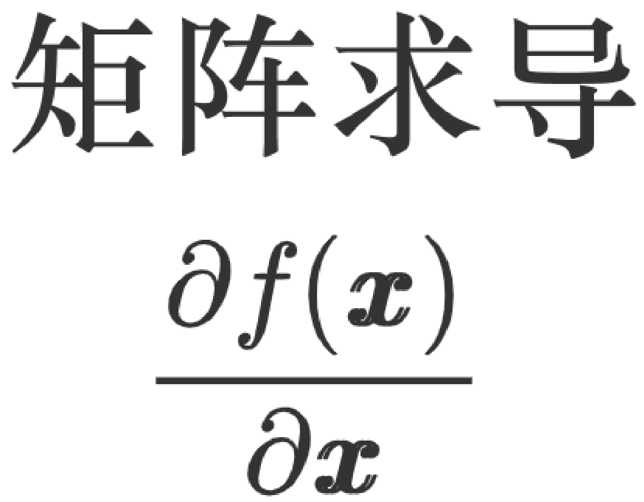


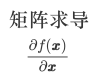
矩阵求导公式的数学推导（矩阵求导——基础篇）

Iterator

O. 前言

1、看本文之前**请务必**先看这篇文章：

矩阵求导的本质与分子布局、分母布局的本质（矩阵求导——本质篇） - Iterator的...
1623 赞同 · 67 评论 文章



下文以"本质篇"指代上面这篇文章。

2、本文介绍**向量变元**的**实值标量函数**、**矩阵变元**的**实值标量函数**中**最基础**的矩阵求导公式的**数学推导**。掌握了这些**最基础**的推导，才能理解之后的那些千变万化的**技巧**。

3、进阶的技巧（**矩阵的迹** $\mathrm{tr}(\boldsymbol{A})$ 与**一阶实矩阵微分** $\mathrm{d}\boldsymbol{X}$ ）会在**下一篇**讲，本篇**不**涉及。

4、本文使用的符号与本质篇**相同。**

5、看懂本文需要了解**本质篇**所提及的知识，以及了解本科阶段线性代数中**矩阵乘法**、**向量内积**的知识，**无需任何其他知识。**

6、有一个矩阵求导的网站，大家可以验证自己算的结果是否正确。

Matrix Calculus
www.matrixcalculus.org/

一. 向量变元的实值标量函数

$$f(\boldsymbol{x}), \boldsymbol{x} = [x_1, x_2, \cdots, x_n]^T$$

我们使用**梯度向量**形式，即 (本质篇_6) 式

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x})=\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}}=\left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right]^T \quad (\text{本质篇\_6})$$

1、四个法则

1.1 常数求导[1]**：**

与一元函数常数求导相同：结果为零向量

$$\frac{\partial c}{\partial \boldsymbol{x}}=\mathbf{0}_{n \times 1} \tag{1}$$

其中， c 为常数。

证明：

$$\begin{aligned}
\frac{\partial c}{\partial \boldsymbol{x}} &= \begin{bmatrix} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \\ \vdots \\ \frac{\partial c}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \mathbf{0}_{n \times 1}
\end{aligned} \tag{2}$$

证毕。

1.2 线性法则[1]

与一元函数求导线性法则相同：相加再求导等于求导再相加，常数提外面

$$\frac{\partial [c_1 f(\boldsymbol{x})+c_2 g(\boldsymbol{x})]}{\partial \boldsymbol{x}}=c_1 \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}}+c_2 \frac{\partial g(\boldsymbol{x})}{\partial \boldsymbol{x}} \tag{3}$$

其中， c_1, c_2 为常数。

证明：

$$
\begin{aligned}
\frac{\partial[c_1 f(\boldsymbol{x})+c_2 g(\boldsymbol{x})]}{\partial \boldsymbol{x}} &= \begin{bmatrix} \frac{\partial(c_1 f+c_2 g)}{\partial x_1} \\ \frac{\partial(c_1 f+c_2 g)}{\partial x_2} \\ \vdots \\ \frac{\partial(c_1 f+c_2 g)}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} c_1 \frac{\partial f}{\partial x_1}+c_2 \frac{\partial g}{\partial x_1} \\ c_1 \frac{\partial f}{\partial x_2}+c_2 \frac{\partial g}{\partial x_2} \\ \vdots \\ c_1 \frac{\partial f}{\partial x_n}+c_2 \frac{\partial g}{\partial x_n} \end{bmatrix} \\
&= c_1 \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{bmatrix} \\
&= c_1 \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}}+c_2 \frac{\partial g(\boldsymbol{x})}{\partial \boldsymbol{x}}
\end{aligned} \tag{4}
$$

证毕。

1.3 乘积法则[1]

与一元函数求导乘积法则相同：前导后不导 **加** 前不导后导

$$
\frac{\partial[f(\boldsymbol{x}) g(\boldsymbol{x})]}{\partial \boldsymbol{x}}=\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} g(\boldsymbol{x})+f(\boldsymbol{x}) \frac{\partial g(\boldsymbol{x})}{\partial \boldsymbol{x}} \tag{5}
$$

