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Brief paper

Global finite-time stabilization of a class of uncertain nonlinear systems☆

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Abstract

This paper studies the problem of finite-time stabilization for nonlinear systems. We prove that *global finite-time* stabilizability of uncertain nonlinear systems that are dominated by a lower-triangular system can be achieved by Hölder continuous state feedback. The proof is based on the finite-time Lyapunov stability theorem and the nonsmooth feedback design method developed recently for the control of inherently nonlinear systems that cannot be dealt with by any smooth feedback. A recursive design algorithm is developed for the construction of a Hölder continuous, global finite-time stabilizer as well as a C^1 positive definite and proper Lyapunov function that guarantees finite-time stability.

Keywords: Global stabilization; Finite-time convergence; Hölder continuous state feedback; Lyapunov stability; Uncertain nonlinear systems

1. Introduction

In this paper, we consider a family of uncertain nonlinear systems of the form

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x, u, t),\\
\dot{x}_2 &= x_3 + f_2(x, u, t),\\
&\vdots\\
\dot{x}_n &= u + f_n(x, u, t),
\end{aligned} \tag{1.1}$$

where $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state and input, respectively, and $f_i : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$, are C^1 uncertain functions with $f_i(0, 0, t) = 0$, $\forall t$.

The objective of this paper is to address the questions:

(i) Under what kind of conditions, is there a state feedback control law that renders the trivial solution $x = 0$ of (1.1) *finite-time* globally stable (i.e. global stability in the sense of Lyapunov plus finite-time convergence)?
(ii) How to design systematically a finite-time, globally stabilizing controller if it exists?

Our interest in these two questions is motivated by several papers and books in the literature (Athans & Falb, 1966; Bhat & Bernstein, 1998, 2000; Haimo, 1986; Hong, Huang, & Xu, 2001; Hong, 2002), which discussed how finite-time stabilization problems can arise naturally in practice and how they can be addressed by using finite-time stability theory. In classical control engineering, there is an important control design technique known as dead-beat control. As we shall see, what is studied in this paper is indeed a nonlinear enhancement of the well-known dead-beat control technique that has found wide applications, for instance, in process control and digital control, just to name a few. On the other hand, the concept of finite-time stability also arises naturally in time-optimal control. A classical example is the double integrator with bang–bang time-optimal feedback control (Athans & Falb, 1966). Using the maximal principle, a time-optimal controller can be obtained, steering all the trajectories of the closed-loop system to the origin in a minimum time from any initial condition. The time-optimal control system exhibits a very special property, namely,

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finite-time convergence rather than infinite settling time. In contrast to the commonly used notion of asymptotic stability, finite-time stability requires essentially that a control system be stable in the sense of Lyapunov and its trajectories tend to zero in *finite time.*

The problem of finite-time stabilization has been studied, for instance, in the papers (Bhat & Bernstein, 1998, 1997, 2000; Ryan, 1979; Haimo, 1986; Hong et al., 2001; Hong, 2002), in which it was demonstrated that finite-time stable systems might enjoy not only faster convergence but also better robustness and disturbance rejection properties. In the recent work (Bhat & Bernstein, 2000), a Lyapunov stability theorem has been presented for testing finite-time stability of continuous autonomous systems. This result provides a basic tool for analysis and synthesis of nonlinear control systems. The Lyapunov theory for finite-time stability was employed in Bhat and Bernstein (1998), resulting in C^0 finite-time stabilizing state feedback controllers for the double integrator. Later, finite-time output feedback stabilizers were also derived for the double integrator (Hong et al., 2001) by means of the Lyapunov finite-time stability theorem (Bhat & Bernstein, 2000). This, together with the homogeneous systems theory (Bacciotti, 1992; Coron & Praly, 1991; Dayawansa, 1992; Dayawansa, Martin, & Knowles, 1990; Hermes, 1991; Kawski, 1989, 1990; Rosier, 1992), particularly, the robust stability theorem of homogeneous systems and the homogeneous approximation technique, led to a local result on output feedback stabilization of feedback linearizable systems in the plane (Hong et al., 2001). In Praly, Andrea-Novel, and Coron (1991), the ideas of control Lyapunov function and universal formula were used to design a continuous controller for affine systems. The approach was illustrated via the planar system studied in Kawski (1989) but very hard to be applied to three- or higher-dimensional systems.

Most of the finite-time stabilization results available in the literature (Bhat & Bernstein, 1998, 1997, 2000; Haimo, 1986; Hong et al., 2001; Ryan, 1979) are only applicable to two- or three-dimensional control systems. Moreover, these results are *local* because of the use of a homogeneous approximation. In the higher-dimensional case, the paper (Hong, 2002) derives continuous state feedback control laws achieving *local* finite-time stabilization for triangular systems and certain class of nonlinear systems. It also contains some interesting global finite-time stabilization results for certain class of nonlinear systems. However, a nontrivial but important issue on whether *global finite-time* stabilization of *n*-dimensional nonlinear systems can be achieved by continuous state feedback remains unknown and unanswered.

