Stability Analysis of Switched Linear Systems under Persistent Dwell-Time Constraints

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***Abstract*—Persistent dwell-time (PDT) constraint is demonstrated to be powerful for modeling the dynamics of switched systems with nonuniform time-dependent scheduling constraints. However, the study of switched systems under PDT constraints presently struggles to provide nonconservative and convex stability conditions for stabilization. In this note, we present a novel concept called *dictionary* to precisely characterize PDT constraints. It outperforms the existing methods for PDT constraints which undesirably involve inadmissible or redundant switching sequences. By building an advanced dictionary equivalent to any concerned PDT constraint, we develop a nonconservative stability criterion which is further lifted to preserve convexity. Two numerical examples verify the derived theories.**

***Index Terms*—Persistent dwell-time, stability, switched system.**

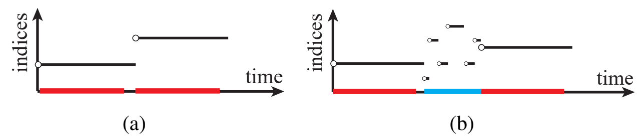

Fig. 1: Time-dependent scheduling constraints that decide the indices of activated subsystems with time. (a) A dwell-time constraint. (b) A persistent dwell-time constraint. Red lines indicate the non-switching intervals. The blue line implies the arbitrary-switching interval.

I. INTRODUCTION

Persistent dwell-time (PDT) constraint is a powerful tool to represent a class of nonuniform time-dependent scheduling constraints in the field of switched systems. Compared to the dwell-time constraint which uniformly requires any non-switching intervals to be not less than a constant value, a PDT constraint not only accepts non-switching disjoint intervals, but also accommodates arbitrary-switching intervals [1]. Fig. 1 compares these two constraints. This flexible feature enables the PDT constraint to model the systems with intermittent faults [2], or mechanical systems that involve low-frequency and high-frequency motions [3].

Several significant results on the stability analysis problem of PDT switched systems have been reported. In [4], the authors present a sufficient stability criterion which confines the established Lyapunov function respectively on non-switching and arbitrary-switching intervals. This result is further generalized in [5] where the authors relax the stability conditions by presenting the idea of stage and applying the lifting approach [6]. Some extensions and applications of this idea can be found in [7]–[9]. However, these conditions are non-convex and conservative due to the introduction of tuning parameters. Recently, a promising method called the meta sequence list approach is proposed in [10], which can achieve a convex and less conservative stability criterion. However, this approach involves redundant PDT switching sequences and thus only provides sufficiency. Hence, the main problem to be addressed in this note is: Can the nonconservative stability criterion of PDT switched systems be derived by presenting a precise characterization of PDT constraints, and also preserve convexity for stabilization?

The stability analysis problem for switched systems under uniform time-dependent scheduling constraints has attracted a considerable attention [11]–[14]. Particularly in the discrete-time domain, convex and nonconservative stability conditions of switched systems with arbitrary and dwell-time switchings are obtained where the nonconservativeness is achieved by capturing fixed-length patterns of switching sequences [15], [16]. However, the PDT constraint exhibits weaker uniformity which cannot be equivalently represented by certain fixed-length patterns. In [17], [18], a novel framework is presented for the stability analysis of switched systems with switching sequences constrained by an automaton, but it cannot be explicitly extended for stabilization. Moreover, how to determine nodes and paths in a directed graph for PDT constraints still needs to be discussed.

To address the aforementioned problem, we present a novel concept called "dictionary" that can remove inadmissible and redundant PDT switching sequences undesirably introduced in [5], [10]. Moreover, by building an equivalent advanced dictionary to any concerned PDT constraint, we develop a convex and nonconservative stability criterion that also remains effective when the PDT constraint reduces to unconstrained

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or dwell-time constraints. The convex feature can significantly facilitate extensions from stability analysis to stabilization.

Notations: $\mathbb{R}^n$ denotes the vector space for n-tuples of real numbers; $\mathbb{N}_+$ denotes the set of non-negative integers; $\mathbb{N}[t_1, t_2] := \{k \in \mathbb{N}_+ | t_1 \leq k \leq t_2\}$. The concepts called "dictionary" and "list" are borrowed from Python programming language. We use $|\cdot|$ to denote the length of a list. For a list $\mathcal{P}$, $\mathcal{P}_i$ denotes the ith element, and $\mathcal{P}_i(j)$ denotes the jth element of $\mathcal{P}_i$, $i \in \mathbb{N}[0, |\mathcal{P}|-1]$, $j \in \mathbb{N}[0, |\mathcal{P}_i|-1]$. i^N denotes a sequence of element i of length N. $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ denote the largest and smallest singular values. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a class $\mathcal{KL}$ function if for each s, the function $\beta(r, s)$ is a class $\mathcal{K}$ function with respect to r, and for each r, the function $\beta(r, s)$ decreases with respect to s, and $\beta(r, s) \to 0$ as $s \to \infty$. The set of $n \times n$ symmetric positive definite matrices is denoted by $\mathbb{S}^n_{\succ 0}$.

II. Preliminaries and Problem Formulation

Consider a class of discrete-time switched linear systems:

$$x(k+1) = A_{\sigma(k)} x(k), x(k_0) = x_0 \tag{1}$$

where $x(k) \in \mathbb{R}^n$ denotes the system state, $A_{\sigma(k)} \in \mathbb{R}^{n \times n}$ dictates system dynamics at the kth sampling instant, $\sigma(k) : \mathbb{N}_+ \to \mathcal{I}_N := \mathbb{N}[1, N]$ indicates the activated subsystem among N possible modes. Let $k_0, k_1, \ldots, k_s, \ldots,$ be the switching instants, $s \in \mathbb{N}_+$, and $x_0 \in \mathbb{R}^n$ be the initial state.

