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Stability analysis of two-dimensional switched non-linear continuous-time systems

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Abstract: This study addresses the problem of stability analysis of two-dimensional (2-D) switched non-linear continuous-time systems based on the Roesser model. First, two sufficient conditions for asymptotic stability of 2-D switched non-linear systems are established via the methods of common Lyapunov function and multiple Lyapunov functions, respectively. Then, the definitions of dwell time and average dwell time in one-dimensional switched systems are extended to the 2-D case, a stability criterion is proposed for the 2-D switched non-linear system with average dwell time switching. Finally, three numerical examples are provided to illustrate the effectiveness of the proposed results.

1 Introduction

It is well known that stability is one of the important issues in the study of dynamical systems (see [1–5]). Recently, two dimensional (2-D) systems have drawn a lot of attention due to their broad applications in many areas such as multi-dimensional digital filtering, linear image processing, signal processing, and process control [6–8]. Many significant results on stability of 2-D linear systems [9–15] and 2-D non-linear systems [16–22] have been obtained in the literature. To name a few, several Lyapunov theorems for checking asymptotic and exponential stability of 2-D non-linear discrete systems based on the Roesser model were developed in [20], stability of a system described by the time-varying non-linear 2-D Fornasini–Marchesini model was investigated in [21], and a sufficient condition for exponential stability of 2-D non-linear systems described by the continuous-time Roesser model was established in [22].

A switched system is a dynamical system that consists of a finite number of subsystems and a logical rule that orchestrates switching between these subsystems. This class of systems has numerous applications in the control of mechanical systems, the automotive industry, aircraft and air traffic control, switching power converters, and many other fields [23]. Stability is one of the key issues in the study of switched systems. Several methods have been developed for stability analysis of such systems, such as common Lyapunov function method [24], multiple Lyapunov functions method [25], average dwell time approach [26], and so on. Recently, 2-D switched systems in the discrete-time domain have been received growing attention. Some stability results on such systems have been obtained in [27–31]. However, to the best of the authors' knowledge, there is no stability criterion for 2-D switched non-linear continuous-time systems to date, which motivate us to carry out this work.

In this paper, we are interested in stability analysis of 2-D switched non-linear systems in the continuous-time domain. The main contributions of this paper are as follows: (i) a common Lyapunov function is used to derive a sufficient condition for asymptotic stability of 2-D switched non-linear continuous-time systems under arbitrary switching; (ii) by using the multiple Lyapunov functions method, an asymptotic stability condition for the addressed system under a state-dependent switching law is established; (iii) the notion of dwell time and average dwell time for the system is introduced, with the help of the average dwell time approach, an asymptotic stability criterion is developed.

This paper is organised as follows. In Section 2, problem formulation and some necessary definitions are given. In Section 3, some stability results on 2-D switched non-linear continuous-time systems are developed. Three numerical examples are provided to illustrate the effectiveness of the proposed results in Section 4. Concluding remarks are given in Section 5.

Notations: Throughout this paper, the superscript 'T' denotes the transpose. $\|\cdot\|$ denotes the Euclidean norm. R is the set of all real numbers. R^n is the n-dimensional real vector space.

2 Problem formulation and preliminaries

Consider the following 2-D switched non-linear continuous-time system in Roesser model

$$\begin{bmatrix} \dfrac{\partial x^{\mathrm{h}}(t_1,t_2)}{\partial t_1} \\ \dfrac{\partial x^{\mathrm{v}}(t_1,t_2)}{\partial t_2} \end{bmatrix} = f_{\sigma(t_1,t_2)}(x^{\mathrm{h}}(t_1,t_2), x^{\mathrm{v}}(t_1,t_2)), \tag{1}$$

where $x^{\mathrm{h}}(t_1,t_2) \in R^{n_1}$ and $x^{\mathrm{v}}(t_1,t_2) \in R^{n_2}$ are the horizontal and the vertical states, respectively,

$$x(t_1,t_2) = \begin{bmatrix} x^{\mathrm{h}}(t_1,t_2) \\ x^{\mathrm{v}}(t_1,t_2) \end{bmatrix}$$

is the whole state in R^n with $n = n_1 + n_2$. $\sigma(t_1,t_2) : [0,\infty) \times [0,\infty) \to \underline{N} = \{1,2,\ldots,N\}$ is a piecewise right continuous function, called the switching signal. N is the number of subsystems. For each $i \in \underline{N}$,

$$f_i(x^{\mathrm{h}},x^{\mathrm{v}}) = \begin{bmatrix} f_i^{\mathrm{h}}(x^{\mathrm{h}},x^{\mathrm{v}}) \\ f_i^{\mathrm{v}}(x^{\mathrm{h}},x^{\mathrm{v}}) \end{bmatrix} \in R^n$$

is a Lipschitz continuous non-linear function with $f_i^{\mathrm{h}}(0,0) = 0$ and $f_i^{\mathrm{v}}(0,0) = 0$. The state of the system does not jump at switching instants, i.e. the trajectory $x(t_1,t_2)$ is everywhere continuous.

The initial conditions are given by

$$x^{\mathrm{h}}(0,t_2) = \zeta(t_2), \quad x^{\mathrm{v}}(t_1,0) = \varsigma(t_1), \tag{2}$$

where $\zeta(t_2)$ and $\varsigma(t_1)$ are bounded continuous functions satisfying

$$\zeta(t_2) = 0, \quad \forall t_2 > T_2^*, \quad \varsigma(t_1) = 0, \quad \forall t_1 > T_1^*, \tag{3}$$

$T_1^* < \infty$ and $T_2^* < \infty$ are given positive constants.

