International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

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Version of record first published: 08 Nov 2010.

INT. J. CONTROL, 1998, VOL. 71, NO. 5, 959–979

Adaptive dynamic surface control: a simplified algorithm for adaptive backstepping control of nonlinear systems

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In this paper, we propose a new algorithm for adaptive backstepping control of non-linear uncertain systems. Current backstepping algorithms require repeated differentiations of the modelled non-linearities. The addition of n first order low pass filters allows the algorithms to be implemented without differentiating any model non-linearities. The uncertainties are assumed to be linear in the unknown constant parameters. The combined adaptive backstepping/first order filter system is proven to be semi-globally stable for sufficiently fast filters by a singular perturbation approach.

1. Introduction

A major contribution to the control of uncertain nonlinear systems, particularly those systems that do not satisfy matching conditions, is the use of 'backstepping' methods (Kanellakapoulous *et al.* 1991, Krstic *et al.* 1995). A systematic approach has been developed for a class of systems where the uncertainty can be linearly parameterized (Sastry and Isidori 1989, Kanellakapoulous *et al.* 1991, Slotine and Li 1991, Krstic *et al.* 1995), e.g.

$$f(x) = a\bar{f}(x)$$

where a is an unknown constant and $\bar{f}(x)$ is a known nonlinear function. Traditional backstepping algorithms, although systematic, suffer from an 'explosion of complexity' due to the necessity to perform repeated differentiations of the nonlinear functions. A third order example will be given in the next section to illustrate this phenomenon. A recent paper (Swaroop *et al.* 1997) introduced the concept of 'dynamic sliding surfaces' to eliminate this problem for non-adaptive systems. This paper introduced first-order filtering of the synthetic inputs at each level of the traditional backstepping approach. Stability was proven in Swaroop *et al.* (1997) for uncertain nonlinear systems with Lipschitz or non-Lipschitz nonlinearities. In this paper, we extend this technique to adaptive systems.

2. Adaptive backstepping

Consider the third order system

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$$\left.\begin{aligned} \dot{x}_1 &= a_1 f_1(x_1) + x_2 \\ \dot{x}_2 &= a_2 f_2(x_1, x_2) + x_3 \\ \dot{x}_3 &= u \end{aligned}\right\} \tag{1}$$

where a_1 and a_2 are unknown constants and the goal is to regulate x_1 at 0. The adaptive backstepping approach is to sequentially define synthetic values for x_2 and x_3. Following the tuning functions design procedure in Krstic *et al.* (1995), define

$$z_1 = x_1 \tag{2}$$

$$\Rightarrow \dot{z}_1 = x_2 + a_1 f_1(x_1)$$

The synthetic value for x_2 is defined to stabilize z_1:

$$\alpha_1(x_1, \hat{a}_1) = -c_1 z_1 - \hat{a}_1 f_1 \tag{3}$$

Next define

$$z_2 = x_2 - \alpha_1 \tag{4}$$

$$\Rightarrow \dot{z}_2 = x_3 + \alpha_2 + a_2 f_2 - \dot{\alpha}_1$$

The synthetic value for x_3 is defined to stabilize z_2:

$$\alpha_2(x_1, x_2, \hat{a}_1, \hat{a}_2) = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1}(x_2 + \hat{a}_1 f_1) - \hat{a}_2 f_2 \tag{5}$$

Finally, define

$$z_3 = x_3 - \alpha_2 \tag{6}$$

and the control u is chosen to stabilize z_3:

$$\begin{aligned} u = &-z_2 - c_3 z_3 + \frac{\partial \alpha_2}{\partial x_1}(x_2 + \hat{a}_1 f_1) + \frac{\partial \alpha_2}{\partial x_2}(x_3 + \hat{a}_2 f_2) \\ &+ \frac{\partial \alpha_2}{\partial \hat{a}_1}\dot{\hat{a}}_1 + \frac{\partial \alpha_2}{\partial \hat{a}_2}\dot{\hat{a}}_2 - z_2\left(\rho_1 \frac{\partial \alpha_2}{\partial \hat{a}_1}\frac{\partial \alpha_2}{\partial x_1} f_1 + \rho_2 \frac{\partial \alpha_2}{\partial \hat{a}_2}\frac{\partial \alpha_2}{\partial x_2} f_2\right) \end{aligned} \tag{7}$$

where c_1, c_2, c_3, ρ_1, and ρ_2 are positive design parameters while $\hat{a}_1$ and $\hat{a}_2$ are estimates for the constant parameters a_1 and a_2 respectively.

The parameter update law is chosen as

$$\left.\begin{aligned} \dot{\hat{a}}_1 &= \rho_1 f_1\left(z_1 - \frac{\partial \alpha_1}{\partial x_1} z_2 - \frac{\partial \alpha_2}{\partial x_1} z_3\right) \\ \dot{\hat{a}}_2 &= \rho_2 f_2\left(z_2 - \frac{\partial \alpha_2}{\partial x_2} z_3\right) \end{aligned}\right\} \tag{8}$$

Using the Lyapunov function

$$V = \frac{1}{2}\left(\sum_{i=1}^{3} z_i^2 + \sum_{i=1}^{2} \frac{1}{\rho_i}\tilde{a}_i^2\right)$$

the above control and update laws result in:

$$\dot{V} = -\sum_{k=1}^{3} c_k z_k^2$$

Barbalat's lemma can be used to show that $\dot{V} \to 0$ as $t \to \infty$. Thus z_1, z_2, and $z_3 \to 0$ as $t \to \infty$. This 'explosion of complexity' comes from the evaluations of the partial derivatives of α_1 and α_2. The explosion becomes untenable as the order of the system increases.

3. Adaptive dynamic surface control

A procedure to deal with the explosion of terms for the non-adaptive case has been presented in Swaroop *et al.* (1997) and is called 'dynamic surface control' (DSC). In this paper, we are extending this technique to adaptive control. To illustrate the procedure, we treat the same third order example as in §2. We proceed as before by defining the first surface

$$S_1 = x_1 \tag{9}$$

$\bar{x}_2$ (equivalent to α_1 before) is chosen to drive $S_1 \to 0$:

$$\bar{x}_2 = -\hat{a}_1 f_1(x_1) - K_1 S_1 \tag{10}$$

However, departing from the approach in the previous section, we pass $\bar{x}_2$ through a first order filter, with time constant τ_2, to obtain x_{2d},

$$\tau_2 \dot{x}_{2d} + x_{2d} = \bar{x}_2, \qquad x_{2d}(0) = \bar{x}_2(0) \tag{11}$$

Define the second surface to be

$$S_2 = x_2 - x_{2d} \tag{12}$$

We choose $\bar{x}_3$ to drive $S_2 \to 0$,

$$\bar{x}_3 = -\hat{a}_2 f_2(x_1, x_2) - K_2 S_2 + \dot{x}_{2d} \tag{13}$$

Comparing $\bar{x}_3$ with α_2 in the previous section, we notice that $\bar{x}_3$ is a lot simpler because the first-order filter for x_{2d} has allowed us to eliminate the partial derivatives associated with the model nonlinearities. Once again, we pass $\bar{x}_3$ through a first-order filter, with time constant τ_3, to obtain x_{3d},

$$\tau_3 \dot{x}_{3d} + x_{3d} = \bar{x}_3, \qquad x_{3d}(0) = \bar{x}_3(0) \tag{14}$$

Finally, define the surface

$$S_3 = x_3 - x_{3d} \tag{15}$$

The control u is chosen to be

$$u = \dot{x}_{3d} - K_3 S_3 \tag{16}$$

The update law for the parameter estimate is as follows:

$$\begin{aligned} \dot{\hat{a}}_1 &= \rho_1 S_1 f_1 \\ \dot{\hat{a}}_2 &= \rho_2 S_2 f_2 \end{aligned} \tag{17}$$

which is a lot simpler than before.

Summarizing our control structure:

$$\left.\begin{aligned} u &= \frac{\bar{x}_3 - x_{3d}}{\tau_3} - K_3(x_3 - x_{3d}) \\ \bar{x}_3 &= -\hat{a}_2 f_2 + \frac{\bar{x}_2 - x_{2d}}{\tau_2} - K_2(x_2 - x_{2d}) \\ \bar{x}_2 &= -\hat{a}_1 f_1 - K_1 x_1 \\ \dot{\hat{a}}_1 &= \rho_1 x_1 f_1 \\ \dot{\hat{a}}_2 &= \rho_2 (x_2 - x_{2d}) f_2 \\ \tau_2 \dot{x}_{2d} + x_{2d} &= \bar{x}_2, \qquad x_{2d}(0) = \bar{x}_2(0) \\ \tau_3 \dot{x}_{3d} + x_{3d} &= \bar{x}_3, \qquad x_{3d}(0) = \bar{x}_3(0) \end{aligned}\right\} \tag{18}$$

One can see from equations (18) that no model differentiation is required. The final two first-order filters in equations (18) have eliminated the need for such differentiation. We will show in the next section that systems employing filtered adaptive backstepping are semi-globally stable for small enough τ_2, τ_3, and big enough K_1, K_2, and K_3.