Definition 1: [1] The set $S_{\text{p-dwell}}[\tau, T]$, $\tau > 0$, $T \in [0, \infty]$, is a PDT-constrained set of switching signals, if there is an infinite number of disjoint intervals of length no smaller than τ on which σ is constant, and consecutive intervals with this property are separated by no more than T. The constant τ is called the persistent dwell-time and T the period of persistence.

Remark 1: In the discrete-time domain, the sets of switching signals under the dwell-time constraint and unconstrained can be denoted by $S_{\text{p-dwell}}[\tau, 0]$ and $S_{\text{p-dwell}}[1, 0]$, respectively.

Definition 2: [1] The switched system in (1) is globally uniformly asymptotically stable (GUAS) under certain switching signals σ if for any initial condition x_0, there exists a class $\mathcal{KL}$ function β such that the solution of the system in (1) satisfies the inequality $\|x(k)\| \leq \beta(\|x_0\|, k)$, $\forall k \in \mathbb{N}_+$.

Problem. The problem we consider in this technical note is to establish a nonconservative and convex stability criterion for the system in (1) under PDT constraints.

Several significant concepts have been reported to characterize switching signals for stability analysis, such as the automaton, the M-sequence, and the virtual clock [15]–[17]. However, turning a PDT constraint into an automaton still needs to determine appropriate nodes and paths, and the computation of the constrained joint spectral radius cannot be explicitly extended for stabilization; the other two methods perform well for the uniform time-dependent scheduling constraints, but fail for the nonuniform PDT constraints. Moreover, the idea of stage presented in [5] involves a mass of inadmissible PDT switching sequences due to the introduction of tuning parameters, resulting in nonconvex conditions and conservative H_∞ disturbance attenuation level. The meta sequence list approach inevitably appends some redundant sequences to the established list [10] which can only provide sufficient stability conditions. To develop a precise characterization of PDT constraints for nonconservative stability conditions, we present a concept called dictionary which can be established equivalently to any concerned PDT constraint.

Definition 3: A dictionary $\mathcal{D}_S$ of a PDT-constrained set S is defined by

$$\begin{aligned} &\text{dictionary } \mathcal{D}_S = \{ \\ &\qquad \text{words: } \mathcal{W}(S), \ \text{\# finite sequences} \\ &\qquad \text{grammar: } \mathcal{G}(\mathcal{W}) \ \ \text{\# concatenation rules} \\ &\} \end{aligned} \tag{2}$$

which consists of two keys, i.e., "words" and "grammar". $\mathcal{W}(S)$ is a list that contains a finite number of sequences. $\mathcal{G}(\mathcal{W})$ is a list that regulates the concatenation rules among sequences in $\mathcal{W}(S)$.

Definition 4: The relationship between a dictionary $\mathcal{D}_S$ and a PDT-constrained set S is defined as follows:

- $\mathcal{D}_S$ is said to be sufficient to S, if any sequences in S belong to $\mathcal{D}_S$-derived sequences.
- $\mathcal{D}_S$ is said to be necessary to S, if any $\mathcal{D}_S$-derived sequences belong to S.
- $\mathcal{D}_S$ is said to be equivalent to S if $\mathcal{D}_S$ is sufficient and necessary to S.

Definition 5: Consider a PDT-constrained set S and a dictionary $\mathcal{D}_S$ equivalent to S. An advanced dictionary $\mathcal{A}_S(L)$ is defined by

$$\begin{aligned} &\text{advanced dictionary } \mathcal{A}_S(L) = \{ \\ &\qquad \text{phrases: } \mathcal{P}(\mathcal{W}(S), L) \ \text{\# finite sequences} \\ &\qquad \text{usages: } \mathcal{U}(\mathcal{P}) \qquad \text{\# concatenation rules} \\ &\} \end{aligned} \tag{3}$$

which consists of two keys, i.e., "phrases" and "usages". $\mathcal{P}(\mathcal{W}(S), L)$ is a list of sequences, and each element contains L "words" in $\mathcal{D}_S$ under "grammar", $L \in \mathbb{N}$. $\mathcal{U}(\mathcal{P})$ is a list that regulates the concatenation rules among "phrases". $\mathcal{P}_i$ can be concatenated to the end of $\mathcal{P}_j$ if the first "word" in $\mathcal{P}_i$ is allowed to be concatenated to the end of the last "word" in $\mathcal{P}_j$ under "grammar".

Inspired by [10], we can group the "words" $\mathcal{W}(S)$ into two categories, denoted by $\bar{\mathcal{W}}(\tau)$ and $\hat{\mathcal{W}}(T)$. $\bar{\mathcal{W}}(\tau)$ is used to construct the non-switching intervals by $\bar{\mathcal{W}}(\tau) := \{\bar{\mathcal{W}}_0, \bar{\mathcal{W}}_1, \ldots, \bar{\mathcal{W}}_{N\tau-1}\} := \{i^\tau, i^{\tau+1}, \ldots, i^{2\tau-1}\}, i \in \mathcal{I}_N$. $\hat{\mathcal{W}}(T)$ is used to construct the arbitrary-switching intervals in the period of persistence by $\hat{\mathcal{W}}(T) := \{\hat{\mathcal{W}}_0, \hat{\mathcal{W}}_1, \ldots, \hat{\mathcal{W}}_{M-1}\}$ where M denotes the length of $\hat{\mathcal{W}}(T)$, and complies with:

1) The lengths of sequences in $\hat{\mathcal{W}}(T)$ belong to $\mathbb{N}[1, T]$.
2) The sequence does not belong to $\hat{\mathcal{W}}(T)$, if it contains at least one interval of length no smaller than τ on which σ is constant.

