its resolvent operator. For the remaining cases satisfying (35) the authors suggest in the conclusion solving numerically an equation that is related to the eigenvalues of the system. Although Theorem III.2 excludes the case $k_1^2 = h_1 v_2/v_1 h_2$ and $k_2^2 = h_2 v_1/v_2 h_1$, it has the advantage of treating all the remaining cases satisfying (35) using only a matrix inequality.

V. Conclusion

We provided tools that facilitate checking the exponentially stability property of a class of BCS. We showed that by using results of [14] and [16] it is easy to select the input and outputs of a BCS. Therefore, we use those results on boundary port Hamiltonian systems to define inputs and outputs for our class of BCS. Once this is done, checking for exponential stability follows easily. The main idea behind the proof consists in using a multiplier common to the whole class of BCS. This multiplier only depends on the norm of the co-energy variables at the boundary of the spatial domain. In this way one avoids searching for different multipliers every time the system or the boundary conditions are changed. This simplifies drastically the verification of the exponential stability property, as can be seen already from the examples in Section IV. Also the proof of the results of [22] and [23] can be simplified by using our results.

Even though the results are only valid for a class of one-dimensional systems, the authors believe that the approach has potential to be extended to 2-D and 3-D systems. The key point being the definition and selection of the boundary port variables. Some ideas about this are presented in [16, Ch. 8]. However, this still requires more research.

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Delay-Dependent Exponential Stability of Neutral Stochastic Delay Systems

Lirong Huang and Xuerong Mao

***Abstract*—This technical note studies stability of neutral stochastic delay systems by linear matrix inequality approach. Delay-dependent criterion for exponential stability is presented and numerical examples are conducted to verify the effectiveness of the proposed method.**

***Index Terms*—Exponential stability, linear matrix inequalitys (LMIs), neutral systems, stochastic systems, time delay.**

I. Introduction

Many dynamical systems are described with neutral functional differential equations that include neutral delay differential equations [19]. These systems are called neutral-type systems or neutral systems. Motivated by chemical engineering systems as well as theory of aero elasticity, studies on deterministic neutral systems have been of research interest over the past decades [3]–[11], [21]. As stochastic modelling has come to play an important role in many branches of science and industry, neutral stochastic delay systems have been intensively studied over recent year [10]–[17]. Mao [14]–[17] initiated the study of exponential stability of neutral stochastic functional equations, developed the Razumikhin-type theorems further for exponential stability of neutral stochastic functional equations and studied asymptotic properties of neutral stochastic delay differential equations [1]. More recently, Luo *et al.* [12] proposed new criteria on exponential stability of neutral stochastic delay differential equations while Chen *et al.* [2] studied delay-dependent stability of neutral stochastic delay systems. However, the stability result in [2] employed an

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assumption on the difference operator matrix, which is also assumed in other results [4] and [18] but may be restrictive in many cases (see Examples 1 and 2). As is known, delay-independent results may be conservative when the size of time delay is small. This technical note studies problem of delay-dependent stability of neutral stochastic delay systems. An exponential stability criterion is established by linear matrix inequality (LMI) approach. Numerical examples are conducted to verify the effectiveness of our proposed method.

II. Problem Statement

Throughout the technical note, unless otherwise specified, we will employ the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\mathbb{E}[\cdot]$ be the expectation operator with respect to the probability measure. Let $w(t)$ be a scalar Brownian motion defined on the probability space. If A is a vector or matrix, its transpose is denoted by A^T. If P is a square matrix, $P > 0$ ($P < 0$) means that P is a symmetric positive (negative) definite matrix of appropriate dimensions while $P \geq 0$ ($P \leq 0$) is a symmetric positive (negative) semidefinite matrix. I stands for the identity matrix of appropriate dimensions. Denote by $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ the maximum and minimum eigenvalue of a matrix respectively. Let $|\cdot|$ denote the Euclidean norm of a vector and its induced norm of a matrix. Unless explicitly specified, matrices are assumed to have real entries and compatible dimensions. Let $h \geq 0$ and $C([-h,0];R^n)$ denote the family of all continuous R^n-valued functions φ on $[-h,0]$ with the norm $\|\varphi\| = \sup\{|\varphi(\theta)| : -h \leq \theta \leq 0\}$. Let $C^b_{\mathcal{F}_0}([-h,0];R^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-h,0];R^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$.

