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Abstract

abstract

Key words: LOI; switched system; neutral system.

1 Introduction

Recently,a novel L-K functional based on the Partial-Integral operator.The new L-K functional denotes the complete L-K functional by inner product.For the Linear Time Delay system with time-invariant delay,we can obtain less conservative results even necessary and sufficient results.

(SOS)provide a computationally tractable test for Zeno stability in hybrid systems with semi-algebraic guard sets.develop a polynomial-time algorithm for construciton of the Lyapunov-like funtion proposed in (23) and (24).extend this method to the verification of Zeno stability for system with parameetric uncertainly.

(wangyibo)A complete quadratic L-K functional was built in theform of an inner product of the PI operator, which has the potential to obtain accurate stability conditions.The novel stability criteria for the multiarea LFC systems with time delays were proposed in this article, which were less conservative than existing literature. Furthermore, the relationships between controller parameters and delay margins were further studied

(wushuangshuang) use PIE-based methods to analyze stability and H_∞ performance problem of linear TDSs with uncertain delays.

(chaitanya Murti) provide a computationally tractable test for Zeno stability in hybrid system with semi-algebraic guard sets.use Sum-of-Squares to find a convex approach for construciton of Lyapunov functions for Zeno stability.

(PIETOOLS)present PIETOOLS matlab toolbox for construciton and handling of Partial Integral(PI) operator.After several refinements, PIETOOLS comes to the latest version of 2022. In the 2022 version, user manuals and many demos were added.Simulations of this paper based on PIETOOLS2022.

(Minimal Differential Difference) Propose a an algorithm for constructing minimal DDF relization of DDE systems. The algorithm can dramatically reduce the computational complexity of analysis and control problems for delayed networks.And it extended this result to a algorithm for minimal DDF relizations of DDFs-thus also solving the problem of inefficient DDF representing of NDSs.

The LOI method in (article Representation) is further extended to the neutral system by (Representation),comparing with the method used by Delay-Dependent Stability for Load Frequency Control System via Linear Operator Inequality, the new method can convert not only multiple time-delay system but also the nuetral delay system(NDS) even the neutral system with Integral term. provide formulae for conversion between representations under which solutions are equivalent.

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The LOI method in (article Representation) was already further extended to the neutral system by (Representation), however, those theorems proposed for linear time delay system(muti delay?multi-delay system) have not been extended to neutral system.

In neutral delay system,the delay not only exsit in the state of s but also exsit in the derivative of state.The delay in (derivative term) calls neutral delay.

The traditional method for linear neutral switched system globally stability constructing Lyapunov function integral term with exponential coefficient.It is necessary to ensure that after the derivation and the addition of the original function, the complex quadratic integral term can be eliminated and only the simple first integral term can be left. Finally, a very complex matrix negative definite condition is obtained by Jensen inequality and the global stability condition of the switched system is obtained by using the hypothesis condition.

In this article,we use the extended conversion formulae to convert NDS to PIE.The previous LOI criterion which can only deal with constant delay systems is extended to neutral delay systems.In addition, the complete L-K functional based on inner product is applied to neutral switched systems to obtain a simpler and less conservative global stability criterion.

2 Problem formulation and preliminaries

Consider a neutral switched system described by the following equation:

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t-r) \\ &\quad + C_{\sigma(t)}\dot{x}(t-h) + D_{\sigma(t)}u(t) \\ x(\theta) &= \Psi(\theta), \forall\theta \in [-H,0], H = max\{t,h\} \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state,$u(t) \in R^p$ denote the system state and control input.$\sigma(t) : [0, +\infty) \to \mathcal{P}$ is a piecewise constant function denote switching signal. h and r are the state delay and neutral delay. $\Psi(\theta)$ is a initial vector function on $[-H, 0]$.

The Zeno and impulsive conditions are assumed to be excluded consideration.The switching sequence is expressed as

$$\begin{aligned} \aleph_p = \{&(\sigma(t_0), t_0), \dots (\sigma(t_k), t_k), \dots \\ &|\sigma(t_k) \in \mathcal{P}, k = 0, 1, \dots\} \end{aligned} \tag{2}$$

where t_0 is the initial time and t_k is the switching instant. The $\sigma(t_k)$th subsystem is active when $t \in [t_k, t_{k+1})$.

Considering the asynchronous switching, the candidate controller presented in the form of

$$u(t) = K_{\sigma(t-\tau(t))}x(t) \tag{3}$$

where $\tau(t)$ Represents the switching delay satisfying $0 < \tau(t) < \tau_d < t_{k+1} - t_k, \forall k \in \mathbb{N}$.

Operator

If we also do not consider the output $y(t)$,we can convert Equation (1) to standard NDS(Neutral Delay System) (4) by Lemma (NDStoDDF)

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} A_{\sigma(t)} & 0 & D_{\sigma(t)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \\ &+ \begin{bmatrix} B_{\sigma(t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t-r) \\ w(t-r) \\ u(t-r) \\ \dot{x}(t-r) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & 0 & C_{\sigma(t)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t-h) \\ w(t-h) \\ u(t-h) \\ \dot{x}(t-h) \end{bmatrix} \end{aligned} \tag{4}$$

