Delay-Dependent and Order-Dependent Stability and Stabilization of Fractional-Order Linear Systems With Time-Varying Delay

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***Abstract*—Stability of fractional-order (FO) delayed systems remains a formidable problem. In this brief, asymptotic stability and stabilization of FO linear systems with time-varying structured uncertainties and time-varying delay are discussed. By employing FO Razumikhin theorem, two new delay-dependent and order-dependent stability criteria in form of LMI for FO nominal systems and FO uncertain sytems are first formulated, respectively. Moreover, state feedback controllers of stabilization are also derived with the help of the proposed stability criteria. The numerical simulation demonstrates the effectiveness of the theoretical formulation.**

***Index Terms*—Fractional-order system, stability, stabilization, delay.**

I. Introduction

THE STABILITY analysis of FO time-delay systems (FOTDS) is a challenging issue, especially FO time-varying delay systems (FOTVDS). On the one hand, the characteristic equation of a FOTDS involves exponential type transcendental terms and fractional (non-integer) orders. Therefore, there exists an infinite number of characteristic roots and it is not straightforward to determine whether all roots of the characteristic equations lie on the left half-plane or not. On the other hand, since calculating the FO derivatives of Lyapunov functions poses problems, the time-domain stability analysis methods for integer-order systems, such as the Lyapunov functional method, can not be easily generalized to FOTDS and FOTVDS. Researches about the stability of FOTDS have been developed in two directions. The first is to find all possible stability regions in the parameter space [1], [2], [3], [4]. The second is to explore some conditions ensuring the stability of the system [5], [6], [7], [8].

The stability regions and the stability conditions of FOTDS can be checked through different schemes, namely the Rouche's theorem [9], numerical algorithms [1], the Lambert W function [2], the Argument Principle for complex functions and the Hassard's technique [7] and frequency domain (Nyquist plot). Regardless of which method is used for stability analysis, the obtained stability conditions can be divided into being delay-independent and delay-dependent. When the delay is small, it is obvious that delay-dependent stability conditions possess less conservativeness than delay-independent ones. Nevertheless, these contributions do not provide delay-dependent stability criteria or algorithms for testing the stability of a given general FOTDS, not to mention FOTVDS. Therefore, the main goal of this brief is to present a simple and easily verifiable delay-dependent and order-dependent criterion for the stability testing of FOTVDS. Two new delay-dependent and order-dependent stability criteria are derived for FO linear systems and FO uncertain linear systems in form of LMI, respectively. Moreover, stabilization controller design methods are presented. The results have the simple forms and are convenient for calculation and application. This brief can also be extended to the problem of stability and stabilization of such FO nonlinear systems with time-varying delays.

II. Preliminaries and Model Description

The following symbols stand for: $C([a, b], R^n)$ the set of continuous functions mapping the interval $[a, b]$ to R^n, I identity matrix with the appropriate dimension, and $\star$ the elements below the main diagonal of a symmetric block matrix. The superscript T the transpose, $diag\{\cdot\}$ the diagonal matrix, $X > 0 (< 0)$ a symmetric positive definite (negative definite) matrix. $C = C([-h, 0], R^n)$, $x_t(\theta) \in C$ be a segment of function defined as $x_t(\theta) = x(t+\theta)$, $-h \le \theta \le 0$.

Let us now consider the following n-dimensional nominal FOTVDS

$$\begin{cases} D^{\alpha}x(t) = A_0x(t) + A_1x(t-h(t)) + Bu(t), & t > 0, \\ x(t) = \phi(t), & t \in [-h, 0], \end{cases} \quad (1)$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T \in R^n$ represents the state vector of the system, the FO α belongs to the interval $(0, 1)$, $A_0 \in R^{n\times n}$ and $A_1 \in R^{n\times n}$ are the nominal matrix and the state delayed matrix, $h(t)$ denotes the time delay, being a

Manuscript received April 30, 2019; revised June 14, 2019; accepted June 27, 2019. Date of publication July 1, 2019; date of current version June 4, 2020. This work was supported by the National Natural Science Funds of China under Grant 61403115 and Grant 11571016. This brief was recommended by Associate Editor X. Wu. *(Corresponding author: Liping Chen.)*

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Digital Object Identifier 10.1109/TCSII.2019.2926135

time-varying continuous function that satisfies $0 \leq h(t) \leq h$ and $x(t) = \phi(t)$ is a continuous vector-valued initial condition on $[-h, 0]$. Solution forms of system (1) was given in [10]. The Caputo derivative of FO α of function $x(t)$ is defined as follows:

$$D^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}x^{(n)}(\tau)d\tau,$$

where $n-1 < \alpha < n \in Z^{+}$, $\Gamma(\cdot)$ is the Gamma function, $\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt$.

