C}_6+M_0\mathcal{C}_9\\ &+\mathcal{C}_{11}^T Y_1+\mathcal{C}_{10}^T Y_2\Big\}+\mathcal{C}_3^T(Q_1+Q_2)\mathcal{C}_3-(1-\mu_2)\mathcal{C}_4^T Q_1\mathcal{C}_4\\ &-\mathcal{C}_5^T Q_2\mathcal{C}_5+\bar{h}(A\bar{e}_1+A_d\bar{e}_2)^T R(A\bar{e}_1+A_d\bar{e}_2)\end{aligned} \tag{87}$$

and the other notations are defined in Proposition 4.

Remark 2. *Propositions* 4 *and* 5 *provides two stability criteria for the system* (1) *subject to* (48)*. Compared with [64, Theorem 1], the main difference lies in that Propositions* 4 *and* 5 *are derived based on such a condition as* $\Xi(h(t)) < 0$ *for* $h(t) \in [0, \bar{h}]$*, where* $\Xi(h(t))$ *is a linear matrix-valued function on* $h(t)$*, leading to a necessary and sufficient condition* $\Xi(0) < 0$ *and* $\Xi(\bar{h}) < 0$ *such that* $\Xi(h(t) < 0$ *for* $h(t) \in [0, \bar{h}]$*. This linear matrix-valued function contributes to the introduction of the vectors* $\nu_{3i}(t)$ *and* $\nu_{4i}(t)$ $(i = 1, 2)$ *in* (81)*. However, in the proof of [64, Theorem 1],* $\Xi(h(t))$ *is a quadratic function on* $h(t)$*. Thus, applying the quadratic convex approach in* (13) *only gives a sufficient condition such that* $\Xi(h(t)) < 0$ *for* $h(t) \in [0, \bar{h}]$.

Remark 3. *The purpose of introducing the vectors* $\nu_{3i}(t)$ *and* $\nu_{4i}(t)$ $(i = 1, 2)$ *in* (81) *is to absorb* $h(t)$ *such that* $\Xi(h(t))$ *is linear on* $h(t)$*; Otherwise,* $\Xi(h(t))$ *will be a triple matrix-valued polynomial function on* $h(t)$*, which is difficult in deriving a stability criterion for the system* (1) *in Case 2. The number of decision variables required is* $76n^2 + 5n$ *for Proposition* 4 *and* $114n^2 + 5n$ *for Proposition* 5.

4.3 Stability criteria for Case 3

Under Case 3, the time-varying delay $h(t)$ is known to be continuous, but no any information on the derivative of the time-varying delay is available in the stability analysis. In this case, the augmented Lyapunov-Krasovskii functional can be chosen as

$$\bar{V}(t,x_t) := \vartheta_1^T(t)P\vartheta_1(t) + \int_{t-\bar{h}}^{t}\vartheta_2^T(s,t)Q\vartheta_2(s,t)ds + \bar{h}\int_{-\bar{h}}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R\dot{x}(s)dsd\theta \quad (88)$$

where $\vartheta_1(t)$ and $\vartheta_2(t)$ are defined in (74). Using the integral inequality (40) and its affine version (46), similar to the proof of Propositions 4 and 5, we have the following two stability criteria.

Proposition 6. *For a constant $\bar{h}$ (>0), the system* (1) *subject to* (49) *is asymptotically stable if there exist real matrices $P>0$, $Q>0$, and $R>0$ and real matrices Y_1, Y_2 and M_0 with appropriate dimensions such that one of the following two statements is true.*

i) The LMIs in (75) *and* (76) *are satisfied, where $\Xi_1(h(t))$ is replaced with $\tilde{\Xi}_1(h(t))$ as*

$$\tilde{\Xi}_1(h(t)) := He\{\mathcal{C}_1^T P\mathcal{C}_2 + \mathcal{C}_8^T Q\mathcal{C}_6 + M_0\mathcal{C}_9\} + \mathcal{C}_3^T Q\mathcal{C}_3 - \mathcal{C}_5^T Q\mathcal{C}_5 + \bar{h}^2(A\bar{e}_1 + A_d\bar{e}_2)R(A\bar{e}_1 + A_d\bar{e}_2)$$

ii) The LMIs in (86) *are satisfied, where $\Xi_1(h(t))$ is replaced with $\hat{\Xi}_1(h(t))$ as*

$$\hat{\Xi}_1(h(t)) := He\{\mathcal{C}_1^T P\mathcal{C}_2 + \mathcal{C}_8^T Q\mathcal{C}_6 + M_0\mathcal{C}_9 + \mathcal{C}_{11}^T Y_1\} + He\{\mathcal{C}_{10}^T Y_2\} + \mathcal{C}_3^T Q\mathcal{C}_3 - \mathcal{C}_5^T Q\mathcal{C}_5 + \bar{h}(A\bar{e}_1 + A_d\bar{e}_2)R(A\bar{e}_1 + A_d\bar{e}_2)$$

The other notations are defined in Proposition 4.