证明：

$$
\begin{aligned}
\frac{\partial[f(\boldsymbol{x})g(\boldsymbol{x})]}{\partial \boldsymbol{x}} &= \begin{bmatrix} \frac{\partial(fg)}{\partial x_1} \\ \frac{\partial(fg)}{\partial x_2} \\ \vdots \\ \frac{\partial(fg)}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f}{\partial x_1}g + f\frac{\partial g}{\partial x_1} \\ \frac{\partial f}{\partial x_2}g + f\frac{\partial g}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n}g + f\frac{\partial g}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} g + f \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{bmatrix} \\
&= \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} g(\boldsymbol{x}) + f(\boldsymbol{x})\frac{\partial g(\boldsymbol{x})}{\partial \boldsymbol{x}}
\end{aligned} \tag{6}
$$

证毕。

1.4 商法则[1]

与一元函数求导商法则相同：（上导下不导 **减** 上不导下导）**除以**（下的平方）：

$$
\frac{\partial\left[\frac{f(\boldsymbol{x})}{g(\boldsymbol{x})}\right]}{\partial \boldsymbol{x}} = \frac{1}{g^2(\boldsymbol{x})}\left[\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} g(\boldsymbol{x}) - f(\boldsymbol{x})\frac{\partial g(\boldsymbol{x})}{\partial \boldsymbol{x}}\right] \tag{7}
$$

其中， $g(\boldsymbol{x}) \neq 0$ 。

证明：

$$
\begin{aligned}
\frac{\partial\left[\frac{f(\boldsymbol{x})}{g(\boldsymbol{x})}\right]}{\partial \boldsymbol{x}} &= \begin{bmatrix} \frac{\partial(\frac{f}{g})}{\partial x_1} \\ \frac{\partial(\frac{f}{g})}{\partial x_2} \\ \vdots \\ \frac{\partial(\frac{f}{g})}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{g^2}\left(\frac{\partial f}{\partial x_1}g - f\frac{\partial g}{\partial x_1}\right) \\ \frac{1}{g^2}\left(\frac{\partial f}{\partial x_2}g - f\frac{\partial g}{\partial x_2}\right) \\ \vdots \\ \frac{1}{g^2}\left(\frac{\partial f}{\partial x_n}g - f\frac{\partial g}{\partial x_n}\right) \end{bmatrix} \\
&= \frac{1}{g^2}\left(\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} g - f \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{bmatrix}\right) \\
&= \frac{1}{g^2(\boldsymbol{x})}\left[\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} g(\boldsymbol{x}) - f(\boldsymbol{x})\frac{\partial g(\boldsymbol{x})}{\partial \boldsymbol{x}}\right]
\end{aligned} \tag{8}
$$

证毕。

2、几个公式

2.1

$$
\frac{\partial(\boldsymbol{x}^T\boldsymbol{a})}{\partial \boldsymbol{x}} = \frac{\partial(\boldsymbol{a}^T\boldsymbol{x})}{\partial \boldsymbol{x}} = \boldsymbol{a} \tag{9}
$$

其中， $\boldsymbol{a}$ 为常数向量， $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)^T$ 。

证明：

$$
\begin{aligned}
\frac{\partial(\boldsymbol{x}^T\boldsymbol{a})}{\partial\boldsymbol{x}} = \frac{\partial(\boldsymbol{a}^T\boldsymbol{x})}{\partial\boldsymbol{x}}
&= \frac{\partial(a_1x_1+a_2x_2+\cdots+a_nx_n)}{\partial\boldsymbol{x}} \\
&= \begin{bmatrix} \frac{\partial(a_1x_1+a_2x_2+\cdots+a_nx_n)}{\partial x_1} \\ \frac{\partial(a_1x_1+a_2x_2+\cdots+a_nx_n)}{\partial x_2} \\ \vdots \\ \frac{\partial(a_1x_1+a_2x_2+\cdots+a_nx_n)}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \\
&= \boldsymbol{a}
\end{aligned} \tag{10}
$$

证毕。

2.2

$$
\frac{\partial(\boldsymbol{x}^T\boldsymbol{x})}{\partial\boldsymbol{x}} = 2\boldsymbol{x} \tag{11}
$$

证明：

$$
\begin{aligned}
\frac{\partial(\boldsymbol{x}^T\boldsymbol{x})}{\partial\boldsymbol{x}} &= \frac{\partial(x_1^2+x_2^2+\cdots+x_n^2)}{\partial\boldsymbol{x}} \\
&= \begin{bmatrix} \frac{\partial(x_1^2+x_2^2+\cdots+x_n^2)}{\partial x_1} \\ \frac{\partial(x_1^2+x_2^2+\cdots+x_n^2)}{\partial x_2} \\ \vdots \\ \frac{\partial(x_1^2+x_2^2+\cdots+x_n^2)}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{bmatrix} \\
&= 2\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= 2\boldsymbol{x}
\end{aligned} \tag{12}
$$

证毕。

$$\frac{\partial(\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})}{\partial \boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{A}^T\boldsymbol{x} \tag{13}$$

其中， $\boldsymbol{A}_{n\times n}$ 是常数矩阵， $\boldsymbol{A}_{n\times n}=(a_{ij})_{i=1,j=1}^{n,n}$ 。

