SIAM J. CONTROL OPTIM.
Vol. 58, No. 3, pp. 1547–1578

A CONVEX SOLUTION OF THE H_∞-OPTIMAL CONTROLLER SYNTHESIS PROBLEM FOR MULTIDELAY SYSTEMS*

MATTHEW M. PEET†

Abstract. Optimal controller synthesis is a bilinear problem and hence difficult to solve in a computationally efficient manner. We are able to resolve this bilinearity for systems with delay by first convexifying the problem in infinite dimensions, i.e., formulating the H_∞-optimal state-feedback controller synthesis problem for distributed-parameter systems as a linear operator inequality, which a form of convex optimization with operator variables. Next, we use positive matrices to parameterize positive "complete quadratic" operators, allowing the controller synthesis problem to be solved using semidefinite programming (SDP). We then use the solution to this SDP to calculate the feedback gains and provide effective methods for real-time implementation. Finally, we use several test cases to verify that the resulting controllers are *optimal* to several decimal places as measured by the minimal achievable closed-loop H_∞ norm, and as compared against controllers designed using high-order Padé approximations.

Key words. delay systems, LMIs, controller synthesis

AMS subject classifications. 68Q25, 68R10, 68U05

DOI. 10.1137/19M1252016

1. Introduction.

To control systems with delay, we must account for the transportation and flow of information. Although solutions to equations of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + Bu(t)$$

appear to be functions of time, they are better understood as functions of both time and space:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_1v(t,-\tau) + Bu(t),\\ \partial_t v(t,s) &= \partial_s v(t,s), \quad v(t,0) = x(t).\end{aligned}$$

That is, instead of being lost, the state information, $x(t)$, is preserved as $v(t,0)$, transported through a hidden process ($\partial_t v = \partial_s v$), moving at fixed velocity ($-1m/s$), through a pipe of fixed length (τm), emerges a fixed time later ($t+\tau$) as $v(t+\tau,-\tau)$, and influences the evolution at that future time ($\dot{x}(t+\tau)$).

The implication is that feedback controllers for systems with delay must account for both the visible part of the state, $x(t)$, and the hidden process, $v(t,s)$. This concept is well established and is expressed efficiently in the use of Lyapunov–Krasovskii (LK) functions, a concept dating back to at least 1959 [8]. LK functionals $V(x,v)$ map $V : \mathbb{R}^n \times L_2^n \to \mathbb{R}^+$ and offer a method for combining the states, both current (x) and hidden (v), into a single energy metric.

While the concept of an LK functional may seem obvious, this same logic has been relatively neglected in the design of feedback controllers for time-delay systems.

*Received by the editors March 25, 2019; accepted for publication (in revised form) February 18, 2020; published electronically June 11, 2020. This is an extended and revised version of a preliminary conference report that was presented at ACC 2019 [15].
https://doi.org/10.1137/19M1252016
Funding: This work was supported by the National Science Foundation under grants 1301660, 1538374, and 1739990.

†School for the Engineering of Matter, Transport and Energy, Arizona State University, Tempe, AZ 85287 (mpeet@asu.edu, http://control.asu.edu).

That is, a controller should not only account for the present state, $x(t) \in \mathbb{R}^n$, but also react to the hidden state $v(t) \in L_2[-\tau, 0]$.

The reason for the relative neglect of the hidden state lies in the development of linear matrix inequality (LMI) methods for control in the mid 1990s. Specifically, Ricatti equations and later LMIs were shown to be reliable and practical computational tools for designing optimal and robust controllers for finite-dimensional systems. As a result, research on stability and control of time-delay systems focused on developing clever ways to suppress the infinite-dimensional nature of the hidden state and apply LMIs to a resulting problem in $\mathbb{R}^n$ for which these tools were originally designed. For example, model transformations were used in [2, 10, 13], resulting in a Lyapunov function of the form $V(x, v) = z^T M z$, where

$$z(t) = x(t-\tau) + \int_{t-\tau}^{t} \left(A_0 x(s) + A_1 x(s-\tau)\right) ds.$$

More recently, Jenson's inequality and free-weighting matrices have been used to parameterize ever more complex Lyapunov functions by projecting the distributed hidden state v onto a finite-dimensional vector. Indeed, this approach was recently formalized and made infinitely scalable in [22] using a projection-based approach so that for any set of basis functions $L_i(s)$, we may define an expanded finite-dimensional vector

$$z_i(t) = \int_{-\tau}^{0} L_i(s) v(t, s) ds$$

so that the resulting Lyapunov function becomes $V(x, v) = z^T M z$, where the size of M increases with the number of basis functions.

Given that LMIs were developed for finite-dimensional systems, the desire to project the hidden state $v \in L_2$ onto a finite-dimensional vector space is understandable. However, this approach severely limits our ability to perform controller synthesis. Specifically, these projections from $\mathcal{P} : (x, v) \mapsto z$ are not invertible. This is problematic, since standard methods for controller synthesis require the state transformation $\mathcal{P}$ to be invertible from primal state (x, v) to dual state $(\hat{x}, \hat{v})$. In this approach, the controllers are then designed for the dual state $u(t) = \mathcal{Z}(\hat{x}, \hat{v})$ and then implemented on the original state using the inverse transformation $u(t) = \mathcal{Z}\mathcal{P}^{-1}(x, v)$.

In contrast to projection-based approaches, in this paper and its companion [16], we initially ignore the limitations of the LMI framework and directly formulate convex controller synthesis conditions on an infinite-dimensional space. Specifically, in [16] we formulated convex stabilizing controller synthesis conditions directly in terms of the existence of an invertible state transformation $\mathcal{P} : (x, v) \mapsto (\hat{x}, \hat{v})$ and a dual control operator $\mathcal{Z} : (\hat{x}(t), \hat{v}(t)) \mapsto u(t)$. In section 3, these results are extended to provide a convex formulation of the H_∞-optimal full-state feedback controller synthesis problem for a general class of distributed parameter systems (DPS).

Having developed a convex formulation of the controller synthesis problem, the question becomes how to test feasibility of these conditions using LMIs, a tool developed for optimization of positive matrix variables (NOT positive operators). As discussed above, a natural approach is to find a way to project these operator-valued inequalities onto a finite-dimensional state space (wherein they become matrices); indeed, one can view the work of [7, 20] (or in the PDE case, [9]) as an attempt to do exactly this. However, these works were unable to recover controller gains. Moreover, the operator inequality conditions for controller synthesis that we propose in [16] and in Theorem 3 explicitly prohibit solutions using projected operators, as the synthesis conditions require the solution to be coercive, and a projected operator by definition has a nontrivial null-space.

Because projection is not an option, in this paper and in [16], we have proposed reversing the dominant paradigm by not *narrowing* the control problem to a finite-dimensional space (where we can apply LMIs) but instead *expanding* the LMI toolset to explicitly allow for parameterization and optimization of operator variables. To understand how this works, let us now discard ODE-based LK functions of the form $V(x,v) = x^T M x$ and instead focus on LK functions of the form

$$V(x,v) := \int_{-\tau}^{0} v(s) M v(s) ds,$$

where the LK function is positive if $M \geq 0$. Now, following the same logic presented above, we increase the complexity of the Lyapunov function by replacing $v(s) : s \mapsto \mathbb{R}^n$ with $z(s) : s \mapsto \mathbb{R}^q$ defined as

$$z(s) = \begin{bmatrix} x \\ Z(s)v(s) \\ \int_{-\tau}^{0} Z(s,\theta)v(\theta)d\theta \end{bmatrix},$$

where $Z(s)$ and $Z(s,\theta)$ are vectors of functions and increase the dimension of M and hence the complexity of the LK function, resulting in the well-known class of "complete-quadratic" functions. The advantage of this approach, then, is that the resulting LK function can also be represented as

$$V(x,v) := \int_{-\tau}^{0} \begin{bmatrix} x \\ v(s) \end{bmatrix} \left(\mathcal{P} \begin{bmatrix} x \\ v(\cdot) \end{bmatrix} \right)(s) ds,$$

where

$$\left(\mathcal{P} \begin{bmatrix} x \\ v \end{bmatrix} \right)(s) = \begin{bmatrix} Px + \int_{-\tau}^{0} Q(\theta)v(\theta)d\theta \\ Q(s)^T x + S(s)v(s) + \int_{-\tau}^{0} R(s,\theta)v(\theta)d\theta \end{bmatrix}$$

for some P, Q, S, and R (defined in Theorem 7). In this way, positive matrices represent not just positive LK functions (of the complete-quadratic type) but also positive operators in a standardized form—these are denoted as $\mathcal{P}\left[\begin{smallmatrix} P, & Q \\ S, & R \end{smallmatrix}\right]$. This means that if we constrain our operators to have this standard form, we can enforce positivity using LMI constraints. Furthermore, linear constraints on the matrix P and the functions Q, R, and S translate into linear constraints on the elements of the positive matrix M.

The contribution of section 4, then, is to assume all operators have the $PQRS$ form and state conditions on the functions P, Q, R, and S such that the resulting operators satisfy the conditions of Theorem 3. Positivity is then formulated as an LMI constraint in section 8.

One of the drawbacks of the proposed approach is that the resulting controllers are expressed as operators of the form $u(t) = \mathcal{Z}\mathcal{P}\left[\begin{smallmatrix} P, & Q \\ S, & R \end{smallmatrix}\right]^{-1}(x(t), v(t))$. The solution to the LMI yields numerical values of operator $\mathcal{Z}$ (also a $PQRS$ operator) and functions P, Q, R, and S. However, in order to compute the controller gains,

$$u(t) = K_1 x(t) + K_2 v(t,-\tau) + \int_{-\tau}^{0} K_3(s) v(t,s) ds, \tag{1.1}$$

we need to find $\hat{P}$, $\hat{Q}$, $\hat{R}$, and $\hat{S}$ such that $\mathcal{P}\left[\begin{smallmatrix}\hat{P}, & \hat{Q}\\ \hat{S}, & \hat{R}\end{smallmatrix}\right] = \mathcal{P}\left[\begin{smallmatrix}P, & Q\\ S, & R\end{smallmatrix}\right]^{-1}$. This problem is solved in section 6 (the inversion formula is a generalization of the result in [12]) by derivation of an analytic expression for $\hat{P}$, $\hat{Q}$, $\hat{R}$, and $\hat{S}$ in terms of P, Q, R, and S. Finally, practical implementation requires an efficient numerical scheme for calculating $u(t)$ in real time. This issue is resolved in section 7.

To make the results of this paper more broadly useful, we have developed efficient implementations for solving the LMI, calculating the feedback gains, and simulated the closed-loop response. These are available online via Code Ocean [17] and from [14]. In section 9, the results are shown to be nonconservative to several decimal places by calculation of the minimal achievable closed-loop H_∞-norm bound for several systems and by comparison to results obtained using high-order Padé approximations of the same systems. Obviously, the results presented in this paper are significantly better than any known algorithm for controller synthesis with provable performance metrics. Furthermore, these results can be extended in obvious ways to robust control with uncertainty in system parameters or in delay.

1.1. Notation. Shorthand notation used throughout this paper includes the Hilbert spaces $L_2^m[X] := L_2(X;\mathbb{R}^m)$ of square integrable functions from X to $\mathbb{R}^m$ and $W_2^m[X] := W^{1,2}(X;\mathbb{R}^m) = H^1(X;\mathbb{R}^m) = \{x \,:\, x, \dot{x} \in L_2^m[X]\}$. We use L_2^m and W_2^m when domains are clear from the context. We also use the extensions $L_2^{n\times m}[X] := L_2(X;\mathbb{R}^{n\times m})$ and $W_2^{n\times m}[X] := W^{1,2}(X;\mathbb{R}^{n\times m})$ for matrix-valued functions. $S^n \subset \mathbb{R}^{n\times n}$ denotes the symmetric matrices. We say an operator $\mathcal{P} : Z \to Z$ is positive on a subset X of Hilbert space Z if $\langle x, \mathcal{P}x\rangle_Z \geq 0$ for all $x \in X$. $\mathcal{P}$ is coercive on X if $\langle x, \mathcal{P}x\rangle_Z \geq \epsilon \|x\|_Z^2$ for some $\epsilon > 0$ and for all $x \in X$. Given an operator $\mathcal{P} : Z \to Z$ and a set $X \subset Z$, we use the shorthand $\mathcal{P}(X)$ to denote the image of $\mathcal{P}$ on subset X. $I_n \in \mathbb{S}^n$ denotes the identity matrix. $0_{n\times m} \in \mathbb{R}^{n\times m}$ is the matrix of zeros with shorthand $0_n := 0_{n\times n}$. We will occasionally denote the intervals $T_i^j := [-\tau_i, -\tau_j]$ and $T_i^0 := [-\tau_i, 0]$. For $K \in \mathbb{N}$, we adopt the index shorthand notation which denotes $[K] = \{1, \ldots, K\}$. The symmetric completion of a matrix is denoted $*^T$.

2. The LMI for H_∞-optimal controller synthesis for ODEs.

Much of the proof of Theorem 3 is a simple generalization of the proof of the ODE controller synthesis LMI (Lemma 1). Thus, to illustrate the logic behind the main result in Theorem 3, we examine the same well-known result in finite dimensions. One of the advantages of the $PQRS$ framework (described above and in section 5) is that it *simplifies* the development of new results for infinite-dimensional systems—the idea being that almost any LMI developed for estimation and control of ODEs may be generalized and solved for delay systems. To illustrate, we consider the ODE system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \quad x(0) = 0,\\ y(t) &= Cx(t) + D_1 w(t) + D_2 u(t). \end{aligned}$$

Then the following LMI provides a necessary and sufficient condition for existence of an H_∞-optimal full-state feedback controller.

