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Stability and stabilization for a class of fractional-order linear time-delay systems

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Abstract: Using the fractional Razumikhin theorem, this paper investigates the stability and the stabilization of fractional-order linear time-delay systems, and presents a number of new results. First, a sufficient condition of asymptotical stability for the fractional-order linear time-delay systems is given by the fractional Razumikhin theorem. Then, by designing state feedback controller for the fractional linear time-delay controlled systems, asymptotical stabilization for closed-loop systems is discussed. Finally, two illustrative examples are provided to illustrate main results.

Key Words: Fractional-order system, Time-delay system, Stability, Stabilization, Fractional Razumikhin theorem

1 Introduction

Fractional order systems have been of great interest in the last two decades. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications. Apart from different areas of mathematics, fractional order systems play an important role in physics, chemistry, engineering and so on [1, 2]. Meanwhile, the existence of the time delays in dynamical systems is often the cause of instability and poor performance. The stability problem of fractional-order time-delay systems is a true challenge and has received considerable attention.

In the past half a century, several studies have been made to analyze the stability of a class of more general fractional-order systems. The earliest study on the stability of fractional order systems can be traced back to 1996 [3]. There are a lot of papers on the stability and stabilization of fractional-order time-delay systems, see [4–7]. Recently, the Lyapunov function method has also been used to study the stability of fractional-order systems [8–12]. Some Lyapunov functions were constructed in [9] and the classic Lyapunov function method was considered to stabilize fractional-order time-delay systems. Compared with integer-order nonlinear time-delay systems, D. Baleanu et al. [10] extended and presented the fractional Razumikhin theorem for fractional-order nonlinear time-delay systems. It should be pointed out that it is usually difficult to construct a positive definite function and calculate its fractional derivative for a given fractional-order system with/without time delays. To this end, Zhao et al. [11, 12] presented some new and useful properties for Caputo fractional derivative which allowed finding a general Lyapunov candidate function for the given fractional order systems. These results are very important in the sense that they have provided a basic tool for the stability analysis and control design of fractional order control systems. However, much more work needs to be done in this area.

To the authors' best knowledge, fewer works have been considered the stability and stabilization of fractional-order linear time-delay systems using the fractional Razumikhin theorem. In this paper, by the fractional Razumikhin theorem, we investigate the stability and stabilization of fractional-order time-delay systems. The main contributions of this paper are as follows. (i) A sufficient condition of asymptotical stability for the fractional-order linear time-delay systems is given based on the Lyapunov function method. (ii) By designing state feedback controller for fractional order linear time-delay controlled systems, asymptotical stabilization for closed-loop systems is considered.

The rest of this paper is organized as follows. Section 2 presents some necessary preliminaries on fractional calculus. Section 3 discusses stability and stabilization of fractional-order linear time-delay systems, and presents the main results of this paper. Section 4 gives two illustrative examples to illustrate our new results, which is followed by the conclusion in Section 5.

Notation. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{n\times n}$ denotes the set of $n \times n$ real matrices. For real symmetric matrices M and N, the notation $M > N$ ($M \geq N$) means that matrix $M - N$ is positive definite (positive semi-definite), and similarly, $M < N$ ($M \leq N$) means that matrix $M - N$ is negative definite (negative semi-definite). I is the identity matrix with appropriate dimension. P^T and P^{-1} represent the transpose and the inverse of matrix P, respectively. For a symmetric matrix $P \in \mathbb{R}^{n\times n}$, $\bar{\lambda}(P)$ and $\underline{\lambda}(P)$ denote the largest and the smallest eigenvalues of P, respectively. $\|\cdot\|$ denotes the Euclidean norm for a vector, or the induced Euclidean norm for a matrix.

2 Preliminaries

In this section, we give some definitions and lemmas which will be used in the sequel. These definitions and lemmas can be found in the recent literatures, see [2, 11].

Definition 1 *([2]) Caputo fractional derivative of order $\alpha > 0$ of a continuous function $g(t)$ is given by*

$$ {}_0^C D_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, $$

where n is the smallest integer greater than or equal to α and $\Gamma(\cdot)$ denotes the Gamma function, provided that the right side is pointwise defined on $(0, +\infty)$.

The Leibniz rule for fractional differentiation is the following.