证明：

$$\begin{aligned}
\frac{\partial(\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})}{\partial \boldsymbol{x}} &= \frac{\begin{matrix}\partial(a_{11}x_1x_1+a_{12}x_1x_2+\cdots+a_{1n}x_1x_n\\ +a_{21}x_2x_1+a_{22}x_2x_2+\cdots+a_{2n}x_2x_n\\ +\cdots\\ +a_{n1}x_nx_1+a_{n2}x_nx_2+\cdots+a_{nn}x_nx_n)\end{matrix}}{\partial \boldsymbol{x}}\\
&= \begin{bmatrix} \frac{\begin{matrix}\partial(a_{11}x_1x_1+a_{12}x_1x_2+\cdots+a_{1n}x_1x_n\\ +a_{21}x_2x_1+a_{22}x_2x_2+\cdots+a_{2n}x_2x_n\\ +\cdots\\ +a_{n1}x_nx_1+a_{n2}x_nx_2+\cdots+a_{nn}x_nx_n)\end{matrix}}{\partial x_1}\\ \frac{\begin{matrix}\partial(a_{11}x_1x_1+a_{12}x_1x_2+\cdots+a_{1n}x_1x_n\\ +a_{21}x_2x_1+a_{22}x_2x_2+\cdots+a_{2n}x_2x_n\\ +\cdots\\ +a_{n1}x_nx_1+a_{n2}x_nx_2+\cdots+a_{nn}x_nx_n)\end{matrix}}{\partial x_2}\\ \vdots\\ \frac{\begin{matrix}\partial(a_{11}x_1x_1+a_{12}x_1x_2+\cdots+a_{1n}x_1x_n\\ +a_{21}x_2x_1+a_{22}x_2x_2+\cdots+a_{2n}x_2x_n\\ +\cdots\\ +a_{n1}x_nx_1+a_{n2}x_nx_2+\cdots+a_{nn}x_nx_n)\end{matrix}}{\partial x_n} \end{bmatrix}\\
&= \begin{bmatrix} (a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n)+(a_{11}x_1+a_{21}x_2+\cdots+a_{n1}x_n)\\ (a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n)+(a_{12}x_1+a_{22}x_2+\cdots+a_{n2}x_n)\\ \vdots\\ (a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n)+(a_{1n}x_1+a_{2n}x_2+\cdots+a_{nn}x_n) \end{bmatrix}\\
&= \begin{bmatrix} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n\\ \vdots\\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n \end{bmatrix}+\begin{bmatrix} a_{11}x_1+a_{21}x_2+\cdots+a_{n1}x_n\\ a_{12}x_1+a_{22}x_2+\cdots+a_{n2}x_n\\ \vdots\\ a_{1n}x_1+a_{2n}x_2+\cdots+a_{nn}x_n \end{bmatrix}\\
&= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n \end{bmatrix}+\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1}\\ a_{12} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n \end{bmatrix}\\
&= \boldsymbol{A}\boldsymbol{x}+\boldsymbol{A}^T\boldsymbol{x}
\end{aligned} \tag{14}$$

证毕。

2.4

$$\frac{\partial(\boldsymbol{a}^T\boldsymbol{x}\boldsymbol{x}^T\boldsymbol{b})}{\partial \boldsymbol{x}}=\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{x}+\boldsymbol{b}\boldsymbol{a}^T\boldsymbol{x} \tag{15}$$

其中， $\boldsymbol{a},\boldsymbol{b}$ 为常数向量， $\boldsymbol{a}=(a_1,a_2,\cdots,a_n)^T,\boldsymbol{b}=(b_1,b_2,\cdots,b_n)^T$ 。

证明：

因为 $\boldsymbol{a}^T\boldsymbol{x}=\boldsymbol{x}^T\boldsymbol{a},\boldsymbol{x}^T\boldsymbol{b}=\boldsymbol{b}^T\boldsymbol{x}$ ，所以有

$$\frac{\partial(\boldsymbol{a}^T\boldsymbol{x}\boldsymbol{x}^T\boldsymbol{b})}{\partial \boldsymbol{x}}=\frac{\partial(\boldsymbol{x}^T\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{x})}{\partial \boldsymbol{x}} \tag{16}$$

$$\frac{\partial(\boldsymbol{a}^T\boldsymbol{x}\boldsymbol{x}^T\boldsymbol{b})}{\partial\boldsymbol{x}}=\frac{\partial(\boldsymbol{x}^T\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{x})}{\partial\boldsymbol{x}}=\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{x}+\boldsymbol{b}\boldsymbol{a}^T\boldsymbol{x} \tag{17}$$

证毕。

二. 矩阵变元的实值标量函数

$$f(\boldsymbol{X}),\boldsymbol{X}_{m\times n}=(x_{ij})_{i=1,j=1}^{m,n}$$

我们使用**梯度矩阵**形式，即 $(本质篇\_11)$ 式

$$\begin{aligned}\nabla_{\boldsymbol{X}}f(\boldsymbol{X})&=\frac{\partial f(\boldsymbol{X})}{\partial\boldsymbol{X}_{m\times n}}\\&=\begin{bmatrix}\frac{\partial f}{\partial x_{11}}&\frac{\partial f}{\partial x_{12}}&\cdots&\frac{\partial f}{\partial x_{1n}}\\\frac{\partial f}{\partial x_{21}}&\frac{\partial f}{\partial x_{22}}&\cdots&\frac{\partial f}{\partial x_{2n}}\\\vdots&\vdots&\vdots&\vdots\\\frac{\partial f}{\partial x_{m1}}&\frac{\partial f}{\partial x_{m2}}&\cdots&\frac{\partial f}{\partial x_{mn}}\end{bmatrix}_{m\times n}\end{aligned} \tag{本质篇\_11}$$

