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New finite-time stability for fractional-order time-varying time-delay linear systems: A Lyapunov approach

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Abstract

The primary goal of this paper is to examine the finite-time stability and finite-time contractive stability of the linear systems in fractional domain with time-varying delays. We develop some sufficient criteria for finite-time contractive stability and finite-time stability utilizing fractional-order Lyapunov-Razumikhin technique. To validate the proposed conditions, two different types of dynamical systems are taken into account, one is general time-delay fractional-order system and another one is fractional-order linear time-varying time-delay system, furthermore the efficacy of the stability conditions is demonstrated numerically.

1. Introduction

A new conceptualization of stability was proposed by the Russian literature known as the finite-time stability (FTS). FTS is an idea to learn the action of a system with predefined time interval and it considers the behavior of the system after time T as irrelevant. A number of adequate and essential conditions of FTS were discussed in the literature for nonlinear systems, dynamic linear systems with impulses, and time-varying linear systems [1–5]. Also the fixed/finite-time stabilization problems for discontinuous impulsive systems were discussed

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with differential equations [6–8]. The foundational work of A.M. Lyapunov's provided a basic concept of Lyapunov function for stability analysis but that is extremely powerful. A lot of results are found and proved for stability via the sense of Lyapunov for different systems like parameter-varying linear system with non-monotonic terms, switched linear system with 3 dimensional and dynamical autonomous system [9–11]. The concept of FTS examined via the Lyapunov function (LF) is more natural than Lyapunov stability (LS).

In a finite interval, the notations of traditional FTS are described through the characteristics of boundedness. Whereas in the field of rockets, network controlled systems and robot controls also characterized the concept of contraction via these notations. Weiss and Infantane [12] proposed the FTCS concept demonstrating that the state, in addition to stay within a specified threshold for a particular time-frame, will also lie within a provided bound that is less than the original configuration bound before attaining the end time. In [13], discussed the concept of FTCS for Markovian jump linear systems and also in [14], researchers examined that the state had a fixed settling time within a predetermined constraint and created the idea of finite time contractive stability with fixed settling time (FTCSwFST) based on Lyapunov's FTCSwFST. The authors of [15], with the help of LFs, developed some results for FTS and FTCS concepts via the conditions of Lyapunov-Razumikhin for linear system with time delay. The Authors also added the concepts of LS, FTS or FTCS are independent. If the stability of solutions for time-delay equations is assessed using the Lyapunov function through finite-dimensional functions, it is considered to be a Lyapunov-Razumikhin (LR) function and the theory of Razumikhin has been utilized more extensively to demonstrate the stability of systems with time-delay [16–20].

The fundamental benefit of fractional-order models over integer-order models is that the fractional derivative is a powerful tool for describing memory and hereditary aspects of diverse processes. As a result, many real issues are characterized by fractional-order systems rather than integer-order systems. Also, the notion of fractional-order systems can be used to model physical systems with greater flexibility and accuracy when compared to integer-order systems. For this reason, it is wise to use fractional-order dynamical systems rather than integer-order systems to analyze the concept of stability (see the interesting monographs [21–24]). References [25–27] examined a fractional-order system's stability with time delays, as well as various fractional calculus applications. The advantage of the fractional-order derivative system over integer-order system is that the stability will be attained sooner in the former when compared to the latter. In addition, the global stability results in terms of linear matrix inequality (LMI) were presented in [28] and [29] using fractional-order neural networks without or with impulses, respectively. Furthermore, the Lyapunov technique is critical in the research of fractional-order system stability and stabilization findings [30–32]. The generic form of quadratic Lyapunov function (QLF) $x^T(t)\mathbb{P}x(t)$ is constructed to derive the basic stability results in the literature. The major conclusion of the fractional derivative is that the chain rule and the general Leibniz rule are not fulfilled. A novel technique known as product of three term functions (PTTFs) $x^T(t)\mathbb{P}(t)x(t)$ is handled fruitfully and the proper Leibniz rule for fractional-order system has been successfully applied in [33]. It should be noted that the product of QLFs cannot be directly extended of PTTFS.

As we all know, the time delay is uncontrollable in many domains of engineering and science and it is a source of a systems' performance deterioration and instability. Over the past few decades, in the area of control theory, control and stability has been an interesting area for researchers. Designing the suitable control and investigating the stability is always a complete analsysis, in this regard, authors in [34] have explored the effect fractional-order derivatives in the time-delay system and validated their results. Yet a question of how to define criteria for FTS and FTCS of fractional-order systems with time-delays via LR condition remains unknown and unsolved, these motivate our research work. The major contribution of this study is to examine the linear time-varying (LTV) time-delay system through analyzing their stability, which cannot be obtained directly via its coefficient even in the rare case because of the periodicity [35]. Besides, traditional stability analysis, we focus on deriving FTS and FTCS for a system with fractional-order derivative function. In this aspect, Caputo-sense fractional derivative and Lyapunov-Razmukihin based sufficient conditions are compatible with time-delay system. To exhibit the practical perspective of the proposed stability conditions, numerical examples are also provide.

The paper is sorted out as follows: Section 2 elaborates some sufficient definitions and notations. Section 3 presents the derived results. Section 4 provides numerical examples that demonstrate the theoretical findings. Section 5 sums up the paper to present the conclusion.

