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Robust finite-Time control and reachable set estimation for uncertain switched neutral systems with time delays and input constraints

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ABSTRACT

This work focuses on robust finite-time fault-tolerant control and reachable set estimation for uncertain switched neutral systems subject to time delays as well as input constraints. There are few attempts to investigate robust finite-time boundedness and reachable set estimation for uncertain switched neutral systems. At the same time, input-output finite-time stability is also investigated. Compared with the existing studies, this system has wider application. A dynamic output feedback nonlinear controller is researched. An augmented closed-loop system is provided. Moreover, the sufficient conditions of finite-time boundedness, reachable set estimation and input-output finite-time stability for the closed-loop system are obtained in the framework of linear matrix inequalities via piecewise Lyapunov function. Ultimately, two examples are provided to demonstrate this validity of this approach in this paper.

1. Introduction

Switched systems caused by the changing environment are omnipresent in production and life such as in power electronics systems, communication systems as well as flight and air traffic control systems [1–3]. Considerable attention has been paid to the research on switched systems [4–7]. In [8], the problem of sampled-data adaptive output feedback fuzzy stabilization was studied for switched uncertain nonlinear systems associated against asynchronous switching. In [9], event-triggered bumpless transfer control was investigated for switched systems and was applied to switched radio link control (RLC) circuits. It is worth pointing out that switched systems still have much research potential.

In many practical systems, due to a variety of propagation and transmission conditions, the phenomenon of time delay is very common when establishing data dynamic models [10–12]. In [13], time-delay compensation method was investigated

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for networked control system according to time-delay prediction as well as implicit proportional-integral-based generalized predictive controller (PIGPC). In [14], this output feedback control was investigated for the linear discrete-time system against an interval time-varying delay by the free-weighting matrix method. In [15], this issue of the adaptive sliding mode fault-tolerant control was investigated for unmanned marine vehicles against signal quantization as well as state time-delay. Lots of considerable attention has been gained to this issue of switched systems with time delay up to now[16]. dealt with this issue of global stabilization for a cluster of switched feedforward nonlinear time-delay systems where both states and input exist time delays. In addition, the research on finite-time boundness (FTB) and input-output finite-time stability (IO-FTS) becomes a hot issue [17–21], because more and more practical systems need to be stable or bounded in finite time. Other than this papers mentioned above, the issues of FTB with IO-FTS for uncertain switched neutral systems with time delays are not sufficiently studied, which is explored in the work.

This reachable set of dynamic systems means this set bounding all trajectories of the system states under some disturbances [22,23]. On the one hand, reachable set estimation is essential from the aspect of robust control [24]. On the other hand, reachable set estimation is achieved and unsafe regions are avoided by controllers in terms of practical significance. In [25], the issues of output reachable estimation with safety verification were investigated for multilayer perceptron (MLP) neural networks. In [26], this reachable set estimation issue was explored for Markovian jump neural networks against time-varying delays as well as bounded peak disturbances. In [27], this reachable set estimation issue was investigated for discrete-time Takagi-Sugeno (T-S) fuzzy systems against bounded input disturbances, time delay as well as nonzero initial conditions. However, the problem of reachable set estimation under robust finite-time control are not sufficiently studied for uncertain switched neutral systems against time delays. This inspires our research enthusiasm.

The paper is mainly devoted to studying robust finite-time fault-tolerant control as well as reachable set estimation for switched neutral system with time delays and input constraints. In this switched neutral system, some elements are considered, such as faults, disturbances and model uncertainties. sufficient conditions of FTB, reachable set estimation and IO-FTS are considered for this system against time delays as well as input constraints. A dynamic output feedback controller is constructed. At last, one numerical example and one practical example are presented to illustrate this validity of this method acquired in the paper. These main features of the study are summarized as follows:

1) This is one of the few attempts to investigate a dynamic output feedback controller to realize fault-tolerant control for switched neutral system against time delays as well as input constraints.
2) This work shows one of the few attempts to handle the issues of FTB, reachable set estimation and IO-FTS simultaneously for switched neutral system against time delays as well as input constraints, indicating that satisfactory results are acquired in finite-time.
3) A novel piecewise Lyapunov function is utilized and then less conservative results are acquired.

The remainder of the study is assigned in the following. In Section 2, this uncertain switched neutral system with input constraints is depicted, at the same time, these preliminaries are described. A dynamic output feedback controller is employed in Section 3. In Section 4, the conditions of FTB, reachable set estimation and IO-FTS for this uncertain switched neutral system are acquired in the form of linear matrix inequalities (LMIs). In Section 5, this effectiveness of this suggested approach is validated via two simulation examples. At last, in Section 6, the conclusions are described.

Notations: R^n represents this n-dimensional real Euclidean space. I means an identity matrix against appropriate dimensions. These superscripts "T'' as well as "$-1''$ stand for the matrix transpose as well as the inverse matrix, separately. $U>0$ or $U\geq 0$ stands for that U is a real symmetric positive definite matrix or a real symmetric positive semidefinite matrix. This symbol $*$ in a matrix denotes this transposed element in the symmetric position. diag$\{U_1,U_2,\cdots,U_n\}$ represents a block diagonal matrix consisting of square matrices $U_1,U_2,\cdots,U_n$. sym(U) is equivalent to U^T+U.

2. Problem formulation

The uncertain switched neutral system is described as follows:

$$\begin{aligned}\dot{x}(t)-\sum_{j=1}^{m}\sigma_j(t)\mathscr{B}^j(t)\dot{x}(t-h)=\sum_{j=1}^{m}\sigma_j(t)[&\mathscr{A}_0^j(t)x(t)+\mathscr{A}_1^j(t)x(t-\tau)\\&+\mathscr{A}_u^j(t)u(t)+\mathscr{A}_a^j(t)f_a(t)+\mathscr{A}_\omega^j(t)\omega(t)]\end{aligned}\tag{1}$$

$$y(t)=\sum_{j=1}^{m}\sigma_j(t)[\mathscr{C}_0^j(t)x(t)+\mathscr{C}_1^j(t)x(t-\tau)+\mathscr{C}_s^j(t)f_s(t)+\mathscr{C}_\omega^j(t)\omega(t)]\tag{2}$$

$$y_z(t)=\sum_{j=1}^{m}\sigma_j(t)[\mathscr{D}_0^jx(t)+\mathscr{D}_\omega^j\omega(t)]\tag{3}$$

$$x(t)=\varphi(t),\ \ t\in[-\bar{\tau},0]\tag{4}$$

where $x(t) \in R^{n_x}$, $u(t) \in R^{n_u}$, $f_a(t) \in R^{n_a}$, $\omega(t) \in R^{n_\omega}$, $y(t) \in R^{n_y}$, $f_s(t) \in R^{n_s}$ and $y_z(t) \in R^{n_z}$ stand for the state, the control input, the actuator fault, the disturbance input, the measurement output, the sensor fault and the controlled output, separately. At the same time, τ and h both denote time delays. $\bar{\tau} = \max\{h, \tau\}$. $\varphi(t)$ means the vector-valued initial function. $\mathscr{D}_0^j$ and $\mathscr{D}_\omega^j$, $j \in \{1, \cdots, m\}$, stand for known constant matrices. $\mathscr{B}^j(t)$, $\mathscr{A}_0^j(t)$, $\mathscr{A}_1^j(t)$, $\mathscr{A}_u^j(t)$, $\mathscr{A}_a^j(t)$, $\mathscr{A}_\omega^j(t)$, $\mathscr{C}_0^j(t)$, $\mathscr{C}_1^j(t)$, $\mathscr{C}_s^j(t)$ and $\mathscr{C}_\omega^j(t)$, $j \in \{1, \cdots, m\}$, stand for uncertain matrices obeying

$$\begin{aligned}
&\mathscr{B}^j(t) = \mathscr{B}^j + \Delta\mathscr{B}^j(t), \mathscr{A}_0^j(t) = \mathscr{A}_0^j + \Delta\mathscr{A}_0^j(t), \mathscr{A}_1^j(t) = \mathscr{A}_1^j + \Delta\mathscr{A}_1^j(t)\\
&\mathscr{A}_u^j(t) = \mathscr{A}_u^j + \Delta\mathscr{A}_u^j(t), \mathscr{A}_a^j(t) = \mathscr{A}_a^j + \Delta\mathscr{A}_a^j(t), \mathscr{A}_\omega^j(t) = \mathscr{A}_\omega^j + \Delta\mathscr{A}_\omega^j(t)\\
&\mathscr{C}_0^j(t) = \mathscr{C}_0^j + \Delta\mathscr{C}_0^j(t), \mathscr{C}_1^j(t) = \mathscr{C}_1^j + \Delta\mathscr{C}_1^j(t), \mathscr{C}_s^j(t) = \mathscr{C}_s^j + \Delta\mathscr{C}_s^j(t)\\
&\mathscr{C}_\omega^j(t) = \mathscr{C}_\omega^j + \Delta\mathscr{C}_\omega^j(t)
\end{aligned}$$

where $\mathscr{B}^j$, $\mathscr{A}_0^j$, $\mathscr{A}_1^j$, $\mathscr{A}_u^j$, $\mathscr{A}_a^j$, $\mathscr{A}_\omega^j$, $\mathscr{C}_0^j$, $\mathscr{C}_1^j$, $\mathscr{C}_s^j$ and $\mathscr{C}_\omega^j$, $j \in \{1, \cdots, m\}$, stand for known constant matrices. In addition, model uncertainties $\Delta\mathscr{B}^j(t)$, $\Delta\mathscr{A}_0^j(t)$, $\Delta\mathscr{A}_1^j(t)$, $\Delta\mathscr{A}_u^j(t)$, $\Delta\mathscr{A}_a^j(t)$, $\Delta\mathscr{A}_\omega^j(t)$, $\Delta\mathscr{C}_0^j(t)$, $\Delta\mathscr{C}_1^j(t)$, $\Delta\mathscr{C}_s^j(t)$ and $\Delta\mathscr{C}_\omega^j(t)$, $j \in \{1, \cdots, m\}$, stand for time-varying matrices obeying

$$\begin{aligned}
&\begin{bmatrix} \Delta\mathscr{B}^j(t) & \Delta\mathscr{A}_0^j(t) & \Delta\mathscr{A}_1^j(t) & \Delta\mathscr{A}_u^j(t) & \Delta\mathscr{A}_a^j(t) & \Delta\mathscr{A}_\omega^j(t) \end{bmatrix} = \mathscr{E}_x^j \mathscr{F}_x^j(t)\\
&\begin{bmatrix} \mathscr{G}_x^j & \mathscr{G}_{x0}^j & \mathscr{G}_{x1}^j & \mathscr{G}_{xu}^j & \mathscr{G}_{xa}^j & \mathscr{G}_{x\omega}^j \end{bmatrix}\\
&\begin{bmatrix} \Delta\mathscr{C}_0^j(t) & \Delta\mathscr{C}_1^j(t) & \Delta\mathscr{C}_s^j(t) & \Delta\mathscr{C}_\omega^j(t) \end{bmatrix} = \mathscr{E}_y^j \mathscr{F}_y^j(t) \begin{bmatrix} \mathscr{G}_{y0}^j & \mathscr{G}_{y1}^j & \mathscr{G}_{ys}^j & \mathscr{G}_{y\omega}^j \end{bmatrix}
\end{aligned}$$

where $\mathscr{E}^j$, $\mathscr{G}_x^j$, $\mathscr{G}_{x0}^j$, $\mathscr{G}_{x1}^j$, $\mathscr{G}_{xu}^j$, $\mathscr{G}_{xa}^j$, $\mathscr{G}_{x\omega}^j$, $\mathscr{E}_y^j$, $\mathscr{G}_{y0}^j$, $\mathscr{G}_{y1}^j$, $\mathscr{G}_{ys}^j$ and $\mathscr{G}_{y\omega}^j$, $j \in \{1, \cdots, m\}$, stand for known constant matrices. $\mathscr{F}_x^j(t)$ and $\mathscr{F}_y^j(t)$, $j \in \{1, \cdots, m\}$, stand for unknown matrix functions and are Lebesgue measurable satisfying

$$(\mathscr{F}_x^j(t))^T \mathscr{F}_x^j(t) \le I, (\mathscr{F}_y^j(t))^T \mathscr{F}_y^j(t) \le I$$

$\sigma_j(t)$, $j = 1, \cdots, m$, mapping from $[0, \infty)$ to $\{0, 1\}$, denotes the switching signal and means the activated subsystem at the switching instant. This implies if this jth switched subsystem is activated between this time interval $[\bar{t}, \tilde{t})$, then $\sigma_j(t) = 1$ for $t \in [\bar{t}, \tilde{t})$. We can find that $\sum_{j=1}^{m} \sigma_j(t) = 1$.