LEMMA 1 (Full-State Feedback Controller Synthesis). *Define the map from the closed-loop (using $u(t) = Kx(t)$) state-space representation to the associated transfer function as*

$$\hat{G} = \left[\begin{array}{c|c} A + B_2K & B_1 \\ \hline C + D_2K & D_1 \end{array}\right].$$

The following are equivalent:

- *There exists a K such that $\|\hat{G}\|_{H_\infty} < \gamma$.*
- *There exists a $P > 0$ and Z such that*

$$\begin{bmatrix} PA^T + AP + Z^T B_2^T + B_2 Z & B_1 & PC_1^T + Z^T D_2^T \\ B_1^T & -\gamma I & D_1^T \\ C_1 P + D_2 Z & D_1 & -\gamma I \end{bmatrix} < 0.$$

Proof. The proof is a straightforward application of the Kalman–Yakubovich–Popov (KYP) lemma, a duality transformation, and the Schur complement lemma. However, since the purpose of this proof is to motivate the proof of Theorem 3, we only prove sufficiency of the nonstrict inequality. First, define the storage function $V(x) = x^T P^{-1} x$. Let $K = ZP^{-1}$ and $u(t) = Kx(t)$. Then if $x(t)$ is a solution of system $\hat{G}$,

$$\begin{aligned} \dot{V}(t) &= x(t)^T P^{-1}(Ax(t) + B_2 ZP^{-1}x(t) + B_1 w(t)) \\ &\quad + (Ax(t) + B_2 ZP^{-1}x(t) + B_1 w(t))^T P^{-1} x(t) \\ &= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} P^{-1}(A + B_2 ZP^{-1}) + *^T & *^T \\ B_1^T P^{-1} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\ &= \begin{bmatrix} P^{-1}x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} AP + B_2 Z + *^T & *^T \\ B_1^T & 0 \end{bmatrix} \begin{bmatrix} P^{-1}x(t) \\ w(t) \end{bmatrix}. \end{aligned}$$

Now let $z(t) = P^{-1}x(t)$. Then for any v, the matrix inequality implies

$$\begin{bmatrix} z \\ w \\ v \end{bmatrix}^T \begin{bmatrix} AP + B_2 Z + *^T & *^T & *^T \\ B_1^T & -\gamma I & *^T \\ C_1 P + D_2 Z & D_1 & -\gamma I \end{bmatrix} \begin{bmatrix} z \\ w \\ v \end{bmatrix} \le 0.$$

Following the proof of the Schur complement lemma, we let $v = \frac{1}{\gamma}((C_1 P + D_2 Z)z + D_1 w)$, which implies

$$\begin{aligned} \begin{bmatrix} z \\ w \\ v \end{bmatrix}^T \begin{bmatrix} AP + B_2 Z + *^T & *^T & *^T \\ B_1^T & -\gamma I & *^T \\ C_1 P + D_2 Z & D_1 & -\gamma I \end{bmatrix} \begin{bmatrix} z \\ w \\ v \end{bmatrix} &= \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} AP + B_2 Z + *^T & *^T \\ B_1^T & -\gamma I \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \\ &\quad + \frac{1}{\gamma} \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} (C_1 P + D_2 Z)^T \\ D_1^T \end{bmatrix} \begin{bmatrix} C_1 P + D_2 Z & D_1 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \le 0. \end{aligned}$$

Applying this to $\dot{V}$, we find

$$\begin{aligned} &\dot{V}(t) = \begin{bmatrix} P^{-1}x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} AP + B_2 Z + *^T & *^T \\ B_1^T & 0 \end{bmatrix} \begin{bmatrix} P^{-1}x(t) \\ w(t) \end{bmatrix} \\ &\le \gamma \|w\|^2 - \frac{1}{\gamma} \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} (C_1 P + D_2 Z)^T \\ D_1^T \end{bmatrix} \begin{bmatrix} C_1 P + D_2 Z & D_1 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \\ &= \gamma \|w\|^2 - \frac{1}{\gamma} \|(C_1 P + D_2 Z)z(t) + D_1 w(t)\|^2 \\ &= \gamma \|w\|^2 - \frac{1}{\gamma} \|C_1 x(t) + D_2 Kx(t) + D_1 w(t)\|^2 = \gamma \|w\|^2 - \frac{1}{\gamma} \|y(t)\|^2 . \end{aligned}$$

If $w = 0$, the LMI implies the system is exponentially stable, and hence $\lim_{t\to\infty} V(t) = 0$. Since $V(0) = 0$, by integrating the inequality forward in time, we obtain

$$\frac{1}{\gamma} \|y\|_{L_2}^2 \le \gamma \|w\|_{L_2}^2 ,$$

which completes the proof. ∎

In the following section, we replicate these steps, simply expanding

$$\begin{bmatrix} z \\ w \\ v \end{bmatrix}^T \begin{bmatrix} AP + B_2Z + *^T & *^T & *^T \\ B_1^T & -\gamma I & *^T \\ C_1P + D_2Z & D_1 & -\gamma I \end{bmatrix} \begin{bmatrix} z \\ w \\ v \end{bmatrix} < 0$$

and replacing terms such as $z^T APz$ with inner products on the appropriate function space as in $\langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle$. Here the bold version of $\mathbf{z}$ emphasizes that this term lies in a function space, and the calligraphic notation $\mathcal{A}$ indicates that $\mathcal{A}$ is an operator.

3. A convex formulation of the controller synthesis problem for distributed-parameter systems. Consider the generic distributed-parameter system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \quad \mathbf{x}(0) = 0, \\ y(t) &= \mathcal{C}\mathbf{x}(t) + D_1 w(t) + \mathcal{D}_2 u(t), \end{aligned} \tag{3.1}$$

where $\mathcal{A} : X \to Z$, $\mathcal{B}_1 : \mathbb{R}^m \to Z$, $\mathcal{B}_2 : U \to Z$, $\mathcal{C} : X \to \mathbb{R}^q$, $D_1 : \mathbb{R}^m \to \mathbb{R}^q$, and $\mathcal{D}_2 : U \to \mathbb{R}^q$.

We begin with the following mathematical result on duality, which is a reduced version of Theorem 3 in [16].

THEOREM 2. *Suppose $\mathcal{P}$ is a bounded, coercive linear operator $\mathcal{P} : X \to X$ with $\mathcal{P}(X) = X$ and is self-adjoint with respect to the Z inner product. Then $\mathcal{P}^{-1}$: exists, is bounded, and is self-adjoint; $\mathcal{P}^{-1} : X \to X$; and $\mathcal{P}^{-1}$ is coercive.*

Using Theorem 2, we give a convex formulation of the H_∞-optimal full-state feedback controller synthesis problem. This result combines: (a) a relatively simple extension of the Schur complement lemma to infinite dimensions, with (b) the dual synthesis condition in [16]. We note that the ODE equivalent (Lemma 1) of this theorem is necessary and sufficient, and the proof structure can be credited to, e.g., [1].

THEOREM 3. *Suppose there exists an $\epsilon > 0$, an operator $\mathcal{P} : Z \to Z$ which satisfies the conditions of Theorem 2, and an operator $\mathcal{Z} : X \to U$ such that*

$$\begin{aligned} &\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z \\ &\quad \le \gamma w^T w - v^T(\mathcal{C}\mathcal{P}\mathbf{z}) - (\mathcal{C}\mathcal{P}\mathbf{z})^T v - v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) - (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v \\ &\qquad - v^T(D_1 w) - (D_1 w)^T v + \gamma v^T v - \epsilon \|z\|_Z^2 \end{aligned}$$

for all $\mathbf{z} \in X$, $w \in \mathbb{R}^m$, and $v \in \mathbb{R}^q$. Then for any $w \in L_2$, if $\mathbf{x}(t)$ and $y(t)$ satisfy $\mathbf{x}(t) \in X$ and

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \\ y(t) &= \mathcal{C}\mathbf{x}(t) + D_1 w(t) + \mathcal{D}_2 u(t), \\ u(t) &= \mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t) \end{aligned} \tag{3.2}$$

for all $t \ge 0$, then $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$.

Proof. By Theorem 2, $\mathcal{P}^{-1}$: exists, is bounded, and is self-adjoint; $\mathcal{P}^{-1} : X \to X$; and is coercive.

For $w \in L_2$, let $\mathbf{x}(t)$ and $y(t)$ be a solution of

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t) = (\mathcal{A} + \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1})\mathbf{x}(t) + \mathcal{B}_1 w(t), \\ y(t) &= \mathcal{C}\mathbf{x}(t) + D_1 w(t) + \mathcal{D}_2 u(t) = (\mathcal{C} + \mathcal{D}_2\mathcal{Z}\mathcal{P}^{-1})\mathbf{x}(t) + D_1 w(t) \end{aligned}$$

such that $\mathbf{x}(t) \in X$ for any finite t.

Define the storage function $V(t) = \left\langle \mathbf{x}(t), \mathcal{P}^{-1}\mathbf{x}(t) \right\rangle_Z$. Then $V(t) \geq \delta \left\| \mathbf{x}(t) \right\|_Z^2$ for some $\delta > 0$. Define $\mathbf{z}(t) = \mathcal{P}^{-1}\mathbf{x}(t) \in X$. Differentiating the storage function in time, we obtain

$$
\begin{aligned}
\dot{V}(t) &= \left\langle \mathbf{x}(t), \mathcal{P}^{-1}(\mathcal{A}\mathbf{x}(t) + \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t) + \mathcal{B}_1 w(t)) \right\rangle_Z \\
&\quad + \left\langle \mathcal{P}^{-1}(\mathcal{A}\mathbf{x}(t) + \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t) + \mathcal{B}_1 w(t)), \mathbf{x}(t) \right\rangle_Z \\
&= \left\langle \mathcal{P}^{-1}\mathbf{x}(t), \mathcal{A}\mathbf{x}(t) \right\rangle_Z + \left\langle \mathcal{P}^{-1}\mathbf{x}(t), \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t) \right\rangle_Z + \left\langle \mathcal{P}^{-1}\mathbf{x}(t), \mathcal{B}_1 w(t) \right\rangle_Z \\
&\quad + \left\langle \mathcal{A}\mathbf{x}(t), \mathcal{P}^{-1}\mathbf{x}(t) \right\rangle_Z + \left\langle \mathcal{B}_2\mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t), \mathcal{P}^{-1}\mathbf{x}(t) \right\rangle_Z + \left\langle \mathcal{B}_1 w(t), \mathcal{P}^{-1}\mathbf{x}(t) \right\rangle_Z \\
&= \left\langle \mathbf{z}(t), \mathcal{A}\mathcal{P}\mathbf{z}(t) \right\rangle_Z + \left\langle \mathcal{B}_2\mathcal{Z}\mathbf{z}(t), \mathbf{z}(t) \right\rangle_Z + \left\langle \mathbf{z}(t), \mathcal{B}_1 w(t) \right\rangle_Z \\
&\quad + \left\langle \mathcal{A}\mathcal{P}\mathbf{z}(t), \mathbf{z}(t) \right\rangle_Z + \left\langle \mathbf{z}(t), \mathcal{B}_2\mathcal{Z}\mathbf{z}(t) \right\rangle_Z + \left\langle \mathcal{B}_1 w(t), \mathbf{z}(t) \right\rangle_Z \\
&\leq \gamma w(t)^T w(t) - v(t)^T(\mathcal{C}\mathcal{P}\mathbf{z}(t)) - (\mathcal{C}\mathcal{P}\mathbf{z}(t))^T v(t) - v(t)^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}(t)) \\
&\quad - (\mathcal{D}_2\mathcal{Z}\mathbf{z}(t))^T v(t) - v(t)^T(D_1 w(t)) - (D_1 w(t))^T v(t) + \gamma v(t)^T v(t) - \epsilon \left\| \mathbf{z}(t) \right\|_Z^2 \\
&= \gamma w(t)^T w(t) - v(t)^T \left((\mathcal{C} + \mathcal{D}_2\mathcal{Z}\mathcal{P}^{-1})\mathbf{x}(t) + D_1 w(t) \right) \\
&\quad - \left((\mathcal{C} + \mathcal{D}_2\mathcal{Z}\mathcal{P}^{-1})\mathbf{x}(t) + D_1 w(t) \right)^T v(t) + \gamma v(t)^T v(t) - \epsilon \left\| \mathbf{z}(t) \right\|_Z^2 \\
&= \gamma w(t)^T w(t) - v(t)^T y(t) - y(t)^T v(t) + \gamma v(t)^T v(t) - \epsilon \left\| \mathbf{z}(t) \right\|_Z^2
\end{aligned}
$$

for any $v(t) \in \mathbb{R}^q$ and all $t \geq 0$. Choosing $v(t) = \frac{1}{\gamma} y(t)$, we get

$$
\begin{aligned}
\dot{V}(t) &\leq \gamma \left\| w(t) \right\|^2 - \frac{2}{\gamma} \left\| y(t) \right\|^2 + \frac{1}{\gamma} \left\| y(t) \right\|^2 - \epsilon \left\| \mathbf{z}(t) \right\|_Z^2 \\
&= \gamma \left\| w(t) \right\|^2 - \frac{1}{\gamma} \left\| y(t) \right\|^2 - \epsilon \left\| \mathbf{z}(t) \right\|_Z^2 .
\end{aligned}
$$

Since $\mathcal{P}$ is bounded, there exists a $\sigma > 0$ such that

$$
V(t) = \left\langle \mathbf{x}(t), \mathcal{P}^{-1}\mathbf{x}(t) \right\rangle_Z = \left\langle \mathbf{z}(t), \mathcal{P}\mathbf{z}(t) \right\rangle_Z \leq \sigma \left\| \mathbf{z}(t) \right\|_Z^2 .
$$

We conclude, therefore, that

$$
\dot{V}(t) \leq -\frac{\epsilon}{\sigma} V(t) + \gamma \left\| w(t) \right\|^2 - \frac{1}{\gamma} \left\| y(t) \right\|^2 .
$$

Therefore, since $w \in L_2$, we may conclude by the Gronwall–Bellman inequality that $\lim_{t\to\infty} V(t) = 0$. Integrating this expression forward in time and using $V(0) = V(\infty) = 0$, we obtain

$$
\frac{1}{\gamma} \left\| y \right\|_{L_2}^2 \leq \gamma \left\| w \right\|_{L_2}^2 ,
$$

which concludes the proof. ∎

4. Theorem 3 applied to multidelay systems. Theorem 3 gives a convex formulation of the controller synthesis problem for a generic class of distributed-parameter systems. In this section and the next, we apply Theorem 3 to the case of systems with multiple delays. Specifically, we consider solutions to the system of equations given by

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t),$$
$$y(t) = C_0 x(t) + \sum_{i=1}^{K} C_i x(t-\tau_i) + D_1 w(t) + D_2 u(t), \tag{4.1}$$

where $w(t) \in \mathbb{R}^m$ is the disturbance input, $u(t) \in \mathbb{R}^p$ is the controlled input, $y(t) \in \mathbb{R}^q$ is the regulated output, $x(t)$ are the state variables, and $\tau_i > 0$ for $i \in [1, \ldots, K]$ are the delays ordered by increasing magnitude. We assume $x(s) = 0$ for $s \in [-\tau_K, 0]$.

Our first step, then, is to express system (4.1) in the abstract form of (3.1). Following the mathematical formalism developed in [16], we define the inner-product space $Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$, and for $\{x, \phi_1, \ldots, \phi_K\} \in Z_{m,n,K}$, we define the following shorthand notation:

$$\begin{bmatrix} x \\ \{\phi_i\}_i \end{bmatrix} := \{x, \phi_1, \ldots, \phi_K\}.$$

For convenience, we use uppercase boldface to denote the sets $\mathbf{\Phi} := \{\phi_i\}_{i=1}^K$ and $\mathbf{\Psi} := \{\psi_i\}_{i=1}^K$, which allows us to simplify expression of the inner product on $Z_{m,n,K}$, which we define as

$$\left\langle \begin{bmatrix} y \\ \mathbf{\Psi} \end{bmatrix}, \begin{bmatrix} x \\ \mathbf{\Phi} \end{bmatrix} \right\rangle_{Z_{m,n,K}} = \tau_K y^T x + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \psi_i(s)^T \phi_i(s) ds.$$

When $m = n$, we simplify the notation using $Z_{n,K} := Z_{n,n,K}$. The state space for system (4.1) is defined as

$$X := \left\{ \begin{bmatrix} x \\ \mathbf{\Phi} \end{bmatrix} \in Z_{n,K} \,:\, \phi_i \in W_2^n[-\tau_i, 0] \text{ and } \phi_i(0) = x \quad \forall\, i \in [K] \right\}.$$

Note that X is a subspace of $Z_{n,K}$ and inherits the norm of $Z_{n,K}$. In order to conveniently assign values to elements of $Z_{m,n,K}$, we further extend this notation to say that

$$\begin{bmatrix} x \\ \{\phi_i\}_i \end{bmatrix}(s) = \begin{bmatrix} y \\ \{\psi_i(s)\}_i \end{bmatrix}$$

if $x = y$ and $\phi_i(s) = \psi_i(s)$ for $s \in [-\tau_i, 0]$ and $i \in [K]$.

