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Stability Analysis of Switched Systems with Stable and Unstable Subsystems: An Average Dwell Time Approach

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Abstract

We study the stability properties of linear switched systems consisting of both Hurwitz stable and unstable subsystems using an average dwell time approach. We show that if the average dwell time is chosen sufficiently large and the total activation time of unstable subsystems is relatively small compared with that of Hurwitz stable subsystems, then exponential stability of a desired degree is guaranteed. We also derive a piecewise Lyapunov function for the switched system subjected to the switching law and the average dwell time scheme under consideration, and we extend these results to the case for which nonlinear norm-bounded perturbations exist in the subsystems. We show that when the norms of the perturbations are small, we can modify the switching law appropriately to guarantee that the solutions of the switched system converge to the origin exponentially with large average dwell time.

1. Introduction

By a switched system, we mean a hybrid dynamical system that is composed of a family of continuous-time subsystems and a rule orchestrating the switching between the subsystems. Recently, there has been increasing interest in the stability analysis and switching control design of such systems (see, e.g., [1] and the references cited therein). The motivation for studying such switched systems comes from the fact that many practical systems are inherently multimodal in the sense that several dynamical systems are required to describe their behavior which may depend on various environmental factors [1, 2, 3], and from the fact that the methods of intelligent control design are based on the idea of switching between different controllers [1, 4, 5].

In the present paper, we first study some of the stability properties of switched systems composed of a family of linear time-invariant subsystems. For such systems, it is shown in [4] that when all subsystem matrices are *Hurwitz stable* (i.e., all eigenvalues lie in the left half complex plane), then the entire system is exponentially stable for any switching signal if the time between consecutive switchings (called "dwell time") is sufficiently large. In [6], a dwell time scheme is analyzed for local asymptotic stability of nonlinear switched systems with the activation time being used as a dwell time. In a recent paper [7], Hespanha extends the "dwell time" concept to "average dwell time", which means that the average time interval between consecutive switchings is no less than a specified constant, and he proves that if such a constant is sufficiently large, then the switched system is exponentially stable.

Motivated primarily by the above works, we consider switched systems with both Hurwitz stable and unstable subsystems. The reason for considering unstable subsystems is theoretical as well as the fact that unstable subsystems cannot be avoided in many applications. A switching law is proposed which ensures that the entire switched system is exponentially stable for any switching signal provided that the average dwell time is sufficiently large. In the switching law, the total activation time ratio between Hurwitz stable subsystems and unstable subsystems is required to be no less than a specified constant which is computed using the desired stability degree of the switched system. Two numerical examples are presented to demonstrate the applicability of the results. We point out that the idea of specifying the total activation time period ratio between Hurwitz stable subsystems and unstable subsystems is motivated in [8], where all subsystem matrices are assumed to be pairwise commutative.

We extend the above viewpoint by considering a piecewise Lyapunov function for the switched system, since generally no quadratic Lyapunov function exists for stable as well as unstable subsystems. By considering a positive stability margin for Hurwitz stable subsystems (positive λ_i such that $A_i + \lambda_i I$ remains Hurwitz stable) and a negative stability margin for unstable subsystems (positive λ_i that makes $A_i - \lambda_i I$ Hurwitz stable), we propose a piecewise Lyapunov function which proves to be effective in guaranteeing exponential stability of the entire switched system under the given switching law with large average dwell time. It turns out that our considerations concerning piecewise Lyapunov functions are consistent with those that use differential equation theory directly.

Finally, we extend the above results to the case where nonlinear norm-bounded perturbations exist in all subsystems. The perturbations that we consider are vanishing or non-vanishing [9]. We show that when the perturbations are small, a desired convergence of the solutions can be obtained if we adopt a more restrictive switching law with the average dwell time chosen sufficiently large.

2. Problem Formulation

We consider linear switched systems described by equations of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad x(t_0) = x_0, \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state, $t_0 \geq 0$ is the initial time and x_0 is the initial state. $\sigma(t) : [t_0, \infty) \to \mathcal{I}_N = \{1, 2, \cdots, N\}$ is a piecewise constant function of time, called a *switching signal*, which will be determined later, and $\{A_i : i \in \mathcal{I}_N\}$ constitute a family of constant matrices describing the subsystems where $N > 1$ denotes the number of subsystems. Throughout this paper, we assume that both Hurwitz stable and unstable subsystem matrix A_i's exist in the switched system (1).