Let us consider an n-dimensional neutral stochastic delay system

$$\begin{aligned} \mathrm{d}[x(t) - Cx(t-h_1)] &= [A_0x(t) + A_1x(t-h_1) + A_2x(t-h_2)]\,\mathrm{d}t \\ &\quad + [H_0x(t) + H_1x(t-h_1) + H_2x(t-h_2)]\mathrm{d}w(t) \end{aligned} \tag{1}$$

on $t \geq 0$ with initial data $x_0 = \{x(\theta) : -h \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-h,0];R^n)$, where $x(t) \in R^n$ is the state vector; positive scalar constants h_1, h_2 are time delays of the system and $h = \max\{h_1, h_2\}$; C, A_i and H_i, $i = 0, 1, 2$, are known matrices.

Denote

$$\begin{aligned} f(t) &= A_0x(t) + A_1x(t-h_1) + A_2x(t-h_2) \\ g(t) &= H_0x(t) + H_1x(t-h_1) + H_2x(t-h_2) \end{aligned} \tag{2}$$

for all $t \geq 0$. One can observe that

$$|f(t)|^2 \leq K_f\|x_t\|^2, \quad |g(t)|^2 \leq K_g\|x_t\|^2 \tag{3}$$

for all $t \geq 0$, where $x_t = \{x(t+\theta) : -h \leq \theta \leq 0\}$, $K_f = 3\sum_{i=0}^{2}|A_i|^2$ and $K_g = 3\sum_{i=0}^{2}|H_i|^2$. This implies that both $f(\varphi,t)$ and $g(\varphi,t)$ satisfy the local Lipschitz condition and the linear growth condition. It is easy to verify, by the way of induction proposed in the proof of Theorem 3.1, p208, [16], that there exists a unique continuous solution denoted by $x(t;\xi)$ to neutral stochastic delay differential (1).

The objective of this technical note is to establish sufficient conditions for robust exponential stability of system (1). It should be pointed out that, for simplicity only, we do not consider uncertainties in our models. The proposed method can be easily extended to those cases with norm-bounded uncertainties in parameters A_i and H_i. The method can also be applied to systems with multiple and distributed delays.

At the end of this section, let us introduce the following definitions and lemmas that are useful for the development of our results.

Definition 1: The neutral stochastic delay system (1) is said to be exponentially stable in mean square if there is a positive constant λ such that [16]

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}|x(t;\xi)|^2 \leq -\lambda\,. \tag{4}$$

Definition 2: The neutral stochastic delay system (1) is said to be almost surely exponentially stable if there is a positive constant λ such that [16]

$$\limsup_{t\to\infty} \frac{1}{t}\log |x(t;\xi)| \leq -\lambda\,. \tag{5}$$

Lemma 1: ([20]) For any constant matrix $M \in R^{q\times l}$, inequality

$$2u^TMv \leq ru^TMGM^Tu + \frac{1}{r}v^TG^{-1}v\,, \quad u \in R^q\,,\ v \in R^l$$

holds for any pair of symmetric positive definite matrix $G \in R^{l\times l}$ and positive number $r > 0$.

Lemma 2: ([6]) For any pair of symmetric positive definite constant matrix $G \in R^{l\times l}$ and scalar $r > 0$, if there exists a vector function $v : [0,r] \to R^l$ such that integrals $\int_0^r v^T(s)Gv(s)\mathrm{d}s$ and $\int_0^r v(s)\mathrm{d}s$ are well defined, then the following inequality holds:

$$r\int_0^r v^T(s)Gv(s)\mathrm{d}s \geq \left(\int_0^r v(s)\mathrm{d}s\right)^T G\left(\int_0^r v(s)\mathrm{d}s\right).$$

III. Delay-Dependent Exponential Stability

Delay-dependent stability of neutral deterministic delay systems has been intensively studied over recent years [3]–[5], [8], [11], [18]. However, relatively little is known about delay-dependent stability of neutral stochastic delay systems. Denote $\bar{A}_0 = A_0$, $\bar{A}_1 = A_0C + A_1$, $\bar{A}_2 = A_2$, $\bar{H}_0 = H_0$, $\bar{H}_1 = H_0C + H_1$, $\bar{H}_2 = H_2$, $\bar{A} = \sum_{i=0}^{2}\bar{A}_i$ and $\bar{H} = \sum_{i=0}^{2}\bar{H}_i$. Sufficient conditions for delay-dependent exponential stability of system (1) are proposed as follows.