When the system contains time-varying structured uncertainties, it can be described by

$$\begin{cases} D^{\alpha}x(t) = (A_0 + \Delta A_0)x(t) \\ \qquad\qquad + (A_1 + \Delta A_1)x(t-h(t)) + Bu(t), \\ x(t) = \phi(t), \quad t \in [-h, 0], \end{cases} \tag{2}$$

where the uncertain matrices $\Delta A_0, \Delta A_1$ are time-varying uncertain matrices with appropriate dimensions subject to the following form:

$$\begin{bmatrix} \Delta A_0 & \Delta A_1 \end{bmatrix} = E_1 H(t) \begin{bmatrix} F_1 & F_2 \end{bmatrix},$$

where E_1, F_1 and F_2 are known constant real matrices with appropriate dimensions, and $H(t)$ is the unknown time-varying matrix satisfying

$$H^T(t)H(t) \leq I.$$

Lemma 1 [11]: Given matrices $Q = Q^T$, H, E and $R = R^T > 0$ of appropriate dimensions,

$$Q + HFE + E^T F^T H^T < 0,$$

for all F satisfying $F^T F \leq R$, if and only if there exists some $\lambda > 0$ such that

$$Q + \lambda HH^T + \lambda^{-1}E^T RE < 0.$$

III. Main Results

A. Robust Stability

In this subsection, the delay-dependent and order-dependent stability criterion for system (1) with $u(t) = 0$ will be presented, which are stated in the following theorem.

Theorem 1: Given scalars h and $\alpha \in (0, 1]$. If there exist a symmetric positive-definite matrix P, symmetric semi-positive-definite matrices X, Z and any appropriately dimensioned matrix Y such that the following LMIs hold

$$\Psi = \begin{bmatrix} \Psi_{11} & PA_1 + h^{\alpha}\alpha^{-1}YA_1 & h^{\alpha}\alpha^{-1}A_0^T Z \\ \star & -P & h^{\alpha}\alpha^{-1}A_1^T Z \\ \star & \star & -h^{\alpha}\alpha^{-1}Z \end{bmatrix} < 0,$$

$$\Pi = \begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0, \tag{3}$$

where $\Psi_{11} = PA_0 + A_0^T P + h^{\alpha}\alpha^{-1}X + h^{\alpha}\alpha^{-1}(YA_0 + A_0^T Y) + P$, then the nominal system (1) with $u(t) = 0$ is asymptotically stable.

Proof: Constructing the Lyapunov function candidate $V(x(t)) = x^T(t)Px(t)$ and calculating the derivative of $V(x(t))$ along system (1) yields in view of [12, Lemma 1].

$$\begin{aligned} D^{\alpha}V(x(t)) \leq\ & x^T(t)(PA_0 + A_0^T P)x(t) \\ & + 2x^T(t)PA_1 x(t-h(t)). \end{aligned} \tag{4}$$

On the other hand, for any and real matrices $X = X^T$, Y and $Z = Z^T$, satisfying

$$\Pi = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0,$$

the following holds,

$$\begin{aligned} h^{\alpha}\alpha^{-1}\zeta^T(t)\Pi\zeta(t) - \int_{t-h(t)}^{t}(t-s)^{\alpha-1}\zeta^T(t)\Pi\zeta(t)ds \\ \geq 0, \end{aligned} \tag{5}$$

where $\zeta(t) = [x^T(t), (D^{\alpha}x(t))^T]^T$.

