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Asynchronously switched control of switched linear systems with average dwell time[☆]

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ABSTRACT

This paper concerns the asynchronously switched control problem for a class of switched linear systems with average dwell time (ADT) in both continuous-time and discrete-time contexts. The so-called asynchronous switching means that the switchings between the candidate controllers and system modes are asynchronous. By further allowing the Lyapunov-like function to increase during the running time of active subsystems, the extended stability results for switched systems with ADT in nonlinear setting are first derived. Then, the asynchronously switched stabilizing control problem for linear cases is solved. Given the increase scale and the decrease scale of the Lyapunov-like function and the maximal delay of asynchronous switching, the minimal ADT for admissible switching signals and the corresponding controller gains are obtained. A numerical example is given to show the validity and potential of the developed results.

1. Introduction

Switched systems, which are used to model many physical or man-made systems displaying switching features, have been extensively studied in past years (Hespanha, 2004; Johansson & Rantzer, 1998; Liberzon, 2003; Lu, Wu, & Kim, 2006; Morse, 1996). The systems, typically, contain a finite number of subsystems and a switching signal governing the switching among them. The diverse switching signals differentiate switched systems from general time-varying systems, since the solutions of the former are dependent on not only the system's initial conditions but also the switching signals (Hespanha, 2004). As a class of switching signals, average dwell time (ADT) switching means that the number of switches in a finite interval is bounded and the average time between consecutive switching is not less than a constant (Hespanha & Morse, 1999). Rapid progress in the field has shown that ADT switching is more general and flexible than dwell time (DT) switching (Morse, 1996) in related stability analyses and control syntheses (Hespanha, 2004; Sun, Zhao, & Hill, 2006; Vu, Chatterjee, & Liberzon, 2007).

In switched systems, we often call each subsystem a mode, and say that control problems are to design a set of mode-dependent controllers or a mode-independent controller for the unforced system and find admissible switching signals such that the resulting system is stable and satisfies certain performance criteria. Many related reports on this issue are available; see, e.g., Lee and Dullerud (2006), Lu et al. (2006), Rinehart, Dahleh, Reed, and Kolmanovsky (2008), Sun and Ge (2005), Xu and Antsaklis (2004) and the references therein. With an adaptation sense, mode-dependent controller design is less conservative. However, a very common assumption in the "mode-dependent" context is that the controllers are switched synchronously with the switching of system modes, which is quite unpractical. In reality, it takes time to identify the system modes and apply the matched controller, and so phenomena of asynchronous switching between system modes and controller candidates generally exist.[1] The necessities of considering asynchronous switching for efficient control design have been shown in a class of chemical systems (Mhaskar, El-Farra, & Christofides, 2008). Also, a recent study on a class of

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[1] In this paper, we slightly abuse the notion of synchronous (or asynchronous) switching to mean that the switchings of system modes and the switchings of desired mode-dependent controllers are synchronous (respectively, asynchronous). Correspondingly, the delay of asynchronous switching is the time lag from the controllers switching to the system modes switching.

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Markov jump linear systems (MJLSs) shows that the designed controllers based on synchronous switching fail to stabilize the system in the presence of asynchronous switching (Xiong & Lam, 2006) (MJLSs can be viewed as a category of switched systems with autonomous switching determined by an attached stochastic process; see, e.g., Wang, Lam, and Liu (2003)). Other studies on the issue, such as state feedback stabilization (Xie & Wang, 2005), input-to-state stabilization (Xie, Wen, & Li, 2001) and output feedback stabilization (Masubuchi & Tsutsui, 2001), have also been reported. However, the switching signals adopted there are still restricted to DT switching: the DT of each mode also has to be greater than the delay of asynchronous switching. Surprisingly, no attention has been paid to incorporate the advanced ADT switching rule to investigate the problem, even in a linear context.

Thus, in this paper, we aim at addressing the asynchronous control problem for a class of switched linear systems with ADT in both continuous-time and discrete-time contexts. The main contributions of the paper are twofold. First, the extended results of Hespanha and Morse (1999) and Zhang and Shi (2008) on exponential stability for switched nonlinear systems are obtained by relaxing the requirement that the Lyapunov-like function must be always decreasing for each active mode. Second, the asynchronously switched stabilization problem of switched linear systems with ADT is solved by designing a set of mode-dependent stabilizing controllers and finding a set of switching signals with admissible ADT. The existence criteria of the desired controllers are formulated, and the minimal ADT is obtained as well for given maximal delay of asynchronous switching and increase and decrease scales of the Lyapunov-like function.

Notation: The notation used in this paper is fairly standard. The superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{N}$ represents the set of nonnegative integers; the notation $\| \ \|$ refers to the Euclidean vector norm. $\mathcal{C}^1$ denotes the space of continuously differentiable functions, and a function $\beta : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, unbounded, and $\beta(0) = 0$. In addition, in symmetric block matrices or long matrix expressions, we use $*$ as an ellipsis for the terms that are introduced by symmetry and $diag\{\cdots\}$ stands for a block-diagonal matrix. The notation $P > 0$ (≥ 0) means that P is real symmetric and positive definite (semi-positive definite).