3. Controller design

The dynamic output feedback controller with zero initial value is designed as follows

$$\dot{\eta}(t) = \sum_{j=1}^{m} \sigma_j(t)[A^j \eta(t) + B^j y(t)] \tag{5}$$

$$u(t) = \sum_{j=1}^{m} \sigma_j(t) C^j \eta(t) \tag{6}$$

where $\eta(t) \in R^{n_x}$ stands for the controller state. A^j, B^j and C^j, $j \in \{1, \cdots, m\}$, denote the gain matrices of the controller.

Denote $\bar{\zeta}(t) = \begin{bmatrix} x^T(t) & \eta^T(t) \end{bmatrix}^T$ and $\bar{\omega}(t) = \begin{bmatrix} f_a^T(t) & f_s^T(t) & \omega^T(t) \end{bmatrix}^T$. Combining (1)-(4) with (5)-(6), the following closed-loop system can be obtained

$$\begin{aligned}
\dot{\bar{\zeta}}(t) - \sum_{j=1}^{m} \sigma_j(t) \mathcal{B}^j(t) \dot{\bar{\zeta}}(t-h) = \sum_{j=1}^{m} \sigma_j(t)[&\mathcal{A}_0^j(t)\bar{\zeta}(t) + \mathcal{A}_1^j(t)\bar{\zeta}(t-\tau)\\
&+ \mathcal{A}_{\bar{\omega}}^j(t)\bar{\omega}(t)]
\end{aligned} \tag{7}$$

$$y_z(t) = \sum_{j=1}^{m} \sigma_j(t)[\mathcal{D}_0^j \bar{\zeta}(t) + \mathcal{D}_{\bar{\omega}}^j \bar{\omega}(t)] \tag{8}$$

$$\bar{\zeta}(t) = \bar{\varphi}(t), \ t \in [-\bar{\tau}, 0] \tag{9}$$

where

$$\mathcal{B}^j(t) = \mathrm{diag}\{\mathscr{B}^j(t), 0\}, \mathcal{A}_0^j(t) = \begin{bmatrix} \mathscr{A}_0^j(t) & \mathscr{A}_u^j(t) C^j \\ B^j \mathscr{C}_0^j(t) & A^j \end{bmatrix}$$

$$\mathcal{A}_1^j(t) = \begin{bmatrix} \mathscr{A}_1^j(t) & 0 \\ B^j\mathscr{C}_1^j(t) & 0 \end{bmatrix}, \mathcal{A}_{\bar{\omega}}^j(t) = \begin{bmatrix} \mathscr{A}_a^j(t) & 0 & \mathscr{A}_\omega^j(t) \\ 0 & B^j\mathscr{C}_s^j(t) & B^j\mathscr{C}_\omega^j(t) \end{bmatrix}$$

$$\mathcal{D}_0^j = \begin{bmatrix} \mathscr{D}_0^j & 0 \end{bmatrix}, \mathcal{D}_{\bar{\omega}}^j = \begin{bmatrix} 0 & 0 & \mathscr{D}_\omega^j \end{bmatrix}, \ \bar{\varphi}(t) = \begin{bmatrix} \varphi(t) \\ 0 \end{bmatrix}.$$

Assumption 1. Assume that $\mathbb{T} > 0$ is a time scalar. For the given scalar $\beta_1 > 0$, positive definite matrix $\mathbb{J}_1$ and time interval $[0, \mathbb{T}]$, it is assumed that

$$\mathbb{H}(\mathbb{T}, \mathbb{J}_1, \beta_1) =: \left\{ \bar{\omega}(t) \in L_2[0, \mathbb{T}] : \bar{\omega}^T(t)\mathbb{J}_1\bar{\omega}(t) \le \frac{\beta_1}{\mathbb{T}} \right\}.$$

Further, the following equation holds

$$\int_0^{\mathbb{T}} \bar{\omega}^T(t)\mathbb{J}_1\bar{\omega}(t)dt \le \beta_1.$$

Assumption 2. Assume that $\|u(t)\| < \bar{u}$, where the scalar $\bar{u}$ is a known positive constant.

Definition 1 ([28]). In this model (7)-(9), if the following equation holds

$$\mathbb{N}(\bar{t}, \tilde{t}) \le \mathbb{N}_0 + \frac{\tilde{t} - \bar{t}}{\mathbb{T}_a}.$$

$\mathbb{T}_a > 0$ stands for an average dwell time along with $\mathbb{N}_0 \ge 0$, the scalar $\tilde{t} > \bar{t} \ge 0$ and $\mathbb{N}(\bar{t}, \tilde{t})$ stands for this switching number over this interval $[\bar{t}, \tilde{t})$. According to [28], assume $\mathbb{N}_0 = 0$ in the work.

Definition 2 ([29]). Suppose that there exist positive scalars $\mathbb{T}$, c_1, c_2, as well as one matrix $\mathbb{R} > 0$. For the given time interval $t \in [0, \mathbb{T}]$, the system (7)-(9) can be called FTB with respect to $(c_1, c_2, \mathbb{R}, \mathbb{T})$, if it yields that

$$\max_{l \in [-\bar{\tau}, 0]} \{\bar{\zeta}^T(l)\mathbb{R}\bar{\zeta}(l), \dot{\bar{\zeta}}^T(l)\mathbb{R}\dot{\bar{\zeta}}(l)\} \le c_1$$

such that

$$\bar{\zeta}^T(t)\mathbb{R}\bar{\zeta}(t) < c_2.$$

Definition 3 ([29]). For a series of parameters $\mathbb{H}(\mathbb{T}, \mathbb{J}_1, \beta_1)$ as well as one given matrix $\mathbb{J}_2 > 0$, this closed-loop system (7)-(9) can be called IO-FTS with regard to $(\mathbb{H}(\mathbb{T}, \mathbb{J}_1, \beta_1), \mathbb{J}_2, \beta_2)$, if it yields that

$$\bar{\omega}(t) \in \mathbb{H}(\mathbb{T}, \mathbb{J}_1, \beta_1) \Longrightarrow y_z^T(t)\mathbb{J}_2 y_z(t) < \beta_2.$$

Definition 4. [[30]] The system (7)-(9) can be called robust FTB and IO-FTB against H_∞ performance, if the system (7)-(9) can be FTB with respect to $(c_1, c_2, \mathbb{R}, \mathbb{T})$, can be IO-FTB with respect to $(\mathbb{H}(\mathbb{T}, \mathbb{J}_1, \beta_1), \mathbb{J}_2, \beta_2)$ and it follows for $\bar{\omega}(t)$ in Assumption 1

$$\int_0^{\mathbb{T}} e^{-2\mu t} y_z^T(t) y_z(t)dt < V(\bar{\zeta}(0), 0) + \gamma^2 \int_0^{\mathbb{T}} \bar{\omega}^T(t)\bar{\omega}(t)dt \tag{10}$$

where $V(\cdot)$ is a real function with $V(0, 0) = 0$.

Definition 5 ([31]). Assume that all of the states for this closed-loop system can be bounded by the reachable set as follows:

$$\{\bar{\zeta}(t) \in R^{n_{2x}} | \bar{\zeta}(t) \text{satisfies system (7)-(9)}, \bar{\omega}(t) \text{satisfies Assumption 1 and} u(t) \text{satisfies Assumption 2}, \ t \ge 0\}$$

An ellipsoid bounding this reachable set of the closed-loop system (7)-(9) can be described as

$$\{\bar{\zeta}(t) | \bar{\zeta}^T(t)\mathbb{P}\bar{\zeta}(t) < 1, \bar{\zeta}(t) \in R^{n_{2x}}\}$$

were $\mathbb{P}$ is a positive definite matrix.

4. Main results

In the following, the sufficient conditions of FTB, reachable set estimation and IO-FTS are given for the closed-loop system (7)-(9).

Theorem 1. *For a set of parameters $(c_1, c_2, \mathbb{R}, \mathbb{T})$, $(\mathbb{H}(\mathbb{T}, \mathbb{J}_1, \beta_1), \mathbb{J}_2, \beta_2)$, these assigned scalars $\mu > 0$, $\bar{\beta}_1 \ge \beta_1$, $\bar{\beta}_2 > 0$, $\tilde{\beta}_2 > 0$, $\bar{\beta}_2 + \tilde{\beta}_2 \le \beta_2$, $\gamma > 0$, $\bar{\upsilon}^j > 0$, $\theta > 1$, as well as every switching signal against average dwell time satisfying $\mathbb{T}_a > \frac{\ln\theta}{\mu}$, this closed-loop system (7)-(9) against input constraints can be called robust FTB and IO-FTS against H_∞ performance level γ and the* *controller (5)-(6) can ensure that this reachable set of this system (7)-(9) can be bounded through this intersection of ellipsoids in Definition 5, if there are matrices $P^j > 0$, $Q^j > 0$, $R^j > 0$, $i, j \in \{1, \cdots, m\}$, such that these inequalities can be feasible as follows*

$$P^j < \theta P^i, Q^j < \theta Q^i, R^j < \theta R^i \tag{11}$$

$$\Lambda^j(t) < 0 \tag{12}$$

$$\frac{e^{2\mu\mathbb{T}}[\nu_1 c_1 + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})]}{\nu_2} < c_2 \tag{13}$$

$$\begin{bmatrix} -\frac{\bar{\beta}_2}{e^{2\mu\mathbb{T}}[\nu_1 c_1+\bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})]}P^j & 0 & (\mathcal{D}_0^j)^T \\ & -\frac{\bar{\beta}_2\mathbb{T}}{\bar{\beta}_1}\mathbb{J}_1 & (\mathcal{D}_{\bar{\omega}}^j)^T \\ & * & \mathbb{J}_2^{-1} \end{bmatrix} < 0 \tag{14}$$

$$\begin{bmatrix} -\frac{1}{e^{2\mu\mathbb{T}}[\nu_1 c_1+\bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})]}P^j & (\mathcal{C}^j)^T \\ & -\bar{u}^2 I \end{bmatrix} < 0 \tag{15}$$

