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# Delay-dependent criteria for robust stability and stabilization of fractional-order time-varying delay systems

Xiao-Chuang Jin[a,b], Jun-Guo Lu[a,b,*]

[a] Department of Automation, Shanghai Jiao Tong University, Shanghai 200240, PR China
[b] Key Laboratory of System Control and Information Processing, Ministry of Education of China, Shanghai 200240, PR China

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ABSTRACT

In this paper, the robust stability and stabilization problems for fractional-order time-varying delay systems are investigated. Firstly, a new fractional-order Razumikhin theorem is given. By using the proposed fractional-order Razumikhin theorem, novel delay-dependent stability conditions for both nominal and uncertain linear fractional-order time-varying delay systems are derived. The results are in form of linear matrix inequalities, which are convenient for application and calculation. Then, the obtained stability conditions are utilized to derive a state feedback stabilization controller. To tackle the computational difficulty of the controller design method, a local optimization algorithm is proposed. Finally, three examples are provided to illustrate that the proposed results are valid and less conservative.



## 1. Introduction

Fractional-order calculus has become a hot topic and has been widely used in control theory [17,23], electric motors [31], viscoelastic systems [21] and some other interdisciplinary fields [24,26] in the last few decades. Since fractional-order calculus possesses advantages in depicting phenomenon with memory and heredity [5,30], the study for fractional-order systems has caught extensive attention from academia and many intriguing results have been reported [11–13,20].

Stability and stabilization are two essential and fundamental issues in studying fractional-order systems. Various criteria for stability and stabilization of fractional-order systems have been proposed [19,20,25,29]. In [20], the fractional Lyapunov direct method was introduced to prove the Mittag-Leffler stability of fractional-order systems. In [25], a linear matrix inequality (LMI) stability condition for fractional-order systems with order $\delta \in (0,1)$ was proposed. In [19], a finite-time stability condition was derived using a Gronwall inequality. In [29], a feedback control law that robustly stabilizes singular fractional-order systems with order $\delta \in (0,2)$ was obtained.

Time delay is inevitable in real-world systems. Considering the time-delay effect on stability and stabilization of fractional-order systems is of importance. However, the study on stability and stabilization of fractional-order time-delay systems (FOTDSs) is challenging. Because time delay arising in fractional-order systems incurs exponential type transcendental terms for the characteristic equation and brings obstacles to time-domain analysis. Recently, some methods have been proposed to tackle the stability and stabilization problems of FOTDS. A comparison theorem for FOTDS was proved In [28]. In [14–16], the stability and stabilization conditions were derived using fractional Halanay inequality. A Lyapunov indirect method was used to give the stability criteria In [4]. In [2], the stability condition of FOTDS was obtained by using Lyapunov functions from integer-order counterparts. In [8], a fractional-order Razumikhin theorem was proposed. Though the fractional-order Razumikhin theorem was widely used [9,10,22], the statements and the arguments of the fractional-order Razumikhin for asymptotic stability methods are not rigorous. Besides, the fractional-order Razumikhin theorem proposed by [8] cannot be used to derive delay-dependent stability conditions for FOTDS. For these reasons, the fractional-order Razumikhin theorem still needs investigating.

Notice that few of the above contributions can provide delay-dependent criteria. Delay-dependent criteria are obviously less conservative than the delay-independent ones when delay is small. In [5–7], delay-dependent stability criteria of several different

* Corresponding author at: Department of Automation, Shanghai Jiao Tong University, Shanghai 200240, PR China.
*E-mail address:* jglu@sjtu.edu.cn (J.-G. Lu).

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FOTDS were given. But these results can only be applied to scalar systems. In [10], the authors claimed delay-dependent and order-dependent stability and stabilization conditions. However, their results are in fact delay-independent and very conservative. There is still a lack of delay-dependent stability conditions for FOTDS.

Motivated by the above comments, this paper aims to investigate the delay-dependent stability and stabilization conditions for fractional-order time-varying delay systems (FOTVDSs). Stability criteria for both nominal and uncertain FOTDVS are proposed respectively and a method of constructing stabilizing control law is derived. The main contributions of this paper are highlighted as follows:

1. Fractional-order Razumikhin theorem is first given correctly for asymptotic stability criteria for FOTDS and FOTVDS;
2. Real delay-dependent stability and stabilization conditions for FOTVDS are given;
3. The proposed criteria are less conservative than the latest results of [10];
4. The results are easily verifiable and convenient for application.

*Notations:* Throughout the paper, $X^T$ denotes the transpose of $X$, $Sym\{X\}$ denotes $X^T+X$. The symbol $\star$ represents symmetric component in matrix. Define function $k_\delta(t,\varsigma,\nu)$ as $k_\delta(t,\varsigma,\nu)=(t-\varsigma-\nu)^{\delta-1}-(t-\nu)^{\delta-1}$.

## 2. Preliminaries and problem description

Some related definitions and lemmas need to be given before giving problem description.

**Definition 1** ([23]). The $\delta$ order fractional-order integral for function $g$ is given by

$$D_{t_0}^{-\delta}g(t)=\frac{1}{\Gamma(\delta)}\int_{t_0}^{t}(t-\nu)^{\delta-1}g(\nu)d\nu, \tag{1}$$

where $t\geq t_0$ and $\delta>0$, $\Gamma(\cdot)$ is the gamma function, defined as $\Gamma(\delta)=\int_0^\infty \nu^{\delta-1}e^{-\nu}d\nu$ when $\delta>0$.

**Definition 2** ([23]). The $\delta$ order Riemann-Liouville fractional-order derivative for function $g$ is given by

$$D_{t_0}^{\delta}g(t)=\frac{1}{\Gamma(n-\delta)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-\nu)^{n-\delta-1}g(\nu)d\nu, \tag{2}$$

where $t\geq t_0$, $\delta>0$, $n=\lfloor\delta\rfloor+1$.

**Definition 3** ([23]). The $\delta$ order Caputo fractional-order derivative for function $g$ is given by

$${}^{C}D_{t_0}^{\delta}g(t)=\frac{1}{\Gamma(n-\delta)}\int_{t_0}^{t}(t-\nu)^{n-\delta-1}g^{(n)}(\nu)d\nu, \tag{3}$$

where $t\geq t_0$, $\delta>0$, $n=\lfloor\delta\rfloor+1$.

The Caputo derivative is adopted in the paper. Because it utilizes initial values of classical integer-order derivatives, which has clear physical meanings.

**Lemma 1** ([23]). *$D_{t_0}^{-\delta\,C}D_{t_0}^{\delta}g(t)=g(t)-g(t_0)$, where $g(t)$ is continuous and $\delta\in(0,1)$.*

**Lemma 2** ([23]). *${}^{C}D_{t_0}^{\delta}g(t)=D_{t_0}^{\delta}g(t)-\frac{(t-t_0)^{-\delta}}{\Gamma(1-\delta)}g(t_0)$, where $g(t)$ is continuous and $\delta\in(0,1)$.*

**Lemma 3** ([23]). *$\Gamma(\delta+1)=\delta\Gamma(\delta)$, where $\delta\neq -n, n=0,1,2,\cdots$.*

**Lemma 4** ([23]). *The gamma function can be represented by the limit $\Gamma(\delta)=\lim_{\lambda\to\infty}\frac{\lambda!\lambda^\delta}{\delta(\delta+1)\cdots(\delta+\lambda)}$, where $\delta\neq -n, n=0,1,2,\cdots$.*

**Lemma 5** ([23]). *If $g(t)$ is continuous in $[t_0,t]$ and $\varphi(t)$ has $n+1$ continuous derivatives in $[t_0,t]$, the Leibniz rule for the Riemann-Liouville differentiation has the following form:*

$$D_{t_0}^{\delta}(\varphi(t)g(t))=\sum_{\lambda=0}^{n}\binom{\delta}{\lambda}\frac{d^\lambda\varphi(t)}{dt^\lambda}D_{t_0}^{\delta-\lambda}g(t)-R_n^\delta(t),$$

*where $n\geq\delta+1$,*

$$R_n^\delta(t)=\frac{(-1)^n(t-t_0)^{n+1-\delta}}{n!\Gamma(-\delta)}\int_0^1\int_0^1(1-\mu)^{n-\delta}\nu^n g(t_0+\nu(t-t_0))$$
$$\varphi^{(n+1)}(t_0+(t-t_0)(\mu+\nu-\mu\nu))d\mu d\nu,$$

*and*

$$\binom{\delta}{\lambda}=\frac{\Gamma(\delta+1)}{\lambda!\Gamma(\delta-\lambda+1)}.$$

*Especially, if $g(t)$ is continuous in $[t_0,t]$ and $\varphi(t)$ alone with all its derivatives are in $[t_0,t]$,*

$$D_{t_0}^{\delta}(\varphi(t)g(t))=\sum_{\lambda=0}^{\infty}\binom{\delta}{\lambda}\frac{d^\lambda\varphi(t)}{dt^\lambda}D_{t_0}^{\delta-\lambda}g(t),$$

**Lemma 6** ([1]). *Let $g(t):\mathbb{R}^n\to\mathbb{R}^n$ be a vector of continuous differentiable function. The following inequality holds for any time $t\geq t_0$.*

$${}^{C}D_{t_0}^{\delta}(g^T(t)Qg(t))\leq({}^{C}D_{t_0}^{\delta}g(t))^TQg(t)+g^T(t)Q\,{}^{C}D_{t_0}^{\delta}g(t),$$

*where $\delta\in(0,1)$, and $Q>0$ is a positive definite matrix.*

**Lemma 7** ([3]). *Let $g(t):\mathbb{R}^n\to\mathbb{R}^n$ be a vector of integrable function. Then, the following inequality holds*

$$D_{t-\varsigma(t)}^{-\delta}(g^T(t)Qg(t))\geq\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}(D_{t-\varsigma(t)}^{-\delta}(g(t)))^TQD_{t-\varsigma(t)}^{-\delta}(g(t)),$$

*where $\varsigma_M$ is a constant, $\varsigma(t)\in[0,\varsigma_M]$, $\delta\in(0,1)$, and $Q>0$.*

**Lemma 8.** *Let $\chi(t):\mathbb{R}^n\to\mathbb{R}^n$ be a vector of integrable function. Then, the following inequality holds*

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)}k_\delta(t,\varsigma(t),\nu)(\chi^T(\nu)Q\chi(\nu))d\nu\\&\quad\geq\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}\left(\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)}k_\delta(t,\varsigma(t),\nu)\chi(\nu)d\nu\right)^T\\&\quad Q\left(\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)}k_\delta(t,\varsigma(t),\nu)\chi(\nu)d\nu\right),\end{aligned}$$

*where $\varsigma_M$ is a constant, $\varsigma(t)\in[0,\varsigma_M]$, $t>\varsigma_M$, $\delta\in(0,1)$, and $Q>0$.*