Note that the elements in $\mathcal{W}(S)$ have different lengths, and fixed-length patterns such as the virtual clock approach in [16] cannot precisely characterize the PDT constraints. Different from the meta sequence list approach in [10] where redundant switching sequences are involved by appending each element in $\hat{\mathcal{W}}(T)$ to certain elements in $\bar{\mathcal{W}}(\tau)$, here we modify the concatenation rules together with the so-called meta sequence list so as to remove all the redundant sequences:

i) Rule 1: $\bar{\mathcal{W}}_i$ can be concatenated to the end of $\bar{\mathcal{W}}_j$, $\forall i,j \in \mathbb{N}[0, N\tau-1]$.
ii) Rule 2: $\bar{\mathcal{W}}_i$ can be concatenated to the end of $\hat{\mathcal{W}}_j$ if $\bar{\mathcal{W}}_i(0) \neq \hat{\mathcal{W}}_j\left(\left|\hat{\mathcal{W}}_j\right|-1\right)$, $\forall i \in \mathbb{N}[0, N\tau-1]$, $\forall j \in \mathbb{N}[0, M-1]$.
iii) Rule 3: $\hat{\mathcal{W}}_i$ can be concatenated to the end of $\bar{\mathcal{W}}_j$ if $\bar{\mathcal{W}}_j(0) \neq \hat{\mathcal{W}}_i(0)$, $\forall i \in \mathbb{N}[0, M-1]$, $\forall j \in \mathbb{N}[0, N\tau-1]$.

Example 1: A dictionary equivalent to a set $S_{\text{p-dwell}}[2,3]$ can be established as follows:

$$
\begin{aligned}
&\text{dictionary } \mathcal{D}_{S_{\text{p-dwell}}[2,3]} = \{\\
&\quad \text{words:}\\
&\quad \mathcal{W}(S) = \{\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_5, \mathcal{W}_6, \mathcal{W}_7, \mathcal{W}_8, \mathcal{W}_9\}\\
&\qquad = \{1^2, 1^3, 2^2, 2^3, \{1\}, \{2\}, \{1,2\}, \{2,1\}, \{1,2,1\},\\
&\qquad\quad \{2,1,2\}\},\\
&\quad \text{grammar:}\\
&\quad \mathcal{G}(\mathcal{W}) = \{\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_6, \mathcal{G}_7, \mathcal{G}_8, \mathcal{G}_9\}\\
&\qquad = \{\mathbb{N}[0,9]\backslash\{4,6,8\}, \mathbb{N}[0,9]\backslash\{4,6,8\},\\
&\qquad\quad \mathbb{N}[0,9]\backslash\{5,7,9\}, \mathbb{N}[0,9]\backslash\{5,7,9\},\\
&\qquad\quad \{2,3\}, \{0,1\}, \{0,1\}, \{2,3\}, \{2,3\}, \{0,1\}\}\\
&\}
\end{aligned} \tag{4}
$$

It can be checked that the dictionary $\mathcal{D}_{S_{\text{p-dwell}}[2,3]}$ in (4) is equivalent to $S_{\text{p-dwell}}[2,3]$. Moreover, the dictionary can be further extended to construct an equivalent advanced dictionary $\mathcal{A}_S(L)$ with a positive integer L.

III. Main Results

In this section, we consider the stability analysis problem of the system in (1) and an associated system derived from an equivalent advanced dictionary of form (3) as follows:

$$\hat{x}(l+1) = \hat{A}_{\eta(l)}\hat{x}(l), \hat{x}(l_0) = x_0 \tag{5}$$

where $\hat{x}(l) \in \mathbb{R}^n$ denotes the system state, and

$$\begin{cases} \hat{A}_{\eta(l)} \in \left\{ \prod_{p=0}^{|\mathcal{P}_i|-1} A_{\mathcal{P}_i(p)} : \mathcal{P}_i \in \mathcal{P}(\mathcal{W}(S), L), i \in \mathcal{I}_{\mathcal{P}} \right\} \\ \eta(l) : \mathbb{N}_+ \to \mathcal{I}_{\mathcal{P}} := \mathbb{N}[0, |\mathcal{P}(\mathcal{W}(S), L)| - 1] \end{cases}$$

Note that $\prod_{p=k_1}^{k_2} A_p := A_{k_2}A_{k_2-1}\cdots A_{k_1}$, and $\mathcal{P}_i(p)$ denotes the pth subsystem in the ith "phrase" $\mathcal{P}_i$. Each subsystem $\hat{A}_i$ is the product of system matrices inside a "phrase", and the switching signal $\eta(l)$ indicates the possible "phrase" under "usages", denoted by $\eta \in \mathcal{U}$. Let $\hat{k}_0, \hat{k}_1, \ldots, \hat{k}_s, \ldots,$ be the starting instants of each "phrase" for the system in (1), so it holds that $\hat{x}(s) = x(\hat{k}_s)$, $s \in \mathbb{N}_+$.

