A Switching Adaptive Controller for Feedback Linearizable Systems

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***Abstract*— One of the main open problems in the area of adaptive control of linear-in-the-parameters feedback linearizable systems is the computation of the feedback control law when the identification model becomes uncontrollable. In this paper, the authors propose a switching adaptive control strategy that overcomes this problem. The proposed strategy is applied to nth-order feedback linearizable systems in canonical form. The closed-loop system is proved to be globally stable in the sense that all the closed-loop signals are bounded and the tracking error converges arbitrarily close to zero. No assumptions are made about the type of nonlinearities of the system, except that such nonlinearities are smooth. However, the proposed controller requires knowledge of the sign and lower bound of the input vector field.**

***Index Terms*—Adaptive estimation, nonlinear systems, switching control.**

I. Introduction

A significant problem that arises in adaptive control of linear-in-the-parameters feedback linearizable systems is the computation of the feedback control law when the identification model becomes uncontrollable although the actual system is controllable; so far, there is no known solution for overcoming such a problem. For instance, consider the simple scalar system

$$\dot{x} = \theta^\tau f(x) + \theta^\tau g(x)u \tag{1}$$

where x, u are the scalar state and input of the system, θ is a vector of unknown parameters, and $f(\cdot), g(\cdot)$ are smooth vector functions, and, moreover, $|\theta^\tau g(x)| > 0$ for all x, i.e., system (1) is feedback linearizable and, thus, controllable. If $\hat{\theta}(t)$ denotes the estimate of θ at time t, the parameter estimation techniques used in adaptive control cannot guarantee, in general, that $|\hat{\theta}(t)^\tau g(x(t))| > 0$ for each time t, that is, they cannot guarantee that the identification model is controllable. Another example is the system of the form (parametric-pure-form system)

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \theta^T f_i(x_1, \cdots, x_{i+1}), \qquad 1 \le i \le n-1 \\ \dot{x}_n &= \theta^T f_n(x) + [\theta^T g(x) + g_0(x)]u \end{aligned} \tag{2}$$

where θ is the vector of the unknown parameters; the procedures proposed in [8], [18], and [11] are applicable if both $|\theta^T g(x) + g_0(x)| > 0$ and $|\hat{\theta}^T g(x) + g_0(x)| > 0$, where $\hat{\theta}$ denotes the estimate of θ; moreover, these procedures guarantee global stability only in the case where the input vector field $\theta^T g(x) + g_0(x)$ is independent of θ, i.e., in the case where $g(x) = 0$ and the functions $f_i(\cdot)$ are independent of x_{i+1}. Similar restrictions are made in many other works (see, e.g., [17], [2], and [7]). Such restrictions are made due to the fact that the computation of the adaptive control law depends on the existence of the inverse of the matrix that consists of the estimated input vector fields (or the Lie derivatives of the output functions along those vector fields[1]). Even in the case of known parameters where the inverse of the corresponding matrix exists (this is trivially satisfied for feedback linearizable systems) the inverse of the estimate of this matrix might not exist at each time due to insufficiently rich regressor signals, large initial parameter estimation errors, etc. In fact, when the estimated decoupling matrix becomes noninvertible the identification model becomes uncontrollable, and thus the certainty equivalence controller cannot be applied.

The problem of loss of controllability of the identification model even though the actual plant is controllable appears also in linear systems and several solutions [5], [13], [14], [19] have been proposed that overcome such a problem. Most of these solutions are based on switching and persistence of excitation in order to guarantee the computability of the feedback adaptive control law [5]. These methods cannot be extended to nonlinear systems where the persistence of excitation of the regressor vector cannot be guaranteed by the use of rich external reference signals. The cyclic switching strategies used in [13], [14], and [19] avoid the use of excitation and overcome the problem of uncontrollability of the identification model at certain instants of time. These strategies exploit the linear properties of the plant and it is not clear how to extend them to the nonlinear case. In [1] a switching strategy used in the linear case is extended to a first-order nonlinear plant. Global stability is established under the assumption the nonlinearities of the system satisfy certain sector-boundedness conditions.

In this paper, we propose and analyze an adaptive control scheme with switching that completely overcomes the problem of computability of the control law. The switching takes place between two different control laws, the standard Certainty Equivalent Feedback Linearizing (CEFL) control and a new control law referred to as the Adaptive Derivative Feedback

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[1] This is the so-defined *decoupling matrix* $A(x)$ [3], [17]; the ijth entry of this matrix is given by $L_{g_i}(L_f^{\gamma_j - 1} h_j)$, where $L_{(\cdot)}(\cdot)$ denotes the Lie derivative, h is the output function, and γ_i is the so-defined relative degree.

(ADF) control. The proposed strategy is applied to an nth-order feedback linearizable system in canonical form. The closed-loop system is shown to be globally stable in the sense that all the closed-loop signals are bounded and the tracking error converges arbitrarily close to zero. No assumptions are made about the type of nonlinearities of the system, except that such nonlinearities are smooth. However, the proposed controller requires knowledge of the sign of the input vector field and its lower bounds.

The drawback of the proposed scheme is that it does not guarantee zero residual tracking errors. Furthermore, the controller may exhibit high-gain behavior with discontinuities. Therefore, stability and performance is traded-off with the possibility of having large but bounded control inputs that may also have a high-frequency content. The use of more severe high-gain controllers to handle nonlinearities and/or parametric uncertainties can be found in [9], [16], and [12].

Finally, we mention that in [10], the reader can find a solution to the universal stabilization of nonlinear systems; in that work the switching strategy proposed in this paper is appropriately combined with Control Lyapunov Function techniques and neural networks in order to solve the universal stabilization problem.

A. Notations and Preliminaries

$|\cdot|$ denotes the standard Euclidean vector norm; when x is a scalar $|x|$ denotes its absolute value. $\mathcal{L}_2$ denotes the space of all square integrable functions of time; $\mathcal{L}_\infty$ denotes the space of all bounded functions of time. If $f_K(\cdot)$ is a function parameterized by the constant real K, we will say that $f_K(\cdot)$ is $\mathcal{O}(c(K))$ where $c(\cdot)$ is a positive function, if for every x and every K there exists a nonnegative real c such that $f_K(x) \le c(K)$. If $\mathcal{S}_1, \mathcal{S}_2$ are two subsets of $\Re^n$, then $\partial\mathcal{S}_1$ denotes the boundary of $\mathcal{S}_1$ and when $\mathcal{S}_2 \subset \mathcal{S}_1$, $\mathcal{S}_1/\mathcal{S}_2$ denotes the set of all $x \in \mathcal{S}_1$ satisfying $x \notin \mathcal{S}_2$.

II. Switching Adaptive Control of Feedback Linearizable Systems in Canonical Form

Consider an nth-order single-input single-output (SISO) feedback linearizable system in canonical form, whose dynamics are as follows:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_n &= f(x_1, x_2, \cdots, x_n) + g(x_1, x_2, \cdots, x_n)u\\ y &= x_1\end{aligned} \tag{3}$$

where $y, u \in \Re$ are the scalar system output and input, respectively, f, g, are smooth vector fields, and $x \triangleq [x_1, x_2, \cdots, x_n]^\tau$ is the state vector of the system. In order for system (3) to be controllable and feedback linearizable we assume that:

A1) A lower bound ϵ^* for $g(x)$, i.e., $|g(x)| > \epsilon^* > 0, \forall x \in \Re^n$, and the sign of $g(x)$ are known.