Remark 1: The switching signal $\sigma(t_1, t_2)$ in system (1) is only dependent upon $t_1 + t_2 = t$ (see [29, 30]). When $N = 1$, system (1) reduces to the 2-D continuous-time system studied in [22].

Remark 2: The condition (3) means that the initial conditions given here are in a finite set.

Definition 1 (Asymptotic stability): System (1) with the initial conditions (2) and (3) is said to be asymptotically stable if $\lim_{t_1+t_2\to\infty} \|x(t_1, t_2)\| = 0$.

Now we present a lemma which will be used in the later section.

Lemma 1: If $f(t-s, s) \geq 0$ is continuous with respect to s and piecewise right continuous with respect to t, and

$$\lim_{t\to\infty} \int_0^t f(t-s, s)\,\mathrm{d}s = 0, \tag{4}$$

then $\lim_{t\to\infty} f(t-s, s) = 0$, $\forall s \in [0, \infty)$.

Proof: Assume that $\lim_{t\to\infty} f(t-s_1, s_1) \neq 0$ or the limit does not exist for some $s_1 \geq 0$, then for every $T > 0$, there exist a $\theta > T$ and $\varepsilon(\theta) > 0$ such that $f(\theta - s_1, s_1) > \varepsilon(\theta)$. Suppose that $f(\theta - s, s)$ is right continuous with respect to s at s_1. According to the integral mean value theorem, there exist s_2, s_3 with $s_3 > s_2 > s_1$ such that

$$\begin{aligned}
\int_0^{s_3} f(\theta - s, s)\,\mathrm{d}s &\geq \int_{s_1}^{s_3} f(\theta - s, s)\,\mathrm{d}s \\
&= f(\theta - s_2, s_2)(s_3 - s_1) \\
&\geq \varepsilon(\theta)(s_3 - s_1) > 0.
\end{aligned}$$

Note that θ can be arbitrarily large and be selected as $\theta > s_3$. As a result

$$\int_0^{\theta} f(\theta - s, s)\,\mathrm{d}s \geq \int_0^{s_3} f(\theta - s, s)\,\mathrm{d}s > 0. \tag{5}$$

Suppose that $f(\theta - s, s)$ is left continuous with respect to s at s_1. there exist s_2, s_3 with $s_3 < s_2 < s_1$ such that

$$\begin{aligned}
\int_0^{s_1} f(\theta - s, s)\,\mathrm{d}s &\geq \int_{s_3}^{s_1} f(\theta - s, s)\,\mathrm{d}s \\
&= f(\theta - s_2, s_2)(s_1 - s_3) \\
&\geq \varepsilon(\theta)(s_1 - s_3) > 0.
\end{aligned}$$

Since θ can be arbitrarily large and be selected as $\theta > s_1$, we have

$$\int_0^{\theta} f(\theta - s, s)\,\mathrm{d}s \geq \int_0^{s_1} f(\theta - s, s)\,\mathrm{d}s > 0. \tag{6}$$

Both (5) and (6) contradict the condition (4). The proof is completed. □

3 Main results

3.1 Common Lyapunov function

For the stability analysis, the first question is whether the switched system is stable when there is no restriction on the switching signals. This problem is usually called stability analysis under arbitrary switching.

Theorem 1: 2-D system (1) with the initial conditions (2) and (3) is asymptotically stable for any switching signals $\sigma(t_1, t_2)$ if there exist continuously differentiable functions $V^{\mathrm{h}} : R^{n_1} \to R$ and $V^{\mathrm{v}} : R^{n_2} \to R$ such that

$$c_1\|x^{\mathrm{h}}(t_1, t_2)\|^2 \leq V^{\mathrm{h}}(x^{\mathrm{h}}(t_1, t_2)) \leq c_2\|x^{\mathrm{h}}(t_1, t_2)\|^2, \tag{7}$$

$$c_3\|x^{\mathrm{v}}(t_1, t_2)\|^2 \leq V^{\mathrm{v}}(x^{\mathrm{v}}(t_1, t_2)) \leq c_4\|x^{\mathrm{v}}(t_1, t_2)\|^2, \tag{8}$$

$$\begin{aligned}
&\frac{\partial V^{\mathrm{h}}(x^{\mathrm{h}})}{\partial x^{\mathrm{h}}} f_i^{\mathrm{h}}(x^{\mathrm{h}}, x^{\mathrm{v}}) + \frac{\partial V^{\mathrm{v}}(x^{\mathrm{v}})}{\partial x^{\mathrm{v}}} f_i^{\mathrm{v}}(x^{\mathrm{h}}, x^{\mathrm{v}}) \\
&\quad \leq -c_5(\|x^{\mathrm{h}}\|^2 + \|x^{\mathrm{v}}\|^2), \quad \forall i \in \underline{N},
\end{aligned} \tag{9}$$

where c_g $(g = 1, \ldots, 5)$ are positive constants.