Theorem 1: The neutral stochastic delay system (1) is mean-square exponentially stable and is also almost surely exponentially stable provided that there exist matrices $P_{11} > 0$, $Q_k > 0$, $R_k > 0$, $S > 0$, $T_k > 0$, P_{21}, P_{22}, P_{23}, P_{31}, P_{32}, P_{33} and $k = 1, 2$ such that LMI is (6), shown at the bottom of the next page, where

$$\begin{aligned}
\Gamma_{11} &= P_{21}^T\bar{A} + \bar{A}^TP_{21} + P_{31}^T\bar{H} + \bar{H}^TP_{31} + S + T_1 + T_2 \\
\Gamma_{12} &= \bar{A}^TP_{22} + \bar{H}^TP_{32} + P_{11} - P_{21}^T \\
\Gamma_{13} &= \bar{A}^TP_{23} + \bar{H}^TP_{33} - P_{31}^T\,, \quad \Gamma_{18} = (S + T_1 + T_2)C \\
\Gamma_{22} &= -P_{22}^T - P_{22} + h_1Q_1 + h_2Q_2\,, \quad \Gamma_{23} = -P_{23} - P_{32}^T \\
\Gamma_{33} &= -P_{33}^T - P_{33} + P_{11} + h_1R_1 + h_2R_2 \\
\Gamma_{88} &= -S + C^T(S + T_1 + T_2)C \\
L_{11} &= P_{21}^T\bar{A}_1 + P_{31}^T\bar{H}_1\,, \quad L_{12} = P_{21}^T\bar{A}_2 + P_{31}^T\bar{H}_2 \\
L_{21} &= P_{22}^T\bar{A}_1 + P_{32}^T\bar{H}_1\,, \quad L_{22} = P_{22}^T\bar{A}_2 + P_{32}^T\bar{H}_2 \\
L_{31} &= P_{23}^T\bar{A}_1 + P_{33}^T\bar{H}_1\,, \quad L_{32} = P_{23}^T\bar{A}_2 + P_{33}^T\bar{H}_2
\end{aligned}$$

and entries denoted by $*$ can be readily inferred from symmetry of the matrix.

Proof: To simplify the expression, we define

$$\eta(t) = x(t) - Cx(t-h_1) \tag{7}$$

for all $t \geq 0$. With notations (2) and (7), we can rewrite the unforced system (1) as

$$\mathrm{d}\eta(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}w(t) \tag{8}$$

on $t \geq 0$ with initial data ξ.

So we have

$$\eta(t_2) - \eta(t_1) = \int_{t_1}^{t_2} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) \tag{9}$$

for all $t_2 \geq t_1 \geq 0$.

By (2) and (9), we can observe that

$$\begin{aligned} f(t) &= \sum_{i=0}^{2} \bar{A}_i \eta(t) - \sum_{i=1}^{2} \bar{A}_i \Big[\eta(t) - \eta(t-h_i)\Big] \\ &\quad + \sum_{i=1}^{2} \bar{A}_i C x(t - h_1 - h_i) \\ &= \bar{A}\,\eta(t) - \sum_{i=1}^{2} \bar{A}_i \int_{t-h_i}^{t} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) \\ &\quad + \sum_{i=1}^{2} \bar{A}_i C x(t - h_1 - h_i), \end{aligned} \tag{10}$$

$$\begin{aligned} g(t) &= \bar{H}\eta(t) - \sum_{i=1}^{2} \bar{H}_i \int_{t-h_i}^{t} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) \\ &\quad + \sum_{i=1}^{2} \bar{H}_i C x(t - h_1 - h_i) \end{aligned} \tag{11}$$

for all $t \geq h$. Choose a Lyapunov–Krasovskii functional candidate for system (8) as follows:

$$V(t) = \sum_{j=1}^{5} V_j(t), \quad t \geq h \tag{12}$$

where

$$\begin{aligned} V_1(t) &= \eta^T(t) P_{11} \eta(t) \\ V_2(t) &= \sum_{i=1}^{2} \int_{t-h_i}^{t} (s - t + h_i) f^T(s) Q_i f(s)\mathrm{d}s \\ V_3(t) &= \sum_{i=1}^{2} \int_{t-h_i}^{t} (s - t + h_i) g^T(s) R_i g(s)\mathrm{d}s \\ V_4(t) &= \int_{t-h_1}^{t} x^T(s) S x(s)\mathrm{d}s \\ V_5(t) &= \sum_{i=1}^{2} \int_{t-h_1-h_i}^{t} x^T(s) T_i x(s)\mathrm{d}s\,. \end{aligned}$$