where

$$\Lambda^j(t) = \begin{bmatrix} \Lambda_1^j(t) & P^j\mathcal{A}_1^j(t) & P^j\mathcal{B}^j(t) & P^j\mathcal{A}_{\bar{\omega}}^j(t) & \left(P^j\mathcal{A}_0^j(t)\right)^T & \left(\mathcal{D}_0^j\right)^T \\ & -e^{\mu\tau}Q^j & 0 & 0 & \left(P^j\mathcal{A}_1^j(t)\right)^T & 0 \\ & * & -e^{\mu h}R^j & 0 & \left(P^j\mathcal{B}^j(t)\right)^T & 0 \\ & * & * & -\gamma^2 I & \left(P^j\mathcal{A}_{\bar{\omega}}^j(t)\right)^T & \left(\mathcal{D}_{\bar{\omega}}^j\right)^T \\ & * & * & * & -2\bar{\upsilon}^j P^j + (\bar{\upsilon}^j)^2 R^j & 0 \\ & * & * & * & * & -I \end{bmatrix}$$

$$\Lambda_1^j(t) = \mathrm{sym}\{P^j\mathcal{A}_0^j(t)\} + Q^j - \mu P^j,\ \mathcal{C}^j = \begin{bmatrix} 0 \\ C^j \end{bmatrix}$$

$$\nu_1 = \max_{j\in\{1,\cdots,m\}}\{\lambda_{\max}(\mathbb{R}^{-\frac{1}{2}}P^j\mathbb{R}^{-\frac{1}{2}}) - \frac{1}{\mu}(1-e^{\mu\tau})\lambda_{\max}(\mathbb{R}^{-\frac{1}{2}}Q^j\mathbb{R}^{-\frac{1}{2}})$$
$$- \frac{1}{\mu}(1-e^{\mu h})\lambda_{\max}(\mathbb{R}^{-\frac{1}{2}}R^j\mathbb{R}^{-\frac{1}{2}})$$
$$\nu_2 = \min_{j\in\{1,\cdots,m\}}\lambda_{\min}(\mathbb{R}^{-\frac{1}{2}}P^j\mathbb{R}^{-\frac{1}{2}})$$

Proof. Choose a piecewise Lyapunov function candidate with the following form

$$V(\bar{\zeta}(t),t) = V_1(\bar{\zeta}(t),t) + V_2(\bar{\zeta}(t),t) + V_3(\bar{\zeta}(t),t) \tag{16}$$

where

$$V_1(\bar{\zeta}(t),t) = \sum_{j=1}^m \sigma_j(t)\bar{\zeta}^T(t)P^j\bar{\zeta}(t)$$
$$V_2(\bar{\zeta}(t),t) = \sum_{j=1}^m \sigma_j(t)\int_{t-\tau}^t e^{\mu(t-\alpha)}\bar{\zeta}^T(\alpha)Q^j\bar{\zeta}(\alpha)d\alpha$$
$$V_3(\bar{\zeta}(t),t) = \sum_{j=1}^m \sigma_j(t)\int_{t-h}^t e^{\mu(t-\alpha)}\dot{\bar{\zeta}}^T(\alpha)R^j\dot{\bar{\zeta}}(\alpha)d\alpha$$

Based on the trajectory of (7)-(9), it follows from (16)

$$\dot{V}(\bar{\zeta}(t),t) = \dot{V}_1(\bar{\zeta}(t),t) + \dot{V}_2(\bar{\zeta}(t),t) + \dot{V}_3(\bar{\zeta}(t),t) \tag{17}$$

where

$$\dot{V}_1(\bar{\zeta}(t),t) = 2\sum_{j=1}^m \sigma_j(t)\bar{\zeta}^T(t)P^j\dot{\bar{\zeta}}(t)$$
$$\dot{V}_2(\bar{\zeta}(t),t) = \sum_{j=1}^m \sigma_j(t)[\bar{\zeta}^T(t)Q^j\bar{\zeta}(t) - e^{\alpha\tau}\bar{\zeta}^T(t-\tau)Q^j\bar{\zeta}(t-\tau)$$

$$+\mu\int_{t-\tau}^{t} e^{\mu(t-\alpha)}\bar{\zeta}^T(\alpha)Q^j\bar{\zeta}(\alpha)d\alpha]$$

$$\dot{V}_3(\bar{\zeta}(t),t)=\sum_{j=1}^{m}\sigma_j(t)[\dot{\bar{\zeta}}^T(t)R^j\dot{\bar{\zeta}}(t)-e^{\alpha h}\dot{\bar{\zeta}}^T(t-h)R^j\dot{\bar{\zeta}}(t-h)$$

$$+\mu\int_{t-h}^{t} e^{\mu(t-\alpha)}\dot{\bar{\zeta}}^T(\alpha)R^j\dot{\bar{\zeta}}(\alpha)d\alpha]$$

Combining with (16) and (17), one has

$$\begin{aligned}&\dot{V}(\bar{\zeta}(t),t)-\mu V(\bar{\zeta}(t),t)-\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\bar{\omega}^T(t)\mathbb{J}_1\bar{\omega}(t)\\ \leq&\sum_{j=1}^{m}\sigma_j(t)\bar{\bar{\zeta}}^T(t)\bar{\Lambda}^j(t)\bar{\bar{\zeta}}(t)\end{aligned} \tag{18}$$

where

$$\bar{\bar{\zeta}}(t)=\begin{bmatrix}\bar{\zeta}^T(t) & \bar{\zeta}^T(t-\tau) & \bar{\zeta}^T(t-h) & \bar{\omega}^T(t)\end{bmatrix}^T$$

$$\bar{\Lambda}^j(t)=\begin{bmatrix}\Lambda_1^j(t) & P^j\mathcal{A}_1^j(t) & P^j\mathcal{B}^j(t) & P^j\mathcal{A}_{\bar{\omega}}^j(t)\\ & -e^{\mu\tau}Q^j & 0 & 0\\ & * & -e^{\mu h}R^j & 0\\ & * & * & -\gamma^2 I\end{bmatrix}$$

$$+\begin{bmatrix}(\mathcal{A}_0^j(t))^T\\(\mathcal{A}_1^j(t))^T\\(\mathcal{B}^j(t))^T\\(\mathcal{A}_{\bar{\omega}}^j(t))^T\end{bmatrix}R^j\begin{bmatrix}(\mathcal{A}_0^j(t))^T\\(\mathcal{A}_1^j(t))^T\\(\mathcal{B}^j(t))^T\\(\mathcal{A}_{\bar{\omega}}^j(t))^T\end{bmatrix}^T$$

Using Schur complement, based on (12) and (18), it follows that

$$\begin{aligned}\left(e^{-\mu t}V(\bar{\zeta}(t),t)\right)' =&e^{-\mu t}[\dot{V}(\bar{\zeta}(t),t)-\mu V(\bar{\zeta}(t),t)]\\ <&e^{-\mu t}\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\bar{\omega}^T(t)\mathbb{J}_1\bar{\omega}(t)\end{aligned} \tag{19}$$

Assume t_k as the kth switching time and t_k^- is used to stand for the left limit value of time t_k. According to (11), one has

$$V(\bar{\zeta}(t_k),t_k)<\theta V(\bar{\zeta}(t_k^-),t_k^-) \tag{20}$$

According to Definition 1, it can be calculated that $\mathbb{N}(0,t)\leq\frac{t-0}{\mathbb{T}_a}$. According to $\mathbb{T}_a>\frac{\ln\theta}{\mu}$, it can be obtained $\mu>\frac{\ln\theta}{\mathbb{T}_a}$. Integrating (19) from 0 to t, combining with (20), the following equation holds for $t\in[t_k,t_{k+1})$ that

$$\begin{aligned}&V(\bar{\zeta}(t),t)\\ <&\theta^{\mathbb{N}(0,t)}e^{\mu t}V(\bar{\zeta}(0),0)+e^{\mu t}\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\int_{t_k}^{t}e^{-\mu\alpha}\bar{\omega}^T(\alpha)\mathbb{J}_1\bar{\omega}(\alpha)d\alpha\\ &+\theta e^{\mu t}\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\int_{t_{k-1}}^{t_k}e^{-\mu\alpha}\bar{\omega}^T(\alpha)\mathbb{J}_1\bar{\omega}(\alpha)d\alpha\\ &+\theta^2 e^{\mu t}\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\int_{t_{k-2}}^{t_{k-1}}e^{-\mu\alpha}\bar{\omega}^T(\alpha)\mathbb{J}_1\bar{\omega}(\alpha)d\alpha\\ &\cdots\\ &+\theta^{\mathbb{N}(0,t)}e^{\mu t}\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\int_{0}^{t_1}e^{-\mu\alpha}\bar{\omega}^T(\alpha)\mathbb{J}_1\bar{\omega}(\alpha)d\alpha\\ =&\theta^{\mathbb{N}(0,t)}e^{\mu t}V(\bar{\zeta}(0),0)+e^{\mu t}\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\int_{0}^{t}e^{-\mu\alpha}\theta^{\mathbb{N}(\alpha,t)}\bar{\omega}^T(\alpha)\mathbb{J}_1\bar{\omega}(\alpha)d\alpha\\ \leq&e^{(\mu+\frac{\ln\theta}{\mathbb{T}_a})t}[V(\bar{\zeta}(0),0)+\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\int_{0}^{t}\bar{\omega}^T(\alpha)\mathbb{J}_1\bar{\omega}(\alpha)d\alpha]\\ <&e^{2\mu t}[V(\bar{\zeta}(0),0)+\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\int_{0}^{t}\bar{\omega}^T(\alpha)\mathbb{J}_1\bar{\omega}(\alpha)d\alpha]\end{aligned} \tag{21}$$

Reconsidering (16), it follows that

$$V(\bar{\zeta}(0),0)\leq\nu_1\max_{l\in[-\bar{\tau},0]}\{\bar{\zeta}^T(t)\mathbb{R}\bar{\zeta}(t),\dot{\bar{\zeta}}^T(t)\mathbb{R}\dot{\bar{\zeta}}(t)\}\leq\nu_1 c_1 \tag{22}$$

$$V(\bar{\zeta}(t),t) \geq \nu_2 \bar{\zeta}^T(t)\mathbb{R}\bar{\zeta}(t) \tag{23}$$

Next, according to (21)-(23), it follows that

$$\bar{\zeta}^T(t)\mathbb{R}\bar{\zeta}(t) < \frac{e^{2\mu\mathbb{T}}[\nu_1 c_1 + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})]}{\nu_2} \tag{24}$$

where $t \in [0, \mathbb{T}]$

According to (13), it is straightforward that $\bar{\zeta}^T(t)\mathbb{R}\bar{\zeta}(t) < c_2$. Based on Definition 2, FTB holds.

