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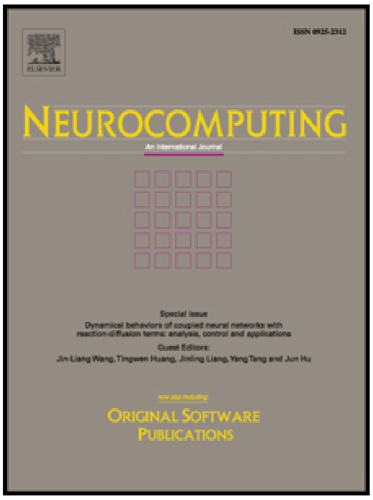

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Iterative learning based consensus control for distributed parameter multi-agent systems with time-delay

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Abstract

This paper concerns about the iterative learning consensus control scheme for a class of multi-agent systems (MAS) with delay distributed parameter models. First, based on the framework of network topologies and using the nearest neighbor knowledge, a second-order iterative learning consensus control protocol is proposed. Next, under the proposed controller, a discrete dynamic system with respect to the iteration number variable is established and the consensus control problem is then converted to a stability problem of a discrete dynamic system. Furthermore, a sufficient condition for the convergence of the consensus errors between any two agents is obtained. Finally, the simulation examples illustrate the validity of the proposed method.

Keywords: multi-agent systems, iterative learning control, delay distributed parameter system, Gronwall inequality.

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1 Introduction

Iterative learning control (ILC) is an effective technique for repetitive processes [1]. One of the most outstanding characteristics of ILC is that the tasks to be accomplished by the controlled system are repeatable [2]. By efficiently using the information from past trials of a repetitive process, it is possible to improve the control performance in the current trial. Thanks to its simplicity and effectiveness, ILC plays an important role in many fields and applications [3, 4]

Distributed parameter systems (DPS) are a class of complicated infinite-dimensional systems modeled by partial differential equations, whose states depend not only on spatial position but also on time [5–8]. In recent years, the application of ILC to DPS has become a new topic [9, 10]. For a class of first order hyperbolic DPS, a open-loop P-type ILC scheme was designed in [11]. For parabolic DPS, the P-type and D-type ILC algorithms were studied in [12]. For a class of single-input single-output coupling nonlinear DPS, the P-type ILC as well as the convergence condition and robustness of were discussed in [13]. By employing special norm, the P-type ILC was investigated for uncertain nonlinear and state time-delay DPS, in [14] and [15], respectively. In [16], a D-type anticipatory ILC scheme was applied to the boundary control for inhomogeneous heat equations. By exploiting the properties of the embedded Jacobi Theta functions, the learning convergence of ILC was guaranteed. In [17], the frequency domain ILC design and analysis was constructed for linear inhomogeneous DPS, which may be hyperbolic, parabolic, or elliptic. In [18], ILC with forgetting factor was proposed and the conditions for convergence of algorithm were established. Using the eigenspectrum and reduced-order model, the ILC was proposed for a class of parabolic DPS in [19].

On the other hand, in recent years, more and more researchers are interested in networked control systems (NCS) [20, 21] and multi-agent systems (MAS) [22]. In [23], a general ILC scheme was enhanced with a neural network controller to reduce the uncertainty of the model. The learning consensus problem for heterogenous high-order nonlinear multi-agent systems with output constraints was considered in [24]. In [25], a consensus-based ILC protocol was proposed for a class of MAS described with DPS. Note that the above existing ILC methods for DPS almost focus on the lower-order ILC. As we all known, the higher-order ILC algorithm can provide better performance in terms of both robustness and convergence rate [26–28].

Motivated by the above works, the purpose of this paper is to present a second-order P-type ILC algorithm for MAS which expressed by DPS with time-delay. The main contributions are as follows. 1) By using the nearest neighbor knowledge, a second-order P-type ILC protocol is proposed for MAS with time-delay distributed parameter models, and the consensus conrtrol problem is converted to a stability problem for a discrete dynamic system; 2) Using Gronwall inequality and contraction mapping theorem, the convergence analysis for consensus errors is given in detail. The obtained condition is less conservative than the existing one in the case of lower-order ILC; 3) The proposed method can be applied to ILC for MAS governed by the parabolic or hyperbolic distributed parameter dynamics.

The rest of this paper is organized as follows: Some preliminaries and problem formulation are presented in Section 2. The convergent condition for P-type ILC algorithm is derived in Section 3. Section 4 present the numerical examples to demonstrates the effectiveness of the method. Finally, some conclusions are drawn in Section 5.

Throughout this paper, $\mathbb{R}^n$ denotes an n-dimensional Euclidean space, I_m means an $m \times m$ dimensional identity matrix. $\mathbf{1}_N$ denotes an N dimensional column vector with all components 1. For the n dimensional vector $W = (w_1, w_2, \cdots, w_n)^T$, its 2-norm for the n-dimensional vector $w = (w_1,\ w_2,\ \cdots,\ w_n)$ is defined as $\|w\| = \sqrt{\sum_{i=1}^{n} w_i^2}$ and the spectrum norm of the $n \times n$ -order square matrix A is $\|A\| = \sqrt{\lambda_{max}(A^T A)}$, where λ_{max} represents the maximum eigenvalue. Let $L^2(\Omega)$ be Hilbert space. If $Q_i \in L^2(\Omega)(i = 1,\ 2,\ \cdots, n)$, we define $Q = (Q_1,\ Q_2,\ \cdots,\ Q_n) \in \mathbf{R}^n \bigcap L^2(\Omega)$, then $\|Q\|_{L^2} = \{\int_\Omega (Q(x)^T Q(x))dx\}^{\frac{1}{2}}$.

For the function $f(x, t) :\ \Omega \times [0, T] \to \mathbf{R}^n$, $f(x, t) \in \mathbf{R}^n \bigcap L^2(\Omega),\ t \in [0, T]$, we define the norm of $(L_2,\ \lambda)$ as

$$\|f\|_{(L_2,\ \lambda)} = \sup_{t\in[0,\ T]} \{e^{-\lambda t}\|f\|_{L^2}^2\},\ \lambda > 0.$$

2 Preliminaries and problem statement

In this paper, we consider a set of N agents, whose dynamics are described by the following delay distributed parameter systems

$$\begin{cases} \dfrac{\partial^a q_{i,k}(x,t)}{\partial t^a} = D\triangle q_{i,k}(x,t) + Aq_{i,k}(x,t), \\ \qquad\qquad + A_1 q_{i,k}(x, t-\tau) + Bu_{i,k}(x,t), \\ y_{i,k}(x,t) = Cq_{i,k}(x,t) + Gu_{i,k}(x,t), \end{cases} \tag{1}$$

where $a = 1$ or 2, $i \in \{1, 2, \ldots, N\} = \bar{N}$, subscript k denotes the iterative number of the process; x and t respectively denote space and time variables, $(x, t) \in \Omega \times [0, T]$; Ω is a bounded open subset with smooth boundary $\partial\Omega$; $q_{i,k}(\cdot,\cdot) \in \mathbb{R}$, $u_{i,k}(\cdot,\cdot) \in \mathbb{R}$, $y_{i,k}(\cdot,\cdot) \in \mathbb{R}$ are the state vector, input vector and output vector of the ith agent at the kth iteration, respectively. $\tau > 0$ denotes the constant time-delay and $q_{i,k}(x,t) = \phi(x,t)$ when $(x, t) \in \Omega \times [-\tau, 0]$. $A,\ B,\ C,\ D,\ G$ are constants and $D > 0,\ G \neq 0$. $\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}$ is a Laplace operator on Ω.