Theorem 1: Consider the switched system in (1), a set $S = S_{\text{p-dwell}}[\tau, T]$, and an equivalent dictionary $\mathcal{D}_S$. The following two statements are equivalent:

a) The switched system in (1) is GUAS with $\sigma \in S$.
b) There exists an equivalent advanced dictionary $\mathcal{A}_S(L)$, $L \in \mathbb{N}$, derived from $\mathcal{D}_S$ such that the switched system in (5) is GUAS with $\eta \in \mathcal{U}$.

Proof. Proof of (a)⇒(b). Consider an arbitrary interval $[\tau_1, \tau_2)$, $\forall \tau_2 \geq \tau_1$, $\tau_1, \tau_2 \in \mathbb{N}_+$. Take an arbitrary unit-norm vector $z \in \mathbb{R}^n$ and let $\bar{x} \in \mathbb{R}^n$ be the solution of the system in (1) with any initial condition $\bar{x}(\tau_1) = z$. Since the system in (1) is GUAS with $\sigma \in S$, and the switching is independent on the trajectory, there exists a class $\mathcal{KL}$ function β_1 such that

$$\begin{aligned} \|\bar{x}(\tau_2)\| &= \left\|\prod_{k=\tau_1}^{\tau_2-1} A_{\sigma(k)}\bar{x}(\tau_1)\right\| = \left\|\prod_{k=\tau_1}^{\tau_2-1} A_{\sigma(k)}z\right\| \\ &\leq \beta_1(\|z\|, \tau_2-\tau_1) = \beta_1(1, \tau_2-\tau_1) \end{aligned} \tag{6}$$

Since z can be arbitrarily chosen with $\|z\| = 1$, we have

$$\left\|\prod_{p=0}^{|\mathcal{P}_i|-1} A_{\mathcal{P}_i(p)}\right\| \leq \beta_1(1, |\mathcal{P}_i|) \leq \bar{\beta}, \forall i \in \mathcal{I}_P \tag{7}$$

where $\bar{\beta} := \max_{i\in\mathcal{I}_P}\{\beta_1(1, |\mathcal{P}_i|)\} < 1$ can be guaranteed by selecting a sufficiently large L so as to enlarge $|\mathcal{P}_i|$. From (6) and (7), we obtain

$$\begin{aligned} \|\hat{x}(l)\| &= \left\| \prod_{k=\hat{k}_{l-1}}^{\hat{k}_l-1} A_{\sigma(k)} \prod_{k=\hat{k}_{l-2}}^{\hat{k}_{l-1}-1} A_{\sigma(k)} \cdots \prod_{k=\hat{k}_0}^{\hat{k}_1-1} A_{\sigma(k)}x_0 \right\| \\ &\leq \beta_1\left(1, \hat{k}_l - \hat{k}_{l-1}\right)\cdots\beta_1\left(1, \hat{k}_1-\hat{k}_0\right)\|x_0\| \\ &\leq \bar{\beta}^l \|x_0\| \end{aligned}$$

Letting $\beta_2(r,s) := \bar{\beta}^s r$, we get the desired $\mathcal{KL}$ function β_2 such that $\|\hat{x}(l)\| \leq \beta_2(\|x_0\|, l)$ holds.

Proof of (b)⇒(a). Since the system in (5) is GUAS with $\eta \in \mathcal{U}$, there exists a class $\mathcal{KL}$ function β_1 such that $\|\hat{x}(l)\| \leq \beta_1(\|x_0\|, l)$ holds. We can select a sufficiently large $\bar{T}$ so as to ensure $\beta_1(1, \bar{T}) < \delta$, for some $0 < \delta < 1$. Similar to (6), we can obtain

$$\left\|\prod_{p=t}^{t+\bar{T}-1} \hat{A}_{\eta(p)}\right\| \leq \beta_1(1, \bar{T}) < \delta, t \in \mathbb{N}_+.$$

We can continue to increase $\bar{T}$ such that $0 < \lambda := -\frac{\ln\delta}{\bar{T}} < 1$ holds. Note that increasing $\bar{T}$ does not change the inequality $\beta_1(1, \bar{T}) < \delta$, $0 < \delta < 1$, since $\beta_1(1, \bar{T})$ decreases as $\bar{T}$ increases. Then, for any $l \in [(d-1)\bar{T}, d\bar{T})$, $d \in \mathbb{N}$, we have

$$\begin{aligned} \|\hat{x}(l)\| &= \left\|\prod_{p=(d-1)\bar{T}}^{l-1} \hat{A}_{\eta(p)} \cdots \prod_{p=0}^{\bar{T}-1} \hat{A}_{\eta(p)}x_0\right\| \\ &\leq \beta_1(1,0)\delta^{d-1}\|x_0\| \\ &\leq \beta_1(1,0)e^{-\lambda(d-1)\bar{T}}\|x_0\| \\ &\leq \beta_1(1,0)e^{-\lambda(l-\bar{T})}\|x_0\| = \frac{1}{\delta}\beta_1(1,0)e^{-\lambda l}\|x_0\| \end{aligned}$$