Reconsidering (21), one has for $t \in [0, \mathbb{T}]$ that

$$\begin{aligned}
&\sum_{j=1}^{m}\sigma_j(t)\frac{1}{e^{2\mu\mathbb{T}}[\nu_1 c_1 + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})]}\bar{\zeta}^T(t)P^j\bar{\zeta}(t)\\
\leq &\frac{1}{e^{2\mu\mathbb{T}}[\nu_1 c_1 + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})]}V(\bar{\zeta}(t),t)\\
< &\frac{1}{\nu_1 c_1 + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})}[V(\bar{\zeta}(0),0) + \gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\int_0^{\mathbb{T}}\bar{\omega}^T(t)\mathbb{J}_1\bar{\omega}(t)dt]\\
\leq &1
\end{aligned} \tag{25}$$

Based on (25), the following equation holds

$$\begin{aligned}
&y_z^T(t)\mathbb{J}_2 y_z(t) - \frac{\tilde{\beta}_2\mathbb{T}}{\bar{\beta}_1}\bar{\omega}^T(t)\mathbb{J}_1\bar{\omega}(t)\\
<&\sum_{j=1}^{m}\sigma_j(t)\tilde{\bar{\zeta}}^T(t)\Xi^j\tilde{\bar{\zeta}}(t) + \frac{\bar{\beta}_2}{\nu_1 c_1 + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})}V(\bar{\zeta}(0),0)\\
&+\frac{\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\bar{\beta}_2}{\nu_1 c_1 + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})}\int_0^{\mathbb{T}}\bar{\omega}^T(t)\mathbb{J}_1\bar{\omega}(t)dt
\end{aligned} \tag{26}$$

where

$$\tilde{\bar{\zeta}}(t) = \begin{bmatrix}\bar{\zeta}^T(t) & \bar{\omega}^T(t)\end{bmatrix}^T$$

$$\Xi^j = \begin{bmatrix}(\mathcal{D}_0^j)^T\mathbb{J}_2\mathcal{D}_0^j - \frac{\bar{\beta}_2}{e^{2\mu\mathbb{T}}[\nu_1 c_1 + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})]}P^j & (\mathcal{D}_0^j)^T\mathbb{J}_2\mathcal{D}_{\bar{\omega}}^j\\ & (\mathcal{D}_{\bar{\omega}}^j)^T\mathbb{J}_2\mathcal{D}_{\bar{\omega}}^j - \frac{\tilde{\beta}_2\mathbb{T}}{\bar{\beta}_1}\mathbb{J}_1\end{bmatrix}$$

Using Schur complement, (14) indicates that $\Xi^j < 0$. Then we have

$$\begin{aligned}
&y_z^T(t)\mathbb{J}_2 y_z(t) < \frac{\tilde{\beta}_2\mathbb{T}}{\bar{\beta}_1}\bar{\omega}^T(t)\mathbb{J}_1\bar{\omega}(t) + \frac{\bar{\beta}_2}{\nu_1 c_1 + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})}V(\bar{\zeta}(0),0)\\
&+\frac{\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})\bar{\beta}_2}{\nu_1 c_1 + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})}\int_0^{\mathbb{T}}\bar{\omega}^T(t)\mathbb{J}_1\bar{\omega}(t)dt \leq \tilde{\beta}_2 + \bar{\beta}_2 \leq \beta_2
\end{aligned} \tag{27}$$

Based on Definition 1, we can find that IO-FTS holds.

Reconsidering (16) and (17), it follows

$$\begin{aligned}
&\dot{V}(\bar{\zeta}(t),t) - \mu V(\bar{\zeta}(t),t) + y_z^T(t)y_z(t) - \gamma^2\bar{\omega}^T(t)\bar{\omega}(t)\\
\leq &\sum_{j=1}^{m}\sigma_j(t)\bar{\zeta}^T(t)\tilde{\Lambda}^j(t)\bar{\zeta}(t)
\end{aligned} \tag{28}$$

where

$$\tilde{\Lambda}^j(t) = \bar{\Lambda}^j(t) + \begin{bmatrix}(\mathcal{D}_0^j)^T\\0\\0\\(\mathcal{D}_{\bar{\omega}}^j)^T\end{bmatrix}\begin{bmatrix}(\mathcal{D}_0^j)^T\\0\\0\\(\mathcal{D}_{\bar{\omega}}^j)^T\end{bmatrix}^T$$

Using Schur complement, based on (12) and (28), one has that

$$(e^{-\mu t}V(\bar{\zeta}(t),t))' < -e^{-\mu t}[y_z^T(t)y_z(t) - \gamma^2\bar{\omega}^T(t)\bar{\omega}(t)] \tag{29}$$

According to (11) and (20), integrating (29) from 0 to t, it follows that

$$\begin{aligned} & e^{-\mu t} V(\bar{\zeta}(t), t) \\ < & \theta^{\mathbb{N}(0,t)} V(\bar{\zeta}(0), 0) - \int_0^t e^{-\mu\alpha} \theta^{\mathbb{N}(\alpha,t)} [y_z^T(\alpha) y_z(\alpha) - \gamma^2 \bar{\omega}^T(\alpha) \bar{\omega}(\alpha)] d\alpha \\ = & \theta^{\mathbb{N}(0,t)} \Big[V(\bar{\zeta}(0), 0) - \int_0^t e^{-\mu\alpha - \mathbb{N}(0,\alpha) \ln\theta} [y_z^T(\alpha) y_z(\alpha) \\ & - \gamma^2 \bar{\omega}^T(\alpha) \bar{\omega}(\alpha)] d\alpha \Big] \end{aligned} \tag{30}$$

Recalling the fact that $V(\bar{\zeta}(t), t) \geq 0$, it follows that

$$\begin{aligned} & \int_0^t e^{-\mu\alpha - \mathbb{N}(0,\alpha) \ln\theta} y_z^T(\alpha) y_z(\alpha) d\alpha \\ < & V(\bar{\zeta}(0), 0) + \gamma^2 \int_0^t e^{-\mu\alpha - \mathbb{N}(0,\alpha) \ln\theta} \bar{\omega}^T(\alpha) \bar{\omega}(\alpha) d\alpha \end{aligned} \tag{31}$$

According to $\mathbb{N}(0, \alpha) \leq \frac{\alpha - 0}{\mathbb{T}_a}$ and $\mathbb{T}_a > \frac{\ln\theta}{\mu}$, $\theta > 1$, it can be obtained $-\mathbb{N}(0, \alpha) \ln\theta > -\mu\alpha \geq -\mu t$, $e^{-\mathbb{N}(0,\alpha) \ln\theta} < 1$. And then, one has

$$\int_0^t e^{-\mu\alpha - \mathbb{N}(0,\alpha) \ln\theta} y_z^T(\alpha) y_z(\alpha) d\alpha > \int_0^t e^{-2\mu\alpha} y_z^T(\alpha) y_z(\alpha) d\alpha \tag{32}$$

$$\int_0^t e^{-\mu\alpha - \mathbb{N}(0,\alpha) \ln\theta} \bar{\omega}^T(\alpha) \bar{\omega}(\alpha) d\alpha < \int_0^t \bar{\omega}^T(\alpha) \bar{\omega}(\alpha) d\alpha \tag{33}$$

Further one has

$$\int_0^{\mathbb{T}} e^{-2\mu t} y_z^T(t) y_z(t) dt < V(\bar{\zeta}(0), 0) + \gamma^2 \int_0^{\mathbb{T}} \bar{\omega}^T(t) \bar{\omega}(t) dt \tag{34}$$

implying that Definition 4 is satisfied. Thus the closed-loop system (7)-(9) can be called robust FTB and IO-FTS against H_∞ performance level γ.

According to (25), we have

$$\bar{\zeta}^T(t) \mathbb{P} \bar{\zeta}(t) < 1 \tag{35}$$

where $\mathbb{P} = \sum_{j=1}^{m} \sigma_j(t) \frac{1}{e^{2\mu\mathbb{T}}[\nu_1 c_1 + \bar{\beta}_1 \gamma^2 \lambda_{\max}(\mathbb{J}_1^{-1})]} P^j$.

Therefore, the controller (5)-(6) can ensure that this reachable set of this system (7)-(9) can be bounded through this intersection of ellipsoids in Definition 5.

Using Schur complement, it can be found that (15) and (35) mean this following inequality holds.

$$\frac{u^T(t) u(t)}{\bar{u}^2} < \sum_{j=1}^{m} \sigma_j(t) \frac{1}{e^{2\mu\mathbb{T}}[\nu_1 c_1 + \bar{\beta}_1 \gamma^2 \lambda_{\max}(\mathbb{J}_1^{-1})]} \bar{\zeta}^T(t) P^j \bar{\zeta}(t) < 1 \tag{36}$$

which means that Assumption 2 is satisfied. That is, the input constraint condition holds.

This proof is completed. □

In Theorem 1, the proof process of robust FTB, IO-FTS and reachable set estimation against H_∞ performance is shown for the closed-loop system (7)-(9) against input constraints.

In order to make this reachable set have the "smallest" bound by constructing the controller, an optimization problem is involved. That is, maximize χ s.t. $\sum_{j=1}^{m} \sigma_j(t) \chi^j < \mathbb{P}$, that is $\chi^j < \frac{1}{e^{2\mu\mathbb{T}}[\nu_1 c_1 + \bar{\beta}_1 \gamma^2 \lambda_{\max}(\mathbb{J}_1^{-1})]} P^j$. The following corollary is described to solve this above optimization problem by Schur complement.

Corollary 1. *For a set of parameters* $(c_1, c_2, \mathbb{R}, \mathbb{T})$, $(\mathbb{H}(\mathbb{T}, \mathbb{J}_1, \beta_1), \mathbb{J}_2, \beta_2)$, *these assigned constants* $\mu > 0$, $\bar{\beta}_1 \geq \beta_1$, $\bar{\beta}_2 > 0$, $\tilde{\beta}_2 > 0$, $\bar{\beta}_2 + \tilde{\beta}_2 \leq \beta_2$, $\gamma > 0$, $\bar{\upsilon}^j > 0$, $\theta > 1$, $\check{\chi}^j > 0$, *as well as every switching signal against average dwell time satisfying* $\mathbb{T}_a > \frac{\ln\theta}{\mu}$, *this closed-loop system (7)-(9) against input constraints can be called robust FTB and IO-FTS against* H_∞ *performance level* γ *and the controller (5)-(6) can ensure that this reachable set of this system (7)-(9) can be bounded and have the "smallest" bound through this intersection of ellipsoids in Definition 5, if there are matrices* $P^j > 0$, $Q^j > 0$, $R^j > 0$, $i, j \in \{1, \cdots, m\}$, *such that this optimization problem is feasible as followed*

$$\begin{aligned} & \min \ \check{\chi}^j \\ & \text{s.t.} \begin{bmatrix} \check{\chi}^j I & I \\ & \frac{1}{e^{2\mu\mathbb{T}}[\frac{c_1}{\nu_1} + \bar{\beta}_1 \gamma^2 \lambda_{\max}(\mathbb{J}_1^{-1})]} P^j \end{bmatrix} > 0 \end{aligned} \tag{37}$$

$$(11)-(15)$$

Proof. By setting $\breve{\chi}^j = (\chi^j)^{-1}$, similar to Theorem 1, the conclusion could be obtained. Thus the proof can be omitted. □

In Theorem 1 and Corollary 1, (12) is a bilinear matrix inequality (BMI) instead of a LMI, so can not be directly solved. Converting the BMI into a LMI is a difficult point. The form of LMIs for the closed-loop system (7)-(9) and the gain matrices of the controller are given in the following.