Assumption 2.1 *For any subsystem i switching to j where $i, j \in \mathcal{P}$, there exsits a scalar $\mu_{ij} > 0$, the following inequality holds:*

$$V_j(x(t)) \le \mu_{ij}V_i(x(t)) \quad \forall x(t) \in R^n \tag{5}$$

Definition 2.2 *Let $N_\sigma(t)$ denote the switching times in the time interval $(0, t)$,then we define the h-frequency of switching at t as*

$$v_t(t) := \frac{N_{\sigma(t)}}{h(t)} \tag{6}$$

Definition 2.3 *We divide the time interval into the mismatched period $\mathcal{M}_1$ and matched period $\mathcal{M}_2$.*

Let $\mathcal{P}_{m1}$ denote the index set systems during the mode-identifying period or mismatched period. Meanwhile, denote $\mathcal{P}^s_{m2}$, $\mathcal{P}^u_{m2} \in \mathcal{P}_{m2}$ as the stable and unstable indices sets during the normal-working period, respectively. Namely,

$$\begin{aligned} \mathcal{P}_{m1} = \{&i \in \mathcal{P} | \sigma(t) = i, \\ &\forall t \in [t_k, t_k + \tau(t_k)), k = 0, 1, \dots\} \\ \mathcal{P}_{m1} = \{&i \in \mathcal{P} | \sigma(t) = i, \\ &\forall t \in [t_k + \tau(t_k, t_k)), k = 0, 1, \dots\} \end{aligned} \tag{7}$$

Clearly, it holds that $\mathcal{P} = \mathcal{P}_{m1} = \mathcal{P}_{m2} = \mathcal{P}_{m2}^{s} \bigcup \mathcal{P}_{m2}^{u}$

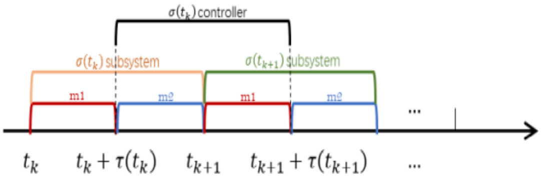

Fig. 1. A S

Remark 2.1 *reamrk1*

Definition 2.4 *Define the kth holding time of a switching signal* σ *in the matched period as*

$$S_{k+1} = t_{k+1} - (t_k + \tau(t_k)), \quad k = 0, 1 \ldots \tag{8}$$

Then, for each $j \in \mathcal{P}$, *define the h-frequency of activation of subsystem* j *in mismatched period as*

$$\eta_1^h(j,t) := \sum_{i:\sigma(t_i)=j} \frac{\tau_{t_i}}{h(t)}, \quad t > 0 \tag{9}$$

In matched period, the corresponding h-frequency of activation of subsystem j *is defined as*

$$\eta_2^h(j,t) := \sum_{i:\sigma(t_i)=j} \frac{S_{i+1}}{h(t)}, \quad t > 0 \tag{10}$$

Let $E(\mathcal{P})$ *be the set for admissible switch from subsystem* m *to subsystem* n, $\forall m, n \in \mathcal{P}$, *which is represented by a sequential pair* (m, n). *For each pair* $(m, n) \in E(\mathcal{P})$, *we define the transition frequency from the mth subsystem to the nth subsystem as*

$$\rho_{mn}(t) := \frac{\#\{m \to n\}}{N_\sigma}, \quad t > 0 \tag{11}$$

where$\#\{m \to n\}_t$ *is the transition number from subsystem* m *to subsystem* n *in the time interval* $[0, t)$

Now, define the asymptotic upper density of v_h,ρ_{mn} *as*

$$\hat{v}_h := \lim_{t \to +\infty} \sup v_h(t) \tag{12}$$

$$\hat{\rho}_{mn} := \lim_{t \to +\infty} \sup \rho_{mn}(t) \tag{13}$$

Similarly, the asymptotic upper densities of $\eta_1^h(j,t)$,$\eta_2^h(j,t)$ *are defined as*

$$\hat{\eta}_1^h(j) := \lim_{t \to +\infty} \sup \eta_1^h(j,t) \tag{14}$$

$$\hat{\eta}_2^h(j) := \lim_{t \to +\infty} \sup \eta_2^h(j,t) \tag{15}$$

In addition, we also give the definition of asymptotic lower densities of $\eta_2^h(j,t)$ *as*

$$\check{\eta}_2^h(j) := \lim_{t \to +\infty} \inf \eta_2^h(j,t) \tag{16}$$

Z inner produced

$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} = y^T x + \int_{-1}^{0} \psi_i(s)^T \phi_i(s) ds \tag{17}$$

To establish the main result of this paper, we review four useful lemmas that will be used in the proof of this paper .

Lemma 2.1 (c) *For given time delays* $\tau_i (i = 0, 1, \cdots N)$ *the system* $\mathcal{T}\dot{x}_f(t) = \mathcal{A}x_f(t)$ *is asymptotically stable if there exsit a positive,self-adjoint PI operator* $\mathcal{P} := \mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{bmatrix}$ *such that the following inequality holds:*

$$\mathcal{T}^* \mathcal{P} \mathcal{A} + \mathcal{A}^* \mathcal{T} \mathcal{P} < 0 \tag{18}$$

The proof of lemma 1 will be found in Appendixes

Lemma 2.2 (c) *then we get the standard DDF form*

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \\ r_i(t) \end{bmatrix} = \begin{bmatrix} A_{\sigma(t)} & 0 & D_{\sigma(t)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{ri} & B_{r1i} & B_{r2i} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} B_v \\ D_{1v} \\ D_{2v} \\ D_{rvi} \end{bmatrix} v(t)$$
$$v(t) = C_{v1} r_1(t - r) + C_{v2} r_2(t - h) \tag{19}$$

