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Analysis on robust $\mathcal{H}_\infty$ performance and stability for linear systems with interval time-varying state delays via some new augmented Lyapunov–Krasovskii functional

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ABSTRACT

In this paper, the problem of $\mathcal{H}_\infty$ performance and stability analysis for linear systems with interval time-varying delays is considered. First, by constructing a newly augmented Lyapunov–Krasovskii functional which has not been proposed yet, an improved $\mathcal{H}_\infty$ performance criterion with the framework of linear matrix inequalities (LMIs) is introduced. Next, the result and method are extended to the problem of a delay-dependent stability criterion. Finally, numerical examples are given to show the superiority of proposed method.

1. Introduction

The stability analysis of dynamic systems with time delays has been one of the hottest issues in the field of control society since time-delays occur in many systems including neural networks, biological systems, chemical engineering systems, synchronization between two chaotic systems, transportation systems, multi-agent systems and so on. As a basic and essential work of stability analysis for systems with time delays, various delay-dependent stability criteria for linear systems have been proposed by many researchers and some related studies are still ongoing. For details, see [1–11] and references therein.

The main concern for checking the conservatism of stability criteria for systems with time delays is to find maximum delay bounds for guaranteeing the asymptotic stability of the concerned systems. By comparison of obtained maximum delay bounds with the existing results, the superiority and less conservatism can be shown by emphasizing the enlargement of maximum delay bounds since a stability criterion which is larger than those of other literature means that it provides larger feasible region. With a stability criterion having an enhanced feasible region, the applicability such as stabilization, guaranteed cost control, filtering, state estimation, synchronization, and so on, can be increased in various systems. Thus, many times and efforts have been put into the development of some techniques and new Lyapunv Krasovskii functional because how to choose Lyapunov–Krasovskii functional and estimate an upper bound of time-derivative of Lyapunov–Krasovskii functional play key roles to improve the feasible region of stability criteria. Park's Lemma [12], model transformation [1,13,14], free-weighting matrices techniques [15], delay partitioning method [16], and reciprocally convex optimization approach [17] are the well-known and main techniques in reducing the conservatism of stability criteria.

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Another approach to reduce the conservatism of stability criteria is to utilize integral terms of states as augmented vectors. Thus, more information of system can be utilized, which can increase the feasible region of stability criteria. In [18] and [19], some triple-integral terms in Lyapunov–Krasovskii functional were proposed to reduce the conservatism of stability criteria. Since then, various problems [20–33] have been tacked by utilizing triple-integral terms of Lyapunov–Krasovskii functional. Recently, by constructing the quadrable-integral terms in Lyapunov–Krasovskii functional, new delay-dependent stability criteria for linear systems with interval time-varying delays have been reported in [34]. In [35], by utilization of the idea in [34], improved stability criteria for neural networks with time-varying delays were proposed based on quadratic convex combination.

On the other hand, since the theory of $\mathcal{H}_\infty$ control was presented by Zames [36], $\mathcal{H}_\infty$ performance analysis and control for various systems have been reported in [37–41]. The $\mathcal{H}_\infty$ control technique has been used to minimize the effects of the external disturbances. It is the objective of $\mathcal{H}_\infty$ control to design the controllers such that the closed-loop system is internally stable and its $\mathcal{H}_\infty$-norm of the transfer function between the controlled output and the disturbances will not exceed a given level γ. Very recently, improved $\mathcal{H}_\infty$ performance analysis and stability for uncertain systems with interval time-varying delays were proposed in [42]. However, there are rooms for further improvements in the feasible region of criteria for $\mathcal{H}_\infty$ performance and stability.

With this motivation, the problem of $\mathcal{H}_\infty$ performance and stability analysis for linear systems with interval time-varying delays is revisited. The parameter uncertainties are assumed to be norm-bounded. Based on Lyapunov stability theory, an improved $\mathcal{H}_\infty$ performance criterion for linear systems with interval time-varying delays and external disturbances is derived by the framework of LMIs which will be introduced in Theorem 1. The main contribution of this paper lies in three aspects.

1. The main contribution of this paper is that some new augmented Lyapunov–Krasovskii functional which have not been considered yet in stability analysis of $\mathcal{H}_\infty$ performance criterion are introduced. With the proposed Lyapunov–Krasovskii functional, some new cross terms are considered in upper bounds of the time-derivative of Lyapunov–Krasovskii functional to enhance the feasible region.
2. The double integral term such as $\int_{t-h_U}^{t-h_L}\int_s^{t-h_L} x(u)duds$ is utilized as an element of augmented vector in estimating the time-derivative of Lyapunov–Krasovskii functional.
3. Inspired by the work of [41], one new zero equality is proposed. This zero equality is merged into the results of the time-derivative of Lyapunov–Krasovskii functional to reduce the conservatism of $\mathcal{H}_\infty$ performance criterion.

Based on the result of Theorem 1, a robust stability criterion for uncertain systems with interval time-varying delays will be proposed in Corollary 1. Four numerical examples are included to show the improvement of feasible region over the existing ones.

Notation 1. In this presentation, the following notations will be used. Lebesgue space $\mathcal{L}_{2+} = \mathcal{L}_2[0,\infty)$ consists of square-integral functions on $[0,\infty)$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ is the set of all $m\times n$ real matrix. $\star$ denotes the symmetric part. $X > 0$ $(X \geqslant 0)$ means that X is a real symmetric positive definite matrix (positive semi-definite). I denotes the identity matrix with appropriate dimensions. $X^\perp$ denotes a basis for the null-space of X. $I_n, 0_n$ and $0_{m\times n}$ denotes $n\times n$ identity matrix, $n\times n$ and $m\times n$ zero matrices, respectively. $\|\cdot\|$ refers to the induced matrix 2-norm. $diag\{\cdots\}$ denotes the block diagonal matrix. $\mathcal{C}_{n,h} = \mathcal{C}([-h,0],\mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-h,0]$ into $\mathbb{R}^n$, with the topology of uniform convergence. $X_{[f(t)]} \in \mathbb{R}^{m\times n}$ means that the elements of matrix $X_{[f(t)]}$ include the scalar value of $f(t)$. For any matrix M, $Sym\{M\}$ means $M + M^T$. $col\{x_1, x_2, \ldots, x_n\}$ means $[x_1^T, x_2^T, \ldots, x_n^T]^T$.

2. Problem statement and preliminaries

Consider the following linear systems with interval time-varying delays:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t-h(t)) + B_1 w(t), \\ z(t) &= Cx(t) + C_d x(t-h(t)) + B_2 w(t), \\ x(s) &= \phi(s), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^m$ denotes the disturbance input such that $w(t) \in \mathcal{L}_{2+}$, $z(t) \in \mathbb{R}^q$ is the performance output, $A, A_d, B_i (i = 1,2), C$, and C_d are known constant matrices with appropriate dimensions, $\phi(s) \in \mathcal{C}_{n,h_U}$ is a vector-valued initial function, $\Delta A(t)$ and $\Delta A_d(t)$ are the uncertainties of system matrices of the form

$$[\Delta A(t),\ \Delta A_d(t)] = DF(t)[E, E_d], \tag{2}$$

in which $D \in \mathbb{R}^{n\times l}$, $E \in \mathbb{R}^{l\times n}$, and $E_d \in \mathbb{R}^{l\times n}$ are known constant matrices and the time-varying nonlinear function $F(t) \in \mathbb{R}^{l\times l}$ satisfies $F^T(t)F(t) \leqslant I_{l\times l}$. The delay, $h(t)$, is time-varying continuous function that satisfies

$$h_L \leqslant h(t) \leqslant h_U, \quad \dot{h}(t) \leqslant h_D, \tag{3}$$

where $h_U > h_L > 0$, and h_D are constant values.

Now, with the defined uncertainties (2), the system (1) can be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-h(t)) + B_1 w(t) + Dp(t), \\ p(t) &= F(t)q(t), \\ q(t) &= Ex(t) + E_d x(t-h(t)), \\ z(t) &= Cx(t) + C_d x(t-h(t)) + B_2 w(t), \\ x(s) &= \phi(s). \end{aligned} \tag{4}$$

The objective of this paper is to investigate $\mathcal{H}_\infty$ performance and stability analysis for system (4).

Before deriving our main results, we need the following lemma.