In this paper, we shall address this issue and provide an affirmative answer for a family of uncertain nonlinear systems. In particular, we show that for the nonlinear system (1.1) dominated by a lower-triangular system, it is possible to achieve global finite-time stabilization by *non-Lipschitz continuous* state feedback. This conclusion is proved based on the Lyapunov theory for finite-time stability (Bhat & Bernstein, 2000) and a new feedback design method that leads to a construction of C^0 finite-time global stabilizers. Our finite-time feedback control scheme is inspired by the two recent papers (Qian & Lin, 2001a,b), where non-Lipschitz continuous state feedback controllers were constructed via the adding a power integrator technique, achieving global asymptotic stabilization for a wide class of inherently nonlinear systems that cannot be dealt with, even locally, by any smooth feedback. The new ingredient of the proposed finite-time control strategy is the explicit construction of subtle homogeneous-like Lyapunov functions and non-Lipschitz continuous state feedback controllers, so that global finite-time stabilization of the closed-loop system can be concluded from the finite-time stability theorem (Bhat & Bernstein, 2000). In contrast to adding a power integrator design (Qian & Lin, 2001a,b), the feedback design method in this paper is more subtle and delicate because to guarantee global finite-time stability of the closed-loop system, the derivative of the control Lyapunov function $V(x)$ along the trajectories of the closed-loop system must be not only negative definite but also less than $-cV^{\alpha}(x)$, for suitable real numbers $c > 0$ and $0 < \alpha < 1$. The contribution of this work is to show how to find such a control Lyapunov function and a finite-time global stabilizer simultaneously for the whole family of nonlinear systems (1.1), under appropriate conditions.

The rest of this paper is organized as follows. Section 2 reviews briefly the Lyapunov theory for finite-time stability of continuous autonomous systems. The main result of this paper is presented in Section 3, where sufficient conditions are given under which finite-time globally stabilizing, non-Lipschitz continuous state feedback control laws can be explicitly constructed for a family of nonlinear systems (1.1), using the tool of adding a power integrator (Qian & Lin, 2001a,b) with a subtle construction of homogeneous-like Lyapunov functions. Concluding remarks are given in Section 4.

2. Lyapunov theory for finite-time stability and useful lemmas

In this section, we review some basic concepts and terminologies related to the notion of *finite-time stability* and the corresponding Lyapunov stability theory. We also recall Lyapunov theorem and the converse theorem for finite-time stability of autonomous systems, which were discussed previously in the paper (Bhat & Bernstein, 2000).

The classical Lyapunov stability theory is only applicable to a differential equation whose solution from any initial condition is unique. A well-known sufficient condition for the existence of a unique solution of the autonomous system

$$\dot{x} = f(x), \quad \text{with } f(0) = 0, \quad x \in \mathbb{R}^n \tag{2.1}$$

is that the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous. The solution trajectories of the locally Lipschitz continuous system (2.1) can have at most *asymptotic* convergent rate. However, in many practical situations such as dead-beat control, sliding mode and time-optimal control, it is not only necessary but also rather important to achieve *finite-time* convergence. It should be observed that only *nonsmooth* or *non-Lipschitz continuous* autonomous systems can have a finite-time convergent property. The simplest example may be the scalar system

$$\dot{x} = -x^{1/3}, \quad x(0) = x_0,$$

whose solution trajectories are unique in forward time and described by

$$x(t) = \begin{cases} \mathrm{sgn}(x_0)(x_0^{2/3} - \frac{2}{3}t)^{3/2}, & 0 \leqslant t < \frac{3}{2}x_0^{2/3}, \\ 0, & t \geqslant \frac{3}{2}x_0^{2/3}. \end{cases} \tag{2.2}$$

Clearly, all the solutions converge to the equilibrium $x = 0$ in finite time. This example suggests that in order to achieve finite-time stabilizability, nonsmooth or at least non-Lipschitz continuous feedback must be employed, even if the controlled plant $\dot{x} = f(x, u, t)$ is smooth.

In what follows, we recall the Lyapunov stability theorems for finite-time stability, which will be used in the next section. In a series of papers (Bhat & Bernstein, 1997, 1998, 2000), the notion of finite-time stability was introduced and a necessary and sufficient condition was given for non-Lipschitz continuous autonomous systems to be finite-time stable.

Definition 2.1 (*Bhat and Bernstein, 2000*). Consider the autonomous system (2.1), where $f : D \to \mathbb{R}^n$ is non-Lipschitz continuous on an open neighborhood D of the origin $x = 0$ in $\mathbb{R}^n$. The equilibrium $x = 0$ of (2.1) is *finite-time* convergent if there are an open neighborhood U of the origin and a function $T_x : U\backslash\{0\} \to (0, \infty)$, such that every solution trajectory $x(t, x_0)$ of (2.1) starting from the initial point $x_0 \in U\backslash\{0\}$ is well-defined and unique in forward time for $t \in [0, T_x(x_0))$, and $\lim_{t\to T_x(x_0)} x(t, x_0) = 0$. Here $T_x(x_0)$ is called the *settling time* (of the initial state x_0). The equilibrium of (2.1) is finite-time stable if it is Lyapunov stable and finite-time convergent. If $U = D = \mathbb{R}^n$, the origin is a globally finite-time stable equilibrium.