**Proof.** Since $Q>0$, for all $\mu\geq 0$ and $\nu\geq 0$,

$$(\chi(\mu)-\chi(\nu))^TQ(\chi(\mu)-\chi(\nu))\geq 0. \tag{4}$$

Hence,

$$\chi^T(\mu)Q\chi(\mu)+\chi^T(\nu)Q\chi(\nu)\geq\chi^T(\mu)Q\chi(\nu)+\chi^T(\nu)Q\chi(\mu). \tag{5}$$

Now, multiply both sides by $k_\delta(t,\varsigma(t),\mu)/\Gamma(\delta)$ and integrate $\mu$ over $(t_0,t-\varsigma(t))$, one has

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)}k_\delta(t,\varsigma(t),\mu)\chi^T(\mu)Q\chi(\mu)d\mu\\&\quad+\frac{\varsigma(t)^\delta+(t-t_0-\varsigma(t))^\delta-(t-t_0)^\delta}{\Gamma(\delta+1)}\chi^T(\nu)Q\chi(\nu)\\&\quad\geq\left(\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)}k_\delta(t,\varsigma(t),\mu)\chi(\mu)d\mu\right)^TQ\chi(\nu)\\&\quad+\chi^T(\nu)Q\left(\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)}k_\delta(t,\varsigma(t),\mu)\chi(\mu)d\mu\right).\end{aligned} \tag{6}$$

Then, multiply both sides by $k_\delta(t,\varsigma,\nu)/\Gamma(\delta)$ and integrate $\nu$ over $(t_0, t-\varsigma(t))$, one has

$$\begin{aligned}
&\frac{\varsigma(t)^\delta+(t-t_0-\varsigma(t))^\delta-(t-t_0)^\delta}{\Gamma(\delta+1)}\frac{1}{\Gamma(\delta)}\\
&\int_{t_0}^{t-\varsigma(t)} k_\delta(t,\varsigma(t),\mu)\chi^T(\mu)Q\chi(\mu)d\mu\\
&+\frac{\varsigma(t)^\delta+(t-t_0-\varsigma(t))^\delta-(t-t_0)^\delta}{\Gamma(\delta+1)}\frac{1}{\Gamma(\delta)}\\
&\int_{t_0}^{t-\varsigma(t)} k_\delta(t,\varsigma(t),\nu)\chi^T(\nu)Q\chi(\nu)d\nu\\
&\geq\left(\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)} k_\delta(t,\varsigma(t),\mu)\chi(\mu)d\mu\right)^T\\
&Q\left(\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)} k_\delta(t,\varsigma(t),\mu)\chi(\nu)d\nu\right)\\
&+\left(\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)} k_\delta(t,\varsigma(t),\nu)\chi(\mu)d\mu\right)^T\\
&Q\left(\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)} k_\delta(t,\varsigma(t),\nu)\chi(\nu)d\nu\right). \qquad (7)
\end{aligned}$$

For any $t>\varsigma_M$, $\varsigma(t)^\delta+(t-t_0-\varsigma(t))^\delta-(t-t_0)^\delta\leq\varsigma_M^\delta$. Therefore,

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)} k_\delta(t,\varsigma(t),\nu)\chi^T(\nu)Q\chi(\nu)d\nu\\
&\geq\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}\left(\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)} k_\delta(t,\varsigma(t),\nu)\chi(\nu)d\nu\right)^T\\
&\times Q\left(\frac{1}{\Gamma(\delta)}\int_{t_0}^{t-\varsigma(t)} k_\delta(t,\varsigma(t),\nu)\chi(\nu)d\nu\right). \qquad (8)
\end{aligned}$$

This completes the proof. □

**Lemma 9** ([8]). *Suppose $\varrho$ and $\omega$ are $\mathcal{K}$ class functions. Consider the Caputo fractional-order system ${}^CD_{t_0}^\delta\chi(t)=g(t,\chi(t))$, where $g(t,\chi(t))$ is locally Lipschitz with respect to $\chi$ and $\delta\in(0,1)$. The Caputo fractional-order system is uniformly stable if there exists a continuous and derivable function $V$ such that*

$$\varrho(\|\chi(t)\|)\leq V(t,\chi(t))\leq\omega(\|\chi(t)\|), \qquad \forall t\geq t_0,$$

*and*

$$\begin{cases}{}^CD_{t_0}^\delta V(t,\chi(t))\leq 0,\\ \text{whenever} \sup\limits_{t_0-\varsigma\leq\nu\leq t} V(\nu,\chi(\nu))=V(t,\chi(t)).\end{cases}$$

**Lemma 10** ([18]). *For matrices $P$ and $Q$ with appropriate dimensions,*

$$P^TQ+Q^TP\leq\xi P^TP+\xi^{-1}Q^TQ, \quad \text{for any } \xi>0.$$

**Lemma 11.** *Consider the following linear integral equation*

$$\begin{aligned}
y(t)&=F(t)+\frac{1}{\Gamma(\delta)}\int_{t_0}^t(t-\nu)^{\delta-1}h_1(\nu)y(\nu)d\nu\\
&+\frac{1}{\Gamma(\delta)}\int_{t_0}^t(t-\nu)^{\delta-1}h_2(\nu)y(\nu-\varsigma(\nu))d\nu, \qquad (9)
\end{aligned}$$

*where $\varsigma(t)\in[0,\varsigma_M]$ and $y(t)$ is integrable and bounded when $t\leq t_0$. For some $T>t_0$, assume $F\in C(t_0,T]\cap L^1(t_0,T)$ and $h_1,h_2$ are bounded functions. Then, the integral Eq. (9) has a unique solution $y\in C(t_0,T]\cap L^1(t_0,T)$.*

**Proof.** Please see Appendix A. □

**Lemma 12.** *Suppose $f,g$ are Lipschitz continuous functions and $\varsigma(t)$ is a continuously differentiable positive function satisfying $\varsigma(t)\in[0,\varsigma_M]$. The solution of the system*

$$\begin{cases}{}^CD_{t_0}^\delta x(t)= & f(x(t))+g(x(t-\varsigma(t))),\\ x(t)= & \phi(t), \quad t\in[t_0-\varsigma_M,t_0],\end{cases} \qquad (10)$$

*has a continuous derivative in $(t_0,T]$ and $\dot{x}(t)\in L^1(t_0,T)$ for some $T>t_0$.*

**Proof.** Please see Appendix B. □

Consider the following nominal FOTVDS,

$$\begin{cases}{}^CD_{t_0}^\delta\chi(t) &=A\chi(t)+A_d\chi(t-\varsigma(t))+Bu(t),\\ \chi(t) &=\phi(t), \quad t\in[t_0-\varsigma_M,t_0],\end{cases} \qquad (11)$$

where $\delta\in(0,1)$, $\chi(t)\in\mathbb{R}^n$ is the state vector; $u(t)\in\mathbb{R}^m$ is the control signal; $A$, $A_d\in\mathbb{R}^{n\times n}$; $B\in\mathbb{R}^{n\times m}$; $\varsigma(t)$ denotes the time-varying delay satisfying $\varsigma(t)\in[0,\varsigma_M]$, $\forall t\geq t_0$. $\phi(t)\in\mathcal{C}[t_0-\varsigma_M,t_0]$ is the initial condition. When there exist time-varying norm-bounded uncertainties, the uncertain FOTVDS is modeled as

$$\begin{cases}{}^CD_{t_0}^\delta\chi(t) &=\bar{A}(t)\chi(t)+\bar{A_d}(t)\chi(t-\varsigma(t))+Bu(t),\\ \chi(t) &=\phi(t), \quad t\in[t_0-\varsigma_M,t_0],\end{cases} \qquad (12)$$

where $\bar{A}(t)=A+\Delta A(t)$ and $\bar{A_d}(t)=A_d+\Delta A_d(t)$. $A$ and $A_d$ are constant matrices. $\Delta A(t)$ and $\Delta A_d(t)$ are the uncertainties satisfying $\Delta A(t)=DF(t)E$, $\Delta A_d(t)=D_dF_d(t)E_d$. $D$, $D_d$, $E$ and $E_d$ are known constant matrices. $F(t)$ and $F_d(t)$ are time-varying matrices satisfying $F^T(t)F(t)\leq I$ and $F_d^T(t)F_d(t)\leq I$, $\forall t\geq t_0$. According to Lemma 11 and Lemma 12, the existence, uniqueness and differentiability of the states of (11) is guaranteed.

Our objective is to investigate the asymptotic stability conditions of the nominal FOTVDS (11) and the stability and stabilization conditions of the uncertain FOTVDS (12).

## 3. Main results

The fractional-order Razumikhin theorem, proposed by Chen and Chen [8], has been widely used to derive stability conditions for FOTDS. However, the statements and arguments of the fractional-order Razumikhin theorem for asymptotic stability in [8] is not rigorous. To prove the fractional-order Razumikhin theorem, the authors proposed an inequality

$$\begin{aligned}
{}^CD_{t_0}^\delta W(t)&=D_{t_0}^\delta W(t)-\frac{r^\delta V(t_0)}{\Gamma(1-\delta)}(t-t_0)\\
&\leq(t-t_0+r)^\delta D_{t_0}^\delta V(t)+\delta^2(t-t_0+r)^{\delta-1}D_{t_0}^{\delta-1}V(t)\\
&-\frac{r^\delta V(t_0)}{\Gamma(1-\delta)}(t-t_0), \ \forall t\geq t_0, \qquad (13)
\end{aligned}$$

in (3.11) of [8]. $r>0$ is a positive real number, $W(t)=(t-t_0+r)^\delta V(t)$ and $V(t)\geq 0$, $\forall t\geq t_0$. But according to Lemma 5,

$$\begin{aligned}
D_{t_0}^\delta W(t)=&(t-t_0+r)^\delta D_{t_0}^\delta V(t)+\delta^2(t-t_0+r)^{\delta-1}D_{t_0}^{\delta-1}V(t)\\
&+\sum_{\lambda=2}^n\binom{\delta}{\lambda}\left((t-t_0+r)^\delta\right)^{(\lambda)}D_{t_0}^{\delta-\lambda}V(t)-R_n^\delta(t),
\end{aligned}$$

where $n\geq\lfloor\delta\rfloor+1$. For any $\lambda\geq 2$

$$\begin{aligned}
&\binom{\delta}{\lambda}\left((t-t_0+r)^\delta\right)^{(\lambda)}D_{t_0}^{\delta-\lambda}V(t)\\
&=\frac{\Gamma(\delta+1)(\delta(\delta-1)\cdots(\delta-\lambda+1))}{\lambda!\Gamma(\delta-\lambda+1)(t-t_0+r)^{\lambda-\delta}}D_{t_0}^{\delta-\lambda}V(t)\\
&=\frac{(\delta(\delta-1)\cdots(\delta-\lambda+1))^2}{\lambda!(t-t_0+r)^{\lambda-\delta}}D_{t_0}^{\delta-\lambda}V(t)\geq 0.
\end{aligned}$$

Hence, $\sum_{\lambda=2}^{n}\binom{\delta}{\lambda}\Big((t-t_0+r)^{\delta}\Big)^{(\lambda)}D_{t_0}^{\delta-\lambda}V(t)\ge 0$. For any $n\ge 2$,

$$\begin{aligned}R_n^{\delta}(t)=&\frac{(-1)^n(t-t_0)^{n+1-\delta}\Big(\delta(\delta-1)\cdots(\delta-n)\Big)}{n!\Gamma(-\delta)}\\&\times\int_0^1\int_0^1(\mu+\nu-\mu\nu)^{n+1}(1-\mu)^{n-\delta}\nu^n\\&(t_0+(t-t_0)(\mu+\nu-\mu\nu))^{\delta-n-1}V(t_0+\nu(t-t_0))d\mu d\nu.\end{aligned}$$

Since $\Gamma(-\delta)<0$, $R_n^{\delta}(t)\le 0$. It follows that

$$D_{t_0}^{\delta}W(t)\ge(t-t_0+r)^{\delta}D_{t_0}^{\delta}V(t)+\delta^2(t-t_0+r)^{\delta-1}D_{t_0}^{\delta-1}V(t).$$

Therefore, the inequality (13) does not always hold. The proof of [8] is not rigorous. Besides, the fractional-order Razumikhin theorem of [8] cannot be used to derive delay-dependent stability conditions. For the above reasons, a new fractional-order Razumikhin theorem must be given.