Lemma 1. *For a positive matrix M, the following inequality holds:*

$$-(\alpha-\beta)\int_\beta^\alpha \dot{x}^T(s)M\dot{x}(s)ds \leqslant \begin{bmatrix} x(\alpha) \\ x(\beta) \end{bmatrix}^T \begin{bmatrix} -M & M \\ \star & -M \end{bmatrix} \begin{bmatrix} x(\alpha) \\ x(\beta) \end{bmatrix}. \tag{5}$$

Proof. According to Lemma 1 in [43], one can see

$$-(\alpha-\beta)\int_\beta^\alpha \dot{x}^T(s)M\dot{x}(s)ds \leqslant -\left(\int_\beta^\alpha \dot{x}(s)ds\right)^T M\left(\int_\beta^\alpha \dot{x}(s)ds\right). \tag{6}$$

Then, the inequality (5) can be obtained. □

Lemma 2. *For a positive matrix M, the following inequality holds:*

$$-\frac{(\alpha-\beta)^2}{2}\int_\beta^\alpha\int_s^\alpha x^T(u)Mx(u)duds \leqslant -\left(\int_\beta^\alpha\int_s^\alpha x(u)duds\right)^T M\left(\int_\beta^\alpha\int_s^\alpha x(u)duds\right). \tag{7}$$

Proof. From Lemma 1, the following inequality holds:

$$-(\alpha-s)\int_s^\alpha x^T(u)Mx(u)du \leqslant -\left(\int_s^\alpha x(u)du\right)^T M\left(\int_s^\alpha x(u)du\right). \tag{8}$$

By Schur complement, inequality (8) is equivalent to

$$\begin{bmatrix} -\int_s^\alpha x^T(u)Mx(u)du & \left(\int_s^\alpha x(u)du\right)^T \\ \star & -(\alpha-s)M^{-1} \end{bmatrix} \leqslant 0. \tag{9}$$

By integrating inequality (9) from β to α, we have

$$\begin{bmatrix} -\int_\beta^\alpha\int_s^\alpha x^T(u)Mx(u)duds & \left(\int_\beta^\alpha\int_s^\alpha x(u)duds\right)^T \\ \star & -\int_\beta^\alpha(\alpha-s)M^{-1}ds \end{bmatrix} \leqslant 0. \tag{10}$$

By Schur complement, inequality (10) is equivalent to inequality (7). This completes the proof of Lemma 2. □

Lemma 3. *For a positive matrix M, the following inequality holds:*

$$-\frac{(\alpha-\beta)^3}{6}\int_\beta^\alpha\int_s^\alpha\int_u^t x^T(v)Mx(v)dvduds \leqslant -\left(\int_\beta^\alpha\int_s^\alpha\int_u^t x(v)duds\right)^T M\left(\int_\beta^\alpha\int_s^\alpha\int_u^t x(v)duds\right). \tag{11}$$

Proof. By using the similar method presented in the proof of Lemma 2, the inequality (11) can be easily obtained. Thus, it is omitted. □

Lemma 4 [44]. *Let $\zeta \in \mathbb{R}^n, \Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{m\times n}$ such that $\mathrm{rank}(B) < n$. Then, the following two statements are equivalent:*

(1) $\zeta^T\Phi\zeta < 0, \quad B\zeta = 0, \quad \zeta \neq 0,$
(2) $(B^\perp)^T\Phi B^\perp < 0$, *where $B^\perp$ is a right orthogonal complement of B.*

Lemma 5 [45]. *For the symmetric appropriately dimensional matrices $\Omega > 0, \Xi$, matrix Λ, the following two statements are equivalent:*

(1) $\Xi - \Lambda^T\Omega\Lambda < 0$,
(2) *There exists a matrix of appropriate dimension Ψ such that*

$$\begin{bmatrix} \Xi + \Lambda^T\Psi + \Psi^T\Lambda & \Psi^T \\ \Psi & -\Omega \end{bmatrix} < 0. \tag{12}$$

3. Main results

In this section, a new $\mathcal{H}_\infty$ performance and stability analysis for system (4) with interval time-varying delays will be introduced. For simplicity of matrix representation, $e_i (i = 1, \ldots, 12) \in \mathbb{R}^{(12n+l+m)\times n}, e_{13} \in \mathbb{R}^{(12n+l+m)\times l}$, and $e_{14} \in \mathbb{R}^{(12n+l+m)\times m}$ are defined as block entry matrices. For example,

$$e_3 = \Big[0_n, 0_n, I_n, \underbrace{0_n \ldots, 0_n}_{9}, 0_{n\times l}, 0_{n\times m}\Big]^T,$$
$$e_{13} = \Big[\underbrace{0_{l\times n}, \ldots, 0_{l\times n}}_{12}, I_{l\times l}, 0_{l\times m}\Big]^T,$$
$$e_{14} = \Big[\underbrace{0_{m\times n}, \ldots, 0_{m\times n}}_{12}, 0_{m\times l}, I_{m\times m}\Big]^T \tag{13}$$

And some of vectors and matrices are defined as

$$\zeta(t) = col\Big\{x(t), x(t-h(t)),\ x(t-h_L), x(t-h_U), \dot{x}(t), \dot{x}(t-h_L), \dot{x}(t-h_U), \int_{t-h_L}^{t} x(s)ds, \int_{t-h(t)}^{t-h_L} x(s)ds, \int_{t-h_U}^{t-h(t)} x(s)ds,$$
$$\int_{t-h_L}^{t}\int_{s}^{t} x(u)duds, \int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L} x(u)duds, p(t), w(t)\Big\},$$

$$\alpha(t) = col\Big\{x(t), x(t-h_L), x(t-h_U), \int_{t-h_L}^{t} x(s)ds, \int_{t-h_U}^{t-h_L} x(s)ds, \int_{t-h_L}^{t}\int_{s}^{t} x(u)duds, \int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L} x(u)duds\Big\},$$

$$\beta(t) = col\{x(t), \dot{x}(t)\},$$

$$\mathcal{P}_1 = \begin{bmatrix} P_3 & P_1 & 0_n \\ P_1 & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix}, \quad \mathcal{P}_2 = \begin{bmatrix} P_3 & P_2 & 0_n \\ P_2 & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix}, \quad \mathcal{P}_3 = \begin{bmatrix} 0_n & P_3 \\ P_3 & 0_n \end{bmatrix},$$

$$\Xi_1 = Sym\Big\{[e_1, e_3, e_4, e_8, e_9 + e_{10}, e_{11}, e_{12}]\mathcal{R}[e_5, e_6, e_7, e_1 - e_3, e_3 - e_4, h_L e_1 - e_8, (h_U - h_L)e_3 - e_9 - e_{10}]^T\Big\},$$
$$\Xi_2 = [e_1, e_5]\mathcal{N}[e_1, e_5]^T + [e_3, e_6](-\mathcal{N} + \mathcal{M})[e_3, e_6]^T - [e_4, e_7]\mathcal{M}[e_4, e_7]^T,$$
$$\Xi_3 = e_3 G_{11} e_3^T - (1 - h_D)[e_2, e_3 - e_2]\mathcal{G}[e_2, e_3 - e_2]^T + Sym\{e_9(G_{12} - G_{22})e_6^T\},$$

$$\Xi_4 = h_L^2[e_1, e_5]\begin{bmatrix} Q_{1,11} & Q_{1,12} \\ \star & Q_{1,22} \end{bmatrix}[e_1, e_5]^T - [e_8, e_1 - e_3, h_L e_1 - e_8]\mathcal{Q}_1[e_8, e_1 - e_3, h_L e_1 - e_8]^T$$
$$+ h_L Sym\Big\{e_{11}Q_{1,13}e_5^T + (h_L e_1 - e_8)Q_{1,23}e_5^T + ((h_L^2/2)e_1 - e_{11})Q_{1,33}e_5^T\Big\},$$

$$\Xi_5 = (h_U - h_L)^2[e_3, e_6]\begin{bmatrix} Q_{2,11} & Q_{2,12} \\ \star & Q_{2,22} \end{bmatrix}[e_3, e_6]^T$$
$$+ (h_U - h_L)Sym\Big\{e_{12}Q_{2,13}e_6^T + ((h_U - h_L)e_3 - e_9 - e_{10})Q_{2,23}e_6^T + (((h_U - h_L)^2/2)e_3 - e_{12})Q_{2,33}e_6^T\Big\},$$

$$\Xi_6 = ((h_L)^2/2)^2[e_1, e_5]\mathcal{Q}_3[e_1, e_5]^T - [e_{11}, h_Le_1 - e_8]\mathcal{Q}_3[e_{11}, h_Le_1 - e_8]^T,$$

$$\begin{aligned}\Xi_7 =&((h_U - h_L)^2/2)^2[e_3, e_6]\mathcal{Q}_4[e_3, e_6]^T - [e_{12}, (h_U - h_L)e_3 - e_9 - e_{10}]\\ &\times \left(\mathcal{Q}_4 + (4/(h_U - h_L))\begin{bmatrix} 0_n & P_3 \\ P_3 & 0_n \end{bmatrix}\right)[e_{12}, (h_U - h_L)e_3 - e_9 - e_{10}]^T,\end{aligned}$$