Theorem 2. *For a set of parameters $(c_1, c_2, \mathbb{R}, \mathbb{T})$, $(\mathbb{H}(\mathbb{T}, \mathbb{J}_1, \beta_1), \mathbb{J}_2, \beta_2)$, these assigned scalars $\mu > 0$, $\bar{\beta}_1 \geq \beta_1$, $\bar{\beta}_2 > 0$, $\tilde{\beta}_2 > 0$, $\bar{\beta}_2 + \tilde{\beta}_2 \leq \beta_2$, $\gamma > 0$, $\bar{\upsilon}^j > 0$, $\tilde{\upsilon}^j > 0$, $\check{\upsilon}^j > 0$, $\theta > 1$, $\check{\nu}_1 > 0$, the given matrix $\mathbb{X}^j$, as well as every switching signal against average dwell time satisfying $\mathbb{T}_a > \frac{\ln\theta}{\mu}$, this closed-loop system (7)-(9) against input constraints can be called robust FTB and IO-FTS against H_∞ performance level γ and the controller (5)-(6) can ensure that this reachable set of this system (7)-(9) can be bounded through this intersection of ellipsoids in Definition 5, if there are matrices $\mathbb{Y}^j > 0$, $\bar{Q}^j > 0$, $\bar{R}^j > 0$, $\mathbb{U}^j$, and a scalar $\check{\nu}_2$, $i, j \in \{1, \cdots, m\}$, such that these following inequalities can be satisfied*

$$\bar{P}^j < \theta\bar{P}^i, \bar{Q}^j < \theta\bar{Q}^i, \bar{R}^j < \theta\bar{R}^i \tag{38}$$

$$\begin{bmatrix} \Pi_1^j & \Pi_2^j \\ & \Pi_3^j \end{bmatrix} < 0 \tag{39}$$

$$\bar{P}^j - \frac{1}{\mu}(1-e^{\mu\tau})\bar{Q}^j - \frac{1}{\mu}(1-e^{\mu h})\bar{R}^j - \frac{1}{\check{\nu}_1}\Omega^j < 0 \tag{40}$$

$$\begin{bmatrix} -\bar{P}^j & (\Upsilon_1^j)^T \\ & -\check{\nu}_2\mathbb{R}^{-1} \end{bmatrix} < 0 \tag{41}$$

$$\check{\nu}_2 e^{2\mu\mathbb{T}}[\frac{c_1}{\check{\nu}_1} + \bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})] < c_2 \tag{42}$$

$$\begin{bmatrix} -\frac{\bar{\beta}_2}{e^{2\mu\mathbb{T}}[\frac{c_1}{\check{\nu}_1}+\bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})]}\bar{P}^j & 0 & (\bar{\mathcal{D}}_0^j)^T \\ & -\frac{\tilde{\beta}_2\mathbb{T}}{\bar{\beta}_1}\mathbb{J}_1 & (\mathcal{D}_{\bar{\omega}}^j)^T \\ & * & \mathbb{J}_2^{-1} \end{bmatrix} < 0 \tag{43}$$

$$\begin{bmatrix} -\frac{1}{e^{2\mu\mathbb{T}}[\frac{c_1}{\check{\nu}_1}+\bar{\beta}_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})]}\bar{P}^j & (\bar{\mathcal{C}}^j)^T \\ & -\bar{u}^2 I \end{bmatrix} < 0 \tag{44}$$

where

$$\Pi_1^j = \begin{bmatrix} \Pi_{11}^j & \bar{\mathcal{A}}_1^j & \bar{\mathcal{B}}^j & \bar{\mathcal{A}}_{\bar{\omega}}^j & (\bar{\mathcal{A}}_0^j)^T & (\bar{\mathcal{D}}_0^j)^T \\ & -e^{\mu\tau}\bar{Q}^j & 0 & 0 & (\bar{\mathcal{A}}_1^j)^T & 0 \\ & * & -e^{\mu h}\bar{R}^j & 0 & (\bar{\mathcal{B}}^j)^T & 0 \\ & * & * & -\gamma^2 I & (\bar{\mathcal{A}}_{\bar{\omega}}^j)^T & (\mathcal{D}_{\bar{\omega}}^j)^T \\ & * & * & * & -2\bar{\upsilon}^j\bar{P}^j + (\bar{\upsilon}^j)^2 R^j & 0 \\ & * & * & * & * & -I \end{bmatrix}$$

$$\Pi_{11}^j = \mathrm{sym}\{\bar{\mathcal{A}}_0^j\} + \bar{Q}^j - \mu\bar{P}^j$$

$$\bar{\mathcal{A}}_0^j = \begin{bmatrix} \mathscr{A}_0^j\mathbb{X}^j + \mathscr{A}_u^j\bar{C}^j & \mathscr{A}_0^j \\ \bar{A}^j & \mathbb{Y}^j\mathscr{A}_0^j + \bar{B}^j\mathscr{C}_0^j \end{bmatrix}, \bar{P}^j = \begin{bmatrix} \mathbb{X}^j & I \\ & \mathbb{Y}^j \end{bmatrix}$$

$$\bar{\mathcal{A}}_1^j = \begin{bmatrix} \mathscr{A}_1^j\mathbb{X}^j & \mathscr{A}_1^j \\ \mathbb{Y}^j\mathscr{A}_1^j\mathbb{X}^j + \bar{B}^j\mathscr{C}_1^j\mathbb{X}^j & \mathbb{Y}^j\mathscr{A}_1^j + \bar{B}^j\mathscr{C}_1^j \end{bmatrix}, \bar{\mathcal{B}}^j = \begin{bmatrix} \mathscr{B}^j\mathbb{X}^j & \mathscr{B}^j \\ \mathbb{Y}^j\mathscr{B}^j\mathbb{X}^j & \mathbb{Y}^j\mathscr{B}^j \end{bmatrix}$$

$$\bar{\mathcal{A}}_{\bar{\omega}}^j = \begin{bmatrix} \mathscr{A}_a^j & 0 & \mathscr{A}_\omega^j \\ \mathbb{Y}^j\mathscr{A}_a^j & \bar{B}^j\mathscr{C}_s^j & \mathbb{Y}^j\mathscr{A}_\omega^j + \bar{B}^j\mathscr{C}_\omega^j \end{bmatrix}, \bar{\mathcal{D}}_0^j = \begin{bmatrix} \mathscr{D}_0^j\mathbb{X}^j & \mathscr{D}_0^j \end{bmatrix}$$

$$\Pi_2^j = \begin{bmatrix} \tilde{\upsilon}^j\mathcal{F}^j & (\mathcal{G}_0^j)^T \\ 0 & (\mathcal{G}_1^j)^T \\ 0 & (\mathcal{G}_2^j)^T \\ 0 & (\mathcal{G}_3^j)^T \\ \tilde{\upsilon}^j\mathcal{F}^j & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{F}^j=\begin{bmatrix}\mathscr{E}_x^j & 0\\ \mathbb{Y}^j\mathscr{E}_x^j & \bar{B}^j\mathscr{E}_y^j\end{bmatrix},\ \mathcal{G}_0^j=\begin{bmatrix}\mathscr{G}_{x0}^j\mathbb{X}^j+\mathscr{G}_{xu}^j\bar{C}^j & \mathscr{G}_{x0}^j\\ \mathscr{G}_{y0}^j\mathbb{X}^j & \mathscr{G}_{y0}^j\end{bmatrix}$$

$$\mathcal{G}_1^j=\begin{bmatrix}\mathscr{G}_{x1}^j\mathbb{X}^j & \mathscr{G}_{x1}^j\\ \mathscr{G}_{y1}^j\mathbb{X}^j & \mathscr{G}_{y1}^j\end{bmatrix},\ \mathcal{G}_2^j=\begin{bmatrix}\mathscr{G}_x^j\mathbb{X}^j & \mathscr{G}_x^j\\ 0 & 0\end{bmatrix},\ \mathcal{G}_3^j=\begin{bmatrix}\mathscr{G}_{xa}^j & 0 & \mathscr{G}_{x\omega}^j\\ 0 & \mathscr{G}_{ys}^j & \mathscr{G}_{y\omega}^j\end{bmatrix}$$

$$\Pi_3^j=\mathrm{diag}\{-\tilde{\upsilon}^jI,-\tilde{\upsilon}^jI\}$$

$$\Omega^j=\begin{bmatrix}\mathbb{X}^j\mathbb{R}_{11}\mathbb{X}^j+\mathrm{sym}(\mathbb{X}^j\mathbb{R}_{12}(\mathbb{U}^j)^T)-(\check{\upsilon}^j)^2\mathbb{R}_{22}^{-1}+\check{\upsilon}^j\mathrm{sym}(\mathbb{U}^j) & \mathbb{X}^j\mathbb{R}_{11}+\mathbb{U}^j\mathbb{R}_{12}^T\\ & \mathbb{R}_{11}\end{bmatrix}$$

$$\Upsilon_1^j=\begin{bmatrix}\mathbb{X}^j & I\\ (\mathbb{U}^j)^T & 0\end{bmatrix},\ \hat{C}^j=\begin{bmatrix}\bar{C}^j & 0\end{bmatrix}$$

Moreover, in the following, these gain matrices of the controller are provided

$$A^j=(\mathbb{S}^j)^{-1}(\bar{A}^j-\bar{B}^j\mathscr{C}_0^j\mathbb{X}^j-\mathbb{Y}^j\mathscr{A}_u^j\bar{C}^j-\mathbb{Y}^j\mathscr{A}_0^j\mathbb{X}^j)(\mathbb{U}^j)^{-T} \tag{45}$$

$$B^j=(\mathbb{S}^j)^{-1}\bar{B}^j \tag{46}$$

$$C^j=\bar{C}^j(\mathbb{U}^j)^{-T} \tag{47}$$

where the matrix $\mathbb{S}^j$ *satisfies* $\mathbb{S}^j=(I-\mathbb{Y}^j\mathbb{X}^j)(\mathbb{U}^j)^{-T}$.

Proof. Define P^j and $(P^j)^{-1}$ as follows

$$P^j=\begin{bmatrix}\mathbb{Y}^j & \mathbb{S}^j\\ & Z_1^j\end{bmatrix},\quad (P^j)^{-1}=\begin{bmatrix}\mathbb{X}^j & \mathbb{U}^j\\ & Z_2^j\end{bmatrix} \tag{48}$$

where Z_1^j and Z_2^j are arbitrary matrices with the condition that $P^j(P^j)^{-1}=I$.

Choose two matrices as follows

$$\Upsilon_1^j=\begin{bmatrix}\mathbb{X}^j & I\\ (\mathbb{U}^j)^T & 0\end{bmatrix},\ \Upsilon_2^j=\begin{bmatrix}I & \mathbb{Y}^j\\ 0 & (\mathbb{S}^j)^T\end{bmatrix} \tag{49}$$

Based on the condition that $P^j(P^j)^{-1}=I$, (48) and (49) indicate that $P^j\Upsilon_1^j=\Upsilon_2^j$. And according to (45)-(47), define $\bar{A}^j$, $\bar{B}^j$ and $\bar{C}^j$ as follows

$$\bar{A}^j=\mathbb{S}^jA^j(\mathbb{U}^j)^T+\mathbb{S}^jB^j\mathscr{C}_0^j\mathbb{X}^j+\mathbb{Y}^j\mathscr{A}_u^jC^j(\mathbb{U}^j)^T+\mathbb{Y}^j\mathscr{A}_0^j\mathbb{X}^j \tag{50}$$

$$\bar{B}^j=\mathbb{S}^jB^j \tag{51}$$

$$\bar{C}^j=C^j(\mathbb{U}^j)^T \tag{52}$$

Applying the congruence transformation $(\Upsilon_1^j)^{-T}$ to (38) produces (11).