Remark 1 *When $a = 1$, every agent in system (1) is described by the parabolic DPS in [6]. When $a = 2$, every agent system (1) is the hyperbolic DPS in [25].*

For system (1), it is assumed that the network topology at the kth iteration is known and represented by adjacency matrix $\mathtt{A}$. In addition, the corresponding boundary condition of system (1) is

$$\alpha q_{i,k}(x,t)+\beta\frac{\partial q_{i,k}(x,t)}{\partial\nu}=0,\qquad(\text{for a}=1),\tag{2}$$

where $(x,t)\in\partial\Omega\times[0,T]$, $\frac{\partial}{\partial\nu}$ is the unit outward normal derivative on $\partial\Omega$, α and β are known positive constants.

Assume that in the learning process, the system states start from the same initial value, or more generally, for the case of $a=1$,

$$q_{i,k}(x,0)=\varphi_{i,k}(x),\ x\in\Omega,\ k=0,\ 1,\ 2,\ \cdots,\tag{3}$$
$$\|\varphi_{i,k+1}(x)-\varphi_{i,k}(x)\|_{L^2}^2\leq lr^k,\ r\in[0,1),\ l>0.\tag{4}$$

For the case of $a=2$,

$$q_{i,k}(x,0)=\varphi_{i,k}(x),\ x\in\Omega,\ k=0,\ 1,\ 2,\ \cdots\tag{5}$$
$$q_{i,k}(0,t)=f_{i,1}(t),\quad q_{i,k}(1,t)=f_{i,2}(t),\tag{6}$$
$$\left\|\frac{\partial q_{i,k+1}(x,t)}{\partial t}\Big|_{t=0}-\frac{\partial q_{i,k}(x,t)}{\partial t}\Big|_{t=0}\right\|_{L^2}^2\leq\varepsilon,\tag{7}$$
$$\frac{\partial q_{i,k+1}(x,t)}{\partial x}\Big|_{t=0}=\frac{\partial q_{i,k}(x,t)}{\partial x}\Big|_{t=0},\tag{8}$$
$$\|\varphi_{i,k+1}(x)-\varphi_{i,k}(x)\|_{L^2}^2\leq\varepsilon_1.\tag{9}$$

where $\varepsilon>0$, $\varepsilon_1>0$ are positive constants.

Remark 2 *The initial value conditions (4) and (9) are essentially the same. That is, the initial value can be varied in a certain range.*

Definition 1 ([29]) *For MAS (1), protocols $u_{i,k}(x,t)$ are said to solve consensus if and only if for any initial and boundary values, the states of agents satisfy*

$$\lim_{k\to\infty}\|y_{j,k}(x,t)-y_{i,k}(x,t)\|=0,\ i,j\in\bar{N}.$$

In order to describe the connection among these agents, a directed graph $\mathtt{G}=(\mathtt{V},\mathtt{E},\mathtt{A})$ will be used, where $\mathtt{V}=\{1,2,\cdots,N\}$ and $\mathtt{E}\subseteq\mathtt{V}\times\mathtt{V}$ are the sets of vertices and edges of the graph $\mathtt{G}$, respectively. In $\mathtt{G}$, we use the ith vertex to represent the ith agent, and use a directed edge from i to j to represent an ordered pir $(i,j)\in\mathtt{E}$, which means that agent j can directly receive information from agent i. The communication graph can be represented by two types of matrices: the adjacency matrix $\mathtt{A}=(a_{i,j})\in R^{N\times N}$ with $a_{i,j}>0$ if $(i,j)\in\mathtt{E}$ and $a_{i,j}=0$, otherwise. Besides, we assume that $a_{i,i}=0$. $\mathtt{D}=\text{diag}\{d_1,d_2,\cdots,d_N\}$ is the degree matrix whose diagonal elements are defined by $d_i=\sum_{j=1}^N a_{ij}$. The Laplacian matrix $L=[l_{ij}]\in\mathbb{R}^{N\times N}$ with $l_{ij}=\sum_{j=1,i\neq j}^N a_{ij}$ and $l_{ij}=-a_{ij},\ i\neq j$, and $L=\mathtt{D}-\mathtt{A}$. It is well known that Laplace matrix L has a simple zero eigenvalue and all the other eigenvalues have positive real parts if and only if $\mathtt{G}$ has a directed spanning tree.

The following lemmas show important properties of Laplacian matrix.

Lemma 1 *(See [29])) L has at least one zero eigenvalue with $\mathbf{1}_N$ as its eigenvector, and each non-zero eigenvalue of L has positive real parts. Moreover, L has a simple zero eigenvalue if and only if $\mathtt{G}$ contains a spanning tree.*

Lemma 2 *(See [30])) If digraph $\mathtt{G}$ contains a spanning tree, then each eigenvalue of $L_{22}+\mathbf{1}_N\cdot\alpha_1^T$ has positive real part, where*

$$L_{22}=\begin{pmatrix}d_2&-a_{23}&\dots&-a_{2N}\\-a_{32}&d_3&\dots&-a_{3N}\\\vdots&\vdots&\ddots&\vdots\\-a_{N2}&-a_{N3}&\dots&d_N\end{pmatrix},$$
$$\alpha_1=(\alpha_{12},\ \alpha_{13},\ \cdots,\ \alpha_{1N})^T.$$

In this paper, we construct the following distributed P-type ILC protocol for each agent $i\in\bar{N}$:

$$\begin{aligned}&u_{i,k+1}(x,t)\\=\;&\gamma u_{i,k}(x,t)+(1-\gamma)u_{i,k-1}(x,t)\\+\;&c\sum_{j=1}^N a_{ij}(y_{j,k}(x,t)-y_{i,k}(x,t)),\end{aligned}\tag{10}$$

where $u_{i,0}(x,t)$, $u_{i,1}(x,t)$ are two given initial inputs, $0<\gamma\leq1$ is a constant, a_{ij} are the elements of the adjacency matrix and c is the learning gain to be determined later.

Our control objective is to design the above learning scheme (10) to guarantee that the outputs from all agents asymptotically achieve to the desired trajectory over a finite time interval.

Remark 3 *When $\gamma = 1$, the distributed P-type ILC law* (10) *is degenerated to the case of [25]. In fact, if $\gamma \neq 1$, the prior information of the control input will be fully utilized. As a result, the better performance in terms of both robustness and convergence rate can be provided [26].*

3 Main results

To obtain the main results, some useful lemmas are introduced.

Lemma 3 *(see [18]). Suppose constant $0 < \theta < 1$, real sequences $\{V_k\}_{k\geq 0}$ and $\{Z_k\}_{k\geq 0}$ satisfy*

i $\lim_{k\to\infty} V_k = 0$,

ii $Z_{k+1} \leq \theta Z_k + V_K$.