4. Stability analysis

Consider the following nonlinear system in parametric strict-feedback form:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + a_1 f_1(x_1) \\ \dot{x}_2 &= x_3 + a_2 f_2(x_1, x_2) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + a_{n-1} f_{n-1}(x_1, \ldots, x_{n-1}) \\ \dot{x}_n &= u + a_n f_n(x_1, \ldots, x_n) \\ y &= x_1 \end{aligned}\right\} \tag{19}$$

where a_i, $i = 1, \ldots, n$, are the unknown constant parameters.

The tracking problem is to synthesize a control law for u such that x_1 follows a desired trajectory $x_{1\text{des}}(t)$. For regulation to the origin, $x_{1\text{des}}(t) = 0$ identically. For set-point control, $x_{1\text{des}}$ can be an arbitrary constant but $\dot{x}_{1\text{des}} = \ddot{x}_{1\text{des}} = 0$ identically.

4.1. *Controller design procedure*

The design procedure for the Adaptive Dynamic Surface Controller is described below. Let the error in tracking a desired trajectory x_{1d} be S_1.

$$S_1 := x_1 - x_{1d} \tag{20}$$

$\bar{x}_2$ is chosen to drive $S_1 \to 0$:

$$\bar{x}_2 = -\hat{a}_1 f_1(x_1) - K_1 S_1 \tag{21}$$

where $\hat{a}_1$ is the estimate of the unknown parameter a_1.

Filtering $\bar{x}_2$, x_{2d} is obtained.

$$\tau_2 \dot{x}_{2d} + x_{2d} = \bar{x}_2 \tag{22}$$

Define the second surface to be

$$S_2 := x_2 - x_{2d} \tag{23}$$

We choose $\bar{x}_3$ to drive $S_2 \to 0$,

$$\bar{x}_3 = -\hat{a}_2 f_2(x_1, x_2) - K_2 S_2 + \dot{x}_{2d} \tag{24}$$

and obtain x_{3d} by filtering $\bar{x}_3$,

$$\tau_3 \dot{x}_{3d} + x_{3d} = \bar{x}_3 \tag{25}$$

Proceeding similarly, define the ith surface as

$$S_i := x_i - x_{id} \tag{26}$$

and $\bar{x}_{i+1}$ is chosen to drive $S_i \to 0$,

$$\bar{x}_{i+1} = -\hat{a}_i f_i(x_1, \ldots, x_i) - K_i S_i + \dot{x}_{id} \tag{27}$$

Filtering $\bar{x}_{i+1}$, x_{i+1d} is obtained.

$$\tau_{i+1} \dot{x}_{i+1d} + x_{i+1d} = \bar{x}_{i+1} \tag{28}$$

Finally, define

$$S_n := x_n - x_{nd} \tag{29}$$

The control u is chosen to be

$$u = \dot{x}_{nd} - \hat{a}_n f_n(x_1, \ldots, x_n) - K_n S_n \tag{30}$$

The update law for the parameter estimates is as follows:

$$\left.\begin{aligned} \dot{\hat{a}}_1 &= \rho_1 S_1 f_1 \\ &\vdots \\ \dot{\hat{a}}_i &= \rho_i S_i f_i \\ &\vdots \\ \dot{\hat{a}}_n &= \rho_n S_n f_n \end{aligned}\right\} \tag{31}$$

where $\rho_i > 0, i = 1, \ldots, n$, are design parameters that can be adjusted for the rate of convergence of the parameter estimates.

4.2. *Stability analysis for regulation to the origin*

In this section, a stability analysis is performed on the closed-loop system regulating the output $y = x_1$ to the origin. In other words, $x_{1d}(t) = 0$ identically. Techniques and terminology from Singular Perturbation Theory would be utilized heavily here. Interested readers are referred to Kokotovic *et al.* (1986) and Khalil (1992). The following assumption on (19) is used in the analysis.

Assumption 1:

- *The map* $f_i : R^i \to R$ *is* C^1 *and* $f_i(0, \ldots, 0) = 0$.
- $a_1, \ldots, a_n$ *are unknown constants with known bounds, i.e.* $|a_i| \le \gamma_i$, $i = 1, \ldots, n$.

We will show semi-global asymptotic stability for the state vector $\left[x_1 \cdots x_n\right]^T$, i.e. $x_i \to 0, i = 1, \ldots, n$.

First, we need to set up some terminology and notations. Define the boundary layer error as:

$$y_i := x_i - \bar{x}_i, \quad i = 2, \ldots, n \tag{32}$$

and let the ith surface error be:

$$S_i := x_i - x_{id}, \quad i = 1, \ldots, n \tag{33}$$

The estimate error is defined as:

$$\tilde{a}_i := a_i - \hat{a}_i, \quad i = 1, \ldots, n \tag{34}$$

Let

$$V_{is} := \tfrac{1}{2} S_i^2, \quad i = 1, \ldots, n \tag{35}$$

$$V_{iy} := \tfrac{1}{2} y_i^2, \quad i = 2, \ldots, n \tag{36}$$

$$V_{ia} := \frac{1}{2\rho_i} \tilde{a}_i^2, \quad i = 1, \ldots, n \tag{37}$$

and

$$V := \sum_{i=1}^{n} V_{is} + \sum_{i=2}^{n} V_{iy} + \sum_{i=1}^{n} V_{ia} \tag{38}$$

We will show that given any $p > 0$, there exist K_i and τ_i such that the closed-loop system is stable and $x_i \to 0, i = 1, \ldots, n$, for all $\mathbf{x}(0) \in \{\mathbf{x} : V(\mathbf{x}) \leq p\}$. The outline of the analysis is as follows. First, we derive analytic expressions of the closed-loop dynamics in terms of the surface (S_i), boundary layer (y_i) and estimate error ($\tilde{a}_i$) coordinates. We then obtain upper bounds of some of the time derivatives. Finally, stability results are established through a Lyapunov analysis utilizing these time derivative bounds.

4.2.1. *Closed-loop dynamics*. The closed-loop dynamics can be expressed as follows:

$$\dot{x}_1 = \dot{S}_1 \tag{39}$$

because $x_{1d} = \dot{x}_{1d} = 0$. For $i = 2, \ldots, n$,

$$\dot{x}_i = \dot{S}_i - \frac{y_i}{\tau_i} \tag{40}$$

For the surface errors,

$$\left.\begin{aligned}
\dot{S}_1 &= \dot{x}_1 \\
&= x_2 + a_1 f_1(x_1) \\
&= S_2 + x_{2d} + f_1(x_1) \\
&= S_2 + y_2 - K_1 S_1 + \tilde{a}_1 f_1(x_1) \\
&\vdots \\
\dot{S}_i &= \dot{x}_i - \dot{x}_{id} \\
&= x_{i+1} + f_i(x_1, \ldots, x_i) - \dot{x}_{id} \\
&= S_{i+1} + x_{i+1d} + f_i(x_1, \ldots, x_i) - \dot{x}_{id} \\
&= S_{i+1} + y_{i+1} - K_i S_i + \tilde{a}_{n-1} f_{n-1}(x_1, \ldots, x_{n-1}) \\
&\vdots \\
\dot{S}_n &= -K_n S_n + \tilde{a}_n f_n(x_1, \ldots, x_n)
\end{aligned}\right\} \tag{41}$$

For the boundary layers,

$$
\left.\begin{aligned}
\dot{y}_2 &= -\frac{y_2}{\tau_2} - \dot{\bar{x}}_2 \\
&\ \vdots \\
\dot{y}_i &= -\frac{y_i}{\tau_i} - \dot{\bar{x}}_i \\
&\ \vdots \\
\dot{y}_n &= -\frac{y_n}{\tau_n} - \dot{\bar{x}}_n
\end{aligned}\right\} \tag{42}
$$

For the estimate errors,

$$
\left.\begin{aligned}
\dot{\tilde{a}}_1 &= -\rho_1 S_1 f_1 \\
&\ \vdots \\
\dot{\tilde{a}}_i &= -\rho_i S_i f_i \\
&\ \vdots \\
\dot{\tilde{a}}_n &= -\rho_n S_n f_n
\end{aligned}\right\} \tag{43}
$$

4.2.2. *Upper bound estimates.* Here, we estimate upper bounds for $|\dot{\bar{x}}_i|$ and $|\dot{S}_i|$. We will show that an upper bound on $|\dot{S}_i|$ is given by:

$$
\begin{aligned}
|\dot{S}_i| \le |S_{i+1}| + |y_{i+1}| + (n_{ii} + K_i)|S_i| + n_{i,2i-1}|y_i| \\
+ n_{i1}|S_1| + \cdots + n_{i,i-1}|S_{i-1}| \\
+ n_{i,i+1}|y_2| + \cdots + n_{i,2i-2}|y_{i-1}|
\end{aligned} \tag{44}
$$

where the coefficients n_{ij} depend only on $K_1, \ldots, K_{i-1}$, and $\tau_2, \ldots, \tau_{i-1}$.