$$\begin{aligned}\Xi_8 = &(h_L^3/6)^2e_5Q_5e_5^T - \left[(h_L^2/2)e_1 - e_{11}\right]Q_5\left[(h_L^2/2)e_1 - e_{11}\right]^T + ((h_U - h_L)^3/6)^2e_6Q_6e_6^T\\ &- \left[((h_U - h_L)^2/2)e_3 - e_{12}\right]Q_6\left[((h_U - h_L)^2/2)e_3 - e_{12}\right]^T,\end{aligned}$$

$$\Xi_9 = e_3((h_U - h_L)P_1)e_3^T - e_2((h_U - h_L)P_1)e_2^T + e_2((h_U - h_L)P_2)e_2^T - e_4((h_U - h_L)P_2)e_4^T + e_3((h_U - h_L)^2P_3)e_3^T,$$

$$\Xi_{10} = \varepsilon((e_1E^T + e_2E_D^T)(e_1E^T + e_2E_D^T)^T - e_{13}e_{13}^T),$$

$$\begin{aligned}\Xi_{11} = &((e_1C^T + e_2C_D^T + e_{14}B_2^T)(e_1C^T + e_2C_D^T + e_{14}B_2^T)^T - \gamma e_{14}e_{14}^T), \Gamma = [A, A_d, 0_n, 0_n, -I_n, 0_n, 0_n, 0_n, 0_n, 0_n, 0_n, 0_n, D, B_1], \Omega_{[h(t)]}\\ &= Sym\{(h(t) - h_L)e_3G_{22}e_6^T\},\end{aligned}$$

$$\Lambda_{[h(t)]} = \begin{bmatrix} 0_n & I_n & (h(t) - h_L)I_n & 0_n & 0_n & (h_U - h(t))I_n \\ 0_n & -I_n & 0_n & 0_n & I_n & 0_n \\ 0_n & 0_n & 0_n & 0_n & -I_n & 0_n \\ I_n & 0_n & -I_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & I_n & 0_n & -I_n \end{bmatrix}^T \times [e_3, e_2, e_4, e_9, e_{10}]^T. \tag{14}$$

Now, we have the following theorem.

Theorem 1. *For given scalars $h_U > h_L > 0$, and h_D, system (4) is asymptotically stable with $\mathcal{H}_\infty$ performance γ for $h_L \leqslant h(t) \leqslant h_U$ and $\dot{h}(t) \leqslant h_D$ if there exist positive scalar ε, positive definite matrices $\mathcal{R} \in \mathbb{R}^{7n\times 7n}$, $\mathcal{N} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{M} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{G} = [G_{ij}]_{2\times 2} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{Q}_1 = [Q_{1,ij}]_{3\times 3} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_2 = [Q_{2,ij}]_{3\times 3} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_3 \in \mathbb{R}^{2n\times 2n}$, $\mathcal{Q}_4 \in \mathbb{R}^{2n\times 2n}$, $Q_5 \in \mathbb{R}^{n\times n}$, $Q_6 \in \mathbb{R}^{n\times n}$, any matrices $\mathcal{S} \in \mathbb{R}^{3n\times 3n}$, $\Psi \in \mathbb{R}^{6n\times(11n+l+m)}$, and any symmetric matrices $P_i \in \mathbb{R}^{n\times n} (i = 1, 2, 3)$ satisfying the following two LMIs:*

$$\begin{bmatrix} \begin{pmatrix} (\Gamma^\perp)^T\left(\sum_{i=1}^{11}\Xi_i + \Omega_{[h_L]}\right)\Gamma^\perp \\ +Sym\{(\Gamma^\perp)^T(\Lambda_{[h_L]})^T\Psi\} \end{pmatrix} & \Psi^T \\ \Psi & -\begin{pmatrix} \mathcal{Q}_2 + \mathcal{P}_1 & S \\ S^T & \mathcal{Q}_2 + \mathcal{P}_2 \end{pmatrix} \end{bmatrix} < 0, \tag{15}$$

$$\begin{bmatrix} \begin{pmatrix} (\Gamma^\perp)^T\left(\sum_{i=1}^{11}\Xi_i + \Omega_{[h_U]}\right)\Gamma^\perp \\ +Sym\{(\Gamma^\perp)^T(\Lambda_{[h_U]})^T\Psi\} \end{pmatrix} & \Psi^T \\ \Psi & -\begin{pmatrix} \mathcal{Q}_2 + \mathcal{P}_1 & S \\ S^T & \mathcal{Q}_2 + \mathcal{P}_2 \end{pmatrix} \end{bmatrix} < 0, \tag{16}$$

$$\mathcal{Q}_4 + (4/(h_U - h_L))\mathcal{P}_3 \geqslant 0. \tag{17}$$

Proof. For positive definite matrices $\mathcal{R} \in \mathbb{R}^{7n\times 7n}$, $\mathcal{N} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{M} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{G} = [G_{ij}]_{2\times 2} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{Q}_1 = [Q_{1,ij}]_{3\times 3} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_2 = [Q_{2,ij}]_{3\times 3} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_3 \in \mathbb{R}^{2n\times 2n}$, $\mathcal{Q}_4 \in \mathbb{R}^{2n\times 2n}$, $Q_5 \in \mathbb{R}^{n\times n}$, $Q_6 \in \mathbb{R}^{n\times n}$, let us consider the Lyapunov–Krasovskii functional candidate:

$$V(t) = \sum_{i=1}^{8}V_i(t) \tag{18}$$

where

$$\begin{aligned}
V_1(t) &= \alpha^T(t)\mathcal{R}\alpha(t)\\
V_2(t) &= \int_{t-h_L}^{t}\beta^T(s)\mathcal{N}\beta(s)ds + \int_{t-h_U}^{t-h_L}\beta^T(s)\mathcal{M}\beta(s)ds,\\
V_3(t) &= \int_{t-h(t)}^{t-h_L}\begin{bmatrix} x(s)\\ \int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}^T\mathcal{G}\begin{bmatrix} x(s)\\ \int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}ds,\\
V_4(t) &= h_L\int_{t-h_L}^{t}\int_s^t\begin{bmatrix}\beta(u)\\ \int_u^t\dot{x}(v)dv\end{bmatrix}^T\mathcal{Q}_1\begin{bmatrix}\beta(u)\\ \int_u^t\dot{x}(v)dv\end{bmatrix}duds,\\
V_5(t) &= (h_U-h_L)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}\begin{bmatrix}\beta(u)\\ \int_u^{t-h_L}\dot{x}(v)dv\end{bmatrix}^T\mathcal{Q}_2\begin{bmatrix}\beta(u)\\ \int_u^{t-h_L}\dot{x}(v)dv\end{bmatrix}duds,\\
V_6(t) &= \left((h_L)^2/2\right)\int_{t-h_L}^{t}\int_s^t\int_u^t\beta^T(v)\mathcal{Q}_3\beta(v)dvduds,\\
V_7(t) &= \left((h_U-h_L)^2/2\right)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}\int_u^{t-h_L}\beta^T(v)\mathcal{Q}_4\beta(v)dvduds,\\
V_8(t) &= \left((h_L)^3/6\right)\int_{t-h_L}^{t}\int_s^t\int_u^t\int_v^t\dot{x}^T(\lambda)Q_5\dot{x}(\lambda)d\lambda dvduds + \left((h_U-h_L)^3/6\right)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}\int_u^{t-h_L}\int_v^{t-h_L}\dot{x}^T(\lambda)Q_6\dot{x}(\lambda)d\lambda dvduds.
\end{aligned} \tag{19}$$

The time derivative of $V_1(t)$ can be represented as

$$\dot{V}_1(t) = 2\alpha^T(t)\mathcal{R}\dot{\alpha}(t). \tag{20}$$

Here, it should be noted that

$$\dot{\alpha}(t) = \begin{bmatrix}\dot{x}(t)\\ \dot{x}(t-h_L)\\ \dot{x}(t-h_U)\\ x(t)-x(t-h_L)\\ x(t-h_L)-x(t-h_U)\\ h_Lx(t)-\int_{t-h_L}^t x(s)ds\\ (h_U-h_L)x(t-h_L) - \underbrace{\int_{t-h_U}^{t-h_L}x(s)ds}_{\int_{t-h(t)}^{t-h_L}x(s)ds+\int_{t-h_U}^{t-h(t)}x(s)ds}\end{bmatrix} = [e_5, e_6, e_7, e_1-e_3, e_3-e_4, h_Le_1-e_8, (h_U-h_L)e_3-e_9-e_{10}]^T\zeta(t) \tag{21}$$

and

$$\alpha(t) = [e_1, e_3, e_4, e_8, e_9+e_{10}, e_{11}, e_{12}]^T\zeta(t). \tag{22}$$