The control objective is to find the control input u that guarantees signal boundedness and forces y to follow the output y_m of the reference model

$$\begin{aligned}\dot{x}_m &= Ax_m + br\\ y_m &= c^\tau x_m\end{aligned} \tag{4}$$

where A is a Hurwitz $n \times n$ matrix, $r \in \mathcal{L}_\infty$ and therefore, $x_m \in \mathcal{L}_\infty$. In order to have a well-posed problem, it is assumed that the relative degree of the reference model is equal to n, which in turn implies that [6]

$$c^\tau b = c^\tau Ab = \cdots = c^\tau A^{n-2} b = 0. \tag{5}$$

If $e \triangleq y_m - y$ is the tracking error, then its nth time derivative satisfies

$$\begin{aligned}e^{(n)} &= y_m^{(n)} - y^{(n)}\\ &= c^\tau A^n x_m + c^\tau A^{n-1} br - f(x) - g(x)u.\end{aligned} \tag{6}$$

It is not difficult to see that

$$e^{(i)} = c^\tau A^i x_m - x_{i+1}, \qquad i = 0, 1, \cdots, n-1.$$

Let $\overline{h}(s) = s^n + k_1 s^{n-1} + \cdots + k_n$ be a Hurwitz polynomial (here s denotes the d/dt operator). Also let $\varepsilon \triangleq [e, \dot{e}, \cdots, e^{(n-1)}]^\tau$. Under Assumption A1), system (3) is a feedback linearizable system. Therefore, if we know the vector fields f and g we can apply the static feedback

$$u = \frac{-f(x) + c^\tau A^n x_m + c^\tau A^{n-1} br + k^\tau \varepsilon}{g(x)} \tag{7}$$

where $k \triangleq [k_n, k_{n-1}, \cdots, k_1]^\tau$. Then the error system (6) becomes

$$e^{(n)} = -k^\tau \varepsilon$$

or equivalently

$$\overline{h}(s)[e] = 0$$

which implies that $e, \varepsilon \in \mathcal{L}_\infty$ and therefore all closed-loop signals are bounded, and $\lim_{t\to\infty} e(t) = 0$. Note that, after the application of the control (7), the x_n part of (3) becomes

$$\dot{x}_n = c^\tau A^n x_m + c^\tau A^{n-1} br + k^\tau \varepsilon. \tag{8}$$

In many cases, the vector fields f and g are not completely known and thus adaptive versions of the feedback law (7) have to be applied. For instance, using the usual assumption of linear parameterization, if the vector fields f and g are of the form

$$\begin{aligned}f(x) &= \theta_1^\tau \phi_f(x)\\ g(x) &= \theta_2^\tau \phi_g(x)\end{aligned} \tag{9}$$

where $\theta_i, i = 1, 2$ are vectors with unknown constant parameters, one may replace the feedback law (7) with the "certainty equivalent" one (*Certainty-Equivalent Feedback Linearizing* (*CEFL*) controller)

$$u = \frac{-\hat{\theta}_1^\tau \phi_f(x) + c^\tau A^n x_m + c^\tau A^{n-1} br + k^\tau \varepsilon}{\hat{\theta}_2^\tau \phi_g(x)} \tag{10}$$

where $\hat{\theta}_i\ i=1,2$ are the estimates of the unknown parameter vectors $\theta_i, i=1,2$. The estimates $\hat{\theta}_i\ i=1,2$ of the vectors $\theta_i, i=1,2$ are generated by an online adaptive law. However, one can easily observe that there is a danger the denominator of (10) to become equal to zero at certain instants of time leading to an unbounded or noncomputable control input u.

Instead of the control law (7) let us now consider the control law

$$u = \frac{a_K(x)K}{1+a_K(x)Kg(x)}(-f(x)+c^\tau A^n x_m + c^\tau A^{n-1}br + k^\tau \varepsilon) \tag{11}$$

where K and $a_K(\cdot)$ are design terms.

Lemma 1: Assume that the design term $a_K(\cdot)$ in (11) satisfies

$$\frac{|f(x)|+2}{|g(x)|} < a_K(x), \qquad \forall x \in \Re^n. \tag{12}$$

Then, there exists a constant $K^* > 1$ such that for every $K > K^*$ the control law (11) guarantees that the solutions of the closed-loop system (3), (11) are bounded and the error ε converges to the residual set $\mathcal{D}_\varepsilon$ where

$$\mathcal{D}_\varepsilon \triangleq \left\{\varepsilon\colon |\varepsilon| \le \frac{c}{K}\right\}$$

and $c > 0$ is a constant.

Proof: Let us consider the x_n part of (3). We have that for the feedback law (119)

$$\begin{aligned}\dot{x}_n &= f(x) + g(x)\frac{a_K(x)K}{1+a_K(x)Kg(x)} \\ &\quad \cdot (-f(x)+c^\tau A^n x_m + c^\tau A^{n-1}br + k^\tau\varepsilon) \\ &= \frac{1}{1+a_K(x)Kg(x)}[f(x) + a_K(x)Kg(x)f(x) \\ &\quad + a_K(x)Kg(x)(-f(x)+c^\tau A^n x_m + c^\tau A^{n-1}br \\ &\quad + k^\tau\varepsilon)] \\ &= \frac{f(x)}{1+a_K(x)Kg(x)} + \frac{a_K(x)Kg(x)}{1+a_K(x)Kg(x)} \\ &\quad \cdot (c^\tau A^n x_m + c^\tau A^{n-1}br + k^\tau\varepsilon) \\ &= c^\tau A^n x_m + c^\tau A^{n-1}br + k^\tau\varepsilon + \beta_K(x) + \delta_K(x) \\ &\quad \cdot (c^\tau A^n x_m + c^\tau A^{n-1}br + k^\tau\varepsilon)\end{aligned} \tag{13}$$

where

$$\begin{aligned}\beta_K(x) &\triangleq \frac{f(x)}{1+g(x)a_K(x)K} \\ \delta_K(x) &\triangleq -1 + \frac{g(x)a_K(x)K}{1+g(x)a_K(x)K} \\ &\equiv \frac{-1}{1+g(x)a_K(x)K}.\end{aligned} \tag{14}$$

Using (6) and (13) we obtain the error equation

$$e^{(n)} = -k^\tau\varepsilon + \beta_K(x) + \delta_K(x)(c^\tau A^n x_m + c^\tau A^{n-1}br + k^\tau\varepsilon)$$

or, equivalently

$$\overline{h}(s)[e] = -\beta_K(x) - \delta_K(x)(c^\tau A^n x_m + c^\tau A^{n-1}br + k^\tau\varepsilon).$$

The above equation can be rewritten as

$$\begin{aligned}\dot{\varepsilon} &= \Lambda_c\varepsilon + b_c[-\beta_K(x) - \delta_K(x)(c^\tau A^n x_m + c^\tau A^{n-1}br \\ &\quad + k^\tau\varepsilon)] \\ e &= C_c\varepsilon\end{aligned} \tag{15}$$

where $\det(sI - \Lambda_c) = \overline{h}(s), b_c \triangleq [0, \cdots, 0, 1]^\tau$, $C_c \triangleq [1, 0, 0, \cdots, 0]$, and (Λ_c, b_c) is in the controllable canonical form. We will now prove that the terms $\beta_K(\cdot)$ and $\delta_K(\cdot)$ are bounded from above by $1/K$, i.e.,

$$|\beta_K(x)|, |\delta_K(x)| < \frac{1}{K}, \qquad \forall x. \tag{16}$$

In order to do so, first observe that from (12) we have that $|g(x)a_K(x)| > 2$ and therefore the denominator $1 + g(x)a_K(x)K$ never becomes zero since $K > 1$. Let us now prove that $\beta_K(\cdot)$ is bounded from above by $1/K$. Since $K > 1$ we have that $(|f(x)| + 2/|g(x)|) > (|f(x)| + 1/K/|g(x)|)$ and thus, from (12) we obtain

$$\begin{aligned}&\frac{|f(x)|+1/K}{|g(x)|} < a_K(x) \\ &\quad \Rightarrow |f(x)| < \frac{1}{K}|Ka_K(x)g(x)| - \frac{1}{K} \\ &\quad \Rightarrow |f(x)| < \frac{1}{K}|Ka_K(x)g(x)+1| \\ &\quad \Rightarrow |f(x)| < \frac{1}{K}|Ka_K(x)g(x)+1| \\ &\quad \Rightarrow |\beta_K(x)| = \frac{|f(x)|}{|Ka_K(x)g(x)+1|} \le \frac{1}{K}\end{aligned}$$

and thus $\beta_K(\cdot)$ is bounded from above by $1/K$. The proof for the case of $\delta_K(\cdot)$ can be done in a similar way.

