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Mathematical and Computer Modelling 43 (2006) 1393–1403

MATHEMATICAL AND COMPUTER MODELLING

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Control parameterization enhancing transform for optimal control of switched systems

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Received 15 August 2005; accepted 17 August 2005

Abstract

In this paper, a class of optimal switching control problems with prespecified order of the sequence of subsystems is considered, where the switching instants are included in the cost functional. Both the switching instants and the control function are to be chosen such that the cost functional is minimized. Through the discretization of the control space, each control component is approximated by a piecewise constant function. The partition points and the heights of each of these piecewise constant functions are taken as decision varibles. Using the control parameterization enhancing transform, we map both types of switching instants into preassigned knot points via the introduction of an additional control, known as the enhancing control. In this way, we construct a sequence of approximate optimal parameter selection problems with fixed switching time points. We then show that these approximate optimal parameter selection problems are solvable as mathematical programming problems. The convergence analysis of this approximation is investigated. Two examples are solved using the proposed method so as to demonstrate the effectiveness of the method proposed.

Keywords: Switched systems; Optimal control computation; Control parameterization; Control parameterization enhancing transform

1. Introduction

A switched system consists of a number of subsystems and a switching law. The switching law is to define the order of the subsystems to be activated at certain specified switching instants during the planning horizon. Switched systems arise in many real world applications, such as the control of mechanical systems, the automotive industry, aircraft and air traffic control, and switching power converters. Some details on these applications can be found in [1].

For problems of the optimal control of switched systems, the objective is to seek a switching law and a control function such that some performance criterion is minimized subject to some constraints on the state and the control variables. These optimal control problems have attracted increasing attention (see, for example, [2–7]) because of their practical significance and theoretical challenge. However, there are many open issues yet to be answered.

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0895-7177/$ - see front matter
doi:10.1016/j.mcm.2005.08.012

For example, even for problems involving only linear subsystems and quadratic costs, a closed form solution of the optimal switching instants is still unavailable.

In this paper, we consider a class of optimal switching problems where the order of the sequence of the subsystems is known, while the switching instants and the control function are to be obtained optimally. Our aim is to provide an efficient computational algorithm for solving this optimal switching problem based on the control parameterization technique and the control parameterization enhancing transform (CPET) introduced in [8,9].

The rest of the paper is organized as follows. The basic problem is formulated in Section 2. In Section 3, each control component is approximated by a piecewise constant function. The switching instants and heights of each of these approximate piecewise constant functions are both taken as decision variables to be chosen optimally. Then, the CPET is applied to this approximate optimal control problem. Subsequently, an equivalent optimal parameter selection problem is obtained, where both types of the switching instants, one for the scheduling of the subsystems and the other for the switching instants of the approximate piecewise constant control function, are mapped into preassigned knot points. This optimal parameter selection problem, which involves multiple characteristic times, is in the form considered in [10]. Thus, the gradient formulae of the cost functional and the constraints are available from [11]. Convergence analysis of the approximation scheme is also included. On the basis of the results obtained in Section 3, an efficient computational method is developed in Section 4. Two examples are solved in Section 5 using the proposed method. Section 6 concludes the paper.

2. Problem formulation

Consider the following switched system, which consists of N subsystems:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f^i(t, x(t), u(t)), \quad t \in [0, T], i \in \{1, 2, \dots, N\} \tag{2.1}$$

$$x(t_0) = x^0, \tag{2.2}$$

where $x = [x_1, x_2, \dots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ and $u = [u_1, u_2, \dots, u_r]^{\mathrm{T}} \in \mathbb{R}^r$ are, respectively, the state and control vectors; $f^i : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$, $i = 1, \dots, N$, and $x^0 = [x_1^0, x_2^0, \dots, x_n^0]^{\mathrm{T}} \in \mathbb{R}^n$ is a given n-vector. Let the switching instants, t_i, $i = 1, \dots, N$, in the sequence be such that

$$0 = t_0, \quad t_{i-1} \le t_i, \quad i = 1, \dots, N, \quad \text{and} \quad t_N = T. \tag{2.3}$$

Define

$$\Lambda = \left\{ t \in \mathbb{R}^{N-1} : 0 = t_0 \le t_1 \le \cdots \le t_{N-1} \le t_N = T \right\}, \tag{2.4}$$

where

$$t = [t_1, t_2, \dots, t_{N-1}]^{\mathrm{T}} \in \mathbb{R}^{N-1}. \tag{2.5}$$

Elements from Λ are to be referred to as switching vectors.

Let

$$U_1 = \left\{ v = [v_1, v_2, \dots, v_r]^{\mathrm{T}} \in \mathbb{R}^r : \left(E^i\right)^{\mathrm{T}} v \le b_i, i = 1, \dots, q \right\}, \tag{2.6}$$

where $E^i \in \mathbb{R}^r$, $i = 1, \dots, q$, and b_i, $i = 1, \dots, q$, are real numbers; and furthermore, let

$$U_2 = \left\{ v = [v_1, v_2, \dots, v_r]^{\mathrm{T}} \in \mathbb{R}^r : \alpha_i \le v_i \le \beta_i, i = 1, \dots, r \right\}, \tag{2.7}$$

where α_i, $i = 1, \dots, r$, and β_i, $i = 1, \dots, r$, are real numbers.

