Optimal Control of Switched System

Equivalent Problem Formulation

Problem 1

Given:

- subsystem $\dot{x} = f_i(x, u)$
- a fixed time interval $[t_0, t_f]$
- a prespecified squence of active subsystems $\sigma = ((t_0, i_0), (t_1, i_1), \ldots, (t_K, i_K))$

find a continuous input $u \in U_{[t_0, t_f]}$ and switching instants $t_1, \ldots, t_K$

such that

- $x(t_0) = x_0$
- meet S_f at t_f
- minimize cost function

$$J = \Psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t)) dt$$

Ψ为终端部分,L为积分部分.

Two Stage Decomposition

Decomposition Problem 1 into two stages.

Stage(a)

find an optimal continuous input u and the corresponding minimum J.

Seek $J_1(\hat{t})$ for the corresponding $\hat{t} = (t_1, \ldots, t_k)^T$ is conventional since these intervals are fixed.

Only difference is system dynamics changes with respect to different time intervals.

Theorem 1Necessary conditions for stage (a)

Assume:

- the subsystem k is active in $[t_{k-1}, t_k)$,$k \in [1, K]$.
- subsystem K+1 is active in $[t_K, t_{K+1}]$,$t_{K+1} = t_f$.
- $u \in U_{[t_0, t_f]}$ continuous input such that $x(t_0) = x_0$ and meets $S_f = \{x | \mathit{\Phi}_f(x) = 0, \mathit{\Phi} : R^n \to R^{l_f}\}$ at t_f.

> such that $\Phi_f(x(t_f)) = 0$

In order for u is optimal :

- exist a vector function $p(t) = [p_1(t), \ldots, p_n(t)]^T, t \in [t_0, t_f]$.

Such that following conditions:

- $H(x, p, u) = L(x, u) + p^T f_k(x, u), \text{for any } t \in [t_0, t_f]$
    1. 状态方程state equations: $\frac{dx(t)}{dt} = (\frac{\partial H}{\partial p}(x(t), p(t), u(t)))^T$
    2. 协态方程costate equations: $\frac{dp(t)}{dt} = -(\frac{\partial H}{\partial x}(x(t), p(t), u(t)))^T$
    3. 控制方程 stationarity condition: $0 = (\frac{\partial H}{\partial u}(x(t), p(t), u(t)))^T$
    4. 横截条件 $p(t_f) = \frac{\partial \mathit{\Psi}}{\partial x}(x(t_f))^T + \frac{\partial \Phi_f}{\partial x}(x(t_f))^T \lambda$
    5. 连续性条件$p(t_{k^-}) = p(t_{k^+})$.

Stage(b)

solve the constrained nonlinear optimization problem

$$
\begin{gathered}
min_{\tilde{t}} J_1(\hat{t}) \\
\text{subject to } \tilde{t} \in T
\end{gathered}
$$

Problem 2

Given:

- a switched system

$$\dot{x} = f_1(x, u), t_0 \le t \le t_1$$

$$\dot{x} = f_2(x, u), t_1 \le t \le t_f$$

- t_0,t_f and $x(t_0) = x_0$

find a switching instant t_1 and $u(t)$

such that

- minimize the cost functional

$$J = \Psi(x(t_f)) + \int_{t_0}^{t_f} L(x, u)dt$$

Problem 3 (an Equivalent Problem)

introduce a state variable x_{n+1} corresponding to t_1.
x_{n+1} satisfy

$$\frac{x_{n+1}}{dt} = 0$$

$$x_{n+1}(0) = t_1$$

这里 x_{n+1} 为一常量 t_1 ,不过会在下一节中看作未知参数.

introduce a new independent time variable τ.
t will become τ and u_{n+1}

$$t=\begin{cases} t_0+(x_{n+1}-t_0)\tau & 0\le\tau\le 1 \\ x_{n+1}+(t_f-x_{n+1})(\tau-1) & 1\le\tau\le 2 \end{cases}$$

显然$t=t_0,\tau=0$;$t=t_1,\tau=1$;$t=t_f,\tau=2$

Given:

- a system

in the interval $\tau\in[0,1)$

$$\frac{dx(\tau)}{d\tau}=(x_{n+1}-t_0)f_1(x,u)$$

$$\frac{dx_{n+1}}{d\tau}=0$$

in the interval $\tau\in[1,2]$

$$\frac{dx(\tau)}{d\tau}=(t_f-x_{n+1})f_2(x,u)$$

$$\frac{dx_{n+1}}{d\tau}=0$$

- t_0,t_f and $x(0)=x_0$

such that:

- minimize the cost functional

$$J=\Psi(x(2))+\int_0^1(x_{n+1}-t_0)L(x,u)d\tau+\int_1^2(t_f-x_{n+1})L(x,u)d\tau$$

Q:引入τ的作用是什么？

1. 切换时刻不再是时变的,整个问题变为传统问题
2. 将x_{n+1}看作参数,Problem2 和 Problem3 维数相同

Method Based on Solving a Boundary Value Differential Algebraic Equation

Define:

- $\tilde{f}_1(x,u,x_{n+1}) = (x_{n+1} - t_0)f_1(x,u)$
- $\tilde{f}_2(x,u,x_{n+1}) = (t_f - x_{n+1})f_2(x,u)$
- $\tilde{L}_1(x,u,x_{n+1}) = (x_{n+1} - t_0)L(x,u)$
- $\tilde{L}_2(x,u,x_{n+1}) = (t_f - x_{n+1})L(x,u)$

Regarding x_{n+1} as a parameter,$x(\tau) \to x(\tau, x_{n+1})$.