Then we have

$$\lim_{k\to\infty} Z_k = 0.$$

Lemma 4 *(Greens formula [31]) Let $u,\ v \in c^2(\Omega)$, then*

$$\int_\Omega \bigtriangledown u \cdot \bigtriangledown v dx = -\int_\Omega u \bigtriangleup v dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} u ds,$$

where $\bigtriangledown u = \frac{\partial u}{\partial x}$, $\frac{\partial}{\partial \nu}$ is the unit outward normal derivative on $\partial\Omega$.

Lemma 5 *(Gronwall inequality [32]). Let $M \in R$, $u(t)$, $a \geq 0$ and $w(t)$ be continuous functions on $t \in [t_0, \infty)$. If*

$$u(t) \leq M + \int_0^t \Big[au(s) + bw(s)\Big]ds,$$

then

$$u(t) \leq Me^{at} + \int_0^t e^{a(t-s)} bw(s) ds.$$

Lemma 6 *For MAS (1), denote*

$$\begin{aligned}
e_{i,k}(x,t) &= y_{i,k}(x,t) - y_{1,k}(x,t),\\
\delta q_{i,k+1}(x,t) &= q_{i,k+1}(x,t) - q_{i,k}(x,t),\\
\delta u_{i,k+1}(x,t) &= u_{i,k+1}(x,t) - u_{i,k}(x,t),\\
\bar{\Delta} u_{i,k}(x,t) &= \delta u_{i,k}(x,t) - \delta u_{1,k}(x,t),
\end{aligned}$$

where $y_{1,k}(x,t)$ is the virtual leader, and

$$\begin{aligned}
& e_k(x,t)\\
=\ & [e_{2,k}^T(x,t)\ e_{3,k}^T(x,t)\ \cdots\ e_{N,k}^T(x,t))]^T,\\
& \delta q_k(x,t)\\
=\ & [\delta q_{2,k}^T(x,t)\ \delta q_{3,k}^T(x,t)\ \cdots\ \delta q_{N,k}^T(x,t)]^T,\\
& \bar{\Delta} u_k(x,t)\\
=\ & [\bar{\Delta} u_{2,k}^T(x,t)\ \bar{\Delta} u_{3,k}^T(x,t)\ \cdots\ \bar{\Delta} u_{N,k}^T(x,t))]^T,\\
& Q_k(x,t) = [e_k^T(x,t)\ \bar{\Delta} u_k^T(x,t)]^T,\\
& \tilde{\delta} q_{k+1}(x,t)\\
=\ & \delta q_{k+1}(x,t) - \mathbf{1}_{N-1} \otimes \delta q_{1,k+1}(x,t).
\end{aligned}$$

Then

$$Q_{k+1}(x,t) = \Pi Q_k(x,t) + F_k(x,t), \tag{11}$$

where

$$\begin{aligned}
\Pi &= \begin{pmatrix} \Pi_{11} & (\gamma-1)G \cdot I_{N-1} \\ \Pi_{21} & (\gamma-1)I_{N-1} \end{pmatrix},\\
\Pi_{11} &= I_{N-1} - cG(L_{22} + \mathbf{1}_{N-1} \cdot \alpha_1^T),\\
\Pi_{21} &= -c(L_{22} + \mathbf{1}_{N-1} \cdot \alpha_1^T),\\
F_k(x,t) &= \begin{bmatrix} C\tilde{\delta} q_{k+1}(x,t) \\ 0 \end{bmatrix}.
\end{aligned}$$

Proof. For $i = 1, 2, \ldots, N$, it follows from (1), we can get

$$\begin{aligned}
& e_{i,k+1}(x,t)\\
=\ & y_{i,k+1}(x,t) - y_{1,k+1}(x,t)\\
=\ & y_{i,k}(x,t) - y_{1,k}(x,t)\\
+\ & y_{i,k+1}(x,t) - y_{i,k}(x,t)\\
-\ & (y_{1,k+1}(x,t) - y_{1,k}(x,t))\\
=\ & e_{i,k}(x,t) + C\delta q_{i,k+1}(x,t) + G\delta u_{i,k+1}(x,t)\\
-\ & C\delta q_{1,k+1}(x,t) - G\delta u_{1,k+1}(x,t).
\end{aligned} \tag{12}$$

From (10), it can be obtained that

$$\begin{aligned} & \delta u_{i,k+1}(x,t) \\ = \ & (\gamma-1)\delta u_{i,k}(x,t) \\ + \ & c\sum_{j=1}^{N} a_{ij}(y_{j,k}(x,t)-y_{i,k}(x,t)) \\ = \ & (\gamma-1)\delta u_{i,k}(x,t) \\ + \ & c\sum_{j=1}^{N} a_{ij}(e_{j,k}(x,t)-e_{i,k}(x,t)). \end{aligned} \quad (13)$$

Note that $e_{1,k}(x,t)=0$. Then, from (13), we directly obtain

$$\begin{aligned} & \delta u_{i,k+1}(x,t)-\delta u_{1,k+1}(x,t) \\ = \ & (\gamma-1)[\delta u_{i,k}(x,t)-\delta u_{1,k}(x,t)] \\ + \ & c\sum_{j=1}^{N} a_{ij}(e_{j,k}(x,t)-e_{i,k}(x,t)) \\ - \ & c\sum_{j=1}^{N} a_{1j}(e_{j,k}(x,t)). \quad i=2,3,\cdots,N. \end{aligned} \quad (14)$$

The above equalities can be written in the compact form as

$$\begin{aligned} \bar{\Delta}u_{k+1}(x,t) &= (\gamma-1)I_{N-1}\bar{\Delta}u_k(x,t) \\ &- c(L_{22}+\mathbf{1}_{N-1}\cdot\alpha_1^T)e_k(x,t). \end{aligned} \quad (15)$$

Combining (12) and (15), by some calculation, we have

$$\begin{aligned} & e_{k+1}(x,t) \\ = \ & (I_{N-1}-cG(L_{22}+\mathbf{1}_{N-1}\cdot\alpha_1^T))e_k(x,t) \\ + \ & (\gamma-1)G\cdot I_{N-1}\Delta u_k(x,t) \\ + \ & C\tilde{\delta}q_{k+1}(x,t). \end{aligned} \quad (16)$$

The proof is completed.

From Lemma 6, it can be see that the consensus problem of a second order P-type protocol (10) for MAS (1) is equivalent to the stability problem of the dynamic system (11).

In order to investigate the stability of system (11), we first give an estimation of $F_k(x,t)$ as following.