Combining (4) and (5), one has

$$\begin{aligned} D^{\alpha}V(x(t)) &\leq x^T(t)(PA_0 + A_0^T P)x(t) \\ &\quad + 2x^T(t)PA_1 x(t-h(t)) \\ &\leq x^T(t)(PA_0 + A_0^T P)x(t) \\ &\quad + 2x^T(t)PA_1 x(t-h(t)) + h^{\alpha}\alpha^{-1}\zeta^T(t)\Pi\zeta(t) \\ &\quad - \int_{t-h(t)}^{t}(t-s)^{\alpha-1}\zeta^T(t)\Pi\zeta(t)ds \\ &= x^T(t)(PA_0 + A_0^T P + h^{\alpha}\alpha^{-1}X \\ &\quad + h^{\alpha}\alpha^{-1}(YA_0 + A_0^T Y) + A_0^T ZA_0)x(t) \\ &\quad + 2x^T(t)(PA_1 + h^{\alpha}\alpha^{-1}(YA_0 + A_0^T ZA_1)) \\ &\quad \times x(t-h(t)) + x^T(t-h(t)) \\ &\quad \times h^{\alpha}\alpha^{-1}A_1^T ZA_1 x(t-h(t)) \\ &\quad - \int_{t-h(t)}^{t}(t-s)^{\alpha-1}\zeta^T(t)\Pi\zeta(t)ds. \end{aligned} \tag{6}$$

Whenever x_t satisfies, for $-h \leq \theta \leq 0$,

$$V(t+\theta, x(t+\theta)) < pV(t, x(t)),$$

for some $p > 1$, one can conclude that

$$px^T(t)Px(t) - x^T(t-h(t))Px(t-h(t)) \geq 0. \tag{7}$$

Using (6) and (7) will supply

$$\begin{aligned} D^{\alpha}V(x(t)) &\leq x^T(t)(PA_0 + A_0^T P + h^{\alpha}\alpha^{-1}X \\ &\quad + h^{\alpha}\alpha^{-1}(YA_0 + A_0^T Y) + A_0^T ZA_0 \\ &\quad + pP)x(t) + 2x^T(t)(PA_1 + h^{\alpha}\alpha^{-1} \\ &\quad \times (YA_1 + A_0^T ZA_1)x(t-h(t)) + x^T(t-h) \\ &\quad \times (h^{\alpha}\alpha^{-1}A_1^T ZA_1 - P)x(t-h(t)) \\ &\quad - \int_{t-h(t)}^{t}(t-s)^{\alpha-1}\zeta^T(t)\Pi\zeta(t)ds \\ &=: \eta^T(t)\Omega\eta(t) \\ &\quad - \int_{t-h(t)}^{t}(t-s)^{\alpha-1}\zeta^T(t)\Pi\zeta(t)ds, \end{aligned} \tag{8}$$

where $\eta(t) = [x^T(t), x^T(t-h)]^T$,

$$\Omega = \begin{bmatrix} \Omega_{11} + h^{\alpha}\alpha^{-1}A_0^T ZA_0 & \Omega_{12} + h^{\alpha}\alpha^{-1}A_0^T ZA_1 \\ \star & \Omega_{22} + h^{\alpha}\alpha^{-1}A_1^T ZA_1 \end{bmatrix} < 0, \tag{9}$$

$\Omega_{11} = PA_0 + A_0^T P + h^\alpha \alpha^{-1} X + h^\alpha \alpha^{-1}(YA_0 + A_0^T Y) + pP$, $\Omega_{12} = PA_1 + h^\alpha \alpha^{-1} YA_1$, $\Omega_{22} = -P$.

If $\Omega < 0$, applying the Schur complement to (9), $\Omega < 0$ is equivalent to

$$\begin{bmatrix} \Sigma_{11} + pP & PA_1 + h^\alpha \alpha^{-1} YA_1 & h^\alpha \alpha^{-1} A_0^T Z \\ \star & -P & h^\alpha \alpha^{-1} A_1^T Z \\ \star & \star & -h^\alpha \alpha^{-1} Z \end{bmatrix} < 0, \quad (10)$$

where $\Sigma_{11} = PA_0 + A_0^T P + h^\alpha \alpha^{-1} X + h^\alpha \alpha^{-1}(YA_0 + A_0^T Y)$.