2. Problem formulation and preliminaries

Consider a class of switched linear systems given by

$$\delta x(t) = A_\sigma x(t) + B_\sigma u(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, the symbol δ denotes the derivative operator in the continuous-time context ($\delta x(t) = \frac{d}{dt}x(t)$) and the shift forward operator in the discrete-time case ($\delta x(t) = x(t+1)$). σ is a piecewise constant function of time, called a switching signal, which takes its values in a finite set $\mathcal{I} = \{1, \dots, N\}$; $N > 1$ is the number of subsystems. At an arbitrary time t, σ is dependent on t or $x(t)$, or both, or other logic rules. Also, for a switching time sequence $0 < t_1 < t_2 < \cdots$, σ is continuous from the right everywhere and may be either autonomous or controlled. When $t \in [t_l, t_{l+1})$, we say that the $\sigma(t_l)$th subsystem is active and therefore the trajectory x_t of system (1) is the trajectory of the $\sigma(t_l)$th subsystem. The two-matrix pair (A_i, B_i), $\forall \sigma = i \in \mathcal{I}$, represents the ith subsystem or ith mode of (1). As commonly assumed in the literature, we exclude Zeno behavior for all types of switching signal here. Also, we assume that the state of the system (1) does not jump at the switching instants, i.e., the trajectory $x(t)$ is everywhere continuous (in the discrete-time context, this means that a continuous signal $x(t)$ cannot be reconstructed everywhere).

For the purpose of this paper, the definition of the ADT property used to restrict a class of switching signals is recalled as follows.

Definition 1 (*Hespanha & Morse, 1999*). For a switching signal σ and any $t_2 > t_1 > t_0$, let $N_\sigma(t_1, t_2)$ be the switching numbers of $\sigma(t)$ over the interval $[t_1, t_2)$. If $N_\sigma(t_1, t_2) \leq N_0 + (t_2 - t_1)/\tau_a$ holds for $N_0 \geq 1$, $\tau_a > 0$, then τ_a and N_0 are called the average dwell time and the chatter bound, respectively.

Remark 1. Definition 1 means that if there exists a positive number τ_a such that a switching signal has the ADT property, the average time interval between consecutive switching is at least τ_a. Therefore, a basic problem for systems with such switching is to identify or determine the minimal τ_a and the set of switching signals such that the system is stable, provided the system dynamics meet some associated conditions.

Also, we introduce the following exponential stability definition of system (1) for later development and we denote time by k in the discrete-time case.

Definition 2 (*Liberzon, 2003*). The equilibrium $x = 0$ of system (1) is globally uniformly exponentially stable (GUES) under certain switching signals σ if, for $u(t)$ (or $u(k)$) $\equiv 0$ and initial condition x_{t_0} (or x_{k_0}), there exist constants $K > 0, \delta > 0$ (respectively, $0 < \zeta < 1$) such that the solution of the system satisfies $\|x_t\| \leq K e^{-\delta(t-t_0)}\|x_{t_0}\|$, $\forall t \geq t_0$ (respectively, $\|x_k\| \leq K\zeta^{(k-k_0)}\|x_{k_0}\|$, $\forall k \geq k_0$).

The control input $u(t)$ (or $u(k)$) in (1) is used to achieve system stability or certain performances for certain switching signals, and usually, the mode-dependent control pattern is considered and formed as (if the state feedback is used) $u(t) = K_\sigma x(t)$, or $u(k) = K_\sigma x(k)$, where K_i ($\forall \sigma = i \in \mathcal{I}$) is the controller gain to be determined. In the literature, however, a common assumption is that the switches of K_σ coincide in *real time* with those of the system modes, which is hard to satisfy in practice. Then, if the time lag of switched controllers to system modes (asynchronous switching) is T, the control input will become $u(t) = K_{\sigma(t-T)}x(t)$, or $u(k) = K_{\sigma(k-T)}x(k)$, and hence the resulting closed-loop system is given by

$$\dot{x}(t) = \left(A_\sigma + B_\sigma K_{\sigma(t-T)}\right) x(t) \tag{2}$$

$$x(k+1) = \left(A_\sigma + B_\sigma K_{\sigma(k-T)}\right) x(k) \tag{3}$$

in the continuous-time context and the discrete-time context, respectively. Obviously, the mode-unmatched (probably wrong) controllers in the loop, together with the switching signals designed/found in the case of synchronous switching, may cause instability or a worse performance for the underlying system.