Moreover, for any $k \in [\hat{k}_{l-1}, \hat{k}_l)$, $l \in \mathbb{N}$, it holds that

$$\begin{aligned} \|x(k)\| &\leq b\left\|x\left(\hat{k}_{l-1}\right)\right\| \leq \frac{b}{\delta}\beta_1(1,0)e^{-\lambda(l-1)}\|x_0\| \\ &= \frac{b}{\delta}\beta_1(1,0)e^{-\frac{\lambda}{\bar{X}}(l-1)\bar{X}}\|x_0\| \\ &\leq \frac{b}{\delta}\beta_1(1,0)e^{-\frac{\lambda}{\bar{X}}\left(k-\bar{X}\right)}\|x_0\| \end{aligned}$$

$$= \frac{b}{\delta}\beta_1(1,0)\, e^{\lambda} e^{-\frac{\lambda}{\bar{X}}k} \|x_0\|$$

where $b := \max_{i\in\mathcal{I}_P}\{\sigma_{\max}(\prod_{p=0}^{|\mathcal{P}_i|-1} A_{\mathcal{P}_i(p)}), \sigma_{\max}(\prod_{p=0}^{|\mathcal{P}_i|-2} A_{\mathcal{P}_i(p)}), \ldots, \sigma_{\max}(A_{\mathcal{P}_i(0)})\}$ and $\bar{X} := \max_{i\in\mathcal{I}_P}\{|\mathcal{P}_i|\}$.

Letting $\beta_2(r,s) := \frac{b}{\delta}\beta_1(1,0)\, e^{\lambda} e^{-\frac{\lambda}{\bar{X}}s} r$, the desired class $\mathcal{KL}$ function is determined. This completes the proof. ■

Theorem 1 demonstrates the equivalence of the global uniform asymptotic stability between the systems in (1) and (5). In the following theorem, we will present the nonconservative stability criterion for the system in (5) which is further lifted to preserve convexity by the lifting approach [6].

Theorem 2: Consider the switched system in (1), a set $S = S_{\text{p-dwell}}[\tau, T]$, and an equivalent dictionary $\mathcal{D}_S$. The following three statements are equivalent:

a) The switched system in (1) is GUAS with $\sigma \in S$.

b) There exists an equivalent advanced dictionary $\mathcal{A}_S(L)$, $L \in \mathbb{N}$, derived from $\mathcal{D}_S$, and matrix sequences $O(i) \in \mathbb{S}^n_{\succ 0}$, $i \in \mathcal{I}_{\mathcal{P}}$, such that

$$\hat{A}_i^{\mathrm{T}} O(i)\hat{A}_i - O(j) \prec 0, \forall j \in \mathcal{I}_{\mathcal{P}}, \forall i \in \mathcal{U}_j \quad (8)$$

c) There exists an equivalent advanced dictionary $\mathcal{A}_S(L)$, $L \in \mathbb{N}$, derived from $\mathcal{D}_S$, and matrix sequences $R_i(p) \in \mathbb{S}^n_{\succ 0}$, $p \in \mathbb{N}[0, |\mathcal{P}_i|]$, $i \in \mathcal{I}_{\mathcal{P}}$, such that

$$A^{\mathrm{T}}_{\mathcal{P}_i(p)} R_i(p+1) A_{\mathcal{P}_i(p)} - R_i(p) \prec 0, \quad (9)$$
$$\forall i \in \mathcal{I}_{\mathcal{P}}, p \in \mathbb{N}[0, |\mathcal{P}_i| - 1]$$

$$R_i(0) - R_j(|\mathcal{P}_j|) \prec 0, \forall j \in \mathcal{I}_{\mathcal{P}}, \forall i \in \mathcal{U}_j \quad (10)$$

Proof. Proof of (a)⇒(b). If the system in (1) is GUAS with $\sigma \in S$, there exists a class $\mathcal{KL}$ function β such that $\|x(k)\| \le \beta(\|x_0\|, k)$ holds, where k can be selected such that the sequence

$$\{\sigma(0), \ldots, \sigma(k)\} = \{\mathcal{P}_i(0), \ldots, \mathcal{P}_i(|\mathcal{P}_i| - 1)\}$$

is the ith "phrase" in $\mathcal{A}_S(L)$. This means

$$\left\|x\left(\hat{k}_1\right)\right\| = \|\hat{x}(1)\| = \left\|\hat{A}_{\eta(\hat{k}_0)} x_0\right\|$$
$$= \left\|\prod_{p=0}^{\left|\mathcal{P}_{\eta(\hat{k}_0)}\right|-1} A_{\mathcal{P}_{\eta(\hat{k}_0)}(p)} x_0\right\| \le \beta\left(\|x_0\|, \hat{k}_1\right).$$

Note that we can increase $\hat{k}_1 = \left|\mathcal{P}_{\eta(\hat{k}_0)}\right|$ by increasing L.

Since β is a class $\mathcal{KL}$ function, it implies that $\lim_{L\to\infty}\beta\left(\|x_0\|, \hat{k}_1\right) = 0$, which leads to

$$\lim_{L\to\infty} \hat{A}_i = 0, \forall i \in \mathcal{I}_{\mathcal{P}}.$$

Thus, for any arbitrarily chosen $O(i) \in \mathbb{S}^n_{\succ 0}$, $i \in \mathcal{I}_{\mathcal{P}}$, there exists a constant $\varepsilon > 0$ such that $\forall j \in \mathcal{I}_{\mathcal{P}}, \forall i \in \mathcal{U}_j$,

$$\lim_{L\to\infty} \hat{A}_i^{\mathrm{T}} O(i)\hat{A}_i - O(j) = -O(j) \prec -\varepsilon I \prec 0.$$

It also implies that we can select an integer L^* to build the advanced dictionary such that, for any $L \ge L^*$, the following inequality holds

$$\hat{A}_i^{\mathrm{T}} O(i)\hat{A}_i - O(j) \prec 0, \forall j \in \mathcal{I}_{\mathcal{P}}, \forall i \in \mathcal{U}_j$$

which implies (8).