Proof: Denote

$$\begin{aligned}
W(t) &= \int_{t_1+t_2=t} V(x^{\mathrm{h}}, x^{\mathrm{v}})\,\mathrm{d}s \\
&= \int_{t_1+t_2=t} [V^{\mathrm{h}}(x^{\mathrm{h}}(t_1, t_2)) + V^{\mathrm{v}}(x^{\mathrm{v}}(t_1, t_2))]\,\mathrm{d}s.
\end{aligned}$$

Following the proof line of Theorem 1 in [22], one obtains from (7) to (9) that, for $t \in [T_l,\ T_{l+1})$,

$$\begin{aligned}
W(t) \leq{}& e^{-\tilde{c}(t-T_l)} W(T_l) + e^{-\tilde{c}t} \int_{T_l}^{t} \sqrt{2} e^{\tilde{c}\tau} [V^{\mathrm{h}}(x^{\mathrm{h}}(0, \tau)) \\
&+ V^{\mathrm{v}}(x^{\mathrm{v}}(\tau, 0))]\,\mathrm{d}\tau,
\end{aligned} \tag{10}$$

where T_l is the lth switching instant, $l = 1, 2, \ldots$, and $\tilde{c} = c_5/\max\{c_2, c_4\}$.

Noting that $W(T_l) = W(T_l^-)$, we obtain from (10) that, for $t \in [T_l,\ T_{l+1})$,

$$\begin{aligned}
W(t) \leq{}& e^{-\tilde{c}(t-T_l)} W(T_l) + e^{-\tilde{c}t} \int_{T_l}^{t} \sqrt{2} e^{\tilde{c}\tau} [V^{\mathrm{h}}(x^{\mathrm{h}}(0, \tau)) \\
&+ V^{\mathrm{v}}(x^{\mathrm{v}}(\tau, 0))]\,\mathrm{d}\tau \\
\leq{}& e^{-\tilde{c}(t-T_l)} W(T_l^-) + e^{-\tilde{c}t} \int_{T_l}^{t} \sqrt{2} e^{\tilde{c}\tau} [V^{\mathrm{h}}(x^{\mathrm{h}}(0, \tau)) \\
&+ V^{\mathrm{v}}(x^{\mathrm{v}}(\tau, 0))]\,\mathrm{d}\tau \\
\leq{}& e^{-\tilde{c}(t-T_{l-1})} W(T_{l-1}) \\
&+ e^{-\tilde{c}t} \int_{T_{l-1}}^{t} \sqrt{2} e^{\tilde{c}\tau} [V^{\mathrm{h}}(x^{\mathrm{h}}(0, \tau)) \\
&+ V^{\mathrm{v}}(x^{\mathrm{v}}(\tau, 0))]\,\mathrm{d}\tau \\
\leq{}& \cdots \\
\leq{}& e^{-\tilde{c}(t)} W(0) + e^{-\tilde{c}t} \int_0^t \sqrt{2} e^{\tilde{c}\tau} [V^{\mathrm{h}}(x^{\mathrm{h}}(0, \tau)) \\
&+ V^{\mathrm{v}}(x^{\mathrm{v}}(\tau, 0))]\,\mathrm{d}\tau.
\end{aligned} \tag{11}$$

From the definition of $W(t)$, we have

$$W(t) = \int_0^t \sqrt{2} [V^{\mathrm{h}}(x^{\mathrm{h}}(t-\tau, \tau)) + V^{\mathrm{v}}(x^{\mathrm{v}}(t-\tau, \tau))]\,\mathrm{d}\tau.$$

From (2) and (3), we obtain $V^{\mathrm{h}}(x^{\mathrm{h}}(0, t)) = V^{\mathrm{v}}(x^{\mathrm{v}}(t, 0)) = 0$, $\forall t > T^* = \max\{T_1^*, T_2^*\}$. Since $W(0) = 0$, it follows from (11) that

$$\begin{aligned}
&\int_0^t \sqrt{2} [V^{\mathrm{h}}(x^{\mathrm{h}}(t-\tau, \tau)) + V^{\mathrm{v}}(x^{\mathrm{v}}(t-\tau, \tau))]\,\mathrm{d}\tau \\
&\quad \leq e^{-\tilde{c}t} \int_0^{T^*} \sqrt{2} e^{\tilde{c}\tau} [V^{\mathrm{h}}(x^{\mathrm{h}}(0, \tau)) + V^{\mathrm{v}}(x^{\mathrm{v}}(\tau, 0))]\,\mathrm{d}\tau,
\end{aligned} \tag{12}$$

which implies

$$\lim_{t\to\infty}\int_0^t [V^{\mathrm{h}}(x^{\mathrm{h}}(t-\tau,\tau))+V^{\mathrm{v}}(x^{\mathrm{v}}(t-\tau,\tau))]\,\mathrm{d}\tau = 0. \tag{13}$$

Note that $V^{\mathrm{h}}(x^{\mathrm{h}}(t-\tau,\tau))+V^{\mathrm{v}}(x^{\mathrm{v}}(t-\tau,\tau))\ge 0$ is continuous with respect to τ and piecewise right continuous with respect to t. By Lemma 1, we obtain

$$\lim_{t\to\infty}[V^{\mathrm{h}}(x^{\mathrm{h}}(t-\tau,\tau))+V^{\mathrm{v}}(x^{\mathrm{v}}(t-\tau,\tau))]=0,\quad \forall\tau\in[0,\ \infty), \tag{14}$$

i.e.

$$\lim_{t_1+t_2\to\infty}[V^{\mathrm{h}}(x^{\mathrm{h}}(t_1,t_2))+V^{\mathrm{v}}(x^{\mathrm{v}}(t_1,t_2))]=0.$$

From (7) and (8), we get $\lim_{t_1+t_2\to\infty}\|x(t_1,t_2)\|=0$. This completes the proof. □

Remark 3: We would like to point out that the function $W(t)=\int_{t_1+t_2=t}V(x^{\mathrm{h}},x^{\mathrm{v}})\,\mathrm{d}s$ instead of $V(x^{\mathrm{h}},x^{\mathrm{v}})$ is utilised as the energy function, which is different from that in the 1-D case. It can be seen from Theorem 1 that if the unidirectional derivative of the Lyapunov function $V(x^{\mathrm{h}},x^{\mathrm{v}})$ [see (9)] is negative, then the energy function $W(t)$ is decreasing and the system is asymptotically stable under arbitrary switching. It implies that the energy stored along the lines $t_1+t_2=t$ of the asymptotically stable system is decreasing.