Lemma 2.3 *And then we try to (cite Representation of Networks and Systems with Delay:DDEs, DDFs, ODE-PDEs and PIEs) convert standard DDF form Equation (7) to PIE from Equation (8)*

$$\mathcal{T}\dot{\mathbf{x}} + \mathcal{B}_{T_2}\dot{u} = \mathcal{A}\mathbf{x} + \mathcal{B}_2 u \tag{20}$$

$$\mathbf{x(t)} := \begin{bmatrix} x(t) \\ \Phi(t, .) \end{bmatrix}$$

remark? It should be noticed the difference between $\mathbf{x}$ and x. The reason why $\mathbf{x}$ terms and x terms can be added up is that there is no difference for the operator $\mathcal{B}_{T_2}$ to dual with $\mathbf{x}$ and x.

remark $\mathcal{B}_{T_2}$,$\mathcal{B}_2$ was minimalized by converting program in PIETOOLS2022.

And then (cite Representation of Networks and Systems with Delay:DDEs, DDFs, ODE-PDEs and PIEs) ,we can convert standard NDS form Equation (3) to DDF(Differential Difference Equations) form Eqution (7) by Eqution (4)

3 Stability Analysis

Cite Delay-Dependent Stability for Load FrequencyControl System via Linear Operator Inequality

Theorem 3.1 *For given time delays r, h and controller $u = \mathcal{K}\S(t)$,the neutral system (1) is asymptotically stable if there exsit a positive, self-adjoint PI operator $\mathcal{H} :=$ $\mathcal{H}\begin{bmatrix} P & Q_1 \\ Q_2 & \{R_i\} \end{bmatrix}$ such that the following inequality*

$$\mathcal{T}'^{*}\mathcal{H}\mathcal{A}' + \mathcal{A}'^{*}\mathcal{H}\mathcal{T}' < 0 \tag{24}$$

where

$$\begin{aligned} \mathcal{T}' &= \mathcal{T} + \mathcal{B}_{T_2}K \\ \mathcal{A}' &= \mathcal{A} + \mathcal{B}_2 K \end{aligned}$$

PROOF.

$$u(t) = K\mathbf{x}(\mathbf{t}) \tag{25}$$

$$\mathbf{x}(\mathbf{t}) := \begin{bmatrix} x(t) \\ \Phi(t,.) \end{bmatrix}$$

then we got the standard PIE form

$$\mathcal{T}\dot{\mathbf{x}} + \mathcal{B}_{T_2}K\dot{x} = \mathcal{A}\mathbf{x} + \mathcal{B}_2 K x \tag{26}$$

Next we will do some variable substitution

$$\mathcal{T}' = \mathcal{T} + \mathcal{B}_{T_2}K \tag{27}$$

$$\mathcal{A}' = \mathcal{A} + \mathcal{B}_2 K \tag{28}$$

then we got a more concise PIE form

$$\mathcal{T}'\dot{\mathbf{x}} = \mathcal{A}'\mathbf{x} \tag{29}$$

L-K funtional

$$V(\mathbf{x}) = < \mathcal{T}\mathbf{x}, \mathcal{H}\mathcal{A}\mathbf{x} >_Z \tag{30}$$

$$\begin{aligned} \dot{V}(\mathbf{x}) &= < \mathcal{T}'\mathbf{x}, \mathcal{H}\mathcal{T}'\mathbf{x} >_Z + < \mathcal{A}'\mathbf{x}, \mathcal{H}\mathcal{T}'\mathbf{x} >_Z \\ &= < \mathbf{x}, (\mathcal{T}'^{*}\mathcal{H}\mathcal{A}' + \mathcal{A}'^{*}\mathcal{H}\mathcal{T}')\mathbf{x} >_Z \end{aligned} \tag{31}$$

remark? The procedures for calculating the delay margin are provided as follows.?

Conversion Formula from NDS to DDF:

$$\begin{gathered} D_{rvi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}, \qquad [C_{ri}B_{r1i}B_{r2i}] = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_p \\ A_0 & 0 & B_2 \end{bmatrix}, \qquad I_{n+q+r} = \begin{bmatrix} B_v \\ D_{1v} \\ D_{2v} \end{bmatrix} \\ C_{vi} = \begin{bmatrix} A_i & 0 & B_{2i} & E_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C_{vdi}(s) = 0 \end{gathered} \tag{21}$$

Conversion Formula from ODE-PDE to DDF to PIE part1:

$$\begin{gathered} \mathcal{A} = \mathcal{P}\begin{bmatrix} A_0 & A \\ 0 & \{I_\tau, 0, 0\} \end{bmatrix}, \quad \mathcal{T} = \mathcal{P}\begin{bmatrix} I & 0 \\ T_0 & \{0, T_a, T_b\} \end{bmatrix}, \quad \mathcal{B}_{T_1} = \mathcal{P}\begin{bmatrix} 0 & \varnothing \\ T_1 & \{\varnothing\} \end{bmatrix}, \quad \mathcal{B}_{T_2} = \mathcal{P}\begin{bmatrix} 0 & \varnothing \\ T_2 & \{\varnothing\} \end{bmatrix}, \\ \mathcal{B}_1 = \mathcal{P}\begin{bmatrix} \mathbf{B_1} & \varnothing \\ 0 & \{\varnothing\} \end{bmatrix}, \quad \mathcal{B}_2 = \mathcal{P}\begin{bmatrix} \mathbf{B_2} & \varnothing \\ 0 & \{\varnothing\} \end{bmatrix}, \quad \mathcal{C}_1 = \mathcal{P}\begin{bmatrix} \mathbf{C_{10}} & \mathbf{C_{11}} \\ \varnothing & \{\varnothing\} \end{bmatrix}, \quad \mathcal{C}_2 = \mathcal{P}\begin{bmatrix} \mathbf{C_{20}} & \mathbf{C_{21}} \\ \varnothing & \{\varnothing\} \end{bmatrix} \end{gathered} \tag{22}$$