Now taking $p \to 1^+$ in (10), (10) leads to (3). From (8), the following inequality holds for some small $\varepsilon > 0$.

$$D^\alpha V(t, x(t)) \le -\varepsilon \|x\|^2 - \int_{t-h(t)}^{t} (t-s)^{\alpha-1} \times \zeta^T(t) \Pi \zeta(t) ds \le -\varepsilon \|x\|^2.$$

It follows from [13, Th. 3.1] that the nominal FOTVDS with $u(t) = 0$ is asymptotically stable. This completes the proof. Now, by employing Theorem 1 to system (2) with time-varying structured uncertainties. One can yield the following theorem. ■

Theorem 2: Given scalars h and $\alpha \in (0, 1]$. If there exists a symmetric positive-definite matrix P, symmetric semi-positive-definite matrices X, Z, a scalar λ and any appropriately dimensioned matrix Y such that the following LMIs hold

$$\Psi = \begin{bmatrix} \Psi_{11} & PA_1 + h^\alpha \alpha^{-1} YA_1 & h^\alpha \alpha^{-1} A_0^T Z & F_1^T & \lambda PE_1 + \lambda h^\alpha \alpha^{-1} YE_1 \\ \star & -P & h^\alpha \alpha^{-1} A_1^T Z & F_2^T & 0 \\ \star & \star & -h^\alpha \alpha^{-1} Z & 0 & \lambda h^\alpha \alpha^{-1} ZE_1 \\ \star & \star & \star & -\lambda I & 0 \\ \star & \star & \star & \star & -\lambda I \end{bmatrix} < 0, \quad (11)$$

$$\Pi = \begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} > 0, \quad (12)$$

where $\Psi_{11} = PA_0 + A_0^T P + h^\alpha \alpha^{-1} X + h^\alpha \alpha^{-1}(YA_0 + A_0^T Y) + P$, then the uncertain FOTVDS (2) with $u(t) = 0$ is asymptotically stable.

Proof: Replacing A_0 and A_1 in (3) with $A_0 + E_1 H(t) F_1$ and $A_1 + E_1 H(t) F_2$, respectively, one can observe that Ψ is equivalent to the following condition:

$$\Psi = \begin{bmatrix} \Psi_{11} & PA_1 + h^\alpha \alpha^{-1} YA_1 & h^\alpha \alpha^{-1} A_0^T Z \\ \star & -P & h^\alpha \alpha^{-1} A_1^T Z \\ \star & \star & -h^\alpha \alpha^{-1} Z \end{bmatrix} + \begin{bmatrix} PE_1 + h^\alpha \alpha^{-1} YE_1 \\ 0 \\ h^\alpha \alpha^{-1} ZE_1 \end{bmatrix} H(t) \begin{bmatrix} F_1 & F_2 & 0 \end{bmatrix} + \begin{bmatrix} F_1^T \\ F_2^T \\ 0 \end{bmatrix} H^T(t) \times \begin{bmatrix} E_1^T P + h^\alpha \alpha^{-1} E_1^T Y & 0 & h^\alpha \alpha^{-1} E_1^T Z \end{bmatrix}. \quad (13)$$

Applying Lemma 3 to (13), there exists a positive number λ such that

$$\Psi = \begin{bmatrix} \Psi_{11} & PA_1 + h^\alpha \alpha^{-1} YA_1 & h^\alpha \alpha^{-1} A_0^T Z \\ \star & -P & h^\alpha \alpha^{-1} A_1^T Z \\ \star & \star & -h^\alpha \alpha^{-1} Z \end{bmatrix} + \lambda \begin{bmatrix} PE_1 + h^\alpha \alpha^{-1} YE_1 \\ 0 \\ h^\alpha \alpha^{-1} ZE_1 \end{bmatrix} \times \begin{bmatrix} E_1^T P + h^\alpha \alpha^{-1} E_1^T Y & 0 & h^\alpha \alpha^{-1} E_1^T Z \end{bmatrix} + \lambda^{-1} \begin{bmatrix} F_1^T \\ F_2^T \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & F_2 & 0 \end{bmatrix} < 0,$$