We first introduce some of the notation used in [4] and [7]. Given a positive constant τ_d, let $\mathcal{S}_d[\tau_d]$ denote the set of all switching signals with interval between consecutive switchings being no smaller than τ_d. The constant τ_d is called the "dwell time" [4]. When all subsystem matrices A_i are Hurwitz stable, it has been shown in [4] that we can choose τ_d sufficiently large so that the switched system (1) is exponentially stable for every $\sigma(t) \in \mathcal{S}_d[\tau_d]$. In [7], Hespanha introduced a new concept of "average dwell time". For any switching signal $\sigma(t)$ and any $t \geq \tau$, let $N_\sigma(\tau, t)$ denote the number of switchings of $\sigma(t)$ over the interval $[\tau, t)$. For given $N_0, \tau_d > 0$, let $\mathcal{S}_a[\tau_a, N_0]$ denote the set of all switching signals satisfying

$$N_\sigma(\tau, t) \leq N_0 + \frac{t-\tau}{\tau_a}, \tag{2}$$

where the constant τ_a is called the "average dwell time" and N_0 denotes the "chatter bound". The idea there is that there may exist some consecutive switchings separated by less than τ_a, but the average interval between consecutive switchings is no less than τ_a. In fact, (2) indicates that if we ignore the first N_0 switchings, then the average time interval between consecutive switchings is at least τ_a. It has been shown in [7] that if τ_a is sufficiently large, then the switched system (1) is exponentially stable for any switching signal $\sigma(t) \in \mathcal{S}_a[\tau_a, N_0]$.

The papers [4, 7] deal with switched systems consisting of only Hurwitz stable subsystems; however, in practice we frequently encounter cases where unstable subsystems have to be dealt with. For this reason, we consider switched systems which are composed of both Hurwitz stable and unstable subsystems, and we aim to derive a switching law that incorporates an average dwell time approach so that the switched system (1) is exponentially stable. It is clear that the switching signals that switch only among Hurwitz stable subsystems while ignoring unstable ones will be special cases in our discussion. We point out that in [8] a detailed stability analysis of the case where both Hurwitz stable and unstable subsystems exist is also given. However, in that work, all subsystem matrices are assumed to be pairwise commutative and thus the results of [8] are not directly applicable to the control problem on hand.

3. Time-Switched Control

We first give a difinition on stability property of the switched system (1).

Definition. For certain switching signal $\sigma(t)$, the switched system (1) is said to be *globally exponentially stable with stability degree* $\lambda \geq 0$ if $\|x(t)\| \leq e^{\alpha - \lambda(t-t_0)}\|x_0\|$ holds for all $t \geq t_0$ and a known constant α. □

Since both Hurwitz stable and unstable subsystems exist in (1), we assume without loss of generality that $A_1, \cdots, A_r$ $(r < N)$ are unstable and the remaining matrices are Hurwitz stable. Then, there always exist a set of scalars $\lambda_i > 0$ and a_i such that

$$\begin{cases} \|e^{A_i t}\| \leq e^{a_i + \lambda_i t}, & 1 \leq i \leq r \\ \|e^{A_i t}\| \leq e^{a_i - \lambda_i t}, & r < i \leq N. \end{cases} \tag{3}$$

The scalar a_i's and λ_i's are easy to compute using algebraic matrix theory.

Now, for any piecewise constant switching signal $\sigma(t)$ and any $t > \tau$, let $T^+(\tau, t)$ (resp., $T^-(\tau, t)$) denote the total activation time of the unstable subsystems (resp., the Hurwitz stable subsystems) during $[\tau, t)$, and define $\lambda^+ = \max_{1\leq q\leq r} \lambda_q$, $\lambda^- = \min_{r+1\leq q\leq N} \lambda_q$. Then, for any given $\lambda \in (0, \lambda^-)$, we choose a scalar $\lambda^* \in (\lambda, \lambda^-)$ arbitrarily to propose the following switching law:

(S1) Determine the switching signal $\sigma(t)$ so that $\frac{T^-(t_0,t)}{T^+(t_0,t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*}$ holds for any $t > t_0$. □

We are now in the position to state the main result in this section.

