Dynamic Surface Control for a Class of Nonlinear Systems

D. Swaroop, J. K. Hedrick, P. P. Yip, and J. C. Gerdes

***Abstract*—A new method is proposed for designing controllers with arbitrarily small tracking error for uncertain, mismatched nonlinear systems in the strict feedback form. This method is another "synthetic input technique," similar to backstepping and multiple surface control methods, but with an important addition, $r-1$ low pass filters are included in the design where r is the relative degree of the output to be controlled. It is shown that these low pass filters allow a design where the model is not differentiated, thus ending the complexity arising due to the "explosion of terms" that has made other methods difficult to implement in practice. The backstepping approach, while suffering from the problem of "explosion of terms" guarantees boundedness of tracking errors globally; however, the proposed approach, while being simpler to implement, can only guarantee boundedness of tracking error semiglobally, when the nonlinearities in the system are non-Lipschitz.**

***Index Terms*—Integrator backstepping, nonlinear control system design, semiglobal tracking, sliding mode control, strict feedback form.**

I. INTRODUCTION

Tremendous strides have been made in the past 25 years in the area of controller design for nonlinear systems. Variable structure control or sliding mode control [5], [25], uses a discontinuous control structure to guarantee perfect tracking for a class of systems satisfying "matching" conditions. Retaining the concept of an "attractive" surface but eliminating the control discontinuities, the method of sliding control [21] is currently being applied in many different applications.

The works of Brockett [2], Hunt *et al.* [9], Isidori [10], Jakubczyk and Respondek [12] initiated a surge of interest in feedback linearization and more generally in the application of differential geometry to nonlinear control [10], [18], [20].

Recently, the area of robust nonlinear control has received a great deal of attention in the literature. Many methods employ a synthesis approach where the controlled variable is chosen to make the time derivative of a Lyapunov function candidate negative definite. Corless and Leitmann [3] have applied this approach to open-loop stable mismatched nonlinear systems.

A design methodology that has received a great deal of interest recently is "integrator backstepping." The recent book by Krstic *et al.* [16] develops the backstepping approach to the point of a step by step design procedure. The integrator backstepping (IB) technique suffers from the problem of "explosion of terms." The following example illustrates the backstepping approach as well as the difficulty that this paper seeks to solve:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + x_2 + \Delta f_1(x_1)\\ \dot{x}_2 &= u\end{aligned} \tag{1}$$

where $f_1,\ \Delta f_1(x_1)$ are non-Lipschitz nonlinearities. The function $f_1(x_1)$ is assumed completely known, while Δf_1 is uncertain. However, it is bounded by a known nonlinearity, $\rho_1(x_1)$. The goal is to regulate the system in the presence of mismatched non-Lipschitz uncertainty. Let

$$z_1 = x_1 \tag{2}$$
$$\dot{z}_1 = f_1 + x_2 + \Delta f_1(x_1) \tag{3}$$
$$z_2 := x_2 - x_{2d} \tag{4}$$
$$\Rightarrow \dot{z}_1 = f_1 + z_2 + x_{2d} + \Delta f_1(x_1) \tag{5}$$
$$\Rightarrow \dot{z}_2 = u - \dot{x}_{2d}. \tag{6}$$

Using the idea of nonlinear damping [16] in integrator backstepping, set

$$x_{2d} = -f_1(x_1) - z_1\frac{\rho_1^2(x_1)}{2\epsilon} - Kz_1 \tag{7}$$
$$\phi_1 = \frac{\partial f_1}{\partial x_1} + \frac{z_1\rho_1}{\epsilon}\frac{\partial\rho_1(x_1)}{\partial x_1} + \frac{\rho_1^2(x_1)}{2\epsilon} + K \tag{8}$$
$$u = -(x_2+f_1)\phi_1 - \frac{\phi_1^2\rho_1^2 z_2}{2\epsilon} - Kz_2 - z_1. \tag{9}$$

We have deliberately set the K to be the same in (7) and (9) for simplicity of illustration. Consequently, the closed-loop system is governed by

$$\dot{z}_1 = z_2 - Kz_1 - \frac{z_1\rho_1^2(x_1)}{2\epsilon} + \Delta f_1(x_1) \tag{10}$$
$$\dot{z}_2 = -Kz_2 - z_1 + \phi_1\Delta f_1(x_1) - \frac{z_2\rho_1^2\phi_1^2}{2\epsilon}. \tag{11}$$

Define a Lyapunov function candidate,

$$V := \frac{z_1^2 + z_2^2}{2}.$$

Therefore,

$$\begin{aligned}\dot{V} = {} & z_1\left(z_2 - Kz_1 - \frac{z_1\rho_1^2(x_1)}{2\epsilon} + \Delta f_1(x_1)\right)\\ & + z_2\left(-z_1 - Kz_2 + \phi_1\Delta f_1(x_1) - \frac{z_2\rho_1^2\phi_1^2}{2\epsilon}\right).\end{aligned}$$

From Young's inequality

$$\frac{z_1^2\rho_1^2}{2\epsilon} + \frac{\epsilon}{2} \ge |z_1|\rho_1 \ge z_1\Delta f_1(x_1)$$

and

$$\frac{z_2^2\phi_1^2\rho_1^2}{2\epsilon} + \frac{\epsilon}{2} \ge z_2\phi_1\Delta f_1(x_1).$$

As a result of the above control law, $\dot{V} \le -2KV + \epsilon$ and this results in ultimately uniformly bounded regulation of the state. Since ϵ is arbitrary, the ultimate error bound in regulation can be made arbitrarily small. In the above example, we see the beginning of the "complexity due to the explosion of terms" arising from the calculation of $\dot{x}_{2d}$ as well as the presence of uncertainty in $\dot{x}_{2d}$. The requirement on the nonlinear functions f_1 and ρ_1 is very clear—they must be at least C^1 functions. For a nonlinear system in strict feedback form with a relative degree n, the requirement on f_1, ρ_1 is more stringent—they must be at least C^n functions.