Thus, $\dot{V}_1(t)$ can be represented as

$$\dot{V}_1(t) = \zeta^T(t)\Xi_1\zeta(t). \tag{23}$$

Calculating $\dot{V}_2(t)$ gives

$$\dot{V}_2(t) = \beta^T(t)\mathcal{N}\beta(t) + \beta^T(t-h_L)(-\mathcal{N}+\mathcal{M})\beta(t-h_L) - \beta^T(t-h_U)\mathcal{M}\beta(t-h_U) = \zeta^T(t)\Xi_2\zeta(t). \tag{24}$$

By the well-known relation [46], $\frac{d}{dt}\int_{a(t)}^{b(t)}f(t,s)ds = \frac{d}{dt}(b(t))f(t,b(t)) - \frac{d}{dt}(a(t))f(t,a(t)) + \int_{a(t)}^{b(t)}\frac{\partial}{\partial t}\{f(t,s)\}ds$, calculation of $\dot{V}_3(t)$ leads to

$$\begin{aligned}
\dot{V}_3(t) = {} & \begin{bmatrix} x(s)\\ \int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}^T\mathcal{G}\begin{bmatrix} x(s)\\ \int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}\Bigg|_{s=t-h_L}\times\frac{d}{dt}(t-h_L) - \begin{bmatrix} x(s)\\ \int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}^T\mathcal{G}\begin{bmatrix} x(s)\\ \int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}\Bigg|_{s=t-h(t)}\times\frac{d}{dt}(t-h(t)) + \int_{t-h(t)}^{t-h_L}\frac{\partial}{\partial t}\\
& \left(\begin{bmatrix} x(s)\\ \int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}^T\mathcal{G}\begin{bmatrix} x(s)\\ \int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}\right)ds \leqslant \begin{bmatrix} x(t-h_L)\\ 0_{n\times 1}\end{bmatrix}^T\mathcal{G}\begin{bmatrix} x(t-h_L)\\ 0_{n\times 1}\end{bmatrix}\\
& -(1-h_D)\begin{bmatrix} x(t-h(t))\\ x(t-h_L)-x(t-h(t))\end{bmatrix}^T\mathcal{G}\begin{bmatrix} x(t-h(t))\\ x(t-h_L)-x(t-h(t))\end{bmatrix} + 2\int_{t-h(t)}^{t-h_L}\begin{bmatrix} x(s)\\ \int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}^T\mathcal{G}
\end{aligned}$$

$$\begin{bmatrix} 0_{n\times 1} \\ \dot{x}(t-h_L) \end{bmatrix} ds = \zeta^T(t)\left(\Xi_3 + \Omega_{[h(t)]}\right)\zeta(t). \tag{25}$$

$\dot{V}_4(t)$ can be obtained as

$$\begin{aligned} \dot{V}_4(t) &= \underbrace{h_L\left(\int_s^t \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix} du\right)\Bigg|_{s=t}}_{0} \\ &\times \frac{d}{dt}(t) - h_L\left(\int_s^t \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix} du\right)\Bigg|_{s=t-h_L} \frac{d}{dt}(t-h_L) + h_L \int_{t-h_L}^t \frac{\partial}{\partial t}\left(\int_s^t \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix} du\right) ds. \end{aligned} \tag{26}$$

Here, it should be noted that

$$\begin{aligned} &-h_L\left(\int_s^t \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix} du\right)\Bigg|_{s=t-h_L} \frac{d}{dt}(t-h_L) = -h_L \int_{t-h_L}^t \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix} du \\ &\leqslant -\begin{bmatrix} \int_{t-h_L}^t \beta(s)ds \\ \int_{t-h_L}^t \int_s^t \dot{x}(u)duds \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \int_{t-h_L}^t \beta(s)ds \\ \int_{t-h_L}^t \int_s^t \dot{x}(u)duds \end{bmatrix} = -\begin{bmatrix} \int_{t-h_L}^t x(s)ds \\ x(t)-x(t-h_L) \\ h_L x(t) - \int_{t-h_L}^t x(s)ds \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \int_{t-h_L}^t x(s)ds \\ x(t)-x(t-h_L) \\ h_L x(t) - \int_{t-h_L}^t x(s)ds \end{bmatrix} \\ &= -\zeta^T(t)\left([e_8, e_1-e_3, h_L e_1 - e_8]\mathcal{Q}_1[e_8, e_1-e_3, h_L e_1 - e_8]^T\right)\zeta(t), \end{aligned} \tag{27}$$

$$\begin{aligned} &h_L \int_{t-h_L}^t \frac{\partial}{\partial t}\left(\int_s^t \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix} du\right) ds = h_L \int_{t-h_L}^t \left(\begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}\right)\Bigg|_{u=t} \\ &\times \underbrace{\frac{d}{dt}(t)}_{1} \times ds - h_L \int_{t-h_L}^t \left(\begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}\right)\Bigg|_{u=s} \\ &\times \underbrace{\frac{d}{dt}(s)}_{0} \times ds + h_L \int_{t-h_L}^t \int_s^t \frac{\partial}{\partial t}\left(\begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}\right) duds \\ &= h_L \int_{t-h_L}^t \left(\begin{bmatrix} \beta(t) \\ 0_{n\times 1} \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \beta(t) \\ 0_{n\times 1} \end{bmatrix}\right) ds + h_L \int_{t-h_L}^t \int_s^t 2\begin{bmatrix} \beta(u) \\ \int_u^t \dot{x}(v)dv \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} 0_{2n\times 1} \\ \dot{x}(t) \end{bmatrix} duds \\ &= h_L^2 \beta^T(t)\begin{bmatrix} Q_{1,11} & Q_{1,12} \\ Q_{1,12}^T & Q_{1,22} \end{bmatrix}\beta(t) + 2h_L\left(\int_{t-h_L}^t \int_s^t x(u)duds\right)^T Q_{1,13}\dot{x}(t) + 2h_L\left(h_L x(t) - \int_{t-h_L}^t x(s)ds\right)^T Q_{1,23}\dot{x}(t) \\ &+ 2h_L\left((h_L^2/2)x(t) - \int_{t-h_L}^t \int_s^t x(u)duds\right)^T Q_{1,33}\dot{x}(t) \\ &= \zeta^T(t)\{h_L^2[e_1, e_5]\begin{bmatrix} Q_{1,11} & Q_{1,12} \\ Q_{1,12}^T & Q_{1,22} \end{bmatrix}[e_1, e_5]^T + h_L \times Sym\{e_{11}Q_{1,13}e_5^T + (h_L e_1 - e_8)Q_{1,23}e_5^T + \left((h_L^2/2)e_1 - e_{11}\right)Q_{1,33}e_5^T\}\}\zeta(t). \end{aligned} \tag{28}$$

From (27) to (28), an upper bound of $\dot{V}_4(t)$ can be

$$\dot{V}_4(t) \leqslant \zeta^T(t)\Xi_4\zeta(t). \tag{29}$$

Inspired by the work of [41], the following two zero equalities with symmetric matrices P_1 and P_2 are considered:

$$0 = (h_U - h_L)\left\{x^T(t-h_L)P_1 x(t-h_L) - x^T(t-h(t))P_1 x(t-h(t)) - 2\int_{t-h(t)}^{t-h_L} x^T(s)P_1\dot{x}(s)ds\right\}, \tag{30}$$

$$0 = (h_U - h_L)\left\{x^T(t-h(t))P_2 x(t-h(t)) - x^T(t-h_U)P_2 x(t-h_U) - 2\int_{t-h_U}^{t-h(t)} x^T(s)P_2\dot{x}(s)ds\right\}. \tag{31}$$

Furthermore, the following zero equality with symmetric matrix P_3 are newly introduced:

$$0=(h_U-h_L)\left\{(h_U-h_L)x^T(t-h_L)P_3x(t-h_L)-\int_{t-h(t)}^{t-h_L}x^T(s)P_3x(s)ds-\int_{t-h_U}^{t-h(t)}x^T(s)P_3x(s)ds-2\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}x^T(u)P_3\dot{x}(u)duds\right\}. \tag{32}$$

By summing the three zero equalities in (30)–(32), it can be obtained

$$\begin{aligned}0&=\zeta^T(t)\Xi_9\zeta(t)-2(h_U-h_L)\int_{t-h(t)}^{t-h_L}x^T(s)P_1\dot{x}(s)ds-2(h_U-h_L)\int_{t-h_U}^{t-h(t)}x^T(s)P_2\dot{x}(s)ds-(h_U-h_L)\int_{t-h(t)}^{t-h_L}x^T(s)P_3x(s)ds\\&\quad-(h_U-h_L)\int_{t-h_U}^{t-h(t)}x^T(s)P_3x(s)ds-2(h_U-h_L)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}x^T(u)P_3\dot{x}(u)duds.\end{aligned} \tag{33}$$