We will also show that an upper bound for $|\dot{\bar{x}}_{i+1}|$ is given by:

$$
\begin{aligned}
|\dot{\bar{x}}_{i+1}| \le k_{i+1,1}|S_i| + \cdots + k_{i+1,i+1}|S_{i+1}| \\
+ k_{i+1,i+2}|y_2| + \cdots + k_{i+1,2i+1}|y_{i+1}|
\end{aligned} \tag{45}
$$

where the coefficients $k_{i+1,j}$ are functions of $K_1, \ldots, K_i$, and $\tau_2, \ldots, \tau_i$.

Because of the tedious algebra involved, the derivations of these upper bound estimates are shown in Appendix A.

4.2.3. *Lyapunov stability analysis.* With bounds on the time derivatives $\dot{S}_i$ and $\dot{\bar{x}}_i$, we are ready to perform a Lyapunov analysis to study the stability properties of the closed-loop system. Consider the Lyapunov function,

$$
V = (S_1^2 + \cdots + S_n^2 + y_2^2 + \cdots + y_n^2 + \tilde{a}_1^2/\rho_1 + \cdots + \tilde{a}_n^2/\rho_n)/2 \tag{46}
$$

we get

$$
\dot{V} = S_1\dot{S}_1 + \cdots + S_n\dot{S}_n + y_2\dot{y}_2 + \cdots + y_n\dot{y}_n + \tilde{a}_1\dot{\tilde{a}}_1/\rho_1 + \cdots + \tilde{a}_n\dot{\tilde{a}}_n/\rho_n \tag{47}
$$

For $1 \le i \le n-1$,

$$S_i\dot{S}_i \leq -K_iS_i^2 + |S_i||y_{i+1}| + |S_i||S_{i+1}| + S_i\tilde{a}_if_i(x_1,\ldots,x_i) \tag{48}$$

where the coefficients n_{ij} depend only on $K_1,\ldots,K_{i-1}$, and $\tau_2,\ldots,\tau_{i-1}$. For $S_n\dot{S}_n$, we have

$$S_n\dot{S}_n = -K_nS_n^2 + S_n\tilde{a}_nf_n \tag{49}$$

For $i = 2,\ldots,n$,

$$\begin{aligned} y_i\dot{y}_i &= y_i(\dot{x}_{id} - \dot{\bar{x}}_i) \\ &\leq (-1/\tau_i + k_{i,2i-1})y_i^2 + \left(\sum_{j=1}^{i} k_{ij}|S_j| + \sum_{j=2}^{i-1} k_{i,i+j-1}|y_j|\right)|y_i| \end{aligned} \tag{50}$$

where the coefficients k_{ij} depend only on $K_1,\ldots,K_{i-1}$, and $\tau_2,\ldots,\tau_{i-1}$.

For $\tilde{a}_i\dot{\tilde{a}}_i/\rho_i$, we have

$$\frac{\tilde{a}_i\dot{\tilde{a}}_i}{\rho_i} = -\tilde{a}_iS_if_i, \quad i = 1,\ldots,n \tag{51}$$

which cancels out the $S_i\tilde{a}_if_i$ cross term in $S_i\dot{S}_i$.

For the surface gains, choose

$$K_i \geq 2.5, \quad i = 1,\ldots,n \tag{52}$$

Note that k_{2j} does not depend on τ_2, and $k_{ij}, i = 3,\ldots,n$, depend only on $\tau_2,\ldots,\tau_{i-1}$. Thus, the filter time constants can be chosen recursively as follows:

$$\frac{1}{\tau_2} \geq (k_{2,1}^2 + k_{2,2}^2)n/2 + 2 + k_{2,3} \tag{53}$$

$$\frac{1}{\tau_{i+1}} \geq \left(\sum_{j=1}^{2i} k_{i+1,j}^2\right)n/2 + 2 + k_{i+1,2i+1}, \quad i = 2,\ldots,n-1 \tag{54}$$

With the above choice of K_i and τ_i, and using the inequality $|ab| \leq (\epsilon a^2/2) + (b^2/2\epsilon)$, $(\epsilon > 0)$, we get $\dot{V} \leq -(S_1^2 + \cdots + S_n^2 + y_2^2 + \cdots + y_n^2) \leq 0$. Thus, $\dot{V}$ is negative semi-definite, and $V \leq p$ is indeed an invariant set. Barbalat's lemma can be used to show that $\dot{V} \to 0$ as $t \to \infty$. Thus, $S_i, y_i \to 0$ as $t \to \infty$. Hence $x_i \to 0$. Note that we only have the system state (x_i) convergence. There is no guarantee that the parameter estimates $(\hat{a}_i)$ will converge to the correct values.

The result of the above analysis on the closed-loop system regulating the output $y = x_1$ to 0 can be stated in the following theorem:

Theorem 1 (semi-global stability for adaptive DSC regulation): *Consider the non-linear system in parametric strict-feedback form described by* (19). *Given any nonlinearities and uncertainties that satisfy Assumption* 1, *for any particular initial state, there exists a set of surface gains* $K_1,\ldots,K_n$ *and filter time constants* $\tau_2,\ldots,\tau_n$ *such that the Adaptive Dynamic Surface Controller guarantees the convergence of the state* $[x_1 \cdots x_n]^T$ *to the origin.*

4.3. *Stability analysis for set point control*

The result we obtained in the previous section is rather limited in the sense that we can only regulate the output to one particular value, namely 0. For set-point control, we would like to maintain the output to any desired constant value. In that case, x_{1d} would be an arbitrary constant, but we still have $\dot{x}_{1d} = \ddot{x}_{1d} = 0$ identically. This objective can be accomplished if we modify the assumption we imposed on (19).

Assumption 2:

- *The map* $f_i : \mathcal{R}^i \to \mathcal{R}$ *is* C^1 *and* $f_i(\bar{x}_{1d}, \ldots, \bar{x}_{id}) \neq 0$, $i = 1, \ldots, n$, *where* $\bar{x}_{1d} = x_{1d}$, $\bar{x}_{j+1d} = -a_j f_j(\bar{x}_{1d}, \ldots, \bar{x}_{jd})$, $j = 1, \ldots, n-1$, *and* x_{1d} *is a constant.*
- $a_1, \ldots, a_n$ *are unknown constants.*

The definitions on the surface, boundary layer, and parameter estimate errors are the same as before:

$$
\begin{aligned}
y_i &:= x_{id} - \bar{x}_i \\
S_i &:= x_i - x_{id} \\
\tilde{a}_i &:= a_i - \hat{a}_i
\end{aligned}
$$

We will show that given any $p > 0$, there exist K_i and τ_i such that the closed-loop system is exponentially stable for all $\mathbf{x}(0) \in \{\mathbf{x} : \|\mathbf{x}\| \leq p\}$ when Assumption 2 and some other conditions are met for the functions f_i, $i = 1, \ldots, n$.

The closed-loop dynamics is almost the same as before. The only difference is that x_{1d} is no longer zero. For completeness, the closed-loop dynamics is repeated here again.

$$\dot{x}_1 = \dot{S}_1 \tag{55}$$

and

$$\dot{x}_i = \dot{S}_i - \frac{y_i}{\tau_i} \tag{56}$$

for $i = 2, \ldots, n$.

