Stability Analysis of Linear Neutral Delay Systems With Two Delays via Augmented Lyapunov–Krasovskii Functionals

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***Abstract*— Stability analysis is considered in this paper for linear neutral delay systems subject to two different delays in both the state variables and the retarded derivatives of state variables. By choosing a suitable state vector indexed by an integer k, a new augmented Lyapunov-Krasovskii functional (LKF) is constructed, and a stability criterion based on linear matrix inequalities is developed accordingly. It is shown that the proposed condition is less conservative than the existing methods due to the introduction of the delay-product-type integral terms in the LKF. The resulting stability criterion is then applied to the robust stability analysis of neutral delay systems with norm-bounded uncertainty. Moreover, a delay-independent stability criterion is developed based on the proposed LKF, and its frequency-domain interpretation is also given. These developed stability criteria indexed by an integer k exhibit a hierarchical character: the larger the integer k, the less conservatism of the resulting stability criterion. Finally, two numerical examples are carried out to illustrate the effectiveness of the proposed method.**

***Index Terms*— Neutral delay systems, augmented Lyapunov-Krasovskii functionals, hierarchical stability criteria, robust stability.**

I. Introduction

NEUTRAL delay systems, which contain delays both in system states and derivatives of states [13], can be used to model numerous engineering systems, including partial element equivalent circuits [1], [2], [8], controlled constrained manipulators [24], complex dynamical networks [22], etc. The stability is the basic requirement of systems, thus the stability analysis problem of neutral delay systems has important theoretical and practical significance [28], [29]. As we know, the LKF method is an effective tool for the stability analysis of time-delay systems [7], [11], [16], [17], [18], [25], [31], [32]. By using this method, neutral delay systems with only a single delay have been extensively studied in [12], [14], and [15]. For neutral delay systems with multiple delays, a simple extension of the ideas for a single time delay problem has been adopted in [9]. This extension is simple and thus might lead to certain conservatism. Thus, for neutral delay systems with multiple delays, an important issue is how to find some proper LKFs, by which one can obtain less conservative results. In the current paper, we mainly focus on linear neutral delay systems with two different delays.

Over the past few decades, efforts have been made to derive stability criteria with less conservatism for linear neutral delay systems with two delays. In [27], a stability criterion was established by choosing a proper LKF and adding some appropriate zero terms to the deviation of LKF. In [5], by choosing an LKF whose derivative considers the relationships among delays, a stability criterion which is less conservative than the result in [27] was obtained. Recently, an augmented LKF was constructed in [19]. Since some delayed states and time delays information were introduced in the constructed LKF, the resulting stability criterion greatly improves the results in [27] and [5]. It is noted that the constructed LKF in [19] only involve a small number of delayed states, which leads to a requirement for further improvement of the constructed augmented LKF. Besides, it should be mentioned that augmented LKFs have also been used in the stability analysis of systems with a time-varying delay to provide some less conservative results [6], [20], [21], [23].

In [4], a nonconservative linear matrix inequalities (LMIs) condition was established by using a frequency-domain approach for neutral delay system. It is worthy mention that such an LMI condition can also be derived by an augmented LKF with a state variable $[x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-h_1), \ldots, x^{\mathrm{T}}(t-(k-1)h_1))]^{\mathrm{T}}$, in which k is a positive integer. Such kind of augmented LKFs have been extended to studying the delay-independent stability analysis problem in [3], [26], and [30]. However, to the best of our knowledge, there is little related discussions regarding the delay-dependent stability analysis by using such kind of LKFs.

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Motivated by the above analysis, we aim to find a new LKF with an augmented state variable associated with an integer k, to solve the stability analysis problem of linear neutral systems with two delays. The main feature of the considered system is that there are two different delays in both the left-hand (the time-derivative term) and the right-hand of the system. Such a feature makes the problem challenging in two aspects. On the one hand, for a chosen augmented state variable indexed by an integer k, all possible quadratic integral functions may contain redundant elements that must be eliminated since, otherwise, these redundant elements will greatly increase the computational burden. On the other hand, the constructed LKF will contain absolute value terms, which are not easy to handle. In this paper, by solving these two challenging problems mentioned above, we will construct a proper LKF and then derive some new stability criteria for the considered system. To summarize, the main contributions are listed as follows.

- By choosing a suitable state vector indexed by a positive integer k, a new augmented LKF is constructed. Compared with the augmented LKF in [19], the proposed one utilizes more information of the system states and time delays, which helps to reduce conservatism. Moreover, the constructed LKF contains all possible yet the minimal number of delay-product-type quadratic integral functionals (QIFs), which contributes to realizing the reduction of both the conservatism and computational complexity of the resulting stability criterion.
- By using the proposed LKF, a hierarchical delay-dependent stability criterion is established. Since more delay-product-type integral terms are involved in the constructed LKF, the obtained stability criterion is less conservative than the existing ones [5], [9], [19], and [27].
- The established stability criterion is used to obtain a robust stability criterion for neutral delay systems with norm-bounded uncertainty.
- Considering a special case of the constructed LKF, a delay-independent stability criterion is derived for neutral delay systems with two delays. Also, a frequency-domain interpretation of this criterion is given.

The remainder of this article is arranged as follows. The problem to be solved is formulated in Section II. In Section III, a novel LKF is constructed and stability criteria are proposed. In Section IV, the robust stability analysis problem is discussed. Numerical simulations are posted in Section V to verify the correctness of the given scheme and conclusions are given in Section VI.

Notation: We denote by $\mathbb{C}$ and $\mathbb{R}$, $\mathbb{N}^+$, $\mathbb{S}^n$ sets of complex and real numbers, positive integers, $n \times n$ symmetric real matrices, respectively. Let $\overline{\mathbb{C}}_+$ denote the closed right half plane of the complex plane, and $\mathbb{D}$ denote the closed unit disc on the complex plane, respectively. Let P^{T} and P^{H} represent the transpose and conjugate-transpose of P respectively. For $A \in \mathbb{R}^{n\times n}$, let $\rho(A)$ denote the spectral radius of A. Denote $\mathbf{I}[p_1, p_2] = \{p_1, p_1+1, \ldots, p_2\}$ for two integers p_1 and p_2 with $p_2 \geq p_1$. The notation $\lceil b \rceil$, $b \in \mathbb{R}$ signifies the minimum integer greater than b. Define an operator $\triangledown$ on the set of functions $x(t), t \in \mathbb{R}$ by $(\triangledown_h x)(t) = x(t-h)$ for time delay h. For $u \in \mathbb{C}$, let

$$u^{[k]} \triangleq \begin{bmatrix} 1 & u & \cdots & u^{k-1} \end{bmatrix}^{\mathrm{T}}. \tag{1}$$

II. Problem Formulation

Consider the neutral system formulated as

$$\dot{x}(t) - \sum_{i=1}^{2} B_i \dot{x}(t-h_i) = \sum_{i=0}^{2} A_i x(t-h_i), \ t \geq 0, \tag{2}$$

where $0 = h_0 < h_1 < h_2$ are the constant delays; A_0, A_1, A_2, B_1, B_2 are $n \times n$ constant matrices. Let $x(t) = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ be the unique solution to system (2) under the initial value $x(\theta) = \phi(\theta), \theta \in [-h_2, 0]$. Define $x_t = x(t+\theta)$, $\forall\ \theta \in [-h_2, 0]$, and the operator $\mathscr{D}x_t = x(t) - B_1 x(t-h_1) - B_2 x(t-h_2)$. Throughout this paper, we impose the following assumption.

Assumption 1 ([10], [13]): $\sup_{\theta_1 \in [0,2\pi]} \rho\left(B_1 + B_2 e^{\mathrm{j}\theta_1}\right) < 1$.

We know from Theorem 6.1 (Chapter 9, page 286) in [13] that Assumption 1 guarantees that the operator $\mathscr{D}$ is strongly stable, namely, the delay difference equation

$$x(t) = B_1 x(t-h_1) + B_2 x(t-h_2),$$

is stable independent of delays h_1, h_2.

In this paper, we aim to find a novel quadratic integral LKF to investigate the stability analysis problem of system (2). In order to greatly reduce the conservatism of some existing results, we expect to construct an LKF which can make fully usage of the information of the system states and time delays. To achieve this expectation, different from the existing methods, we choose an augmented state variable indexed by an integer k for the construction of LKFs. With the augmented state variable, we will construct a proper LKF and then derive a novel stability criterion guaranteeing the stability of system (2). Besides, we will provide a stability criterion to solve the robust stability analysis problem of system (2) with norm-bounded uncertainty.

III. Stability Criteria

A. Construction of the LKF

The main objective of this subsection is to construct a new augmented LKF for system (2). To begin with, we need to choose a proper state vector $\xi(t)$. As stated in [19], a possible state vector $\xi(t)$ should consist of two parts, one part is point delays (denoted as $\xi_1(t)$) and the other part is distributed delays (denoted as $\xi_2(t)$). The main idea of choosing $\xi_1(t)$ in the present paper is to replace the usual state variable, say $\{x(t+\theta), -h_2 \leq \theta \leq 0\}$, with the augmented state $\{x(t+\theta), -2kh_2 \leq \theta \leq 0\}$ for some positive integer k. Following this idea, we define an augmented state variable

$X_{k,k}(t)$ as

$$X_{k,k}(t) = \left(\nabla_{h_1}^{[k]} \otimes \nabla_{h_2}^{[k]}\right) x(t) = \begin{bmatrix} x(t) \\ x(t-h_2) \\ \vdots \\ x(t-(k-1)h_2) \\ \hline x(t-h_1) \\ x(t-h_1-h_2) \\ \vdots \\ x(t-h_1-(k-1)h_2) \\ \hline \vdots \\ \hline x(t-(k-1)h_1) \\ x(t-(k-1)h_1-h_2) \\ \vdots \\ x(t-(k-1)h_1-(k-1)h_2) \end{bmatrix}, \tag{3}$$

where we have used (1). From (2) and (3) we obtain the following augmented system

$$\dot{\varphi}_{k,k}(t) = \sum_{i=0}^{2} \left(I_{k^2} \otimes A_i\right) X_{k,k}(t-h_i), \tag{4}$$

where

$$\varphi_{k,k}(t) = X_{k,k}(t) - \sum_{i=1}^{2} \left(I_{k^2} \otimes B_i\right) X_{k,k}(t-h_i).$$

Consider the structure of system (4), we hope that the vectors $X_{k,k}(t), X_{k,k}(t-h_1)$ and $X_{k,k}(t-h_2)$ can be expressed as linear combinations of $\xi_1(t)$. Thus a possible state vector $\xi_1(t)$ should be

$$\xi_1(t) = X_{k+1,k+1}(t). \tag{5}$$

In order to ensure that the derivative of $\xi_2(t)$ can be expressed as a linear function of $\xi_1(t)$, a possible state vector $\xi_2(t)$ should be

$$\xi_2(t) = \begin{bmatrix} \int_{t-h_1}^{t} X_{k,k+1}(s)\mathrm{d}s \\ \int_{t-h_2}^{t} X_{1,k}(s)\mathrm{d}s \end{bmatrix}. \tag{6}$$

By (5) and (6), we conclude that a suitable $\xi(t)$ can be

$$\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} X_{k+1,k+1}(t) \\ \int_{t-h_1}^{t} X_{k,k+1}(s)\mathrm{d}s \\ \int_{t-h_2}^{t} X_{1,k}(s)\mathrm{d}s \end{bmatrix}. \tag{7}$$

Remark 1: Noticed that, for a specified augmented state vector $\xi_1(t)$, there are many vectors with distributed delays whose derivative can be expressed as a linear function of $\xi_1(t)$. It should be pointed out here that the selected $\xi_2(t)$ in (6) has the smallest dimension among those variables whose derivative can be expressed as a linear function of $\xi_1(t)$ and can cover all the state variables $\xi_1(t)$. This can help us reduce the complexity of computation.