A procedure similar to backstepping, called multiple surface sliding (MSS) control [7], [26] was developed to simplify the controller design of systems where model differentiation was difficult. (Earlier work on

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variable structure controller based on a similar hierarchical and block control principle can also be found in [4], [17], and [25].) Let us apply MSS control to the previous example. Let

$$S_1 := z_1 = x_1 \tag{12}$$
$$\Rightarrow \dot{S}_1 = f_1 + x_2 + \Delta f_1(S_1) \tag{13}$$
$$S_2 := z_2 = x_2 - x_{2d} \tag{14}$$
$$\Rightarrow \dot{S}_1 = f_1 + S_2 + x_{2d} + \Delta f_1(S_1). \tag{15}$$

We now choose x_{2d} to make $S_1\dot{S}_1 < 0$ assuming S_2 will be driven to zero. A reasonable choice for x_{2d} is

$$x_{2d} = -f_1 - KS_1 - \rho_1 \operatorname{sgn}(S_1). \tag{16}$$

The dynamics of S_1 is given by:

$$\dot{S}_1 = S_2 - KS_1 + \Delta f_1(S_1) - \rho_1 \operatorname{sgn}(S_1) \tag{17}$$
$$\dot{S}_2 = u - \dot{x}_{2d} := -KS_2. \tag{18}$$

For simplicity of illustration, we have picked the same value of K in (16) and (18). Thus, $u = \dot{x}_{2d} - KS_2$. As a result of the control law that is chosen, let

$$V = \frac{S_1^2 + S_2^2}{2} \tag{19}$$
$$\Rightarrow \dot{V} = S_1\dot{S}_1 + S_2\dot{S}_2 \le -K\left(S_1^2 + S_2^2\right) + S_1S_2 \tag{20}$$

which can be made negative definite for a choice of $K > 1/2$. The difficulty with this scheme is obtaining $\dot{x}_{2d}$, since $\dot{S}_1$ involves $\Delta f_1(S_1)$, which is not known exactly. This problem has been dealt with in an ad hoc way by numerical differentiation, i.e.,

$$\dot{x}_{2d}(n) \approx \frac{x_{2d}(n) - x_{2d}(n-1)}{\Delta T}$$

where ΔT is the sample time. Reference [7] also discusses the uses of a low pass filter to smooth the signal produced by the above equation. This ad hoc approach has worked well in many experimental applications ranging from active suspension control [1] to fuel-injection control [8] to throttle/brake control on automated vehicles [6].

In this paper we introduce a dynamic extension to MSS control that overcomes the problem of explosion of terms associated with the IB technique and the problem of finding derivatives of reference (desired) trajectories for the ith state for the MSS scheme. The first structured approach to the use of dynamic filters can be found in the dissertation of Gerdes [6]. To illustrate how the proposed method overcomes the shortcomings of the previous methods, a controller is designed for the example discussed earlier in this section

$$S_1 = x_1, \qquad \Rightarrow \dot{S}_1 = f_1(x_1) + x_2 + \Delta f_1(x_1) \tag{21}$$
$$\overline{x}_2 := -f_1(x_1) - K_1S_1 - S_1\frac{\rho_1^2}{2\epsilon}. \tag{22}$$

If x_2 were to track $\overline{x}_2$ asymptotically, S_1 would converge to a neighborhood about 0. In order to avoid the problem faced by the MSS scheme, $\overline{x}_2$ is passed through a first order filter, i.e.,

$$\tau\dot{x}_{2d} + x_{2d} = \overline{x}_2, \qquad x_{2d}(0) := \overline{x}_2(0). \tag{23}$$

The choice of a first order filter stems from simplicity of implementation. We define the term S_2 as

$$S_2 := x_2 - x_{2d}.$$

The differentiation of x_{2d} is now possible and u is chosen to drive S_2 to zero

$$u = \dot{x}_{2d} - K_2S_2 = \frac{\overline{x}_2 - x_{2d}}{\tau} - K_2S_2.$$

We note that this control law does not involve model differentiation and thus has prevented the explosion of terms. There are two important advantages associated with dynamic surface controller (DSC). It prevents the problem of "explosion of terms" and the requirement on the smoothness of f_1, ρ_1 is relaxed. In order to design a DSC, f_1, ρ_1 are required to be C^1 functions, irrespective of the order of the system. The following sections will provide details of the DSC for a system with non-Lipschitz nonlinearities (corresponding details for a Lipschitz nonlinear system may be found in [23]). In summary, the DSC algorithm, while providing the ease in designing and implementing nonlinear control laws, guarantees global exponential regulation and arbitrarily bounded tracking, if the nonlinearities in the system are Lipschitz, and semiglobal, arbitrarily bounded regulation and tracking, if the nonlinearities in the system are not Lipschitz. This is a trade-off in performance between ease in control law design and implementation, and global stability.

II. Other Considerations

While the focus of this paper is on designing controllers for a class of non-Lipschitz systems, it is illustrative to look at the following Lipschitz nonlinear system. For a detailed discussion on the design of a DSC for Lipschitz systems, the reader is referred to [23] and [22]. For a Lipschitz nonlinear system in strict feedback form, the complexity of explosion of terms can be circumvented using the design in Teel and Praly [24] or Khalil [27] without the addition of filters suggested by DSC. However, the following example brings out a heuristic concerning when a nonlinear controller, as opposed to a linear controller, is apt for a nonlinear system:

$$\dot{x}_1 = x_2 + l\sin(wx_1) \tag{24}$$
$$\dot{x}_2 = x_3 + l\sin(wx_2) \tag{25}$$
$$\dot{x}_3 = u(t) + l\sin(wx_3). \tag{26}$$

We will assume that l and w are known exactly.