Notations: Let $\mathbb{C}$ and $\mathbb{D}$ be symmetric matrices that are positive or negative definite. $\mathbb{C}^T$ and $\mathbb{C}^{-1}$ represent transpose and inverse of a matrix $\mathbb{C}$ respectively and the identity matrix with the required dimensions is represented by $\mathbf{I}$. Let i denote the interval which is a subset of $\mathbb{R}$ that contains the origin and let E be a subset of $\mathbb{R}$, $\mathscr{C}(i, E) = \{\hat{\phi} : i \to E$ is continuous $\}$ and particularly, for a given delay $\mathfrak{r} > 0$, the set $\mathscr{C}_{\mathfrak{r}} = \mathscr{C}([-\mathfrak{r}, 0], \mathbb{R}^n)$. $\mathrm{K} = \{e(.) \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}_+) | e(0) = 0$ and $e(r) > 0$ for $r > 0$ and r is strictly increasing in r $\}$. Let n-dimensional real space be denoted as $\mathbb{R}^n$ that has the Euclidean norm $\| \bullet \|$. The norm function with the initial condition is defined as $\|\theta\| = sup_{r\in[-\mathfrak{r},0]}|\theta(r)|$. Consider the set of real and positive values which are denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively. $e \bigvee h = \max\{e, h\}$ where e and h are constants.

2. Preliminaries

Let us begin with the following distinct fractional-order systems for the investigation of FTS and FTCS.

Firstly, the system with time-delay in fractional-order general form is considered as following

$$\begin{cases} {}^{C}_{t}\mathcal{D}^{q}_{0}y(t) = f(t, y_t), \quad t \geq t_0, \\ y_{t_0} = \theta. \end{cases} \tag{1}$$

Here $y \in \mathbb{R}^n$ denotes the state variable of fractional-order time-delay system (1); $y_t \in \mathscr{C}_{\mathfrak{r}}$ is defined by $y_t(r) = y(t + r)$ for every $t \geq 0$ where $r \in [-\mathfrak{r}, 0]$, $\mathfrak{r}$ denotes the time-delay; $t_0 = 0$; ${}^{C}_{t}\mathcal{D}^{q}_{0}y(t)$ represents the fractional-order derivative of y in Caputo sense and if $q = 1$, then the system becomes integer-order system. $\theta \in \mathscr{C}_{\mathfrak{r}}$ is the initial value for that system and then for every t, $f(t, \theta) : [-\mathfrak{r}, T] \times \mathscr{C}_{\mathfrak{r}} \to \mathbb{R}$ is considered as a function which is continuous and lipschitz locally in θ.

Secondly, a linear time-varying delay system for fractional-order

$$\begin{cases} {}^{C}_{t}\mathcal{D}^{q}_{0}y(t) = \mathbb{C}(t)y(t) + \mathbb{D}(t)y(t - \mathfrak{r}), \quad t > 0 \\ y(t) = \theta(t), \quad t \in [-\mathfrak{r}, 0] \end{cases} \tag{2}$$

where $y \in \mathbb{R}^n$, $\mathbb{C}(t)$ and $\mathbb{D}(t) \in \mathbb{R}^{n\times n}$ are matrices for the above system.

Property 1 [33] (Leibeniz rule for fractional-order derivative): Consider two functions $\mathbb{A}_1(t)$ and $\mathbb{A}_2(t)$ are continuous with their derivatives in the interval $[0, t]$, then the fractional derivative of their product $\mathbb{A}_1(t)\mathbb{A}_2(t)$ that are given by

$$ {}_t^C\mathcal{D}_0^q(\mathbb{A}_1(t)\mathbb{A}_2(t)) = \sum_{l=0}^{\infty}\left(\begin{array}{c} q \\ l \end{array}\right)\mathbb{A}_1^l(t)\,{}_t^C\mathcal{D}_0^{q-l}\mathbb{A}_2(t). $$

Property 2 [33]: Consider $\mathfrak{J}(t, \mathfrak{h}(t)) = \mathfrak{h}^T(t)\mathbb{P}(t)\mathfrak{h}(t)$ where $\mathfrak{h}(t) \in \mathbb{R}^n$ is derivable and continuous function, then we obtain

$$ {}_t^C\mathcal{D}_0^q\mathfrak{J}(t, \mathfrak{h}(t)) \leq 2\mathfrak{h}^T(t)\mathbb{P}(t)\,{}_t^C\mathcal{D}_0^q\mathfrak{h}(t) + \mathfrak{h}^T(t)({}_t^C\mathcal{D}_0^q\mathbb{P}(t))\mathfrak{h}(t), $$

where $\mathbb{P}(t)$ is a positive definite matrix belonging to $\mathbb{R}^{n\times n}$.