For the surface errors,

$$
\begin{aligned}
\dot{S}_1 &= \dot{x}_1 \\
&= x_2 + a_1 f_1(x_1) \\
&= S_2 + x_{2d} + f_1(x_1) \\
&= S_2 + y_2 - K_1 S_1 + \tilde{a}_1 f_1(x_1)
\end{aligned} \tag{57}
$$

$$\vdots$$

$$
\begin{aligned}
\dot{S}_i &= \dot{x}_i - \dot{x}_{id} \\
&= x_{i+1} + f_i(x_1, \ldots, x_i) - \dot{x}_{id} \\
&= S_{i+1} + x_{i+1d} + f_i(x_1, \ldots, x_i) - \dot{x}_{id} \\
&= S_{i+1} + y_{i+1} - K_i S_i + \tilde{a}_i f_i(x_1, \ldots, x_i)
\end{aligned} \tag{58}
$$

$$\vdots$$

$$\dot{S}_n = -K_n S_n + \tilde{a}_n f_n(x_1, \ldots, x_n) \tag{59}$$

For the boundary layers,

$$\left.\begin{aligned} \dot{y}_2 &= -\frac{y_2}{\tau_2} - \tilde{x}_2 \\ &\vdots \\ \dot{y}_i &= -\frac{y_i}{\tau_i} - \tilde{x}_i \\ &\vdots \\ \dot{y}_n &= -\frac{y_n}{\tau_n} - \tilde{x}_n \end{aligned}\right\} \tag{60}$$

For the estimate errors,

$$\left.\begin{aligned} \dot{\tilde{a}}_1 &= -\rho_1 S_1 f_1 \\ &\vdots \\ \dot{\tilde{a}}_i &= -\rho_i S_i f_i \\ &\vdots \\ \dot{\tilde{a}}_n &= -\rho_n S_n f_n \end{aligned}\right\} \tag{61}$$

Before diving into the stability analysis of the closed-loop system with set-point control, we would like to state without proof a stability theorem from Singular Perturbation Theory. This theorem is taken from Chapter 8 of Khalil (1992). It will be used later in the sequel.

Theorem 2 (stability theorem on singularly perturbed systems): *Consider the singularly perturbed system*

$$\dot{x} = f(t, x, z, \epsilon) \tag{62}$$

$$\epsilon \dot{z} = g(t, x, z, \epsilon) \tag{63}$$

Assume that the following assumptions are satisfied for all

$$(t, x, \epsilon) \in [0, \infty) \times B_r \times [0, \epsilon_0]$$

- $f(t, 0, 0, \epsilon) = 0$ *and* $g(t, 0, 0, \epsilon) = 0$.
- *The equation*

$$0 = g(t, x, z, 0)$$

  has an isolated root $x = h(t, z)$ *such that* $h(t, 0) = 0$.
- *The functions* f, g, *and* h *and their partial derivatives up to the second order are bounded for* $z - h(t, x) \in B_\rho$.
- *The origin of the reduced system*

$$\dot{x} = f(t, x, h(t, x), 0)$$

  is exponentially stable.
- *The origin of the boundary-layer system*

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = g(t, x, y + h(t, x), 0)$$

where $y = z - h(x)$ and $\tau = t/\epsilon$, is exponentially stable uniformly in (t, x). Then there exists $\epsilon^* > 0$ such that, for all $\epsilon < \epsilon^*$, the origin of (62)–(63) is exponentially stable.

Next, we also need the following lemma on semi-global exponential stability.

Lemma 1 (lemma on semi-global exponential stability): *Consider the autonomous system*

$$\dot{x} = f(x) \tag{64}$$

Let $x = 0$ be an equilibrium point of (64). If $x = 0$ is globally or semi-globally asymptotically stable and the Jacobian matrix $[\mathrm{d}f/\mathrm{d}x]$ is Hurwitz at the origin, then $x = 0$ is semi-globally exponentially stable.

The proof of the lemma is shown in Appendix B.

4.3.1. *Lyapunov stability analysis for the adaptive DSC regulator.* Here, we show that the origin of the closed-loop system in the S_i, y_i, and $\tilde{a}_i$ coordinates is exponentially stable through induction.

Consider the case $n = 2$, we have

$$\left.\begin{aligned}
\dot{S}_1 &= -K_1 S_1 + S_2 + y_2 + \tilde{a}_1 f_1(S_1 + x_{1d}) \\
\dot{S}_2 &= -K_2 S_2 + \tilde{a}_2 f_2(S_1 + x_{1d}, S_2 + y_2 + \bar{x}_2) \\
\dot{\tilde{a}}_1 &= -\rho_1 f_1(S_1 + x_{1d}) S_1 \\
\dot{\tilde{a}}_2 &= -\rho_2 f_2(S_1 + x_{1d}, S_2 + y_2 + \bar{x}_2) S_2 \\
\dot{y}_2 &= -y_2/\tau_2 + \left[\left(K_1 + \hat{a}_1 \frac{\mathrm{d}f_1}{\mathrm{d}x_1}\right)\dot{S}_1 + \rho_1 f_1^2 S_1\right]
\end{aligned}\right\} \tag{65}$$

Let $x = [S_1\ S_2\ \tilde{a}_1\ \tilde{a}_2]^T$, $z = y_2$, and $\epsilon = \tau_2$. Denote $\dot{x}$ by $f(x, z, \epsilon)$, and $\epsilon\dot{z}$ by $g(x, z, \epsilon)$. Note that $f(0, 0, \epsilon) = 0$, and $g(0, 0, \epsilon) = 0$. Solving $g(x, z, 0) = 0$ gives $z = h(x) = 0$. The origin of the boundary layer system $\mathrm{d}y/\mathrm{d}\tau = g(x, y + h(x), 0)$, where $y = z - h(x)$ and $\tau = t/\epsilon$, is obviously exponentially stable since $g(x, y + h(x), 0) = -y_2 = -y$. The origin of the reduced system $\dot{x} = f(x, h(x), 0)$ is globally asymptotically stable if $K_1 > 1$ and $K_2 > 1$. This can be shown using La-Salle's Invariance Theorem with the Lyapunov function candidate $V(x) = (1/2)(S_1^2 + S_2^2 + \tilde{a}_1^2/\rho_1 + \tilde{a}_2^2/\rho_2)$. In addition, if the Jacobian matrix $[\mathrm{d}f/\mathrm{d}x]$ at the origin is Hurwitz, then the origin of the reduced system is semi-globally exponentially stable from Lemma 1. In that case, we would need the following matrix to be Hurwitz.

$$A = \begin{bmatrix} -K_1 & 1 & a & 0 \\ 0 & -K_2 & 0 & b \\ -\rho_1 a & 0 & 0 & 0 \\ 0 & -\rho_2 b & 0 & 0 \end{bmatrix} \tag{66}$$

where $a = f_1(x_{1d})$, and $b = f_2(x_{1d}, -a_1 f_1(x_{1d}))$.

After setting up the two sub-systems and checking their stability properties, we can now invoke Theorem 2 to claim that there exists $\tau_2 > 0$ such that the closed-loop system for the adaptive DSC is exponentially stable. Thus, we have established exponential stability for the base case with $n = 2$.

Next, suppose the adaptive DSC is exponentially stable with $n = k$. Consider the case $n = k + 1$. The closed-loop dynamics are as follows:

$$\begin{aligned}
\dot{S}_1 &= -K_1 S_1 + S_2 + y_2 + \tilde{a}_1 f \\
&\vdots \\
\dot{S}_i &= -K_i S_i + S_{i+1} + y_{i+1} + \tilde{a}_i f_i \\
&\vdots \\
\dot{S}_{k+1} &= -K_{k+1} S_{k+1} + \tilde{a}_{k+1} f_{k+1} \\
\dot{\tilde{a}}_1 &= -\rho_1 f_1 S_1 \\
&\vdots \\
\dot{\tilde{a}}_{k+1} &= -\rho_{k+1} f_{k+1} S_{k+1} \\
\dot{y}_2 &= -\frac{y_2}{\tau_2} + \left[\left(K_1 + \hat{a}_1 \frac{\mathrm{d} f_1}{dx_1}\right)\dot{S}_1 + \rho_1 f_1^2 S_1\right] \\
\dot{y}_3 &= -\frac{y_3}{\tau_3} + \frac{\dot{y}_2}{\tau_2} + \left(K_2 + \hat{a}_2 \frac{\partial f_2}{\partial x_2}\right)\dot{S}_2 + \hat{a}_2 \frac{\partial f_2}{\partial x_1}\dot{S}_1 + \rho_2 f_2^2 S_2 - \hat{a}_2 \frac{\partial f_2}{\partial x_2}\frac{y_2}{\tau_2} \\
&\vdots \\
\dot{y}_{i+1} &= -\frac{y_{i+1}}{\tau_{i+1}} + \frac{\dot{y}_i}{\tau_i} + K_i \dot{S}_i + \rho_i f_i^2 S_i + \hat{a}_i \frac{\partial f_i}{\partial x_1}\dot{S}_1 + \sum_{j=2}^{i} \hat{a}_i \frac{\partial f_i}{\partial x_j}(\dot{S}_j - y_j/\tau_j) \\
&\vdots \\
\dot{y}_{k+1} &= -\frac{y_{k+1}}{\tau_{k+1}} + \frac{\dot{y}_k}{\tau_k} + K_k \dot{S}_k + \rho_k f_k^2 S_k + \hat{a}_k \frac{\partial f_k}{\partial x_1}\dot{S}_1 \\
&\quad + \sum_{j=2}^{k} \hat{a}_k \frac{\partial f_k}{\partial x_j}(\dot{S}_j - y_j/\tau_j)
\end{aligned} \tag{67}$$