Define

$$U = U_1 \cap U_2. \tag{2.8}$$

Clearly, U is a compact and convex subset of $\mathbb{R}^r$. Let $u = [u_1, u_2, \dots, u_N]^{\mathrm{T}}$ be a Boral measurable function from $[0, T]$ into $\mathbb{R}^r$ such that $u(t) \in U$ for almost all $t \in [0, T]$. Such a u is referred to as an admissible control. Denote by $\mathcal{D}$ the class of all such admissible controls.

Our optimal switching control problem may now be stated formally as:

Problem 1.1. Given the dynamical system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f^i(t, x(t), u(t)), \quad t \in [t_{i-1}, t), i = 1, 2, \ldots, N \tag{2.9a}$$

$$x(0) = x^0 \tag{2.9b}$$

find a control $u \in \mathcal{D}$ and a switching vector $t \in \Lambda$ such that the cost functional

$$J = \Phi_0(x(t_1), x(t_2), \ldots, x(T)) + \int_0^{\mathrm{T}} \mathcal{L}_0(t, x(t), u(t))\mathrm{d}t \tag{2.10}$$

is minimized.

We point out that the cost functional (2.10) includes multiple switching instants. This problem is referred to as the optimal control problem with multiple characteristic time points in the literature. See, for example, [11]. However, these characteristic time points are considered as fixed in [11], while those in Problem 1.1 are decision variables to be chosen optimally.

For the functions Φ_0, $\mathcal{L}_0$, and f^i, $i = 1, 2, \ldots, N$, we assume that the following conditions are satisfied:

(A1) $f^i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$, $i = 1, \ldots, N$,

$$\Phi_0 : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R},$$

where $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ denotes the product of N copies of $\mathbb{R}^n$;

$$\mathcal{L}_0 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R};$$

(A2) for any compact subset $V \subset \mathbb{R}^r$, there exists a positive constant K such that, for $i = 1, \ldots, N$,

$$\left| f^i(t, x, u) \right| \le K \{1 + |x|\}, \tag{2.11}$$

for all $(t, x, u) \in [0, T] \times \mathbb{R}^n \times V$, where $|\cdot|$ denotes a usual Euclidean norm;

(A3) $\mathcal{L}_0$ and f^i, $i = 1, \ldots, N$, together with their partial derivatives with respect to each of the components of x and u, are piecewise continuous on $[0, T]$ for each $(x, u) \in \mathbb{R}^n \times \mathbb{R}^r$ and continuous on $\mathbb{R}^n \times \mathbb{R}^r$ for each $t \in [0, T]$;

(A4) Φ_0 is continuously differentiable with respect to all its arguments.

3. Problem transformation

For each $p_i \ge 1$, $i = 1, 2, \ldots, N$, let the time subinterval $[t_{i-1}, t_i]$ be partitioned into n_{p_i} subintervals with $n_{p_i} + 1$ partition points denoted by

$$\tau_0^{p_i}, \tau_1^{p_i}, \ldots, \tau_{n_{p_i}}^{p_i} \tag{3.1}$$

$$\tau_0^{p_i} = t_{i-1}, \quad \tau_{n_{p_i}}^{p_i} = t_i \quad \text{and} \quad \tau_{k-1}^{p_i} \le \tau_k^{p_i}. \tag{3.2}$$

Let the number n_{p_i} of the partition points be chosen such that

$$n_{p_i+1} \ge n_{p_i}. \tag{3.3}$$

We now approximate the control function in the form of the piecewise constant function as:

$$u^p(t|\sigma^p, \tau^p) = \sum_{i=1}^{N} \sum_{k=1}^{n_{p_i}} \sigma^{p_i,k} \chi_{[\tau_{k-1}^{p_i}, \tau_k^{p_i})}(t). \tag{3.4}$$

Here, $\chi_{[\tau_{k-1}^{p_i}, \tau_k^{p_i})}$ denotes the indicator function of the interval $[\tau_{k-1}^{p_i}, \tau_k^{p_i})$ defined by

$$\chi_I(t) = \begin{cases} 1, & t \in I \\ 0, & \text{elsewhere.} \end{cases} \tag{3.5}$$

Let

$$\sigma^p = [(\sigma^{p_1})^{\mathrm{T}}, \ldots, (\sigma^{p_N})^{\mathrm{T}}]^{\mathrm{T}} \tag{3.6}$$

where

$$\sigma^{p_i} = [(\sigma^{p_i,1})^{\mathrm{T}}, \ldots, (\sigma^{p_i,n_{p_i}})^{\mathrm{T}}]^{\mathrm{T}}$$
$$\sigma^{p_i,k} = [\sigma_1^{p_i,k}, \ldots, \sigma_r^{p_i,k}]^{\mathrm{T}},$$