Lemma 7 *Set*

$$\bar{h} = 2|A|+|A_1|+|B|,$$
$$m = (N-1)|(\gamma-1)|+\|c(L_{22}+\mathbf{1}_{N-1}\cdot\alpha_1^T)\|.$$

Then, for MAS (1) in the case of $a=1$, it is possible to choose $\lambda>\bar{h}$ sufficiently large such that

$$\begin{aligned} & \| F_k(x,t) \|_{(L^2,\lambda)} \\ \leq \ & 2(N-1)lr^k|C| \\ + \ & \frac{m|B||C|}{\lambda-\bar{h}}\|Q_k(x,t)\|_{(L^2,\lambda)}. \end{aligned} \quad (17)$$

Proof. From the definition of $F_k(x,t)$, we have

$$\| F_k(x,t) \|_{(L^2,\lambda)} \leq |C|\cdot\|\tilde{\delta}q_{k+1}(x,t)\|_{(L^2,\lambda)}. \quad (18)$$

Clearly,

$$\begin{aligned} & \frac{\partial \delta q_{i,k+1}(x,t)}{\partial t} \\ = \ & D\triangle\delta q_{i,k+1}(x,t)+A\delta q_{i,k+1}(x,t), \\ + \ & A_1\delta q_{i,k+1}(x,t-\tau)+B\delta u_{i,k+1}(x,t). \end{aligned} \quad (19)$$

Therefore,

$$\begin{aligned} & \frac{\partial[\delta q_{i,k+1}(x,t)-\delta q_{1,k+1}(x,t)]}{\partial t} \\ = \ & D\triangle[\delta q_{i,k+1}(x,t)-\delta q_{1,k+1}(x,t)] \\ + \ & A[\delta q_{i,k+1}(x,t)-\delta q_{1,k+1}(x,t)] \\ + \ & A_1[\delta q_{i,k+1}(x,t-\tau)-\delta q_{1,k+1}(x,t-\tau)] \\ + \ & B[\delta u_{i,k+1}(x,t)-\delta u_{1,k+1}(x,t)]. \end{aligned} \quad (20)$$

Equation (20) can be written in the following compact form

$$\begin{aligned} & \frac{\partial\tilde{\delta}q_{k+1}(x,t)}{\partial t} \\ = \ & D\triangle\tilde{\delta}q_{k+1}(x,t)+A\tilde{\delta}q_{k+1}(x,t) \\ + \ & A_1\tilde{\delta}q_{k+1}(x,t-\tau)+B\bar{\Delta}u_{k+1}(x,t). \end{aligned} \quad (21)$$

Note that

$$\begin{aligned} & \frac{\partial(\|\tilde{\delta}q_{k+1}(x,t)\|_{L^2}^2)}{\partial t} \\ = \ & \int_\Omega \frac{\partial\left(\tilde{\delta}q_{k+1}^T(x,t)\tilde{\delta}q_{k+1}(x,t)\right)}{\partial t}dx \\ = \ & 2\int_\Omega \tilde{\delta}q_{k+1}^T(x,t)\frac{\partial\tilde{\delta}q_{k+1}(x,t)}{\partial t}dx. \end{aligned} \quad (22)$$

Substituting (21) into (22), we obtain

$$\begin{aligned}
&\frac{\partial(\|\tilde{\delta}q_{k+1}(x,t)\|_{L^2}^2)}{\partial t}\\
= &\ 2\int_\Omega \tilde{\delta}q_{k+1}^T(x,t)D\triangle\tilde{\delta}q_{k+1}(x,t)dx\\
+ &\ 2A\int_\Omega \tilde{\delta}q_{k+1}^T(x,t)\tilde{\delta}q_{k+1}(x,t)dx\\
+ &\ 2A_1\int_\Omega \tilde{\delta}q_{k+1}^T(x,t-\tau)\tilde{\delta}q_{k+1}(x,t-\tau)dx\\
+ &\ 2\int_\Omega \tilde{\delta}q_{k+1}^T(x,t)B\bar{\Delta}u_{k+1}(x,t)dx\\
:= &\ I_1+I_2+I_3+I_4. \qquad (23)
\end{aligned}$$

Using the Green formula and the boundary condition (2) to I_1, we have

$$\begin{aligned}
I_1 &= 2D\int_\Omega \tilde{\delta}q_{k+1}^T(x,t)\triangle\tilde{\delta}q_{k+1}(x,t)dx\\
&= 2D\int_\Omega \tilde{\delta}q_{k+1}^T(x,t)\frac{\partial\tilde{\delta}q_{k+1}(x,t)}{\partial\nu}dx\\
&- 2D\int_\Omega \nabla\tilde{\delta}q_{k+1}^T(x,t)\nabla\tilde{\delta}q_{k+1}(x,t)dx\\
&= 2D\int_\Omega \tilde{\delta}q_{k+1}^T(x,t)(-\frac{\alpha}{\beta}\tilde{\delta}q_{k+1}(x,t))dx\\
&- 2D\int_\Omega \nabla\tilde{\delta}q_{k+1}^T(x,t)\nabla\tilde{\delta}q_{k+1}(x,t)dx\\
&\le 0. \qquad (24)
\end{aligned}$$

Obviously,

$$\begin{aligned}
I_2 &\le 2|A|\int_\Omega \tilde{\delta}q_{k+1}^T(x,t)\tilde{\delta}q_{k+1}(x,t)dx\\
&= 2|A|\cdot\|\tilde{\delta}q_{k+1}(x,t)\|_{L^2}^2, \qquad (25)\\
I_3 &\le 2|A_1|\cdot\|\tilde{\delta}q_{k+1}(x,t-\tau)\|_{L^2}^2. \qquad (26)
\end{aligned}$$

Using the Hölder inequality to I_4, it yields

$$I_4 \le |B|\cdot\|\tilde{\delta}q_{k+1}(x,t)\|_{L^2}^2+|B|\cdot\|\bar{\Delta}u_{k+1}(x,t)\|_{L^2}^2. \quad (27)$$

Thus, from (23) to (27), we obtain

$$\begin{aligned}
&\frac{\partial(\|\tilde{\delta}q_{k+1}(x,t)\|_{L^2}^2)}{\partial t}\\
\le &\ (2|A|+|B|)\cdot\|\tilde{\delta}q_{k+1}(x,t)\|_{L^2}^2\\
+ &\ |A_1|\cdot\|\tilde{\delta}q_{k+1}(x,t-\tau)\|_{L^2}^2\\
+ &\ |B|\cdot\|\bar{\Delta}u_{k+1}(x,t)\|_{L^2}^2. \qquad (28)
\end{aligned}$$

Integrating both sides of (28) with respect to t, we can get

$$\begin{aligned}
&\|\tilde{\delta}q_{k+1}(x,t)\|_{L^2}^2\\
\le &\ \|\tilde{\delta}q_{k+1}(x,0)\|_{L^2}^2\\
+ &\ (2|A|+|B|)\int_0^t \|\tilde{\delta}q_{k+1}(x,s)\|_{L^2}^2ds\\
+ &\ |A_1|\int_0^t \|\tilde{\delta}q_{k+1}(x,s-\tau)\|_{L^2}^2ds\\
+ &\ |B|\int_0^t \|\bar{\Delta}u_{k+1}(x,s)\|_{L^2}^2ds. \qquad (29)
\end{aligned}$$

Estimating the delay term in (29), it leads to

$$\begin{aligned}
&\int_0^t \|\tilde{\delta}q_{k+1}(x,s-\tau)\|_{L^2}^2ds\\
= &\begin{cases}\int_{-\tau}^{t-\tau}\|\tilde{\delta}q_{k+1}(x,s)\|_{L^2}^2ds=0, & t\in[0,\ \tau]\\ \int_0^{\tau}\|\tilde{\delta}q_{k+1}(x,s-\tau)\|_{L^2}^2ds & \\ +\int_0^{t-\tau}\|\tilde{\delta}q_{k+1}(x,s)\|_{L^2}^2ds, & t\in[\tau,\ T]\end{cases}\\
\le &\int_0^t \|\tilde{\delta}q_{k+1}(x,s)\|_{L^2}^2ds. \qquad (30)
\end{aligned}$$

Substituting (30) into (29) and using condition (5), it yields

$$\begin{aligned}
&\|\tilde{\delta}q_{k+1}(x,t)\|_{L^2}^2\\
\le &\ 2(N-1)lr^k\\
+ &\ \bar{h}\int_0^t \|\tilde{\delta}q_{k+1}(x,s)\|_{L^2}^2ds\\
+ &\ |B|\int_0^t \|\bar{\Delta}u_{k+1}(x,s)\|_{L^2}^2ds, \qquad (31)
\end{aligned}$$

where $\bar{h}=2|A|+|A_1|+|B|$.

