Technical Notes and Correspondence

Adaptive Backstepping Controller Design for Stochastic Jump Systems

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***Abstract*—In this technical note, we improve the results in a paper by Shi *et al.*, in which problems of stochastic stability and sliding mode control for a class of linear continuous-time systems with stochastic jumps were considered. However, the system considered is switching stochastically between different subsystems, the dynamics of the jump system can not stay on each sliding surface of subsystems forever, therefore, it is difficult to determine whether the closed-loop system is stochastically stable. In this technical note, the backstepping techniques are adopted to overcome the problem in a paper by Shi *et al.*. The resulting closed-loop system is bounded in probability. It has been shown that the adaptive control problem for the Markovian jump systems is solvable if a set of coupled linear matrix inequalities (LMIs) have solutions. A numerical example is given to show the potential of the proposed techniques.**

***Index Terms*—Adaptive control, backstepping control, linear matrix inequality, Markovian jump system, stochastic stability.**

I. Introduction

It is well known that many physical systems have different structures due to random abrupt changes, which may be caused by random failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, modification of the operating point of a linearized model of a nonlinear system, etc. The *hybrid* systems, which involve both time-evolving and event-driven mechanisms, may be employed to model the above problems. One special class of hybrid systems is the so-called Markowina jump linear system (MJLS). A MJLS is a hybrid one with many operation modes, and every mode corresponds to a deterministic system. The system mode switching is governed by a Markov process. A number of control problems related to MJLS systems has been analyzed by several authors; see, e.g., [1]–[10] and the references therein.

Moreover, the sliding-mode control (SMC) has received relatively a lot of attention since it has various attractive features such as fast response, good transient performance, order-reduction and so on. In particular, SMC laws are robust with respect to the so-called matched uncertainty, see, e.g., [11]–[18]. Recently, the sliding mode control is proposed to stabilize MJLS with matched uncertainties and disturbances [1]. However, system is switching stochastically between different subsystems, the dynamics of the jump systems can not stay on each sliding surface of subsystems forever, therefore, it can not be determined whether the closed-loop system is stochastically stable. This motivated us to study the above systems with Markovian jumps further.

In this technical note, we consider the problem of adaptive backstepping controller design for stochastic jump systems with matched uncertainties and disturbances. The jumping parameters are treated as continuous-time, discrete-state Markov process. Note that backstepping method is one of the most popular techniques of nonlinear control design [19]–[27]. In [22], the backstepping method is proposed to design a memoryless state feedback controller for a class of uncertain time-delay systems, but it can not solve the control problem for system with matched disturbances and Markov jumping. In this technical note, adaptive backstepping controller for the system will be designed. Unknown upper bounds of uncertainties and disturbances can be estimated by adaptive control method ([28]–[30]). The above problems are solved in terms of a finite set of coupled linear matrix inequalities (LMIs). Finally, a numerical example is included to demonstrate the effectiveness of the theoretical results obtained.

Notations: The notation used in this technical note is quite standard. In the sequel, the Euclidean norm is used for vectors. We use W^T, W^{-1}, $\lambda(W)$, $Tr(W)$ and $\|W\|$ to denote, respectively, the transpose, the inverse, the eigenvalues, the trace and the induced norm of any square matrix W. We use $W > 0$ $(\geq, <, \leq 0)$ to denote a symmetric positive definite (positive semi-definite, negative, negative semi-definite) matrix W with $\lambda_{\min}(W)$ and $\lambda_{\max}(W)$ being the minimum and maximum eigenvalues of W and I to denote the $n \times n$ identity matrix. C^k denotes the space of k-times continuously differentiable functions. The Lebesgue space $\mathcal{L}_2[0,\ T]$ consists of square-integrable functions on the interval $[0,\ T]$ equipped with the norm $\|.\|_2$. $\mathcal{E}[\cdot]$ stands for mathematical expectation. Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where Ω is the sample space, $\mathcal{F}$ is the algebra of events and $\mathbf{P}$ is the probability measure defined on $\mathcal{F}$. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem Formulation and Preliminaries

We consider a class of stochastic systems with Markovian jump parameters in a fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$

$$\dot{x}(t) = A(\eta_t)x(t) + B(\eta_t)[u(t) + F(\eta_t)w(x,t)], \quad \eta_o = i, \quad t \geq 0 \tag{1}$$

where $x(t) \in R^n$ is the state vector; $u(t) \in R^m$ is the control input, $w \in R^l$ is the disturbance, while $\{\eta_t,\ t \in [0, \mathcal{T}]\}$ is a finite-state Markovian process having a state space $\mathcal{S} \triangleq \{1, 2, \ldots, \nu\}$, generator (α_{ij}) with transition probability from mode i at time t to mode j at time $t+\delta,\ i, j \in \mathcal{S}$

$$\begin{aligned} p_{ij} &= Pr(\eta_{t+\delta} = j \mid \eta_t = i) \\ &= \begin{cases} \alpha_{ij}\delta + o(\delta), & \text{if } i \neq j \\ 1 + \alpha_{ii}\delta + o(\delta), & \text{if } i = j \end{cases} \end{aligned} \tag{2}$$

$$\alpha_{ii} = -\sum_{m=1, m\neq i}^{\nu} \alpha_{im}, \quad \alpha_{ij} \geq 0\ \forall\, i, j \in \mathcal{S},\ i \neq j \tag{3}$$

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where $\delta > 0$ and $\lim_{\delta \downarrow 0} o(\delta)/\delta = 0$.

For $\mathcal{V}(t,x) \in C^1$, let us introduce the weak infinitesimal operator $\mathfrak{F}_k^x[\cdot]$ of the process $\{x(t), \eta_t, t \geq 0\}$ at the point $\{t, x, k\}$ [31], [32]

$$\mathfrak{F}_k^x[\mathcal{V}] = \frac{\partial \mathcal{V}}{\partial t} + \frac{\partial \mathcal{V}}{\partial x}\dot{x}(t) + \sum_{j=1}^{\nu} \alpha_{kj} \mathcal{V}(x, j). \tag{4}$$

For each possible value $\eta_t = k,\ k \in \mathcal{S}$, we will denote the system matrices associated with mode i by

$$A(\eta_t) \triangleq A(k), \quad B(\eta_t) \triangleq B(k), \quad F(\eta_t) \triangleq F(k)$$

where $A(k)$, $B(k)$ and $F(k)$ are known real constant matrices of appropriate dimensions which describe the nominal system.