We now represent the infinitesimal generator, $\mathcal{A} : X \to Z_{n,K}$, of (4.1) as [3]

$$\mathcal{A} \begin{bmatrix} x \\ \mathbf{\Phi} \end{bmatrix}(s) := \begin{bmatrix} A_0 x + \sum_{i=1}^K A_i \phi_i(-\tau_i) \\ \left\{ \dot{\phi}_i(s) \right\}_i \end{bmatrix}.$$

Furthermore, $\mathcal{B}_1 : \mathbb{R}^m \to Z_{n,K}$, $\mathcal{B}_2 : \mathbb{R}^p \to Z_{n,K}$, $\mathcal{D}_1 : \mathbb{R}^m \to \mathbb{R}^q$, $\mathcal{D}_2 : \mathbb{R}^p \to \mathbb{R}^q$, and $\mathcal{C} : Z_{n,K} \to \mathbb{R}^p$ are defined as

$$(\mathcal{B}_1 w)(s) := \begin{bmatrix} B_1 w \\ 0 \end{bmatrix}, \quad (\mathcal{B}_2 u)(s) := \begin{bmatrix} B_2 u \\ 0 \end{bmatrix}, \quad \left(\mathcal{C} \begin{bmatrix} \psi \\ \mathbf{\Phi} \end{bmatrix} \right) := \left[C_0 \psi + \sum_{i=1}^K C_i \phi_i(-\tau_i) \right],$$
$$(\mathcal{D}_1 w)(s) := \left[D_1 w \right], \quad (\mathcal{D}_2 u)(s) := \left[D_2 u \right].$$

Having defined these operators, we note that for any solution $x(t)$ of (4.1), using the above notation, if we define

$$(\mathbf{x}(t))\,(s) = \begin{bmatrix} x(t) \\ \{x(t+s)\}_i \end{bmatrix},$$

then $\mathbf{x}$ satisfies (3.1) using the operator definitions given above. The converse statement is also true.

4.1. A parameterization of operators. We now introduce a class of operators $\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right] : Z_{m,n,K} \to Z_{m,n,K}$, parameterized by matrix P and the tuples $\mathbf{Q} := \{Q_i\}_{i=1}^K$, $\mathbf{S} := \{S_i\}_{i=1}^K$, and $\mathbf{R} := \{R_{ij}\}_{i,j=1}^K$, which are defined by the matrix-valued functions $Q_i \in W_2^{m\times n}[-\tau_i, 0]$, $S_i \in W_2^{n\times n}[-\tau_i, 0]$, $R_{ij} \in W_2^{n\times n}\left[[-\tau_i, 0] \times [-\tau_j, 0]\right]$, yielding

$$\left(\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right] \begin{bmatrix} x \\ \mathbf{\Phi} \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s)\phi_i(s) ds \\ \left\{\tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)\, d\theta\right\}_i \end{bmatrix}.$$

For this class of operators, the following lemma combines Lemmas 3 and 4 in [16] and gives conditions under which $\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]$ satisfies the conditions of Theorem 2.

LEMMA 4. *Suppose that $S_i \in W_2^{n\times n}[T_i^0]$, $R_{ij} \in W_2^{n\times n}\left[T_i^0 \times T_j^0\right]$ and $S_i(s) = S_i(s)^T$, $R_{ij}(s,\theta) = R_{ji}(\theta, s)^T$, $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$, and $Q_j(s) = R_{ij}(0, s)$ for all $i, j \in [K]$. Further suppose $\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]$ is coercive on $Z_{n,K}$. Then $\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]$: is a self-adjoint bounded linear operator with respect to the inner product defined on $Z_{n,K}$; maps $\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right] : X \to X$; and $\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right](X) = X$.*

Starting in section 5, we will assume Q_i, S_i, and R_{ij} are polynomial and give LMI conditions for positivity of operators of the form $\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]$.

4.2. The controller synthesis problem for systems with delay. Theorem 3 gives a convex formulation of the controller synthesis problem, where the data are the six operators $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{C}$, $\mathcal{D}_1$, and $\mathcal{D}_2$, and the variables are the operators $\mathcal{P}$ and $\mathcal{Z}$. For multidelay systems, we have defined the six operators and parameterized the decision variables $\mathcal{P}$ using $\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]$. Likewise, now we parameterize the decision variables $\mathcal{Z} : Z_{n,k} \to \mathbb{R}^p$ using matrices Z_0, Z_{1i} and functions Z_{2i} as

$$\left(\mathcal{Z}\begin{bmatrix} \psi \\ \mathbf{\Phi}\end{bmatrix}\right) := \left[Z_0\psi + \sum_{i=1}^K Z_{1i}\phi_i(-\tau_i) + \sum_i \int_{-\tau_i}^0 Z_{2i}(s)\phi_i(s) ds\right].$$

The following theorem gives convex constraints on the variables P, Q_i, S_i, R_{ij}, Z_0, Z_{1i}, and Z_{2i} under which Theorem 3 is satisfied when $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{C}$, $\mathcal{D}_1$, and $\mathcal{D}_2$ are as defined above.

THEOREM 5. *Suppose that there exist $S_i \in W_2^{n\times n}[T_i^0]$, $R_{ij} \in W_2^{n\times n}\left[T_i^0 \times T_j^0\right]$, and $S_i(s) = S_i(s)^T$ such that $R_{ij}(s,\theta) = R_{ji}(\theta, s)^T$, $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$, and $Q_j(s) = R_{ij}(0, s)$ for all $i, j \in [K]$, and matrices $Z_0 \in \mathbb{R}^{p\times n}$, $Z_{1i} \in \mathbb{R}^{p\times n}$, and $Z_{2i} \in W_2^{p\times n}[T_i^0]$ such that $\left\langle \mathbf{x}, \mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]\mathbf{x}\right\rangle_{Z_{n,K}} \geq \epsilon \left\|\mathbf{x}\right\|_{Z_{n,K}}^2$ for all $\mathbf{x} \in Z_{n,K}$ and*

$$\left\langle \begin{bmatrix} \begin{bmatrix} v \\ w \\ y_1 \\ y_2 \end{bmatrix} \\ \mathbf{\Phi} \end{bmatrix}, \mathcal{P}\left[\begin{smallmatrix} D, & \{E_i\}_i \\ \{\dot{S}_i\}_i, & \{G_{ij}\}_{i,j}\end{smallmatrix}\right] \begin{bmatrix} \begin{bmatrix} v \\ w \\ y_1 \\ y_2 \end{bmatrix} \\ \mathbf{\Phi} \end{bmatrix} \right\rangle_{Z_{q+m+n(K+1),n,K}} \leq -\epsilon \left\| \begin{bmatrix} y_1 \\ \mathbf{\Phi} \end{bmatrix} \right\|_{Z_{n,K}}^2$$

for all $y_1 \in \mathbb{R}^n$ *and*

$$\begin{bmatrix}\begin{bmatrix} v \\ w \\ y_1 \\ y_2 \end{bmatrix} \\ \mathbf{\Phi}\end{bmatrix} \in Z_{q+m+n(K+1),n,K},$$

where

$$L_0 = A_0 P + \sum_{i=1}^{K}\left(\tau_K A_i Q_i(-\tau_i)^T + \frac{1}{2}S_i(0)\right) + B_2 Z_0,$$

$$L_1 = \frac{1}{\tau_K}C_0 P + \sum_{i=1}^{K} C_i Q_i(-\tau_i)^T + \frac{1}{\tau_K}D_2 Z_0,$$

$$L_{2i} = C_i S_i(-\tau_i) + \frac{1}{\tau_K}D_2 Z_{1i}, \quad L_{3i} := \tau_K A_i S_i(-\tau_i) + B_2 Z_{1i},$$

$$D = \begin{bmatrix} -\frac{\gamma}{\tau_K}I & \frac{1}{\tau_K}D_1 & L_1 & L_{21} & \dots & L_{2K} \\ *^T & -\frac{\gamma}{\tau_K}I & B_1^T & 0 & \dots & 0 \\ *^T & *^T & L_0 + L_0^T & L_{31} & \dots & L_{3K} \\ *^T & *^T & *^T & -S_1(-\tau_1) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ *^T & *^T & *^T & *^T & \dots & -S_k(-\tau_K) \end{bmatrix},$$

$$E_i(s) = \frac{1}{\tau_K}\begin{bmatrix} C_0 Q_i(s) + \sum_{j=1}^K C_j R_{ji}(-\tau_j, s) + D_2 Z_{2i}(s) \\ 0 \\ \tau_K\left(A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^K A_j R_{ji}(-\tau_j, s) + B_2 Z_{2i}(s)\right) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$G_{ij}(s,\theta) = \frac{\partial}{\partial s}R_{ij}(s,\theta) + \frac{\partial}{\partial \theta}R_{ji}(s,\theta)^T, \quad i,j \in [K].$$

Then if

$$u(t) = \mathcal{Z}\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]^{-1}\mathbf{x}(t), \qquad \textit{where} \qquad (\mathbf{x}(t))(s) = \begin{bmatrix} x(t) \\ \{x(t+s)\}_i \end{bmatrix}$$

and where

$$\left(\mathcal{Z}\begin{bmatrix} x \\ \mathbf{\Phi}\end{bmatrix}\right)(s) := Z_0 x + \sum_{i=1}^K Z_{1i}\phi_i(-\tau_i) + \sum_{i=1}^K \int_{-\tau_i}^0 Z_{2i}(s)\phi_i(s)ds,$$

then for any $w \in L_2$, *if* $x(t)$ *and* $y(t)$ *satisfy* (4.1), $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$.

Proof. For any $w \in L_2$, using the definitions of $u(t)$, and $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{C}$, $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{Z}$ given above, recall that $y(t)$ and $x(t)$ satisfy (4.1) if and only if $y(t)$ and

$$\mathbf{x}(t) := \begin{bmatrix} x(t) \\ \{x(t+s)\}_i \end{bmatrix}$$

satisfy (3.1). Therefore, $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$ if

$$\begin{aligned}&\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z \\ &\qquad \le \gamma w^T w - v^T(\mathcal{C}\mathcal{P}\mathbf{z}) - (\mathcal{C}\mathcal{P}\mathbf{z})^T v - v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) - (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v \\ &\qquad - v^T(D_1 w) - (D_1 w)^T v + \gamma v^T v - \epsilon \|z\|_Z^2\end{aligned}$$

for all $\mathbf{z} \in X$, $w \in \mathbb{R}^m$, and $v \in \mathbb{R}^q$. The rest of the proof is lengthy but straightforward. We simply show that if we define

$$f = \begin{bmatrix} \mathbf{z}_{2,1}(-\tau_1)^T & \cdots & \mathbf{z}_{2,K}(-\tau_K)^T \end{bmatrix}^T,$$

then

(4.2)

$$\begin{aligned}&\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z \\ &+ \langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma w^T w + v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v \\ &+ v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \\ &= \left\langle \begin{bmatrix} \begin{bmatrix} v \\ w \\ \mathbf{z}_1 \\ f \end{bmatrix} \\ \{\mathbf{z}_{2i}\}_i \end{bmatrix}, \mathcal{P}_{\left[\begin{smallmatrix} D, & \{E_i\}_i \\ \{S_i\}_i, & \{G_{ij}\}_{ij}\end{smallmatrix}\right]} \begin{bmatrix} \begin{bmatrix} v \\ w \\ \mathbf{z}_1 \\ f \end{bmatrix} \\ \{\mathbf{z}_{2i}\}_i \end{bmatrix} \right\rangle_{Z_{q+m+n(K+1),n,K}} \le -\epsilon \left\| \begin{bmatrix} \mathbf{z}_1 \\ \{\mathbf{z}_{2i}\}_i \end{bmatrix} \right\|_{Z_{n,K}}^2 = -\epsilon \|\mathbf{z}\|_{Z_{n,K}}^2 .\end{aligned}$$

Before we begin, for convenience and efficiency of presentation, we will denote $m_0 := q + m + n(K+1)$ and

$$h := \begin{bmatrix} v \\ w \\ \mathbf{z}_1 \\ f \end{bmatrix}.$$

It may also be helpful to note that the quadratic form defined by a $\mathcal{P}_{\left[\begin{smallmatrix} D, & \{E_i\}_i \\ \{F_i\}_i, & \{G_{ij}\}_{ij}\end{smallmatrix}\right]}$ operator expands out as

$$\begin{aligned}(4.3)\qquad &\left\langle \begin{bmatrix} h \\ \{\mathbf{z}_{2i}\}_i \end{bmatrix}, \mathcal{P}_{\left[\begin{smallmatrix} D, & \{E_i\}_i \\ \{F_i\}_i, & \{G_{ij}\}_{ij}\end{smallmatrix}\right]} \begin{bmatrix} h \\ \{\mathbf{z}_{2i}\}_i \end{bmatrix} \right\rangle_{Z_{m_0,n,K}} \\ &= \tau_K h^T D h + \tau_K \sum_{i=1}^K \int_{-\tau_i}^0 h^T E_i(s) \mathbf{z}_{2i}(s) ds + \tau_K \sum_{i=1}^K \int_{-\tau_i}^0 \mathbf{z}_{2i}(s)^T E_i(s)^T h ds \\ &+ \tau_K \sum_{i=1}^K \int_{-\tau_i}^0 \mathbf{z}_{2i}(s)^T F_i(s) \mathbf{z}_{2i}(s) ds + \sum_{i,j=1}^K \int_{-\tau_i}^0 \int_{-\tau_j}^0 \mathbf{z}_{2i}(s)^T G_{ij}(s,\theta) \mathbf{z}_{2j}(\theta)\, d\theta ds.\end{aligned}$$

Our task, therefore, is simply to write all the terms we find in (4.2) in the form of (4.3) for an appropriate choice of matrix D and functions E_i, F_i, and G_{ij}. Fortunately, the most complicated part of this operation has already been completed. Indeed, from Theorem 5 in [16], we have that the first two terms can be represented as

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_{Z_{n,K}} + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_{Z_{n,K}} = \left\langle \begin{bmatrix} h \\ \{\mathbf{z}_{2i}\}_i \end{bmatrix}, \mathcal{D} \begin{bmatrix} h \\ \{\mathbf{z}_{2i}\}_i \end{bmatrix} \right\rangle_{Z_{m_0,n,K}},$$

where $\mathcal{D} := \mathcal{P}\left[\begin{smallmatrix} D_1, & \{E_{1i}\}_i \\ \{\dot{S}_i\}_i, & \{G_{ij}\}_i \end{smallmatrix}\right]$ (do not confuse this D_1 with the D_1 in (3.1)) and

$$D_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & C_0 + C_0^T & C_1 & \cdots & C_k \\ 0 & 0 & C_1^T & -S_1(-\tau_1) & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & C_{k^T} & 0 & 0 & -S_k(-\tau_K) \end{bmatrix},$$

$$C_0 := A_0 P + \tau_K \sum_{i=1}^{K} \left(A_i Q_i(-\tau_i)^T + \frac{1}{2} S_i(0) \right), \qquad C_i := \tau_K A_i S_i(-\tau_i), \quad i \in [K],$$

$$E_{1i}(s) := \begin{bmatrix} 0 & 0 & B_i(s)^T & 0 & \cdots & 0 \end{bmatrix}^T, \quad i \in [K],$$

$$B_i(s) := A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^{K} A_j R_{ji}(-\tau_j, s), \quad i \in [K],$$

$$G_{ij}(s,\theta) := \frac{\partial}{\partial s} R_{ij}(s,\theta) + \frac{\partial}{\partial \theta} R_{ji}(s,\theta)^T, \quad i,j \in [K].$$