Based on the augmented state vector in (7), we will construct a suitable augmented LKF for system (2). Generally, the LKF for time-delay systems is the sum of a quadratic functional $\tilde{V}(\xi(t))$ and some nonnegative QIFs. With the state vector $\xi(t)$ in (7), the quadratic functional $\tilde{V}(\xi(t))$ can be naturally expressed as

$$\tilde{V}(\xi(t)) = \begin{bmatrix} \varphi_{k,k}(t) \\ \xi_2(t) \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} \varphi_{k,k}(t) \\ \xi_2(t) \end{bmatrix}, \tag{8}$$

where $P > 0$. In what follows, we need to determine the nonnegative QIFs. We know that the general form of a nonnegative QIF is

$$\int_{t-h_2}^{t-h_1} x^{\mathrm{T}}(s)Ux(s)\mathrm{d}s, \ \ h_1 < h_2, \tag{9}$$

where $U > 0$. The derivative of (9) can be expressed as a quadratic function of $[x^{\mathrm{T}}(t-h_1), x^{\mathrm{T}}(t-h_2)]^{\mathrm{T}}$. Following this process, to ensure that the derivative of a QIF can be written as a quadratic function of $\xi_1(t)$, the QIF should be in the form of

$$\mathcal{V}_{ij}(x_t) = \left| \int_{t-jh_2}^{t-ih_1} X_{k+1-i,k+1-j}^{\mathrm{T}}(s) U_{ij} X_{k+1-i,k+1-j}(s)\mathrm{d}s \right|, \tag{10}$$

where $i, j \in \mathbf{I}[0, k]$, and U_{ij} are some positive definite matrices with appropriate dimensions. Since $i, j \in \mathbf{I}[0, k]$ in (10), all possible nonnegative QIFs contains $(k+1)^2$ different integral lengths. If the constructed LKF involves all these functions with $(k+1)^2$ different integral lengths, the computation burden is heavy, especially, when k is large. Fortunately, we notice that not all possible $\mathcal{V}_{ij}$ are needed to be included. We take the $\mathcal{V}_{20}(x_t)$ (namely, $i = 2$ and $j = 0$) for example. By a simple computation, we get

$$\begin{aligned} \mathcal{V}_{20}(x_t) &= \int_{t-2h_1}^{t} X_{k-1,k+1}^{\mathrm{T}}(s) U_{20} X_{k-1,k+1}(s)\mathrm{d}s \\ &= \int_{t-h_1}^{t} X_{k,k+1}^{\mathrm{T}}(s) \left(\lambda_1^{\mathrm{T}} U_{20} \lambda_1 + \lambda_2^{\mathrm{T}} U_{20} \lambda_2\right) X_{k,k+1}(s)\mathrm{d}s, \end{aligned}$$

where $U_{20} > 0$, and $\lambda_1 = [I_{k-1} \ 0_{(k-1)\times 1}] \otimes I_{(k+1)n}$, $\lambda_2 = [0_{(k-1)\times 1} \ I_{k-1}] \otimes I_{(k+1)n}$. It is clear that $\lambda_1^{\mathrm{T}} U_{20} \lambda_1$ and $\lambda_2^{\mathrm{T}} U_{20} \lambda_2$ are some positive semi-definite matrices. Thus $\mathcal{V}_{20}(x_t)$ can be absorbed by $\mathcal{V}_{10}(x_t)$ ($i = 1$ and $j = 0$) with an integral length h_1.

In what follows, inspired by the observation above, we will make a significant effort in finding all redundant QIFs and eliminating them. In order to remove the absolute value sign of functions in (10), we discuss this problem by cases.

Case {1}: $i \le j$. In this case, removing the absolute value sign of $\mathcal{V}_{ij}(x_t)$ in (10) gives

$$\mathcal{V}_{ij}(x_t) = \int_{t-jh_2}^{t-ih_1} X_{k+1-i,k+1-j}^{\mathrm{T}}(s) U_{ij} X_{k+1-i,k+1-j}(s)\mathrm{d}s.$$

Then we can calculate

$$\begin{aligned}\mathcal{V}_{ij}(x_t) &= \int_{t-ih_2}^{t-ih_1} X^{\mathrm{T}}_{k+1-i,k+1-j}(s)U_{ij}X_{k+1-i,k+1-j}(s)\mathrm{d}s\\ &\quad+\int_{t-jh_2}^{t-ih_2} X^{\mathrm{T}}_{k+1-i,k+1-j}(s)U_{ij}X_{k+1-i,k+1-j}(s)\mathrm{d}s\\ &=\sum_{m=1}^{i}\int_{t-h_2}^{t-h_1} X^{\mathrm{T}}_{k,k}(s)\lambda^{\mathrm{T}}_{3m}U_{ij}\lambda_{3m}X_{k,k}(s)\mathrm{d}s\\ &\quad+\sum_{m=i+1}^{j}\int_{t-h_2}^{t} X^{\mathrm{T}}_{k+1,k}(s)\lambda^{\mathrm{T}}_{4m}U_{ij}\lambda_{4m}X_{k+1,k}(s)\mathrm{d}s,\end{aligned}$$

where

$$\begin{aligned}\lambda_{3m} &= [\ 0_{(k+1-i)\times(m-1)}\quad I_{k+1-i}\quad 0_{(k+1-i)\times(i-m)}\ ]\\ &\quad\otimes[\ 0_{(k+1-j)\times(i-m)}\quad I_{k+1-j}\quad 0_{(k+1-j)\times(m-1+j-i)}\ ]\otimes I_n,\\ \lambda_{4m} &= [\ I_{k+1-i}\quad 0_{(k+1-i)\times i}\ ]\\ &\quad\otimes[\ 0_{(k+1-j)\times(m-1)}\quad I_{k+1-j}\quad 0_{(k+1-j)\times(j-m)}\ ]\otimes I_n.\end{aligned}$$

Notice that $\lambda^{\mathrm{T}}_{3m}U_{ij}\lambda_{3m}$ and $\lambda^{\mathrm{T}}_{4m}U_{ij}\lambda_{4m}$ are some positive semi-definite matrices. Thus the functionals $\mathcal{V}_{ij}(x_t)$ in Case $\{1\}$ can be absorbed by the following two nonnegative QIFs

$$\mathcal{V}_{01}(x_t) = \int_{t-h_2}^{t} X^{\mathrm{T}}_{k+1,k}(s)U_{01}X_{k+1,k}(s)\mathrm{d}s, \tag{11}$$

$$\mathcal{V}_{11}(x_t) = \int_{t-h_2}^{t-h_1} X^{\mathrm{T}}_{k,k}(s)U_{11}X_{k,k}(s)\mathrm{d}s. \tag{12}$$

Case $\{2\}$: $i > j$.

In this case, in order to remove the absolute value sign of $\mathcal{V}_{ij}(x_t)$, we introduce an integer p. For further use, we define $\check{\mu}_p = \left\lceil \frac{ph_2}{h_1}\right\rceil$ and $\check{\mu}_{p-1} = \left\lceil \frac{(p-1)h_2}{h_1}\right\rceil$ for $p \in \mathbb{N}^+$. Obviously, for a given integer k there must exist a $p \in \mathbb{N}^+$ satisfying

$$k \in \mathbf{I}[\check{\mu}_{p-1}+1, \check{\mu}_p]. \tag{13}$$

From (13), we have $k \geq p$. With the help of (13), we take off the absolute value sign of some integral functionals $\mathcal{V}_{ij}(x_t)$, which will be discussed in three cases.

Case $\{2.1\}$: For $j \in \mathbf{I}[0, p-1]$, $i \in \mathbf{I}[\check{\mu}_j+1, k]$ with $k \geq 1$, $p \geq 1$.

In Case $\{2.1\}$, we have $jh_2 < ih_1$ since $\check{\mu}_j \geq \frac{jh_2}{h_1}$ and $i \in \mathbf{I}[\check{\mu}_j+1, k]$. Taking off the absolute value sign of $\mathcal{V}_{ij}(x_t)$ gives

$$\mathcal{V}_{ij}(x_t) = \int_{t-ih_1}^{t-jh_2} X^{\mathrm{T}}_{k+1-i,k+1-j}(s)U_{ij}X_{k+1-i,k+1-j}(s)\mathrm{d}s.$$

Then we can compute

$$\begin{aligned}\mathcal{V}_{ij}(x_t) &= \int_{t-\check{\mu}_jh_1}^{t-jh_2} X^{\mathrm{T}}_{k+1-i,k+1-j}(s)U_{ij}X_{k+1-i,k+1-j}(s)\mathrm{d}s\\ &\quad+\int_{t-ih_1}^{t-\check{\mu}_jh_1} X^{\mathrm{T}}_{k+1-i,k+1-j}(s)U_{ij}X_{k+1-i,k+1-j}(s)\mathrm{d}s\\ &=\sum_{m=\check{\mu}_j}^{i-1}\int_{t-h_1}^{t} X^{\mathrm{T}}_{k,k+1}(s)\lambda^{\mathrm{T}}_{5m}U_{ij}\lambda_{5m}X_{k,k+1}(s)\mathrm{d}s\\ &\quad+\int_{t-\check{\mu}_jh_1}^{t-jh_2} X^{\mathrm{T}}_{k+1-\check{\mu}_j,k+1-j}(s)\lambda^{\mathrm{T}}_{6}U_{ij}\lambda_{6}\\ &\quad\times X_{k+1-\check{\mu}_j,k+1-j}(s)\mathrm{d}s,\end{aligned} \tag{14}$$

where

$$\begin{aligned}\lambda_{5m} &= [\ 0_{(k+1-i)\times m}\quad I_{k+1-i}\quad 0_{(k+1-i)\times(i-m-1)}\ ]\\ &\quad\otimes[\ I_{k+1-j}\quad 0_{(k+1-j)\times j}\ ]\otimes I_n,\\ \lambda_6 &= [\ I_{k+1-i}\quad 0_{(k+1-i)\times(i-\check{\mu}_j)}\ ]\otimes I_{(k+1-j)n}.\end{aligned}$$