$\sigma^p = \left[\begin{matrix} [\sigma^{p:1} \cdots] \\ [\quad] \\ \vdots \\ [\quad] \end{matrix} \right]$

and $\sigma^{p_i,k} \in \mathbb{R}^r$, $k = 1, 2, \ldots, n_{p_i}$. Furthermore, let

$$\tau^p = [(\tau^{p_1})^{\mathrm{T}}, (\tau^{p_2})^{\mathrm{T}}, \ldots, (\tau^{p_N})^{\mathrm{T}}]^{\mathrm{T}} \tag{3.7}$$

where $\tau^{p_i} = [\tau_1^{p_i}, \tau_2^{p_i}, \ldots, \tau_{n_{p_i}}^{p_i}]^{\mathrm{T}}$, $i = 1, \ldots, N$, and $\tau_j^{p_i}$, $j = 1, 2, \ldots, n_{p_i}$, are decision variables such that the following constraints are satisfied:

$$t_{i-1} = \tau_0^{p_i} \le \tau_1^{p_i} \le \cdots \le \tau_{n_{p_i}}^{p_i} = t_i. \tag{3.8}$$

Let Γ^p be the set of all vectors τ^p defined by (3.7) such that (3.8) is satisfied. Let $r\sum_{i=1}^{N} n_{p_i} = \kappa$ and let $\widetilde{\mathcal{V}}^p(\Gamma^p)$ be the set of all those $u^p(\cdot|\sigma^p, \tau^p)$ expressed by (3.4) with $\sigma^p \in \mathbb{R}^\kappa$ and $\tau^p \in \Gamma^p$.

For the control constraints (2.6) and (2.7), the corresponding constraints on the control parameter vectors are:

$$l_j(\sigma^{p_i,k}) = (E^j)^{\mathrm{T}} \sigma^{p_i,k} - b_j \le 0, \quad i = 1, \ldots, N; j = 1, \ldots, q; k = 1, \ldots, n_{p_i} \tag{3.9}$$

$$\alpha_j \le \sigma_j^{p_i,k} \le \beta_j, \quad j = 1, \ldots, r; k = 1, \ldots, n_{p_i}. \tag{3.10}$$

Let $\mathcal{B}^p$ be the set of all those control parameter vectors $\sigma^p \in \mathbb{R}^\kappa$ such that the constraints (3.10) are satisfied, and let $\widetilde{\mathcal{D}}^p(\Gamma^p)$ be the set of all those corresponding $u^p(\cdot|\sigma^p, \tau^p) \in \widetilde{\mathcal{V}}^p(\Gamma^p)$. Furthermore, let Ξ^p be the set of all those $\sigma^p \in \mathcal{B}^p$ such that the constraints (3.9) are also satisfied, and let $\widetilde{\mathcal{U}}^p(\Gamma^p)$ be the set of all those corresponding $u^p(\cdot|\sigma^p, \tau^p) \in \widetilde{\mathcal{D}}^p(\Gamma^p)$.

Restricting controls in $\widetilde{\mathcal{U}}^p(\Gamma^p)$, the system of differential equations (2.9) takes the form:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \tilde{f}(t, x(t), \sigma^p, \tau^p), \tag{3.11}$$

with the initial condition:

$$x(0) = x^0, \tag{3.12}$$

where

$$\tilde{f}(t, x(t), \sigma^p, \tau^p) = \sum_{i=1}^{N} f^i\left(t, x(t), \sum_{k=1}^{n_{p_i}} \sigma^{p_i,k} \chi_{[\tau_{k-1}^{p_i}, \tau_k^{p_i})}(t)\right) \chi_{[t_{i-1}, t_i)}(t). \tag{3.13}$$

Let $x(\cdot|\sigma^p, \tau^p)$ be the solution of the system (3.11) and (3.12) corresponding to the combined control parameter vector and the switching vector $(\sigma^p, \tau^p) \in \Xi^p \times \Gamma^p$.

We may now specify the approximate Problem 1.1a as follows:

Problem 1.1a. Subject to the dynamical system (3.11) and (3.12), find a combined control parameter vector and switching vector $(\sigma^p, \tau^p) \in \Xi^p \times \Gamma^p$ such that the cost functional

$$\tilde{J}(\sigma^p, \tau^p) = \Phi_0(x(t_1), x(t_2), \ldots, x(T)) + \int_0^{\mathrm{T}} \tilde{\mathcal{L}}_0(t, x(t), \sigma^p, \tau^p)\mathrm{d}t \tag{3.14}$$

is minimized, where

$$\tilde{\mathcal{L}}_0(t, x(t), \sigma^p, \tau^p) = \mathcal{L}_0\left(t, x(t), \sum_{i=1}^{N} \sum_{k=1}^{n_{p_i}} \sigma^{p_i,k} \chi_{[\tau_{k-1}^{p_i}, \tau_k^{p_i})}(t)\right). \tag{3.15}$$

Note that, for each p, the approximate Problem 1.1a is an optimal parameter selection problem involving multiple characteristic time points, which are decision variables to be chosen optimally. In this paper, we shall develop a new computational method for solving this unconventional optimal parameter selection problem as a mathematical programming problem. For this, we need the gradient formulae of the cost functional with respect to the control parameter vector and the switching vector. The gradient formulae of the cost functional with respect to the control parameter vector are easy to be obtained. In fact, it is also possible to derive the gradient formulae of the cost functional with respect to the switching vector by using an argument similar to that given for Theorem 3.1 of [10]. However, these gradient formulae are not effective for numerical calculation. This statement is valid even for problems without involving multiple characteristic time points. For details, see [9]. In this paper, we will employ the idea of the control parameterization enhancing transform (CPET) to map these variable switching time points into preassigned fixed knots.