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Lemma 1 *([2]) If f and g along with all its derivatives are continuous in $(0,+\infty)$, then the Leibniz rule for Caputo fractional derivative takes the form*

$$\begin{aligned} & {}_0^C D_t^\alpha \left(f(t)g(t)\right) \\ = & \sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} f^{(k)}(t)\, {}_0^C D_t^{\alpha-k} g(t), \end{aligned}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Remark 1 *By Lemma 1, we can easily see that*

$${}_0^C D_t^\alpha \left(f(t)g(t)\right) \neq {}_0^C D_t^\alpha f(t) g(t) + f(t) {}_0^C D_t^\alpha g(t),$$

where $\alpha \in (0,1)$. Obviously, the Leibniz rule for Caputo fractional derivative does not have the form like that for classical derivative.

Some properties for Caputo fractional derivative of a general quadratic function are given in the following.

Lemma 2 *([11]) Let $\alpha \in (0,1]$, $x(t) = (x_1(t),\ x_2(t),\ \cdots,\ x_n(t))^T \in \mathbb{R}^n$ and $x_i(t)$ $(i = 1,\ 2,\ \cdots,\ n)$ be continuous and derivable functions. Then, for any time instant $t \geq 0$, there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$, such that*

$$\frac{1}{2} {}_0^C D_t^\alpha x^T(t) P x(t) \leq x^T(t) P {}_0^C D_t^\alpha x(t). \tag{1}$$

Remark 2 *([11]) Inequality (1) is equivalent to one of the following inequalities, respectively:*

$$\frac{1}{2} {}_0^C D_t^\alpha x^T(t) P x(t) \leq \left({}_0^C D_t^\alpha x(t)\right)^T P x(t),$$

$${}_0^C D_t^\alpha x^T(t) P x(t) \leq \left({}_0^C D_t^\alpha x(t)\right)^T P x(t) + x^T(t) P {}_0^C D_t^\alpha x(t). \tag{2}$$

Next, we recall a recent useful result on the fractional Razumikhin theorem.

Lemma 3 *([10]) Consider the fractional-order nonlinear time-delay systems*

$$\begin{aligned} {}_0^C D_t^\alpha x(t) &= f(t, x(t-\tau)), \\ x(\theta) &= \varphi(\theta),\ \theta \in [-r, 0], \end{aligned} \tag{3}$$

where ${}_0^C D_t^\alpha$ denotes Caputo fractional derivative, $0 < \alpha \leq 1$, $\tau = \tau(t)$ is a continuous function satisfying $0 \leq \tau(t) \leq r$. Suppose that $f:\ \mathbb{R}\times C \to \mathbb{R}^n$ in (3) maps $\mathbb{R}\times$(bounded sets in $C([-r,0],\mathbb{R}^n)$ into bounded sets in $\mathbb{R}^n$, and $\gamma_1,\ \gamma_2,\ \gamma_3 :$ $[0,+\infty) \to [0,+\infty)$ are continuous nondecreasing functions, $\gamma_1(s),\ \gamma_2(s)$ are positive for $s > 0$, and $\gamma_1(s) = \gamma_2(s) = 0$, γ_2 is strictly increasing. If there exists a continuously differentiable function $V :\ \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ such that

$$\gamma_1(\|x\|) \leq V(t,x) \leq \gamma_2(\|x\|),$$

for $t \in \mathbb{R},\ x \in \mathbb{R}^n$, and the Caputo fractional derivative of V along the solution $x(t)$ of (3) satisfies

$${}_0^C D_t^\alpha V(t, x(t)) \leq -\gamma_3(\|x\|),$$

whenever $V(t+\theta, x(t+\theta)) \leq V(t, x(t))$, for $0 < \alpha \leq 1$ and $\theta \in [-r, 0]$, then system (3) is uniformly stable. If, in addition, $\gamma_3(s) > 0$ for $s > 0$ and there exists a continuous nondecreasing function $l(s) > s$ for $s > 0$ such that

$${}_0^C D_t^\alpha V(t, x(t)) \leq -\gamma_3(\|x\|),$$

if $V(t+\theta, x(t+\theta)) \leq l(V(t, x(t)))$, for $0 < \alpha \leq 1$ and $\theta \in [-r, 0]$, then system (3) is uniformly asymptotically stable. If in addition $\lim_{s\to\infty}\gamma_1(s) = \infty$, then system (3) is globally uniformly asymptotically stable.

