Uniform Tube Based Stabilization of Switched Linear Systems With Mode-Dependent Persistent Dwell-Time

Lixian Zhang, Songlin Zhuang, Peng Shi, and Yanzheng Zhu

***Abstract*—In this note, the stabilization problem for a class of discrete-time switched linear systems with additive disturbances is investigated. The considered switching signals are of mode-dependent persistent dwell-time (MPDT) property and the disturbances are assumed to be amplitude-bounded. By constructing a quasi-time-varying (QTV) Lyapunov function, a QTV stabilizing controller is designed for the nominal system such that the resulting closed-loop system is globally uniformly asymptotically stable. In the presence of bounded additive disturbances, a MPDT robust positive invariant set is determined for the error system between the nominal system and disturbed system. A concept of generalized robust positive invariant (GRPI) set under admissible MPDT switching is further proposed for the error system. It is demonstrated that the disturbed system is also asymptotically stable in the sense of converging to the MPDT GRPI set that can be regarded as the cross section of a uniform tube of the disturbed system. A numerical example is provided to verify the theoretical findings.**

***Index Terms*—Generalized robust positive invariant set, mode-dependent persistent dwell-time, stabilization, switched systems.**

I. Introduction

The past decades have witnessed a rapid advance on theories and applications of switched systems. The systems can efficiently model multiple-mode systems or processes, and hybrid control systems in closed-loop via different controllers. Basic stability analysis and other issues for switched systems with various switching signals have been broadly addressed, either autonomous switching or controlled switching [10]. It has been well demonstrated in the literature that fast switching may lead to poor system performance or even instability. An important subject has been therefore developed, i.e., to determine the minimum *dwell-time* or minimum *average dwell-time* (ADT), joint with the switched controller design in certain scenarios, such that the resulting switched system is asymptotically stable. So far, the two classes of switching signals have been considerably intensively studied, and many useful results on control of the corresponding switched systems have been obtained, see e.g., [3], [5], [9], [12], [14]. Particularly, it is worth mentioning that a class of quasi-time-varying (QTV) Lyapunov function approach recently developed in [1], [15] is shown to be less conservative in obtaining shorter dwell-time.

As pointed out in [8], another important switching signal, the persistent dwell-time (PDT) switching is more general than the dwell-time switching and the ADT switching. This type of switching signal means that there exists an infinite number of disjoint intervals of length no smaller than a dwell-time on which the system remains unswitched, and the consecutive intervals with this property are separated by no more than a period of persistence. Therefore, the PDT switching has the capability of describing a switched system with hybrid slow and fast switching feature, such as the systems that may encounter abrupt and intermittent faults [13]. Certain conditions ensuring the uniform asymptotic stability of switched linear systems with PDT switching property have been primarily explored in [8], and the results have also been further extended to nonlinear cases in [6]. These works provide a solid foundation for investigations on switched systems with PDT switching property, however, the developed stability criteria are not yet parameterized with both the dwell-time and the period of persistence. In addition, how to carry out the analysis and synthesis of switched systems with PDT switching by using the QTV Lyapunov function has not been addressed up to date, either in linear or nonlinear context.

Moreover, control of switched systems with disturbances is also of great importance as various disturbances are unavoidable in practice. As for the energy-bounded disturbances (the l_2 norm of the disturbances is finite), the investigations are not rare, especially for the switched linear systems with dwell-time or ADT switching, see e.g., [7], [16]. On the contrary, for switched systems with amplitude-bounded disturbances (the l_∞ norm of the disturbances is finite), the results are relatively seldom reported. Recently, a study on switched linear systems with l_∞ additive disturbances under dwell-time switching has been carried out in [2]. A so-called dwell-time robust positive invariant (RPI) set is determined such that the state trajectory of the switched system can always remain within the dwell-time RPI set at switching instants, and within an outer set related to this RPI set during the subsystems. Allowing for the more general PDT switching, however, the determination of the corresponding PDT RPI set for the corresponding switched systems will be more challenging and still largely open nowadays.

Motivated by the above considerations, this note investigates the stabilization problem for a class of discrete-time switched linear systems with bounded additive disturbances. The disturbances are considered to be l_∞ finite and the switching signals are considered to belong to the set of mode-dependent persistent dwell-time (MPDT) switching. The existence conditions of a stabilizing controller for nominal switched linear system are first obtained such that the resulting closed-loop system is asymptotically stable under MPDT switching for a given period of persistence. Then, allowing for the l_∞ additive disturbances input, a concept of generalized robust positive invariant (GRPI) set under MPDT switching is proposed such that the stability analysis of the underlying disturbed closed-loop systems can be carried out in the sense of set stability. An algorithm capable of determining MPDT RPI set for

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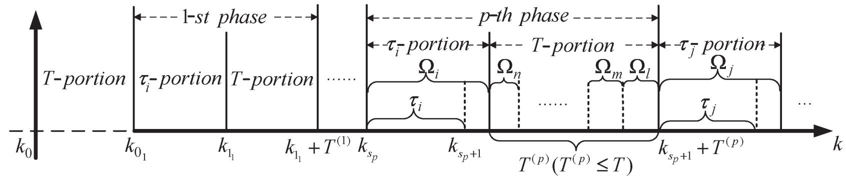

Fig. 1. Illustration of MPDT, where the period of persistence is T, $i \neq n$, $m \neq l$, $j \neq l$ and $T^{(p)} \leq T$.

the switched error systems between the nominal and disturbed systems is developed, upon which the MPDT GRPI set is obtained as the cross section of a uniform tube to demonstrate the system uniform stability.

Notations: In this note, $\mathbb{R}^n$ denotes the n -dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; the $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the set of non-negative real numbers and set of non-negative integers, respectively; and $\mathbb{Z}_{\geq s_1}$ and $\mathbb{Z}_{[s_1,s_2]}$ denote the sets $\{k \in \mathbb{Z}_+ | k \geq s_1\}$ and $\{k \in \mathbb{Z}_+ | s_1 \leq k \leq s_2\}$, respectively, for some $s_1, s_2 \in \mathbb{Z}_+$. The set of $n \times n$ (positive definite) symmetric matrices is denoted by $\mathbb{S}^n_{>0}$. A function $\kappa : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, unbounded, and $\kappa(0) = 0$. The distance of a vector x to set $\mathcal{S}$, $x \in \mathbb{R}^n$, $\mathcal{S} \subset \mathbb{R}^n$, be denoted by $\|x\|_{\mathcal{S}} := \inf_{y \in \mathcal{S}} \|x - y\|$. The Pontryagin difference, Minkowski sum of two arbitrary sets $\mathcal{S}_1 \subseteq \mathbb{R}^n$, $\mathcal{S}_2 \subseteq \mathbb{R}^n$, are denoted as $\mathcal{S}_1 \ominus \mathcal{S}_2 := \{s \in \mathbb{R}^n | s + s_2 \in \mathcal{S}_1,\ \forall s_2 \in \mathcal{S}_2\}$ and $\mathcal{S}_1 \oplus \mathcal{S}_2 := \{s_1 + s_2 \in \mathbb{R}^n | s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2\}$, respectively; $co\{\mathcal{S}\}$ denotes the convex-hull of $\mathcal{S}$. Let $\mathcal{B}^n := \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$.