Using Gronwall inequality to inequality (31), it yields

$$\begin{aligned}
&\|\tilde{\delta}q_{k+1}(x,t)\|_{L^2}^2\\
\le &\ \bar{M}+|B|\int_0^t e^{\bar{h}(t-s)}\|\bar{\Delta}u_{k+1}(x,s)\|_{L^2}^2ds. \qquad (32)
\end{aligned}$$

where $\bar{M}=2(N-1)lr^ke^{\bar{h}t}$.

Selecting appropriate large of λ such that $\lambda>\bar{h}$, and multiplying both sides of inequality (32) by $e^{-\lambda t}$, we can derive

$$
\begin{aligned}
& \|\tilde{\delta} q_{k+1}(x,t)\|_{L^2}^2 e^{-\lambda t} \\
\leq\ & 2(N-1)lr^k e^{-(\lambda-\bar{h})t} \\
+\ & |B| \int_0^t e^{-(\lambda-\bar{h})(t-s)} \|\bar{\Delta} u_{k+1}(x,s)\|_{L^2}^2 e^{-\lambda s} ds \\
\leq\ & 2(N-1)lr^k \\
+\ & |B| \|\bar{\Delta} u_{k+1}(x,t)\|_{(L^2,\lambda)} \int_0^t e^{-(\lambda-\bar{h})(t-s)} ds \\
=\ & 2(N-1)lr^k + |B| \frac{e^{-(\lambda-\bar{h})t}}{\lambda-\bar{h}} \|\bar{\Delta} u_{k+1}(x,t)\|_{(L^2,\lambda)} \\
\leq\ & 2(N-1)lr^k + \frac{|B|}{\lambda-\bar{h}} \|\bar{\Delta} u_{k+1}(x,t)\|_{(L^2,\lambda)}. \qquad (33)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \|\tilde{\delta} q_{k+1}(x,t)\|_{(L^2,\lambda)} \\
\leq\ & 2(N-1)lr^k \\
+\ & \frac{|B|}{\lambda-\bar{h}} \|\bar{\Delta} u_{k+1}(x,t)\|_{(L^2,\lambda)}. \qquad (34)
\end{aligned}
$$

On the other hand, form (15), we can see that

$$
\begin{aligned}
& \|\ \bar{\Delta} u_{k+1}(x,t)\|_{(L^2,\lambda)} \\
\leq\ & (N-1)|(\gamma-1)||\bar{\Delta} u_k(x,t)|_{(L^2,\lambda)} \\
+\ & \|c(L_{22} + \mathbf{1}_{N-1} \cdot \alpha_1^T)\| \cdot \|e_k(x,t)\|_{(L^2,\lambda)} \\
\leq\ & m\|Q_k(x,t)\|_{(L^2,\lambda)}, \qquad (35)
\end{aligned}
$$

where $m = (N-1)|(\gamma-1)| + \|c(L_{22} + \mathbf{1}_{N-1} \cdot \alpha_1^T)\|$. Substituting (35) into (34), then we get

$$
\begin{aligned}
& \|\tilde{\delta} q_{k+1}(x,t)\|_{(L^2,\lambda)} \\
\leq\ & 2(N-1)lr^k + \frac{m|B|}{\lambda-\bar{h}} \|Q_k(x,t)\|_{(L^2,\lambda)}. \qquad (36)
\end{aligned}
$$

Combining (36) and (18), we can obtain (17). The proof is completed.

Lemma 8 *Set $\bar{g} = |A| + |A_1| + |B|$. Then, for MAS (1) in the case of $a = 2$, it is possible to choose $\lambda >$* $\max\{\bar{g}, 1\}$ *sufficiently large such that*

$$
\begin{aligned}
& \|\ F_k(x,t)\ \|_{(L^2,\lambda)} \\
\leq\ & \frac{|C|}{\xi}[(N-1)\varepsilon_1 e^T + \frac{1-e^{-(\lambda-1)T}}{\lambda-1}(N-1)\varepsilon e^{\bar{g}T}] \\
+\ & \frac{m|B||C|(1-e^{-(\lambda-\bar{g})T})}{\xi(\lambda-\bar{g})} \cdot \Big\|Q_k(x,t)\Big\|_{(L^2,\lambda)}, \qquad (37)
\end{aligned}
$$

where m is the same as that in Lemma 7 and

$$
\xi = 1 - \frac{1-e^{-(\lambda-1)T}}{\lambda-1} \cdot \frac{(\bar{g}-|B|)\cdot(1-e^{-(\lambda-\bar{g})T})}{\lambda-\bar{g}}. \qquad (38)
$$

Proof. Similar to the proof of Lemma 7, we have

$$
\begin{aligned}
& \frac{\partial^2 \delta \tilde{q}_{k+1}(x,t)}{\partial t^2} \\
=\ & D\triangle\tilde{\delta} q_{k+1}(x,t) + A\tilde{\delta} q_{k+1}(x,t) \\
+\ & A_1\tilde{\delta} q_{k+1}(x,t-\tau) + B\bar{\Delta} u_{k+1}(x,t). \qquad (39)
\end{aligned}
$$

Since

$$
\begin{aligned}
& \frac{d}{dt}\Big\|\frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial t}\Big\|_{L^2}^2 \\
=\ & 2\int_\Omega \Big\{\Big(\frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial t}\Big)^T \frac{\partial^2\tilde{\delta} q_{k+1}(x,t)}{\partial t^2}\Big\} dx,
\end{aligned}
$$

we can get

$$
\begin{aligned}
& \frac{d}{dt}\Big\|\frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial t}\Big\|_{L^2}^2 \\
=\ & 2D\int_\Omega \Big(\frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial t}\Big)^T \triangle\tilde{\delta} q_{k+1}(x,t) dx \\
+\ & 2A\int_\Omega \Big(\frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial t}\Big)^T \tilde{\delta} q_{k+1}(x,t) dx \\
+\ & 2A_1\int_\Omega \Big(\frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial t}\Big)^T \tilde{\delta} q_{k+1}(x,t-\tau) dx \\
+\ & 2B\int_\Omega \Big(\frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial t}\Big)^T \bar{\Delta} u_{k+1}(x,t) dx \\
:=\ & I_1 + I_2 + I_3 + I_4. \qquad (40)
\end{aligned}
$$