3 Main Results

In this section, we will give sufficient conditions of stability and stabilization for fractional-order linear time-delay system by the fractional Razumikhin theorem.

3.1 Stability

In this section, we will present a sufficient condition of stability for fractional-order linear time-delay system by the fractional Razumikhin theorem.

Consider the following fractional-order linear time-delay system

$${}_0^C D_t^\alpha x(t) = Ax(t) + Bx(t-\tau), \tag{4}$$

where ${}_0^C D_t^\alpha$ is the Caputo fractional derivative, $\alpha \in (0,1]$, $x(t) \in \mathbb{R}^n$, $\tau = \tau(t)$ is a continuous function satisfying $0 \leq \tau(t) \leq r$, $x(t-\tau) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$.

Theorem 1 *The system (4) is asymptotically stable if the following conditions hold:*

(i) There exists a $\lambda > 0$ and a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A^{\mathrm{T}}P + PA \leq -\lambda I.$$

(ii) There exists a $q > 1$ such that

$$x^{\mathrm{T}}(\xi)Px(\xi) < qx^{\mathrm{T}}(t)Px(t), \quad for\ t - r \leq \xi \leq t.$$

(iii) $2q_1\|PB\| + \eta < \lambda$, where $q_1 = \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}q}$, $\eta > 0$.

Proof. Let $V(x(t)) = x^{\mathrm{T}}(t)Px(t)$, where $P \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Then V is positive definite. By using inequality (2), then we obtain

$$\begin{aligned} & {}_0^C D_t^\alpha V(x(t))|_{(4)} = {}_0^C D_t^\alpha x^T(t) P x(t) \\ \leq\ & \left({}_0^C D_t^\alpha x(t)\right)^T P x(t) + x^T(t) P {}_0^C D_t^\alpha x(t) \\ =\ & (Ax(t) + Bx(t-\tau))^T P x(t) \\ & + x^T(t) P (Ax(t) + Bx(t-\tau)) \\ =\ & x^T(t)\left(A^T P + PA\right) x(t) + 2x^T(t) PBx(t-\tau). \end{aligned}$$

By (ii), then we get

$$\|x(\xi)\| \leq \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)} q} \|x(t)\| = q_1\|x(t)\|.$$

Choose $l(s) = qs$. And from (i) and (iii), then we have

$$\begin{aligned} & {}_0^C D_t^\alpha V(x(t))|_{(4)} \\ <\ & -\lambda\|x(t)\|^2 + 2\|x(t)\|\cdot\|PB\|\cdot\|x(t-\tau)\| \\ \leq\ & -\lambda\|x(t)\|^2 + 2q_1\|PB\|\cdot\|x(t)\|^2 \\ <\ & -\eta\|x(t)\|^2. \end{aligned}$$

Therefore, according to Lemma 3, the system (4) is asymptotically stable. ∎

3.2 Stabilization

In this section, asymptotical stabilization for closed-loop systems is studied by designing state feedback controller for fractional order linear time-delay controlled system.

Consider the fractional-order linear time-delay controlled system

$$ {}_0^C D_t^\alpha x(t) = Ax(t) + Bx(t-\tau) + \tilde{B}u(t), \tag{5}$$

where ${}_0^C D_t^\alpha$ is the Caputo fractional derivative, $\alpha \in (0,1]$, $x(t) \in \mathbb{R}^n$ is the state, $\tau = \tau(t)$ is a continuous function satisfying $0 \leq \tau(t) \leq r$, $x(t-\tau) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the input of the system. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, and $\tilde{B} \in \mathbb{R}^{n\times m}$.

If one uses a state feedback controller

$$u(t) = Kx(t), \quad K \in \mathbb{R}^{m\times n},$$

then the closed-loop system consisting of (5) and the controller becomes

$$ {}_0^C D_t^\alpha x(t) = (A + \tilde{B}K)x(t) + Bx(t-\tau). \tag{6}$$

The aim is to design the controller which ensures asymptotically stable.

Theorem 2 *If condition (i) in Theorem (1) is replaced by*

(iv) There exists a $\lambda > 0$, a state feedback matrix $K \in \mathbb{R}^{m\times n}$, and a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A^{\mathrm{T}}P + K^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P + PA + P\tilde{B}K \leq -\lambda I.$$

Then the closed-loop system (6) is asymptotically stable.