Therefore, in this paper, we are interested in finding a set of mode-dependent state-feedback controllers and a set of admissible switching signals with ADT such that the resulting closed-loop system (2) or (3) is GUES in the presence of asynchronous switching. It is worth noting that for a practical system, it also takes time to measure the system state besides identifying the system modes. Thus the corresponding state-feedback problem needs to take both the switching delay and the state delays into account. This general case and other more complex cases considering output-feedback control, H_∞ control, etc., can be further studied based on the basic state-feedback stabilization methods to be developed in this paper.

Before ending this section, let us revisit the stability results for switched nonlinear systems with ADT in the absence of asynchronous switching for later discussion.

Lemma 1 (*Hespanha & Morse, 1999*). *Consider the switched system $\dot{x}_t = f_\sigma(x_t)$, $\sigma \in \mathcal{I}$ and let $\alpha > 0$, $\mu > 1$ be given constants. Suppose that there exist $\mathcal{C}^1$ functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(t) \in \mathcal{I}$, and two class $\mathcal{K}_\infty$ functions β_1 and β_2 such that $\forall i \in \mathcal{I}$, $\beta_1(|x_t|) \leq V_i(x_t) \leq \beta_2(|x_t|)$, $\dot{V}_i(x_t) \leq -\alpha V_i(x_t)$ and $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,*

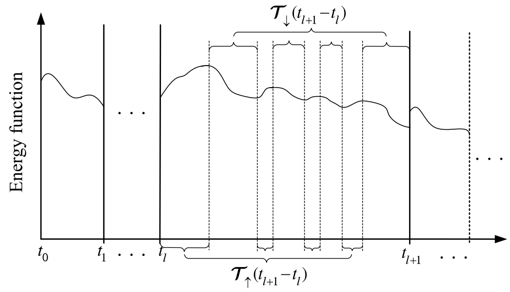

Fig. 1. Extended Lyapunov-like function.

$V_i(x_{t_l}) \le \mu V_j(x_{t_l})$; then the system is globally uniformly asymptotically stable (GUAS) for any switching signal with ADT

$$\tau_a > \tau_a^* = \ln \mu / \alpha. \tag{4}$$

Lemma 2 (*Zhang & Shi, 2008, Discrete-time Version*). *Consider the switched system $x_{k+1} = f_{\sigma(k)}(x_k)$, $\sigma(k) \in \mathcal{I}$ and let $0 < \gamma < 1$, $\mu > 1$ be given constants. Suppose that there exist $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, and two class $\mathcal{K}_\infty$ functions β_1 and β_2 such that $\forall i \in \mathcal{I}$, $\beta_1(|x_k|) \le V_i(x_k) \le \beta_2(|x_k|)$, $\Delta V_i(x_k) \triangleq V_i(x_{k+1}) - V_i(x_k) \le -\gamma V_i(x_k)$ and $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \ne j$, $V_i(x_{k_l}) \le \mu V_j(x_{k_l})$; then the system is GUAS for any switching signal with ADT*

$$\tau_a > \tau_a^* = -\ln \mu / \ln(1-\gamma). \tag{5}$$

3. Main results

It has been well recognized that the multiple Lyapunov-like function (MLF) is an efficient stability analysis tool for switched systems (Branicky, 1998; DeCarlo, Branicky, Pettersson, & Lennartson, 2000; Peleties & DeCarlo, 1991; Ye, Michel, & Hou, 1998), especially for slowly switched systems with DT or ADT (Liberzon, 2003). In MLF theory, each Lyapunov-like function constructed for each active subsystem is generally considered to be decreasing. An interesting extension also gives the so-called weak Lyapunov function, where the Lyapunov-like function can rise to a limited extent (Ye et al., 1998).

In the presence of asynchronous switching, the mode-unmatched controller will be applied in a control loop for a certain time; then the energy function to evaluate the system may be increased. This, together with the inspiration from Ye et al. (1998), motivates us to consider a class of Lyapunov-like functions allowed to increase but for which the increase rate is bounded.

For concise notation, respectively, let t_l and t_{l+1}, $\forall l \in \mathbb{N}$ denote the starting time and ending time of some active subsystem, while $\mathcal{T}_{\uparrow}(t_l, t_{l+1})$ and $\mathcal{T}_{\downarrow}(t_l, t_{l+1})$ represent the unions of the dispersed intervals during which Lyapunov function is increasing and decreasing within the interval $[t_l, t_{l+1})$. The separation gives that $[t_l, t_{l+1}) = \mathcal{T}_{\uparrow}(t_l, t_{l+1}) \cup \mathcal{T}_{\downarrow}(t_l, t_{l+1})$ and Fig. 1 illustrates the considered Lyapunov-like function. Also, we use $\mathcal{T}_{\uparrow}(t_{l+1} - t_l)$ and $\mathcal{T}_{\downarrow}(t_{l+1} - t_l)$ to denote the length of $\mathcal{T}_{\uparrow}(t_l, t_{l+1})$ and $\mathcal{T}_{\downarrow}(t_l, t_{l+1})$, respectively.