We introduce the new time variable $s \in [0, N]$. We rescale $t \in [0, T]$ into $s \in [0, N]$:

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \upsilon^p(s), \tag{3.16}$$

with the initial condition

$$t(0) = 0, \tag{3.17}$$

where υ^p, with possible discontinuity points at $s = i - 1 + \frac{j}{n_{p_i}}$, $j = 1, \ldots, n_{p_i}$; $i = 1, \ldots, N$, is called an enhancing control and is given by

$$\upsilon^p(s) = \sum_{i=1}^{N} \sum_{j=1}^{n_{p_i}} \delta_j^{p_i} \chi_{\left[i-1+\frac{j-1}{n_{p_i}}, i-1+\frac{j}{n_{p_i}}\right)}(s). \tag{3.18}$$

Clearly,

$$t(s) = \int_0^s \upsilon^p(\tau)\mathrm{d}\tau = \sum_{l=1}^{i-1} \sum_{k=1}^{n_{p_l}} \frac{\delta_k^{p_l}}{n_{p_l}} + \sum_{k=1}^{j-1} \frac{\delta_k^{p_i}}{n_{p_i}} + \delta_j^{p_i} \left(s - i + 1 - \frac{j-1}{n_{p_i}} \right)$$

$$s \in \left[i - 1 + \frac{j-1}{n_{p_i}}, i - 1 + \frac{j}{n_{p_i}} \right) \tag{3.19}$$

and

$$t(N) = \sum_{i=1}^{N} \sum_{j=1}^{n_{p_i}} \frac{\delta_j^{p_i}}{n_{p_i}} = T. \tag{3.20}$$

Define

$$w^p(s) = u^p(t(s)). \tag{3.21}$$

Then

$$w^p(s) = \sum_{i=1}^{N} \sum_{k=1}^{n_{p_i}} \sigma^{p_i,k} \chi_{\left[i-1+\frac{j-1}{n_{p_i}}, i-1+\frac{j}{n_{p_i}}\right)}(s). \tag{3.22}$$

Let $\delta_j^{p_i}$, $j = 1, \ldots, n_{p_i}$; $i = 1, \ldots, N$, be referred to collectively as δ^p, and let Ω^p be the set of all such δ^p. Furthermore, let $\mathcal{A}^p$ be the set containing all the corresponding enhancing controls υ^p obtained by elements from Ω^p via (3.18). Let $\widehat{\mathcal{D}^p}$ (respectively, $\widehat{\mathcal{U}^p}$) be the set containing all piecewise constant controls given by (3.22) with $\sigma^p \in \mathcal{B}^p$ (respectively, $\sigma^p \in \Xi^p$).

Define

$$\widehat{\mathcal{L}}_0(s, \widehat{x}(s), \delta^p, \sigma^p) = \widetilde{\mathcal{L}}_0(t(s), x(t(s)), \sigma^p, \tau^p)\upsilon^p(s), \tag{3.23}$$

$$\widehat{x}(s) = \begin{bmatrix} x(s) \\ t(s) \end{bmatrix} \tag{3.24}$$

and

$$\widehat{f}(s, \widehat{x}(s), \sigma^p, \delta^p) = \begin{bmatrix} \upsilon^p(s)\tilde{f}(t(s), x(t(s)), \sigma^p, \tau^p) \\ \upsilon^p(s) \end{bmatrix}. \tag{3.25}$$

Now, applying the CPET to Problem 1.1a, we obtain the following equivalent problem:

Problem 1.2b. Subject to the system of differential equations

$$\frac{\mathrm{d}\widehat{x}(s)}{\mathrm{d}s} = \widehat{f}(s, \widehat{x}(s), \sigma^p, \delta^p), \tag{3.26}$$

with initial condition

$$\widehat{x}(0) = \widehat{x}^0, \tag{3.27}$$

where

$$\widehat{x}^0 = \begin{bmatrix} x^0 \\ 0 \end{bmatrix}, \tag{3.28}$$

find a combined vector $(\sigma^p, \delta^p) \in \Xi^p \times \Omega^p$ such that the cost functional

$$\widehat{J}(\sigma^p, \delta^p) = \Phi_0(\widehat{x}(1), \widehat{x}(2), \ldots, \widehat{x}(N)) + \int_0^N \widehat{\mathcal{L}}_0(s, \widehat{x}(s), \sigma^p, \delta^p)\mathrm{d}s \tag{3.29}$$

is minimized.

By similar arguments as those given in [11], we obtain the following two theorems on the convergence properties of the sequence of approximate optimal costs and optimal controls.