**Theorem 1.** *Suppose $\varrho$ and $\omega$ are $\mathcal{K}$ class functions. Consider the Caputo fractional-order system ${}^{C}D_{t_0}^{\delta}\chi(t)=g(t,\chi(t))$, where $g(t,\chi(t))$ is locally Lipschitz with respect to $\chi$ and $\delta\in(0,1)$. The Caputo fractional-order system is uniformly asymptotically stable if there exist a positive constant $\epsilon$ and a continuously differentiable function $V$ such that*

$$\varrho(\|\chi(t)\|)\le V(t,\chi(t))\le\omega(\|\chi(t)\|),\qquad\forall t\ge t_0,$$

*and*

$$\begin{cases}{}^{C}D_{t_0}^{\delta}V(t,\chi(t))\le-\epsilon V(t,\chi(t)),\\ \textit{whenever}\ \sup\limits_{t_0-\varsigma\le\nu\le t}V(\nu,\chi(\nu))=V(t,\chi(t)).\end{cases}$$

**Proof.** Let

$$\begin{aligned}W(t)&=(t-t_0+r)^{\delta}V(t,\chi(t)),\\ \bar{W}(t)&=\sup_{t_0-\varsigma\le\nu\le t}W(\nu,\chi(\nu)),\\ \bar{V}(t)&=\sup_{t_0\le\nu\le t}V(\nu,\chi(\nu)),\end{aligned}$$

where $r>0$ is a positive real number. According to Lemma 2 and Lemma 5, one has that whenever $\sup_{t_0-\varsigma\le\nu\le t}V(\nu,\chi(\nu))=V(t,\chi(t))$,

$$\begin{aligned}{}^{C}D_{t_0}^{\delta}W(t)&=D_{t_0}^{\delta}W(t)-\frac{r^{\delta}(t-t_0)^{-\delta}}{\Gamma(1-\delta)}V(t_0)\\&=(t-t_0+r)^{\delta}D_{t_0}^{\delta}V(t)+\sum_{\lambda=1}^{\infty}\binom{\delta}{\lambda}\\&\quad\Big((t-t_0+r)^{\delta}\Big)^{(\lambda)}D_{t_0}^{\delta-\lambda}V(t)-\frac{r^{\delta}(t-t_0)^{-\delta}}{\Gamma(1-\delta)}V(t_0)\\&=(t-t_0+r)^{\delta C}D_{t_0}^{\delta}V(t)+\sum_{\lambda=1}^{\infty}\binom{\delta}{\lambda}\\&\quad\Big((t-t_0+r)^{\delta}\Big)^{(\lambda)}D_{t_0}^{\delta-\lambda}V(t)\\&\quad+\frac{(t-t_0+r)^{\delta}}{\Gamma(1-\delta)(t-t_0)^{\delta}}V(t_0)-\frac{r^{\delta}}{\Gamma(1-\delta)(t-t_0)^{\delta}}V(t_0)\\&\le-\epsilon W(t)+\frac{V(t_0)}{\Gamma(1-\delta)}+\sum_{\lambda=1}^{\infty}\binom{\delta}{\lambda}\\&\quad\Big((t-t_0+r)^{\delta}\Big)^{(\lambda)}D_{t_0}^{\delta-\lambda}V(t).\end{aligned}$$

By using Lemma 3, one has that whenever $\sup_{t_0-\varsigma\le\nu\le t}V(\nu,\chi(\nu))=V(t,\chi(t))$,

$$\begin{aligned}&\sum_{\lambda=1}^{\infty}\binom{\delta}{\lambda}\Big((t-t_0+r)^{\delta}\Big)^{(\lambda)}D_{t_0}^{\delta-\lambda}V(t)\\&=\sum_{\lambda=1}^{\infty}\frac{\Gamma(\delta+1)\Big(\delta(\delta-1)\cdots(\delta-\lambda+1)\Big)}{\lambda!\Gamma(\delta-\lambda+1)}\\&(t-t_0+r)^{\delta-\lambda}\left(\frac{1}{\Gamma(\lambda-\delta)}\int_{t_0}^{t}(t-\nu)^{\lambda-\delta-1}V(\nu)d\nu\right)\\&\le\sum_{\lambda=1}^{\infty}\frac{\Gamma(\delta+1)\Big(\delta(\delta-1)\cdots(\delta-\lambda+1)\Big)}{\lambda!\Gamma(\delta-\lambda+1)\Gamma(\lambda-\delta)}\\&(t-t_0+r)^{\delta-\lambda}\bar{V}(t)\int_{t_0}^{t}(t-\nu)^{\lambda-\delta-1}d\nu\\&=\sum_{\lambda=1}^{\infty}\frac{\Gamma(\delta+1)\Big(\delta(\delta-1)\cdots(\delta-\lambda+1)\Big)}{\lambda!\Gamma(\delta-\lambda+1)\Gamma(\lambda-\delta+1)}\\&\frac{(t-t_0)^{\lambda-\delta}}{(t-t_0+r)^{\lambda-\delta}}\bar{V}(t)\\&\le\sum_{\lambda=1}^{\infty}\frac{\Big(\delta(\delta-1)\cdots(\delta-\lambda+1)\Big)^2}{\lambda!\Gamma(\lambda-\delta+1)}\bar{V}(t)\\&=\sum_{\lambda=1}^{\infty}\frac{(-\delta)(1-\delta)\cdots(\lambda-1-\delta)}{\lambda!\Gamma(-\delta)(\lambda-\delta)}\bar{V}(t).\end{aligned}$$

Based on Lemma 4, $\lim_{\lambda\to\infty}\frac{(-\delta)(1-\delta)\cdots(\lambda-1-\delta)}{\lambda!\Gamma(-\delta)(\lambda-\delta)}=\lim_{\lambda\to\infty}\frac{1}{\Gamma(-\delta)^2\lambda^{\delta}(\lambda-\delta)^2}$. According to the positive series comparison method, $\sum_{\lambda=1}^{\infty}\frac{(-\delta)(1-\delta)\cdots(\lambda-1-\delta)}{\lambda!\Gamma(-\delta)(\lambda-\delta)}$ is convergent. Define $m_{\delta}=\sum_{\lambda=1}^{\infty}\frac{(-\delta)(1-\delta)\cdots(\lambda-1-\delta)}{\lambda!\Gamma(-\delta)(\lambda-\delta)}$. Then, one has

$$\begin{aligned}{}^{C}D_{t_0}^{\delta}W(t)\le&-\epsilon W(t)+\frac{W(t_0)}{r^{\delta}\Gamma(1-\delta)}+m_{\delta}\bar{V}(t)\\\le&-\epsilon W(t)+\frac{\bar{W}(t)}{r^{\delta}\Gamma(1-\delta)}+\frac{m_{\delta}\bar{W}(t)}{r^{\delta}}.\end{aligned}$$

Whenever $\bar{W}(t)=W(t)$, one has

$${}^{C}D_{t_0}^{\delta}W(t)\le\left(-\epsilon+\frac{1+m_{\delta}\Gamma(1-\delta)}{r^{\delta}\Gamma(1-\delta)}\right)W(t).$$

For any $\epsilon>0$, there always exists some $r>0$ such that $(-\epsilon+\frac{1+m_{\delta}\Gamma(1-\delta)}{r^{\delta}\Gamma(1-\delta)})<0$. Thus, for appropriate $r$, $W(t)$ satisfies that

$$\begin{cases}{}^{C}D_{t_0}^{\delta}W(t)\le 0,\\ \textit{whenever}\ \sup\limits_{t_0-\varsigma\le\nu\le t}W(\nu)=W(t).\end{cases}$$

According to Lemma 9, $W(t)$ is bounded. Let $Ub$ be a positive real number such that $W(t)\le Ub,\forall t\ge t_0$. It follows that

$$\rho(\|\chi(t)\|)\le V(t,\chi(t))\le\frac{Ub}{(t-t_0+r)^{\delta}},$$

which implies the Caputo fractional-order system is asymptotically stable. □

Theorem 1 is very useful. It can be used to derived stability conditions for both linear and nonlinear FOTDS. Moreover, with proper Lyapunov functional, delay-dependent stability conditions can be obtained by Theorem 1.

Now, based on the new fractional-order Razumikhin theorem, delay-dependent stability condition for the nominal system (11) is derived.

**Theorem 2.** *The nominal system (11) with $u(t) \equiv 0$ is asymptotically stable if there exist a symmetric positive definite matrix P, a free matrix N and constants $\gamma_i > 0$ $(i = 0, 1, \ldots, 4)$, satisfying the following matrix inequality:*

$$\begin{bmatrix} \Omega_{11} & \star & \star & \star & \star & \star \\ (A_d - N)^T P & -\gamma_0 P & \star & \star & \star & \star \\ A^T N^T P & 0 & -\gamma_1 \frac{\Gamma(\delta+1)}{\varsigma_M^\delta} P & \star & \star & \star \\ A^T N^T P & 0 & 0 & -\gamma_2 \frac{\Gamma(\delta+1)}{\varsigma_M^\delta} P & \star & \star \\ A_d^T N^T P & 0 & 0 & 0 & -\gamma_3 \frac{\Gamma(\delta+1)}{\varsigma_M^\delta} P & \star \\ A_d^T N^T P & 0 & 0 & 0 & 0 & -\gamma_4 \frac{\Gamma(\delta+1)}{\varsigma_M^\delta} P \end{bmatrix} < 0, \tag{14}$$

*where* $\Omega_{11} = \mathrm{Sym}\{(A+N)^T P\} + (\gamma_0 + \sum_{i=1}^4 \gamma_i \frac{\varsigma_M^\delta}{\Gamma(\delta+1)})P$.