Remark 4: From (11), we can see that the initial conditions (2) and (3) are necessary for guaranteeing the asymptotic stability of system (1). For those initial states whose norms are bounded above by decreasing exponential functions, i.e. $\|x^{\mathrm{h}}(0,t_2)\|\le\varphi_{\mathrm{h}}e^{-at_2}$ and $\|x^{\mathrm{v}}(t_1,0)\|\le\varphi_{\mathrm{v}}e^{-bt_1}$ with φ_{h}, φ_{v}, a and b being positive constants, a similar result can be obtained.

3.2 Multiple Lyapunov functions

Although the existence of a common Lyapunov function for various subsystems guarantees the asymptotic stability of system (1), such a function is not always possible and might lead to conservative results. Thus, this subsection studies the use of multiple Lyapunov functions.

Theorem 2: If there exist continuously differentiable functions $V_i^{\mathrm{h}}: R^{n_1}\to R$ and $V_i^{\mathrm{v}}: R^{n_2}\to R$, $i\in\underline{N}$, such that

$$c_{1,i}\|x^{\mathrm{h}}(t_1,t_2)\|^2\le V_i^{\mathrm{h}}(x^{\mathrm{h}}(t_1,t_2))\le c_{2,i}\|x^{\mathrm{h}}(t_1,t_2)\|^2, \tag{15}$$

$$c_{3,i}\|x^{\mathrm{v}}(t_1,t_2)\|^2\le V_i^{\mathrm{v}}(x^{\mathrm{v}}(t_1,t_2))\le c_{4,i}\|x^{\mathrm{v}}(t_1,t_2)\|^2, \tag{16}$$

$$\frac{\partial V_i^{\mathrm{h}}(x^{\mathrm{h}})}{\partial x^{\mathrm{h}}}f_i^{\mathrm{h}}(x^{\mathrm{h}},x^{\mathrm{v}})+\frac{\partial V_i^{\mathrm{v}}(x^{\mathrm{v}})}{\partial x^{\mathrm{v}}}f_i^{\mathrm{v}}(x^{\mathrm{h}},x^{\mathrm{v}}) \le -c_{5,i}(\|x^{\mathrm{v}}\|^2+\|x^{\mathrm{v}}\|^2),\quad \forall i\in\underline{N}, \tag{17}$$

where $c_{g,i}$ $(g=1,\ldots,5,\ i\in\underline{N})$ are positive constants, then 2-D system (1) with the initial conditions (2) and (3) is asymptotically stable under the following switching signal

$$\sigma(t_1,t_2)=\arg\min_{i\in\underline{N}}\int_{t_1+t_2=t}[V_i^{\mathrm{h}}(x^{\mathrm{h}}(t_1,t_2)) + V_i^{\mathrm{v}}(x^{\mathrm{v}}(t_1,t_2))]\,\mathrm{d}s. \tag{18}$$

Proof: Denote

$$\tilde W(t)=\int_{t_1+t_2=t}[V^{\mathrm{h}}_{\sigma(t_1,t_2)}(x^{\mathrm{h}}(t_1,t_2)) + V^{\mathrm{v}}_{\sigma(t_1,t_2)}(x^{\mathrm{v}}(t_1,t_2))]\,\mathrm{d}s.$$

Under the switching signal (18), one gets that, at the switching instants $t=T_l$, $l=1,2,\ldots$

$$\tilde W(t)\le\tilde W(t^-). \tag{19}$$

Following the proof line of Theorem 1 in [22], one obtains from (15) to (17) that, for $t\in[T_l,\ T_{l+1})$

$$\tilde W(t)\le e^{-\bar c(t-T_l)}\tilde W(T_l)+e^{-\bar ct}\int_{T_l}^t\sqrt{2}e^{\bar c\tau}[V^{\mathrm{h}}_{\sigma(0,\tau)}(x^{\mathrm{h}}(0,\tau)) + V^{\mathrm{v}}_{\sigma(\tau,0)}(x^{\mathrm{v}}(\tau,0))]\,\mathrm{d}\tau, \tag{20}$$

where $\bar c=\min_{i\in\underline{N}}\{c_{5,i}/\max_{i\in\underline{N}}\{c_{2,i},c_{4,i}\}\}$.