Proof: The proof can be completed by following the proof of Proposition 4. □

Remark 4. *In Case 3, Proposition* 6 *presents two stability criteria for the system* (1)*. By using the second-order Bessel-Legendre inequality, a stability criterion for the system* (1) *with* (49) *is also reported in [52, Theorem 1]. The main difference between them lies in the chosen Lyapunov-Krasovskii functional. In Proposition* 6*, an augmented vector $\vartheta_2(s,t)$ is introduced in $\bar{V}(t,x_t)$ in* (88)*, but not in [52, Theorem 1]. As a result, taking the derivative of the augmented term yields*

$$\frac{d}{dt}\int_{t-\bar{h}}^{t}\vartheta_2^T(s,t)Q\vartheta_2(s,t)ds = \bar{\xi}^T(t)\big(2\mathcal{C}_8^T Q\mathcal{C}_6 + \mathcal{C}_3^T Q\mathcal{C}_3 - \mathcal{C}_5^T Q\mathcal{C}_5\big)\bar{\xi}(t)$$

That is, the vectors $x(t), x(t-h(t)), x(t-\bar{h})$ and ν_{3i}, ν_{4i} $(i=1,2)$ in (81) *are coupled by Q, which enhances the feasibility of the stability conditions in Proposition* 6.

Remark 5. *The number of decision variables required in Proposition* 6 *is $74.5n^2+3.5n$ for the condition i) and $119.5n^2+3.5n$ for the condition ii), which are smaller than $154.5n^2+4.5n$ in [52, Theorem 2].*

Remark 6. *It should be pointed out that, the proposed results in this section can be easily extended to a linear system with an interval time-varying delay $d(t)\in[h_0,\bar{h}]$ provided that one modifies the chosen Lyapunov-Krasovskii functionals by taking the lower bound h_0 into account. Because of their similarities, those results are omitted in the paper.*

Table 1 The AMUB $\bar{h}$ for μ $(\mu_2=-\mu_1=\mu)$ (Case 1 for Example 1)

Method \ μ	0	0.1	0.5	0.8
[21]	6.117	4.714	2.280	1.608
[104]	6.117	4.794	2.682	1.957
[69]	6.165	4.714	2.608	2.375
[67]	6.059	4.788	3.055	2.615
[102]	6.0593	4.71	2.48	2.30
[105]	6.0593	4.8313	3.1487	2.7135
[72]	6.168	4.910	3.233	2.789
[94]*	6.1725	5.01	3.19	2.70
Prop 1 ($N=1$)	6.0593	4.8344	3.1422	2.7131
Prop 1 ($N=2$)	6.1689	4.9192	3.1978	2.7656
Prop 1 ($N=3$)	6.1725	4.9203	3.2164	2.7875
Prop 1 ($N=4$)	6.1725	4.9246	3.2230	2.7900
Prop 2 ($N=1$)	6.0593	4.8377	3.1521	2.7278
Prop 2 ($N=2$)	6.1689	4.9217	3.2211	2.7920
Prop 2 ($N=3$)	6.1725	4.9239	3.2405	2.8159
Prop 2 ($N=4$)	6.1725	4.9297	3.2527	2.8230

The symbol '*' means that the results in this line are obtained from Theorem 8 with $N=4$ in [94].

4.4 Illustrative examples

In this section, we compare the above stability criteria with some existing ones through two numerical examples.

Example 1. *Consider the system* (1)*, where*

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \; A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \quad (89)$$

The time-varying delay $h(t)$ is a differentiable function on $[t_0,\infty)$.

Example 1 is well used to calculate the admissible maximum upper bound (AMUB) $\bar{h}$ for the time-varying delay $h(t)$. In order to make a comparison, we consider two cases of $h(t)$.

Case 1: $h(t)$ satisfies (47) *with $\mu_2=-\mu_1=\mu$.*

We compare the stability criteria with some existing ones obtained for $(h(t),\dot{h}(t))\in\mathcal{H}_1$ defined in (54). For different values of μ, Tab. 1 lists the obtained AMUBs of $\bar{h}$ by [102, Theorem 1], [72, Proposition 1], [69, Theorem 2], [105, Theorem 1], [67, Corollary 1], [94, Theorem 8 with $N=4$], the IQC approach [21], the quadratic separation approach [104] and Propositions 1 and 2 with $N\in\{1,2,3,4\}$ in this paper. From Tab. 1, one can see that

- Propositions 1 and 2 with $N\in\{2,3,4\}$ obtain a larger upper bound $\bar{h}$ than the criteria in [67, 69, 72, 102, 105], the IQC approach [21] and the quadratic separation approach [104]. Even for $N=1$ and $\mu\in\{0.1,0.5,0.8\}$, Propositions 1 and 2 outperform [69, Theorem 2], [105, Theorem 1], [67, Corollary 1], the IQC approach [21] and the quadratic separation approach [104];
- For $\mu=0.1$, [94, Theorem 8 with $N=4$] gives a larger delay upper bound than Propositions 1 and 2 with $N=4$ due to that the positive definiteness of the matrices P_N, Q_1 and Q_2 are relaxed. However, for $\mu\in\{0.5,0.8\}$ and $N\geq 2$, Propositions 1 and 2 offer better results that [94, Theorem 8]; and
- For the same N, Proposition 2 delivers a larger upper bound $\bar{h}$ than Proposition 2 at the cost of higher computation burden, which means that a stability criterion using the affine integral inequality (46) can derive a larger upper bound $\bar{h}$ than that using the integral inequality (40) and the improved reciprocally convex inequality (21).

Case 2: The time-varying delay $h(t)$ satisfies (48).