Since finite-time stability requires that every solution trajectory reaches the origin in finite time, finite-time stability is therefore a much stronger requirement than asymptotic stability. The following theorem (Bhat & Bernstein, 2000) provides sufficient conditions for the origin of system (2.1) to be a finite-time stable equilibrium.

Theorem 2.2. *Consider the non-Lipschitz continuous autonomous system* (2.1). *Suppose there are C^1 function $V(x)$ defined on a neighborhood $\hat{U} \subset \mathbb{R}^n$ of the origin, and real numbers $c > 0$ and $0 < \alpha < 1$, such that*

(1) *$V(x)$ is positive definite on $\hat{U}$;*
(2) $\dot{V}(x) + cV^{\alpha}(x) \leqslant 0, \ \forall x \in \hat{U}$.

Then, the origin of system (2.1) *is locally finite-time stable. The settling time, depending on the initial state $x(0) = x_0$, satisfies*

$$T_x(x_0) \leqslant \frac{V(x_0)^{1-\alpha}}{c(1-\alpha)},$$

for all x_0 in some open neighborhood of the origin. If $\hat{U} = \mathbb{R}^n$ and $V(x)$ is also radially unbounded (i.e., $V(x) \to +\infty$ as $\|x\| \to +\infty$), the origin of system (2.1) *is globally finite-time stable.*

In the case of asymptotic stability, the conventional Lyapunov stability theorem requires only $\dot{V}(x)$ be negative definite and $V(x)$ be positive definite. On the contrary, the finite-time stability theorem above requires a much stronger condition such as assumption (2). In Bhat and Bernstein (2000), it has been shown that condition (2) is also necessary for continuous autonomous systems to be finite-time stable. For this reason, the problem of finite-time stabilization is far more difficult than the asymptotic stabilization problem.

In the next section, we shall prove that under an appropriate condition, global finite-time stabilization can be achieved for a family of nonlinear systems (1.1), by means of *non-Lipschitz continuous* state feedback. This will be done by explicitly constructing a non-Lipschitz C^0 controller $u(x)$ and a C^1 Lyapunov function $V(x)$, such that the closed-loop system satisfies Theorem 2.2. To this end, we introduce the following lemmas that will be used in the sequel.

Lemma 2.3. *For any real numbers x_i, $i = 1, \ldots, n$* and $< 0b \leqslant 1$, *the following inequality holds*:

$$(|x_1| + \cdots + |x_n|)^b \leqslant |x_1|^b + \cdots + |x_n|^b. \tag{2.3}$$

When $b = p/q \leqslant 1$, where $p > 0$ and $q > 0$ are odd *integers,*

$$|x^b - y^b| \leqslant 2^{1-b}|x - y|^b. \tag{2.4}$$

The proof of this lemma is not difficult and hence left to the reader as an exercise. The next lemma is a direct consequence of the Young's inequality. Its proof can be found in Qian and Lin (2001b).

Lemma 2.4. *Let c, d be positive real numbers and $\gamma(x, y) > 0$ a real-valued function. Then,*

$$|x|^c|y|^d \leqslant \frac{c\gamma(x, y)|x|^{c+d}}{c+d} + \frac{d\gamma^{-c/d}(x, y)|y|^{c+d}}{c+d}. \tag{2.5}$$

3. Nonsmooth feedback stabilization in finite time

Using Theorem 2.2, together with Lemmas 2.3–2.4, we, we are able to prove the following theorem that is the main result of this paper. The proof is constructive and gives a systematic procedure for the design of C^0 global finite-time stabilizers for the nonlinear system (1.1).

Theorem 3.1. *The uncertain nonlinear system* (1.1) *is globally finite-time stabilizable by non-Lipschitz continuous state feedback if the following conditions hold*:

for $i=1,\ldots,n$, *and for all* (x,u,t),

$$|f_i(x,u,t)| \leqslant (|x_1|+\cdots+|x_i|)\gamma_i(x_1,\ldots,x_i), \tag{3.1}$$

where $\gamma_i(x_1,\ldots,x_i)\geqslant 0$ *is a known* C^1 *function.*

Proof. Let $d=4n/(2n+1)$ and choose the Lyapunov function $V_1(x_1)=x_1^2/2$. Using (3.1), a simple computation gives

$$\begin{aligned}\dot V_1(x_1) &\leqslant x_1x_2 + x_1^2\gamma_1(x_1)\\ &\leqslant x_1(x_2-x_2^*)+x_1x_2^*+x_1^d\tilde\rho_1(x_1),\end{aligned} \tag{3.2}$$

where $\tilde\rho_1(x_1)\geqslant x_1^{2/(2n+1)}\gamma_1(x_1)\geqslant 0$ is a C^1 function. For instance, one can simply choose $\tilde\rho_1(x_1)=(1+x_1^2)\gamma_1(x_1)$.