By using the similar method presented in (27) and (28), the calculation of $\dot{V}_5(t)$ can be represented as

$$\begin{aligned}\dot{V}_5(t)&=(h_U-h_L)^2\beta^T(t-h_L)\begin{bmatrix}Q_{2,11}&Q_{2,12}\\Q_{2,12}^T&Q_{2,22}\end{bmatrix}\beta(t-h_L)+2(h_U-h_L)\left(\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}x(u)duds\right)^TQ_{2,13}\dot{x}(t-h_L)\\&\quad+2(h_U-h_L)\left((h_U-h_L)x(t-h_L)-\int_{t-h(t)}^{t-h_L}x(s)ds-\int_{t-h_U}^{t-h(t)}x(s)ds\right)^TQ_{2,23}\dot{x}(t-h_L)\\&\quad+2(h_U-h_L)\left(((h_U-h_L)^2/2)x(t-h_L)-\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}x(u)duds\right)^TQ_{2,33}\dot{x}(t-h_L)\\&\quad-(h_U-h_L)\int_{t-h_U}^{t-h_L}\begin{bmatrix}\beta(s)\\\int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}^T\mathcal{Q}_2\begin{bmatrix}\beta(s)\\\int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}ds\\&=\zeta^T(t)\Xi_5\zeta(t)-(h_U-h_L)\int_{t-h_U}^{t-h_L}\begin{bmatrix}\beta(s)\\\int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}^T\mathcal{Q}_2\begin{bmatrix}\beta(s)\\\int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}ds.\end{aligned} \tag{34}$$

With the consideration of the four integral terms in (33), if $\begin{bmatrix}\mathcal{Q}_2+\mathcal{P}_1&\mathcal{S}\\\mathcal{S}^T&\mathcal{Q}_2+\mathcal{P}_2\end{bmatrix}>0$, then the last integral term at Eq. (34) can be estimated by reciprocally confex optimization approach [17] as

$$\begin{aligned}&-(h_U-h_L)\int_{t-h_U}^{t-h_L}\begin{bmatrix}\beta(s)\\\int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}^T\mathcal{Q}_2\begin{bmatrix}\beta(s)\\\int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}ds-2(h_U-h_L)\int_{t-h(t)}^{t-h_L}x^T(s)P_1\dot{x}(s)ds-2(h_U-h_L)\int_{t-h_U}^{t-h(t)}x^T(s)P_2\dot{x}(s)ds\\&-(h_U-h_L)\int_{t-h(t)}^{t-h_L}x^T(s)P_3x(s)ds-(h_U-h_L)\int_{t-h_U}^{t-h(t)}x^T(s)P_3x(s)ds\\&=-(h_U-h_L)\int_{t-h(t)}^{t-h_L}\begin{bmatrix}x(s)\\\dot{x}(s)\\\int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}^T\left(\mathcal{Q}_2+\underbrace{\begin{bmatrix}P_3&P_1&0_n\\P_1&0_n&0_n\\0_n&0_n&0_n\end{bmatrix}}_{\mathcal{P}_1}\right)\begin{bmatrix}x(s)\\\dot{x}(s)\\\int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}ds\\&-(h_U-h_L)\int_{t-h_U}^{t-h(t)}\begin{bmatrix}x(s)\\\dot{x}(s)\\\int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}^T\left(\mathcal{Q}_2+\underbrace{\begin{bmatrix}P_3&P_2&0_n\\P_2&0_n&0_n\\0_n&0_n&0_n\end{bmatrix}}_{\mathcal{P}_2}\right)\begin{bmatrix}x(s)\\\dot{x}(s)\\\int_s^{t-h_L}\dot{x}(u)du\end{bmatrix}ds\\&\leqslant-\begin{bmatrix}\int_{t-h(t)}^{t-h_L}x(s)ds\\x(t-h_L)-x(t-h(t))\\(h(t)-h_L)x(t-h_L)-\int_{t-h(t)}^{t-h_L}x(s)ds\\\int^{t-h(t)}x(s)ds\\x(t-h(t))-x(t-h_U)\\(h_U-h(t))x(t-h_L)-\int_{t-h_U}^{t-h(t)}x(s)ds\end{bmatrix}^T\begin{bmatrix}\mathcal{Q}_2+\mathcal{P}_1&\mathcal{S}\\\mathcal{S}^T&\mathcal{Q}_2+\mathcal{P}_2\end{bmatrix}\times\begin{bmatrix}\int_{t-h(t)}^{t-h_L}x(s)ds\\x(t-h_L)-x(t-h(t))\\(h(t)-h_L)x(t-h_L)-\int_{t-h(t)}^{t-h_L}x(s)ds\\\int^{t-h(t)}x(s)ds\\x(t-h(t))-x(t-h_U)\\(h_U-h(t))x(t-h_L)-\int_{t-h_U}^{t-h(t)}x(s)ds\end{bmatrix}\end{aligned}$$

$$= -\begin{bmatrix} x(t-h_L) \\ x(t-h(t)) \\ x(t-h_U) \\ \int_{t-h(t)}^{t-h_L} x(s)ds \\ \int_{t-h_U}^{t-h(t)} x(s)ds \end{bmatrix}^T \begin{bmatrix} 0_n & I_n & (h(t)-h_L)I_n & 0_n & 0_n & (h_U-h(t))I_n \\ 0_n & -I_n & 0_n & 0_n & I_n & 0_n \\ 0_n & 0_n & 0_n & 0_n & -I_n & 0_n \\ I_n & 0_n & -I_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & I_n & 0_n & -I_n \end{bmatrix}$$
$$\times \begin{bmatrix} \mathcal{Q}_2+\mathcal{P}_1 & \mathcal{S} \\ \mathcal{S}^T & \mathcal{Q}_2+\mathcal{P}_2 \end{bmatrix} \begin{bmatrix} 0_n & I_n & (h(t)-h_L)I_n & 0_n & 0_n & (h_U-h(t))I_n \\ 0_n & -I_n & 0_n & 0_n & I_n & 0_n \\ 0_n & 0_n & 0_n & 0_n & -I_n & 0_n \\ I_n & 0_n & -I_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & I_n & 0_n & -I_n \end{bmatrix}^T \times \begin{bmatrix} x(t-h_L) \\ x(t-h(t)) \\ x(t-h_U) \\ \int_{t-h(t)}^{t-h_L} x(s)ds \\ \int_{t-h_U}^{t-h(t)} x(s)ds \end{bmatrix}$$
$$= -\zeta^T(t)\left(\Lambda_{[h(t)]}\right)^T \begin{bmatrix} \mathcal{Q}_2+\mathcal{P}_1 & \mathcal{S} \\ \mathcal{S}^T & \mathcal{Q}_2+\mathcal{P}_2 \end{bmatrix} \left(\Lambda_{[h(t)]}\right)\zeta(t). \tag{35}$$

From (34) and (35), an upper bound of $\dot{V}_5(t)$ with the four integral term shown in (35) can be

$$\dot{V}_5(t) - 2(h_U-h_L)\int_{t-h(t)}^{t-h_L} x^T(s)P_1\dot{x}(s)ds - 2(h_U-h_L)\int_{t-h_U}^{t-h(t)} x^T(s)P_2\dot{x}(s)ds - (h_U-h_L)\int_{t-h(t)}^{t-h_L} x^T(s)P_3x(s)ds$$
$$-(h_U-h_L)\int_{t-h_U}^{t-h(t)} x^T(s)P_3x(s)ds \leqslant \zeta^T(t)\left(\Xi_5 - \left(\Lambda_{[h(t)]}\right)^T \begin{bmatrix} \mathcal{Q}_2+\mathcal{P}_1 & \mathcal{S} \\ \mathcal{S}^T & \mathcal{Q}_2+\mathcal{P}_2 \end{bmatrix} \left(\Lambda_{[h(t)]}\right)\right)\zeta(t). \tag{36}$$

By utilizing Lemma 2, an upper bound of $\dot{V}_6(t)$ can be estimated as

$$\dot{V}_6(t) \leqslant ((h_L)^2/2)^2\beta^T(t)\mathcal{Q}_3\beta(t) - \begin{bmatrix} \int_{t-h_L}^t \int_s^t x(u)duds \\ h_Lx(t) - \int_{t-h_L}^t x(s)ds \end{bmatrix}^T \mathcal{Q}_3 \begin{bmatrix} \int_{t-h_L}^t \int_s^t x(u)duds \\ h_Lx(t) - \int_{t-h_L}^t x(s)ds \end{bmatrix} = \zeta^T(t)\Xi_6\zeta(t). \tag{37}$$