Conversion Formula from ODE-PDE to DDF to PIE part2:

$$
\hat{C}_{vi} = C_{vi}, \qquad D_I = \begin{bmatrix} (I - \sum_{i=1}^{K} C_{vi} D_{rvi})^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix},
$$

$$
C_{Ii} = -D_I C_{vi} = \begin{bmatrix} -(I - \sum_{i=1}^{K} E_i)^{-1} A_i & 0 & -(I - \sum_{i=1}^{K} E_i)^{-1} B_{2i} & -(I - \sum_{i=1}^{K} E_i)^{-1} E_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
[T_0 \quad T_1 \quad T_2] = \begin{bmatrix} C_{r1} + D_{rv1} C_{vx} & C_{r1} + D_{rv1} D_{vw} & C_{r1} + D_{rv1} D_{vu} \\ C_{r2} + D_{rv2} C_{vx} & C_{r2} + D_{rv2} D_{vw} & C_{r2} + D_{rv2} D_{vu} \\ \vdots & \vdots & \vdots \\ C_{rK} + D_{rvK} C_{vx} & C_{rK} + D_{rvK} D_{vw} & C_{rK} + D_{rvK} D_{vu} \end{bmatrix},
$$

$$
C_{r1} + D_{rv1} C_{vx} = \begin{bmatrix} I_n \\ 0 \\ A_0 + (I - \sum_{i=1}^{K} E_i)^{-1} (\sum_{i=1}^{K} (A_i + E_i A_0) \end{bmatrix}, T_a(s,\theta) = \begin{bmatrix} D_{rv1} C_{I1} & D_{rv1} C_{I2} & \dots & D_{rv1} C_{IK} \\ D_{rv2} C_{I1} & D_{rv2} C_{I2} & \dots & D_{rv2} C_{IK} \\ \vdots & \vdots & \vdots & \vdots \\ D_{rvK} C_{I1} & D_{rvK} C_{I2} & \dots & D_{rvK} C_{IK} \end{bmatrix},
$$

$$
[C_{vx} \quad D_{vw} \quad D_{vu}] = \begin{bmatrix} (I - \sum_{i=1}^{K} E_i)^{-1} (\sum_{i=1}^{K} (A_i + E_i A_0) & (I - \sum_{i=1}^{K} E_i)^{-1} (E_i B_1) & (I - \sum_{i=1}^{K} E_i)^{-1} (E_i B_2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

$$
D_{rv1} C_{I1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(I - \sum_{i=1}^{K} E_i)^{-1} A_1 & 0 & -(I - \sum_{i=1}^{K} E_i)^{-1} B_{21} & -(I - \sum_{i=1}^{K} E_i)^{-1} E_1 \end{bmatrix},
$$

$$
T_b(s,\theta) = -I_{\sum_i p_i} + T_a(s,\theta) = \begin{bmatrix} -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ -(I - \sum_{i=1}^{K} E_i)^{-1} A_1 & 0 & -(I - \sum_{i=1}^{K} E_i)^{-1} B_{21} & -I - (I - \sum_{i=1}^{K} E_i)^{-1} E_1 \end{bmatrix},
$$

$$
I_\tau = \begin{bmatrix} \frac{1}{\tau_1} I_{p1} & 0 & 0 & 0 \\ 0 & \frac{1}{\tau_2} I_{p2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \frac{1}{\tau_K} I_{pK} \end{bmatrix}, \quad \begin{bmatrix} A \\ C_{11} \\ C_{21} \end{bmatrix} = \begin{bmatrix} B_v \\ D_{1v} \\ D_{2v} \end{bmatrix} [C_{I1} \dots C_{IK}], \quad \begin{bmatrix} \mathbf{A_0} & \mathbf{B_1} & \mathbf{B_2} \\ \mathbf{C_{10}} & \mathbf{D_{11}} & \mathbf{D_{12}} \\ \mathbf{C_{20}} & \mathbf{D_{21}} & \mathbf{D_{22}} \end{bmatrix} = \begin{bmatrix} A_0 & B_1 & B_2 \\ C_{10} & D_{11} & D_{12} \\ C_{20} & D_{21} & D_{22} \end{bmatrix} [C_{vx} \quad D_{vw} \quad D_{vu}], \tag{23}
$$

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Theorem 3.2 *Given the controller under asynchronous switching:*

$$u(t) = \mathcal{K}_{\sigma(t-\tau(t))}\mathbf{x}(t) \tag{32}$$

Let α_i, β_i be given constants. The neutral switched system(1) is global asymptotic stablility if ther exists positive, self-adjoint PI operators $\mathcal{H}_i$, let following PI operator $\mathcal{M}_i, \mathcal{N}_i$satisfies:

In the stage of stable normal operation, $i \in \mathcal{P}^s_{m2}, \alpha_i > 0$, and in the stage of unstable normal operation, $i \in \mathcal{P}^u_{m2}, \alpha_i < 0$

$$\mathcal{M}_i = \alpha_i \tilde{\mathcal{T}}_i^* \mathcal{H}_i \tilde{\mathcal{T}}_i + \tilde{\mathcal{T}}_i^* \mathcal{H}_i \tilde{\mathcal{A}}_i + \tilde{\mathcal{A}}_i^* \mathcal{H}_i \tilde{\mathcal{T}}_i < 0 \tag{33}$$

Where $\tilde{\mathcal{T}}_i = \mathcal{T}_i + \mathcal{K}_i \mathcal{B}_{T_2 i}$, $\tilde{\mathcal{A}}_i = \mathcal{A}_i + \mathcal{K}_i \mathcal{B}_{2i}$.