From (3.2), it is easy to see that the C^0 virtual controller $x_2^* = -x_1^{(2n-1)/(2n+1)}(n+\tilde\rho_1(x_1)) := -\xi_1^{q_2}\beta_1(x_1)$ with $\beta_1(x_1)>0$ being C^1, results in

$$\dot V_1(x_1)\leqslant -nx_1^d + x_1(x_2-x_2^*).$$

Inductive step: Suppose at step $k-1$, there are a C^1 Lyapunov function $V_{k-1}(x_1,\ldots,x_{k-1})$, which is positive definite and proper, satisfying

$$V_{k-1}(x_1,\ldots,x_{k-1})\leqslant 2(\xi_1^2+\cdots+\xi_{k-1}^2), \tag{3.3}$$

and a set of parameters $q_1=1>q_2>\cdots>q_k := (2n+3-2k)/(2n+1)>0$, and C^0 virtual controllers $x_1^*,\ldots,x_k^*$, defined by

$$\begin{aligned}
&x_1^*=0, && \xi_1 = x_1^{1/q_1}-x_1^{*1/q_1},\\
&x_2^*=-\xi_1^{q_2}\beta_1(x_1), && \xi_2 = x_2^{1/q_2}-x_2^{*1/q_2},\\
&\vdots && \vdots\\
&x_k^*=-\xi_{k-1}^{q_k}\beta_{k-1}(x_1,\ldots,x_{k-1}), && \xi_k = x_k^{1/q_k}-x_k^{*1/q_k},
\end{aligned}$$

with $\beta_1(\cdot)>0,\ldots,\beta_{k-1}(\cdot)>0$ being C^1, such that

$$\dot V_{k-1}(\cdot)\leqslant \xi_{k-1}^{2-q_{k-1}}(x_k-x_k^*)-(n-k+2)\left(\sum_{l=1}^{k-1}\xi_l^d\right). \tag{3.4}$$

We claim that (3.3) and (3.4) also hold at step k. To prove this claim, consider

$$V_k(x_1,\ldots,x_k)=V_{k-1}(\cdot)+W_k(x_1,\ldots,x_k), \tag{3.5}$$

where

$$W_k(x_1,\ldots,x_k)=\int_{x_k^*}^{x_k}(s^{1/q_k}-x_k^{*1/q_k})^{2-q_k}\mathrm{d}s. \tag{3.6}$$

The Lyapunov function $V_k(x_1,\ldots,x_k)$ thus defined has several useful properties collected in the following two propositions.

Proposition 1. $W_k(x_1,\ldots,x_k)$ is C^1. *Moreover,*

$$\frac{\partial W_k}{\partial x_k}=\xi_k^{2-q_k} \quad \text{and for } l=1,\ldots,k-1,$$

$$\frac{\partial W_k}{\partial x_l}=-(2-q_k)\frac{\partial(x_k^{*1/q_k})}{\partial x_l}\int_{x_k^*}^{x_k}(s^{1/q_k}-x_k^{*1/q_k})^{1-q_k}\mathrm{d}s.$$

Proposition 2. $V_k(x_1,\ldots,x_k)$ *is* C^1, *positive definite and proper satisfying* $V_k(\cdot)\leqslant 2(\xi_1^2+\cdots+\xi_k^2)$.

The proofs of Propositions 1 and 2 are quite straightforward and therefore are left to the reader as an exercise. Using Proposition 1, it is deduced from (3.4) that

$$\begin{aligned}\dot V_k(x_1,\ldots,x_k)\leqslant{}& \xi_{k-1}^{2-q_{k-1}}(x_k-x_k^*)-(n-k+2)\left(\sum_{l=1}^{k-1}\xi_l\right)\\ &+\xi_k^{2-q_k}(x_{k+1}-x_{k+1}^*)+\xi_k^{2-q_k}x_{k+1}^*\\ &+\xi_k^{2-q_k}f_k(x,u,t)+\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot x_l.\end{aligned} \tag{3.7}$$

Now we estimate each term on the right-hand side of (3.7). First, it follows Lemma 2.3 that

$$|x_k-x_k^*|\leqslant 2^{1-q_k}|x_k^{1/q_k}-(x_k^*)^{1/q_k}|^{q_k}\leqslant 2|\xi_k|^{q_k}.$$

Consequently, the identity $q_k=q_{k-1}-2/(2n+1)$ yields

$$\begin{aligned}|\xi_{k-1}^{2-q_{k-1}}(x_k-x_k^*)| &\leqslant 2|\xi_{k-1}|^{2-q_{k-1}}|\xi_k|^{q_k}\\ &\leqslant \frac{\xi_{k-1}^d}{3}+c_k\xi_k^d,\end{aligned} \tag{3.8}$$

where $c_k>0$ is a fixed constant.

To continue the proof and facilitate the construction of a finite-time stabilizer, we introduce two additional propositions whose proofs are given in the appendix. They are very useful when estimating the last two terms in inequality (3.7).