Now, the following lemmas present the extended stability results for switched nonlinear systems with ADT.

Lemma 3. *Consider the continuous-time switched system $\dot{x}_t = f_\sigma(x_t)$, $\sigma \in \mathcal{I}$ and let $\alpha > 0$, $\beta > 0$ and $\mu > 1$ be given constants. Suppose that there exist $\mathcal{C}^1$ functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(t) \in \mathcal{I}$, and two class $\mathcal{K}_\infty$ functions κ_1 and κ_2 such that, $\forall \sigma(t) = i \in \mathcal{I}$,*

$$\kappa_1(|x_t|) \le V_i(x_t) \le \kappa_2(|x_t|) \tag{6}$$

$$\dot{V}_i(x_t) \le \begin{cases} -\alpha V_i(x_t), & \forall t \in \mathcal{T}_{\downarrow}(t_l, t_{l+1}) \\ \beta V_i(x_t), & \forall t \in \mathcal{T}_{\uparrow}(t_l, t_{l+1}) \end{cases} \tag{7}$$

and $\forall (\sigma(t_l) = i, \sigma(t_l^-) = j) \in \mathcal{I} \times \mathcal{I}$, $i \ne j$,

$$V_i(x_{t_l}) \le \mu V_j(x_{t_l}); \tag{8}$$

then the system is GUAS for any switching signal with ADT

$$\tau_a > \tau_a^* = [\mathcal{T}_{\max}(\alpha + \beta) + \ln \mu] / \alpha, \tag{9}$$

where $\mathcal{T}_{\max} \triangleq \max_l \mathcal{T}_{\uparrow}(t_{l+1} - t_l)$, $\forall l \in \mathbb{N}$.

Proof. By integrating (7) for $t \in [t_l, t_{l+1})$, it holds that

$$\begin{aligned} V_{\sigma(t)}(x_t) &\le e^{-\alpha \mathcal{T}_{\downarrow}(t_l,t) + \beta \mathcal{T}_{\uparrow}(t_l,t)} V_{\sigma(t_l)}(x_{t_l}) \\ &\le e^{-\alpha[\mathcal{T}_{\downarrow}(t_l,t) + \mathcal{T}_{\uparrow}(t_l,t)]} \frac{e^{\beta \mathcal{T}_{\uparrow}(t_l,t)}}{e^{-\alpha \mathcal{T}_{\uparrow}(t_l,t)}} V_{\sigma(t_l)}(x_{t_l}) \\ &= e^{-\alpha(t-t_l)} \left(e^{(\beta+\alpha)\mathcal{T}_{\uparrow}(t_l,t)} \right) V_{\sigma(t_l)}(x_{t_l}). \end{aligned} \tag{10}$$

Then, according to (8), (10) and $N_\sigma(t_0, t)$ in Definition 1, one can get

$$\begin{aligned} V_{\sigma(t)}(x_t) &\le e^{-\alpha(t-t_l)} \left(e^{(\beta+\alpha)\mathcal{T}_{\uparrow}(t_l,t)} \right) \mu V_{\sigma(t_l^-)}(x_{t_l}) \\ &\le e^{-\alpha(t-t_l)} \left(e^{(\beta+\alpha)\mathcal{T}_{\max}} \right) \mu V_{\sigma(t_l^-)}(x_{t_l}) \le \cdots \\ &\le e^{-\alpha(t-t_0)} \left(e^{(\beta+\alpha)\mathcal{T}_{\max}} \right)^{N_\sigma(t_0,t)} \mu^{N_\sigma(t_0,t)} V_{\sigma(t_0)}(x_{t_0}) \\ &\le e^{N_0[(\beta+\alpha)\mathcal{T}_{\max} + \ln \mu]} \\ &\quad \times \left(e^{-\alpha} e^{\frac{1}{\tau_a}(\beta+\alpha)\mathcal{T}_{\max}} e^{\frac{1}{\tau_a}\ln \mu} \right)^{(t-t_0)} V_{\sigma(t_0)}(x_{t_0}) \\ &\le e^{N_0[(\beta+\alpha)\mathcal{T}_{\max} + \ln \mu]} \\ &\quad \times e^{-\left(\alpha - \frac{1}{\tau_a}(\beta+\alpha)\mathcal{T}_{\max} - \frac{1}{\tau_a}\ln \mu\right)(t-t_0)} V_{\sigma(t_0)}(x_{t_0}). \end{aligned} \tag{11}$$

Therefore, if the ADT satisfies (9), we conclude that $V_{\sigma(t)}(x_t)$ converges to zero as $t \to \infty$; then the asymptotic stability can be deduced with the aid of (6). □

Remark 2. The proof of Lemma 3 actually follows that of Lemma 1. It is noted that hypothesis (7) relaxes the requirement of the counterpart in Lemma 1, i.e., the Lyapunov-like function is allowed to rise with a bounded rate. Accordingly, the engaged subsystem may be unstable (but bounded) for the prescribed intervals.