Proof of (b)⇒(a). We can assign $O(i) \in \mathbb{S}^n_{\succ 0}$, $i \in \mathcal{I}_{\mathcal{P}}$, to every "phrase" in $\mathcal{A}_S(L)$, and establish the Lyapunov function for the system in (5) in form of

$$\hat{V}(\hat{x}(l)) := \mathrm{s}\left(\hat{x}^{\mathrm{T}}(l)\, O(\eta(l-1))\,\hat{x}(l)\right), l \in \mathbb{N} \quad (11)$$

where $\mathrm{s}(\cdot)$ denotes the arithmetic square root and $\eta(l-1)$ takes values in $\mathcal{I}_{\mathcal{P}}$. So it holds that

$$\mathrm{s}(\sigma_1)\|\hat{x}(l)\| \le \hat{V}(\hat{x}(l)) \le \mathrm{s}(\sigma_2)\|\hat{x}(l)\|, \forall i \in \mathcal{I}_{\mathcal{P}} \quad (12)$$

where

$$\sigma_1 := \min_{i\in\mathcal{I}_P}\{\sigma_{\min}(O(i))\}, \sigma_2 := \max_{i\in\mathcal{I}_P}\{\sigma_{\max}(O(i))\}. \quad (13)$$

We can select a sufficiently small ε satisfying

$$0 < \varepsilon < \sigma_2 + \mathrm{s}(\delta\sigma_2) \quad (14)$$

with

$$\delta := \max_{i\in\mathcal{I}_{\mathcal{P}}}\{\sigma_{\max}(\hat{A}_i^{\mathrm{T}} O(i)\hat{A}_i)\}$$

such that

$$\hat{A}_i^{\mathrm{T}} O(i)\hat{A}_i - O(j) \prec -\varepsilon I \prec 0, \forall j \in \mathcal{I}_{\mathcal{P}}, \forall i \in \mathcal{U}_j. \quad (15)$$

Suppose that $\eta(l-1) = j \in \mathcal{I}_{\mathcal{P}}$, and $\eta(l) = i \in \mathcal{U}_j$. From (15) we can get

$$\begin{aligned}
&\hat{V}(\hat{x}(l+1)) - \hat{V}(\hat{x}(l)) \\
&= \mathrm{s}\left(\hat{x}^{\mathrm{T}}(l+1)\, O(i)\,\hat{x}(l+1)\right) - \mathrm{s}\left(\hat{x}^{\mathrm{T}}(l)\, O(j)\,\hat{x}(l)\right) \\
&= \frac{\hat{x}^{\mathrm{T}}(l+1)\, O(i)\,\hat{x}(l+1) - \hat{x}^{\mathrm{T}}(l)\, O(j)\,\hat{x}(l)}{\mathrm{s}(\hat{x}^{\mathrm{T}}(l+1)\, O(i)\,\hat{x}(l+1)) + \mathrm{s}(\hat{x}^{\mathrm{T}}(l)\, O(j)\,\hat{x}(l))} \\
&= \frac{\hat{x}^{\mathrm{T}}(l)\,\hat{A}_i^{\mathrm{T}} O(i)\,\hat{A}_i\hat{x}(l) - \hat{x}^{\mathrm{T}}(l)\, O(j)\,\hat{x}(l)}{\mathrm{s}\left(\hat{x}^{\mathrm{T}}(l)\,\hat{A}_i^{\mathrm{T}} O(i)\,\hat{A}_i\hat{x}(l)\right) + \mathrm{s}(\hat{x}^{\mathrm{T}}(l)\, O(j)\,\hat{x}(l))} \\
&< \frac{-\varepsilon\|\hat{x}(l)\|^2}{\mathrm{s}(\delta)\|\hat{x}(l)\| + \mathrm{s}(\sigma_2)\|\hat{x}(l)\|} = -\lambda\|\hat{x}(l)\|
\end{aligned} \quad (16)$$

where $\lambda := \varepsilon/(\mathrm{s}(\delta) + \mathrm{s}(\sigma_2)) > 0$ is a positive constant. By (12) and (16), we can further obtain

$$\hat{V}(\hat{x}(l+1)) < \left(1 - \frac{\lambda}{\mathrm{s}(\sigma_2)}\right)\hat{V}(\hat{x}(l))$$

where $0 < 1 - \lambda/\mathrm{s}(\sigma_2) < 1$ is guaranteed by (14). Together with (12), we can get

$$\|\hat{x}(l)\| \le \frac{\mathrm{s}(\sigma_2)}{\mathrm{s}(\sigma_1)}\left(1 - \frac{\lambda}{\mathrm{s}(\sigma_2)}\right)^l \|\hat{x}(0)\| = \beta(\|x_0\|, l)$$

where $\beta(r,s) := \mathrm{s}(\sigma_2)/\mathrm{s}(\sigma_1)\,(1 - \lambda/\mathrm{s}(\sigma_2))^s\, r$. Thus, the switched system in (5) is GUAS with $\eta \in \mathcal{U}$. The system in (1) is also GUAS with $\sigma \in S$ according to Theorem 1.