Theorem 1. *Under the switching law* (S1)*, there is a finite positive constant τ_a^* such that the switched system* (1) *is globally exponentially stable with stability degree λ over $\mathcal{S}_a[\tau_a, N_0]$ for any average dwell time $\tau_a \geq \tau_a^*$ and any chatter bound $N_0 > 0$.*

Proof. Let $t_1, t_2, \cdots$ denote the time points at which switching occurs, and write p_j for the value of $\sigma(t)$ on $[t_{j-1}, t_j)$. Then, for any t satisfying $t_0 < \cdots < t_i \leq t < t_{i+1}$, we obtain

$$x(t) = e^{A_{p_{i+1}}(t-t_i)} e^{A_{p_i}(t_i - t_{i-1})} \cdots e^{A_{p_1}(t_1-t_0)} x_0. \tag{4}$$

From the inequality (3), we get the following estimate by collecting the terms of Hurwitz stable and unstable subsystems, respectively,

$$\begin{aligned} \|x(t)\| &\leq \left(\prod_{q=1}^{i+1} e^{a_{p_q}}\right) e^{\lambda^+ T^+(t_0,t) - \lambda^- T^-(t_0,t)} \|x_0\| \\ &\leq e^{(i+1)a + \lambda^+ T^+(t_0,t) - \lambda^- T^-(t_0,t)} \|x_0\| \\ &= c e^{a N_\sigma(t_0,t) + \lambda^+ T^+(t_0,t) - \lambda^- T^-(t_0,t)} \|x_0\|, \end{aligned} \tag{5}$$

where $a = \max_{q\in \mathcal{I}_N} a_q$, $c = e^a$, and $N_\sigma(t_0, t)$, which was defined in the previous section, denotes the number of switchings of $\sigma(t)$ over the interval (t_0, t).

Now for any given $\lambda < \lambda^-$, we require for the switching law (S1) that $\frac{T^-(t_0,t)}{T^+(t_0,t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*}$ holds for some $\lambda^* \in (\lambda, \lambda^-)$. Since such a switching condition is equivalent to

$$\begin{aligned} &\lambda^+ T^+(t_0, t) - \lambda^- T^-(t_0, t) \\ &\quad \leq -\lambda^* \left(T^+(t_0, t) + T^-(t_0, t)\right) = -\lambda^*(t - t_0), \end{aligned} \tag{6}$$

we obtain from (5) that

$$\|x(t)\| \leq c e^{a N_\sigma(t_0,t) - \lambda^*(t-t_0)} \|x_0\|. \tag{7}$$

When $a \leq 0$, we obtain from (7) that

$$\|x(t)\| \leq e^{-\lambda^*(t-t_0)} \|x_0\| \leq e^{-\lambda(t-t_0)} \|x_0\|, \tag{8}$$

which implies that the switched system is globally exponentially stable with stability degree λ for any average dwell time and any chatter bound.

When $a > 0$, we set

$$a N_\sigma(t_0, t) - \lambda^*(t - t_0) \leq \alpha - \lambda(t - t_0) \tag{9}$$

so as to obtain the desired stability degree λ. This is equivalent to

$$N_\sigma(t_0, t) \leq N_0 + \frac{t - t_0}{\tau_a^*} \tag{10}$$

with $\tau_a^* = \frac{a}{\lambda^* - \lambda}$ and $N_0 = \frac{\alpha}{a}$. Since α is arbitrary, N_0 can also be specified arbitrarily. We conclude that

$$\|x(t)\| \leq c e^{\alpha - \lambda(t-t_0)} \|x_0\| \tag{11}$$

and thus the switched system (1) is globally exponentially stable with stability degree λ over $\mathcal{S}_a[\tau_a, N_0]$ for any average dwell time $\tau_a \ge \tau_a^*$ and any chatter bound N_0. □