By Schur complement and using (49)-(52), the following inequality can be got from (39)

$$(\Upsilon^j)^T\Lambda^j(t)\Upsilon^j<0 \tag{53}$$

where

$$\Upsilon^j=\mathrm{diag}\{\Upsilon_1^j,\Upsilon_1^j,\Upsilon_1^j,I,\Upsilon_1^j,I\}.$$

Applying the congruence transformation $(\Upsilon^j)^{-T}$ to (53) produces (12).

Apply the congruence transformation $(\Upsilon_1^j)^{-T}$ to (40). Apply the congruence transformation $\mathrm{diag}\{(\Upsilon_1^j)^{-T},I\}$ to (41) and then use Schur complement. The following inequality can be obtained

$$\mathbb{R}^{-\frac{1}{2}}P^j\mathbb{R}^{-\frac{1}{2}}-\frac{1}{\mu}(1-e^{\mu\tau})\mathbb{R}^{-\frac{1}{2}}Q^j\mathbb{R}^{-\frac{1}{2}}-\frac{1}{\mu}(1-e^{\mu h})\mathbb{R}^{-\frac{1}{2}}R^j\mathbb{R}^{-\frac{1}{2}}<\frac{1}{\breve{\nu}_1} \tag{54}$$

$$\mathbb{R}^{-\frac{1}{2}}P^j\mathbb{R}^{-\frac{1}{2}}>\frac{1}{\breve{\nu}_2} \tag{55}$$

From the definition of ν_1 and (54), it follows that $\nu_1<\frac{1}{\breve{\nu}_1}$. From the definition of ν_2 and (55), it follows that $\nu_2>\frac{1}{\breve{\nu}_2}$. It is obvious that (42) holding means that (13) is satisfied.

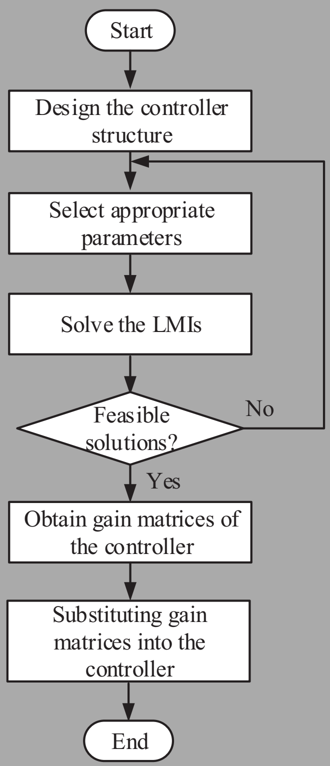

Fig. 1. The flowchart of robust finite-time control algorithm.

Applying the congruence transformation diag$\{(\Upsilon^j)^{-T}, I, I\}$ to (43) produces (14). Applying the congruence transformation diag$\{(\Upsilon^j)^{-T}, I\}$ to (44) produces (15).

In Theorem 2, the sufficient conditions are given in the form of LMIs by Schur complement and congruence transformation. Accordingly, in the following corollary, the optimization problem is described in the form of LMIs to make this reachable set have the "smallest" bound. □

Corollary 2. *For a set of parameters* $(c_1, c_2, \mathbb{R}, \mathbb{T})$, $(\mathbb{H}(\mathbb{T}, \mathbb{J}_1, \beta_1), \mathbb{J}_2, \beta_2)$, *these assigned scalars* $\mu > 0$, $\bar\beta_1 \geq \beta_1$, $\bar\beta_2 > 0$, $\tilde\beta_2 > 0$, $\bar\beta_2 + \tilde\beta_2 \leq \beta_2$, $\gamma > 0$, $\bar\upsilon^j > 0$, $\tilde\upsilon^j > 0$, $\check\upsilon^j > 0$, $\theta > 1$, $\check\nu_1 > 0$, $\check\chi^j > 0$, *the given matrix* $\mathbb{X}^j$, *as well as every switching signal against average dwell time satisfying* $\mathbb{T}_a > \frac{\ln\theta}{\mu}$, *this closed-loop system (7)-(9) against input constraints can be called robust FTB and IO-FTS against* H_∞ *performance level* γ *and the controller (5)-(6) can ensure that this reachable set of this system (7)-(9) can be bounded and have the "smallest" bound through this intersection of ellipsoids in Definition 5, if there are matrices* $\mathbb{Y}^j > 0$, $\bar Q^j > 0$, $\bar R^j > 0$, $\mathbb{U}^j$, *and a scalar* $\check\nu_2$, $i, j \in \{1, \cdots, m\}$, *such that this following optimization problem is feasible*

$$\begin{aligned} &\min \ \check\chi^j \\ &\text{s.t.} \begin{bmatrix} \check\chi^j I & (\Upsilon_1^j)^T \\ & \frac{1}{e^{2\mu\mathbb{T}}[\frac{c_1}{\check\nu_1} + \bar\beta_1\gamma^2\lambda_{\max}(\mathbb{J}_1^{-1})]}\bar P^j \end{bmatrix} > 0 \\ &(38) - (44) \end{aligned} \tag{56}$$

Moreover, these gain matrices of controller are provided as (45)-(47).

Proof. Applying the congruence transformation diag$\{I, (\Upsilon^j)^{-T}\}$ to (56) produces (37). Similar to Theorem 2, the conclusion can be acquired. So the proof is omitted. □

In Theorem 2 and Corollary 2, the finite-time fault-tolerant control and reachable set estimation are realized. Sufficient stability conditions in LMIs are handled by Matlab. Take Theorem 2 for example, the flowchart of robust finite-time control algorithm is presented in Fig. 1. The controller design algorithm is presented in Algorithm 1.

Remark 1. The model considered in this paper is novel. Different from switched systems where time delays appear in system state [32], this system studied in this paper is a switched neutral model with complicated structure, where time-delays not only appear in the state of the system, but also appear in the derivative of the system state, so this model in the paper is suitable for more time-delay scenarios. This paper considers the condition of input constraints, which are not considered in many systems such as [33]. If input constraints are not handled, it could result in the decline of the system performance, undesired inaccuracy, or instability. physical input saturation issue is considered in this paper and then the system performance can be improved. These models are explored rarely, reflect the reality more closely and are handled well by the Theorems 1–2 and Corollaries 1–.

Remark 2. The controller investigated in this paper is novel. Compared with static output feedback controllers [34], the dynamic output feedback controller is explored to realize robust finite-time control in the paper and the control performance can be improved by designing more adjusted gain matrices. Compared to just realizing control [30], FTB, reachable set estimation and IO-FTS are all realized by the controller designed in this paper. In addition, different from [35] which only considers the bound not the "smallest" bound for the reachable set, the controller designed in this paper can ensure that this reachable set can be bounded and have the "smallest" bound.

Remark 3. The method addressed in this paper is novel. Compared with asymptotic stability which means that the system is stable in infinite-time [27]. FTB is considered in this paper, which means many practical problems that need to be bounded in finite time can be solved by this method. In contrast to only considering FTB [30], only considering reachable set estimation or only considering IO-FTS [21], FTB, reachable set estimation and IO-FTS are all considered in this paper, which means that more complex problems can be solved by the method in this paper. In contrast to the general Lyapunov function [36], piecewise Lyapunov function is investigated in the study, leading to that the conservatism of the stability results can be reduced greatly in theory. In contrast to using slack matrices [37], which enhances the freedom of this solution space for the suggested stability criteria and then makes the stability analysis complex, this study does not need slack matrices, which reduces the computational complexity. In the whole process of obtaining LMIs, each term is considered and not be ignored without the help of slack matrices by involving a novel piecewise Lyapunov function and then the conservatism and the computational complexity are reduced in theory.

5. Simulation examples

In this subsection, this feasibility of this approach proposed in this work is illustrated through two examples.

Example 1. The uncertain switched neutral systems against time delays and input constraints modeled as (1)-(4) is described with the following data:

$$
\begin{aligned}
&\tau = 0.9, h = 0.2\\
&\mathscr{B}^1 = \begin{bmatrix} -0.01 & 0 \\ 0 & -0.03 \end{bmatrix}, \mathscr{B}^2 = \begin{bmatrix} -0.02 & 0 \\ 0 & -0.01 \end{bmatrix}\\
&\mathscr{A}_0^1 = \begin{bmatrix} -6.9 & 0.02 \\ 0.2 & -9.1 \end{bmatrix}, \mathscr{A}_0^2 = \begin{bmatrix} -6.8 & 0 \\ 0.01 & -8.3 \end{bmatrix}\\
&\mathscr{A}_1^1 = \begin{bmatrix} 0.24 & 0.02 \\ 0 & 0.57 \end{bmatrix}, \mathscr{A}_1^2 = \begin{bmatrix} 0.18 & -0.01 \\ 0.23 & 0.41 \end{bmatrix}\\
&\mathscr{A}_u^1 = \begin{bmatrix} 0.34 & 0 \\ 0 & 0.2 \end{bmatrix}, \mathscr{A}_u^2 = \begin{bmatrix} 1.16 & 0.02 \\ 0 & 0.45 \end{bmatrix}\\
&\mathscr{A}_a^1 = \begin{bmatrix} 0.26 \\ 0.19 \end{bmatrix}, \mathscr{A}_a^2 = \begin{bmatrix} 0.17 \\ 0.26 \end{bmatrix}\\
&\mathscr{A}_\omega^1 = \begin{bmatrix} 0.001 \\ 0.002 \end{bmatrix}, \mathscr{A}_\omega^2 = \begin{bmatrix} 0.009 \\ 0 \end{bmatrix}\\
&\mathscr{C}_0^1 = \begin{bmatrix} 0.16 & 0.01 \\ 0 & 0.2 \end{bmatrix}, \mathscr{C}_0^2 = \begin{bmatrix} 0.18 & 0 \\ 0.02 & 0.35 \end{bmatrix}\\
&\mathscr{C}_1^1 = \begin{bmatrix} 0.3 & 0 \\ 0.005 & 0.04 \end{bmatrix}, \mathscr{C}_1^2 = \begin{bmatrix} 0.58 & 0.02 \\ -0.001 & 0.04 \end{bmatrix}\\
&\mathscr{C}_s^1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \mathscr{C}_s^2 = \begin{bmatrix} 0.7 \\ 0 \end{bmatrix}\\
&\mathscr{C}_\omega^1 = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}, \mathscr{C}_\omega^2 = \begin{bmatrix} 0 \\ 0.04 \end{bmatrix}\\
&\mathscr{D}_0^1 = \begin{bmatrix} 0.02 & 0.01 \end{bmatrix}, \mathscr{D}_0^2 = \begin{bmatrix} 0 & 0.01 \end{bmatrix}\\
&\mathscr{D}_\omega^1 = 0.018, \mathscr{D}_\omega^2 = 0.023.
\end{aligned}
$$

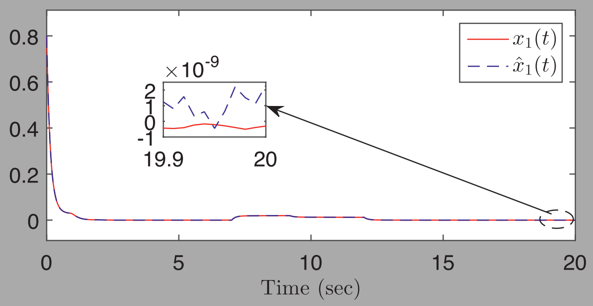

Fig. 2. The curves of states $x(t)$ and $\hat{x}(t)$.