Lemma 4 (*Discrete-time Version*). *Consider the discrete-time switched system $x_{k+1} = f_\sigma(x_k)$, $\sigma \in \mathcal{I}$ and let $0 < \gamma < 1$, $\eta > -1$ and $\mu > 1$ be given constants. Suppose that there exist $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, and two class $\mathcal{K}_\infty$ functions κ_1 and κ_2 such that $\forall \sigma(k) = i \in \mathcal{I}$,*

$$\kappa_1(|x_k|) \le V_i(x_k) \le \kappa_2(|x_k|) \tag{12}$$

$$\Delta V_i(x_k) \le \begin{cases} -\gamma V_i(x_k), & \forall k \in \mathcal{T}_{\downarrow}(k_l, k_{l+1}) \\ \eta V_i(x_k), & \forall k \in \mathcal{T}_{\uparrow}(k_l, k_{l+1}) \end{cases} \tag{13}$$

and $\forall (\sigma(k_l) = i, \sigma(k_l - 1) = j) \in \mathcal{I} \times \mathcal{I}$, $i \ne j$,

$$V_i(x_{k_l}) \le \mu V_j(x_{k_l}); \tag{14}$$

then, letting $\tilde{\eta} \triangleq 1 + \eta$ and $\bar{\gamma} \triangleq 1 - \gamma$, the system is GUAS for any switching signal with ADT

$$\tau_a > \tau_a^* = -\left\{ \mathcal{T}_{\max} \left[\ln \tilde{\eta} - \ln \bar{\gamma} \right] + \ln \mu \right\} / \ln \bar{\gamma}, \tag{15}$$

where $\mathcal{T}_{\max} \triangleq \max_l \mathcal{T}_{\uparrow}(k_{l+1} - k_l)$, $\forall l \in \mathbb{N}$.

Proof. Similar to the proof for Lemma 1. □

Remark 3. The energy function considered in Lemmas 3 and 4 (from a whole perspective) can be increased both at switching instants and during the running time of active subsystems. However, the increment will be finally compensated by an increased decrement such that the energy function is decreasing as a whole and hereby the system stability is guaranteed.

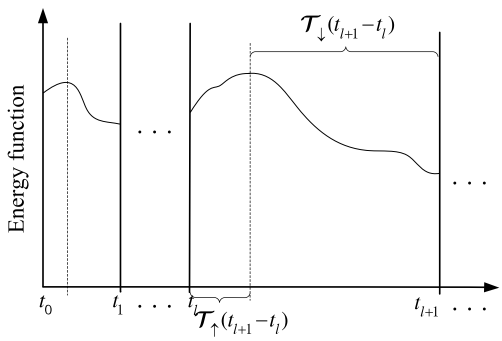

Fig. 2. Typical case of the extended Lyapunov-like function in Fig. 1.

Remark 4. In Lemmas 3 and 4, if $\mathcal{T}_{\max} = 0$, one can readily get Lemmas 1 and 2, respectively. Therefore, Lemmas 3 and 4 present a more general version of stability results for the switched systems with ADT in continuous-time and discrete-time cases, respectively. In addition, if one regards the increasing and decreasing intervals in one mode as two different modes (one stable and one unstable), a similar study for linear cases in the continuous-time context can be found in Zhai, Hu, Yasuda, and Michel (2000). In view of this, Lemma 3 addressing continuous-time nonlinear cases in our paper is more general than Theorem 1 in Zhai et al. (2000).

Note that, for Lemmas 3 and 4, a natural question is how $\mathcal{T}_{\max}$ is known in advance. Generally, this is hard since, within $[t_l, t_{l+1})$, $\forall l \in \mathbb{N}^+$, $\mathcal{T}_{\uparrow}(t_l, t_{l+1})$ includes all the randomly dispersed intervals during which the Lyapunov function is increasing. However, for the asynchronously switched control problem, the corresponding $\mathcal{T}_{\uparrow}(t_l, t_{l+1})$ will be only the interval close to the switching instants of subsystems as shown in Fig. 2, depending on the running time of the unmatched controller. In practice, the interval rests with the identification and scheduling process among all the candidates of stabilizing controllers, which may be different in different environments. Here we assume that the maximal delay of asynchronous switching, $\mathcal{T}_{\max}$, is known *a priori* without loss of generality.

Now, we are in a position to give the existence conditions of a set of asynchronous mode-dependent stabilizing controllers for system (1).