Property 3 [22]: Let $\mathfrak{g}(t) \in \mathbb{R}^n$ be a derivable and continuous vector function defined in $[0, \infty]$ and $q \in (0, 1)$. Then for any time $t \geq 0$ it implies

$$ \frac{1}{2}\,{}_t^C\mathcal{D}_0^q(\mathfrak{g}^T(t)\mathfrak{g}(t)) \leq \mathfrak{g}^T(t)\,{}_t^C\mathcal{D}_0^q\mathfrak{g}(t). $$

Definition 2.1. [22] Let a function $y(t) \in \mathscr{C}^n([0, +\infty), \mathbb{R})$. The fractional derivative in Caputo sense of this function is given as

$$ {}_t^C\mathcal{D}_0^q y(t) = \frac{1}{\Gamma(1-q)}\int_0^t y'(\mathfrak{r})(t-\mathfrak{r})^{-q}\mathrm{d}\mathfrak{r}, \tag{3} $$

where the gamma function is represented as $\Gamma(\cdot)$ such that $\Gamma(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ and $0 < q < 1$.

Definition 2.2. [22] Let us consider an integral function $y(t) : [0, +\infty) \to \mathbb{R}$ then the fractional integral in Caputo of this function is given as

$$ {}_t^C\mathcal{I}_0^q y(t) = \frac{1}{\Gamma(q)}\int_0^t y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}, \tag{4} $$

where the gamma function is represented as $\Gamma(\cdot)$ and $0 < q < 1$.

Definition 2.3. [15] Let T, k_1 and k_2 are three positive constants with $k_2 > k_1$ then the system (1) is FTS corresponding to the conditions of (k_1, k_2, T) only when $||\theta||_{\mathfrak{r}} < k_1$ implies $|y(t)| < k_2$, $\forall\ t \in [0, T]$.

Definition 2.4. [15] There exist constants $T, k_1, k_2, \delta, \varepsilon > 0$ with $\varepsilon \in (0, T)$ and $k_2 > k_1 > \delta$ then the system (1) is FTCS corresponding to the conditions of $(k_1, k_2, \delta, \varepsilon, T)$ if $||\theta||_{\mathfrak{r}} < k_1$ implies $|y(t)| < k_2$, $\forall\ t \in [0, T]$ and additionally $|y(t)| < \delta$, $\forall\ t \in [T-\varepsilon, T]$.

3. Main results

FTS and FTCS for system (1) in fractional-order with Lyapunov-Razumikhin conditions are analyzed below.

Theorem 3.1. *Let us take the positive values k_1, k_2, δ, ε, T with $\delta < k_1 < k_2$, $\varepsilon \in (0, T)$, $\varrho(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is real valued function, the functions $d_1, d_2 \in$ K and the fractional-order integrable function $F : [-\mathfrak{r}, T] \times \mathbb{R}^n \to \mathbb{R}_+$*

(i) $d_1(|y|) \leq F(t, y) \leq d_2(|y|)$, $\forall (t, y) \in [-\mathfrak{r}, T] \times \mathbb{R}^n$;
(ii) ${}_t^C\mathcal{D}_0^q F(t, \hat{\phi}(0)) \leq \varrho(t)F(t, \hat{\phi}(0))$ *at any time* $F(t+k, \hat{\phi}(k)) \leq \chi(t, k)F(t, \hat{\phi}(0))$, $\forall t \in [0, T]$, $k \in [-\mathfrak{r}, 0]$ *and* $\hat{\phi} \in \mathscr{C}_{\mathfrak{r}}$, *where* $\chi(t, k)$ *is defined as follows* $\chi(t, k) = \int_{(t+k)\bigtriangledown 0}^t y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}$.

(iii) $\frac{1}{\Gamma(q)}({}_t^C\mathcal{D}_0^q y(t)) \leq \frac{d_1(k_2)}{d_2(k_1)}$, $\forall\ t \in [0, T]$ *then system (1) satisfies FTS corresponding to the conditions of the constants* (k_1, k_2, T). *In addition* $\frac{1}{\Gamma(q)}({}_t^C\mathcal{D}_0^q y(t)) \leq \frac{d_1(\delta)}{d_2(k_1)}$, $\forall t \in [T-\varepsilon, T]$, *then system (1) satisfies FTCS corresponding to the conditions of constants* $(k_1, k_2, \delta, \varepsilon, T)$.

Proof. Consider $F(t) = F(t, x(t))$, $F_0 = sup_{r\in[-\mathfrak{r},0]}F(E)$ and even if $(0, \theta)$, $\theta \in \mathscr{C}_{\mathfrak{r}}$, $y(t) = y(t, 0, \theta)$ is a solution to the system (1). We assert that

$$\left(\int_0^t y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t) \leq F_0, \tag{5}$$

for all $t \in [0, T]$. Now define $\varsigma(t)$ to claim the above inequality by contradiction method $\varsigma(t) = \left(-\int_0^{t\,\nabla 0} y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t)$, for all $t \in [-\mathfrak{r}, T]$.

To display (5), it is enough to prove $\varsigma(t) \leq F_0$, for all $t \in [0, T]$. Suppose the condition does not hold for certain values on $(0, T]$, then there exists a $t^* \in (0, T]$, such that ${}_{t^*}^C\mathcal{D}_0^q\varsigma(t) \geq 0$, $\varsigma(t) \leq F_0$ and $\varsigma(t^*) = F_0$ for all $t \in [0, t^*]$ (Note: $\varsigma(0) = F(0) \leq F_0$ is obtained when $t = 0$). As a result of the definition of $\varsigma(t)$, it follows that $\varsigma(t) \leq F_0$ for all $t \in [-\mathfrak{r}, 0]$ which means $\varsigma(t) \leq F_0 = \varsigma(t^*)$, $\forall t \in [-\mathfrak{r}, t^*]$.