We will begin with the design in [24] in [27]. Let K be a positive constant that will be determined later. Consider the following linear state feedback controller:

$$\psi_1 := x_1; \quad \psi_2 := \frac{x_2}{K}; \quad \psi_3 := \frac{x_3}{K^2}; \quad v := \frac{u}{K^3}$$
$$\Rightarrow \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{pmatrix} = K\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{27}$$
$$+ \begin{pmatrix} l\sin(w\psi_1) \\ l\,\dfrac{\sin(Kw\psi_2)}{K} \\ l\,\dfrac{\sin(K^2w\psi_3)}{K^2} \end{pmatrix} + K\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v. \tag{28}$$

Let $0 < \lambda_1 \le \lambda_2 \le \lambda_3$. A feedback controller of the form

$$v = -\lambda_1\lambda_2\lambda_3\psi_1 - (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\psi_2 - (\lambda_1 + \lambda_2 + \lambda_3)\psi_3$$

will exponentially regulate the state with an appropriate choice of K. To see this, set λ_1 to unity without any loss of generality, because the closed-loop eigenvalues are $-K\lambda_1, -K\lambda_2, -K\lambda_3$ and can always be obtained with a new choice of $K_{\text{new}} = K\lambda_1$. New values of λ_2, λ_3

can be similarly determined. Let $M \geq 1$ be the condition number of the generalized eigen vector matrix of A_c, where

$$A_c := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda_2\lambda_3 & -(\lambda_2+\lambda_3+\lambda_2\lambda_3) & -(1+\lambda_2+\lambda_3) \end{bmatrix}.$$

From the Bellman–Gronwall inequality, it can be shown that a choice of K greater than wlM will exponentially stabilize the system. This implies that the control gains vary as $(wlM)^3$. For an nth order system, the gains will vary as $(wlM)^n$. Notice that the control gains grow faster with the frequency w as well as the Lipschitz constant l of the nonlinearity with this scheme. This is the price for simplifying the controller design without completely incorporating the knowledge of the model in the high gain controller.

We will now apply IB technique to design a controller for this system. Such a design is illustrated as follows:

$$\begin{aligned}
z_1 &:= x_1 \\
\dot{z}_1 &= x_2 + l\sin(wx_1) \\
z_2 &:= x_2 + l\sin(wx_1) + K_1 z_1 \\
\Rightarrow \dot{z}_1 &= z_2 - K_1 z_1 \\
\dot{z}_2 &= x_3 + l\sin(wx_2) + (K_1 + lw\cos(wz_1)) \\
&\quad \times (z_2 - K_1 z_1) \\
z_3 &:= x_3 + l\sin(wx_2) + (K_1 + lw\cos(wz_1)) \\
&\quad \times (z_2 - K_1 z_1) + K_2 z_2 \\
\Rightarrow \dot{z}_2 &= z_3 - K_2 z_2 \\
\dot{z}_3 &= u + \phi(x_1, x_2, x_3) \\
\phi(x_1, x_2, x_3) &= l\sin(wx_3) + wl\cos(wx_2)(x_3 + l\sin(wx_2)) \\
&\quad - w^2 l\sin(wz_1)(z_2 - K_1 z_1)^2 \\
&\quad + (wl\cos(wz_1) + K_1)\big(z_3 - (K_1+K_2)z_2 \\
&\quad + K_1^2 z_1\big) + K_2(z_3 - K_2 z_2) \\
u &= -\phi(x_1, x_2, x_3) - K_3 z_3 - K_4 z_2 - K_5 z_1.
\end{aligned}$$

In the above controller design, $K_1,\ K_2,\ K_3,\ K_4,\ K_5$ are control gains that can be chosen appropriately to guarantee that $z_1,\ z_2,\ z_3$ decay exponentially. At this stage, it is important to note that the control effort is proportional to w^2. For an nth order system, the control effort will be proportional to w^{n-1}, even if l is very small. In addition to explosion of terms, such a dependence on the frequency w is a drawback of feedback linearizing controllers also.

We will use the design procedure for a DSC for Lipschitz nonlinear systems illustrated in [22]:

$$S_1 := x_1 \tag{29}$$
$$\dot{S}_1 = x_2 + l\sin(wx_1) \tag{30}$$
$$\tau_2\dot{x}_{2d} + x_{2d} := -l\sin(wx_1) - K_1 S_1 \tag{31}$$
$$x_{2d}(0) := x_2(0) \tag{32}$$
$$S_2 := x_2 - x_{2d} \tag{33}$$
$$\dot{S}_2 = x_3 + l\sin(wx_2) - \dot{x}_{2d} \tag{34}$$
$$\tau_3\dot{x}_{3d} + x_{3d} := -l\sin(wx_2) + \dot{x}_{2d} - K_2 S_2 \tag{35}$$
$$x_{3d}(0) := x_3(0) \tag{36}$$
$$S_3 := x_3 - x_{3d} \tag{37}$$
$$\dot{S}_3 = u + l\sin(wx_3) - \dot{x}_{3d} \tag{38}$$
$$\Rightarrow u = -l\sin(wx_3) + \dot{x}_{3d} - K_3 S_3. \tag{39}$$

For any positive τ and a smooth p such that $\tau\dot{y} + y = p$, $\|\dot{y}\|_\infty \leq \|\dot{p}\|_\infty$. From this fact, it follows that u increases as w^2 in the worst case in this case and as w^n in the general case and yet is free of the problem of explosion of terms.

The above controller designs for this example bring forth an important heuristic: If the Lipschitz constant is small and the frequency term w is big, it is better to use a high gain linear state feedback controller. If the Lipschitz constant is big and the frequency term is small, then it is better to use backstepping and/or dynamic surface control techniques.

For Lipschitz nonlinear systems of the form, $\dot{x} = Ax + Bu + \phi(x)$, where (A, B) form a controllable pair and $\phi(x)$ is Lipschitz and not necessarily lower triangular as in a strict feedback representation, the result due to Rajamani and Cho [19] provides a sufficient condition for the existence of a linear static state feedback controller—if A is Hurwitz and the distance to uncontrollability of the pair, (A, B), is greater than the Lipschitz constant, such a controller exists. This condition is quite restrictive. It may be viewed as a test for determining whether the given Lipschitz nonlinear system is "weakly" nonlinear or "strongly" nonlinear.