Remark 1. The switching condition $\frac{T^-(t_0,t)}{T^+(t_0,t)} \ge \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*}$ in (S1) is quite easy to satisfy. For example, we can first activate the Hurwitz stable subsystems with a time period of about $2(\lambda^+ + \lambda^*)T_0$, and then activate the unstable subsystems with a time period of about $(\lambda^- - \lambda^*)T_0$, where $T_0 > 0$ is a positive time unit sufficiently large to satisfy the average dwell time condition (10). □

Remark 2. The idea of specifying the activation time period ratio of Hurwitz stable subsystems to unstable subsystems has appeared in [8], where all subsystem matrices are assumed to be pairwise commutative. In fact, if all subsystem matrices are pairwise commutative, then each subsystem's activation time period during $[t_0, t)$ can be grouped in (4) and thus the term $\left(\prod_{q=1}^{i+1} e^{a_{p_q}}\right)$ in (5) can be replaced by a constant $\left(\prod_{q=1}^{N} e^{a_q}\right)$, which has appeared in [8], and thus the consideration of average dwell time is not necessary from (5) and (6). In this sense, we have extended the results of [8] to a more general class of switched systems. □

Next, we present two examples which demonstrate the two cases given in the proof of the theorem.

Example 1. Consider the switched system (1) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}. \tag{12}$$

It is obvious that A_1 is unstable while A_2 is Hurwitz stable. The scalars a_i and λ_i in (3) are easily computed as $a_1 = a_2 = 0$ and $\lambda^+ = \lambda^- = 1$ from

$$P_1^{-1}A_1P_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{13}$$

$$P_2^{-1}A_2P_2 = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{14}$$

and $\log\left(\frac{\sigma_{max}(P_i)}{\sigma_{min}(P_i)}\right) = \log(1) = 0$, $i = 1, 2$.

Now since $a = 0$, the average dwell time and the chatter bound can be arbitrary. If we let $\lambda = 0.45$ and $\lambda^* = 0.5$, then the switching law (S1) will require $\frac{T^-(t_0,t)}{T^+(t_0,t)} \ge \frac{1+0.5}{1-0.5} = 3$. Here, we activate A_2 and A_1 alternatively with the time periods 0.3 and 0.1, respectively. Figure 1 depicts the solution trajectory of the above switched system with the initial condition $x_0 = [1 \;\; 2]^T$.

Example 2. Consider the switched system (1) with

$$A_1 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \tag{15}$$

where A_1 is unstable while A_2 is Hurwitz stable, and $\lambda(A_1) = \{1, 4\}$, $\lambda(A_2) = \{-1, -3\}$, from which $\lambda^+ = 4, \lambda^- = 1$ is obtained. We choose $a_1 = 0$ and $a_2 = 0.6$ since

$$P_1^{-1}A_1P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \tag{16}$$

$$P_2^{-1}A_2P_2 = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{17}$$

and $\log\left(\frac{\sigma_{max}(P_1)}{\sigma_{min}(P_1)}\right) = 0.57$, $\log\left(\frac{\sigma_{max}(P_2)}{\sigma_{min}(P_2)}\right) = 0$.

We now choose $\lambda = 0.25$, $\lambda^* = 0.5$. Then, the switching law (S1) will require $\frac{T^-(t_0,t)}{T^+(t_0,t)} \ge \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} = 9$, and the average dwell time is computed as $\ge \tau_a^* = \frac{a}{\lambda^* - \lambda} = 2.4$. In order to satisfy both the switching law and the average dwell time condition, we choose to activate A_2 and A_1 with time periods 4.5 and 0.5, respectively. The solution trajectory of the above switched system is shown in Figure 2, where the initial condition is $x_0 = [10 \;\; 20]^T$.

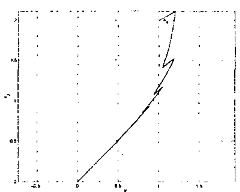

Figure 1: The solution trajectory of Example 1 (arbitrary average dwell time)

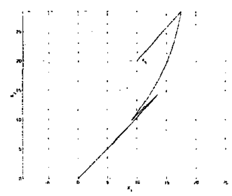

Figure 2: The solution trajectory of Example 2 (specified average dwell time)

Piecewise Lyapunov function.