Having already dealt with the most difficult terms, we now start with the easiest. Recalling that

$$(\mathcal{B}_1 w)(s) := \begin{bmatrix} B_1 w \\ 0 \end{bmatrix}, \quad (\mathcal{D}_1 w)(s) := \begin{bmatrix} D_1 w \end{bmatrix},$$

we have $\langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z = \tau_K z_1^T B_1 w$ and hence

$$\begin{aligned}
&\langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z - \gamma w^T w + v^T D_1 w + (D_1 w)^T v - \gamma v^T v \\
&= \tau_K \begin{bmatrix} v \\ w \\ \mathbf{z}_1 \\ \mathbf{z}_{21}(-\tau_1) \\ \vdots \\ \mathbf{z}_{2K}(-\tau_K) \end{bmatrix}^T \underbrace{\frac{1}{\tau_K} \begin{bmatrix} -\gamma I & D_1 & 0 & 0 & \dots & 0 \\ D_1^T & -\gamma I & \tau_K B_1^T & 0 & \dots & 0 \\ 0 & \tau_K B_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}}_{D_0} \begin{bmatrix} v \\ w \\ \mathbf{z}_1 \\ \mathbf{z}_{21}(-\tau_1) \\ \vdots \\ \mathbf{z}_{2K}(-\tau_K) \end{bmatrix} \\
&= \left\langle \begin{bmatrix} h \\ \{\mathbf{z}_{2i}\}_i \end{bmatrix}, \mathcal{P}\left[\begin{smallmatrix} D_0, & 0 \\ 0, & 0 \end{smallmatrix}\right] \begin{bmatrix} h \\ \{\mathbf{z}_{2i}\}_i \end{bmatrix} \right\rangle_{Z_{m_0,n,K}}.
\end{aligned}$$

Next, we consider the terms

$$v^T(\mathcal{CP}\mathbf{z}) + (\mathcal{CP}\mathbf{z})^T v.$$

If we recall that

$$\left(\mathcal{C} \begin{bmatrix} \psi \\ \mathbf{\Phi} \end{bmatrix} \right) := \left[C_0 \psi + \textstyle\sum_{i=1}^K C_i \phi_i(-\tau_i) \right],$$

then we have the expansion

$$\begin{aligned}
2v^T(\mathcal{C}\mathcal{P}\mathbf{z}) &= 2v^T\Bigg[C_0\left(P\mathbf{z}_1+\sum_{i=1}^K\int_{-\tau_i}^0 Q_i(s)\mathbf{z}_{2i}(s)ds\right)\\
&\quad+\sum_{i=1}^K C_i\left(\tau_K Q_i(-\tau_i)^T\mathbf{z}_1+\tau_K S_i(-\tau_i)\mathbf{z}_{2i}(-\tau_i)+\sum_{j=1}^K\int_{-\tau_j}^0 R_{ij}(-\tau_i,\theta)\mathbf{z}_{2j}(\theta)\,d\theta\right)\Bigg]\\
&=2v^T\Bigg[\left(C_0P+\sum_{i=1}^K\tau_K C_iQ_i(-\tau_i)^T\right)\mathbf{z}_1+\tau_K\sum_{i=1}^K C_iS_i(-\tau_i)\mathbf{z}_{2i}(-\tau_i)\\
&\quad+\sum_{i=1}^K\int_{-\tau_i}^0\left(C_0Q_i(s)\right)\mathbf{z}_{2i}(s)ds+\sum_{i=1}^K\int_{-\tau_i}^0\sum_{j=1}^K C_jR_{ji}(-\tau_j,s)\mathbf{z}_{2i}(s)\,ds\Bigg]\\
&=2v^T\tau_K\Bigg[\left(\frac{1}{\tau_K}C_0P+\sum_{i=1}^K C_iQ_i(-\tau_i)^T\right)\mathbf{z}_1+\sum_{i=1}^K C_iS_i(-\tau_i)\mathbf{z}_{2i}(-\tau_i)\\
&\quad+\frac{1}{\tau_K}\sum_{i=1}^K\int_{-\tau_i}^0\left(C_0Q_i(s)+\sum_{j=1}^K C_jR_{ji}(-\tau_j,s)\right)\mathbf{z}_{2i}(s)ds\Bigg]\\
&=\tau_K\begin{bmatrix}v\\ w\\ \mathbf{z}_1\\ \mathbf{z}_{21}(-\tau_1)\\ \vdots\\ \mathbf{z}_{2K}(-\tau_K)\end{bmatrix}^T\underbrace{\frac{1}{\tau_K}\begin{bmatrix}0 & *^T & *^T & *^T & \dots & *^T\\ 0 & 0 & *^T & *^T & \dots & *^T\\ (C_0P+\sum_{i=1}^K\tau_KC_iQ_i(-\tau_i)^T)^T & 0 & 0 & *^T & \dots & *^T\\ (\tau_KC_1S_1(-\tau_1))^T & 0 & 0 & 0 & \dots & *^T\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ (\tau_KC_KS_K(-\tau_K))^T & 0 & 0 & 0 & \dots & 0\end{bmatrix}}_{D_2}\begin{bmatrix}v\\ w\\ \mathbf{z}_1\\ \mathbf{z}_{21}(-\tau_1)\\ \vdots\\ \mathbf{z}_{2K}(-\tau_K)\end{bmatrix}\\
&\quad+2\tau_K\sum_{i=1}^K\int_{-\tau_i}^0\begin{bmatrix}v\\ w\\ \mathbf{z}_1\\ \mathbf{z}_{21}(-\tau_1)\\ \vdots\\ \mathbf{z}_{2K}(-\tau_K)\end{bmatrix}^T\underbrace{\frac{1}{\tau_K}\begin{bmatrix}C_0Q_i(s)+\sum_{j=1}^K C_jR_{ji}(-\tau_j,s)\\ 0\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix}}_{E_{2i}(s)}\mathbf{z}_{2i}(s)ds.
\end{aligned}$$

We therefore conclude that

$$v^T(\mathcal{C}\mathcal{P}\mathbf{z})+(\mathcal{C}\mathcal{P}\mathbf{z})^Tv=\left\langle\begin{bmatrix}h\\ \{\mathbf{z}_{2i}\}_i\end{bmatrix},\mathcal{P}_{\left[\begin{smallmatrix}D_2, & \{E_{2i}\}_i\\ 0, & 0\end{smallmatrix}\right]}\begin{bmatrix}h\\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}\right\rangle_{Z_{m_0,n,K}}.$$

We now examine the final set of terms which contain $\mathcal{Z}$:

$$\langle\mathcal{B}_2\mathcal{Z}\mathbf{z},\mathbf{z}\rangle_Z+\langle\mathbf{z},\mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z+v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z})+(\mathcal{D}_2\mathcal{Z}\mathbf{z})^Tv.$$

If we recall that

$$(\mathcal{B}_2u)(s):=\begin{bmatrix}B_2u\\ 0\end{bmatrix},\quad(\mathcal{D}_2u)(s):=\begin{bmatrix}D_2u\end{bmatrix},$$

then we have the expansion

$$
\begin{aligned}
&2\left\langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\right\rangle_Z + 2v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z})\\
&= 2\tau_K \mathbf{z}_1^T\left[B_2Z_0\mathbf{z}_1 + \sum_{i=1}^K B_2Z_{1i}\mathbf{z}_{2i}(-\tau_i) + \sum_{i=1}^K\int_{-\tau_i}^0 B_2Z_{2i}(s)\mathbf{z}_{2i}(s)ds\right]\\
&\quad + 2v^T\left[D_2Z_0\mathbf{z}_1 + \sum_{i=1}^K D_2Z_{1i}\mathbf{z}_{2i}(-\tau_i) + \sum_{i=1}^K\int_{-\tau_i}^0 D_2Z_{2i}(s)\mathbf{z}_{2i}(s)ds\right]\\
&= \tau_K\begin{bmatrix} v\\ w\\ \mathbf{z}_1\\ \mathbf{z}_{21}(-\tau_1)\\ \vdots\\ \mathbf{z}_{2K}(-\tau_K)\end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & *^T & *^T & *^T & \dots & *^T\\ 0 & 0 & *^T & *^T & \dots & *^T\\ (\frac{1}{\tau_K}D_2Z_0)^T & 0 & B_2Z_0+Z_0^TB_2^T & *^T & \dots & *^T\\ (\frac{1}{\tau_K}D_2Z_{11})^T & 0 & (B_2Z_{11})^T & 0 & \dots & *^T\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ (\frac{1}{\tau_K}D_2Z_{1K})^T & 0 & (B_2Z_{1K})^T & 0 & \dots & 0\end{bmatrix}}_{D_3}\begin{bmatrix} v\\ w\\ \mathbf{z}_1\\ \mathbf{z}_{21}(-\tau_1)\\ \vdots\\ \mathbf{z}_{2K}(-\tau_K)\end{bmatrix}\\
&\quad + 2\tau_K\sum_{i=1}^K\int_{-\tau_i}^0 \begin{bmatrix} v(t)\\ w(t)\\ \mathbf{z}_1(t)\\ \mathbf{z}_{21}(t,-\tau_1)\\ \vdots\\ \mathbf{z}_{2K}(t,-\tau_K)\end{bmatrix}^T \underbrace{\frac{1}{\tau_K}\begin{bmatrix} D_2Z_{2i}(s)\\ 0\\ \tau_KB_2Z_{2i}(s)\\ 0\\ \vdots\\ 0\end{bmatrix}}_{E_{3i}(s)}\mathbf{z}_{2i}(s)ds.
\end{aligned}
$$

We therefore conclude that

$$
\langle \mathcal{B}_2\mathcal{Z}\mathbf{z},\mathbf{z}\rangle + \langle \mathbf{z},\mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^Tv = \left\langle \begin{bmatrix} h\\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}, \mathcal{P}_{\left[\begin{smallmatrix} D_3, & \{E_{3i}\}_i\\ 0, & 0\end{smallmatrix}\right]}\begin{bmatrix} h\\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}\right\rangle_{Z_{m_0,n,K}}.
$$

Summing all the terms, we have

$$D = D_0 + D_1 + D_2 + D_3$$

and

$$E_i(s) = E_{1i}(s) + E_{2i}(s) + E_{3i}(s).$$

We conclude, therefore, that for any $\mathbf{z}\in X$,

$$
\begin{aligned}
&\langle \mathcal{A}\mathcal{P}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w,\mathbf{z}\rangle_Z - \gamma\left\|w\right\|^2\\
&\quad + v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^Tv + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^Tv + v^T(D_1w) + (D_1w)^Tv - \gamma\left\|v\right\|^2\\
&= \left\langle \begin{bmatrix} h\\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}, \mathcal{P}_{\left[\begin{smallmatrix} D, & \{E_i\}_i\\ \{\dot S_i\}_i, & \{G_{ij}\}_{i,j}\end{smallmatrix}\right]}\begin{bmatrix} h\\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}\right\rangle_{Z_{m_0,n,K}} \le -\epsilon\left\|\begin{bmatrix}\mathbf{z}_1\\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}\right\|^2_{Z_{n,K}} = -\epsilon\left\|\mathbf{z}\right\|^2_{Z_{n,K}}.
\end{aligned}
$$

Thus, by Lemma 4 and Theorem 3, we have that for any $w\in L_2$, if $x(t)$ and $y(t)$ satisfy (4.1), then $\left\|y\right\|_{L_2} \le \gamma\left\|w\right\|_{L_2}$. ∎

Theorem 5 provides a convex formulation of the controller synthesis problem for systems with multiple delays. However, the theorem does not provide a way to enforce the operator inequalities or reconstruct the optimal controller. In section 5 we will review how the operator inequalities can be represented using LMIs. In sections 6 and 7, we discuss how to invert operators of the $\mathcal{P}_{\left[\begin{smallmatrix} P, & \mathbf{Q}\\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]}$ class and reconstruct the controller gains in a numerically reliable manner.

5. Enforcing operator inequalities in the *PQRS* framework. The problem of enforcing operator positivity on $Z_{m,n,K}$ in the $\mathcal{P}\left[\begin{smallmatrix}P, & \mathbf{Q}\\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]$ framework was solved in [16] by using a two-step approach. First, we construct an operator $\mathcal{P}\left[\begin{smallmatrix}\tilde{P}, & \hat{\tilde{Q}}\\ \tilde{S}, & \tilde{R}\end{smallmatrix}\right]$ whose positivity on $Z_{m,nK,1}$ is equivalent to positivity of the original operator on $Z_{m,n,K}$. Then, assuming that $\tilde{Q}, \tilde{R}, \tilde{S}$ are polynomials, we give an LMI condition on $\tilde{P}$ and the coefficients of $\tilde{Q}, \tilde{R}, \tilde{S}$, which ensures positivity of $\mathcal{P}\left[\begin{smallmatrix}\tilde{P}, & \hat{\tilde{Q}}\\ \tilde{S}, & \tilde{R}\end{smallmatrix}\right]$ on $Z_{m,nK,1}$. Because the transformation from $\{P, Q_i, R_{ij}, S_i\}$ to $\{\tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}\}$ is linear, if Q_i, R_{ij}, S_i are polynomials, the result is an LMI constraint on the coefficients of these original polynomials. For ease of implementation, these two results are combined into a single MATLAB function, which is described in section 9.

First, we give the following transformation. Specifically, we say that

$$\{\tilde{P}, \tilde{Q}, \tilde{S}, \tilde{R}\} := \mathcal{L}_1(P, \{Q_i\}_i, \{S_i\}_i, \{R_{ij}\}_{ij}) \tag{5.1}$$

if $a_i = \frac{\tau_i}{\tau_K}$, $\tilde{P} = P$, and

$$\tilde{Q}(s) := \begin{bmatrix}\sqrt{a_1}Q_1(a_1 s) & \cdots & \sqrt{a_K}Q_K(a_K s)\end{bmatrix}, \quad \tilde{S}(s) := \begin{bmatrix}S_1(a_1 s) & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & S_K(a_K s)\end{bmatrix},$$

$$\tilde{R}(s,\theta) := \begin{bmatrix}\sqrt{a_1 a_1}R_{11}(sa_1, \theta a_1) & \cdots & \sqrt{a_1 a_K}R_{1K}(sa_1, \theta a_K)\\ \vdots & \cdots & \vdots\\ \sqrt{a_K a_1}R_{K1}(sa_K, \theta a_1) & \cdots & \sqrt{a_K a_K}R_{KK}(sa_K, \theta a_K)\end{bmatrix}.$$

Then we have the following result [16].

LEMMA 6. *Let* $\{\tilde{P}, \tilde{Q}, \tilde{S}, \tilde{R}\} := \mathcal{L}_1(P, \{Q_i\}_i, \{S_i\}_i, \{R_{ij}\}_{ij})$. *Then*

$$\left\langle \begin{bmatrix}x\\ \mathbf{\Phi}\end{bmatrix}, \mathcal{P}\left[\begin{smallmatrix}P, & \mathbf{Q}\\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]\begin{bmatrix}x\\ \mathbf{\Phi}\end{bmatrix}\right\rangle_{Z_{m,n,K}} \geq \alpha \left\| \begin{bmatrix}x\\ \mathbf{\Phi}\end{bmatrix}\right\|_{Z_{m,n,K}}$$

for all $\begin{bmatrix}x\\ \mathbf{\Phi}\end{bmatrix} \in Z_{m,n,K}$ *if and only if*

$$\left\langle \begin{bmatrix}x\\ \tilde{\phi}\end{bmatrix}, \mathcal{P}\left[\begin{smallmatrix}\tilde{P}, & \tilde{Q}\\ \tilde{S}, & \tilde{R}\end{smallmatrix}\right]\begin{bmatrix}x\\ \tilde{\phi}\end{bmatrix}\right\rangle_{Z_{m,nK,1}} \geq \alpha \left\| \begin{bmatrix}\tilde{x}\\ \tilde{\phi}\end{bmatrix}\right\|_{Z_{m,nK,1}}$$

for all $\begin{bmatrix}\tilde{x}\\ \tilde{\phi}\end{bmatrix} \in Z_{m,nK,1}$.

To enforce positivity of $\mathcal{P}\left[\begin{smallmatrix}\tilde{P}, & \tilde{Q}\\ \tilde{S}, & \tilde{R}\end{smallmatrix}\right]$ on $Z_{m,nK,1}$ as an LMI, we use the following result [16].