Let

$$x := \left[S_1 \cdots S_k y_2 \cdots y_k \tilde{a}_1 \cdots \tilde{a}_k\right]^T \tag{68}$$

and

$$\dot{x} = p(x, S_{k+1}, y_{k+1}, \tilde{a}_{k+1}) = p_1(x) + p_2(S_{k+1}, y_{k+1}, \tilde{a}_{k+1}) \tag{69}$$

Note that $p(x, 0, 0, 0) = p_1(x)$ and

$$\begin{aligned}
p(x, S_{k+1}, y_{k+1}, \tilde{a}_{k+1}) - p(x, 0, 0, 0) &= p_2(S_{k+1}, y_{k+1}, \tilde{a}_{k+1}) \\
&= \left[0 \cdots 0 \ (S_{k+1} + y_{k+1}) \ 0 \cdots 0\right]^T
\end{aligned} \tag{70}$$

From the inductive assumption, $\dot{x} = p_1(x)$ is exponentially stable for some $\tau_i > 0$, $i = 1, \ldots, k$. Thus, there exists $V(x)$ such that

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$
$$\frac{\mathrm{d}V}{\mathrm{d}x}p(x,0,0,0) \le -c_3\|x\|^2$$
$$\left\|\frac{\mathrm{d}V}{\mathrm{d}x}\right\| \le c_4\|x\|$$

Let $g(x, y_{k+1}, S_{k+1}, \tau_{k+1}) = \tau_{k+1}\dot{y}_{k+1}$. Note that $g(0,0,0,\tau_{k+1}) = 0$, and it is Lipschitz in τ_{k+1} linearly in the state (x, y_{k+1}, S_{k+1}). In particular,

$$\|g(x, y_{k+1}, S_{k+1}, \tau_{k+1}) - g(x, y_{k+1}, S_{k+1}, 0)\| \le \tau_{k+1}L_1(\|x\| + \|y_{k+1}\| + \|S_{k+1}\|) \tag{71}$$

After some algebraic manipulation, we can get $g(x, y_{k+1}, S_{k+1}, 0) = -y_{k+1}$.

Let

$$W(y_{k+1}) := \tfrac{1}{2}y_{k+1}^2 \tag{72}$$
$$Z(S_{k+1}, \tilde{a}_{k+1}) := \tfrac{1}{2}(S_{k+1}^2 + \tilde{a}_{k+1}^2/\rho_{k+1}) \tag{73}$$

We are going to use

$$U(x, y_{k+1}, S_{k+1}, \tilde{a}_{k+1}) := V(x) + W(y_{k+1}) + Z(S_{k+1}, \tilde{a}_{k+1}) \tag{74}$$

as a Lyapunov function candidate. Hence

$$\begin{aligned}
\dot{U} &= \dot{V} + \dot{W} + \dot{Z}\\
&= \frac{\mathrm{d}V}{\mathrm{d}x}p(x, S_{k+1}, y_{k+1}, \tilde{a}_{k+1}) + \frac{\mathrm{d}W}{\mathrm{d}y_{k+1}}\frac{1}{\tau_{k+1}}g(x, y_{k+1}, \tau_{k+1}) - K_{k+1}S_{k+1}^2\\
&= \frac{\mathrm{d}V}{\mathrm{d}x}(p(x,0,0,0) + p(x, S_{k+1}, y_{k+1}, \tilde{a}_{k+1}) - p(x,0,0,0))\\
&\quad + \frac{\mathrm{d}W}{\mathrm{d}y_{k+1}}\frac{1}{\tau_{k+1}}g(x, y_{k+1}, \tau_{k+1}) - K_{k+1}S_{k+1}^2\\
&\le -c_3\|x\|^2 + c_4(\|S_{k+1}\| + \|y_{k+1}\|)\|x\|\\
&\quad + \frac{\mathrm{d}W}{\mathrm{d}y_{k+1}}\frac{1}{\tau_{k+1}}g(x, y_{k+1}, \tau_{k+1}) - K_{k+1}S_{k+1}^2\\
&\le -c_3\|x\|^2 + c_4\|S_{k+1}\|\|x\| + c_4\|y_{k+1}\|\|x\| - K_{k+1}S_{k+1}^2\\
&\quad + \frac{y_{k+1}}{\tau_{k+1}}(g(x, y_{k+1}, S_{k+1}, 0) + g(x, y_{k+1}, S_{k+1}, \tau_{k+1}) - g(x, y_{k+1}, S_{k+1}, 0))\\
&\le -c_3\|x\|^2 + c_4\|S_{k+1}\|\|x\| + c_4\|y_{k+1}\|\|x\| - K_{k+1}S_{k+1}^2\\
&\quad - \frac{\|y_{k+1}\|^2}{\tau_{k+1}} + \frac{\|y_{k+1}\|}{\tau_{k+1}}(\tau_{k+1}L_1(\|x\| + \|y_{k+1}\| + \|S_{k+1}\|))\\
&\le -b_1\|x\|^2 + b_2\|S_{k+1}\|\|x\| + b_3\|y_{k+1}\|\|x\| + b_4\|y_{k+1}\|\|S_{k+1}\| + b_5\|y_{k+1}\|^2\\
&\quad - \frac{\|y_{k+1}\|^2}{\tau_{k+1}} - K_{k+1}S_{k+1}^2
\end{aligned} \tag{75}$$

The non-negative definite cross terms can be dominated by the negative definite terms by choosing τ_{k+1} small enough and K_{k+1} large enough. Thus, we get $\dot{U} \le 0$. Using LaSalle's Invariance Theorem, we can prove asymptotic stability with $\left[x^{\mathrm{T}}\ y_{k+1}\ S_{k+1}\ \tilde{a}_{k+1}\right]^{\mathrm{T}} \to 0$. In addition, if the Jacobian matrix of the closed-loop system is Hurwitz at the origin, we get semi-global exponential stability from Lemma 1 and the induction process continues. Note that unlike the previous case of regulating the system to the origin, we also get convergence of the parameter estimates here. We now summarize the stability analysis of the adaptive DSC set-point control problem in the following theorem.

Theorem 3 (semi-global exponential stability for set-point control): *Consider the nonlinear system in parametric strict-feedback form described by* (19). *Given any nonlinearities and uncertainties that satisfy Assumption* 2, *if the Jacobian matrix of the closed-loop subsystem, with* $k = 0, \ldots, n-1$, *in* (67) *is Hurwitz at the origin, then there exists a set of surface gains* $K_1, \ldots, K_n$ *and filter time constants* $\tau_2, \ldots, \tau_n$ *such that the Adaptive Dynamic Surface Controller guarantees semi-global exponential stability for set-point control of the output* $y = x_1$ *to any desired constant value* x_{1d}. *In addition, the parameter estimates* $\hat{a}_i$ *are also guaranteed to converge to the correct values.*

5. Numerical example

We consider the third order example of equations (1) with,

$$f_1(x_1) = x_1^3$$
$$f_2(x_1, x_2) = x_1^2 + x_2^2$$

The values for the unknown constants are

$$a_1 = a_2 = 1$$

The adaptive dynamic surface controller was implemented with

$$\hat{a}_1(0) = 0$$
$$\hat{a}_2(0) = 0$$
$$\rho_1 = 1$$
$$\rho_2 = 0.01$$

The time constants are chosen to be

$$\tau_2 = \tau_3 = 0.001$$

The control gains are selected to be

$$K_1 = K_2 = K_3 = 10$$

Simulation results are shown in figures 1–3 regulating the output x_1 to 0 with initial conditions:

$$x_1(0) = 2$$
$$x_2(0) = x_3(0) = 0$$

Figure 1 shows that the output x_1 converges nicely to 0. Looking at figure 2, we notice that the estimates for a_1 and a_2 do not converge to the correct values. The control effort is shown in figure 3. The large control effort is probably due to the structure of the plant, with the strong linearity x_1^3 in $\dot{x}_1$ and that the control u is separated from the output by two levels of integration. If we are not careful in controlling the growth of x_1, the x_1^3 term in $\dot{x}_1$ can actually drive the system to infinity in finite time.