There is a philosophical issue associated with the design of controllers for non-Lipschitz systems, that of uniqueness of solutions of the closed loop system.[1] Consider the following first-order system:

$$\dot{x} = u - \alpha(x)x^{1/3}. \tag{40}$$

The function, $\alpha(x)$ is piecewise smooth, but not known exactly; however, it is known that it is bounded, i.e., there exist α_l, α_u such that $\alpha_l \leq \alpha(x) \leq \alpha_u$ for all x. As long as u is smooth, solutions exist for the closed-loop system; however, their uniqueness may not be guaranteed. Consequently, stability of closed-loop solutions may not be guaranteed. We do not require uniqueness of solutions of the closed loop system when designing controllers for uncertain, non-Lipschitz systems in this paper; we only require boundedness of solutions of the closed loop system.

III. DSC for Non-Lipschitz Systems

In this paper, we are primarily concerned with the design of a DSC for non-Lipschitz systems. The corresponding case for Lipschitz systems can be found in [22]. Consider a nonlinear system of the following form:

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1) + \Delta f_1(x_1) \\
\dot{x}_2 &= x_3 + f_2(x_1, x_2) + \Delta f_2(x_1, x_2) \\
&\cdots \\
\dot{x}_{n-1} &= x_n + f_{n-1}(x_1, \cdots, x_{n-1}) \\
&\quad + \Delta f_{n-1}(x_1, \cdots, x_{n-1}) \\
\dot{x}_n &= u.
\end{aligned} \tag{41}$$

We make the following assumptions for analysis.

- $|\Delta f_i(x_1, \cdots, x_i)| \leq \rho_i(x_1, \cdots, x_i)$ where ρ_i is a C^1 function in its arguments. ρ_i is not required to be globally Lipschitz and Δf_i is not required to be smooth or even locally Lipschitz; we will, however, assume that Δf_i to be a C^0 function, to guarantee the existence of solutions.
- f_i is a smooth function in its arguments, and $f_i(0, \cdots, 0) = 0$.

For example, $\Delta f_1(x)$ may be $C\,\mathrm{sign}(x)\sqrt{|x|}$, which is a non-Lipschitz function; however, it can always be bounded by a locally Lipschitz $|Dx + E|$, where D and E are appropriate positive constants.

The objective of the controller is to make $x_1(t)$ track $x_{1d}(t)$.

[1]We thank Prof. M. Krstic for raising this question.

A. Set of Feasible Output Trajectories

We are interested in semiglobal regulation and tracking. For this reason, the desired ball of operation is centered at origin and is of radius R in $\Re^n$.

We will refer to $x_{1d}(t)$ as a feasible output trajectory in the desired ball of operation, if $x_{1d}(t)$ is a C^1 function and if there exist n C^1 functions, $\psi_1(t), \cdots, \psi_n(t)$, such that

$$\begin{aligned}
&\psi_1(t) \equiv x_{1d}(t)\\
&\dot{\psi}_i(t) = \psi_{i+1} + f_i(\psi_1, \cdots, \psi_i), \qquad i = 1, \cdots, n-1,\\
&\psi_1^2 + \psi_2^2 + \cdots + \psi_n^2 < (R-\delta)^2,\\
&\qquad \forall t \geq 0, \quad \text{and for some } \delta > 0.
\end{aligned}$$

In the absence of any uncertainty in the plant, the state of the system corresponding to the feasible output trajectory will lie entirely inside the desired ball of operation.

From the smoothness of the functions, $f_i,\ i = 1, 2, \cdots, n-1$, it follows that for some $K_0(R) > 0$, $x_{1d}^2 + (dx_{1d}/dt)^2 + \cdots + (d^{n-1}x_{1d}/dt^{n-1})^2 \leq K_0$ implies $\|\psi\| < R - \delta$. Here, ψ is a vector whose ith element is ψ_i.

We will, henceforth, use this alternative definition of a feasible trajectory, i.e., a trajectory, x_{1d} is a feasible output trajectory in the desired ball of radius R, if $x_{1d}^2 + \dot{x}_{1d}^2 + \cdots + (d^{n-1}x_{1d}/dt^{n-1})^2 \leq K_0(R)$.

B. Controller Design

$$S_1 := x_1 - x_{1d} \tag{42}$$

$$\tau_2\dot{x}_{2d} + x_{2d} = -f_1(x_1) - \frac{S_1\rho_1^2}{2\epsilon} - K_1S_1 + \dot{x}_{1d}. \tag{43}$$

Here, ϵ is an arbitrarily small positive constant which will be chosen later. It is a measure of the regulation (or tracking) accuracy that one desires. Continuing this procedure, for $2 \leq i \leq n-1$

$$\begin{aligned}
S_i &:= x_i - x_{id}\\
\tau_{i+1}\dot{x}_{i+1d} + x_{i+1d} &= -f_i(x_1, \cdots, x_i) - \frac{S_i\rho_i^2}{2\epsilon} - K_iS_i + \dot{x}_{id}\\
S_n &= x_n - x_{nd}\\
u_n &= \dot{x}_{nd} - K_nS_n.
\end{aligned}$$

C. Boundedness of Tracking Error Using a DSC

Theorem III.1: Consider any non-Lipschitz nonlinear system in strict feedback form, described in this section. Given *any* uncertain non-Lipschitz nonlinearity with a known C^1 function as its upper bound, and given any $\epsilon > 0$, there exists a set of surface gains, $K_1, \cdots, K_n$ and filter time constants, $\tau_2, \cdots, \tau_n$ such that the dynamic surface controller guarantees:

- There exists a $R_0 \geq 0$, such that for any desired ball of operation of radius $R > R_0$ and for the set of feasible output trajectories considered in this paper, there exists a set, I_c, of initial conditions inside the desired ball, such that the state of the system is regulated within the desired ball at all instants of time whenever the initial condition is in I_c.
- The tracking (regulation) error eventually resides in a ball of radius ϵ.