Combining (19) and (20) leads to, for $t\in[T_l,\ T_{l+1})$

$$\begin{aligned}\tilde W(t)&\le e^{-\bar c(t-T_l)}\tilde W(T_l)+e^{-\bar ct}\int_{T_l}^t\sqrt{2}e^{\bar c\tau}[V^{\mathrm{h}}_{\sigma(0,\tau)}(x^{\mathrm{h}}(0,\tau)) + V^{\mathrm{v}}_{\sigma(\tau,0)}(x^{\mathrm{v}}(\tau,0))]\,\mathrm{d}\tau\\ &\le e^{-\bar c(t-T_l)}\tilde W(T_l^-)+e^{-\bar ct}\int_{T_l}^t\sqrt{2}e^{\bar c\tau}[V^{\mathrm{h}}_{\sigma(0,\tau)}(x^{\mathrm{h}}(0,\tau)) + V^{\mathrm{v}}_{\sigma(\tau,0)}(x^{\mathrm{v}}(\tau,0))]\,\mathrm{d}\tau\\ &\le e^{-\bar c(t-T_{l-1})}\tilde W(T_{l-1}) + e^{-\bar ct}\int_{T_{l-1}}^t\sqrt{2}e^{\bar c\tau}[V^{\mathrm{h}}_{\sigma(0,\tau)}(x^{\mathrm{h}}(0,\tau)) + V^{\mathrm{v}}_{\sigma(\tau,0)}(x^{\mathrm{v}}(\tau,0))]\,\mathrm{d}\tau\\ &\le e^{-\bar ct}\int_0^t\sqrt{2}e^{\bar c\tau}[V^{\mathrm{h}}_{\sigma(0,\tau)}(x^{\mathrm{h}}(0,\tau)) + V^{\mathrm{v}}_{\sigma(\tau,0)}(x^{\mathrm{v}}(\tau,0))]\,\mathrm{d}\tau,\end{aligned}$$

which, together with (2) and (3), yields

$$\lim_{t\to\infty}\int_0^t[V^{\mathrm{h}}_{\sigma(t-\tau,\tau)}(x^{\mathrm{h}}(t-\tau,\tau)) + V^{\mathrm{v}}_{\sigma(t-\tau,\tau)}(x^{\mathrm{v}}(t-\tau,\tau))]\,\mathrm{d}\tau=0. \tag{21}$$

Note that the switching signal $\sigma(t_1,t_2)$ in system (1) is only dependent upon $t_1+t_2=t$. It is clear that $V^{\mathrm{h}}_{\sigma(t-\tau,\tau)}(x^{\mathrm{h}}(t-\tau,\tau))+V^{\mathrm{v}}_{\sigma(t-\tau,\tau)}(x^{\mathrm{v}}(t-\tau,\tau))\ge 0$ is continuous with respect to τ and piecewise right continuous with respect to t. According to Lemma 1, we obtain

$$\lim_{t\to\infty}[V^{\mathrm{h}}_{\sigma(t-\tau,\tau)}(x^{\mathrm{h}}(t-\tau,\tau))+V^{\mathrm{v}}_{\sigma(t-\tau,\tau)}(x^{\mathrm{v}}(t-\tau,\tau))]=0,\quad \forall\tau\in[0,\ \infty). \tag{22}$$

Combining (15), (16) and (22) leads to $\lim_{t_1+t_2\to\infty}\|x(t_1,t_2)\|=0$. This completes the proof. □

Remark 5: In Theorem 2, the multiple functions $V_i(x^{\mathrm{h}},x^{\mathrm{v}})=V_i^{\mathrm{h}}(x^{\mathrm{h}})+V_i^{\mathrm{v}}(x^{\mathrm{v}})$, which correspond to each subsystem, are concatenated together to produce a piecewise differentiable Lyapunov function $\tilde W(t)$. Conditions (15)–(17) ensure that the energy function $\tilde W(t)$ is decreasing when the corresponding mode is active, and the condition (18) implies that the energy function $\tilde W(t)$ is non-increasing at each switching instant. This is consistent with the idea of multiple Lyapunov functions in the 1-D case.

Remark 6: When conditions (15)–(17) hold for $V_i^{\mathrm{h}}(x^{\mathrm{h}})=V_j^{\mathrm{h}}(x^{\mathrm{h}})$ and $V_i^{\mathrm{v}}(x^{\mathrm{v}})=V_j^{\mathrm{v}}(x^{\mathrm{v}})$, $\forall i,j\in\underline{N}$, Theorem 2 reduces to Theorem 1.

3.3 Average dwell time

In Theorems 1 and 2, it is required that the energy function at each switching instant is non-increasing for stability of system (1). However, the energy function may be increasing at some switching instants. In this case, if the system switches too frequently, the stability may be lost. Therefore, a natural question is what if we restrict the switching signal to some constrained subclasses. Intuitively, if one switches less frequently, i.e. slow switching, one may trade off the energy increase caused by switching, and maintain stability. These ideas are proved to be reasonable and are captured by concepts like dwell time and average dwell time switchings [26]. In this paper, we will extend the concepts of dwell time and average dwell time developed in the 1-D system to the 2-D case, and give a stability criterion for system (1) with average dwell time switching.

Definition 2 (Dwell time): A positive constant τ_d is called the dwell time of a switching signal $\sigma(t_1, t_2)$ if the time interval between any two consecutive switchings is no smaller than τ_d, i.e. $T_{l+1} - T_l \geq \tau_d$, $\forall l = 0, 1, 2, \ldots$, where $T_0 = 0$.

Definition 3 (Average dwell time): A positive constant τ_a is called the average dwell time for a switching signal $\sigma(t_1, t_2)$ if

$$N_\sigma(t, \tau) \leq N_0 + \frac{t - \tau}{\tau_a} \tag{23}$$

holds for all $0 \leq \tau \leq t$ and a scalar $N_0 \geq 0$, where $N_\sigma(t, \tau)$ denotes the number of mode switches of a given switching signal $\sigma(t_1, t_2)$ from $t_1 + t_2 = \tau$ to $t_1 + t_2 = t$.

Remark 7: Here N_0 is called the chatter bound. Equation (23) means that on average the 'dwell time' between any two consecutive switchings is no smaller than τ_a.