Calculation of $\dot{V}_7(t)$ leads to

$$\dot{V}_7(t) = ((h_U-h_L)^2/2)^2\beta^T(t-h_L)\mathcal{Q}_4\beta(t-h_L) - ((h_U-h_L)^2/2)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L} \beta^T(u)\mathcal{Q}_4\beta(u)duds. \tag{38}$$

Here, with the consideration of the last term in (33), if the inequality (17) holds, the last integral term at Eq. (38) can be bounded as

$$-((h_U-h_L)^2/2)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L} \beta^T(u)\mathcal{Q}_4\beta(u)duds - 2(h_U-h_L)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L} x^T(u)P_3\dot{x}(u)duds$$
$$= -((h_U-h_L)^2/2)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L} \begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix}^T \left(\mathcal{Q}_4 + (4/(h_U-h_L))\underbrace{\begin{bmatrix} 0_n & P_3 \\ P_3 & 0_n \end{bmatrix}}_{\mathcal{P}_3}\right)\begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix}duds$$
$$\leqslant -\begin{bmatrix} \int_{t-h_U}^{t-h_L}\int_s^{t-h_L} x(u)duds \\ (h_U-h_L)x(t-h_L) - \int_{t-h(t)}^{t-h_L} x(s)ds - \int_{t-h_U}^{t-h(t)} x(s)ds \end{bmatrix}^T (\mathcal{Q}_4 + (4/(h_U-h_L))\mathcal{P}_3)$$
$$\times \begin{bmatrix} \int_{t-h_U}^{t-h_L}\int_s^{t-h_L} x(u)duds \\ (h_U-h_L)x(t-h_L) - \int_{t-h(t)}^{t-h_L} x(s)ds - \int_{t-h_U}^{t-h(t)} x(s)ds \end{bmatrix}$$
$$= -\zeta^T(t)\left([e_{12}, (h_U-h_L)e_3 - e_9 - e_{10}](\mathcal{Q}_4 + (4/(h_U-h_L))\mathcal{P}_3)[e_{12}, (h_U-h_L)e_3 - e_9 - e_{10}]^T\right)\zeta(t). \tag{39}$$

Therefore, an estimation of $\dot{V}_7(t)$ with the term $-2(h_U-h_L)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L} x^T(u)P_3\dot{x}(u)duds$ can be

$$\dot{V}_7(t) - 2(h_U-h_L)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L} x^T(u)P_3\dot{x}(u)duds \leqslant \zeta^T(t)\Xi_7\zeta(t). \tag{40}$$

Lastly, by Lemma 3, an upper bound $\dot{V}_8(t)$ can be

$$\begin{aligned}
\dot{V}_8(t) &\leqslant ((h_L)^3/6)^2\dot{x}^T(t)Q_5\dot{x}(t) - \left(\int_{t-h_L}^{t}\int_{s}^{t}\int_{u}^{t}\dot{x}(v)dvduds\right)^T Q_5\left(\int_{t-h_L}^{t}\int_{s}^{t}\int_{u}^{t}\dot{x}(v)dvduds\right) \\
&+((h_U-h_L)^3/6)^2\dot{x}^T(t-h_L)Q_6\dot{x}(t-h_L) \\
&-\left(\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}\int_{u}^{t-h_L}\dot{x}(v)dvduds\right)^T Q_6\left(\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}\int_{u}^{t-h_L}\dot{x}(v)dvduds\right) \\
&\leqslant ((h_L)^3/6)^2\dot{x}^T(t)Q_5\dot{x}(t) - \left(((h_L)^2/2)x(t) - \int_{t-h_L}^{t}\int_{s}^{t}x(u)duds\right)^T Q_5\left(((h_L)^2/2)x(t) - \int_{t-h_L}^{t}\int_{s}^{t}x(u)duds\right) \\
&+((h_U-h_L)^3/6)^2\dot{x}^T(t-h_L)Q_6\dot{x}(t-h_L) - \left(((h_U-h_L)^2/2)x(t-h_L) - \int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}x(u)duds\right)^T Q_6 \\
&\times\left(((h_U-h_L)^2/2)x(t-h_L) - \int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}x(u)duds\right) = \zeta^T(t)\Xi_8\zeta(t).
\end{aligned} \tag{41}$$

Since the following inequalities holds from Eqs. (2) and (4),

$$p^T(t)p(t) \leqslant q^T(t)q(t), \tag{42}$$

there exists a positive scalar ε satisfying the following inequality:

$$\varepsilon[q^T(t)q(t) - p^T(t)p(t)] = \zeta^T(t)\Xi_{10}\zeta(t) \geqslant 0. \tag{43}$$

Therefore, from Eqs. (18) to (43), an upper bound of $\dot{V}(t) = \sum_{i=1}^{8}\dot{V}_i(t)$ with the addition of (33) can be written as

$$\begin{aligned}
&\dot{V}(t) + \zeta^T(t)\Xi_9\zeta(t) - 2(h_U-h_L)\int_{t-h(t)}^{t-h_L}x^T(s)P_1\dot{x}(s)ds \\
&-2(h_U-h_L)\int_{t-h_U}^{t-h(t)}x^T(s)P_2\dot{x}(s)ds - (h_U-h_L)\int_{t-h(t)}^{t-h_L}x^T(s)P_3x(s)ds \\
&-(h_U-h_L)\int_{t-h_U}^{t-h(t)}x^T(s)P_3x(s)ds - 2(h_U-h_L)\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}x^T(u)P_3\dot{x}(u)duds \\
&\leqslant \zeta^T(t)\left(\sum_{i=1}^{10}\Xi_i + \Omega_{[h(t)]} - (\Lambda_{[h(t)]})^T\begin{bmatrix}\mathcal{Q}_2+\mathcal{P}_1 & \mathcal{S}\\ \mathcal{S}^T & \mathcal{Q}_2+\mathcal{P}_2\end{bmatrix}(\Lambda_{[h(t)]})\right)\zeta(t).
\end{aligned} \tag{44}$$

Now, $\mathcal{H}_\infty$ performance for system (4) such that $\sup_{w\in\mathcal{L}_{2+}}(\|z(t)\|_2/\|w(t)\|) < \gamma$ will be discussed. As is well known, the $\mathcal{H}_\infty$ performance and stability analysis can be derived from the condition $\dot{V}(t) + z^T(t)z(t) - \gamma^2w^T(t)w(t) < 0$. It should be noted that $z^T(t)z(t) - \gamma^2w^T(t)w(t) = \zeta^T(t)\Xi_{11}\zeta(t)$. Thus, the condition $\dot{V}(t) + z^T(t)z(t) - \gamma^2w^T(t)w(t) < 0$ with zero equality (33) can be written as

$$\zeta^T(t)\left(\sum_{i=1}^{11}\Xi_i + \Omega_{[h(t)]} - (\Lambda_{[h(t)]})^T\begin{bmatrix}\mathcal{Q}_2+\mathcal{P}_1 & \mathcal{S}\\ \mathcal{S}^T & \mathcal{Q}_2+\mathcal{P}_2\end{bmatrix}(\Lambda_{[h(t)]})\right)\zeta(t) < 0. \tag{45}$$

By Lemma 4, the inequality (45) with $0 = \Gamma\zeta(t)$ is equivalent to

$$(\Gamma^\perp)^T\left(\sum_{i=1}^{11}\Xi_i + \Omega_{[h(t)]} - (\Lambda_{[h(t)]})^T\begin{bmatrix}\mathcal{Q}_2+\mathcal{P}_1 & \mathcal{S}\\ \mathcal{S}^T & \mathcal{Q}_2+\mathcal{P}_2\end{bmatrix}(\Lambda_{[h(t)]})\right)(\Gamma^\perp) < 0. \tag{46}$$

Then, by Lemma 5, the condition (46) is equivalent to the following condition with any matrix $\Psi \in \mathbb{R}^{6n\times(11n+l+m)}$

$$\begin{bmatrix}
\left(\begin{array}{c}(\Gamma^\perp)^T\left(\sum_{i=1}^{11}\Xi_i + \Omega_{[h(t)]}\right)(\Gamma^\perp)\\ +Sym\left\{(\Gamma^\perp)^T(\Lambda_{[h(t)]})^T\Psi\right\}\end{array}\right) & \Psi^T \\
\Psi & -\begin{bmatrix}\mathcal{Q}_2+\mathcal{P}_1 & \mathcal{S}\\ \mathcal{S}^T & \mathcal{Q}_2+\mathcal{P}_2\end{bmatrix}
\end{bmatrix} < 0. \tag{47}$$

The above condition is affinely dependent on $h(t)$. Therefore, if inequalities (15) and (16) hold, then system (4) is asymptotically stable with $\mathcal{H}_\infty$ performance γ for $h_L \leqslant h(t) \leqslant h_U$. It should be noted that the inequality $\begin{bmatrix} \mathcal{Q}_2+\mathcal{P}_1 & \mathcal{S} \\ \mathcal{S}^T & \mathcal{Q}_2+\mathcal{P}_2 \end{bmatrix} > 0$ utilized in (35) is satisfied if the inequality (15) or (16) hold. This completes our proof. □

Remark 1. The utilized augmented vector $\zeta(t)$ includes not only single integral terms of states but also double integral terms such as $\int_{t-h_L}^{t}\int_{s}^{t} x(u)duds$ and $\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L} x(u)duds$. By defining $\alpha(t)$ as

$$\alpha(t) = col\left\{x(t), x(t-h_L), x(t-h_U), \int_{t-h_L}^{t} x(s)ds, \int_{t-h_U}^{t-h_L} x(s)ds, \int_{t-h_L}^{t}\int_{s}^{t} x(u)duds, \int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L} x(u)duds\right\} \tag{48}$$

and considering $V_1(t) = \alpha^T(t)\mathcal{R}\alpha(t)$ as Lyapunov functional candidate, more information on states were utilized in the criterion presented in Theorem 1.