3.1. Continuous-time case

Theorem 1. *Consider the switched linear system (1) and let $\alpha > 0$, $\beta > 0$ and $\mu > 1$ be given constants. If there exist matrices $S_i > 0$ and U_i, $\forall i \in \mathcal{I}$, such that, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,*

$$A_iS_i + B_iU_i + S_iA_i^T + U_i^TB_i^T + \alpha S_i \leq 0 \tag{16}$$

$$A_iS_j + B_iU_j + S_jA_i^T + U_j^TB_i^T - \beta S_j \leq 0 \tag{17}$$

$$S_j \leq \mu S_i, \tag{18}$$

then there exists a set of mode-dependent stabilizing controllers with asynchronous delay $\mathcal{T}_{\max}$ such that system (2) is GUES for any switching signal with ADT satisfying (9). Moreover, if (16)–(18) have a solution, the admissible controller can be given by

$$K_i = U_iS_i^{-1}. \tag{19}$$

Proof. Consider the mode-dependent controller input $u_t = K_ix_t$ in the asynchronous switching case, namely, when the subsystem i has been switched, the controller K_j, $\forall i \neq j \in \mathcal{I}$, is still active instead of K_i for $\mathcal{T}_{\max}$. Thus, we have, $\forall \sigma(t_l) = i \in \mathcal{I}, i \neq j$,

$$\dot{x}_t = \begin{cases} \hat{A}_{i,j}x_t, & \forall t \in [t_l\ t_l + \mathcal{T}_{\max}) \\ \bar{A}_ix_t, & \forall t \in [t_l + \mathcal{T}_{\max}\ t_{l+1}), \end{cases} \tag{20}$$

where

$$\hat{A}_{i,j} \triangleq A_i + B_iK_j, \quad \bar{A}_i \triangleq A_i + B_iK_i. \tag{21}$$

Now consider the extended Lyapunov-like function given by the following quadratic form:

$$V_i(x) = x^TP_ix, \quad \forall \sigma(t) = i \in \mathcal{I}, \tag{22}$$

where P_i is a positive definite matrix with $P_i^{-1} = S_i$ satisfying (16)–(18). Then, from (7), (20) and (22), we have, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,

$$\dot{V}_i(t) + \alpha V_i(t) = x_t^T\left[\bar{A}_i^TP_i + P_i\bar{A}_i + \alpha P_i\right]x_t,$$

$$\dot{V}_i(t) - \beta V_i(t) = x_t^T\left[\hat{A}_{i,j}^TP_i + P_i\hat{A}_{i,j} - \beta P_i\right]x_t,$$

$$V_i(x_{t_l}) - \mu V_j(x_{t_l}) = x_{t_l}^T\left[P_i - \mu P_j\right]x_{t_l}.$$

Thus, if

$$\bar{A}_i^TP_i + P_i\bar{A}_i + \alpha P_i \leq 0 \tag{23}$$

$$\hat{A}_{i,j}^TP_i + P_i\hat{A}_{i,j} - \beta P_i \leq 0 \tag{24}$$

$$P_i - \mu P_j \leq 0, \tag{25}$$

system (1) is GUAS for any switching signal with ADT (9) according to Lemma 3. Replacing $\bar{A}_i$, $\hat{A}_{i,j}$ in (23)–(24) by (21), setting $S_i \triangleq P_i^{-1}$ and $U_i \triangleq K_iS_i$, we can know that, if (16)–(17) hold, (23)–(24) are satisfied. Moreover, if (18) holds, we have $S_j - \mu S_i \leq 0$. Note that by the Schur complement, $S_j - \mu S_i \leq 0$ is equivalent to $\Phi \triangleq \begin{bmatrix} -\mu S_i & I \\ I & -S_j^{-1} \end{bmatrix} \leq 0$. Then, further considering the Schur complement of $-\mu S_i$ in Φ, we obtain that $\Phi \leq 0$ is equivalent to $-S_j^{-1} - I^T(-\mu S_i)^{-1}I \leq 0$, which is $-S_j^{-1} + \mu^{-1}S_i^{-1} \leq 0$, i.e., (25) holds. In addition, if the inequalities (16)–(18) have a feasible solution, the admissible mode-dependent controller gain is given by $K_i = U_iS_i^{-1}$. Finally, by denoting $\delta = \frac{1}{2}\left(\alpha - \frac{1}{\tau_a}(\beta + \alpha)\mathcal{T}_{\max} - \frac{1}{\tau_a}\ln\mu\right)$, it is not hard to obtain from (6) and (22) that the system state satisfies $\|x_t\| \leq K\mathrm{e}^{-\delta(t-t_0)}\|x_{t_0}\|$ for a certain $K > 0$, i.e., the underlying system is GUES. □

In the absence of asynchronous switching, i.e., $\mathcal{T}_{\max} = 0$ in Theorem 1, we can easily get the following corollary.