Proposition 3. *For* $k=1,\ldots,n$, *there are* C^1 *functions* $\tilde\gamma_k(x_1,\ldots,x_k)\geqslant 0$ *such that*

$$|f_k(x,u,t)|\leqslant(|\xi_1|^{q_k}+\cdots+|\xi_k|^{q_k})\tilde\gamma_k(x_1,\ldots,x_k).$$

Proposition 4. *For* $l=1,\ldots,k-1$, *there are* C^1 *functions* $C_{k,l}(x_1,\ldots,x_k)\geqslant 0$, *such that*

$$\begin{aligned}\left|\frac{\partial(x_k^{*1/q_k})}{\partial x_l}\dot x_l\right| \leqslant{}& (|\xi_1|^{(2n-1)/(2n+1)}+\cdots\\ &+|\xi_k|^{(2n-1)/(2n+1)})C_{k,l}(\cdot).\end{aligned}$$

Using Proposition 3 and Lemma 2.4, we have

$$\begin{aligned}|\xi_k^{2-q_k}f_k(x,u,t)| &\leqslant |\xi_k|^{2-q_k}\left(\sum_{i=1}^k|\xi_i|^{q_k-2/(2n+1)}\right)\bar\gamma_k(\cdot)\\ &\leqslant \frac{1}{3}\left(\sum_{i=1}^{k-1}\xi_i^d\right)+\xi_k^d\tilde\rho_k(x_1,\ldots,x_k)\end{aligned} \tag{3.9}$$

for C^1 functions $\bar\gamma_k(\cdot),\tilde\rho_k(\cdot)\geqslant 0$.

To estimate the last term in (3.7), we observe from Propositions 1 and 4 that

$$\begin{aligned}&\left|\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot{x}_l\right|\\ &\quad\leqslant 2(2-q_k)|\xi_k|\left(\sum_{l=1}^{k}|\xi_l|^{(2n-1)/(2n+1)}\right)\left(\sum_{l=1}^{k-1}C_{k,l}(\cdot)\right)\\ &\quad\leqslant \frac{1}{3}\left(\sum_{i=1}^{k-1}\xi_i^d\right)+\xi_k^d\overline{\rho}_k(x_1,\ldots,x_k),\end{aligned}\tag{3.10}$$

where $\overline{\rho}_k(\cdot)\geqslant 0$ is a C^1 function.

Substituting (3.8)–(3.10) into (3.7) yields

$$\begin{aligned}\dot{V}_k\leqslant&-(n-k+1)\left(\sum_{i=1}^{k-1}\xi_i^d\right)+\xi_k^{2-q_k}(x_{k+1}-x_{k+1}^*)\\ &+\xi_k^{2-q_k}x_{k+1}^*+\xi_k^d(c_k+\tilde{\rho}_k(\cdot)+\overline{\rho}_k(\cdot)).\end{aligned}$$

Clearly, the C^0 virtual controller

$$\begin{aligned}x_{k+1}^*=&-\xi_k^{q_k-2/(2n+1)}(n-k+1+c_k+\tilde{\rho}_k(\cdot)+\overline{\rho}_k(\cdot))\\ :=&-\xi_k^{q_{k+1}}\beta_k(x_1,\ldots,x_k)\end{aligned}$$

with $\beta_k(\cdot)>0$ being C^1 and $0<q_{k+1}:=q_k-2/(2n+1)<q_k$, results in

$$\dot{V}_k(\cdot)\leqslant-(n-k+1)\left(\sum_{i=1}^{k}\xi_i^d\right)+\xi_k^{2-q_k}(x_{k+1}-x_{k+1}^*).$$

This completes the proof of the inductive step.

Using the inductive argument above, one concludes that at the nth step, there exist a *non-Lipschitz continuous* state feedback control law of the form

$$\begin{aligned}u=x_{n+1}^*=&-\xi_n^{q_{n+1}}\beta_n(x_1,\ldots,x_n)\\ :=&-\xi_n^{1/(2n+1)}\beta_n(\cdot)\end{aligned}\tag{3.11}$$

with $\beta_n(\cdot)>0$ being C^1, and a C^1 positive definite and proper Lyapunov function $V_n(x_1,\ldots,x_n)$ of the form (3.5)–(3.6), such that

$$\begin{aligned}&V_n(x_1,\ldots,x_n)\leqslant 2(\xi_1^2+\cdots+\xi_n^2),\\ &\dot{V}_n(x_1,\ldots,x_n)\leqslant-(\xi_1^{4n/(2n+1)}+\cdots+\xi_n^{4n/(2n+1)}).\end{aligned}$$

Let $\alpha:=2n/(2n+1)\in(0,1)$. By Lemma 2.3, one has

$$V_n^\alpha(x_1,\ldots,x_n)\leqslant 2(\xi_1^{4n/(2n+1)}+\cdots+\xi_n^{4n/(2n+1)}).$$

With this in mind, it is easy to see that

$$\dot{V}_n+\tfrac{1}{4}V_n^\alpha\leqslant-\tfrac{1}{2}(\xi_1^{4n/(2n+1)}+\cdots+\xi_n^{4n/(2n+1)})\leqslant 0.$$

By Theorem 2.2, the closed-loop system (1.1)–(3.11) is globally finite-time stable. □

As a consequence of Theorem 3.1, it is immediate to conclude that *global finite-time* stabilization of the triangular system

$$\begin{aligned}\dot{x}_1&=x_2+f_1(x_1),\\ &\vdots\\ \dot{x}_n&=u+f_n(x_1,\ldots,x_n)\end{aligned}\tag{3.12}$$

is achievable by non-Lipschitz continuous state feedback, as long as $f_i:\mathbb{R}^i\to\mathbb{R}^1$, $i=1,2,\ldots,n$, are C^1 functions with $f_i(0,\ldots,0)=0$.