1、四个法则

1.1 常数求导[1]**：**

与一元函数常数求导相同：结果为零矩阵

$$\frac{\partial c}{\partial\boldsymbol{X}}=\boldsymbol{0}_{m\times n} \tag{18}$$

其中， c 为常数。

证明：

$$\begin{aligned}\frac{\partial c}{\partial\boldsymbol{X}}&=\begin{bmatrix}\frac{\partial c}{\partial x_{11}}&\frac{\partial c}{\partial x_{12}}&\cdots&\frac{\partial c}{\partial x_{1n}}\\\frac{\partial c}{\partial x_{21}}&\frac{\partial c}{\partial x_{22}}&\cdots&\frac{\partial c}{\partial x_{2n}}\\\vdots&\vdots&\vdots&\vdots\\\frac{\partial c}{\partial x_{m1}}&\frac{\partial c}{\partial x_{m2}}&\cdots&\frac{\partial c}{\partial x_{mn}}\end{bmatrix}_{m\times n}\\&=\begin{bmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\vdots&\vdots\\0&0&\cdots&0\end{bmatrix}_{m\times n}\\&=\boldsymbol{0}_{m\times n}\end{aligned} \tag{19}$$

证毕。

1.2 线性法则[1]

与一元函数求导线性法则相同：相加再求导等于求导再相加，常数提外面

$$\frac{\partial[c_1f(\boldsymbol{X})+c_2g(\boldsymbol{X})]}{\partial\boldsymbol{X}}=c_1\frac{\partial f(\boldsymbol{X})}{\partial\boldsymbol{X}}+c_2\frac{\partial g(\boldsymbol{X})}{\partial\boldsymbol{X}} \tag{20}$$

其中， c_1, c_2 为常数。

证明：

$$
\begin{aligned}
\frac{\partial[c_1 f(\boldsymbol{X})+c_2 g(\boldsymbol{X})]}{\partial \boldsymbol{X}} &= \begin{bmatrix} \frac{\partial(c_1f+c_2g)}{\partial x_{11}} & \frac{\partial(c_1f+c_2g)}{\partial x_{12}} & \cdots & \frac{\partial(c_1f+c_2g)}{\partial x_{1n}} \\ \frac{\partial(c_1f+c_2g)}{\partial x_{21}} & \frac{\partial(c_1f+c_2g)}{\partial x_{22}} & \cdots & \frac{\partial(c_1f+c_2g)}{\partial x_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial(c_1f+c_2g)}{\partial x_{m1}} & \frac{\partial(c_1f+c_2g)}{\partial x_{m2}} & \cdots & \frac{\partial(c_1f+c_2g)}{\partial x_{mn}} \end{bmatrix} \\
&= \begin{bmatrix} c_1\frac{\partial f}{\partial x_{11}}+c_2\frac{\partial g}{\partial x_{11}} & c_1\frac{\partial f}{\partial x_{12}}+c_2\frac{\partial g}{\partial x_{12}} & \cdots & c_1\frac{\partial f}{\partial x_{1n}}+c_2\frac{\partial g}{\partial x_{1n}} \\ c_1\frac{\partial f}{\partial x_{21}}+c_2\frac{\partial g}{\partial x_{21}} & c_1\frac{\partial f}{\partial x_{22}}+c_2\frac{\partial g}{\partial x_{22}} & \cdots & c_1\frac{\partial f}{\partial x_{2n}}+c_2\frac{\partial g}{\partial x_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ c_1\frac{\partial f}{\partial x_{m1}}+c_2\frac{\partial g}{\partial x_{m1}} & c_1\frac{\partial f}{\partial x_{m2}}+c_2\frac{\partial g}{\partial x_{m2}} & \cdots & c_1\frac{\partial f}{\partial x_{mn}}+c_2\frac{\partial g}{\partial x_{mn}} \end{bmatrix} \\
&= c_1\begin{bmatrix} \frac{\partial f}{\partial x_{11}} & \frac{\partial f}{\partial x_{12}} & \cdots & \frac{\partial f}{\partial x_{1n}} \\ \frac{\partial f}{\partial x_{21}} & \frac{\partial f}{\partial x_{22}} & \cdots & \frac{\partial f}{\partial x_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f}{\partial x_{m1}} & \frac{\partial f}{\partial x_{m2}} & \cdots & \frac{\partial f}{\partial x_{mn}} \end{bmatrix} + c_2\begin{bmatrix} \frac{\partial g}{\partial x_{11}} & \frac{\partial g}{\partial x_{12}} & \cdots & \frac{\partial g}{\partial x_{1n}} \\ \frac{\partial g}{\partial x_{21}} & \frac{\partial g}{\partial x_{22}} & \cdots & \frac{\partial g}{\partial x_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g}{\partial x_{m1}} & \frac{\partial g}{\partial x_{m2}} & \cdots & \frac{\partial g}{\partial x_{mn}} \end{bmatrix} \\
&= c_1\frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}} + c_2\frac{\partial g(\boldsymbol{X})}{\partial \boldsymbol{X}}
\end{aligned} \tag{21}
$$