Assumption 2.1: The matched uncertainties $w(x,t)$ are assumed to satisfy the following condition:

$$\|F(\eta_t)w(x,t)\| \leq c + \kappa\|x(t)\| = \rho \tag{5}$$

where c and κ are constants, but it may not be easily obtained due to the complexity of the structure of the uncertainty.

Remark 2.1: The model of the form (1) is a hybrid system in which one state $x(t)$ takes values continuously and another state η_t, referred to as the mode or operating form, takes values discretely in $\mathcal{S}$. This kind of system can be used to represent many important physical systems subject to random failures and structure changes, such as electric power systems [33], control systems of a solar thermal central receiver [34], communications systems [35], aircraft flight control [36], control of nuclear power plants [37] and manufacturing systems [38], [39].

For convenience, it is assumed that

$$B(\eta_t) = \begin{bmatrix} 0_{(n-m)\times m} \\ B_2(\eta_t) \end{bmatrix}$$

where $B_2(\eta_t) \in R^{m\times m}$ is nonsingular. Let

$$\begin{aligned} y_1(t) &= [\,I_{n-m} \quad 0\,]\,x(t) \\ y_2(t) &= -K(\eta_t)\,[\,I_{n-m} \quad 0\,]\,x(t) + [\,0 \quad I_m\,]\,x(t) \end{aligned} \tag{6}$$

that is

$$\begin{aligned} y(t) &= \begin{bmatrix} I_{n-m} & 0 \\ -K(\eta_t) & I_m \end{bmatrix} x(t) \\ x(t) &= \begin{bmatrix} I_{n-m} & 0 \\ K(\eta_t) & I_m \end{bmatrix} y(t) \end{aligned} \tag{7}$$

where $K(\eta_t)$ is the virtual control input matrix to be determined later.

Let us recall the definition proposed in [40].

Definition 2.1: A stochastic process $y(t)$ is said to be bounded in probability if the random variables $|y(t)|$ are bounded in probability uniformly in t, i.e.,

$$\lim_{r \longrightarrow \infty} \sup_{t>0} P\{|y(t)| > r\} = 0. \tag{8}$$

The criterion for boundedness in probability is given as follows.

Lemma 2.1: Assume that there exists a function $V \in C^2$ and parameters $d_c > 0$ such that content-announce

$$\mathcal{E}V(y) \leq d_c \tag{9}$$

$$R \to \infty \Longrightarrow V_R = \inf_{|y|>R} V(y(t)) \to \infty \tag{10}$$

Then for any $y_0 \in \mathbb{R}^n$ and $i_0 \in S$, $y(t)$ is bounded in probability.

Proof: By Lemma 1.4.1 of [40], from (9), it follows that:

$$P\{|y(t)| > R\} \leq \frac{\mathcal{E}V(y(t))}{\inf\limits_{|y|>R} V(y(t))} \leq \frac{d_c}{V_R} \tag{11}$$

which, together with (10), means that (8) holds. ■

III. Main Results

In this section, the design results of backstepping controller will be presented.

Taking a symmetric positive-definite matrix variable $P(\eta(t)) \in R^{(n-m)\times(n-m)}$ and choosing the Lyapunov function candidate as

$$\mathcal{V}_1(x, \eta_t) = y_1^T(t)P(\eta(t))y_1(t). \tag{12}$$

In order to show the stochastic stability of system, two steps will be presented.

Step 1: To solve the virtual control $K(k),\ k \in \mathcal{S}$.

Letting (13) and (14), as shown at the bottom of the page.

Lemma 3.1: For given positive definite matrices $\bar{R}(k)$ and $\bar{Q}(k)$, if there exist positive definite matrices $\bar{P}(k)$ and general matrices $C(k)$ such that the following coupled of set of LMIs hold for each $k \in \mathcal{S}$:

$$\begin{bmatrix} \Theta(k) & A_{12}(k)\bar{P}(k) & \mathcal{W}_k(\bar{P}) \\ \bar{P}(k)A_{12}^T(k) & -\bar{Q}(k) & 0 \\ \mathcal{W}_k^T(\bar{P}) & 0 & -\mathcal{X}_k(\bar{P}) \end{bmatrix} < 0 \tag{15}$$

where $\Theta(k) = A_{11}(k)\bar{P}(k) + A_{12}(k)C(k) + [A_{11}(k)\bar{P}(k) + A_{12}(k)C(k)]^T + \bar{R}(k) + \alpha_{kk}\bar{P}(k)$, then, $K(k) = C(k)\bar{P}^{-1}(k)$, and the weak infinitesimal operator $\mathfrak{F}_k^x[\cdot]$ of the process $\{x(t), \eta_t, t \geq 0\}$ at the point $\{t, x, k\}$

$$\begin{aligned} \mathfrak{F}_k^x[\mathcal{V}_1] \leq &-y_1^T(t)\bar{P}^{-1}(k)\bar{R}(k)\bar{P}^{-1}(k)y_1(t) \\ &+y_2^T(t)\bar{P}^{-1}(k)\bar{Q}(k)\bar{P}^{-1}(k)y_2(t). \end{aligned} \tag{16}$$

Proof: Taking

$$\mathcal{V}_1(x, \eta_t) = y_1^T(t)P(\eta_t)y_1(t) \tag{17}$$

applying (4) yields that (18), as shown at the bottom of the next page. From (6), it follows that we have (19), as shown at the bottom of the next page, where

$$A(k) = \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ A_{21}(k) & A_{22}(k) \end{bmatrix}$$

$$\Pi(k) = \begin{bmatrix} \Pi_{11}(k) & P(k)A_{12}(k) \\ A_{12}^T(k)P(k) & -Q(k) \end{bmatrix}$$

$$\mathcal{W}_k(\bar{P}) = [\,\sqrt{\alpha_{k1}}\bar{P}(k) \quad \sqrt{\alpha_{k2}}\bar{P}(k) \quad \cdots \quad \sqrt{\alpha_{k(k-1)}}\bar{P}(k) \quad \sqrt{\alpha_{k(k+1)}}\bar{P}(k) \quad \cdots \quad \sqrt{\alpha_{k\nu}}\bar{P}(k)\,]^T \tag{13}$$

$$\mathcal{X}_k(\bar{P}) = \mathrm{diag}\left\{\bar{P}(1), \bar{P}(2), \ldots, \bar{P}(k-1), \bar{P}(k+1) \ldots, \bar{P}(\nu)\right\} \tag{14}$$

$$\begin{aligned}\Pi_{11}(k) &= P(k)(A_{11}(k)+A_{12}(k)K(k)) \\ &\quad + (A_{11}(k)+A_{12}(k)K(k))^T P(k) \\ &\quad + R(k) + \sum_{j=1}^{\nu}\alpha_{kj}P(j) \\ Q(k) &= \bar{P}^{-1}(k)\bar{Q}(k)\bar{P}^{-1}(k), R(k) \\ &= \bar{P}^{-1}(k)\bar{R}(k)\bar{P}^{-1}(k). \end{aligned} \tag{20}$$