Integrating by parts and using the boundary condition (6), we have

$$
\begin{aligned}
I_1 &= 2D\int_\Omega \Big\{\Big(\frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial t}\Big)^T \frac{\partial^2\tilde{\delta} q_{k+1}(x,t)}{\partial x^2}\Big\} dx \\
&= 2D\Big(\frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial t}\Big)^T \frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial x}\Big|_\Omega \\
&- 2D\int_\Omega \Big\{\Big(\frac{\partial^2\tilde{\delta} q_{k+1}(x,t)}{\partial t\partial x}\Big)^T \frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial x}\Big\} dx \\
&= -D\frac{d}{dt}\Big\|\frac{\partial\tilde{\delta} q_{k+1}(x,t)}{\partial x}\Big\|_{L^2}^2. \qquad (41)
\end{aligned}
$$

Using the Hölder inequality to I_2, I_3 and I_4, we can obtain

$$\begin{aligned} I_2 &\leq |A| \cdot \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{L^2}^2 \\ &+ |A| \cdot \|\tilde{\delta} q_{k+1}(x,t)\|_{L^2}^2. \end{aligned} \tag{42}$$

$$\begin{aligned} I_3 &\leq |A_1| \cdot \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{L^2}^2 \\ &+ |A_1| \cdot \|\tilde{\delta} q_{k+1}(x,t-\tau)\|_{L^2}^2, \end{aligned} \tag{43}$$

$$\begin{aligned} I_4 &\leq |B| \cdot \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{L^2}^2 \\ &+ |B| \cdot \|\bar{\Delta} u_{k+1}(x,t)\|_{L^2}^2. \end{aligned} \tag{44}$$

Thus, from (40) to (44), it yields

$$\begin{aligned} & \frac{d}{dt} \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{L^2}^2 \\ &+ D \frac{d}{dt} \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial x} \right\|_{L^2}^2 \\ &\leq \bar{g} \cdot \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{L^2}^2 \\ &+ |A| \cdot \|\tilde{\delta} q_{k+1}(x,t)\|_{L^2}^2 \\ &+ |A_1| \cdot \|\tilde{\delta} q_{k+1}(x,t-\tau)\|_{L^2}^2 \\ &+ |B| \cdot \|\bar{\Delta} u_{k+1}(x,t)\|_{L^2}^2, \end{aligned} \tag{45}$$

where $\bar{g} = |A| + |A_1| + |B|$.

Integrating both sides of (45) above t and combining with the boundary conditions (7), (8) and inequality (30), we can get

$$\begin{aligned} & \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{L^2}^2 \\ \leq & \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{L^2}^2 + D \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial x} \right\|_{L^2}^2 \\ \leq & (N-1)\varepsilon + \bar{g} \cdot \int_0^t \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,s)}{\partial s} \right\|_{L^2}^2 ds \\ + & (|A| + |A_1|) \int_0^t \| \tilde{\delta} q_{k+1}(x,s)\|_{L^2}^2 ds \\ + & |B| \int_0^t \|\bar{\Delta} u_{k+1}(x,s)\|_{L^2}^2 ds. \end{aligned} \tag{46}$$

Applying Gronwall inequality, we have

$$\begin{aligned} & \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{L^2}^2 \\ \leq & (N-1)\varepsilon e^{\bar{g}t} \\ + & \int_0^t (|A| + |A_1|) e^{\bar{g}(t-s)} \| \tilde{\delta} q_{k+1}(x,s)\|_{L^2}^2 ds \\ + & \int_0^t (|B|) e^{\bar{g}(t-s)} \|\bar{\Delta} u_{k+1}(x,s)\|_{L^2}^2 ds. \end{aligned} \tag{47}$$

As a result, for $\lambda > \bar{g}$,

$$\begin{aligned} & \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{(L^2,\lambda)}^2 \\ \leq & (N-1)\varepsilon e^{\bar{g}t} \\ + & (|A| + |A_1|) \cdot \frac{1 - e^{-(\lambda - \bar{g})T}}{\lambda - \bar{g}} \cdot \| \tilde{\delta} q_{k+1}(x,t)\|_{(L^2,\lambda)}^2 \\ + & |B| \cdot \frac{1 - e^{-(\lambda - \bar{g})T}}{\lambda - \bar{g}} \cdot \|\bar{\Delta} u_{k+1}(x,t)\|_{(L^2,\lambda)}^2. \end{aligned} \tag{48}$$

On the other hand, by using the basic inequality, we can get

$$\begin{aligned} & \frac{\partial (\|\tilde{\delta} q_{k+1}(x,t)\|_{L^2}^2)}{\partial t} \\ = & 2 \int_0^t (\tilde{\delta} q_{k+1}(x,t))^T \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} dx \\ \leq & \|\tilde{\delta} q_{k+1}(x,t)\|_{L^2}^2 + \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{L^2}^2. \end{aligned} \tag{49}$$

Using Gronwall inequality and combing with condition (9), it yields,

$$\begin{aligned} & \|\tilde{\delta} q_{k+1}(x,t)\|_{L^2}^2 \\ \leq & (N-1)\varepsilon_1 e^t \\ + & \int_0^t e^{(t-s)} \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,s)}{\partial s} \right\|_{L^2}^2 ds. \end{aligned} \tag{50}$$

Therefore, for $\lambda > \max\{\bar{g}, 1\}$,

$$\begin{aligned} & \|\tilde{\delta} q_{k+1}(x,t)\|_{(L^2,\lambda)}^2 \\ \leq & (N-1)\varepsilon_1 e^T \\ + & \frac{1 - e^{-(\lambda - 1)T}}{\lambda - 1} \left\| \frac{\partial \tilde{\delta} q_{k+1}(x,t)}{\partial t} \right\|_{(L^2,\lambda)}^2. \end{aligned} \tag{51}$$

Noting that $\xi < 1$, from inequality (48) and (51), we can obtain

$$\begin{aligned} & \|\tilde{\delta} q_{k+1}(x,t)\|^2_{(L^2,\lambda)} \\ \leq & \frac{1}{\xi}[(N-1)\varepsilon_1 e^T + \frac{1-e^{-(\lambda-1)T}}{\lambda-1}(N-1)\varepsilon e^{\bar{g}T}] \\ + & \frac{|B|(1-e^{-(\lambda-\bar{g})T})}{\xi(\lambda-\bar{g})} \cdot \left\|\bar{\Delta} u_{k+1}(x,t)\right\|^2_{(L^2,\lambda)}. \quad (52) \end{aligned}$$

The rest proof is the same lines as that after inequality (34) in the proof of Lemma 7.

In the following, based on Lemma 6, Lemma 7 and Lemma 8, a consensus result is derived for MAS (1).

Theorem 1 *Consider the MAS (1) under the P-type ILC protocol (10). If*

$$\rho(\Pi) = \rho_0 < 1, \quad (53)$$

where Π *is the same as in Lemma 6, then the consensus of MAS (1) can be achieved asymptotically.*

Proof. For $a = 1$, it follows form (11) that for all $k > 1$

$$\begin{aligned} & Q^T_{k+1}(x,t)Q_{k+1}(x,t) \\ = & (\Pi Q_k(x,t) + F_k(x,t))^T(\Pi Q_k(x,t) + F_k(x,t))^T \\ = & (\Pi Q_k(x,t))^T(\Pi Q_k(x,t)) + (\Pi Q_k(x,t))^T F_k(x,t) \\ + & F^T_k(x,t)(\Pi Q_k(x,t)) + F^T_k(x,t)F_k(x,t) \\ \leq & (1+\eta)(\Pi Q_k(x,t))^T(\Pi Q_k(x,t)) \\ + & (1+\frac{1}{\eta})F^T_k(x,t)F_k(x,t) \\ \leq & \rho(\Pi)^2(1+\eta)Q^T_k(x,t)Q_k(x,t) \\ + & (1+\frac{1}{\eta})F^T_k(x,t)F_k(x,t), \quad (54) \end{aligned}$$

where $\eta > 0$.