证毕。

1.3 乘积法则[1]

与一元函数求导乘积法则相同：前导后不导 **加** 前不导后导

$$
\frac{\partial[f(\boldsymbol{X})g(\boldsymbol{X})]}{\partial \boldsymbol{X}} = \frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}}g(\boldsymbol{X}) + f(\boldsymbol{X})\frac{\partial g(\boldsymbol{X})}{\partial \boldsymbol{X}} \tag{22}
$$

证明：

$$
\begin{aligned}
\frac{\partial[f(\boldsymbol{X})g(\boldsymbol{X})]}{\partial \boldsymbol{X}} &= \begin{bmatrix} \frac{\partial(fg)}{\partial x_{11}} & \frac{\partial(fg)}{\partial x_{12}} & \cdots & \frac{\partial(fg)}{\partial x_{1n}} \\ \frac{\partial(fg)}{\partial x_{21}} & \frac{\partial(fg)}{\partial x_{22}} & \cdots & \frac{\partial(fg)}{\partial x_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial(fg)}{\partial x_{m1}} & \frac{\partial(fg)}{\partial x_{m2}} & \cdots & \frac{\partial(fg)}{\partial x_{mn}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f}{\partial x_{11}}g+f\frac{\partial g}{\partial x_{11}} & \frac{\partial f}{\partial x_{12}}g+f\frac{\partial g}{\partial x_{12}} & \cdots & \frac{\partial f}{\partial x_{1n}}g+f\frac{\partial g}{\partial x_{1n}} \\ \frac{\partial f}{\partial x_{21}}g+f\frac{\partial g}{\partial x_{21}} & \frac{\partial f}{\partial x_{22}}g+f\frac{\partial g}{\partial x_{22}} & \cdots & \frac{\partial f}{\partial x_{2n}}g+f\frac{\partial g}{\partial x_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f}{\partial x_{m1}}g+f\frac{\partial g}{\partial x_{m1}} & \frac{\partial f}{\partial x_{m2}}g+f\frac{\partial g}{\partial x_{m2}} & \cdots & \frac{\partial f}{\partial x_{mn}}g+f\frac{\partial g}{\partial x_{mn}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f}{\partial x_{11}} & \frac{\partial f}{\partial x_{12}} & \cdots & \frac{\partial f}{\partial x_{1n}} \\ \frac{\partial f}{\partial x_{21}} & \frac{\partial f}{\partial x_{22}} & \cdots & \frac{\partial f}{\partial x_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f}{\partial x_{m1}} & \frac{\partial f}{\partial x_{m2}} & \cdots & \frac{\partial f}{\partial x_{mn}} \end{bmatrix} g + f\begin{bmatrix} \frac{\partial g}{\partial x_{11}} & \frac{\partial g}{\partial x_{12}} & \cdots & \frac{\partial g}{\partial x_{1n}} \\ \frac{\partial g}{\partial x_{21}} & \frac{\partial g}{\partial x_{22}} & \cdots & \frac{\partial g}{\partial x_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g}{\partial x_{m1}} & \frac{\partial g}{\partial x_{m2}} & \cdots & \frac{\partial g}{\partial x_{mn}} \end{bmatrix} \\
&= \frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}}g(\boldsymbol{X}) + f(\boldsymbol{X})\frac{\partial g(\boldsymbol{X})}{\partial \boldsymbol{X}}
\end{aligned} \tag{23}
$$

1.4 商法则[1]

与一元函数求导商法则相同：（上导下不导 **减** 上不导下导）**除以**（下的平方）：

$$\frac{\partial\left[\frac{f(\boldsymbol{X})}{g(\boldsymbol{X})}\right]}{\partial \boldsymbol{X}}=\frac{1}{g^2(\boldsymbol{X})}\left[\frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}}g(\boldsymbol{X})-f(\boldsymbol{X})\frac{\partial g(\boldsymbol{X})}{\partial \boldsymbol{X}}\right] \tag{24}$$

其中， $g(\boldsymbol{X})\neq 0$ 。

证明：

[公式]

证毕。

2、几个公式

2.1

$$\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}\boldsymbol{b})}{\partial \boldsymbol{X}}=\boldsymbol{a}\boldsymbol{b}^T \tag{26}$$

其中， $\boldsymbol{a}_{m\times 1},\boldsymbol{b}_{n\times 1}$ 为常数向量， $\boldsymbol{a}_=(a_1,a_2,\cdots,a_m)^T,\boldsymbol{b}=(b_1,b_2,\cdots,b_n)^T$ 。