Remark 2. The main differences of proposed Lyapunov–Krasovskii functionals comparing with the previous ones in the literature are $V_3(t), V_4(t)$ and $V_5(t)$ which have not been proposed yet. By including the integral terms of derivative of state as the integrands of $V_3(t), V_4(t)$ and $V_5(t)$, some cross terms such as

$$2\int_{t-h(t)}^{t-h_L} \begin{bmatrix} x(s) \\ \int_{s}^{t-h_L} \dot{x}(u)du \end{bmatrix}^T \mathcal{G}\begin{bmatrix} 0_{n\times 1} \\ \dot{x}(t-h_L) \end{bmatrix} ds, \tag{49}$$

$$\zeta^T(t) h_L Sym\left\{e_{11}Q_{1,13}e_5^T + (h_L e_1 - e_8)Q_{1,23}e_5^T + ((h_L^2/2)e_1 - e_{11})Q_{1,33}e_5^T\right\}\zeta(t) \tag{50}$$

and

$$\zeta^T(t)\left\{(h_U-h_L)Sym\left(e_{12}Q_{2,13}e_6^T + ((h_U-h_L)e_3 - e_9 - e_{10})Q_{2,13}e_6^T + (((h_U-h_L)^2/2)e_3 - e_{12})Q_{2,13}e_6^T\right)\right\}\zeta(t) \tag{51}$$

are utilized in estimating the time-derivative of $V(t)$.

Remark 3. The two zero equalities at Eqs. (30) and (31) are proposed inspired by the work of [41] and utilized in Theorem 1 to enhance the feasible region of $\mathcal{H}_\infty$ performance criterion. As presented in Eqs. (30) and (31), the quadratic terms such as $(h_U-h_L)(x^T(t-h_L)P_1x(t-h_L) - x^T(t-h(t))P_1x(t-h(t)))$ and $(h_U-h_L)(x^T(t-h(t))P_2x(t-h(t)) - x^T(t-h_U)P_2x(t-h_U))$ play roles to enhance the feasible region of the $\mathcal{H}_\infty$ performance criterion. Also, by merging the two integral terms $-2(h_U-h_L)\int_{t-h(t)}^{t-h_L}\dot{x}^T(s)P_1x(s)ds$ and $-2(h_U-h_L)\int_{t-h_U}^{t-h(t)}\dot{x}^T(s)P_2x(s)ds$ into the terms $-(h_U-h_L)\int_{t-h_U}^{t-h_L}\begin{bmatrix} \beta(s) \\ \int_{s}^{t-h_L}\dot{x}(u)du \end{bmatrix}^T \mathcal{Q}_2 \begin{bmatrix} \beta(s) \\ \int_{s}^{t-h_L}\dot{x}(u)du \end{bmatrix} ds$ as shown in (35), the conservatism of the $\mathcal{H}_\infty$ performance criterion can be reduced. Furthermore, the zero quality (32) is proposed for the first time to increase the feasible region of the criterion. This zero equality can be obtained from the fact $\int_{t-h}^{t}\int_{s}^{t}\dot{f}(u)duds = hf(t) - \int_{t-h}^{t} f(s)ds$ with $f(t) = x^T(t)Px(t)$. The term $-(h_U-h_L)\int_{t-h_U}^{t-h_L} x^T(s)P_3x(s)ds$ is merged into the results of $\dot{V}_5(t)$ and the term $-2(h_U-h_L)\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}\dot{x}^T(u)P_3x(u)duds$ is into the result of $\dot{V}_7(t)$.

Remark 4. In [25,27], improved stability criteria for dynamic systems with interval time-varying delays were proposed by taking tight interval of integral terms (for details, see Remark 1 of [25]). Inspired by the work of [25,27], instead of $\int_{t-h_U}^{t-h_L}\int_{s}^{t} x(u)duds$, the term $\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L} x(u)duds$ is utilized as augmented vector for the first time. Thus, $V_5(t), V_7(t)$, and $V_8(t)$ having the forms of $\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}(\cdot)duds$, $\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}\int_{u}^{t-h_L}(\cdot)d\upsilon duds$ and $\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}\int_{u}^{t-h_L}\int_{\upsilon}^{t-h_L}(\cdot)d\lambda d\upsilon duds$ respectively are proposed.

Remark 5. When h_D is unknown, the corresponding criterion can be easily obtained by deleting $V_3(t)$ in Lyapunov–Krasovskii functional $V(t)$.

As a special case of Theorem 1, when $w(t) = 0$, system (4) can be represented as

$$\begin{aligned} &\dot{x}(t) = Ax(t) + A_d x(t-h(t)) + Dp(t), \\ &p(t) = F(t)q(t), \\ &q(t) = Ex(t) + E_d x(t-h(t), \\ &x(s) = \phi(s), s \in [-h_M, 0]. \end{aligned} \tag{52}$$

For the system mentioned above, a delay dependent stability will be introduced as Corollary 1. Before introducing Corollary 1, let us define $e_i(i = 1, \ldots, 12) \in \mathbb{R}^{(12n+l)\times n}$ and $e_{13} \in \mathbb{R}^{(12n+l)\times l}$ as block entry matrices. And some of vectors are defined as

$$\Gamma = \left[A, A_d, 0_n, 0_n, -I_n, \underbrace{0_n, \ldots, 0_n}_{7}, D\right],$$
$$\zeta(t) = col\{x(t), x(t-h(t)), x(t-h_L), x(t-h_U), \dot{x}(t), \dot{x}(t-h_L), \dot{x}(t-h_U),$$
$$\int_{t-h_L}^{t} x(s)ds, \int_{t-h(t)}^{t-h_L} x(s)ds, \int_{t-h_U}^{t-h(t)} x(s)ds, \int_{t-h_L}^{t}\int_{s}^{t} x(u)duds, \int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L} x(u)duds, p(t)\Big\}. \tag{53}$$

Except e_i, Γ, and $\zeta(t)$, all the notations defined in Eq. (14) will be used in Corollary 1. Now, the following result is given by Corollary 1.

Corollary 1. *For given scalars $h_U > h_L > 0$, and h_D, system (52) is asymptotically stable for $h_L \leqslant h(t) \leqslant h_U$ and $\dot{h}(t) \leqslant h_D$ if there exist positive scalar ε, positive definite matrices $\mathcal{R} \in \mathbb{R}^{7n\times 7n}$, $\mathcal{N} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{M} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{G} = [G_{ij}]_{2\times 2} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{Q}_1 = [Q_{1,ij}]_{3\times 3} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_2 = [Q_{2,ij}]_{3\times 3} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_3 \in \mathbb{R}^{2n\times 2n}$, $\mathcal{Q}_4 \in \mathbb{R}^{2n\times 2n}$, $Q_5 \in \mathbb{R}^{n\times n}$, $Q_6 \in \mathbb{R}^{n\times n}$, any matrices $\mathcal{S} \in \mathbb{R}^{3n\times 3n}$, $\Psi \in \mathbb{R}^{6n\times(11n+l)}$, and any symmetric matrices $P_i \in \mathbb{R}^{n\times n}(i = 1, 2, 3)$ satisfying the following two LMIs:*

$$\begin{bmatrix} \begin{pmatrix} (\Gamma^{\perp})^T\left(\sum_{i=1}^{10}\Xi_i + \Omega_{[h_L]}\right)\Gamma^{\perp} \\ +Sym\{(\Gamma^{\perp})^T(\Lambda_{[h_L]})^T\Psi\} \end{pmatrix} & \Psi^T \\ \Psi & -\begin{pmatrix} \mathcal{Q}_2 + \mathcal{P}_1 & S \\ S^T & \mathcal{Q}_2 + \mathcal{P}_2 \end{pmatrix} \end{bmatrix} < 0, \tag{54}$$

$$\begin{bmatrix} \begin{pmatrix} (\Gamma^{\perp})^T\left(\sum_{i=1}^{10}\Xi_i + \Omega_{[h_U]}\right)\Gamma^{\perp} \\ +Sym\{(\Gamma^{\perp})^T(\Lambda_{[h_U]})^T\Psi\} \end{pmatrix} & \Psi^T \\ \Psi & -\begin{pmatrix} \mathcal{Q}_2 + \mathcal{P}_1 & S \\ S^T & \mathcal{Q}_2 + \mathcal{P}_2 \end{pmatrix} \end{bmatrix} < 0, \tag{55}$$

$$\mathcal{Q}_4 + (4/(h_U - h_L))\mathcal{P}_3 \geqslant 0. \tag{56}$$

Proof. It should be noted that unlike the term $(\Gamma^{\perp})^T\left(\sum_{i=1}^{11}\Xi_i\right)\Gamma^{\perp}$ in (15) of Theorem 1, $(\Gamma^{\perp})^T\left(\sum_{i=1}^{10}\Xi_i\right)\Gamma^{\perp}$ was used in (54) of Corollary 1. With the same Lyapunov–Krasovskii functional and similar method in the proof of Theorem 1, LMIs (54)–(56) can be easily obtained. So, it is omitted. □

