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Stabilizing controller design for nonlinear fractional order systems with time varying delays

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Abstract: To deal with stabilizing of nonlinear affine fractional order systems subject to time varying delays, two methods for finding an appropriate pseudo state feedback controller are discussed. In the first method, using the Mittag-Lefler function, Laplace transform and Gronwall inequality, a linear stabilizing controller is derived, which uses the fractional order of the delayed system and the upper bound of system nonlinear functions. In the second method, at first a sufficient stability condition for the delayed system is given in the form of a simple linear matrix inequality (LMI) which can easily be solved. Then, on the basis of this result, a stabilizing pseudo-state feedback controller is designed in which the controller gain matrix is easily computed by solving an LMI in terms of delay bounds. Simulation results show the effectiveness of the proposed methods.

Keywords: fractional order nonlinear system, time varying delay, state feedback control, linear matrix inequality (LMI), stabilizing.

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1. Introduction

A lot of real dynamic systems are more accurate to be described by fractional order equations instead of classical integer order ones [1–4], such as electrochemistry systems [5], diffusion [6], viscoelastic systems [7], biological systems [8] and so on.

Fractional order calculations have been applied in many engineering fields due to the new tools which they have provided in order to describe the detail properties of various real systems. The reader can refer to [9–11] in order to have a review on the theory and application of fractional order systems and calculation. Fractional order systems have been also studied from different aspects such as stability analysis [12], identification [13], control [14], synchronization [15] and so on.

On the other hand, time delay is one of the undeniable phenomena in the problem of control of real dynamic systems. Time delay often exists in different technical systems and usually has a negative impact on the system performance, so many methods have been proposed in the control theory to deal with the time delay in the stability problem of dynamic systems [16,17]. In addition, it should be noted that considering the time delay as a fixed value in dynamic systems is only an unrealistic approximation. In most systems including fractional order ones, the delay is variable in practice depending on system states and operating conditions of the system.

Likewise, the fractional-order time-delay systems also present a class of behaviors in real applications [18–20] and the stability analysis and stabilizing of fractional-order systems in the presence of time delays have attracted great attention of researchers in recent years [21–26].

Till now, many studies have been presented in order to deal with time delays in the stability problem of the fractional order systems. Stability analysis of the linear fractional differential system with multiple time delays was considered in [21]. Fractional-order chaotic systems in the presence of delay was considered in [22] and stability of delayed linear continuous-time fractional order systems was studied in [26,27]. Finite-time stability of fractional order systems with multi-state time delay was also presented in [28].

Recently, in [24] linear fractional order distributed delay systems were presented and sufficient conditions to check the stability of a fractional commensurate order nonlinear time-varying delay system was concluded in [25]. By using a new functional transformation, a new stability criterion for time-varying delay fractional-order financial systems was presented in [22]. Finally, in [23] for a particular class of fractional nonlinear systems called fractional-order hopfield neural networks, an adaptive sliding mode controller was designed for achieving synchronization in the presence of model uncertainties and time delays.

By reviewing the previous researches, it can be concluded that stability analysis of nonlinear fractional order systems with time varying-delay is studied only for a spe-

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cific type of fractional order systems and to the best of authors knowledge, stabilizing of a more general form of fractional order nonlinear systems in the presence of time varying delays has not been reported earlier in the literature.

The main contribution of this paper is the design of a stabilizing feedback controller for a more general form of nonlinear fractional order systems that are in the presence of time varying delay. To do this, two different methods are presented.

In the first method, using Laplace transformation, Mittag-Leler function and Gronwall inequality, a linear controller is derived for asymptotic stability of nonlinear fractional order delayed systems. In the second method, a sufficient condition for stability is presented in a simple linear matrix inequality (LMI) form. In addition, on the basis of this stability condition, the existence of a stable state feedback controller is proved and the feedback controller gain matrix is computed by solving another LMI.

This paper is organized as follows: In Section 2 problem formulation along with some definitions is given. The designed fractional-order controllers are presented in Section 3. Simulation results in Section 4 are presented to confirm the proposed methods. Finally, conclusions are presented in Section 5.

2. Problem formulation

Consider the following nonlinear fractional order (NFO) input affine system:

$$D_t^{\alpha}\boldsymbol{x}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{f}(\boldsymbol{x}(t))+\boldsymbol{g}(\boldsymbol{x}(t-\tau(t)))+\boldsymbol{B}\boldsymbol{u}(t) \tag{1}$$

where $0<\alpha<1$ is the fractional order, $\boldsymbol{f}(\boldsymbol{x}(t))\in\mathbf{R}^n$ and $\boldsymbol{g}(\boldsymbol{x}(t))\in\mathbf{R}^{n\times m}$ are nonlinear functions and $\boldsymbol{f}(\boldsymbol{x}(t))$ satisfies the Lipschitz condition. $\boldsymbol{x}(t)\in\mathbf{R}^n$ where $\boldsymbol{x}(t)=\boldsymbol{\psi}(t), t\in[-\tau_{\max},0]$ is the pseudo state and $\boldsymbol{u}(t)\in\mathbf{R}^m$ is the control input vector. The time-delay function $\tau(t)\in\mathbf{R}^+$ is continuous, bounded and satisfies the following inequalities:

$$\begin{cases} 0<\tau(t)<\tau_{\max}<\infty \\ 0<\partial\tau_{\min}\leqslant\dfrac{\partial\tau(t)}{\partial t}\leqslant\partial\tau_{\max}<1 \end{cases} \tag{2}$$

where $\tau_{\max}$ and $\partial\tau_{\max}$ are two constant scalars.

The objective of this paper is to find a stabilizing input controller for the system in (1) in the presence of time varying delay in (2).