Proof: The proof uses the technique of singular perturbations; interested readers are referred to [14] and [15]. In this constructive proof, $\phi_i, \Theta_i, \rho_i, \psi_i, \eta_i$ are used to denote functions at the ith step of the induction. Define

$$\begin{aligned}
y_2 &:= x_{2d} + f_1(x_1) + \frac{S_1\rho_1^2}{2\epsilon} + K_1S_1 - \dot{x}_{1d}\\
y_{i+1} &:= x_{i+1d} + f_i(x_1, \cdots, x_i) + \frac{S_i\rho_i^2(x_1, \cdots, x_i)}{2\epsilon}\\
&\quad + \frac{y_i}{\tau_i} + K_iS_i, \qquad \forall i \geq 2.
\end{aligned}$$

Since $S_i := x_i - x_{id}$, it follows that

$$\begin{aligned}
x_1 &= S_1 + x_{1d}\\
x_2 &= S_2 + y_2 - f_1(x_1) - \frac{S_1\rho_1^2}{2\epsilon} - K_1S_1 + \dot{x}_{1d}\\
x_{i+1} &= S_{i+1} + y_{i+1} - f_i(x_1, \cdots, x_i) - \frac{S_i\rho_i^2(x_1, \cdots, x_i)}{2\epsilon}\\
&\quad - \frac{y_i}{\tau_i} - K_iS_i, \qquad \forall i \geq 2.
\end{aligned}$$

By induction, for all $i \geq 2$,

$$\begin{aligned}
x_{i+1} = \psi_i(&S_1, \cdots, S_{i+1}, y_2, \cdots, y_{i+1},\\
&K_1, \cdots, K_i, \tau_2, \cdots, \tau_i, x_{1d}, \dot{x}_{1d}).
\end{aligned}$$

From the definition of S_i, it follows that

$$\begin{aligned}
\dot{S}_i &= x_{i+1} + f_i(x_1, \cdots, x_i) + \Delta f_i(x_1, \cdots, x_i) - \dot{x}_{id}\\
&= S_{i+1} + y_{i+1} - K_iS_i - \frac{S_i\rho_i^2}{2\epsilon} + \Delta f_i\\
|\dot{S}_i| &\leq \Theta_i(S_1, \cdots, S_{i+1}, y_2, \cdots, y_{i+1},\\
&\qquad K_1, \cdots, K_i, \tau_2, \cdots, \tau_i, x_{1d}, \dot{x}_{1d}).
\end{aligned}$$

Thus, the bound on $\dot{S}_i$ depends only on $S_1, \cdots, S_{i+1}$, $y_2, \cdots, y_{i+1}$, $K_1, \cdots, K_i, \tau_2, \cdots, \tau_i, x_{1d}$ and $\dot{x}_{1d}$. Similarly, one can show that

$$\begin{aligned}
\frac{d\rho_i}{dt} &= \sum_1^i \frac{\partial\rho_i}{\partial x_j}\dot{x}_j\\
\left|\frac{d\rho_i}{dt}\right| &\leq \phi_i(S_1, \cdots, S_{i+1}, y_2, \cdots, y_{i+1},\\
&\qquad K_1, \cdots, K_i, \tau_2, \cdots, \tau_i, x_{1d}, \dot{x}_{1d})
\end{aligned}$$

and

$$\begin{aligned}
\dot{y}_2 = &-\frac{y_2}{\tau_2} - \frac{\partial f_1}{\partial x_1}\dot{x}_1 - \dot{S}_1\frac{\rho_1^2}{2\epsilon} - \frac{S_1\rho_1}{\epsilon}\frac{\partial\rho_1}{\partial x_1}\dot{x}_1\\
&+ K_1\dot{S}_1 - \ddot{x}_{1d}.
\end{aligned}$$

Clearly,

$$\left|\dot{y}_2 + \frac{y_2}{\tau_2}\right| \leq \eta_2(S_1, S_2, y_2, K_1, x_{1d}, \dot{x}_{1d}, \ddot{x}_{1d})$$

for some continuous function η_2. This follows from the fact that f_1 and ρ_1 are C^1 functions.

$$\begin{aligned}
\dot{y}_{i+1} = &-\frac{y_{i+1}}{\tau_{i+1}} - \sum_1^i \frac{\partial f_i}{\partial x_j}\dot{x}_j - \frac{\rho_i^2}{2\epsilon}\dot{S}_i\\
&- \frac{\rho_iS_i}{\epsilon}\sum_1^i \frac{\partial\rho_i}{\partial x_j}\dot{x}_j + \frac{\dot{y}_i}{\tau_i} + K_i\dot{S}_i.
\end{aligned} \tag{44}$$

By induction, for some continous function, η_{i+1},

$$\begin{aligned}
\left|\dot{y}_{i+1} + \frac{y_{i+1}}{\tau_{i+1}}\right| \leq \eta_{i+1}(&S_1, \cdots, S_{i+1}, y_2, \cdots, y_{i+1}, K_1, \cdots, K_i,\\
&\tau_2, \cdots, \tau_i, x_{1d}, \dot{x}_{1d}, \ddot{x}_{1d}).
\end{aligned}$$

Define

$$V_{is} := \frac{S_i^2}{2}, \qquad i = 1, \cdots, n$$

and

$$\begin{aligned}
V_{iy} &:= \frac{y_{i+1}^2}{2}, \qquad i = 1, \cdots, n-1\\
\Rightarrow \dot{V}_{is} &= S_i\left(S_{i+1} + y_{i+1} - K_iS_i - \frac{S_i\rho_i^2}{2\epsilon} + \Delta f_i\right)\\
&\leq |S_i|(|S_{i+1}| + |y_{i+1}|) - K_iS_i^2 + \frac{\epsilon}{2},\\
&\qquad i = 1, \cdots, n-1\\
\dot{V}_{ns} &= -K_nS_n^2.
\end{aligned}$$

The last inequality follows from Young's inequality, $(S_i^2\rho_i^2/2\epsilon) + (\epsilon/2) \geq |S_i|\rho_i \geq |S_i||\Delta f_i|$. Also

$$\begin{aligned}\dot{V}_{iy} &= y_{i+1}\dot{y}_{i+1}\\ &\leq -\frac{y_{i+1}^2}{\tau_{i+1}} + |y_{i+1}|\eta_{i+1}(S_1, \cdots, S_{i+1}, y_2, \cdots, y_{i+1}, K_1, \cdots,\\ &\qquad K_i, \tau_2, \cdots, \tau_i, x_{1d}, \dot{x}_{1d}, \ddot{x}_{1d}).\end{aligned}$$

Let $V := V_{1s} + \cdots + V_{ns} + V_{1y} + \cdots + V_{n-1y}$.