Proof. Let $V(x(t)) = x^{\mathrm{T}}(t)Px(t)$, where $P \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Then V is positive definite. By using inequality (2), then we have

$$\begin{aligned} & {}_0^C D_t^\alpha V(x(t))\big|_{(6)} = {}_0^C D_t^\alpha \left(x^{\mathrm{T}}(t)Px(t)\right) \\ \leq\ & \left({}_0^C D_t^\alpha x(t)\right)^{\mathrm{T}} Px(t) + x^{\mathrm{T}}(t)P\, {}_0^C D_t^\alpha x(t) \\ =\ & ((A+\tilde{B}K)x(t) + Bx(t-\tau))^{\mathrm{T}}Px(t) \\ & +x^{\mathrm{T}}(t)P((A+\tilde{B}K)x(t) + Bx(t-\tau)) \\ =\ & x^{\mathrm{T}}(t)\left(A^{\mathrm{T}}P + K^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P + PA + P\tilde{B}K\right)x(t) \\ & +2x^{\mathrm{T}}(t)PBx(t-\tau). \end{aligned}$$

From (ii), then we obtain

$$\|x(\xi)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}q\|x(t)\| = q_1\|x(t)\|.$$

Choose $l(s) = qs$. And by (i) and (iii), then we get

$$\begin{aligned} & {}_0^C D_t^\alpha V(x(t))\big|_{(6)} \\ <\ & -\lambda\|x(t)\|^2 + 2\|x(t)\|\cdot\|PB\|\cdot\|x(t-\tau)\| \\ \leq\ & -\lambda\|x(t)\|^2 + 2q_1\|PB\|\cdot\|x(t)\|^2 < -\eta\|x(t)\|^2 \end{aligned}$$

Therefore, according to Lemma 3, the closed-loop system (6) is asymptotically stable. ∎

4 Examples

In this section, we will present two examples to illustrate the main results.

Example 1 *Consider the following fractional-order linear time-delay system:*

$$ {}_0^C D_t^\alpha x = Ax + Bx(t-\tau), \tag{7}$$

where $\alpha \in (0,1]$, $x = (x_1,\ x_2)^{\mathrm{T}}$, $x(t-\tau) = \big(x_1(t-\tau), x_2(t-\tau)\big)^{\mathrm{T}}$, $\tau > 0$,

$$A = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix},\ B = \begin{pmatrix} 0.05 & -0.06 \\ -0.2 & 0.1 \end{pmatrix}.$$

Let the following Lyapunov candidate function

$$V(x_1,x_2) = (x_1,x_2)P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where

$$P = \begin{pmatrix} 14 & 5 \\ 5 & 3 \end{pmatrix}$$

is a positive definite matrix. Then we have

$$\begin{aligned} A^{\mathrm{T}}P + PA &= \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} 14 & 5 \\ 5 & 3 \end{pmatrix} \\ &\quad + \begin{pmatrix} 14 & 5 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix} \\ &= -8I. \end{aligned}$$

Choose $\lambda = 8$, $q = 1.1$, then we get $q_1 = 4.0529$,

$$\eta = 3 < \lambda - 2q_1\|PB\| = 3.7739.$$

Therefore, by Theorem 1, the system (7) is asymptotically stable.

Fig 4.1 shows the state response of the system (7) with $\alpha = 0.8$ and $\tau = 1$ for the initial function $(x_1(t), x_2(t)) = (2,3)$ $(t \in [-1,0])$, which clearly demonstrate the asymptotic stability of the system.

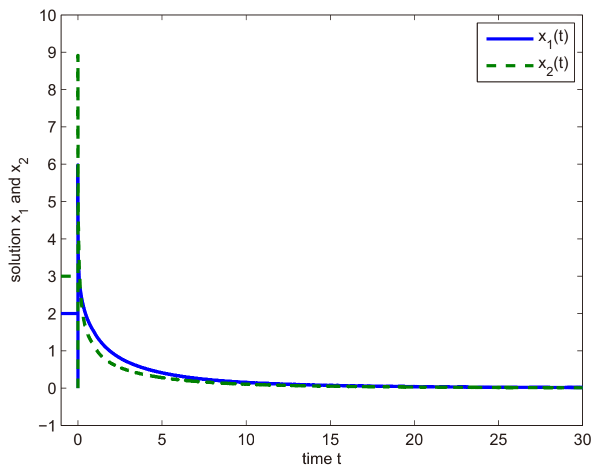

Fig 4.1 The state of system (7) with $\alpha = 0.8$.