By Itô's lemma, we have

$$\mathrm{d}V(t) = \mathcal{L}V(t)\mathrm{d}t + \sigma(t)\mathrm{d}w(t) \tag{13}$$

where

$$\begin{aligned} \mathcal{L}V(t) &= \sum_{j=1}^{5} \mathcal{L}V_j(t) = 2\eta^T(t) P_{11} f(t) + g^T(t) P_{11} g(t) \\ &\quad + \sum_{j=2}^{5} \dot{V}_j(t) \\ \sigma(t) &= 2\eta^T(t) P_{11} g(t)\,. \end{aligned} \tag{14}$$

Denote

$$y(t) = \begin{bmatrix} \eta(t) \\ f(t) \\ g(t) \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} P_{11} & 0 & 0 \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}. \tag{15}$$

By equalities (10) and (11), we have

$$\begin{aligned} &2\eta^T(t) P_{11} f(t) \\ &= y^T(t)(P^T A + A^T P) y(t) - 2y^T(t) \sum_{i=1}^{2} P^T [0 \quad \bar{A}_i^T \quad \bar{H}_i^T]^T \\ &\quad \cdot \left(\int_{t-h_i}^{t} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) + Cx(t - h_1 - h_i) \right) \end{aligned} \tag{16}$$

where

$$A = \begin{bmatrix} 0 & I & 0 \\ \bar{A} & -I & 0 \\ \bar{H} & 0 & -I \end{bmatrix}$$

$$P^T A + A^T P = \begin{bmatrix} P_{A1} & P_{A2} & P_{A3} \\ * & -P_{22}^T - P_{22} & -P_{32}^T - P_{23} \\ * & * & -P_{33}^T - P_{33} \end{bmatrix}$$

with $P_{A1} = P_{21}^T \bar{A} + \bar{A}^T P_{21} + P_{31}^T \bar{H} + \bar{H}^T P_{31}$, $P_{A2} = \bar{A}^T P_{22} + \bar{H}^T P_{32} + P_{11} - P_{21}^T$, $P_{A3} = \bar{A}^T P_{23} + \bar{H}^T P_{33} - P_{31}^T$ and $P^T\,[0 \quad \bar{A}_i^T \quad \bar{H}_i^T]^T = [\,L_{1i}^T \quad L_{2i}^T \quad L_{3i}^T]^T$ for $i = 1, 2$.

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & h_1 L_{11} & h_2 L_{12} & L_{11} & L_{12} & \Gamma_{18} & L_{11}C & L_{12}C \\ * & \Gamma_{22} & \Gamma_{23} & h_1 L_{21} & h_2 L_{22} & L_{21} & L_{22} & 0 & L_{21}C & L_{22}C \\ * & * & \Gamma_{33} & h_1 L_{31} & h_2 L_{32} & L_{31} & L_{32} & 0 & L_{31}C & L_{32}C \\ * & * & * & -h_1 Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & -h_2 Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & -R_1 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & -R_2 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{88} & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & -T_1 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & -T_2 \end{bmatrix} < 0 \tag{6}$$

Direct computations with Lemma 2 and (7) give

$$\dot{V}_2(t) \le \sum_{i=1}^{2}\left[f^T(t)h_iQ_if(t) - \int_{t-h_i}^{t}\frac{1}{h_i}f^T(s)\mathrm{d}s\cdot(h_iQ_i)\cdot\int_{t-h_i}^{t}\frac{1}{h_i}f(s)\mathrm{d}s\right], \tag{17}$$

$$\dot{V}_3(t) = \sum_{i=1}^{2}\left[g^T(t)h_iR_ig(t) - \int_{t-h_i}^{t}g^T(s)R_ig(s)\mathrm{d}s\right], \tag{18}$$

$$\dot{V}_4(t) = \begin{bmatrix}\eta(t)\\ x(t-h_1)\end{bmatrix}^T\begin{bmatrix}S & SC\\ C^TS & -S+C^TSC\end{bmatrix}\begin{bmatrix}\eta(t)\\ x(t-h_1)\end{bmatrix} \tag{19}$$

$$\dot{V}_5(t) = \sum_{i=1}^{2}\begin{bmatrix}\eta(t)\\ x(t-h_1)\\ x(t-h_1-h_i)\end{bmatrix}^T\begin{bmatrix}T_i & T_iC & 0\\ C^TT_i & C^TT_iC & 0\\ 0 & 0 & -T_i\end{bmatrix}\cdot\begin{bmatrix}\eta(t)\\ x(t-h_1)\\ x(t-h_1-h_i)\end{bmatrix}. \tag{20}$$