In order to show the effectiveness of Propositions 4 and 5, the AMUBs of $\bar{h}$ are listed in Tab. 2 for different values of μ_2. From this table, one can see that i) Propositions 4 and 5 indeed can derive some larger upper bounds of $\bar{h}$ than [64, Theorem 1] and [79, Theorem 1], while Propositions 4 and 5 require more decision variables than [79,

Table 2 The AMUB $\bar{h}$ for different μ_2 (Case 2 for Example 1)

Method \ μ_2	0	0.1	0.5	0.8	1
[79]	6.059	4.704	2.420	2.113	2.113
[64]	6.168	4.733	2.429	2.183	2.182
Proposition 4	6.168	4.800	2.533	2.231	2.231
Proposition 5	6.168	4.800	2.558	2.269	2.263

Table 3 The AMUB $\bar{h}$ for Example 2

Method	$\bar{h}$	Number of decision variables
[47]	1.59	$10.5n^2 + 3.5n$
[53]	1.64	$21n^2 + 6n$
[52]	2.39	$154.5n^2 + 4.5n$
Prop 6-i)	2.39	$74.5n^2 + 3.5n$
Prop 6-ii)	2.53	$119.5n^2 + 3.5n$

Theorem 1] ($9n^2 + 3n$) and [64, Theorem 1] ($27n^2 + 4n$); and ii) The affine integral inequality (46) can result in a larger upper bound $\bar{h}$ than the integral inequality (40) plus the improved reciprocally convex inequality (21).

Example 2. *Consider the system* (1) *subject to* (49)*, where*

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} \tag{90}$$

The time delay $h(t)$ is continuous but not differentiable on $[t_0, \infty)$.

This example is taken to illustrate the validity of Proposition 6.

For comparison, we calculate the upper bound of $\bar{h}$ such that the system remains stable. Applying [47, Theorem 7], [53, Theorem 1], [52, Theorem 2] and Proposition 6, the obtained results and the required number of decision variables are listed in Tab. 3, from which one can see that Proposition 6 outperforms those methods in [47, 52, 53]. Moreover, it is clear that using the affine integral inequality (46) can yield a larger upper bound $\bar{h}$ than that using the integral inequality (40) though more decision variables are required.

In summary, through two well-used numerical examples, it is shown that, the obtained stability criteria in this paper are more effective than some existing ones in deriving a larger upper bound for a linear system with a time-varying delay.

As a counterpart of integral inequalities, finite-sum inequalities for stability analysis of discrete-time systems with time-varying delays also have gained much attention. A large number of finite-sum inequalities and stability criteria have been reported in the published literature, see, [106–115]. Since discrete-time systems with time-varying delays are not the focus of the paper, stability criteria based on finite-sum inequalities developed recently are not mentioned in the paper.

5 Conclusion and some challenging issues

An overview and in-depth analysis of recent advances in stability analysis of time-delay systems has been provided, including recent developments of integral inequalities, convex delay analysis approaches, reciprocally convex approaches and augmented Lyapunov-Krasovskii functionals. Then, some existing stability conditions have been reviewed by taking into consideration three cases of time-varying delay, where information on the upper and lower bounds of the delay-derivative are totally known, partly known and completely unknown. Furthermore, a number of stability criteria have been developed by employing the recent canonical Bessel-Legendre integral inequalities and an augmented Lyapunov-Krasovskii functional. When information on the lower and upper bounds of both $h(t)$ and $\dot{h}(t)$ is known, the obtained stability criteria have been proven to be hierarchical.

However, although there has been significant progress in stability analysis of time-delay systems, the following issues are still challenging.

- If the positive integer N approaches to infinity, the canonical N-order Bessel-Legendre inequality can provide an accurate estimate on the integral term. Thus, using such an integral inequality, it is possible to derive a necessary and sufficient condition on stability for linear systems with time-varying delays, which is interesting but challenging. Moreover, extending the canonical N-order Bessel-Legendre inequality to multi-dimensional systems like 2-D systems with time-varying delays is also an interesting topic [116, 117];
- For the system (1) subject to (48) or (49), no N-dependent stability criteria are derived using the integral inequality (40) due to the vectors γ_k $(k = 1, 2, \cdots, N)$ in which the scalar $(b-a)^k$ appears in the form of its inverse. Applying the integral inequality (40) to the system (1) possibly yields such a stability condition as $\sum_{j=0}^{N} h^j(t)\Gamma_j < 0$, where Γ_j $(j = 0, 1, 2, \cdots, N)$ are real matrices irrespective of the time-varying delay $h(t)$. How to obtain a necessary and sufficient feasible condition such that $\sum_{j=0}^{N} h^j(t)\Gamma_j < 0$ for $h(t) \in [0, \bar{h}]$ is a significant problem;
- In the proof of Proposition 4, four extra vectors $\nu_{3i}(t)$ and $\nu_{4i}(t)$ $(i = 1, 2)$ are introduced to absorb $h(t)$ such that the obtained stability condition $\Xi(h(t)) < 0$ is dependent *linearly* on the time-varying delay $h(t) \in [0, \bar{h}]$. If not doing so, $\Xi(h(t))$ will be a triple matrix-valued polynomial function on $h(t)$. As a result, a necessary and sufficient condition, i.e. $\Xi(0) < 0$ and $\Xi(\bar{h}) < 0$, can be derived such that $\Xi(h(t)) < 0$ for $h(t) \in [0, \bar{h}]$. How to extend this technique to a general case is an interesting issue;
- The integral inequality (40) is established based on a sequence of orthogonal polynomials. Is it possible to formulate some integral inequality based on a sequence of non-orthogonal polynomials such that the scalars $(b-a)^k$ $(k = 1, 2, \cdots, N)$ disappear or appears linearly? Answering this question is beneficial for deriving less conservative stability criteria for linear systems with time-varying delay, which is significant and challenging;
- Simulation in this paper shows that the canonical Bessel-Legendre inequality approach can yield some nice results on stability. However, how to apply it to deal with control problems of a number of practical systems, such as networked control systems [118, 119], event-triggered control systems [120–122], vibration control systems [123–125], formation control systems [126] and multi-agent systems [127–130], deserves much effort of researchers.