Letting $\bar{P}(k) = P^{-1}(k)$, pre- and post-multiplying $\Pi(k)$ by $\begin{bmatrix}\bar{P}(k) & 0\\ 0 & \bar{P}(k)\end{bmatrix}$ gives

$$\begin{aligned}\Psi(k) :&= \begin{bmatrix}\bar{P}(k) & 0\\ 0 & \bar{P}(k)\end{bmatrix}\Pi(k)\begin{bmatrix}\bar{P}(k) & 0\\ 0 & \bar{P}(k)\end{bmatrix} \\ &= \begin{bmatrix}\Psi_{11}(k) & A_{12}(k)\bar{P}(k)\\ \bar{P}(k)A_{12}^T(k) & -\bar{Q}(k)\end{bmatrix}\end{aligned} \tag{21}$$

where

$$\begin{aligned}\Psi_{11}(k) &= \Pi_{11}(k) \\ &= (A_{11}(k)+A_{12}(k)K(k))\bar{P}(k) \\ &\quad + \bar{P}(k)(A_{11}(k)+A_{12}(k)K(k))^T + \bar{R}(k) \\ &\quad + \bar{P}(k)\left(\sum_{j=1}^{\nu}\alpha_{kj}\bar{P}^{-1}(k)(j)\right)\bar{P}(k). \end{aligned} \tag{22}$$

Letting $C(k) = K(k)\bar{P}(k)$, and noting that (14), $\Psi(k) < 0,\ k\in\mathcal{S}$ are equivalent to (15) based on Schur complement formula. It follows from (4) that:

$$\mathfrak{F}_k^x[\mathcal{V}_1] \le -y_1^T(t)R(k)y_1(t) + y_2^T(t)Q(k)y_2(k). \tag{23}$$

1) Step 2: To obtain control u in the following theorem, let:

$$\begin{aligned}&\begin{bmatrix}\Omega_{11}(k) & \Omega_{12}(k)\\ \Omega_{12}^T(k) & 0\end{bmatrix} \\ &\quad = \left(\begin{bmatrix}I_{n-m} & 0\\ K(k) & I_m\end{bmatrix}\right)^T \\ &\quad\quad \times\left(\sum_{j=1}^{\nu}\alpha_{kj}\begin{bmatrix}-K^T(j)\\ I\end{bmatrix}[\,-K(j)\quad I\,]\right) \\ &\quad\quad \times\begin{bmatrix}I_{n-m} & 0\\ K(k) & I_m\end{bmatrix} \\ &[\,\Lambda_{11}(k)\quad \Lambda_{12}(k)\,] \\ &\quad = [\,-K(k)\quad I\,]A(k)\begin{bmatrix}I_{n-m} & 0\\ K(k) & I_m\end{bmatrix}. \end{aligned} \tag{24}$$

Theorem 3.1: Assume the condition in Lemma 3.1 holds, i.e., inequalities (15) have solutions $\bar{P}(k)\in R^{m\times m}$, $K(k)\in R^{m\times(n-m)}$, $k\in\mathcal{S}$, $Q(k)\in R^{m\times(n-m)}$, $k\in\mathcal{S}$, and there exists a constant scalar $\lambda > 0$ such that the following inequalities hold:

$$R(k) - \Omega_{11}(k) - \sum_{j=1}^{\nu}\alpha_{kj}P(j) > \lambda P(k) \tag{25}$$

then the following control makes the closed—loop system is bounded in probability:

$$u = -B_2^{-1}(k)[(\Lambda_{11}(k)+\Omega_{12}(k))y_1(t)+(\Lambda_{12}(k)+Q(k))y_2(t)]+u_N \tag{26}$$

$$u_N = \begin{cases}-\frac{B_2^T y_2(t)}{\|B_2^T y_2(t)\|}\hat{\rho}, & \text{if } \|B_2^T y_2(t)\|\hat{\rho} > \epsilon\\ -\frac{B_2^T y_2(t)}{\epsilon}\hat{\rho}^2, & \text{if } \|B_2^T y_2(t)\|\hat{\rho} \le \epsilon\end{cases} \tag{27}$$

and the adaptation laws are

$$\hat{\rho} = \hat{c}(y(t),t) + \hat{\kappa}(y(t),t)\|y(t)\| \tag{28}$$

$$\dot{\hat{c}}(t,y) = q_1(-\epsilon_0\hat{c} + \|B_2^T y_2(t)\|) \tag{29}$$

$$\dot{\hat{\kappa}}(t,y) = q_2(-\epsilon_1\hat{\kappa} + \|B_2^T y_2(t)\|\|y(t)\|) \tag{30}$$

where q_1, q_2, ϵ_0 and ϵ_1 are design parameters.

Proof: Let us consider the function

$$\mathcal{V}(x,\tilde{c},\tilde{\kappa},\eta_t) = y_2^T(t)y_2(t) + \mathcal{V}_1(y_1(t),\eta_t) + \frac{1}{2q_1}\tilde{c}^2 + \frac{1}{2q_2}\tilde{\kappa}^2 \tag{31}$$

where $\tilde{c} = c - \hat{c}(y(t))$ and $\tilde{\kappa} = \kappa - \hat{\kappa}(y(t))$.