- Parameterized Hamiltonian

$$H(x,p,u,x_{n+1}) = \begin{cases} \tilde{L}_1(x,u,x_{n+1}) + p^T\tilde{f}_1(x,u,x_{n+1}) & 0 \le \tau \le 1 \\ \tilde{L}_2(x,u,x_{n+1}) + p^T\tilde{f}_2(x,u,x_{n+1}) & 1 \le \tau \le 2 \end{cases}$$

Assume:

- x_{n+1} is a given fixed unknown parameter

Apply Theorem 1 to Problem 3:

1. 状态方程state equ: $\frac{\partial x}{\partial \tau} = (\frac{\partial H}{\partial p})^T = \tilde{f}_k(x,u,x_{n+1})$
2. 协态方程costate function: $\frac{\partial p}{\partial \tau} = -(\frac{\partial H}{\partial x})^T = -(\frac{\partial \tilde{f}_k}{\partial x})^T p - (\frac{\partial \tilde{L}_k}{\partial x})^T$
3. 控制方程 stationarity equ: $0 = (\frac{\partial H}{\partial u})^T = (\frac{\partial \tilde{f}_k}{\partial u})^T p + (\frac{\partial \tilde{L}_k}{\partial u})^T$
4. 边界条件 $x(0,x_{n+1}) = x_0$; $p(2,x_{n+1}) = (\frac{\partial \Psi}{\partial x}(x(2,x_{n+1})))^T$.
5. 连续性条件 $p(1^-,x_{n+1}) = p(1^+,x_{n+1})$
6. cost function

$$J(x_{n+1}) = \Psi(x(2,x_{n+1})) + \int_0^1 \tilde{L}(x,u,x_{n+1})d\tau + \int_1^2 \tilde{L}(x,u,x_{n+1})d\tau$$

differentiating above function with respect to x_{n+1}

Problem 4General Switched Linear Quadratic

Problem

Given:

- a switched system

$$\dot{x} = A_1 x + B_1 u, t_0 \leq t \leq t_1$$

$$\dot{x} = A_2 x + B_2 u, t_1 \leq t \leq t_f$$

find a switching instant t_{1} and a continous input u

such that:

- minimize cost functional

$$J = \underbrace{\frac{1}{2}x(t_f)^T Q_f x(t_f) + M_f x(t_f) + W_f)}_{\Psi} + \int_{t_0}^{t_f} \underbrace{(\frac{1}{2}x^T Q x + x^T V u + \frac{1}{2}u^T R u + M}_{L(x,u)}$$

Problem 5Equivalent GSLQ problem

Given:

- a system

in the interval $\tau \in [0, 1)$

$$\frac{dx(\tau)}{d\tau} = (x_{n+1} - t_0)(A_1 x + B_1 u)$$

$$\frac{dx_{n+1}}{d\tau} = 0$$

in the interval $\tau \in [1, 2]$

$$\frac{dx(\tau)}{d\tau} = (t_f - x_{n+1})(A_2 x + B_2 u)$$

$$\frac{dx_{n+1}}{d\tau} = 0$$

find a x_{n+1} and u_τ
such that:

- minimize

$$J = \underbrace{\frac{1}{2}x(2)^T Q_f x(2) + M_f x(2) + W_f)}_{\Psi} + \int_0^1 (x_{n+1} - t_0)L(x,u)d\tau + \int_1^2 (t_f - x_{n+1}$$

aassume:

- the optimal value function 值函数:

$$V^*(x, \tau, x_{n+1}) = \frac{1}{2}x^T P(\tau, x_{n+1})x + S(\tau, x_{n+1})x + T(\tau, x_{n+1})$$

- 计算 HJB function:
  - HJB计算公式

$$-\frac{\partial V^\star}{\partial t}(x,t) = min_u\{F + \frac{\partial V^\star}{\partial t}f\}$$

  - in the interval $\tau \in [0, 1]$

$$-\frac{\partial V^\star}{\partial \tau}(x, \tau, x_{n+1}) = min_{u(\tau)}\{(x_{n+1} - t_0)(L(x,u) + \frac{\partial V^\star}{\partial x}(x, \tau, x_{n+1})f_1$$

  - in the interval $\tau \in [1, 2]$

$$-\frac{\partial V^\star}{\partial \tau}(x, \tau, x_{n+1}) = min_{u(\tau)}\{(t_f - x_{n+1})(L(x,u) + \frac{\partial V^\star}{\partial x}(x, \tau, x_{n+1})f_2$$

the solution to the above HJB equation:

$$u(x, \tau, x_{n+1}) = R^{-1}(B_k^T P(\tau, x_{n+1}) + V^T)x(\tau, x_{n+1}) - R^{-1}(B_k^T S^T(\tau, x_{n+1}) + N^T)$$

Q.
什么是HJB function
什么是the optimal value function

如何解HJB function

ＰＳＴＶＮ是什么？

Question

1. 引入x_{n+1}参数化作用是什么?参数化方法是指什么方法?
2. a two point boundary value DAE 是指t_0,t_f给定吗？
3. 引入独立参数τ的作用是什么?