**Proof.** By introducing a free matrix $N$, (11) is written as

$$\begin{aligned} {}^C D_{t_0}^\delta \chi(t) = &(A+N)\chi(t) + (A_d - N)\chi(t - \varsigma(t)) \\ &-N(\chi(t) - \chi(t - \varsigma(t))). \end{aligned} \tag{15}$$

According to Lemma 1, one has

$$\begin{aligned} \chi(t) - \chi(t - \varsigma(t)) = &D_{t_0}^{-\delta C} D_{t_0}^\delta \chi(t) - \frac{1}{\Gamma(\delta)} \int_{t_0}^{t-\varsigma(t)} \\ &(t - \varsigma(t) - \nu)^{\delta-1} ({}^C D_{t_0}^\delta g(\nu)) d\nu \\ = &\frac{1}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t (t-\nu)^{\delta-1C} D_{t_0}^\delta \chi(\nu) d\nu \\ &- \frac{1}{\Gamma(\delta)} \int_{t_0}^{t-\varsigma(t)} k_\delta(t, \varsigma(t), \nu) {}^C D_{t_0}^\delta \chi(\nu) d\nu. \end{aligned} \tag{16}$$

Combining (11), (15) and (16), (11) is rewritten as

$$\begin{aligned} {}^C D_{t_0}^\delta \chi(t) = &(A+N)\chi(t) + (A_d - N)\chi(t - \varsigma(t)) \\ &-NA \frac{1}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t (t-\nu)^{\delta-1} \chi(\nu) d\nu \\ &+NA \frac{1}{\Gamma(\delta)} \int_{t_0}^{t-\varsigma(t)} k_\delta(t, \varsigma(t), \nu) \chi(\nu) d\nu \\ &-MA_d \frac{1}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t (t-\nu)^{\delta-1} \chi(\nu - \varsigma(\nu)) d\nu \\ &+NA_d \frac{1}{\Gamma(\delta)} \int_{t_0}^{t-\varsigma(t)} k_\delta(t, \varsigma(t), \nu) \chi(\nu - \varsigma(\nu)) d\nu, \end{aligned} \tag{17}$$

with initial condition $\chi(t) = \psi(t)$, $t \in [t_0 - \varsigma_M, t_0 + 2\varsigma(t)]$. $\psi(t)$ equals to $\phi(t)$ when $t \in [t_0 - \varsigma_M, t_0]$. When $t \in [t_0, t_0 + 2\varsigma(t)]$, $\psi(t)$ equals to the solution of (11) with initial condition $\chi(t) = \phi(t)$, $t \in [t_0 - \varsigma_M, t_0]$. Clearly, the stability of the system represented by (17) implies the stability of (11). To obtain the delay-dependent stability of (17), the Lyapunov functional is constructed as $V(t, \chi(t)) = \chi^T(t) P \chi(t)$. According to Lemma 6, one has

$$\begin{aligned} {}^C D_{t_0}^\delta V(t, \chi(t)) \le &2\chi^T(t) P (A+N) \chi(t) \\ &+2\chi^T(t) P (A_d - N) \chi(t - \varsigma(t)) \\ &-2\chi^T(t) PNA \left( \frac{1}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t (t-\nu)^{\delta-1} \chi(\nu) d\nu \right) \\ &+2\chi^T(t) PNA \left( \frac{1}{\Gamma(\delta)} \int_{t_0}^{t-\varsigma(t)} k_\delta(t, \varsigma(t), \nu) \chi(\nu) d\nu \right) \\ &-2\chi^T(t) PNA_d \left( \frac{1}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t (t-\nu)^{\delta-1} \right. \\ &\left. \chi(\nu - \varsigma(\nu)) d\nu \right) \\ &+2\chi^T(t) PNA_d \left( \frac{1}{\Gamma(\delta)} \int_{t_0}^{t-\varsigma(t)} k_\delta(t, \varsigma(t), \nu) \right. \\ &\left. \chi(\nu - \varsigma(\nu)) d\nu \right) = \Lambda. \end{aligned}$$

Whenever $\sup_{t_0 - \varsigma \le \nu \le t} V(\nu, \chi(\nu)) = V(t, \chi(t))$

$$\begin{aligned} &{}^C D_{t_0}^\delta V(t, \chi(t)) \\ &\le \Lambda + \gamma_0 \big( \chi^T(t) P \chi(t) - \chi^T(t - \varsigma(t)) P \chi(t - \varsigma(t)) \big) \\ &+ \frac{\gamma_1}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t (t-\nu)^{\delta-1} \big( \chi^T(t) P \chi(t) - \chi^T(\nu) P \chi(\nu) \big) d\nu \\ &+ \frac{\gamma_2}{\Gamma(\delta)} \int_{t_0}^{t-\varsigma(t)} k_\delta(t, \varsigma(t), \nu) \big( \chi^T(t) P \chi(t) - \chi^T(\nu) P \chi(\nu) \big) d\nu \\ &+ \frac{\gamma_3}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t (t-\nu)^{\delta-1} \big( \chi^T(t) P \chi(t) \\ &\quad - \chi^T(\nu - \varsigma(\nu)) P \chi(\nu - \varsigma(\nu)) \big) d\nu \\ &+ \frac{\gamma_4}{\Gamma(\delta)} \int_{t_0}^{t-\varsigma(t)} k_\delta(t, \varsigma(t), \nu) \big( \chi^T(t) P \chi(t) \\ &\quad - \chi^T(\nu - \varsigma(\nu)) P \chi(\nu - \varsigma(\nu)) \big) d\nu. \end{aligned}$$

By applying Lemma 7 and Lemma 8, one has

$$\begin{aligned} {}^C D_{t_0}^\delta V(t, \chi(t)) \le &\Lambda + \chi^T(t) \left( \left( \gamma_0 + \sum_{i=1}^4 \gamma_i \frac{\varsigma_M^\delta}{\Gamma(\delta+1)} \right) P \right) \chi(t) \\ &-\gamma_0 \chi^T(t - \varsigma(t)) P \chi(t - \varsigma(t)) \\ &-\gamma_1 \frac{\Gamma(\delta+1)}{\varsigma_M^\delta} \left( \frac{1}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t (t-\nu)^{\delta-1} \chi(\nu) d\nu \right)^T P \\ &\left( \frac{1}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t (t-\nu)^{\delta-1} \chi(\nu) d\nu \right) \\ &-\gamma_2 \frac{\Gamma(\delta+1)}{\varsigma_M^\delta} \\ &\left( \frac{1}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t k_\delta(t, \varsigma(t), \nu) \chi(\nu) d\nu \right)^T P \\ &\left( \frac{1}{\Gamma(\delta)} \int_{t-\varsigma(t)}^t k_\delta(t, \varsigma(t), \nu) \chi(\nu) d\nu \right) \\ &-\gamma_3 \frac{\Gamma(\delta+1)}{\varsigma_M^\delta} \end{aligned}$$

$$\left(\frac{1}{\Gamma(\delta)}\int_{t-\varsigma(t)}^{t}(t-\nu)^{\delta-1}\chi(\nu-\varsigma(\nu))d\nu\right)^{T}P$$
$$\left(\frac{1}{\Gamma(\delta)}\int_{t-\varsigma(t)}^{t}(t-\nu)^{\delta-1}\chi(\nu-\varsigma(\nu))d\nu\right)$$
$$-\gamma_4\frac{\Gamma(\delta+1)}{\varsigma_M^{\delta}}$$
$$\left(\frac{1}{\Gamma(\delta)}\int_{t-\varsigma(t)}^{t}k_\delta(t,\varsigma(t),\nu)\chi(\nu-\varsigma(\nu))d\nu\right)^{T}P$$
$$\left(\frac{1}{\Gamma(\delta)}\int_{t-\varsigma(t)}^{t}k_\delta(t,\varsigma(t),\nu)\chi(\nu-\varsigma(\nu))d\nu\right). \quad (18)$$

Define

$$\zeta(t)=\begin{bmatrix} x(t)\\ x(t-\varsigma(t))\\ \frac{1}{\Gamma(\delta)}\int_{t-\varsigma(t)}^{t}(t-\nu)^{\delta-1}\chi(\nu)d\nu\\ \frac{1}{\Gamma(\delta)}\int_{t-\varsigma(t)}^{t}k_\delta(t,\varsigma(t),\nu)\chi(\nu)d\nu\\ \frac{1}{\Gamma(\delta)}\int_{t-\varsigma(t)}^{t}(t-\nu)^{\delta-1}\chi(\nu-\varsigma(\nu))d\nu\\ \frac{1}{\Gamma(\delta)}\int_{t-\varsigma(t)}^{t}k_\delta(t,\varsigma(t),\nu)\chi(\nu-\varsigma(\nu))d\nu \end{bmatrix}.$$

Then, (18) is equivalent to

$${}^{C}D_{t_0}^{\delta}V(t,\chi(t))\le\zeta^{T}(t)\tilde{\Omega}\zeta(t),$$

where

$$\Omega=\begin{bmatrix} \Omega_{11} & \star & \star & \star & \star & \star\\ (A_d-N)^TP & -\gamma_0P & \star & \star & \star & \star\\ -A^TN^TP & 0 & -\gamma_1\frac{\Gamma(\delta+1)}{\varsigma_M^{\delta}}P & \star & \star & \star\\ A^TN^TP & 0 & 0 & -\gamma_2\frac{\Gamma(\delta+1)}{\varsigma_M^{\delta}}P & \star & \star\\ -A_d^TN^TP & 0 & 0 & 0 & -\gamma_3\frac{\Gamma(\delta+1)}{\varsigma_M^{\delta}}P & \star\\ A_d^TN^TP & 0 & 0 & 0 & 0 & -\gamma_4\frac{\Gamma(\delta+1)}{\varsigma_M^{\delta}}P \end{bmatrix}.$$

Since (14) holds, $\Omega<0$. Let $\epsilon$ be the minimum singular value of $\Omega$. Then, ${}^{C}D_{t_0}^{\delta}V(t,\chi(t))\le-\epsilon V(t,\chi(t))$. Based on Theorem 1, the nominal system (11) with $u(t)\equiv0$ is asymptotically stable. □

**Remark 1.** Notice that when $\gamma_i$ is pre-selected, condition (14) can be transformed to an LMI simply by replacing $PN$ with a free matrix $L$. Hence, the proposed stability criterion is computationally efficient.

**Remark 2.** Letting $N$ be 0, the delay-dependent stability condition (14) becomes delay-independent one:

$$\begin{bmatrix} \mathrm{Sym}\{A^TP\}+\gamma_0P & \star\\ A_d^TP & -\gamma_0P \end{bmatrix}<0. \quad (19)$$

Obviously, the delay-independent condition is a special case of the delay-dependent stability condition (14). The delay-dependent stability condition utilizes additional information of the time delay, and is less conservative than delay-independent ones.