Claim: Given any $p > 0, K_0 > 0$, such that for all $V(0) \leq p$ and $x_{1d}^2 + \dot{x}_{1d}^2 + \ddot{x}_{1d}^2 \leq K_0$, there exists a set of gains, $K_1, \cdots, K_n$, and filter time constants, $\tau_2, \cdots, \tau_n$, such that $V(t) \leq p \ \forall t > 0$ and $\dot{V} \leq -2\alpha_0 V + n\epsilon$.

Proof of Claim: Consider the set $Q := \{(x_{1d}, \dot{x}_{1d}, \ddot{x}_{1d}){:}\ x_{1d}^2 + \dot{x}_{1d}^2 + \ddot{x}_{1d}^2 \leq K_0\}$. Clearly, Q is compact in $\Re^3$. Consider the sets $A_i := \{S_1^2 + y_2^2 + S_2^2 + \cdots + y_i^2 + S_i^2 \leq 2p\}$. A_i is compact in $\Re^{2i-1}$. Also, $A_i \times Q$ is compact in $\Re^{2i+2}$. Therefore, η_{i+1} has a maximum, say M_{i+1} on $A_i \times Q$. Fix the gains, K_i as $K_i = 2 + \alpha_0$, where $\alpha_0 > n\epsilon/2p$. Choose the time constants of the filters inductively: $(1/\tau_{i+1}) := 1 + (M_{i+1}^2/2\epsilon) + \alpha_0$. Therefore

$$\begin{aligned}\dot{V} &\leq -(2+\alpha_0)\sum_{i=1}^{n} S_i^2 + \sum_{i=1}^{n-1}\left[\frac{2S_i^2 + S_{i+1}^2 + y_{i+1}^2}{2}\right.\\ &\quad \left. - \left(1 + \frac{M_{i+1}^2}{2\epsilon} + \alpha_0\right)y_{i+1}^2 + \frac{M_{i+1}^2 y_{i+1}^2}{2\epsilon}\,\frac{\eta_{i+1}^2}{M_{i+1}^2}\right]\\ &\quad + (n-1)\epsilon\\ &\leq -2\alpha_0 V + n\epsilon - \sum_{i=1}^{n-1}\left(1 - \frac{\eta_{i+1}^2}{M_{i+1}^2}\right)\frac{M_{i+1}^2 y_{i+1}^2}{2\epsilon}.\end{aligned}$$

On $V(S_1, \cdots, S_n, y_2, \cdots, y_n) = p$, $\eta_{i+1} \leq M_{i+1}$. Therefore, $\dot{V} \leq -2\alpha_0 p + n\epsilon$. Since $\alpha_0 > n\epsilon/2p$, it follows that $\dot{V} \leq 0$ on $V = p$. Therefore, $V \leq p$ is an invariant set, i.e., if $V(0) \leq p$, then $V(t) \leq p$ for all $t > 0$.

To conclude semiglobal boundedness of errors, we must provide, for each feasible output trajectory, $x_{1d}(t)$, a corresponding set of initial conditions, $x_i(0), i = 1, \cdots, n$, such that the state of the system lies within a ball of radius R at all instants of time and the output error, S_1 is ultimately bounded in a ball of radius, ϵ.

We want to show that if $V(t) \leq p$, and $x, d^2 + \dot{x}, d^2 + \ddot{x}, d^2 \leq K_0$, $\|x(t)\| \leq h(p)$, for some continuous, radially unbounded function of p.

It can be inductively shown that x_{i+1d} is bounded by showing that the input to the filter is bounded uniformly and is a continuous function of p. Consequently, x_i is bounded for all i and for all instants of time. Let

$$h(p) := \sup_{V(t)\leq p, x_{1d}^2 + \dot{x}_{1d}^2 + \ddot{x}_{1d}^2 \leq K_0} \sqrt{x_1^2(t) + \cdots + x_n^2(t)}.$$

For all $p \leq p_0$, for some $p_0 > 0$, we will choose gains and filter constants corresponding to p_0. Doing so will guarantee that for all $V(0) \leq p \leq p_0 \Rightarrow V(t) \leq p_0$ for all $t \geq 0$. Define $R_0 := h(p_0)$. The constant p_0 can be chosen to minimize R_0.

The bound, $h(p)$, on $\|x\|$ is also a continuous, radially unbounded function of p. If the desired ball of operation is $R > R_0$, the equation $h(p^*) = R$ has a solution.

The gains and filter time constants corresponding to p^* will be used in the controller.

Now, we have to provide a set of initial conditions in the desired ball so that the claims of the theorem are proven.

Since $x_{id}(0)$ can be chosen, we will choose $x_{id}(0)$ so as to make $y_{i+1}(0) = 0,\ i = 1, \cdots, n-1$. To determine the set of initial conditions, $x(0)$ that will ensure the trajectory will always remain in the desired ball of operation, the set of equations, obtained by setting, $y_{i+1}(0) = 0$, is considered, given at the bottom of the page.

Observing that x_{1d} is a feasible trajectory and setting $S_i = 0, i = 1, \cdots, n$ in the above equation yields a solution ψ_0 to $x(0)$ that lies inside the open ball of operation (of radius $R > R_0$).