6 Acknowledgments

This work is supported in part by the Australian Research Council Discovery Project under Grant DP160103567.

7 References

1 Hale, J.K., Verduyn Lunel, S.M.: 'Introduction to Functional Differential Equations' (New York: Springer-Verlag 1993)
2 Gu, K., Niculescu, S. I.: 'Survey on recent results in the stability and control of time-delay systems', J. Dyn. Syst. Meas. Control, 2003, 125, (2), pp. 158-165
3 Richard, J.: 'Time-delay systems: An overview of some recent advances and open problems', Automatica, 2003, 39, (10), pp. 1667-1694
4 Sipahi, R., Niculescu, S., Abdallah, C. T., Michiels, W., Gu, K.: 'Stability and stabilization of systems with time delay', IEEE Control Syst. Mag. 2011, 31, (1), pp. 38-65
5 Zhang, X.-M., Han, Q.-L.: 'Global asymptotic stability for a class of generalized neural networks with interval time-varying delay', IEEE Trans. Neural Netw., 2011, 22, (8), pp. 1180-1192
6 Zhang, X.-M., Han, Q.-L., Wang, Z., Zhang, B.-L.: 'Neuronal state estimation for neural networks with two additive time-varying delay components', IEEE Trans. Cybern., 2017, 47, (10), pp. 3184-3194
7 Briat, C.: 'Linear parameter-varying and time-delay systems' (Heidelberg: Springer 2014)
8 Fridman, E.: 'Introduction to time-delay systems: analysis and control'. (Birkhäuser, Basel, 2014)
9 Gu, K., Kharitonov, L., Chen, J.: 'Stability of time-delay systems' (Boston: Birkhäuser 2003)