It is well known that Lyapunov function theory is the most general and useful approach for studying stability of various control systems. However, for switched systems, common quadratic Lyapunov functions for all subsystems exist only in a few situations [10, 11]. Instead of utilizing single Lyapunov functions, the use of piecewise Lyapunov functions (or multiple Lyapunov functions) has been proposed recently [12]~[15]. The switched system (1) is composed of both Hurwitz stable and unstable subsystems, and thus no single quadratic Lyapunov function will exist. Moreover, at this time, it is not known how to construct a piecewise Lyapunov function for the switched system (1). In the remainder of this section, we briefly consider some theoretical aspects of piecewise Lyapunov functions for the switched system (1) under the switching law (S1).

Since $A_1, \cdots, A_r$ are unstable and $A_{r+1}, \cdots, A_N$ are Hurwitz stable, there always exist a set of positive scalars $\lambda_1, \lambda_2, \cdots, \lambda_N$ such that $A_i - \lambda_i I$ $(i \le r)$ and $A_i + \lambda_i I$ $(i > r)$ are Hurwitz stable, and thus there are positive definite matrices $P_1, P_2, \cdots, P_N$ such that

$$\begin{cases} (A_i - \lambda_i I)^T P_i + P_i(A_i - \lambda_i I) < 0 & i \le r \\ (A_i + \lambda_i I)^T P_i + P_i(A_i + \lambda_i I) < 0 & i > r \end{cases} \tag{18}$$

holds for all i. Note that the above inequalities are LMIs [16] with respect to P_i, and thus easily solved using any existing software, such as the LMI Control Toolbox [17].

Using the solution P_i's of (18), we define the following *piecewise Lyapunov function* candidate

$$V(t) = x^T P_{\sigma(t)} x, \quad t \ge t_0 \tag{19}$$

for the switched system (1), where $P_{\sigma(t)}$ is switched among the solution P_i's of (18) in accordance with the piecewise constant switching signal $\sigma(t)$. Then, the following properties of (19) are obtained:

1. Each $V_i = x^T P_i x$ in (19) is continuous and its derivative along the solutions of the corresponding subsystem satisfies

$$\dot{V}_i = \frac{\partial V_i}{\partial x} A_i x \le \begin{cases} 2\lambda_i V_i & i \le r \\ -2\lambda_i V_i & i > r. \end{cases} \tag{20}$$

2. There exist constant scalars $\alpha_2 \geq \alpha_1 > 0$ such that
$$\alpha_1 \|x\|^2 \leq V_i(x) \leq \alpha_2 \|x\|^2 \,,\ \forall x \in \mathcal{R}^n \,,\ \forall i \in \mathcal{I}_N \,. \quad (21)$$
3. There exists a constant scalar $\mu \geq 1$ such that
$$V_i(x) \leq \mu V_j(x) \,, \quad \forall x \in \mathcal{R}^n \,, \quad \forall i,j \in \mathcal{I}_N \,. \quad (22)$$

The first property is a straightforward consequence of (18) while the second and third properties hold, for example, with $\alpha_1 = \inf_{i \in \mathcal{I}_N} \lambda_m(P_i)$, $\alpha_2 = \sup_{i \in \mathcal{I}_N} \lambda_M(P_i)$, and $\mu = \sup_{k,l \in \mathcal{I}_N} \frac{\lambda_M(P_k)}{\lambda_m(P_l)}$, respectively. Here, $\lambda_M(P)$ ($\lambda_m(P)$) denotes the largest (smallest) eigenvalue of the positive definite matrix P.