From the continuity of functions used in the above set of equations, it is clear that if $\|x(0) - \psi_0\|$ is a continuous function of $S_i(0), i = 1, 2, \cdots, n$ and is zero when $S_i(0) = 0,\ i = 1, 2, \cdots, n$. Therefore, given a $\delta > 0$, there exists some positive q, such that $S_1^2(0) + \cdots + S_n^2(0) < 2q$ implies that $\|x(0) - \psi_0\| < \delta$. If $q > p^*$, there exists a $\delta^* < \delta$ such that $S_1^2(0) + \cdots + S_n^2(0) < 2p^*$ implies $\|x(0) - \psi_0\| < \delta^*$. In the case when $q \leq p^*$, a desired set of initial conditions is the ball centered at ψ_0 and of radius q. Otherwise, a desired set of initial conditions is the ball centered at ψ_0 and of radius p^*. In either cases, desired initial condition set is entirely inside the desired ball of operation.

From the choice of initial conditions for the auxillary filters, $y_i(0) = 0, i = 2, \cdots, n$. Therefore, $2V(0) = S_1^2(0) + \cdots + S_n^2(0) \leq 2p^* \Rightarrow V(0) \leq p^*$. We have already shown that $V < p^*$ is an invariant set. We also know that if $V(t) < p^* \Rightarrow \|x(t)\| < h(p^*) = R$. Therefore, semiglobal boundedness of the state is guaranteed.

Since $\dot{V} \leq -\alpha_0 V + n\epsilon$, it follows that V and consequently, the tracking error is ultimately confined to a ball of radius ϵ.

This completes the proof for semiglobal regulation and tracking of DSC for non-Lipschitz systems. □

IV. Illustrative Example

In this section, the following example is considered:

$$\dot{x}_1 = x_2 + \Delta f_1(x_1) \tag{45}$$

$$\dot{x}_2 = x_3 \tag{46}$$

$$\dot{x}_3 = u \tag{47}$$

$$y = x_1. \tag{48}$$

The control objective is to synthesize a state feedback law for u such that the output, $y(t)$, tracks a reference signal, $r(t)$. The uncertainty $\Delta f_1(x_1)$ is assumed to be bounded by x_1^2. For the simulation study,

$$\begin{pmatrix} x_1(0) - x_d(0) \\ x_{2d}(0) - \dot{x}_d(0) + S_1(0)\left(\dfrac{\rho_1(x_1(0))^2}{2\epsilon} + K_1\right) + f_1(x_1(0)) \\ x_{3d}(0) + S_2(0)\left(\dfrac{\rho_2(x_1(0), x_2(0))^2}{2\epsilon} + K_2\right) + f_2(x_1(0), x_2(0)) - \dot{x}_{2d}(0) \\ \vdots \\ x_{nd}(0) + S_{n-1}(0)\left(\dfrac{\rho_{n-1}(x_1(0), x_2(0), \cdots, x_{n-1}(0))^2}{2\epsilon} + K_{n-1}\right) + f_{n-1}(x_1(0), x_2(0), \cdots, x_{n-1}(0)) \end{pmatrix} = \begin{pmatrix} S_1(0) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

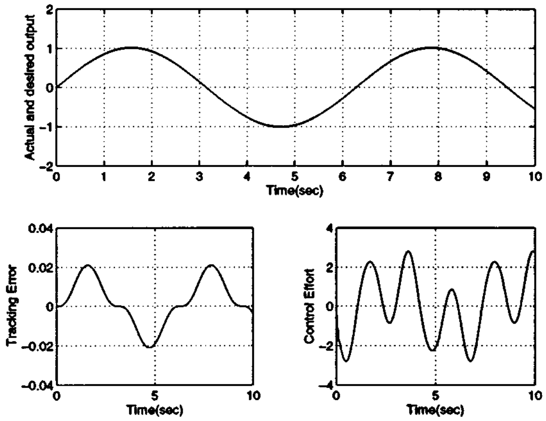

Fig. 1. Performance of DSC with filter time constants $= 0.01$ s.

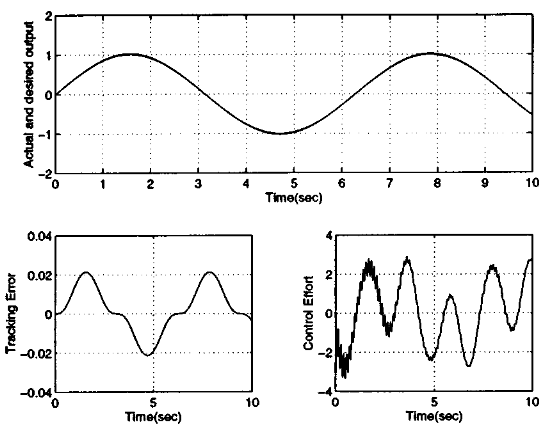

Fig. 2. Performance of DSC with filter time constants $= 0.024$ s.

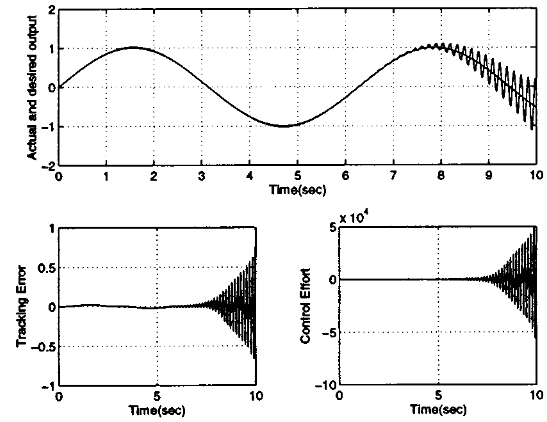

Fig. 3. Performance of DSC with filter time constants $= 0.028$ s.