Firstly, some preliminaries on the fractional order calculus are presented, which appear in our study. ${}_0D_t^{\alpha}$ in (1) denotes the Caputo fractional derivative operator [29] which is defined as

$${}_0D_t^{\alpha}\boldsymbol{x}(t)=\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^t\frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\mathrm{d}\tau,\ \ 0<\alpha\notin\mathbf{Z}^+ \tag{3}$$

where $\Gamma(\cdot)$ denotes the Gamma function and $\lceil\alpha\rceil$ maps α to the least integer which is greater than or equal to α [29].

Remark 1 It is worth to mention that there are a number of definitions for fractional order derivatives including Riemann-Liouville, Grunwald-Letnikov and Caputo. However, the Caputo fractional derivative is used mostly compared with the others to describe the model of fractional order dynamic systems because the initial conditions for a fractional order differential equation which is defined by the Caputo derivatives are the same as those for the integer-order counterpart. Thus, using Caputo definition, we can describe fractional order dynamic systems in real applications.

The asymptotic stability for an NFO system can be defined as follows.

Definition 1[29] Stability of NFO system

The equilibrium solution of ${}_0D_t^{\alpha}\boldsymbol{x}(t)=\boldsymbol{f}(\boldsymbol{x}(t))$ is said to be stable if and only if, for any initial conditions $\boldsymbol{x}(0)\in\mathbf{R}^n$, there exists $\delta>0$ such that any solution $\boldsymbol{x}(t)$ of ${}_0D_t^{\alpha}\boldsymbol{x}(t)=\boldsymbol{f}(\boldsymbol{x}(t))$ satisfies $\|\boldsymbol{x}(t)\|<\delta$ for all $t>t_0$. Further, the equilibrium solution of the fractional system is asymptotically stable if and only if the system is stable and $\|\boldsymbol{x}(t)\|\to 0$ as $t\to\infty$.

Using Definition 1, the definition of a nonlinear stable function is also presented as follows.

Definition 2 A nonlinear vector function $\boldsymbol{L}(\boldsymbol{x})\in\mathbf{R}^{n\times 1}$ is asymptotically stable if and only if the NFO system ${}_0D_t^{\alpha}\boldsymbol{x}(t)=\boldsymbol{L}(\boldsymbol{x}(t))$ in (1) is asymptotically stable.

In the next section, two different methods for designing a stabilizing controller for the NFO delayed system in (1) are proposed.

3. Stabilizing controller design

In this part, the main theorem of this paper, which is designing two new controllers for stabilizing the NFO delayed system in (1) is presented. The designed controllers are in the form of state-feedback. Therefore, it is assumed that all the states of system are available. The conditions for obtaining the state feedback gain are obtained through two main theorems.

3.1 The first method: controller design using upper bound of pseudo states

In the first method, the stablizing controller is designed using the upper bounds of the delayed nonlinear functions of the system. First, consider the following assumption. In this assumption, the upper bound of vector functions in system (1) is specified.

Assumption 1 Suppose the nonlinear vector functions $\boldsymbol{f}(\boldsymbol{x})\in\mathbf{R}^n$ and $\boldsymbol{g}(\boldsymbol{x})\in\mathbf{R}^{n\times m}$ satisfy the following conditions:

$$\begin{cases} \|\boldsymbol{f}(\boldsymbol{x}(t))\| \leqslant M_1 \|\boldsymbol{x}(t)\| \\ \|\boldsymbol{g}(\boldsymbol{x}(t-\tau(t))\| \leqslant M_2 \|\boldsymbol{x}(t)\| \end{cases}. \tag{4}$$

It should be noted that Assumption 1 is not a conservative assumption, because it is not assumed that these nonlinear functions are limited or with constraints. It is only assumed that considering time varying delay, the upper bound of each function is a linear function of the norm of the pseudo states of the system.

Now, before presenting the main part of this section, consider the following two lemmas.

Lemma 1 [29] If $0 < \alpha \leqslant 1, \boldsymbol{A} \in \mathbf{R}^{n\times n}$, β is an arbitrary real value and $c > 0$ is a constant value, then

$$E_{\alpha,\beta}(\boldsymbol{A}) \leqslant \frac{c}{1+\|\boldsymbol{A}\|} \tag{5}$$

where $\mu \leqslant \left|\arg(eig(\boldsymbol{A})\right| \leqslant \pi$ with $\mu \in \mathbf{R}$ satisfying $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$.

Lemma 2 [30] Gronwall inequality

If $t \in [0, T]$ and

$$x(t) \leqslant h(t) + \int_0^t k(\tau)x(\tau)\mathrm{d}\tau \tag{6}$$

where $k(t) \geqslant 0$ and all the other functions are continuous in the interval $[0, T]$, then we can obtain

$$x(t) \leqslant h(t) + \int_0^t k(\tau)h(\tau)\exp\left[\int_\tau^t k(u)\mathrm{d}u\right]\mathrm{d}\tau. \tag{7}$$

Now, consider the pseudo state-feedback as

$$u(t) = kx(t). \tag{8}$$

We are ready to express one of the main ideas of this paper via the following theorem.

Theorem 1 Consider the NFO delayed system in (1) and suppose that Assumption 1 is satisfied. If $\alpha\|\boldsymbol{A}\| > d$ where $d = c_2(M_1+M_2) + \|\boldsymbol{B}\|\|\boldsymbol{K}\|$ and M_1 and M_2 are obtained from (4), then the controller in (8) will asymptotically stablize the system.