By isometry property, for $i = 1, 2$, we have

$$\begin{aligned}\mathbb{E}\left[\int_{t-h_i}^{t}g^T(s)R_ig(s)\mathrm{d}s\right] &= \int_{t-h_i}^{t}\mathbb{E}\left[g^T(s)R_ig(s)\right]\mathrm{d}s\\ &= \mathbb{E}\left[\int_{t-h_i}^{t}g^T(s)\mathrm{d}w(s)\,R_i\int_{t-h_i}^{t}g(s)\mathrm{d}w(s)\right].\end{aligned}$$

Therefore, substituting inequalities (16)–(20) into (14) and taking expectation on the both sides of (14) yield

$$\mathbb{E}\mathcal{L}V(t) \le \mathbb{E}\left[z^T(t)\Gamma z(t)\right] \tag{21}$$

where

$$z^T(t) = \left[\eta^T(t)\ \ f^T(t)\ \ g^T(t)\ \ \frac{1}{h_1}\int_{t-h_1}^{t}f^T(s)\mathrm{d}s\frac{1}{h_2}\cdot\int_{t-h_2}^{t}f^T(s)\mathrm{d}s\int_{t-h_1}^{t}g^T(s)\mathrm{d}w(s)\int_{t-h_2}^{t}g^T(s)\mathrm{d}w(s)\cdot x^T(t-h_1)\ \ x^T(t-2h_1)\ \ x^T(t-h_1-h_2)\right]^T.$$

By LMI (6), we have

$$\mathbb{E}\mathcal{L}V(t) \le -\lambda_\Gamma\mathbb{E}|z(t)|^2 \le -\lambda_\Gamma\mathbb{E}\left[|\eta(t)|^2 + |x(t-h_1)|^2\right] \tag{22}$$

with $\lambda_\Gamma = \lambda_m(-\Gamma)$ and

$$C^TSC - S < 0\,. \tag{23}$$

For any $\kappa \in (0,1)$, (7), inequalities (22)–(23) and Lemma 1 give

$$\begin{aligned}&\mathbb{E}\mathcal{L}V(t)\\ &\le -(1-\kappa)\lambda_\Gamma\mathbb{E}|\eta(t)|^2 - \kappa\lambda_\Gamma\left(\mathbb{E}|\eta(t)|^2 + \frac{1}{\kappa}\mathbb{E}|x(t-h_1)|^2\right)\\ &\le -(1-\kappa)\lambda_\Gamma\mathbb{E}|\eta(t)|^2 - \frac{\kappa\lambda_\Gamma}{\lambda_M(S)}\mathbb{E}\left[\Big(x(t)-Cx(t-h_1)\Big)^TS\cdot\Big(x(t)-Cx(t-h_1)\Big) + \frac{1}{\kappa}x^T(t-h_1)Sx(t-h_1)\right]\\ &\le -(1-\kappa)\lambda_\Gamma\mathbb{E}|\eta(t)|^2 - \frac{\kappa\lambda_\Gamma}{\lambda_M(S)}\mathbb{E}\left[x^T(t)Sx(t) - 2x^T(t)SCx(t-h_1) + \frac{1+\kappa}{\kappa}x^T(t-h_1)C^TSCx(t-h_1)\right]\\ &\le -(1-\kappa)\lambda_\Gamma\mathbb{E}|\eta(t)|^2 - \frac{\kappa\lambda_\Gamma\lambda_m(S)}{(1+\kappa)\lambda_M(S)}\mathbb{E}|x(t)|^2\,.\\ &\le -\lambda_0\mathbb{E}\left[|\eta(t)|^2 + |x(t)|^2\right]\end{aligned}$$

where $\lambda_0 = \min\left\{(1-\kappa)\lambda_\Gamma,\ \kappa\lambda_\Gamma\lambda_m(S)[(1+\kappa)\lambda_M(S)]^{-1}\right\} > 0$.