It follows from (14) that $\mathcal{V}_{ij}(x_t)$ in Case $\{2.1\}$ can be absorbed by

$$\begin{aligned}\mathcal{V}_{10}(x_t) &= \int_{t-h_1}^{t} X^{\mathrm{T}}_{k,k+1}(s)U_{10}X_{k,k+1}(s)\mathrm{d}s,\\ \mathcal{V}_{\check{\mu}_jj}(x_t) &= \int_{t-\check{\mu}_jh_1}^{t-jh_2} X^{\mathrm{T}}_{k+1-\check{\mu}_j,k+1-j}(s)U_{\check{\mu}_jj}\\ &\quad\times X_{k+1-\check{\mu}_j,k+1-j}(s)\mathrm{d}s,\ \ j \in \mathbf{I}[0, p-1].\end{aligned}$$

Case $\{2.2\}$: For $j \in \mathbf{I}[2, p]$, $i \in \omega_l$ with $l \in \mathbf{I}[1, j-1]$, $p \geq 2$, $k \geq 4$, where if $j = 2$,

$$\omega_1 = \mathbf{I}\left[j+1, \check{\mu}_1\right],$$

and if $j \geq 3$,

$$\begin{aligned}\omega_1 &= \mathbf{I}[j+1, \check{\mu}_1+j-2],\\ \omega_2 &= \mathbf{I}\left[\check{\mu}_1+j-1, \check{\mu}_2+j-3\right],\\ \omega_3 &= \mathbf{I}\left[\check{\mu}_2+j-2, \check{\mu}_3+j-4\right],\\ &\ \ \vdots\\ \omega_{j-1} &= \mathbf{I}\left[\check{\mu}_{j-2}+2, \check{\mu}_{j-1}\right].\end{aligned}$$

In this case, it is clear that $jh_2 > ih_1$. Then one can get

$$\begin{aligned}\mathcal{V}_{ij}(x_t) &= \int_{t-jh_2}^{t-ih_1} X^{\mathrm{T}}_{k+1-i,k+1-j}(s)U_{ij}X_{k+1-i,k+1-j}(s)\mathrm{d}s\\ &=\int_{t-lh_2}^{t-(i-j+l)h_1}\hat{\mathcal{X}}^{\mathrm{T}}_{ij}U_{ij}\hat{\mathcal{X}}_{ij}\mathrm{d}s+\int_{t-(j-l)h_2}^{t-(j-l)h_1}\check{\mathcal{X}}^{\mathrm{T}}_{ij}U_{ij}\check{\mathcal{X}}_{ij}\mathrm{d}s\\ &=\int_{t-lh_2}^{t-(i-j+l)h_1}\hat{\mathcal{X}}^{\mathrm{T}}_{ij}U_{ij}\hat{\mathcal{X}}_{ij}\mathrm{d}s\\ &\quad+\sum_{m=1}^{j-l}\int_{t-h_2}^{t-h_1} X^{\mathrm{T}}_{k,k}(s)\lambda^{\mathrm{T}}_{7m}U_{ij}\lambda_{7m}X_{k,k}(s)\mathrm{d}s,\end{aligned} \tag{15}$$

in which $\hat{\mathcal{X}}_{ij} = X_{k+1-i,k+1-j}(s-(j-l)h_1)$, $\check{\mathcal{X}}_{ij} = X_{k+1-i,k+1-j}(s-lh_2)$, and

$$\begin{aligned}\lambda_{7m} &= [\ 0_{(k+1-i)\times(m-1)}\quad I_{k+1-i}\quad 0_{(k+1-i)\times(i-m)}\ ]\\ &\quad\otimes[\ 0_{(k+1-j)n\times(j-m)n}\quad I_{(k+1-j)n}\quad 0_{(k+1-j)n\times(m-1)n}\ ].\end{aligned} \tag{16}$$

Let $i' = i-j+l$, $l \in \mathbf{I}[1, j-1]$. Then we have $i' \in \tilde{\omega}_l$, $l \in \mathbf{I}[1, j-1]$, in which $j = 2$,

$$\tilde{\omega}_1 = \mathbf{I}[2, \check{\mu}_1-1], \tag{17}$$

and if $j \geq 3$,

$$\begin{cases}\tilde{\omega}_1 = \mathbf{I}[2, \check{\mu}_1-1],\\ \tilde{\omega}_2 = \mathbf{I}[\check{\mu}_1+1, \check{\mu}_2-1],\\ \tilde{\omega}_3 = \mathbf{I}[\check{\mu}_2+1, \check{\mu}_3-1],\\ \vdots\\ \tilde{\omega}_{j-1} = \mathbf{I}\left[\check{\mu}_{j-2}+1, \check{\mu}_{j-1}-1\right].\end{cases} \tag{18}$$

Therefore (15) can be expressed as

$$\begin{aligned}\mathcal{V}_{ij}(x_t) &= \sum_{m=1}^{j-l}\int_{t-h_2}^{t-h_1} X_{k,k}^{\mathrm{T}}(s)\lambda_{7m}^{\mathrm{T}}U_{ij}\lambda_{7m}X_{k,k}(s)\mathrm{d}s \\ &\quad + \int_{t-lh_2}^{t-i'h_1} X_{k+1-i',k+1-l}^{\mathrm{T}}(s)\lambda_8^{\mathrm{T}}U_{ij}\lambda_8 X_{k+1-i',k+1-l}(s)\mathrm{d}s,\end{aligned} \tag{19}$$

where

$$\begin{aligned}\lambda_8 &= [0_{(k+1-i)\times(i-i')} \quad I_{(k+1-i)}] \\ &\quad \otimes[I_{k+1-j} \quad 0_{(k+1-j)\times(j-l)}]\otimes I_n.\end{aligned}$$

From (19), we know that $\mathcal{V}_{ij}(x_t)$ in Case $\{2.2\}$ can be absorbed by

$$\begin{aligned}\mathcal{V}_{11}(x_t) &= \int_{t-h_2}^{t-h_1} X_{k,k}^{\mathrm{T}}(s)U_{11}X_{k,k}(s)\mathrm{d}s, \\ \mathcal{V}_{i'l}(x_t) &= \int_{t-lh_2}^{t-i'h_1} X_{k+1-i',k+1-l}^{\mathrm{T}}(s)U_{i'l}X_{k+1-i',k+1-l}(s)\mathrm{d}s,\end{aligned} \tag{20}$$

where $i' \in \tilde{\omega}_l$, and $\tilde{\omega}_l$ is defined in (17)/(18).

Case {2.3}: For $j \in \mathbf{I}[p+1, k-1]$, $i \in \mathbf{I}[j+1, k]$ with $k \geq 3$, $1 \leq p \leq k-2$.

Since $k \leq \left\lceil \frac{ph_2}{h_1}\right\rceil < \frac{ph_2}{h_1}+1$, we have $kh_1 < (p+1)h_2$, which yields $jh_2 > ih_1$ in Case $\{2.3\}$. Taking off the absolute value sign of $\mathcal{V}_{ij}(x_t)$ gives

$$\mathcal{V}_{ij}(x_t) = \int_{t-jh_2}^{t-ih_1} X_{k+1-i,k+1-j}^{\mathrm{T}}(s)U_{ij}X_{k+1-i,k+1-j}(s)\mathrm{d}s.$$

Then we can compute

$$\begin{aligned}&\mathcal{V}_{ij}(x_t) \\ &= \nu_{ij} + \int_{t-ph_2-(j-p)h_2}^{t-ph_2-(j-p)h_1} X_{k+1-i,k+1-j}^{\mathrm{T}}(s) \\ &\quad \times U_{ij}X_{k+1-i,k+1-j}(s)\mathrm{d}s \\ &= \nu_{ij} + \sum_{m=1}^{j-p}\int_{t-h_2}^{t-h_1} X_{k,k}^{\mathrm{T}}(s)\lambda_{7m}^{\mathrm{T}}U_{ij}\lambda_{7m}X_{k,k}(s)\mathrm{d}s,\end{aligned}$$

in which λ_{7m} is defined in (16) and

$$\nu_{ij} = \int_{t-ph_2-(j-p)h_1}^{t-ih_1} X_{k+1-i,k+1-j}^{\mathrm{T}}(s)U_{ij}X_{k+1-i,k+1-j}(s)\mathrm{d}s.$$

Notice that $i-j+p \in \mathbf{I}[p+1, k-1]$. When $p=1$, we have $i-j+1 \in \mathbf{I}[2, k-1]$. Obviously, the functional

$$\nu_{ij} = \int_{t-h_2-(j-1)h_1}^{t-ih_1} X_{k+1-i,k+1-j}^{\mathrm{T}}(s)U_{ij}X_{k+1-i,k+1-j}(s)\mathrm{d}s$$

can not be removed. When $p \geq 2$, we divide the region $\mathbf{I}[p+1, k-1]$ into two parts for analysis. For $q_p \triangleq i-j+p \in \mathbf{I}[\check{\mu}_{p-1}+1, k-1]$, the functional

$$\nu_{ij} = \int_{t-ph_2}^{t-q_p h_1} X_{k+1-q_p,k+1-p}^{\mathrm{T}}(s)\lambda_9^{\mathrm{T}}U_{ij}\lambda_9 X_{k+1-q_p,k+1-p}(s)\mathrm{d}s,$$

in which

$$\begin{aligned}\lambda_9 &= [0_{(k+1-i)\times(j-p)} \quad I_{k+1-i}] \\ &\quad \otimes[I_{k+1-j} \quad 0_{(k+1-j)\times(j-p)}]\otimes I_n.\end{aligned}$$

For $i-j+p \in \mathbf{I}[p+1, \check{\mu}_{p-1}]$, it follows from the Case $\{2.2\}$ that the functional ν_{ij} can be eliminated by $\mathcal{V}_{11}(x_t)$ and (20).

Now, by removing the redundant functionals in Cases $\{1\}$ $\{2.1\}$, $\{2.2\}$, $\{2.3\}$, we get all nonnegative QIFs $\mathcal{V}_{ij}(x_t)$ which should be included in the constructed LKF. For clarity, we write them below

$$\mathcal{V}_{10}(x_t) = \int_{t-h_1}^{t} X_{k,k+1}^{\mathrm{T}}(s)U_{10}X_{k,k+1}(s)\mathrm{d}s, \tag{21}$$

$$\mathcal{V}_{01}(x_t) = \int_{t-h_2}^{t} X_{k+1,k}^{\mathrm{T}}(s)U_{01}X_{k+1,k}(s)\mathrm{d}s, \tag{22}$$

$$\mathcal{V}_{il}(x_t) = \left|\int_{t-lh_2}^{t-ih_1} X_{k+1-i,k+1-l}^{\mathrm{T}}(s)U_{il}X_{k+1-i,k+1-l}(s)\mathrm{d}s\right|, \tag{23}$$

where U_{10}, U_{01}, U_{il} are some positive matrices with appropriate dimensions, $l \in \mathbf{I}[1, p]$, $i \in q_l$, and