So far we have shown that global finite-time stabilization of the nonlinear systems such as (1.1) and (3.12) is possible using non-Lipschitz continuous state feedback, under condition (3.1). It is worth pointing out that the condition can be slightly relaxed and Theorem 3.1 can be extended to a class of nonsmooth but Hölder continuous nonlinear systems. The details are omitted for the reason of space.

Finally, we present, as suggested by one of the referees, an application to a double integrator for which finite-time stabilizers were already obtained from different points of view (Athans & Falb, 1966; Bhat & Bernstein, 1998; Hong et al., 2001).

Example 3.2. Consider a chain of integrators in the case when $n=2$, i.e.

$$\dot{x}_1=x_2,\quad \dot{x}_2=u.\tag{3.13}$$

Following the proof of Theorem 3.1, one can construct (in two steps) the Hölder continuous controller

$$u=-1.5(x_2^{5/3}+0.15x_1)^{1/5},\tag{3.14}$$

such that the closed-loop system (3.13)–(3.14) satisfies

$$\dot{V}_2\leqslant-0.1(x_1^{8/5}+(x_2^{5/3}+0.15x_1)^{8/5}),$$

where V_2 is a C^1 positive definite and proper function defined by

$$V_2=\frac{x_1^2}{2}+15\int_{x_2^*}^{x_2}(s^{5/3}-x_2^{*5/3})^{7/5}\,\mathrm{d}s$$

with $x_2^*=-(0.15x_1)^{3/5}$. In other words, the double integrator (3.13) is globally finite-time stabilizable by the C^0 control law (3.14).

It is worth pointing out that the finite-time stabilizer (3.14) is very similar to the one proposed in Bhat and Bernstein (1998), where the following finite-time stabilizer

$$u=-x_2^{1/3}-(x_1+\tfrac{3}{5}x_2^{5/3})^{1/5}\tag{3.15}$$

was derived based on a different design method. Therefore, it appears that in the case of a double integrator, our nonsmooth feedback design method has led to a similar solution that

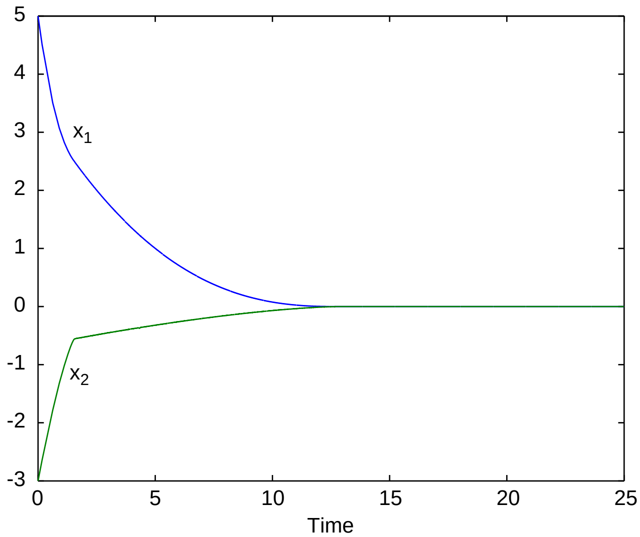

Fig. 1. Trajectories of the closed-loop system (3.13)–(3.14) with $x_1(0) = 5, x_2(0) = -3$.

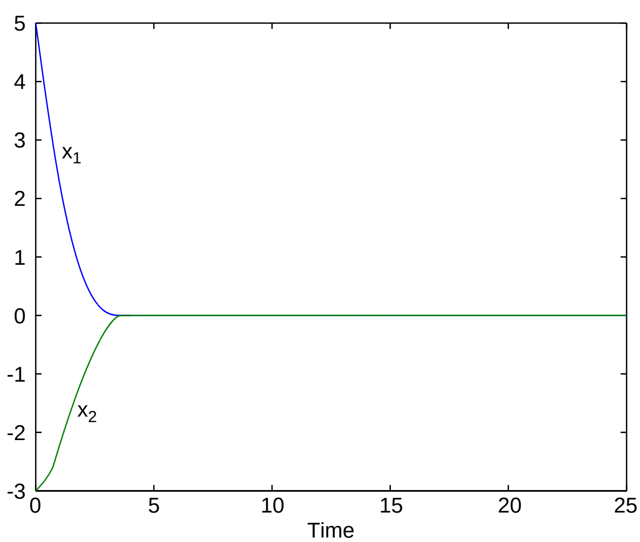

Fig. 2. Trajectories of the closed-loop system (3.13)–(3.15) with $x_1(0) = 5, x_2(0) = -3$.

was obtained previously. By comparison, a new feature of our finite-time stabilizer is its capability of achieving *global finite-time* stabilization for a family of uncertain nonlinear systems that go beyond planar systems.