It is obvious from the definition of $V(t)$ that

$$\alpha_0|\eta(t)|^2 \le V(t) \le \alpha_1|\eta(t)|^2 + \alpha_2\int_{t-2h}^{t}|x(s)|^2\mathrm{d}s \tag{24}$$

where $\alpha_0 = \lambda_m(P_{11})$, $\alpha_1 = \lambda_M(P_{11})$

$$\alpha_2 = \sum_{i=1}^{2}h_i[\lambda_M(Q_i)K_f + \lambda_M(R_i)K_g] + \lambda_M(S) + \sum_{i=1}^{2}\lambda_M(T_i).$$

Choose $\varepsilon > 0$ such that

$$\max\{\varepsilon\alpha_1, 2h\varepsilon\alpha_2e^{2h\varepsilon}\} \le \lambda_0 \quad\text{and}\quad e^{2h\varepsilon}C^TSC - S < 0. \tag{25}$$

By Itô's lemma, we have

$$\mathrm{d}[e^{\varepsilon s}V(s)] = e^{\varepsilon s}[\varepsilon V(s) + \mathcal{L}V(s)]\mathrm{d}s + e^{\varepsilon s}\sigma(s)\mathrm{d}w(s),\ \forall s \ge 0. \tag{26}$$

Let $t_0 = h$, then integrating from t_0 to t and taking expectation on (28) give

$$\begin{aligned}&e^{\varepsilon t}\mathbb{E}V(t) - e^{\varepsilon t_0}\mathbb{E}V(t_0)\\ &= \mathbb{E}\int_{t_0}^{t}e^{\varepsilon s}\left[\varepsilon V(s) + \mathcal{L}V(s)\right]\mathrm{d}s\\ &\le \mathbb{E}\int_{t_0}^{t}e^{\varepsilon s}\left[\varepsilon\alpha_1|\eta(s)|^2 + \varepsilon\alpha_2\int_{s-2h}^{s}|x(v)|^2\mathrm{d}v - \lambda_0(|\eta(s)|^2 + |x(s)|^2)\right]\mathrm{d}s\\ &\le \mathbb{E}\int_{t_0}^{t}e^{\varepsilon s}\left[\varepsilon\alpha_2\int_{s-2h}^{s}|x(v)|^2\mathrm{d}v - \lambda_0|x(s)|^2\right]\mathrm{d}s.\end{aligned} \tag{27}$$

Since

$$\begin{aligned}&\int_{t_0}^{t}e^{\varepsilon s}\mathrm{d}s\int_{s-2h}^{s}|x(v)|^2\mathrm{d}v\\ &\le \int_{t_0-2h}^{t}|x(v)|^2\mathrm{d}v\int_{v}^{v+2h}e^{\varepsilon s}\mathrm{d}s\\ &\le 2he^{2h\varepsilon}\int_{t_0-2h}^{t}|x(s)|^2e^{\varepsilon s}\mathrm{d}s\end{aligned}$$

$$\leq 2he^{2h\varepsilon}\int_{t_0}^{t}|x(s)|^2e^{\varepsilon s}\mathrm{d}s + 2he^{2h\varepsilon}\int_{t_0-2h}^{t_0}|x(s)|^2\mathrm{d}s$$

it follows:

$$\alpha_0 e^{\varepsilon t}\mathbb{E}|\eta(t)|^2 \leq e^{\varepsilon t}\mathbb{E}V(t) \leq \alpha_0 C_h \text{ or } \mathbb{E}|\eta(t)|^2 \leq C_h e^{-\varepsilon t} \quad (28)$$

where $C_h = \alpha_h \sup_{-h\leq\theta\leq h}\mathbb{E}|x(\theta)|^2$ with $\alpha_h = \alpha_0^{-1}e^{\varepsilon h}[\alpha_1 + 2h\alpha_2(1+2h\varepsilon e^{2h\varepsilon})] \geq 1$. Since neutral stochastic delay differential (1) has a unique continuous solution, C_h is a nonnegative finite number for any $0 \leq h < \infty$.