10 Seuret, A., Gouaisbaut, F., Baudouin, L.: 'D1.1 - Overview of Lyapunov methods for time-delay systems', [Research Report] Rapport LAAS n$^\circ$16308, LAAS-CNRS. 2016. <hal-01369516>
11 Sun, J., Chen, J.: 'A survey on Lyapunov-based methods for stability of linear time-delay systems', Front. Comput. Sci., 2017, 11, (4), pp. 555-567
12 Gu, K., Han, Q.-L.: 'Discretized Lyapunov functional for linear uncertain systems with time-varying delay'. Proc. Amer. Control Conf., 2000, pp. 1375-1379
13 Hetel, L., Fiter, C., Omran, H., Seuret, A., Fridman, E., Richard, J.P., Niculescu, S.I.: 'Recent developments on the stability of systems with aperiodic sampling: an overview', Automatica, 2017, 76, pp. 309-335.
14 Yue, D., Han, Q.-L., Lam, J.: 'Network-based robust H_∞ control of systems with uncertainty', Automatica, 2005, 41, (6), pp. 999-1007
15 Yue, D., Tian, E., Han, Q.-L.: 'A delay system method for designing event-triggered controllers of networked control systems', IEEE Trans. Autom. control, 2013, 58, (2), pp. 475-481
16 Zhang, D., Han, Q.-L., Jia, X.: 'Network-based output tracking control for TŨS fuzzy systems using an event-triggered communication scheme', Fuzzy Sets Syst., 2015, 273, pp. 26–48
17 Peng, C., Han, Q.-L.: 'On designing a novel self-triggered sampling scheme for networked control systems with data losses and communication delays,' IEEE Trans. Ind. Electron., 2016, 63, (2), pp. 1239–1248
18 Zhang, X.-M., Han, Q.-L., Zhang, B.-L.: 'An overview and deep investigation on sampled-data-based event-triggered control and filtering for networked systems', IEEE Trans. Ind. Informat., 2017, 13, (1), pp. 4-16
19 Zhang, X.-M., Han, Q.-L., Yu, X.: 'Survey on recent advances in networked control systems', IEEE Trans. Ind. Informat., 2016, 12, (10), pp. 1740-1752
20 Wang, Y.-L., Han, Q.-L.: 'Network-based modelling and dynamic output feedback control for unmanned marine vehicles in network environments', Automatica, 2018, 91, pp. 43–53
21 Kao, C.-Y., Rantzer, A.: 'Stability analysis of systems with uncertain time-varying delays', Automatica, 2007, 43, (5), pp. 959-970
22 Megretski, A., Jönsson, U., Kao, C., Rantzer, A.: 'Chapter 41: Integral quadratic constraints'. In Control systems handbook. Boca Raton: CRC Press (Chapter), 2010
23 Megretski, A., Rantzer, A.: 'System analysis via integral quadratic constraints', IEEE Trans. Autom. Control, 1997, 42, (6), pp. 819-830
24 Pfifer, H., Seiler, P.: 'Integral quadratic constraints for delayed nonlinear and parameter-varying systems', Automatica, 2015, 56, pp. 36-43
25 Gu, K.: 'Discretized LMI set in the stability problem of linear uncertain time-delay systems', Int. J. Control, 1997, 68, (4), pp. 923-934
26 Gu, K.: 'A further refinement of discretized Lyapunov functional method for the stability of time-delay systems', Int. J. Control, 2001, 74, (10), pp. 967-976
27 Gu, K., Han, Q.-L., Luo, A., Niculescu, S. I.: 'Discretized Lyapunov functional for systems with distributed delay and piecewise constant coefficients', Int. J. Control, 2001, 74, (7), pp. 737-744
28 Han, Q.-L., Gu, K.: 'On robust stability of time-delay systems with norm-bounded uncertainty', IEEE Trans. Autom. Control, 2001, 46, (9), pp. 1426-1431
29 Han, Q.-L.: 'A discrete delay decomposition approach to stability of linear retarded and neutral systems', Automatica, 2009, 45, (2), pp. 517-524
30 Park, P. G.: 'A delay-dependent stability criterion for systems with uncertain time-invariant delays', IEEE Trans. Autom. Control, 1999, 44, pp. 876-877
31 Moon, Y. S., Park, P. G., Kwon, W. H., Lee, Y. S.: 'Delay-dependent robust stabilization of uncertain statedelayed systems', Int. J. Control, 2001, 74, pp. 1447-1455
32 Li, X., de Souza, C.: 'Criteria for robust stability and stabilization of uncertain linear systems with state delay', Automatica, 1997, 33, pp. 1657-1662
33 Kolmanovskii, V. B., Richard, J.-P., Tchangani, A. P.: 'Some model transformation for the stability study of linear systems with delays'. Proc. IFAC Workshop Linear Time-Delay Systems, Grenoble, France, 1998, pp. 75-80
34 Kolmanovskii, V. B., Richard, J.-P.: 'Stability of some linear systems with delays', IEEE Trans. Autom. Control, 1999, 44, pp. 984-989
35 Niculescu, S.-I.: 'On delay-dependent stability under model transformations of some neutral linear systems', Int. J. Control, 2001, 74, pp. 609-617
36 Fridman, E.: 'New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems', Syst. Control Lett., 2001, 43, pp. 309-319
37 Fridman, E., Shaked, U.: 'A descriptor system approach to H_∞ control of linear time-delay systems' IEEE Trans. Autom. control, 2002, 47, (2), pp. 253-270
38 Gu, K., Niculescu S.-I.: 'Additional dynamics in transformed time-delay systems', IEEE Trans. Autom. Control, 2000, 45, pp. 572-575
39 Gu, K., Niculescu, S. I.: 'Further remarks on additional dynamics in various model transformations of linear delay systems', IEEE Trans. Autom. Control, 2001, 46, pp. 497-500
40 He, Y., Wu, M., She, J. H., Liu, G. P.: 'Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays', Syst. Control Lett., 2004, 51, pp. 57-65
41 Wu, M., He, Y., She, J. H., Liu, G. P.: 'Delay-dependent criteria for robust stability of time-varying delay systems', Automatica, 40, pp. 1435-1439
42 Briat, C.: 'Convergence and equivalence results for the Jensen's inequality – application to time-delay and sampled-data systems', IEEE Trans. Autom. Control, 2011, 56, (7), pp. 1660-1665
43 Zhang, X.-M., Han, Q.-L.: 'Robust H_∞ filtering for a class of uncertain linear systems with time-varying delay', Automatica, 2008, 44, (1), pp. 157-166
44 Zhang, X.-M., Wu, M., She, J., He, Y.: 'Delay-dependent stabilization of linear systems with time-varying state and input delays', Automatica, 2005, 41, (8), pp. 1405-1412
45 Gu, K.: 'An integral inequality in the stability problem of time-delay systems'. Proc. 39th IEEE Conf. Deci. Control, 2000, pp. 2805-2810
46 Han, Q.-L.: 'Absolute stability of time-delay systems with sector-bounded nonlinearity', Automatica, 2005, 41, (12), pp. 2171-2176