Proof First, taking the Laplace transform on both sides of (1), one can obtain

$$\begin{aligned} &\boldsymbol{s}^\alpha \boldsymbol{X}(\boldsymbol{s}) - \boldsymbol{s}^{\alpha-1}\boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(\boldsymbol{s}) + \boldsymbol{F}(\boldsymbol{s}) + \boldsymbol{G}(\boldsymbol{s},\tau) + \boldsymbol{B}\boldsymbol{U}(\boldsymbol{s}) \Rightarrow \\ &(\boldsymbol{s}^\alpha \boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(\boldsymbol{s}) = \boldsymbol{s}^{\alpha-1}\boldsymbol{x}(0) + \boldsymbol{F}(\boldsymbol{s}) + \boldsymbol{G}(\boldsymbol{s},\tau) + \boldsymbol{B}\boldsymbol{U}(\boldsymbol{s}) \Rightarrow \\ &\boldsymbol{X}(\boldsymbol{s}) = (\boldsymbol{s}^\alpha \boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{s}^{\alpha-1}\boldsymbol{x}(0) + \boldsymbol{F}(\boldsymbol{s}) + \boldsymbol{G}(\boldsymbol{s},\tau) + \boldsymbol{B}\boldsymbol{U}(\boldsymbol{s})) \end{aligned} \tag{9}$$

where

$$\boldsymbol{F}(\boldsymbol{s}) = \boldsymbol{L}\{\boldsymbol{f}(\boldsymbol{x}(t))\},$$
$$\boldsymbol{G}(\boldsymbol{s},\tau) = \boldsymbol{L}\{\boldsymbol{g}(\boldsymbol{x}(t-\tau(t)))\}.$$

Then, obtain the pseudo state of the system and find its upper bound. Tacking the inverse Laplace transform from (9), one can obtain the solution of the system in (1) as

$$\begin{aligned} \boldsymbol{x}(t) = {} & \boldsymbol{E}_{\alpha,1}(\boldsymbol{A}(t^\alpha))\boldsymbol{x}(0) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\boldsymbol{A}(t-\tau)^\alpha)\boldsymbol{f}(\boldsymbol{x}(t))\mathrm{d}\tau + \\ & \int_0^t (t-\tau)^{\alpha-1}\boldsymbol{E}_{\alpha,\alpha}(\boldsymbol{A}(t-\tau)^\alpha)\boldsymbol{g}(\boldsymbol{x}(t-\tau(t)))\mathrm{d}\tau + \\ & \int_0^t (t-\tau)^{\alpha-1}\boldsymbol{E}_{\alpha,\alpha}(\boldsymbol{A}(t-\tau)^\alpha)\boldsymbol{B}\boldsymbol{K}\boldsymbol{x}(t)\mathrm{d}\tau. \end{aligned} \tag{10}$$

According to Lemma 1 and Lemma 2, we know that there exist some constants such that the upper bound of the system states can be obtained as

$$\begin{aligned} \|\boldsymbol{x}(t)\| \leqslant {} & \frac{c_1\|\boldsymbol{x}_0\|}{1+\|\boldsymbol{A}\|t^\alpha} + \int_0^t \frac{(t-\tau)^{\alpha-1}c_2}{1+\|\boldsymbol{A}\|(t-\tau)^\alpha}\|\boldsymbol{f}(\boldsymbol{x}(\tau))\|\mathrm{d}\tau + \\ & \int_0^t \frac{(t-\tau)^{\alpha-1}c_2}{1+\|\boldsymbol{A}\|(t-\tau)^\alpha}\left\|\boldsymbol{g}(\boldsymbol{x}(\tau-\tau'(\tau)))\right\|\mathrm{d}\tau + \\ & \int_0^t \frac{(t-\tau)^{\alpha-1}c_2}{1+\|\boldsymbol{A}\|(t-\tau)^\alpha}\|\boldsymbol{B}\|\|\boldsymbol{K}\|\|\boldsymbol{x}(\tau)\|\mathrm{d}\tau. \end{aligned} \tag{11}$$

Using Assumption 1, the inequality in (11) can be written as

$$\begin{aligned} \|\boldsymbol{x}(t)\| \leqslant {} & \frac{c_1\|\boldsymbol{x}_0\|}{1+\|\boldsymbol{A}\|t^\alpha} + M_1\int_0^t \frac{(t-\tau)^{\alpha-1}c_2}{1+\|\boldsymbol{A}\|(t-\tau)^\alpha}\|\boldsymbol{x}(\tau)\|\mathrm{d}\tau + \\ & M_2\int_0^t \frac{(t-\tau)^{\alpha-1}c_2}{1+\|\boldsymbol{A}\|(t-\tau)^\alpha}\|\boldsymbol{x}(\tau)\|\mathrm{d}\tau + \\ & \int_0^t \frac{(t-\tau)^{\alpha-1}c_2}{1+\|\boldsymbol{A}\|(t-\tau)^\alpha}\|\boldsymbol{B}\|\|\boldsymbol{K}\|\|\boldsymbol{x}(\tau)\|\mathrm{d}\tau. \end{aligned} \tag{12}$$

By defining $d = c_2(M_1+M_2) + \|\boldsymbol{B}\|\|\boldsymbol{K}\|$, the inequality in (12) can be simplified as

$$\|\boldsymbol{x}(t)\| \leqslant \frac{c_1\|\boldsymbol{x}_0\|}{1+\|\boldsymbol{A}\|t^\alpha} + d\int_0^t \frac{(t-\tau)^{\alpha-1}}{1+\|\boldsymbol{A}\|(t-\tau)^\alpha}\|\boldsymbol{x}(\tau)\|\mathrm{d}\tau. \tag{13}$$