Consider now the Lyapunov function

$$V_d = \varepsilon^\tau P_c \varepsilon$$

where P_c is the positive definite solution of the Lyapunov equation $P_c\Lambda_c + \Lambda_c P_c = -qI, q > 0$. Then, using (15) we obtain that

$$\begin{aligned}\dot{V}_d &= -q|\varepsilon|^2 + 2\varepsilon^\tau P_c b_c[-\beta_K(x) - \delta_K(x) \\ &\quad \cdot (c^\tau A^n x_m + c^\tau A^{n-1}br + k^\tau\varepsilon)].\end{aligned}$$

Using now (16) and the facts that $r, x_m \in \mathcal{L}_\infty$ it can be seen that

$$\dot{V}_d \le -q(1 - c_1/K^2)|\varepsilon|^2 + c_2/K^2 \le -c_3|\varepsilon|^2 + c_2/K^2$$

for some positive constants c_1, c_2, c_3 and for $K > K^* \triangleq \max[1, \sqrt{c_1}]$. By choosing $c = \sqrt{c_2/c_3}$, we have that $\dot{V}_d$ is negative whenever $\varepsilon \notin \mathcal{D}_e$; applying now standard Lyapunov stability arguments [4] we can establish boundedness of all the closed-loop signals and convergence of ε to the residual set $\mathcal{D}_\varepsilon$. ∎

Remark 1: The control law (11) does no longer guarantee that the tracking error e will converge to zero as $t \to \infty$. The error, however, can be made as small as possible by increasing the value of K. Large K does not imply high-gain feedback. The improvement in the tracking performance as K increases is due to the fact that the modified term $Ka_K(x)/1 + Ka_K(x)g(x)$ approaches the term $1/g(x)$ as $K \to \infty$, which is the one that leads to zero residual tracking error.

The reason for considering the modified control law (11) is that in the adaptive case where $f(\cdot), g(\cdot)$ are unknown, the adaptive controller based on (11) will be shown to have certain important advantages over the one described by (10). $\diamondsuit$

Remark 2: The control law (11) is the same as the control law

$$u = a_K(x)K(-\dot{x}_n + c^\tau A^n x_m + c^\tau A^{n-1}br + k^\tau \varepsilon). \quad (17)$$

That is, from the above equation we have that

$$\begin{aligned} u &= a_K(x)K(-f(x) - g(x)u + c^\tau A^n x_m \\ &\quad + c^\tau A^{n-1}br + k^\tau\varepsilon) \Rightarrow \\ u(1 + a_K(x)Kg(x)) &= a_K(x)K(-f(x) + c^\tau A^n x_m \\ &\quad + c^\tau A^{n-1}br + k^\tau\varepsilon) \Rightarrow \\ u &= \frac{a_K(x)K}{1 + a_K(x)Kg(x)}(-f(x) + c^\tau A^n \\ &\quad \cdot x_m + c^\tau A^{n-1}br + k^\tau\varepsilon) \end{aligned}$$

which is the control law (11). The control law (17) involves the use of $\dot{x}_n$ which is not available for measurement and thus it is not an implementable control law. Due to the equivalence of (11) with (17) we refer to (11) as the Derivative Feedback Controller (DFC). $\diamondsuit$

Remark 3: Note that the construction of the function $a_K(\cdot)$ does not require explicit knowledge of the vector fields f and g. In fact, in the case where the vector fields f and g are linear combinations of unknown constant vectors and known functions, we can easily design a function $a_K(\cdot)$ satisfying the conditions of Lemma 1 (see Example 1 below). $\diamondsuit$

The control law (11) is implementable provided the vector fields $f(\cdot)$ and $g(\cdot)$ are exactly known. In the case where $f(x) = \theta_1^\tau \phi_f(x), g(x) = \theta_2^\tau \phi_g(x)$, and $\phi_f(\cdot), \phi_g(\cdot)$ are known functions and the constant vectors θ_1, θ_2 are unknown, instead of (11) we use the certainty equivalent control law referred to as the *ADF* controller

$$\begin{aligned} u = \frac{a_K(x)K}{1 + a_K(x)K\hat{\theta}_2^\tau \phi_g(x)}&(-\hat{\theta}_1^\tau \phi_f(x) + c^\tau A^n x_m \\ &+ c^\tau A^{n-1}br + k^\tau\varepsilon). \end{aligned} \quad (18)$$

While the control law (10) was nonimplementable when $\hat{\theta}_2^\tau \phi_g(x) \approx 0$, the above control law becomes nonimplementable when $\hat{\theta}_2^\tau \phi_g(x) \approx -1/a_K(x)K$.

Our approach for avoiding these singularities or nonimplementable conditions is described as follows. We use the following assumption for $a_K(\cdot)$:

A2) $a_K(x) < K, \qquad \forall x;$

and define the set

$$\mathcal{S}_1 \equiv \mathcal{S}_1(\theta_2, t) \triangleq \left\{(\theta_2, t)\colon |\theta_2^\tau \phi_g(x(t))| < \frac{1}{K^2}\right\}.$$

It follows from Assumption A2) that if $\hat{\theta}_2(t) \in \mathcal{S}_1$ then $1 + Ka_K(x)\hat{\theta}_2^\tau \phi_g(x) \neq 0$ and therefore (18) can be used whenever $\hat{\theta}_2(t) \in \mathcal{S}_1$. If $\hat{\theta}_2(t) \notin \mathcal{S}_1$ it follows that $|\theta_2^\tau \phi_f(x(t))| \geq (1/K^2) > 0$ and therefore (10) can be used whenever $\hat{\theta}_2(t) \notin \mathcal{S}_1$.

A reasonable control strategy is to switch between the control laws (10) and (18) depending on whether $\hat{\theta}_2(t)$ belongs or does not belong to $\mathcal{S}_1$. The details of the switching approach are given in Section II-A together with the analysis.

Remark 4: Assumption A2) may appear to be restrictive since clearly no $a_K(\cdot)$ can be found to satisfy

$$\frac{|f(x)| + 2}{|g(x)|} < a_K(x) < K$$

when $|f(x)|$ grows faster than $|g(x)|$ as $|x| \to \infty$. A small modification of the control law will take care of such situation, which allow us to use A2) without loss of generality. The modification is explained as follows: When $a_K(\cdot)$ is not bounded from above by K we decompose it as

$$a_K(x) = \tilde{a}_K(x)\overline{a}_K(x) \quad (19)$$

where $\overline{a}_K(\cdot) < K$ and $\tilde{a}_K(\cdot)$ contains all the possibly unbounded terms. We then use the prefeedback

$$u = \tilde{a}_K(x)\overline{u} \quad (20)$$

to system (3) to obtain

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= f(x_1, x_2, \cdots, x_n) + \overline{g}(x_1, x_2, \cdots, x_n)\overline{u} \\ y &= x_1 \end{aligned} \quad (21)$$

where $\overline{g}(x) \triangleq g(x)\tilde{a}_K(x)$. For the system (21) we can now establish that

$$\frac{|f(x)| + 2}{|\overline{g}(x)|} < \overline{a}_K(x) < K.$$

Therefore, since the general system (3) can be transformed into (21) where the corresponding $a_K(\cdot)$ is bounded from above by K, we can use A2) without loss of generality.