The parameter uncertainties can be described as follows

$$
\begin{aligned}
&\mathscr{E}_x^1 = \begin{bmatrix} 0.002 \\ 0 \end{bmatrix}, \mathscr{E}_x^2 = \begin{bmatrix} 0 \\ 0.003 \end{bmatrix}, \mathscr{G}_x^1 = \begin{bmatrix} 0.015 & 0 \end{bmatrix}, \mathscr{G}_x^2 = \begin{bmatrix} 0.021 & 0 \end{bmatrix}\\
&\mathscr{G}_{x0}^1 = \begin{bmatrix} 0.027 & 0 \end{bmatrix}, \mathscr{G}_{x0}^2 = \begin{bmatrix} 0.017 & 0 \end{bmatrix}, \mathscr{G}_{x1}^1 = \begin{bmatrix} 0.02 & 0.005 \end{bmatrix}\\
&\mathscr{G}_{x1}^2 = \begin{bmatrix} 0.019 & 0 \end{bmatrix}, \mathscr{G}_{xu}^1 = \begin{bmatrix} 0.02 & 0.01 \end{bmatrix}, \mathscr{G}_{xu}^2 = \begin{bmatrix} 0.05 & 0.01 \end{bmatrix}\\
&\mathscr{G}_{xa}^1 = 0.02, \mathscr{G}_{xa}^2 = 0.06, \mathscr{G}_{x\omega}^1 = 0.02, \mathscr{G}_{x\omega}^2 = 0.01\\
&\mathscr{E}_y^1 = \begin{bmatrix} 0 \\ 0.0025 \end{bmatrix}, \mathscr{E}_y^2 = \begin{bmatrix} 0.0027 \\ 0 \end{bmatrix}, \mathscr{G}_{y0}^1 = \begin{bmatrix} 0 & 0.023 \end{bmatrix}, \mathscr{G}_{y0}^2 = \begin{bmatrix} 0.017 & 0 \end{bmatrix}\\
&\mathscr{G}_{y1}^1 = \begin{bmatrix} 0.031 & 0 \end{bmatrix}, \mathscr{G}_{y1}^2 = \begin{bmatrix} 0 & 0.027 \end{bmatrix}, \mathscr{G}_{ys}^1 = 0.018, \mathscr{G}_{ys}^2 = 0.006\\
&\mathscr{G}_{y\omega}^1 = 0.024, \mathscr{G}_{y\omega}^2 = 0.001.
\end{aligned}
$$

It is assumed that $\mathscr{F}_x^1(t) = \sin(0.2t)$, $\mathscr{F}_x^2(t) = \sin(0.3t)$, $\mathscr{F}_y^1(t) = \cos(0.5t)$, $\mathscr{F}_y^2(t) = \cos(0.1t)$.

It is assumed that the faults satisfy

$$
f_a(t) = \begin{cases} 0, & 0 \le t < 7 \\ 0.4 + 0.1\sin(0.2t), & 7 \le t \le 12 \\ 0, & 12 < t \le 20 \end{cases}
$$
$$
f_s(t) = \begin{cases} 0, & 0 \le t < 7 \\ 0.85 + 0.05\cos(0.5t), & 7 \le t \le 12 \\ 0, & 12 < t \le 20. \end{cases}
$$

Set $c_1 = 2$, $c_2 = 12$, $\mathbb{R} = I$, $\mathbb{T} = 5$, $\mathbb{J}_1 = I$, $\mathbb{J}_2 = I$, $\beta_1 = 5.4$, $\beta_2 = 1 \times 10^{-3}$, $\bar{u} = 1 \times 10^{-4}$ and choose $\bar{\beta}_1 = 5.4$, $\bar{\beta}_2 = 5 \times 10^{-4}$, $\tilde{\beta}_2 = 5 \times 10^{-4}$, $\gamma = 0.01$, $\mu = 0.08$, $\theta = 1.4$. The switching time is 9.20594s, 18.4118s.

The results are exhibited in Figs. 2–7. These curves of state $x(t)$, output $y(t)$ as well as controller state $\eta(t)$ are presented in Figs. 2–4, which means that the curves of state $x(t)$, output $y(t)$ and controller state $\eta(t)$ are FTB. And the controller has good fault-tolerant control performance. The response of reachable set $\bar{\zeta}^T(t)\mathbb{P}\bar{\zeta}(t)$ in the closed-loop system is shown in Fig. 5, which means that the controller can ensure that this reachable set of this closed-loop system can be bounded through this intersection of ellipsoids. The spectral norm of control input $\|u(t)\|$ is described in Fig. 6, which means that the condition of input constraints is satisfied. The signal $y_z^T(t)\mathbb{J}_2 y_z(t)$ is described in Fig. 7, which implies that this closed-loop system can be called IO-FTS.

In order to demonstrate the merits of the method in this paper, a comparative experiment is carried out in the following. Piecewise Lyapunov function designed in this paper is compared with a general Lyapunov function [36]. $\hat{x}$ along with $\hat{y}$ represents the system state along with output obtained by the contrast method. The comparison results are shown in

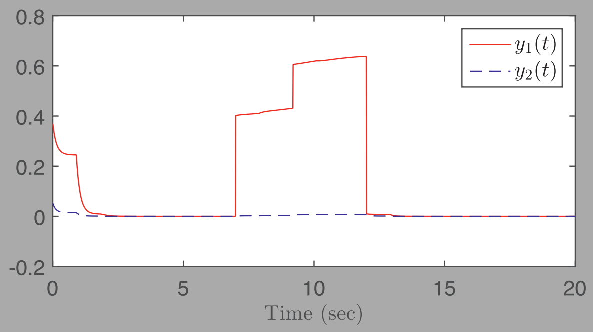

Fig. 3. The curves of output $y(t)$.

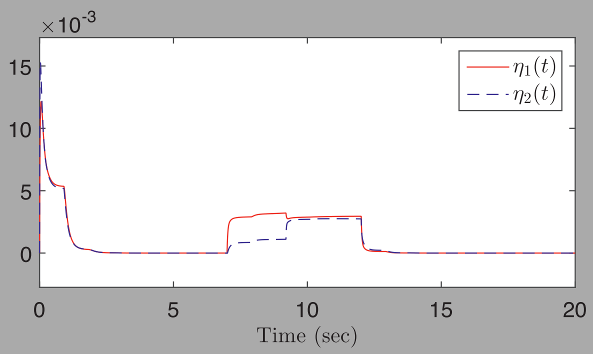

Fig. 4. The curves of controller state $\eta(t)$.

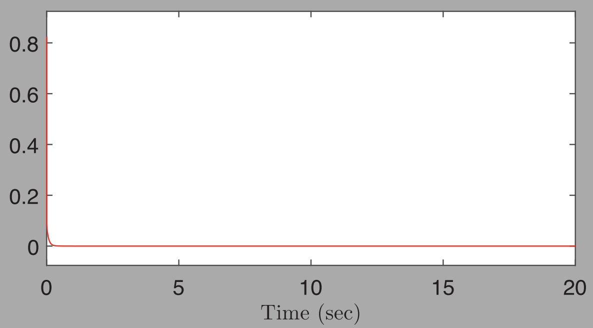

Fig. 5. The response of reachable set $\bar{\zeta}^T(t)\mathbb{P}\bar{\zeta}(t)$ in the closed-loop system.

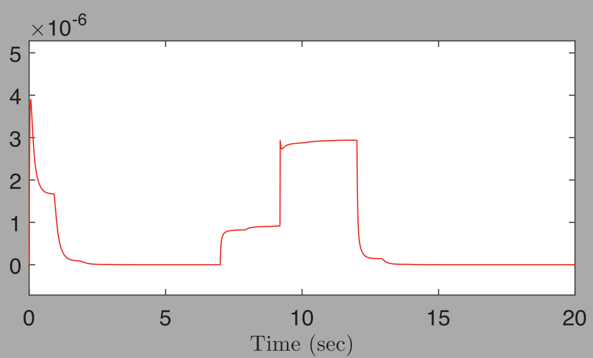

Fig. 6. The spectral norm of control input $\|u(t)\|$.

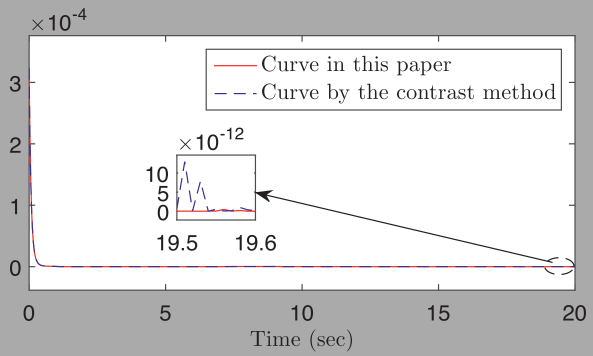

Fig. 7. The signals $y_z^T(t)\mathbb{J}_2 y_z(t)$ and $\hat{y}_z^T(t)\mathbb{J}_2\hat{y}_z(t)$.

Figs. 2 and 7. These curves of state $\hat{x}(t)$ obtained by the contrast method are presented in Fig. 2, which means that the curves of state $\hat{x}(t)$ obtained by the contrast method are FTB. However, the curves of state $x(t)$ obtained in this paper are closer to zero so the FTB performance in this paper is better. The signal $\hat{y}_z^T(t)\mathbb{J}_2\hat{y}_z(t)$ obtained by the contrast method is described in Fig. 7, which implies that this closed-loop system obtained by the contrast method can be called IO-FTS. However, the signal $y_z^T(t)\mathbb{J}_2 y_z(t)$ obtained in this paper is closer to zero so the IO-FTS performance in this paper is better. And it implies that the controller in this paper has a better fault-tolerant control performance. So the advantages of the method in this paper are proved powerfully.