Upon applying (4) to (31) yields

$$\begin{aligned}\mathfrak{F}_k^{x,\tilde{c},\tilde{\kappa}}[\mathcal{V}] &= x^T(t)[\,-K(k)\quad I\,]^T[\,-K(k)\quad I\,]\dot{x}(t) \\ &+ \dot{x}^T(t)[[\,-K(k)\quad I\,]^T[\,-K(k)\quad I\,]x(t) \\ &+ x^T(t)\left(\sum_{j=1}^{\nu}\alpha_{kj}\begin{bmatrix}-K^T(j)\\ I\end{bmatrix}[\,-K(j)\quad I\,]\right)x(t) \\ &+ y_1^T(t)P(k)[\,I_{n-m}\quad 0\,]\dot{x}(t) + [[\,I_{n-m}\quad 0\,]\dot{x}(t)]^T P(k)y_1(t) \\ &+ y_1^T(t)\left(\sum_{j=1}^{\nu}\alpha_{kj}P(j)\right)y_1(t) - \frac{1}{q_1}\tilde{c}\dot{\hat{c}} - \frac{1}{q_2}\tilde{\kappa}\dot{\hat{\kappa}}. \end{aligned} \tag{32}$$

Based on system (1), (7), and inequality (16), it follows that we have (33), as shown at the bottom of the next page, it follows from (24) that we get (34), also shown at the bottom of the next page.

If $\|B_2^T y_2(t)\|\hat{\rho} > \epsilon$, $y_2^T(t)B_2 u_N = -y_2^T B_2 B_2^T y_2(t)/\|B_2^T y_2(t)\|\hat{\rho} = -\|B_2^T y_2(t)\|(\hat{\kappa}\|z\| + \hat{c})$, with the

$$\mathfrak{F}_k^x[\mathcal{V}_1] = y_1^T(t)P(k)[\,I_{n-m}\quad 0\,]\dot{x}(t) + [[\,I_{n-m}\quad 0\,]\dot{x}(t)]^T P(k)y_1(t) + y_1^T(t)\left(\sum_{j=1}^{\nu}\alpha_{kj}P(j)\right)y_1(t) \tag{18}$$

$$\begin{aligned}\mathfrak{F}_k^x[\mathcal{V}_1] &= 2y_1^T(t)P(k)[\,A_{11}(k)\quad A_{12}(k)\,]\begin{bmatrix}y_1(t)\\ K(k)y_1+y_2\end{bmatrix} + y_1^T(t)\left(\sum_{j=1}^{\nu}\alpha_{kj}P(j)\right)y_1(t) \\ &= y_1^T(t)\left[P(k)(A_{11}(k)+A_{12}(k)K(k)) + (A_{11}(k)+A_{12}(k)K(k))^T P(k) + \left(\sum_{j=1}^{\nu}\alpha_{kj}P(j)\right)\right] \\ &\quad \times y_1(t) + 2y_1^T P(k)A_{12}(k)y_2 \\ &= [\,y_1^T(t)\quad y_2^T(t)\,]\Pi(k)\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} - y_1^T(t)R(k)y_1(t) + y_2^T(t)Q(k)y_2(k) \end{aligned} \tag{19}$$

control law defined in (26) and adaptation laws defined in (29)–(30), we have (35), as shown at the bottom of the page.

Note that for any positive scalars $\delta_0 > 1/2$ and $\delta_1 > 1/2$, the following inequalities hold:

$$\begin{aligned}\epsilon_0\tilde{c}\hat{c} &= \epsilon_0\tilde{c}(-\tilde{c}+c)\\ &= \epsilon_0(-\tilde{c}^2+\tilde{c}c)\\ &\le \epsilon_0\left(-\tilde{c}^2+\frac{1}{2\delta_0}\tilde{c}^2+\frac{\delta_0}{2}c^2\right)\\ &= \frac{-\epsilon_0(2\delta_0-1)}{2\delta_0}\tilde{c}^2+\frac{\epsilon_0\delta_0}{2}c^2\end{aligned} \tag{36}$$

$$\epsilon_1\tilde{\kappa}\hat{\kappa} \le \frac{-\epsilon_1(2\delta_1-1)}{2\delta_1}\tilde{\kappa}^2+\frac{\epsilon_1\delta_1}{2}\kappa^2. \tag{37}$$

From (25), letting $\beta = \lambda_{\min}(Q(k))$, $k \in \mathcal{S}$, $\gamma = \min(\lambda,\beta)$, $q_1 = \delta_0\gamma/\epsilon_0(2\delta_0-1)$, $q_2 = \delta_1\gamma/\epsilon_1(2\delta_1-1)$ and a constant $\sigma_1 = \epsilon_0\delta_0/2c^2+\epsilon_1\delta_1/2\kappa^2$ and then

$$\mathfrak{F}_k^{x,\tilde{c},\tilde{\kappa}}[\mathcal{V}] \le -\lambda\mathcal{V}_1(y_1(t),k)-\beta y_2^Ty_2-\frac{\gamma}{2q_1}\tilde{c}^2-\frac{\gamma}{2q_2}\tilde{\kappa}^2+\sigma_1 \tag{38}$$

we have (39), as shown at the bottom of the page. If $\|B_2^Ty_2(t)\|\hat{\rho}^2 \le \epsilon$, with the control law defined in (26) and adaptation laws defined in (29)–(30), we obtain (40), as shown at the bottom of the page. From

$$\begin{aligned}\mathfrak{F}_k^{x,\tilde{c},\tilde{\kappa}}[\mathcal{V}] \le{}& 2y_2^T(t)\begin{bmatrix}-K(k) & I\end{bmatrix}\left[A(k)\begin{bmatrix}I_{n-m} & 0\\ K(k) & I_m\end{bmatrix}y(t)+B(k)[u(t)+F(k)w(x,t)]\right]\\ &+\left(\begin{bmatrix}I_{n-m} & 0\\ K(k) & I_m\end{bmatrix}y(t)\right)^T\left(\sum_{j=1}^{\nu}\alpha_{kj}\begin{bmatrix}-K^T(j)\\ I\end{bmatrix}\begin{bmatrix}-K(j) & I\end{bmatrix}\right)\begin{bmatrix}I_{n-m} & 0\\ K(k) & I_m\end{bmatrix}y(t)\\ &-y_1^T(t)R(k)y_1(t)+y_2^T(t)Q(k)y_2(t)+\sum_{j=1}^{\nu}\alpha_{kj}y_1^TP(j)y_1(t)-\frac{1}{q_1}\tilde{c}\dot{\tilde{c}}-\frac{1}{q_2}\tilde{\kappa}\dot{\tilde{\kappa}}\end{aligned} \tag{33}$$