Since $e^{2\varepsilon h}C^T SC < S$, there exists a number $\mu \in (0,1)$ such that

$$e^{2\varepsilon h}C^T SC < \mu S < S. \quad (29)$$

Note that

$$\eta^T(t)S\eta(t) = x^T(t)Sx(t) \\ -2x^T(t)SCx(t-h_1) + x^T(t-h_1)C^T SCx(t-h_1)$$

for all $t \geq 0$. By Lemma 1, we have

$$e^{\varepsilon t}x^T(t)Sx(t) \leq \frac{e^{\varepsilon t}}{1-\mu}\eta^T(t)S\eta(t) \\ +\frac{e^{\varepsilon t}}{\mu}x^T(t-h_1)C^T SCx(t-h_1). \quad (30)$$

Let ρ be any nonnegative real number. For all $0 \leq t \leq \rho$, we have

$$\begin{aligned} &e^{\varepsilon t}\mathbb{E}\left[x^T(t)Sx(t)\right] \\ &\leq \frac{1}{1-\mu}\sup_{0\leq t\leq\rho}\mathbb{E}\left[e^{\varepsilon t}\eta^T(t)S\eta(t)\right] \\ &\quad+\frac{1}{\mu}\sup_{0\leq t\leq\rho}\mathbb{E}\left[e^{\varepsilon t}x^T(t-h_1)C^T SCx(t-h_1)\right] \\ &\leq \frac{1}{1-\mu}\lambda_M(S)\sup_{0\leq t\leq\rho}\mathbb{E}[e^{\varepsilon t}|\eta(t)|^2] \\ &\quad+\frac{e^{\varepsilon h_1}}{\mu}\sup_{-h_1\leq t\leq\rho}\mathbb{E}\left[e^{\varepsilon t}x^T(t)C^T SCx(t)\right] \\ &\leq \frac{1}{1-\mu}\lambda_M(S)C_h + e^{-\varepsilon h}\sup_{-h\leq t\leq\rho}\left\{e^{\varepsilon t}\mathbb{E}\left[x^T(t)Sx(t)\right]\right\}. \end{aligned}$$

But this holds for all $-h \leq t \leq \rho$. So

$$\sup_{-h\leq t\leq\rho}\left\{e^{\varepsilon t}\mathbb{E}\left[x^T(t)Sx(t)\right]\right\} \leq \frac{\lambda_M(S)C_h}{(1-e^{-\varepsilon h})(1-\mu)}. \quad (31)$$

Since ρ is an arbitary nonnegative number, we have

$$\mathbb{E}|x(t)|^2 \leq \frac{\lambda_M(S)C_h e^{-\varepsilon t}}{(1-e^{-\varepsilon h})(1-\mu)\lambda_m(S)}, \quad \forall\, t \geq -h. \quad (32)$$

The mean-square exponential stability has been proven.

Now let us proceed to discuss the almost sure exponential stability. Let $\gamma \in (0,\varepsilon)$ be arbitrary. We claim that there is a finite positive number t_h such that for all $t \geq t_h$

$$|\eta(t)|^2 \leq e^{-(\varepsilon-\gamma)t} \quad a.s. \quad (33)$$

Therefore, for all $t \geq t_h$, inequality (30) implies

$$e^{(\varepsilon-\gamma)t}x^T(t)Sx(t) \\ \leq \frac{\lambda_M(S)e^{(\varepsilon-\gamma)t}}{1-\mu} + \frac{e^{(\varepsilon-\gamma)t}}{\mu}x^T(t-h_1)C^T SCx(t-h_1) \ a.s.$$

Using the similar reasoning as above and letting $\gamma \to 0$, we have $|x(t)|^2 \leq \lambda_M(S)e^{-\varepsilon t}[(1-e^{-\varepsilon h})(1-\mu)\lambda_m(S)]^{-1}$ $a.s.$ for all $t \geq t_h - h$. This implies immediately

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq -\frac{\varepsilon}{2} \quad a.s.$$

We complete the proof by showing that inequality (33) is true. Note that

$$\mathbb{E}|f(t)|^2 \leq K_f \sup_{-h\leq\theta\leq 0}\mathbb{E}|x(t-\theta)|^2$$
$$\text{and} \quad \mathbb{E}|g(t)|^2 \leq K_g \sup_{-h\leq\theta\leq 0}\mathbb{E}|x(t-\theta)|^2$$

for all $t \geq 0$. For any integer $k \geq 1$, by Hölder's inequality and Burkholder–Davis–Gundy inequality, one can derive that

$$\begin{aligned} &\mathbb{E}\left[\sup\nolimits_{0\leq\theta\leq h}|\eta(kh+\theta)|^2\right] \\ &\leq 3\left[\mathbb{E}|\eta(kh)|^2 + h\int_{kh}^{(k+1)h}\mathbb{E}|f(s)|^2\mathrm{d}s \right. \\ &\quad \left. + \mathbb{E}\left(\sup_{0\leq\theta\leq h}\left|\int_{kh}^{kh+\theta}g(s)\mathrm{d}w(s)\right|\right)\right] \\ &\leq \beta_h e^{-kh\varepsilon} \end{aligned} \quad (34)$$

where $\beta_h = 3C_h(1 + K_f h^2 e^{h\varepsilon} + 4K_g h e^{h\varepsilon})$. But, by Chebyshev's inequality, this implies

$$\mathbb{P}\left\{\omega : \sup_{0\leq\theta\leq h}|\eta(kh+\theta)|^2 > e^{-(\varepsilon-\gamma)kh}\right\} \leq \beta_h e^{-\gamma kh}.$$