Theorem 3.1. *Suppose that u^* is an optimal control of Problem 1.1. Let $(\sigma^{p,*}, \delta^{p,*})$ be an optimal parameter vector of Problem 1.2b and let $(u^{p,*}, \tau^{p,*})$ be the corresponding combined optimal control and switching vector of Problem 1.1a such that*

$$u^{p,*}(t|\sigma^{p,*}, \tau^{p,*}) = \sum_{i=1}^{N}\sum_{k=1}^{n_{p_i}} \sigma^{p_i,k,*}\chi_{[\tau_{k-1}^{p_i,*}, \tau_k^{p_i,*})}(t), \tag{3.30}$$

where $\sigma^{p_i,*} = [(\sigma^{p_i,1,*})^{\mathrm{T}}, \ldots, (\sigma^{p_i,n_{p_i},*})^{\mathrm{T}}]^{\mathrm{T}}$, and $\tau^{p_i,*} = [\tau_1^{p_i,*}, \tau_2^{p_i,*}, \ldots, \tau_{n_{p_i}}^{p_i,*}]^{\mathrm{T}}$. If $\{u^{p,*}\}_{p=1}^{\infty}$ is a bounded sequence of controls in L_∞^r, then there exists a subsequence, denoted by the original sequence, such that

$$\lim_{n_{p_i}\to\infty} \widehat{J}(u^{p,*}) = J(u^*). \tag{3.31}$$

Theorem 3.2. *Let $(\sigma^{p,*}, \delta^{p,*})$ and $(u^{p,*}, \tau^{p,*})$ be defined as in Theorem 3.1. Suppose that*

$$\lim_{n_{p_i}\to\infty} u^{p,*}(t) = \widehat{u}(t), \quad \textit{a.e. in } [0, T]. \tag{3.32}$$

Then, $\widehat{u}$ is also an optimal control of Problem 1.1.

4. A computational procedure

The procedure for solving Problem 1.1 may be stated as follows.

For each $i = 1, \ldots, N$, let $p_i = 1, 2, \ldots$. Then, we use the control parametrization enhancing transform to obtain Problem 1.2b. Thus, Problem 1.1 can be solved by solving a sequence of approximate Problem 1.2b. Each of these approximate Problem 1.2b is an optimal parameter selection problem, which can be viewed as a nonlinear mathematical programming problem. This nonlinear mathematical programming problem in the control and switching parameter vectors can be solved by using any efficient optimization technique, such as the sequential quadratic programming routine (see, for example, [10]). For this, we need, for each $(\sigma^p, \delta^p) \in \Xi^p \times \Omega^p$, the information on the values of the cost functional $\widehat{J}(\sigma^p, \delta^p)$ and the constraint functionals $l_j(\sigma^{p_i,k})$, $j = 1, \ldots, q$, as well as their respective gradients with respect to each $(\sigma^p, \delta^p) \in \Xi^p \times \Omega^p$.

The gradient formulae of the cost functional with respect to $(\sigma^p, \delta^p) \in \Xi^p \times \Omega^p$ are given in the following theorem.

Theorem 4.1. *Consider Problem 1.2b. Then, it holds that*

$$\frac{\partial \widehat{J}(\sigma^p, \delta^p)}{\partial \sigma_l^{p_i,j}} = \int_{i-1+(j-1)/n_{p_i}}^{i-1+j/n_{p_i}} \frac{\partial \widehat{H}(s, \widehat{x}(s), \sigma^p, \delta^p, \lambda(s))}{\partial \sigma_l^{p_i,j}} \mathrm{d}s, \tag{4.1}$$

$$\frac{\partial \widehat{J}(\sigma^p, \delta^p)}{\partial \delta_l^{p_i,j}} = \int_{i-1+(j-1)/n_{p_i}}^{i-1+j/n_{p_i}} \frac{\partial \widehat{H}(s, \widehat{x}(s), \sigma^p, \delta^p, \lambda(s))}{\partial \delta_l^{p_i,j}} \mathrm{d}s, \tag{4.2}$$

where

$$\widehat{H}(s, \widehat{x}(s), \sigma^p, \delta^p, \lambda(s)) = \widehat{\mathcal{L}}_0(s, \widehat{x}(s), \sigma^p, \delta^p) + \lambda^{\mathrm{T}}(s) \widehat{f}(s, \widehat{x}(s), \sigma^p, \delta^p), \tag{4.3}$$

and

$$\lambda^{\mathrm{T}}(s|\sigma^p, \delta^p) = [\lambda_1(s|\sigma^p, \delta^p), \ldots, \lambda_n(s|\sigma^p, \delta^p), \lambda_{n+1}(s|\sigma^p, \delta^p)]^{\mathrm{T}} \tag{4.4}$$

is the solution of the co-state system

$$\frac{(\mathrm{d}\lambda(s))^{\mathrm{T}}}{\mathrm{d}s} = -\frac{\partial \widehat{H}(s, \widehat{x}(s), \sigma^p, \delta^p, \lambda(s))}{\partial \widehat{x}}, \tag{4.5}$$

with the boundary condition

$$\lambda^{\mathrm{T}}(k^+) - \lambda^{\mathrm{T}}(k^-) = -\frac{\partial \Phi_0(\widehat{x}(1), \widehat{x}(2), \ldots, \widehat{x}(N))}{\partial \widehat{x}(k)}, \quad k = 1, 2, \ldots, N-1, \tag{4.6}$$

$$\lambda^{\mathrm{T}}(N) = \frac{\partial \Phi_0(\widehat{x}(1), \widehat{x}(2), \ldots, \widehat{x}(N))}{\partial \widehat{x}(N)}. \tag{4.7}$$