THEOREM 7. *For any functions* $Y_1 : [-\tau_K, 0] \to \mathbb{R}^{m_1 \times n}$ *and* $Y_2 : [-\tau_K, 0] \times [-\tau_K, 0] \to \mathbb{R}^{m_2 \times n}$, *square integrable on* $[-\tau_K, 0]$ *with* $g(s) \geq 0$ *for* $s \in [-\tau_K, 0]$, *suppose that*

$$P = M_{11} \cdot \frac{1}{\tau_K}\int_{-\tau_K}^{0} g(s)ds, \qquad S(s) = \frac{1}{\tau_K} g(s)Y_1(s)^T M_{22} Y_1(s),$$

$$Q(s) = \frac{1}{\tau_K}\left(g(s)M_{12}Y_1(s) + \int_{-\tau_K}^{0} g(\eta)M_{13}Y_2(\eta,s)d\eta \right),$$

$$R(s,\theta) = g(s)Y_1(s)^T M_{23} Y_2(s,\theta) + g(\theta)Y_2(\theta,s)^T M_{32} Y_1(\theta) + \int_{-\tau_K}^{0} g(\eta) Y_2(\eta,s)^T M_{33} Y_2(\eta,\theta) d\eta,$$

where $M_{11} \in \mathbb{R}^{m\times m}$, $M_{22} \in \mathbb{R}^{m_1 \times m_1}$, $M_{33} \in \mathbb{R}^{m_2\times m_2}$, *and*

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \geq 0.$$

Then $\left\langle \mathbf{x}, \mathcal{P}\left[\begin{smallmatrix} P, & Q \\ S, & R\end{smallmatrix}\right]\mathbf{x}\right\rangle_{Z_{m,n,1}} \geq 0$ *for all* $\mathbf{x} \in Z_{m,n,1}$.

For notational convenience, we use $\{P, Q, S, R\} \in \Xi_{d,m,n}$ to denote the LMI constraints associated with Theorem 7 as

$$\Xi_{d,m,n} := \left\{ \{P,Q,S,R\} : \begin{smallmatrix} \{P,Q,S,R\} = \{P_1,Q_1,S_1,R_1\} + \{P_2,Q_2,S_2,R_2\}, \\ \text{where } \{P_1,Q_1,S_1,R_1\} \text{ and } \{P_2,Q_2,S_2,R_2\} \text{ satisfy} \\ \text{Thm. 7 with } g=1 \text{ and } g = -s(s+\tau_K), \text{ respectively} \end{smallmatrix} \right\}.$$

We now have the single unified result.

COROLLARY 8. *Suppose there exist* $d \in \mathbb{N}$, *constant* $\epsilon > 0$, *matrix* $P \in \mathbb{R}^{m\times m}$, *and polynomials* Q_i, S_i, R_{ij} *for* $i, j \in [K]$ *such that*

$$\mathcal{L}_1(P, \{Q_i\}_i, \{S_i\}_i, \{R_{ij}\}_{ij}) \in \Xi_{d,m,nK}.$$

Then $\left\langle \mathbf{x}, \mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]\mathbf{x}\right\rangle_{Z_{m,n,K}} \geq 0$ *for all* $\mathbf{x} \in Z_{m,n,K}$.

A more detailed discussion of these LMI-based methods can be found in [16].

6. An analytic inverse of *PQRS* operators. Having taken Q_i, R_{ij}, S_i to be polynomials and having given an LMI which enforces strict positivity of the operator $\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]$, we now give an analytical representation of the inverse of operators of this class. The inverse of $\mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]$ is also of the form $\mathcal{P}\left[\begin{smallmatrix} \hat{P}, & \hat{\mathbf{Q}} \\ \hat{\mathbf{S}}, & \hat{\mathbf{R}}\end{smallmatrix}\right]$, where expressions for the matrix $\hat{P}$ and functions $\hat{Q}_i, \hat{R}_{ij}, \hat{S}_i$ are given in the following theorem, which is a generalization of the result in [12] to the case of multiple delays. In this result, we first extract the coefficients of the polynomials Q_i and R_{ij} as $Q_i(s) = H_i Z(s)$ and $R_{ij}(s,\theta) = Z(s)^T \Gamma_{ij} Z(\theta)$, where $Z(s)$ is a vector of bases for vector-valued polynomials (typically a monomial basis). The theorem then gives an expression for the coefficients of $\hat{Q}_i$ and $\hat{R}_{ij}$ using a similar representation. Note that the results of the theorem are still valid even if the basis functions in $Z(s)$ are not monomials or even polynomials.

THEOREM 9. *Suppose that* $Q_i(s) = H_i Z(s)$ *and* $R_{ij}(s,\theta) = Z(s)^T \Gamma_{ij} Z(\theta)$ *and that* $\mathcal{P} := \mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]$ *is a coercive operator where* $\mathcal{P} : X \to X$ *and* $\mathcal{P} = \mathcal{P}^*$. *Define*

$$H = \begin{bmatrix} H_1 & \dots & H_K \end{bmatrix} \qquad \text{and} \qquad \Gamma = \begin{bmatrix} \Gamma_{11} & \dots & \Gamma_{1K} \\ \vdots & & \vdots \\ \Gamma_{K,1} & \dots & \Gamma_{K,K} \end{bmatrix}.$$

Now let

$$K_i = \int_{-\tau_i}^{0} Z(s)S_i(s)^{-1}Z(s)^T ds, \quad K = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & K_K \end{bmatrix},$$

$$\hat{H} = P^{-1}H\left(KH^TP^{-1}H - I - K\Gamma\right)^{-1}, \quad \hat{\Gamma} = -(\hat{H}^TH + \Gamma)(I + K\Gamma)^{-1},$$

$$\begin{bmatrix}\hat{H}_1 & \dots & \hat{H}_K\end{bmatrix} = \hat{H}, \qquad \begin{bmatrix} \hat{\Gamma}_{11} & \dots & \hat{\Gamma}_{1K} \\ \vdots & & \vdots \\ \hat{\Gamma}_{K,1} & \dots & \hat{\Gamma}_{K,K} \end{bmatrix} = \hat{\Gamma}.$$

If we define

$$\hat{P} = \left(I - \hat{H}KH^T\right)P^{-1}, \quad \hat{Q}_i(s) = \hat{H}_iZ(s)S_i(s)^{-1},$$
$$\hat{S}_i(s) = S_i(s)^{-1}, \quad \hat{R}_{ij}(s,\theta) = S_i(s)^{-1}Z(s)^T\hat{\Gamma}_{ij}Z(\theta)S_j(\theta)^{-1},$$

then for

$$\hat{\mathcal{P}} := \mathcal{P}\begin{bmatrix} \hat{P}, & \left\{\frac{1}{\tau_K}\hat{Q}_i\right\}_i \\ \left\{\frac{1}{\tau_K^2}\hat{S}_i\right\}_i, & \left\{\frac{1}{\tau_K}\hat{R}_{ij}\right\}_{i,j} \end{bmatrix},$$

we have that $\hat{\mathcal{P}} = \hat{\mathcal{P}}^*$, $\hat{\mathcal{P}} : X \to X$, *and* $\hat{\mathcal{P}}\mathcal{P}\mathbf{x} = \mathcal{P}\hat{\mathcal{P}}\mathbf{x} = \mathbf{x}$ *for any* $\mathbf{x} \in Z_{m,n,K}$.

Proof. One approach to proving this theorem is to let $\hat{\mathcal{P}}$ be as defined and show that this implies $\hat{\mathcal{P}}\mathcal{P}\mathbf{x} = \mathbf{x}$ for any $x \in Z_{m,n,K}$. Although this is clearly the most direct path towards establishing the theorem statement, it is not the easiest to understand, due to the intensely algebraic nature of the calculations. Thus, in order to help the reader understand the derivation of the results and to encourage generalization, we will, as much as possible, show how these results were obtained. Specifically, we start by assuming that the inverse has the following structure:

$$\left(\hat{\mathcal{P}}\begin{bmatrix}x\\ \mathbf{\Phi}\end{bmatrix}\right)(s) := \begin{bmatrix} \hat{P}x + \frac{1}{\tau_K}\sum_{i=1}^K \int_{-\tau_i}^0 \hat{Q}_i(s)\phi_i(s)ds \\ \left\{\hat{Q}_i(s)^Tx + \frac{1}{\tau_K}\hat{S}_i(s)\phi_i(s) + \frac{1}{\tau_K}\sum_{j=1}^K \int_{-\tau_j}^0 \hat{R}_{ij}(s,\theta)\phi_j(\theta)\,d\theta\right\}_i \end{bmatrix}.$$

Our approach to finding $\hat{P}, \hat{Q}_i, \hat{S}_i$, and $\hat{R}_{ij}$ is then to calculate $\mathbf{y} = \hat{\mathcal{P}}\mathcal{P}\mathbf{x}$ and use the five equality constraints implied by $\mathbf{y} = \mathbf{x}$ to solve for the variables $\hat{P}, \hat{Q}_i, \hat{S}_i$, and $\hat{R}_{ij}$. To do this, we define

$$\mathbf{y}(s) := \begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix} := \begin{bmatrix} y \\ \{\psi_i\}_i\end{bmatrix} := \left(\hat{\mathcal{P}}\mathcal{P}\begin{bmatrix}x\\ \mathbf{\Phi}\end{bmatrix}\right)(s)$$

and start by expanding the first term $y = \mathbf{y}_1$:

$$\begin{aligned}
y &= \hat{P}Px + \sum_{i=1}^K \int_{-\tau_i}^0 \hat{P}Q_i(s)\phi_i(s)ds + \sum_{i=1}^K \int_{-\tau_i}^0 \hat{Q}_i(s)Q_i(s)^T x ds \\
&\quad + \sum_{i=1}^K \int_{-\tau_i}^0 \hat{Q}_i(s)S_i(s)\phi_i(s)ds + \sum_{i=1}^K \int_{-\tau_i}^0 \hat{Q}_i(s) \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)\, d\theta ds \\
&= \left(\hat{P}P + \sum_{i=1}^K \int_{-\tau_i}^0 \hat{Q}_i(s)Q_i(s)^T ds\right) x + \sum_{i=1}^K \int_{-\tau_i}^0 \left(\hat{P}Q_i(s) + \hat{Q}_i(s)S_i(s)\right)\phi_i(s)ds \\
&\quad + \sum_{j=i}^K \int_{-\tau_i}^0 \sum_{j=1}^K \int_{-\tau_j}^0 \hat{Q}_j(\theta)R_{ji}(\theta,s)\phi_i(s)\, dsd\theta \\
&= \left(\hat{P}P + \sum_{i=1}^K \int_{-\tau_i}^0 \hat{Q}_i(s)Q_i(s)^T ds\right) x \\
&\quad + \sum_{i=1}^K \int_{-\tau_i}^0 \left(\hat{P}Q_i(s) + \hat{Q}_i(s)S_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{Q}_j(\theta)R_{ji}(\theta,s)d\theta\right)\phi_i(s)ds.
\end{aligned}$$

From this expansion, we conclude that a sufficient condition for $y = x$ (i.e., $\mathbf{y}_1 = \mathbf{x}_1$) is that

$$\hat{P}P + \sum_{i=1}^K \int_{-\tau_i}^0 \hat{Q}_i(s)Q_i(s)^T ds = I$$

and

$$\hat{P}Q_i(s) + \hat{Q}_i(s)S_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{Q}_j(\theta)R_{ji}(\theta,s)d\theta = 0$$

for all $i \in [K]$. This provides two sets of equality constraints which will help us determine $\hat{P}$ and $\hat{Q}$. We next examine the more complicated terms $\mathbf{y}_2 = \{\psi_i\}_i$:

$$\begin{aligned}
\psi_i(s) &= \hat{Q}_i(s)^T P x + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{Q}_i(s)^T Q_j(\theta)\phi_j(\theta) d\theta + \hat{S}_i(s) Q_i(s)^T x + \hat{S}_i(s) S_i(s)\phi_i(s)\\
&\quad + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{S}_i(s) R_{ij}(s,\theta)\phi_j(\theta)\, d\theta + \left(\sum_{j=1}^K \int_{-\tau_j}^0 \hat{R}_{ij}(s,\theta) Q_j(\theta)^T d\theta\right) x\\
&\quad + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{R}_{ij}(s,\theta) S_j(\theta)\phi_j(\theta) d\theta + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{R}_{ij}(s,\theta) \sum_{k=1}^K \int_{-\tau_k}^0 R_{jk}(\theta,\eta)\phi_k(\eta)\, d\eta d\theta\\
&= \left(\hat{Q}_i(s)^T P + \hat{S}_i(s) Q_i(s)^T + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{R}_{ij}(s,\theta) Q_j(\theta)^T d\theta\right) x + \hat{S}_i(s) S_i(s)\phi_i(s)\\
&\quad + \sum_{j=1}^K \int_{-\tau_j}^0 \left(\hat{Q}_i(s)^T Q_j(\theta) + \hat{S}_i(s) R_{ij}(s,\theta) + \hat{R}_{ij}(s,\theta) S_j(\theta)\right)\phi_j(\theta)\, d\theta\\
&\quad + \sum_{j=1}^K \int_{-\tau_j}^0 \sum_{k=1}^K \int_{-\tau_k}^0 \hat{R}_{ik}(s,\eta) R_{kj}(\eta,\theta)\, d\eta \phi_j(\theta) d\theta\\
&= \left(\hat{Q}_i(s)^T P + \hat{S}_i(s) Q_i(s)^T + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{R}_{ij}(s,\theta) Q_j(\theta)^T d\theta\right) x + \hat{S}_i(s) S_i(s)\phi_i(s)\\
&+\sum_{j=1}^K \int_{-\tau_j}^0 \left(\hat{Q}_i(s)^T Q_j(\theta) + \hat{S}_i(s) R_{ij}(s,\theta) + \hat{R}_{ij}(s,\theta) S_j(\theta) + \sum_{k=1}^K \int_{-\tau_k}^0 \hat{R}_{ik}(s,\eta) R_{kj}(\eta,\theta)\, d\eta\right)\phi_j(\theta)\, d\theta.
\end{aligned}$$

From this expansion, we conclude that a sufficient condition for $\psi_i(s) = \phi_i(s)$ (i.e., $\mathbf{y}_2 = \mathbf{x}_2$) is that

$$\begin{aligned}
&\hat{S}_i(s) S_i(s) = I,\\
&\hat{Q}_i(s)^T P + \hat{S}_i(s) Q_i(s)^T + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{R}_{ij}(s,\theta) Q_j(\theta)^T d\theta = 0,\\
&\hat{Q}_i(s)^T Q_j(\theta) + \hat{S}_i(s) R_{ij}(s,\theta) + \hat{R}_{ij}(s,\theta) S_j(\theta) + \sum_{k=1}^K \int_{-\tau_k}^0 \hat{R}_{ik}(s,\eta) R_{kj}(\eta,\theta)\, d\eta = 0.
\end{aligned}$$

We now have five constraints which $\hat{P}, \hat{Q}_i, \hat{S}_i$, and $\hat{R}_{ij}$ must satisfy if $\hat{\mathcal{P}}$ is to be an inverse of $\mathcal{P}$:

$$\begin{aligned}
&\hat{S}_i(s) S_i(s) = I, \qquad \hat{P} P + \sum_{i=1}^K \int_{-\tau_i}^0 \hat{Q}_i(s) Q_i(s)^T ds = I,\\
&\hat{P} Q_i(s) + \hat{Q}_i(s) S_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{Q}_j(\theta) R_{ji}(\theta,s) d\theta = 0 \qquad \forall i \in [K],\\
&\hat{Q}_i(s)^T P + \hat{S}_i(s) Q_i(s)^T + \sum_{j=1}^K \int_{-\tau_j}^0 \hat{R}_{ij}(s,\theta) Q_j(\theta)^T d\theta = 0 \quad \forall i \in [K],\\
&\hat{Q}_i(s)^T Q_j(\theta) + \hat{S}_i(s) R_{ij}(s,\theta) + \hat{R}_{ij}(s,\theta) S_j(\theta) + \sum_{k=1}^K \int_{-\tau_k}^0 \hat{R}_{ik}(s,\eta) R_{kj}(\eta,\theta)\, d\eta = 0 \quad \forall i \in K.
\end{aligned}$$