Theorem 3: If there exist continuously differentiable functions $V_i^h : R^{n_1} \to R$ and $V_i^v : R^{n_2} \to R$, $i \in \underline{N}$, such that (15), (16) and

$$\begin{aligned} &\frac{\partial V_i^h(x^h)}{\partial x^h} f_i^h(x^h, x^v) + \frac{\partial V_i^v(x^v)}{\partial x^v} f_i^v(x^h, x^v) \\ &\quad \leq -\lambda(V_i^h(x^h) + V_i^v(x^v)), \end{aligned} \tag{24}$$

$$V_i^h(x^h) + V_i^v(x^v) \leq \mu(V_j^h(x^h) + V_j^v(x^v)), \quad \forall i, j \in \underline{N}, \tag{25}$$

where $\lambda > 0$ and $\mu \geq 1$ are given constants, then 2-D system (1) with the initial conditions (2) and (3) is asymptotically stable for any switching signals $\sigma(t_1, t_2)$ with average dwell time satisfying $\tau_a > \tau_a^* = \ln \mu / \lambda$.

Proof: Denote

$$V(\tau) = \sqrt{2}[V_{\sigma(0,\tau)}^h(x^h(0, \tau)) + V_{\sigma(\tau,0)}^v(x^v(\tau, 0))].$$

Following the proof line of Theorem 2, we can obtain from (15), (16), (24) and (25) that, $\forall t \in [T_l, T_{l+1})$

$$\begin{aligned} \tilde{W}(t) &\leq e^{-\lambda(t - T_l)} \tilde{W}(T_l) + \int_{T_l}^{t} e^{-\lambda(t-\tau)} V(\tau)\, d\tau \\ &\leq \mu e^{-\lambda(t - T_l)} \tilde{W}(T_l^-) + \int_{T_l}^{t} e^{-\lambda(t-\tau)} V(\tau)\, d\tau \\ &\leq \cdots \\ &\leq \mu^{N_\sigma(0,t)} e^{-\lambda t} \tilde{W}(0) + \int_0^{T_1} \mu^{N_\sigma(0,t)} e^{-\lambda(t-\tau)} V(\tau)\, d\tau \\ &\quad + \cdots + \int_{T_l}^{t} \mu^{N_\sigma(T_l,t)} e^{-\lambda(t-\tau)} V(\tau)\, d\tau \\ &\leq \mu^{N_\sigma(0,t)} e^{-\lambda t} \tilde{W}(0) + \int_0^{t} \mu^{N_\sigma(\tau,t)} e^{-\lambda(t-\tau)} V(\tau)\, d\tau, \end{aligned} \tag{26}$$

where $\tilde{W}(t)$ is defined in Theorem 2.

Noting that $\tilde{W}(0) = 0$ and $N_\sigma(t, \tau) \leq N_0 + (t - \tau)/\tau_a$, we have

$$\tilde{W}(t) \leq \int_0^t \mu^{N_0} e^{(\ln \mu/\tau_a - \lambda)(t-\tau)} V(\tau)\, d\tau. \tag{27}$$

Since $\tau_a > \tau_a^* = \ln \mu / \lambda$, there exists a scalar $\lambda^* > 0$ such that

$$\tilde{W}(t) \leq \int_0^t \mu^{N_0} e^{-\lambda^*(t-\tau)} V(\tau)\, d\tau. \tag{28}$$

By taking the initial conditions (2) and (3) into account, it yields $V(\tau) \leq M$, $0 \leq \tau \leq T^*$, and $V(\tau) = 0$, $\tau > T^*$, where M is a positive constant. It follows from (28) that

$$\tilde{W}(t) \leq M \mu^{N_0} (e^{\lambda^* T^*} - 1) e^{-\lambda^* t} / \lambda^*,$$

which gives

$$\begin{aligned} \lim_{t \to \infty} \int_0^t &[V_{\sigma(t-\tau,\tau)}^h(x^h(t - \tau, \tau)) \\ &+ V_{\sigma(t-\tau,\tau)}^v(x^v(t - \tau, \tau))]\, d\tau = 0. \end{aligned} \tag{29}$$

Similar to the final part of the proof of Theorem 2, we obtain from (15), (16) and (29) that $\lim_{t_1+t_2 \to \infty} \|x(t_1, t_2)\| = 0$ holds. The proof is completed. □

Remark 8: Conditions (15), (16) and (24) mean that all subsystems in system (1) are asymptotically stable. Equation (25) implies that the energy function $\tilde{W}(t)$ at switching instants is allowed to be increasing. However, the possible increment will be compensated by the more specific decrement (by limiting the lower bound of average dwell time); therefore, the system energy is decreasing from a whole perspective and the system stability is guaranteed accordingly.

Remark 9: When all conditions in Theorem 3 hold for $\mu = 1$, it yields $V_i^h(x^h) + V_i^v(x^v) = V_j^h(x^h) + V_j^v(x^v)$, $\forall i, j \in \underline{N}$, and $\tau_a > \tau_a^* = 0$, which means that the switching signal $\sigma(t_1, t_2)$ can be arbitrary. Thus, Theorem 1 is a special case of Theorem 3.

4 Numerical examples

In this section, three examples will be utilised to illustrate the effectiveness of the proposed results.