Next we give an example on how to design the function $a_K(\cdot)$ and how to decompose it according to the above. $\diamondsuit$

Example 1: Consider the case where we know a constant ϵ^* such that $|g(x)| > \epsilon^*$ for all $x \in \Re^n$ and a function $\overline{f}(x)$ satisfying $|f(x)| \leq \overline{f}(x)$. Then $a_K(\cdot)$ can be chosen as

$$a_K(x) = \frac{\overline{f}(x) + 2}{\epsilon^*}.$$

Since $a_K(\cdot)$ is not bounded from above by a constant, we choose

$$\tilde{a}_K(x) = \frac{2}{K}\frac{\overline{f}(x) + 2}{\epsilon^*}$$

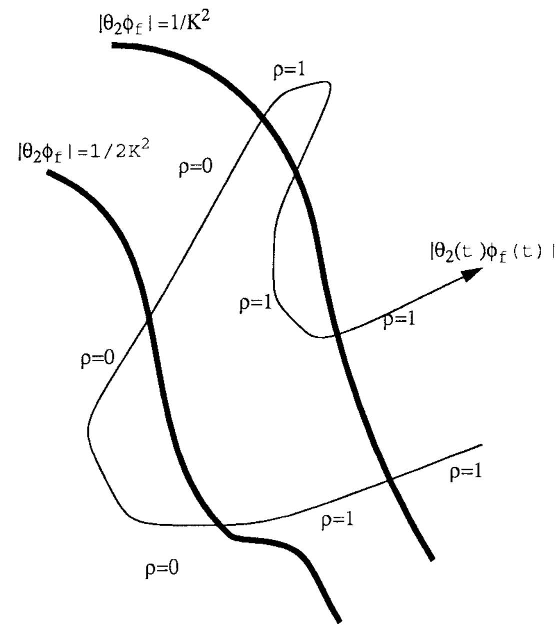

Fig. 1. Definition of the variable ϱ.

and

$$\overline{a}_K(x) = \tfrac{1}{2}K.$$

We then use $u = \tilde{a}_K(x)\overline{u}$ to put the system in the form of (21) where $\overline{a}_K(x) < K$ for all $x \in \Re^n$. Note that the above design for the functions $a_K(\cdot), \tilde{a}_K(\cdot), \overline{a}_K(\cdot)$ ignores the growth properties of the function $g(\cdot)$ and thus may result in the design of a conservative controller. Less conservative controllers may be obtained by incorporating the growth properties of the function $g(\cdot)$ in the design of the functions $a_K(\cdot), \tilde{a}_K(\cdot), \overline{a}_K(\cdot)$. $\diamondsuit$

A. The New Adaptive Controller

In this subsection, we will present and analyze the new adaptive controller. As we have already mentioned, we will use a controller which switches between the CEFL and ADF controllers depending whether $\hat{\theta}_2(t)$ belongs or does not belong to $\mathcal{S}_1$. In order to avoid any possibility of *sliding motions* [15], we will use a *hysteresis switching* [13], [19] described as follows: Let us define the function $\varrho(\cdot)$ as[2] follows:

$$\varrho(t) = \begin{cases} 0, & \text{if } (\hat{\theta}_2(t), t) \in \mathcal{S}_2 \\ 0, & \text{if } (\hat{\theta}_2(t), t) \in \mathcal{S}_1/\mathcal{S}_2 \text{ and } \varrho_-(t) = 0 \\ 1, & \text{if } (\hat{\theta}_2(t), t) \in \mathcal{S}_1/\mathcal{S}_2 \text{ and } \varrho_-(t) = 1 \\ 1, & \text{if } (\hat{\theta}_2(t), t) \notin \mathcal{S}_1 \end{cases}$$

where

$$\mathcal{S}_2 \equiv \mathcal{S}_2(\theta_2, t) \triangleq \left\{(\theta_2, t) \colon |\theta_2^\tau \phi_g(x(t))| \le \frac{1}{2K^2}\right\}.$$

The definition of the variable ϱ can be easily understood by looking at Fig. 1. More precisely, in Fig. 1 we have plotted a possible trajectory of the term $|\hat{\theta}_2^\tau \phi_g(x)|$; in the case where $(\hat{\theta}_2^\tau, t) \notin \mathcal{S}_1$ then $\varrho = 1$ and moreover ϱ remains one until $(\hat{\theta}_2^\tau, t)$ enters the set $\mathcal{S}_2$. After $(\hat{\theta}_2^\tau, t)$ enters the set $\mathcal{S}_2, \varrho$ switches to zero and remains zero until $(\hat{\theta}_2^\tau, t)$ exits the set $\mathcal{S}_1$. After $(\hat{\theta}_2^\tau, t)$ exits the set $\mathcal{S}_1$, the variable ϱ switches to one and remains equal to one, although $(\hat{\theta}_2^\tau, t)$ enters the set $\mathcal{S}_1$ (since $(\hat{\theta}_2^\tau, t)$ does not enter $\mathcal{S}_2$).

[2] $\varrho_-(t) \triangleq \lim_{\tau \to t^-} \varrho(\tau)$, where $\tau < t$.

The variable ϱ controls the switching policy of the proposed controller. In particular, if $\varrho = 0$ then the controller is an ADF controller, while when $\varrho = 1$, the controller is a CEFL controller.

Let

$$v(x_m, r, \varepsilon) \triangleq c^\tau A^n x_m + c^\tau A^{n-1} br + k^\tau \varepsilon. \tag{22}$$

The proposed switching control law is

$$u = \varrho \frac{-\hat{\theta}_1^\tau \phi_f + v}{\hat{\theta}_2^\tau \phi_g} + (1-\varrho)\{-Ka_K(x)\{\hat{\theta}_1^\tau \phi_f + \hat{\theta}_2^\tau \phi_g u - v\}\}$$

or

$$\begin{aligned} &u(1 + Ka_K(x)(1-\varrho)\hat{\theta}_2^\tau \phi_g) \\ &\quad = \varrho \frac{-\hat{\theta}_1^\tau \phi_f + v}{\hat{\theta}_2^\tau \phi_g} + (1-\varrho)\{-Ka_K(x)\{\hat{\theta}_1^\tau \phi_f - v\}\} \end{aligned}$$

and thus

$$u = \frac{\varrho \dfrac{-\hat{\theta}_1^\tau \phi_f + v}{\hat{\theta}_2^\tau \phi_g}}{1 + Ka_K(x)(1-\varrho)\hat{\theta}_2^\tau \phi_g} + \frac{(1-\varrho)\{-Ka_K(x)\{\hat{\theta}_1^\tau \phi_f - v\}\}}{1 + Ka_K(x)(1-\varrho)\hat{\theta}_2^\tau \phi_g}. \tag{23}$$

In fact when $\varrho = 0$ the above control law becomes equal to (18), while when $\varrho = 1$ the above control law is equal to (10).