II. Preliminaries and Problem Formulation

Consider a class of discrete-time switched linear systems with bounded additive disturbances

$$\left(\Omega_{\sigma(k)}\right) : \ x_{k+1} = A_{\sigma(k)} x_k + B_{\sigma(k)} u_k + w_k \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}, u_k \in \mathbb{R}^{n_u}, k \in \mathbb{Z}_+$, are the system state and control input, respectively; $w_k \in \mathbb{W} \subseteq \mathbb{R}^{n_w}$ is the additive disturbance and $\mathbb{W}$ is a compact polyhedral set containing the origin in its interior. The switching signal $\sigma(k)$ is a piecewise constant function of time k, taking values at the sampling times in a finite set $\mathcal{I} = \{1, \ldots, M\}$, where $M > 1$ is the number of subsystems. The switching sequences $k_0, k_1, k_2, \ldots, k_s, \ldots$ are unknown *a priori*, but are known instantly, in which the switching instant is denoted as k_s, $s \in \mathbb{Z}_+$. When $k \in [k_s, k_{s+1})$, the $\sigma(k_s)$-th subsystem (or system mode) is said to be *activated* and the length of the current running time of the subsystem is $k_{s+1} - k_s$.

The switching signals are considered to have *mode-dependent persistent dwell-time* (MPDT) property in this note. We first give the concept of mode-dependent dwell-time as below.

Definition 1: Consider switching instants $k_0, k_1, \ldots, k_s, \ldots$ with $k_0 = 0$. A positive constant τ_i is said to be mode-dependent dwell-time (MDT) associated with subsystem Ω_i, if $k_{s+1} - k_s \geq \tau_i$ when $\sigma(k) = i$ for $k \in [k_s, k_{s+1})$, $s \in \mathbb{Z}_+$.

Then, combining Definition 1 and the definition on PDT given in [8], the concept of MPDT is given as follows.

Definition 2: Consider switching instants $k_0, k_1, \ldots, k_s, \ldots$ with $k_0 = 0$. A positive constant τ_i is said to be the mode-dependent persistent dwell-time (MPDT) if there exists an infinite number of disjoint intervals of length no smaller than τ_i on which σ is constant at subsystem Ω_i, and consecutive intervals with this property are separated by no more than T, where T is called the period of persistence.

A switching sequence satisfying the MPDT property defined in Definition 2 is called MPDT switching sequence. Further, let the set of MPDT τ_i's be denoted by $\boldsymbol{\tau}^{[T]} := \{\tau_1, \tau_2, \ldots, \tau_M\}$, and the set of all admissible MPDT switching sequences with $\boldsymbol{\tau}^{[T]}$ till time k be denoted by $\xi_{\boldsymbol{\tau}^{[T]}}(k)$.

The example below is used to show what an admissible MPDT switching sequence means (we use $\bar{\xi}_{\boldsymbol{\tau}^{[T]}}(k)$ denote inadmissible switching sequences). Consider a switched system consisting of three subsystems. The admissible MPDT set, with the period of persistence $T = 3$, is supposed to be $\boldsymbol{\tau}^{[3]} = \{4, 3, 5\}$. Then $\xi^a_{\boldsymbol{\tau}^{[3]}}(11) = \{1, 1, 1, 1, 2, 1, 3, 2, 2, 2, 2\}$ is an admissible sequence, but both $\bar{\xi}^a_{\boldsymbol{\tau}^{[3]}}(11) = \{1, 1, 1, 3, 2, 1, 3, 3, 3, 3, 3\}$ and $\bar{\xi}^b_{\boldsymbol{\tau}^{[3]}}(11) = \{1, 1, 1, 1, 2, 1, 3, 1, 2, 2, 2\}$ are not since the requirements of $\tau_1 \geq 4$ and $T \leq 3$ are not satisfied in the former and latter cases, respectively.

An illustration on MPDT is given in Fig. 1, where the interval consisting of the running time (τ_i-portion) of a certain subsystem and the period of persistence (T-portion) is considered as a *MPDT phase*,[1] and k_{s_p} is denoted as the initial instant of the p-th phase, $p \in \mathbb{Z}_{\geq 1}$ with $k_{s_1} \geq k_0$ (here "$\geq$" means a period of persistence may exist before the 1-st phase). Let the actual running time of the T-portion at the p-th phase be denoted as $T^{(p)}$, $p \in \mathbb{Z}_{\geq 1}$, it holds that $T^{(p)} := \sum_{r=1}^{\mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})} T_{\sigma(k_{s_p+r})} \leq T$ where $T_{\sigma(k_{s_p+r})} < \tau_i$ denotes the running time of the subsystem activated at the switching instant $k_{s_p+r} \in [k_{s_p+1}, k_{s_{p+1}}), r \in \mathbb{Z}_{\geq 1}$, and $\mathcal{Q}(k_{s_p}, k_{s_{p+1}})$ stands for the switching times within $[k_{s_p+1}, k_{s_{p+1}})$.

To handle the l_∞ additive disturbances, in this note, we shall consider the tube-based control methodology (cf. [11]), in which the key idea is to steer the nominal state trajectory converge to the origin, and the error between states of the nominal and disturbed systems within a robust positive invariant (RPI) set. Due to the switching dynamics, however, the conventional RPI set obtained for non-switched systems will be no longer competent in stability analysis of the underlying system. A concept of generalized robust positive invariant (GRPI) set for the switched error system with MPDT switching will be needed to apply for the methodology. For notational simplicity, let us rewrite the nominal system as

$$z_{k+1} = A_{\sigma(k)} z_k + B_{\sigma(k)} v_k \tag{2}$$

where $v_k = \mathcal{F}_{\sigma(k)}(z_k)$ is a certain control policy with $\mathcal{F}_{\sigma(k)}(\cdot)$ to be designed. Consider the classical structure of tube-based control law

$$u_k = v_k + K_{\sigma(k)}(x_k - z_k) \tag{3}$$

which contains a stabilizing state-feedback gain K_i, $\forall \sigma(k) = i \in \mathcal{I}$ to be also determined, and let $e_k := x_k - z_k$, the switched error system between (1) and (2) becomes

$$\left(\Xi_{\sigma(k)}\right) : e_{k+1} = \bar{A}_{\sigma(k)} e_k + w_k \tag{4}$$

where $\bar{A}_{\sigma(k)} := A_{\sigma(k)} + B_{\sigma(k)} K_{\sigma(k)}$ and $e_0 = 0$ is supposed.

Definition 3: [11] A set $\mathcal{O} \subseteq \mathbb{R}^{n_x}$ is said to be a robust positive invariant (RPI) set for system $x_{k+1} = f(x_k, w_k)$, $w_k \in \mathbb{W}$, if $x_k \in \mathcal{O}$ implies $x_t \in \mathcal{O}$ for any $w_t \in \mathbb{W}$, $t \in \mathbb{Z}_{\geq k+1}$.

Definition 4: A set $\mathcal{O}(\boldsymbol{\tau}^{[T]}) \subseteq \mathbb{R}^n$ is said to be a MPDT RPI set for system (4) with MPDT set $\boldsymbol{\tau}^{[T]} := \{\tau_1, \tau_2, \ldots, \tau_M\}$, if $e_0 \in \mathcal{O}(\boldsymbol{\tau}^{[T]})$ implies $e_k \in \mathcal{O}(\boldsymbol{\tau}^{[T]})$ for every admissible switching $\xi_{\boldsymbol{\tau}^{[T]}}(k)$ and for $w_t \in \mathbb{W}$, $t \in \mathbb{Z}_{[0,k-1]}$.