4. Numerical examples

In this section, four numerical examples are introduced to show the improvements of the proposed results. In examples, MATLAB, YALMIP 3.0 and SeDuMi 1.3 are used to solve LMI problems.

Example 1. Consider the system (4) with the parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.01 \\ 0.05 \end{bmatrix}, \quad C = [0.1\ 0.2], \quad C_d = [0\ 0], \quad B_2 = 0, \quad E = \begin{bmatrix} 1.6 & 0 \\ 0 & 0.05 \end{bmatrix},$$
$$E_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}. \tag{57}$$

When $\gamma = 1$, and h_D is unknown, maximum delay bounds obtained by Theorem 1 and recent works in the literature is listed in Table 1 as two cases. From the results of Table 1, it can be confirmed that Theorem 1 in case of both $D = 0.5I$ and I enhances the feasible region of $\mathcal{H}_\infty$ performance criterion comparing with the results of [37] and [42]. This shows the proposed Lyapunov–Krasovskii functional and some utilized techniques in Theorem 1 effectively enhance the feasible region of $\mathcal{H}_\infty$ performance criterion.

Table 1
Delay bounds h_U with different h_L for fixed $\gamma = 1$ (Example 1).

Methods	D	h_L					
		0	0.2	0.4	0.6	0.8	1
[37]	$D = 0.5I$	0.7844	0.8791	0.9925	1.1179	1.2518	1.3920
[42]	$D = 0.5I$	1.3768	1.4106	1.4486	1.4900	1.5459	1.618
Theorem 1	$D = 0.5I$	1.5903	1.6060	1.6130	1.6322	1.6766	1.7456
[37]	$D = I$	0.6695	0.7343	0.8118	0.8962	0.9852	1.0784
[42]	$D = I$	1.040	1.0404	1.0411	1.0426	1.0508	1.0794
Theorem 1	$D = I$	1.0736	1.0995	1.0955	1.0986	1.1254	1.174

Example 2. Consider the system (4) having the following system matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 1 \\ -0.375 & -1.15 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad C = [0\ 1], \quad C_d = [-0.375\ 1.15], \quad B_2 = 0, \quad D = E = E_d = 0_2. \tag{58}$$

When h_D is unknown, under the same values of h_L and h_U utilized in [37] and [42], the comparison of minimized $\mathcal{H}_\infty$ performance is conducted in Table 2. It shows that Theorem 1 provides much less γ which means that the feasible region of $\mathcal{H}_\infty$ performance criterion is significantly enhanced, which also shows the advantage of Theorem 1.

Example 3. Consider the following system with interval time-varying delays

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)). \tag{59}$$

When h_D is unknown and $h_L = 0, 0.3, 0.6, 0.9, 1.2, 1.5$, the results of the maximum delay bounds for guaranteeing the system (59) asymptotically stable are shown in Table 3 with the comparison of [5], [30]–[31], and [42]. When h_D is unknown and $h_L = 1, 2, 3, 4, 5$, the maximum delay bounds for the addressed system are also listed in Table 4 with the comparison of [5], [30], and [42]. From Tables 3 and 4, one can see that the proposed Corollary 1 significantly enhance maximum delay bounds as listed in Tables 3 and 4.

Example 4. Consider the following system with interval time-varying delays

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t - h(t)). \tag{60}$$

When $h_D = 0.3$ and $h_L = 0.3, 0.5, 0.8, 1$, the obtained results by Corollary 1 are shown in Table 5. Also, with unknown h_D and various h_L, the obtained maximum delay bounds are included in Table 6. From Tables 5 and 6, it can be surly confirmed that Corollary 1 provides larger delay bounds than those of the recent works [5,30,31] which supports the proposed idea in this paper is very effective in reducing the conservatism of $\mathcal{H}_\infty$ performance and stability criteria.

Table 2
Minimized $\mathcal{H}_\infty$ performance γ with different h_L and h_U (Example 2).

Methods	$h_U = 0.8695$		$h_U = 1$		
	$h_L = 0$	$h_L = 0.5695$	$h_L = 0$	$h_L = 0.5$	$h_L = 0.9$
[37]	6.82	1.26	–	–	3.98
[42]	0.87	0.81	4.05	3.27	2.59
Theorem 1	0.7810	0.7348	2.6153	1.7607	1.8788

Table 3
Upper bounds of time-varying delays with unknown h_D and various $h_L (h_L = 0, 0.3, \ldots, 1.5)$ (Example 3).

Methods	h_L	0	0.3	0.6	0.9	1.2	1.5
[30]	h_U	1.5296	1.5962	1.6992	1.8451	2.0202	2.2145
[31]	h_U	1.7093	1.7867	1.8997	2.0430	2.2013	2.3775
[5] (m = 4)	h_U	1.7994	1.8238	1.8811	1.9848	2.1220	2.2837
[42]	h_U	1.9579	2.0257	2.0819	2.1543	2.2540	2.3824
Corollary 1	h_U	2.2329	2.2571	2.2634	2.2926	2.3666	2.4816

Table 4
Upper bounds of time-varying delays with unknown h_D and various $h_L(h_L = 1, 2, \ldots, 5)$ (Example 3).

Methods	h_L	1	2	3	4	5
[30]	h_U	1.9008	2.5663	3.3408	4.1690	5.0275
[5] (m = 4)	h_U	2.0273	2.5915	3.3010	4.0855	–
[31]	h_U	2.0921	2.6987	3.4186	4.2097	5.0440
Corollary 1	h_U	2.3124	2.7462	3.4263	4.2103	5.0440

Table 5
Upper bounds of time-varying delays with $h_D = 0.3$ and various h_L (Example 4).

Methods	h_L	0.3	0.5	0.8	1
[30]	h_U	2.2634	2.2858	2.3078	2.3167
[5] ($m = 4$)	h_U	2.2923	2.2983	2.3064	2.3110
[31]	h_U	2.2887	2.3094	2.3370	2.3516
Corollary 1	h_U	2.4503	2.4756	2.5069	2.5279

Table 6
Upper bounds of time-varying delays with unknown h_D and various h_L (Example 4).

Methods	h_L	0	0.3	0.5	0.8	1	2	3	4	5
[30]	h_U	0.8720	1.0717	1.2198	1.4558	1.6198	2.4884	3.3403	4.3424	5.2970
[5] ($m = 4$)	h_U	1.0208	1.2043	1.3429	1.5663	1.7228	2.5608	3.4542	4.3787	5.3228
[31]	h_U	1.0462	1.2463	1.3903	1.6177	1.7753	2.6134	3.5046	4.4271	5.3696
Corollary 1	h_U	1.2896	1.4210	1.5240	1.7078	1.8446	2.6344	3.5124	4.4304	5.3709

5. Conclusion

In this paper, $\mathcal{H}_\infty$ performance and stability criteria for linear systems with interval time-varying delays have been proposed by the use of Lyapunov method and LMI framework. In Theorem 1, by constructing the newly augmented Lyapunov–Krasovskii functional shown at Eq. (18) and adding new zero equality in estimating the time-derivative of Lyapunov–Krasovskii functional, it was shown that improved $\mathcal{H}_\infty$ performance can be obtained via two numerical examples. Also, Corollary 1 shows the proposed idea of Theorem 1 can effectively reduce the conservatism of stability criteria by comparing maximum delay bounds with the results of [5,30,31], [37,42] in two numerical examples. By extending the proposed idea of this paper, future works will focus on stability, stabilization and synchronization for various systems such as neural networks, complex networks, multi-agent systems, and so on.

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