Integrating by x both sides of (54) on Ω and taking the (L^2, λ)-norm, it yields,

$$\begin{aligned} & \|Q_{k+1}(x,t)\|_{(L^2,\lambda)} \\ \leq & \rho(\Pi)^2(1+\eta)\|Q_k(x,t)\|_{(L^2,\lambda)} \\ + & (1+\frac{1}{\eta})\|F_k(x,t)\|_{(L^2,\lambda)}. \quad (55) \end{aligned}$$

From Lemma 7 and inequality (55), we can obtain

$$\begin{aligned} & \|Q_{k+1}(x,t)\|_{(L^2,\lambda)} \\ \leq & \rho(\Pi)^2(1+\eta)\|Q_k(x,t)\|_{(L^2,\lambda)} \\ + & (1+\frac{1}{\eta})\frac{m|B||C|}{\lambda-\bar{h}}\|Q_k(x,t)\|_{(L^2,\lambda)} \\ + & 2(N-1)lr^k|C| \\ = & \rho_1\|Q_k(x,t)\|_{(L^2,\lambda)} \\ + & 2(N-1)lr^k|C|, \quad (56) \end{aligned}$$

where

$$\rho_1 = \rho(\Pi)^2(1+\eta) + (1+\frac{1}{\eta})\frac{m|B||C|}{\lambda-\bar{h}}.$$

According to the condition (53) in Theorem 1, we can select an appropriate large $\lambda > \bar{h}$ and $\eta > 0$ such that

$$\rho(\Pi) = \rho_0 \leq \sqrt{1/(1+\eta)} < 1. \quad (57)$$

So, from (56), (57) and Lemma 3, it can be seen that

$$\lim_{k\to\infty}\|Q_k(x,t)\|_{(L^2,\lambda)} = 0. \quad (58)$$

Note that

$$\begin{aligned} & \|e_k(x,t)\|^2_{L^2} \\ \leq & \|Q_k(x,t)\|^2_{L^2} \\ \leq & \|Q_k(x,t)\|^2_{L^2}e^{-\lambda t}e^{\lambda T} \\ \leq & \|Q_k(x,t)\|_{(L^2,\lambda)}e^{\lambda T}. \quad (59) \end{aligned}$$

Then, it follows from (58), (59), we have

$$\lim_{k\to\infty} \| e_k(x,t) \|_{L^2} = 0. \quad (60)$$

The proof in the case of $a = 2$ is on exactly the same argument as that in the case of $a = 1$. Thus, we omitted it. The proof is complete.

Remark 4 *The proposed method can be extended to iterative learning based consensus control for MAS with delay distributed parameter models involving bounded external disturbance. With a slight modification and the same argument of Theorem 1, the convergence and robustness of the ILC can be guaranteed.*

The following corollary is obvious.

Corollary 1 *Consider MAS (1) under the P-type ILC protocol*

$$\begin{aligned} & u_{i,k+1}(x,t) \\ = \ & u_{i,k}(x,t) \\ + \ & c\sum_{j=1}^{N} a_{ij}(y_{j,k}(x,t) - y_{i,k}(x,t)). \end{aligned} \tag{61}$$

If

$$\rho(\Pi_1) = \rho\{I_{N-1} - cG(L_{22} + \mathbf{1}_{N-1}\cdot\alpha_1^T)\} < 1, \tag{62}$$

then the consensus of MAS (1) can be achieved asymptotically.

Proof. Setting $\gamma = 1$, from Theorem 1, it can be concluded that if

$$\rho(\Xi_1) < 1, \tag{63}$$

where

$$\Xi_1 = \begin{pmatrix} I_{N-1} - cG(L_{22} + \mathbf{1}_{N-1}\cdot\alpha_1^T) & 0 \\ -c(L_{22} + \mathbf{1}_{N-1}\cdot\alpha_1^T) & 0 \end{pmatrix},$$

then the P-type ILC protocol (61) guarantees that $\lim_{k\to\infty}\|e_k(x,t)\|_{L^2} = 0$.

Noting that

$$\begin{aligned} & \det(\Xi_1) \\ = & \begin{vmatrix} \lambda I - (I_{N-1} - cG(L_{22} + \mathbf{1}_{N-1}\cdot\alpha_1^T) & 0 \\ c(L_{22} + \mathbf{1}_{N-1}\cdot\alpha_1^T) & \lambda I \end{vmatrix}, \end{aligned}$$

it follows from the definition of the spectral radius that

$$\rho(\Xi_1) = \rho(\Pi_1).$$

The proof is completed.

The following corollary is also holds.

Corollary 2 *Consider MAS (1) with $\tau = 0$, i.e.,*

$$\begin{cases} \frac{\partial^a q_{i,k}(x,t)}{\partial t^a} = D\triangle q_{i,k}(x,t) + Aq_{i,k}(x,t), \\ \qquad + Bu_{i,k}(x,t), \\ y_{i,k}(x,t) = Cq_{i,k}(x,t) + Gu_{i,k}(x,t), \end{cases} \tag{64}$$

under the P-type ILC protocol (61). If the condition (62) holds, then the consensus of MAS (64) can be achieved asymptotically.

Remark 5 *Note that the consensus analysis under ILC protocol (61) for MAS (64) has been investigated in [25], in which the convergence condition is*

$$\|I_{N-1} - cG(L_{22} + \mathbf{1}_{N-1}\cdot\alpha_1^T)\| < 1. \tag{65}$$

Since

$$\begin{aligned} & \rho\{I_{N-1} - cG(L_{22} + \mathbf{1}_{N-1}\cdot\alpha_1^T)\} \\ \leq \ & \|I_{N-1} - cG(L_{22} + \mathbf{1}_{N-1}\cdot\alpha_1^T)\|, \end{aligned} \tag{66}$$

the convergence condition (65) is less conservative than the condition (62)(See, e.g., the Example 2 in Section 4).

4 Numerical Examples

In this section, two numerical examples are presented to demonstrate the validity of the design method.

Example 1. Consider MAS (1) with $a = 1$, $(x,t) \in [0,2]\times[0,1]$ and

$$D = 1.1,\ A = B = 1, A_1 = 0.5, C = 1.5,\ G = 1.$$

Assume that the MAS consist of three agents and the communication topology among these agents is shown as in Fig. 2. It is clear that there is a directed spanning tree in the topology. The weighted adjacency matrix of the communication topology is given as follows:

$$\mathtt{A} = \begin{bmatrix} 0 & 0 & 0.1 \\ 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}. \tag{67}$$

We use the ILC protocol (10) and take the feedback gain $c = 0.50$ and $\gamma = 0.2$. In this case, we can calculate that $\rho(\Pi) = 0.9738 < 1$.