Corollary 1. *Consider the switched linear system (1) and let $\alpha > 0$ and $\mu > 1$ be given constants. If there exist matrices $S_i > 0$ and U_i, $\forall i \in \mathcal{I}$, such that, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,*

$$A_iS_i + B_iU_i + S_iA_i^T + U_i^TB_i^T + \alpha S_i \leq 0 \tag{26}$$

$$S_j \leq \mu S_i, \tag{27}$$

then there exists a set of mode-dependent stabilizing controllers such that the closed-loop system is GUES for any switching signal with ADT satisfying (4). Moreover, if (26)–(27) have a solution, the admissible controller can be given by (19).

By further adopting the technique of slack variables (de Oliveira, Bernussou, & Geromel, 1999) and other linear matrix inequality (LMI) manipulations, the discrete-time case can be given as follows.

3.2. Discrete-time case

Theorem 2. *Consider the switched linear system (1) and let $0 < \gamma < 1$, $\eta > -1$, $\mu > 1$ be given constants. If there exist matrices $S_i > 0$ and U_i, $\forall i \in \mathcal{I}$, such that, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,*

$$\begin{bmatrix} -S_i & A_iS_i + B_iU_i \\ * & -(1-\gamma)S_i \end{bmatrix} \leq 0 \tag{28}$$

$$\begin{bmatrix} -S_i & A_iS_j + B_iU_j \\ * & (1+\eta)(S_i - S_j - S_j^T) \end{bmatrix} \leq 0 \tag{29}$$

$$S_j \leq \mu S_i, \tag{30}$$

then there exists a set of mode-dependent stabilizing controllers with asynchronous delay $\mathcal{T}_{\max}$ such that system (3) is GUES for any switching signal with ADT satisfying (15). Moreover, if (28)–(30) have a solution, the admissible controller can be given by (19).

Proof. Analogous to the proof in Theorem 1, for mode-dependent controller input $u_k = K_i x_k$ in the asynchronous switching case, namely, when the subsystem i has been switched, the controller K_j will be still active instead of K_i for $\mathcal{T}_{\max}$; then we have

$$\dot{x}_k = \begin{cases} \hat{A}_{i,j}x_k, & \forall k \in [k_l\ k_l + \mathcal{T}_{\max}) \\ \bar{A}_i x_k, & \forall k \in [k_l + \mathcal{T}_{\max}\ k_{l+1}), \end{cases} \tag{31}$$

where $\bar{A}_i, \hat{A}_{i,j}$ are denoted as in (21).

Then, considering the extended Lyapunov-like function (22), together with (13) and (31), we have, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,

$$\Delta V_i(x_k) + \gamma V_i(x_k) = x_k^T \left[\bar{A}_i^T P_i \bar{A}_i + \gamma P_i - P_i\right] x_k$$

$$\Delta V_i(x_k) - \eta V_i(x_k) = x_k^T \left[\hat{A}_{i,j}^T P_i \hat{A}_{i,j} - \eta P_i - P_i\right] x_k$$

$$V_i(k_l) - \mu V_j(k_l) = x_k^T \left[P_i - \mu P_j\right] x_k.$$

Thus, if

$$\bar{A}_i^T P_i \bar{A}_i + \gamma P_i - P_i \leq 0 \tag{32}$$

$$\hat{A}_{i,j}^T P_i \hat{A}_{i,j} - \eta P_i - P_i \leq 0 \tag{33}$$

$$P_i - \mu P_j \leq 0, \tag{34}$$

system (1) is GUAS for any switching signal with ADT (15) according to Lemma 4. Replacing $\bar{A}_i, \hat{A}_{i,j}$ in (31) and by the Schur complement, we have

$$\begin{bmatrix} -P_i & P_iA_i + P_iB_iK_i \\ * & -(1-\gamma)P_i \end{bmatrix} \leq 0 \tag{35}$$

$$\begin{bmatrix} -P_i & P_iA_i + P_iB_iK_j \\ * & -(1+\eta)P_i \end{bmatrix} \leq 0. \tag{36}$$

Setting $S_i \triangleq P_i^{-1}$, $U_i \triangleq K_i S_i$ and performing a congruence transformation to (35) via $diag\{S_i, S_i\}$, we can obtain (28). In addition, from the fact that $(S_i - S_j)^T S_i (S_i - S_j) \geq 0$, we have $S_i - S_j - S_j^T \geq -S_j^T S_i^{-1} S_j$. Then, if (29) holds, one has

$$\begin{bmatrix} -S_i & A_iS_j + B_iU_j \\ * & -(1+\eta)S_j^T S_i^{-1} S_j \end{bmatrix} \leq 0.$$

Performing a congruence transformation to the above inequality via $diag\{S_i^{-1}, S_j^{-1}\}$, we can obtain (36). Therefore, (28)–(30) ensure that (32)–(34) are satisfied. In addition, by denoting $\zeta = \sqrt{(1-\gamma)\theta^{\mathcal{T}_{\max}/\tau_a}\mu^{1/\tau_a}}$, the system state satisfies $\|x_k\| \leq K\zeta^{(k-k_0)}\|x_{k_0}\|$ for a certain $K > 0$, i.e., the underlying system is GUES. $\square$

Likewise, the following corollary gives the case of synchronous switching in the discrete-time context.