Proof. The proof is similar to that given for Theorem 4.3. of Chapter 4 in [11]. □

For each $(\sigma^p, \delta^p) \in \Xi^p \times \Omega^p$, the values of the functions $l_j(\sigma^{p_i,k})$, $j = 1, \ldots, q$, as well as their respective gradients with respect to (σ^p, δ^p), are straightforward to calculate. In view of these and Theorem 4.1, Problem 1.2b can be solved as a mathematical programming problem. On this basis, we can obtain an approximate optimal control for Problem 1.1, as shown in the following algorithm.

Algorithm 1.

Step 1 Solve Problem 1.2b as a standard mathematical programming problem to give $(\sigma^{p,*}, \delta^{p,*})$. Then, we obtain the corresponding piecewise constant control $(w^{p,*}(s), \upsilon^{p,*}(s))$.

Step 2 If $\min n_{p_i} \geq M$, $i = 1, \ldots, N$, where M is a pre-specified positive constant, go to Step 3. Otherwise go to Step 1 with n_{p_i} increased to n_{p_i+1} for each $i = 1, \ldots, N$.

Step 3 Stop. Construct $(u^{p,*}, \tau^{p,*})$ from $(w^{p,*}, \upsilon^{p,*})$ such that

$$u^{p,*}(t) = \sum_{i=1}^{N} \sum_{k=1}^{n_{p_i}} \sigma^{p_i,k,*} \chi_{[\tau_{k-1}^{p_i,*}, \tau_k^{p_i,*})}(t). \tag{4.8}$$

The piecewise constant control $u^{p,*}$ obtained is an approximate optimal solution of Problem 1.1.

Remark 4.1. Note that each $(\sigma^p, \delta^p) \in \Xi^p \times \Omega^p$ defines uniquely a $(w^p, \upsilon^p) \in \widehat{\mathcal{D}^p} \times \mathcal{A}^p$ via (3.18) and (3.22), and vice versa. We further note that, for each $(w^p, \upsilon^p) \in \widehat{\mathcal{D}^p} \times \mathcal{A}^p$, there exists a unique (u^p, τ^p) defined in the original time horizon $[0, T]$ such that

$$u^p(t) = \sum_{i=1}^{N} \sum_{k=1}^{n_{p_i}} \sigma^{p_i,k} \chi_{[\tau_{k-1}^{p_i}, \tau_k^{p_i})}(t), \tag{4.9}$$

where $\sigma^{p_i,k} = [\sigma_1^{p_i,k}, \ldots, \sigma_r^{p_i,k}]^{\mathrm{T}}$ is the same as that appearing in (3.22) given for w^p, and $\tau^{p_i} = [\tau_1^{p_i}, \tau_2^{p_i}, \ldots, \tau_{n_{p_i}}^{p_i}]^{\mathrm{T}}$ is determined uniquely by δ^p via evaluating (3.19) at $s = i - 1 + j/n_{p_i}$, $j = 1, \ldots, n_{p_i}$, $i = 1, \ldots, N$.

5. Illustrative examples

We first consider the numerical example given in [6].

Example 5.1. Consider a switched system, which consists of two subsystems described as follows:

Subsystem 1:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \begin{bmatrix} 0.6 & 1.2 \\ -0.8 & 3.4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t). \tag{5.1}$$

Subsystem 2:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u(t). \tag{5.2}$$

Assume that $t_0 = 0$, $t_f = 2$ and that the system switches once at $t = t_1$, where $0 \le t_1 \le 2$, from subsystem 1 to subsystem 2. We want to find an optimal switching instant t_1 and an optimal input u such that the cost functional

$$J = (1/2)(x_1(2) - 4)^2 + (1/2)(x_2(2) - 2)^2 + (1/2)\int_0^2 \{(x_2(t) - 2)^2 + u^2(t)\}\mathrm{d}t \tag{5.3}$$

is minimized. Here, $x(0) = [0, 2]^{\mathrm{T}}$.

We apply the approach developed in Section 4 to this problem using 10 partitions for the control space in the time interval [0, 2], and the optimal switching instant obtained is $t_1 = 0.2165$. The corresponding optimal cost is 10.0911, which is almost the same as the one obtained in [6]. The corresponding control and state trajectory are shown in Fig. 1.

We now consider a multiple characteristic time points problem as follows:

Example 5.2. In this problem, the switched system, which consists of three subsystems, is defined on the planning horizon [0, 3].