47 Seuret, A., Gouaisbaut, F.: 'Wirtinger-based integral inequality: Application to time-delay systems', Automatica, 2013, 49, (9), pp. 2860-2866
48 Park, P. G., Ko, J. W.: 'Stability and robust stability for systems with a time-varying delay', Automatica, 2007, 43, (10), pp. 1855-1858
49 Park, P. G., Ko, J. W., Jeong, J. W.: 'Reciprocally convex approach to stability of systems with time-varying delays', Automatica, 2011, 47, (1), pp. 235-238
50 Seuret, A., Gouaisbaut, F.: 'Complete quadratic Lyapunov functionals using Bessel-Legendre inequality'. Proc. 13th Europ. control conf., Strasborug, France, 2014
51 Gyurkovics, É, Takács, T.: 'Multiple integral inequalities and stability analysis of time delay systems', Syst. Control Lett., 2016, 96, pp. 72-80
52 Liu, K., Seuret, A., Xia, Y.: 'Stability analysis of systems with time-varying delays via the second-order Bessel-Legendre inequality', Automatica, 2017, 76, pp. 138-142
53 Park, P. G., Lee, W., Lee, S.: 'Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems', J. Frankl. Inst., 2015, 352, (4), pp. 1378-1396
54 Seuret, A., Gouaisbaut, F.: 'Hierarchy of LMI conditions for the stability of time delay systems', Syst. Control Lett. 2015, 81, pp. 1-7
55 Seuret, A., Gouaisbaut, F., Ariba, Y.: 'Complete quadratic Lyapunov functionals for distributed delay systems', Automatica, 2015, 62, pp. 168-176
56 Hien, L., Trinh, H. 'Refined Jensen-based inequality approach to stability analysis of time-delay systems', IET Control Theory Appl., 2015, 9, (14), pp. 2188-2194
57 Hien, L., Trinh, H. 'An enhanced stability criterion for time-delay systems via a new bounding technique', J. Franklin Inst., 2015, 352, (10), pp. 4407-4422
58 Gyurkovics, É.: 'A note on Wirtinger-type integral inequalities for time-delay systems', Automatica, 2015, 61, pp. 44-46.
59 Gyurkovics, É., Szabó-Varga, G., Kiss, K. 'Stability analysis of linear systems with interval time-varying delays utilizing multiple integral inequalities', Appl. Math. Comput., 2017, 311, pp. 164-177.
60 Lee, S. Y., Lee, W. I., Park, P. G. 'Polynomials-based integral inequality for stability analysis of linear systems with time-varying delays', J. Franklin Inst., 2017, 354, (4), pp. 2053-2067
61 Fridman, E., Shaked, U., Liu, K.: 'New conditions for delay-derivative-dependent stability', Automatica, 2009, 45, (11), pp. 2723-2727
62 Kim, J. H.: Note on stability of linear systems with time-varying delay', Automatica, 2011, 47, (9), pp. 2118-2121
63 Zhang, X.-M., Han, Q.-L.: 'Global asymptotic stability analysis for delayed neural networks using a matrix quadratic convex approach', Neural Netw. 2014, 54, pp. 57-69
64 Kim, J. H.: 'Further improvement of Jensen inequality and application to stability of time-delayed systems', Automatica, 2016, 64, pp. 121-125
65 Zhang, C.-K., He, Y., Jiang, L., Lin, W., Wu, M.: 'Delay-dependent stability analysis of neural networks with time-varying delay: A generalized free-weighting-matrix approach', Appl. Math. Comput., 2017, 294, pp. 102-120
66 Lee, T. H., Park, J. H.: 'A novel Lyapunov functional for stability of time-varying delay systems via matrix-refined-function', Automatica, 2017, 80, pp. 239-242
67 Zeng, H.-B., He, Y., Wu, M., She, J.: 'Free-matrix-based integral inequality for stability analysis of systems with time-varying delay', IEEE Trans. Autom. control, 2015, 60, (10), pp. 2768-2772
68 Seuret, A., Gouaisbaut, F. Liu, K.: 'Discretized Jensen's inequality: an alternative vision of the reciprocally convex combination lemmas', IFAC Workshop on Time-Delay Systems, 2016, 49, (10), pp. 136-140
69 Zhang, C.-K., He, Y., Jiang, L., Wu, M., Zeng, H.-B.: 'Stability analysis of systems with time-varying delay via relaxed integral inequalities', Syst. Control Lett., 2016, 92, pp. 52-61
70 Zhang, X.-M., Han, Q.-L.: 'State estimation for static neural networks with time-varying delays based on an improved reciprocally convex inequality', IEEE Trans. Neural Netw. Learn. Syst., 2018, 29, (4), pp. 1376-1381
71 Seuret, A., Gouaisbaut, F.: 'Delay-dependent reciprocally convex combination lemma', Rapport LAAS n16006, 2016
72 Zhang, X.-M., Han, Q.-L., Seuret, A., Gouaisbaut F.: 'An improved reciprocally convex inequality and an augmented Lyapunov-Krasovskii functional for stability of linear systems with time-varying delay', Automatica, 2017, 84, pp. 221-226
73 Liu, K., Seuret, A.: 'Comparison of bounding methods for stability analysis of systems with time-varying delays', J. Frankl. Inst., 2017, 354, pp. 2979-2993
74 Zhang, X.-M., Han, Q.-L.: 'Event-based H_∞ filtering for sampled-data systems', Automatica, 2015, 51, pp. 55-69
75 Zhang, X.-M., Han, Q.-L., Wang, J.: 'Admissible delay upper bounds for global asymptotic stability of neural networks with time-varying delays', IEEE Trans. Neural Netw. Learn. Syst., DOI: 10.1109/TNNLS.2018.2797279
76 Xu, S., Lam, J., Zhang, B., Zou, Y.: 'New sight into delay-dependent stability of time-delay systems', Int. J. Robust Nonlin. Control, 2015, 25, pp. 961-970
77 He, Y., Wang, Q.-G., Lin, C., Wu, M.: 'Augmented Lyapunov functional and delay-dependent stability criteria for neutral systems', Int. J. Robust Nonlin. Control, 2005, 15, (18), pp. 923-933
78 Lin, C., Wang, Q.-G., Lee, T.: 'A less conservative robust stability test for linear uncertain time-delay systems', IEEE Trans. Autom. Control, 2006, 51, (1), pp. 87-91
79 Kwon, O. M., Park, M. J., Park, J. H., Lee, S. M., Cha, E. J.: 'Improved results on stability of linear systems with time-varying delays via Wirtinger-based integral inequality', J. Frankl. Inst. 2014, 351, pp. 5386-5398
80 Zhang, C.-K., He, Y., Jiang, L., Wu, M.: 'Notes on stability of time-delay systems: Bounding inequalities and augmented Lyapunov-Krasovskii functionals', IEEE Trans. Autom. control, 2017, 62, (10), pp. 5331-5336
81 Park, M. Kwon, O., Park, J. H., Lee, S., Cha E.: 'Stability of time-delay systems via Wirtinger-based double integral inequality', Automatica, 2015, 55, pp. 204-208
82 Sun, J., Liu, G.-P., Chen, J., Rees, D.: 'Improved delay-range-dependent stability criteria for linear systems with time-varying delays', Automatica, 2010, 46, (2), pp. 466-470