Corollary 2. *Consider the switched linear system (1) and let $0 < \gamma < 1$ and $\mu > 1$ be given constants. If there exist matrices $S_i > 0$ and U_i, $\forall i \in \mathcal{I}$, such that, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,*

$$\begin{bmatrix} -S_i & A_iS_i + B_iU_i \\ * & -(1-\gamma)S_i \end{bmatrix} \leq 0 \tag{37}$$

$$S_j \leq \mu S_i, \tag{38}$$

then there exists a set of mode-dependent stabilizing controllers such that system (1) is GUES for any switching signal with ADT satisfying (5). Moreover, if (37)–(38) have a solution, the admissible controller can be given by (19).

Remark 5. The conditions derived in the above Theorems and Corollaries are LMIs for given α, β (or α only) and μ. Then, by providing two parameters (or one, respectively) *a priori*, the optimum for the other one can be approximately obtained by the bisection method when a feasible solution of the corresponding LMIs is guaranteed. This is actually due to the latent monotonicity of all of them, e.g., a bigger β corresponding to more possibilities of feasible solutions in Theorem 1.

Table 1
Minimal ADT and the corresponding controllers for different β.

β	τ_a^*	Controller gains
1.50	3.0660	$K_1 = [2.1124\quad -0.9336]$ $K_2 = [-1.3111\quad -1.3158]$
1.40	2.8993	$K_1 = [2.1013\quad -1.1047]$ $K_2 = [-1.2700\quad -1.4545]$
1.3765	2.8602	Infeasible

4. Numerical example

In this section, a numerical example in the continuous-time context will be presented to demonstrate the potential and validity of our developed theoretical results. In the same vein, the corresponding validation can be done for discrete-time cases, which we omit due to space limitations.

Consider the continuous-time switched linear systems consisting of two subsystems described by

$$A_1 = \begin{bmatrix} 0.2 & -0.5 \\ 0.5 & -0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0.3 \\ -1 & 0.4 \end{bmatrix}, \quad B_1 = [-0.4 \quad 1.8]^T,$$

$B_2 = [0.1 \quad 1.5]^T$. The maximal delay of asynchronous switching $\mathcal{T}_{\max} = 0.5$. Our purpose here is to design a set of mode-dependent stabilizing controllers and find the admissible switching signals such that the resulting closed-loop system is stable in the presence of asynchronous switching. Firstly, if one studies the control problem based on Corollary 2, i.e., assuming synchronous switching, feasible solutions can be found even if $\mu \to 1.0$ and $\alpha \leq 4.9047$. As a result, the minimal ADT may tend to 0, which means that the admissible switching signals can be approximately arbitrary.

Then, assigning $\mu = 1.02$ and $\alpha = 0.3$, we can obtain $\tau_a^* = 0.0660$ and a set of controllers correspondingly. Generating a possible switching sequence satisfying $\tau_a = 1 > 0.0660$, one can test that the resulting closed-loop system is stable for the initial state $x_0 = [0.8, -1.2]^T$. Now if there exists asynchronous switching in practice with $\mathcal{T}_{\max} = 0.5$, one can check that the state responses of the resulting systems are either diverging or with larger overshoots (although converging as the ADT is gradually increased). This means that the smaller ADT is unacceptable in the presence of asynchronous switching. Also, it is insufficient to only increase ADT to get the corresponding switching signals such that the underlying system is stable, since the controller designed using Corollary 2 may also be wrong.

Now, turning to using Theorem 1, for the same α, μ, we can obtain a different minimal ADT and the corresponding controller gains in Table 1 for different β. It can be seen that feasible solutions of the desired controllers are guaranteed when $\beta \geq 1.3765$, which also implies that a smaller ADT, proportional to the lower β, may be incompetent in the presence of asynchronous switching. Despite the occurrence of asynchronous switching, the solved controllers can be verified to be effective under the obtained admissible switching signals. The illustrative figures are omitted here due to space limitations.

5. Conclusions

The so-called asynchronously switched control problem for a class of switched linear systems with ADT is investigated.

Improved results of stability analysis for switched systems with ADT in nonlinear setting are given by relaxing the requirement of the Lyapunov-like function decreasing during each running time of each active subsystem. The asynchronously switched stabilizing control problem is therefore solved by finding a set of mode-dependent controllers and a set of admissible switching signals. The criteria for the existence of the desired controllers are derived. Also, the minimal ADT of the acceptable switching signals is obtained for the given increase and decrease scales of the Lyapunov-like function and the maximal delay of asynchronous switching. The developed results are expected to extend to issues such as output feedback, H_∞ control and estimation in the presence of asynchronous switching for the underlying systems.

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