Before we present the update laws for $\hat{\theta}_i$, let us analyze the above control law. Note that using the above feedback law, the x_n part of (3) becomes

$$\begin{aligned} \dot{x}_n &= \theta_1^\tau \phi_f + \theta_2^\tau \phi_g u \\ &= \theta_1^\tau \phi_f + \theta_2^\tau \phi_g \varrho \frac{-\hat{\theta}_1^\tau \phi_f + v}{\hat{\theta}_2^\tau \phi_g} \\ &\quad + \theta_2^\tau \phi_g (1-\varrho)\{-Ka_K(x)\{\hat{\theta}_1^\tau \phi_f + \hat{\theta}_2^\tau \phi_g u - v\}\} \\ &= \theta_1^\tau \phi_f + \hat{\theta}_2^\tau \phi_g \varrho \frac{-\hat{\theta}_1^\tau \phi_f + v}{\hat{\theta}_2^\tau \phi_g} + \tilde{\theta}_2^\tau \phi_g \varrho \frac{-\hat{\theta}_1^\tau \phi_f + v}{\hat{\theta}_2^\tau \phi_g} \\ &\quad + \theta_2^\tau \phi_g (1-\varrho)\{-Ka_K(x)\{\dot{x}_n - \tilde{\dot{x}}_n - v\}\}, \\ &\qquad \text{Note that } \hat{\theta}_2^\tau \phi_g \varrho \frac{1}{\hat{\theta}_2^\tau \phi_g} = \varrho \\ &= \theta_1^\tau \phi_f - \varrho \hat{\theta}_1^\tau \phi_f + \varrho v + \tilde{\theta}_2^\tau \phi_g \varrho \frac{-\hat{\theta}_1^\tau \phi_f + v}{\hat{\theta}_2^\tau \phi_g} \\ &\quad + \theta_2^\tau \phi_g (1-\varrho)\{-Ka_K(x)\{\dot{x}_n - \tilde{\theta}_1^\tau \phi_f - \tilde{\theta}_2^\tau \phi_g \\ &\qquad \cdot u - v\}\} \\ &= (1-\varrho)\theta_1^\tau \phi_f + \varrho \tilde{\theta}_1^\tau \phi_f + \varrho v + \varrho \tilde{\theta}_2^\tau \psi_1(\hat{\theta}_1, \hat{\theta}_2, x, v) \\ &\quad + \theta_2^\tau \phi_g (1-\varrho)\{-Ka_K(x)\{\dot{x}_n - \tilde{\theta}_1^\tau \phi_f - \tilde{\theta}_2^\tau \phi_g \\ &\qquad \cdot u - v\}\} \end{aligned}$$

where $\tilde{\theta}_i \triangleq \theta_i - \hat{\theta}_i$ and $\tilde{\dot{x}}_n \triangleq \dot{x}_n - \hat{\theta}_1^\tau \phi_f - \hat{\theta}_2^\tau \phi_g u = \tilde{\theta}_1^\tau \phi_f + \tilde{\theta}_2^\tau \phi_g u, \psi_1(\hat{\theta}_1, \hat{\theta}_2, x, v) \triangleq -\phi_g(\hat{\theta}_1^\tau \phi_f - v/\hat{\theta}_2^\tau \phi_g)$. Note now that the above equation can be rewritten as follows:

$$\begin{aligned}&\dot{x}_n(1+(1-\varrho)Ka_K(x)\theta_2^\tau\phi_g)\\&\quad=(1-\varrho)\theta_1^\tau\phi_f+\varrho\tilde{\theta}_1^\tau\phi_f+\varrho v+\varrho\tilde{\theta}_2^\tau\psi_1(\hat{\theta}_1,\hat{\theta}_2,x,v)\\&\qquad+\theta_2^\tau\phi_g(1-\varrho)\{Ka_K(x)\{\tilde{\theta}_1^\tau\phi_f+\tilde{\theta}_2^\tau\phi_g u+v\}\}\end{aligned}$$

and therefore

$$\begin{aligned}\dot{x}_n&=\frac{(1-\varrho)\theta_1^\tau\phi_f+\varrho\tilde{\theta}_1^\tau\phi_f+\varrho v+\varrho\tilde{\theta}_2^\tau\psi_1(\hat{\theta}_1,\hat{\theta}_2,x,v)}{1+(1-\varrho)Ka_K(x)\theta_2^\tau\phi_g}\\&\quad+\frac{\theta_2^\tau\phi_g(1-\varrho)Ka_K(x)}{1+(1-\varrho)Ka_K(x)\theta_2^\tau\phi_g}\{\tilde{\theta}_1^\tau\phi_f+\tilde{\theta}_2^\tau\phi_g u+v\}\\&=\frac{(1-\varrho)f(x)+\varrho\tilde{\theta}_1^\tau\phi_f+\varrho v+\varrho\tilde{\theta}_2^\tau\psi_1}{1+(1-\varrho)Ka_K(x)g(x)}\\&\quad+(1-\varrho)\tilde{\theta}_1^\tau\phi_f+(1-\varrho)\tilde{\theta}_2^\tau\phi_g u+(1-\varrho)v\\&\quad+(1-\varrho)\overline{\delta}_K(x)\{\tilde{\theta}_1^\tau\phi_f+\tilde{\theta}_2^\tau\phi_g u+v\}\end{aligned}\tag{24}$$

where $\overline{\delta}_K(x) \triangleq (-1/1+(1-\varrho)Ka_K(x)g(x))$.

Using now (6), (24), (22) and the fact that $\varrho \in \{0,1\}$ we can easily see that

$$\begin{aligned}e^{(n)}&=v-k^\tau\varepsilon-\dot{x}_n\\&=v-k^\tau\varepsilon-\frac{(1-\varrho)f(x)+\varrho\tilde{\theta}_1^\tau\phi_f+\varrho v+\varrho\tilde{\theta}_2^\tau\psi_1}{1+(1-\varrho)Ka_K(x)g(x)}\\&\quad-(1-\varrho)\tilde{\theta}_1^\tau\phi_f-(1-\varrho)\tilde{\theta}_2^\tau\phi_g u-(1-\varrho)v\\&\quad-(1-\varrho)\overline{\delta}_K(x)\{\tilde{\theta}_1^\tau\phi_f+\tilde{\theta}_2^\tau\phi_g u+v\}\\&=-k^\tau\varepsilon-\frac{(1-\varrho)f(x)+\varrho\tilde{\theta}_1^\tau\phi_f+\varrho\tilde{\theta}_2^\tau\psi_1}{1+(1-\varrho)Ka_K(x)g(x)}\\&\quad-(1-\varrho)\tilde{\theta}_1^\tau\phi_f-(1-\varrho)\tilde{\theta}_2^\tau\phi_g u-(1-\varrho)\\&\quad\cdot\overline{\delta}_K(x)\{\tilde{\theta}_1^\tau\phi_f+\tilde{\theta}_2^\tau\phi_g u+v\}\\&=-k^\tau\varepsilon-(1-\varrho)\overline{\beta}_K(x)-\tilde{\theta}_1^\tau\phi_f-\tilde{\theta}_2^\tau\phi_g u\\&\quad-(1-\varrho)\overline{\delta}_K(x)\{\tilde{\theta}_1^\tau\phi_f+\tilde{\theta}_2^\tau\phi_g u+v\}\end{aligned}$$

where $\overline{\beta}_K(x) \triangleq (f(x)/1+(1-\varrho)Ka_K(x)g(x))$. The above equation can be rewritten as

$$\begin{aligned}h(s)[e]&=-(1-\varrho)\overline{\beta}_K(x)-\tilde{\theta}_1^\tau\phi_f-\tilde{\theta}_2^\tau\phi_g u-(1-\varrho)\\&\quad\cdot\overline{\delta}_K(x)\{\tilde{\theta}_1^\tau\phi_f+\tilde{\theta}_2^\tau\phi_g u+v\}\end{aligned}$$

or, equivalently

$$\begin{aligned}\dot{\varepsilon}&=\Lambda_c\varepsilon+\tilde{\theta}_1^\tau\overline{\phi}_f+(1+(1-\varrho)\overline{\delta}_K(x))\tilde{\theta}_2^\tau\overline{\phi}_g+\overline{d}\\e&=C_c\varepsilon\end{aligned}\tag{25}$$

where $\theta \triangleq (\theta_1^\tau, \theta_2^\tau)^\tau$ and $\overline{\phi}_f \triangleq -b_c\phi_f^\tau$, $\overline{\phi}_g \triangleq -ub_c\phi_g^\tau$ and, finally

$$\overline{d} \triangleq b_c[-(1-\varrho)\overline{\beta}_K(x)-(1-\varrho)\overline{\delta}_K(x)\{\tilde{\theta}_1^\tau\phi_f+v\}].\tag{26}$$