If all five constraints are satisfied, we can conclude that $\hat{\mathcal{P}}\mathcal{P}\mathbf{x} = \mathbf{x}$. Clearly, the first constraint is satisfied if $\hat{S}_i(s) = S_i(s)^{-1}$. We now parameterize the variables $\hat{Q}_i$ and

$\hat{R}_{ij}$ using parameters $\hat{H}_i$ and $\hat{\Gamma}_{ij}$ as

$$\hat{Q}_i(s) = \hat{H}_i Z(s)\hat{S}_i(s), \;\; \hat{R}_{ij}(s,\theta) = \hat{S}_i(s)^T Z(s)\hat{\Gamma}_{ij} Z(\theta)\hat{S}_j(\theta)$$

and examine the second constraint, which is equivalent to

$$\hat{P}P = I - \sum_{i=1}^{K}\int_{-\tau_i}^{0} \hat{Q}_i(s)Z(s)^T ds H_i^T.$$

Solving this expression for $\hat{P}$ in terms of $\hat{H}$, we obtain

$$\begin{aligned}\hat{P} &= \left(I - \sum_{i=1}^{K}\int_{-\tau_i}^{0} \hat{Q}_i(s)Z(s)^T ds H_i^T\right)P^{-1}\\ &= \left(I - \sum_{i=1}^{K}\hat{H}_i\left(\int_{-\tau_i}^{0} Z(s)\hat{S}_i(s)Z(s)^T ds\right)H_i^T\right)P^{-1}\\ &= \left(I - \sum_{i=1}^{K}\hat{H}_i K_i H_i^T\right)P^{-1} = \left(I - \hat{H}KH^T\right)P^{-1}.\end{aligned}$$

We now examine the third set of constraints, indexed by $i \in [K]$:

$$\begin{aligned}&\hat{P}Q_i(s) + \hat{Q}_i(s)S_i(s) + \sum_{j=1}^{K}\int_{-\tau_j}^{0}\hat{Q}_j(\theta)R_{ji}(\theta,s)d\theta\\ &= \hat{P}H_i Z(s) + \hat{H}_i Z(s) + \sum_{j=1}^{K}\hat{H}_j\int_{-\tau_j}^{0} Z(\theta)\hat{S}_j(\theta)Z(\theta)^T\Gamma_{ji}Z(s)d\theta\\ &= \left(\hat{P}H_i + \hat{H}_i + \sum_{j=1}^{K}\hat{H}_j K_j \Gamma_{ji}\right)Z(s) = 0.\end{aligned}$$

Combining these K constraints into a single expression yields

$$\hat{P}H + \hat{H} + \hat{H}K\Gamma = 0.$$

Substituting our expression for $\hat{P}$ now yields the constraint

$$\begin{aligned}&\hat{P}H + \hat{H} + \hat{H}K\Gamma = \left(I - \hat{H}KH^T\right)P^{-1}H + \hat{H} + \hat{H}K\Gamma\\ &= P^{-1}H - \hat{H}\left(KH^TP^{-1}H - I - K\Gamma\right) = 0,\end{aligned}$$

which is equivalent to

$$\hat{H} = P^{-1}H\left(KH^TP^{-1}H - I - K\Gamma\right)^{-1}.$$

Thus, we have found an expression for $\hat{H}$. Furthermore, since we have already found an expression for $\hat{\mathcal{P}}$ in terms of $\hat{H}$, all that now remains is to solve for $\hat{\Gamma}$. For this result, we turn to the fifth set of constraints,

$$\begin{aligned}
&\hat{Q}_i(s)^T Q_j(\theta) + \hat{S}_i(s) R_{ij}(s,\theta) + \hat{R}_{ij}(s,\theta) S_j(\theta) + \sum_{k=1}^{K} \int_{-\tau_k}^{0} \hat{R}_{ik}(s,\eta) R_{kj}(\eta,\theta)\, d\eta \\
&= \hat{S}_i(s)^T Z(s)^T \hat{H}_i^T H_j Z(\theta) + \hat{S}_i(s) Z(s)^T \Gamma_{ij} Z(\theta) + \hat{S}_i(s)^T Z(s) \hat{\Gamma}_{ij} Z(\theta) \\
&\quad + \sum_{k=1}^{K} \int_{-\tau_k}^{0} \hat{S}_i(s)^T Z(s)^T \hat{\Gamma}_{ik} Z(\eta) \hat{S}_k(\eta) Z(\eta) d\eta \Gamma_{kj} Z(\theta) \\
&= \hat{S}_i(s)^T Z(s)^T \left(\hat{H}_i^T H_j + \Gamma_{ij} + \hat{\Gamma}_{ij} + \sum_{k=1}^{K} \hat{\Gamma}_{ik} K_k \Gamma_{kj} \right) Z(\theta) = 0 \qquad \forall i,j \in [K].
\end{aligned}$$

Combining these K^2 constraints into a single expression yields

$$\hat{H}^T H + \Gamma + \hat{\Gamma} + \hat{\Gamma} K \Gamma = 0.$$

Solving this expression for $\hat{\Gamma}$, we find

$$\hat{\Gamma} = -(\hat{H}^T H + \Gamma)(I + K\Gamma)^{-1}.$$

We have now derived expressions for $\hat{P}$, $\hat{S}$, $\hat{H}$, and $\hat{\Gamma}$. However, to show that $\hat{\mathcal{P}}\mathcal{P}\mathbf{x} = \mathbf{x}$, we must verify that the fourth constraint is also satisfied. Namely,

$$\begin{aligned}
&\hat{Q}_i(s)^T P + \hat{S}_i(s) Q_i(s)^T + \sum_{j=1}^{K} \int_{-\tau_j}^{0} \hat{R}_{ij}(s,\theta) Q_j(\theta)^T d\theta \\
&= \hat{S}_i(s) Z(s)^T \hat{H}_i^T P + \hat{S}_i(s) Z(s)^T H_i^T + \sum_{j=1}^{K} \int_{-\tau_j}^{0} \hat{S}_i(s) Z(s)^T \hat{\Gamma}_{ij} Z(\theta) \hat{S}_j(\theta) Z(\theta)^T d\theta H_j^T \\
&= \hat{S}_i(s) Z(s)^T \left(\hat{H}_i^T P + H_i^T + \sum_{j=1}^{K} \hat{\Gamma}_{ij} K_j H_j^T \right) = 0
\end{aligned}$$

for all $i \in [K]$, which is satisfied if

$$\hat{H}_i^T P + H_i^T + \sum_{j=1}^{K} \hat{\Gamma}_{ij} K_j H_j^T = 0 \qquad \forall i \in [K].$$

Combining these K constraints into a single expression yields

$$\hat{H}^T P + H^T + \hat{\Gamma} K H^T = 0.$$

To verify that this is satisfied, we let $L = H^T P^{-1} H$ and $T = (I + K\Gamma - KL)^{-1}$. Then $\hat{H} = -P^{-1} H T$ and thus

$$\hat{H}^T P + H^T + \hat{\Gamma} K H^T = -T^T H^T P^{-1} P + H^T + \hat{\Gamma} K H^T = \left(-T^T + I + \hat{\Gamma} K \right) H^T.$$

Substituting in $\hat{\Gamma} = (T^T L - \Gamma)(I + K\Gamma)^{-1}$, and observing that Γ, L, and K are symmetric (for Γ, this is due to $\mathcal{P} = \mathcal{P}^*$), we have that

$$\begin{aligned}
&- T^T + I + \hat{\Gamma} K = I - T^T + (T^T L - \Gamma)(I + K\Gamma)^{-1} K \\
&= I - T^T (I + \Gamma K)(I + \Gamma K)^{-1} + (T^T L - \Gamma) K (I + \Gamma K)^{-1} \\
&= I - T^T (I + \Gamma K)(I + \Gamma K)^{-1} + T^T L K (I + \Gamma K)^{-1} - \Gamma K (I + \Gamma K)^{-1} \\
&= I - T^T (I + \Gamma K + L K)(I + \Gamma K)^{-1} - \Gamma K (I + \Gamma K)^{-1} \\
&= I - (I + \Gamma K)^{-1} - \Gamma K (I + \Gamma K)^{-1} = I - (I + \Gamma K)(I + \Gamma K)^{-1} = I - I = 0.
\end{aligned}$$

In a similar manner, it can be shown that $\mathcal{P}\hat{\mathcal{P}}\mathbf{x} = \mathbf{x}$. It can be likewise shown directly that $\hat{\mathcal{P}} : X \to X$ through a lengthy series of algebraic manipulations. However, this property is also established by Theorem 2. ∎

7. Controller reconstruction and numerical implementation. In this section, we reconstruct the controller using $\mathcal{Z}$ and $\mathcal{P}^{-1}$ and explain how this can be implemented numerically. First, we have the following obvious result.

LEMMA 10. *Suppose that*

$$\left(\mathcal{Z}\begin{bmatrix} y \\ \mathbf{\Psi}\end{bmatrix}\right) := \left[Z_0 y + \sum_{i=1}^K Z_{1i}\psi_i(-\tau_i) + \sum_i \int_{-\tau_i}^0 Z_{2i}(s)\psi_i(s)ds\right] \tag{7.1}$$

and

$$\left(\hat{\mathcal{P}}\begin{bmatrix} x \\ \mathbf{\Phi}\end{bmatrix}\right)(s) := \begin{bmatrix} \hat{P}x + \frac{1}{\tau_K}\sum_{i=1}^K \int_{-\tau_i}^0 \hat{Q}_i(s)\phi_i(s)ds \\ \hat{Q}_i(s)^T x + \frac{1}{\tau_K}\hat{S}_i(s)\phi_i(s) + \frac{1}{\tau_K}\sum_{j=1}^K \int_{-\tau_j}^0 \hat{R}_{ij}(s,\theta)\phi_j(\theta)\,d\theta \end{bmatrix}.$$

Then if $u(t) = \mathcal{Z}\hat{P}\mathbf{x}(t)$,

$$u(t) = K_0 x(t) + \sum_{i=1}^K K_{1i}x(t-\tau_i) + \sum_{i=1}^K \int_{-\tau_i}^0 K_{2i}(s)x(t+s)ds,$$

where

$$K_0 = Z_0\hat{P} + \sum_{j=1}^K \left(Z_{1j}\hat{Q}_j(-\tau_j)^T + \int_{-\tau_j}^0 Z_{2j}(s)\hat{Q}_j(s)^T ds\right),$$

$$K_{1i} = \frac{1}{\tau_K}Z_{1i}\hat{S}_i(-\tau_i),$$

$$K_{2i}(s) = \frac{1}{\tau_K}\left(Z_0\hat{Q}_i(s) + Z_{2i}(s)\hat{S}_i(s) + \sum_{j=1}^K\left(Z_{1j}\hat{R}_{ji}(-\tau_j,s) + \int_{-\tau_j}^0 Z_{2j}(\theta)\hat{R}_{ji}(\theta,s)d\theta\right)\right).$$

Proof. Suppose that $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1} = \mathcal{Z}\hat{\mathcal{P}}$. Then

$$
\begin{aligned}
\mathcal{Z}\hat{\mathcal{P}} \begin{bmatrix} x \\ \mathbf{\Phi} \end{bmatrix} &= Z_0 \left(\hat{P}x + \frac{1}{\tau_K} \sum_{i=1}^{K} \int_{-\tau_i}^{0} \hat{Q}_i(s)\phi_i(s)ds \right) \\
&\quad + \sum_{i=1}^{K} Z_{1i} \left(\hat{Q}_i(-\tau_i)^T x + \frac{1}{\tau_K} \hat{S}_i(-\tau_i)\phi_i(-\tau_i) + \frac{1}{\tau_K} \sum_{j=1}^{K} \int_{-\tau_j}^{0} \hat{R}_{ij}(-\tau_i, \theta)\phi_j(\theta)\, d\theta \right) \\
&\quad + \sum_i \int_{-\tau_i}^{0} Z_{2i}(s) \left(\hat{Q}_i(s)^T x + \frac{1}{\tau_K} \hat{S}_i(s)\phi_i(s) + \frac{1}{\tau_K} \sum_{j=1}^{K} \int_{\theta=-\tau_j}^{0} \hat{R}_{ij}(s,\theta)\phi_j(\theta)\, d\theta \right) ds \\
&= Z_0\hat{P}x + \frac{1}{\tau_K} \sum_{j=1}^{K} \int_{-\tau_j}^{0} Z_0 \hat{Q}_j(s)\phi_j(s)ds + \sum_{i=1}^{K} Z_{1i}\hat{Q}_i(-\tau_i)^T x + \frac{1}{\tau_K} \sum_{i=1}^{K} Z_{1i}\hat{S}_i(-\tau_i)\phi_i(-\tau_i) \\
&\quad + \frac{1}{\tau_K} \sum_{j=1}^{K} \int_{-\tau_j}^{0} \sum_{i=1}^{K} Z_{1i}\hat{R}_{ij}(-\tau_i, s)\phi_j(s)\, ds + \sum_{i=1}^{K} \int_{-\tau_i}^{0} Z_{2i}(s)\hat{Q}_i(s)^T x ds \\
&\quad + \frac{1}{\tau_K} \sum_j \int_{-\tau_j}^{0} Z_{2j}(s)\hat{S}_j(s)\phi_j(s)ds + \frac{1}{\tau_K} \sum_{i,j=1}^{K} \int_{\theta=-\tau_i}^{0} \int_{s=-\tau_j}^{0} Z_{2i}(\theta)\hat{R}_{ij}(\theta,s)\phi_j(s)\, d\theta ds \\
&= \left(Z_0\hat{P} + \sum_{j=1}^{K} \left(Z_{1j}\hat{Q}_j(-\tau_j)^T + \int_{-\tau_j}^{0} Z_{2j}(s)\hat{Q}_j(s)^T ds \right) \right) x + \frac{1}{\tau_K} \sum_{i=1}^{K} Z_{1i}\hat{S}_i(-\tau_i)\phi_i(-\tau_i) \\
&\quad + \frac{1}{\tau_K} \sum_i \int_{-\tau_i}^{0} \left(Z_0\hat{Q}_i(s) + Z_{2i}(s)\hat{S}_i(s) + \sum_{j=1}^{K} \left(Z_{1j}\hat{R}_{ji}(-\tau_j, s) + \int_{\theta=-\tau_j}^{0} Z_{2j}(\theta)\hat{R}_{ji}(\theta,s)d\theta \right) \right) \phi_i(s)ds.
\end{aligned}
$$

We conclude that the controller $\mathcal{K}$ has the form

$$
\left(\mathcal{K} \begin{bmatrix} x \\ \mathbf{\Phi} \end{bmatrix} \right) := \left[K_0 x + \textstyle\sum_{i=1}^{K} K_{1i}\phi_i(-\tau_i) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} K_{2i}(s)\phi_i(s)ds \right]. \qquad \square
$$

We conclude that given $\hat{P}, \hat{Q}_i, \hat{S}_i$, and $\hat{R}_{ij}$, it is possible to compute the controller gains K_0, K_{1i}, and K_{2i}. In practice, however, if S is polynomial, then $\hat{S}_i(s) = S(s)^{-1}$ will be a rational matrix-valued function. This implies that $\hat{Q}_i$ and $\hat{R}_{ij}$ are likewise rational. Computing and analytically integrating such rational functions poses serious challenges. Fortunately, however, this task can be largely avoided. Specifically, if we use the formulae from Theorem 9 and substitute into the expression for $u(t)$, we obtain the following.