Subsystem 1:

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = x_1(t) + u(t)\sin x_1(t) \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -x_2(t) + u(t)\cos x_2(t) \end{cases} \quad t \in [0, t_1). \tag{5.4}$$

Subsystem 2:

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = x_2(t) + u(t)\sin x_2(t) \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -x_1(t) + u(t)\cos x_1(t) \end{cases} \quad t \in [t_1, t_2). \tag{5.5}$$

Subsystem 3:

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -x_1(t) - u(t)\sin x_1(t) \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = x_2(t) + u(t)\cos x_2(t) \end{cases} \quad t \in [t_2, 3]. \tag{5.6}$$

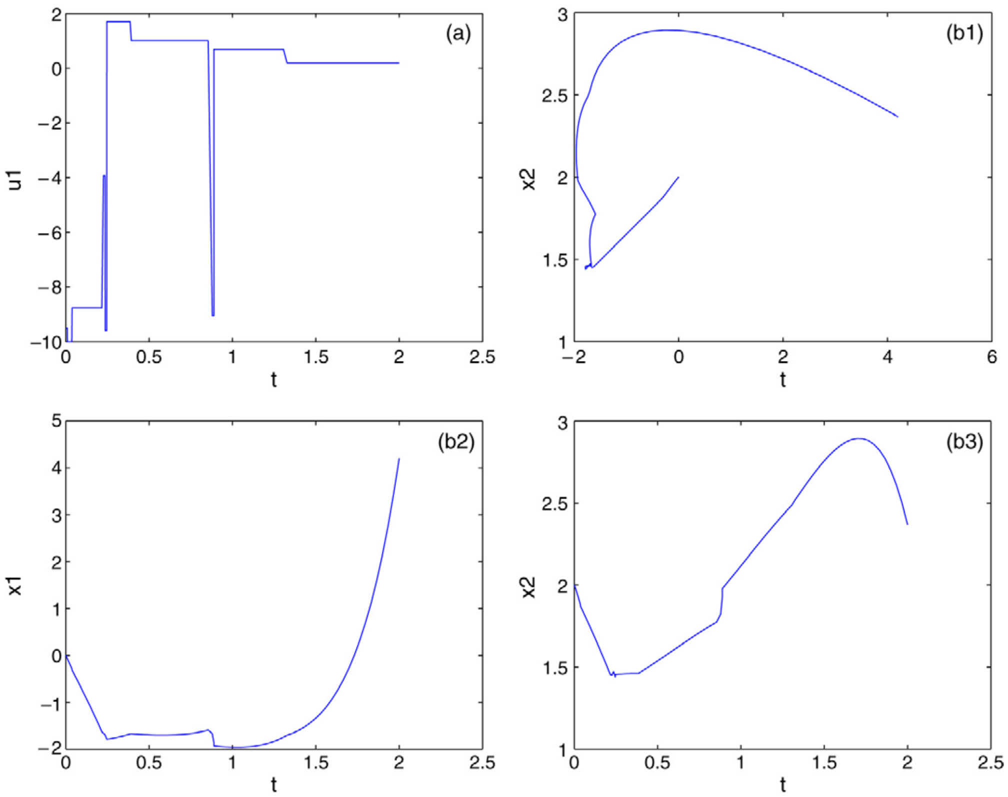

Fig. 1. (a) Computed optimal control input for Example 5.1. (b) Corresponding optimal state trajectory.

The initial conditions are:

$$x_1(0) = 2 \quad \text{and} \quad x_2(0) = 3. \tag{5.7}$$

Assume that the system switches at t_1 and t_2, where $0 \leq t_1 \leq t_2 \leq 3$. We want to find the optimal switching instants t_1, t_2 and an optimal control u such that the cost functional

$$J = 0.5(x_1(t_1) - 1)^2 + 0.5(x_2(3) + 1)^2 + 0.5 \int_0^3 \{(x_1(t) - 1)^2 + (x_2(t) + 1)^2 + u^2(t)\}\mathrm{d}t \tag{5.8}$$

is minimized.

We introduce the enhancing transform:

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \upsilon(s), \quad t(0) = 0 \tag{5.9}$$

and create a new state variable

$$x_3(s) = t(s). \tag{5.10}$$

Then, the variable switching times t_1, t_2 are mapped into the fixed time points 1, 2. The systems (5.4)–(5.6) are transformed into:

$$\begin{cases} \dfrac{\mathrm{d}x_1(s)}{\mathrm{d}s} = (x_1(s) + u(s)\sin x_1(s))\upsilon(s) \\ \dfrac{\mathrm{d}x_2(s)}{\mathrm{d}s} = (-x_2(s) - u(s)\cos x_2(s))\upsilon(s), \qquad s \in [0, 1) \\ \dfrac{\mathrm{d}x_3(s)}{\mathrm{d}s} = \upsilon(s) \end{cases} \tag{5.11}$$