$$\begin{aligned}\mathfrak{F}_k^{x,\tilde{c},\tilde{\kappa}}[\mathcal{V}] \le{}& y_2^T(t)[\Lambda_{11}(k)y_1(t)+\Lambda_{12}(k)y_2(t)+\Omega_{12}(k)y_1(t)+B_2(k)(u(t)+F(k)w(x(t),t))\\ &+[\Lambda_{11}(k)y_1(t)+\Lambda_{22}(k)y_2(t)+\Omega_{12}(k)y_1(t)+B_2(k)(u(t)+F(k)w(x(t),t))]^Ty_2(t)\\ &+-y_1^T(t)R(k)y_1(t)+y_2^T(t)Q(k)y_2(t)+y_1^T(t)\Omega_{11}(k)y_1(t)\\ &+\sum_{j=1}^{\nu}\alpha_{kj}y_1^T(t)P(j)y_1(t)-\frac{1}{q_1}\tilde{c}\dot{\tilde{c}}-\frac{1}{q_2}\tilde{\kappa}\dot{\tilde{\kappa}}\end{aligned} \tag{34}$$

$$\begin{aligned}\mathfrak{F}_k^{x,\tilde{c},\tilde{\kappa}}[\mathcal{V}] \le{}& -y_1^T(t)(R(k)-\Omega_{11}(k))y_1(t)-y_2^T(t)Q(k)y_2(t)+\sum_{j=1}^{\nu}\alpha_{kj}y_1^TP(j)y_1(t)\\ &-y_2^T(t)B_2u_N+y_2^T(t)B_2F(k)w(y(t)),t)-\tilde{c}(-\epsilon_0\hat{c}+\|B_2^Ty_2(t)\|)\\ &-\tilde{\kappa}(-\epsilon_1\hat{\kappa}+\|B_2^Ty_2(t)\|\|y\|)\\ \le{}& -y_1^T(t)\left(R(k)-\Omega_{11}(k)-\sum_{j=1}^{\nu}\alpha_{kj}P(j)\right)y_1(t)-y_2^T(t)Q(k)y_2(t))\\ &+\|B_2^Ty_2(t)\|(\kappa\|y\|+c)-\|B_2^Ty_2(t)\|(\hat{\kappa}\|y\|+\hat{c})-\tilde{c}(-\epsilon_0\hat{c}+\|B_2^Ty_2(t)\|)\\ &-\tilde{\kappa}(-\epsilon_1\hat{\kappa}+\|B_2^Ty_2(t)\|\|y\|)\\ ={}& -y_1^T(t)\left(R(k)-\Omega_{11}(k)-\sum_{j=1}^{\nu}\alpha_{kj}P(j)\right)y_1(t)-y_2^T(t)Q(k)y_2(t))+\epsilon_0\tilde{c}\hat{c}+\epsilon_1\tilde{\kappa}\hat{\kappa}\end{aligned} \tag{35}$$

$$\mathfrak{F}_k^{x,\tilde{c},\tilde{\kappa}}[\mathcal{V}] \le -\gamma\left(\mathcal{V}_1(y_1(t),k)+y_2^Ty_2+\frac{1}{2q_1}\tilde{c}^2+\frac{1}{2q_2}\tilde{\kappa}^2\right)+\sigma_1 \tag{39}$$

$$\begin{aligned}\mathfrak{F}_k^{x,\tilde{c},\tilde{\kappa}}[\mathcal{V}] \le{}& -(\gamma\mathcal{V}_1(t,y,k)+\beta y_2^Ty_2)-\frac{\|B_2^Ty_2(t)\|^2}{\epsilon}\hat{\rho}^2+\|B_2^Ty_2(t)\|(\kappa\|y\|+c)\\ &-\tilde{c}(-\epsilon_0\hat{c}+\|B_2^Ty_2(t)\|)-\tilde{\kappa}(-\epsilon_1\hat{\kappa}+\|B_2^Ty_2(t)\|\|y\|)\\ ={}& -\gamma(\mathcal{V}_1(y_1(t),k)+y_2^Ty_2)-\frac{\|B_2^Ty_2(t)\|^2}{\epsilon}\hat{\rho}^2+\|B_2^Ty_2(t)\|\hat{\rho}+\epsilon_0\tilde{c}\hat{c}+\epsilon_1\tilde{\kappa}\hat{\kappa}\\ ={}& -\gamma(\mathcal{V}_1(y_1(t),k)+y_2^Ty_2)-\left(\frac{\|B_2^Ty_2(t)\|^2}{\sqrt{\epsilon}}\hat{\rho}-\frac{\sqrt{\epsilon}}{2}\right)^2+\frac{\epsilon}{4}+\epsilon_0\tilde{c}\hat{c}+\epsilon_1\tilde{\kappa}\hat{\kappa}\end{aligned} \tag{40}$$

(36) and (37), we have (41), shown at the bottom of the page. Letting a constant $\sigma_2 = \epsilon/4 + \epsilon_0\delta_0/2c^2 + \epsilon_1\delta_1/2\kappa^2$, we have (42), as shown at the bottom of the page. Based on inequalities (39) and (42), we have

$$\mathfrak{F}_k^{x,\tilde{c},\tilde{\kappa}}[\mathcal{V}] \le -\gamma\mathcal{V} + \sigma_3 \tag{43}$$

where $\sigma_3 = \max(\sigma_1, \sigma_2)$, it results in

$$\frac{d\mathcal{E}(V)}{dt} \le -\gamma\mathcal{E}(V) + \sigma_3. \tag{44}$$

It follows from Lemma 2.1, we know that the solution of the closed-loop system is bounded in probability.

Remark 3.1: Note that Theorem 3.1 provides a solution to the problem of adaptive control for stochastic system. It is worth mentioning that the work conducted in this technical note is the attempt to overcome the problem arising in the sliding mode control for Markov jumping systems and adopt adaptive backstepping controller for systems with Markovian jump parameters. The results obtained in this technical note could be extended to general systems with other forms of stochastic jumps.