As in the proof of Theorem 1, we let $t_1 < t_2 < \cdots < t_i$ denote the switching points of $\sigma(t)$ over the interval (t_0, t). Then, we know from (20) that the piecewise Lyapunov function candidate (19) satisfies
$$V(t) \leq \begin{cases} e^{2\lambda^+(t-t_i)} V(t_i) & \text{if } p_{i+1} \leq r \\ e^{-2\lambda^-(t-t_i)} V(t_i) & \text{if } p_{i+1} > r \,. \end{cases} \quad (23)$$
Noting that $V(t_j) \leq \mu V(t_j^-)$ holds on all the switching point t_j's according to (22), where $t_j^- = \lim_{\tau \uparrow t_j} \tau$, we obtain by induction that
$$V(t) \leq \mu^{N_\sigma(t_0,t)} e^{2\lambda^+ T^+(t_0,t) - 2\lambda^- T^-(t_0,t)} V(t_0) \,, \quad (24)$$
and by (21),
$$\|x(t)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{\lambda^+ T^+(t_0,t) - \lambda^- T^-(t_0,t) + \frac{\ln \mu}{2} N_\sigma(t_0,t)} \|x_0\| \,. \quad (25)$$
Comparing this inequality with (5), we see that the remaining discussion is almost the same as in Theorem 1, if we replace a by $\frac{\ln \mu}{2}$ there. Therefore, we conclude that (19) with (18) is a piecewise Lyapunov function for the switched system (1) when adopting the switching law (S1) with the average dwell time sufficiently large.

4. Nonlinear Perturbations

In this section, we consider the nonlinear switched system described by
$$\dot{x}(t) = A_{\sigma(t)} x(t) + f_{\sigma(t)}(x(t), t) \,, \quad x(t_0) = x_0 \quad (26)$$
where $x(t)$ and $\sigma(t)$ are the same as in Section 2. A_i ($i \in \mathcal{I}_N$) is a known constant matrix describing the linear part of the i-th subsystem while $f_i(x(t), t)$ is the nonlinear part which is only known to satisfy the norm condition
$$\|f_i(x(t), t)\| \leq \gamma \|x(t)\| + \beta(t) \,, \quad i \in \mathcal{I}_N \,, \quad (27)$$
where $\gamma \geq 0$ is a known scalar, $\beta(t) \geq 0$ is a Lebesgue integrable function such that $\int_{t_0}^{\infty} e^{\lambda \tau} \beta(\tau) d\tau < \infty$, and λ is the desired stability degree as in Section 3. The perturbations satisfying (27) are called *vanishing* when $\beta(t) \equiv 0$, and *non-vanishing* when $\beta(t) \not\equiv 0$ [9].

For the switched system (26) with (27), we propose the following switching law, which is more restrictive than (S1) proposed in Section 3.

(S2) Let $t_0 = \delta_0 < \delta_1 < \cdots$ ($\lim_{i \to \infty} \delta_j = \infty$) be a sequence of time instants satisfying $\sup_j \{\delta_{j+1} - \delta_j\} = T < \infty$. For any given $\lambda \in (0, \lambda^-)$, determine the switching signal $\sigma(t)$ so that $\frac{T^-(\delta_j, \delta_{j+1})}{T^+(\delta_j, \delta_{j+1})} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*}$ holds on every time interval $[\delta_j, \delta_{j+1})$ ($j = 0, 1, \cdots$), where λ^+, λ^- and λ^* are the same as given in (S1). □