Then, the MPDT GRPI set for the switched system (4) with MPDT switching is defined as follows.

[1]We will slightly abuse the concept as a phase in this technical note.

Definition 5: A set $\mathcal{G}(\boldsymbol{\tau}^{[T]}) \subseteq \mathbb{R}^{n_x}$ is said to be a MPDT generalized robust positive invariant (GRPI) set for system (4) with MPDT set $\boldsymbol{\tau}^{[T]} := \{\tau_1, \tau_2, \ldots, \tau_M\}$, if $e_k \in \mathcal{O}(\boldsymbol{\tau}^{[T]}) \subseteq \mathcal{G}(\boldsymbol{\tau}^{[T]})$ implies $e_t \in \mathcal{G}(\boldsymbol{\tau}^{[T]})$ for any $w_t \in \mathbb{W}, t \in \mathbb{Z}_{\geq k+1}$, where $\mathcal{O}(\boldsymbol{\tau}^{[T]})$ is a MPDT RPI set for system (4).

To state the objectives of the note, the following definitions are also needed.

Definition 6: [10] System (2) is globally uniformly asymptotically stable (GUAS) under certain switching signals σ if for initial condition z_{k_0}, there exists a class of $\mathcal{K}_\infty$ function κ such that the solution of the system satisfies $\|z_k\| \leq \kappa(\|z_{k_0}\|)$, $\forall k \in \mathbb{Z}_{\geq k_0}$ and $\|z_k\| \to 0$ as $k \to \infty$.

Definition 7: A MPDT GRPI set $\mathcal{G}(\boldsymbol{\tau}^{[T]}) \subseteq \mathbb{R}^n$ is said to be GUAS for system (1) with MPDT switching, if for all $k \in \mathbb{Z}_+$, $\|x_k\|_{\mathcal{G}(\boldsymbol{\tau}^{[T]})} \leq \kappa(\|x_0\|_{\mathcal{G}(\boldsymbol{\tau}^{[T]})})$ and $\|x_k\|_{\mathcal{G}(\boldsymbol{\tau}^{[T]})} \to 0$ as $k \to \infty$, where $\kappa \in \mathcal{K}_\infty$.

Then, the objectives of this note are to develop a control policy $\mathcal{F}_{\sigma(k)}(\cdot)$, determine the state-feedback gain K_i, find a set of switching signals with admissible MPDT, and obtain a MPDT GRPI set for the resulting closed-loop (2) such that it is asymptotically stable in the sense of Definition 7.

III. Main Results

A. Stabilization of Nominal Systems

The subsection is concerned with the design of control policy $\mathcal{F}_{\sigma(k)}(z_k)$ that can be even advanced control methodologies such as model-based optimal control. Here, we are interested in developing a fundamental stabilizing state-feedback policy, but with the QTV form below, as adopted in [1], [15]

$$\mathcal{F}_{\sigma(k)}(z_k) := F_{\sigma(k)}(\vartheta) z_k \tag{5}$$

where ϑ is a scheduled index for the activated subsystem and can be computed online according to the following rules: $\forall \sigma(k) = i \in \mathcal{I}$:

i) in the τ_i-portion,

$$\vartheta = \begin{cases} k - k_{s_p}, & k \in \left[k_{s_p}, k_{s_p} + \tau_i\right) \\ \tau_i, & k \in \left[k_{s_p} + \tau_i, k_{s_p+1}\right) \end{cases} \tag{6}$$

ii) in the T-portion

$$\vartheta = k - H_r, | k \in \left[k_{s_p+1}, k_{s_{p+1}}\right) \tag{7}$$

where $H_r := \arg\{\max(k_{s_p+r}, r \in \mathbb{Z}_{\geq 1} | k_{s_p+r} \leq k, k_{s_p+r} \in [k_{s_p+1}, k_{s_{p+1}}))\}$.

It has been demonstrated in [15] for switched systems with dwell-time switching that the QTV state-feedback law outperforms the conventional one with less conservatism in achieving minimum dwell-time ensuring the stability of the underlying system. In order to obtain the stabilization criterion by using (5) for system under MPDT switching, we consider the corresponding QTV Lyapunov function as $V_{\sigma(k)}(z_k, \vartheta)$. Then the stability conditions for the nominal system in nonlinear case can be first arrived at.

Lemma 1: Consider a discrete-time switched system $z_{k+1} = f_{\sigma(k)}(z_k)$, and $0 < \alpha_i < 1$, $\mu_i > 0$ to be given constants. For a prescribed period of persistence T, suppose that there exist functions $V_{\sigma(k)} : (\mathbb{R}^{n_x}, \mathbb{Z}_{[0, \tau_{\sigma(k)}]}) \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, and two class $\mathcal{K}_\infty$ functions κ_1 and κ_2 such that $\forall \sigma(k) = i \in \mathcal{I}$:

i) $\forall \vartheta \in \mathbb{Z}_{[0, \tau_i]}$,

$$\kappa_1(\|z_k\|) \leq V_i(z_k, \vartheta) \leq \kappa_2(\|z_k\|); \tag{8}$$

ii) $\forall k \in [k_{s_p}, k_{s_p} + \tau_i)$,

$$V_i\left(z_{k+1}, k + 1 - k_{s_p}\right) \leq \alpha_i V_i\left(z_k, k - k_{s_p}\right); \tag{9}$$

iii) $\forall k \in [k_{s_p} + \tau_i, k_{s_p+1})$,

$$V_i(z_{k+1}, \tau_i) \leq \alpha_i V_i(z_k, \tau_i); \tag{10}$$

iv) $\forall k \in [k_{s_p+1}, k_{s_{p+1}}), r \in \mathbb{Z}_{[1, \mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})]}$

$$V_i(z_{k+1}, k + 1 - H_r) \leq \alpha_i V_i(z_k, k - H_r); \tag{11}$$

v) $\forall \sigma(k_{s_p+1}) = i \neq j = \sigma(k_{s_p+1} - 1)$,

$$V_i\left(z_{k_{s_p+1}}, 0\right) \leq \mu_j V_j\left(z_{k_{s_p+1}}, \tau_j\right); \tag{12}$$

vi) $\forall \sigma(k_{s_p+r}) = i \neq j = \sigma(k_{s_p+r} - 1)$,

$$V_i\left(z_{k_{s_p+r}}, 0\right) \leq \mu_j V_j\left(z_{k_{s_p+r}}, T_j\right); \tag{13}$$

where $T_j \in [1, \min(\tau_i - 1, T^{(p)})]$, $\forall i \in \mathcal{I}$, $T^{(p)} \in \mathbb{Z}_{[1,T]}$ and $r \in \mathbb{Z}_{[2, \mathcal{Q}(k_{s_p+1}, k_{s_{p+1}})+1]}$. Then the switched nonlinear system is GUAS for MPDT switching signals satisfying (8)–(13) and

$$\tau_i \geq -\left((T+1)\ln \mu_{\max} + T \ln \alpha_{\max}\right) / \ln \alpha_i \tag{14}$$

where $\mu_{\max} := \max_{i \in \mathcal{I}} \mu_i$, $\alpha_{\max} := \max_{i \in \mathcal{I}} \alpha_i$.