which can be rearranged as

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & PA_1 + h^\alpha \alpha^{-1} YA_1 + \lambda^{-1} F_1^T F_2 & \Upsilon_{13} \\ \star & -P + \lambda^{-1} F_2^T F_2 & h^\alpha \alpha^{-1} A_1^T Z \\ \star & \star & \Upsilon_{33} \end{bmatrix} < 0, \quad (14)$$

$$\Upsilon_{11} = \Psi_{11} + \lambda(PE_1 + h^\alpha \alpha^{-1} YE_1) \times (E_1^T P + h^\alpha \alpha^{-1} E_1^T Y) + \lambda^{-1} F_1^T F_1,$$

$$\Upsilon_{13} = h^\alpha \alpha^{-1} A_0^T Z + \lambda(PE_1 + h^\alpha \alpha^{-1} YE_1)(h^\alpha \alpha^{-1} E_1^T Z),$$

$$\Upsilon_{33} = -h^\alpha \alpha^{-1} Z + \lambda(h^\alpha \alpha^{-1} ZE_1)(h^\alpha \alpha^{-1} E_1^T Z).$$

By Schur complement, (14) is equivalent to (10). This completes the proof. ■

B. Robust Stabilization

In this subsection, as an extension of the method in Section III-A, we shall develop a delay-dependent robust state feedback stabilization method for systems (1) and (2), respectively. State feedback controller is designed as the following form:

$$u(t) = Kx(t), \quad (15)$$

where $K \in R^{m \times n}$ is the feedback gain.

Under the controller (15), the closed-loop control systems (1) and (2) are given by

$$\begin{cases} D^\alpha x(t) = (A_0 + BK)x(t) + A_1 x(t-h), & t > 0, \\ x(t) = \phi(t), & t \in [-h, 0], \end{cases} \quad (16)$$

$$\begin{cases} D^\alpha x(t) = (A_0 + BK + \Delta A_0)x(t) \\ \qquad + (A_1 + \Delta A_1)x(t-h), \\ x(t) = \phi(t), \quad t \in [-h, 0]. \end{cases} \quad (17)$$

Now, the design problem of systems (1) and (2) can be transformed to the stability and robust stability of systems (16) and (17), respectively.

Theorem 3: Given scalars h and $\alpha \in (0, 1]$. The controlled system (16) with controller (15) is asymptotically stable if there exist symmetric positive-definite matrices $\bar{P}$ and any appropriately dimensioned matrix $\bar{X}$ such that the following LMI holds

$$\Psi = \begin{bmatrix} \Psi_{11} & A_1 \bar{P} & h^\alpha \alpha^{-1}(\bar{P} A_0^T + \bar{X}^T B_0^T) \\ \star & -\bar{P} & h^\alpha \alpha^{-1} \bar{P} A_1^T \\ \star & \star & -h^\alpha \alpha^{-1} I \end{bmatrix} < 0, \quad (18)$$

where $\Psi_{11} = A_0 \bar{P} + B_0 \bar{X} + \bar{P} A_0^T + \bar{X} B_0^T + h^\alpha \alpha^{-1} \bar{P} + \bar{P}$.

Moreover, a state-feedback gain matrix is given by

$$K = \bar{X}\bar{P}^{-1}. \tag{19}$$

Proof: By replacing A_0 with $(A_0 + B_0K)$ in (3), one has

$$\Psi = \begin{bmatrix} \Psi_{11} & PA_1 + h^{\alpha}\alpha^{-1}YA_1 & h^{\alpha}\alpha^{-1}(A_0+B_0K)^TZ \\ \star & -P & h^{\alpha}\alpha^{-1}A_1^TZ \\ \star & \star & -h^{\alpha}\alpha^{-1}Z \end{bmatrix} < 0, \tag{20}$$

where $\Psi_{11} = P(A_0 + B_0K) + (A_0 + B_0K)^TP + h^{\alpha}\alpha^{-1}X + h^{\alpha}\alpha^{-1}(Y(A_0 + B_0K) + (A_0 + B_0K)^TY) + P$.