The numerical simulation is done with initial state values $q_1(x,0) = -0.82$, $q_2(x,0) = 0.45$ and $q_3(x,0) = 0.40$, the boundary conditions: $\frac{\partial q_1(0,t)}{\partial t} = 0$, $q_1(2,t) = 1$ and $q_i(0,t) = 0$, $\frac{\partial q_i(2,t)}{\partial t} = 0$, $i = 2,3$. The control input value at the beginning of learning is set 0. We take the sampling period $T = 0.01s$ and employ the following performance index

$$E = \sqrt{\frac{1}{2\times 10^4}\sum_{i=0}^{200}\sum_{j=0}^{100} e_s(x(i),j)^2}, \quad s = 2,3.$$

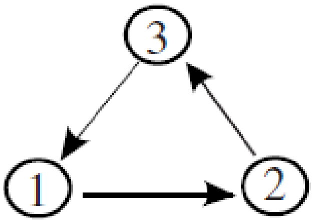

Figure 1: Communication topology among agents.

to evaluate the tracking error in each iteration. Fig. 2 - 4 show the system output of the ninth iterations. The simulation results of $e_i(x,t)$, $i = 2,3$ with the change of k is shown in Fig. 5. Therefore, Figs. 2-5 show that the consensus of $q_i(x,t)(i = 1,2,3)$ is reached as time increases.

Example 2. Consider MAS (1) with the same parameters and conditions in Example 1.

The communication topology among the agents is just shown as in Fig. 2. But the weighted adjacency matrix of the communication topology is given as

$$\mathtt{A} = \begin{bmatrix} 0 & 0 & 0.4 \\ 0.4 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}. \tag{68}$$

We use the ILC protocol (61) and take the feedback gain $c = 2.50$. By some calculation, we can find that

$$\|I_2 - cG(L_{22} + \mathbf{1}_{N-1} \cdot \alpha_1^T)\| = 1.0757 > 1, \tag{69}$$

which implies the convergence condition in [25] is not satisfied.

However, we can get

$$\rho\{I_{N-1} - cG(L_{22} + \mathbf{1}_{N-1} \cdot \alpha_1^T)\} = 0.433 < 1. \tag{70}$$

So, we can use the ILC protocol (61) to consensus control MAS (1), which indicates that the convergence condition (65) is less conservative than that in [25].

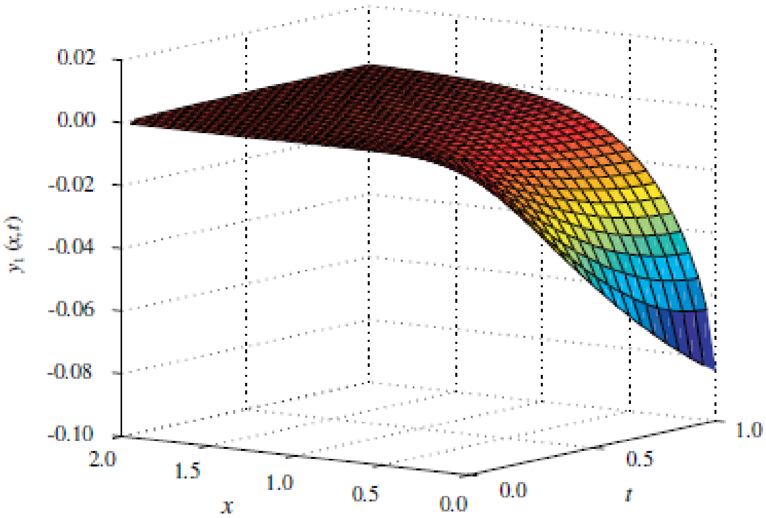

Figure 2: The trajectory of $y_1(x,t)$.

5 Conclusions

In this paper, a second-order P-type ILC protocol was applied to the MAS described by delay distributed parameter models. The consensus condition was derived, which is less conservative than the existing one in the case of without time-delay. The validity of the proposed design method was demonstrated by a numerical example. Our future work includes ILC with initial state learning and D-type ILC protocol for consensus control of MAS with distributed parameter models.

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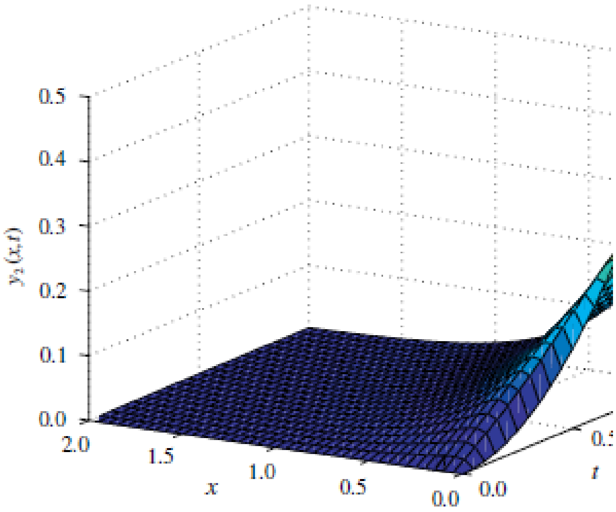

Figure 3: The trajectory of $y_2(x,t)$.

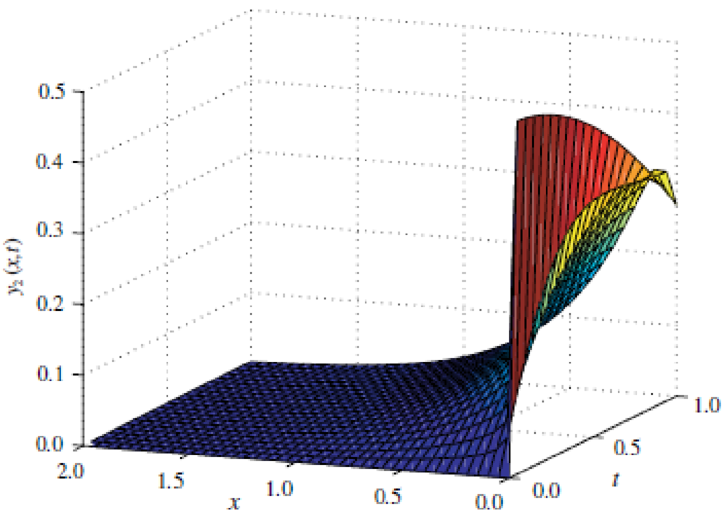

Figure 4: The trajectory of $y_3(x,t)$.

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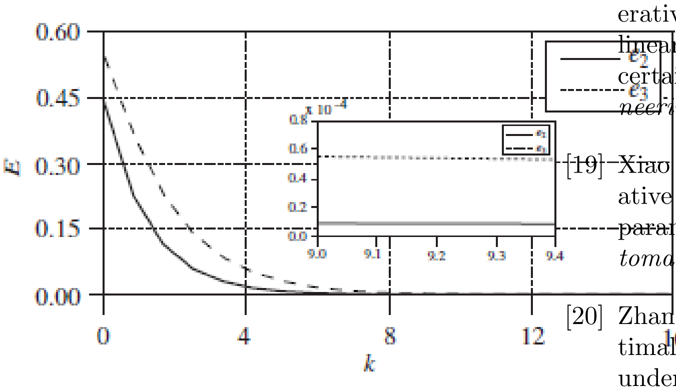

Figure 5: The simulation results of $e_i(x,t),\ i=2,3$.

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