**Remark 3.** In [10], the authors claimed a delay-dependent and order-dependent stability criterion of FOTVDS (11). However, the stability condition is not necessarily delay-dependent or order-dependent. The stability condition of [10] is the nominal system (11) with $u(t)\equiv0$ is asymptotically stable if

$$\begin{bmatrix} \mathrm{Sym}\{A^TP+YA\}+X+P & \star & \star\\ A_d^TP+\tau_M^{\delta}\delta^{-1}A_d^TY^T & -P & \star\\ \tau_M^{\delta}\delta^{-1}ZA & \tau_M^{\delta}\delta^{-1}ZA_d & -\tau_M^{\delta}\delta^{-1}Z \end{bmatrix}<0, \quad (20)$$

where $P$ is symmetric positive definite matrix and $X$, $Y$, $Z$ are appropriately dimensioned matrices satisfying

$$\begin{bmatrix} X & \star\\ Y^T & Z \end{bmatrix}\ge0.$$

By the Schur complement principal, (20) is equivalent to

$$\begin{bmatrix} \mathrm{Sym}\{A^TP\}+P & \star\\ A_d^TP & -P \end{bmatrix}+\tau_M^{\delta}\delta^{-1}\begin{bmatrix} X+A^TY^T+YA+A^TZA & YA_d+A^TZA_d\\ A_d^TY^T+A_d^TZA & A_d^TZA_d \end{bmatrix}<0.$$

That is

$$\begin{bmatrix} \mathrm{Sym}\{A^TP\}+P & \star\\ A_d^TP & -P \end{bmatrix}+\tau_M^{\delta}\delta^{-1}\begin{bmatrix} I & 0\\ A & B \end{bmatrix}^T\begin{bmatrix} X & \star\\ Y^T & Z \end{bmatrix}\begin{bmatrix} I & 0\\ A & B \end{bmatrix}<0.$$

Apparently, the $\tau_M^{\delta}\delta^{-1}$ is not helpful. It can be eliminated by replacing $X$, $Y$, $Z$ with $\tilde{X}=\tau_M^{\delta}\delta^{-1}X$, $\tilde{Y}=\tau_M^{\delta}\delta^{-1}Y$, $\tilde{Z}=\tau_M^{\delta}\delta^{-1}Z$ respectively. Hence, the stability condition of [10] is equivalent to

$$\begin{bmatrix} \mathrm{Sym}\{A^TP\}+P & \star\\ A_d^TP & -P \end{bmatrix}+\begin{bmatrix} I & 0\\ A & B \end{bmatrix}^T\begin{bmatrix} \tilde{X} & \star\\ \tilde{Y}^T & \tilde{Z} \end{bmatrix}\begin{bmatrix} I & 0\\ A & B \end{bmatrix}<0, \quad (21)$$

and

$$\begin{bmatrix} \tilde{X} & \star\\ \tilde{Y}^T & \tilde{Z} \end{bmatrix}\ge0. \quad (22)$$

As can be seen, the stability condition is not necessarily delay-dependent or order-dependent. In fact, the LMIs (21) and (22) imply that (19) holds, which means the stability criterion of [10] is even more conservative than the delay-independent stability condition. As mentioned above, the delay-independent condition is only a special case of Theorem 2. Therefore, Theorem 2 is less conservative than the stability condition of [10].

Now, by employing Theorem 2, the robust stability condition for the uncertain FOTVDS (12) can be obtained.

**Theorem 3.** *The uncertain system (12) with $u(t)\equiv0$ is asymptotically stable if there exist a positive definite matrix $P$, a free matrix $N$, constants $\gamma_i>0$ $(i=0,1,\dots,4)$ and $\xi_i>0$ $(i=1,2,\dots,6)$, satisfying the following matrix inequality:*

$$\begin{bmatrix} \Phi_{11}+\Psi & \star\\ \Phi_{21} & \Phi_{22} \end{bmatrix}<0, \quad (23)$$

*where*

$$\Phi_{11}=\begin{bmatrix}\Omega_{11} & \star & \star & \star & \star & \star\\ (A_d-N)^TP & -\gamma_0 P & \star & \star & \star & \star\\ A^TN^TP & 0 & -\gamma_1\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}P & \star & \star & \star\\ A^TN^TP & 0 & 0 & -\gamma_2\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}P & \star & \star\\ A_d^TN^TP & 0 & 0 & 0 & -\gamma_3\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}P & \star\\ A_d^TN^TP & 0 & 0 & 0 & 0 & -\gamma_4\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}P\end{bmatrix},$$

$$\Phi_{21}=\begin{bmatrix}D^TP & 0 & 0 & 0 & 0 & 0\\ D_d^TP & 0 & 0 & 0 & 0 & 0\\ D^TN^TP & 0 & 0 & 0 & 0 & 0\\ D^TN^TP & 0 & 0 & 0 & 0 & 0\\ D_d^TN^TP & 0 & 0 & 0 & 0 & 0\\ D_d^TN^TP & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\Phi_{22}=\mathrm{diag}\{-\xi_1 I,-\xi_2 I,-\xi_3 I,-\xi_4 I,-\xi_5 I,-\xi_6 I\},$$

$$\Psi=\mathrm{diag}\left\{\xi_1E^TE,\xi_2E_d^TE_d,\xi_3E^TE,\xi_4E^TE,\xi_5E_d^TE_d,\xi_6E_d^TE_d\right\},$$

*and* $\Omega_{11}=\mathrm{Sym}\{(A+N)^TP\}+(\gamma_0+\sum_{i=1}^4\gamma_i\frac{\varsigma_M^\delta}{\Gamma(\delta+1)})P.$

**Proof.** According to Theorem 2, the uncertain system (12) is asymptotically stable if

$$\Phi_{11}+\begin{bmatrix}\mathrm{Sym}\{E^TF^T(t)D^TP\} & \star & \star & \star & \star & \star\\ E_d^TF_d^T(t)D_d^TP & 0 & \star & \star & \star & \star\\ E^TF^T(t)D^TN^TP & 0 & 0 & \star & \star & \star\\ E^TF^T(t)D^TN^TP & 0 & 0 & 0 & \star & \star\\ E_d^TF_d^T(t)D_d^TN^TP & 0 & 0 & 0 & 0 & \star\\ E_d^TF_d^T(t)D_d^TN^TP & 0 & 0 & 0 & 0 & 0\end{bmatrix}<0,\quad \forall t\ge t_0.$$

By using Lemma 10, the uncertain system (12) is asymptotically stable if

$$\Phi_{11}+\begin{bmatrix}\Pi+\xi_1E^TE & \star & \star & \star & \star & \star\\ 0 & \xi_2E_d^TE_d & \star & \star & \star & \star\\ 0 & 0 & \xi_3E^TE & \star & \star & \star\\ 0 & 0 & 0 & \xi_4E^TE & \star & \star\\ 0 & 0 & 0 & 0 & \xi_5E_d^TE_d & \star\\ 0 & 0 & 0 & 0 & 0 & \xi_6E_d^TE_d\end{bmatrix}<0, \tag{24}$$

where $\Pi=P(\xi_1^{-1}DD^T+\xi_2^{-1}D_dD_d^T+\xi_3^{-1}NDD^TN^T+\xi_4^{-1}NDD^TN^T+\xi_5^{-1}ND_dD_d^TN^T+\xi_6^{-1}ND_dD_d^TN^T)P$. By the Schur Complement Principal, (24) and (23) are equivalent. The proof is completed. □

**Remark 4.** Like (14) in Theorem 2, (23) with fixed $\gamma_i>0$ $(i=0,1,\ldots,4)$ can also be transformed to an LMI by replacing $PN$ with a free matrix $L$. This shows that the proposed delay-dependent robust stability condition is also computationally efficient. Besides, Theorem 3 is less conservative than the robust stability condition in [10].

Based on the obtained robust stability conditions, a state feedback stabilization controller for the uncertain FOTVDS (12) is developed. The state feedback control law is designed as

$$u(t)=Kx(t), \tag{25}$$

where $K$ is the control gain. (12) is rewritten as

$$\begin{cases}{}^CD_{t_0}^\delta\chi(t) & =(\bar{A}+BK)\chi(t)+\bar{A}_d\chi(t-\tau),\\ \chi(t) & =\phi(t),\qquad t\in[t_0-\tau_M,t_0].\end{cases} \tag{26}$$

Then, the state feedback controller can be designed as follows.

**Theorem 4.** *The uncertain system (12) is robustly stabilizable if there exist a positive definite matrix $Q$, free matrices $N$, $V$, constants $\gamma_i>0$ $(i=0,1,\ldots,4)$ and $\xi_i>0$ $(i=1,2,\ldots,6)$, satisfying the following matrix inequality:*

$$\begin{bmatrix}\tilde{\Phi}_{11}+\tilde{\Psi} & \star\\ \tilde{\Phi}_{21} & \tilde{\Phi}_{22}\end{bmatrix}<0, \tag{27}$$

*where*

$$\tilde{\Phi}_{11}=\begin{bmatrix}\tilde{\Omega}_{11} & \star & \star & \star & \star & \star\\ Q(A_d-N)^T & -\gamma_0 Q & \star & \star & \star & \star\\ (AQ+BV)^TN^T & 0 & -\gamma_1\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}Q & \star & \star & \star\\ (AQ+BV)^TN^T & 0 & 0 & -\gamma_2\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}Q & \star & \star\\ QA_d^TN^T & 0 & 0 & 0 & -\gamma_3\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}Q & \star\\ QA_d^TN^T & 0 & 0 & 0 & 0 & -\gamma_4\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}Q\end{bmatrix},$$

$$\tilde{\Phi}_{21}=\begin{bmatrix}D^T & 0 & 0 & 0 & 0 & 0\\ D_d^T & 0 & 0 & 0 & 0 & 0\\ D^TN^T & 0 & 0 & 0 & 0 & 0\\ D^TN^T & 0 & 0 & 0 & 0 & 0\\ D_d^TN^T & 0 & 0 & 0 & 0 & 0\\ D_d^TN^T & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\tilde{\Phi}_{22}=\mathrm{diag}\{-\xi_1 I,-\xi_2 I,-\xi_3 I,-\xi_4 I,-\xi_5 I,-\xi_6 I\},$$

$$\tilde{\Psi}=\mathrm{diag}\big\{\xi_1QE^TEQ,\xi_2QE_d^TE_dQ,\xi_3QE^TEQ,\xi_4QE^TEQ,\xi_5QE_d^TE_dQ,\xi_6QE_d^TE_dQ\big\},$$

*and* $\mathrm{Sym}\{(A+N)Q+BV\}+(\gamma_0+\sum_{i=1}^4\gamma_i\frac{\varsigma_M^\delta}{\Gamma(\delta+1)})Q$. *Furthermore, the control gain is obtained as* $K=VQ^{-1}$.