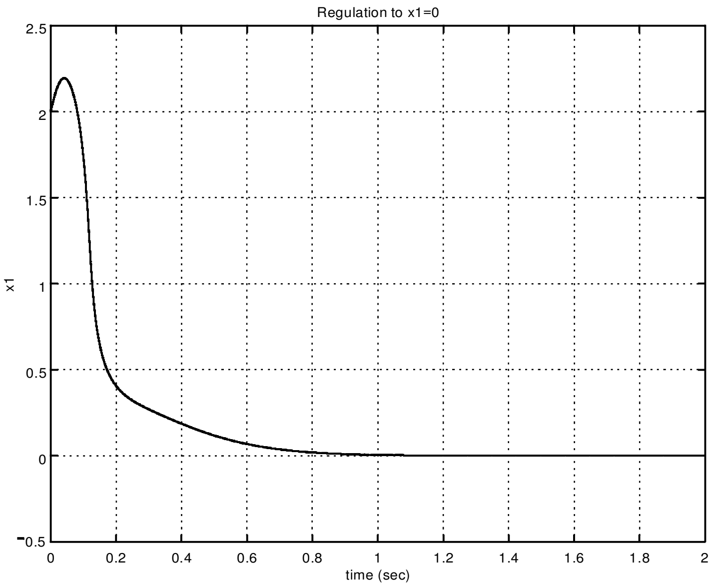

Figure 1. Regulation of output x_1 to 0: output vs. time.

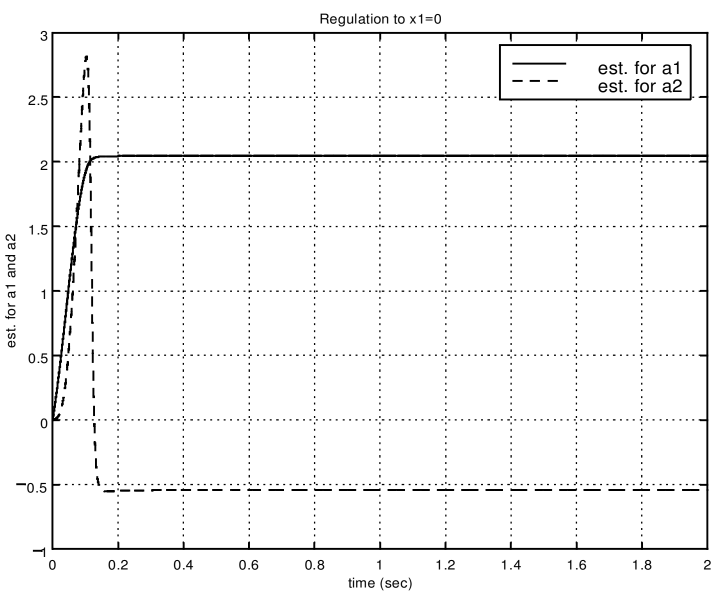

Figure 2. Regulation of output x_1 to 0: parameter estimates vs. time.

Another simulation is performed for set-point control regulating the output x_1 to 1. The initial conditions are set to be

$$x_1(0) = 1.5$$

$$x_2(0) = x_3(0) = 0$$

A different set of gains is used for the parameter update law:

$$\rho_1 = 10$$
$$\rho_2 = 1$$

The simulation results are shown in figures 4–6. Figure 4 shows that the output x_1 converges to the desired value of 1. Figure 5 shows that the estimates for a_1 and a_2

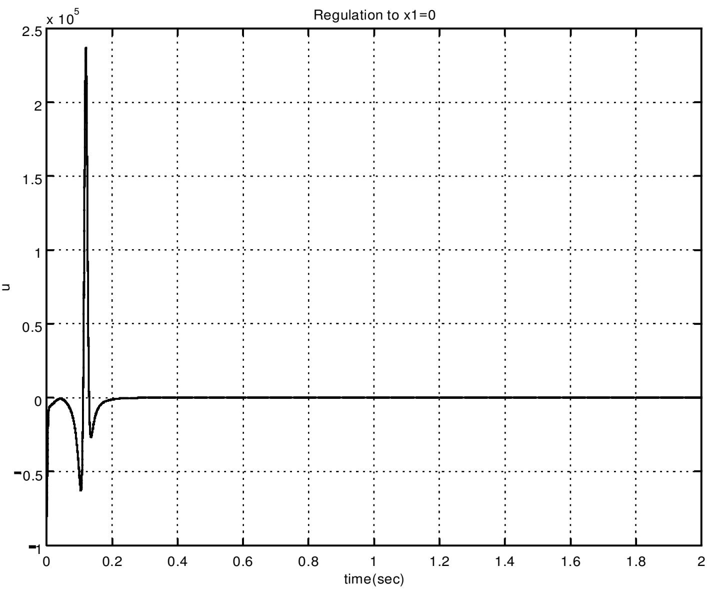

Figure 3. Regulation of output x_1 to 0: control vs. time.

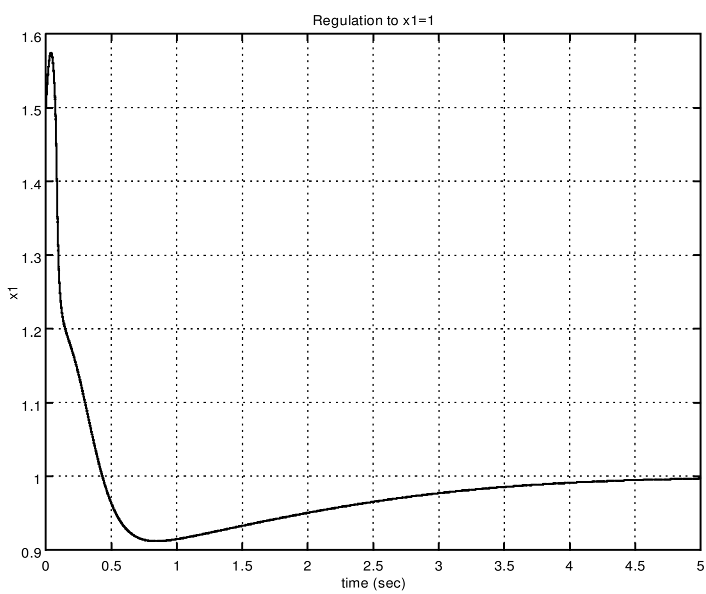

Figure 4. Set-point control of output x_1 to 1: output vs. time.

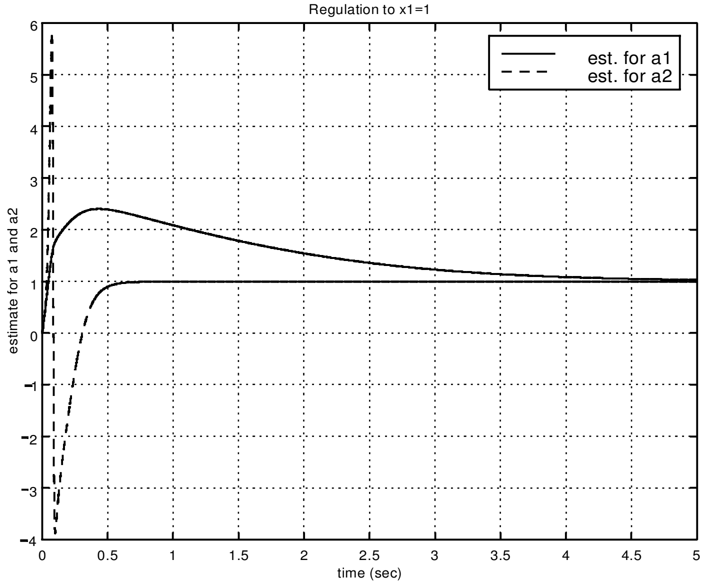

Figure 5. Set-point control of output x_1 to 1: parameter estimates vs. time.