83 Sun, J., Han, Q.-L., Chen, J., Liu, G.-P.: 'Less conservative stability criteria for linear systems with interval time-varying delays', Int. J. Robust Nonlin. Control, 2015, 25, (4), pp. 475-485
84 Fridman, E.: 'Tutorial on Lyapunov-based methods for time-delay systems', Europ. J. Control, 2014, 20, (6), pp. 271-283
85 He, Y., Wang, Q.-G., Lin, C., Wu, M.: 'Delay-range-dependent stability for systems with time-varying delay', Automatica, 2007, 43, (2), pp. 371-376
86 He, Y., Wang, Q.-G., Xie, L., Lin, C.: 'Further improvement of free-weighting matrices technique for systems with time-varying delay', IEEE Trans. Autom. Control, 2007, 52, (2), pp. 293-299
87 Jiang, X., Han, Q.-L.: 'On H_∞ control for linear systems with interval time-varying delay', Automatica, 2005, 41, (12), pp. 2099-2106
88 Jiang, X., Han, Q.-L.: 'Delay-dependent robust stability for uncertain linear systems with interval time-varying delay', Automatica, 2006, 42, (6), pp. 1059-1065
89 Zhang, X.-M., Han, Q.-L.: 'Novel delay-derivative-dependent stability criteria using new bounding techniques', Int. J. Robust Nonlin. Control, 2013, 23, (13), pp. 1419-1432
90 Gouaisbaut, F., Peaucelle, D.: 'Delay-dependent stability analysis of linear time delay systems', IFAC Proc. Vol., 2006, 39, (10), pp. 54-59
91 Zeng, H.-B., He, Y., Wu, M., Zhang, C.: 'Complete delay-decomposing approach to asymptotic stability for neural networks with time-varying delays', IEEE Trans. Neural Netw., 2011, 22, (5), pp. 806-812
92 Zhang, X.-M., Han, Q.-L.: 'New Lyapunov-Krasovskii functionals for global asymptotic stability of delayed neural networks', IEEE Trans. Neural Netw., 2009, 20, (3), pp. 533-539
93 Yue, D., Tian, E., Zhang, Y. 'A piecewise analysis method to stability analysis of linear continuous/discrete systems with time-varying delay', Int. J. Robust Nonlin. Control, 2009, 19, (13), pp. 1493-1518
94 Seuret, A., Gouaisbaut, F.: 'Stablity of linear systems with time-varying delays using Bessel-Legendre inequalities', IEEE Trans. Autom. Control, 2018, 63, (1), pp. 225-232
95 Zhang, X.-M., Han, Q.-L., Zeng, Z.: 'Hierarchical type stability criteria for delayed neural networks via canonical Bessel-Legendre inequalities', IEEE Trans. Cybern., 2018, 48, (5), pp. 1660-1671
96 Johnson, L. W., Riess, R. D.: 'Numerical analysis' (Addison-Wesley, New York 1982)
97 Chen, J., Xu, S., Zhang, B., Liu, G.: 'A note on relationship between two classes of integral inequalities', IEEE Trans. Autom. Control, 2017, 62, (8), pp. 4044-4049
98 Zeng, H.-B., He, Y., Wu, M., She, J.: 'New results on stability analysis for systems with discrete distributed delay', Automatica, 2015, 60, pp. 189-192
99 Gyurkovics, E.: 'A note on wirtinger-type integral inequalities for time-delay systems', Automatica, 2015, 61, pp. 44-46
100 Xiao, S.-P., Lian, H., Zeng, H., Chen, G., Zheng, W.: 'Analysis on robust passivity of uncertain neural networks with time-varying delays via free-matrix-based integral inequality', Int. J. Control Autom. Syst., 2017, 15, (5), pp. 2385-2394
101 Zhang, X.-M., Han, Q.-L.: 'Event-triggered H_∞ control for a class of nonlinear networked control systems using novel integral inequalities', Int. J. Robust Nonlin. Control, 2017, 27, (4), pp. 679-700
102 Seuret, A. Gouaisbaut, F.: 'Allowable delay sets for the stability analysis of linear time-varying delay systems using a delay-dependent reciprocally convex lemma', Rapport LAAS n 17075. 2017
103 Trinh, H., Nam, P., Pathirana, P., Le, H. 'On backwards and forwards reachable sets bounding for perturbed time-delay systems', Appl. Math. Computat., 2015, 269, pp. 664-673
104 Ariba, Y., Gouaisbaut, F.: 'Input-output framework for robust stability of time-varying delay systems'. Proc. 48th IEEE Conf. Decision Control, Shanghai, China, 2009
105 Lee, T., Park, J.-H., Xu, S.: 'Relaxed conditions for stability of time-varying delay systems', Automatica, 2017, 75, pp. 11-15
106 Seuret, A., Gouaisbaut, F., Fridman, E.: 'Stability of discrete-time systems with time-varying delays via a novel summation inequality', IEEE Trans. Autom. Control, 2015, 60, (10), pp. 2740-2745
107 Zhang, X.-M. Han, Q.-L.: 'Abel lemma-based finite-sum inequality and its application to stability analysis for linear discrete time-delay systems', Automatica, 2015, 57, pp. 199-202