We now propose a gradient adaptive law with constant σ-modification [4] for adjusting $\tilde{\theta}_i$

$$\dot{\hat{\theta}}_i=-\kappa_i\Gamma_i\overline{\phi}_i^\tau P_c\varepsilon-\sigma\Gamma_i\hat{\theta}_i\tag{27}$$

where $\Gamma_i, i=1,2$ are symmetric positive definite matrices, and $\sigma > 0$ is a design constant.

The parameters κ_i are chosen as follows:

$$\begin{aligned}\kappa_1&=1\\\kappa_2&=1+\delta_{max}(1-\varrho)\operatorname{sgn}(g(x))\tanh(K\varepsilon^\tau P_c b_c u)\end{aligned}\tag{28}$$

where $\delta_{\max} \in (\max_x |\overline{\delta}_K(x)|, 1)$; $\tanh(\cdot)$ denotes the hyperbolic tangent function and, for large K, it can be thought as a smooth approximation of the signum function. We are now ready to present our first main result.

Theorem 1: Consider system (3) and the switching feedback (23), (27), (28). Assume that A1) and A2) hold. Then there exists a $K^* > 1$ such that, for all $K > K^*$, all the signals of the closed-loop system are bounded, and moreover, the tracking error converges to the residual set $\mathcal{D}$ given by

$$\mathcal{D} \triangleq \{\varepsilon\colon |\varepsilon|^2 \le c_0[1/K^2+\sigma]\}$$

where c_0 is a constant independent of K and σ.

Proof: Using similar arguments as those in [13] (see also [19, Lemma A.1]) we can establish that there exists an interval $[0,T)$ of maximal length on which the hysteresis switching closed-loop system possesses a unique solution with ϱ piecewise constant on $[0,T)$ and, moreover, that on each strictly proper subinterval $[0,\tau) \subset [0,T)$ ϱ can switch at most finite times. Since the closed-loop system possesses a unique solution on $[0,T)$, we can apply Lyapunov stability arguments. Consider the Lyapunov function

$$V=\varepsilon^\tau P_c\varepsilon+\sum_{i=1}^{2}\tilde{\theta}_i^\tau\Gamma_i^{-1}\tilde{\theta}_i.\tag{29}$$

Differentiating V along the solutions of (25), (27), we obtain

$$\begin{aligned}\dot{V}&=-q|\varepsilon|^2+2\varepsilon^\tau P_c\overline{d}-2\sigma\sum_{i=1}^{2}\tilde{\theta}_i^\tau\hat{\theta}_i\\&\quad-2(1-\varrho)\overline{\delta}_K(x)\tilde{\theta}_2^\tau\phi_g u\varepsilon^\tau P_c b_c\\&\quad-2\delta_{\max}(1-\varrho)\operatorname{sgn}(g(x))\tanh(K\varepsilon^\tau P_c b_c u)\\&\quad\cdot\tilde{\theta}_2^\tau\phi_g u\varepsilon^\tau P_c b_c.\end{aligned}\tag{30}$$

By noticing now that $(1-\varrho)\overline{\beta}_K(x) \equiv (1-\varrho)\beta_K(x)$ and $(1-\varrho)\overline{\delta}_K(x) \equiv (1-\varrho)\delta_K(x)$, we obtain—from the proof of Lemma 1—that $(1-\varrho)\overline{\beta}_K(x)$ and $(1-\varrho)\overline{\delta}_K(x)$ are bounded from above by $(1-\varrho)/K$; moreover, we have that

$$\begin{aligned}&(1-\varrho)\overline{\delta}_K(x)\tilde{\theta}_1^\tau\phi_f(x)\\&\quad=(1-\varrho)\overline{\delta}_K(x)f(x)-(1-\varrho)\overline{\delta}_K(x)\hat{\theta}_1^\tau\phi_f(x)\\&\quad\le(1-\varrho)|\beta_K(x)|+(1-\varrho)|\overline{\delta}_K(x)|\,|\hat{\theta}_1|\,|\phi_f(x)|\\&\quad\le(1-\varrho)|\beta_K(x)|+c(1-\varrho)|\beta_K(x)|\,|\hat{\theta}_1|\\&\quad=(1-\varrho)\mathcal{O}(1/K)|\hat{\theta}_1|+(1-\varrho)\mathcal{O}(1/K)\end{aligned}$$

for some $c > 0$, and therefore $(1-\varrho)\overline{\delta}_K(x)\tilde{\theta}_1^\tau\phi_f$ is $(1-\varrho)\mathcal{O}(1/K)|\hat{\theta}_1| + (1-\varrho)\mathcal{O}(1/K)$. Therefore, by using (22) and the fact that $x_m, r \in \mathcal{L}_\infty$, we can see that

$$\begin{aligned}2\varepsilon^\tau P_c\overline{d}&=2\varepsilon^\tau P_c b_c[-(1-\varrho)\overline{\beta}_K(x)-(1-\varrho)\overline{\delta}_K(x)\\&\qquad\cdot\{\tilde{\theta}_1^\tau\phi_f+v\}]\\&\le c_1(1-\varrho)\frac{1}{K}|\varepsilon|^2+(1-\varrho)\mathcal{O}(1/K^2)|\hat{\theta}_1|^2\\&\quad+(1-\varrho)\mathcal{O}(1/K^2)\end{aligned}\tag{31}$$

for some positive constant c_1.

Let

$$K^* \triangleq \max_{x\in\Re^n} [1, c_1/q, a_K(x), \sqrt{c_3/\sigma}] \tag{32}$$

where c_3 is a positive constant to be defined later. From (30), (31) we have that, since $K > c_1/q$

$$\begin{aligned}\dot V \le & -c_2|\varepsilon|^2 - 2\sigma \sum_{i=1}^{2} \tilde\theta_i^\tau \hat\theta_i + (1-\varrho)\mathcal{O}(1/K^2)|\hat\theta_1|^2 \\ & + (1-\varrho)\mathcal{O}(1/K^2) - 2(1-\varrho)\overline{\delta}_K(x)\tilde\theta_2^\tau \phi_g u\varepsilon^\tau P_c b_c \\ & - 2\delta_{\max}(1-\varrho)\,\mathrm{sgn}\,(g(x))\tanh\,(K\varepsilon^\tau P_c b_c u)\tilde\theta_2^\tau\phi_g \\ & \cdot u\varepsilon^\tau P_c b_c \end{aligned} \tag{33}$$