Example 2 *Consider the following controlled fractional-order linear time-delay system:*

$$
{}_0^C D_t^{\alpha} x = Ax + Bx(t-\tau) + \tilde{B}u, \tag{8}
$$

where $\alpha \in (0,1]$, $x = (x_1,\ x_2)^{\mathrm{T}}$, $x(t-\tau) = \big(x_1(t-\tau), x_2(t-\tau)\big)^{\mathrm{T}}$, $\tau > 0$, u *is the input of the system,*

$$
A = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix},\ B = \begin{pmatrix} 2 & -0.3 \\ -0.2 & 1 \end{pmatrix},
$$

$\tilde{B} = (0,\ 1)^{\mathrm{T}}$.

We use a state feedback controller

$$
u = Kx,
$$

where $K = (-2\ \ -4)$, then the closed-loop system becomes

$$
{}_0^C D_t^{\alpha} x = (A + \tilde{B}K)x + Bx(t-\tau). \tag{9}
$$

where

$$
A + \tilde{B}K = \begin{pmatrix} -3 & -1 \\ -1 & -5 \end{pmatrix}.
$$

Let the following Lyapunov candidate function

$$
V(x_1, x_2) = (x_1, x_2)P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
$$

where

$$
P = \begin{pmatrix} 5/28 & -1/28 \\ -1/28 & 3/28 \end{pmatrix}
$$

is positive definite. Then

$$
\begin{aligned}
& A^{\mathrm{T}}P + K^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P + PA + P\tilde{B}K \\
= & \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 5/28 & -1/28 \\ -1/28 & 3/28 \end{pmatrix} \\
& + \begin{pmatrix} -2 \\ -4 \end{pmatrix}(0\ \ 1)\begin{pmatrix} 5/28 & -1/28 \\ -1/28 & 3/28 \end{pmatrix} \\
& + \begin{pmatrix} 5/28 & -1/28 \\ -1/28 & 3/28 \end{pmatrix}\begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} \\
& + \begin{pmatrix} 5/28 & -1/28 \\ -1/28 & 3/28 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}(-2\ \ -4) \\
= & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}
$$

Choose $\lambda = 1$, $q = 1.1$, then we get $q_1 = 1.5176$,

$$
\eta = 0.1 < \lambda - 2q_1\|PB\| = 0.1325.
$$

Therefore, by Theorem 2, the system (8) is asymptotically stable.

Fig 4.2 shows the state response of the system (8) with $\alpha = 0.8$ and $\tau = 0.5$ for the initial function $(x_1(t), x_2(t)) = (1, 4)$ $(t \in [-0.5, 0])$, which clearly demonstrate the asymptotic stability of the system.

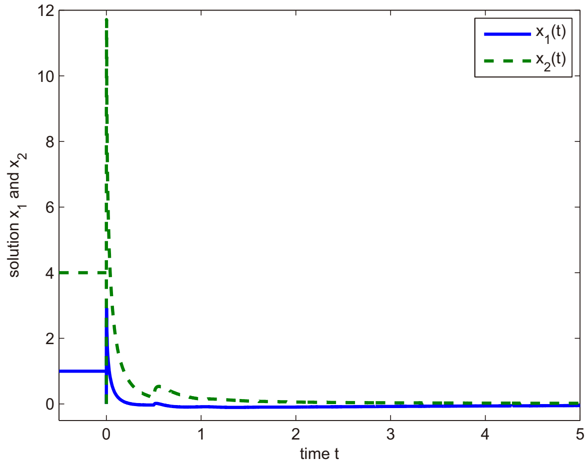

Fig 4.2 The state of system (8) with $\alpha = 0.8$.

5 Conclusion

In this paper, we have investigated the stability and the stabilization of fractional-order linear time-delay system by the fractional Razumikhin theorem. Based on the fractional Razumikhin theorem, a sufficient condition of asymptotical stability for fractional-order linear time-delay system was obtained. Asymptotical stabilization for closed-loop systems was considered by designing state feedback controller for fractional-order linear time-delay controlled system. Two illustrative examples were provided to illustrate the main results.

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