IV. A Numerical Example

Let us consider the following system with generator for the Markov process governing the mode switching being

$$\Im = \begin{bmatrix} -1.4 & 1.4 \\ 1.2 & -1.2 \end{bmatrix}.$$

For the two operating conditions (modes), the associated data is:

Mode 1

$$A(1) = \begin{bmatrix} -2.9 & 0.3 & 0.4 & 1.2 \\ -0.1 & -0.2 & 0.6 & 1.5 \\ 1.2 & 2.1 & 2.8 & 3.4 \\ 1 & -2 & -2.5 & -2.5 \end{bmatrix}$$

$$B(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1.0 & 0.5 \\ -0.1 & 0.2 \end{bmatrix}$$

$$w(t) = 0.1 + 0.01 * \|y(t)\|.$$

Mode 2

$$A(2) = \begin{bmatrix} -1.3 & -0.1 & 0.21 & 0.3 \\ -0.8 & 0 & 0.2 & 1.2 \\ -0.6 & 0.2 & 1.4 & -0.9 \\ 0.5 & 0.5 & 0.3 & 1.2 \end{bmatrix},$$

$$B(2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.1 & 0.1 \\ 0.5 & -0.2 \end{bmatrix},$$

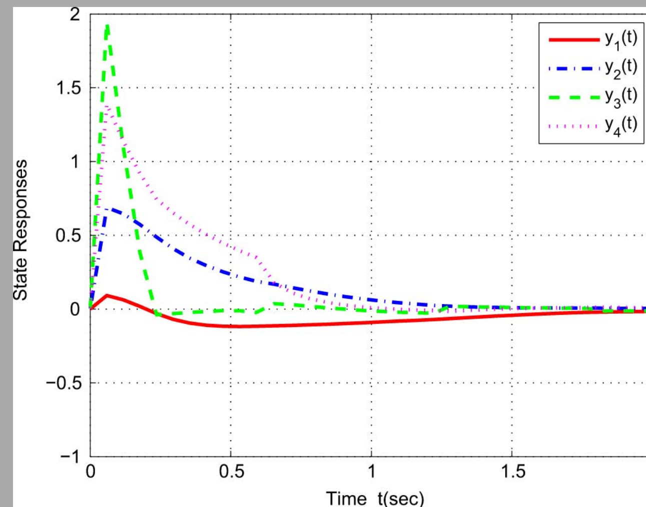

Fig. 1. States $(y_1(t),\ y_2(t),\ y_3(t),\ y_4(t))$.

$$w(t) = 0.1 + 0.01 * \|y(t)\|.$$

Using Theorem 3.1 and LMI tool box in Matlab, we have

$$\bar{P}(1) = \begin{bmatrix} 0.7041 & -0.1596 \\ -0.1596 & 0.3376 \end{bmatrix}$$
$$C(1) = \begin{bmatrix} -2.7156 & 0.0526 \\ 1.4592 & -1.1077 \end{bmatrix}$$
$$\bar{P}(2) = \begin{bmatrix} 0.8125 & -0.0660 \\ -0.0660 & 0.4975 \end{bmatrix}$$
$$C(2) = \begin{bmatrix} -4.1284 & -1.03950 \\ 1.6207 & -1.2202 \end{bmatrix}.$$

Then

$$K(1) = \begin{bmatrix} -4.2799 & -1.8670 \\ 1.4882 & -2.5778 \end{bmatrix}$$
$$K(2) = \begin{bmatrix} -5.3081 & -2.7934 \\ 1.8150 & -2.2118 \end{bmatrix}.$$

Taking $F(1) = F(2) = [0.7071 \quad 0.7071]^T$, it can be shown that $\|Fw(y,t)\| \le 0.1 + 0.01\|y(t)\|$. Letting $\lambda = 0.5$, $\epsilon = 0.35$, $\epsilon_0 = 0.0844$, $\epsilon_1 = 0.011$, $q_1 = 0.8$, $q_2 = 13$, we have the following simulation results.

The closed-loop dynamic responses are given in Figs. 1–3 under the following initial conditions $y = [-0.0003 \quad 0.0036 \quad 0.0049 \quad 0.0083]^T$, $c(0) = 0.5$ and $k(0) = 0.1$. Fig. 1 shows that the transformed system states are bounded in probability. Fig. 2 depicts the input control signal. The adaptive parameters are shown in Fig. 3, it can be shown that $\hat{c}$ and $\hat{\kappa}$ convergent to the upper bounds of disturbances and uncertainties

$$\mathfrak{F}_k^{x,\tilde{c},\tilde{\kappa}}[\mathcal{V}] \le -\gamma\left(\mathcal{V}_1(y_1(t),k) + y_2^T y_2 + \frac{1}{2q_1}\tilde{c}^2 + \frac{1}{2q_2}\tilde{\kappa}^2\right) + \frac{\epsilon}{4} + \frac{\epsilon_0\delta_0}{2}c^2 + \frac{\epsilon_1\delta_1}{2}\kappa^2 \tag{41}$$

$$\mathfrak{F}_k^{x,\tilde{c},\tilde{\kappa}}[\mathcal{V}] \le -\gamma\left(\mathcal{V}_1(y_1(t),k) + y_2^T y_2 + \frac{1}{2q_1}\tilde{c}^2 + \frac{1}{2q_2}\tilde{\kappa}^2\right) + \sigma_2 \tag{42}$$

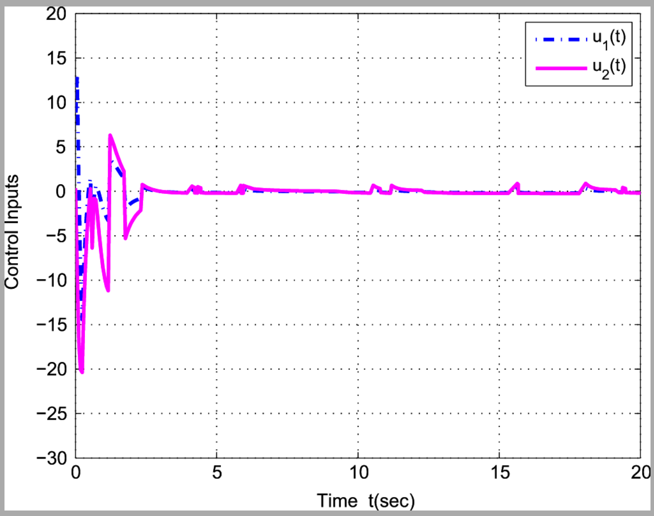

Fig. 2. Control $(u_1(t),\ u_2(t))$.

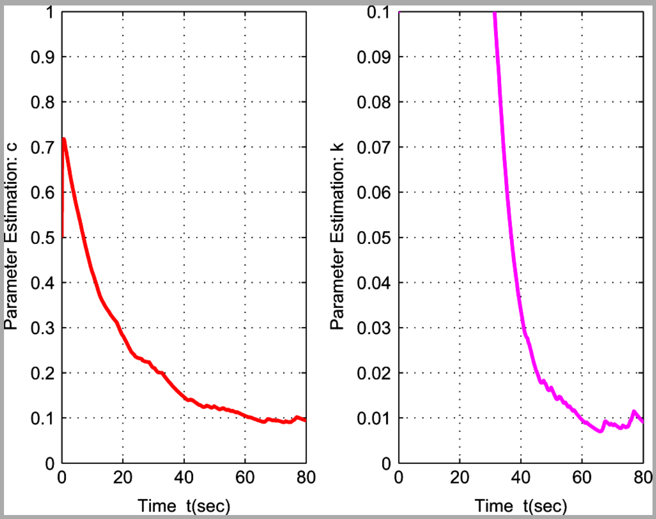

Fig. 3. Adaptive parameters c and κ.