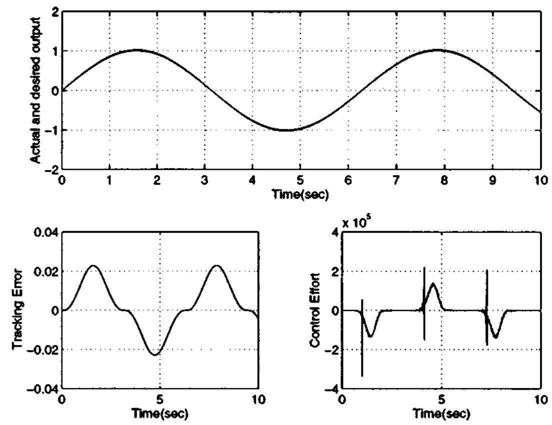

Fig. 4. Performance of a Backstepping controller.

$\Delta f_1(x_1) = x_1^2 \sin(x_1)$. The reference signal, $r(t)$, the output is required to track in the simulation is $\sin(t)$. The details of designing a backstepping controller and a DSC for this system are omitted for saving space.

For this system, $r(t) = A\sin(wt)$ is a feasible output trajectory in a ball of radius R, if for some $\delta > 0$, $A^2(1+w^4) < (R-\delta)^2$.

Figs. 1–3 illustrate how the filter time constants affect the performance of the system. Three sets of time constants chosen for simulation are: $\{\tau_2 = \tau_3 = 0.01\}$, $\{\tau_2 = \tau_3 = 0.024\}$, and $\{\tau_2 = \tau_3 = 0.028\}$. The surface gains used for all those simulations are $\{K_1 = 40.0, K_2 = 60.0, K_3 = 60.0\}$. $\tau_2 K_1$ equals 0.4 for the first set, 0.96 for the second set and 1.12 for the third set. The tracking parameter, ϵ, is set to 0.2 for all the DSC cases. With the first set of gains and filter time constants, the amplitude of the tracking error is about 2% of the reference signal. With the second set of gains, the output oscillates around the desired trajectory before converging to a steady state. With the third set of time constants, the controller goes unstable. There are two figures showing control effort for a set of gains. The latter is a blow-up of the former and it excludes the transients.

Fig. 4 shows the corresponding performance with a backstepping controller. The gains chosen for the backstepping controller are $K_1 = 40, K_2 = 60, K_3 = 60$ and the tracking parameter, ϵ, is chosen to be 1.0 in this case. It can be seen that dynamic surface controller and backstepping controller have a similar performance, and that the control effort of a DSC is orders of magnitude smaller than the backstepping controller in this specific case.

V. Conclusions

In this paper, Dynamic Surface Controller design is proposed. This controller design is intuitively appealing, and it has "$r-1$" lowpass filters, where r is the relative degree of the output to be controlled. These low pass filters allow a design where the model is not differentiated, at the same time, avoiding the complexity that arises due to the explosion of terms. In our earlier paper [22], dynamic surface controller has been shown to guarantee exponential regulation and bounded tracking error in the presence of Lipschitz mismatched uncertainties in strict feedback form. In this paper, we have designed dynamic surface controller for non-Lipschitz systems. We have shown that Dynamic Surface Controller guarantees arbitrarily tight semiglobal regulation. Backstepping algorithm guarantees arbitrarily tight regulation globally. This is the tradeoff in performance. In particular, the key feature of the algorithm, which removes the need for differentiations in the controller design and reduces the explosion of terms, is that due to the presence of the auxiliary first-order filters, none of the nonlinearities are ever differentiated more than once. This is a crucial point, because it implies that any C^1 nonlinearity can be used. In some of the previous control design schemes, the assumption of globally continuity was needed not only on the original nonlinearities, but also on all the derivatives that were generated during the design process, thus limiting the allowable nonlinearities so much that sometimes only linear functions could be dealt with.

In this paper, we have shown that there exists a set of gains and filter time constants that guarantee semiglobal stability for the nonlinear system in strict feedback form. While the proof follows a constructive approach to determine the time constants, in practice, such a constructive approach exposes a control engineer to the problem of explosion of terms at the design stage. Future work is underway to systematize the choice of time constants for the filters. Real-time implementation of the software filters poses a hard performance limitation. The maximum bandwidth of the software filters is bounded by the control sampling frequency. In other words, the filter time constants cannot be made arbitrarily small in real-time implementation.

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Convergence Behavior of the Schur Recursion in the Krein Space for the J-Spectral Factorization

Kyungsup Kim and Joohwan Chun

***Abstract*—We present a "Krein-space version" of the Schur recursion for the J-spectral factorization which arises in H^∞-related problems. The most notable difference of the proposed Schur recursion from the ordinary one is that the proposed recursion can handle temporary changes of the inertia during the process. We show that the Schur recursion in the Krein-space converges to a J-spectral factor exponentially under a suitable condition.**

***Index Terms*—H^∞ problem, J-spectral factorization, Krein space, Riccati equation, Schur recursion in the Krein space.**

I. INTRODUCTION

The goal of this paper is to show how one could use a Schur-like recursion to find a J-spectral factor $G(z)$, where $G(z)$ and its inverse are analytic outside the unit circle, of a given $p \times p$ para-Hermitian rational matrix-valued function $\Pi(z, z^{-1})$ such that

$$\Pi(z, z^{-1}) = G(z) j \tilde{G}(z), \tag{1}$$

where j is a suitable signature matrix.

This J-spectral factorization problem is encountered in various H^∞ problems [2], [10]–[13]. To provide a concrete example, we shall consider a discrete-time system and a measurement model of the form,

$$\begin{aligned} y(z) &= H(z)u(z) + G(z)v(z), \\ s(z) &= L(z)u(z) + M(z)v(z), \end{aligned}$$

where $u(z)$ is the (z-transform of the) process noise, $v(z)$ is the measurement noise, $y(z)$ is the measurement, and $s(z)$ is the unknown to be estimated. We wish to find $K(z)$ such that $\hat{s}(z) \triangleq K(z)y(z)$ is as "close" to $s(z)$ as possible for the "worst" choices of $u(z)$ and $v(z)$. More precisely, we wish to find $K(z)$ which minimizes $\|A(z) - K(z)B(z)\|_\infty$, where $A(z) \triangleq [L(z) \quad M(z)]$ and

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