According to Lemma 2, the upper bound of $\|\boldsymbol{x}(\tau)\|$ is written as

$$\begin{aligned} \|\boldsymbol{x}(t)\| \leqslant {} & \frac{c_1\|\boldsymbol{x}_0\|}{1+\|\boldsymbol{A}\|t^\alpha} + \int_0^t \frac{dc_1(t-\tau)^{\alpha-1}\|\boldsymbol{x}_0\|}{(1+\|\boldsymbol{A}\|(t-\tau)^\alpha)(1+\|\boldsymbol{A}\|(\tau)^\alpha)}\cdot\exp\left(\int_0^t \frac{d(t-y)^{\alpha-1}}{(1+\|\boldsymbol{A}\|(t-y)^\alpha)}\mathrm{d}y\right)\mathrm{d}\tau = \\ & \frac{c_1\|\boldsymbol{x}_0\|}{1+\|\boldsymbol{A}\|t^\alpha} + \int_0^t \frac{dc_1(t-\tau)^{\alpha-1}\|\boldsymbol{x}_0\|}{(1+\|\boldsymbol{A}\|\tau^\alpha)(1+\|\boldsymbol{A}\|(t-\tau)^\alpha)^{1-d/\alpha\|\boldsymbol{A}\|}}\mathrm{d}\tau \leqslant \frac{c_1\|\boldsymbol{x}_0\|}{1+\|\boldsymbol{A}\|t^\alpha} + dc_1\|\boldsymbol{x}_0\|\|\boldsymbol{A}\|^{d/\alpha\|\boldsymbol{A}\|-2}\int_0^t (t-\tau)^{d/\|\boldsymbol{A}\|-1}\tau^{-\alpha} = \\ & \frac{c_1\|\boldsymbol{x}_0\|}{1+\|\boldsymbol{A}\|t^\alpha} + dc_1\|\boldsymbol{x}_0\|\|\boldsymbol{A}\|^{d/\alpha\|\boldsymbol{A}\|-2}\cdot\frac{\Gamma(d/\|\boldsymbol{A}\|)\Gamma(1-\alpha)}{\Gamma(1+d/\|\boldsymbol{A}\|-\alpha)}t^{d/\|\boldsymbol{A}\|-\alpha}, \end{aligned} \tag{14}$$

and finally

$$\|\boldsymbol{x}(t)\| \leqslant \frac{c_1\|\boldsymbol{x}_0\|}{1+\|\boldsymbol{A}\|t^{\alpha}} + dc_1\|\boldsymbol{x}_0\|\|\boldsymbol{A}\|^{d/\alpha\|\boldsymbol{A}\|-2}\cdot$$

$$\frac{\Gamma(d/\|\boldsymbol{A}\|)\Gamma(1-\alpha)}{\Gamma(1+d/\|\boldsymbol{A}\|-\alpha)}t^{d/\|\boldsymbol{A}\|-\alpha}. \tag{15}$$

Therefore, since $\alpha\|\boldsymbol{A}\|>d$, the power $t^{d/\|\boldsymbol{A}\|-\alpha}$ will be negative and the right side of (15) will converge to zero and it is easily concluded that

$$\lim_{t\to\infty}\|\boldsymbol{x}(t)\| = 0. \tag{16}$$

□

Remark 2 Theorem 1 states that the matrix corresponding to the linear part in the system (1) is important in the optimal stabilization controller for the system with delay. In this case, the sufficient conditions to obtain asymptotic stability of the NFO delayed system in (1) are achieved from the inequality $\alpha\|\boldsymbol{A}\|>d$. These conditions depend on both the fractional order α and the parameter d which expresses the upper bound of nonlinear functions of the system.

3.2 The second method: controller design using LMI

In this subsection, the stabilizing condition of the system is obtained via LMI. To do this, the method presented in [31] for stabilizing the linear delayed fractional order systems, is extended to the nonlinear delayed fractional order systems as in (1).

Lemma 3 The following fractional order nonlinear system

$$D_t^{\alpha}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) \tag{17}$$

can be expressed as

$$\begin{cases} \dfrac{\partial z(w,t)}{\partial t} = -wz(w,\boldsymbol{t}) + \boldsymbol{f}(\boldsymbol{x}(t)) \\ \boldsymbol{x}(t) = \displaystyle\int_0^{\infty}\mu(w)z(w,t)\mathrm{d}w \end{cases} \tag{18}$$

where $\mu(w)$ is given by

$$\mu(w) = \frac{\sin(\alpha\pi)}{\pi}w^{-\alpha}. \tag{19}$$

Then, consider a more general form of the system in (1) as

$${}_0^C D_t^{\alpha}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t-\tau(t))) + \boldsymbol{Bu}(t),$$

$$0<\tau(t)<\tau_{\max}<\infty; 0<\partial\tau_{\min}\leqslant\frac{\partial\tau(\mathrm{t})}{\partial t}\leqslant\partial\tau_{\max}<1. \tag{20}$$

In order to carry out stability analysis, we first consider an unforced form of the system in (20) that is without input as

$${}_0^C D_t^{\alpha}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t-\tau(t))). \tag{21}$$

Before expressing the main theorems, consider the below assumption.

Assumption 2 Suppose that functions $\boldsymbol{f}(\boldsymbol{x}(t))$ and $\boldsymbol{g}(\boldsymbol{x}(t-\tau(t)))$ satisfy the following inequalities:

$$\begin{cases} \boldsymbol{x}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{x}(t)) \leqslant \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x} \\ \boldsymbol{x}^{\mathrm{T}}\boldsymbol{g}(\boldsymbol{x}(t-\tau(t))) \leqslant \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}_2\boldsymbol{x}(t-\tau(t)) \end{cases} \tag{22}$$

where $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are matrices with appropriate dimensions and $\boldsymbol{P}\in\mathbf{R}^{n\times n}$ is a symmetric positive definite matrix.