108 Nam, P., Pathirana, P., Trinh, H.: 'Discrete Wirtinger-based inequality and its application', J. Franklin Inst., 2015, 352, (5), pp. 1893-1905
109 Nam, P., Trinh, H., Pathirana, P.: 'Discrete inequalities based on multiple auxiliary functions and their applications to stability analysis of time-delay systems', J. Franklin Inst., 2015, 352, (12), pp. 5810-5831
110 Hien, L., Trinh, H.: 'New finite-sum inequalities with applications to stability of discrete time-delay systems', Automatica, 2016, 71, pp. 197-201
111 Xiao, S.-P., Xu, L., Zeng, H., Teo, K.: 'Improved stability criteria for discrete-time delay systems via novel summation inequalities', Int. J. Control Autom. Syst., 2018, 16, (4), pp. 1592-1602
112 Zhang, X.-M., Han, Q.-L., Ge, X.: 'A novel finite-sum inequality-based method for robust H_∞ control of uncertain discrete-time Takagi-Sugeno fuzzy systems with interval-like time-varying delays', IEEE Trans. Cybern., 2018, 48, (9), pp. 2569-2582
113 Gyurkovics, É., Kiss, K., Nagy, I., Takács, T.: 'Multiple summation inequalities and their application to stability analysis of discrete-time delay systems', J. Franklin Inst. 2017, 354, (1), pp. 123-144.
114 Liu, X., Wang, F., Tang, M.: 'Auxiliary function-based summation inequalities and their applications to discrete-time systems', Automatica, 2017, 78, pp. 211-215
115 Zhang, C.-K., He, Y., Jiang, L., Wu, M., Zeng, H.: 'Summation inequalities to bounded real lemmas of discrete-time systems with time-varying delay', IEEE Trans. Autom. Control, 2017, 62, (5), 2582-2588
116 Hien, L., Trinh, H.: 'Stability analysis and control of two-dimensional fuzzy systems with directional time-varying delays', IEEE Trans. Fuzzy Syst., 2018, 26, (3), pp. 1550-1564
117 Hien, L., Trinh, H.: 'Exponential stability of two-dimensional homogeneous monotone systems with bounded directional delays', IEEE Trans. Autom. Control, 2018, 63, (8), pp. 2694-2700
118 Gupta, R., Chow, M.: 'Networked control system: Overview and research trends', IEEE Trans. Ind. Electron., 2010, 57, (7), pp. 2527-2535
119 Zhang, X.-M. Han, Q.-L.: 'Network-based H_∞ filtering using a logic jumping-like trigger', Automatica, 2013, 49, (5), pp. 1428-1435
120 Zhang, X.-M. Han, Q.-L.: 'A decentralized event-triggered dissipative control scheme for systems with multiple sensors to sample the system outputs', IEEE Trans. Cybern., 2016, 46, (12), pp. 2745-2757
121 Zhang, X.-M. Han, Q.-L.: 'Event-triggered dynamic output feedback control for networked control systems', IET Control Theory Appl., 2014, 8, (4), pp. 226-234
122 Wang, J., Zhang, X.-M., Han, Q.-L.: 'Event-triggered generalized dissipativity filtering for neural networks with time-varying delays', IEEE Trans. Neural Netw. Learn. Syst., 2016, 27, (1), pp. 77-88
123 Zhang, B.-L., Han, Q.-L., Zhang, X.-M., Yu., X.: 'Sliding mode control with mixed current and delayed states for offshore steel jacket platform', IEEE Trans. Control Syst. Technol., 2014, 22, (5), pp. 1769-1783
124 Zhang, B.-L.,Han, Q.-L., Zhang, X.-M.: 'Event-triggered H_∞ reliable control for offshore structures in network environments', J. Sound Vib., 2016, 368, pp. 1-21
125 Zhang, B.-L.,Han, Q.-L., Zhang, X.-M.: 'Recent advances in vibration control of offshore platforms', Nonlin. Dyn. 2017, 89, (2), pp. 755-771
126 Ge, X., Han, Q.-L., Zhang, X.-M.: 'Achieving cluster formation of multi-agent systems under aperiodic sampling and communication delays', IEEE Trans. Ind. Electron., 2018, 65, (4), pp. 3417-3426
127 Ding, L., Han, Q.-L., Ge, X., Zhang, X.-M.: 'An overview of recent advances in event-triggered consensus of multiagent systems', IEEE Trans. Cybern. 2018, 48, (4), pp. 1110-1123
128 Ge, X., Han, Q.-L., Ding, D., Zhang, X.-M., Ning, B.: 'A survey on recent advances in distributed sampled-data cooperative control of multi-agent systems', Neurocomputing, 2018, 275, pp. 1684-1701
129 Ding, D., Han, Q.-L., Xiang, Y., Ge, X., Zhang, X.-M.: 'A survey on security control and attack detection for industrial cyber-physical systems', Neurocomputing, 2018, 275, pp. 1674-1683
130 Zuo, Z., Han, Q.-L., Ning, B., Ge, X., Zhang, X.-M.: 'An overview of recent advances in fixed-time cooperative control of multi-agent systems', IEEE Trans. Ind. Informat., 2018, 14, (6), pp. 2322-2334