COROLLARY 11. *If $\mathcal{Z}$ is as defined in Lemma* 10, *$\hat{\mathcal{P}}$ is as defined in Theorem* 9, *and*

$$
u(t) = \mathcal{Z}\hat{\mathcal{P}} \begin{bmatrix} x(t) \\ \{x(t+s)\}_i \end{bmatrix},
$$

then

$$
u(t) = K_0 x(t) + \sum_{i=1}^{K} K_{1i} x(t-\tau_i) + \sum_i \int_{-\tau_i}^{0} K_{2i}(s) x(t+s) ds,
$$

where, for $i \in [K]$,

$$K_0 = Z_0\hat{P} + \sum_{j=1}^{K}\left(Z_{1j}S_j(-\tau_j)^{-1}Z(-\tau_j)^T + O_j\right)\hat{H}_j^T, \qquad K_{1i} = \frac{1}{\tau_K}Z_{1i}S_i(-\tau_i)^{-1},$$

$$K_{2i}(s) = \frac{1}{\tau_K}\Bigg(\left(Z_0\hat{H}_iZ(s) + Z_{2i}(s)\right) + \sum_{j=1}^{K}\left(Z_{1j}S_j(-\tau_j)^{-1}Z(-\tau_j)^T + O_j\right)\hat{\Gamma}_{ji}Z(s)\Bigg)S_i(s)^{-1},$$

$$O_i = \int_{-\tau_j}^{0} Z_{2j}(s)S_j(s)^{-1}Z(s)^T ds.$$

The proof is straightforward.

The advantage of this representation is that the matrices O_i can be numerically calculated a priori to machine precision using trapezoidal integration without an analytic expression for S^{-1}. Naturally, implementation still requires integration of $\int_{-\tau_i}^0 K_{2i}(s)\phi_i(s)ds$ in real time. However, practical implementation of such controllers is typically based on a sampling $\{t_i\}$ of the largest delay interval, meaning computation of $\int_{-\tau_i}^0 K_{2i}(s)\phi_i(s)ds$ can be reduced to matrix multiplication based on numerical evaluations of $S(t_i)^{-1}$. This implementation can be further simplified if the state-feedback controller is combined with an H_∞-optimal estimator, as described in [18].

8. An LMI formulation of the H_∞-optimal controller synthesis problem for multidelay systems. In this section, we combine all previous results to give a concise formulation of the controller synthesis problem in the LMI framework.

THEOREM 12. *For any $\gamma > 0$, suppose there exist $d \in \mathbb{N}$; constant $\epsilon > 0$; matrix $P \in \mathbb{R}^{n\times n}$; polynomials $S_i, Q_i \in W_2^{n\times n}[T_i^0]$, $R_{ij} \in W_2^{n\times n}\left[T_i^0 \times T_j^0\right]$ for $i,j \in [K]$; matrices $Z_0, Z_{1i} \in \mathbb{R}^{p\times n}$; and polynomials $Z_{2i}[T_i^0] \in W_2^{p\times n}$ for $i \in [K]$ such that*

$$\mathcal{L}_1(P - \epsilon I_n, \{Q_i\}_i, \{S_i - \epsilon I_n\}_i, \{R_{ij}\}_{ij}) \in \Xi_{d,n,nK}$$

and

$$-\mathcal{L}_1(D + \epsilon\hat{I}, \{E_i\}_i, \{\dot{S}_i + \epsilon I_n\}_i, \{G_{ij}\}_{ij}) \in \Xi_{d,q+m+n(K+1),nK},$$

where D, E_i, G_{ij} are as defined in Theorem 5, $\hat{I} = \operatorname{diag}(0_{q+m}, I_n, 0_{nK})$, and $\mathcal{L}_1$ is as defined in (5.1). Furthermore, suppose P, Q_i, S_i, R_{ij} satisfy the conditions of Lemma 4. Let

$$u(t) = K_0x(t) + \sum_{i=1}^{K}K_{1i}x(t-\tau_i) + \sum_{i=1}^{K}\int_{-\tau_i}^{0}K_{2i}(s)x(t+s)ds,$$

where $\hat{P}$, $\hat{H}_i$, and $\hat{\Gamma}_{ji}$ for $Z(s)$ are as defined in Theorem 9, and

$$K_0 = Z_0\hat{P} + \sum_{j=1}^{K}\left(Z_{1j}S_j(-\tau_j)^{-1}Z(-\tau_j)^T + O_j\right)\hat{H}_j^T, \qquad K_{1i} = \frac{1}{\tau_K}Z_{1i}S_i(-\tau_i)^{-1},$$

$$K_{2i}(s) = \frac{1}{\tau_K}\left(\left(Z_0\hat{H}_iZ(s) + Z_{2i}(s)\right) + \sum_{j=1}^{K}\left(Z_{1j}S_j(-\tau_j)^{-1}Z(-\tau_j)^T + O_j\right)\hat{\Gamma}_{ji}Z(s)\right)S_i(s)^{-1},$$

$$O_i = \int_{-\tau_j}^{0} Z_{2j}(s)S_j(s)^{-1}Z(s)^T ds.$$

Thus, for any $w \in L_2$, if $y(t)$ and $x(t)$ satisfy (4.1), then $\|y\|_{L_2} \le \gamma\|w\|_{L_2}$.

Proof. Define $\mathcal{P} := \mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]$. By assumption, $\mathcal{P}$ satisfies the conditions of Lemma 4. By Corollary 8, we have

$$\left\langle \mathbf{x}, \mathcal{P}\left[\begin{smallmatrix} P-\epsilon I_n, & \{Q_i\}_i \\ \{S_i-\epsilon I_n\}_i, & \{R_{ij}\}_{ij}\end{smallmatrix}\right]\mathbf{x}\right\rangle_{Z_{n,K}} = \left\langle \mathbf{x}, \mathcal{P}\left[\begin{smallmatrix} P, & \mathbf{Q} \\ \mathbf{S}, & \mathbf{R}\end{smallmatrix}\right]\mathbf{x}\right\rangle_{Z_{n,K}} - \epsilon\|\mathbf{x}\|^2_{Z_{n,K}} \ge 0$$

for all $\mathbf{x} \in Z_{n,K}$. Similarly, we have

$$\left\langle \begin{bmatrix}\begin{bmatrix} v \\ w \\ \mathbf{z}_1 \\ f\end{bmatrix} \\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}, \mathcal{P}\left[\begin{smallmatrix} D+\epsilon\hat{I}, & \{E_i\}_i \\ \{\dot{S}_i+\epsilon I_n\}_i, & \{G_{ij}\}_{ij}\end{smallmatrix}\right]\begin{bmatrix}\begin{bmatrix} v \\ w \\ \mathbf{z}_1 \\ f\end{bmatrix} \\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}\right\rangle_{Z_{q+m+n(K+1),n,K}}$$
$$= \left\langle \begin{bmatrix}\begin{bmatrix} v \\ w \\ \mathbf{z}_1 \\ f\end{bmatrix} \\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}, \mathcal{P}\left[\begin{smallmatrix} D, & \{E_i\}_i \\ \{\dot{S}_i\}_i, & \{G_{ij}\}_{ij}\end{smallmatrix}\right]\begin{bmatrix}\begin{bmatrix} v \\ w \\ \mathbf{z}_1 \\ f\end{bmatrix} \\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}\right\rangle_{Z_{q+m+n(K+1),n,K}} + \epsilon\left\|\begin{bmatrix}\mathbf{z}_1 \\ \{\mathbf{z}_{2i}\}_i\end{bmatrix}\right\|^2_{Z_{n,K}} \le 0$$

for all $\mathbf{z}_1 \in \mathbb{R}^n$ and

$$\begin{bmatrix}\begin{bmatrix} v \\ w \\ \mathbf{z}_1 \\ f\end{bmatrix} \\ \{\mathbf{z}_{2i}\}_i\end{bmatrix} \in Z_{q+m+n(K+1),n,K}.$$

Furthermore, by Theorem 9 and Corollary 11,

$$u(t) = \mathcal{Z}\mathcal{P}^{-1}\begin{bmatrix} x(t) \\ x(t+s)\end{bmatrix},$$

where

$$\left(\mathcal{Z}\begin{bmatrix} x \\ \mathbf{\Phi}\end{bmatrix}\right)(s) := Z_0x + \sum_{i=1}^{K} Z_{1i}\phi_i(-\tau_i) + \sum_{i=1}^{K}\int_{-\tau_i}^{0} Z_{2i}(s)\phi_i(s)ds.$$

Therefore, by Theorem 5, if $y(t)$ and $x(t)$ satisfy (4.1), then $\|y\|_{L_2} \le \gamma\|w\|_{L_2}$. ∎

9. Numerical testing, validation, and practical implementation. The algorithms described in this paper have been implemented in MATLAB within the DelayTOOLs framework, which is based on SOSTOOLS and the `pvar` framework. Several supporting functions were described in [16], and these are sufficient to enforce the conditions of Theorem 12. For all examples, the computation time is in CPU seconds on an Intel i7-5960X 3.0GHz processor. This time corresponds to the IPM iteration in SeDuMi and does not account for preprocessing, postprocessing, or time spent on polynomial manipulations formulating the SDP using SOSTOOLS. Such polynomial manipulations can significantly exceed SDP computation time for small problems.

For simulation and practical use, some additional functionality has been added to facilitate calculation of controller gains and real-time implementation. The most significant new function introduced in this paper is `P_PQRS_Inverse_joint_sep_ndelay`, which takes the matrix P and polynomials Q_i, S_i, and R_{ij} and computes $\hat{P}$, $\hat{H}_i$, and $\hat{\Gamma}_{ij}$ as described in Theorem 9. In addition, the script `solver_ndelay_opt_control` combines all aspects of this paper and simulates the resulting controller in closed loop. For simulation, a fixed-step forward difference method is used, with a different set of states representing each delay channel. In the simulation results given below, 200 spatial discretization points are used for each delay channel.

9.1. Bounding the H_∞ norm of a system with delay. Naturally, the results of this paper can be used to bound the H_∞ norm of a time-delay system by simply setting $B_2 = 0$. In this subsection, we take this approach and verify that the resulting H_∞ norm bounds are accurate to several decimal places, as compared with a high-order Padé-based approximation scheme, and compare favorably with existing results in the literature. In each case, the Padé estimate is calculated using a tenth-order Padé approximation combined with the MATLAB `norm` command. The minimum H_∞ norm bound is indicated by $\gamma_{\min}$.

Example A.1.

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} w(t), \qquad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

d	1	2	3	Padé	[5]	[23]
$\gamma_{\min}$	0.2373	0.2365	0.2365	0.2364	0.32	2

Example A.2. Here, we consider the following well-studied example, which is known to be stable for delays in the interval $\tau \in [0.100173, 1.71785]$:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t), \qquad y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t).$$

We use the algorithm to compute bounds for the open-loop H_∞ norm of this system as the delay varies within this interval. The results are illustrated in Figure 1. Note that, as expected, the H_∞ norm approaches infinity quickly as we approach the limits of the stable region.

9.2. Validation of H_∞-optimal controller synthesis. We now apply the controller synthesis algorithm to several problems. Unfortunately, there are very few challenging example problems available in the literature. When these examples do exist, they are often trivial in the sense that the dynamics can be entirely eliminated by the controller—meaning only the control effort is to be minimized, and the achievable norms do not change significantly with delay or other parameters. The problems listed below were found to be the most challenging as measured by either significant variation of the closed-loop norm with delay or the requirement for a degree of more than one to achieve optimal performance. In each case, the results are compared to existing results in the literature (when available) and to an H_∞-optimal controller designed for the ODE obtained by using a tenth-order Padé approximation of the delay terms.

Example B.1.

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}x(t) + \begin{bmatrix}-1 & -1\\0 & -0.9\end{bmatrix}x(t-\tau) + \begin{bmatrix}1\\1\end{bmatrix}w(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t),\\ y(t) &= \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}x(t) + \begin{bmatrix}0\\0.1\end{bmatrix}u(t),\end{aligned}$$

$\gamma_{\min}$	$d=1$	$d=2$	$d=3$	Padé	[6]	[10]
$\tau=0.99$	0.10001	0.10001	0.10001	0.1000	0.2284	1.882
$\tau=2$	1.438	1.353	1.332	1.339	∞	∞
CPU sec	0.478	0.879	2.48	2.78	N/A	N/A

Example B.2. This example comes from [4]. In that work, the authors set $D_1 = D_2 = 0$ and for, e.g., $\tau = 0.3$, obtained a closed-loop H_∞ bound of $\gamma = 0.3983$. Theorem 12 was able to find a closed-loop controller for an arbitrarily small closed-loop norm bound ($< 10^{-6}$). This is because the control effort is not included in the regulated output. We remedy this and add a second regulated output to obtain

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}x(t) + \begin{bmatrix}-1 & 0\\-1 & 1\end{bmatrix}x(t-\tau) + \begin{bmatrix}-0.5\\1\end{bmatrix}w(t) + \begin{bmatrix}3\\1\end{bmatrix}u(t),\\ y(t) &= \begin{bmatrix}1 & -0.5\\0 & 0\end{bmatrix}x(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t),\end{aligned}$$

$\gamma_{\min}$	$d=1$	$d=2$	$d=3$	Padé	[4]
$\tau=0.3$	0.3953	0.3953	0.3953	0.3953	N/A
CPU sec	0.655	1.248	2.72	N/A	N/A

Example B.3. This example is a modified version of the example in [2] (B_2 was modified to make the problem more difficult, and regulated outputs and disturbances were added). In that work, the authors were able to find a stabilizing controller for a maximum delay of $\tau_1 = 0.1934$ and $\tau_2 = 0.2387$. We are able to find a controller for any τ_1 and τ_2. The results here are for $\tau_1 = 1$ and $\tau_2 = 2$. The closed-loop system response is illustrated in Figure 2.

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}-1 & 2\\0 & 1\end{bmatrix}x(t) + \begin{bmatrix}0.6 & -.4\\0 & 0\end{bmatrix}x(t-\tau_1) + \begin{bmatrix}0 & 0\\0 & -0.5\end{bmatrix}x(t-\tau_2) + \begin{bmatrix}1\\1\end{bmatrix}w(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t),\\ y(t) &= \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}x(t) + \begin{bmatrix}0\\0\\0.1\end{bmatrix}u(t),\end{aligned}$$

$\gamma_{\min}$	$d=1$	$d=2$	$d=3$	Padé
$\tau_1=1, \tau_2=2$	0.6104	0.6104	0.6104	0.6104
CPU sec	2.07	7.25	25.81	N/A

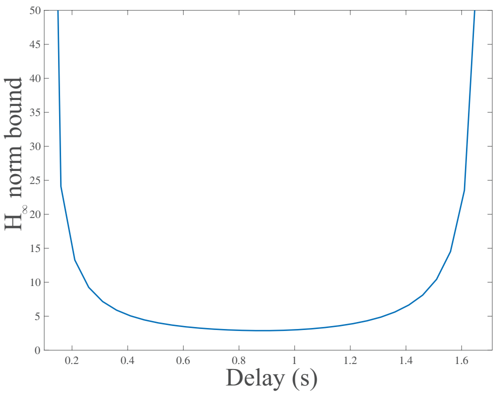

FIG. 1. *Calculated open-loop H_∞ norm bound versus delay for Example* A.2.