Simulation results shown in Fig. 1 illustrate effectiveness of the finite-time controller (3.14) as well as a transient response of the closed-loop system, while Fig. 2 shows the simulation result of controller (3.15) by Bhat and Bernstein (1998).

In the previous papers (Athans & Falb, 1966; Ryan, 1979; Bhat & Bernstein, 1998; Hong et al., 2001), finite-time stabilization of the double integrators was studied. A bang–bang controller was derived based on the maximal principle (Athans & Falb, 1966). Although the bang–bang control law is only one of finite-time stabilizers for the double integrator, it has two nice features: (1) it is optimal in terms of time; (2) it is bounded. However, the bang–bang controller has a serious drawback, i.e., it is discontinuous. As a result, this type of finite-time stabilizers leads to the so-called "chattering" phenomenon and thus may not be desirable in practical applications. In the works (Bhat & Bernstein, 1998; Hong et al., 2001; Hong, 2002), finite-time stabilizers were derived by using the theory of homogeneous systems. An obvious advantage of such design methods is its simplicity. In fact, one can design a finite-time controller in such a way that the resulting system is homogeneous with a *negative degree*. However, only *local* finite-time stabilization results can be established for nonlinear systems such as (1.1), due to the use of a homogeneous approximation. In contrast, the finite-time controller developed in this paper relies on the Lyapunov theory for finite-time stability, and hence yields a *global* result for a family of n-dimensional uncertain systems. The price we paid is, similar to most of the Lyapunov-based approaches, the use of a series of subtle estimations for the construction of global finite-time stabilizers, making the design not straightforward. However, in the case of smooth feedback stabilization (i.e., $q_1 = q_2 = \cdots = q_n = 1$), our approach is nothing but the standard backstepping design (Freeman & Kokotovic, 1996).

4. Conclusion

In this paper, we have presented a systematic design method for achieving *global finite-time* stabilization of a family of uncertain nonlinear systems (1.1) under condition (3.1), which turns out to be naturally fulfilled in the case of a lower-triangular system (3.12). Motivated by the adding a power integrator design approach (Qian & Lin, 2001a,b), an iterative algorithm was developed, making it possible to simultaneously construct a globally finite-time, non-Lipschitz continuous stabilizer as well as a C^1 control Lyapunov function that satisfies the Lyapunov theory for finite-time stability, i.e. Theorem 2.2, particularly, the Lyapunov inequality $\dot{V}(x) \leqslant -cV^{\alpha}(x)$, for suitable real numbers $c > 0$ and $0 < \alpha < 1$.

Appendix

Proof of Proposition 3. Recall that $\xi_l = x_l^{1/q_l} - x_l^{*1/q_l}$ and $x_l^* = -\xi_{l-1}^{q_l}\beta_{l-1}(x_1, \ldots, x_{l-1})$. By Lemma 2.3, for $l = 2, \ldots, k$,

$$|x_l| \leqslant |\xi_l + x_l^{*1/q_l}|^{q_l} \leqslant |\xi_l|^{q_l} + |\xi_{l-1}|^{q_l}|\beta_{l-1}(\cdot)|. \qquad \text{(A.1)}$$

Using (3.1) and $0 < q_k < \cdots < q_2 < q_1 = 1$, we have

$$\begin{aligned}|f_k(x,u,t)| &\leqslant \left[|\xi_1| + \sum_{l=2}^{k}[|\xi_l|^{q_l} + |\xi_{l-1}|^{q_l}\beta_{l-1}(\cdot)]\right]\gamma_k(\cdot)\\ &\leqslant (|\xi_1|^{q_k} + \cdots + |\xi_k|^{q_k})\tilde{\gamma}_k(x_1,\ldots,x_k), \quad (A.2)\end{aligned}$$

where $\tilde{\gamma}_k(\cdot) \geqslant 0$ is a C^1 function. □

Proof of Proposition 4. Using (A.1)–(A.2) and $q_{l+1} = q_l - 2/(2n+1)$, one has (for $l = 1, \ldots, k-1$)

$$\begin{aligned}|\dot{x}_l| &\leqslant (|\xi_{l+1}|^{q_{l+1}} + |\xi_l|^{q_{l+1}}\beta_l(\cdot)) + \left(\sum_{i=1}^{l}|\xi_i|^{q_l}\right)\tilde{\gamma}_l(\cdot)\\ &\leqslant \left(\sum_{i=1}^{l+1}|\xi_i|^{q_l-2/(2n+1)}\right)\rho_l(x_1,\ldots,x_l), \quad (A.3)\end{aligned}$$

for a C^1 function $\rho_l(\cdot) \geqslant 0$.

The estimate of $|\partial(x_k^{*1/q_k})/\partial x_l|$ can be done by using an inductive argument. First of all, it is clear that

$$\left|\frac{\partial(x_2^{*1/q_2})}{\partial x_1}\right| \leqslant \left|\frac{\partial[x_1\beta_1^{1/q_2}(x_1)]}{\partial x_1}\right| \leqslant \tilde{C}_{2,1}(x_1),$$

where $\tilde{C}_{2,1}(x_1) \geqslant 0$ is a C^1 function.