$c = 0.1$ and $\kappa = 0.01$, respectively. Moreover, it should be pointed out that the Markov switching is generated on-line. From the above figures, we can see that the proposed control methods work well.

V. Conclusion

In this technical note, the well-known backstepping method is used to overcome the problem in [1]. The adaptive backstepping controller design problem is investigated by using LMI technique and adaptive control approach. Numerical example has been given to demonstrate the applicability of the theoretical results obtained in this technical note.

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References

[1] P. Shi, Y. Xia, G.-P. Liu, and D. Rees, “On designing of sliding-mode control for stochastic jump system,” *IEEE Trans. Autom. Control*, vol. 51, no. 1, pp. 97–103, Jan. 2006.

[2] E. K. Boukas and Z. Liu, *Deterministic and Stochastic Time-Delay Systems*. Boston, MA: Birkhauser, 2002.

[3] E. K. Boukas, P. Shi, and S. K. Nguang, “Robust H_∞ control for linear Markovian jump systems with unknown nonlinearities,” *J. Mathemat. Anal. and Applic.*, vol. 282, no. 1, pp. 241–255, 2003.

[4] E. K. Boukas, Z. Liu, and P. Shi, “Delay-dependent stability and output feedback stabilization of Markov jump systems with time-delay,” *IEE-Part D, Control Theory and Applications*, vol. 149, no. 5, pp. 379–386, 2002.

[5] M. Mahmoud and P. Shi, *Methodologies for Control of Jump Time-Delay Systems*. Boston, MA: Kluwer, 2003.

[6] P. Shi, E. K. Boukas, and R. K. Agarwal, “Kalman filtering for continuous-time uncertain systems with Markovian jumping parameters,” *IEEE Trans. Autom. Control*, vol. 44, no. 8, pp. 1592–1597, Aug. 1999.

[7] P. Shi, E. K. Boukas, and R. K. Agarwal, “Control of Markovian jump discrete-time systems with norm bounded uncertainty and unknown delays,” *IEEE Trans. Autom. Control*, vol. 44, no. 11, pp. 2139–2144, Nov. 1999.

[8] P. Shi and E. K. Boukas, “H_∞ control for Markovian jumping linear systems with parametric uncertainty,” *J. Optimiz. Theory and Applic.*, vol. 95, no. 1, pp. 75–99, 1997.

[9] Y. Niu, D. W. C. Ho, and X. Wang, “Sliding mode control for itô stochastic systems with Markovian switching,” *Automatica*, vol. 43, pp. 1784–1790, 2007.

[10] J.-F. Yao and J.-G. Attali, “On stability of nonlinear AR processes with Markov switching,” *Adv. in Appl. Probab.*, vol. 32, no. 2, pp. 394–407, 2000.

[11] Y. Xia and Y. Jia, “Robust sliding mode control of uncertain time-delay systems: An LMI approach,” *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 1986–1092, Jun. 2003.

[12] C. Y. Chan, “Discrete adaptive sliding mode control of a class of stochastic systems,” *Automatica*, vol. 35, no. 8, pp. 1491–1498, 1999.

[13] Y. Niu, D. W. C. Ho, and J. Lam, “Robust integral sliding mode control mode control for uncertain stochastic systems with time-varying delay,” *Automatica*, vol. 41, no. 5, pp. 873–880, 2005.

[14] X. Zhong, H. Xing, and K. Fujimoto, “Sliding mode variable structure control for uncertain stochastic systems,” *Int. J. Innov. Comput., Inform., and Control*, vol. 3, pp. 397–406, 2007.

[15] X. Sun, J. Zhao, and W. Wang, “Two design schemes for robust adaptive control of a class of linear uncertain neutral delay systems,” *Int. J. Innov. Comput., Inform., and Control*, vol. 3, pp. 385–396, 2007.

[16] Y. Xia, G.-P. Liu, P. Shi, J. Chen, and D. Rees, “Robust delay-dependent sliding mode control for uncertain time-delay systems,” *Int. J. Robust Nonlin. Control*, vol. 18, pp. 1142–1161, 2008.

[17] Y. Xia, G.-P. Liu, P. Shi, J. Chen, D. Rees, and J. Liang, “Sliding mode control of uncertain linear discrete time systems with input delay,” *IET Control Theory Appl.*, vol. 1, pp. 1169–1175, 2007.

[18] M. Ahmed, P. Zhang, and Y.-J. Wu, “Position control of synchronous motor drive by modified adaptive two-phase sliding mode controller,” *Int. J. Autom. and Comput.*, vol. 5, pp. 406–412, 2008.

[19] Z. Wu, X. Xie, and S. Zhang, “Stochastic adaptive backstepping controller design by introducing dynamic signal and changing supply function,” *Int. J. Control*, vol. 79, pp. 1635–1646, 2006.

[20] S. Tong and Y. Li, “Direct adaptive fuzzy backstepping control for a class of nonlinear systems,” *Int. J. Innov. Comput., Inform., and Control*, vol. 3, pp. 887–896, 2007.

[21] Z. Wu, X. Xie, and S. Zhang, “Adaptive backstepping controller design using stochastic small-gain theorem,” *Automatica*, vol. 43, pp. 608–620, 2007.

[22] C. Hua, Q. Wang, and X. Guan, “Memoryless state feedback controller design for mismatched nonlinear time delay systems,” *IEEE Trans. Autom. Control*, vol. 53, no. 3, pp. 801–807, Mar. 2008.

[23] C. Hua, G. Feng, and X. Guan, “Robust controller design of a class of nonlinear time delay systems via backstepping method,” *Automatica*, vol. 44, pp. 567–573, 2008.

[24] M. Krstić, I. Kanellakopoulos, and P. Kokotović, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.

[25] M. Krstić and H. Deng, *Stability of Nonlinear Uncertain Systems*. New York: Springer, 1998.

[26] R. Marino and P. Tomei, *Nonlinear Control Design: Geometric, Adaptive and Robust*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[27] M. Chen, C.-S. Jiang, and Q.-X. Wu, “Sensor fault diagnosis for a class of time delay uncertain nonlinear systems using neural network,” *Int. J. Autom. and Comput.*, vol. 4, pp. 401–405, 2008.