By following the above inequality, we get

$$\left(\int_0^{t\,\nabla 0} y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t) \leq \left(\int_0^{t^*} y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t^*),$$

$$F(t) \leq \left(\int_{t\,\nabla 0}^{t^*} y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t^*).$$

after that, it follows for all $h \in [-\mathfrak{r}, 0]$

$$F(t^*+h) \leq \left(\int_{(t^*+h)\,\nabla 0}^{t^*} y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t^*),$$

comparing with (ii) then it is satisfies ${}_t^C\mathcal{D}_0^q F(t) \leq \varrho(t)F(t)$ at time $t = t^*$.

$$\begin{aligned}
\varsigma(t)|_{t=t^*} &\leq \left(-\int_0^{t^*} y(\mathfrak{r})(t^*-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t^*),\\
&= \left(\frac{\Gamma(q)}{\Gamma(q)}\right)\left(-\int_0^{t^*} y(\mathfrak{r})(t^*-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t^*),\\
&= -\Gamma(q)\left({}_{t^*}^C\mathcal{I}_0^q y(t^*)\right)F(t^*).
\end{aligned}$$

Now we calculate fractional derivative,

$$\begin{aligned}
{}_t^C\mathcal{D}_0^q\varsigma(t)|_{t=t^*} &\leq -(\Gamma(q))\,{}_{t^*}^C\mathcal{D}_0^q[\left({}_{t^*}^C\mathcal{I}_0^q y(t^*)\right)F(t^*)],\\
&\leq -(\Gamma(q))\sum_{l=0}^{\infty}\begin{pmatrix} q \\ l \end{pmatrix}({}_{t^*}^C\mathcal{I}_0^q y(t^*))^l({}_{t^*}^C\mathcal{D}_0^{q-l}F(t^*)).
\end{aligned}$$

Finally we get contradiction ${}_t^C\mathcal{D}_0^q\varsigma(t)|_{t=t^*} \leq 0$ to our assumption ${}_t^C\mathcal{D}_0^q\varsigma(t)|_{t=t^*} \geq 0$. Hence, fulfilled the inequality (5) by the outcome of that contradiction result. It follows from condition (iii) and (5) that when $\|\theta\|_{\mathfrak{r}} < k_1$,

$$d_1(|y(t)|) \leq \left(\int_0^t y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t) \leq F_0,$$

$$d_1(|y(t)|) \leq \Gamma(q)\big({}_t^C\mathcal{I}_0^q y(t)\big)F(t) \leq F_0,$$

$$\begin{aligned} F(t) &\leq \frac{1}{\Gamma(q)}\big({}_t^C\mathcal{D}_0^q y(t)\big)d_2(\|\theta\|)_{\mathfrak{r}}, \\ &< \frac{1}{\Gamma(q)}({}_t^C\mathcal{D}_0^q y(t))d_2(k_1) \leq d_1(k_2), \end{aligned} \tag{6}$$

$\forall t \in [0, T]$. From the inequality (6) we obtain $|y(t)| < k_2, \forall t \in [0, T]$. Hence from the result, system (1) satisfies FTS corresponding to the conditions of constants (k_1, k_2, δ). System (1) will fulfill the FTCS, as shown by inequality (6).

Therefore,

$$d_1(|y(t)|) \leq F(t) < \frac{1}{\Gamma(q)}({}_t^C\mathcal{D}_0^q y(t))d_2(k_1) \leq d_1(\delta),$$

for all $t \in [T-\varepsilon, T]$. From the above inequality, system (1) is FTCS when $|y(t)| < \delta$, $\forall t \in [T-\varepsilon, T]$ corresponding to the conditions of the constants $(k_1, k_2, T, \delta, \varepsilon)$. $\square$

Remark : It should be noted that condition (ii) in Theorem 3.1 is of the Razumikhin type. The notable difference between the FTCS and FTS is, in the view of FTCS it is necessary to consider the information about the boundedness and the contraction properties, whereas FTS is not required to have those information. In Theorem 3.1, displayed FTS and FTCS of fractional-order time-delay system by Lyapunov-Razumikhin method with adequate conditions. Especially, predefined bounds (k_1, k_2, δ) and the function $\varrho(t)$ both are established their relationship over a finite period of time.

Theorem 3.2. *Suppose there exist non zero positive scalars T, k_1, k_2, δ and ε in addition to $\delta < k_1 < k_2$ and $\varepsilon \in (0, T)$, real-valued function $\varrho(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ and $\rho(t) : [0, T] \to \mathbb{R}_+$ in such a way*

(i) $\mathbb{C}(t) + \mathbb{C}^T(t) + \frac{1}{\rho(t)}\mathbb{D}(t)\mathbb{D}^T(t) + \rho(t)\chi(t, -\mathfrak{r})\mathbf{I} \leq \varrho(t)\mathbf{I}$, $\forall\ t \in [0, T]$. *Where* $\chi(t, -\mathfrak{r})$ *defined as*
$\chi(t, -\mathfrak{r}) = \int_{(t-\mathfrak{r})\,\nabla\, 0}^{t} y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}$.