In the stage of asynchronous switching, $i \in \mathcal{P}_{m1}, \beta_i > 0$. For any $j \in \mathcal{P}_{m2}, i \neq j$

$$\mathcal{N}_i = -\beta_i \hat{\mathcal{T}}_i^* \mathcal{H}_i \hat{\mathcal{T}}_i + \hat{\mathcal{T}}_i^* \mathcal{H}_i \hat{\mathcal{A}}_i + \hat{\mathcal{A}}_i^* \mathcal{H}_i \hat{\mathcal{T}}_i < 0 \tag{34}$$

Where $\hat{\mathcal{T}}_i = \mathcal{T}_i + \mathcal{K}_j \mathcal{B}_{T_2 i}$, $\hat{\mathcal{A}}_i = \mathcal{A}_i + \mathcal{K}_j \mathcal{B}_{2i}$.

Then when the switched signal satisfies

$$\breve{v}_h := \lim_{t\to+\infty} \inf v_h(t) > 0 \tag{35}$$

and

$$\begin{aligned} &\hat{v}_h \sum_{(m,n)\in E(\mathcal{P})} \hat{\rho}_{mn} \ln \mu_{mn} + \sum_{i\in\mathcal{P}^u_{m2}} |\alpha_i| \,\hat{\eta_2}^h(i,t) \\ &- \sum_{i\in\mathcal{P}^s_{m2}} |\alpha_i| \,\breve{\eta_2}^h(i,t) + \sum_{i\in\mathcal{P}} |\beta_i| \,\hat{\eta_1}^h(i,t) < 0 \end{aligned} \tag{36}$$

PROOF.
The complete quadratic L-K functional candidate is established as follows :

$$\begin{aligned} V_{\sigma(t)}(\mathbf{x}) = \big\langle &\left(\mathcal{T}_{\sigma(t)} + \mathcal{K}_{\sigma(t-\tau(t))}\mathcal{B}_{T_2\sigma(t)}\right)\mathbf{x}, \\ &\mathcal{H}\left(\mathcal{A}_{\sigma(t)} + \mathcal{K}_{\sigma(t-\tau(t))}\mathcal{B}_{T_2\sigma(t)}\right)\mathbf{x}\big\rangle_Z \end{aligned} \tag{37}$$

Condition 1:

Considering the stage of stable normal operation $i \in \mathcal{P}^s_{m2}, \alpha_i > 0$, and the stage of unstable Normal operation $i \in \mathcal{P}^u_{m2}, \alpha_i < 0$.

Let $\tilde{\mathcal{T}}_i = \mathcal{T}_i + \mathcal{K}_i \mathcal{B}_{T_2 i}$, $\tilde{\mathcal{A}}_i = \mathcal{A}_i + \mathcal{K}_i \mathcal{B}_{2i}$, then

$$\begin{aligned} V_i(\mathbf{x}) &= \left\langle \tilde{\mathcal{T}}_i \mathbf{x}, \mathcal{H}_i \tilde{\mathcal{A}}_i \mathbf{x} \right\rangle_Z \\ &= \left\langle \mathbf{x}, \tilde{\mathcal{T}}_i^* \mathcal{H}_i \tilde{\mathcal{A}}_i \mathbf{x} \right\rangle_Z \end{aligned} \tag{38}$$

$$\dot{V}_i(\mathbf{x}) = \left\langle \mathbf{x}, \left(\hat{\mathcal{T}}_i^* \mathcal{H}_i \hat{\mathcal{A}}_i + \hat{\mathcal{A}}_i^* \mathcal{H}_i \hat{\mathcal{T}}_i\right)\mathbf{x} \right\rangle_Z \tag{39}$$

$$\dot{V}_i + \alpha_i V_i = \left\langle \mathbf{x}, \left(\alpha_i \tilde{\mathcal{T}}_i^* \mathcal{H}_i \tilde{\mathcal{A}}_i + \tilde{\mathcal{T}}_i^* \mathcal{H}_i \tilde{\mathcal{A}}_i + \tilde{\mathcal{A}}_i^* \mathcal{H}_i \tilde{\mathcal{T}}_i\right)\mathbf{x} \right\rangle_Z \tag{40}$$

Condition 2:

Considering in the stage of Asynchronous switching $i \in \mathcal{P}_{m1}, \beta_i > 0$. For any $j \in \mathcal{P}_{m2}, i \neq j$.