Inductive assumption: For $l = 1, \ldots, k-2$, there exist smooth functions $\tilde{C}_{k-1,l}(\cdot) \geqslant 0$ such that

$$\left|\frac{\partial(x_{k-1}^{*1/q_{k-1}})}{\partial x_l}\right| \leqslant \left(\sum_{i=1}^{k-2}\xi_i^{1-q_l}\right)\tilde{C}_{k-1,l}(x_1,\ldots,x_{k-2}). \quad (A.4)$$

Our objective is to prove that there are C^1 functions $\tilde{C}_{k,l}(\cdot) \geqslant 0$, $l = 1, \ldots, k-1$, such that

$$\left|\frac{\partial(x_k^{*1/q_k})}{\partial x_l}\right| \leqslant \left(\sum_{i=1}^{k-1}\xi_i^{1-q_l}\right)\tilde{C}_{k,l}(x_1,\ldots,x_{k-1}). \quad (A.5)$$

First, we consider the case where $l = 1, \ldots, k-2$. Note that $(x_k^*)^{1/q_k} = -\xi_{k-1}[\beta_{k-1}^{1/q_k}(\cdot)] := -\xi_{k-1}\tilde{\beta}_{k-1}(\cdot)$ and $\xi_{k-1} = x_{k-1}^{1/q_{k-1}} - x_{k-1}^{*1/q_{k-1}}$. This, together with (A.4), results in

$$\begin{aligned}\left|\frac{\partial(x_k^{*1/q_k})}{\partial x_l}\right| &\leqslant \left|\xi_{k-1}\frac{\partial\tilde{\beta}_{k-1}(\cdot)}{\partial x_l}\right| + \left|\tilde{\beta}_{k-1}(\cdot)\frac{\partial(x_{k-1}^{*1/q_{k-1}})}{\partial x_l}\right|\\ &\leqslant \left|\xi_{k-1}\frac{\partial\tilde{\beta}_{k-1}(\cdot)}{\partial x_l}\right|\\ &\quad + \tilde{\beta}_{k-1}(\cdot)\left(\sum_{i=1}^{k-2}\xi_i^{1-q_l}\right)\tilde{C}_{k-1,l}(\cdot)\\ &\leqslant \left(\sum_{i=1}^{k-1}\xi_i^{1-q_l}\right)\tilde{C}_{k,l}(x_1,\ldots,x_{k-1}), \quad (A.6)\end{aligned}$$

where $\tilde{C}_{k,l}(\cdot) \geqslant 0$ is a C^1 function.

Next, we shall prove that (A.6) also holds for $l = k-1$. Recall that $\xi_{k-1} = x_{k-1}^{1/q_{k-1}} - x_{k-1}^{*1/q_{k-1}}$ and $x_{k-1}^{*1/q_{k-1}}$ is not related to x_{k-1}. We have

$$\begin{aligned}\left|\frac{\partial(x_k^{*1/q_k})}{\partial x_{k-1}}\right| &\leqslant \left|\xi_{k-1}\frac{\partial\tilde{\beta}_{k-1}(\cdot)}{\partial x_{k-1}}\right| + \left|\frac{\tilde{\beta}_{k-1}(\cdot)}{q_{k-1}}x_{k-1}^{1/q_{k-1}-1}\right|\\ &\leqslant \left|\xi_{k-1}\frac{\partial\tilde{\beta}_{k-1}(\cdot)}{\partial x_{k-1}}\right|\\ &\quad + \frac{\tilde{\beta}_{k-1}(\cdot)}{q_{k-1}}[\xi_{k-1}^{1-q_{k-1}} + \xi_{k-2}^{1-q_{k-1}}\tilde{\beta}_{k-2}^{1/q_{k-1}-1}(\cdot)]\\ &\leqslant \left(\sum_{i=1}^{k-1}\xi_i^{1-q_l}\right)\tilde{C}_{k,k-1}(x_1,\ldots,x_{k-1}), \quad (A.7)\end{aligned}$$

where $\tilde{C}_{k,k-1}(\cdot) \geqslant 0$ is a C^1 function.

Putting (A.6) and (A.7) together, one arrives at (A.5) which, together with (A.3), implies that for $1 \leqslant l \leqslant k-1$,

$$\begin{aligned}&\left|\frac{\partial(x_k^{*1/q_k})}{\partial x_l}\dot{x}_l\right|\\ &\leqslant \left(\sum_{i=1}^{l+1}|\xi_i|^{q_l-2/(2n+1)}\right)\rho_l(\cdot)\left(\sum_{i=1}^{k-1}\xi_i^{1-q_l}\right)\tilde{C}_{k,l}(\cdot)\\ &\leqslant \left(\sum_{i=1}^{k}|\xi_i|^{(2n-1)/(2n+1)}\right)C_{k,l}(x_1,\ldots,x_k),\end{aligned}$$

where $C_{k,l}(\cdot) \geqslant 0$ are C^1 functions. □

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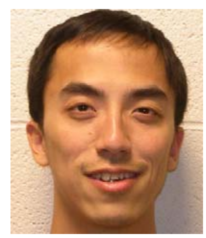

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