(ii) *If we obtained* $\frac{1}{\Gamma(q)}{}_t^C\mathcal{D}_0^q y(t) \leq \frac{k_2}{k_1}$, $\forall t \in [0, T]$, *then the system (2) is FTS corresponding to the conditions of the constants* (k_1, k_2, T). $\frac{1}{\Gamma(q)}{}_t^C\mathcal{D}_0^q y(t) \leq \frac{\delta}{k_1}$, $\forall t \in [T-\varepsilon, T]$ *then the system (2) is FTCS corresponding to the conditions of the constants* $(T, k_1, k_2, \delta, \varepsilon)$.

Proof. In this theorem, first we discuss about Razumikhin condition. Let the Lyapunov function $F(t, y(t)) = y^T(t)y(t)$. Consider the condition like (ii) in Theorem 3.1 is supplied by $y^T(t-\mathfrak{r})y(t-\mathfrak{r}) \leq \chi(t, -\mathfrak{r})y^T(t)y(t), \forall t \in [0, T]$. If the vectors $j, h \in \mathbb{R}^n$ satisfies the condition $\pm 2j^T h \leq \frac{1}{k}j^T j + kh^T h$ where $k > 0$.

$$\begin{aligned} {}_t^C\mathcal{D}_0^q F(t, y(t)) &= {}_t^C\mathcal{D}_0^q(y^T(t)y(t)) \\ &\leq 2y^T(t){}_t^C\mathcal{D}_0^q y(t) \\ &= 2y^T(t)[\mathbb{C}(t)y(t) + \mathbb{D}(t)y(t-\mathfrak{r})], \\ &= y^T(t)[\mathbb{C}(t) + \mathbb{C}^T(t)]y(t) + 2[\mathbb{D}^T(t)y(t)]^T y(t-\mathfrak{r}), \\ &\leq y^T(t)[\mathbb{C}(t) + \mathbb{C}^T(t)]y(t) + \frac{1}{\rho(t)}[y^T(t)\mathbb{D}(t)\mathbb{D}^T(t)y(t)] \\ &\quad + \rho(t)[y^T(t-\mathfrak{r})y(t-\mathfrak{r})], \end{aligned}$$

$$= y^T(t)[\mathbb{C}(t) + \mathbb{C}^T(t) + \frac{1}{\rho(t)}\mathbb{D}(t)\mathbb{D}^T(t) + \rho(t)\chi(t-\mathfrak{r})\mathbf{I}]y(t),$$
$$\leq y^T(t)\varrho(t)\mathbf{I}y(t),$$

$${}^C_0\mathcal{D}^q_t F(t, y(t)) \leq \varrho(t)F(t, y(t)).$$

So here after we proceed the procedure same as Theorem 3.1 to prove system (2) that has FTS corresponding to the conditions of the constants (T, k_1, k_2) and FTCS corresponding to the conditions of the constants $(T, k_1, k_2, \delta, \varepsilon)$. For this purpose, next we want to claim

$$\left(\int_0^t y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t) \leq F_0. \tag{7}$$

Equation (7) will be obtained like Theorem 3.1. This may produce a contradiction proof same as in Theorem 3.1, i.e., ${}^C_q\mathcal{D}^0_t\varsigma(t)|_{t=t^*} \geq 0$ and consequently (7) holds.

Here $\varsigma(t)$ is define as

$\varsigma(t) = \left(\int_0^t y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t), \ \forall t \in [0, T].$

And the outcome of that contradiction result is satisfied with the following condition

$F(t^* - h) \leq \left(\int_{(t^*-h)\nabla 0}^{t^*} y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)(F(t^*)) \ , \forall h \in [-\mathfrak{r}, 0].$

The above inequality satisfies the condition like (ii) in Theorem 3.1 at $t = t^*$. From the above procedure, system (2) will obtain the Razumikhin condition. From (7) and condition (ii) we obtained $||\omega||_{\mathfrak{r}} < k_1$,

$$(|y(t)|) \leq \frac{\Gamma(q)}{\Gamma(q)}\left(\int_0^t y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}\right)F(t) \leq F_0,$$
$$(|y(t)|) \leq \Gamma(q)({}^C_0I^q_t y(t))F(t) \leq F_0,$$
$$(|y(t)|) \leq F(t) \leq \frac{1}{\Gamma(q)}{}^C_0D^q_t y(t)F_0,$$
$$\leq \frac{1}{\Gamma(q)}{}^C_0D^q_t y(t)(||\omega||)_{\mathfrak{r}},$$
$$< \frac{1}{\Gamma(q)}{}^C_0D^q_t y(t)(k_1) \leq (|y(t)|),$$
$$(|y(t)|) < \frac{1}{\Gamma(q)}{}^C_0D^q_t y(t)(k_1) \leq k_2,$$
$$\frac{1}{\Gamma(q)}{}^C_0D^q_t y(t) \leq \frac{k_2}{k_1}, \forall t \in [0, T],$$

which yields that $|y(t)| < k_2$, $\forall\ t \in [0, T]$. Hence the fractional-order linear system (2) is FTS corresponding to the conditions of the constants (T, k_1, k_2).