**Proof.** By replacing $A$ with $A+BK$ in (24) and by multiplying $\mathrm{diag}\{P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}\}$ on both sides, one has that (26) is asymptotically stable if

$$\tilde{\Phi}_{11} + \begin{bmatrix} Sym\{QE^TF^T(t)D^T\} & \star & \star & \star & \star & \star \\ QE_d^TF_d^T(t)D_d^T & 0 & \star & \star & \star & \star \\ QE^TF^T(t)D^TM^T & 0 & 0 & \star & \star & \star \\ QE^TF^T(t)D^TM^T & 0 & 0 & 0 & \star & \star \\ QE_d^TF_d^T(t)D_d^TM^T & 0 & 0 & 0 & 0 & \star \\ QE_d^TF_d^T(t)D_d^TM^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \quad \forall t \geq t_0, \tag{28}$$

where $Q = P^{-1}$ and $V = KQ$. According to Lemma 10 and the Schur Complement Principal, (27) implies (28). This completes the proof. □

Unlike the condition (23), the condition (27) cannot be transformed to an LMI. The stabilization problem is non-convex and solving it is NP-hard [27]. Global optimization approaches are computationally inefficient and difficult to implement. To cope with the computational problem, a local optimization is proposed. Notice that (27) is also equivalent to

$$\begin{bmatrix} \hat{\Phi}_{11} & \star \\ \hat{\Phi}_{21} & \hat{\Phi}_{22} \end{bmatrix} < 0, \tag{29}$$

where

$$\hat{\Phi}_{11} = \begin{bmatrix} \hat{\Omega}_{11} & \star & \star & \star & \star & \star \\ Q(A_d-N)^T & -\gamma_0 Q & \star & \star & \star & \star \\ (AQ+BV)^TN^T & 0 & -\gamma_1\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}Q & \star & \star & \star \\ (AQ+BV)^TN^T & 0 & 0 & -\gamma_2\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}Q & \star & \star \\ QA_d^TN^T & 0 & 0 & 0 & -\gamma_3\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}Q & \star \\ QA_d^TN^T & 0 & 0 & 0 & 0 & -\gamma_4\frac{\Gamma(\delta+1)}{\varsigma_M^\delta}Q \end{bmatrix},$$

$$\hat{\Phi}_{21} = \mathrm{diag}\{EQ, E_dQ, EQ, EQ, E_dQ, E_dQ\},$$

$$\hat{\Phi}_{22} = \mathrm{diag}\{-\hat{\xi}_1 I, -\hat{\xi}_2 I, -\hat{\xi}_3 I, -\hat{\xi}_4 I, -\hat{\xi}_5 I, -\hat{\xi}_6 I\},$$

and $\hat{\Omega} = Sym\{(A+N)Q+BV\} + (\gamma_0 + \sum_{i=1}^4 \gamma_i \frac{\varsigma_M^\delta}{\Gamma(\delta+1)})Q + \hat{\xi}_1 DD^T + \hat{\xi}_2 D_dD_d^T + (\hat{\xi}_3+\hat{\xi}_4)NDD^TN^T + (\hat{\xi}_5+\hat{\xi}_6)ND_dD_d^TN^T$. $\hat{\xi}_i$ $(i = 1, 2, \ldots, 6)$ are positive. Obviously, (29) with fixed $\gamma_i > 0$ $(i = 0, 1, \ldots, 4)$ and $N$ is an LMI, and (27) with fixed $Q$ and $V$ is an LMI. The proposed local optimization algorithm split the stabilization problem into two quasi-convex optimization problems, which are computationally efficient. The algorithm procedure is shown in Algorithm 1. It can provide a suboptimal delay margin.

**Algorithm 1** Local Optimization Algorithm.

**Input:** $A$, $A_d$, $B$, $D$, $D_d$, $E$, $E_d$ and $\delta$
**Output:** $Q$ and $V$.
1: Choose an initial $\gamma_i$ $(i = 0, 1, \cdots, 4)$ and $M$. For instance, $\gamma_i = 1$ $(i = 0, 1, \cdots, 4)$ and $M = A_d$
2: **while** $\varsigma_M$ does not converge to a desired precision within a specified number of iterations **do**
3: For given $\gamma_i > 0$ $(i = 0, 1, \ldots, 4)$ and $M$, maximize $\varsigma_M$ over $Q > 0$ and $V$, subject to the condition (29)
4: For $Q$ and $V$ found from previous step, maximize $\varsigma_M$ over $\gamma_i > 0$ $(i = 0, 1, \ldots, 4)$ and $M$, subject to the condition (27);
5: **end while**
6: **return** $Q$ and $V$

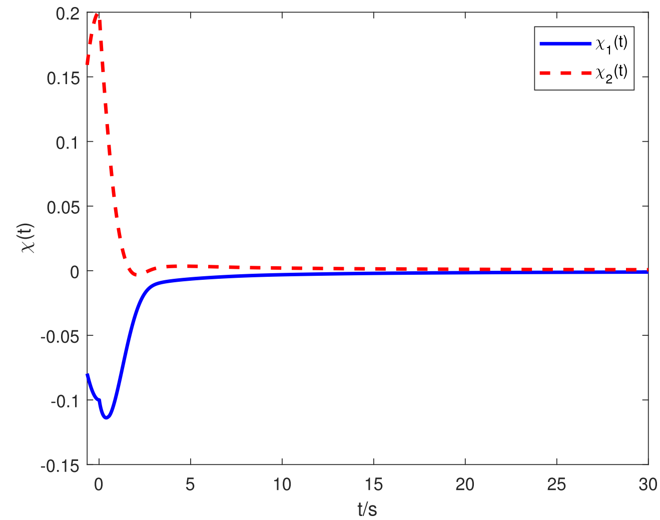


**Fig. 1.** The states of (30) with $\varsigma(t) = 0.3229(1+\sin(t))$.

## 4. Numerical examples

In the section, three examples are provided to illustrate that the proposed stability and stabilization criteria are effective and less conservative than the latest result in [10].

### 4.1. Example 1

Consider the following nominal FOTVDS

$$\begin{cases} {}^CD_{t_0}^\delta \chi(t) = & \begin{bmatrix} -1 & -0.6 \\ 0.5 & -0.2 \end{bmatrix}\chi(t) \\ & +\begin{bmatrix} -0.3 & -0.4 \\ 0 & -0.5 \end{bmatrix}\chi(t-\varsigma(t)), \ t > t_0, \\ \chi(t) = & \cos(t), \ t \in [t_0 - \varsigma_M, t_0], \end{cases} \tag{30}$$

with $\delta = 0.9$. $\varsigma(t)$ is time-varying delay. There exist

$$P = \begin{bmatrix} 1.9965 & -0.7609 \\ -0.7609 & 2.0281 \end{bmatrix}, M = \begin{bmatrix} -0.2700 & -0.4071 \\ 0.0222 & -0.4837 \end{bmatrix},$$

$$\gamma = \begin{bmatrix} 0.0335 & 0.4419 & 0.4419 & 0.4287 & 0.4287 \end{bmatrix}^T,$$

where $\gamma$ is defined as $[\gamma_0, \gamma_1, \cdots, \gamma_4]^T$, such that (14) holds for any $\varsigma(t)$ bounded by 0.6458. Therefore, based on Theorem 2, (30) is asymptotically stable for any $\varsigma(t)$ bounded by 0.6458. Figure 1 depicts the state trajectories of (30) with $\varsigma(t) = 0.3229(1+\sin(t))$. As shown in Fig. 1, the states of (30) converges to 0, which demonstrates the effectiveness of Theorem 2. On the other hand, [10, Theorem 3.1] cannot be applied to the example (30) with any $\varsigma(t)$. Hence, [10, Theorem 3.1] cannot assert that (30) is asymptotically stable, even though it is asymptotically stable as shown in Fig. 1. This shows that our results are less conservative than [10, Theorem 3.1].

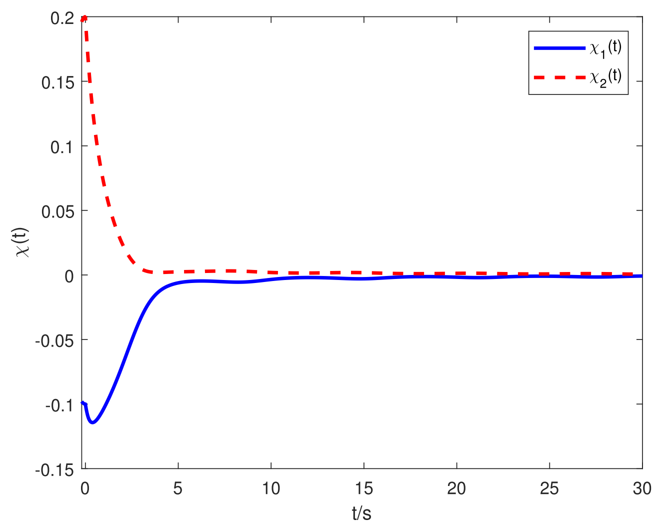


**Fig. 2.** The states of (31) with $\varsigma(t) = 0.1011(1 + \sin(10t))$ and $\Delta A(t) = \Delta A_d(t) = \mathrm{diag}\{0.2\sin(t), 0.2\sin(t)\}$.

### 4.2. Example 2

Now, consider the FOTVDS (30) with norm-bounded uncertainties, namely

$$\begin{cases} {}^C D_{t_0}^{\delta}\chi(t) = (A + \Delta A(t))\chi(t) + (A_d + \Delta A_d(t))\chi(t - \varsigma(t)), & t > t_0, \\ \chi(t) = \cos(t), & t \in [t_0 - \varsigma_M, t_0], \end{cases} \tag{31}$$

where $\delta$, $A$ and $A_d$ are as in (30). $\Delta A(t)$ and $\Delta A_d(t)$ satisfy that

$$\|\Delta A(t)\| \le 0.2, \qquad \|\Delta A_d(t)\| \le 0.2, \qquad \forall t \ge t_0.$$

Let

$$D = D_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad E = E_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, there exist

$$P = \begin{bmatrix} 1.0506 & 0.5259 \\ 0.5259 & 1.2898 \end{bmatrix}, M = \begin{bmatrix} -0.2976 & -0.3853 \\ -0.0318 & -0.5441 \end{bmatrix},$$

$$\gamma = \begin{bmatrix} 1.3057 & 0.6399 & 0.6399 & 0.6410 & 0.6410 \end{bmatrix}^T,$$

$$\xi = \begin{bmatrix} 0.2638 & 0.1920 & 0.1503 & 0.1503 & 0.1682 & 0.1682 \end{bmatrix},$$

where $\gamma$ is defined as $[\gamma_0, \gamma_1, \cdots, \gamma_4]^T$ and $\xi$ is defined as $[\xi_1, \xi_2, \cdots, \xi_6]$, such that (23) holds for any $\varsigma(t)$ bounded by 0.2022. Based on Theorem 3, (31) is guaranteed to be asymptotically stable for $\varsigma(t)$ bounded by 0.2022. Figure 2 depicts the state trajectories of (31) with $\varsigma(t) = 0.1011(1 + \sin(10t))$ and $\Delta A(t) = \Delta A_d(t) = \mathrm{diag}\{0.2\sin(t), 0.2\sin(t)\}$. As can be seen, (31) is asymptotically stable, which illustrates that Theorem 3 is valid.