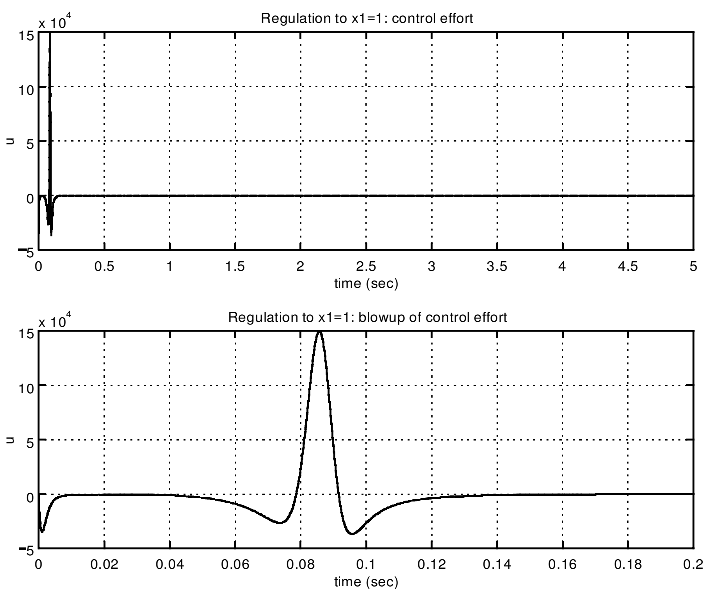

Figure 6. Set-point control of output x_1 to 1: control vs. time.

actually converge to the correct value. The corresponding control effort is shown in figure 6. The lower portion of figure 6 is a blowup of the control effort during the first 0.2 second. The large control effort can also be attributed to the structure of the model as mentioned earlier.

6. Conclusions

A simplified adaptive backstepping algorithm (adaptive dynamic surface control) has been introduced which greatly simplifies the amount of computations required. In particular, the addition of n first order filters has eliminated the need to differentiate the model. Semi-global stability results have been obtained using a singular perturbation approach for the cases of regulation to the origin and set-point control. Further research is needed to establish stability results for tracking control.

Appendix A. Derivations of upper bound estimates

Here, we estimate upper bounds for $|\dot{\bar{x}}_i|$ and $|\dot{S}_i|$. We will show that an upper bound on $|\dot{S}_i|$ is given by:

$$\begin{aligned}|\dot{S}_i| \le |S_{i+1}| + |y_{i+1}| + (n_{ii} + K_i)|S_i| + n_{i,2i-1}|y_i| \\ + n_{i1}|S_1| + \cdots + n_{i,i-1}|S_{i-1}| \\ + n_{i,i+1}|y_2| + \cdots + n_{i,2i-2}|y_{i-1}|\end{aligned}$$

where the coefficients n_{ij} depend only on $K_1, \ldots, K_{i-1}$, and $\tau_2, \ldots, \tau_{i-1}$.

We will also show that an upper bound for $|\dot{\bar{x}}_{i+1}|$ is given by:

$$\begin{aligned}|\dot{\bar{x}}_{i+1}| \le k_{i+1,1}|S_i| + \cdots + k_{i+1,i+1}|S_{i+1}| \\ + k_{i+1,i+2}|y_2| + \cdots + k_{i+1,2i+1}|y_{i+1}|\end{aligned}$$

The coefficients $k_{i+1,j}$ are functions of $K_1, \ldots, K_i$, and $\tau_2, \ldots, \tau_i$.

Given $p > 0$. Let q_{ij} denote $\max_{V \le p} |(\partial f_i / \partial x_j)|$. Then, we have

$$|f_i| \le \sum_{j=1}^{i} q_{ij}|x_j| \tag{76}$$

Note that if $V \le p$ is an invariant set, we can bound $|\hat{a}_i|$, $|\tilde{a}_i|$, and $|f_i|$ by known constants. From (22),

$$|\bar{x}_2| \le K_1|S_1| + |\hat{a}_1||f_1(x_1)|$$

Substituting the bound on $|f_1|$ from (76), we get

$$\begin{aligned}|\bar{x}_2| &\le (q_{11}|\hat{a}_1| + K_1)|S_1| \\ &= e_{21}|S_1|\end{aligned} \tag{77}$$

where e_{21} is a positive constant bounding $q_{11}|\hat{a}_1| + K_1$.

From (33), and (32), $x_2 = S_2 + y_2 + \bar{x}_2$. Hence $|x_2| \le |S_2| + |y_2| + |\bar{x}_2|$. Substituting the bound for $|\bar{x}_2|$ from above, we get

$$|x_2| \le e_{21}|S_1| + |S_2| + |y_2| \tag{78}$$

From the closed-loop dynamics presented in §4.2.1, $|\dot{S}_1|$ is bounded by $|S_2| + |y_2| + K_1|S_1| + |\tilde{a}_1||f_1(x_1)|$. Bounding $|f_1|$ by $q_{11}|S_1|$, we obtain the following bound for $\dot{S}_1$,

$$\begin{aligned}|\dot{S}_1| &\le (q_{11}|\tilde{a}_1| + K_1)|S_1| + |S_2| + |y_2| \\ &\le (K_1 + n_{11})|S_1| + |S_2| + |y_2|\end{aligned} \tag{79}$$

where n_{11} is a positive constant bounding $q_{11}|\tilde{a}_1|$.

Next, using $\bar{x}_2 = -\hat{a}_1 f_1(x_1) - K_1 S_1$, an upper bound for $|\dot{\bar{x}}_2|$ is obtained:

$$|\dot{\bar{x}}_2| \leq K_1|\dot{S}_1| + |\dot{\hat{a}}_1||f_1(x_1)| + |\hat{a}_1|\left|\frac{\mathrm{d}f_1}{\mathrm{d}x_1}\right||\dot{x}_1|$$
$$\leq (K_1 + |\hat{a}_1|q_{11})|\dot{S}_1| + \rho_1 f_1^2|S_1|$$

From inequality (79), we get

$$|\dot{\bar{x}}_2| \leq (K_1 + |\hat{a}_1|q_{11})((K_1 + n_{11})|S_1| + |S_2| + |y_2|) + \rho_1 f_1^2|S_1|$$
$$\leq k_{21}|S_1| + k_{22}|S_2| + k_{23}|y_2| \tag{80}$$

where k_{21} is a positive constant bounding $(e_{21}(K_1 + n_{11}) + \rho_1 f_1^2)$ and $k_{22} = k_{23} = e_{21}$.

From Assumption 1,

$$|f_2(x_1, x_2)| \leq q_{21}|x_1| + q_{22}|x_2|$$
$$\leq (q_{21} + e_{21})|S_1| + q_{22}|S_2| + q_{22}|y_2| \tag{81}$$

Since $\bar{x}_3 = -K_2S_2 - \hat{a}_2 f_2 - y_2/\tau_2$, we have

$$|\bar{x}_3| \leq K_2|S_2| + |\hat{a}_2||f_2(x_1, x_2)| + |y_2/\tau_2|$$
$$\leq K_2|S_2| + |\hat{a}_2|((q_{21} + e_{21})|S_1| + q_{22}|S_2| + q_{22}|y_2|) + |y_2/\tau_2|$$
$$\leq e_{31}|S_1| + e_{32}|S_2| + e_{33}|y_2| \tag{82}$$

where e_{31}, k_{22}, and k_{33} are positive constants bounding $(q_{21} + e_{21})|\hat{a}_2|$, $K_2 + q_{22}|\hat{a}_2|$, and $q_{22}|\hat{a}_2| + 1/\tau_2$, respectively.

Since $x_3 = S_3 + y_3 + \bar{x}_3$, we get

$$|x_3| \leq |S_3| + |y_3| + |\bar{x}_3|$$
$$\leq |S_3| + |y_3| + e_{31}|S_1| + e_{32}|S_2| + e_{33}|y_2| \tag{83}$$

From the closed-loop dynamics presented in §4.2.1, $|\dot{S}_2|$ is bounded by $|S_3| + |y_3| + K_2|S_2| + |\tilde{a}_2||f_2(x_1, x_2)|$. Substituting the bound of $|f_2|$ from (81), we obtain the following bound for $\dot{S}_2$,

$$|\dot{S}_2| \leq |S_3| + |y_3| + K_2|S_2| + |\tilde{a}_2||f_2(x_1, x_2)|$$
$$\leq n_{21}|S_1| + (n_{22} + K_2)|S_2| + |S_3| + n_{23}|y_2| + |y_3| \tag{84}$$

where n_{21} is a positive constant bounding $(q_{21} + e_{21})|\tilde{a}_2|$, n_{22} and n_{23} are positive constants bounding $q_{22}|\tilde{a}_2|$.