Let $\hat{\mathcal{T}}_i = \mathcal{T}_i + \mathcal{K}_j \mathcal{B}_{T_2 i}$, $\hat{\mathcal{A}}_i = \mathcal{A}_i + \mathcal{K}_j \mathcal{B}_{2i}$, then

$$\begin{aligned} V_i(\mathbf{x}) &= \left\langle \hat{\mathcal{T}}_i \mathbf{x}, \mathcal{H}_i \hat{\mathcal{A}}_i \mathbf{x} \right\rangle_Z \\ &= \left\langle \mathbf{x}, \hat{\mathcal{T}}_i^* \mathcal{H}_i \hat{\mathcal{A}}_i \mathbf{x} \right\rangle_Z \end{aligned} \tag{41}$$

$$\dot{V}_i(\mathbf{x}) = \left\langle \mathbf{x}, \left(\hat{\mathcal{T}}_i^* \mathcal{H}_i \hat{\mathcal{A}}_i + \hat{\mathcal{A}}_i^* \mathcal{H}_i \hat{\mathcal{T}}_i\right)\mathbf{x} \right\rangle_Z \tag{42}$$

$$\dot{V}_i - \beta_i V_i = \left\langle \mathbf{x}, \left(-\beta_i \hat{\mathcal{T}}_i^* \mathcal{H}_i \hat{\mathcal{A}}_i + \hat{\mathcal{T}}_i^* \mathcal{H}_i \hat{\mathcal{A}}_i + \hat{\mathcal{A}}_i^* \mathcal{H}_i \hat{\mathcal{T}}_i\right)\mathbf{x} \right\rangle_Z \tag{43}$$

Combining the above two situations,we have

$$\dot{V}_{\sigma(t)}(x(t)) \leq \begin{cases} -\alpha_{\sigma(t)} V_{\sigma(t)}(x(t)) & t \in [t_k + \tau(t_k), t_{k+1}) \\ \beta_{\sigma(t)} V_{\sigma(t)}(x(t)) & t \in [t_k, t_k + \tau(t_k)) \end{cases} \tag{44}$$

From (above) we can obtain

$$V_{\sigma(t)}(x(t)) \leq \mathbf{exp}\left(-\alpha_{\sigma(T_{N_\sigma})}\right) V_{\sigma(T_{N_\sigma})}(x(T_{N_\sigma})) \tag{45}$$

Using the left continuity of Lyapunov function $V_{\sigma(t)}(\mathbf{x}(t^-)) = V_{\sigma(t)}(\mathbf{x}(t))$

$$\begin{aligned} V_{\sigma(t)}(x(t)) \leq \mathbf{exp} \big[&-\alpha_{\sigma(T_{N_\sigma})}(t - T_{N_\sigma}) \\ &+ \beta_{\sigma(t_{N_\sigma})}\tau(t_{N_\sigma})\big] V_{\sigma(t_{N_\sigma})}(x(t_{N_\sigma})) \end{aligned} \tag{46}$$

Then by (Assum 1) iterating the above equation, we obtain

$$V_{\sigma(t)}(x(t)) \leq \mathbf{exp}\left(\Phi(t)) V_{\sigma(0)}(x(0)\right) \tag{47}$$

in which

$$\phi(t) = \sum_{k=0}^{N_\sigma-1} \ln \mu_{\sigma(t_k)\sigma(t_{k+1})} - \alpha_{\sigma(T_{N_\sigma})}(t - T_{N_\sigma}) - \sum_{k=0}^{N_\sigma-1} \alpha_{\sigma(T_k)}(t_{k+1} - T_k) + \sum_{k=0}^{N_\sigma-1} \beta_{\sigma(t_k)}\tau(t_k) \tag{48}$$

We notice that

$$\begin{aligned}\sum_{k=0}^{N_\sigma-1} \ln \mu_{\sigma(t_k)\sigma(t_{k+1})} &= \sum_{m\in\mathcal{P}} \sum_{k=0}^{N_\sigma-1} \sum_{m\to n} \ln \mu_{mn} \\ &= N_\sigma \sum_{(m,n)\in E(\mathcal{P})} \ln \mu_{mn} \frac{\#\{m \to n\}_t}{N_\sigma}\end{aligned} \tag{49}$$

Hence

$$\begin{aligned}\sum_{k=0}^{N_\sigma-1} \alpha_{\sigma(T_k)}(t_{k+1} - T_k) &= \sum_{k=0}^{N_\sigma-1} \alpha_{\sigma(T_k)} S_{k+1} \\ &= \sum_{k=0}^{N_\sigma-1} \left(\sum_{i\in\mathcal{P}} 1_{(i)}(\sigma(T_k))\alpha_i S_{k+1} \right) \\ &= -\sum_{i\in\mathcal{P}^u_{m2}} |\alpha_i| \sum_{k:\sigma(T_k)=i} S_{k+1} \\ &\quad + \sum_{i\in\mathcal{P}^s_{m2}} |\alpha_i| \sum_{k:\sigma(T_k)=i} S_{k+1}\end{aligned} \tag{50}$$

$$\begin{aligned}\phi(t) = \ h(t) \Bigg(\frac{N_\sigma}{h(t)} &\sum_{(m,n)\in E(\mathcal{P})} \ln \mu_{mn} \frac{\#\{m \to n\}}{N_\sigma} \\ &+ \sum_{i\in\mathcal{P}^u_{m2}} |\alpha_i| \sum_{k:\sigma(T_k)=i} \frac{S_{k+1}}{h(t)} \\ &- \sum_{i\in\mathcal{P}^s_{m2}} |\alpha_i| \sum_{k:\sigma(T_k)=i} \frac{S_{k+1}}{h(t)} \\ &- \alpha_{\sigma(T_{N_\sigma})} \frac{(t - T_{N_\sigma})}{h(t)} \\ &+ \sum_{k=0}^{N_\sigma-1} \beta_{\sigma(t_k)} \frac{\tau_{t_k}}{h(t)} \Bigg)\end{aligned} \tag{51}$$