- if $1 \leq l \leq p-1$,

$$q_l = \mathbf{I}[\check{\mu}_{l-1}+1, \check{\mu}_l]; \tag{24}$$

- if $l = p$,

$$q_p = \mathbf{I}[\check{\mu}_{p-1}+1, k]. \tag{25}$$

On the other hand, in order to ensure that the time-derivative of $\mathcal{V}_{10}, \mathcal{V}_{01}, \mathcal{V}_{il}$ still contains integral functions [19], the sum of nonnegative QIFs can be expressed as

$$\begin{aligned}&\check{V}(x_t) \\ &= \int_{t-h_1}^{t} X_{k,k+1}^{\mathrm{T}}(s)\left(Q_1 + \frac{s-t+h_1}{h_1}W_1\right)X_{k,k+1}(s)\mathrm{d}s \\ &\quad + \int_{t-h_2}^{t} X_{k+1,k}^{\mathrm{T}}(s)\left(Q_2 + \frac{s-t+h_2}{h_2}W_2\right)X_{k+1,k}(s)\mathrm{d}s \\ &\quad + \sum_{l=1}^{p} V_l(x_t),\end{aligned} \tag{26}$$

where $Q_1, Q_2, W_1, W_2 \in \mathbb{S}^{k(1+k)n}$ are four positive definite matrices and

- if $1 \leq l \leq p-1$,

$$V_l(x_t) = \sum_{i=\check{\mu}_{l-1}+1}^{\check{\mu}_l-1} \hat{\vartheta}_{k,k}(i,l) + \check{\vartheta}_{k,k}(\check{\mu}_l, l),$$

- if $l = p$,

$$V_p(x_t) = \begin{cases} \sum_{i=\check{\mu}_{p-1}+1}^{k} \hat{\vartheta}_{k,k}(i,p), & k < \check{\mu}_p, \\ \sum_{i=\check{\mu}_{p-1}+1}^{k-1} \hat{\vartheta}_{k,k}(i,p) + \check{\vartheta}_{k,k}(k,p), & k = \check{\mu}_p,\end{cases}$$

in which

$$\begin{aligned}\hat{\vartheta}_{\varsigma,\varepsilon}(i,l) &= \int_{t-lh_2}^{t-ih_1} X_{\varsigma+1-i,\varepsilon+1-l}^{\mathrm{T}}(s) \\ &\quad \times \frac{s-t+lh_2}{lh_2-ih_1}W_{i+2}X_{\varsigma+1-i,\varepsilon+1-l}(s)\mathrm{d}s,\end{aligned}$$

$$\check{\vartheta}_{\varsigma,\varepsilon}(i,l) = \int_{t-lh_2}^{t-ih_1} X^{\mathrm{T}}_{\varsigma+1-i,\varepsilon+1-l}(s) \times \frac{s-t+ih_1}{lh_2-ih_1} W_{i+2} X_{\varsigma+1-i,\varepsilon+1-l}(s)\mathrm{d}s,$$

with $W_3, W_4, \ldots, W_{k+2}$ being some positive definite matrices.

Finally, it follows from (8) and (26) that the new augmented LKF for system (2) can be written as

$$V(x_t) = \tilde{V}(x_t) + \check{V}(x_t), \tag{27}$$

in which $\tilde{V}(x_t) = \tilde{V}(\xi(t))$ is defined in (8) and $\check{V}(x_t)$ is defined in (26). Noticed that, in $V_l(x_t)$ of (26), we have taken the case of $lh_2 \neq ih_1$, $i \in q_l$, $l \in \mathbf{I}[1, p]$ into consideration.

Remark 2: In $V_l(x_t)$, if $lh_2 = ih_1$ for some $i \in q_l$, $l \in \mathbf{I}[1, p]$, the quadratic integral term $\hat{\vartheta}_{k,k}(i,l)$ or $\check{\vartheta}_{k,k}(i,l)$ in (26) is absent, namely, $W_{i+2} = 0$.

Remark 3: Compared with the existing LKFs in [5], [9], [19], and [27], we see from (26) that the proposed LKF (27) includes more information about the delayed states and time delays since the introduction of the augmented state variable $X_{k,k}(t)$ and the delay-product-type integral terms $V_l(x_t), l \in \mathbf{I}[1, p]$. They are essential for deriving less conservative results.

Remark 4: It can be seen from Section III(A) that the proper LKF (27) is constructed by taking 4 different cases into account. However, for the case of more delays, it is obvious that much more than 4 different cases should be taken into account to construct a suitable LKF, which will result in a great challenge in the stability analysis. Such a problem deserves a further study.

B. Stability Criteria

Based on the LKF (27), we propose a stability criterion for system (2), which exhibits a hierarchical character: the larger the integer k, the less conservatism of the resulting stability criterion, since more decision variables are involved in (27). To keep the representation simple, we define

$$\mathcal{A}_k = \begin{bmatrix} \mathcal{A}_{k11} & 0_{k^2n\times gn} \\ \mathcal{A}_{k21} & 0_{gn} \end{bmatrix}, \mathcal{B}_k = \begin{bmatrix} \mathcal{B}_{k11} & 0_{k^2n\times gn} \\ 0_{gn\times(k+1)^2n} & I_{gn} \end{bmatrix}, \tag{28}$$

in which, for $k \in \mathbb{N}^+$, $g = k(k+2)$ and

$$\begin{aligned} \mathcal{A}_{k11} &= L_k \otimes L_k \otimes A_0 + R_k \otimes L_k \otimes A_1 + L_k \otimes R_k \otimes A_2, \\ \mathcal{B}_{k11} &= L_k \otimes L_k \otimes I_n - R_k \otimes L_k \otimes B_1 - L_k \otimes R_k \otimes B_2, \\ \mathcal{A}_{k21} &= \begin{bmatrix} L_k \otimes I_{(k+1)n} - R_k \otimes I_{(k+1)n} \\ \left[\,1\ 0_{1\times k}\,\right] \otimes L_k \otimes I_n - \left[\,1\ 0_{1\times k}\,\right] \otimes R_k \otimes I_n \end{bmatrix}, \end{aligned}$$

with

$$L_k = \left[\, I_k\ 0_{k\times 1} \,\right], \ R_k = \left[\, 0_{k\times 1}\ I_k \,\right].$$

Define

$$\begin{aligned} \mathcal{C}_{\varsigma,\varepsilon}(i,l) &= [0_{(\varsigma+1-i)\times i} \quad I_{\varsigma+1-i}] \\ &\quad \otimes [I_{(\varepsilon+1-l)} \quad 0_{(\varepsilon+1-l)\times l}] \otimes I_n, \end{aligned} \tag{29}$$

$$\begin{aligned} \mathcal{D}_{\varsigma,\varepsilon}(i,l) &= [I_{\varsigma+1-i} \quad 0_{(\varsigma+1-i)\times i}] \\ &\quad \otimes [0_{(\varepsilon+1-l)\times l}\ \ I_{(\varepsilon+1-l)}\ ] \otimes I_n, \end{aligned} \tag{30}$$

for any $l \in \mathbf{I}[1, \varepsilon]$ and $i \in \mathbf{I}[1, \varsigma]$ with $\varepsilon, \varsigma \in \mathbb{N}^+$. With the help of (29) and (30), we define

$$\begin{aligned} \Psi^*_{\varsigma,\varepsilon} &= \mathcal{D}^{\mathrm{T}}_{\varsigma,\varepsilon}(1,0)\left(Q_1 + h_1^2 W_1\right)\mathcal{D}_{\varsigma,\varepsilon}(1,0) \\ &\quad + \mathcal{C}^{\mathrm{T}}_{\varsigma,\varepsilon}(0,1)(Q_2 + h_2^2 W_2)\mathcal{C}_{\varsigma,\varepsilon}(0,1) \\ &\quad - \mathcal{C}^{\mathrm{T}}_{\varsigma,\varepsilon}(1,0)Q_1\mathcal{C}_{\varsigma,\varepsilon}(1,0) - \mathcal{D}^{\mathrm{T}}_{\varsigma,\varepsilon}(0,1)Q_2\mathcal{D}_{\varsigma,\varepsilon}(0,1), \end{aligned} \tag{31}$$

and, if $1 \le l \le p-1$,

$$\begin{aligned} \Psi_{\varsigma,\varepsilon}(i,l) &= \sum_{i=\check{\mu}_{l-1}+1}^{\check{\mu}_l - 1} \zeta_{il}^2 \mathcal{C}^{\mathrm{T}}_{\varsigma,\varepsilon}(i,l) W_{i+2} \mathcal{C}_{\varsigma,\varepsilon}(i,l) \\ &\quad + \zeta^2_{\check{\mu}_l l}\mathcal{D}^{\mathrm{T}}_{\varsigma,\varepsilon}(\check{\mu}_l, l) W_{\check{\mu}_l+2}\mathcal{D}_{\varsigma,\varepsilon}(\check{\mu}_l, l), \end{aligned} \tag{32}$$

if $l = p$,

$$\Psi_{\varsigma,\varepsilon}(i,p) = \begin{cases} \displaystyle\sum_{i=\check{\mu}_{p-1}+1}^{k} \zeta_{ip}^2 \mathcal{C}^{\mathrm{T}}_{\varsigma,\varepsilon}(i,p) W_{i+2}\mathcal{C}_{\varsigma,\varepsilon}(i,p), & k < \check{\mu}_p, \\ \displaystyle\sum_{i=\check{\mu}_{p-1}+1}^{\check{\mu}_p - 1} \zeta_{ip}^2 \mathcal{C}^{\mathrm{T}}_{\varsigma,\varepsilon}(i,p) W_{i+2}\mathcal{C}_{\varsigma,\varepsilon}(i,p) & k = \check{\mu}_p, \\ \quad + \zeta^2_{kp}\mathcal{D}^{\mathrm{T}}_{\varsigma,\varepsilon}(k,p) W_{k+2}\mathcal{D}_{\varsigma,\varepsilon}(k,p), & \end{cases} \tag{33}$$

in which $\zeta_{il} = lh_2 - ih_1$, $l \in \mathbf{I}[1, p]$, $i \in \mathbf{I}[1, k]$.