where $c_2 = q - c_1/K$; note that from (32) we have that $c_2 > 0$. Let us now examine the last two terms of the right-hand side (RHS) of the above inequality. First, observe that $\mathrm{sgn}\,(\tilde\theta_2^\tau \phi_g(x)) = \mathrm{sgn}\,(g(x) - \hat\theta_2^\tau\phi_g(x))$, and, thus, we have that $\mathrm{sgn}\,(\tilde\theta_2^\tau\phi_g(x)) = \mathrm{sgn}\,(g(x))$ provided that $|\hat\theta_2^\tau\phi_g(x)| < \epsilon^*$; however, from A2), (12), and (32) it can be easily seen that $\epsilon^* > 1/K^2$ and thus, in the case where $\varrho = 0$ (i.e., in the case where $|\hat\theta_2^\tau\phi_g(x)| < 1/K^2$), we have that $\mathrm{sgn}\,(\tilde\theta_2^\tau\phi_g(x)) = \mathrm{sgn}\,(g(x))$. Therefore, by using the property of the function $\tanh\,(\cdot)$ that $\tanh\,(Kx)x = |x| + \mathcal{O}(1/K)$, we have that

$$\begin{aligned} & -2(1-\varrho)\overline{\delta}_K(x)\tilde\theta_2^\tau\phi_g u\varepsilon^\tau P_c b_c - 2\delta_{\max}(1-\varrho)\,\mathrm{sgn}\,(g(x)) \\ & \quad \cdot\tanh\,(K\varepsilon^\tau P_c b_c u)\tilde\theta_2^\tau\phi_g u\varepsilon^\tau P_c b_c \\ & \le 2(1-\varrho)|\overline{\delta}_K(x)|\,|\tilde\theta_2^\tau\phi_g|\,|u\varepsilon^\tau P_c b_c| - 2\delta_{\max}(1-\varrho) \\ & \quad \cdot \mathrm{sgn}\,(g(x))\tanh\,(K\varepsilon^\tau P_c b_c u)\tilde\theta_2^\tau\phi_g u\varepsilon^\tau P_c b_c \\ & \le 2(1-\varrho)|\overline{\delta}_K(x)|(|\tilde\theta_2^\tau\phi_g|\,|u\varepsilon^\tau P_c b_c| - |\tilde\theta_2^\tau\phi_g| \\ & \quad \cdot\tanh\,(K\varepsilon^\tau P_c b_c u)(u\varepsilon^\tau P_c b_c)) \\ & \le 2(1-\varrho)|\overline{\delta}_K(x)|\,|\tilde\theta_2^\tau\phi_g|\mathcal{O}(1/K) \\ & \equiv (1-\varrho)\mathcal{O}(1/K^2)|\hat\theta_2| + (1-\varrho)\mathcal{O}(1/K^2). \end{aligned}$$

Hence, (33) becomes

$$\begin{aligned}\dot V \le & -c_2|\varepsilon|^2 - 2\sum_{i=1}^{2}\sigma_i\tilde\theta_i^\tau\hat\theta_i + (1-\varrho)\mathcal{O}(1/K^2) \\ & \cdot\sum_{i=1}^{2}|\hat\theta_i|^2 + (1-\varrho)\mathcal{O}(1/K^2) \\ \le & -c_2|\varepsilon|^2 - 2\sum_{i=1}^{2}\sigma_i\tilde\theta_i^\tau\hat\theta_i + \frac{c_3}{K^2}\sum_{i=1}^{2}|\hat\theta_i|^2 \\ & + (1-\varrho)\mathcal{O}(1/K^2) \end{aligned} \tag{34}$$

for some positive constant c_3. We will first prove that $\varepsilon, \hat\theta_i$ are bounded. As it is shown in [4]

$$-2\sigma\sum_{i=1}^{2}\tilde\theta_i^\tau\hat\theta_i \le -\sigma\sum_{i=1}^{2}|\hat\theta_i|^2 + \sigma\sum_{i=1}^{2}|\theta_i|^2$$

and, therefore, we have that

$$\begin{aligned}\dot V \le & -c_2|\varepsilon|^2 - \sigma\sum_{i=1}^{2}|\hat\theta_i|^2 + \mathcal{O}(\sigma) + \frac{c_3}{K^2}\sum_{i=1}^{2}|\hat\theta_i|^2 \\ & + (1-\varrho)\mathcal{O}(1/K^2) \\ \le & -c_2|\varepsilon|^2 - c_4\sum_{i=1}^{2}|\hat\theta_i|^2 + \mathcal{O}(\sigma) + (1-\varrho)\mathcal{O}(1/K^2) \end{aligned} \tag{35}$$

where $c_4 = \sigma - (c_3/K^2)$ which is positive from (32). Thus, we have that $\dot V$ is negative whenever $c_2|\varepsilon|^2 + c_4\sum_{i=1}^{2}|\hat\theta_i|^2 > (1-\varrho)\mathcal{O}(1/K^2) + \mathcal{O}(\sigma)$, which implies [4] that $\varepsilon, \hat\theta_i$ are bounded; this, in turn, implies that all the closed-loop signals are bounded on the interval $[0, T)$. Since the bounds are independent of T the interval of existence can be extended to $[0, \infty)$ as shown in [4].

We will now prove convergence of ε. Since $\hat\theta_i$ is bounded, inequality (35) may be rewritten as

$$\dot V \le -c_2|\varepsilon|^2 + (1-\varrho)\mathcal{O}(1/K^2) + \mathcal{O}(\sigma).$$

Therefore, we have that $\dot V < 0$ for $\varepsilon \notin \mathcal{D}$ where $\mathcal{D}$ denotes the residual set

$$\mathcal{D} \triangleq \{\varepsilon\colon |\varepsilon|^2 \le (1-\varrho)\mathcal{O}(1/K^2) + \mathcal{O}(\sigma)\}.$$

Applying now standard Lyapunov stability arguments [4], we conclude that ε converges—in finite time—in the set $\mathcal{D}$, and remain on this set thereafter. ■

Remark 5: Due to the use of the hysteresis switching control variable ϱ, and the fact that one has to make $Ka_K(x)$ large enough to ensure stability and small steady-state error, the proposed controller may exhibit high-gain behavior with discontinuities. Therefore stability and performance is traded-off with the possibility of having large but bounded control inputs that may also have a high-frequency content. On the other hand, in many practical situations $Ka_K(x)$ cannot be made arbitrarily large due to various factors, like sampling rates, limited control authority, unmodeled high-frequency dynamics. These issues as well as the robustness of our controller to bounded disturbances and unmodeled dynamics is the subject of our current research. It is worth noticing that similar problems occur to the high-gain controllers proposed in [9], [16], and [12].

We close this remark by mentioning that the constant K^* and c_0 in Theorem 1 in general depend on initial conditions, the reference signal $r(t)$, the choice of the reference model (4) as well as the functions f, g of the plant. ◇

Remark 6: Using similar arguments with those of [4, Th. 8.5.2], we can show that if, instead of the constant σ-modification, we use a continuous switching σ-modification, then Theorem 1 is still valid and, moreover, the tracking error ε is $1/K^2$-small in the mean square sense, that is

$$\int_t^{t+T} |\varepsilon(\tau)|^2\,d\tau \le \overline{c}_0\frac{1}{K^2}T + \overline{c}_1, \qquad \forall t, T \ge 0$$

for some $\overline{c}_0, c_1$ independent of K. In the case of continuous switching σ-modification, the parameter σ is time-varying and

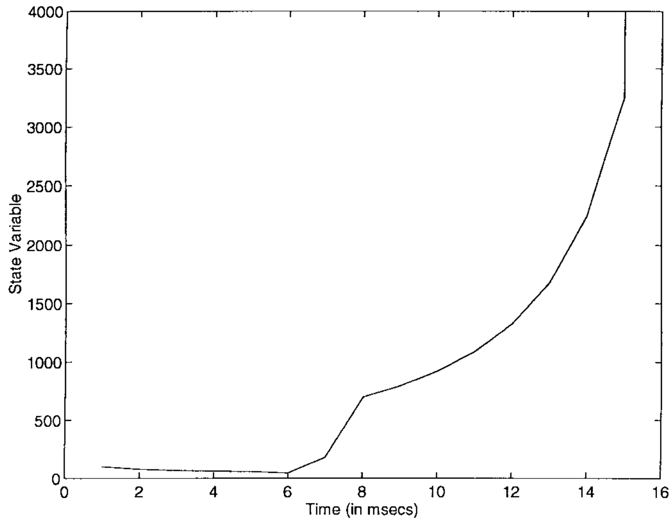

Fig. 2. Unstable state trajectory produced by CEFL controller.