$$|y(t)| \leq F(t) < \frac{1}{\Gamma(q)}{}^C_0D^q_t y(t)(k_1) \leq \delta$$
$$\frac{1}{\Gamma(q)}{}^C_tD^q_0 y(t) \leq \frac{\delta}{k_1}, \forall t \in [T-\varepsilon, T]$$

which claims $|y(t)| < \delta, \forall t \in [T-\varepsilon, T]$. Thus the fractional-order linear system (2) is FTCS corresponding to the conditions of the constants $(T, k_1, k_2, \delta, \varepsilon)$. □

Theorem 3.3. *Suppose there exist non zero positive scalars T, k_1, k_2, δ and ε with $\delta < k_1 < k_2$ and $\varepsilon \in (0, T)$, then $\aleph(t) : [0, T] \to \mathbb{R}_+$ and $\varrho(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ in such a way*

(i)

$$\mathbb{P}^{-1}(t)\,{}_t^C\mathcal{D}_0^q\mathbb{P}(t) + \mathbb{P}^{-1}(t)\mathbb{C}^T(t)\mathbb{P}(t) + \mathbb{C}(t) + \mathbb{P}^{-1}(t)\aleph(t)\chi(t, -\mathfrak{r}) + \frac{1}{\aleph(t)}\mathbb{D}(t)\mathbb{D}^T(t)\mathbb{P}^T(t) \le \varrho(t)\mathbf{I},$$

where $t \in [0, T]$ and $\chi(t, -\mathfrak{r})$ defined as $\chi(t, -\mathfrak{r}) = \int_{(t-\mathfrak{r})\nabla 0}^{t} y(\mathfrak{r})(t-\mathfrak{r})^{q-1}\mathrm{d}\mathfrak{r}$.

(ii) *If we obtain $\frac{1}{\Gamma(q)}({}_t^C\mathcal{D}_0^q y(t)) \le \frac{k_2}{k_1}$, $\forall t \in [0, T]$, then the system (2) is FTS corresponding to the conditions of the constants (k_1, k_2, T). $\frac{1}{\Gamma(q)}({}_t^C\mathcal{D}_0^q y(t)) \le \frac{\delta}{k_1}$, $\forall t \in [T-\varepsilon, T]$, then the system (2) is FTCS corresponding to the conditions of constants $(T, k_1, k_2, \delta, \varepsilon)$.*

Proof. In this Theorem, via the Lyapunov function $F(t, y(t)) = y^T(t)\mathbb{P}(t)y(t)$, first we discuss about the Lyapunov-Razumikhin condition with positive definite matrix $\mathbb{P}(t) \in \mathbb{R}^{n\times n}$. The Lyapunov-Razumikhin condition is supplied by the inequality $y^T(t-\mathfrak{r})y(t-\mathfrak{r}) \le \chi(t, -\mathfrak{r})y^T(t)y(t), \forall t \in [0, T]$ follows from condition (ii) in Theorem 3.1. If the vectors $j, h \in \mathbb{R}^n$, satisfies the condition $\pm 2j^T h \le \frac{1}{k} j^T j + kh^T h$ where $k > 0$.

$$\begin{aligned}
&{}_t^C\mathcal{D}_0^q F(t, y(t)) \\
&= {}_t^C\mathcal{D}_0^q(y^T(t)\mathbb{P}(t)y(t)), \\
&\le 2y^T(t)\mathbb{P}(t)\,{}_t^C\mathcal{D}_0^q y(t) + y^T(t)({}_t^C\mathcal{D}_0^q\mathbb{P}(t))y(t) \\
&= 2y^T(t)\mathbb{P}(t)[\mathbb{C}(t)y(t) + \mathbb{D}(t)y(t-\mathfrak{r})] + y^T(t)({}_t^C\mathcal{D}_0^q\mathbb{P}(t))y(t), \\
&\le y^T(t)\mathbb{P}(t)[\mathbb{C}(t) + \mathbb{C}^T(t)]y(t) + 2(\mathbb{D}^T(t)\mathbb{P}^T(t)y(t))^T y(t-\mathfrak{r}) + y^T(t)({}_t^C\mathcal{D}_0^q\mathbb{P}(t))y(t), \\
&\le y^T(t)[\mathbb{P}(t)\mathbb{C}(t) + \mathbb{C}^T(t)\mathbb{P}(t)]y(t) + \frac{1}{\aleph(t)}y^T(t)\mathbb{P}(t)\mathbb{D}(t)\mathbb{D}^T(t)\mathbb{P}^T(t)y(t) \\
&\quad + \aleph(t)y^T(t-\mathfrak{r})y(t-\mathfrak{r}) + y^T(t)({}_t^C\mathcal{D}_0^q\mathbb{P}(t))y(t), \\
&= y^T(t)\mathbf{I}[\mathbb{P}(t)\mathbb{C}(t) + \mathbb{C}^T(t)\mathbb{P}(t) + \frac{1}{\aleph(t)}\mathbb{P}(t)\mathbb{D}(t)\mathbb{D}^T(t)\mathbb{P}^T(t) + \aleph(t)\chi(t, -\mathfrak{r}) + {}_t^C\mathcal{D}_0^q\mathbb{P}(t)]y(t), \\
&= y^T(t)\mathbb{P}(t)\mathbb{P}^{-1}(t)[\mathbb{P}(t)\mathbb{C}(t) + \mathbb{C}^T(t)\mathbb{P}(t) + \frac{1}{\aleph(t)}\mathbb{P}(t)\mathbb{D}(t)\mathbb{D}^T(t)\mathbb{P}^T(t) \\
&\quad + \aleph(t)\chi(t, -\mathfrak{r}) + {}_t^C\mathcal{D}_0^q\mathbb{P}(t)]y(t), \\
&= y^T(t)\mathbb{P}(t)[\mathbb{P}^{-1}(t)\,{}_t^C\mathcal{D}_0^q\mathbb{P}(t) + \mathbb{C}(t) + \mathbb{P}^{-1}(t)\mathbb{C}^T(t)\mathbb{P}(t) + \mathbb{P}^{-1}(t)\aleph(t)\chi(t, -\mathfrak{r}) \\
&\quad + \frac{1}{\aleph(t)}\mathbb{D}(t)\mathbb{D}^T(t)\mathbb{P}^T(t)]y(t), \\
&\le y^T(t)\mathbb{P}(t)\varrho(t)\mathbf{I}y(t), \\
&{}_t^C\mathcal{D}_0^q F(t, y(t)) \le \varrho(t)F(t, y(t)).
\end{aligned}$$