Example 2. A water-quality dynamic model for the Nile River was studied [38] with the following data:

$$
\begin{aligned}
&\mathscr{B}^2 = \mathscr{B}^1 = 0,\ \mathscr{A}_0^1 = \begin{bmatrix} -1.0 & 0.0 \\ -3.0 & -2.0 \end{bmatrix}, \mathscr{A}_0^2 = \begin{bmatrix} 1.0 & 0.0 \\ -3.0 & -2.0 \end{bmatrix} \\
&\mathscr{A}_1^1 = \begin{bmatrix} -0.55 & 0.70 \\ -0.25 & -0.30 \end{bmatrix}, \mathscr{A}_1^2 = \begin{bmatrix} -0.45 & -0.50 \\ -0.15 & -0.10 \end{bmatrix} \\
&\mathscr{A}_u^1 = \begin{bmatrix} 1.4 & 0.0 \\ 0.0 & 1.5 \end{bmatrix}, \mathscr{A}_u^2 = \begin{bmatrix} 1.2 & 0.0 \\ 0.0 & 1.4 \end{bmatrix} \\
&\mathscr{C}_0^1 = \begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 2.0 \end{bmatrix}, \mathscr{C}_0^2 = \begin{bmatrix} 1.0 & 1.4 \\ 1.5 & 1.0 \end{bmatrix} \\
&\mathscr{C}_1^1 = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \mathscr{C}_1^2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.1 \end{bmatrix} \\
&\mathscr{E}_x^2 = \mathscr{E}_x^1 = 0.1\begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix} \\
&\mathscr{F}_x^2(t) = \mathscr{F}_x^1(t) = \mathrm{diag}\{\sin(0.1t), \sin(0.1t)\} \\
&\mathscr{G}_{x0}^1 = \begin{bmatrix} 0.5 & -0.3 \\ -0.2 & 0.8 \end{bmatrix}, \mathscr{G}_{x0}^2 = \begin{bmatrix} 0.7 & -0.3 \\ -0.6 & 0.7 \end{bmatrix} \\
&\mathscr{G}_{x1}^1 = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & -0.2 \end{bmatrix}, \mathscr{G}_{x1}^2 = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & -0.1 \end{bmatrix} \\
&\mathscr{G}_{xu}^1 = \begin{bmatrix} -0.2 & 0.0 \\ 0.0 & -0.1 \end{bmatrix}, \mathscr{G}_{xu}^2 = \begin{bmatrix} -0.1 & 0.1 \\ 0.0 & -0.1 \end{bmatrix}.
\end{aligned}
$$

Assume that there exist actuator faults, sensor faults and disturbance input in the system. These parameter matrices are assumed as follows

$$
\begin{aligned}
&\tau = 0.5, h = 0.3 \\
&\mathscr{A}_a^1 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, \mathscr{A}_a^2 = \begin{bmatrix} 0.02 \\ 0 \end{bmatrix} \\
&\mathscr{A}_\omega^1 = \begin{bmatrix} 0.001 \\ 0 \end{bmatrix}, \mathscr{A}_\omega^2 = \begin{bmatrix} 0.002 \\ 0 \end{bmatrix}
\end{aligned}
$$

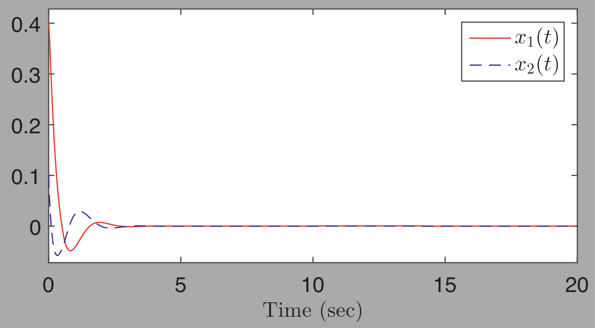

Fig. 8. The curves of state $x(t)$.

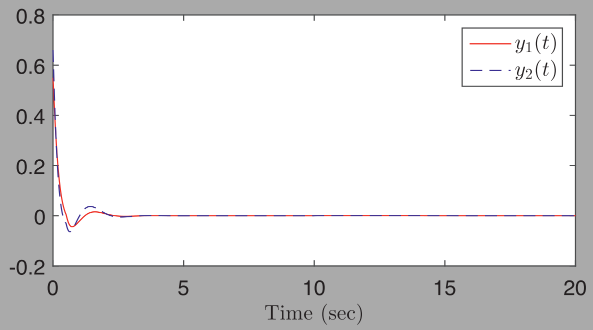

Fig. 9. The curves of output $y(t)$.

$$\begin{aligned}
&\mathscr{C}_s^1 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, \mathscr{C}_s^2 = \begin{bmatrix} 0.03 \\ 0 \end{bmatrix} \\
&\mathscr{C}_\omega^1 = \begin{bmatrix} 0 \\ 0.001 \end{bmatrix}, \mathscr{C}_\omega^2 = \begin{bmatrix} 0.003 \\ 0 \end{bmatrix} \\
&\mathscr{D}_0^1 = \begin{bmatrix} 0.01 & 0 \end{bmatrix}, \mathscr{D}_0^2 = \begin{bmatrix} 0.03 & 0 \end{bmatrix} \\
&\mathscr{D}_\omega^1 = 0.002, \mathscr{D}_\omega^2 = 0.003 \\
&\mathscr{G}_x^1 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \mathscr{G}_x^2 = \begin{bmatrix} 0.01 & 0 \\ 0 & -0.01 \end{bmatrix}.
\end{aligned}$$

It is assumed that there are no other parameter uncertainties.

It is assumed that the faults satisfy

$$f_a(t) = \begin{cases} 0, & 0 \le t < 10 \\ 1 + 0.2\sin(0.1t), & 10 \le t \le 14 \\ 0, & 14 < t \le 20 \end{cases}$$
$$f_s(t) = \begin{cases} 0, & 0 \le t < 10 \\ 0.02 + 0.01\cos(0.1t), & 10 \le t \le 14 \\ 0, & 14 < t \le 20. \end{cases}$$

Set $c_1 = 2$, $c_2 = 14$, $\mathbb{R} = I$, $\mathbb{T} = 5$, $\mathbb{J}_1 = I$, $\mathbb{J}_2 = I$, $\beta_1 = 7.7$, $\beta_2 = 2 \times 10^{-4}$, $\bar{u} = 1$ and choose $\bar{\beta}_1 = 7.7$, $\bar{\beta}_2 = 1 \times 10^{-4}$, $\tilde{\beta}_2 = 1 \times 10^{-4}$, $\gamma = 0.02$, $\mu = 0.1$, $\theta = 3$. The switching time is 18.9861s.

The results are exhibited in Figs. 8–13. These curves of state $x(t)$, output $y(t)$ as well as controller state $\eta(t)$ are presented in Figs. 8–10, which means that the curves of state $x(t)$, output $y(t)$ and controller state $\eta(t)$ are FTB. And the controller has good fault-tolerant control performance. The response of reachable set $\bar{\zeta}^T(t)\mathbb{P}\bar{\zeta}(t)$ in the closed-loop system is shown in Fig. 11, which means that the controller can ensure that this reachable set of this closed-loop system can be bounded through this intersection of ellipsoids. The spectral norm of control input $\|u(t)\|$ is described in Fig. 12, which means that the condition of input constraints is satisfied. The signal $y_z^T(t)\mathbb{J}_2 y_z(t)$ is described in Fig. 13, which implies that this closed-loop system can be called IO-FTS.

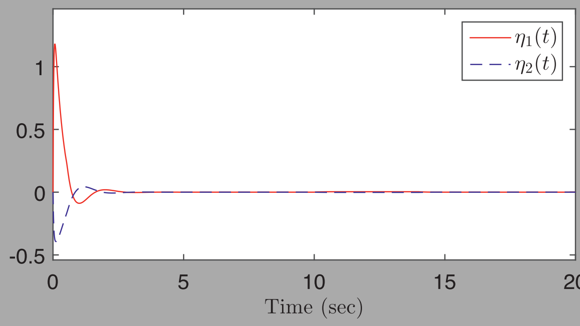

Fig. 10. The curves of controller state $\eta(t)$.

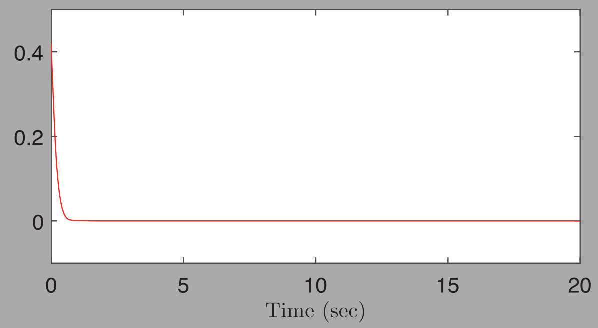

Fig. 11. The response of reachable set $\bar{\zeta}^T(t)\mathbb{P}\bar{\zeta}(t)$ in the closed-loop system.

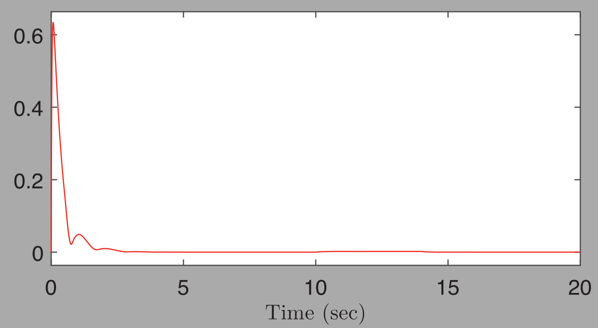

Fig. 12. The spectral norm of control input $\|u(t)\|$.

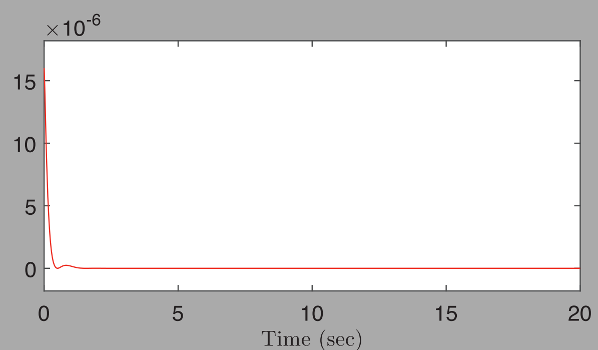

Fig. 13. The signal $y_z^T(t)\mathbb{J}_2 y_z(t)$.

Algorithm 1 The controller design algorithm.

Input: The data of system (1)-(4) such as h, τ, $\mathscr{A}_0^j$ and a set of parameters $(c_1, c_2, \mathbb{R}, \mathbb{T})$, $(\mathbb{H}(\mathbb{T}, \mathbb{J}_1, \beta_1), \mathbb{J}_2, \beta_2)$, where $j \in \{1, \cdots, m\}$.
Output: Gain matrices of the controller
Step 1: Design the structure of the controller (5)-(6).
Step 2: Set the assigned scalars μ, $\bar{\beta}_1$, $\bar{\beta}_2$, $\tilde{\beta}_2$, γ, $\bar{\upsilon}^j$, $\tilde{\upsilon}^j$, $\check{\upsilon}^j$, θ, $\check{\nu}_1$, the given matrix $\mathbb{X}^j$, which all have the same definitions as the ones in Theorem 2.
Step 3: Solve the LMIs in Theorem 2, and find the feasible solutions. If there are no feasible solutions, return Step 2, if there are feasible solutions, go to Step 4.
Step 4: Calculate the gain matrices of the controller by (38)-(44).
Step 5: Substituting A^j, B^j and C^j into the controller (5)-(6), then robust finite-time control and reachable set estimation can be realized.

6. Conclusions

The study has focused on robust finite-time fault-tolerant control as well as reachable set estimation for uncertain switched neutral systems against time delays and input constraints. There have been few attempts to investigate FTB and reachable set estimation for uncertain switched neutral systems. At the same time, IO-FTS also has been researched. Compared with the existing works, this system can be applied more widely. A nonlinear dynamic output feedback controller has been researched. An augmented closed-loop system has been provided. Moreover, the sufficient conditions of FTB, reachable set estimation as well as IO-FTS for the closed-loop system have been obtained via piecewise Lyapunov function in terms of LMIs. Ultimately, two examples have been provided to illustrate this validity of this method in this paper. Moreover, robust finite-time fault-tolerant control as well as reachable set estimation will be explored for uncertain fuzzy semi-Markovian jump neutral systems against time delays and input constraints.

CRediT authorship contribution statement

Shuowei Jin: Conceptualization, Methodology, Software, Validation, Writing - original draft. **Yongheng Pang:** Validation, Formal analysis, Software. **Xiaoming Zhou:** Writing - review & editing, Supervision, Validation. **Aiyun Yan:** Writing - review & editing, Supervision, Validation. **Wei Wang:** Writing - review & editing, Supervision, Validation. **Wenbo Hu:** Writing - review & editing, Supervision, Validation.

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