Proof: First of all, if $\mu_{\max}\alpha_{\max} < 1$, then it is straightforward that a switched system is GUAS with $\tau_i = 1$, i.e., under arbitrarily switching. If (14) holds, τ_i is at least 1 in discrete-time domain. Then the proof boils down to the case $\mu_{\max}\alpha_{\max} \geq 1$.

Consider $\sigma(k_{s_p}) = i$, $\sigma(k_{s_p+1} + T^{(p)}) = j$ in the p-th phase of MPDT switching (see Fig. 1), and suppose an arbitrary switching occurs within $T^{(p)}$, it follows from (9)–(13) that

$$\begin{aligned}
&V_j\left(z_{k_{s_p+1+T^{(p)}}}, 0\right) \\
&\leq \mu_l V_l\left(z_{k_{s_p+1+T^{(p)}}}, T_l\right) \leq \mu_l \alpha_l^{T_l} V_l\left(z_{k_{s_p+1+T^{(p)}-T_l}}, 0\right) \\
&\leq \mu_m \mu_l \alpha_l^{T_l} V_m\left(z_{k_{s_p+1+T^{(p)}-T_l}}, T_m\right) \\
&\leq \mu_i \mu_n \cdots \mu_m \mu_l \alpha_l^{T_l} \alpha_m^{T_m} \cdots \alpha_n^{T_n} \alpha_i^{k_{s_p+1} - k_{s_p}} V_i\left(z_{k_{s_p}}, 0\right) \\
&\leq \mu_{\max}^{Q\left(k_{s_p}, k_{s_p+1+T^{(p)}}\right)} \alpha_{\max}^{T_l + T_m + \cdots + T_n} \alpha_i^{\tau_i} V_i\left(z_{k_{s_p}}, 0\right) \\
&\leq \mu_{\max}^{T^{(p)}+1} \alpha_{\max}^{T^{(p)}} \alpha_i^{\tau_i} V_i\left(z_{k_{s_p}}, 0\right)
\end{aligned} \tag{15}$$

where $l, m, \ldots, n$ denote all the possible indices of subsystems being switched within $T^{(p)}$.

Thus since $\mu_{\max}\alpha_{\max} \geq 1$, $\mu_{\max}^{T^{(p)}+1}\alpha_{\max}^{T^{(p)}} \leq \mu_{\max}^{T+1}\alpha_{\max}^{T^{(p)}}$ holds. From (15), it follows that $V_j(z_{k_{s_p+1+T^{(p)}}}, 0) \leq \lambda_i V_i(z_{k_{s_p}}, 0)$, where $\lambda_i := \mu_{\max}^{T+1}\alpha_{\max}^{T}\alpha_i^{\tau_i}$. Then, if (14) is satisfied, $\lambda_i \leq 1$ holds. Let $\lambda_{\max} := \max_{i \in \mathcal{I}} \lambda_i$, and consider the fact that a period of persistence may exist before the 1-st phase, it follows $V_{\sigma(k_{s_p})}(z_{k_{s_p}}, 0) \leq \lambda_{\max} V_{\sigma(k_{s_{p-1}})}(z_{k_{s_{p-1}}}, 0) \leq \cdots \leq \lambda_{\max}^{p-1} V_{\sigma(k_{s_1})}(z_{k_{s_1}}, 0) \leq \lambda_{\max}^{p-1} \times \mu_{\max}^{T}\alpha_{\max}^{T} V_{\sigma(k_0)}(z_{k_0}, 0)$. From (8), $\|z_{k_{s_p}}\| \leq \kappa_1^{-1}(\lambda_{\max}^{p-1}\mu_{\max}^{T} \times \alpha_{\max}^{T}\kappa_2(\|z_{k_0}\|))$ holds. Thus, due to (8)–(13), $\|z_k\| \leq \kappa_3(\|z_{k_0}\|)$ holds $\forall k \in (k_{s_p}, k_{s_{p+1}}]$, where $\kappa_3(\cdot) := \kappa_1^{-1}(\mu_{\max}^{T}\alpha_{\max}^{T}\kappa_2(\kappa_1^{-1} \times (\lambda_{\max}^{p-1}\mu_{\max}^{T}\alpha_{\max}^{T}\kappa_2(\cdot))))$. Thus the global uniform asymptotic stability can be inferred by the denotation of $\lambda_{\max}$ and Definition 6. □

Remark 1: It should be noted that since the running time of each activated subsystem during the period of persistence is unknown *a priori*, the worst case of using $\mu_{\max}$, $\alpha_{\max}$ in the derivations of (15) is taken into account, as well as the consideration of T times of switching during the period of persistence.

Then, by considering the QTV Lyapunov function as $V_{\sigma(k)}(z_k, \vartheta) := z_k^T P_i(\vartheta) z_k$, the stabilization criterion for nominal system (2) can be readily established in the following theorem.

Theorem 1: Consider system (2) and let $0 < \alpha_i < 1$, $\mu_i > 0$ be given constants, $i \in \mathcal{I}$. Suppose there exist matrices $S_i(\vartheta) \in \mathbb{S}_{>0}^{n_x}$ and

$U_i(\vartheta), \vartheta = 0, 1, \dots, \tau_i, \forall i \in \mathcal{I}$, such that $\forall \vartheta = 0, 1, \dots, \tau_i - 1$

$$\begin{bmatrix} -S_i(\tau_i) & A_i S_i(\tau_i) + B_i U_i(\tau_i) \\ \star & -\alpha_i S_i(\tau_i) \end{bmatrix} \leq 0 \tag{16}$$

$$\begin{bmatrix} -S_i(\vartheta+1) & A_i S_i(\vartheta) + B_i U_i(\vartheta) \\ \star & -\alpha_i S_i(\vartheta) \end{bmatrix} \leq 0 \tag{17}$$

and $\forall (i \times j) \in \mathcal{I} \times \mathcal{I},\ i \neq j$

$$S_j(T_j) - \mu_j S_i(0) \leq 0 \tag{18}$$

$$S_j(\tau_j) - \mu_j S_i(0) \leq 0 \tag{19}$$

hold, where $T_j \in \mathbb{Z}_{[1, \min(\tau_j - 1, T^{(p)})]}$, $T^{(p)} \in \mathbb{Z}_{[1,T]}$. Then, the resulting closed-loop system is GUAS for MPDT switching signals satisfying (14). Moreover, the QTV stabilizing controller gain is given by $F_i(\vartheta) = U_i(\vartheta) S_i^{-1}(\vartheta)$.

Proof: Omitted. Based on Lemma 1, the proof can be obtained by basic matrix manipulations and Schur complement, cf. [16]. □

In Theorem 1, a small τ_i corresponding to fast switching may not guarantee a feasible solution of admissible controller, then considering α_i and μ_i to be variables, the MPDT can be minimized by solving the following minimization procedure:

$$\min_{\mu_i, \alpha_i, S_i, U_i} \tau_i, \quad \text{s.t. (14), (16)–(19).} \tag{20}$$

Remark 2: As pointed out in [3], [4], the optimization in (20) forms a typical bilinear matrix inequality problem in $\mu_i, \alpha_i, S_i, U_i$ and is not easy to solve. However, a subminimal MPDT can be obtained by bisection method (i.e., varying μ_i and α_i within their respective range while guaranteeing a feasible solution of admissible controller). Note that, for a fixed T, the minimum MPDT means to be the one with both the smallest $\|\boldsymbol{\tau}^{[T]}\|_1$ and the smallest variance of $\boldsymbol{\tau}^{[T]}$.