From the above procedure, we will obtain the Razumikhin condition for system (2). So here after we proceed the procedure same as Theorem 3.2 to prove system (2) has FTS and FTCS corresponding to the conditions of constants $(T, k_1, k_2, \delta, \varepsilon)$. □

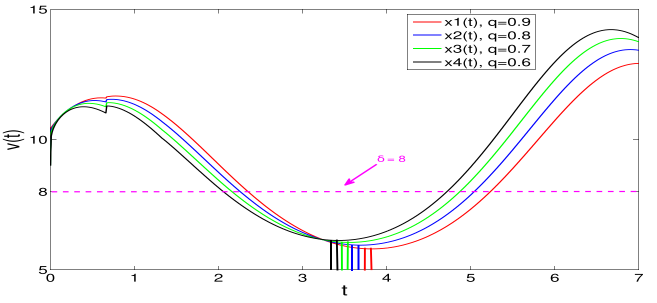

Fig. 1. State trajectories of different fractional-orders in Example 1.

4. Numerical examples

In this section, we discuss the above theoretical result's potency by three examples.

Example 4.1. Basic ball motions model are depicted in Fig. 1. Consider a ball whose velocity is denoted by the variable v (unit ms^{-1}) with the condition $v(0) \leq 10(ms^{-1})$. The system with time-delay $\forall t \in [0, T]$ is defined as

$$ {}_0^C\mathcal{D}_t^q v(t) = (0.5)[(\cos t - 1)v(t) + v(t - \mathfrak{r})]. \tag{8}$$

By using the concept of FTS, the main objective is to determine the velocity of the ball over a finite interval, say, $[0, T]$ and also find their minimum values with respect to different fractional-orders in (8).

Assume that $\mathfrak{r} = 0.55$ and $T = 4$. In Fig. 1, shows the different trajectories of the ball for different fractional-orders $q = 0.9$, $q = 0.8$, $q = 0.7$ and $q = 0.6$. From the concept of FTS in Theorem 3.1, Velocity of the ball can be evaluated as $v(t) \leq 16\ ms^{-1}$ for $t \in [0, 4]$. Also, velocity of the ball is evaluated by the concept of FTCS is $v(t) \leq 8\ ms^{-1}$ in $t \in [3, 4]$ but in the sense of Lyapunov stability it's not stable. When compared to the integer-order system's velocity result of [15], the velocity of the ball is evaluated quicker within a finite interval. The minimum value of the trajectories is obtained faster as we reduce the fractional-order in $t \in [3, 4]$ as shown in Fig. 1. At order $q = 0.9$, $min(x1(t))$ lies in $[3.77, 3.87]$, order $q = 0.8$, $min(x2(t))$ lies in $[3.64, 3.74]$, order $q = 0.7$, $min(x3(t))$ lies in $[3.50, 3.60]$ and order $q = 0.6$, $min(x4(t))$ lies in $[3.38, 3.48]$.

Example 4.2. Let LTV system (2) with parameters:

$$\mathbb{C}(t) = \begin{pmatrix} \frac{1}{20} + 0.7\cos 0.5t & -0.6 \\ 0.6 & \frac{1}{20} + 0.7\cos 0.5t \end{pmatrix},$$

$$\mathbb{D}(t) = \begin{pmatrix} 0.01\cos 0.5t & 0 \\ -0.01 & -0.01 \end{pmatrix}.$$

Take $k_1 = 3.62$, $k_2 = 22$, $T = 9$, $\delta = 3.5$ and $\varepsilon = 1$. Pick $\aleph(t) = 0.01$ for $t \in [0, T]$ and $\mathfrak{r} = 0.1$. Then, the concept of Theorem 3.2: System (2) is FTCS and FTS by the considered parameters and the stability trajectories are shown in Fig. 2 for different fractional-orders. The chosen parameters satisfy the conditions which are considered in Theorem 3.2. By simulating the state, one can see that the system becomes divergent as time T increases. In the case of FTCS, it does not a matter what happens after time T. Expressing the stability criteria for fractional-order linear time-varying systems is more effective when compared with integer-order systems [15].