### 4.3. Example 3

Consider the following uncertain FOTVDS

$$\begin{cases} {}^C D_{t_0}^{\delta}\chi(t) = (A + \Delta A(t))\chi(t) \\ \quad + (A_d + \Delta A_d(t))\chi(t - \varsigma(t)) + Bu(t), \quad t > t_0, \\ \chi(t) = \cos(t), \quad t \in [t_0 - \varsigma_M, t_0], \end{cases} \tag{32}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_d = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

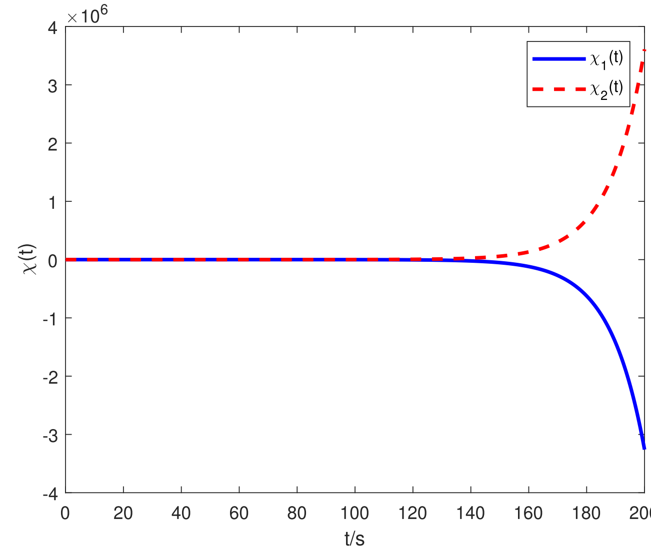


**Fig. 3.** The states of (32) with $\varsigma(t) \equiv 0$, $\Delta A \equiv 0$ and $\Delta A_d \equiv 0$.

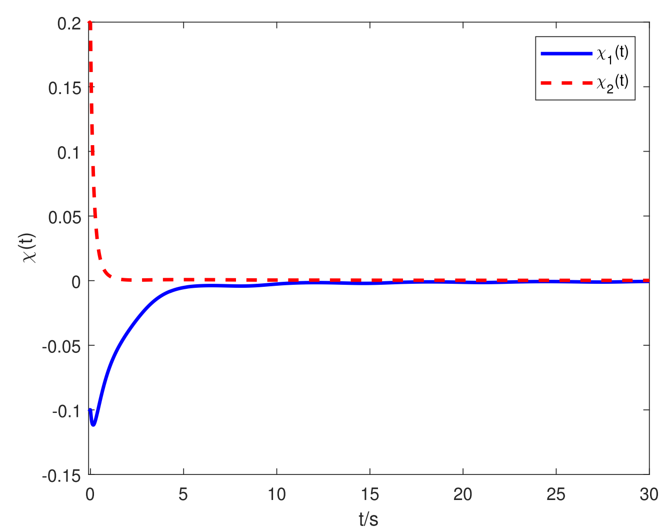


**Fig. 4.** The states of (32) with $\varsigma(t) = 0.0976|\sin(10t)|$ and $\Delta A(t) = \Delta A_d(t) = \mathrm{diag}\{0.2\sin(t), 0.2\sin(t)\}$ under the control law $u(t) = [0.2755, -2.4914]\chi(t)$.

$$D = D_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, E = E_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Figure 3 depicts the state trajectories of (32) without delay, uncertainties and control input. It is observed that (32) is not asymptotically stable. Applying Theorem 4, it is found that for $\varsigma(t)$ bounded by 0.0976, (32) can be robustly stabilized via a state feedback controller, with control gain $K = [0.2755, -2.4914]$. Note that the controller design method of [10] is not applicable to this case. Figure 4 shows that (32) is asymptotically stable with the help of the designed state feedback controller. Therefore, the proposed stabilization controller is effective.

## 5. Conclusions

Delay-dependent stability and stabilization conditions for FOTVDS with norm-bounded uncertainties have been studied in this paper. A fractional-order Razumikhin theorem has been given. Novel delay-dependent criteria for stability of both nominal and uncertain linear fractional-order systems with time-varying delay has been established using the proposed fractional-order Razumikhin theorem. Then, a state feedback stabilization method has been derived based on the obtained criteria. The proposed stability and stabilization conditions are easily verifiable and computationally efficient. Finally, numerical examples have been given to demonstrate that the established results are valid and less conservative.

### Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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### Appendix A. Proof of Lemma 11

**Proof.** Define $h_0$ as $h_0 = \max\{\|h_1(t)\|_{\infty,\infty}, \|h_1(t)\|_{\infty,\infty}\}$ on $t \in [t_0, T]$, where $\|\cdot\|_{\infty,\infty}$ denotes the $L_\infty$-induced norm. Given $a = (\frac{\Gamma(\delta+1)}{3h_0})^{1/\delta}$, one has that

$$\frac{h_0}{\Gamma(\delta+1)}\left(\int_{t_0}^{t_0+a}(t-\nu)^{\delta-1}d\nu + \int_{t_0}^{t_0+a}(t-\nu)^{\delta-1}d\nu\right) = \frac{2}{3} < 1.$$

Based on the principle of contraction mappings on $L^1(t_0, t_0+a)$, there exists a unique $y(t) \in L^1(t_0, t_0+a)$. Replacing $t$ by $t+a$ yields that

$$\begin{aligned} y(t+a) = F(t+a) &+ \frac{1}{\Gamma(\delta)}\int_{t_0}^{t_0+a}(t+a-\nu)^{\delta-1} \\ &(h_1(\nu)y(\nu) + h_2(\nu)y(\nu-\varsigma(\nu)))d\nu \\ &+ \frac{1}{\Gamma(\delta)}\int_{t_0}^{t}(t-\nu)^{\delta-1}(h_1(\nu+a)y(\nu+a) \\ &+ h_2(\nu+a)y(\nu+a-\varsigma(\nu+a)))d\nu. \end{aligned}$$

Since $y(t) \in L^1(t_0, t_0+a)$ and $y(t)$ is integrable and bounded, $F(t+a) + \frac{1}{\Gamma(\delta)}\int_{t_0}^{t_0+a}(t+a-\nu)^{\delta-1}\Big(h_1(\nu)y(\nu) + h_2(\nu)y(\nu-\varsigma(\nu))\Big)d\nu$ belongs to $L^1(t_0, t_0+a)$. By the principle of contraction mappings, one has that there exists a unique $y(t) \in L^1(t_0, t_0+2a)$. Continue by induction, $y(t) \in L^1(t_0, T)$ is proved. Now, let $\xi$ be any positive number satisfying $\xi \in (0, T-t_0)$. Then, one has that

$$\begin{aligned} y(t+\xi) = F(t+\xi) &+ \frac{1}{\Gamma(\delta)}\int_{t_0}^{t_0+\xi}(t+\xi-\nu)^{\delta-1} \\ &(h_1(\nu)y(\nu) + h_2(\nu)y(\nu-\varsigma(\nu)))d\nu \\ &+ \frac{1}{\Gamma(\delta)}\int_{t_0}^{t}(t-\nu)^{\delta-1}(h_1(\nu+\xi)y(\nu+\xi) \\ &+ h_2(\nu+\xi)y(\nu+\xi-\varsigma(\nu+\xi)))d\nu. \end{aligned}$$

Obviously, $F(t+\xi)$ is continuous on $[t_0, T-\xi]$ and $\frac{1}{\Gamma(\delta)}\int_{t_0}^{t_0+\xi}(t+\xi-\nu)^{\delta-1}\Big(h_1(\nu)y(\nu) + h_2(\nu)y(\nu-\varsigma(\nu))\Big)d\nu$ is continuous on $(t_0, T-\xi]$. Besides, for any sequence $t_n$ tending monotonically to $t_0$, $(t_n+\xi-\nu)^{\delta-1} \to (t_0+\xi-\nu)^{\delta-1}$ monotonically. By the dominated convergence theorem, $\frac{1}{\Gamma(\delta)}\int_{t_0}^{t_0+\xi}(t+\xi-\nu)^{\delta-1}\Big(h_1(\nu)y(\nu) + h_2(\nu)y(\nu-\varsigma(\nu))\Big)d\nu$ is continuous on $[t_0, T-\xi]$. Hence, $y(t+\xi)$ is continuous on $[t_0, T-\xi]$. Since $\xi$ can be arbitrarily small, $y(t)$ is continuous on $(t_0, T]$. This completes the proof. □

### Appendix B. Proof of Lemma 12

**Proof.** Define $y(t, \eta)$ as a solution of

$$\begin{aligned} y(t,\eta) = &\frac{(t-t_0)^{\delta-1}}{\Gamma(\delta)}(f(x(t_0)) + g(x(t_0-\varsigma(t_0)))) \\ &+ \frac{1}{\Gamma(\delta)}\int_{t_0}^{t}(t-\nu)^{\delta-1}[f_1(x(\nu))y(\nu,\eta) \\ &+ g_1(x(\nu-\varsigma(\nu)))y(\nu-\varsigma(\nu),\eta)]d\nu, \end{aligned}$$

where $f_1(x(t)) = \frac{df(x(t))}{dx(t)}$ and $g_1(x(t-\varsigma(t))) = \frac{dg(x(t-\varsigma(t)))}{dx(t-\varsigma(t))}$. Suppose $t - \varsigma(t) < t_0$ and $\eta$ is small enough, $y(t-\varsigma(t), \eta)$ equals to $\frac{x(t+\eta-\varsigma(t+\eta)) - x(t-\varsigma(t))}{\eta-\varsigma(t+\eta)+\varsigma(t)}$. According to Lemma 11, $y(t, \eta)$ exists and $y(t, \eta) \in C(t_0, T] \cap L^1(t_0, T)$, where $T > t_0$ is a constant. Let $z(t, \eta) = \frac{x(t+\eta)-x(t)}{\eta}$ and $|\cdot|$ represent the vector norm. Then one has that