[28] K. S. Narenda and J. D. Boskovic, “A combined direct, indirect, and variable structure method for robust adaptive control,” *IEEE Trans. Autom. Control*, vol. 37, no. 2, pp. 262–268, Feb. 1992.

[29] G. Wheeler, C. Y. Su, and Y. Stepanenko, "A sliding mode controller with improved adaptation laws for the upper bounds on the norm of uncertainties," *Automatica*, vol. 34, pp. 1657–1661, 1998.

[30] W.-S. Chen and J.-M. Li, "Adaptive output-feedback regulation for nonlinear delayed systems using neural network," *Int. J. Autom. and Comput.*, vol. 5, pp. 103–108, 2008.

[31] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*. London, U.K.: Springer-Verlag, 1993.

[32] H. J. Kushner, *Stochastic Stability and Control*. New York: Academic, 1967.

[33] A. S. Willsky, "A survey of design methods for failure detection in dynamic systems," *Automatica*, vol. 12, no. 5, pp. 601–611, 1976.

[34] D. D. Sworder and R. O. Rogers, "An LQ-solution to a control problem associated with a solar thermal central receiver," *IEEE Trans. Autom. Control*, vol. AC-28, no. 8, pp. 971–978, Aug. 1983.

[35] M. Athans, "Command and control (C2) theory: A challenge to control science," *IEEE Trans. Autom. Control*, vol. AC-32, no. 4, pp. 286–293, Apt. 1987.

[36] D. D. Moerder, N. Halyo, J. R. Broussard, and A. K. Caglayan, "Application of precomputed control laws in a reconfigurable aircraft flight control system," *J. Guid., Control and Dynam.*, vol. 12, no. 3, pp. 325–333, 1989.

[37] D. B. Petkovski, "Multivariable control system design: A case study of robust control of nuclear power plants," *Fault Detect. and Reliab.*, vol. 9, pp. 239–246, 1987.

[38] E. K. Boukas, Q. Zhang, and G. Yin, "Robust production and maintenance planning in stochastic manufacturing systems," *IEEE Trans. Autom. Control*, vol. 40, no. 6, pp. 1098–1102, Jun. 1995.

[39] E. K. Boukas and A. Haurie, "Manufacturing flow control and preventive maintenance: A stochastic control approach," *IEEE Trans. Automat. Control*, vol. 35, no. 7, pp. 1024–1031, 1990.

[40] R. Z. Khas'minskii, *Stochastic Stability of Differential Equations*. Rockville, MD: S&N International Publisher, 1980.

On the Control and Estimation Over Relative Sensing Networks

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***Abstract*—In this note, we consider certain structural aspects of estimation and control over relative sensing networks (RSNs). In this venue, using tools from basic algebraic graph theory—namely, the incidence matrix and cut and cycle spaces of a connect graph- we examine transformations among relative sensing topologies for a networked system. These transformations are parameterized for noise-free as well as noisy networks and their utility in the context of network-centric robust control and control reconfigurations is explored.**

***Index Terms*—Control and estimation over networks, cut and cycle spaces of a graph, relative sensing networks (RSNs).**

I. INTRODUCTION

Networked dynamic systems are collections of dynamical units that interact over an information exchange network for their operation.

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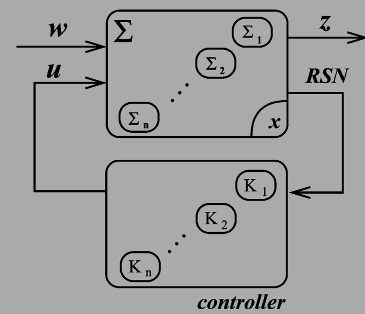

Fig. 1. Feedback configuration over an RSN.

These systems are ubiquitous in diverse areas of science and engineering; examples include physiological systems and gene networks, large scale energy systems, and multiple space, air, and land vehicles [2], [3], [8], [11], [12]; see also [16]–[18]. The present work is motivated by examining how structural features of a particular class of information-exchange networks, namely, relative sensing networks (RSNs), influence the system-theoretic properties of networked systems. RSNs are common in a wide variety of applications, including multi-agent coordination, clock synchronization, and sensor networks [1], [6], [9], [10]. For example, multiple spacecraft formation flying over RSNs have recently been considered by Smith and Hadaegh [15]; this reference has in fact motivated our studies on RSNs in [13] and [14].

The system configuration of interest in this technical note is shown in Fig. 1 where the signal z captures the coordination state among multiple dynamic systems; signals x, w, and u, denote, respectively, the system state- comprised of states of the individual dynamic elements, the exogenous signal, and the control input. In this figure, the measured signal designated as 'RSN,' denotes the information vector (sensed or communicated over the network), that is available to the controller. Since the control objective is the coordination of relative states among dynamic units, it is assumed that $z(t)$ in Fig. 1 consists of components that are functions of vector differences $x_i(t) - x_j(t)$ $(i \neq j)$. Likewise, the information available to the controller consists of a subset of these relative states. We refer to such a feedback system setup, and the resulting system-theoretic issues, as the problem of control and estimation over RSNs.

Our contribution in this technical note to the general area of control and estimation over RSNs is as follows. First, we parameterize transformations that would allow mapping between structurally distinct RSNs. We then explore the utility of such parameterizations in the context of network-centric robustness analysis and a control reconfiguration mechanism when the underlying RSN undergoes structural changes. In Section IV, we proceed to derive analogous transformations among RSNs with noisy links. Examples are incorporated throughout the technical note to complement the theoretical analysis.

Notation and Preliminaries

Our networked dynamic system consists of dynamic units with indices from the set $\{1, 2, \ldots, n\}$; this set will be denoted by $[n]$. We will write $|S|$ for the cardinality of set S; thus, $|[n]| = n$. The matrix I_p denotes the $p \times p$ identity matrix and $\mathbf{Diag}\{x\}$ is the diagonal matrix whose diagonal entries orderly correspond to the entries of x. $\mathbf{E}\{x\}$ denotes the expectation of a random variable x; $f \circ g$ refers to the composition of two maps f and g. Lastly, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are, respectively, the range and the null spaces of the matrix A.

Our graph-theoretic constructs and notation are standard and follow [4]. A graph $G = (V, E)$ consists of a vertex set $V(G)$ and an edge set