Samely as in our previous discussion, for any $t > 0$, we write $t_0 < t_1 < \cdots < t_i \leq t < t_{i+1}$ where t_j's ($j = 1, \cdots, i$) are the switching points, and we suppose that the p_j-th subsystem is activated during the interval $[t_{j-1}, t_j)$. Then, using the ordinary differential equation theory for (26), we have
$$x(t_1) = e^{A_{p_1}(t_1 - t_0)} x_0 + \int_{t_0}^{t_1} e^{A_{p_1}(t_1 - \tau)} f_{p_1}(\tau, x(\tau)) d\tau$$
$$\begin{aligned} x(t_2) &= e^{A_{p_2}(t_2 - t_1)} x(t_1) + \int_{t_1}^{t_2} e^{A_{p_2}(t_2 - \tau)} f_{p_2}(\tau, x(\tau)) d\tau \\ &= e^{A_{p_2}(t_2 - t_1)} e^{A_{p_1}(t_1 - t_0)} x_0 + \int_{t_0}^{t_1} e^{A_{p_2}(t_2 - t_1)} e^{A_{p_1}(t_1 - \tau)} \\ &\quad \times f_{p_1}(\tau, x(\tau)) d\tau + \int_{t_1}^{t_2} e^{A_{p_2}(t_2 - \tau)} f_{p_2}(\tau, x(\tau)) d\tau, \end{aligned} \quad (28)$$
and by induction,
$$\begin{aligned} x(t) &= e^{A_{p_{i+1}}(t - t_i)} e^{A_{p_i}(t_i - t_{i-1})} \cdots e^{A_{p_1}(t_1 - t_0)} x_0 \\ &+ \int_{t_0}^{t_1} e^{A_{p_{i+1}}(t - t_i)} e^{A_{p_i}(t_i - t_{i-1})} \cdots e^{A_{p_1}(t_1 - \tau)} f_{p_1}(\tau, x(\tau)) d\tau \\ &+ \int_{t_1}^{t_2} e^{A_{p_{i+1}}(t - t_i)} e^{A_{p_i}(t_i - t_{i-1})} \cdots e^{A_{p_2}(t_2 - \tau)} f_{p_2}(\tau, x(\tau)) d\tau \\ &+ \cdots + \int_{t_{i-1}}^{t_i} e^{A_{p_{i+1}}(t - t_i)} e^{A_{p_i}(t_i - \tau)} f_{p_i}(\tau, x(\tau)) d\tau \\ &+ \int_{t_i}^{t} e^{A_{p_{i+1}}(t - \tau)} f_{p_{i+1}}(\tau, x(\tau)) d\tau \,. \end{aligned} \quad (29)$$
We note that in the right hand side of the above equation, the first term corresponds to the case of switchings between linear subsystem A_i's during $[t_0, t)$, while the integral term $e^{A_{p_{i+1}}(t - t_i)} \cdots e^{A_{p_j}(t_j - \tau)}$ ($1 \leq j \leq i$) corresponds to the switchings during $[\tau, t)$. We also note that for any τ_1, τ_2 satisfying $\delta_{j-1} < \tau_1 \leq \delta_j < \delta_{j+1} < \cdots < \delta_k \leq \tau_2$, the state transition matrix $\Phi(\tau_1, \tau_2)$ from τ_1 to τ_2 (i.e., $x(\tau_2) = \Phi(\tau_1, \tau_2) x(\tau_1)$) will always satisfy, in the absence of perturbations,
$$\begin{aligned} \|\Phi(\tau_1, \tau_2)\| &\leq c e^{a N_\sigma(\tau_1, \tau_2)} e^{\lambda^+(\tau_2 - \delta_k)} \\ &\quad \times e^{\sum_{q=j}^{k-1} [\lambda^+ T^+(\delta_q, \delta_{q+1}) - \lambda^- T^-(\delta_q, \delta_{q+1})]} e^{\lambda^+(\delta_j - \tau_1)} \\ &\leq c e^{a N_\sigma(\tau_1, \tau_2)} e^{\lambda^+(\tau_2 - \delta_k)} e^{-\lambda^*(\delta_k - \delta_j)} e^{\lambda^+(\delta_j - \tau_1)} \\ &\leq \bar{c} e^{a N_\sigma(\tau_1, \tau_2) - \lambda^*(\tau_2 - \tau_1)} \end{aligned} \quad (30)$$
where $\bar{c} = c e^{2(\lambda^+ + \lambda^*) T}$, and the second inequality is due to action of the switching law (S2) over every interval $[\delta_q, \delta_{q+1})$. It is not difficult to see that (30) is also valid for any τ_1, τ_2 satisfying $\delta_q \leq \tau_1 \leq \tau_2 < \delta_{q+1}$ ($q \geq 0$). Then, with the average dwell time scheme: for any $\tau_1 < \tau_2$,
$$N_\sigma(\tau_1, \tau_2) \leq N_0 + \frac{\tau_2 - \tau_1}{\tau_a^*} \quad (31)$$
where $\tau_a^* = \frac{a}{\lambda^* - \lambda}$ and $N_0 = \frac{\alpha}{a}$ as in (10), we obtain
$$\|\Phi(\tau_1, \tau_2)\| < \bar{c} e^{\alpha - \lambda(\tau_2 - \tau_1)} \quad (32)$$
for any $\tau_1 \leq \tau_2$. From this fact and (29), we have
$$\begin{aligned} \|x(t)\| &\leq \bar{c} e^{\alpha - \lambda(t - t_0)} \|x_0\| \\ &+ \int_{t_0}^{t_1} \bar{c} e^{\alpha - \lambda(t - \tau)} (\gamma \|x(\tau)\| + \beta(\tau)) d\tau \\ &+ \cdots + \int_{t_{k-1}}^{t_k} \bar{c} e^{\alpha - \lambda(t - \tau)} (\gamma \|x(\tau)\| + \beta(\tau)) d\tau \\ &+ \int_{t_k}^{t} \bar{c} e^{\alpha - \lambda(t - \tau)} (\gamma \|x(\tau)\| + \beta(\tau)) d\tau \,, \end{aligned} \quad (33)$$