If setting $U_i(\vartheta) \equiv U_i$ and $S_i(\vartheta) \equiv S_i$ in Theorem 1, one can obtain the corresponding control policy with "non-QTV" controller gains $F_i = U_i S_i^{-1}, i \in \mathcal{I}$. As a result, for a certain switched system, the minimum MPDT obtained by an optimization procedure similar to (20), denoted by θ_i, will be generally greater than the minimum τ_i derived from the QTV control policy. Nevertheless, such non-QTV F_i can be directly used as the stabilizing state-feedback gains for the error system (4) as demanded in the structure of control law (3).

B. Systems With Bounded Additive Disturbances

In this subsection, a MPDT GRPI set of system (4) under MPDT switching will be determined to address the system stability in the sense of Definition 7. To this end, a MPDT RPI set of system (4) needs to be firstly determined.

We adopt $K_i := F_i, i \in \mathcal{I}$ in (3) so that system (4) is GUAS with MPDT switching and we suppose the admissible MPDT to be θ_i. The following theorem demonstrates the existence of a MPDT RPI set for system (4).

Theorem 2: Suppose that system (4) with MPDT θ_i for a given T is GUAS, then a MPDT RPI set $\mathcal{O}(\boldsymbol{\theta}^{[T]})$ exists for system (4).

Proof: Let $R_{t,i,g_i} := (\prod_{l=1}^{t} \bar{A}_{[l]}) \bar{A}_i^{g_i - 1} \mathbb{W} \oplus (\prod_{l=1}^{t} \bar{A}_{[l]}) \times \bar{A}_i^{g_i - 2} \mathbb{W} \oplus \cdots \oplus (\prod_{l=1}^{t} \bar{A}_{[l]}) \mathbb{W} \oplus (\prod_{l=2}^{t} \bar{A}_{[l]}) \mathbb{W} \oplus \cdots \oplus \bar{A}_{[t]} \mathbb{W} \oplus \mathbb{W}$, where $\prod_{l=1}^{t} \bar{A}_{[l]}$ stands for $\bar{A}_{[t]} \bar{A}_{[t-1]} \cdots \bar{A}_{[1]}$ and $\bar{A}_{[l]}$ varying with $l, l \in \mathbb{Z}_{\geq 1}$ denotes a matrix taken in set $\mathcal{A}_1 := \{\bar{A}_1, \bar{A}_2, \dots, \bar{A}_M\}$. Define $\Theta_i := \{\theta_i, \theta_i + 1, \dots, 2\theta_i - 1\}$ and consider

$$\mathcal{O}_{v+1} := co\left\{\left(\prod_{l=1}^{t} \bar{A}_{[l]}\right) \bar{A}_i^{g_i} \mathcal{O}_v \oplus R_{t,i,g_i} : t \in \mathbb{Z}_{[0,T]},\ i \in \mathcal{I},\ g_i \in \Theta_i\right\} \tag{21}$$

where $\mathcal{O}_0$ is $\Lambda := co\{(\prod_{l=1}^{t-1} \bar{A}_{[l]}) \mathbb{W} \oplus (\prod_{l=2}^{t-1} \bar{A}_{[l]}) \mathbb{W} \oplus \cdots \oplus \bar{A}_{[t-1]} \mathbb{W} \oplus \mathbb{W} : t \in \mathbb{Z}_{[1,T]}\}$ or $\{0\}$, respectively, for the cases that a period of persistence exists or not before the 1-st phase. Let $\mathcal{R} := co\{R_{t,i,g_i} : t \in \mathbb{Z}_{[0,T]},\ i \in \mathcal{I},\ g_i \in \Theta_i\}$, $\mathcal{O}_{v+1}$ satisfies

$$\mathcal{O}_{v+1} \subseteq co\left\{\left(\prod_{l=1}^{t} \bar{A}_{[l]}\right) \bar{A}_i^{g_i} \mathcal{O}_v \oplus \mathcal{R} : t \in \mathbb{Z}_{[0,T]},\ i \in \mathcal{I},\ g_i \in \Theta_i\right\}. \tag{22}$$

Then, iterating (22) from v to 0 yields that, $\forall v \in \mathbb{Z}_{\geq 1}, \mathcal{O}_v \subseteq \Psi_v := co\{(\prod_{l=1}^{t_v} \bar{A}_{[l]}) \bar{A}_h^{g_h} (\prod_{l=1}^{t_{v-1}} \bar{A}_{[l]}) \bar{A}_i^{g_i} \cdots (\prod_{l=1}^{t_2} \bar{A}_{[l]}) \bar{A}_j^{g_j} (\prod_{l=1}^{t_1} \bar{A}_{[l]}) \times \bar{A}_k^{g_k} \mathcal{O}_0 \oplus (\prod_{l=1}^{t_v} \bar{A}_{[l]}) \bar{A}_h^{g_h} \cdots (\prod_{l=1}^{t_2} \bar{A}_{[l]}) \bar{A}_j^{g_j} \mathcal{R} \oplus \cdots \oplus (\prod_{l=1}^{t_v} \bar{A}_{[l]}) \times \bar{A}_h^{g_h} \mathcal{R} \oplus \mathcal{R} : t_c \in \mathbb{Z}_{[0,T]}, c \in \mathbb{Z}_{[1,v]}, g_d \in \Theta_d, d \in \mathcal{I},\ (h \times i \times \cdots \times j \times k) \in \mathcal{I} \times \mathcal{I} \times \cdots \times \mathcal{I} \times \mathcal{I}\}$. Since system (4) under MPDT θ_i for a given T is GUAS, then the system $\hat{z}_{k+1} = \hat{A} \hat{z}_k$ is asymptotically stable under arbitrary switching, where $\hat{A} \in \Phi(\Theta_i, T) := \{(\prod_{l=1}^{t} \bar{A}_{[l]}) \bar{A}_i^r : t \in \mathbb{Z}_{[0,T]}, i \in \mathcal{I}, r \in \Theta_i\}$. Here, the finite set Θ_i is invoked with a similar usage in [2] such that the subsystems evolution under all the admissible switching sequences during $[k_{s_p}, k_{s_{p+1}})$, $p \in \mathbb{Z}_+$ can be represented equivalently by combinations of matrices in $\Phi(\Theta_i, T)$ where t can be zero. Therefore, there exists a constant $\varepsilon \in (0, 1)$ and $\eta > 0$ satisfying $\mathcal{R} \subseteq \eta \mathcal{B}^n$ such that $\hat{A} \mathcal{R} \subseteq \eta \varepsilon \mathcal{B}^n$. Then

$$\mathcal{O}_v \subseteq \Psi_v \subseteq \eta(\varepsilon^n + \varepsilon^{n-1} + \cdots + \varepsilon + 1) \mathcal{B}^n \tag{23}$$

where $n = v$ or $n = v - 1$ corresponds to the case $\mathcal{O}_0 = \Lambda (\subseteq \mathcal{R})$ or $\mathcal{O}_0 = \{0\}$, respectively. Hence, from (21) and (23), it holds that $\mathcal{O}_v \subseteq \mathcal{O}_{v+1}$ and $\mathcal{O}_v$ is bounded above by $(\eta / (1 - \varepsilon)) \mathcal{B}^n$ as $v \to \infty$, respectively. Thus the set sequence $\{\mathcal{O}_v : v \in \mathbb{Z}_+\}$ has a limit $\mathcal{O}_\infty$ that is dependent on the MPDT θ_i and T. Therefore, for system (4), it follows from the computations of $\mathcal{O}_v$ that for any $e_0 \in \mathcal{O}(\boldsymbol{\theta}^{[T]}) := \mathcal{O}_\infty, e_k \in \mathcal{O}(\boldsymbol{\theta}^{[T]})$ for the admissible MPDT switching with $\xi_{\boldsymbol{\theta}^{[T]}}(k)$ and for $w_t \in \mathbb{W}, t \in \mathbb{Z}_{[0,k-1]}$. □