Theorem 2 The delayed NFO system in (21) is asymptotically stable if there exists a positive definite matrix $\boldsymbol{P}$ that satisfies the LMI in (23).

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{PA}_1 + \boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P} + 2\boldsymbol{\Lambda}_1 & \boldsymbol{PA}_2 \\ * & -\boldsymbol{\Lambda}_2 \end{bmatrix} \tag{23}$$

where

$$\boldsymbol{\Lambda}_1 = \frac{1}{2(1-\partial\tau_{\min})}\boldsymbol{I},\ \boldsymbol{\Lambda}_2 = \frac{1}{(1-\partial\tau_{\min})}\boldsymbol{I}$$

and $*$ stands for the corresponding part of a symmetric matrix.

Proof Using Lemma 3, (21) can be written as

$$\begin{cases} \dfrac{\partial z(w,t)}{\partial t} = -wz(w,t) + \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t-\tau(t))) \\ \boldsymbol{x} = \displaystyle\int_0^{\infty}\mu(w)z(w,t)\mathrm{d}w \end{cases}. \tag{24}$$

In order to check the stability of the system in (21), consider the following Lyapunov function:

$$v_1(w,t) = z^{\mathrm{T}}(w,t)\boldsymbol{P}z(w,t) \tag{25}$$

where $\boldsymbol{P}\in\mathbf{R}^{n\times n}$ is a symmetric positive definite matrix. The time derivatives of $v_1(\omega,t)$ in (25) with respect to $z(w,t)$ is obtained as

$$\frac{\partial v_1(w,t)}{\partial z(w,t)} = 2z^{\mathrm{T}}(w,t)\boldsymbol{P} \tag{26}$$

and as a result,

$$\frac{\partial v_1(w,t)}{\partial t} = \frac{\partial v_1(w,t)}{\partial z(w,t)}\frac{\partial z(w,t)}{\partial t} =$$

$$2z^{\mathrm{T}}(w,t)\boldsymbol{P}(-wz(w,t)) + \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t-\tau(t)))). \tag{27}$$

Another Lyapunov function candidate $V_1(t)$ can be defined by adding all the monochromatic functions $v_1(w,t)$ with weighting function $\mu(w)$ as

$$V_1(t) = \int_0^{\infty}\mu(w)v_1(w,t)\mathrm{d}w. \tag{28}$$

The derivative of the Lyapunov function (28) along the trajectories of (24) yields

$$\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}=\int_0^\infty \mu(w)\frac{\partial v_1(w,t)}{\partial t}\mathrm{d}w=$$
$$\int_0^\infty \mu(w)2\boldsymbol{z}^{\mathrm{T}}(w,t)\boldsymbol{P}(-w\boldsymbol{z}(w,t))+\boldsymbol{f}(\boldsymbol{x}(t))+\boldsymbol{g}(\boldsymbol{x}(t-\tau(t)))\mathrm{d}w=$$
$$-2\int_0^\infty w\mu(w)\boldsymbol{z}^{\mathrm{T}}(w,t)\boldsymbol{P}(z(w,t))\mathrm{d}w+$$
$$2\int_0^\infty \mu(w)\boldsymbol{z}^{\mathrm{T}}(w,t)\boldsymbol{P}\mathrm{d}w\boldsymbol{f}(\boldsymbol{x}(t))+\boldsymbol{g}(\boldsymbol{x}(t-\tau(t))). \tag{29}$$

By substituting the pseudo-state from (24) into (29), one can obtain

$$\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}=-2\int_0^\infty w\mu(w)\boldsymbol{z}^{\mathrm{T}}(w,t)\boldsymbol{P}(z(w,t))\mathrm{d}w+$$
$$\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}(\boldsymbol{f}(\boldsymbol{x}(t))+\boldsymbol{g}(\boldsymbol{x}(t-\tau(t)))). \tag{30}$$

As a result, in order to achieve positivity of the Lyapunov function and negativity of its derivative, the following sufficient conditions are given:

$$\begin{cases}\boldsymbol{P}>0\\ \boldsymbol{P}(\boldsymbol{f}(\boldsymbol{x}(t))+\boldsymbol{g}(\boldsymbol{x}(t-\tau(t))))<0\end{cases}. \tag{31}$$

Another Lyapunov-Krasovskii function $V_2(t)$ is also defined as

$$V_2=\frac{1}{1-\partial\tau_{\min}}\int_{t-\tau(t)}^{t}\boldsymbol{x}^{\mathrm{T}}(s)\boldsymbol{x}(s)\mathrm{d}s. \tag{32}$$

The time derivative of (32) yields

$$\frac{\mathrm{d}V_2}{\mathrm{d}t}=\frac{1}{1-\partial\tau_{\min}}\{\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{x}(t)\}-$$
$$\frac{1}{1-\partial\tau_{\min}}\left(1-\frac{\partial\tau(t)}{\partial t}\right)\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))\boldsymbol{x}(t-\tau(t)). \tag{33}$$

which is bounded as

$$\frac{\mathrm{d}V_2}{\mathrm{d}t}\leqslant\frac{1}{1-\partial\tau_{\min}}\{\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{x}(t)-\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))\boldsymbol{x}(t-\tau(t))\}. \tag{34}$$

Using Assumption 2, the upper bound of the first Lyapunov function in (30) is obtained as

$$\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}=-2\int_0^\infty w\mu(w)\boldsymbol{z}^{\mathrm{T}}(w,t)\boldsymbol{P}(z(w,t))\mathrm{d}w+$$
$$2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}(\boldsymbol{A}_1\boldsymbol{x}(t)+\boldsymbol{A}_2(\boldsymbol{x}(t-\tau(t))). \tag{35}$$