For $l \in \mathbf{I}[1, \varepsilon]$ and $i \in \mathbf{I}[1, \varsigma]$, we define

$$\begin{aligned} \varXi_{\varsigma,\varepsilon}(i,l) &= \sum_{m=1}^{i} [\ 0_{(\varsigma+1-i)\times(m-1)} \quad I_{\varsigma+1-i} \quad 0_{(\varsigma+1-i)\times(i-m)}\ ] \\ &\quad \otimes [\ I_{\varepsilon+1-l} \quad 0_{(\varepsilon+1-l)\times l}\ ] \otimes I_n, \\ \pi_{\varsigma,\varepsilon}(i,l) &= \sum_{d=1}^{l} [\ I_{\varsigma+1-i} \quad 0_{(\varsigma+1-i)\times i}\ ] \\ &\quad \otimes [\ 0_{(\varepsilon+1-l)\times(d-1)} \quad I_{\varepsilon+1-l} \quad 0_{(\varepsilon+1-l)\times(l-d)}\ ] \otimes I_n, \end{aligned}$$

and $\alpha_\varepsilon = -L_\varepsilon \otimes I_n + R_\varepsilon \otimes I_n$,

$$\gamma_\varsigma = \begin{bmatrix} 0_{1\times\varsigma} \\ 1\ 0_{1\times(\varsigma-1)} \\ 1_{1\times 2}\ 0_{1\times(\varsigma-2)} \\ \vdots \\ 1_{1\times\varsigma} \end{bmatrix}, \beta_{\varsigma,\varepsilon} = \begin{bmatrix} I_{\varepsilon n} \\ I_{\varepsilon n} \\ \vdots \\ I_{\varepsilon n} \end{bmatrix} \in \mathbb{R}^{(\varsigma+1)\varepsilon n\times\varepsilon n}.$$

Then we define

$$\begin{aligned} \Phi^*_{\varsigma,\varepsilon} &= -\left[\,\gamma_\varsigma \otimes \alpha_\varepsilon\ \ \beta_{\varsigma,\varepsilon}\,\right]^{\mathrm{T}} W_2 \left[\,\gamma_\varsigma \otimes \alpha_\varepsilon\ \ \beta_{\varsigma,\varepsilon}\,\right] \\ &\quad - \left[I_{\varsigma(\varepsilon+1)n}\ \ 0_{\varsigma(\varepsilon+1)n\times\varepsilon n}\right]^{\mathrm{T}} W_1 \left[I_{\varsigma(\varepsilon+1)n}\ \ 0_{\varsigma(\varepsilon+1)n\times\varepsilon n}\right], \end{aligned} \tag{34}$$

and, if $1 \le l \le p-1$,

$$\Phi_{\varsigma,\varepsilon}(i,l) = \sum_{i=\check{\mu}_{l-1}+1}^{\check{\mu}_l} \mathcal{H}^{\mathrm{T}}_{\varsigma,\varepsilon}(i,l) W_{i+2}\mathcal{H}_{\varsigma,\varepsilon}(i,l), \tag{35}$$

and, if $l = p$,

$$\Phi_{\varsigma,\varepsilon}(i,l) = \sum_{i=\check{\mu}_{p-1}+1}^{k} \mathcal{H}^{\mathrm{T}}_{\varsigma,\varepsilon}(i,l) W_{i+2}\mathcal{H}_{\varsigma,\varepsilon}(i,l), \tag{36}$$

where

$$\mathcal{H}_{\varsigma,\varepsilon}(i,l)=\left[\, \pi_{\varsigma,\varepsilon}(i,l)\left(\gamma_{\varsigma}\otimes\alpha_{\varepsilon}\right)-\varXi_{\varsigma,\varepsilon}(i,l),\ \pi_{\varsigma,\varepsilon}(i,l)\beta_{\varsigma,\varepsilon}\,\right]. \tag{37}$$

We now state the following hierarchical stability criterion.

Theorem 1: Let Assumption 1 be satisfied and $lh_2 \neq ih_1$ for $l \in \mathbf{I}[1, p]$ with p satisfying (13) and $i \in q_l$ where q_l is defined in (24) and (25). Then system (2) is asymptotically stable, if there exists a positive definite matrix $P \in \mathbb{S}^{(g+k^2)n}$, four positive definite matrices $Q_1, Q_2, W_1, W_2 \in \mathbb{S}^{k(1+k)n}$, and k positive definite matrices $W_{i+2} \in \mathbb{S}^{(k+1-i)(k+1-l)n}$, $i \in q_l$, $l \in \mathbf{I}[1, p]$, such that

$$\mathscr{R}_k = \mathcal{A}_k^{\mathrm{T}} P\mathcal{B}_k + \mathcal{B}_k^{\mathrm{T}} P\mathcal{A}_k + \Gamma_k < 0, \tag{38}$$

where $\mathcal{A}_k$ and $\mathcal{B}_k$ are defined in (28) and

$$\Gamma_k = \begin{bmatrix} \Psi_{k,k}^* + \sum_{l=1}^{p}\Psi_{k,k}(i,l) & 0_{(k+1)^2n\times gn} \\ 0_{gn\times(k+1)^2n} & \Phi_{k,k}^* - \sum_{l=1}^{p}\Phi_{k,k}(i,l) \end{bmatrix}$$

with $\Psi_{k,k}^*, \Psi_{k,k}(i,l)$ and $\Phi_{k,k}^*, \Phi_{k,k}(i,l)$ being defined by (31)-(33) and (34)-(36), respectively.

Proof: Simple computations give that

$$\begin{aligned}
&\dot{\check{V}}(x_t)\\
&= \begin{bmatrix}\dot{\varphi}_{k,k}(t)\\ \dot{\xi}_2(t)\end{bmatrix}^{\mathrm{T}} P \begin{bmatrix}\varphi_{k,k}(t)\\ \xi_2(t)\end{bmatrix} + \begin{bmatrix}\varphi_{k,k}(t)\\ \xi_2(t)\end{bmatrix}^{\mathrm{T}} P \begin{bmatrix}\dot{\varphi}_{k,k}(t)\\ \dot{\xi}_2(t)\end{bmatrix}\\
&= \xi^{\mathrm{T}}(t)\left(\mathcal{A}_k^{\mathrm{T}} P\mathcal{B}_k + \mathcal{B}_k^{\mathrm{T}} P\mathcal{A}_k\right)\xi(t).
\end{aligned} \tag{39}$$

Notice that

$$\begin{aligned}
\int_{t-h_1}^{t} X_{k,k+1}(s)\mathrm{d}s &= \left[\, I_{k(k+1)n}\ 0_{k(k+1)n\times kn}\,\right]\xi_2(t) \triangleq \mathbf{\Omega}_1\xi_2(t),\\
\int_{t-h_2}^{t} X_{k+1,k}(s)\mathrm{d}s &= \left[\, \gamma_k\otimes\alpha_k\ \ \beta_{k,k}\,\right]\xi_2(t) \triangleq \mathbf{\Omega}_2\xi_2(t).
\end{aligned}$$

Then the time derivative of the first two terms in $\check{V}(x_t)$ can be evaluated as

$$\begin{aligned}
&X_{k+1,k+1}^{\mathrm{T}}(t)\tilde{\Psi}_{k,k}^* X_{k+1,k+1}(t) - \int_{t-h_1}^{t} X_{k,k+1}^{\mathrm{T}}(s)\\
&\quad\times\frac{W_1}{h_1}X_{k,k+1}(s)\mathrm{d}s - \int_{t-h_2}^{t} X_{k+1,k}^{\mathrm{T}}(s)\frac{W_2}{h_2}X_{k+1,k}(s)\mathrm{d}s\\
&\le X_{k+1,k+1}^{\mathrm{T}}(t)\tilde{\Psi}_{k,k}^* X_{k+1,k+1}(t)\\
&\quad-\left(\int_{t-h_1}^{t} X_{k,k+1}^{\mathrm{T}}(s)\mathrm{d}s\right)\frac{W_1}{h_1^2}\int_{t-h_1}^{t} X_{k,k+1}(s)\mathrm{d}s\\
&\quad-\left(\int_{t-h_2}^{t} X_{k+1,k}^{\mathrm{T}}(s)\mathrm{d}s\right)\frac{W_2}{h_2^2}\int_{t-h_2}^{t} X_{k+1,k}(s)\mathrm{d}s\\
&\le X_{k+1,k+1}^{\mathrm{T}}(t)\tilde{\Psi}_{k,k}^* X_{k+1,k+1}(t) + \xi_2^{\mathrm{T}}(t)\tilde{\Phi}_{k,k}^*\xi_2(t),
\end{aligned} \tag{40}$$

where we have used the well known Jensen inequality and

$$\begin{aligned}
\tilde{\Psi}_{k,k}^* &= \mathcal{D}_{k,k}^{\mathrm{T}}(1,0)\left(Q_1+W_1\right)\mathcal{D}_{k,k}(1,0)\\
&\quad+\mathcal{C}_{k,k}^{\mathrm{T}}(0,1)\left(Q_2+W_2\right)\mathcal{C}_{k,k}(0,1)\\
&\quad-\mathcal{C}_{k,k}^{\mathrm{T}}(1,0)Q_1\mathcal{C}_{k,k}(1,0) - \mathcal{D}_{k,k}^{\mathrm{T}}(0,1)Q_2\mathcal{D}_{k,k}(0,1),\\
\tilde{\Phi}_{k,k}^* &= -\frac{1}{h_2^2}\Omega_2^{\mathrm{T}}W_2\Omega_2 - \frac{1}{h_1^2}\Omega_1^{\mathrm{T}}W_1\Omega_1.
\end{aligned}$$

The time derivative of $\sum_{l=1}^{p} V_l(x_t)$ is estimated as

$$\begin{aligned}
\sum_{l=1}^{p}\dot{V}_l(x_t) &= \sum_{l=1}^{p} X_{k+1,k+1}^{\mathrm{T}}(t)\tilde{\Psi}_{k,k}(i,l)X_{k+1,k+1}(t)\\
&\quad-\sum_{l=1}^{p-1}\sum_{i=\check{\mu}_{l-1}+1}^{\check{\mu}_l}\int_{t-lh_2}^{t-ih_1} X_{k+1-i,k+1-l}^{\mathrm{T}}(s)\\
&\quad\times\frac{W_{i+2}}{\zeta_{il}}X_{k+1-i,k+1-l}(s)\mathrm{d}s\\
&\quad-\sum_{i=\check{\mu}_{p-1}+1}^{k}\int_{t-ph_2}^{t-ih_1} X_{k+1-i,k+1-p}^{\mathrm{T}}(s)\\
&\quad\times\frac{W_{i+2}}{\zeta_{ip}}X_{k+1-i,k+1-p}(s)\mathrm{d}s\\
&\le \sum_{l=1}^{p} X_{k+1,k+1}^{\mathrm{T}}(t)\tilde{\Psi}_{k,k}(i,l)X_{k+1,k+1}(t)\\
&\quad-\sum_{l=1}^{p-1}\sum_{i=\check{\mu}_{l-1}+1}^{\check{\mu}_l}\left(\int_{t-lh_2}^{t-ih_1} X_{k+1-i,k+1-l}^{\mathrm{T}}(s)\mathrm{d}s\right)\\
&\quad\times\frac{W_{i+2}}{\zeta_{il}^2}\left(\int_{t-lh_2}^{t-ih_1} X_{k+1-i,k+1-l}(s)\mathrm{d}s\right)\\
&\quad-\sum_{i=\check{\mu}_{p-1}+1}^{k}\left(\int_{t-ph_2}^{t-ih_1} X_{k+1-i,k+1-p}^{\mathrm{T}}(s)\mathrm{d}s\right)\\
&\quad\times\frac{W_{i+2}}{\zeta_{ip}^2}\left(\int_{t-ph_2}^{t-ih_1} X_{k+1-i,k+1-p}(s)\mathrm{d}s\right),
\end{aligned} \tag{41}$$

where $\zeta_{il} = lh_2 - ih_1$, and if $1 \le l \le p-1$,

$$\begin{aligned}
\tilde{\Psi}_{k,k}(i,l) &= \sum_{i=\check{\mu}_{l-1}+1}^{\check{\mu}_l-1}\mathcal{C}_{k,k}^{\mathrm{T}}(i,l)W_{i+2}\mathcal{C}_{k,k}(i,l)\\
&\quad+\mathcal{D}_{k,k}^{\mathrm{T}}(\check{\mu}_l,l)W_{\check{\mu}_l+2}\mathcal{D}_{k,k}(\check{\mu}_l,l),
\end{aligned}$$

if $l = p$,

$$\tilde{\Psi}_{k,k}(i,p) = \begin{cases} \sum\limits_{i=\check{\mu}_{l-1}+1}^{k}\mathcal{C}_{k,k}^{\mathrm{T}}(i,p)W_{i+2}\mathcal{C}_{k,k}(i,p), & k < \check{\mu}_p,\\ \sum\limits_{i=\check{\mu}_{p-1}+1}^{\check{\mu}_p-1}\mathcal{C}_{k,k}^{\mathrm{T}}(i,p)W_{i+2}\mathcal{C}_{k,k}(i,p) & k = \check{\mu}_p.\\ \quad+\mathcal{D}_{k,k}^{\mathrm{T}}(k,p)W_{k+2}\mathcal{D}_{k,k}(k,p), & \end{cases}$$