Remark 3: A direct question will be why not the QTV stabilizing controller gain $F_i(\vartheta)$ is used instead of F_i in the stabilization of system (4) and also the computation of MPDT RPI set. In fact, when using $F_i(\vartheta)$, we expect that we could find a counterpart to $\Phi(\Theta_i, T)$ to represent all the admissible switching sequences during $[k_{s_p}, k_{s_{p+1}})$, say $\hat{\Phi}(\Theta_i, T) := \{(\prod_{l=1}^{t} \hat{A}_{[l]}) \hat{A}_{i,\theta_i}^r \prod_{\vartheta=0}^{\theta_i - 1} \hat{A}_{i,\vartheta} : t \in \mathbb{Z}_{[0,T]}, i \in \mathcal{I},\ r \in \hat{\Theta}_i\}$, where $\prod_{l=1}^{t} \hat{A}_{[l]}$ stands for $\bar{A}_{[t]} \bar{A}_{[t-1]} \cdots \bar{A}_{[1]}$ and $\hat{A}_{[l]}$ varying with $l, l \in \mathbb{Z}_{\geq 1}$ denotes a matrix $\hat{A}_{\sigma(k_{s_p+1}+l-1), k_{s_p+1}+l-1-H_r}$ taken in set $\mathcal{A}_2 := \{\hat{A}_{i,\vartheta} : i \in \mathcal{I}, \vartheta \in \mathbb{Z}_{[0,\theta_i - 1]}\}$ with H_r being denoted in (7) and $\hat{A}_{i,\vartheta} := A_i + B_i F_i(\vartheta)$. Yet, one can not find a finite set $\hat{\Theta}_i$ comparable to Θ_i to represent an infinite number of admissible switching sequences during $[k_{s_p}, k_{s_p+1})$ due to the existence of $\prod_{\vartheta=0}^{\theta_i - 1} \hat{A}_{i,\vartheta}$.

Now, let one step reachable set from a set $\mathcal{X}$ along subsystem Ω_i be denoted as $\mathcal{P}_1^i(\mathcal{X}, \mathbb{W}) := \{\bar{A}_i x + w : x \in \mathcal{X}, w \in \mathbb{W}\} = \bar{A}_i \mathcal{X} \oplus \mathbb{W}$, then the H-step reachable set $\mathcal{P}_H^i(\mathcal{X}, \mathbb{W})$ is defined as $\mathcal{P}_{y+1}^i \times (\mathcal{X}, \mathbb{W}) := \mathcal{P}_1^i(\mathcal{P}_y^i(\mathcal{X}, \mathbb{W}), \mathbb{W}), y \in \mathbb{Z}_{[0,H-1]}$, where $\mathcal{P}_0^i(\mathcal{X}, \mathbb{W}) := \mathcal{X}$. Thus $\mathcal{P}_H^i(\mathcal{X}, \mathbb{W}) = \bar{A}_i^H \mathcal{X} \oplus \bar{A}_i^{H-1} \mathbb{W} \oplus \cdots \oplus \bar{A}_i \mathbb{W} \oplus \mathbb{W}$. Define two operators $\bar{\mathcal{P}}(\cdot, \mathbb{W})$ and $\hat{\mathcal{P}}(\cdot, \mathbb{W})$ as

$$\bar{\mathcal{P}}(\cdot, \mathbb{W}) := \bigcup_{i \in \mathcal{I}} \bigcup_{t_0 \in \Theta_i} \mathcal{P}_{t_0}^i(\cdot, \mathbb{W})$$

$$\hat{\mathcal{P}}(\cdot, \mathbb{W}) := \Bigg\{ \bigcup_{t_Q \in \mathbb{Z}_{[0,T]}} \bigcup_{l \in \mathcal{I}} \mathcal{P}_{t_Q}^l \Bigg(\bigcup_{t_{Q-1} \in \mathbb{Z}_{[0,T]}} \bigcup_{m \in \mathcal{I}} \mathcal{P}_{t_{Q-1}}^m \Bigg(\cdots \bigcup_{t_1 \in \mathbb{Z}_{[0,T]}} \bigcup_{n \in \mathcal{I}} \mathcal{P}_{t_1}^n(\cdot, \mathbb{W}), \dots, \mathbb{W} \Bigg), \mathbb{W} \Bigg) : l \neq m, n \neq i, 0 \leq \sum_{q=1}^{Q} t_q \leq T \Bigg\}.$$

Then based on Theorem 2, an algorithm to compute the MPDT RPI set for system (4) can be obtained as shown in what follows.

Algorithm 1. Computation of $\mathcal{O}(\boldsymbol{\theta}^{[T]})$
Input: $\mathbb{W}, T, M, \boldsymbol{\theta}^{[T]}, \bar{A}_i,\ i \in \mathcal{I}$.

(i) Set $v = 0$ and $\mathcal{O}_v = co\{\hat{\mathcal{P}}(\{0\}, \mathbb{W})\}$.
(ii) Set $\mathcal{O}_{v+1} = co\{\hat{\mathcal{P}}(\bar{\mathcal{P}}(\mathcal{O}_v, \mathbb{W}), \mathbb{W})\}$.
(iii) If $\mathcal{O}_{v+1} \equiv \mathcal{O}_v$, set $\mathcal{O}(\boldsymbol{\theta}^{[T]}) = \mathcal{O}_v$, **exit** and **output** $\mathcal{O}(\boldsymbol{\theta}^{[T]})$; else, set $v = v + 1$ and goto step (ii).

Remark 4: Without loss of generality, in Algorithm 1, $\mathcal{O}_0 = co \times \{\hat{\mathcal{P}}(\{0\}, \mathbb{W})\}(= \Lambda)$ is taken into account since $\{0\} \subseteq co\{\hat{\mathcal{P}} \times (\{0\}, \mathbb{W})\}$, though it brings conservatism to the case that a period of persistence does not exist before the 1-st phase. Also, note that the existence of a MPDT RPI set ensures the convergence of Algorithm 1.