Proof of (b)⇔(c). Following the main idea of the lifting approach in [6], [10], one can turn the nonconservative conditions in (8) equivalently into the convex conditions in (9) and (10). Details are omitted here for simplicity. ■

Remark 2: Theorem 2 is the main result of this note. Statement (b) provides a nonconservative stability criterion based on the construction of an equivalent advanced dictionary, which is lifted to be convex in Statement (c). Compared to the methods in [15], [16] where several fixed-length sequences are used to represent sets of switching signals under dwell-time

Statement	Number of variables	Size of LMIs
(b)	$\frac{1}{2}n\left(n+1\right)n_1$	$n\sum_{i=0}^{n_1-1}n_3\left(i\right)$
(c)	$\frac{1}{2}n\left(n+1\right)\sum_{i=0}^{n_1-1}n_2\left(i\right)$	$n\sum_{i=0}^{n_1-1}n_3\left(i\right)$ $+n\sum_{i=0}^{n_1-1}\left(n_2\left(i\right)-1\right)$

TABLE I: The computational costs of statements (b) and (c), $n_1=\left|\mathcal{P}\left(\mathcal{W}\left(S\right),L\right)\right|$, $n_2\left(i\right)=\left|\mathcal{P}_i\right|+1$, $n_3\left(i\right)=\left|\mathcal{U}_i\right|$, $i\in\mathcal{I}_P$.

constraints or unconstrained, Statement (b) involves sequences of various lengths suitable for the nonuniform PDT constraints. Moreover, by setting $T=0$ or $\tau=1,T=0$, Theorem 2 can be used for the stability analysis of switched systems under the two uniform time-dependent scheduling constraints.

Remark 3: The main improvements reported in this note, compared to [10], are three-fold: (1) the equivalent dictionary removes all the redundant PDT switching sequences; (2) the advanced dictionary relaxes the constraints on the established Lyapunov function; and (3) Theorem 2 provides a necessary and sufficient stability criterion, but necessity can not be ensured by [10]. The idea used in proving the equivalence between statements (b) and (c) is inspired by the lifting approach [6], [10], but the underlying meaning of the term $\prod_{q=0}^{|\mathcal{P}_i|-1}A_{\mathcal{P}_i(q)}$ in this note dictates the switching sequences in certain "phrases", which is more general and can cover the corresponding term in [10] as a particular case.

Remark 4: The computational costs of statements (b) and (c) are shown in TABLE I. It is seen that Statement (c) has a higher cost, because it requires more variables and linear matrix inequalities (LMIs) caused by the lifting approach [6]. This is the price we have to pay for preserving the convexity with respect to the system data.

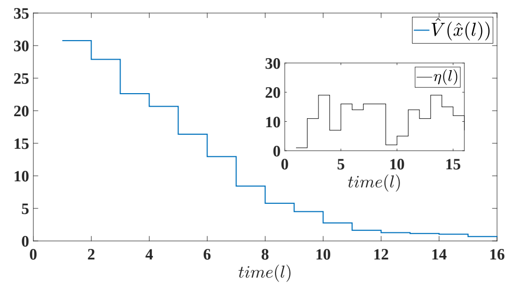
Remark 5: The key drawback of Theorem 2 is that the computational cost grows exponentially as the value of L which is not known *a priori*. The estimation algorithm of L is left for our future work. Intrinsically, Theorem 2 provides a nonconservative verification method to check the stability of the system in (1) with a given $S_{\text{p-dwell}}\left[\tau,T\right]$. The values of τ and T can be selected due to practical concerns.

Remark 6: Compared to the stability criterion that computes the constrained joint spectral radius [17], one of the advantages of Theorem 2 is that the convex feature significantly facilitates the extension to the stabilization of PDT switched systems. The detailed procedures for stabilization are omitted here and one can refer to [10], [16].

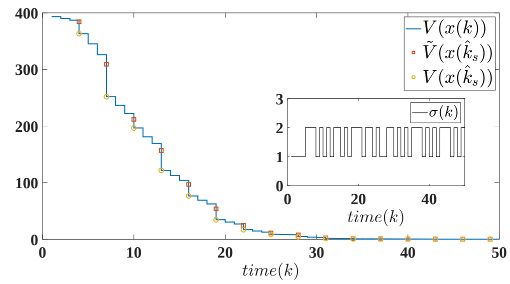
Remark 7: It should be noted that as long as the system in (1) is GUAS, there must exist a Lyapunov function of the form (11) for the system in (5). On the other hand, we can also establish a Lyapunov function for the system in (1) by constructing

$$V\left(x\left(k\right)\right):=x^{\mathrm{T}}(k)\left(R_i(k-\hat{k}_s)\right)x(k), \tag{17}$$
$$k\in\mathbb{N}\left[\hat{k}_s,\hat{k}_{s+1}\right),s\in\mathbb{N}_+,i\in\mathcal{I}_{\mathcal{P}}.$$

This definition is feasible since matrix sequences $R_i\left(p\right)\in\mathbb{S}^n_{\succ 0}$, $p\in\mathbb{N}\left[0,\left|\mathcal{P}_i\right|\right]$, $i\in\mathcal{I}_{\mathcal{P}}$ exist, satisfying the conditions in (9) and (10).



(a) The Lyapunov function (11) of the system in (5).



(b) The Lyapunov function (17) of the system in (1).

Fig. 2: The evolution of the Lyapunov functions by randomly generating an admissible sequence of "phrases" based on an equivalent advanced dictionary $\mathcal{A}_S(3)$, $S=S_{\text{p-dwell}}\left[1,1\right]$.

IV. Numerical Examples

In this section, we provide two examples borrowed from previous literatures to verify the validity of the derived results and make comparisons to some existing methods.