Using $\bar{x}_3 = -K_2S_2 - \hat{a}_2 f_2(x_1, x_2) - y_2/\tau_2$, $|\dot{\bar{x}}_3|$ can be bounded by:

$$|\dot{\bar{x}}_3| \leq K_2|\dot{S}_2| + |\dot{\hat{a}}_2||f_2| + |\hat{a}_2|\left(\left|\frac{\partial f_2}{\partial x_1}\right||\dot{S}_1| + \left|\frac{\partial f_2}{\partial x_2}\right|(|\dot{S}_2| + |y_2/\tau_2|)\right)$$
$$+ |\dot{y}_2/\tau_2|$$
$$= K_2|\dot{S}_2| + \frac{|f_2^2|}{\rho_2}|S_2| + |\hat{a}_2|(q_{21}|\dot{S}_1| + q_{22}(|\dot{S}_2| + |y_2/\tau_2|))$$
$$+ |y_2|/\tau_2^2 + |\dot{\bar{x}}_2|/\tau_2$$
$$\leq k_{31}|S_1| + k_{32}|S_2| + k_{33}|S_3| + k_{34}|y_2| + k_{35}|y_3| \tag{85}$$

Note that the coefficients in inequalities (84) and (85), bounding $|\dot{S}_2|$ and $|\dot{\bar{x}}_3|$, depend only on K_1, K_2, and τ_2.

The bounds for the variables associated with the ith surface are given recursively as,

$$
\begin{aligned}
|f_i(x_1,\ldots,x_n)| &\le q_{i1}|x_1| + \cdots + q_{in}|x_n| \\
&\le q_{i1}|S_1| + q_{i2}(|S_2| + |y_2| + |\bar{x}_2|) + \cdots \\
&\quad + q_{ii}(|S_i| + |y_i| + |\bar{x}_i|) \\
&\le r_{i,1}|S_1| + \cdots + r_{i,i}|S_i| + r_{i,i+1}|y_2| + \cdots + r_{i,2i-1}|y_i| \qquad (86)\\
|\bar{x}_{i+1}| &\le K_i|S_i| + |\hat{a}_i||f_i| + |y_i/\tau_i| \\
&\le e_{i+1,1}|S_1| + \cdots + e_{i+1,i}|S_i| \\
&\quad + e_{i+1,i+1}|y_2| + \cdots + e_{i+1,2i-1}|y_i| \qquad (87)
\end{aligned}
$$

Arguing inductively, one can conclude that the coefficients $e_{i+1,j}$ depend only on the gains $K_1,\ldots,K_i$, and the filter time constants $\tau_2,\ldots,\tau_i$.

Similarly, the bound on $|\dot{S}_i|$ is given by

$$
\begin{aligned}
|\dot{S}_i| &\le |S_{i+1}| + |y_{i+1}| + K_i|S_i| + |\tilde{a}_i||f_i| \\
&\le |S_{i+1}| + |y_{i+1}| + (n_{ii} + K_i)|S_i| + n_{i,2i-1}|y_i| \\
&\quad + n_{i1}|S_1| + \cdots + n_{i,i-1}|S_{i-1}| \\
&\quad + n_{i,i+1}|y_2| + \cdots + n_{i,2i-2}|y_{i-1}| \qquad (88)
\end{aligned}
$$

The coefficients n_{ij} depend only on $K_1,\ldots,K_{i-1}$, and $\tau_2,\ldots,\tau_{i-1}$.

Upper bound for $|\dot{\bar{x}}_{i+1}|$ is given by:

$$
\begin{aligned}
|\dot{\bar{x}}_{i+1}| &\le K_i|\dot{S}_i| + |\hat{a}_i|\left(\left|\frac{\partial f_i}{\partial x_1}\right||\dot{S}_1| + \sum_{j=2}^{i}\left|\frac{\partial f_i}{\partial x_j}\right|(|\dot{S}_j| + |y_j/\tau_j|)\right) \\
&\quad + |\dot{\hat{a}}_i||f_i| + |\dot{y}_i/\tau_i| \\
&\le k_{i+1,1}|S_i| + \cdots + k_{i+1,i+1}|S_{i+1}| \\
&\quad + k_{i+1,i+2}|y_2| + \cdots + k_{i+1,2i+1}|y_{i+1}| \qquad (89)
\end{aligned}
$$

The coefficients $k_{i+1,j}$ are functions of $K_1,\ldots,K_i$, and $\tau_2,\ldots,\tau_i$.

Appendix B. Proof of Lemma 1

Since the Jacobian matrix $[\mathrm{d}f/\mathrm{d}x]$ is Hurwitz at the origin, $x = 0$ is locally exponentially stable. Thus, there exists $\epsilon > 0$, $k_1 > 0$, $\gamma_1 > 0$, such that

$$\|x(t)\| \le k_1\|x(t_1)\|e^{-\gamma_1(t-t_1)}, \qquad \forall t_1 \ge 0, \quad t \ge t_1, \quad \|x(t_1)\| \le \epsilon$$

Given $x(t=0) = x_0$. Since $x = 0$ is globally/semi-globally asymptotically stable, there exists $\tau \ge 0$, such that

$$\|x(t)\| \le \epsilon/2, \qquad \forall t \ge \tau$$

Let $\|x(\tau)\| = c_1$. Then, $\forall t \ge \tau$,

$$\|x(t)\| \le c_1 k_1 \ \mathrm{e}^{-\gamma_1(t-\tau)}$$

Since $c_1 \le \epsilon/2$, we have

$$\|x(t)\| \le \frac{\epsilon}{2}(k_1 \ \mathrm{e}^{\gamma_1\tau}) \ \mathrm{e}^{-\gamma_1 t}$$

Let

$$k_2 = \max_{0 \le t \le \tau} \frac{\|x(t)\|}{\|x(0)\| \ \mathrm{e}^{-\gamma_1 t}}$$

Then, for $0 \le t \le \tau$,

$$k_2 \|x(0)\| \ \mathrm{e}^{-\gamma_1 t} \ge \|x(t)\|$$

Let

$$k = \max\left(k_2, \frac{\epsilon k_1 \, \mathrm{e}^{\gamma_1 \tau}}{2\|x(0)\|}\right)$$

Thus $K\|x(0)\| \, \mathrm{e}^{-\gamma_1 t} \ge \frac{1}{2}\epsilon k_1 \, \mathrm{e}^{\gamma_1 \tau} \, \mathrm{e}^{-\gamma_1 t}$. Hence, for $t \ge \tau$,

$$\begin{aligned} \|x(t)\| &\le \tfrac{1}{2}\epsilon k_1 \, \mathrm{e}^{-\gamma_1 (t-\tau)} \\ &\le k\|x(0)\| \, \mathrm{e}^{-\gamma_1 t} \end{aligned}$$

Since $k \ge k_2$, we have

$$k\|x(0)\| \ \mathrm{e}^{-\gamma_1 t} \ge k_2 \|x(0)\| \ \mathrm{e}^{-\gamma_1 t}$$

But $k_2 \|x(0)\| \ \mathrm{e}^{-\gamma_1 t} \ge \|x(t)\|$ for $0 \le t \le \tau$. Hence, $k\|x(0)\| \ \mathrm{e}^{-\gamma_1 t} \ge \|x(t)\|$ for $0 \le t \le \tau$. Combining the two time intervals, we get

$$\|x(t)\| \le k\|x(0)\| \ \mathrm{e}^{-\gamma_1 t}, \qquad \forall t \ge 0 \qquad \square$$

References

Kanellakapoulous, I., Kokotovic, P., and Morse, A. S., 1991, Systematic design of adaptive controllers for feedback linearizable systems. *IEEE Transactions on Automatic Control*, **36**, 1241–1253.

Khalil, H. K., 1992, *Nonlinear Systems* (New York: Macmillan).

Kokotovic, P., Khalil, H. K., and O'Reilly, J., 1986, *Singular Perturbation Methods in Control: Analysis and Design* (London: Academic Press).

Krstic, M., Kanellakapoulous, I., and Kokotovic, P., 1995, *Nonlinear and Adaptive Control Design* (New York: Wiley Interscience).

Sastry, S., and Isidori, A., 1989, Adaptive control of linearizable systems. *IEEE Transactions on Automatic Control*, Vol. 34, pp. 1123–1131.

Slotine, J. J., and Li, W., 1991, *Applied Nonlinear Control* (Englewood Cliffs, NJ: Prentice Hall).

Swaroop, D., Gerdes, J. C., Yip, P. P., and Hedrick, J. K., 1997, Dynamic surface control of nonlinear systems. *Proceedings of the 1997 American Control Conference*, Albuquerque, NM.