It can be verified that

$$\begin{aligned}
&\int_{t-lh_2}^{t-ih_1} X_{k+1-i,k+1-l}(s)\mathrm{d}s\\
&= -\int_{t-ih_1}^{t} X_{k+1-i,k+1-l}(s)\mathrm{d}s + \int_{t-lh_2}^{t} X_{k+1-i,k+1-l}(s)\mathrm{d}s\\
&= \left[\,\pi_{k,k}(i,l)\left(\gamma_k\otimes\alpha_k\right) - \varXi_{k,k}(i,l),\ \pi_{k,k}(i,l)\beta_{k,k}\,\right]\xi_2(t)\\
&= \mathcal{H}_{k,k}(i,l)\xi_2(t),
\end{aligned} \tag{42}$$

where $\mathcal{H}_{k,k}(i,l)$ is defined by (37). Then it follows from (41) and (42) that

$$\begin{aligned}
\sum_{l=1}^{p}\dot{V}_l(x_t) &\le X_{k+1,k+1}^{\mathrm{T}}(t)\sum_{l=1}^{p}\tilde{\Psi}_{k,k}(i,l)X_{k+1,k+1}(t)\\
&\quad-\xi_2^{\mathrm{T}}(t)\sum_{l=1}^{p}\tilde{\Phi}_{k,k}(i,l)\xi_2(t),
\end{aligned} \tag{43}$$

in which, if $1 \le l \le p-1$,

$$\tilde{\Phi}_{k,k}(i,l) = \sum_{i=\check{\mu}_{l-1}+1}^{\check{\mu}_l} \frac{1}{\zeta_{il}^2} \mathcal{H}_{k,k}^{\mathrm{T}}(i,l) W_{i+2} \mathcal{H}_{k,k}(i,l),$$

if $l = p$,

$$\tilde{\Phi}_{k,k}(i,p) = \sum_{i=\check{\mu}_{p-1}+1}^{k} \frac{1}{\zeta_{ip}^2} \mathcal{H}_{k,k}^{\mathrm{T}}(i,p) W_{i+2} \mathcal{H}_{k,k}(i,p).$$

Combining (39), (40), and (43), we have

$$\dot{V}(x_t) = \dot{\hat{V}}(x_t) + \dot{\check{V}}(x_t) \le \xi^{\mathrm{T}}(t) \tilde{\Pi} \xi(t), \tag{44}$$

where

$$\begin{aligned} \tilde{\Pi} &= \mathcal{A}_k^{\mathrm{T}} P \mathcal{B}_k + \mathcal{B}_k^{\mathrm{T}} P \mathcal{A}_k \\ &\quad + \begin{bmatrix} \tilde{\Psi}_{k,k}^* + \sum_{l=1}^{p} \tilde{\Psi}_{k,k}(i,l) & 0_{(k+1)^2 n \times gn} \\ 0_{gn \times (k+1)^2 n} & \tilde{\Phi}_{k,k}^* - \sum_{l=1}^{p} \tilde{\Phi}_{k,k}(i,l) \end{bmatrix}, \end{aligned}$$

which is equivalent to (38) by setting $\frac{1}{h_1^2} W_1 \to W_1$, $\frac{1}{h_2^2} W_2 \to W_2$, and $\frac{1}{\zeta_{il}^2} W_{i+2} \to W_{i+2}$. Thus we have from (44) and (38) that $\dot{V}(x_t) < 0$. Note that Assumption 1 guarantees that the operator $\mathscr{D}$ is stable. According to Theorem 8.1 of [13], we conclude that system (2) is asymptotically stable. ■

Notice that the range of i and l in the augmented state variable $X_{k+1-i,k+1-l}(s)$ of $\check{V}(x_t)$ is $i \in \mathbf{I}[0,k]$ and $l \in \mathbf{I}[1,p]$. Since $p \le k$, in order to reduced the computational complexity, the augmented state variable $X_{k+1-i,k+1-l}(s)$ in $\check{V}(x_t)$ can be replaced with $X_{k+1-i,p+1-l}(s)$. Then the augmented LKF for system (2) is

$$\begin{aligned} &\mathcal{W}(x_t) \\ &= \begin{bmatrix} \tilde{\varphi}_{k,p}(t) \\ \int_{t-h_1}^{t} X_{k,p+1}(s)\mathrm{d}s \\ \int_{t-h_2}^{t} X_{1,p}(s)\mathrm{d}s \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} \tilde{\varphi}_{k,p}(t) \\ \int_{t-h_1}^{t} X_{k,p+1}(s)\mathrm{d}s \\ \int_{t-h_2}^{t} X_{1,p}(s)\mathrm{d}s \end{bmatrix} \\ &\quad + \int_{t-h_1}^{t} X_{k,p+1}^{\mathrm{T}}(s) \left(Q_1 + \frac{s-t+h_1}{h_1} W_1 \right) X_{k,p+1}(s)\mathrm{d}s \\ &\quad + \int_{t-h_2}^{t} X_{k+1,p}^{\mathrm{T}}(s) \left(Q_2 + \frac{s-t+h_2}{h_2} W_2 \right) X_{k+1,p}(s)\mathrm{d}s \\ &\quad + \sum_{l=1}^{p} \mathcal{W}_l(x_t), \end{aligned} \tag{45}$$

where $\tilde{\varphi}_{k,p}(t) = X_{k,p}(t) - \sum_{i=1}^{2} \left(I_{kp} \otimes B_i \right) X_{k,p}(t-h_i)$ and

- if $1 \le l \le p-1$,

$$\mathcal{W}_l(x_t) = \sum_{i=\check{\mu}_{l-1}+1}^{\check{\mu}_l - 1} \hat{\vartheta}_{k,p}(i,l) + \check{\vartheta}_{k,p}(\check{\mu}_l, l),$$

- if $l = p$,

$$\mathcal{W}_p(x_t) = \begin{cases} \sum_{i=\check{\mu}_{p-1}+1}^{k} \hat{\vartheta}_{k,p}(i,p), & k < \check{\mu}_p, \\ \sum_{i=\check{\mu}_{p-1}+1}^{k-1} \hat{\vartheta}_{k,p}(i,p) + \check{\vartheta}_{k,p}(k,p), & k = \check{\mu}_p. \end{cases}$$

Based on the LKF $\mathcal{W}(x_t)$ in (45) and following a similar analysis of Theorem 1, we derive a hierarchical stability criterion with less decision variable.

Corollary 1: Let Assumption 1 be satisfied and $lh_2 \neq ih_1$ for $l \in \mathbf{I}[1,p]$ with p satisfying (13) and $i \in q_l$ where q_l is defined in (24) and (25). Then system (2) is asymptotically stable, if there exists a positive definite matrix $P \in \mathbb{S}^{(2kp+k+p)n}$, two positive definite matrices $Q_1, W_1 \in \mathbb{S}^{k(1+p)n}$, two positive definite matrices $Q_2, W_2 \in \mathbb{S}^{p(1+k)n}$, and k positive definite matrices $W_{i+2} \in \mathbb{S}^{(k+1-i)(p+1-l)n}$, $i \in q_l$, $l \in \mathbf{I}[1,p]$, such that

$$\hat{\mathcal{A}}_k^{\mathrm{T}} P \hat{\mathcal{B}}_k + \hat{\mathcal{B}}_k^{\mathrm{T}} P \hat{\mathcal{A}}_k + \hat{\Gamma}_k < 0,$$

where

$$\hat{\mathcal{A}}_k = \begin{bmatrix} \hat{\mathcal{A}}_{k11} & 0_{kpn \times \hat{g}n} \\ \hat{\mathcal{A}}_{k21} & 0_{\hat{g}n} \end{bmatrix}, \ \hat{\mathcal{B}}_k = \begin{bmatrix} \hat{\mathcal{B}}_{k11} & 0_{kpn \times \hat{g}n} \\ 0_{\hat{g}n \times \check{g}n} & I_{\hat{g}n} \end{bmatrix},$$

$$\hat{\Gamma}_k = \begin{bmatrix} \Psi_{k,p}^* + \sum_{l=1}^{p} \Psi_{k,p}(i,l) & 0_{\check{g}n \times \hat{g}n} \\ 0_{\hat{g}n \times \check{g}n} & \Phi_{k,p}^* - \sum_{l=1}^{p} \Phi_{k,p}(i,l) \end{bmatrix},$$

in which $\Psi_{k,p}^*$, $\Psi_{k,p}(i,l)$ and $\Phi_{k,p}^*$, $\Phi_{k,p}(i,l)$ are defined by (31)-(33) and (34)-(36), and $\hat{g} = k(p+1)+p$, $\check{g} = (k+1)(p+1)$,

$$\begin{aligned} \hat{\mathcal{A}}_{k11} &= L_k \otimes L_p \otimes A_0 + R_k \otimes L_p \otimes A_1 + L_k \otimes R_p \otimes A_2, \\ \hat{\mathcal{B}}_{k11} &= L_k \otimes L_p \otimes I_n - R_k \otimes L_p \otimes B_1 - L_k \otimes R_p \otimes B_2, \\ \hat{\mathcal{A}}_{k21} &= \begin{bmatrix} L_k \otimes I_{(p+1)n} - R_k \otimes I_{(p+1)n} \\ \begin{bmatrix} 1 & 0_{1\times k} \end{bmatrix} \otimes L_p \otimes I_n - \begin{bmatrix} 1 & 0_{1\times k} \end{bmatrix} \otimes R_p \otimes I_n \end{bmatrix}. \end{aligned}$$

If we set $W_1 = W_2 = \cdots = W_{k+2} = 0$ in (27), then we have

$$\begin{aligned} V(x_t) &= \varphi_{k,k}^{\mathrm{T}}(t) P \varphi_{k,k}(t) + \int_{t-h_1}^{t} X_{k,k+1}^{\mathrm{T}}(s) Q_1 X_{k,k+1}(s)\mathrm{d}s \\ &\quad + \int_{t-h_2}^{t} X_{k+1,k}^{\mathrm{T}}(s) Q_2 X_{k+1,k}(s)\mathrm{d}s. \end{aligned}$$

By computing $\dot{V}(x_t)$, we get a delay-independent stability condition.