证明（右击公式，选择**在新标签页中打开图片**，公式就可以放大了~）：

$$\begin{aligned}
\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}\boldsymbol{b})}{\partial \boldsymbol{X}}&=\frac{\begin{array}{l}\partial(a_1b_1x_{11}+a_1b_2x_{12}+\cdots+a_1b_nx_{1n}\\+a_2b_1x_{21}+a_2b_2x_{22}+\cdots+a_2b_nx_{2n}\\+\cdots\\+a_mb_1x_{m1}+a_mb_2x_{m2}+\cdots+a_mb_nx_{mn})\end{array}}{\partial \boldsymbol{X}}\\
&=\begin{bmatrix}
\frac{\begin{array}{l}\partial(a_1b_1x_{11}+a_1b_2x_{12}+\cdots+a_1b_nx_{1n}\\+a_2b_1x_{21}+a_2b_2x_{22}+\cdots+a_2b_nx_{2n}\\+\cdots\\+a_mb_1x_{m1}+a_mb_2x_{m2}+\cdots+a_mb_nx_{mn})\end{array}}{\partial x_{11}} & \frac{\begin{array}{l}\partial(a_1b_1x_{11}+a_1b_2x_{12}+\cdots+a_1b_nx_{1n}\\+a_2b_1x_{21}+a_2b_2x_{22}+\cdots+a_2b_nx_{2n}\\+\cdots\\+a_mb_1x_{m1}+a_mb_2x_{m2}+\cdots+a_mb_nx_{mn})\end{array}}{\partial x_{12}} & \cdots & \frac{\begin{array}{l}\partial(a_1b_1x_{11}+a_1b_2x_{12}+\cdots+a_1b_nx_{1n}\\+a_2b_1x_{21}+a_2b_2x_{22}+\cdots+a_2b_nx_{2n}\\+\cdots\\+a_mb_1x_{m1}+a_mb_2x_{m2}+\cdots+a_mb_nx_{mn})\end{array}}{\partial x_{1n}}\\
\frac{\begin{array}{l}\partial(a_1b_1x_{11}+a_1b_2x_{12}+\cdots+a_1b_nx_{1n}\\+a_2b_1x_{21}+a_2b_2x_{22}+\cdots+a_2b_nx_{2n}\\+\cdots\\+a_mb_1x_{m1}+a_mb_2x_{m2}+\cdots+a_mb_nx_{mn})\end{array}}{\partial x_{21}} & \frac{\begin{array}{l}\partial(a_1b_1x_{11}+a_1b_2x_{12}+\cdots+a_1b_nx_{1n}\\+a_2b_1x_{21}+a_2b_2x_{22}+\cdots+a_2b_nx_{2n}\\+\cdots\\+a_mb_1x_{m1}+a_mb_2x_{m2}+\cdots+a_mb_nx_{mn})\end{array}}{\partial x_{22}} & \cdots & \frac{\begin{array}{l}\partial(a_1b_1x_{11}+a_1b_2x_{12}+\cdots+a_1b_nx_{1n}\\+a_2b_1x_{21}+a_2b_2x_{22}+\cdots+a_2b_nx_{2n}\\+\cdots\\+a_mb_1x_{m1}+a_mb_2x_{m2}+\cdots+a_mb_nx_{mn})\end{array}}{\partial x_{2n}}\\
\vdots & \vdots & \vdots & \vdots\\
\frac{\begin{array}{l}\partial(a_1b_1x_{11}+a_1b_2x_{12}+\cdots+a_1b_nx_{1n}\\+a_2b_1x_{21}+a_2b_2x_{22}+\cdots+a_2b_nx_{2n}\\+\cdots\\+a_mb_1x_{m1}+a_mb_2x_{m2}+\cdots+a_mb_nx_{mn})\end{array}}{\partial x_{m1}} & \frac{\begin{array}{l}\partial(a_1b_1x_{11}+a_1b_2x_{12}+\cdots+a_1b_nx_{1n}\\+a_2b_1x_{21}+a_2b_2x_{22}+\cdots+a_2b_nx_{2n}\\+\cdots\\+a_mb_1x_{m1}+a_mb_2x_{m2}+\cdots+a_mb_nx_{mn})\end{array}}{\partial x_{m2}} & \cdots & \frac{\begin{array}{l}\partial(a_1b_1x_{11}+a_1b_2x_{12}+\cdots+a_1b_nx_{1n}\\+a_2b_1x_{21}+a_2b_2x_{22}+\cdots+a_2b_nx_{2n}\\+\cdots\\+a_mb_1x_{m1}+a_mb_2x_{m2}+\cdots+a_mb_nx_{mn})\end{array}}{\partial x_{mn}}
\end{bmatrix}_{m\times n}\\
&=\begin{bmatrix}a_1b_1 & a_1b_2 & \cdots & a_1b_n\\ a_2b_1 & a_2b_2 & \cdots & a_2b_n\\ \vdots & \vdots & \vdots & \vdots\\ a_mb_1 & a_mb_2 & \cdots & a_mb_n\end{bmatrix}_{m\times n}\\
&=\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_m\end{bmatrix}[b_1,b_2,\cdots,b_n]\\
&=\boldsymbol{a}\boldsymbol{b}^T
\end{aligned} \tag{27}$$

证毕。

2.2

$$\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}^T\boldsymbol{b})}{\partial \boldsymbol{X}}=\boldsymbol{b}\boldsymbol{a}^T \tag{28}$$

其中， $\boldsymbol{a}_{n\times 1},\boldsymbol{b}_{m\times 1}$ 为常数向量， $\boldsymbol{a}_=(a_1,a_2,\cdots,a_n)^T,\boldsymbol{b}=(b_1,b_2,\cdots,b_m)^T$ 。