$$\lim_{t\to\infty} \mathbf{exp}\{h(t)(v_h(t)f(t) + g(t))\} = 0 \tag{52}$$

$$\lim_{t\to\infty} \mathbf{sup}\ (v_h(t)f(t) + g(t)) < 0 \tag{53}$$

$$\lim_{t\to\infty} \mathbf{sup}\ f(t) \le \sum_{(m,n)\in E(\mathcal{P})} \ln \mu_{mn} \lim_{t\to\infty} \mathbf{sup}\ \rho_{mn}(t) \tag{54}$$

$$\lim_{t\to\infty} \mathbf{sup}\ g(t) = \sum_{(m,n)\in E(\mathcal{P})} \ln \mu_{mn} \lim_{t\to\infty} \mathbf{sup}\ \rho_{mn}(t) \tag{55}$$

$$\begin{aligned}\lim_{t\to\infty} \mathbf{sup}\ g(t) = \lim_{t\to\infty} \mathbf{sup} \Bigg(&\sum_{i\in\mathcal{P}^u_{m2}} |\alpha_i|\, \eta_2^h(i,t) \\ &- \sum_{i\in\mathcal{P}^s_{m2}} |\alpha_i|\, \eta_2^h(i,t) + \sum_{i\in\mathcal{P}} |\beta_i|\, \eta_1^h(i,t) \Bigg)\end{aligned} \tag{56}$$

$$\begin{aligned}\lim_{t\to\infty} \mathbf{sup}\ g(t) \le &\sum_{i\in\mathcal{P}^u_{m2}} |\alpha_i| \lim_{t\to\infty} \mathbf{sup}\ \eta_2^h(i,t) \\ &- \sum_{i\in\mathcal{P}^s_{m2}} |\alpha_i| \lim_{t\to\infty} \mathbf{inf}\ \eta_2^h(i,t) \\ &+ \sum_{i\in\mathcal{P}} |\beta_i| \lim_{t\to\infty} \mathbf{sup}\ \eta_1^h(i,t)\end{aligned} \tag{57}$$

$$\begin{aligned}&\lim_{t\to\infty} \mathbf{sup}\ v_h(t) \sum_{(m,n)\in E(\mathcal{P})} \ln \mu_{mn} \lim_{t\to\infty} \mathbf{sup}\ \rho_{mn}(t) \\ &+ \sum_{i\in\mathcal{P}^u_{m2}} |\alpha_i| \lim_{t\to\infty} \mathbf{sup}\ \eta_2^h(i,t) - \sum_{i\in\mathcal{P}^s_{m2}} |\alpha_i| \lim_{t\to\infty} \mathbf{sup}\ \eta_2^h(i,t) \\ &+ \sum_{i\in\mathcal{P}} |\beta_i| \lim_{t\to\infty} \mathbf{inf}\ \eta_1^h(i,t) < 0\end{aligned} \tag{58}$$

$$V_{\sigma(t)}(x(t)) \le \mathbf{exp}(\phi(t)) V_{\sigma(0)}(x(0)) \tag{59}$$

Remark 3.1 *Due to the use of left continuity for the Lyapunov function, the coupling of two judgment conditions is implicitly involved.*

Remark 3.2 *Due to the use of left continuity for the Lyapunov function, the coupling of two judgment conditions is implicitly involved.*

Remark 3.3 *Due to the use of left continuity for the Lyapunov function, the coupling of two judgment conditions is implicitly involved.*

Remark 3.4 *Due to the use of left continuity for the Lyapunov function, the coupling of two judgment conditions is implicitly involved.*

Remark 3.5 *Due to the use of left continuity for the Lyapunov function, the coupling of two judgment conditions is implicitly involved.*

4 Case Studies Using *PIETOOLS2022a*

Example 4.1 *Consider a neutral system(xia)*

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} A_1 & 0 & 0 & E_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t-\tau_1) \\ w(t-\tau_1) \\ u(t-\tau_1) \\ \dot{x}(t-\tau_1) \end{bmatrix}. \quad (60)$$

where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

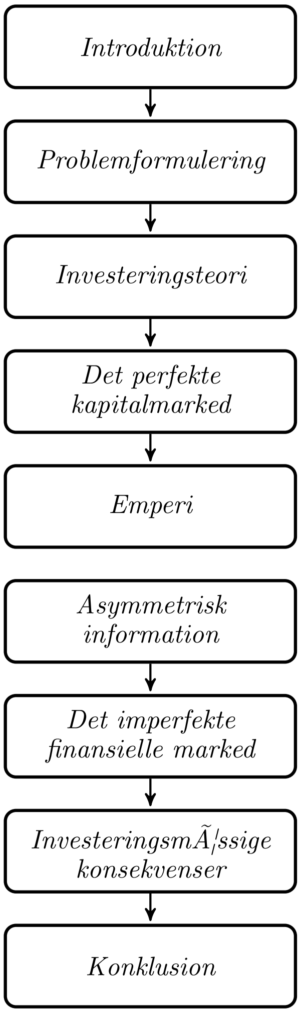

Table 1
DELAY BOUND τ_{max} FOR $c = 0.5$

τ_{max}^{YueHan}	3.69	$\tau_{max}^{Fridman}$	1.14
τ_{max}^{He}	3.67	τ_{max}	4.7274
τ_{max}^{Han}	3.62	$\tau_{max}^{analytical}$	4.7388