and thus,

$$\|x(t)\|e^{\lambda t} \le \bar{c}e^{\alpha}\left[e^{\lambda t_0}\|x_0\| + \int_{t_0}^{\infty} e^{\lambda\tau}\beta(\tau)d\tau\right] + \gamma\bar{c}e^{\alpha}\int_{t_0}^{t}\|x(\tau)\|e^{\lambda\tau}d\tau, \tag{34}$$

where the integral interval of the positive term $e^{\lambda\tau}\beta(\tau)$ has been enlarged from $[t_0, t)$ to $[t_0, \infty)$. By the Gronwall-Bellman Inequality (see, for example, [18]), we obtain

$$\|x(t)\|e^{\lambda t} \le \bar{c}e^{\alpha}\left[e^{\lambda t_0}\|x_0\| + \int_{t_0}^{\infty} e^{\lambda\tau}\beta(\tau)d\tau\right] e^{\gamma\bar{c}e^{\alpha}(t-t_0)}. \tag{35}$$

Multiplying both sides of the above inequality by $e^{-\lambda t}$, we obtain the following result:

Theorem 2. *Under the switching law* (S2), *there is a finite positive constant τ_a^* such that the solutions of the nonlinear switched system* (26) *with* (27) *satisfy*

$$\|x(t)\| \le \bar{c}e^{\alpha}\left[\|x_0\| + \int_{t_0}^{\infty} e^{\lambda\tau}\beta(\tau)d\tau\right] e^{-(\lambda-\gamma\bar{c}e^{\alpha})(t-t_0)} \tag{36}$$

over $S_a[\tau_a, N_0]$ for any average dwell time $\tau_a \ge \tau_a^*$ and any chatter bound $N_0 > 0$. Therefore, under the condition $\gamma < \frac{\lambda}{\bar{c}e^{\alpha}}$,

(1) if $\beta(t) \equiv 0$, then the nonlinear switched system is globally exponentially stable;

(2) if $\beta(t) \not\equiv 0$, then the solutions of the nonlinear switched system are uniformly bounded and converge to the origin exponentially. □

From the condition $\gamma < \frac{\lambda}{\bar{c}e^{\alpha}}$ and $\bar{c} = e^{2(\lambda^+ + \lambda^*)T}$, we see that small α and T are desirable when larger perturbations are admissible.

5. Conclusions

We have studied some stability properties of linear switched systems composed of both Hurwitz stable and unstable subsystems by using an average dwell time approach. We have shown that when the average dwell time is sufficiently large and the total activation time of the unstable subsystems is relatively small compared with that of the Hurwitz stable subsystems, then global exponential stability is guaranteed. For such switched systems under the switching law and the average dwell time proposed herein, a piecewise Lyapunov function has been derived. Finally, we have extended the above results to the case where nonlinear norm-bounded perturbations exist in the subsystems, and we have shown quantitatively that when the perturbations are small in the sense of norm, then the desired exponential convergence of the solutions can be obtained by adding appropriate restrictions to the switching law with the average dwell time sufficiently large. We suggest that the proposed average dwell time approach is practically useful in analyzing other qualitative properties of switched systems as well.

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