By adding the two time derivatives of the two Lyapunov functions $V_1(t)$ and $V_2(t)$ such that $V(t)=V_1(t)+V_2(t)$, the following inequality can be easily obtained:

$$\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}\leqslant-2\int_0^\infty w\mu(w)\boldsymbol{z}^{\mathrm{T}}(w,t)\boldsymbol{P}(z(w,t))\mathrm{d}w+$$
$$2\boldsymbol{x}^{\mathrm{T}}(t)(\boldsymbol{P}\boldsymbol{A}_1+\boldsymbol{\Lambda}_1)\boldsymbol{x}(t)+2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{A}_2\boldsymbol{x}(t-\tau(t))-$$
$$\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))\boldsymbol{\Lambda}_2\boldsymbol{x}(t-\tau(t)) \tag{36}$$

where

$$\boldsymbol{\Lambda}_1=\frac{1}{2(1-\partial\tau_{\min})}\boldsymbol{I},\ \boldsymbol{\Lambda}_2=\frac{1}{(1-\partial\tau_{\min})}\boldsymbol{I}.$$

Finally, the inequality in (36) can be stated as

$$\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\Omega}_1\boldsymbol{x}(t)\leqslant 0 \tag{37}$$

where

$$\boldsymbol{\Omega}_1=\begin{bmatrix}2(\boldsymbol{P}\boldsymbol{A}_1+\boldsymbol{\Lambda}_1) & 2\boldsymbol{P}\boldsymbol{A}_2\\ \boldsymbol{0} & -\boldsymbol{\Lambda}_2\end{bmatrix},\ \boldsymbol{x}(t)=\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}(t-\tau(t))\end{bmatrix}.$$

It should be noted that $\boldsymbol{\Omega}_1$ is not a symmetric matrix and therefore (37) cannot be solved by using LMI. Thus, this matrix can be substituted by an equivalent one as

$$\boldsymbol{\Omega}=\frac{\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_1^{\mathrm{T}}}{2}. \tag{38}$$

Thus, in accordance with the theorem, the final inequality is obtained as

$$\boldsymbol{\Omega}=\begin{bmatrix}\boldsymbol{P}\boldsymbol{A}_1+\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{P}+2\boldsymbol{\Lambda}_1 & \boldsymbol{P}\boldsymbol{A}_2\\ \boldsymbol{A}_2^{\mathrm{T}}\boldsymbol{P} & -\boldsymbol{\Lambda}_2\end{bmatrix}. \tag{39}$$

□

Based on Theorem 2, we prove our method for controller design in the form of the following theorem.

Theorem 3 Consider system (20) with Assumption 2. Then the pseudo state feedback $\boldsymbol{u}(t)=\boldsymbol{K}\boldsymbol{x}(t)$ will stabilize the delayed NFO system in (20) if there exist a positive definite matrix $\boldsymbol{P}$ and a matrix $\boldsymbol{Y}_1$ where the following LMI is satisfied:

$$\boldsymbol{\Omega}=\begin{bmatrix}\boldsymbol{A}_1\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}_1^{\mathrm{T}}+\boldsymbol{B}_1\boldsymbol{Y}_1+\boldsymbol{Y}_1^{\mathrm{T}}\boldsymbol{B}_1^{\mathrm{T}}+2\boldsymbol{\Lambda}_1 & \boldsymbol{A}_2\boldsymbol{P}\\ * & -\boldsymbol{\Lambda}_2\end{bmatrix}<0 \tag{40}$$

where

$$\boldsymbol{\Lambda}_1=\frac{1}{2(1-\partial\tau_{\min})}\boldsymbol{I},\ \boldsymbol{\Lambda}_2=\frac{1}{(1-\partial\tau_{\min})}\boldsymbol{I}.$$

In this case, the pseudo-state feedback controller gain matrix is given as $\boldsymbol{K}=\boldsymbol{Y}_1\boldsymbol{P}$.

Proof The proof is similar to the proof of Theorem 2 and is omitted here for simplicity. Actually, by replacing $\boldsymbol{A}_1$ in the proof of Theorem 1 with $\boldsymbol{A}_1+\boldsymbol{B}\boldsymbol{K}$, the LMI (23) will be changed to LMI (40). □

Remark 3 It should be noted that in Theorem 1, the stability of the system is checked by using the upper bound of the delayed and non-delayed nonlinear functions of the system. However, in Theorem 2, the stability conditions are based on the upper bound of the delay. Therefore, by using Theorem 2, one can easily find the maximum allowed delay of the fractional order nonlinear system, in the presence of which the system is still stable.

4. Simulation results

In this part two illustrated examples are presented to show the effectiveness of the proposed controllers.

4.1 The first example

Consider the chaotic Genesio-Tesi system [32] as

$$\begin{cases} {}_0^C D_t^{\alpha} x_1(t) = x_2(t) \\ {}_0^C D_t^{\alpha} x_2(t) = x_3(t) \\ {}_0^C D_t^{\alpha} x_3(t) = -b_1 x_1(t) - b_2 x_2(t) - b_3 x_3(t) + b_4 x_1^2(t-\tau(t)) \end{cases} \tag{41}$$

where $b_i (i = 1, \cdots, 4)$ is a system parameter. In this case considering

$$\boldsymbol{f}(\boldsymbol{x}) = 0,$$

$$\boldsymbol{g}(\boldsymbol{x}) = [0, 0, b_4 x_1^2(t-\tau(t))]^{\mathrm{T}},$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -b_1 & -b_2 & -b_3 \end{bmatrix},$$

$$\tau(t) = \frac{1}{1+t^2}.$$

The system in (41) is in the form of system in (1). In this system, the capacitors voltages are the pseudo states of the system. An implementation of this system is in the form of a circuit as shown in Fig.1.