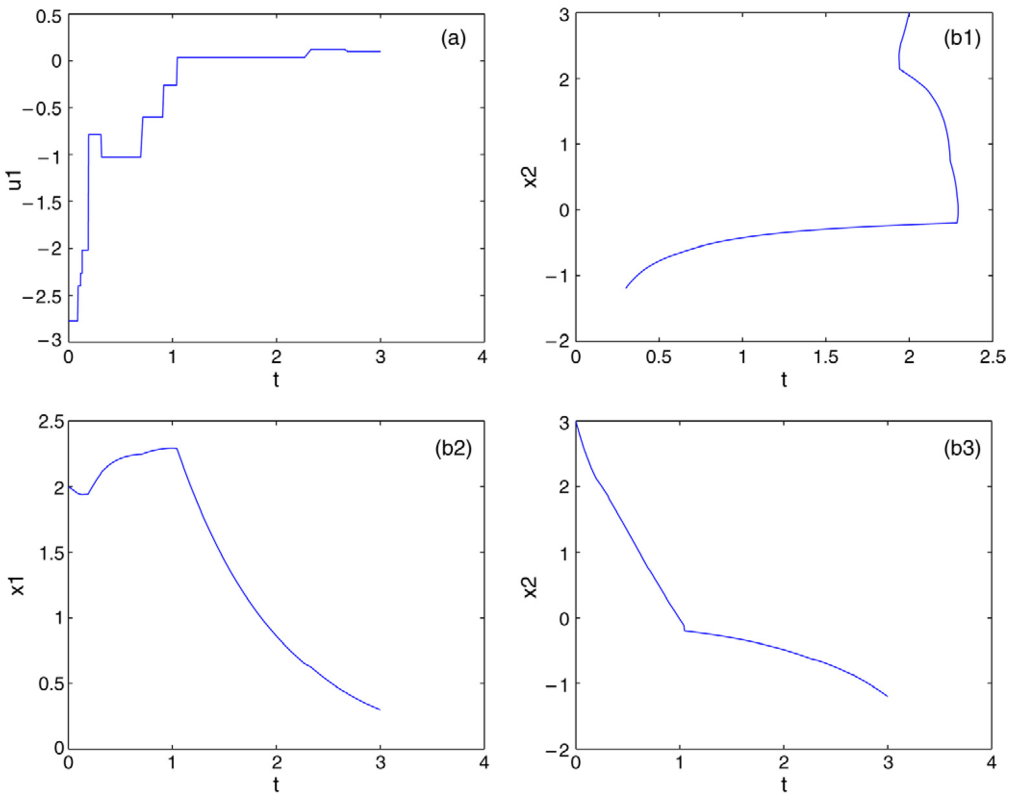

Fig. 2. (a) Computed optimal control input for Example 5.2. (b) Corresponding optimal state trajectory.

$$\begin{cases} \dfrac{\mathrm{d}x_1(s)}{\mathrm{d}s} = (x_2(s) + u(s)\sin x_2(s))\upsilon(s) \\ \dfrac{\mathrm{d}x_2(s)}{\mathrm{d}s} = (-x_1(s) - u(s)\cos x_1(s))\upsilon(s), \qquad s \in [1, 2) \\ \dfrac{\mathrm{d}x_3(s)}{\mathrm{d}s} = \upsilon(s) \end{cases} \tag{5.12}$$

$$\begin{cases} \dfrac{\mathrm{d}x_1(s)}{\mathrm{d}t} = (-x_1(s) - u(s)\sin x_1(s))\upsilon(s) \\ \dfrac{\mathrm{d}x_2(s)}{\mathrm{d}t} = (x_2(s) + u(s)\cos x_2(s))\upsilon(s), \qquad s \in [2, 3] \\ \dfrac{\mathrm{d}x_3(s)}{\mathrm{d}s} = \upsilon(s). \end{cases} \tag{5.13}$$

The initial conditions are:

$$x_1(0) = 2, \quad x_2(0) = 3, \quad \text{and} \quad x_3(0) = 0. \tag{5.14}$$

The cost functional (5.8) is converted to:

$$J = 0.5(x_1(1) - 1)^2 + 0.5(x_2(3) + 1)^2 + 0.5 \int_0^3 \{(x_1(s) - 1)^2 + (x_2(s) + 1)^2 + u^2(s)\}\upsilon(s)\mathrm{d}s. \tag{5.15}$$

Then, we can use the method developed in Section 4 to solve this equivalent problem. This time we use 12 partitions in the interval $[0, 3]$. The obtained optimal switching instants are $t_1 = 0.2567$ and $t_2 = 2.9600$, while the corresponding optimal cost is 6.51909. The corresponding control and state trajectory are shown in Fig. 2.

6. Conclusions

This paper considered a class of optimal control problems governed by a switched system. This switched system consists of a number of subsystems. The switching law, which specifies the instants at which changes of the subsystems are to take place, is to be chosen together with the control function optimally with respect to a given cost functional. This cost functional involves multiple characteristic time points. A computational method is developed for solving this class of optimal control problems. The method is based on the control parameterization technique, which is used in conjunction with the control parameterization enhancing transform. Two examples were solved using the proposed approach, and the results indicated that the proposed approach is highly efficient.

Acknowledgments

This work was supported by the Chinese Outstanding Youth Foundation under Grant No. 69925308, the Chinese Nature Science Foundation by grant No. 60474015, and a research grant from the Australian Research Council.

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