Example 1: Consider a 2-D switched non-linear system as follows:
Subsystem 1

$$\begin{bmatrix} \dfrac{\partial x^h(t_1, t_2)}{\partial t_1} \\ \dfrac{\partial x^v(t_1, t_2)}{\partial t_2} \end{bmatrix} = \begin{bmatrix} x^h(t_1, t_2) x^v(t_1, t_2) - 2x^h(t_1, t_2) \\ -x^v(t_1, t_2) - (x^h(t_1, t_2))^2 \end{bmatrix},$$

Subsystem 2

$$\begin{bmatrix} \dfrac{\partial x^h(t_1, t_2)}{\partial t_1} \\ \dfrac{\partial x^v(t_1, t_2)}{\partial t_2} \end{bmatrix} = \begin{bmatrix} -x^h(t_1, t_2) x^v(t_1, t_2) - x^h(t_1, t_2) \\ (x^h(t_1, t_2))^2 - x^v(t_1, t_2) \end{bmatrix}.$$

Take the Lyapunov function candidate $V(x^h, x^v) = (x^h)^2 + (x^v)^2 = V^h(x^h) + V^v(x^v)$. It is obvious that $V(x^h, x^v)$ satisfies (7), (8), and

$$\frac{\partial V^{\mathrm{h}}(x^{\mathrm{h}})}{\partial x^{\mathrm{h}}} f_1^{\mathrm{h}}(x^{\mathrm{h}}, x^{\mathrm{v}}) + \frac{\partial V^{\mathrm{v}}(x^{\mathrm{v}})}{\partial x^{\mathrm{v}}} f_1^{\mathrm{v}}(x^{\mathrm{h}}, x^{\mathrm{v}})$$
$$= -4(x^{\mathrm{h}})^2 - 2(x^{\mathrm{v}})^2 \le -2((x^{\mathrm{h}})^2 + (x^{\mathrm{v}})^2),$$
$$\frac{\partial V^{\mathrm{h}}(x^{\mathrm{h}})}{\partial x^{\mathrm{h}}} f_2^{\mathrm{h}}(x^{\mathrm{h}}, x^{\mathrm{v}}) + \frac{\partial V^{\mathrm{v}}(x^{\mathrm{v}})}{\partial x^{\mathrm{v}}} f_2^{\mathrm{v}}(x^{\mathrm{h}}, x^{\mathrm{v}})$$
$$= -2(x^{\mathrm{h}})^2 - 2(x^{\mathrm{v}})^2 \le -2((x^{\mathrm{h}})^2 + (x^{\mathrm{v}})^2).$$

By Theorem 1, the system is asymptotically stable for any switching signals $\sigma(t_1, t_2)$.

Example 2: Consider the following 2-D switched non-linear system Subsystem 1

$$\begin{bmatrix} \dfrac{\partial x^{\mathrm{h}}(t_1,t_2)}{\partial t_1} \\ \dfrac{\partial x^{\mathrm{v}}(t_1,t_2)}{\partial t_2} \end{bmatrix} = \begin{bmatrix} 2x^{\mathrm{h}}(t_1,t_2)x^{\mathrm{v}}(t_1,t_2) - (x^{\mathrm{h}}(t_1,t_2)) \\ -x^{\mathrm{v}}(t_1,t_2) - (x^{\mathrm{h}}(t_1,t_2))^2 \end{bmatrix},$$

Subsystem 2

$$\begin{bmatrix} \dfrac{\partial x^{\mathrm{h}}(t_1,t_2)}{\partial t_1} \\ \dfrac{\partial x^{\mathrm{v}}(t_1,t_2)}{\partial t_2} \end{bmatrix} = \begin{bmatrix} -x^{\mathrm{h}}(t_1,t_2)x^{\mathrm{v}}(t_1,t_2) - x^{\mathrm{h}}(t_1,t_2) \\ 2(x^{\mathrm{h}}(t_1,t_2))^2 - (x^{\mathrm{v}}(t_1,t_2)) \end{bmatrix}.$$

For the above system, it is not easy to find a common Lyapunov function satisfying all conditions in Theorem 1, however, we can choose

$$V_1(x^{\mathrm{h}}, x^{\mathrm{v}}) = (x^{\mathrm{h}})^2 + 2(x^{\mathrm{v}})^2 = V_1^{\mathrm{h}}(x^{\mathrm{h}}) + V_1^{\mathrm{v}}(x^{\mathrm{v}}),$$
$$V_2(x^{\mathrm{h}}, x^{\mathrm{v}}) = 2(x^{\mathrm{h}})^2 + (x^{\mathrm{v}})^2 = V_2^{\mathrm{h}}(x^{\mathrm{h}}) + V_2^{\mathrm{v}}(x^{\mathrm{v}}).$$

It is easy to verify that $V_1(x^{\mathrm{h}}, x^{\mathrm{v}})$ and $V_2(x^{\mathrm{h}}, x^{\mathrm{v}})$ satisfy conditions (15) and (16), and

$$\frac{\partial V_1^{\mathrm{h}}(x^{\mathrm{h}})}{\partial x^{\mathrm{h}}} f_1^{\mathrm{h}}(x^{\mathrm{h}}, x^{\mathrm{v}}) + \frac{\partial V_1^{\mathrm{v}}(x^{\mathrm{v}})}{\partial x^{\mathrm{v}}} f_1^{\mathrm{v}}(x^{\mathrm{h}}, x^{\mathrm{v}})$$
$$= -2(x^{\mathrm{h}})^2 - 4(x^{\mathrm{v}})^2 \le -2((x^{\mathrm{h}})^2 + (x^{\mathrm{v}})^2),$$
$$\frac{\partial V_2^{\mathrm{h}}(x^{\mathrm{h}})}{\partial x^{\mathrm{h}}} f_2^{\mathrm{h}}(x^{\mathrm{h}}, x^{\mathrm{v}}) + \frac{\partial V_2^{\mathrm{v}}(x^{\mathrm{v}})}{\partial x^{\mathrm{v}}} f_2^{\mathrm{v}}(x^{\mathrm{h}}, x^{\mathrm{v}})$$
$$= -4(x^{\mathrm{h}})^2 - 2(x^{\mathrm{v}})^2 \le -2((x^{\mathrm{h}})^2 + (x^{\mathrm{v}})^2).$$

According to Theorem 2, the system is asymptotically stable for $\sigma(t_1, t_2) = \mathrm{argmin}_{i=1,2} \int_{t_1+t_2=t} V_i(x^{\mathrm{h}}, x^{\mathrm{v}})\,\mathrm{d}s$.

Example 3: Consider a 2-D switched non-linear system as follows Subsystem 1

$$\begin{bmatrix} \dfrac{\partial x^{\mathrm{h}}(t_1,t_2)}{\partial t_1} \\ \dfrac{\partial x^{\mathrm{v}}(t_1,t_2)}{\partial t_2} \end{bmatrix} = \begin{bmatrix} 2x^{\mathrm{h}}(t_1,t_2)x^{\mathrm{v}}(t_1,t_2) - x^{\mathrm{h}}(t_1,t_2) \\ -x^{\mathrm{v}}(t_1,t_2) - (x^{\mathrm{h}}(t_1,t_2))^2 \end{bmatrix},$$

Subsystem 2

$$\begin{bmatrix} \dfrac{\partial x^{\mathrm{h}}(t_1,t_2)}{\partial t_1} \\ \dfrac{\partial x^{\mathrm{v}}(t_1,t_2)}{\partial t_2} \end{bmatrix} = \begin{bmatrix} -x^{\mathrm{h}}(t_1,t_2)x^{\mathrm{v}}(t_1,t_2) - x^{\mathrm{h}}(t_1,t_2) \\ 4(x^{\mathrm{h}}(t_1,t_2))^2 - x^{\mathrm{v}}(t_1,t_2) \end{bmatrix}.$$

For the above system, it is not easy to find a common Lyapunov function satisfying all conditions in Theorem 1, however, we can select

$$V_1(x^{\mathrm{h}}, x^{\mathrm{v}}) = (x^{\mathrm{h}})^2 + 2(x^{\mathrm{v}})^2 = V_1^{\mathrm{h}}(x^{\mathrm{h}}) + V_1^{\mathrm{v}}(x^{\mathrm{v}}),$$
$$V_2(x^{\mathrm{h}}, x^{\mathrm{v}}) = 4(x^{\mathrm{h}})^2 + (x^{\mathrm{v}})^2 = V_2^{\mathrm{h}}(x^{\mathrm{h}}) + V_2^{\mathrm{v}}(x^{\mathrm{v}}).$$

It can be verified that $V_1(x^{\mathrm{h}}, x^{\mathrm{v}})$ and $V_2(x^{\mathrm{h}}, x^{\mathrm{v}})$ satisfy conditions (15) and (16), and

$$\frac{\partial V_1^{\mathrm{h}}(x^{\mathrm{h}})}{\partial x^{\mathrm{h}}} f_1^{\mathrm{h}}(x^{\mathrm{h}}, x^{\mathrm{v}}) + \frac{\partial V_1^{\mathrm{v}}(x^{\mathrm{v}})}{\partial x^{\mathrm{v}}} f_1^{\mathrm{v}}(x^{\mathrm{h}}, x^{\mathrm{v}})$$
$$= -2(x^{\mathrm{h}})^2 - 4(x^{\mathrm{v}})^2 \le -2V_1(x^{\mathrm{h}}, x^{\mathrm{v}}),$$
$$\frac{\partial V_2^{\mathrm{h}}(x^{\mathrm{h}})}{\partial x^{\mathrm{h}}} f_2^{\mathrm{h}}(x^{\mathrm{h}}, x^{\mathrm{v}}) + \frac{\partial V_2^{\mathrm{v}}(x^{\mathrm{v}})}{\partial x^{\mathrm{v}}} f_2^{\mathrm{v}}(x^{\mathrm{h}}, x^{\mathrm{v}})$$
$$= -8(x^{\mathrm{h}})^2 - 2(x^{\mathrm{v}})^2 \le -2V_2(x^{\mathrm{h}}, x^{\mathrm{v}}),$$
$$V_1(x^{\mathrm{h}}, x^{\mathrm{v}}) \le 4V_2(x^{\mathrm{h}}, x^{\mathrm{v}}),\ V_2(x^{\mathrm{h}}, x^{\mathrm{v}}) \le 4V_1(x^{\mathrm{h}}, x^{\mathrm{v}}).$$

By Theorem 3, the system is asymptotically stable for any switching signals with average dwell time satisfying $\tau_{\mathrm{a}} > \tau_{\mathrm{a}}^* = \ln \mu/\lambda = \ln 4/2$.

5 Conclusions

This paper has dealt with the problem of stability of 2-D switched non-linear continuous-time systems based on the Roesser model. Some results on asymptotic stability have been obtained by using the common Lyapunov function and multiple Lyapunov functions methods. A sufficient condition for asymptotic stability has also been developed via the average dwell time approach. Three examples have been given to show the effectiveness of the proposed results.

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