By Borel–Cantelli lemma, there is a finite integer k_0 such that

$$\sup_{0\leq\theta\leq h}|\eta(kh+\theta)|^2 \leq e^{-(\varepsilon-\gamma)kh} \quad a.s.$$

for all $k \geq k_0$. Therefore, inequality (33) holds with $t_h \geq k_0 h$. ■

Remark 1: From the proof of Theorem 1, it is observed that, letting

$$z^T(t) = \left[\eta^T(t)\ f^T(t)\ g^T(t)\frac{1}{h_1}\int_{t-h_1}^{t}f^T(s)\mathrm{d}s\frac{1}{h_2}\right. \\ \cdot\int_{t-h_2}^{t}f^T(s)\mathrm{d}s\int_{t-h_1}^{t}g^T(s)\mathrm{d}w(s)\int_{t-h_2}^{t}g^T(s)\mathrm{d}w(s) \\ \left.\cdot\, x^T(t-h_1)\ x^T(t-2h_1)C^T\ x^T(t-h_1-h_2)C^T\right]^T$$

we can have a corollary derived from Theorem 1 with

$$\begin{bmatrix} -T_i & C^T W_i \\ W_i C & -W_i \end{bmatrix} \leq 0, \quad i = 1,2 \quad (35)$$

where $W_i > 0$. This corollary can be easily applied to problems of stabilization by the approach of LMIs.

IV. Examples

Example 1: Let us look at the following neutral stochastic delay system:

$$\mathrm{d}[x(t) - Cx(t-h)] = [A_0x(t) + A_1x(t-h)]\mathrm{d}t \\ + [H_0x(t) + H_1x(t-h)]\mathrm{d}w(s) \quad (36)$$

TABLE I
$h_{\max}$ BY DIFFERENT METHODS

	$\gamma = 2.0$	$\gamma = 2.2$	$\gamma = 2.4$
[3]	0.29	0.25	0.21
[8]	0.40	0.32	0.25
Theorem 1	0.46	0.39	0.30

with

$$C = \begin{bmatrix} -0.2 & 0 \\ 1 & 0.2 \end{bmatrix}, A_0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$
$$H_0 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, H_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

It is easy to verify that the existing results [2], [10], [12]–[17] do not work. But, by Theorem 1, the upper bounds of time delay for exponential stability of system (36) is $h_{\max} = 0.35$.

Example 2: Deterministic systems may be regarded a special class of stochastic systems, e.g., the following deterministic neutral system is exactly system (1) with $A_0 = A$, $A_1 = B$ and $A_2 = H_0 = H_1 = H_2 = 0$, i.e.:

$$\dot{x}(t) - C\dot{x}(t-h) = Ax(t) + Bx(t-h) \tag{37}$$

for all $t \geq 0$, where

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, B = -\begin{bmatrix} 1.1 & 0.2 \\ 0.1 & 1.1 \end{bmatrix}, C = \begin{bmatrix} -0.2 & \gamma \\ 0.2 & -0.1 \end{bmatrix}$$

and γ is a constant real number.

The case of $\gamma = 0$ has been studied by many works [4], [8] and [11]. However, results of [2], [4], [9], [11], and [18] are not (conveniently) applicable when $|\gamma| \geq 1$. For $\gamma \geq 2$, the criterion in [5] does not work, but the upper bounds $h_{\max}$ for exponential stability of (37) by other methods are listed in Table I, which shows that the results obtained by the methods proposed in this technical note are less conservative in these cases.

V. CONCLUSION

In this technical note, delay-dependent criterion for stability of neutral stochastic delay systems has been presented by approach of LMIs. Numerical examples have been given to verify the effectiveness of the method proposed in this technical note. Particularly, Example 2 demonstrates that our result developed for stochastic systems is competitive even when it is specialized to the deterministic cases.

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