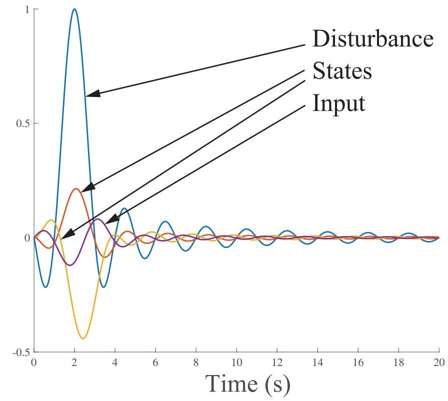

FIG. 2. *Closed-loop system response to a sinc disturbance for Example* B.3.

TABLE 1
CPU sec indexed by # of states (n) and # of delays (K).

$K \downarrow n \rightarrow$	1	2	3	5	10
1	0.438	0.172	0.266	1.24	17.2
2	0.269	0.643	2.932	17.1	647.2
3	0.627	2.634	10.73	91.43	5170
5	1.294	13.12	84.77	1877	65281
10	11.41	469.86	4439	57894	N/A

TABLE 2
Closed-loop norm bound indexed by # of states (n) and # of delays (K).

1	2	3	5	10
0.923	0.979	0.991	0.997	0.999
0.804	0.938	0.971	0.989	0.997
0.766	0.922	0.963	0.986	0.996
0.739	0.910	0.957	0.984	0.996
0.722	0.902	0.953	0.982	N/A

Example B.4. In this example, we rigorously examine the computational complexity of the proposed algorithm. We use a generalized n-D system with K delays, a single disturbance $w(t)$, and a single input $u(t)$,

$$\dot{x}(t) = -\sum_{i=1}^{K} \frac{x(t-i/K)}{K} + \mathbf{1}w(t) + \mathbf{1}u(t), \qquad y(t) = \begin{bmatrix} \mathbf{1}^T \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $\mathbf{1} \in \mathbb{R}^n$ is the vector of all ones. The resulting computation time is listed in Table 1. The achieved closed-loop H_∞ norms are listed in Table 2. As expected, these results indicate that the synthesis problem is not significantly more complex than the stability test. The complexity scales as a function of nK and is possible on a desktop computer when $nK < 50$.

A delayed model of control of Mach number. A linearized model of control of Mach number in a wind tunnel was proposed in [11], wherein the dynamics are given by

(9.1)
$$\dot{x}(t) = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & -2\zeta\omega \end{bmatrix} x(t) + \begin{bmatrix} 0 & ka & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 0 \\ \omega^2 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 10 \end{bmatrix} w(t),$$

where nominal parameter values were listed as $a = \frac{1}{1.964}$, $k = -.0117$, $\zeta = 0.8$, $\omega = 6$, and $\tau = 0.33$, and where the disturbances are to Mach number and a resistant torque in the input motor, as proposed in [21]. The states x_1, x_2, and x_3 represent perturbations to the Mach number, guide vane angle, and guide vane actuator angle,

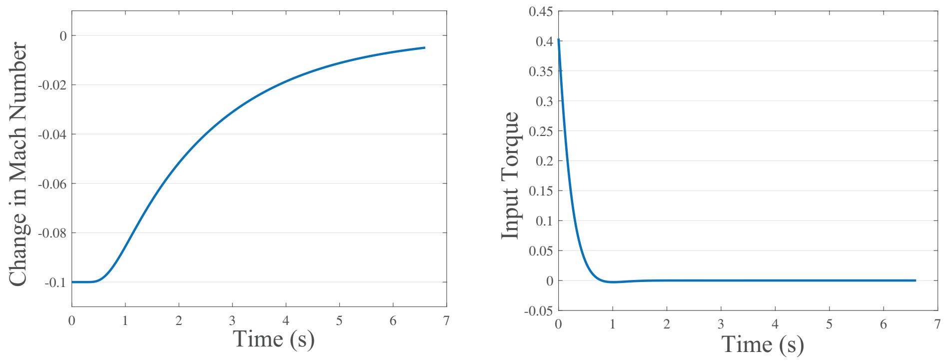

FIG. 3. *A MATLAB simulation of the response to initial condition for system* (9.1) *coupled with the controller from Theorem* 12 *with closed-loop gain of* 1.9639.

respectively. It was shown in [11] that distributed feedback of the form (1.1) was necessary for spectrum assignment. This earlier work posed the LQR optimal control problem, and not the disturbance rejection framework of H_∞-optimal control. Following the work in [21], we have therefore modified the model to add a regulated output which includes Mach number and vane angle. We have, in addition, added control effort to the regulated output, so that

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} u(t).$$

The proposed algorithm now produces a controller with closed-loop H_∞-norm bound of 1.9369 as indicated in the following table:

$\gamma_{\min}$	$d=1$	$d=2$	$d=3$	Padé
$\tau = 0.33$	1.964	1.964	1.9369	1.9369
CPU sec	0.743	1.98	5.629	2.645

Because the original work in [11] considered the LQR framework (a special case of the H_2-optimal control problem), that paper simulated response to an initial condition which was chosen to be $x_1(s) = -0.1$, $x_3(s) = 0$, and $x_2(s) = 8.547$ for $s \in [-\tau, 0]$. The response and control efforts are shown in Figure 3. Note that H_∞-optimal controllers are suboptimal for the LQR problem, and hence the response to initial condition is not a significant improvement over [11].

9.3. A scalable design example with multiple state delays. In this subsection, we demonstrate the scalability and potential applications of the algorithm by considering a practical problem faced in hotel management with a centralized hot water source with multiple showering customers (a generalization of the model proposed in [19]). Specifically, let us first consider a single user attempting to achieve a desired shower temperature by adjusting a hot water tap. In this case, we have a significant transport delay caused by the flow of hot water from the tap to the showerhead. In modeling the dynamics, we assume that a person will adjust the tap at a rate proportional to the difference between current temperature and desired temperature and that the overall flow rate is constant (i.e., does not depend on temperature).

Under these assumptions, we can model the linearized water temperature dynamics at the tap as

$$\dot{T}(t) = -\alpha\left(T(t-\tau) - w(t)\right),$$

where T is the water temperature, $w(t)$ is the desired water temperature, and α is the sensitivity of the user. When multiple users are present and the available hot water pressure is finite, the action of each user will affect the temperature of all other users. In a linearized model, we represent this as

$$\dot{T}_i(t) = -\alpha_i(T_i(t-\tau_i) - w_i(t)) - \sum_{j\neq i}^{K} \gamma_{ij}\dot{T}_j(t),$$

where γ_{ij} represents the fractional reduction of user i's hot water pressure caused by an increase in hot water consumption by user j. Eliminating $\dot{T}_j$ from the right-hand side, we have

$$\dot{T}_i(t) = -\alpha_i\left(T_i(t-\tau_i) - w_i(t)\right) + \sum_{j\neq i}^{K} \gamma_{ij}\alpha_j\left(T_j(t-\tau_j) - w_j(t)\right),$$

where we have neglected products $\gamma_{ij}\gamma_{jk}$, as it is assumed these coupling coefficients are small. Even for a single user, these dynamics are often unstable if the delay is significant. For this reason, we introduce a centralized tracking control system to stabilize the temperature dynamics. Obviously, this controller cannot sense the desired water temperatures, $w_i(t)$. The controller can, however, sense the tap position and the actual water temperature. We account for this by including an augmented state, T_{1i}, which then represents the tap position chosen by user i. Introducing an input into the temperature dynamics yields

$$\begin{aligned}
&\dot{T}_{1i}(t) = T_{2i}(t) - w_i(t),\\
&\dot{T}_{2i}(t) = -\alpha_i\left(T_{2i}(t-\tau_i) - w_i(t)\right) + \sum_{j\neq i}^{K} \gamma_{ij}\alpha_j\left(T_{2j}(t-\tau_j) - w_j(t)\right) + u_i(t),\\
&y_i(t) = \begin{bmatrix} T_{1i}(t) \\ 0.1u_i(t)\end{bmatrix}.
\end{aligned} \tag{9.2}$$

Aggregating these dynamics into the form of (4.1), we have

$$\begin{aligned}
&A_0 = \begin{bmatrix} 0 & I \\ 0 & 0\end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 0 \\ 0 & \hat{A}_i\end{bmatrix},\\
&\hat{A}_i = \Gamma * \operatorname{diag}(e_i) = \Gamma * \operatorname{diag}([0_{1\times i-1},\ 1\ 0_{1\times N-i}]),\\
&B_1 = \begin{bmatrix} -I \\ -\Gamma\end{bmatrix}, \quad \Gamma_{ij} = \begin{cases} \gamma_{ij}\alpha_j, & i\neq j, \\ -\alpha_i, & i = j,\end{cases} \quad i,j = 1,\ldots,N.
\end{aligned}$$

9.3.1. Optimal control of showering users. For numerical implementation with n_u users, we have a system with $2n_u$ states, n_u delays, $2n_u$ regulated outputs, and n_u control inputs. The implementation of this example is included in the code [17], wherein we set $\alpha_i = 1$, $\gamma_{ij} = 1/n$, and $\tau_i = i$. The resulting open-loop dynamics are unstable. For $n_u = 4$, we obtain a closed-loop H_∞ norm bound of $\gamma = 0.38$. For $w_i(t) = i$, the resulting closed-loop dynamics are illustrated in Figure 4, wherein convergence to the desired shower temperature is observed for all users.

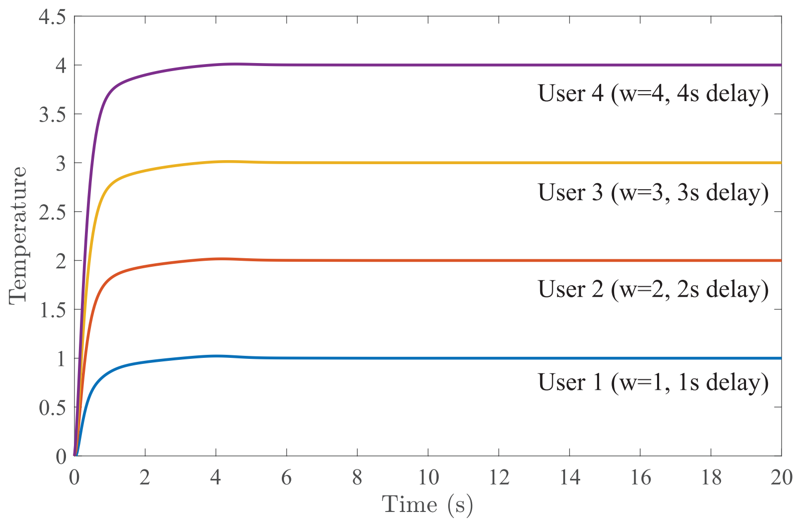

FIG. 4. *A MATLAB simulation of the step response of the closed-loop temperature dynamics* $(T_{2i}(t))$ *for system* (9.2) *with* four *users* (w_i *and* τ_i *as indicated) coupled with the controller from Theorem* 12 *with a closed-loop gain of* 0.36.

10. Conclusion. In this paper, we have shown how the problem of optimal control of systems with multiple delays can be reformulated as a convex optimization problem with operator variables. We have proposed a parameterization of positive operators using positive matrices and verified that the resulting LMIs are accurate to several decimal places when measured by the minimal achievable closed-loop H_∞ norm bound. We have developed an analytic formula for the inverse of the proposed parameterized class of positive operators. Next, we have demonstrated effective methods for real-time computation of the control inputs. Finally, we have implemented the proposed algorithms and gains and simulated the results on a realistic model with eight states and four delays.

REFERENCES

[1] J. Bernussou, P. Peres, and J. C. Geromel, *A linear programming oriented procedure for quadratic stabilization of uncertain systems*, Systems Control Lett., 13 (1989), pp. 65–72.

[2] Y. Cao, Y. Sun, and C. Cheng, *Delay-dependent robust stabilization of uncertain systems with multiple state delays*, IEEE Trans. Automat. Control, 43 (1998), pp. 1608–1612.

[3] R. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, 1995.

[4] E. Fridman and U. Shaked, *H_∞-state-feedback control of linear systems with small state delay*, Systems Control Lett., 33 (1998), pp. 141–150.

[5] E. Fridman and U. Shaked, *New bounded real lemma representations for time-delay systems and their applications*, IEEE Trans. Automat. Control, 46 (2001), pp. 1973–1979.

[6] E. Fridman and U. Shaked, *Delay-dependent stability and H_∞ control: Constant and time-varying delays*, Internat. J. Control, 76 (2003), pp. 48–60.

[7] N. Krasovskii, *On the analytic construction of an optimal control in a system with time lags*, J. Appl. Math. Mech., 26 (1962), pp. 50–67.

[8] N. N. Krasovskiĭ, *Stability of Motion: Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay*, translated from the Russian by J. L. Brenner, Stanford University Press, 1963.

[9] I. Lasiecka, *Control of systems governed by partial differential equations: A historical perspective*, in Proceedings of the 34th Conference on Decision and Control (New Orleans, LA), IEEE, 1995, pp. 2792–2797.

[10] X. Li and C. De Souza, *Criteria for robust stability and stabilization of uncertain linear systems with state delay*, Automatica J. IFAC, 33 (1997), pp. 1657–1662.

[11] A. Manitius, *Feedback controllers for a wind tunnel model involving a delay: Analytical design and numerical simulation*, IEEE Trans. Automat. Control, 29 (1984), pp. 1058–1068.

[12] G. Miao, M. Peet, and K. Gu, *Inversion of separable kernel operators in coupled differential-functional equations and application to controller synthesis*, IFAC-PapersOnLine, 50 (2017), pp. 6513–6518, https://doi.org/10.1016/j.ifacol.2017.08.595.

[13] P. Park, *A delay-dependent stability criterion for systems with uncertain time-invariant delays*, IEEE Trans. Automat. Control, 44 (1999), pp. 876–877.

[14] M. Peet, *Professional Web Site for Matthew M. Peet*, http://control.asu.edu.

[15] M. Peet, *SOS for systems with multiple delays: Part 1: (H_∞)-optimal control*, in Proceedings of the American Control Conference (Philadelphia, PA), IEEE, 2019, pp. 3849–3856.

[16] M. Peet, *A dual to Lyapunov's second method for linear systems with multiple delays and implementation using SOS*, IEEE Trans. Automat. Control, 64 (2019), pp. 944–959.

[17] M. Peet, *A Convex Solution of the H_∞-Optimal Controller Synthesis Problem for Multi-Delay Systems*, https://doi.org/10.24433/CO.eb23c7f2-a1ff-4137-b118-fdead2e82f4a, 2018.

[18] M. Peet and K. Gu, *Synthesis of full-state observers for time-delay systems using SOS*, in Proceedings of the 23rd International Symposium on Mathematical Theory of Networks and Systems, Hong Kong University of Science and Technology, Hong Kong, 2018, pp. 276–283.

[19] M. M. Peet, *Stability and Control of Functional Differential Equations*, Ph.D. thesis, Stanford University, 2006.

[20] D. W. Ross and I. Flügge-Lotz, *An optimal control problem for systems with differential-difference equation dynamics*, SIAM J. Control, 7 (1969), pp. 609–623, https://doi.org/10.1137/0307044.

[21] O. Sename and C. Briat, *Observer-based H_∞ control for time-delay systems: A new LMI solution*, IFAC Proc., 39 (2006), pp. 114–119.

[22] A. Seuret and F. Gouaisbaut, *Complete quadratic Lyapunov functionals using Bessel-Legendre inequality*, in Proceedings of the 18th European Control Conference (Naples, Italy), IEEE, 2014, pp. 448–453.

[23] U. Shaked, I. Yaesh, and C. de Souza, *Bounded real criteria for linear time-delay systems*, IEEE Trans. Automat. Control, 43 (1998), pp. 1016–1022.