$$\begin{aligned} &\Gamma(\delta)|z(t,\eta) - y(t,\eta)| \\ &\leqslant \left|\frac{1}{\eta}\int_{t_0}^{t_0+\eta}(t-\nu)^{\delta-1}[f(x(\nu)) + g(x(\nu-\varsigma(\nu)))]d\nu \right. \\ &\quad \left. -(t-t_0)^{\delta-1}(f(x(t_0)) + g(x(t_0-\varsigma(t_0))))\right| \\ &\quad +\left|\int_{t_0}^{t}(t-\nu)^{\delta-1}\left[\frac{f(x(\nu+\eta)) - f(x(\nu))}{x(\nu+\eta) - x(\nu)}z(\nu,\eta)\right.\right. \\ &\quad \left. +\frac{g(x(\nu+\eta-\varsigma(\nu+\eta))) - g(x(\nu-\varsigma(\nu)))}{x(\nu+\eta-\varsigma(\nu)) - x(\nu-\varsigma(\nu))}z(\nu-\varsigma(\nu),\eta)\right]d\nu \\ &\quad -\int_{t_0}^{t}(t-\nu)^{\delta-1}[f_1(x(\nu))y(\nu,\eta) + g_1(x(\nu-\varsigma(\nu))) \\ &\quad \left.(1-\dot{\varsigma}(\nu))y(\nu-\varsigma(\nu),\eta)]d\nu\right| \\ &\leqslant \left|\frac{1}{\eta}\int_{t_0}^{t_0+\eta}(t-\nu)^{\delta-1}[f(x(\nu)) + g(x(\nu-\varsigma(\nu)))]d\nu \right. \\ &\quad \left. -(t-t_0)^{\delta-1}(f(x(t_0)) + g(x(t_0-\varsigma(t_0))))\right| \\ &\quad +\left|\int_{t_0}^{t}(t-\nu)^{\delta-1}\left[\frac{f(x(\nu+\eta)) - f(x(\nu))}{x(\nu+\eta) - x(\nu)} - f_1(x(\nu))\right]y(\nu,\eta)d\nu\right| \\ &\quad +\left|\int_{t_0}^{t}(t-\nu)^{\delta-1}\left[\frac{g(x(\nu+\eta-\varsigma(\nu+\eta))) - g(x(\nu-\varsigma(\nu)))}{x(\nu+\eta-\varsigma(\nu)) - x(\nu-\varsigma(\nu))}\right.\right. \\ &\quad -g_1(x(\nu-\varsigma(\nu)))(1-\dot{\varsigma}(\nu))]y(\nu-\varsigma(\nu),\eta)d\nu| \\ &\quad +\left|\int_{t_0}^{t}(t-\nu)^{\delta-1}\frac{f(x(\nu+\eta)) - f(x(\nu))}{x(\nu+\eta) - x(\nu)}|z(\nu,\eta) - y(\nu,\eta)|d\nu\right| \\ &\quad +\left|\int_{t_0}^{t}(t-\nu)^{\delta-1}\frac{g(x(\nu+\eta-\varsigma(\nu+\eta))) - g(x(\nu-\varsigma(\nu)))}{x(\nu+\eta-\varsigma(\nu)) - x(\nu-\varsigma(\nu))}\right. \\ &\quad |z(\nu-\varsigma(\nu),\eta) - y(\nu-\varsigma(\nu),\eta)|d\nu|. \end{aligned}$$

Define

$$\begin{aligned} &\zeta(t,\eta) = |z(t,\eta) - y(t,\eta)|, \\ &A(t,\eta) = \frac{1}{\Gamma(\delta)}\left(\left|\frac{1}{\eta}\int_{t_0}^{t_0+\eta}(t-\nu)^{\delta-1}[f(x(\nu)) + g(x(\nu-\varsigma(\nu)))]\right.\right. \\ &\quad \left. d\nu - (t-t_0)^{\delta-1}(f(x(t_0)) + g(x(t_0-\varsigma(t_0))))\right| \\ &\quad +\left|\int_{t_0}^{t}(t-\nu)^{\delta-1}\left[\frac{f(x(\nu+\eta)) - f(x(\nu))}{x(\nu+\eta) - x(\nu)}\right.\right. \\ &\quad -f_1(x(\nu))]y(\nu,\eta)d\nu| \\ &\quad +\left|\int_{t_0}^{t}(t-\nu)^{\delta-1}\left[\frac{g(x(\nu+\eta-\varsigma(\nu+\eta))) - g(x(\nu-\varsigma(\nu)))}{x(\nu+\eta-\varsigma(\nu))-x(\nu-\varsigma(\nu))}\right.\right. \\ &\quad \left.\left. -g_1(x(\nu-\varsigma(\nu)))(1-\dot{\varsigma}(\nu))]y(\nu-\varsigma(\nu),\eta)d\nu\right|\right), \end{aligned}$$

$$F(\nu,\eta) = \left|\frac{f(x(\nu+\eta)) - f(x(\nu))}{x(\nu+\eta) - x(\nu)}\right|,$$
$$G(\nu,\eta) = \left|\frac{g(x(\nu+\eta-\varsigma(\nu+\eta))) - g(x(\nu-\varsigma(\nu)))}{x(\nu+\eta-\varsigma(\nu)) - x(\nu-\varsigma(\nu))}\right|.$$

Then, one has that

$$\zeta(t,\eta) \le A(t,\eta) + \frac{1}{\Gamma(\delta)}\int_{t_0}^{t}(t-\nu)^{\delta-1} [F(\nu,\eta)\zeta(\nu,\eta) + G(\nu,\eta)\zeta(\nu-\varsigma(\nu),\eta)]d\nu, \quad \forall\eta > 0,\ t \in [t_0+\eta, T],$$

where $T > t_0 + \eta$ is a constant. Letting $\eta \to 0$ yields that

$$\tilde{\zeta}(t) \le \tilde{A}(t) + \frac{1}{\Gamma(\delta)}\int_{t_0}^{t}(t-\nu)^{\delta-1}[\tilde{F}(\nu)\tilde{\zeta}(\nu) + \tilde{G}(\nu)\tilde{\zeta}(\nu-\varsigma(\nu))]d\nu, \quad t \in (t_0, T],$$

where $\tilde{\zeta}(t) = \lim_{\eta\to 0}\zeta(t,\eta)$, $\tilde{A}(t) = \lim_{\eta\to 0}A(t,\eta)$, $\tilde{F}(\nu) = \lim_{\eta\to 0}F(\nu,\eta)$ and $\tilde{G}(\nu) = \lim_{\eta\to 0}G(\nu,\eta)$. Obviously, $\tilde{A}(t)$, $\tilde{F}(\nu)$ and $\tilde{G}(\nu)$ are non-negative continuous functions. Besides, $\tilde{\zeta}(t)$ equals to 0 when $t \le t_0$ and $\lim_{t\to t_0}\tilde{\zeta}(t) = 0$. Thus, $\tilde{\zeta}(t)$ is non-negative continuous in $[t_0, T]$. Based on the Hölder inequality and Minkowski inequality, one has that

$$\begin{aligned}\tilde{\zeta}(t) &\leqslant \tilde{A}(t) + \frac{1}{\Gamma(\delta)}\left(\int_{t_0}^{t}(t-\nu)^{p(\delta-1)}d\nu\right)^{1/p} \left(\int_{t_0}^{t}\left[\tilde{F}(\nu)\tilde{\zeta}(\nu) + \tilde{G}(\nu)\tilde{\zeta}(\nu-\varsigma(\nu))\right]^q d\nu\right)^{1/q} \\ &\leqslant \tilde{A}(t) + \frac{(t-t_0)^{\delta-1+1/p}}{\Gamma(\delta)(p\delta-p+1)^{1/p}} \left[\left(\int_{t_0}^{t}\left[\tilde{F}(\nu)\tilde{\zeta}(\nu)\right]^q d\nu\right)^{1/q} + \left(\int_{t_0}^{t}\left[\tilde{G}(\nu)\tilde{\zeta}(\nu-\varsigma(\nu))\right]^q d\nu\right)^{1/q}\right],\end{aligned}$$

where $p, q > 0$ satisfying $1/p + 1/q = 1$ and $q > 1/\delta$. By applying the Jensen inequality, one has that

$$\tilde{\zeta}^q(t) \le 3^{q-1}\tilde{A}^q(t) + \frac{3^{q-1}(t-t_0)^{q\delta-q+q/p}}{\Gamma(\delta)^q(p\delta-p+1)^{q/p}}\int_{t_0}^{t} [\tilde{F}^q(\nu)\tilde{\zeta}^q(\nu) + \tilde{G}^q(\nu)\tilde{\zeta}^q(\nu-\varsigma(\nu))]d\nu.$$

Define

$$B(t) = \frac{3^{q-1}(t-t_0)^{q\delta-q+q/p}}{\Gamma(\delta)^q(p\delta-p+1)^{q/p}},$$
$$\psi(t) = \int_{t_0}^{t}[\tilde{F}^q(\nu)\tilde{\zeta}^q(\nu) + \tilde{G}^q(\nu)\tilde{\zeta}^q(\nu-\varsigma(\nu))]d\nu, \ t \in [t_0, T],$$

and let $\tilde{A}(t) = 0$, $B(t) = 0$ when $t \le t_0$. Then, one has that

$$\tilde{\zeta}^q(t) \le 3^{q-1}\tilde{A}^q(t) + B(t)\psi(t),$$

and

$$\begin{aligned}\dot{\psi}(t) &= [\tilde{F}^q(t)\tilde{\zeta}^q(t) + \tilde{G}^q(t)\tilde{\zeta}^q(\nu-\varsigma(t))] \\ &\leqslant \tilde{F}^q(t)\left(3^{q-1}\tilde{A}^q(t) + B(t)\psi(t)\right) + \tilde{G}^q(t)\left(3^{q-1}\tilde{A}^q(t-\varsigma(t)) + B(t-\varsigma(t))\psi(t)\right) \\ &= 3^{q-1}\left(\tilde{F}^q(t)\tilde{A}^q(t) + \tilde{G}^q(t)\tilde{A}^q(t-\varsigma(t))\right) + \left(\tilde{F}^q(t)B(t) + \tilde{G}^q(t)B(t-\varsigma(t))\right)\psi(t).\end{aligned}$$

According to the Bellman-Gronwall inequality, one has that

$$\psi(t) \le \int_{t_0}^{t}\exp\left(\int_{\nu}^{t}\tilde{F}^q(\mu)B(\mu) + \tilde{G}^q(\mu)B(\mu-\varsigma(\mu))d\mu\right)3^{q-1} \left(\tilde{F}^q(\nu)\tilde{A}^q(\nu) + \tilde{G}^q(\nu)\tilde{A}^q(\nu-\varsigma(\nu))\right)d\nu.$$

It follows that

$$\tilde{\zeta}^q(t) \le 3^{q-1}\tilde{A}^q(t) + B(t)\int_{t_0}^{t} \exp\left(\int_{\nu}^{t}\tilde{F}^q(\mu)B(\mu) + \tilde{G}^q(\mu)B(\mu-\varsigma(\mu))d\mu\right)3^{q-1} \left(\tilde{F}^q(\nu)\tilde{A}^q(\nu) + \tilde{G}^q(\nu)\tilde{A}^q(\nu-\varsigma(\nu))\right)d\nu.$$

Since $\tilde{A}(t) = \lim_{\eta\to 0}A(t,\eta) = 0$ holds for all $t \in (t_0, T]$, $\lim_{\eta\to 0}\zeta(t,\eta) = \tilde{\zeta}(t) = 0, \forall t \in (t_0, T]$ as $\eta \to 0$. Therefore, $\lim_{\eta\to 0}y(t,\eta)$ is the continuous right derivative of $x(t)$. Similarly, it can be proved that $\lim_{\eta\to 0}y(t,\eta)$ is the continuous left derivative of $x(t)$. This completes the proof. □

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