Proposition 1: Let Assumption 1 be satisfied. Then system (2) is asymptotically stable for all delays, if there exists a positive definite matrix $P \in \mathbb{S}^{k^2 n}$, two positive definite matrices $Q_1, Q_2 \in \mathbb{S}^{k(k+1)n}$ such that

$$\tilde{\mathscr{R}}_k = \mathcal{A}_{k11}^{\mathrm{T}} P \mathcal{B}_{k11} + \mathcal{B}_{k11}^{\mathrm{T}} P \mathcal{A}_{k11} + O_k < 0, \tag{46}$$

where

$$O_k = \eta_L^{\mathrm{T}} Q_1 \eta_L - \eta_R^{\mathrm{T}} Q_1 \eta_R + \tilde{\eta}_L^{\mathrm{T}} Q_2 \tilde{\eta}_L - \tilde{\eta}_R^{\mathrm{T}} Q_2 \tilde{\eta}_R,$$

with $\eta_L = L_k \otimes I_{(k+1)n}$, $\eta_R = R_k \otimes I_{(k+1)n}$, $\tilde{\eta}_L = I_{(k+1)} \otimes L_k \otimes I_n$, $\tilde{\eta}_R = I_{(k+1)} \otimes R_k \otimes I_n$.

In fact, Proposition 1 can also be obtained by a frequency-domain method. The details can be found in Appendix.

Remark 5: For the case of $lh_2 = ih_1, i \in q_l$, $l \in \mathbf{I}[1,p]$, according to the discussion in [19] and Remark 2, Theorem 1 is still true when $\mathscr{R}_k < 0$ is replaced by $K^{\mathrm{T}} \mathscr{R}_k K < 0$, where $W_{i+2} = 0$ and K can be obtained by the method in Remark 3 of [19].

Remark 6: It can be seen from (38) in Theorem 1 that for a given k, the computation complexity of (38) depends polynomially on the state dimension n. Obviously, the computation burden grows rapidly with the number of state dimension. So, when applying this approach to large-scale networks, it may induce a huge computational burden and lead to significant memory management problems. Thus, in authors' opinion, this approach is more suitable for system with small state dimension. Besides, it should be mentioned that the stability criterion is less conservative while the number of decision variables also increases. So, in the future, it is a worthwhile work to find a k which gives the best trade-off between maximum allowable delay and number of decision variables.

IV. Robust Stability Criteria

In this section, based on the results in Theorem 1, we will discuss the stability analysis problem of systems described by

$$\dot{x}(t)-\sum_{i=1}^{2}B_i\dot{x}(t-h_i)=\sum_{i=0}^{2}(A_i+\Delta A_i)\,x(t-h_i), \quad (47)$$

where $A_0, B_i, A_i, i=1,2$, are the same as that in (2), and

$$\left[\,\Delta A_0\ \Delta A_1\ \Delta A_2\,\right]=E_0F\left[\,\check{A}_0\ \check{A}_1\ \check{A}_2\,\right], \quad (48)$$

in which $E_0\in\mathbb{R}^{n\times u}, \check{A}_i\in\mathbb{R}^{v\times n}, i=0,1,2$, are constant matrices, and $F\in\mathbb{R}^{u\times v}$ represents the norm bounded uncertainty satisfying

$$F^{\mathrm{T}}F\le I_v. \quad (49)$$

Now, we give a stability criterion guaranteeing system (47) to be asymptotically stable. For convenience presentation, we denote, as shown in the equation at the bottom of the next page, and

$$\Delta\mathcal{A}_{k11}=\begin{bmatrix}\Delta\tilde{\mathcal{A}}_{k11} & 0_{k^2n\times gn}\\ 0_{gn\times(k+1)^2n} & 0_{gn}\end{bmatrix}\in\mathbb{R}^{rn\times cn},$$

in which $g=k(k+2), r=k^2+g, c=(k+1)^2+g$ and

$$\begin{aligned}\Delta\tilde{\mathcal{A}}_{k11}&=L_k\otimes L_k\otimes\Delta A_0+R_k\otimes L_k\otimes\Delta A_1\\&\quad+L_k\otimes R_k\otimes\Delta A_2.\end{aligned}$$

Then a hierarchical stability criterion can be stated as follows.

Theorem 2: Let Assumption 1 be satisfied and $lh_2\neq ih_1$ for $l\in\mathbf{I}[1,p]$ with p satisfying (13) and $i\in q_l$ where q_l is defined in (24) and (25). Then the perturbed neutral delay system (47), with any F satisfying (49), is asymptotically stable, if there exist two positive definite matrices $P, M\in\mathbb{S}^{(g+k^2)n}$, four positive definite matrices $Q_1, Q_2, W_1, W_2\in\mathbb{S}^{k(1+k)n}$, a positive definite matrix $J\in\mathbb{S}^r$, and k positive definite matrices $W_{i+2},\ i\in q_l,\ l\in\mathbf{I}[1,p]$ such that

$$\begin{bmatrix}\mathscr{R}_k+\Upsilon_k & 0_{cn\times rn} & 0_{cn\times ru}\\ 0_{rn\times cn} & -M & P\left(I_r\otimes E_0\right)\\ 0_{ru\times cn} & \left(I_r\otimes E_0^{\mathrm{T}}\right)P & -J\otimes I_u\end{bmatrix}<0, \quad (50)$$

where $\Upsilon_k=T_k^{\mathrm{T}}\left(J\otimes I_v\right)T_k+\mathcal{B}_k^{\mathrm{T}}M\mathcal{B}_k$.

Proof: Let

$$Y_1=\begin{bmatrix}0_{cn\times ru}\\ P\left(I_r\otimes E_0\right)\end{bmatrix},\ Y_2=\begin{bmatrix}T_k^{\mathrm{T}}\\ 0_{rn\times rv}\end{bmatrix}.$$

By Schur complements, we have from (50) that

$$\begin{aligned}0>&\begin{bmatrix}\mathscr{R}_k+\mathcal{B}_k^{\mathrm{T}}M\mathcal{B}_k & 0_{cn\times rn}\\ 0_{rn\times cn} & -M\end{bmatrix}\\&+Y_1\left(J^{-1}\otimes I_u\right)Y_1^{\mathrm{T}}+Y_2\left(J\otimes I_v\right)Y_2^{\mathrm{T}}, \quad (51)\end{aligned}$$

where $\mathscr{R}_k$ is defined in (38). By applying the inequality

$$XY+Y^{\mathrm{T}}X^{\mathrm{T}}\le XZX^{\mathrm{T}}+Y^{\mathrm{T}}Z^{-1}Y, \quad (52)$$

with X, Y being real matrices and $Z>0$, it follows from (49) and (51) that

$$\begin{aligned}&\begin{bmatrix}\mathscr{R}_k+\mathcal{B}_k^{\mathrm{T}}M\mathcal{B}_k & \Delta\mathcal{A}_{k11}^{\mathrm{T}}P\\ P\Delta\mathcal{A}_{k11} & -M\end{bmatrix}\\&=\Theta+\begin{bmatrix}0_{cn} & \Delta\mathcal{A}_{k11}^{\mathrm{T}}P\\ P\Delta\mathcal{A}_{k11} & 0_{rn}\end{bmatrix}\\&=\Theta+Y_1\left(I_r\otimes F\right)Y_2^{\mathrm{T}}+Y_2\left(I_r\otimes F^{\mathrm{T}}\right)Y_1^{\mathrm{T}}\\&\le\Theta+Y_1\left(J^{-1}\otimes I_u\right)Y_1^{\mathrm{T}}\\&\quad+Y_2\left(I_r\otimes F^{\mathrm{T}}\right)\left(J\otimes I_u\right)\left(I_r\otimes F\right)Y_2^{\mathrm{T}}\\&\le\Theta+Y_1\left(J^{-1}\otimes I_u\right)Y_1^{\mathrm{T}}+Y_2\left(J\otimes I_v\right)Y_2^{\mathrm{T}}\\&<0,\end{aligned}$$

where

$$\Theta=\begin{bmatrix}\mathscr{R}_k+\mathcal{B}_k^{\mathrm{T}}M\mathcal{B}_k & 0_{cn\times rn}\\ 0_{rn\times cn} & -M\end{bmatrix}.$$

By a Schur complement again, we have

$$\Delta\mathcal{A}_{k11}^{\mathrm{T}}PM^{-1}P\Delta\mathcal{A}_{k11}+\mathcal{B}_k^{\mathrm{T}}M\mathcal{B}_k+\mathscr{R}_k<0.$$

By using the inequality (52) again, it follows from the above inequality that

$$\begin{aligned}&\Delta\mathcal{A}_{k11}^{\mathrm{T}}P\mathcal{B}_k+\mathcal{B}_k^{\mathrm{T}}P\Delta\mathcal{A}_{k11}+\mathscr{R}_k\\&\quad\le\Delta\mathcal{A}_{k11}^{\mathrm{T}}PM^{-1}P\Delta\mathcal{A}_{k11}+\mathcal{B}_k^{\mathrm{T}}M\mathcal{B}_k+\mathscr{R}_k\\&\quad<0. \quad (53)\end{aligned}$$

By using Theorem 1, we have from (53) that system (47) is asymptotically stable. ■

V. Examples

A. Example 1

Consider system (2) with matrices (borrowed from [19])

$$A_0=0.2\begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix}, A_1=0.3\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},$$
$$A_2=\begin{bmatrix}0.1 & -0.05\\ 0.05 & 0.1\end{bmatrix}, B_1=-0.1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, B_2=0.25I_2.$$

Let $h_1=0.3h_2$, then we have $p=1$ for $k=1,2,3$. We use different methods to obtain the maximum value of h_2 (labeled as h_2^*) such that system (2) is asymptotically stable. From Table I, it is observed that for $k=1$, both Theorem 1 and Corollary 1 provide much less conservative results than the stability criteria in [5], [9], and [27]; for $k=2$, the obtained h_2^* coincides with the results in [19], and for $k=3$, Theorem 1 and Corollary 1 are much less conservative than those results in [5], [9], [19], and [27]. It should be mentioned that the maximum allowed value of h_2 by simulation is 20.3. The obtained h_2^* by Theorem 1 with $k=3$ is close to the simulation result.

The number of decision variables of different methods, which is associated with the computational complexity, is recorded in Table I. It is clear that although h_2^* obtained by

TABLE I
h_2^* BY DIFFERENT METHODS

Methods	h_2^*	Number of Decision Variables
Theorem 1 [9]	0.8279	$4.5n^2 + 4.5n$
Theorem 1 [27]	0.8279	$6.5n^2 + 4.5n$
Theorem 1 [5]	1.0571	$10n^2 + 6n$
Theorem 1 [19]	8.0665	$37n^2 + 10n$
Theorem 1	6.5345 $(k = 1)$	$16.5n^2 + 6.5n$
	8.0665 $(k = 2)$	$154n^2 + 21n$
	18.0524 $(k = 3)$	$639n^2 + 45n$
Corollary 1	6.5345 $(k = 1)$	$16.5n^2 + 6.5n$
	8.0665 $(k = 2)$	$52n^2 + 12n$
	18.0524 $(k = 3)$	$109n^2 + 18n$

Theorem 1 and Corollary 1 is identical, less decision variables is needed in Corollary 1, which implies that Corollary 1 is more effective in this example. Besides, Theorem 1 and Corollary 1 with $k = 3$ is much less conservative than those results in [5], [9], [19], and [27], while more decision variables are required. Clearly, they achieve the goal of reducing conservatism at the cost of increasing the computational complexity. To verify the derived result, for the initial condition $x_0 = [3, -10]^{\mathrm{T}}$, the state responses of system (2) with different h_2 and $h_1 = 0.3h_2$ are presented in Figure 1. From Figure 1, we can see that the larger the time delay, the slower the convergence rate of the system.