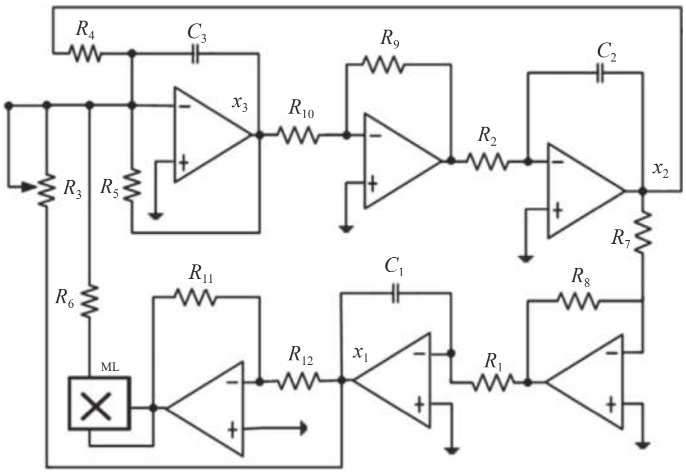

Fig. 1 Implementation of fractional order Genesio-Tesi system [32]

In Fig. 2, the open loop chaotic behavior of the system with parameters $b_1 = 1.1$, $b_2 = 1.1$, $b_3 = 0.45$, $b_4 = 1$, $\alpha = 0.98$ and initial conditions $x(0) = -0.1, y(0) = 0.5$, $z(0) = 0.2$ are shown.

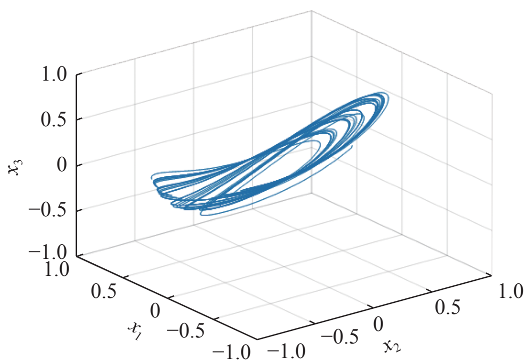

Fig. 2 Chaotic behavior of Genesio-Tesi system

Using Theorem 1, the controller gain in $\boldsymbol{u}(t) = \boldsymbol{K}\boldsymbol{x}(t)$ is obtained as

$$\boldsymbol{K} = \begin{bmatrix} 1.428\,6 & 2.857\,1 & 0 \\ 0 & 1.428\,6 & 5.714\,3 \\ -5.571\,4 & -7 & -9.357\,1 \end{bmatrix}.$$

In this case we have $\|\boldsymbol{A}\| = 11.44$, $|\arg(eig(\boldsymbol{A}))| = [3.141, 1.710, -1.710]^{\mathrm{T}}$, so $|\arg(eig(\boldsymbol{A}))| > \alpha\frac{\pi}{2}$. Therefore, the conditions of Theorem 1 are satisfied as $d = c_2(M_1 + M_2) + \|\boldsymbol{B}\|\|\boldsymbol{K}\| = 8.632$ and $\alpha\|\boldsymbol{A}\| = 12.42 > d$. Also, a saturated constraint by $u_{01} = u_{02} = u_{03} = 1$ is also applied to the system.

Simulation results by applying the controller in the presence of delay are shown in Fig.3. It is evident that the closed-loop Genesio-Tesi system is stable and has a good performance in the presence of time varying delay. In this case, first, all the states are converged to zero and the inputs are not exited from the saturation boundary.