It can be seen from Definition 4 that the MPDT RPI set $\mathcal{O}(\boldsymbol{\theta}^{[T]})$ has the properties that, $\forall s \in \mathbb{Z}_{\geq \theta_i}$, $\mathcal{P}_s^i(\mathcal{O}(\boldsymbol{\theta}^{[T]}), \mathbb{W}) \subseteq \mathcal{O}(\boldsymbol{\theta}^{[T]})$ and

$$\hat{\mathcal{P}}\left(\mathcal{P}_s^i\left(\mathcal{O}\left(\boldsymbol{\theta}^{[T]}\right), \mathbb{W}\right), \mathbb{W}\right) \subseteq \mathcal{O}\left(\boldsymbol{\theta}^{[T]}\right). \tag{24}$$

Then, if letting

$$\begin{aligned}\mathcal{G}^i\left(\boldsymbol{\theta}^{[T]}\right) := co\Big\{&\mathcal{P}^i_{\boldsymbol{\tau}_i-1}\left(\mathcal{O}\left(\boldsymbol{\theta}^{[T]}\right), \mathbb{W}\right),\\ &\mathcal{P}^i_{\boldsymbol{\tau}_i-2}\left(\mathcal{O}\left(\boldsymbol{\theta}^{[T]}\right), \mathbb{W}\right), \cdots, \mathcal{O}\left(\boldsymbol{\theta}^{[T]}\right)\Big\}\end{aligned} \tag{25}$$

it concludes that $e_t \in \mathcal{G}^i(\boldsymbol{\theta}^{[T]}),\ t \in \mathbb{Z}_{\geq k+1}$ for any $e_k \in \mathcal{O}(\boldsymbol{\theta}^{[T]}) \subseteq \mathcal{G}^i(\boldsymbol{\theta}^{[T]})$. Therefore, the MPDT GRPI set for system (4) can be obtained by $\mathcal{G}(\boldsymbol{\theta}^{[T]}) = \bigcup_{i \in \mathcal{I}} \mathcal{G}^i(\boldsymbol{\theta}^{[T]})$, upon which the stability of closed-loop system (1) can be analyzed in the sense of Definition 7. In the following theorem, we shall use an extended set $\mathcal{G}(\boldsymbol{\theta}^{[T]}) \times \{0\}$ to establish the stability criterion of the composite switched system (1) and (2).

Theorem 3: Consider the switched system (1) and (2). Suppose that for a given T, a QTV stabilizing controller and a non-QTV stabilizing controller exist for nominal system (2) and error system (4) with MPDT τ_i and θ_i, respectively. Then the set $\hat{\mathcal{G}} := \mathcal{G}(\boldsymbol{\theta}^{[T]}) \times \{0\}$ is GUAS for the composite switched system (1) and (2) with the admissible MPDT switching satisfying $\Delta_i := max\{\theta_i, \tau_i\}$.

Proof: If there exists a set of controllers such that system (2) is GUAS with MPDT Δ_i satisfying (14) for a given T, then it follows from Definition 6 that $\|z_k\| \leq \kappa(\|z_{k_0}\|),\ \forall k \in \mathbb{Z}_{\geq k_0}$ and $\|z_k\| \to 0$ as $k \to \infty$, where $\kappa \in \mathcal{K}_\infty$. Since for $k \in [k_s, k_{s+1}),\ x_k = z_k + e_k$ where $e_k \in \mathcal{G}(\boldsymbol{\theta}^{[T]})$, it holds that $\|x_k\|_{\mathcal{G}(\boldsymbol{\theta}^{[T]})} = d(z_k + e_k, \mathcal{G}(\boldsymbol{\theta}^{[T]})) \leq d(z_k + e_k, e_k) = \|z_k\| \leq \kappa(\|z_{k_0}\|)$ and $\|x_k\|_{\mathcal{G}(\boldsymbol{\theta}^{[T]})} \to 0$ as $k \to \infty$.

Denote $\|(x_k, z_k)\| := \|x_k\| + \|z_k\|$, it follows that the extended state (x_k, z_k) of the composite system (2) and (3) satisfies $\|(x_k, z_k)\|_{\hat{\mathcal{G}}} = \inf_{\hat{x} \in \mathcal{G}(\boldsymbol{\theta}^{[T]})} \|(x_k, z_k) - (\hat{x}, 0)\| = \inf_{\hat{x} \in \mathcal{G}(\boldsymbol{\theta}^{[T]})} \|(x_k - \hat{x}, z_k)\| = \inf_{\hat{x} \in \mathcal{G}(\boldsymbol{\theta}^{[T]})} \|(x_k - \hat{x})\| + \|z_k\| = \|x_k\|_{\mathcal{G}(\boldsymbol{\theta}^{[T]})} + \|z_k\| \leq 2\kappa \times (\|z_{k_0}\|) \leq 2\kappa(\|x_{k_0}\|_{\mathcal{G}(\boldsymbol{\theta}^{[T]})} + \|z_{k_0}\|) = 2\kappa(\|(x_{k_0}, z_{k_0})\|_{\hat{\mathcal{G}}})$, which implies that $\hat{\mathcal{G}}$ is GUAS for the composite switched system in the sense of Definition 7. □

Remark 5: It can be concluded from Theorem 3 and the definition of $\mathcal{G}^i(\boldsymbol{\theta}^{[T]})$ in (25) that the trajectory of the error system (4) will always remain inside $\mathcal{O}(\boldsymbol{\theta}^{[T]})$ at switching instants k_s and $\mathcal{G}^i(\boldsymbol{\theta}^{[T]})$ within subsystem $\Xi_i,\ \forall i \in \mathcal{I}$, respectively. Such a fact implies that system (1), as well as system (4), possesses a tube whose cross section displays as $\mathcal{O}(\boldsymbol{\theta}^{[T]})$ or $\mathcal{G}^i(\boldsymbol{\theta}^{[T]})$ at each sampling instant. The tube can be therefore viewed as a "uniform tube" as it is uniformly valid for the whole set of switching signals satisfying MPDT property.

Remark 6: Two noteworthy observations can be further made. First, for a concrete MPDT switching sequence (both the order of switching among subsystems and the switching instants are fixed), we can conclude that the error system trajectory will be contained in a tighter tube whose cross section belongs to $\mathcal{O}(\boldsymbol{\theta}^{[T]})$ and $\mathcal{G}^i(\boldsymbol{\theta}^{[T]})$ at k_s and within Ξ_i, respectively. The reason is that $\mathcal{O}(\boldsymbol{\theta}^{[T]})$ is offline determined, which requires all the possible switching within T and all the possible values in Θ_i to be considered (see Algorithm 1 and the definitions of the two operators $\bar{\mathcal{P}}(\cdot, \mathbb{W})$ and $\hat{\mathcal{P}}(\cdot, \mathbb{W})$) to meet the uniformity of the asymptotic stability. Consequently, it will be somewhat conservative to use $\mathcal{G}(\boldsymbol{\theta}^{[T]})$ to evaluate the system stability as far as a concrete switching sequence is concerned. Second, since any activated subsystem will dwell less than the admissible MPDT within T, the tube may expand during T and therefore tends to be rather tighter at the very switching instant of entering T to prevent its subsequent evolution during T getting out of $\mathcal{O}(\boldsymbol{\theta}^{[T]})$ [see (24)].