B. Example 2

Assume that the system in Example 1 is subject to uncertainties, namely

$$\dot{x}(t) - \sum_{i=1}^{2} B_i \dot{x}(t - h_i) = \sum_{i=0}^{2} (A_i + \Delta A_i)\, x(t - h_i), \quad (54)$$

in which A_0, A_1, A_2, B_1, B_2 are given in Example 1, and

$$\Delta A_0 = \begin{bmatrix} a_0 & 0 \\ 0 & 0 \end{bmatrix},\ \Delta A_1 = \begin{bmatrix} 0 & 0 \\ a_1 & 0 \end{bmatrix},\ \Delta A_2 = \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix},$$

in which $a_0 \in [-a, a], a_1 \in [-a, a], a_2 \in [-a, a]$ with a_0, a_1, a_2 being uncertainties. We choose

$$E_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \check{A}_0 = \begin{bmatrix} a & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\check{A}_1 = \begin{bmatrix} 0 & 0 \\ a & 0 \\ 0 & 0 \end{bmatrix},\ \check{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & a \end{bmatrix},$$

which satisfy (48) with $F = \frac{1}{a}\operatorname{diag}\{a_0, a_1, a_2\}$. Let $h_2 = 5$, $h_1 = 0.3h_2 = 1.5$, then we use Theorem 2 for different k to determine the maximal value of a (denoted as $a^*(k)$) which is such that (54) is asymptotically stable. By a linear search technique, it can be found that $a^*(2) = 0.0267, a^*(3) = 0.0745$. Thus, the computed maximum allowable value of a is 0.0745. A simulation result is performed under the initial condition $x_0 = [3, -10]^{\mathrm{T}}$ and $a_0 = 0.0267, a_1 = -0.02, a_2 = -0.024$. The state responses of system (54) are shown in Figure 2, which demonstrates the correctness of our results.

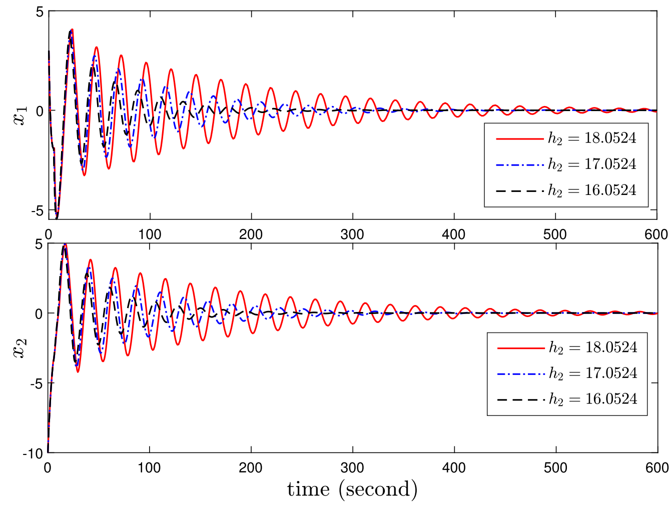

Fig. 1. State responses of system (2) with different h_2 and $h_1 = 0.3h_2$.

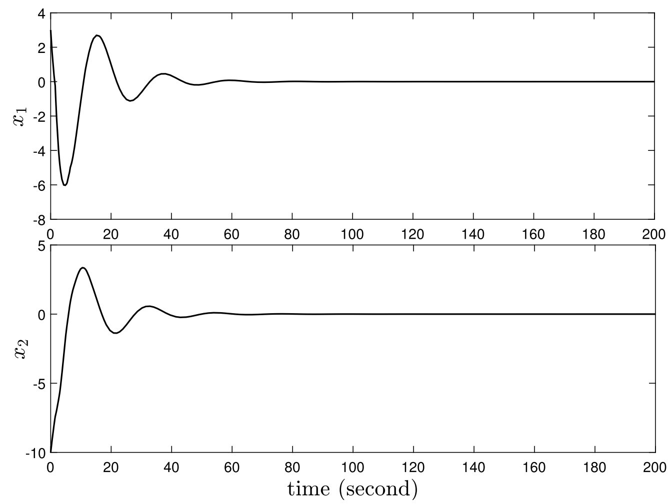

Fig. 2. State responses of system (54) with $h_2 = 5$, $h_1 = 1.5$, $a_0 = 0.0267$, $a_1 = -0.02$, and $a_2 = -0.024$.

VI. CONCLUSION

The stability analysis issues of linear neutral systems with two delays have been investigated. By choosing a suitable

$$T_k = \begin{bmatrix} L_k \otimes L_k \otimes \check{A}_0 + R_k \otimes L_k \otimes \check{A}_1 + L_k \otimes R_k \otimes \check{A}_2 & 0_{k^2 v \times gn} \\ 0_{gv \times (k+1)^2 n} & 0_{gv \times gn} \end{bmatrix}$$

augmented state vector, new augmented LKFs with some delay-product-type terms have been constructed and two hierarchical stability criteria based on LMIs have been derived accordingly. Applying the resulting stability criterion to the neutral delay systems with uncertainties, a hierarchical robust stability criterion has been proposed. Examples have been provided to show that the proposed approach is very effective in reducing conservatism.

Appendix: The Proof of Proposition 1

The equality (46) can be written as

$$\begin{bmatrix} \mathcal{B}_{k11} \\ R_k \otimes L_k \otimes I_n \\ L_k \otimes R_k \otimes I_n \end{bmatrix}^{\mathrm{T}} \Lambda_k \begin{bmatrix} \mathcal{B}_{k11} \\ R_k \otimes L_k \otimes I_n \\ L_k \otimes R_k \otimes I_n \end{bmatrix} + O_k < 0,$$

where

$$\Lambda_k = \begin{bmatrix} I_{k^2 n} \\ 0_{k^2 n} \\ 0_{k^2 n} \end{bmatrix} P \begin{bmatrix} I_{k^2} \otimes A_0^{\mathrm{T}} \\ I_{k^2} \otimes (A_0 B_1 + A_1)^{\mathrm{T}} \\ I_{k^2} \otimes (A_0 B_2 + A_2)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} I_{k^2} \otimes A_0^{\mathrm{T}} \\ I_{k^2} \otimes (A_0 B_1 + A_1)^{\mathrm{T}} \\ I_{k^2} \otimes (A_0 B_2 + A_2)^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} I_{k^2 n} \\ 0_{k^2 n} \\ 0_{k^2 n} \end{bmatrix}^{\mathrm{T}}.$$

Let $\tilde{z}_k = z_1^{[k]} \otimes z_2^{[k]} \otimes I_n$ for any $z_1, z_2 \in \mathbb{D}$. Then we have

$$\begin{bmatrix} \mathcal{B}_{k11} \\ R_k \otimes L_k \otimes I_n \\ L_k \otimes R_k \otimes I_n \end{bmatrix} \tilde{z}_{k+1} = \begin{bmatrix} I_{k^2} \otimes B(z) \\ z_1 I_{k^2 n} \\ z_2 I_{k^2 n} \end{bmatrix} \tilde{z}_k, \tag{55}$$

where $B(z) = I_n - z_1 B_1 - z_2 B_2$. Let

$$\tilde{B}(z) = \begin{bmatrix} I_{k^2} \otimes B(z) \\ z_1 I_{k^2 n} \\ z_2 I_{k^2 n} \end{bmatrix}.$$

By multiplying both side of $\tilde{\mathscr{R}}_k$ in (46) on the right by $\tilde{z}_{k+1} = z_1^{[k+1]} \otimes z_2^{[k+1]} \otimes I_n$ and on the left side by $\tilde{z}_{k+1}^{\mathrm{H}}$, it follows from (55) that

$$\begin{aligned} \tilde{z}_{k+1}^{\mathrm{H}} \tilde{\mathscr{R}}_k \tilde{z}_{k+1} &= \tilde{z}_k^{\mathrm{H}} \tilde{B}^{\mathrm{H}}(z) \Lambda_k \tilde{B}(z) \tilde{z}_k + (1 - |z_1|^2) \varrho_1^{\mathrm{H}} Q_1 \varrho_1 \\ &\quad + (1 - |z_2|^2) \varrho_2^{\mathrm{H}} Q_2 \varrho_2 \\ &< 0, \end{aligned}$$

where $\varrho_1 = z_1^{[k]} \otimes z_2^{[k+1]} \otimes I_n, \varrho_2 = z_1^{[k+1]} \otimes z_2^{[k]} \otimes I_n$. Since $1 - |z_i|^2 \geq 0, i = 1, 2$, it yields

$$\begin{aligned} &\tilde{z}_k^{\mathrm{H}} \tilde{B}^{\mathrm{H}}(z) \Lambda_k \tilde{B}(z) \tilde{z}_k \\ &= B^{\mathrm{H}}(z) \tilde{z}_k^{\mathrm{H}} \left(\left(I_{k^2} \otimes A(z) \right)^{\mathrm{H}} P + P \left(I_{k^2} \otimes A(z) \right) \right) \tilde{z}_k B(z) \\ &= B^{\mathrm{H}}(z) \left(A^{\mathrm{H}}(z) \tilde{z}_k^{\mathrm{H}} P \tilde{z}_k + \tilde{z}_k^{\mathrm{H}} P \tilde{z}_k A(z) \right) B(z) \\ &< 0, \end{aligned} \tag{56}$$

where

$$\begin{aligned} A(z) = A_0 &+ z_1 (A_1 + A_0 B_1) B^{-1}(z) \\ &+ z_2 (A_2 + A_0 B_2) B^{-1}(z). \end{aligned}$$

According to Assumption 1, it follows from (56) that

$$A^{\mathrm{H}}(z) \tilde{z}_k^{\mathrm{H}} P \tilde{z}_k + \tilde{z}_k^{\mathrm{H}} P \tilde{z}_k A(z) < 0.$$

which further implies that

$$\alpha (A(z)) < 0, \tag{57}$$

since $\tilde{z}_k^{\mathrm{H}} P \tilde{z}_k > 0$. Here, $\alpha(A(z))$ denotes the spectral abscissa of matrix $A(z)$.

In view of Assumption 1, it follows from (57) that

$$\begin{aligned} &\det \left(s (I_n - z_1 B_1 - z_2 B_2) - \sum_{i=0}^{2} A_i z_i \right) \\ &= \det \left((s I_n - A_0) B(z) - \sum_{i=1}^{2} z_i (A_0 B_i + A_i) \right) \\ &= \det \left(s I_n - A_0 - \sum_{i=1}^{2} z_i (A_0 B_i + A_i) B^{-1}(z) \right) \det(B(z)) \\ &\neq 0, \forall (s, z_i) \in (\overline{\mathbb{C}}_+, \mathbb{D}), \end{aligned}$$

which shows that system (2) is asymptotically stable independent of h_1, h_2.

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