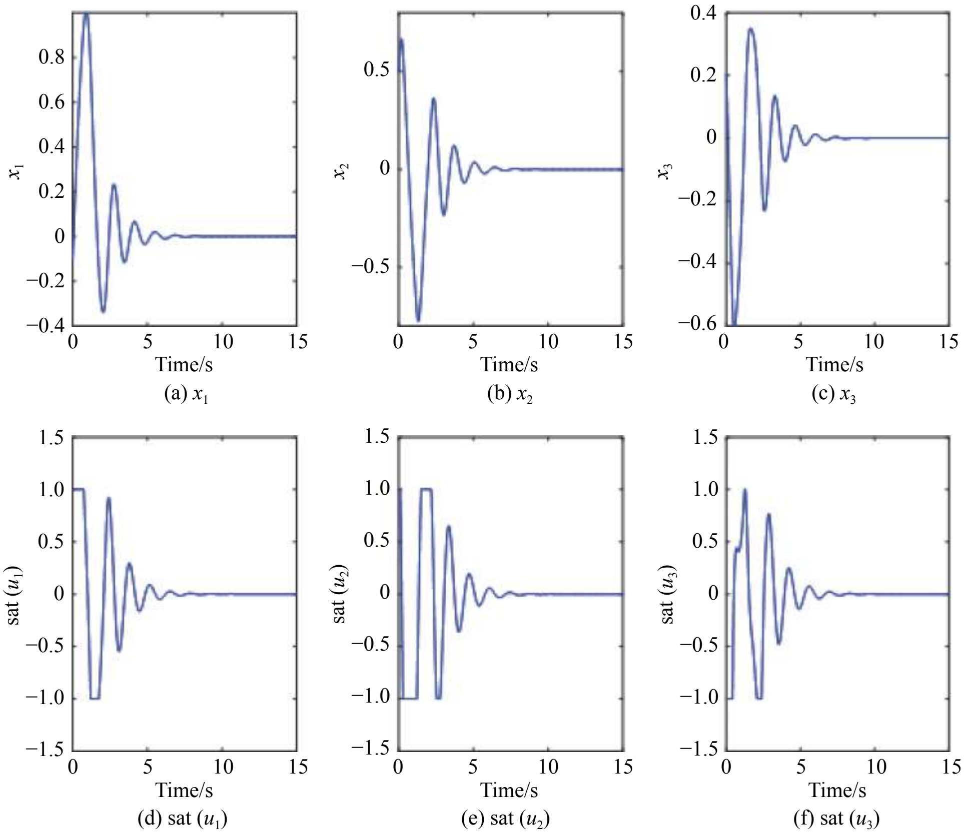

Fig. 3 Time response of state variables and saturated inputs of the controlled system

4.2 The second example

As the second example, consider the following nonlinear delayed fractional order system:

$$ {}_0^C D_t^{\alpha} \boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t-\tau(t))) + \boldsymbol{B}\boldsymbol{u}(t) \tag{42} $$

where

$$ \begin{cases} \boldsymbol{f}(\boldsymbol{x}(t)) = \begin{bmatrix} x_1(t) + x_2(t) - x_1^3(t) \\ x_1(t) + x_2(t) - x_2^3(t) \end{bmatrix} \\ \boldsymbol{g}(\boldsymbol{x}(t - \mathrm{e}^{-0.1t})) = \begin{bmatrix} -x_1(t - \mathrm{e}^{-0.1t}) x_2^2(t - \mathrm{e}^{-0.1t}) \\ -x_2(t - \mathrm{e}^{-0.1t}) x_1^2(t - \mathrm{e}^{-0.1t}) \end{bmatrix} \\ \boldsymbol{B} = \boldsymbol{I}_2 \end{cases} . \tag{43} $$

Considering time delay as $\tau = 3\sin(0.2t) + 2$, the open loop of the system is presented in Fig. 4 that has an unstable behavior in the presence of time varying delay.

The upper bound of time delay and its derivative are

$$ \begin{cases} \tau(t) = 3\sin(0.2t) + 2 \Rightarrow \tau_{\max} = 5 \\ \dfrac{\partial \tau(t)}{\partial t} = 0.6\cos(0.2t) \Rightarrow \begin{cases} \partial \tau_{\min} = 0 \\ \partial \tau_{\max} = 0.6 \end{cases} \end{cases} . \tag{44} $$

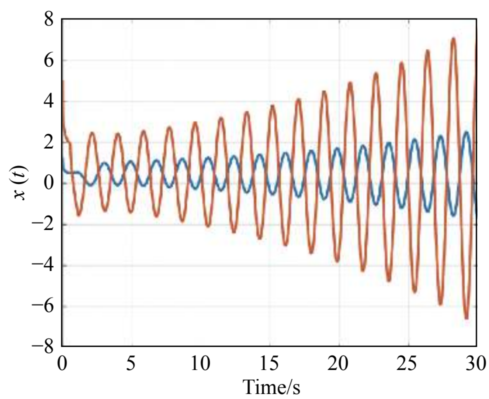

Fig. 4 Time response of open loop of the system in (42) with $\tau = 3\sin(0.2t) + 2$

The derivative of time delay is in the range of Assumption 2. According to Theorem 3 and by solving LMI in (40), matrices $\boldsymbol{P}$ and $\boldsymbol{K}$ are obtained as

$$ \boldsymbol{P} = \begin{bmatrix} 61.872\,1 & 2.023\,1 \\ 0.002\,1 & 31.123\,3 \end{bmatrix}, \boldsymbol{K} = \begin{bmatrix} 132.612\,3 & 0.007\,8 \\ 23.342\,1 & 102.907\,1 \end{bmatrix}. \tag{45} $$

Applying the pseudo state feedback controller, the simulation results are shown in Fig.5. As it is clear from this figure, by establishing the conditions of Theorem 3 and applying the feedback controller $\boldsymbol{u}(t)=\boldsymbol{K}\boldsymbol{x}(t)$ with controller gain in (45), the nonlinear fractional order system with variable time delay in (42) is asymptotically stable.

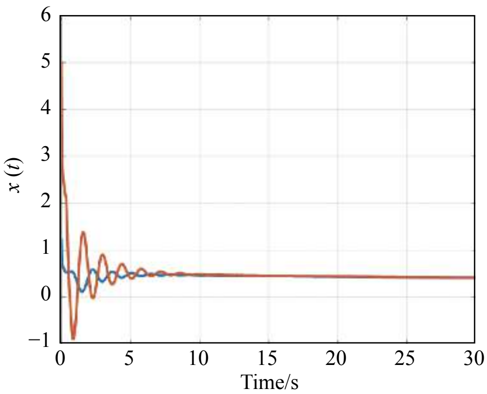

Fig. 5 Time response of closed loop system in (42) with asymptotic stability of system in the presence of time delay $\tau = 3\sin(0.2t)+2$

5. Conclusions

In this paper, the problem of stabilizing fractional-order nonlinear systems in the presence of time varying delays is considered and two different methods are presented in this regards. In the first method, using Laplace transform, Mittag-Leler function and Gronwall inequality, a linear pseudo state feedback controller is derived to stabilize the nonlinear fractional order system in the presence of time varying delay. In the second method, a sufficient stability condition is given in an LMI formulation, which can be easily solved. In addition, a stabilizing pseudo-state feedback controller is also obtained that its gain is computed by solving an LMI. The simulation results on two worked out examples are given to confirm the obtained analytical results.

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