Table 2
DELAY BOUND τ_{max} FOR VARIOUS c

τ_{max}^{YueHan}	3.69	$\tau_{max}^{Fridman}$	1.14
τ_{max}^{He}	3.67	τ_{max}	4.7274
τ_{max}^{Han}	3.62	$\tau_{max}^{analytical}$	4.7388

Example 4.2 *Consider a neutral system(xia)*

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} A_1 & 0 & 0 & E_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t-\tau_1) \\ w(t-\tau_1) \\ u(t-\tau_1) \\ \dot{x}(t-\tau_1) \end{bmatrix}. \tag{61}$$

Subsystem 1:

$$A_0 = \begin{bmatrix} 1.8 & -0.3 \\ 0 & 2.5 \end{bmatrix} \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Subsystem 2:

$$A_0 = \begin{bmatrix} 1.8 & -0.3 \\ 0 & 2.5 \end{bmatrix} \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$\tau(t)$,r,h,α_1,α_2,β_1,β_2

$$K_1 = \begin{bmatrix} 1.8 & -0.3 \\ 0 & 2.5 \end{bmatrix} \quad K_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

Let $h(t) = t$

$$N_t^\sigma$$

$$\eta_1^h(1,t) = \eta_1^h(2,t)$$

$$\eta_2^h(1,t)$$

$$\eta_2^h(2,t)$$

5 Conclusion

cite Delay-Dependent Stability for Load FrequencyControl System via Linear Operator Inequality

$$\mathcal{T}'^*\mathcal{H}\mathcal{A}' + \mathcal{A}'^*\mathcal{H}\mathcal{T}' < 0 \tag{62}$$

PROOF :

$$\begin{aligned} \dot{V}(\mathbf{x}) &= < \mathcal{T}'\mathbf{x}, \mathcal{H}\mathcal{T}'\mathbf{x} >_Z + < \mathcal{A}'\mathbf{x}, \mathcal{H}\mathcal{T}'\mathbf{x} >_Z \\ &= < \mathbf{x}, (\mathcal{T}'^*\mathcal{H}\mathcal{A}' + \mathcal{A}'^*\mathcal{H}\mathcal{T}')\mathbf{x} >_Z \end{aligned} \tag{63}$$

6 References

cite Delay-Dependent Stability for Load FrequencyControl System via Linear Operator Inequality

$$\mathcal{T}'^*\mathcal{H}\mathcal{A}' + \mathcal{A}'^*\mathcal{H}\mathcal{T}' < 0 \tag{64}$$

$$\begin{aligned} \dot{V}(\mathbf{x}) &= < \mathcal{T}'\mathbf{x}, \mathcal{H}\mathcal{T}'\mathbf{x} >_Z + < \mathcal{A}'\mathbf{x}, \mathcal{H}\mathcal{T}'\mathbf{x} >_Z \\ &= < \mathbf{x}, (\mathcal{T}'^*\mathcal{H}\mathcal{A}' + \mathcal{A}'^*\mathcal{H}\mathcal{T}')\mathbf{x} >_Z \end{aligned} \tag{65}$$

7 Appendixes

PROOF. lemma2:
Given NDS form

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_0 & 0 & B_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + \sum_{i=1}^{K} \begin{bmatrix} A_i & 0 & B_{2i} & E_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t-\tau_i) \\ w(t-\tau_i) \\ u(t-\tau_i) \\ \dot{x}(t-\tau_i) \end{bmatrix} \tag{66}$$

assume that

$$v(t) = \sum_{i=1}^{K} C_{vi} r_i(t-\tau_i) \tag{67}$$

then we get

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_0 & 0 & B_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + Iv(t) \tag{68}$$

I is a identity matrix,we define

$$I = \begin{bmatrix} B_v \\ D_{1v} \\ D_{2v} \end{bmatrix} \tag{69}$$

so that

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_0 & 0 & B_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} B_v \\ D_{1v} \\ D_{2v} \end{bmatrix} v(t) \tag{70}$$

from the above eqution,we get

$$\dot{x}(t) = \begin{bmatrix} A_0 & 0 & B_2 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + B_v v(t) \tag{71}$$

we can see $B_v v(t)$ Representing the first part of $v(t)$,so we get

$$B_v v(t) = \begin{bmatrix} I & 0 & 0 \end{bmatrix} v(t) \tag{72}$$

define

$$r_i(t) = \begin{bmatrix} x(t) \\ z(t) \\ y(t) \\ \dot{x}(t) \end{bmatrix} \tag{73}$$

$$\begin{aligned} r_i(t) = \begin{bmatrix} x(t) \\ z(t) \\ y(t) \\ \dot{x}(t) \end{bmatrix} &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{x}(t) \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A_0 & 0 & B_2 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} v(t) \end{aligned} \tag{74}$$

Cite [NDS to DDF] we get

$$r_i(t) = \begin{bmatrix} C_{ri} & B_{r1i} & B_{r2i} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + D_{rvi} v(t) \tag{75}$$

merge eqution (27) and eqution (30),we get the standard DDF form

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \\ r_i(t) \end{bmatrix} = \begin{bmatrix} A_0 & 0 & B_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{ri} & B_{r1i} & B_{r2i} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} B_v \\ D_{1v} \\ D_{2v} \\ D_{rvi} \end{bmatrix} v(t) \tag{76}$$

PROOF. lemma3: