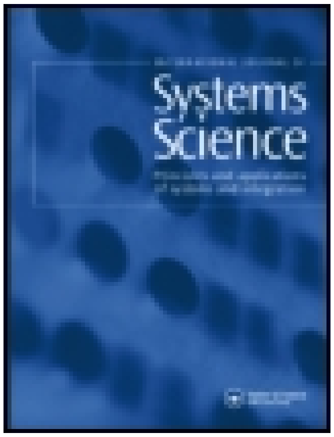

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Global finite-time stabilisation of a class of switched nonlinear systems

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This paper is concerned with the global finite-time stabilisation problem for a class of switched nonlinear systems under arbitrary switchings. All subsystems of the studied switched system under consideration are in lower triangular form. Based on the adding one power integrator technique, both a class of non-Lipschitz continuous state feedback controllers and a common Lyapunov function are simultaneously constructed such that the closed-loop switched system is global finite-time stable under arbitrary switchings. In the controller design process, a common coordinate transformation of all subsystems is exploited to avoid using individual coordinate transformations for subsystems. Finally, two examples are given to show the effectiveness of the proposed method.

Keywords: switched nonlinear systems; global finite-time stabilisation; adding one power integrator; common Lyapunov function

1. Introduction

As an important class of hybrid dynamical systems, the switched systems have become a hot topic due to their significance both in theory and in applications (Ibanez, Suarez-Castanon, & Gutierrez-Frias, 2013; Wang & Zhao, 2013). Considerable effort has been devoted mainly to the analysis and synthesis of switched systems in the last 20 years (Hu & Cheng, 2008; Liu, Liu, & Xie, 2008; Ma, Zhao, & Dimirovski, 2013; Su, Shi, Wu, & Song, 2013; Sun & Wang, 2013; Wu, Shi, Su, & Chu, 2011, 2013; Wu, Su, & Shi, 2012, 2013; Xiang & Xiang, 2009). Many methods, such as a common Lyapunov function (CLF; Ordonez-Hurtado & Duarte-Mermoud, 2012), multiple Lyapunov functions (Branicky, 1998), the average dwell-time method (Zhai, Hu, Yasuda, & Michel, 2001; Zhang & Gao, 2010; Zhang, Han, Zhu, & Huang, 2013), etc., have been proposed to solve the stability and stabilisation problem for switched systems.

Stability under arbitrary switchings is a very desirable property of switched systems. The existence of a CLF for all subsystems was shown to be a necessary and sufficient condition for asymptotic stability of a switched system under arbitrary switchings in Liberzon (2003). A number of conditions have been put forward toward the existence of a CLF guaranteeing the asymptotic stability under arbitrary switchings (Liberzon, 2003). Even though some progress has been made in finding a common quadratic Lyapunov function for a family of subsystems, especially for switched nonlinear systems, such as Sun, Fang, and Huang (2011) and Sun and Wang (2013), it is still an open problem, unless they are in some particular form. For instance, Vu and Liberzon (2005) presented constructions of a CLF for a finite family of pairwise-commuting globally asymptotically stable nonlinear systems. In recent years, switched nonlinear systems in lower triangular form have also drawn considerable attention to study the global stabilisation under arbitrary switchings (Ma & Zhao, 2010; Wu, 2009) and some designed switching signal (Han, Sam Ge, & Heng Lee, 2009; Yu, Zhang, & Fei, 2011).

On the other hand, in many applications, a dynamical system is desired to possess the property that trajectories converge to a Lyapunov stable equilibrium in finite time rather than merely asymptotical. As we all know, the systems with finite-time control may retain not only faster convergence, but also better robustness and disturbance rejection properties (Bhat & Bernstein, 2000; Li, Du, & Lin, 2011). Because of these significant advantages, the finite-time control has been widely used in engineering, for example, in missile systems, communications network systems, robot manipulation, etc. Finite-time stabilisation is one of the most important problems of finite-time control and has been studied in, for example, Huang, Lin, and Yang (2005), Zhang, Feng, and Sun (2012), Khoo, Yin, Man, and Yu (2013), and the references cited therein. Bhat and Bernstein (2000) proposed a Lyapunov stability theorem for finite-time stability analysis of continuous autonomous

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systems, which also provided a basic tool for synthesis of nonlinear control systems. Huang et al. (2005) obtained the finite-time stabilisation result by combining the finite-time stability results with the adding one power integrator technique. Seo, Shim, and Seo (2008) considered the problem of global finite-time stabilisation for a class of triangular nonlinear systems based on backstepping and dynamic exponent scaling. Hong (2002) and Huang et al. (2005) proposed the explicit design scheme of finite-time stable controllers for two classes of nonlinear systems. Hong, Wang, and Cheng (2006) presented the adaptive finite-time control methodology for a class of uncertain nonlinear control systems. A global non-smooth stabilisation scheme is presented in Ding, Li, and Zheng (2012) for a class of nonlinear cascaded systems with uncontrollable linearisation. Naturally, a fundamental and unsolved problem is: *can the finite-time control design for non-switched nonlinear systems be extended to switched nonlinear systems?*

In recent years, the finite-time control problem for switched nonlinear systems has attracted increasing attention because the finite-time control usually demonstrates some desired features. With the help of homogeneous techniques, Orlov (2004) investigated the finite-time stability and robust stability of a class of uncertain switched nonlinear systems. Chen, Yang, and Li (2012) studied the finite-time stability of a class of hybrid dynamical systems with both switching and impulsive effects. The finite-time input-to-state stability problem for switched nonlinear systems was presented in Wang, Hong, and Jiang (2008). Although these results provide some methods for studying the finite-time control for switched nonlinear systems, to the authors' best knowledge, few studies have been carried out for the finite-time stabilisation problem of switched nonlinear systems based on the constructive design method. Therefore, this paper aims to address this problem.

In this paper, we will present a constructive method to design a non-Lipschitz state feedback control law which globally finite-time stabilises a class of switched nonlinear systems in lower triangular form under arbitrary switchings. Such a class of switched systems have attracted a lot of attention in recent years. It is worth pointing out that although some results on global stabilisation under arbitrary switchings for the studied switched systems in this paper have been reported in Wu (2009) and Ma and Zhao (2010) by backstepping, to the authors' best knowledge, no results are available in the literature on finite-time stabilisation for such systems. This is because the non-Lipschitz continuous function should be suggested in designing the common stabilising function, which yields a common coordinate transformation for all subsystems during the iteration process. Hence, such a CLF can be constructed via the common stabilising function. So, how to design such a stabilising function is of great significance, which is solved in this paper by using the adding a power integrator technique to present a systematic approach of constructing such a stabilising function. Then, a recursive design algorithm is developed to construct a global finite-time stabilisation controller as well as a CLF. Compared to the relevant existing results in the literature, this paper has its owns characteristics. First, we are concerned with that the switched nonlinear systems are in lower triangular form where the uncertainties appear in the control channel, which covers more general cases. Second, a sufficient condition on global finite-time stabilisation is derived and a recursive design algorithm for constructing a finite-time stabilisation controller is presented. In addition, a common coordinate transformation at each step of the adding a power integrator technique is employed to avoid using individual coordinate transformations for the subsystems of switched systems.

Notations: The notations used in this paper are standard. Let R denote the field of real numbers and N denote the set of integers. R^n denotes the n-dimensional Euclidean space. A C^1 function means a continuously differentiable function. $B_r(0)$ denotes the ball centred at the origin with radius r. The argument of the functions will be omitted or simplified whenever no confusion arises.

2. Problem formulation and preliminaries

Consider switched nonlinear systems that, after a suitable change of coordinates, can be expressed in the following form:

$$\begin{aligned} \dot{x}_1 &= x_2 + \eta_{1,\sigma(t)}(x, u, t), \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \eta_{n-1,\sigma(t)}(x, u, t), \\ \dot{x}_n &= d(t)u + \eta_{n,\sigma(t)}(x, u, t), \end{aligned} \tag{1}$$

where $x = [x_1, \ldots, x_n]^T \in R^n$ is the state vector, $\bar{x}_i = [x_1, \ldots, x_i]^T$; $\sigma(t) \to M = \{1, \ldots, m\}$ is the switching law, which is a piecewise constant function of time; m is the number of subsystems of the switched system; $u \in R$ is the control input; and $\eta_{i,k}$, $i = 1, \ldots, n$, $\forall k \in M$, are C^1 functions with $\eta_{i,k}(0, 0, t) = 0$, $\forall t \geq 0$. $d(t)$ is an unknown continuous disturbance and/or parameter belonging to a known compact set $\Omega \in R^s$ and $d(t)$ is away from zero, i.e. $d(t) > 0$. Without loss of generality, we assume that there exist constants d_1 and d_2 such that $d_2 \geq d(t) \geq d_1 > 0$. In this paper, we assume that such a switching function $\sigma(t)$ has a finite number of discontinuities on every bounded time interval, and takes a constant value on every interval between two consecutive switching times.

Remark 1: The switched system (1) is more general than many models of the existing results, see e.g. Han et al. (2009), Yu et al. (2011), Wu (2009) and Ma and Zhao (2010). The advantage of the system under consideration is that the uncertainties appear in the channel, which is more realistic. For non-switched nonlinear systems, this structure has been extensively studied (see, e.g. Huang et al., 2005, and references therein).

Now, we give a full characterisation of the switched system (1) via the following assumption.

Assumption 1: *There exist known* C^1 *functions* $\mu_{i,k}(\bar{x}_i) \geq 0$, $i = 1, \ldots, n$, $\forall k \in M$, *such that*

$$\left|\eta_{i,k}(x, u, t)\right| \leq (|x_1| + \cdots + |x_i|)\mu_{i,k}(\bar{x}_i). \quad (2)$$

Remark 2: Assumption 1 is not conservative. In fact, even for non-switched nonlinear systems, Assumption 1 is quite standard in the literatures of global finite-time stabilisation in Huang et al. (2005). In addition, due to the complexity of switched nonlinear systems, this assumption is also reasonable.

In what follows, we review some terminologies and theorems about finite-time stability by considering the autonomous system

$$\dot{x} = f(x); \quad f(0) = 0, \quad x \in R^n. \quad (3)$$

Definition 1 (Bhat & Bernstein, 2000)**:** Consider the nonlinear system (3), where f: $D \to R^n$ is non-Lipschitz continuous on an open neighbourhood D of the origin $x = 0$ in R^n. The equilibrium $x = 0$ of (3) is finite-time convergent if there are an open neighbourhood U of the origin and a function T_x: $U\backslash\{0\} \to (0, \infty)$ such that every solution trajectory $x(t, x_0)$ of (3) starting from the initial point $x_0 \in U\backslash\{0\}$ is well defined and unique in forward time for $t \in [0, T_x(x_0)]$, and $x(t, x_0) \to 0$ as $t \to T_x(x_0)$. Here, $T_x(x_0)$ is called the settling time (of the initial state x_0). The equilibrium $x = 0$ of (3) is finite-time stable if it is Lyapunov stable and finite-time convergent. If $U = D = R^n$, the origin is a globally finite-time stable equilibrium.

The following definition introduces the notion of finite-time stabilisation for switched systems (1) under arbitrary switchings.

Definition 2: The switched system (1) is global finite-time stabilisation under arbitrary switchings if there is, if possible, a control law such that the switched system (1) under arbitrary switchings satisfies the following conditions:

(1) Lyapunov-stable: for $\forall\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for every $x_0 \in B_\delta(0)$, $x(t) \in B_\varepsilon(0)$ for all $t \geq 0$;
(2) Finite-time convergence: $\forall x_0 \in R^n$, if there are $t \in [0, T_x(x_0)]$, such that $x(t, x_0) \to 0$ as $t \to T_x(x_0)$.

Remark 3: It is easy to see that the finite-time stability of the closed-loop switched nonlinear systems can be regarded as a generalisation of one of the non-switched systems (3) in Definition 1. Note that if the switched system (1) is finite-time stable, then it is asymptotically stable, and hence finite-time stability is a stronger notion than asymptotic stability.

Our goal is to design a non-Lipschitz continuous state feedback controller by adding a power integrator to globally finite-time stabilise the switched system (1) under arbitrary switchings.

To achieve our control objective, we introduce the following lemmas.

Lemma 2.1 (Bhat & Bernstein, 2000)**:** *Consider the system* (3). *Suppose these are a* C^1 *function* $V(x)$ *defined in a neighbourhood* $D \subset R^n$ *of the origin, and real numbers* $k > 0$ *and* $\alpha \in (0, 1)$, *such that* (1) $V(x)$ *is positive definite on* D; *and* (2) $\dot{V} + kV^\alpha \leq 0$, $\forall x \in D$. *Then, the origin of the nonlinear system* (3) *is locally finite-time stable. The settling time, depending on the initial state* $x(0) = x_0$, *satisfies* $T_x(x_0) \leq \frac{V(x_0)^{1-\alpha}}{k(1-\alpha)}$ *for all* x_0 *in some open neighbourhood of the origin. If* $D = R^n$ *and* $V(x)$ *is also radially unbounded, the origin of the nonlinear system* (3) *is globally finite-time stable.*

Lemma 2.2 (Lin & Qian, 2000)**:** *For any positive real numbers a, b and any real-valued function* $\omega(x, y) > 0$, $|x|^a|y|^b \leq \frac{a}{a+b}\omega(x, y)|x|^{a+b} + \frac{b}{a+b}\omega(x, y)^{-a/b}|y|^{a+b}$.

Lemma 2.3 (Huang et al., 2005)**:** *For any real numbers* x_i, $i = 1, \ldots, n$, *and* $0 \leq a \leq 1$, $(|x_1| + \cdots + |x_n|)^a \leq |x_1|^a + \cdots + |x_n|^a$. *When* $a = \frac{c}{d} \leq 1$, *where* $c > 0$ *and* $d > 0$ *are odd integers,* $|x^a - y^a| \leq 2^{1-a}|x - y|^a$.

3. Control design for a switched system

The purpose of this section is to show that the global finite-time stabilisation design under arbitrary switchings for the switched system (1) can be solved by a non-Lipschitz continuous state feedback controller via a CLF method. In addition, both a state feedback control law and a CLF can be explicitly constructed by the adding a power integrator technique. In the following, we first construct a CLF and a controller for the switched system (1) and then perform the stability analysis.

Step1. Consider the following collection of first-order subsystems in the switched systems (1):

$$\dot{x}_1 = x_2 + \eta_{1,k}(x, u, t), k = 1, \ldots, m. \quad (4)$$

Define $V_1(x_1) = \frac{1}{2}x_1^2$ and set $c = 4n/(2n + 1)$. With the help of Assumption 1, differentiating $V_1(x_1)$ along the trajectories of all subsystems in (4) gives

$$\begin{aligned}\dot{V}_1(x_1) &= x_1(x_2 + \eta_{1,k}(x, u, t)) \leq x_1x_2 + x_1^2\mu_{1,k}(x_1)\\ &\leq x_1(x_2 - x_2^*) + x_1x_2^* + x_1^c\upsilon_{1,k}, \forall k \in M, \end{aligned} \quad (5)$$

where $\upsilon_{1,k}(x_1) \geq x_1^{2/(2n+1)}\mu_{1,k}(x_1) \geq 0$ is a C^1 function.

Taking x_2 as the virtual control in (4), there exists a continuous feedback controller

$$x_2^* = -x_1^{\rho_2}\varphi_1(x_1), \tag{6}$$

where a C^1 function $\varphi_1(x_1) \geq n + \max_{k\in M}\{\upsilon_{1,k}(x_1)\}$ and $\rho_2 = (2n-1)/(2n+1)$. Similar to Ma and Zhao (2010), such a function x_2^* is called a common stabilisation function in this paper. Note that $\varphi_1(x_1)$ is a function independent of k. Thus, the virtual control x_2^* is also a function independent of k, which will be used to yield a common coordinate transformation of all subsystems in the next step of applying the adding a power integrator technique.

Then, substituting (5) into (6) yields $\dot{V}_1(x_1) \leq -nx_1^c + x_1(x_2 - x_2^*)$.

Inductive step: For the $(i-1)$th step, we will use induction. We assume that, for the following switched system:

$$\dot{x}_j = x_{j+1} + \eta_{j,\sigma(t)}(x,u,t), \quad j = 1,2,\ldots,i-1, \tag{7}$$

there is a C^1, positive-definite and proper function $V_{i-1}(\bar{x}_{i-1})$ satisfying

$$V_{i-1}(\bar{x}_{i-1}) \leq 2\left(\chi_1^2 + \cdots + \chi_{i-1}^2\right), \tag{8}$$

where the common stabilisation functions and the common coordinate transformations are defined by

$$\begin{array}{ll} x_1^* = 0, & \chi_1 = x_1^{1/\rho_1} - x_1^{*1/\rho_1}, \\ x_2^* = -\chi_1^{\rho_2}\varphi_1(x_1), & \chi_2 = x_2^{1/\rho_2} - x_2^{*1/\rho_2}, \\ \vdots & \vdots \\ x_i^* = -\chi_{i-1}^{\rho_i}\varphi_{i-1}(\bar{x}_{i-1}), & \chi_i = x_i^{1/\rho_i} - x_i^{*1/\rho_i} \end{array} \tag{9}$$

with $\rho_1 = 1 > \rho_2 > \cdots > \rho_i \triangleq \frac{2n+3-2i}{2n+1} > 0$, and $\varphi_1(x_1) > 0, \ldots, \varphi_{i-1}(\bar{x}_{i-1}) > 0$ being C^1 functions, such that, for all subsystems of the switched systems (7),

$$\dot{V}_{i-1} \leq \chi_{i-1}^{2-\rho_{i-1}}(x_i - x_i^*) - (n-i+2)\sum_{l=1}^{i-1}\chi_l^c. \tag{10}$$

In the following, our goal is to construct $V_i(\bar{x}_i)$ and a common stabilisation x_{i+1}^* such that the analogue of (8), (9) and (10) holds with i replacing $i-1$.

Define $V_i(\bar{x}_i) = V_{i-1}(\bar{x}_{i-1}) + \Gamma_i(\bar{x}_i)$, where $\Gamma_i(\bar{x}_i) = \int_{x_i^*}^{x_i}\left(\lambda^{1/\rho_i} - x_i^{*1/\rho_i}\right)^{2-\rho_i} d\lambda$.

First, we introduce four propositions for the sake of fulfilling our goal.

Proposition 3.1: *Function $\Gamma_i(\bar{x}_i)$ is C^1. Besides, $\frac{\partial\Gamma_i}{\partial x_i} = \chi_i^{2-\rho_i}$,*

$$\frac{\partial\Gamma_i}{\partial x_l} = -(2-\rho_i)\frac{\partial(x_i^{*1/\rho_i})}{\partial x_l}\int_{x_i^*}^{x_i}\left(\lambda^{1/\rho_i} - x_i^{*1/\rho_i}\right)^{1-\rho_i} d\lambda,$$
$$l = 1,\ldots,i-1. \tag{11}$$

Proposition 3.2: *Function $V_i(\bar{x}_i)$ is a C^1, positive-definite and proper function, which satisfies $V_i(\bar{x}_i) \leq 2(\chi_1^2 + \cdots + \chi_i^2)$.*

Proposition 3.3: *There exist C^1 functions $\tilde{\mu}_{i,k}(\bar{x}_i) \geq 0$, satisfying $\left|\eta_{i,k}(x,u,t)\right| \leq (|\chi_1|^{\rho_i} + \cdots + |\chi_i|^{\rho_i})\tilde{\mu}_{i,k}(\bar{x}_i)$, $i = 1,\ldots,n, \forall k \in M$.*

Proposition 3.4: *There exist non-negative C^1 functions $\Psi_{i,l,k}(\bar{x}_i)$, $l = 1, \ldots, i-1$, satisfying $\left|\frac{\partial(x_i^{*1/\rho_i})}{\partial x_l}\dot{x}_l\right| \leq \Psi_{i,l,k}(\bar{x}_i)\sum_{j=1}^{i}\left|\chi_j\right|^{(2n-1)/(2n+1)}$ for $l = 1, \ldots, i-1$.*

Form Huang et al. (2005), the proofs of Propositions 3.1–3.4 are straightforward and hence left to the reader as an exercise.

With the help of Proposition 3.1, differentiating $V_i(\bar{x}_i)$ along all the trajectories of all subsystems in (1), we obtain that

$$\begin{aligned} \dot{V}_i \leq{} & -(n-i+2)\sum_{l=1}^{i-1}\chi_l^c + \chi_{i-1}^{2-\rho_{i-1}}(x_i - x_i^*) \\ & + \chi_i^{2-\rho_i}(x_{i+1} - x_{i+1}^*) + \chi_i^{2-\rho_i}x_{i+1}^* + \chi_i^{2-\rho_i}\eta_{i,_k} \\ & + \sum_{l=1}^{i-1}\frac{\partial\Gamma_i}{\partial x_l}\dot{x}_l, k = 1,\ldots,m. \end{aligned} \tag{12}$$

In the following, we will estimate the upper bound of each term on the right-hand side of (12).

Using Lemma 2.3 and $\rho_i = \rho_{i-1} - 2/(2n+1)$, we have

$$\begin{aligned} & \left|x_i - x_i^*\right| \leq 2^{1-\rho_i}\left|x_i^{1/\rho_i} - (x_k^*)^{1/\rho_i}\right|^{\rho_i} \leq 2\left|\chi_i\right|^{\rho_i}, \\ & \left|\chi_{i-1}^{2-\rho_{i-1}}(x_i - x_i^*)\right| \leq 2\left|\chi_{i-1}\right|^{2-\rho_{i-1}}\left|\chi_i\right|^{\rho_i} \\ & \leq \frac{\chi_{i-1}^c}{3} + \varepsilon_i\chi_i^c, \end{aligned} \tag{13}$$

where $\varepsilon_i > 0$ is a fixed constant.

Using Lemma 2.2 and Proposition 3.3, for each $k \in M$, the following inequalities hold:

$$\begin{aligned} \left|\chi_i^{2-\rho_i}\eta_{i,_k}(x,u,t)\right| & \leq \left|\chi_i\right|^{2-\rho_i}\sum_{\tau=1}^{i}\left|\chi_\tau\right|^{\rho_i - 2/(2n+1)}\tilde{\mu}_{i,k}(\bar{x}_i) \\ & \leq \frac{1}{3}\sum_{\tau=1}^{i-1}\chi_\tau^c + \chi_i^c\tilde{\zeta}_{i,k}(\bar{x}_i), \end{aligned} \tag{14}$$

where the C^1 function $\tilde{\mu}_{i,k}(\bar{x}_i) \geq 0$, $\tilde{\zeta}_{i,k}(\bar{x}_i) \geq 0$, $\forall \bar{x}_i \in R^i$, $\forall k \in M$.

Now an application of Propositions 3.1 and 3.4 yields that

$$\left|\sum_{l=1}^{i-1} \frac{\partial \Gamma_i}{\partial x_l} \dot{x}_l\right| \leq 2(2-\rho_i)|\chi_i| \sum_{l=1}^{i} |\chi_l|^{(2n-1)/(2n+1)} \times \sum_{l=1}^{i-1} \Psi_{i,l,k}(\bar{x}_i) \leq \frac{1}{3}\sum_{l=1}^{i-1} \chi_l^c + \chi_i^c \hat{\zeta}_{i,k}(\bar{x}_i), \tag{15}$$

where $\hat{\zeta}_{i,k}(\bar{x}_i) \geq 0$ is a C^1 function.

Substituting (13), (14) and (15) into (12) yields

$$\dot{V}_i \leq -(n-i+1)\sum_{l=1}^{i-1} \chi_l^c + \chi_i^{2-\rho_i}(x_{i+1} - x_{i+1}^*) + \chi_i^{2-\rho_i} x_{i+1}^* + \chi_i^c \left(\varepsilon_i + \tilde{\zeta}_{i,k}(\bar{x}_i) + \hat{\zeta}_{i,k}(\bar{x}_i)\right). \tag{16}$$

We choose the common stabilisation function $x_{i+1}^* = -\chi_i^{\rho_{i+1}} \varphi_i(\bar{x}_i)$ with a C^1 function $\varphi_i(\bar{x}_i) \geq n-i+1+\varepsilon_i + \max_{k\in M}\{\tilde{\zeta}_{i,k}(\bar{x}_i) + \hat{\zeta}_{i,k}(\bar{x}_i)\}$ and $0 < \rho_{i+1} \triangleq \rho_i - 2/(2n+1) < \rho_i$. It is easy for us to get $\dot{V}_i \leq -(n-i+1)\sum_{l=1}^{i} \chi_l^c + \chi_i^{2-\rho_i}(x_{i+1} - x_{i+1}^*)$, $\forall k \in M$.

Now, we achieve the proof of the inductive step.

Step n: Using repeatedly the inductive argument above, it is straightforward to see that at the last step (Step n), one can explicitly construct a change of coordinates of the form (20), and a C^1 positive-definite and proper Lyapunov function $V_n(x) = V_{n-1}(\bar{x}_{n-1}) + \Gamma_n(x)$, where $\Gamma_n(x) = \int_{x_n^*}^{x_n} (\lambda^{1/\rho_n} - x_i^{*1/\rho_n})^{2-\rho_n} d\lambda$, satisfying $V_n(x) \leq 2(\chi_1^2 + \cdots + \chi_n^2)$ such that

$$\dot{V}_n \leq -\sum_{l=1}^{n-1} \chi_l^c + \chi_n^{2-\rho_n}\left(d(t)x_{n+1} - d_1 x_{n+1}^*\right) + d_1 \chi_n^{2-\rho_n} x_{n+1}^* + \chi_n^c \left(\varepsilon_n + \tilde{\zeta}_n(x) + \hat{\zeta}_n(x)\right). \tag{17}$$

Then, we choose a static smooth state feedback control law

$$u = x_{n+1}^* = -\chi_n^{\rho_n - 2/(2n+1)}\left(1 + \varepsilon_n + \tilde{\zeta}_n(x) + \hat{\zeta}_n(x)\right)/d_1 \triangleq -\chi_n^{1/(2n+1)} \varphi_n(x)/d_1, \tag{18}$$

which yields $\dot{V}_n \leq = -\sum_{l=1}^{n} \chi_l^c - \chi_n^c(d(t) - d_1)\varphi_n(x)/d_1 \leq -\sum_{l=1}^{n} \chi_l^c$. That is,

$$\dot{V}_n(x) \leq -\left[\chi_1^{4n/(2n+1)} + \cdots + \chi_n^{4n/(2n+1)}\right]. \tag{19}$$

Taking $\alpha = \frac{2n}{2n+1} \in (0, 1)$ and using Lemma 2.3, we have $V_n^\alpha(x) \leq 2\sum_{l=1}^{n} \chi_l^{4n/(2n+1)}$.

Moreover, we can obtain the inequality

$$\dot{V}_n(x) + \frac{1}{4}V_n^\alpha(x) \leq -\frac{1}{2}\sum_{l=1}^{n} \chi_l^{4n/(2n+1)} \leq 0 \tag{20}$$

for each subsystem $k \in M$, and the settling time satisfies $T_x(x_0) \leq \frac{4V(x_0)^{1-\alpha}}{(1-\alpha)}$.

We summarise our main result as follows.

Theorem 3.5: *The close-loop switched system* (1) *with* (18) *is globally finite-time stable under arbitrary switchings with the CLF* $V_n(x)$. *In addition, the settling time satisfies* $T_x(x_0) \leq \frac{4V(x_0)^{1-\alpha}}{1-\alpha}$.

Proof. It follows directly form (19) and (20) . □

4. Simulation studies

In this section, two practical examples are studied to show the effectiveness of the proposed method. One is the continuous stirred-tank reactor (CSTR) with two-mode feed streams and the other is the haptic display system with switched virtual environments, both of which will show potential applications of the proposed method.

4.1. A CSTR system

In this section, we use the proposed method to the CSTR with two-mode feed streams, which is moulded as a switched system and some issues of this switched system have been studied in, e.g. Mhaskar, El-Farra, and Christofides (2005) and Barkhordari Yazdi, Jahed-Motlagh, Attia, and Raisch (2010).

As shown in Ma and Zhao (2010), under a coordinate transformation and smooth feedback, such a CSTR with two-mode feed streams can be changed into the following form:

$$\begin{aligned} \dot{x}_1 &= x_2 + \eta_{1,\sigma(t)}(x_1), \\ \dot{x}_2 &= u, \end{aligned} \tag{21}$$

where $\sigma(t) \to M = \{1, 2\}$, $\eta_{1,1}(x_1) = 0.5x_1$ and $\eta_{1,2}(x_1) = 2x_1$.

For such a switched system (21), the global stabilisation problem under arbitrary switchings has been studied in Ma and Zhao (2010). Here, we will design a controller to address the global finite-time stabilisation problem under arbitrary switchings, which is a stronger property than the global asymptotical stabilisation in Ma and Zhao (2010).

First, consider the x_1-equation of each subsystem and view x_2 as the input. Define $V_1(x_1) = \frac{1}{2}x_1^2$. By Step 1 in the previous section, we can show that $x_2^* = -3x_1^{3/5}(2 + 2x_1^{2/5})$ is a common stabilisation function. In this case, we have $\dot{V}_1(x_1) \leq -2x_1^{8/5} + x_1(x_2 - x_2^*)$.

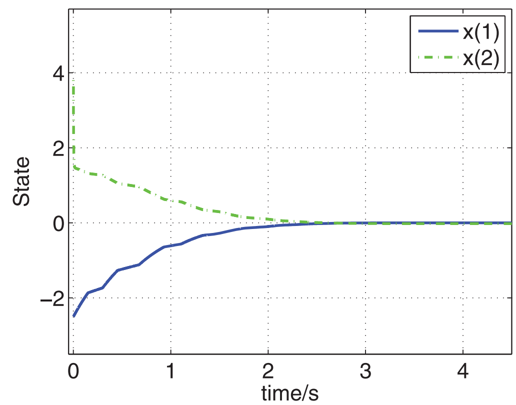

Figure 1. State responses of the system (21)–(22).

Define $V_2(x) = V_1(x_1) + \int_{x_2^*}^{x_2} \left(\lambda^{5/3} - x_2^{*5/3}\right)^{2-3/5} d\lambda$ as a candidate CLF for both subsystems in (21). By (18), we can obtain the finite-time stabilising controller

$$u = -(x_1^{5/3} - x_2^{*5/3})^{1/5} \times (3 + 3(60x_1 + 50x_2 + 6x_2^2 + 10x_1^2)), \tag{22}$$

which results in $\dot{V}_2(x) \leq -\left(x_1^{8/5} + (x_2^{5/3} + x_2^{*5/3})^{8/5}\right)$, and further obtain $\dot{V}_2(x) + \frac{1}{4}V_2^{4/5}(x) \leq -\frac{1}{2}\left(x_1^{8/5} + (x_2^{5/3} + x_2^{*5/3})^{8/5}\right) \leq 0$. Then, according to Theorem 3.5, the closed-loop switched system (21) with (22) is global finite-time stable under arbitrary switchings.

Let $x_1(0) = -2.5$ and $x_2(0) = 3.8$. Figure 1 shows the state trajectories of the closed-loop switched system (21) with (22) under some randomly chosen switching signal shown in Figure 2. It can be seen that the closed-loop switched system is finite-time stable.

It is worth pointing out that the finite-time stabiliser (22) is very different from the asymptotic stabiliser $u = -4(x_2 + 3x_1)$ proposed in Ma and Zhao (2010). Figure 3 shows the simulation result of the controller designed by Ma and Zhao (2010) under the same initial state and the switching signal described in Figure 2. By comparing Figures 1 and

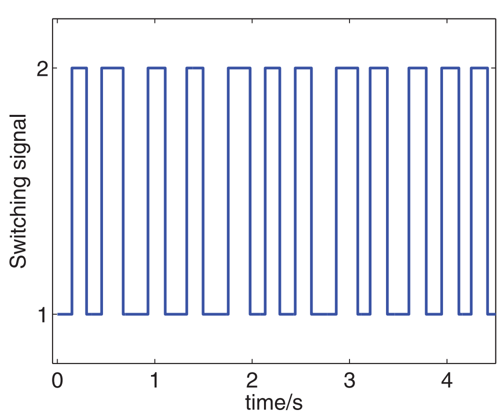

Figure 2. The switching signal.

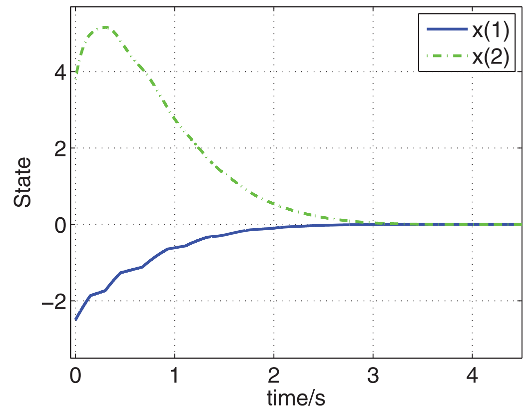

Figure 3. State responses of the system (21) with $u = -4(x_2 + 3x_1)$.

3, one can see that the convergence rate of state responses under our proposed finite-time stabiliser is faster than the one under the controller designed by Ma and Zhao (2010), which obviously shows one of the advantages of the finite-time stabilisation control.

4.2. *A haptic display system*

As shown in Jin, Fu, and Jing (2013) and Mahapatra and Zefran (2003), after some manipulations some haptic display system with switched virtual environments can be modelled as the switched system (1) with the following subsystems:

$$\begin{aligned} \dot{x}_1 &= -12x_1 + x_2, & \dot{x}_1 &= -12x_1 + x_2, \\ \dot{x}_2 &= -x_2 + x_3 & \dot{x}_2 &= -0.5x_2 + x_3, x \\ \dot{x}_3 &= 50x_2 - 0.5x_3 + d(t)u, & \dot{x}_3 &= 20x_2 - x_3 + d(t)u, \end{aligned} \tag{23}$$

where $d(t)$: $d_2 \geq d(t) \geq d_1 > 0$ with $d_1 = 1.5$, $d_2 = 2.5$.

It is easy to obtain that $x_2^* = -3x_1^{5/7}$, $\chi_2 = x_2^{7/5} - x_2^{*7/5}$, $\varepsilon_2 = \frac{5}{12}(\frac{4}{7})^{-\frac{7}{5}}$, $\tilde{\mu}_2 = 3 + 3x_1^{2/7} + \chi_2^{2/7}$, $\tilde{\zeta}_2 = \frac{3}{4}(\frac{4}{3}\tilde{\mu}_2^{-4})^{-\frac{1}{3}} + \tilde{\mu}_2$, $\hat{\zeta}_2 = 7/12(4/5)^{-\frac{5}{7}}\,[18/7\,(15 + 12x_1^{2/7})\,]^{12/7}$, $\varphi_2 = 2 + \varepsilon_2 + \tilde{\zeta}_2 + \hat{\zeta}_2$, $\chi_3 = x_3^{7/3} - x_3^{*7/3}$, $\tilde{\mu}_3 = 150\,(1 + |x_1|^{4/7} + |\chi_2|^{4/7}) + \varphi_2$, $\varepsilon_3 = \frac{1}{4}(\frac{4}{9})^{-9/5}$, $\tilde{\zeta}_3 = \frac{3}{2}(\frac{4}{3})^{-\frac{3}{11}}\tilde{\mu}_3^{12/11} + \tilde{\mu}_3$, $\hat{\zeta}_3 = \frac{7}{6}(\frac{4}{5})^{-5/7}(\frac{22}{7}\Psi_3)^{12/7} + \frac{22}{7}\psi_3$. Finally, we can obtain the following controller by (18): $u = -\frac{2}{3}\chi_3^{1/7}\,[1 + \varepsilon_3 + \tilde{\zeta}_3 + \hat{\zeta}_3]$.

Figure 4 shows the trajectories of the states and the switching signal of the closed-loop switched system (1) with subsystems (23). The switching signal is also generated randomly. From Figure 4, it can be seen that the states of the closed-loop system also converge to zero in finite time. The switching signal is shown in Figure 5.

The above two examples show the effectiveness of the proposed control method.

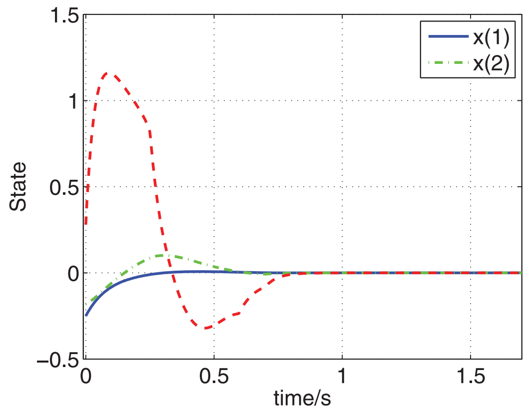

Figure 4. State responses of the switched system (23).

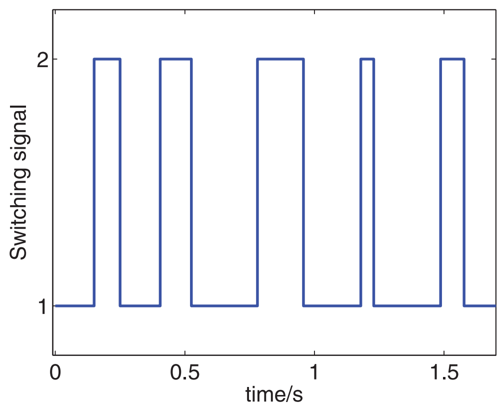

Figure 5. The switching signal.

5. Conclusions

In this paper, we have investigated the global finite-time stabilisation problem for a class of switched nonlinear systems in lower triangular form under arbitrary switchings by the adding one power integrator technique. Both a common state feedback controller independent of switching signals and a CLF are constructed to global finite-time stabilise the switched system under arbitrary switchings. Construction of a common stabilising function plays a crucial role in the method and the underlying idea behind our control design is to adopt a feedback domination approach to control design. Two application examples have been employed to show the effectiveness of the proposed finite-time control design method.

There are relevant problems that need to be investigated in the future. One of such problems is how to generalise the result to a more general class of switched nonlinear systems with parameter uncertainties under arbitrary switchings. Also, the problem of global finite-time stabilisation for a more general class of switched systems whose subsystems are not finite-time stabilisable is a challenging problem and calls for an obvious continuation for future research.

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