证明：

$$\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}^T\boldsymbol{b})}{\partial\boldsymbol{X}}=\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}^T\boldsymbol{b})^T}{\partial\boldsymbol{X}}=\frac{\partial(\boldsymbol{b}^T\boldsymbol{X}\boldsymbol{a})}{\partial\boldsymbol{X}} \tag{29}$$

由 (26) 式得:

$$\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}^T\boldsymbol{b})}{\partial\boldsymbol{X}}=\frac{\partial(\boldsymbol{b}^T\boldsymbol{X}\boldsymbol{a})}{\partial\boldsymbol{X}}=\boldsymbol{b}\boldsymbol{a}^T \tag{30}$$

证毕。

2.3

$$\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{b})}{\partial\boldsymbol{X}}=\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{X}+\boldsymbol{b}\boldsymbol{a}^T\boldsymbol{X} \tag{31}$$

其中， $\boldsymbol{a}_{m\times1},\boldsymbol{b}_{m\times1}$ 为常数向量， $\boldsymbol{a}_=(a_1,a_2,\cdots,a_m)^T,\boldsymbol{b}=(b_1,b_2,\cdots,b_m)^T$ 。

证明（右击公式，选择**在新标签页中打开图片**，公式就可以放大了~）：

[公式]

证毕。

2.4

$$\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{b})}{\partial\boldsymbol{X}}=\boldsymbol{X}\boldsymbol{b}\boldsymbol{a}^T+\boldsymbol{X}\boldsymbol{a}\boldsymbol{b}^T \tag{33}$$

其中， $\boldsymbol{a}_{n\times1},\boldsymbol{b}_{n\times1}$ 为常数向量， $\boldsymbol{a}_=(a_1,a_2,\cdots,a_n)^T,\boldsymbol{b}=(b_1,b_2,\cdots,b_n)^T$ 。

证明:

我们来看一下 $(本质篇\_9)$ 式:

$$\begin{aligned}\mathrm{D}_{\boldsymbol{X}}f(\boldsymbol{X})&=\frac{\partial f(\boldsymbol{X})}{\partial\boldsymbol{X}^T_{m\times n}}\\&=\begin{bmatrix}\frac{\partial f}{\partial x_{11}} & \frac{\partial f}{\partial x_{21}} & \cdots & \frac{\partial f}{\partial x_{m1}}\\ \frac{\partial f}{\partial x_{12}} & \frac{\partial f}{\partial x_{22}} & \cdots & \frac{\partial f}{\partial x_{m2}}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial f}{\partial x_{1n}} & \frac{\partial f}{\partial x_{2n}} & \cdots & \frac{\partial f}{\partial x_{mn}}\end{bmatrix}_{n\times m}\end{aligned} \tag{本质篇\_9}$$

再来看一下 $(本质篇\_11)$ 式:

$$\begin{aligned}\nabla_{\boldsymbol{X}}f(\boldsymbol{X})&=\frac{\partial f(\boldsymbol{X})}{\partial\boldsymbol{X}_{m\times n}}\\&=\begin{bmatrix}\frac{\partial f}{\partial x_{11}} & \frac{\partial f}{\partial x_{12}} & \cdots & \frac{\partial f}{\partial x_{1n}}\\ \frac{\partial f}{\partial x_{21}} & \frac{\partial f}{\partial x_{22}} & \cdots & \frac{\partial f}{\partial x_{2n}}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial f}{\partial x_{m1}} & \frac{\partial f}{\partial x_{m2}} & \cdots & \frac{\partial f}{\partial x_{mn}}\end{bmatrix}_{m\times n}\end{aligned} \tag{本质篇\_11}$$

$$\frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}_{m\times n}^T} = \left(\frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}_{m\times n}}\right)^T \tag{34}$$

所以，我们把 (31) 式中的分母的**矩阵变元**写为**转置**，就有：

$$\begin{aligned}\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{b})}{\partial \boldsymbol{X}^T} &= \left(\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{b})}{\partial \boldsymbol{X}}\right)^T \\ &= (\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{X} + \boldsymbol{b}\boldsymbol{a}^T\boldsymbol{X})^T \\ &= \boldsymbol{X}^T\boldsymbol{b}\boldsymbol{a}^T + \boldsymbol{X}^T\boldsymbol{a}\boldsymbol{b}^T\end{aligned} \tag{35}$$

对于 (33) 式，我们将其写为如下形式：

$$\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{b})}{\partial \boldsymbol{X}} = \frac{\partial(\boldsymbol{a}^T(\boldsymbol{X}^T)(\boldsymbol{X}^T)^T\boldsymbol{b})}{\partial (\boldsymbol{X}^T)^T} \tag{36}$$

然后对 (36) 式使用 (35) 式，得：

$$\begin{aligned}\frac{\partial(\boldsymbol{a}^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{b})}{\partial \boldsymbol{X}} &= \frac{\partial(\boldsymbol{a}^T(\boldsymbol{X}^T)(\boldsymbol{X}^T)^T\boldsymbol{b})}{\partial (\boldsymbol{X}^T)^T} \\ &= (\boldsymbol{X}^T)^T\boldsymbol{b}\boldsymbol{a}^T + (\boldsymbol{X}^T)^T\boldsymbol{a}\boldsymbol{b}^T \\ &= \boldsymbol{X}\boldsymbol{b}\boldsymbol{a}^T + \boldsymbol{X}\boldsymbol{a}\boldsymbol{b}^T\end{aligned} \tag{37}$$