By pre-multiplying and post-multiplying (20) with $diag\{P^{-1}, P^{-1}, I\}$, and let $P^{-1}=\bar{P}$, $Y = 0$, $X = P$, $Z = I$, it results that inequality (18) is equivalent to (3). The proof is completed. ■

Theorem 4: Given scalars h and $\alpha \in (0, 1]$. The controlled system (17) with controller (15) is asymptotically stable if there exist symmetric positive-definite matrices $\bar{P}$, a scalar λ and any appropriately dimensioned matrix $\bar{X}$ such that the following LMI holds

$$\Psi = \begin{bmatrix} \Psi_{11} & A_1\bar{P} & h^{\alpha}\alpha^{-1}(\bar{P}A_0 + \bar{X}^TB_0^T) & \bar{P}F_1^T & \lambda E_1 \\ \star & -\bar{P} & h^{\alpha}\alpha^{-1}\bar{P}A_1^T & \bar{P}F_2^T & 0 \\ \star & \star & -h^{\alpha}\alpha^{-1}I & 0 & \lambda h^{\alpha}\alpha^{-1}E_1 \\ \star & \star & \star & -\lambda I & 0 \\ \star & \star & \star & \star & -\lambda I \end{bmatrix} < 0, \tag{21}$$

where $\Psi_{11} = A_0\bar{P} + B_0\bar{X} + \bar{P}A_0^T + \bar{X}^TB_0^T + h^{\alpha}\alpha^{-1}\bar{P} + \bar{P}$. Moreover, a state-feedback gain matrix is given by

$$K = \bar{X}\bar{P}^{-1}. \tag{22}$$

Proof: Replacing A_0 with $(A_0 + B_0K)$ in (11), it yields

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & h^{\alpha}\alpha^{-1}(A_0+B_0K)^TZ & F_1^T & \lambda PE_1 + \lambda h^{\alpha}\alpha^{-1}YE_1 \\ \star & -P & h^{\alpha}\alpha^{-1}A_1^TZ & F_2^T & 0 \\ \star & \star & -h^{\alpha}\alpha^{-1}Z & 0 & \lambda h^{\alpha}\alpha^{-1}ZE_1 \\ \star & \star & \star & -\lambda I & 0 \\ \star & \star & \star & \star & -\lambda I \end{bmatrix} < 0, \tag{23}$$

where $\Psi_{11} = P(A_0 + B_0K) + (A_0 + B_0K)^TP + h^{\alpha}\alpha^{-1}X + h^{\alpha}\alpha^{-1}(Y(A_0 + B_0K) + (A_0 + B_0K)^TY) + P$, $\Psi_{12} = PA_1 + h^{\alpha}\alpha^{-1}YA_1$.

Similarly, by pre-multiplying and post-multiplying (23) with $diag\{P^{-1}, P^{-1}, I, I\}$, and let $P^{-1}=\bar{P}$, $Y = 0$, $X = P$, $Z = I$, inequality (23) can be rewritten as (21). The proof is completed. ■

Remark 1: Time-varying delay is considered here, if system degenerates into ones with constant time delays, i.e., $h(t) = h$, these results are still new and available.

Remark 2: With the help of the diffusive representation of the Riemann-Liouville integral operator for the continuous frequency distributed model, delayed-dependent stability and stabilization of the nominal FO system (1) were studied in [14]. Reference [15] considered the delay-independent stability based on the detail of the solution of the FO positive delay system, and the delay-independent stabilization problem is studied and solved by using a direct Lyapunov Krasovskii function leading to conditions in terms of a linear program. Here, delayed-dependent and order-dependent (robust) stability and stabilization of the nominal FO system (1) and the uncertain FO system (2) are addressed by employing FO Razumikhin theorem, respectively. Compared with [14], there are less unknown parameters needed to be solved and results are order-dependent. In addition, robust case has been considered.

Remark 3: References [5] and [16] studied finite time stability and asymptotic properties of the following autonomous fractional differential systems with time-invariant delays only involving delayed term, respectively,

$$D^{\alpha}x(t) = Ax(t - \tau). \tag{24}$$

Obviously, the model considered in this brief is more general.

Remark 4: By giving an explicit formula of solutions of linear nonhomogeneous fractional delay differential equations via the variation of constants method, [17] and [16] addressed finite time stability of $D^{x}(t) = Bx(t - \tau) + f(x, x(x))$ and $D^{\alpha}x(t) = Ax(t - \tau)$, respectively. Here, asymptotic stability was studied. It is worth noticing that finite time stability and Lyapunov asymptotic stability are independent concepts, which neither implies nor exclude each other.