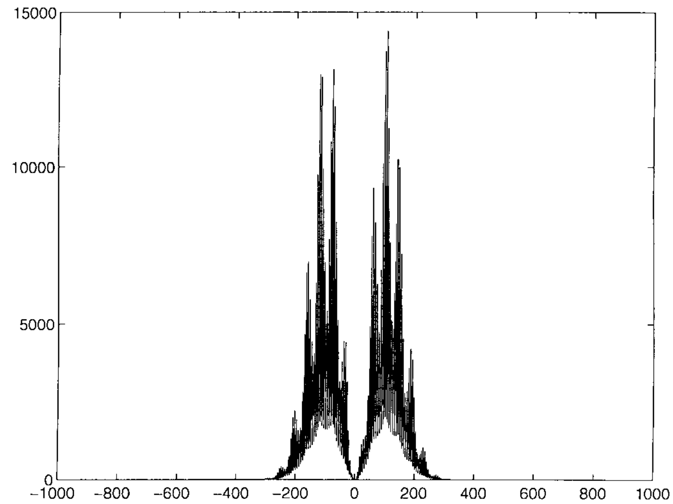

Fig. 3. Plot of the function $|f(x)| + 2/|g(x)|$.

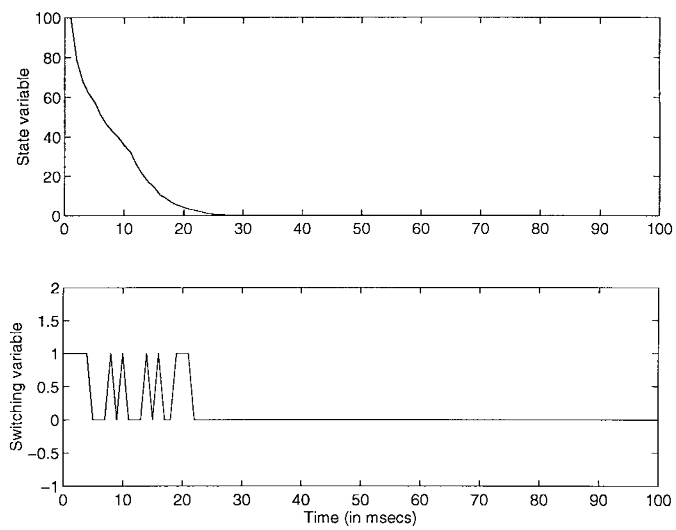

Fig. 4. Performance of the proposed switching controller [upper subplot: time-history of $x(t)$, lower subplot: time-history of $\varrho(t)$].

it is given by

$$\sigma = \begin{cases} 0, & \text{if } |\hat{\theta}| < M_0 \\ \sigma_0\left(\dfrac{\hat{\theta}}{M_0} - 1\right), & \text{if } M_0 \le |\hat{\theta}| \le 2M_0 \\ \sigma_0, & \text{if } |\hat{\theta}| > 2M_0 \end{cases}$$

where $\hat{\theta}^\tau = [\hat{\theta}_1^\tau, \hat{\theta}_2^\tau]^\tau, M_0 > 0, \sigma > 0$ are design parameters and M_0 is large enough so that $|\theta| < M_0$. In the case of the continuous switching σ-modification adaptive law, the tracking error will converge to the residual set

$$\mathcal{D}_1 \triangleq \{\varepsilon\colon |\varepsilon|^2 \le c_0[1/K^2 + \sigma_0]\}$$

where c_0 is a constant independent of K and σ. ◇

III. Simulations

In order to test the applicability of our theoretical results, we performed simulations on the following scalar system:

$$\dot{x} = \theta_1 x^2 + (\theta_{21}\sin(x) \cdot \exp(0.0001x^2) + \theta_{22} \cdot \exp(0.0001x^2) + \theta_{23})u \tag{36}$$

with $[\theta_1, \theta_{21}, \theta_{22}, \theta_{23}] = [2, -2, 2.5, 10^{-3}]$. The initial state of the system was set equal to 100 and the control objective is to regulate x at zero. Note that $g(x) = \theta_{21}\sin(x) * \exp(0.0001x^2) + \theta_{22}\exp(0.0001x^2) + \theta_{23}$ is always positive, and thus the system is feedback-linearizable.

We first attempted to stabilize system (36) using the CEFL control (10). The variable v in (22) was set equal to $-kx$ with $k = 100$. The adaptive law for adjusting $\hat{\theta}_i$ was the same as adaptive law (27) with the difference that κ_2 was set equal to one; the matrix Γ_i was chosen equal to $0.01I$ and $\sigma = 0.02$, where I is the identity matrix. The initial parameter estimates $\hat{\theta}_1(0), \hat{\theta}_{21}(0), \hat{\theta}_{22}(0), \hat{\theta}_{23}(0)$ was set equal to 0.0, -0.1, 0.1, and 10^9, respectively. Note that the above choice for $\hat{\theta}_i(0)$ is such that $\hat{\theta}_2^\tau(0)\phi_g(x) > 0$ for all $x \in \Re^n$. For this particular choice for $\hat{\theta}_i(0)$ the CEFL controller produces unstable trajectories as shown in Fig. 2.

We then attempted to apply the proposed switching controller for the same choice of $\hat{\theta}_i(0)$. Similar to the CEFL controller, the variable v in (22) was set equal to $-kx$ with $k = 100$ and $\Gamma_i = 0.01I, \sigma = 0.02$ in (27); moreover, the parameter $\delta_{\max}$ was set equal to 10^{-8}. The function $a_k(\cdot)$ was chosen as follows: at first, we observed that the function $|f(x)| + 2/|g(x)|$ is bounded from above by 15 000 as it is shown in Fig. 3. For this reason, we chose $a_K(x) = 15{,}000$ for all $x \in \Re^n$. Then, according to Remark 4, we decomposed $a_K(\cdot)$ using (19); in particular, we chose $\tilde{a}_K(x) = 15{,}000 \equiv a_K(x)$ and $\overline{a}_K = 1$. Finally, the parameter K was set equal to 100; note that such a choice for K satisfies the conditions of Theorem 1. Fig. 4 shows the performance of the proposed switching controller. The upper subplot corresponds to the state trajectory of the system, while the lower subplot corresponds to the trajectory of the switching variable ϱ. Clearly, the proposed controller produces stable trajectories.

IV. Conclusions

In this paper we designed and analyzed an adaptive control scheme for a linear-in-the-parameters feedback linearizable system. The scheme involves a switching strategy with hysteresis that overcomes the classical problem of computability of the control law when the identification model is uncontrollable at certain instants of time.

It is shown that the proposed scheme guarantees signal boundedness for all finite initial conditions and convergence of the tracking error to a small residual set. The residual set can be made arbitrarily small by choosing a certain design parameter.

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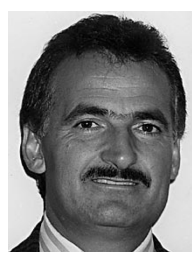

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