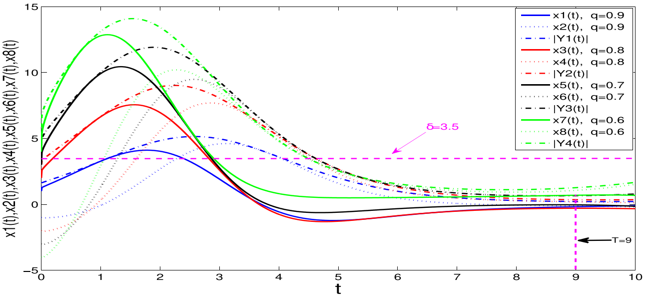

Fig. 2. State trajectories of different fractional-orders in Example 2.

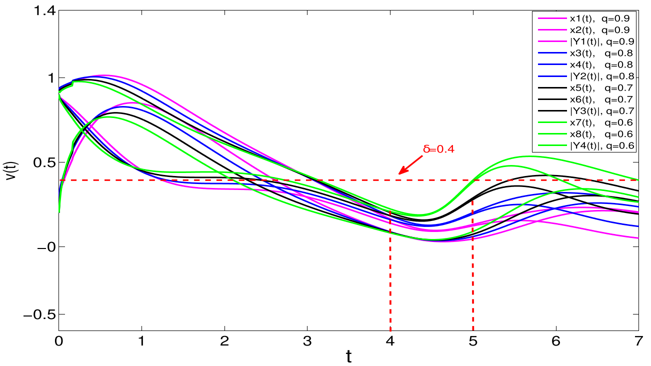

Fig. 3. State trajectories of different fractional-orders in Example 3.

Example 4.3. Take a time-varying fractional-order system in 2-dimension

$$ {}_{0}^{C}\mathcal{D}_t^q x_1(t) = \hat{\hat{\nu}}(t)x_1(t) + \hat{\hat{\Lambda}}(t)x_2(t-\mathfrak{r}) + \hat{\hat{\imath}}(t)x_2(t), $$

$$ {}_{0}^{C}\mathcal{D}_t^q x_2(t) = \breve{\mu}(t). \tag{9} $$

Where $\hat{\hat{\nu}}(t)$, $\hat{\hat{\Lambda}}(t)$ and $\hat{\hat{\imath}}(t)$ are continuous function :$[0, T] \to \mathbb{R}$, $T \leq \mathfrak{r}$ and the control input is $\breve{\mu}(t) \in \mathbb{R}$. Let us take the positive scalars k_1, k_2, δ, ε, T with the condition $\delta < k_1 < k_2$, $\varepsilon \in (0, T)$ and function $\varpi(.) \in \mathscr{C}^1([0, T], \mathbb{R}_+)$. A system (9) is FTCS with corresponding to the conditions of constants $(T, k_1, k_2, \delta, \varepsilon)$ via the law of memoryless control

$$ \breve{\mu}(t) = -\hat{\hat{\imath}}(t)x_1(t) + \left(\hat{\hat{\nu}}(t) + \frac{\hat{\hat{\Lambda}}^2(t)}{2\kappa(t)}\right)x_2(t), $$

now we consider $\mathfrak{r} = T = 5$ and

$$ \hat{\hat{\nu}}(t) = \frac{1}{2}\left[\frac{9\cos t - 10}{10 + 9\sin t} - 1\right], \hat{\hat{\Lambda}}(t) = 1, \hat{\hat{\imath}}(t) = \frac{1}{2}\cos t. $$

Take $\varpi(t) = 0.9\sin t + 1$ and $k_1 = 2$, $k_2 = 2.4$, $\delta = 0.4$ and $\varepsilon = 0.5$. To evaluate FTS and FTCS criteria for fractional-order systems via simple control. The time-varying fractional-order system (9) satisfies the concepts of FTCS and FTS by using the law of memoryless control $\breve{\mu}(t)$ with respect to the chosen parameters $(5, 2, 2.4, 0.4, 0.5)$ and the chosen parameters are selected by satisfy the conditions of FTS and FTCS. Fig. 3 depicts state trajectories for various fractional-orders of system (9), and this technique yields more accurate stability results as compared to integer-order system stability results [15].

5. Conclusion

By the method of LR, some adequate conditions are discussed, and the concepts of FTS and FTCS are analyzed for fractional-order systems. The proposed Razumikhin theorem was extended to fractional-order time-delay systems. The functionality of the theoretical results for fractional-order systems is illustrated by the numerical examples. When comparing the fractional-order systems with the integer-order systems, the stability criteria are more effective and accurate within the finite-time intervals. This study leads to our future work, which will focus on deriving the concepts of FTS and FTCS for switched nonlinear fractional-order systems via the approach of LR.

Declaration of Competing Interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome. We further confirm that any aspect of the work covered in this manuscript that has involved human patients has been conducted with the ethical approval of all relevant bodies and that such approvals are acknowledged within the manuscript.

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