Remark 5: References [1], [2], [3], [4], [5], [6], [7] discussed the stability regions or conditions of FOTDS. However, these results are diagrammatic or numerical, and inconvenient to check. FO Razumikhin theorem provides us the possibility to obtain stability criteria in form of LMI, which can be easy to solve by LMI toolbox in MATLAB. Here, by using FO Razumikhin theorem and constructing appropriate inequality, delayed-dependent and order-dependent (robust) stability and stabilization results of the nominal (uncertain) FOTDS are derived, which are much more convenient to use in engineering application.

IV. Numerical Example

Let us consider uncertain FO system (2) with system matrices

$$A_0 = \begin{bmatrix} -2 & 0 & 0 & 2 \\ 1 & -5 & 0 & -2 \\ -3 & 0 & -2 & -1 \\ 1 & 0 & -3 & 4 \end{bmatrix}, A_1 = \begin{bmatrix} 0.1 & 0 & 1 & -1 \\ -2 & 3 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \\ 0.1 & 0 \\ 0 & 0 \end{bmatrix}, F_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, h = 0.1, \alpha = 0.9.$$

Using LMI toolbox in MATLAB, one obtains the feasible solution from LMI (21) as follows:

$$\bar{P} = \begin{bmatrix} 0.2363 & 0.0826 & 0.1220 & -0.1092 \\ 0.0826 & 0.3439 & -0.0134 & -0.0328 \\ 0.1220 & -0.0134 & 0.2987 & -0.0113 \\ -0.1092 & -0.0328 & -0.0113 & 0.0938 \end{bmatrix},$$

$$\bar{X} = \begin{bmatrix} 0.4594 & -1.2723 & 0.3762 & -2.0400 \end{bmatrix}.$$

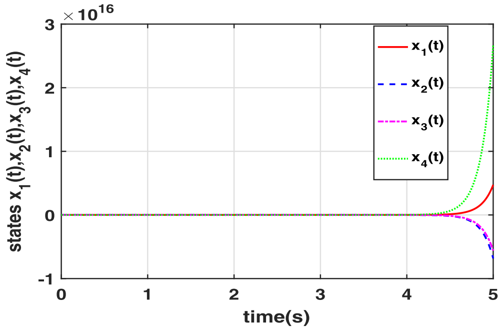

Fig. 1. Time response of the selected system without state feedback controller.

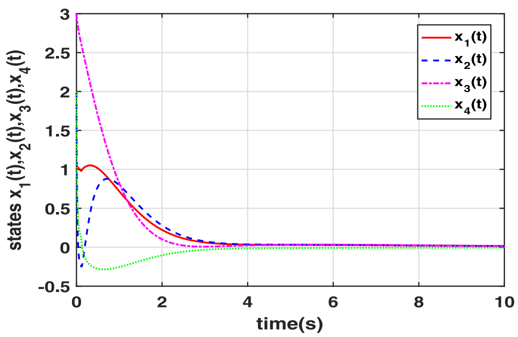

Fig. 2. Time response of the selected system with state feedback controller.

Moreover, the feedback controller gain is given by

$$K = \begin{bmatrix} -27.0006 & -1.8317 & 10.2171 & -52.5811 \end{bmatrix}.$$

Therefore, it follows from Theorem 4 that the controlled system in the example is asymptotically stable. Choose $h(t) = 0.1\cos t$, the initial states $x(s) = [\sin 0.1s, \sin 0.1s, 3\sin 0.1s, 2\sin 0.1s]^T (s \in [-0.1, 0])$. A numerical solution of FOTVDS proposed in [18] is used here. Fig. 1 and Fig. 2 show that states of the system without and with state feedback controller, respectively.

V. Conclusion

Time delays are inevitable in practical plants including FO systems. It is very important and necessary to consider the dynamic behaviour of FO delayed systems. However, to explore simple and useful sufficient stability conditions of FO delayed systems is complex and is still a formidable problem. Here, with the help of FO Razumikhin theorem, two new delay-dependent and order-dependent stability criteria in term of LMIs for FO linear system with time-varying delay are proposed. It is shown that the new criteria are simpler and easier to use.

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