Example 2: [16] Consider the switched system in (1) with two subsystems:

$$A_1=\begin{bmatrix}0.969 & 0.0761\\ -0.7607 & 0.8929\end{bmatrix},A_2=\begin{bmatrix}0.9997 & 0.0685\\ -0.0068 & 0.7259\end{bmatrix}$$

The objective here is to determine a set $S_{\text{p-dwell}}\left[\tau,T\right]$ such that the system in (1) is GUAS with $\sigma\in S_{\text{p-dwell}}\left[\tau,T\right]$.

First, we can build an equivalent dictionary to $S_{\text{p-dwell}}\left[1,1\right]$. Applying Statement (b) of Theorem 2, we find out that the conditions in (8) are feasible when $L=3$. This implies that the system in (1), as well as the system in (5), is GUAS under arbitrary switching. Let x_0 be $\left[-1,5\right]^{\mathrm{T}}$. Fig. 2(a) plots the evolution of the Lyapunov function of the system in (5) where $\eta\left(l\right)$ takes values from the indices of 20 "phrases" in the case of $L=3$. The monotonous decrease of $\hat{V}\left(\hat{x}(l)\right)$ verifies the validity of the condition in (16). Performing Statement (c) of Theorem 2, we can obtain a feasible matrix sequence $R_i(p)$, $i\in\mathcal{I}_{\mathcal{P}}$, $p\in\mathbb{N}\left[0,\left|\mathcal{P}_i\right|\right]$, and accordingly compute the values of the Lyapunov function (17) for the system in (1). Translating η to σ in Fig. 2(b), we find out that the evolution of the Lyapunov function (17) also decreases monotonously.

To verify the validity of the conditions in (10), we define a virtual Lyapunov function $\tilde{V}(x(\hat{k}_s))$ at $\hat{k}_s$, $s\in\mathbb{N}_+$, by $\tilde{V}(x(\hat{k}_s)):=x^{\mathrm{T}}(\hat{k}_s)R_i\left(\left|\mathcal{P}_i\right|\right)x(\hat{k}_s)$, $s\in\mathbb{N}_+,i\in\mathcal{I}_{\mathcal{P}}$, and mark it together with $V(x(\hat{k}_s))$ in Fig. 2(b). $\tilde{V}(x(\hat{k}_s))>V(x(\hat{k}_s))$ always holds, validating the conditions in (10).

Moreover, we apply the meta sequence list approach in [10] to this example, using Remark 2 therein to explore the

T	θ_1	θ_2	τ_1	τ_2	θ
2	4	5	3	4	**2**
3	5	6	4	5	**2**

TABLE II: Minimum dwell-times under given periods of persistence. θ_1, θ_2 and τ_1, τ_2 are mode-dependent dwell-times for the error system and the nominal system computed in [7], respectively. θ is computed by Theorem 2 of this note.

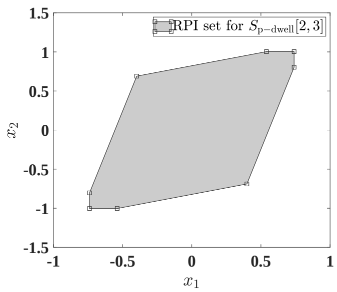


Fig. 3: The robust positive invariant set for $S_{\text{p-dwell}}[2,3]$. x_1 and x_2 denote the two dimensions in $\mathbb{R}^2$.

admissible PDT set. We find out that the minimal dwell-time is 3 under $T = 1$. Thus, the stability criterion in the present Theorem 2 is obviously less conservative than Theorem 2 in [10] since the set $S_{\text{p-dwell}}[3,1]$ is a subset of the set $S_{\text{p-dwell}}[1,1]$.

Example 3: [7] Consider the switched system in (1) with two subsystems:

$$A_1 = \begin{bmatrix} 0.5598 & -0.6162 \\ 0.9402 & -0.7838 \end{bmatrix}, A_2 = \begin{bmatrix} -0.2618 & -0.2517 \\ 0.5002 & 0.5103 \end{bmatrix}$$

The stability of the concerned system can be guaranteed by the quasi-time-dependent method in [7] if the running time of the ith subsystem is no smaller than $\max\{\theta_i, \tau_i\}$, $i = 1, 2$, for a given period of persistence, where θ_i and τ_i are mode-dependent dwell-time for the error system and the nominal system, respectively. TABLE II lists the results of minimum dwell-times computed by Theorem 1 and 2 in [7] (θ_1, θ_2, τ_1, τ_2) and Theorem 2 of this note (θ).

It is seen that θ_1 and θ_2 are the bottlenecks of the minimum dwell-time for the composite system (i.e., the nominal system together with the error system), but actually they can be relaxed to 2 by the presented Theorem 2. To demonstrate that the error system is still GUAS with $\tau = 2$ and $T = 3$ (worse than the case of $\tau = 2, T = 2$), we compute the robust positive invariant (RPI) set for $S_{\text{p-dwell}}[2,3]$, as shown in Fig. 3. The existence of the RPI set implies that the error system is GUAS with a smaller persistent dwell-time.

V. Conclusion

In this note, we investigate the stability analysis problem of switched systems under PDT constraints. To precisely characterize an arbitrarily given PDT constraint, we present a concept called dictionary which can be established equivalently to the constraint. The dictionary can also be turned into an advanced dictionary with a positive integer L. We show the equivalence between the stability of a PDT switched system and the existence of a corresponding advanced dictionary, which further derives convex and nonconservative stability criterion. Yet, we note that the potential drawback of the proposed method is that the complexity of verifying the stability conditions increases exponentially with the integer L which is unknown *a priori*. This is the price we have to pay for the nonconservative feature. Further investigation on estimating the value of L is left for our future work.

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