TABLE I
MINIMUM MPDT BY QTV AND NON-QTV CONTROLLER FOR DIFFERENT T

T	τ_1	τ_2	θ_1	θ_2
2	3	4	4	5
3	4	5	5	6

IV. NUMERICAL EXAMPLE

Consider system (1) consisting of two subsystems described by

$$A_1 = \begin{bmatrix} 1.00 & -0.70 \\ 0.50 & -0.70 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.89 & 0.38 \\ 1.65 & 1.14 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad \|w\|_\infty = 0.1.$$

Our purpose here is to design a QTV stabilizing controller for the nominal system, find out the admissible MPDT switching, and determine a MPDT GRPI set for the error system (4) formed by a non-QTV stabilizing controller. Firstly, it can be checked that the nominal switched system does not admit stabilizing controllers under arbitrary switching. By Theorem 1 and solving (20) for given $\alpha_i = 0.15$, however, the minimum admissible MPDT τ_i can be solved as shown in Table I for given different T, as well as the θ_i corresponding to the non-QTV stabilizing control. It can be seen that the QTV controller has less conservatism in achieving shorter admissible MPDT. The associated controller gains in both cases are omitted here, and θ_i will be used for the QTV control as required in Theorem 3 since $\theta_i \geq \tau_i$ in either case in Table I.

Given $T = 3$ and $x_0 = [-5\ 1.8]^{\mathrm{T}}$ (the superscript "T" here means transpose), consider one admissible switching sequence (shown in the subfigure in Fig. 2(a)) where the running time of subsystems are equivalent to the MPDT and a period of persistence exists before the first MPDT phase, Fig. 2(a) and (b) show the cluster of state trajectories of the practical system, and Fig. 2(c) the error system for 30 realizations of the random disturbance sequences. Also, Fig. 3(d) shows the MPDT RPI set $\mathcal{O}(\boldsymbol{\theta}^{[3]})$ and the two components of the MPDT GRPI set $\mathcal{G}(\boldsymbol{\theta}^{[3]})$, $\mathcal{G}^1(\boldsymbol{\theta}^{[3]})$ and $\mathcal{G}^2(\boldsymbol{\theta}^{[3]})$, which can be obtained by Algorithm 1 and by (25), respectively. The evolution of the uniform tube (displays as $\mathcal{O}(\boldsymbol{\theta}^{[3]})$, $\mathcal{G}^1(\boldsymbol{\theta}^{[3]})$ or $\mathcal{G}^2(\boldsymbol{\theta}^{[3]})$ at each sampling instant, see Remark 6 is also illustrated in Fig. 2(a) and (b) for the practical system, and Fig. 2(c) for the error system. Finally, for one realization of the random disturbance sequences till $k = 1000$, Fig. 2(d) also shows the projection of a state trajectory of the error system at switching instants and within subsystems into one 2-dimension coordinate.

It can be first seen from Fig. 2(a) and (b) that the state trajectory of nominal system converges, verifying the validity of the QTV stabilizing controller. Also, all the subfigures in Fig. 2 show that the state trajectories either at switching instants or within error subsystems remain inside $\mathcal{G}(\boldsymbol{\theta}^{[3]})$, illustrating that the designed non-QTV controller is effective against the random disturbances. Besides, as also shown in Fig. 2(a) (or Fig. 2(c)), all the system trajectories fall

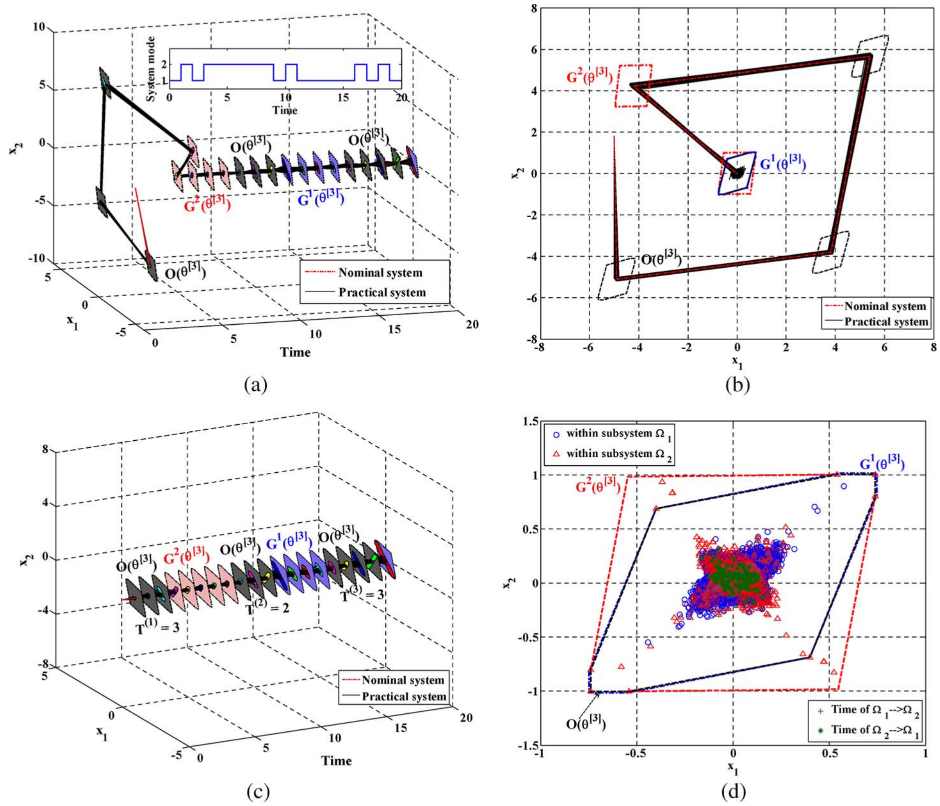

Fig. 2. Tube evolution and state trajectories of practical system and error system under MPDT switching for $T = 3$. (a) Practical system for 30 realizations of random disturbance sequences in 3-D. (b) Practical system for 30 realizations of random disturbance sequences in 2-D. (c) Error system for 30 realizations of random disturbance sequences in 3-D. (d) Projection of state trajectory of error system into one coordinate for one realization of random disturbance sequences.

into a certain inner set of $\mathcal{O}(\boldsymbol{\theta}^{[3]})$ at switching instants, and inner set of $\mathcal{G}^i(\boldsymbol{\theta}^{[3]})$ within Ξ_i, respectively. That is, the designed controller can actually ensure a tighter tube for the error system rather than the uniform tube in the concern of a concrete realization of the MPDT switching sequences, which is consistent to the first observation in Remark 7. The second observation in Remark 7 is also verified in Fig. 2(a) and Fig. 2(c), where the tube expanding during T is relatively "smaller" at switching instants of entering T ($k = 9$ and $k = 16$ within the first and second phase, respectively).

V. Conclusion

In this note, the stabilization problem of a class of discrete-time switched linear systems with l_∞ bounded additive disturbances and MPDT switching was investigated. A stability criterion for general nominal switched nonlinear system with MPDT switching is first presented, and a QTV stabilizing controller for the underlying switched linear system is designed ensuring that the closed-loop system is GUAS. Allowing for the l_∞ additive disturbances, a MPDT RPI set is determined for the switched error system between the nominal system and disturbed system, and the error states remain inside the MPDT RPI set at the switching instants. A MPDT GRPI set is further determined as the cross section of a uniform tube and it is demonstrated that the disturbed system is also asymptotically stable in the sense of converging to the MPDT GRPI set.

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