证毕。

三. 完

本文到这里就结束了，相信大家也和我一样，会觉的后面那几个求导公式，如果**按照定义**去推导的话，**十分的麻烦**，**而且容易出错。**

所以， 在下一篇文章中，我们将介绍**向量变元**的**实值标量函数**、**矩阵变元**的**实值标量函数**进阶的矩阵求导的技巧：**矩阵的迹** $\mathrm{tr}(\boldsymbol{A})$ 与**一阶实矩阵微分** $\mathrm{d}\boldsymbol{X}$ ，它们可以极大地化简我们的推导过程。

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矩阵求导的本质与分子布局、分母布局的本质（矩阵求导——本质篇） - Iterator的文章 - 知乎

参考

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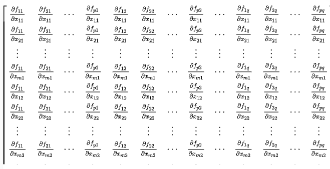
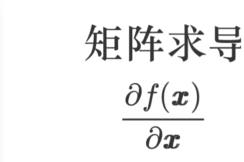


矩阵求导入门学习路线参考

SinclairWang



对称矩阵的求导，以多元
布的极大似然估计为例

Iterator

42 条评论 切换为时间排序

写下你的评论...

精选评论（3）

Iterator (作者) 回复 吃酸奶只舔盖 2021-10-27
z3.ax1x.com/2021/10/27/...
4 查看回复

Iterator (作者) 2021-09-30
知乎的公式渲染好像经常崩溃，有一些比较长的公式会显示不出来，大家可以按照评论区置顶的方法，拿到latex代码后，用typora或者其他支持mathjax的markdown编辑器查看。
赞

Iterator (作者) 2021-01-12

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右击公式，选择在新标签页中打开，然后复制url中的/equation?tex= 后面一堆的代码，然后去 UrlEncode编码/UrlDecode解码 - 站长工具 这个地方选择UrlDecode，就可以看到latex代码了。（如果代码比较长可以按住shift + end可以直接拖到末尾）例如 zhihu.com/equation?...

复制的内容为： f%28x%29%3Dx%2B2+%5C%5C%5C%5C+%5Ctag%7Be.g.1%7D

3

评论（42）

钢铁直男 2021-02-25

感谢感谢，我的机器学习有救了

4

anonymous 03-20

矩阵求导典型的在数学课被直接忽略的内容。国内除了数学系的，其他很多都没管这方面内容

2

吃酸奶只舔盖 2021-09-15

答主，有部分图片挂了，能更新一下么。谢谢

2

Iterator (作者) 回复 吃酸奶只舔盖 2021-10-27

z3.ax1x.com/2021/10/27/...

4 查看回复

ainiweibai 回复 Iterator (作者) 06-09

感谢答主！！

赞

何其速也 2020-11-11

建议你去看下贾松圭的矩阵不等式

2

Iterator (作者) 回复 何其速也 2020-11-11

啊？为什么？

赞

何其速也 回复 Iterator (作者) 2020-11-11

有个转换公式之类的，可以方便解决你文章里的计算

1

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幼安 2021-11-05

感谢作者救命

赞

lll 2021-10-21

赞同 1822 42 条评论 分享 喜欢 收藏 申请转载 ···

Iterator (作者) 回复 lll 2021-10-21
应该就挂了几个，我现在都不敢动它 = = 不过你看置顶评论，能有办法看挂的公式
赞

晨风 2021-10-12
大佬再接再厉！
赞

枝江hku 2021-09-26
瑞思拜
赞

海峰 2021-09-24
作者了不起
赞

呈寻糯米叉烧包 2021-09-19
我大受震撼
赞

Eric 2021-09-12
作者分子、分母布局的定义搞不清楚，很多和教材反了
赞

Iterator (作者) 回复 Eric 2021-09-12
这两个名词"分子布局、分母布局"在一些经典的矩阵分析书籍（张贤达、史荣昌）中压根就没有提过，不过是网上流传的很多讲义上为了方便直观理解而给出的，然后后来的人也不考虑严谨性就直接给沿用了。不知道你说的教材是指专门的机器学习教材还是矩阵分析教材？可以给出书籍名称和页码，我校正一下。
5

雁过无声 2021-08-25
公式2.3的图显示不出来，网页版和APP都看不见。不过还是谢谢答主了
赞

雁过无声 2021-08-24
感谢作者
赞

Perry Yang 2021-05-12
建议亲可以把最后一行放到开头
赞

zerolord 2021-03-25
太贴心了，作者棒棒！
赞

酱二狗 2021-03-17
哈哈我最近也在看张贤达老师的书，答主码字总结辛苦了！！！个人习惯求行向量偏导，感觉答主的例题是不是求的梯度嘞，总和你差个转置~不得不说迹那块的技巧太丰富了，我这几天已经傻了
赞

A导 2021-02-24
答主数学公式用什么软件生成的
赞

Iterator (作者) 回复 A导 2021-02-24

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A导 回复 Iterator (作者) 2021-02-26

好的，我用的AXMath，感觉并不是很方便

赞

1 2 下一页

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